

Jena lectures on generalized global symmetries: principles and applications

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Here we will discuss recent developments in the field of *generalized global symmetries*. These notes are perhaps a bit idiosyncratic and thus use minimal mathematical formalism. I feel that they should be accessible after some basic QFT knowledge. I have inserted quite a few typos so that when you are bored you can find them and email me at nabil.iqbal@durham.ac.uk.

As of this moment the notes are also very much in progress, and this version is really intended only for students of the Jena TPI school.

Lectures will start with some disorganizedly motivational mumbling, which will only be there at the live show.

1 Ordinary global symmetries, phases of matter, and the Landau paradigm

1.1 Ordinary global symmetries in a quaint but hopefully reassuring language

These lectures will be about *new* sorts of global symmetries. So let's start by carefully understanding ordinary global symmetries, of the sort that you presumably already know and love.

To orient ourselves, let's consider a field theory with a $U(1)$ global symmetry. For much of what I will discuss, you can consider any field theory you like, but for concreteness let us consider a charged scalar field:

$$S[\phi] = \int d^d x (\partial\phi^\dagger \partial\phi + V(\phi^\dagger \phi)) \quad (1)$$

We will assume that the potential has at least a mass term, and possibly some self interactions:

$$V(\phi^\dagger \phi) = m^2 \phi^\dagger \phi + \frac{\lambda}{4} (\phi^\dagger \phi)^2 + \dots \quad (2)$$

In almost all of these lectures we will work in Euclidean spacetime with d dimensions, where d will take various interesting values as we proceed.

This is a classical action, but we can make it into a quantum theory by throwing it into a path integral:

$$Z = \int [d\phi] \exp(-S[\phi, \phi^\dagger]) \quad (3)$$

Now we know very well that this theory has a $U(1)$ global symmetry. What does it **act on**? This is a question with an obvious answer: it acts on the fundamental degree of freedom, the ϕ field as

$$\phi(x) \rightarrow \phi'(x) = e^{i\alpha} \phi(x) \quad (4)$$

where $e^{i\alpha} \in U(1)$ is a space-time independent phase, and clearly the action is invariant. $\phi(x)$ is a local operator. Now through the usual Noether procedure we can construct a current which has *one index* for this symmetry:

$$j^\mu(x) = i(\phi^\dagger \partial^\mu \phi - \phi \partial^\mu \phi^\dagger) \quad (5)$$

The current is conserved on-shell. Classically, this just means that if ϕ satisfies the equations of motion we have

$$\partial_\mu j^\mu = 0 . \quad (6)$$

This classical conservation equation is somewhat modified in the quantum theory. If we run through Noether's theorem in the path integral (this is textbook material) then we find an expression of the form:

$$\langle \partial_\mu j^\mu(x) \phi(y) \dots \rangle = i\delta^{(d)}(x-y) \langle \phi(y) \dots \rangle \quad (7)$$

In other words: the current is conserved *up to the insertion of charged operators*. This is the Ward identity, and one should view this expression as telling us what it means to be charged operator.

Finally, global symmetries allow us to define a conserved charge. Normally we do this in Lorentzian signature in the following way:

$$Q(t) = \int_{\mathbb{R}^{d-1}} d^3x j^0(t, \vec{x}) \quad (8)$$

where we do the integral over a time-slice \mathbb{R}^{d-1} . Let's just verify why this is conserved:

$$\frac{d}{dt} Q(t) = \int_{\mathbb{R}^{d-1}} d^3x \partial_t j^0(t, \vec{x}) = - \int_{\mathbb{R}^{d-1}} d^3x \partial_i j^i(t, \vec{x}) = \text{boundary term} = 0 \quad (9)$$

I would like to draw a picture of this equation in Figure 1: imagine you have a bunch of particles sitting around. They trace out worldlines in space-time. The world-lines can't end, because the particle number is conserved. This integral over the time-slice intersects all of the world lines – you can move it up and down in time and get the same answer, which is just the count of the particle number. A key point here is that **ordinary symmetries are associated with counting particles**.

Note also that the local field $\phi(x)$ is the thing that creates a particle or anti-particle. Another point is **the charged object creates the thing that the charge counts**.

One more piece of formalism is that it is often useful to couple in an external gauge field a_μ to the current j_μ . Here this coupling is implemented by demanding that the system is invariant under the combined transformation:

$$a \rightarrow a + d\Lambda \quad \phi \rightarrow \phi e^{i\Lambda} \quad (10)$$

where now $\Lambda(x)$ is a function of spacetime. To do this, we have to replace the normal derivative $\partial_\mu \phi$ with the gauge-covariant one $D_\mu \phi = \partial_\mu \phi - ia_\mu \phi$, and we can consider the action:

$$S[\phi; a] = \int d^d x ((D\phi)^\dagger D\phi + V(\phi^\dagger \phi)) \quad Z[a] = \int [d\phi] \exp(-S[\phi; a]) \quad (11)$$

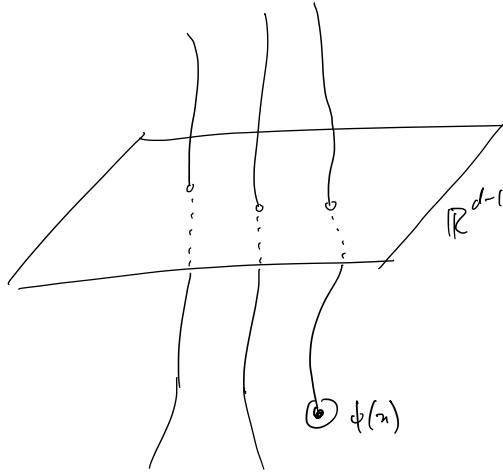


Figure 1: Particle world-lines can't end: no matter at what time-slice we evaluate $Q(t)$ on we get the same number.

Note that a is a fixed external field, and the invariances above mean that the path integral is invariant under external gauge transformations of a :

$$Z[a + d\Lambda] = Z[a] \quad (12)$$

Furthermore, we can compute the current by constructing $j^\mu(x) = \frac{\delta Z[a]}{\delta a_\mu(x)}$.

Now, what are global symmetries **good for**?

1.2 The Old Landau Paradigm

One very important thing is that basically global symmetries are how we classify the **phases of matter**. Note that the theory described above has two different phases, which I will now describe in turn.

1. **The Unbroken Phase.** Here we have $m^2 > 0$, and we have a unique vacuum in which the field ϕ is basically zero, i.e. $\langle \phi(x) \rangle = 0$. Because this vacuum is invariant under the $U(1)$ symmetry, we say that the $U(1)$ symmetry is **unbroken**.

In this phase we can imagine doing a calculation of the following two-point function:

$$\langle \phi^\dagger(x) \phi(0) \rangle \sim \exp(-m|x|) \quad (13)$$

If we turn off the interactions, this is again a standard computation in elementary field theory. Physically, the reason that this decays exponentially in space is that $\phi(0)$ creates a particle at the origin which must then survive until it is annihilated at $\phi^\dagger(x)$. This costs action because the particle is heavy, and thus the exponential decay.

2. **The Spontaneously Broken Phase.** Here we have $m^2 < 0$, and the potential (2) has a family of minima at $\phi^\dagger \phi = \frac{\lambda}{2m^2} \equiv v^2$. Thus the vacuum will have a nonzero value of $\langle \phi \rangle = v$. In this phase we say that the particles have condensed, and the $U(1)$ symmetry is **spontaneously broken**.

We can parametrize the interesting degree of freedom here by writing

$$\phi(x) = v(x)e^{i\theta(x)} \quad (14)$$

describing the modulus and phase of the field respectively. What can we say about the dynamics in the broken phase? Now plugging this into (1) we see that the field $v(x)$ will pick up a mass – so let me ignore it for now – but the field $\theta(x)$ is massless, so the leading low-energy action is

$$S = \int d^d x v^2 (\partial \theta)^2 + \dots \quad (15)$$

θ is the Goldstone mode of the symmetry. Its masslessness is much stronger than the classical argument we just gave: Goldstone's theorem states non-perturbatively that if there is a spontaneously broken symmetry (as defined by $\langle \phi \rangle \neq 0$) then there is necessarily a gapless mode in the spectrum. We will prove this again shortly as a warm-up for fancier things.

Let me describe another way to describe the broken symmetry. Consider computing

$$\lim_{|x| \rightarrow \infty} \langle \phi^\dagger(x) \phi(0) \rangle \sim \langle \phi^\dagger(x) \rangle \langle \phi(0) \rangle = v^2 \quad (16)$$

Here the second equality follows from cluster decomposition, i.e. the fact that things far away should be independent. Thus an equivalent way to state that the symmetry is spontaneously broken is to note that the two-point function of the charged field **saturates** and factorizes, becoming independent of the separation, as contrasted with the exponential decay shown in (13). In this form this is sometimes called off-diagonal long-range order.

We say that ϕ is the **order parameter** for the transition, because it behaves so differently in the two phases.

Now, basically when you have a symmetry, these are the two **phases** that you can have: **unbroken** and **spontaneously broken**. You can also ask what happens right between the two: there you have a **phase transition**.

3. **Phase Transition:** in the model above, the boundary between the two phases is at $m^2 = 0$. It turns out that at such a continuous transition we then have a **conformal field theory** which is characterized by the symmetry group in question (in this case $U(1)$) and the number of dimensions. It seems reasonable to describe the transition by the action we wrote down earlier, but setting $m^2 = 0$:

$$S = \int d^d x \left(\partial \phi^\dagger \partial \phi + \frac{\lambda}{4} (\phi^\dagger \phi)^2 + \dots \right) \quad (17)$$

If we now compute the correlation function of the charged operator we will find something which is *in between* the two behaviors (i.e. constant and exponential) shown above: it is a power law.

$$\langle \phi^\dagger(x) \phi(0) \rangle \sim \frac{1}{|x|^{2\Delta}}, \quad (18)$$

where Δ is a number that is interesting to calculate – it is called an *operator dimension* in CFT or a *critical exponent* in statistical physics. As it turns out, for $d < 4$ this number depends on the interactions in an important way.

Note that from this point of view, the reason why a CFT exists is not to give us hep-th-ers endless theories to write papers about, but to **describe** the phase transition between two distinct phases.

Now the point of all of this is that it gives us a way to describe phases of matter. A phase of matter is a classification of the **long-distance physics** (as captured e.g. by the correlation functions (13) and (16)), and in many cases it is **determined completely by how the symmetries are realized**.

Let me present some examples:

1. **$U(1)$ superfluid:** if we imagine putting a bunch of atoms in a box, then they have a conserved particle number enforced by a $U(1)$ symmetry much like the one we just described¹. Here if the symmetry is *unbroken*, then we call it the normal phase. On the other hand, if you cool the system down enough (see e.g. [1] for a discussion of the experimental situation), then the effective potential for the “field that creates an atom” is spontaneously broken, and now this is called a *superfluid*.
2. **Magnetization:** the simplest way to think about this is to imagine a bunch of spins \vec{s} that are fixed on the sites of a lattice. (I will write down some effective Hamiltonians for this later). There can be a non-Abelian rotational² symmetry that acts on the spins as $\vec{s} \rightarrow R\vec{s}$. We often consider very simplified settings – e.g. you could imagine a simpler symmetry group \mathbb{Z}_2 that simply acts on a scalar spin s as $s \rightarrow \pm s$, which defines what is called the *Ising model*. If this rotational symmetry is unbroken, then the system is called a **paramagnet**, and if it spontaneously broken – i.e. if all the spins are lined up – we call it a **ferromagnet**.
3. **Liquids and solids:** imagine a bunch of atoms moving freely; sometimes they can stop moving and form a crystal, in which case *translational symmetry* is spontaneously broken down to some discrete subgroup (e.g. for a square lattice it would be \mathbb{Z}^d). A **solid** is thus a phase where translational symmetry is spontaneously broken, and a **liquid** is a phase where translational symmetry is *unbroken*. What about the difference between liquid and gas? These have the same symmetries, and thus do not *actually* define different phases – this means that we can connect them continuously in a phase diagram, as is the case e.g. for water.

A further point here is that the transitions between phases can be continuous or discontinuous (i.e. first order). If they are first order there is not much more we can say; but if they are continuous then we can write down an effective theory involving a coarse-grained order parameter to describe the transition basically as we did in (17).

These ideas – that the phases of matter are classified by patterns of spontaneously broken and unbroken symmetry, and that the transitions between them can be described by the symmetries alone – are called the **Landau paradigm** of the phases of matter. It is extremely successful and is one of the cornerstones of our understanding of nature.

It also seems that it doesn’t always work.

1.3 Physics beyond the Landau paradigm

Here is a biased selection of phases that are to some extent outside the simplest Landau paradigm:

1. **Gauge theory:** it seems that gauge theory comes in different phases. For $U(1)$ gauge theory in the continuum we have the Coulomb and Higgs phases, which seem different. What is the order parameter? There is no local order parameter for this, and similarly for non-Abelian gauge theory and confinement.
2. **Topological order:** sometimes there are phases of matter which interact in an interesting manner with the topology of the space where they are defined – e.g. if you put a bunch of electrons in a strong magnetic field, you get the fractional quantum Hall effect, which basically says that the electron somehow splits into three smaller objects with fractional charge, where these objects do odd things if you move them around each other. The ground state degeneracy of this thing is different if you put it on a torus or a sphere.

¹The effective theory is somewhat different because the system is non-relativistic, and the background value of the charge density $\langle j^0 \rangle$ is nonzero.

²You may be worried that in real life this should be a spacetime symmetry and not an internal symmetry. This is correct and manifests in spin-orbit couplings which can couple the spin to the orbital angular momentum of atoms. Often this can be neglected, or we work on explicit lattice systems where it is absent from the construction.

3. **Gravity:** gravity is a thing. It does not fit into the Landau paradigm as far as I can tell.
4. **Fermi (and non-Fermi) liquids**
5. **Deconfined quantum criticality:** [2].

The goal of these lectures is to generalize our understanding of global symmetries in two directions: **higher-form** global symmetries, and **non-invertible** global symmetries. We will see that once we do this, we will actually be able to significantly extend the Landau paradigm in a way that encompasses many of the things above.

1.4 Symmetries as topological operators

To do this, we need to obtain a refined understanding of symmetries. Note that the description above made heavy use of the fundamental field $\phi(x)$: but from a modern point of view, in an age of duality where the notion of “fundamental” has little meaning, this seems quaint. We would like to seek a notion of global symmetry that does not require any particular choice of fundamental fields, and works in a duality-invariant way.

So, take a step back: if we have a symmetry, we have a conserved current:

$$\partial_\mu j^\mu = 0 \quad d \star j = 0 \quad (19)$$

Here I will start using a differential forms notation. Now let's consider the following operation: consider a codimension-1 manifold \mathcal{M}_{d-1} and form the following charge integral:

$$Q(\mathcal{M}_{d-1}) = \int_{\mathcal{M}_{d-1}} d^{d-1}x \sqrt{h} n_\mu j^\mu = \int_{\mathcal{M}_{d-1}} \star j \quad (20)$$

This appears to be a function of the manifold \mathcal{M}_{d-1} – however because the current j^μ is conserved, it is actually independent under small wiggles of \mathcal{M}_{d-1} : it is a **topological defect** in the theory. The notion of topological defect will appear multiple times, so here it is good to make an informal definition:

A topological defect on some submanifold \mathcal{M} is something that does not change under small deformations of \mathcal{M} .

Note that it is exactly the same sort of integral that we do when constructing a usual conserved charge, except there we take \mathcal{M}_{d-1} to be a possibly wiggly manifold rather than a flat time slice \mathbb{R}^{d-1} .

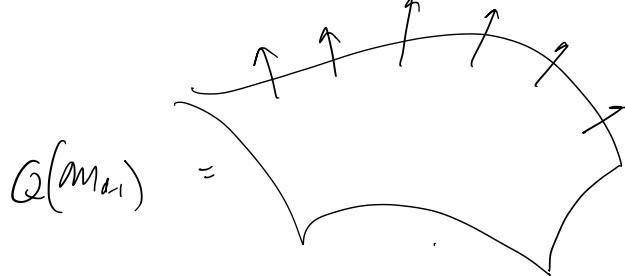


Figure 2: Integrating divergenceless current $\star j$ on a codimension-1 manifold \mathcal{M}_{d-1} .

Thus whenever you have a normal symmetry, you get a **codimension-1 topological defect**. This simple idea now supports many generalizations, which we will now explore.

What happens when you consider the interaction of this charged defect with a charged operator $\mathcal{O}(x)$? In the presence of a charged operator, we know from the Ward identity that the current is not conserved, i.e. we have

$$\partial_\mu j^\mu(x)\mathcal{O}(y) = i\delta^{(d)}(x-y)\mathcal{O}(y) \quad (21)$$

(where it is understood that this expression holds in the path integral). Thus if we try to drag the topological defect \mathcal{M}_{d-1} through the insertion of a charged local operator, it is not topological anymore; instead the operator picks up a phase, i.e. as in Figure 3

$$(Q(\mathcal{M}_{d-1}^1) - Q(\mathcal{M}_{d-1}^2))\mathcal{O}(x) = Q(\mathcal{M}'_{d-1})\mathcal{O}(x) = \oint_{\mathcal{M}'_{d-1}} \star j \mathcal{O}(x) = \int_{B_d} (d\star j)\mathcal{O}(x) = i \int_{B_d} \delta(x-y)\mathcal{O}(x) = i\mathcal{O}(x) \quad (22)$$

There's another way to say this. If we formulate the charge operator on an S^{d-1} that wraps the operator

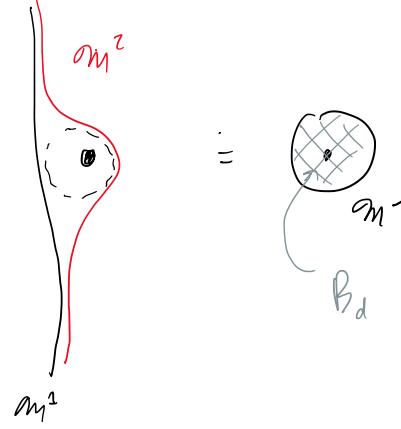


Figure 3: Deforming \mathcal{M}_1 to \mathcal{M}_2 receives a contribution only from the insertion of the local operator $\mathcal{O}(x)$.

and collapse it onto the operator as a point, we find:

$$Q(S^{d-1})\mathcal{O}(x) = i\mathcal{O}(x) \quad (23)$$

It is also convenient to consider the topological defect that constructs a *finite* charge rotation by an angle α . As expected this is the exponential of the infinitesimal charge rotation, so we have:

$$U_\alpha(\mathcal{M}_{d-1}) \equiv \exp(i\alpha Q(\mathcal{M}_{d-1})) \quad U_\alpha(S^{d-1})\mathcal{O}(x) = e^{i\alpha}\mathcal{O}(x) \quad (24)$$

i.e. when we collapse the charge generator onto the local operator it acts as a linear phase rotation on the field.

Note also that if you imagine *composing* two of these charge operators by forming them on parallel manifolds and fusing them together we find:

$$U_\alpha(\mathcal{M}_{d-1})U_\beta(\mathcal{M}_{d-1}) = U_{\alpha+\beta}(\mathcal{M}_{d-1}) \quad (25)$$

Here the last expression $\alpha + \beta$ should be understood in the spirit of group composition of the two $U(1)$ group elements $e^{i\alpha}$ and $e^{i\beta}$.

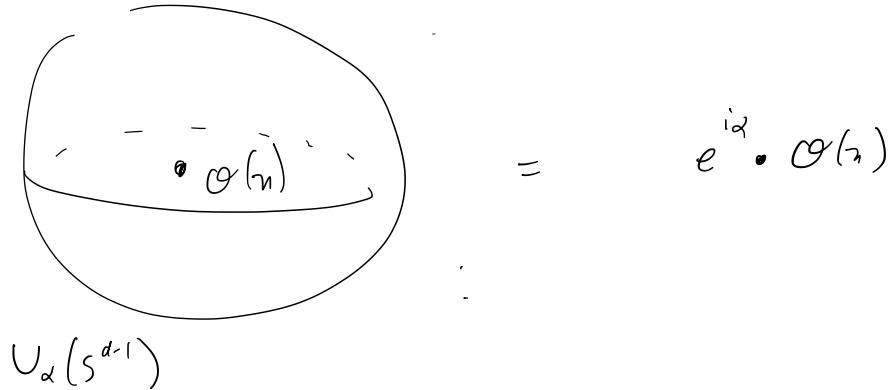


Figure 4: Collapsing the charge generator $U_\alpha(S^{d-1})$ onto $\mathcal{O}(x)$ acts as a phase rotation.

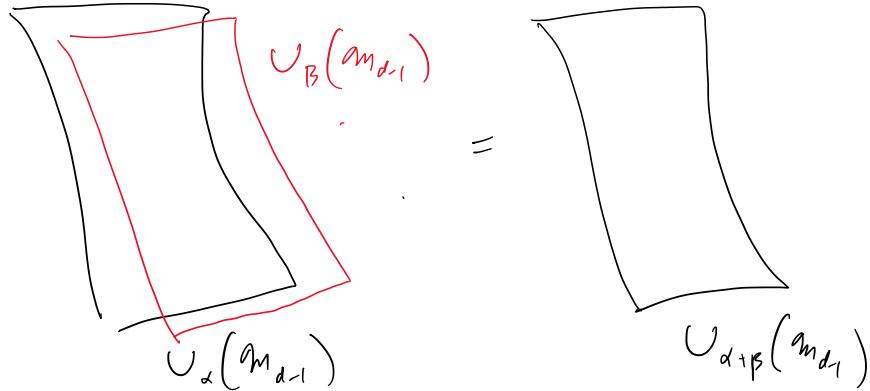


Figure 5: Fusing together two charge defects results U_α and U_β results in a combined charge defect $U_{\alpha+\beta}$.

We may now generalize this somewhat to an arbitrary non-Abelian global symmetry. Consider a symmetry group G , and imagine that we have fields \mathcal{O}_a that transform in some representation r under G , so that for every $g \in G$ there are matrices $V_b^a(g)$ that provide a linear action of the group, so

$$V_b^a(g)V_c^b(g') = V_c^a(gg') \quad (26)$$

The analogue of the above is now that for every element g of the symmetry group G we have topological defect operators $U_g(\mathcal{M}_{d-1})$. They can be composed in the obvious way as follows:

$$U_g(\mathcal{M}_{d-1})U_{g'}(\mathcal{M}_{d-1}) = U_{gg'}(\mathcal{M}_{d-1}) \quad (27)$$

To understand the action on local operators $\mathcal{O}_a(x)$, we imagine forming a defect on an S^{d-1} and collapsing it around the operator as in (24):

$$U_g(S^{d-1})\mathcal{O}_a(x) = V_b^a(g)\mathcal{O}_a(x) \quad (28)$$

I stress that here the V 's are just linear finite-dimensional matrices, and this relation shows how the fancy quantum operator U acts as a finite dimensional rotation on the charged degrees of freedom. So in the $U(1)$

case e.g. we have that r is a one-dimensional representation and the V is just a phase $e^{i\alpha}$. For $SU(2)$ we might take r to be in the fundamental and then the V 's would be exponentials of the Pauli matrices, and so on.

This may seem unfamiliar, but it is simply a restatement of the regular setup of global symmetries.

Exercise

1. Prove, by cutting open the path integral on a time-slice, that (23) is equivalent to the familiar operator equation

$$[Q, \mathcal{O}(\vec{x})] = i\mathcal{O}(\vec{x}) \quad (29)$$

where here Q is defined in the conventional way as in (8).

2. Derive the corresponding operator relation for the finite non-Abelian transformation from (28).

Finally, we should note that essentially the same formalism works for *discrete* symmetries. If we have a theory with a *discrete* symmetry group G , then we do not have a conserved current for it, but we do have a set of topological operators that enact the action of the discrete symmetry as in (28). As an example, imagine a real scalar field theory invariant under the \mathbb{Z}_2 symmetry $\phi \rightarrow -\phi$; then there would exist a topological defect U which has the following action when collapsed on ϕ :

$$U_{\mathbb{Z}_2}(S^{d-1})\phi(x) = -\phi(x) \quad (30)$$

As there is only a single element in \mathbb{Z}_2 and it is its own inverse, we have the simple fusion rule:

$$U_{\mathbb{Z}_2}(\mathcal{M}_{d-1})U_{\mathbb{Z}_2}(\mathcal{M}_{d-1}) = 1 \quad (31)$$

The main point of all of this is just to note that symmetries give us topological operators, often called **charge defects**.

Because the charged objects here are 0-dimensional local operators, ordinary global symmetries are called **0-form symmetries**.

2 Higher form symmetries

2.1 Idea

Once we realize that global symmetries essentially define a *topological sector* of a theory, we can generalize these in many productive ways. We will first consider higher-form symmetries [3], which basically ask the question: if normal symmetries are associated with the conservation of particles, what symmetries are associated with the conservation of **extended objects**, such as strings or branes? This is not a question for esoteric and weird string theorists, as such objects exist in real life.

We will start with the continuous case. Consider an antisymmetric two-index current $J^{\mu\nu} = J^{[\mu\nu]}$ which is conserved:

$$\partial_\mu J^{\mu\nu} = 0 \quad d \star J = 0 \quad (32)$$

Just as ordinary 0-form global symmetries count particles, this object is associated with a $U(1)$ symmetry which counts *strings!* This is called a **1-form global symmetry**.

At some rudimentary level the extra index tells us which way the string points. Let's dig into this a little

bit; first, following (20) we can form the following object

$$Q(\mathcal{M}_{d-2}) = \int_{\mathcal{M}_{d-2}} \sqrt{h} n^{\mu\nu} J_{\mu\nu} = \int_{\mathcal{M}_{d-2}} \star J \quad (33)$$

which we integrate over a codimension-2 manifold. For the same reasons as above, this is a **topological defect**, which is now codimension-2. This is 2-dimensional in 4d. For intuition, imagine a sea of strings – i.e. 2-dimensional worldsheets without end – poking through the blackboard, as in Figure 6. To count all of them we don't need to integrate over the whole room, we just integrate over the blackboard – we can then move the surface \mathcal{M}_{d-2} upwards in time *or* forwards in space and get the same answer, because the strings have no end.

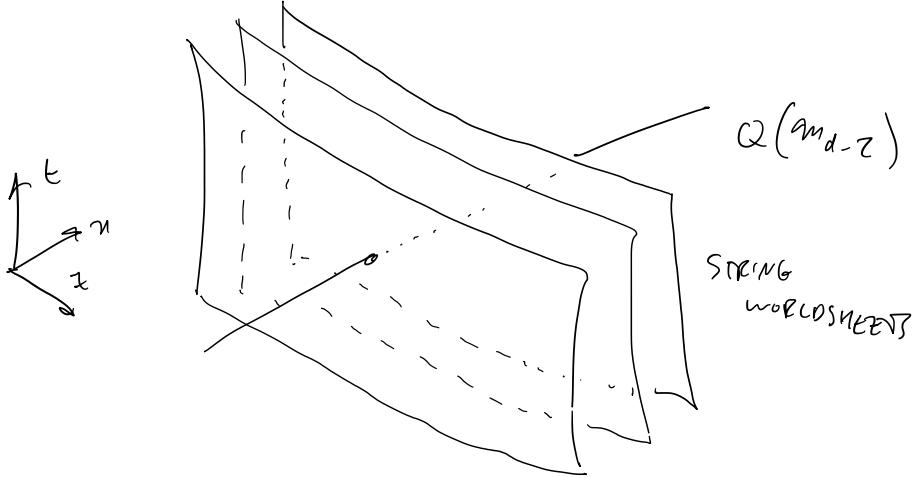


Figure 6: To count 2d string worldsheets, one does an integral over a codimension-2 surface, not a codimension-1 surface as for particle worldlines; compare this to Figure 1.

We have formulated ordinary symmetries in a fancy way above so that we can exploit the fact that much of the formalism carries over very simply. First, what is the charged object? Recall from before that a charged object is the “thing that creates the thing that you count”, in other words it should be the object who creates a string.

This is a **line operator** $W(C)$ defined on a 1d curve $C \subset \mathbb{R}^d$. If it has charge q then the Ward identity in this case now becomes:

$$\partial_\mu J^{\mu\nu}(x) W(C) = iq \int_C ds \frac{dX_C^\nu}{ds} \delta^{(d)}(x - X_C(s)) W(C) \quad (34)$$

which is a clunky way to write that the current is not conserved whenever we evaluate it on the worldline of the extended charged operator. Note that the extra index ν comes from the geometry of the line operator.

We can write this much more elegantly if we define the $(d-p)$ form-valued delta function $\delta_{C_p}(x)$ which depends on a p -submanifold C_p : it is zero whenever $x \notin C_p$ and “integrates to 1”, which in this case means that

$$\int_{\mathbb{R}^d} B_p \wedge \delta_{C_p}(x) = \int_{C_p} B_p \quad (35)$$

for all test forms B_p . The case above is for $p=1$, and we can write it as

$$d \star J(x) W(C) = iq \delta_C(x) W(C) \quad (36)$$

This is just notation to replace the integral in (36).

Now again we can construct the object who generates the finite rotation by an angle α :

$$U_\alpha(\mathcal{M}_{d-2}) = \exp(i\alpha Q(\mathcal{M}_{d-2})) \quad (37)$$

We should now ask what happens if we drag this object through a line operator $W(C)$. The idea is the same as in (22) – $W(C)$ will pick up a phase. This is most cleanly represented if we imagine placing the defect on a S^{d-2} that links the $W(C)$ and then collapsing it, as in Figure 7.

$$U_\alpha(S^{d-2})W(C) = e^{i\alpha} W(C) \quad (38)$$

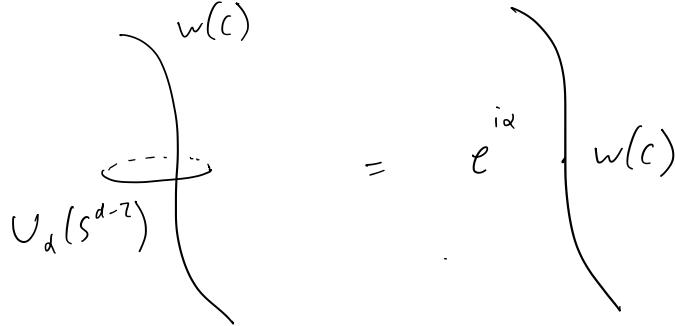


Figure 7: Collapsing the 1-form charge defect $U_\alpha(S^{d-2})$ onto a line operator acts as a phase rotation on the line.

Exercise: Prove (38) from (36).

This is the idea of a 1-form symmetry.

It can clearly be generalized to any p -form symmetry, which in the continuous case means that there exists a $p+1$ -form symmetry current $J^{\mu_1 \dots \mu_{p+1}}$, charge defects $Q(\mathcal{M}_{d-p-1})$, and charged surface operators that exist on submanifolds C_p of dimensionality p . A 0-form symmetry counts particles, a 1-form symmetry strings, a 2-form symmetry bedsheets-type objects, and so on.

It cannot however be generalized to the non-Abelian case: to understand this let's consider doing the group composition (27) for a p -form symmetry.

$$U_g(\mathcal{M}_{d-p-1})U_{g'}(\mathcal{M}_{d-p-1}) = U_{gg'}(\mathcal{M}_{d-p-1}) \quad (39)$$

The ordering on the right hand side of gg' was inherited from the ordering on the left hand side. For codimension 1 objects (i.e. for $p=0$) there is a well-defined notion of ordering – this is the time-ordering that is inherited from the path integral. However for higher codimension objects there isn't – you can always move one around the other. Thus the ordering is ambiguous and the right hand side of the fusion cannot depend on it, and thus g and g' necessarily commute.

We can however easily have *discrete* higher-form symmetries – e.g. \mathbb{Z}_N – which will indeed play an important role below. Just as for 0-form symmetries, we then have a charge operator $U_g(\mathcal{M}_{d-p-1})$ where $g \in \mathbb{Z}_N$.

2.2 Phases of 1-form symmetry

Let us now understand the phases of 1-form symmetries. We will do this as we did around (13) and (16), i.e. by understanding the behavior of the expectation value of the charged object $\langle W(C) \rangle$:

1. **Unbroken phase:** in the unbroken phase, we have that for large closed curves C there is the **area law**:

$$\langle W(C) \rangle \sim \exp(-T \text{Area}[C]) \quad (40)$$

where $\text{Area}[C]$ is the area of the minimal surface that fills in the curve C , i.e. the minimal surface S such that

$$\partial S = C \quad (41)$$

This is the higher-form analogue of the exponential decay shown in (13). The way to understand this is that the objects that we count – i.e. the string worldsheets – are massive with tension T ; thus the line operator creates a 2d worldsheet which must fill in the curve.

You will be familiar with the area law as a diagnostic for confinement in non-Abelian gauge theory; we will return to this below.

2. **Spontaneously broken phase:** here you should imagine that the strings have “condensed”, i.e. that the tension has dropped to zero. What does this mean? Your string theory textbook may have been alarmed by this possibility; however in this case it means that the behavior of the line operator has changed to a **perimeter law**.

$$\langle W(C) \rangle \sim \exp(-mL[C]) \quad (42)$$

where $L[C]$ is just the length of the curve (or more generally any *local* functional of the curve data). The idea here is that just as in (16) the correlation function has factorized as much as possible; however the line operator can always be redefined by a c-number which is a local functional of C , and thus this is as much factorization as possible. The parameter m is thus non-universal.

And, finally, we could imagine the transition between these phases, where we would have

3. **Phase transition:** if the transition is continuous, we would expect to see a conformal field theory describing it, and here we would have

$$\langle W(C) \rangle = \exp(-m'L[C]) f[C] \quad (43)$$

where $f[C]$ is expected to depend on C in a nontrivial manner, and might capture some interesting universal data determined by the transition. Unfortunately I am not aware of too many continuous transitions involving the spontaneous breaking of a 1-form symmetry; they turn out to usually be first-order³. An important exception is \mathbb{Z}_2 gauge theory in 3d, which we will discuss.

The analogy with 0-form symmetries should be clear. As we will see, it will also be **useful** – this will provide a useful classification of phases, and we can do all of the same sorts of things that we did for 0-form symmetries for their higher-form cousins.

It should be clear how the ideas above generalize to an arbitrary $p > 1$.

³For a possible explanation see [4].

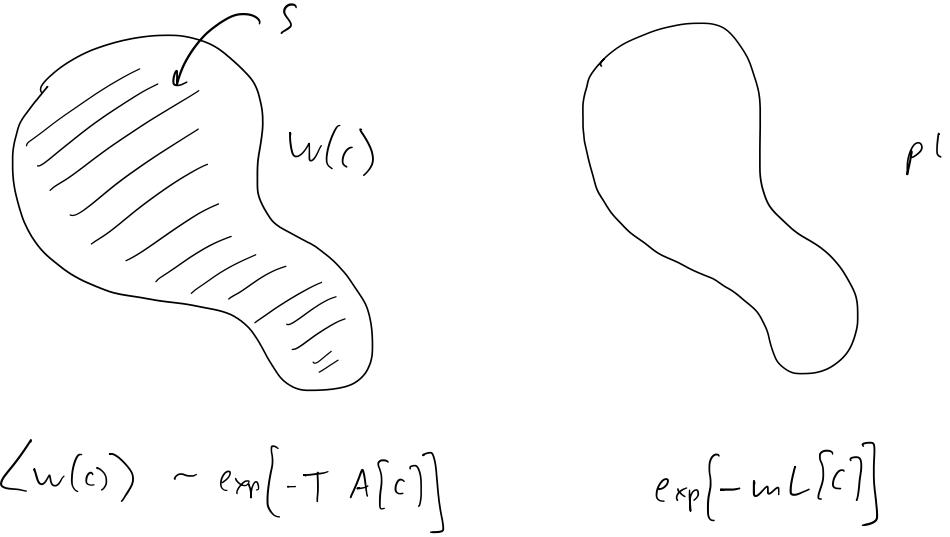


Figure 8: Area vs. perimeter laws in the spontaneous breaking of a 1-form symmetry.

2.3 Genuine and non-genuine operators

At this point we have introduced already a few types of extended objects. This is a good time to define some more terminology.

A **topological** operator (which can be either local or an extended defect) is an object which is invariant under small wiggles of its support. An example is the identity operator **1** and any of the extended symmetry defects defined above. (Indeed, by Schur's lemma, any **local** topological operator is necessarily proportional to the identity).

A **genuine** operator (which can be either local or extended) is an operator who does not have anything tied to it. Examples are most of the operators you know and love.

A **non-genuine** operator (which can be either local or extended) is an operator who **does** have something tied to it. Examples are

1. Fields that are charged under gauge symmetries – e.g. in QED with a charged scalar Φ necessarily has a Wilson line attached to it, so it is not a genuine local operator: we have to write it as $\Phi(x)e^{i\int_x A}$ where the Wilson line defines a curve C that ends on x . Note that in this example the Wilson line is usually not topological, which means that you are generally aware that it is there.
2. A more insidious example of this phenomenon is when the “something tied” to the operator is topological. In that case you might miss it and then be tempted to conflate a genuine operator with a non-genuine one, leading to confusion.
3. Both cases can happen for extended objects: e.g. you can have a 1d line operator with a 2d surface tied to it, where the surface might or might not be topological.

2.3.1 Non-genuine local operators: 2d compact boson

Let me give you an example that you may already know. This section requires you to be familiar with 2d CFT. Consider the compact boson CFT with action

$$\frac{1}{2g^2} \int d^2x (\partial\theta)^2 \quad (44)$$

where $\theta \sim \theta + 2\pi$. Let's formulate it on $S^1 \times \mathbb{R}$. In that case there is the well-known state operator correspondence that maps states on the cylinder to local operators on \mathbb{R}^2 . In particular, the vacuum on the cylinder maps onto the identity on \mathbb{R}^2 .

Now we are free to formulate boundary conditions as we like on the S^1 . It is a reasonable thing to demand that θ wind an integer number of times around the S^1 , i.e. that

$$\theta(\phi) = \theta(\phi + 2\pi) + 2\pi n \quad n \in \mathbb{Z} \quad (45)$$

The vacuum under this boundary condition defines a new state $|n\rangle$. We can then map the vacuum with this boundary condition back to the plane by the state-operator correspondence. This defines a family of local operators $V_n(x)$ labeled by an integer n . These are **genuine** operators.

One could have formulated it directly on the plane by demanding that in a path integral formulation of the theory, the integral of the 1-form $d\theta$ on a circle wrapping the point x be quantized as $2\pi n$, i.e.

$$\oint d\theta = 2\pi n \quad (46)$$

Note that in this formulation we have an operator $V_n(x)$ who is defined not as a polynomial of fundamental fields in the Lagrangian (which is how we normally define fields) but rather as a boundary condition on fields in the path integral. This is an example of a **disorder operator**, which is still **genuine**.

Now consider the following wrong-feeling operation: let's consider a boundary condition on the S^1 so that θ has a *non*-integer winding:

$$\theta(\phi) = \theta(\phi + 2\pi) + 2\pi\alpha \quad \alpha \notin \mathbb{Z} \quad (47)$$

We are free to do this; but what does it mean? Local physics on the cylinder proceeds mostly as normal; but if we map back to the plane the putative local operator that we try to define $\tilde{V}_\alpha(x)$ will have a branch cut emanating out of it. This branch cut is the location of the seam where the field θ jumps by $2\pi\alpha$. It is a topological line, as clearly we can move it around mostly as we like. Thus the operator dual to the non-integer winding vacuum has a tail – i.e. a topological line operator ending on it – and is a **non-genuine operator**.

Exercise: Show that the “tail” above is actually the 1d charge defect for the 0-form $U(1)$ global symmetry associated with θ translation. **Hint:** what is the action of the tail on the operator $e^{im\theta}$?

3 Applications

This has all been rather formal. It is time to look at a few examples and see what this formalism can do for us.

3.1 Free electromagnetism

Consider good old free $U(1)$ electromagnetism in 4d with no matter.

$$S[A] = \int d^4x \left(\frac{1}{4e^2} F^2 \right) \quad F = dA \quad (48)$$

The equation of motion is $\partial_\mu F^{\mu\nu} = 0$.

It will soon be important that we are talking about *compact* electromagnetism with gauge group $U(1)$ and not \mathbb{R} . This means that gauge transformations are labeled by elements $g = e^{i\Lambda(x)} \in U(1)$, with $A \rightarrow g^{-1}(A+d)g$.

What are the symmetries? Introductory textbooks would now talk about gauge symmetry. However we are a very sophisticated audience and we know that gauge “symmetry” is a terrible name – it’s not a symmetry at all, just a redundant way of describing the dynamics. So we can (a tad pretentiously) ask: what is the invariant physical information present in this theory? In other words, what are the **global** symmetries?

This theory has two 1-form symmetries: an **electric one** $U(1)_e^{(1)}$ and **magnetic one** $U(1)_m^{(1)}$ with the following currents:

$$J_e^{\mu\nu} = \frac{1}{e^2} F^{\mu\nu} \quad J_m^{\mu\nu} = \frac{1}{4\pi} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \quad (49)$$

or in forms

$$J_e = \frac{1}{e^2} F \quad J_m = \frac{1}{2\pi} \star F \quad (50)$$

The conservation of $J_e^{\mu\nu}$ follows from the equations of motion and is true only on-shell. This captures the idea that electric field lines are strings that don’t end (there is no electrically charged matter). The conservation of $J_m^{\mu\nu}$ follows from the Bianchi identity $\epsilon^{\mu\nu\rho\sigma} \partial_\nu F_{\rho\sigma} = 0$, and captures the idea that magnetic field lines are strings that don’t end (there are no magnetic monopoles).

What are the charged line operators?

The line operator charged under $U(1)_e^{(1)}$ is the Wilson line, i.e. the worldline of a **test** electric charge:

$$W_q(C) = \exp \left(iq \int_C A \right) \quad (51)$$

In the presence of the Wilson line we obtain the expression:

$$(d \star J_e(x)) W_q(C) = iq \delta_C(x) W_q(C) \quad (52)$$

These charges are integers $q \in \mathbb{Z}$.

The action of $U(1)_e^{(1)}$ on the dynamical degrees of freedom can be easily understood in this case – they correspond to the shift of A by a closed 1-form:

$$A \rightarrow A + \Lambda \quad d\Lambda = 0 \quad (53)$$

which leaves the Lagrangian invariant. Under this the Wilson line operator transforms as a phase

$$W_q(C) \rightarrow e^{iq \int_C \Lambda} W_q(C) . \quad (54)$$

How do we couple an external source $b_{\mu\nu}$ to this 1-form symmetry? In the electric case this takes the following form:

$$S[A; b_e] = \int d^4x \left(-\frac{1}{4e^2} (dA - b_e) \right) \quad (55)$$

and the 1-form invariance is now enhanced to:

$$A \rightarrow A + \Lambda \quad b_e \rightarrow b_e + d\Lambda \quad \Lambda \text{ arbitrary} \quad (56)$$

This is the 1-form analogue of the external coupling to the source that we did back in (10).

We should note that it is often the case that we cannot identify such a simple action on the degrees of freedom. For example, the line operator charged under $U(1)_m^{(1)}$ is called an '**t Hooft line**', which I will denote by $T_p(C)$ —it is the worldline of a *test* magnetic monopole. It is an example of a so-called *disorder operator*, and is defined by demanding that in the path integral over fields, every S^2 wrapping the curve C has:

$$\oint_{S^2(C)} F = 2\pi p \quad (57)$$

i.e. we have a magnetic monopole. Note that naively we have $F = dA$, which clearly means that the integral on the right-hand side of F must vanish; of course this means that A is not well-defined in the presence of a monopole, and must be defined patch-wise and sewn together on overlapping patches by gauge transformations. The consistency of this sewing together with the integer quantized electric charges in (51) enforces the Dirac quantization condition,

$$qp = \mathbb{Z} \quad (58)$$

which you will remind yourself of in the exercise below. (57) is equivalent to the following Ward identity for the magnetic current:

$$(d \star J_m(x))T_q(C) = iq\delta_C(x)T_q(C) \quad (59)$$

For the magnetic symmetry there is no analogue of (53).

Exercise:

1. Prove the Dirac quantization condition in compact $U(1)$ gauge theory (58).
2. Prove the Ward identities (52) and (59).

Exercise:

1. Couple an external two-form source b_m to Maxwell theory for the magnetic 1-form symmetry $U(1)_m^{(1)}$, just as we did for the electric 1-form symmetry in (55).
2. Use this to construct a partition function for Maxwell theory $Z[b_e, b_m]$ that is invariant under both 1-form shifts $b_e \rightarrow b_e + d\Lambda_e, b_m \rightarrow b_m + d\Lambda_m$.
3. Lol you can't. This is a manifestation of a *mixed 't Hooft anomaly* between the two 1-form symmetries. Develop the theory of this anomaly and explain to what extent each current is conserved.

Now that we have understood that we have two 1-form symmetries $U(1)_e^{(1)}$ and $U(1)_m^{(1)}$, we should understand how they are **realized**: i.e. are they spontaneously broken, or unbroken, etc? To do that we compute the expectation values $\langle W_q(C) \rangle$ and $\langle T_p(C) \rangle$.

The calculation of $\langle W_q(C) \rangle$ for a circular loop is something done in textbooks; heuristically speaking the Wilson line emits photons but they don't do too much, and the leading contribution comes from a short-distance singularity along the perimeter of the loop, so one finds:

$$\langle W_q(C) \rangle \sim \exp(-\Lambda L[C]) \quad (60)$$

with Λ a non-universal cutoff. This is a perimeter law. $\langle T_q(C) \rangle$ behaves the same way – one way to understand this is that the action of free electrodynamics is invariant under electric-magnetic duality, which interchanges the Wilson and 't Hooft lines. Thus in the normal phase of “free EM” we see that **both higher form symmetries are spontaneously broken**.

This is an important fact – the way to think about it that the vacuum is a sea of “condensed” strings, where the condensed objects are magnetic and electric field lines. Long-wavelength wiggles in this sea of condensed strings are what we call photons. Indeed this allows us to ask the question: **why is the 4d photon massless?**

When first learning QFT we often hear some words about why the photon is massless due to gauge invariance, which sounds pretty good until the next chapter where we learn about the Higgs mechanism, which writes down a perfectly gauge-invariant action that gives the photon a mass. The modern way of thinking about this is that the photon is the Goldstone mode of a **spontaneously broken** higher-form symmetry. We should thus prove Goldstone’s theorem for higher-form symmetries, which we will now do.

3.2 Formal interlude: Goldstone’s theorem

This section follows [5].

3.2.1 Goldstone’s theorem for 0-form symmetries

We will first present a path-integral proof for ordinary 0-form symmetries. Let us begin by considering the Ward identity for a charged operator $\mathcal{O}(x)$ at the origin.

$$\partial_\mu j^\mu(x)\mathcal{O}(0) = iq\delta^{(d)}(x)\mathcal{O}(0) \quad (61)$$

Let us now integrate x on both sides over a ball B^d which is the interior of a S^{d-1} of radius R centered at the origin, as in Figure 9.

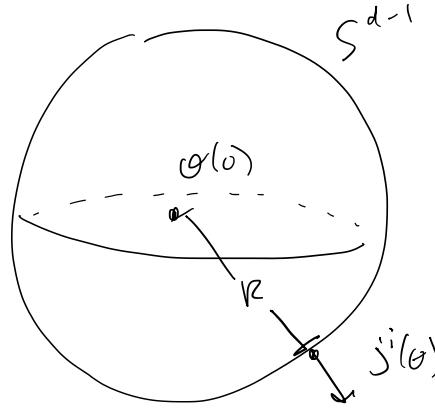


Figure 9: Geometry used for proof of 0-form Goldstone theorem.

We find using the divergence theorem that

$$\oint_{S^{d-1}(R)} d^{d-1}\theta n_i j^i(\theta)\mathcal{O}(0) = iq \int_{B^d(R)} d^d x \delta^{(d)}(x)\mathcal{O}(0) \quad (62)$$

where θ describes the angles parametrizing the S^{d-1} . Now the right hand side integral is trivial and picks up the delta function at the origin, giving us

$$\oint_{S^{d-1}(R)} d^{d-1}\theta n_i j^i(\theta) \mathcal{O}(0) = iq\mathcal{O}(0) \quad (63)$$

So far this has been true in any theory, independent of the phase. Now take the expectation value of both sides. We find

$$\left\langle \oint_{S^{d-1}(R)} d^{d-1}\theta n_i j^i(\theta) \mathcal{O}(0) \right\rangle = iq\langle \mathcal{O} \rangle . \quad (64)$$

There are now two possibilities:

1. The symmetry is **unbroken**, so $\langle \mathcal{O} \rangle = 0$. The left-hand side also vanishes. In this case we can say nothing else.
2. The symmetry is **spontaneously broken**, so $\langle \mathcal{O} \rangle \neq 0$. In this case the left-hand side is both nonzero and independent of the radius R . Thus by spherical symmetry we have that the correlation function between the operator at the origin and the current at a distance R from it satisfies:

$$\langle j^i(\theta) \mathcal{O}(0) \rangle \sim \frac{iq\langle \mathcal{O} \rangle n^i}{R^{d-1}} \quad (65)$$

(Amusingly, this is exactly how we derive the classic r^{-2} potential using Gauss's law in elementary EM, which is not a coincidence). Thus there is a **power-law in distance** correlation in the theory, and there must be at least one gapless mode to generate this power-law correlation. This gapless mode is the Goldstone mode.

This is the fastest proof of the ordinary Goldstone theorem, though to my knowledge it was first written down in [5]. It is different from the one in textbooks in that it is formulated in Euclidean path integral language rather than in a Hamiltonian formulation, but it is conceptually equivalent; if you cut open the path integral then you find the commutators that one normally uses.

3.2.2 Goldstone's theorem for higher form symmetries

We now present the proof for higher form symmetries. It is really exactly the same as above. We begin by considering the Ward identity for a p -form symmetry:

$$d \star J(x) W(C) = iq\delta_C(x) W(C) \quad (66)$$

Let us now assume that we are in a spontaneously broken phase, as that is the case of interest. In that case $W(C)$ obeys a perimeter law as in (42)

$$\langle W(C) \rangle \sim \exp(-m\text{Perimeter}[C]) \quad (67)$$

where here “perimeter” denotes the p -volume of the submanifold C . Note that as this is a local functional of the geometric data characterizing C , we can define a new charged operator $\bar{W}(C)$ that strips this off,

$$\bar{W}(C) \equiv \exp(+m\text{Perimeter}[C]) W(C) \quad (68)$$

$\bar{W}(C)$ satisfies the same Ward identity as $W(C)$, i.e.

$$d \star J(x) \bar{W}(C) = iq\delta_C(x) \bar{W}(C) \quad (69)$$

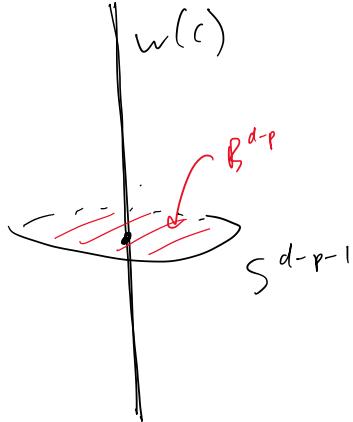


Figure 10: Geometry used for proof of p-form Goldstone theorem.

Now just as above, we take C to be an infinite flat p plane and consider a $(d - p)$ dimensional ball B^{d-p} that intersects C at a single point. This ball has a boundary S^{d-p-1} and the perpendicular distance from the intersection point to the S^{d-p-1} is R , as in Figure 10.

Now we integrate x on both sides of (69) over the B^{d-p} . On the left hand side we have

$$\int_{B^{d-p}} d \star J(x) \bar{W}(C) = \oint_{S^{d-p}(R)} \star J(\theta) \bar{W}(C) . \quad (70)$$

where θ is a point on the S^{d-p} . On the right hand side the delta function contributes from a single point, so we have

$$\oint_{S^{d-p}(R)} \star J(\theta) \bar{W}(C) = iq \bar{W}(C) \quad (71)$$

Finally we take the expectation value:

$$\langle \oint_{S^{d-p}(R)} \star J(\theta) \bar{W}(C) \rangle = iq \langle \bar{W}(C) \rangle \quad (72)$$

By construction $\langle \bar{W}(C) \rangle$ is nonzero, and thus the left hand side is both nonzero and independent of the radius R , which means that the correlation function must behave as

$$\langle \star J(\theta) \bar{W}(C) \rangle \sim \frac{iq}{R^{d-p-1}} \quad (73)$$

i.e. there is a power-law correlation between the current at a distance R from the surface operator. This power law correlation means that there is a single gapless mode in the spectrum, which is the Goldstone mode.

One can now ask what the low-energy theory is that describes this Goldstone mode, i.e. what is the higher-form analogue of (15)? It will not surprise you that this is given by a p -form gauge theory, i.e.

$$S = \int d^d x (v^2 (dB_p)^2) \quad (74)$$

where the extended charge operator is realized in terms of this degree of freedom by

$$W(C) = \exp \left(iq \int_C B_p \right) \quad (75)$$

In the case $p = 1$ this is exactly the structure of ordinary electromagnetism! If you run the argument with J_e then the low-lying Goldstone mode that you get is the ordinary photon $B_{p=1} = A$ and (75) is exactly the ordinary Wilson line. If you run the argument with J_m then the Goldstone mode turns out to be the magnetic photon $B_{p=1} = \tilde{A}$, which is related to the ordinary one by electric-magnetic duality; then (75) turns out to represent the 't Hooft line instead.

Exercise: Prove in the first non-trivial case $p = 1$ that the low-energy effective action (74) saturates the spontaneously broken Ward identity (73). Somewhat cheap extra credit question: do it for arbitrary p .

This is maybe the first case where we see that the structures from ordinary symmetries really do carry over to the higher-form case; we really obtain nontrivial dynamical information, i.e. from the behavior of a certain line operator we can now state that there exists a gapless mode in the spectrum.

3.3 Electrodynamics coupled to matter

Now that we understand that the free photon is the Goldstone mode of a spontaneously broken 1-form symmetry, we should think about what happens if we couple in electric matter. Let's start with a scalar of integer charge q , i.e. the action is now:

$$S[\phi, A] = \int d^4x \left(\frac{1}{4e^2} F^2 + (D\phi)^\dagger D\phi + V(\phi^\dagger \phi) \right) \quad (76)$$

with the usual gauge-covariant derivative $D_\mu \phi = \partial_\mu \phi - iqA_\mu \phi$ and where we take

$$V(\phi^\dagger \phi) = m^2 \phi^\dagger \phi + \frac{\lambda}{4} (\phi^\dagger \phi)^2 + \dots \quad (77)$$

Let us now ask what has happened to the symmetry currents we identified before.

3.3.1 Minimal charge

We begin by studying the case $q = 1$. The equations of motion now say that

$$\partial_\mu J_e^{\mu\nu} = iq (\phi^\dagger D^\mu \phi - \phi D^\mu \phi^\dagger) = j_{\text{el}}^\mu \quad (78)$$

i.e. the electric 1-form symmetry $U(1)_e^{(1)}$ has been **explicitly** broken as the corresponding current is no longer conserved. Physically, the point is that electric field lines can now end on free electric charges.

There is another way to see that $U(1)_e^{(1)}$ is now broken. The charged Wilson line operators can now end on scalar field insertions as $\phi(x) \exp(i \int_x^\infty ds A)$. The line operator then cannot have a nontrivial linking with any topological charge operator as we can just pull the charge operator off the end and take it away **picture**.

The magnetic 1-form symmetry $U(1)_m^{(1)}$, whose current is:

$$J_m^{\mu\nu} = \frac{1}{2} \epsilon^{\mu\nu\rho\sigma} F_{\rho\sigma} \quad (79)$$

remains, as there are no dynamical magnetic monopoles in this theory; of course it may still be spontaneously broken. To understand this, we should separate the two infrared phases of this gauge theory:

1. **Coulomb phase:** this is when the mass in the potential $m^2 > 0$, and so the scalar ϕ is uncondensed. In that case if we go to very long distances we can essentially ignore the ϕ field completely, as the

right hand side of (78) is energetically suppressed. We are then back in the situation of free EM. Note that the electric 1-form symmetry looks like it has come back – it is **emergent** at long distances, and then spontaneously broken. This sort of situation is quite rare for 0-form symmetries but is generic for 1-form symmetries. The magnetic 1-form symmetry is also spontaneously broken.

2. **Higgs phase:** here we imagine adjusting the potential by taking $m^2 < 0$ so that the gauge-charged scalar condenses: this is then the Higgs or superconducting phase.

Then it's convenient to write $\phi(x) = v(x)e^{i\theta(x)}$ and the low-energy physics is described by:

$$S = \int d^4x \left(\frac{1}{4e^2} F^2 + v_0^2(d\theta - A)^2 + (dv)^2 + m_H^2(v - v_0)^2 + \dots \right) \quad (80)$$

It is well-known that in this Higgs phase the spectrum is completely gapped. What has happened to our 1-form symmetries? The electric one is completely broken; in this phase you definitely cannot ignore the right hand side of (78).

To understand how the magnetic symmetry is realized, we should compute $\langle T(C) \rangle$ in this phase. We are thus asking: what happens if we place a probe magnetic monopole into a Higgs phase? In the Coulomb phase the magnetic field lines can spread out. In the Higgs phase, however, they cannot – the very point of the Higgs phase is that magnetic fields are expelled. If you force flux in nevertheless, the magnetic field lines are forced into a tight little bundle called the Abrikosov flux tube.

Let me briefly describe this configuration – consider a static field configuration and try to find a solution to the equations of motion of (80) that supports nonzero magnetic flux. It's helpful to go to cylindrical coordinates

$$ds^2 = d\tau^2 + dz^2 + dr^2 + r^2 d\phi^2 \quad (81)$$

and imagine that we are trying to put a nonzero magnetic field in $F_{r\theta}$. We then find the following configuration. The phase of the scalar field will wind once around:

$$\theta(r \rightarrow \infty) = \phi \quad (82)$$

This is terribly singular at the origin, and thus the scalar field will thus necessarily vanish there and interpolate back to its background value at infinity.

$$v(r \rightarrow 0) \approx 0 \quad v(r \rightarrow \infty) = v_0 \quad \theta(r \rightarrow \infty) = \phi \quad (83)$$

This interpolation will take place over a scale set by the Higgs mass m_H^{-1} . Finally, because the action contains a kinetic term like $(\partial_\phi \theta - A_\phi)^2$, at infinity $A_\phi(r)$ must track θ :

$$A_\phi(r \rightarrow \infty) = 1 \quad (84)$$

Thus the amount of magnetic flux stored in the tube is

$$\int_{\mathbb{R}^2} F = \oint_{S^1(\infty)} d\phi A_\phi = 2\pi \quad (85)$$

and there is thus a single unit of flux.

Note that this quantization can be traced back to the fact that the field θ is winding; by creating an integer winding $\theta = \mathbb{Z}\phi$ one can create configurations supporting an integer number of units of flux, but there is no way within this phase to create (e.g.) half a unit.

This configuration is a 2d worldsheet that fills all of (τ, z) and is localized at $r = 0$; it has an tension t_p per unit worldsheet area which can be calculated numerically and is nonzero.

Finally, let us return to our basic question: what happens if we insert a 't Hooft line into this theory? The 't Hooft line is a magnetic monopole, which is the endpoint of an Abrikosov vortex – in other

words it will create magnetic flux which is then forced into the 2d worldsheet described above, which will result in an *area law*. We thus have

$$\langle T(C) \rangle \sim \exp(-t_p \text{Area}[C]) \quad (86)$$

and the 1-form symmetry in the Higgs phase is **unbroken**.

This may all seem backwards – isn’t the gauge symmetry spontaneously broken in the Higgs phase, or something like that? The issue here is really that the previous statement – though illuminating in the limit of weak gauge coupling $e \rightarrow 0$ – does not really make sense nonperturbatively, because gauge symmetry is not a symmetry. (For example: what would the order parameter be for its breaking? There is nothing charged under the gauge symmetry, and thus the very idea of an order parameter does not make sense).

We can summarize the situation in the following table:

	$U(1)_e^{(1)}$	$U(1)_m^{(1)}$
Free electrodynamics	Spontaneously broken	Spontaneously broken
Massive electrically charged matter	Explicitly broken (emergent in IR)	Spontaneously broken
Higgs phase	Explicitly broken	Unbroken
Massive magnetically charged matter	Spontaneously broken	Explicitly broken (emergent in IR)
Confined phase	Unbroken	Explicitly broken

I have added some last lines: in 4d $U(1)$ electrodynamics we can imagine also coupling in dynamical magnetic monopoles. If they condense then we would end up with a *confined* phase, where it is the Wilson line who has an area law.

3.3.2 Non-minimal charge

Let us now imagine having a nontrivial minimal charge $q > 1 \in \mathbb{Z}$. In this case the electric 1-form symmetry is explicitly broken down to $\mathbb{Z}_q^{(1)}$.

3.4 Gauge theory: $SU(N)$

We now turn to non-Abelian gauge theory.

3.4.1 $SU(N)$ gauge theory in the continuum

We will begin by thinking about pure $SU(N)$ gauge theory. The continuum action depends only on the Lie algebra $\mathfrak{su}(N)$

$$S[A] = \frac{1}{2g^2} \int d^4x \text{Tr}(F_{\mu\nu} F^{\mu\nu}) \quad (87)$$

where the gauge potentials A_μ and field strength $F_{\mu\nu}$ are both valued in $\mathfrak{su}(N)$. As it turns out, the structure of allowed line operators depends intricately not only on the Lie algebra, but also on the global form of the gauge group – e.g. $SU(N)$ and $SU(N)/\mathbb{Z}_N$ are different groups with the same algebra, and result in corresponding Yang-Mills theories with the same action that are nevertheless subtly different.

It turns out that any pure Yang-Mills theory with gauge group G has (at least) a 1-form symmetry given by the *center* of the gauge group. Recall that the center $Z(G)$ of a group G is the set of elements that commute with all elements of G , i.e.

$$Z(G) = \{g \in G \mid gg' = g'g \text{ for every } g' \in G\} \quad (88)$$

For concreteness, we will first discuss pure $SU(N)$ gauge theory. For $SU(N)$, we have that $Z(G) = \mathbb{Z}_N$, written as

$$\begin{pmatrix} e^{\frac{2\pi i k}{N}} & & & \\ & e^{\frac{2\pi i k}{N}} & & \\ & & \dots & \\ & & & e^{\frac{2\pi i k}{N}} \end{pmatrix} \quad k \in \{0, \dots, (N-1)\} \quad (89)$$

which clearly commutes with everything and has unit determinant.

Thus the statement is **that pure $SU(N)$ gauge theory has a \mathbb{Z}_N 1-form symmetry.**

To understand the 1-form symmetry, I will have to do two things: explain the charged line operators, and explain the existence of the topological 2-dimensional surface operator.

The line operators are – unsurprisingly – Wilson lines:

$$W_r(C) = \text{Tr}_r P \left[\exp \left(\oint_C A \right) \right] \quad (90)$$

Here r denotes the representation. We will see in a second that the charged line operator that we want comes from the fundamental representation $r = f$.

The 2d topological surface operator is called a **Gukov-Witten surface operator** $G_g(\mathcal{M}_2)$, and it is labeled by an element of $g \in \mathbb{Z}_N$. I will first give a heuristic explanation of its existence in the continuum: a more convincing argument will require us to study a UV regularization on the lattice.

The heuristic idea is the following: $G_g(\mathcal{M}_2)$ is defined by saying that when it is inserted into the path integral, the holonomy of the gauge field on an infinitesimal loop C that winds around the codimension-2 surface \mathcal{M}_2 is a center element g , i.e.

$$P \exp \left(\oint_C A \right) = g \quad (91)$$

(One can also define these by the conjugacy classes of other group elements that are not in the center; in this case they are however generically not topological objects).

Physically, these objects can be thought of as inserting **vortices** which induce a kind of holonomy for the gauge charged objects that move around them. This holonomy acts as a phase on the fundamental Wilson line, i.e. if $g = e^{\frac{2\pi i k}{N}}$ then we have [6]:

$$G_g(S^2) W_f(C) = \frac{\text{Tr}_f(g)}{\text{Tr}_f(\mathbf{1})} W_f(C) = e^{\frac{2\pi i k}{N}} W_f(C) \quad (92)$$

and thus we see that it generates the desired linking equation (38).

(To understand the second equality, note that the holonomy constraint above simply always contributes an extra constant factor of g inside the path-ordered exponential; as g commutes with everything it is proportional to the identity inside the trace and contributes at most a phase, and we can extract that phase by writing it as the ratio of traces).

However we are looking at pure Yang-Mills theory, where there are no dynamical degrees of freedom transforming in the fundamental. Do any *dynamical* degrees of freedom see this surface operator? The dynamical degrees of freedom are the gluons etc. which are in the adjoint. The holonomy picked up by an adjoint-valued degree of freedom as it traverses the Gukov-Witten operator is given by the Wilson line in the adjoint representation:

$$G_g(S^2) W_{\text{Adj}}(C) = \frac{\text{Tr}_{\text{Adj}}(g)}{\text{Tr}_{\text{Adj}}(\mathbf{1})} W_{\text{Adj}}(C) = W_{\text{Adj}}(C) \quad (93)$$

However, as the adjoint representation acts as conjugation $M \rightarrow U^\dagger M U$, the center doesn't act at all on the adjoint rep, and the trace of any central element in the adjoint is the same as the trace of the identity, $\text{Tr}_{\text{Adj}}(g) = \text{Tr}_{\text{Adj}}(\mathbf{1})$. Thus none of the gluons can see the Gukov-Witten surface operator, and it is topological.

This might feel a little bit unfulfilling, so I would now like to present the same argument with a lattice description of the gauge theory. This has the benefit of making everything extremely transparent, and we will also see how we can identify discrete symmetries in a lattice model.

Exercise: Prove the adjoint trace identity (valid for any $SU(N)$ element):

$$\underset{\text{Adj}}{\text{Tr}}(U) = \underset{f}{\text{Tr}} U \underset{f}{\text{Tr}} U^\dagger - 1 \quad (94)$$

Hint: use the fact that the adjoint representation can be obtained from a tensor product of fundamental and anti-fundamental reps:

$$f \otimes \bar{f} = \text{Adj} \oplus \mathbf{1} \quad (95)$$

From (94) conclude explicitly that $\text{Tr}_{\text{Adj}}(g) = \text{Tr}_{\text{Adj}}(\mathbf{1})$ if $g \in \mathbb{Z}_N$.

3.4.2 $SU(N)$ gauge theory on the lattice

In this section we will formulate pure $SU(N)$ gauge theory on the lattice. Much of the notation is from David Tong's gauge theory notes. Consider a hypercubic lattice \mathbb{Z}^4 . The basic degree of freedom in lattice gauge theory is a group element $U_l \in SU(N)$ defined on each of the directed *links* l of the lattice. I will sometimes use a notation $l = \langle ij \rangle$ to denote a link from site i to site j . Reversing the orientation of the link corresponds to taking the inverse of the group element, i.e.

$$U_{\langle ij \rangle} = U_{\langle ji \rangle}^\dagger \quad (96)$$

Gauge transformations Λ_i are $SU(N)$ valued functions defined on the sites of the lattice, and they act on the link variables as

$$U_{\langle ij \rangle} \rightarrow \Lambda_i U_{\langle ij \rangle} \Lambda_j^\dagger \quad (97)$$

Morally speaking, to make the connection to the continuum gauge potential A , we should view U as an infinitesimal open Wilson line from i to j :

$$U_{\langle ij \rangle} = \exp \left(i \int_i^j A \right) . \quad (98)$$

The analogue of the 2-form field strength is defined on a plaquette, and is constructed by taking the oriented trace of the four links on the plaquette:

$$W_\square = \underset{f}{\text{Tr}} \prod_{l \in \square} U_l \quad (99)$$

Unpacking this for a plaquette in the (x, y) plane with bottom left corner at (x, y) we have:

$$W_\square = \underset{f}{\text{Tr}} (U_{\langle(x,y),(x+1,y)\rangle} U_{\langle(x+1,y),(x+1,y+1)\rangle} U_{\langle(x+1,y+1),(x,y+1)\rangle} U_{\langle(x,y+1),(x,y)\rangle}) \quad (100)$$

Note that this is invariant under (97) because it forms a tiny closed loop and all of the Λ_i 's cancel.

We are now ready to form an action. The **Wilson action** for pure Yang-Mills is the following sum over plaquettes:

$$S[U] = -\frac{1}{g^2} \sum_{\square} (W_\square + W_\square^\dagger) \quad (101)$$

Here g^2 is the Yang-Mills coupling.

Exercise: By assuming that each link has length a and appropriately Taylor expanding the continuum field $A_\mu(x)$ defined in (96), show that to leading order in derivatives

$$W_\square = -\frac{a^2}{2} \text{Tr}_f F_{\mu\nu} F_{\mu\nu} + \dots \quad (102)$$

where F is the usual non-Abelian field strength and there is no sum over indices on the left-hand side (rather the $\mu\nu$ indicate which plaquette we are discussing). Thus show that the lattice Wilson action reduces to the regular continuum action (87) to leading order in derivatives.

The partition function is then given by integrating over all the group elements on each link:

$$Z = \int [dU_l] \exp(-S[U]) \quad (103)$$

There exists a well-defined invariant integration measure on group elements called the *Haar measure*, which basically does exactly what you expect it to do and is invariant under the group action. The partition function is a discretely infinite number of integrals over U and can be (fruitfully) done on a computer.

Now, let us understand the global symmetries of the theory defined by (103). The fundamental Wilson line is defined in the obvious way, by taking products of U along a closed path C :

$$W_r(C) = \text{Tr}_r \left(\prod_{l \in C} U_l \right) \quad (104)$$

Let us now turn to the Gukov-Witten surface operator.⁴

To begin, note that the Wilson action that we defined above has a natural “slot” to which we can couple a \mathbb{Z}_N -valued 2-form source. Consider defining a \mathbb{Z}_N valued function on the plaquettes of the lattice by $b_\square \in \mathbb{Z}_N$. Then we can construct a deformed version of W_\square as

$$W_\square(b_\square) = \text{Tr}_f \left(b_\square \prod_{l \in \square} U_l \right) . \quad (105)$$

i.e. we are twisting the product of U ’s by the appropriate b on the plaquette. It doesn’t matter where we place the b in the string of 4 U ’s, as it commutes with everything. The action and partition function may be defined in the presence of this source.

$$S[U; b] = -\frac{1}{g^2} \sum_\square \left(W_\square(b_\square) + W_\square^\dagger(b_\square) \right) \quad Z[b] = \int [dU_l] \exp(-S[U; b]) \quad (106)$$

Now to define the Gukov-Witten operator $G_g(\mathcal{M}_2)$: let \mathcal{M}_2 be a 2d surface on the *dual-lattice*⁵. A codimension-2 surface naturally intersects a set of plaquettes on the original lattice (as shown in a 3d example in Figure 11).

For each plaquette that intersects \mathcal{M}_2 , set $b_\square = g^6$. (For all other plaquettes, set $b_\square = 1$). The resulting choice of $b_{\mathcal{M}_2}$ can be thought of as defining a new partition function which depends on a choice of surface corresponding to the insertion of the operator.

$$Z[b_{\mathcal{M}_2}] = \langle G_g(\mathcal{M}_2) \dots \rangle \quad (107)$$

⁴I am thankful to T. Sulejmanpasic for discussions on the content of this section.

⁵The dual lattice has a technical definition, but for the cubic lattice its the set of all points that are “halfway between” the points of the lattice, i.e. $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}) + (\text{integer offsets})$

⁶For an interesting study in 2d of related operators see [7].

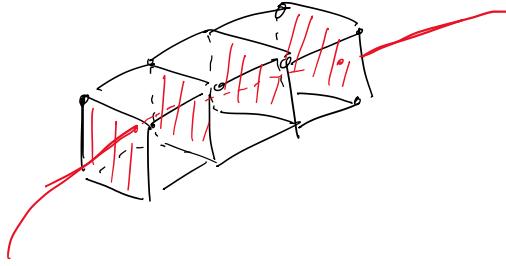


Figure 11: A codimension-2 surface on the dual lattice intersects plaquettes of the original lattice, shown here in a 3d example; red line represents codimension-2 surface, and shaded red plaquettes are intersected by it.

To see that this is topological, let us consider trying to deform the surface. Most of the action is in two directions transverse to the defect, so let us focus on those, as in Figure 12. Note that if we do the following change of variables in the picture:

$$U_4 \rightarrow g^{-1}U_4 \quad (108)$$

it has the effect of setting $b_{\square_1} = 1$ but setting $b_{\square_2} = g$; thus it has moved the surface operator over by one site, but leaves the partition function invariant (as it is just a change of variables in the path integral). By performing operations of this form one can deform the operator however we like; thus there exists a topological operator living on 2d surfaces in the dual lattice. Zooming out to long distances the distinction between dual and original lattices goes away and this becomes a smoothly deformable continuum operator.

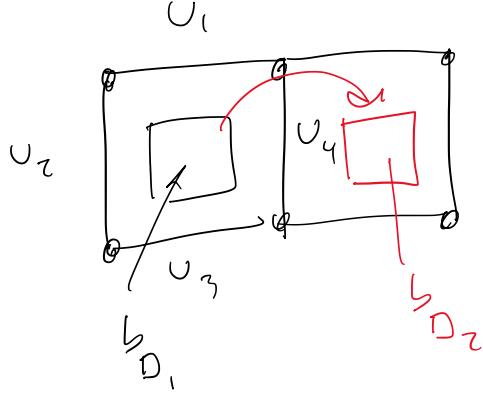


Figure 12: Moving the Gukov-Witten surface operator around by a change of variables in the lattice path integral.

Exercise: Prove that the linking of the lattice Gukov-Witten surface operator $G_g(\mathcal{M}_2)$ with the fundamental Wilson line (104) satisfies the desired linking equation (92).

The existence of a topological operator $G_g(\mathcal{M}_2)$ is a statement that can easily be lifted to the continuum. In fact this is a consequence of a larger invariance of the partition function in the presence of the source b .

Consider a \mathbb{Z}_N valued function g_l on the links l . Assume that this acts on the dynamical link variables U_l and the plaquette variables b_{\square} as follows:

$$U_l \rightarrow U'_l = U_l g_l \quad b_{\square} \rightarrow b'_{\square} = b_{\square} \prod_{l \in \square} g_l^\dagger \quad (109)$$

(where the latter product is again taken in an oriented fashion). This leaves the twisted plaquette product W_{\square} – and thus the whole action $S[U; b]$ – invariant. The partition function is thus invariant under

$$Z[b'] = Z[b] \quad (110)$$

This invariance is equivalent to the existence of the \mathbb{Z}_N 1-form global symmetry. Note that it is a discrete version of the sort of invariance we saw for the $U(1)$ Maxwell theory around (56). Importantly, it is not the same as the $SU(N)$ gauge redundancy discussed around (97), which involved an $SU(N)$ group element on each lattice site, not a \mathbb{Z}_N on the links. We will shortly see that one can preserve the gauge redundancy while breaking the 1-form global symmetry.

3.4.3 Confinement and the 1-form symmetry

Having now carefully understood that pure $SU(N)$ gauge theory has a \mathbb{Z}_N symmetry under which the fundamental Wilson loop is charged, we can ask about the different phases:

1. **Confined phase:** here the 1-form symmetry is *unbroken*, and the Wilson loop behaves as an area law: $\langle W_F(C) \rangle \sim \exp(-T \text{Area}(C))$. The physics interpretation is that the Wilson loop creates a non-Abelian flux tube which has a finite tension T ; it is these flux tubes which the 1-form symmetry counts (mod N). This is the phase where we normally end up for non-Abelian gauge theory.
2. **Deconfined phase:** here the 1-form symmetry is *spontaneously broken*, and the Wilson loop behaves as a perimeter law $\langle W_F(C) \rangle \sim \exp(-TL[C])$. The flux tubes have melted and condensed.

Note that from this point of view the confinement/deconfinement transition is philosophically the same as the phase transitions that we know from the ordinary Landau paradigm, just involving a 1-form symmetry rather than a 0-form one!

We can now ask the following question: what happens if we add matter to the theory, e.g. if we add a scalar charged in the fundamental under $SU(N)$:

$$S[A, \phi] = \int d^4x \left(\frac{1}{2g^2} \text{Tr}(F_{\mu\nu} F^{\mu\nu}) + (D\phi)^\dagger D\phi + m^2 \phi^\dagger \phi \right) \quad (111)$$

where $D_\mu \phi = \partial_\mu \phi + iA_\mu^a t^a \phi$ with t^a the fundamental representation matrices.

In this case the 1-form symmetry is **explicitly broken**, i.e. the putative Gukov-Witten surface operator $G_g(\mathcal{M}_2)$ is no longer topological.

You can see this in the continuum by noting that the quanta of the ϕ field trace out effective worldlines in spacetime, where the coupling of each worldline to the gauge field is given by a fundamental Wilson loop $\langle W_f(C) \rangle$; thus $G_g(\mathcal{M}_2)$ now feels a phase from the *dynamical* degrees of freedom, spoiling its topological-ness.

More concretely, we can also see this on the lattice. To add fundamental matter to the Wilson lattice action, we consider a field $\phi_i \in \mathbb{C}^N$ living on each site i of the lattice. Under the gauge transformation (97) we take ϕ_i and $U_{\langle ij \rangle}$ to transform as

$$U_{\langle ij \rangle} \rightarrow \Lambda_i U_{\langle ij \rangle} \Lambda_j^\dagger \quad \phi_i \rightarrow \Lambda_i \phi_i \quad (112)$$

so that the coupling $\phi_i^\dagger U_{\langle ij \rangle} \phi_j$ is a gauge-invariant lattice kinetic term for ϕ . We can now consider the Wilson action with fundamental matter:

$$S[U, \phi] = -\frac{1}{g^2} \sum_{\square} \left(W_{\square} + W_{\square}^\dagger \right) + t \sum_{\langle ij \rangle} \phi_i^\dagger U_{\langle ij \rangle} \phi_j \quad (113)$$

(We can also add a further on-site mass term which I am neglecting for simplicity). If we now try to construct the topological Gukov-Witten operator through the arguments above, we will see that it is simply not possible; the existence of the ϕ 's spoils renders the manipulation (108) no longer possible.

This means that once fundamental charged matter exists, there is no longer a clean symmetry-based distinction between the confined and deconfined phases; in all phases we will find that the Wilson law exhibits a perimeter law. (Physically, if you try to create an extended flux tube, it is now allowed to *break* into two pieces with a ϕ particle ending each tube). Thus there is no order parameter for confinement in this case, because there is **no global symmetry**. (Note that QCD is also like this – however the situation here is more subtle and intricate.).

3.5 Gauge theory: \mathbb{Z}_2

You might have noticed above that most of the structure of the non-Abelian gauge symmetry played no role in this discussion. One might then reasonably ask: what is the *simplest* theory that has the structure above, i.e. that of a discrete 1-form symmetry which could be spontaneously broken?

I believe the simplest is the \mathbb{Z}_2 lattice gauge theory. This is essentially the same as above, except that instead of having a $SU(N)$ element on each link we have a \mathbb{Z}_2 element, i.e. a single sign which can be either positive or negative:

$$a_l = \pm 1 \quad (114)$$

The analogue of the field strength is also valued in ± 1 :

$$W_{\square} = \prod_{l \in \square} a_l \quad (115)$$

The action is simply

$$S[a] = -\beta \sum_{\square} W_{\square} \quad Z = \sum_{\{a_l\}} \exp(-S[a]) \quad (116)$$

There is a gauge redundancy given by

$$a_{\langle ij \rangle} \rightarrow a'_{\langle ij \rangle} = \Lambda_i a_{\langle ij \rangle} \Lambda_j \quad (117)$$

where Λ_i is a \mathbb{Z}_2 -valued field defined on site.

Importantly, this theory also has a \mathbb{Z}_2 1-form global symmetry. The charged line operator is the *Wegner-Wilson line operator*, given by $W[C] = \prod_{l \in C} a_l$. The co-dimension 2 charge defect – i.e. the analogue of the Gukov-Witten surface operator – can be constructed by an argument similar to that above, and turns out to take the explicit form:

$$G(S) = \prod_{\square \in S} \exp(-2\beta W_{\square}) \quad (118)$$

where again the product over plaquettes is taken over those that intersect the surface in the dual lattice (here the factor of 2 comes from the fact that we are changing the sign of the appropriate plaquette term in the action).

This theory has a confined phase (unbroken 1-form symmetry) at $\beta \approx 0$ and a deconfined phase (spontaneously broken 1-form symmetry) at $\beta \gg 1$, where the precise location of the critical point can easily be found by numerical simulation. Much more can be said about this, and I will just make a few more quick remarks.

3.5.1 Kramers-Wannier duality and the Ising spin model

In 3d, this theory is dual by Kramers-Wannier duality – i.e. a non-local rewriting of the theory – to the regular 3d Ising spin model, which has action

$$S = -\tilde{\beta} \sum_{\langle ij \rangle} \sigma_i \sigma_j \quad (119)$$

where $\sigma_i = \pm 1$.

Physically speaking, this mapping takes the worldsheets of 2d flux tubes in the gauge model to domain walls that separate up from down spins in the spin model.

The global symmetries may be confusing here, as clearly the model (119) has a 0-form global symmetry $\sigma_i \rightarrow -\sigma_i$, and not a 1-form global symmetry. This is correct; the usual statement of Kramers-Wannier duality is somewhat imprecise. The true statement of Kramers-Wannier duality is that the \mathbb{Z}_2 -invariant sectors are mapped into each other, and a genuine charged operator on one side is mapped into a non-genuine charged operator (i.e. something with a tail attached) on the other.

Of course, this is a very well-studied model for magnetism. The action penalizes configurations where σ_i differs from its neighbours, and thus the low-temperature phase ($\tilde{\beta} \rightarrow \infty$) has all of the σ_i with the same sign, and the \mathbb{Z}_2 spin symmetry spontaneously broken, the **ferromagnetic** phase. As we increase temperature (i.e. take $\tilde{\beta} \rightarrow 0$) we eventually hit a phase transition and the model enters a disordered **paramagnetic** phase. The continuous critical point separating these two – the 3d Ising transition – is a strongly coupled fixed point about which we nevertheless know a lot, mostly because of heroic efforts from the bootstrap and from Monte Carlo simulations.⁷

3.5.2 Higgs/confinement continuity

Let me say a few further words about the model in 3d. We can imagine coupling in \mathbb{Z}_2 matter to this gauge field, i.e. introduce $\phi_i = \pm 1$, a \mathbb{Z}_2 -valued field on the *sites* who transforms under the gauge transformations as

$$\phi_i \rightarrow \Lambda_i \phi_i \quad (120)$$

We then have the following model, the \mathbb{Z}_2 analogue of (113):

$$S[a, \phi] = -\beta \sum_{\square} W_{\square} - J \sum_{\langle ij \rangle} \phi_i a_{\langle ij \rangle} \phi_j \quad (121)$$

This model illustrates some important physical points:

1. First, by arguments identical to those around (113), the presence of matter breaks the \mathbb{Z}_2 1-form symmetry; thus at generic points in the (β, J) space the model has no global symmetries.
2. Here J controls the matter fluctuations and β the gauge fluctuations. What is the phase diagram as a function of J and β ? If we set $J \rightarrow 0$ then there is a deconfined phase at larger β and a confined phase at small β .

Let's now consider the limit where $\beta \rightarrow \infty$; this essentially suppresses all the gauge field fluctuations, so we expect to find a setup where $W_{\square} = 1$ on all plaquettes, which means that $a_{\langle ij \rangle} = 1$ (or something gauge-equivalent to that) on all links. We thus end up with the ordinary Ising spin model (119) expressed in terms of the ϕ_i ; as described above this model has a phase transition at some critical point J^* .

⁷An eminent theorist once said that every physicist should at some point in their career try to solve the 3d Ising model.

Note that this phase transition is – morally – condensing the ϕ_i fields. So in conventional language, we would call it the “Higgs” phase. I say morally because the ϕ_i are gauge charged and thus we cannot really speak about a condensate of them, but it is a fact that the phase transition happens.

3. The phase diagram of this model is now somewhat interesting. Note that there are basically two phases; one that is continuously connected to the deconfined phase of the gauge theory – we will call this the **topological** phase – and another, which is continuously connected both to the Higgs phase on one side, and the confined phase on the other. If nothing else happens, this implies the baffling statement that the Higgs and confined phases are **continuously connected** [8].

To someone who has learned gauge theory from perturbation theory in the continuum (as I did, many moons ago), this statement is very surprising. However it is indeed true; the point is simply that there are no global symmetries that can be used to diagnose the phases, and thus, there is nothing *stopping* the Higgs and confined phases from being connected to each other. Whether or not the same is true in QCD is a matter of active and fascinating study, with very recent work suggesting that it is *not* [9].

4. It is not an accident that the phase diagram appears symmetric with respect to reflection about the diagonal. There is in fact a self-duality symmetry that switches the two sides; see [10].

3.6 Effective description of spontaneously broken \mathbb{Z}_N 1-form symmetry: topological order

I now turn to a different question: suppose we have a theory that has a spontaneously broken \mathbb{Z}_N 1-form symmetry, such as deconfined $SU(N)$ gauge theory, or maybe the deconfined phase of the \mathbb{Z}_2 gauge theory described above. What is the *effective description* of the long distance physics? (Here I will focus on the $SU(N)$ and its associated \mathbb{Z}_N 1-form symmetry).

We should first ask: what are the long-distance degrees of freedom? Generically, there will be a gap to all local degrees of freedom, but we should be able to access objects corresponding to the fundamental Wilson line and the Gukov-Witten surface operator, which have a non-trivial braiding phase when they link as in (92)

$$G(S^2)W(C) = e^{\frac{2\pi i}{N}} W(C) \quad (122)$$

where here I have omitted the subscripts and am now referring to the minimal charge object of each class. Note that though $G(S^2)$ was topological by construction, in the extreme infrared limit $W(C)$ is *also* topological – this is because it has vanishing correlation with all other operators (which are all gapped), but is not zero (as it would have been if it had an area law).

3.6.1 BF theory

We thus seek a topological field theory to capture the physics of this braiding. Our discussion will be in 4d with an eye towards the $SU(N)$ gauge theory, but by modifying a few indices here and there it works for \mathbb{Z}_N p-form symmetry in any number of dimensions.

The required object is called **BF theory**, and can be defined in terms of a local action involving a 1-form and 2-form gauge field A_1 and B_2 respectively:

$$S[A, B] = \frac{iN}{2\pi} \int_{\mathcal{M}_4} B \wedge dA = \frac{iN}{2\pi} \int_{\mathcal{M}_4} A \wedge dB \quad (123)$$

This is a topological field theory with a long history; see [11, 12] for discussions in the high-energy literature. It is a kind of mixed, higher-form Chern-Simons theory. The metric does not appear in its construction: it does not care about distances, and cares only about topology.

This action is invariant under the following gauge redundancies:

$$B_2 \rightarrow B_2 + d\Lambda_1 \quad A_1 \rightarrow A_1 + d\lambda_0 . \quad (124)$$

It's important that both of these are compact gauge fields, i.e. the gauge transformations are valued in $U(1)$ and just as around (58) the associated monopole numbers are quantized appropriately, i.e. if we have $F = dA$ and $H = dB$ then over all 2 and 3-cycles respectively we have:

$$\oint_{\mathcal{M}_2} F_2 = 2\pi\mathbb{Z} \quad \oint_{\mathcal{M}_3} H_3 = 2\pi\mathbb{Z} \quad (125)$$

This imposes a restriction on the parameter N . Consider studying the action the theory on $S^1 \times S^3$. Now consider doing a **large gauge transformation** by gauge parameter $e^{i\lambda}$, where λ winds once around the S^1 , i.e. if the S^1 coordinate is $\phi \in [0, 2\pi]$

$$\lambda = \phi . \quad (126)$$

The gauge transformation $e^{i\lambda}$ is single-valued on the circle, and thus this is a perfectly valid gauge transformation to do. Under this transformation $A \rightarrow A' = A + d\lambda = A + d\phi$

$$S[A', B] - S[A, B] = \frac{iN}{2\pi} \int_{S^1 \times S^3} d\phi \wedge H = \frac{iN}{2\pi} \left(\int_{S^1} d\phi \right) \left(\int_{S^3} H \right) = 2\pi i N q \quad (127)$$

where q is the integer-quantized “monopole number” of the integral of H over S^3 . Note the horrifying fact that the action is *not* invariant under this large gauge transformation. We are saved by the fact that we do not study the action itself; rather this is an intrinsically quantum theory which makes sense only when we insert the action into a path integral as

$$Z = \int [dA][dB] \exp(-S[A, B]) \quad (128)$$

and thus what we actually need is for the *exponential* of S to be invariant. This will be the case provided we have

$$N \in \mathbb{Z} . \quad (129)$$

rendering the shift in S an invisible factor of $2\pi i$.

We thus see that the theory defined by (123) has its parameter N quantized (an alternative classic argument involving defining Chern-Simons theory as the boundary of a higher-dimensional manifold can be found in [13]). This is generally the case for topological field theories, which are rigid objects that do not permit continuous deformations.

We now study the dynamics of this theory. The local equations of motion are

$$dA = 0 \quad dB = 0 \quad (130)$$

and thus there are no local degrees of freedom. However there are extended operators we can insert on closed 1-cycles C_1 and 2-cycles \mathcal{M}_2 respectively:

$$W(C_1) = \exp \left(i \oint_{C_1} A \right) \quad G(\mathcal{M}_2) = \exp \left(i \oint_{\mathcal{M}_2} B \right) \quad (131)$$

These objects are invariant under the gauge transformations (124). We now make the main dynamical statement – in BF theory these two objects have a nontrivial \mathbb{Z}_N braiding, i.e.

$$\langle W(C_1)G(\mathcal{M}_2) \rangle \sim \exp \left(\frac{2\pi i}{N} L(C_1, \mathcal{M}_2) \right) \quad (132)$$

where L is the *linking number* of C_1 and \mathcal{M}_2 , a well-defined object in four dimensions.

The proof of this is not completely obvious, so I've decided to make it an exercise.

Exercise: Prove the linking relation (132). **Hint:** consider the equations of motion with the surface operators inserted and study the on-shell action for a simple configuration of C_1 and \mathcal{M}_2 .

Thus we see that we have realized the desired linking algebra (122). A phase described by such a topological field theory is usually called an example of **topological order**, and is often exhibited as an example of a beyond-Landau phase. This name predates the technology of higher-form symmetry: in this section we have just shown that the spontaneous breaking of a 1-form symmetry is an example of topological order, and thus can be understood from the point of view of an enlarged Landau paradigm involving higher-form symmetry.

3.6.2 Ground states in a topologically ordered phase

If we spontaneously break a 0-form \mathbb{Z}_N symmetry, we have N -fold degenerate ground states which are acted upon by the action of the symmetry. They are precisely degenerate in the limit of infinite volume.

Here I will show the higher-form analogue of this phenomena; again if we break a 1-form \mathbb{Z}_N symmetry, we will have degenerate ground states. In this case their number is determined by the **topology** of the spatial manifold.

As an example, let us consider quantizing the theory on $\mathbb{R} \times S^2 \times S^1$, where the first \mathbb{R} is “time”. (Note that this is the first time in these lectures that we have worked in a Hamiltonian framework). It is instructive to quantize the theory carefully. For now I will just note the following points, which you should verify:

1. There are no local degrees of freedom. There may still be “delocalized” states that live on the whole spatial manifold.
2. The Hamiltonian is zero, so all of these states must have vanishing energy: they are “ground states”.
3. The only non-trivial operators that act on the space of states are the following

$$W \equiv \exp\left(i \oint_{S^2} B\right) \quad G \equiv \exp\left(i \oint_{S^1} A\right) \quad (133)$$

These are the only topologically non-trivial operators that we can construct. As the equations of motion set $dA = dB = 0$ any topologically trivial operators can be shrunk to zero and collapsed, and have no action on any state.

Note also that these are the objects that we saw had a non-trivial braiding in Euclidean space (132). Here we are interpreting them as quantum operators acting on the Hilbert space, i.e. they act at a given time. One can directly show from the canonical commutation relation of the local fields A and B that the non-local operators satisfy the following equal-time commutation relation:

$$GW = e^{\frac{2\pi i}{N}} WG, \quad (134)$$

We now seek to find the space of states, i.e. we seek to find a representation of the algebra (134). There is no one-dimensional representation, and thus the ground state is necessarily not unique. The smallest representation is N -dimensional, with basis elements with I label by $|n\rangle$, $n \in \{0, 1, \dots, (N-1)\}$. The operators act as

$$W|n\rangle = |(n+1) \bmod N\rangle \quad G|n\rangle = \exp\left(\frac{2\pi i n}{N}\right) |n\rangle \quad (135)$$

These N states are the ground states of BF theory on $S^2 \times S^1$. Physically, the state $|n\rangle$ corresponds to having n flux tubes threading the S^1 ; acting with the Wilson line on the S^1 creates another flux tube, and G measures the number of tubes (mod N).

Note that this state exists only because of the non-trivial topology; had we played the same game on S^3 we would have found a single unique ground state. This is a hallmark of **topological order**.

Exercise: Prove the commutation relation (134).

Exercise: Consider a situation where there are dynamical string worldsheets – with tension T – that couple to the 2-form B_2 . Compute in some suitable approximation the splitting of the degenerate energy levels we have found above. **Hint:** – you should find $\Delta E \sim \exp(-4\pi TR^2)$, where the S^2 has radius R .

There is much more to say about topological order. I want to make a few more remarks:

1. A simple Hilbert space realization of the topological phase of the \mathbb{Z}_2 gauge theory in $(2+1)d$ is given by the so-called *toric code* [14].
2. In the other series of lectures at this school we are learning about entanglement entropy. In fact, a system that is cursed with topological order will exhibit it in its entanglement entropy. This arises from the fact that there is a reduction of entropy arising from the fact that every one of the un-endable-strings that we discuss that enters a region of space must also exit the same region of space. In the toric code case discussed above the so called “topological entanglement entropy” [15, 16] is a constant contribution that looks like

$$S(\text{disc}) = \text{area law stuff} - \log 2 \quad (136)$$

As I write this on the train to Jena I realize I am not sure if I know the result of a 4d calculation for \mathbb{Z}_N , but I’m pretty sure it would go like $-\log N$. (Exercise?)

3. The canonical example of topological order is actually not anything I have discussed above, but rather the fractional quantum Hall effect; it turns out that if you confine many electrons to a plane and apply a strong magnetic field, the frustrated dynamics arising from the interaction of Landau levels and Coulomb repulsion results in a topologically ordered state which can (sometimes) be described by the following Chern-Simons action:

$$S[A] = \frac{ik}{8\pi} \int_{\mathcal{M}_3} A \wedge dA \quad (137)$$

where I stress that A is not the photon of real life, but rather a new emergent gauge field. There is again an emergent \mathbb{Z}_k 1-form symmetry that is spontaneously broken, but its realization is slightly different due to the different structure of the Chern-Simons term.

I recommend David Tong’s lectures on the fractional quantum Hall effect as an entrance into this fascinating field.

4 Non-invertible symmetries: the ABJ anomaly

We will now discuss a different generalization of the idea of symmetry – *non-invertible symmetries*. Everything we have discussed so far has involved topological charge defects, whose fusion obeys a group composition law (27):

$$U_g(\mathcal{M}_{d-1}) U_{g'}(\mathcal{M}_{d-1}) = U_{gg'}(\mathcal{M}_{d-1}), \quad (138)$$

where $g, g' \in G$ with G some group. In particular, this means that given a particular charge defect labeled by g , there is always an inverse element g^{-1} such that

$$U_g(\mathcal{M}_{d-1})U_{g^{-1}}(\mathcal{M}_{d-1}) = \mathbf{1} \quad (139)$$

However one could ask – is it possible for there to exist topological defects that obey a more general fusion rule, e.g. of the form?

$$U_a(\mathcal{M}_{d-1})U_b(\mathcal{M}_{d-1}) = \sum_c N_{ab}^c U_c(\mathcal{M}_{d-1}) \quad (140)$$

where, e.g. the right hand side involves a non-trivial sum, rather than just a single element? In that case it is possible that some elements will not have an inverse. This is the idea of **non-invertible** symmetries.

Examples of this sort of phenomena have been known for a very long time in 1+1d, where the topological objects are called **Verlinde lines**. Their study in higher dimension has exploded recently, with many new and physically relevant examples found. In these lectures I will tell you about how these symmetries present us with a new way to think about the venerable Adler-Bell-Jackiw anomaly.

4.1 Different types of anomalies

Let us remind ourselves about the physics of anomalies, starting from the example of a massless Dirac fermion in 4d. Consider first the theory of a Dirac fermion alone, which we couple to an **external** $U(1)$ gauge field source a :

$$S[\psi; a] = \int d^4x \bar{\psi} (\not{D} - i\not{a}) \psi \quad (141)$$

Let us remind ourselves of the symmetries. The action above is invariant under two possible phase rotations of the Dirac fermion, $U(1)_V$ and $U(1)_A$:

$$\psi \rightarrow e^{\frac{i\alpha}{2}} \psi \quad \psi \rightarrow e^{\frac{i\alpha}{2}\gamma^5} \psi \quad (142)$$

which have corresponding currents

$$j_V^\mu = \bar{\psi} \gamma^\mu \psi \quad j_A^\mu = \bar{\psi} \gamma^5 \gamma^\mu \psi \quad (143)$$

Above we have coupled an external gauge field source to the **vector** current. Now it is well-known that if we insert this classical action into a path integral as

$$Z[a] = \int [d\psi d\bar{\psi}] \exp(-S[\psi; a]) \quad (144)$$

then in the quantum theory with the external source turned on the axial current is no longer conserved:

$$d \star j_A = \frac{1}{4\pi^2} f \wedge f \quad f = da \quad (145)$$

The result above is one-loop exact.

Note however that it is not conserved in a *very* specific way: the nonconservation of the axial current is *precisely* determined by the external source. This is called an '**t Hooft anomaly**'.

From the point of view of obtaining predictive power over the theory, a '**t Hooft** anomaly is generally *better* than not having any anomaly at all, as one can obtain non-trivial constraints over the dynamics; after all, we can always choose to set the applied source to zero (so that we have a conserved current again), but we can do more. For example, it is clear that a theory with (145) cannot have a trivial unique gapped ground state, as then one could imagine a setup where the right-hand side of the equation (which we can control freely) is not zero, but the left hand side (which is constrained by the gapped dynamics) is.

However, in these lectures we will not be discussing 't Hooft anomalies. Instead I want to consider **not** the theory above – where a was frozen – but rather massless QED. To that end we will promote a to a dynamical gauge field A , with its own kinetic term:

$$S[\psi, A] = \int d^4x \left(\frac{1}{4e^2} F^2 + \bar{\psi} (\not{\partial} - i\not{A}) \psi \right) \quad F = dA, \quad (146)$$

and now *also* integrate over it in the path integral, i.e.

$$Z = \int [dA][d\psi] \exp(-S[\psi, A]) \quad (147)$$

Let us now revisit the anomaly equation (145), which now has a very different character:

$$d \star j_A = \frac{1}{4\pi^2} F \wedge F \quad (148)$$

as the right-hand side is now a dynamical operator, which fluctuates. Thus it now naively seems like we have lost the symmetry entirely – the current is just not conserved any more. I will refer to this kind of anomaly – where the right-hand side is an operator – as an *Adler-Bell-Jackiw* anomaly⁸.

Let us nevertheless try to make sense of this. The first obvious thing to do is consider defining the following shifted current:

$$\star \tilde{j}_A = \star j_A - \frac{1}{4\pi^2} A \wedge F \quad (149)$$

This satisfies $d \star \tilde{j}_A = 0$. However it does not actually make sense as a local operator, because it is not gauge invariant under small gauge transformations $\delta A = d\Lambda$.

However we could try to ignore this, and instead simply integrate the whole thing over a 3-manifold anyway to try and make a conserved charge defect, i.e. imagine constructing the following object:

$$\hat{U}_\alpha(\mathcal{M}_3) = \exp \left(i \frac{\alpha}{2} \int_{\mathcal{M}_3} \left(\star j_A - \frac{1}{4\pi^2} A \wedge dA \right) \right) \quad (150)$$

Because we are considering an integral, this is now gauge-invariant under small gauge transformations $\delta A = d\Lambda$. It is also invariant under small deformations of \mathcal{M}_3 and naively it looks like it forms a topological charge defect; the first term rotates an operator with integer axial charge q by $\mathcal{O} \rightarrow e^{\frac{iq\alpha}{2}} \mathcal{O}$.

The issue however is invariance under *large* gauge transformations. Recall from Section 3.6 that this quantizes the coefficients of Chern-Simons terms in the action; if we want consistency of this object in a situation where \mathcal{M}_3 has a non-trivial 1-cycle, then we require that

$$\alpha \in 2\pi\mathbb{Z} \quad (151)$$

This means that we can no longer perform a continuous rotation, i.e. it looks like there is trivial action on all operators⁹.

Thus we cannot form a topological operator on general 3-manifolds \mathcal{M}_3 , and it really does look like we have lost the axial symmetry. This perhaps “feels wrong”: note that the left hand side has a universal description in terms of the current for the 1-form symmetry $U(1)_m^{(1)}$, and one might hope that this could be given a precise-characterization in terms of some sort of intertwining of these symmetries.

Note also that if we were willing to restrict our attention to topologically trivial 3-manifolds such as \mathbb{R}^3 then technically speaking one could get away with ignoring the issue of large gauge transformations; nevertheless we might worry that we are missing some key physics.

⁸This nomenclature seems a little imperfect, but I believe it is now standard.

⁹There is a possible \mathbb{Z}_2 action depending on whether q is even or odd; this \mathbb{Z}_2 is actually related to fermion parity, which is part of the $U(1)$ gauge symmetry and is not anomalous.

4.2 Charge defect operators for $U(1)_A$

The situation changed in 2022, when the beautiful works [17, 18] explained how to construct topological charge defects for the ABJ anomaly.

Let us begin by considering only rotation angles that are

$$\alpha = \frac{2\pi}{N} \quad (152)$$

where $N \in \mathbb{Z}$, which is an arbitrary integer. For the reasons above \hat{U}_α does not exist. However let us now consider a different operator:

$$D_{\frac{1}{N}}(\mathcal{M}_3) \equiv \int [dB] \exp \left[i \int_{\mathcal{M}_3} \left(\frac{2\pi}{2N} \star j_A + \frac{N}{4\pi} B \wedge dB + \frac{1}{2\pi} B \wedge dA \right) \right] \quad (153)$$

Here we have done something radical – we have introduced a new dynamical compact gauge field B and given it a topological Chern-Simons action with coefficient N , and then we have coupled it to the dynamical photon A . We have then done a path integral over that field.

Let me now state some properties of this object. From the first term, it generates a phase rotation on local operators with axial charge as $\mathcal{O} \rightarrow e^{\frac{2\pi iq}{2N}} \mathcal{O}$. It is perfectly well-defined and gauge-invariant since N is in the numerator everywhere. However if we now try to impose the equations of motion of B we find the following curious equation:

$$B = -\frac{A}{N} \quad (154)$$

If you eliminate a using this relation, then you recover the ill-defined object (150) with a fractional rotation angle $\hat{U}_{\frac{2\pi}{N}}$!

Thus we have succeeded in obtaining a well-defined topological charge defect that generates a non-trivial rotation, at the cost of introducing a new field a on the defect (or equivalently, coupling a TQFT to the bulk degrees of freedom).

Note that (154) does not quite make sense – if you integrate d of it over a non-trivial cycle you find improperly quantized fluxes. Thus there is a more rigorous way than what we have just done to show that the defect is topological, involving gauging a \mathbb{Z}_N subgroup of $U(1)_m^{(1)}$ on half of the space, but the intuitive argument above gives the correct answer.

Let us now explore the consequences of this new degree of freedom. Let's consider fusing two of these objects with opposite rotation angles:

$$D_{\frac{1}{N}}(\mathcal{M}_3) D_{-\frac{1}{N}}(\mathcal{M}_3) = \int [dB_1][dB_2] \exp \left[i \int_{\mathcal{M}_3} \left(\frac{N}{4\pi} B_1 \wedge dB_1 - \frac{N}{4\pi} B_2 \wedge dB_2 + \frac{1}{2\pi} (B_1 - B_2) \wedge dA \right) \right] \quad (155)$$

This object is **not** the identity operator, as it would have been for a normal symmetry – note that the difference gauge field $B_\Delta \equiv B_1 - B_2$ has a nontrivial coupling to the photon magnetic field. (In fact the right hand side is a kind of topological defect called a **condensation defect**).

Thus we cannot invert this symmetry any more, and we see that the true effect of the ABJ anomaly is to turn the ordinary axial symmetry into a **non-invertible symmetry**; there is a topological defect associated with it which counts the charge, but the action of performing a charge rotation can no longer be undone.

How do we use this object? Its action on normal local operators is conventional and invertible, and as such it can be used to derive selection rules on scattering amplitudes and constrain the terms that can appear in RG (this works the normal way on \mathbb{R}^4 but can result in interesting subtleties for other manifolds see [19] for a recent discussion).

However its action on non-local objects such as 't Hooft lines is different. In general if one drags the 't Hooft line through one of these objects it acquires a tail, i.e. it becomes the boundary of a two-dimensional topological surface which is $\exp\left(\frac{i}{N}\int_S F\right)$. This can be understood as a manifestation of the Witten effect, binding a fractionally charged Wilson line to the monopole: I refer you to [17] for more details.

Finally, you might feel worried that we have only constructed a rotation angle by $\frac{2\pi}{2N}$, for any N ; however clearly by composing this p times we can obtain any *rational* fraction $\frac{p}{N}$ of 2π . (In fact there is a simpler route which uses a different and slightly more involved TQFT from the simple Chern-Simons theory to arise at the same rational covering of the unit circle).

The existence of rational numbers may seem peculiar. There is an alternative route to constructing the charge defects [20, 21]; rather than introducing a Chern-Simons field on the defect, you introduce a compact phase field $\theta(x)$ and consider the following gauge invariant object, parametrized by a continuous $U(1)$ element α (and not a rational number).

$$U_\alpha(\mathcal{M}_3) = \int [d\theta] \exp\left(i\frac{\alpha}{2} \int_{\mathcal{M}_3} \left(\star j_A - \frac{1}{4\pi^2}(A - d\theta) \wedge dA\right)\right) \quad (156)$$

Here under a $U(1)$ gauge transformation the scalar θ transforms as:

$$A \rightarrow A + d\lambda \quad \theta \rightarrow \theta + \lambda \quad (157)$$

This object has the advantage that it allows one to construct a local density on the defect

$$\star \tilde{j}_A = \star j_A - \frac{1}{4\pi^2}(A - d\theta) \wedge dA \quad (158)$$

As this object requires the phase field θ it cannot be defined *off* the defect. The existence of this density permits one to prove an analogue of Goldstone's theorem for this non-invertible symmetry, which can be viewed as a perhaps unnecessarily fancy way to understand why $U(1)$ axions are massless (or, in real life, why the π_0 is so light).

5 More applied applications: a biased list

5.1 Finite temperature: static phenomena

It is interesting to consider the realization of these higher-form symmetries at finite temperature T . A very physically relevant example is to study hot QED coupled to massive electrically charged matter (as in Section 3.3), i.e.

$$S[\phi, A] = \int d^4x \left(\frac{1}{4e^2} F^2 + (D\phi)^\dagger D\phi + m^2 \phi^\dagger \phi + \dots \right) \quad (159)$$

If we are interested in static phenomena, we can understand the physics at finite T by placing the system on $S^1 \times \mathbb{R}^3$, where the S^1 has length $\beta = \frac{1}{T}$.

Recall that there is generically only a single magnetic 1-form symmetry with current $J^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. Upon dimensional reduction, from the 3d point of view this will break into a 0-form symmetry $U(1)_m^{(0)}$ with current $j^i = J^{i\tau}$ and a 1-form symmetry $U(1)_m^{(1)}$ with current J^{ij} . We should now ask – as usual – how these symmetries are realized on the \mathbb{R}^3 , i.e. the non-compact directions.

To understand this, let us try and compute an effective action. Let's decompose the vector potential into $A_\mu = (A_\tau, A_i)$ and understand what each of the components *feels*, by integrating out the bosons at finite temperature.

First, A_τ . Let's understand this in two distinct ways.

First, the more honest way: in quantum statistical mechanics, we often consider bosons at finite chemical potential μ , which can be understood as the background value of the real-time vector potential $A_t = \mu$. The physics of a gas of bosons. This typically depends on μ as a Bose-Einstein distribution of the rough form

$$n(E) = \frac{1}{e^{\beta(E \pm q\mu)} - 1} \quad (160)$$

where the sign depends on whether we are considering the particle or the antiparticle, which have opposite sign. Now when we go to Euclidean time we have $A_\tau = iA_t = i\mu$; let's also consider a limit where the mass of the particle is very high, so that $E \approx m$ (i.e. I am ignoring the contribution from the motion of the particles), and also that $\beta \ll E^{-1}, \mu$ (so that we are at very high temperatures).

In that case this distribution takes the form

$$n(E) \sim e^{-\beta m \pm iqA_\tau} \quad (161)$$

And so if we try to construct an effective action for A_τ , the ingredients take the form shown above, and thus in the limit shown we expect an answer of the form:

$$S[A_\tau] = \beta \int d^3x \left(\frac{1}{2e^2} (\partial_i A_\tau)^2 + c_1 e^{-\beta m} \cos(qA_\tau) + \dots \right) \quad (162)$$

where the first term comes from the bare kinetic term, and I have added together the particle and anti-particle contributions and c_1 is a constant that I will not try to calculate.

Note that there is also a slick way to understand this. The gauge-invariant degree of freedom involving A_τ is the holonomy of the gauge field around the thermal circle,

$$\exp \left(iq \oint_{S^1} A_\tau \right) \quad (163)$$

in the low-energy limit we should think about the lowest harmonic of A_τ on the S^1 , which is the constant mode in τ ; thus the holonomy becomes $e^{i\beta A_\tau}$. Particle worldlines that wrap the circle will feel this holonomy (with the particle feeling $e^{i\beta q A_\tau}$ and the anti-particle feeling $e^{-i\beta q A_\tau}$, and each contribution of this worldline will be weighted by $e^{-\beta m}$ (i.e. the exponential of the particle action). Assembling the pieces we find the answer above.

Let us now understand the spatial components A_i ; as it turns out, in the usual thermal equilibrium these components are essentially unaffected and they just inherit their bare kinetic term. (In the plasma literature, this is usually stated as the fact that the “magnetic field is unscreened”).

Assembling the pieces we find the following answer for the full effective action in 3d:

$$S[A_\tau, A_i] = \beta \int d^3x \left(\frac{1}{2e^2} (\partial_i A_\tau)^2 + \frac{1}{4e^2} (\partial_i A_j)(\partial^i A^j) + c_1 e^{-\beta m} \cos(qA_\tau) + \dots \right) \quad (164)$$

(If we move away from very massive limit, then we can still expand the effective action in harmonics of $\cos(qA_\tau)$, in which case you can view this as the leading term, but the the 3d symmetry currents that we previously identified can now be understood in terms of these low-energy fields as:

$$J^{ij} = \epsilon^{ijk} \partial_k A_\tau \quad j^i = \epsilon^{ijk} \partial_j A_k \quad (165)$$

Note the somewhat surprising fact that in the phase just described, the $U(1)_m^{(0)}$ is spontaneously broken – as discussed during the tutorials, it is possible to dualize the 3d vector field into a compact scalar which is the Goldstone mode of the symmetry breaking. This captures the fact that the magnetic field is unscreened –

if you imagine probing the finite temperature medium with two probe magnetic monopoles, they will feel a Coulomb law attraction, mediated by a gapless long-range force. This force arises from the Goldstone mode.

Note that A_τ is gapped out – and therefore probing the system with two probe electric charges will *not* result in a long-range force. This is called Debye screening; the electric field is screened by free charges in the medium.

The situation can be summarized in the table below; the phase we have just described is what is normally called **plasma**.

	$U(1)_m^{(1)}$	$U(1)_m^{(0)}$
Plasma (normal phase)	Unbroken	Spontaneously broken
Superconducting	Unbroken	Unbroken

5.2 Finite temperature: hydrodynamics

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