

Physics for Bangladesh: Quantum Field Theory Lecture Notes

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1 Orientation

These are lecture notes written for the Physics for Bangladesh Quantum Field Theory online school, which took place in Fall 2024. They are based on an abridged version of a course previously taught at Durham University on path integral methods for quantum field theory.

We will use the “mostly-plus” metric:

$$\eta_{\mu\nu} = \eta^{\mu\nu} = \text{diag}(+1, -1, -1, -1) \quad (1.1)$$

This is the opposite to the convention I am used to, so everyone has to remain vigilant for sign errors. We will mostly work in four spacetime dimensions (the “physical” value), but it is nice to keep the spacetime dimension d arbitrary where possible.

Note that Greek indices μ, ν will run over time and space both, whereas i, j will run only over the three spatial coordinates, and thus

$$x^\mu = (x^0, x^i) \quad (1.2)$$

I will sometimes use x^0 to denote the time component and sometimes x^t , depending on what I feel looks better in that particular formula.

I will always set $\hbar = c = 1$.¹ They can be restored if required from dimensional analysis.

2 Path integrals for quantum mechanics

Up till now you have studied quantum field theory by writing down fields as operators $\hat{\phi}$ etc. and then imposing commutation relation on them. This approach – the “usual” one – for understanding a quantum system is called **canonical quantization**. As it turns out, there is an alternative approach, involving what is called a **path integral**: this is a very beautiful formulation of the physics, and like most things in physics that are beautiful it also turns out to be quite practically useful in calculating things.

However before studying path integrals in quantum field theory, we will begin by studying path integrals for quantum mechanics. My discussion here is taken mostly from Peskin and Schroeder Chapter 9 and Anthony Zee Chapter 1².

2.1 Many holes in many screens

I want to motivate this by thinking about the double slit experiment, which everyone knows. To remind you, imagine a slit with two holes A_1 and A_2 , with a source S and an observer O . What is the probability to find a particle after it has gone through the slit? By the superposition principle, we know that the amplitude to find the particle at the end is

$$\mathcal{A}(O) = \mathcal{A}(S \rightarrow A_1 \rightarrow O) + \mathcal{A}(S \rightarrow A_2 \rightarrow O) \quad (2.1)$$

i.e. the sum of the amplitudes to go through either of the slits. (In elementary quantum mechanics we now usually expand out these amplitudes explicitly in terms of plane waves, but we will not do that here).

Now imagine that we add another hole A_3 . Then, well the amplitude is clearly:

$$\mathcal{A}'(O) = \mathcal{A}(S \rightarrow A_1 \rightarrow O) + \mathcal{A}(S \rightarrow A_2 \rightarrow O) + \mathcal{O}(S \rightarrow A_3 \rightarrow O) \quad (2.2)$$

¹ k_B makes an appearance in a homework problem (or rather it would, if I hadn't set it to 1).

²This book is amazing if you want to be motivated to do anything.

But now what if we add A_4 and A_5 ? Clearly we should add those two. But what if we add another screen, with more slits B_i ? (Sum up those paths as well). What if we fill *all of space* with screens? What if we put so many holes in the screen that the screen is no longer there?

What all of this is really suggesting is that in general, should be a way to compute the amplitude in terms of **summing up all the ways that the particle can go from S to O** . Apparently each of them will contribute with some probability, and we should this should seem philosophically soothing: what is quantum mechanics, after all, but the idea that a particle always does everything at once, with a given probability?

This philosophical musing can be elevated to physics we have a formula for it: in other words, how much does each path contribute, and how do we “sum over all paths” anyway?

2.2 A simple finite dimensional integral to warm up

As it turns out, we will sum over paths by doing many integrals. Let’s warm up first. There are very few integrals that we can do in closed form. One of them – a crucial one – is the Gaussian integral in one variable.

$$\int_{-\infty}^{\infty} dx e^{-\frac{ax^2}{2}} = \sqrt{\frac{2\pi}{a}} \quad (2.3)$$

Also, if we add a linear shift in x in this form, the integral is still easy to do:

$$\int_{-\infty}^{\infty} dx e^{-\frac{ax^2}{2} - jx} = \sqrt{\frac{2\pi}{a}} \exp\left(-\frac{j^2}{2a}\right) \quad (2.4)$$

There is a basic generalization of this integral. I promise we will need it later, but I’m going to record it for you now. Imagine that $x_i \in \mathbb{R}^N$ and consider the following integral:

$$I = \int \prod_{i=1}^N dx^i \exp\left(-\frac{x_i A_{ij} x_j}{2}\right) \quad (2.5)$$

where A is an $N \times N$ matrix. After doing a small amount of work (homework problem!) you can show that this integral is

$$I = \int \prod_{i=1}^N dx^i \exp\left(-\frac{x_i A_{ij} x_j}{2}\right) = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det A}} \quad (2.6)$$

Hint: diagonalize the matrix A and work in terms of its eigenvalues. Extra hint: this is worked out in Zee’s book Quantum Field Theory in a Nutshell. Finally, we will need a generalization of this – let me consider adding an extra vector $J_i \in \mathbb{R}^N$ to this, so that we have

$$I[J] = \int \prod_{i=1}^N dx^i \exp\left(-\frac{x_i A_{ij} x_j}{2} + J_i x_i\right) = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det A}} \exp\left(+\frac{1}{2} J_i (A^{-1})_{ij} J_j\right), \quad (2.7)$$

where the last equality can be found by completing the square.

2.3 Derivation of path integral for quantum mechanics

We now present an actual derivation in the case of quantum mechanics. At the end the generalization to quantum field theory will actually be very simple.

Consider a quantum system with one coordinate \hat{q} , canonical momentum \hat{p} , and Hamiltonian:

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{q}) \quad (2.8)$$

where $V(\hat{q})$ is some potential which we don't need to fix yet.

We would like to derive a formula for the following object: the probability for a particle to go from q_a to q_b in time T , i.e.

$$\langle q_b | e^{-i\hat{H}T} | q_a \rangle \quad (2.9)$$

where $|q_{a,b}\rangle$ are position eigenstates, i.e. they satisfy the equation $\hat{q}|q_a\rangle = q_a|q_a\rangle$. We will also need momentum eigenstates later, which satisfy the equation $\hat{p}|p_a\rangle = p_a|p_a\rangle$.

To derive a formula for this, the first thing we will do is introduce a very small time interval ϵ and break the whole time T up into $N \equiv T/\epsilon$ small chunks, so that we write:

$$e^{-i\hat{H}T} = e^{-i\hat{H}\epsilon} e^{-i\hat{H}\epsilon} \dots e^{-i\hat{H}\epsilon} \quad (2.10)$$

The next amazingly clever thing that we will do is, in between *each* of the small ϵ pieces, we will add a complete set of states, i.e.

$$\mathbf{1} = \int dq_k |q_k\rangle \langle q_k| \quad (2.11)$$

where k runs from 1 to $(N-1)$. In other words this now looks like

$$\langle q_b | e^{-i\hat{H}T} | q_a \rangle = \langle q_b | e^{-i\hat{H}\epsilon} \int dq_{N-1} |q_{N-1}\rangle \langle q_{N-1}| e^{-i\hat{H}\epsilon} \int dq_{N-2} |q_{N-2}\rangle \langle q_{N-2}| e^{-i\hat{H}\epsilon} \int dq_{N-3} |q_{N-3}\rangle \langle q_{N-3}| \dots \langle q_1 | e^{-i\hat{H}\epsilon} | q_a \rangle \quad (2.12)$$

We can move all the integrals over to the left – thus we see that the thing to do is to evaluate many matrix elements of the form

$$\langle q_{k+1} | e^{-i\hat{H}\epsilon} | q_k \rangle \quad (2.13)$$

In this notation its natural to rename $q_b = q_N$ and $q_a = q_0$.

The answer depends on the exact form of H . Let's start by doing it for the simple case of a free particle which is just

$$H = \frac{\hat{p}^2}{2m} \quad (2.14)$$

We want to calculate

$$\langle q_{k+1} | e^{-i\frac{\hat{p}^2}{2m}\epsilon} | q_k \rangle \quad (2.15)$$

What is the matrix of the momentum operator in between two position eigenstates? To understand this, let's first recall that

$$\langle q | p \rangle = e^{iqp} \quad \int \frac{dp}{2\pi} |p\rangle \langle p| = \mathbf{1} \quad (2.16)$$

The first equality just reminds us that the momentum eigenstate in the position basis is a plane wave. Now we introduce a complete set of states to find

$$\langle q_{k+1} | e^{-i\frac{\hat{p}^2}{2m}\epsilon} | q_k \rangle = \int \frac{dp}{2\pi} \langle q_{k+1} | e^{-i\frac{\hat{p}^2}{2m}\epsilon} | p \rangle \langle p | q_k \rangle \quad (2.17)$$

Now, we can act with the momentum operator on the eigenstate to just get that $\frac{\hat{p}^2}{2m}|p\rangle = \frac{p^2}{2m}|p\rangle$ (note the loss of the hat!) and then take this number outside. Similarly, we use the inner product between \hat{q} and \hat{p} to find

$$\langle q_{k+1} | e^{-i\frac{\hat{p}^2}{2m}\epsilon} | q_k \rangle = \int \frac{dp}{2\pi} e^{-i\frac{p^2}{2m}\epsilon} e^{ip(q_{k+1}-q_k)} \quad (2.18)$$

We are left with a Gaussian integral over p . But we know how to do Gaussian integrals, so we just refer back to (2.4) find

$$\langle q_{k+1} | e^{-i\frac{\hat{p}^2}{2m}\epsilon} | q_k \rangle = \left(\frac{-im}{2\pi\epsilon} \right)^{\frac{1}{2}} \exp \left(\frac{im(q_{k+1}-q_k)^2}{2\epsilon} \right) = \left(\frac{-im}{2\pi\epsilon} \right)^{\frac{1}{2}} \exp \left(\frac{im\epsilon}{2} \left(\frac{q_{k+1}-q_k}{\epsilon} \right)^2 \right) \quad (2.19)$$

Now recall that this was just one of the many little *bits* that made up the full integral. Assembling all of them together we find

$$\langle q_b | e^{-i\hat{H}T} | q_a \rangle = \left(\frac{-im}{2\pi\epsilon} \right)^{\frac{N}{2}} \int \prod_{k=1}^{N-1} dq_k \exp \left(\frac{im\epsilon}{2} \sum_{k=1}^{N-1} \left(\frac{q_{k+1} - q_k}{\epsilon} \right)^2 \right) \quad (2.20)$$

Now, magically, we take the limit $\epsilon \rightarrow 0$. Recall that ϵ was a small increment in time. Note that from the definition of the derivative and Riemann sum of integrals

$$\lim_{\epsilon \rightarrow 0} \sum \epsilon \rightarrow \int dt \quad \lim_{\epsilon \rightarrow 0} \frac{q_{k+1} - q_k}{\epsilon} \rightarrow \frac{dq}{dt} \quad (2.21)$$

and thus we can write the truly beautiful expression

$$\boxed{\langle q_b | e^{-i\hat{H}T} | q_a \rangle = \int [\mathcal{D}q] \exp \left(i \int_0^T dt \frac{\dot{q}^2}{2} \right)} \quad (2.22)$$

where the “integral over paths” is now formally defined as

$$\int [\mathcal{D}q] = \lim_{N \rightarrow \infty} \left(\frac{-im}{2\pi\epsilon} \right)^{\frac{N}{2}} \int \prod_{k=1}^{N-1} dq_k \quad q(0) = q_a \quad q(T) = q_b \quad (2.23)$$

This is an amazing formula: let’s think about it for a second. What it says is that to compute the propagation amplitude $\langle q_b | e^{-i\hat{H}T} | q_a \rangle$, you should integrate over *all possible paths* that the particle can take from q_a to q_b , where we also have a precise definition of the integral over paths in terms of a discretization (2.23). There is also a precise formula for how much each path contributes – it contributes with a phase factor $e^{i \int_0^T dt \frac{\dot{q}^2}{2}}$.

Now we did this calculation for a free Hamiltonian with $\hat{H} = \frac{\hat{p}^2}{2m}$; however we can now imagine putting the potential $V(q)$ back. It is not too hard to show that if we have the Hamiltonian

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(q) \quad (2.24)$$

then the calculation above is modified to read

$$\langle q_b | e^{-i\hat{H}T} | q_a \rangle = \int [\mathcal{D}q] \exp \left(i \int_0^T dt \left(\frac{\dot{q}^2}{2} - V(q) \right) \right) \quad (2.25)$$

Note the crucial sign difference in front of the $V(q)$. We normally write this as

$$\langle q_b | e^{-i\hat{H}T} | q_a \rangle = \int [\mathcal{D}q] \exp \left(i \int_0^T dt L(q, \dot{q}) \right) \quad L(q, \dot{q}) = \frac{1}{2} \dot{q}^2 - V(q) \quad (2.26)$$

i.e. it is the integral of the Lagrangian – i.e. the *action* S – which appears in the exponent!

This should again feel soothing. After all, classical physics can be studied in either the Lagrangian or the Hamiltonian framework. But when we do quantum mechanics we usually only use the Hamiltonian. You may have wondered what the “Lagrangian” way of doing quantum mechanics is. Now you know.

Now sometimes we want to compute not just the propagation amplitude but also perhaps some expectation values – e.g. how do we calculate this?

$$\langle q_b | e^{-iH(T-t_2)} \hat{q} e^{-iH(t_2-t_1)} \hat{q} e^{-iHt_1} | q_a \rangle \quad T > t_2 > t_1 > 0 \quad (2.27)$$

If you now go through the *exact* same derivation again, you will see that the insertion of the operators means that when you do the path integral you should also evaluate the coordinate being path-integrated over at some specific points, i.e.

$$\langle q_b | e^{-iH(T-t_2)} \hat{q} e^{-iH(t_2-t_1)} \hat{q} e^{-iHt_1} | q_a \rangle = \int [\mathcal{D}q] q(t_2) q(t_1) \exp \left(i \int_0^T dt L(q, \dot{q}) \right) \quad (2.28)$$

There is something important happening here about the operator ordering on the left-hand side – note that on the right hand side $q(t_2)$ and $q(t_1)$ are just numbers (specific parts of a path integral that we are integrating over) – so it doesn't matter what order you put them in. But on the left hand side they were quantum operators where the ordering clearly matters.

How do we figure out what order to put them in? If you think about it, the ordering was actually determined by time – t_2 is to the left of t_1 because it is later in time (i.e. closer to the final endpoint T). In other words, expectation values that we compute from a path integral always give us **time-ordered** correlation functions, and sometimes we express this by saying

$$\langle q_b | T(\hat{q}(t) \hat{q}(t')) | q_a \rangle = \int [\mathcal{D}q] q(t) q(t') \exp \left(i \int_0^T dt L(q, \dot{q}) \right) \quad (2.29)$$

where we are now writing the operators \hat{q} in the Heisenberg picture.

Finally, note that this tells us how to go from q_a to q_b . However, we will often be interested not in going from q_a to q_b but rather in going from one state $|I\rangle$ to another state $|F\rangle$. It's easy to write down a general formula for that (exercise!). Let's ask about something even simpler though – what if we want to go from the vacuum $|0\rangle$ back to the vacuum $|0\rangle$? One way to do this is to note first that every state can be decomposed in energy eigenstates $|n\rangle$ with energies E_n , including the position eigenstate $|q_a\rangle$:

$$|q_a\rangle = \sum_n c_n |n\rangle \quad (2.30)$$

Let us set the zero of energy so that $E_0 = 0$ (i.e. the ground state has energy zero). Now note that if we act on this with $e^{-i\hat{H}T(1-i\epsilon)}$, with ϵ a small infinitesimal that is unrelated to that of the previous section, then we find:

$$e^{-i\hat{H}T(1-i\epsilon)} |q_a\rangle = \sum_n c_n e^{-iE_n T - \epsilon T E_n} |n\rangle \quad (2.31)$$

Finally, consider taking $T \rightarrow \infty$ – in that case the exponential damping by $e^{-\epsilon T E_n}$ kills all of the excited states with $n > 0$, leaving behind only the ground state!

$$\lim_{T \rightarrow \infty} e^{-i\hat{H}T(1-i\epsilon)} |q_a\rangle \rightarrow c_0 |0\rangle \quad (2.32)$$

The point of this is just that it's very easy to get the vacuum – just evolve in *slightly* imaginary time! So this gives us a simple way to evaluate the following object:

$$\langle 0 | 0 \rangle \propto \lim_{T \rightarrow \infty} \langle q_b | e^{-i\hat{H}T(1-i\epsilon)} | q_a \rangle = \lim_{T \rightarrow \infty(1-i\epsilon)} \int [\mathcal{D}q] \exp \left(i \int_0^T dt L(q, \dot{q}) \right) \quad (2.33)$$

At this point it might not be clear why we are doing this – the thing to take away from here is that **if we make time slightly imaginary, then we project onto the vacuum**. This will be useful later. So, for example, if we want to calculate

$$\langle 0 | T(\hat{q}(t_1) \hat{q}(t_2)) | 0 \rangle = \frac{\lim_{T \rightarrow \infty(1-i\epsilon)} \int [\mathcal{D}q] q(t_1) q(t_2) \exp \left(i \int_0^T dt L(q, \dot{q}) \right)}{\lim_{T \rightarrow \infty(1-i\epsilon)} \int [\mathcal{D}q] \exp \left(i \int_0^T dt L(q, \dot{q}) \right)} \quad (2.34)$$

Here the division by the thing without the q 's inserted is important because usually we don't actually manage to keep track of the right normalization of the state.

2.4 Mathematical interlude

Above we have explained how to *integrate* over functions of one variable $\int[\mathcal{D}q]$ – basically we did it by discretizing it into a series of ordinary integrals and then doing each of them. This is sometimes called *functional integration* (though I will usually call it a *path integral*).

There is another thing that we will often need to do, which is take a *derivative* with respect to a function. Let's explain how this works, first from normal calculus. Imagine that we have a vector $x^i \in \mathbb{R}^N$. Then the derivative obviously works like this:

$$\frac{\partial x^i}{\partial x^j} = \delta_{ij} \quad (2.35)$$

This leads to interesting expressions like this:

$$\frac{\partial}{\partial x^j} e^{\sum_i A^i x^i} = \frac{\partial}{\partial x^j} \left(\sum_k A^k x^k \right) e^{\sum_i A^i x^i} \quad (2.36)$$

$$= \left(\sum_k A^k \delta_{kj} \right) e^{\sum_i A^i x^i} \quad (2.37)$$

$$= A_j e^{\sum_i A^i x^i} \quad (2.38)$$

So, note what happened – I took a derivative with respect to one of the variables (x_j), and this brought down the variable multiplying it (A_j). Here I am writing out the sums instead of using the Einstein summation convention for a reason which will be evident shortly.

Now – we now need to move this up slightly by considering how we take derivatives with respect to *functions*. A function $q(t)$ is basically just many many variables. (If this bothers you, imagine discretizing the coordinate t , as we did in the first lecture for space x). So we can imagine a **functional derivative** which acts like this

$$\frac{\delta q(t)}{\delta q(t')} = \delta(t - t') \quad (2.39)$$

This is just the fancy functional version of (2.35). For physicists this expression completely defines the functional derivative and you can use it to calculate anything you need. Note that in this analogy $q(t)$ is just like x^i and $q(t')$ is just like x^j , and the delta function δ_{ij} is like $\delta(t - t')$.

Continuing in this way, note that

$$\frac{\delta}{\delta j(t)} \int dt' j(t') q(t') = \int dt' \delta(t - t') q(t') = q(t) \quad (2.40)$$

And finally, note from the same reasoning that

$$\frac{\delta}{\delta j(t)} \exp \left(\int dt' j(t') q(t') \right) = q(t) \exp \left(\int dt' j(t') q(t') \right) \quad (2.41)$$

We will use this in what follows.

3 Path integrals in free quantum field theory

With this under our belt, we will now move on to quantum field theory. Now that we understand the quantum mechanics case this is really extremely simple. We will begin with a study of the free real scalar field ϕ . This has action

$$S[\phi] = \frac{1}{2} \int d^4x ((\partial\phi(x))^2 - m^2\phi(x)^2) = \frac{1}{2} \int d^4x \phi(x)(-\partial^2 - m^2)\phi(x) \quad (3.1)$$

The classical equations of motion arising from the variation of this action are

$$(\partial^2 + m^2)\phi(x) = 0 \quad (3.2)$$

This is philosophically the same as the quantum mechanics problem studied earlier. To make the transition imagine going from the single quantum-mechanical variable q to a large vector q_a where a runs (say) from 1 to N . Now imagine formally that a runs over all the sites of a lattice that is a discretization of space, and now $q_a(t)$ is basically the same thing as $\phi(x^i; t)$ which is exactly the system we are studying above.

Now we would like to study the quantum theory. First, I point out that we can define a path integral in precisely the same way as before, i.e. we can consider the following path-integral:

$$Z_0 \equiv \int [\mathcal{D}\phi] \exp(iS[\phi]) \quad (3.3)$$

Where $S[\phi]$ is the action written down above, and $[\mathcal{D}\phi]$ now represents the functional integral over all *fields* and not just particle trajectories.

There are two main things that are nice about doing quantum field theory from path integrals the way discussed above. One of them is honest, the other is a bit “secret”.

1. The honest one: all of the symmetries of the problem are manifest. The action S is Lorentz-invariant, and it is fairly easy to see how these symmetries manifest themselves in a particular computation. Compare this to the Hamiltonian methods used in the first half of the course, where you have to pick a time-slice and it always seems like a miracle when final answers are Lorentz-invariant.
2. The secret one: the path integral allows one to be quite cavalier about subtle issues like “what is the structure of the Hilbert space exactly”. This is very convenient when we get to gauge fields, where there are subtle constraints in the Hilbert space (google “Dirac bracket”) that you can more or less not worry about when using the path integral (i.e. one can go quite far in life without knowing exactly what a “Dirac bracket” is).

3.1 The generating functional

Now that the philosophy is out of the way, let us do a computation. We will begin by computing the following two-point function:

$$\langle 0|T(\phi(x)\phi(y))|0\rangle \quad (3.4)$$

This object is called the *Feynman propagator*. It is quite important for many reasons; I will discuss them later, for now let’s just calculate it. By arguments identical to those leading to (2.34), we see that we want

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = Z_0^{-1} \int [\mathcal{D}\phi] \phi(x)\phi(y) \exp(iS[\phi]) \quad (3.5)$$

To calculate this, it is convenient to define the same generating functional as we used for quantum mechanics

$$Z[J] \equiv \int [\mathcal{D}\phi] \exp\left(iS[\phi] + i \int d^4x J(x)\phi(x)\right) \quad (3.6)$$

And we then see from functional differentiation that the two-point function is

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = \frac{1}{Z_0} \left(-i \frac{\delta}{\delta J(x)} \right) \left(-i \frac{\delta}{\delta J(y)} \right) Z[J] \Big|_{J=0} \quad (3.7)$$

Each functional derivative brings down a $\phi(x)$. Now we will evaluate this function. We first note the following crucial identity from a few hours ago, which I have embellished with a few i 's here and there

$$\int \prod_{i=1}^N dx^i \exp \left(-\frac{x_i A_{ij} x_j}{2} + i J_i x_i \right) = \frac{(2\pi)^{\frac{N}{2}}}{\sqrt{\det A}} \exp \left(-\frac{1}{2} J_i (A^{-1})_{ij} J_j \right) \quad (3.8)$$

We note from the form of the action (3.1) that the path integral $Z[J]$ we want to do is of precisely this form, where we do our usual “many integrals” limit and where a labels points in space and the operator A is

$$A = i (\partial^2 + m^2) \quad (3.9)$$

We conclude that the answer for $Z[J]$ is

$$Z[J] = \det \left(\frac{\partial^2 + m^2}{-2\pi i} \right)^{-\frac{1}{2}} \exp \left(-\frac{1}{2} \int d^4x d^4y J(x) D_F(x, y) J(y) \right) \quad (3.10)$$

where I have given the object playing the role of A^{-1} a prescient name D_F . It is the inverse of the differential operator defined in (3.9) and thus satisfies

$$i (\partial^2 + m^2) D_F(x, y) = \delta^{(4)}(x - y) \quad (3.11)$$

This is an important result. Let us first note that the path integral is asking us to compute the *functional determinant* of a differential operator. This is a product over infinitely many eigenvalues; it is quite a beautiful thing but we will not really need it here, so we will return to it later.

The next thing to note is that the dependence on J is quite simple; the exponential of a quadratic. Indeed, inserting this into (3.7) we get

$$\langle 0|T(\phi(x)\phi(y))|0\rangle = D_F(x, y) \quad (3.12)$$

Thus, we have *derived* that the time-ordered correlation function of $\phi(x)$ is given by the inverse of $i(\partial^2 + m^2)$!

Let us now actually calculate this object. We first go to Fourier space:

$$D_F(x, y) = \int \frac{d^4p}{(2\pi)^4} e^{-ip \cdot (x-y)} \tilde{D}_F(p) \quad (3.13)$$

Inserting this into (3.11) we find

$$\int \frac{d^4p}{(2\pi)^4} (-p^2 + m^2) e^{-ip \cdot (x-y)} \tilde{D}_F(p) = -i\delta^{(4)}(x - y) \quad (3.14)$$

We now see that we want the object in momentum space $\tilde{D}_F(p)$ to be proportional to $p^2 - m^2$ so that we can use the identity $\int d^4p e^{ip \cdot x} = (2\pi)^4 \delta^{(4)}(x)$. Getting the factors right, we find the following expression for the propagator in Fourier space:

$$\tilde{D}_F(p) = \frac{i}{p^2 - m^2} \quad (3.15)$$

This is the propagator that we know and love from the earlier part of the course.

Now we can use path integrals to re-derive the Feynman rules – this is a different way of doing it then we did it using the interaction picture. Sadly there is no time to re-explain it here.

End of first lecture.

4 Abelian Gauge Theories

Now that we have understood fermions, we will move on to gauge theories. We warm up with Abelian gauge theories before moving on to the full glory and beauty of non-Abelian gauge theories for the final part of the course.

4.1 Gauge invariance

We will now first understand what a gauge symmetry is, starting from the simplest example. First consider a theory with a global $U(1)$ symmetry: it is nice to consider the Dirac theory to start,

$$S[\psi, \bar{\psi}] = \int d^4x \bar{\psi}(x)(i\not{\partial} - m)\psi(x) \quad (4.1)$$

This theory is clearly invariant under a global $U(1)$ phase rotation, which is

$$\psi(x) \rightarrow \psi'(x) = e^{i\Lambda}\psi(x) \quad \bar{\psi}(x) \rightarrow e^{-i\Lambda}\bar{\psi}(x) \quad (4.2)$$

I emphasize that the symmetry parameter Λ here is a constant in spacetime, which is why we call it a *global* symmetry. I also want to emphasize that two field configurations that are *related* by the symmetry are both *physical field configurations*. If ψ represents something that we integrate over in the path integral, then so does ψ' .

Now here is an idea, which I simply state without a huge amount of motivation: let's do something different. Let us instead *demand* that we have a theory that is invariant under a symmetry where the symmetry parameter *varies* in spacetime:

$$\boxed{\psi(x) \rightarrow \psi'(x) = e^{i\Lambda(x)}\psi(x)} \quad (4.3)$$

This is called a *local* symmetry, or a *gauge symmetry*. As I will explain later, it should probably really be called a *gauge redundancy*, but I will not use that language in this course and continue to call it a gauge symmetry. Things that are invariant under this symmetry are called *gauge-invariant*.

Now, let's understand what is and is not gauge-invariant. For example: $\psi(x)$ itself is *not* gauge invariant, but it transforms under gauge transformations in a simple way (by an overall rotation).

The mass term

$$m\bar{\psi}'(x)\psi'(x) = m\bar{\psi}(x)e^{-i\Lambda(x)}e^{i\Lambda(x)}\psi(x) \rightarrow m\bar{\psi}(x)\psi(x) \quad (4.4)$$

clearly *is* gauge invariant. Note that it would not have been if the field and its conjugate were evaluated at different points, as then the gauge parameter would not have canceled.

Now how about the derivative term in the action? Now we run into an issue that has to do with the local character of the gauge transformation: note that if we try to take a derivative we find

$$\partial_\mu\psi'(x) = \partial_\mu\left(e^{i\Lambda(x)}\psi(x)\right) = e^{i\Lambda(x)}(i\partial_\mu\Lambda + \partial_\mu)\psi(x) \quad (4.5)$$

Something very bad has happened because $\Lambda(x)$ depends on space, we have picked up an extra term $\partial_\mu\Lambda$. There is no obvious way to get rid of this, and thus we conclude that the Dirac action as written (4.12) is *not* gauge invariant.

This is a similar problem: the derivative of ψ no longer has nice gauge transformation properties. To fix it, we will do the same thing that we did previously: we define a new object, called the *gauge covariant derivative*:

$$\boxed{D_\mu\psi \equiv (\partial_\mu + ieA_\mu)\psi} \quad (4.6)$$

where I have introduced a new field A_μ , called the *gauge field* or the *gauge connection*. (GR students may be tempted to compare it to the Christoffel connection: this analogy will get even closer when we do non-Abelian gauge theories soon). I have also chosen to take out a factor e for later convenience.

We will *demand* that the gauge-covariant derivative of ψ has a nice gauge transformation property, i.e. for invariance purposes let us demand:

$$D_\mu \psi \rightarrow D'_\mu \psi' = e^{i\Lambda(x)} D_\mu \psi, \quad (4.7)$$

without any awkward inhomogenous piece. Let's see what we require A'_μ to do to make this happen. Expanding out we have

$$D'_\mu \psi' = (\partial_\mu + ieA'_\mu) e^{i\Lambda(x)} \psi(x) = e^{i\Lambda(x)} (i\partial_\mu \Lambda + ieA'_\mu + \partial_\mu) \psi(x) \quad (4.8)$$

$$= e^{i\Lambda(x)} (\partial_\mu + ieA_\mu) \psi(x) \quad (4.9)$$

where the last equality is our demand. We see that this will work out if and only if:

$$\boxed{A'_\mu = A_\mu - \frac{1}{e} \partial_\mu \Lambda(x) \quad \psi'(x) = e^{i\Lambda(x)} \psi(x)}, \quad (4.10)$$

where I have also written down the transformation property of ψ , as the two are related. This is the *gauge-transformation* of A_μ : if it transforms in this way, then our gauge-covariant derivative does indeed do what we want it to do. Note that now the following term:

$$\bar{\psi}(x) \gamma^\mu D_\mu \psi(x) \rightarrow \bar{\psi}'(x) \gamma^\mu D'_\mu \psi'(x) = \bar{\psi}(x) e^{-i\Lambda(x)} \gamma^\mu e^{i\Lambda(x)} D_\mu \psi(x) = \bar{\psi}(x) \gamma^\mu D_\mu \psi(x) \quad (4.11)$$

is gauge-invariant, and thus we have succeeded in taking a derivative in a gauge-invariant way. With this, we see that we can write a fully gauge-invariantized-Dirac-action:

$$S[\psi, \bar{\psi}, A]_{\text{Dirac}} = \int d^4x \bar{\psi}(x) (i\not{D} - m) \psi(x) \quad (4.12)$$

where we just promote the partial to a gauge-covariant derivative. To do this, we had to invent another field called A_μ . This is the price of gauge-invariance.

But now that we have this field A_μ , we may ask what more we can do with it. To see this, let's first note that $D_\mu \psi$ has a nice gauge-transformation property, and thus so will $D_\mu D_\nu \psi$. It thus makes sense to consider the *commutator* of the two derivatives:

$$[D_\mu, D_\nu] \psi = [\partial_\mu, \partial_\nu] \psi + ie([\partial_\mu, A_\nu] - [\partial_\nu, A_\mu]) \psi \quad (4.13)$$

$$= ie(\partial_\mu A_\nu - \partial_\nu A_\mu) \psi \quad (4.14)$$

Notice that this commutator of derivatives is (when acting on ψ) actually not a derivative at all: it is just a number multiplying ψ . Let's call this number $ieF_{\mu\nu}$, i.e.

$$\boxed{F_{\mu\nu} \equiv \partial_\mu A_\nu - \partial_\nu A_\mu} \quad (4.15)$$

$F_{\mu\nu}$ is called the *field-strength* of the gauge field. Now, purely abstractly, we know that $F_{\mu\nu}$ must be gauge-invariant. That's because we have

$$[D_\mu, D_\nu] = ieF_{\mu\nu} \quad (4.16)$$

Now consider acting this equation on ψ and then doing a gauge-transformation on both sides: we know how both sides transform

$$e^{i\Lambda(x)} [D_\mu, D_\nu] \psi = ieF_{\mu\nu} e^{i\Lambda(x)} \psi \quad (4.17)$$

and we can now cancel the $e^{i\Lambda(x)}$, which immediately implies that $F_{\mu\nu}$ is gauge invariant. Of course we could also simply directly use the gauge transformation property of A_μ (4.10) directly, but this more abstract viewpoint will be useful when we get to non-Abelian gauge theories.

It is not hard to convince yourself that random derivatives of A_μ , i.e. $\partial_\mu A_\nu$ by itself, are actually not gauge-invariant. So let us now try to write down the most general renormalizable gauge-invariant action involving our new friend A_μ and $\psi(x)$. It turns out the most general such action is

$$S[\psi, \bar{\psi}] = \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(x)(i\not{D} - m)\psi(x) + \theta \epsilon^{\alpha\beta\mu\nu} F_{\alpha\beta} F_{\mu\nu} \right) \quad (4.18)$$

The first term is gauge-invariant since $F_{\mu\nu}$ is gauge invariant. We have discussed the second term extensively. The third term looks weird, perhaps – it is weird for two distinct reasons: 1. It has an ϵ symbol, which means that it breaks parity and time reversal, and 2. It is a total derivative, and so does not seem to obviously contribute to the equations of motion – but see *The Uses of Instantons* in S. Coleman’s *Aspects of Symmetry*. We can, however, set θ to 0 if we demand that our theory be parity-invariant³ and from now on we will not worry about it.

How about a mass term for the gauge field, e.g.

$$M^2 A_\mu A^\mu \quad (4.19)$$

It is easy to see that this is not gauge-invariant, and thus one says that gauge-invariance forbids bare mass terms for gauge fields. Thus we conclude that the most general gauge-invariant and P -invariant action with a gauge field and a single fermion is just

$$S[\psi, \bar{\psi}, A]_{\text{QED}} = \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(x)(i\not{D} - m)\psi(x) \right) \quad (4.20)$$

This is the action of QED. Note that we have been led to it purely from symmetry principles: *there is nothing else that we could have written down.*

4.2 Some classical aspects of Abelian gauge theory

For a little while, let us focus on the theory given by just the first part of the action above. This turns out to be ordinary Maxwell electrodynamics:

$$S[A]_{\text{Maxwell}} = \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} \right) \quad (4.21)$$

Let’s figure out the classical equations of motion: varying this action with respect to A_μ we find that these equations of motion are just

$$\partial_\mu F^{\mu\nu} = 0 \quad (4.22)$$

If we pick a time direction and write this down in Lorentz-non-invariant notation $F^{0i} = E^i$, $B^i = \frac{1}{2} \epsilon^{ijk} F_{jk}$ then these are simply Maxwell’s equations, and thus this is the Lagrangian for free EM with no matter.

Now, we should note an interesting fact: recall the Klein-Gordon equation of motion for a free scalar field

$$\partial^2 \phi = 0 \quad (4.23)$$

This defines what is called a *well-defined Cauchy problem*, which means that if you pick a time $t = 0$ and specify initial data $\phi(t = 0, \vec{x})$ as well as $\partial_t \phi(t = 0, \vec{x})$, then you can use the equations of motion to propagate ϕ forwards in time unambiguously. This is nice.

³Ok, this is not strictly true: more formally, P flips $\theta \rightarrow -\theta$; however it turns out that when θ is correctly normalized it is a periodic variable with periodicity 2π , which means that there is *two* P -invariant points, one with $\theta = 0$ and one with $\theta = \pi$, as $-\pi + 2\pi = \pi$. This seems like a ridiculous technicality that nobody should ever worry about, except that actually a lot of beautiful physics associated with something called a “topological insulator” arises from the $\theta = \pi$ point.

The fundamental degree of freedom for the Maxwell equation is $A_\mu(x)$, so this is the analog of $\phi(x)$. The above nice time evolution properties of the Klein-Gordon equation are *not* satisfied by the Maxwell equation. This can be understood from staring at the components of the equation, but there is an indirect route: note that if A_μ is a solution to the equations of motion, then so is $A_\mu - \frac{1}{e}\partial_\mu\Lambda(x)$, where Λ is any arbitrary function of spacetime. If we pick $\Lambda(t, \vec{x})$ to be a function that vanishes near $t = 0$ but is different at late t , it is clear that the same initial data at $t = 0$ can lead to two completely different A_μ 's at late time, where they differ by a gauge transformation.

What do we do with this? It looks like our theory has no predictive power, in that knowing what happens at $t = 0$ does not fix what happens at late t . We can save our classical theory by making the following assertion:

Things that are not gauge-invariant are *not physical*.

So it is okay that we cannot unambiguously solve for their time evolution. It is only things that are gauge-invariant – in this case, the components of $F_{\mu\nu}$ – that are physical, and you know from your elementary EM course that there is absolutely no issue solving for their time evolution.

Let's be a bit more explicit. Writing $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$, we find

$$\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) = 0 \quad (4.24)$$

As mentioned earlier, this is not a well-defined problem for time evolution of A_μ . We need to *fix a gauge*: one choice is Lorenz gauge:

$$\partial_\mu A^\mu = 0 \quad (4.25)$$

(You can show that any A_μ can be placed in this form by a gauge transformation). If we do this, then we find the equation

$$\partial_\mu \partial^\mu A_\nu = 0, \quad (4.26)$$

which is now a perfectly nice wave equation that propagates all components of A_μ in time. It is also a wave equation with no mass term: in Fourier space, we get an equation that looks like:

$$(\omega^2 - k^2)A_\nu = 0 \quad (4.27)$$

and thus the on-shell energy of a single photon is $\omega_k = k$: the photon is massless.

However, it is still not true that all components are physical, as there is a *residual gauge invariance* in Lorenz gauge. I will not review here how this works, but you recall from elementary EM that for a plane wave with momentum k_μ it is the *transverse* components of A_μ to the momentum that are physical, and thus there are two physical polarizations. (One way to check this is to note that only those contribute to the field strength tensor; more formally you can find a way to fix the residual gauge invariance in Lorenz gauge).

With this classical understanding under control, we now turn to the quantum theory, where all these subtleties turn out to play a crucial role.

4.3 Quantizing QED

We turn now to the quantum theory. The new thing to do is to *quantize* the Maxwell action, i.e. we want to perform the following path integral:

$$Z = \int [\mathcal{D}A] \exp \left(i \int d^4x \left(-\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right) \right) \quad (4.28)$$

We will ultimately be interested in quantizing the theory with the fermions also included, but first we will focus on this. As we understand in detail by now, the propagator is always equal to the inverse of the operator appearing in the quadratic part of the Lagrangian: let us thus write this in a useful way:

$$S[A] = \int d^4x \left(-\frac{1}{4} (\partial_\mu A_\nu - \partial_\nu A_\mu) (\partial^\mu A^\nu - \partial^\nu A^\mu) \right) \quad (4.29)$$

$$= \int d^4x \frac{1}{2} (A_\nu \partial_\mu \partial^\mu A_\nu - A_\nu \partial^\nu \partial_\mu A^\mu) \quad (4.30)$$

$$= \int d^4x \frac{1}{2} (A_\mu (\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu) A_\nu) \quad (4.31)$$

Now whenever we do a functional integral, we want to invert the differential operator that appears in the quadratic part of the action. Thus we seek to find a photon Feynman propagator $D_{\mu\nu}^F$ that satisfies:

$$(\partial^2 \eta^{\mu\nu} - \partial^\mu \partial^\nu) D_{\nu\rho}^F(x, y) = i\delta_\rho^\mu \delta^{(4)}(x - y) \quad (4.32)$$

There is just one problem with this: no such $D_{\nu\rho}^F$ exists, because the differential operator *has no inverse*. This is perhaps clear if we go to momentum space, when this equation becomes

$$(p^2 \eta^{\mu\nu} - p^\mu p^\nu) D_{\mu\nu}^F(p) = -i\delta_\nu^\mu \quad (4.33)$$

But the matrix on the left-hand side has many zero eigenvalues: in fact, consider acting with it on any test function of the form $p_\mu \alpha(p)$:

$$(p^2 \eta^{\mu\nu} - p^\mu p^\nu) p_\mu \alpha(p) = (p^2 p^\nu - p^2 p^\nu) \alpha(p) = 0 \quad (4.34)$$

Thus the operator has a non-trivial kernel and cannot be inverted.

Physically, it should be clear that the issue is arising from the gauge transformations. The point is that any field configuration that is *pure gauge*

$$A_\mu = \frac{1}{e} \partial_\mu \Lambda \quad (4.35)$$

has field strength $F_{\mu\nu} = 0$, and thus has vanishing action. Thus when performing the path integral over functions in this “pure gauge” direction, the action does not oscillate and nothing suppresses their contribution to the path integral, which thus ends up diverging badly. This divergence is manifesting itself in the non-invertibility of the differential operator.

This is a serious issue. There is a correct way to fix this, called the *Faddeev-Popov procedure*, but sadly I have no time to really explain it – the end result is that we simply end up adding a term of this form to the action:

$$S_{\text{gauge-fixed}}[A] = \int d^4x \frac{1}{2} \left(A_\mu \left(\partial^2 \eta^{\mu\nu} - \left(1 - \frac{1}{\xi} \right) \partial^\mu \partial^\nu \right) A_\nu \right) \quad (4.36)$$

Thus the kinetic term has been modified, and of course now we can now invert the propagator. We go to Fourier space, and the equation to solve is then:

$$\left(-p^2 \eta_{\mu\nu} + \left(1 - \frac{1}{\xi} \right) p_\mu p_\nu \right) D_F^{\nu\rho}(p) = i\delta_\mu^\rho \quad (4.37)$$

This sort of computation comes up again and again in field theory, so in this case I will work it out: consider the most general form of D_F . It is a tensor, and the only two tensor structures are $\eta^{\mu\nu}$ and $p^\mu p^\nu$, so we the answer must be of the form

$$D_F^{\nu\rho}(p) = A(p) \eta^{\nu\rho} + B(p) \frac{p^\nu p^\rho}{p^2} \quad (4.38)$$

$$\mu \text{ wavy } \nu = -\frac{i}{p^2} \left(\gamma^{\mu\nu} - (1 - \xi) \frac{p^\mu p^\nu}{p^2} \right)$$

$$\text{vertex } \mu = -ie \gamma^\mu$$

Figure 1: New Feynman rules involving photons

where I have taken out some factors of p to make the final thing look nicer. Now we plug this ansatz into the equation and expand. We find

$$-p^2 A(p) \delta_\mu^\rho + p_\mu p^\rho \left(A(p) \left(1 - \frac{1}{\xi} \right) - \frac{B(p)}{\xi} \right) = i \delta_\mu^\rho \quad (4.39)$$

(Note taking away the gauge-fixing term corresponds to $\xi \rightarrow \infty$, in which case there is clearly no solution). We find that $A = -\frac{i}{p^2}$, $B = -\frac{i}{p^2}(\xi - 1)$, and thus:

$$D_F^{\nu\rho}(p) = -\frac{i}{p^2} \left(\eta^{\mu\nu} - (1 - \xi) \frac{p^\mu p^\nu}{p^2} \right) \quad (4.40)$$

This is the Feynman propagator for photons, carefully derived.

What is ξ ? The truth is: it is a gauge-fixing parameter, and so it *does not matter*. You can set it to any value you like, and you have to get the same answer for any calculation. This turns out to be guaranteed by the Ward identity in QED – basically dotting k^μ into any amplitude has to annihilate it. Thus people with truly formidable calculational powers will keep it arbitrary, and its cancellation at the end is a good check on the computation.

People (like me) with less formidable calculational powers will often find it convenient to set it to 1 (this is called Feynman gauge) as this makes the photon propagator have fewer terms in it.

This is it! Now you have carefully derived the photon propagator. This is the only tricky thing: from here we can rederive the full machinery of the Feynman rules. If we don't have the fermion then the theory is free; if we do have the fermion then the action is

$$S[\psi, \bar{\psi}, A]_{\text{QED}} = \int d^4x \left(-\frac{1}{4} F^{\mu\nu} F_{\mu\nu} + \bar{\psi}(x) (i\gamma^\mu (\partial_\mu + ieA_\mu) - m) \psi(x) \right) \quad (4.41)$$

and thus we have the fermion Feynman rules that we wrote down last time and the following new rules for the photon:

- The photon propagator is $-\frac{i}{p^2} \left(\eta^{\mu\nu} - (1 - \xi) \frac{p^\mu p^\nu}{p^2} \right)$
- The photon-electron vertex is $-ie\gamma^\mu$.

(If you like, you can carefully derive the interaction vertex by doing the usual business with the sources, etc – no new effects enter from the gauge field).

It is important to note that the interaction vertex *came from the covariant derivative*, and thus we were not free to fix its form. This is an underlying principle: gauge symmetry fixes the possible interactions that can happen.

In the QED course you have extensively calculated things with these Feynman rules, and now you know where they come from.

5 Non-Abelian gauge theories

This part will be abridged. Now that we understand Abelian gauge theories, we are going to move on to the next and final part of this course: *non-Abelian* gauge theories. They are a powerful and nontrivial generalization of the same basic idea, and play a very important role in the Standard Model.

5.1 Non-Abelian gauge invariance

An abstract way of formulating the discussion above was the following: a gauge transformation in the Abelian case was a map from spacetime to the group $U(1)$, i.e. something like $e^{i\Lambda(x)}$. To be completely concrete, for any field ψ we should specify its *charge* q , and then the gauge transformation acts on the field like

$$\psi(x) \rightarrow e^{iq\Lambda(x)}\psi(x) \quad (5.1)$$

(We actually only discussed things with charge $q = 1$, but it should be clear that you could imagine other values for the charge of a field).

Now we will generalize the whole thing to non-Abelian gauge theory. Consider a (compact, simple) non-Abelian group G : the construction will work for any G , but for concreteness let's take G to be $SU(N)$. (In the back of my mind, I will always be imagining $G = SU(2)$, but if your favorite group is $SU(19)$ or something like that don't let me stop you). We will now try to formulate a theory that is invariant under non-Abelian gauge transformations, which are maps $U(x)$ from spacetime to the group G .

Let's review a little bit of group theory: I denote an abstract set of generators of the group by T^a , where the index a runs over the Lie algebra of the group. Let me denote the dimension of the algebra by $d(G)$ (e.g. $d(SU(N)) = N^2 - 1$). The T^a satisfy the following equation:

$$[T^a, T^b] = if^{abc}T^c \quad (5.2)$$

where the f^{abc} are the *structure constants*. Note that there is the Bianchi identity that all operations obey:

$$[T^a, [T^b, T^c]] + [T^b, [T^c, T^a]] + [T^c, [T^a, T^b]] = 0 \quad (5.3)$$

This identity together with the definition of the structure constants implies that they obey:

$$f^{ade}f^{bcd} + f^{bde}f^{cad} + f^{cde}f^{abd} = 0 \quad (5.4)$$

This is called the *Jacobi identity*.

Now there are many different representations for the group, i.e. the abstract T^a can be represented by concrete matrices. Let me denote the actual matrices in the fundamental representation of the group by little t^a . It is of course also true that

$$[t^a, t^b] = if^{abc}t^c \quad (5.5)$$

e.g. for the group $SU(2)$ we have three generators and the $t^i = \frac{1}{2}\sigma^i$ where the σ^i are the Pauli matrices.

Now let us imagine having a fermion ψ that transforms in the fundamental of our $SU(N)$ gauge group: i.e. if $U(x) \in SU(N)$, we want to have the symmetry property:

$$\boxed{\psi(x) \rightarrow \psi'(x) = U(x)\psi(x),} \quad (5.6)$$

where ψ is now an N -component complex vector, as well as a spinor (so technically it has $4N$ components; we are suppressing all of that information here).

Again, we can imagine writing down the Dirac Lagrangian:

$$S[\psi, \bar{\psi}] = \int d^4x \bar{\psi} (i\not{\partial} - m) \psi \quad (5.7)$$

It is clear that this is invariant under $\psi(x) \rightarrow U\psi(x)$, $\bar{\psi}(x) \rightarrow \bar{\psi}(x)U^\dagger$ if U is constant; however, if $U(x)$ depends on space, then clearly the derivatives will not work out. The solution is clear: we now need to construct a non-Abelian gauge-covariant derivative.

We will do this as follows: the property that we want for the non-Abelian covariant derivative is

$$D_\mu \psi(x) \rightarrow D'_\mu \psi'(x) = U(x) D_\mu \psi(x) \quad (5.8)$$

in other words, we want the whole thing to transform homogeneously. From the $U(1)$ case, it should be clear that we will need a new object to do this: our new object is going to be the *non-Abelian gauge field*:

$$A_\mu^a(x) \quad (5.9)$$

where a runs over the Lie algebra of the group. The covariant derivative acting on ψ is taken to be

$$\boxed{D_\mu \psi(x) \equiv (\partial_\mu - ig A_\mu^a t^a) \psi(x)} \quad (5.10)$$

where g is a number called the *gauge-coupling*, and is analogous to the e that we defined in the $U(1)$ case. The t^a are the group representation matrices in the fundamental: they are in the fundamental because I have taken ψ to be in the fundamental. Now let us see what property we require A_μ^a to have for (5.8) to work out:

$$D'_\mu \psi'(x) = (\partial_\mu - ig A'_\mu^a t^a) U(x) \psi(x) \quad (5.11)$$

$$= (\partial_\mu U) \psi + U(x) \partial_\mu \psi - ig A'_\mu^a t^a U(x) \psi(x) \quad (5.12)$$

$$\equiv U(x) (\partial_\mu - ig A_\mu^a t^a) \psi(x), \quad (5.13)$$

where the last equality is what we *want* to be true. There is an interesting wrinkle here, because on the last line $U(x)$ is all the way on the left, whereas on the previous line it is always on the *right*, and because of the non-Abelian-ness we cannot commute it through. Nevertheless, we can still solve for A'_μ^a to find

$$\partial_\mu U - ig A'_\mu^a t^a U = U(-ig A_\mu^a t^a) \quad \rightarrow \quad \boxed{A'_\mu^a(x) t^a = U(x) \left(A_\mu^a(x) t^a + \frac{i}{g} \partial_\mu \right) U^\dagger(x)} \quad (5.14)$$

This is the transformation of the non-Abelian gauge field under a finite gauge transformation parametrized by $U(x)$. Before discussing what it means, it is instructive to also work out the *infinitesimal* version of the gauge transformation. To do this we parametrize $U(x)$ as

$$U(x) = \exp(i\alpha^a(x)t^a) \quad (5.15)$$

and then work to first order in α^a . The fermion transformation law (5.6) is simply

$$\psi'(x) = (1 + i\alpha^a(x)t^a) \psi(x) \quad (5.16)$$

whereas the gauge field one is slightly more intricate:

$$A'_\mu^a(x) t^a = \exp(i\alpha^b(x)t^b) \left(A_\mu^a(x) t^a + \frac{i}{g} \partial_\mu \right) \exp(-i\alpha^c(x)t^c) \quad (5.17)$$

$$= A_\mu^a(x) t^a + i[\alpha^b(x)t^b, A_\mu^a t^a] + \frac{1}{g} \partial_\mu \alpha^c t^c \quad (5.18)$$

$$= A_\mu^a(x) t^a - \alpha^b f^{bac} t^c A_\mu^a + \frac{1}{g} \partial_\mu \alpha^c t^c \quad (5.19)$$

Relabeling some indices and equating terms we find that the individual components of the gauge field transform as transform as

$$A'^a_\mu(x) = A^a_\mu + f^{abc} A^b_\mu \alpha^c + \frac{1}{g} \partial_\mu \alpha^a \quad (5.20)$$

So, what is going on? Here are a few important points:

1. The last term looks like the transformation of an Abelian gauge field.
2. The first term is different: what it means is that the non-Abelian gauge field points in a direction in group space (i.e. it has an a index), and the non-Abelian transformation rotates the direction that it points in. Let us be a bit more precise about this: A^a_μ is *trying* to be in the adjoint. Recall that the adjoint is a special representation. In a sense it tells us how elements of the Lie algebra themselves transform – i.e. they are a special representation where the dimension is the same as that of the Lie algebra itself, so if Φ is in the adjoint it has $N^2 - 1$ components and I can write Φ^a . The way in which something in the adjoint transforms is:

$$\Phi^a t^a \rightarrow \Phi'^a t^a = U \Phi^a t^a U^\dagger \quad (5.21)$$

This can be understood as a linear transformation on the components Φ^a .

Note from (5.14) now that the gauge field *almost* transforms like this, but not quite because of the inhomogenous bit.

3. Finally, this form should make clear that the transformation of the gauge field has nothing to do with the representation of the field it is acting on. (e.g. our ψ was in the fundamental, but that was just for my notational convenience – it is not important). In general you will have many different fields in different representations of a given group, but you only have one non-Abelian gauge field for all of them.

To emphasize this, let's compute the covariant derivative of a field Φ^a that is in the *adjoint*. In particular, if we denote the representation matrices in the adjoint as T^a_G , then from (5.21) we can deduce that the components of these matrices are themselves the structure constants

$$(T^a_G)_{bc} = if^{bac} \quad (5.22)$$

Let me explain the notation: a tells me which element of the algebra I am talking about, whereas b and c denote the row and column of the matrix itself. Using this expression, the covariant derivative of a field in the adjoint is then

$$D_\mu \Phi^a = \partial_\mu \Phi^a - ig A^b_\mu (T^b_G)_{ac} \Phi^c \quad (5.23)$$

$$= \partial_\mu \Phi^a + gf^{abc} A^b_\mu \Phi^c \quad (5.24)$$

where in the last equality we used the definition of the representation matrix in the adjoint. In the same way you can construct the covariant derivative of a field in any representation of the gauge group G . (It is interesting to note that actually the action depends only on the Lie algebra, not on the global properties of the group).

4. As a brief aside: I will sometimes write A_μ with no superscript to mean $A_\mu \equiv A^a_\mu t^a$, i.e. it is a matrix-valued (more properly, a Lie-algebra valued) vector field (more properly, a one-form).

5.2 The Yang-Mills field-strength and action

So we have understood how to take non-Abelian covariant derivatives, and we have seen that this requires us to introduce a non-Abelian gauge field A^a_μ . We would now like to build an action out of these gauge

fields. We will do this in pretty much the same way as in the Abelian case: in particular, let's consider the commutator of two covariant derivatives acting on ψ . From the above discussion, we know that under a gauge transformation:

$$[D_\mu, D_\nu]\psi(x) \rightarrow U(x)[D_\mu, D_\nu]\psi(x) \equiv U(x) (-igF_{\mu\nu}^a t^a) \psi(x) \quad (5.25)$$

i.e. there are no pesky derivative terms. Thus this object has nice transformation properties and is a natural thing to think about. We will call it the field-strength tensor, and as the notation suggests, it is only a multiplicative operator acting on $\psi(x)$, not a differential one. Let's work it out:

$$[\partial_\mu - igA_\mu^a t^a, \partial_\nu - igA_\nu^b t^b]\psi = (-ig(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) t^a - g^2 A_\mu^a A_\nu^b [t^a, t^b]) \psi \quad (5.26)$$

$$= (-ig(\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) t^a - ig^2 A_\mu^b A_\nu^c f^{bca} t^a) \psi \quad (5.27)$$

Comparing this to the definition, we see that in components the field-strength is

$$F_{\mu\nu}^a = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) + gA_\mu^b A_\nu^c f^{abc} \quad (5.28)$$

Importantly, from our previous study we know that under a gauge transformation we have

$$F_{\mu\nu} = F_{\mu\nu}^a t^a = U(x) F_{\mu\nu}^a t^a U^\dagger(x) . \quad (5.29)$$

Thus $F_{\mu\nu}$ transforms in the adjoint. As explained above, the infinitesimal version of this transformation is

$$F_{\mu\nu}^a = F_{\mu\nu}^a - f^{abc} \alpha^b F_{\mu\nu}^c + \mathcal{O}(\alpha^2) \quad (5.30)$$

It is possible to check these properties explicitly by directly transforming the quantities in (5.28); there is a non-trivial cancellation between the terms that makes this possible, and thus the precise factors of g etc. are important. (The GR-savvy student will recall that the construction here is pretty much how we construct the Riemann tensor in GR).

However, now that we have this object, we can construct a non-Abelian-gauge-invariant action for the gauge field. This is essentially completely unique – just consider taking the trace of the matrix-valued field strength squared.

$$S_{\text{YM}}[A] = \int d^4x \left(-\frac{1}{2} \text{tr} (F_{\mu\nu} F^{\mu\nu}) \right) \quad (5.31)$$

This is the celebrated Yang-Mills action. Let's check its properties: due to (5.29), under a gauge transformation we have

$$\text{tr}(F_{\mu\nu}' F'^{\mu\nu}) = \text{tr}(U(x) F_{\mu\nu} U^\dagger(x) F^{\mu\nu} U^\dagger(x)) = \text{tr}(F_{\mu\nu} F^{\mu\nu}) \quad (5.32)$$

and so it is clearly invariant. It's also convenient to work it out in components: we have

$$-\frac{1}{2} \text{tr} (F_{\mu\nu}^a t^a F^{b\mu\nu} t^b) = -\frac{1}{2} F_{\mu\nu}^a F^{b\mu\nu} \text{tr} (t^a t^b) \quad (5.33)$$

From your group theory course, it is always possible to pick the generators of $SU(N)$ so that we have

$$\text{tr} (t^a t^b) = \frac{1}{2} \delta^{ab} \quad (5.34)$$

which is a convention we will use – we then have that the action is

$$S_{\text{YM}}[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \right) \quad (5.35)$$

where we sum over all the indices of the gauge field. This is it. Note that since $F_{\mu\nu}^a$ is non-linear in the field A_μ , this is a non-trivial interacting theory already. In the next section we will study the quantum theory – but first, let's study a few classical aspects.

Consider varying this action: we have

$$\delta S_{\text{YM}}[A] = \int d^4x \left(-\frac{1}{2} F^{a\mu\nu} \delta F_{\mu\nu}^a \right) \quad (5.36)$$

$$= \int d^4x \left(-\frac{1}{2} F^{a\mu\nu} (\partial_\mu \delta A_\nu^a - \partial_\nu \delta A_\mu^a + g(\delta A_\mu^b A_\nu^c + A_\mu^b \delta A_\nu^c) f^{abc}) \right) \quad (5.37)$$

$$= \int d^4x (\partial_\mu F^{a\mu\nu} + g f^{abc} A_\mu^b F^{c\mu\nu}) \delta A_\nu^a \quad (5.38)$$

From here we can read off that the equations of motion are

$$\partial_\mu F^{a\mu\nu} + g f^{abc} A_\mu^b F^{c\mu\nu} = 0 \quad (5.39)$$

Note that this equation is actually quite elegant – we are trying to take the derivative of an object $F_{\mu\nu}^a$ that transforms in the adjoint, so this is actually the gauge-covariant divergence $F_{\mu\nu}^a$:

$$\boxed{D_\mu F^{a\mu\nu} = 0} \quad (5.40)$$

This is the classical Yang-Mills equation. Despite the deceptive notation, it's a very nonlinear and messy equation.

We can of course also couple matter to these gauge fields. Let's then recap the rules for making a gauge theory: pick a gauge group G . Pick some matter content (e.g. some fermions ψ). Pick a representation for the fermions to transform in. Then write down the action:

$$S[A, \psi, \bar{\psi}] = \int d^4x \left(-\frac{1}{2} \text{tr}(F^2) + \bar{\psi} (i\not{D} - m) \psi \right) \quad (5.41)$$

where the gauge covariant derivative for the fermions is

$$D_\mu \psi = \partial_\mu \psi - ig A_\mu^a T^a \psi \quad (5.42)$$

and you pick the matrices T^a depending on which representation you're in. That's it! Note that again this is the most general renormalizable action, and it depends on only a single parameter g , the gauge coupling (which is hidden in the covariant derivatives).

The classical equations of motion following from (5.41) are

$$D_\mu F^{a\mu\nu} = -g (\bar{\psi} \gamma^\nu T^a \psi) \quad (i\not{D} - m) \psi = 0 \quad (5.43)$$

The new term on the right-hand side of the gauge field equation of motion can be understood as the Noether current that you would have gotten if you had treated the gauge symmetry as a global symmetry.

Let's just discuss a few properties of this *classical* theory. Note that it appears that we have $N^2 - 1$ types of massless gauge boson, or *gluon*, one for each Lie algebra element.

Next, we note that the action above is *exactly* the action for QCD, with $N = 3$ and with a sum over a bunch of fermions called *quarks*. In this context we call the non-Abelian gauge fields “gluons”.

$$\boxed{S_{\text{QCD}}[\bar{q}, q, A] = \int d^4x \left(-\frac{1}{2} \text{tr}(F^2) + \sum_i \bar{q}_i (i\not{D} - m_i) q_i \right)} \quad (5.44)$$

where in this equation i runs over the different types of quarks (up, down, charm, strange, top, bottom), they all have different masses m_i that you can look up, $N = 3$, and each quark is in the fundamental, i.e. q_i^a , where $a \in 1 \cdots 3$.

Finally, let us study some simple solutions to the Yang-Mills equation. As the quarks are in the fundamental they come in 3 types called *color*: pick a direction in this $SU(3)$ space and call it *red*. Now imagine (classically) getting one of the quarks to sit still at a point x . In that case the time-component of the current on the right-hand side of (5.43) is a delta function, and to lowest order in g , we find something that looks like

$$D_\mu F^{a\mu t} = g(t^a)_{11}\delta^{(3)}(\vec{x}) \quad (5.45)$$

The right-hand side is proportional to a particular component of the representation matrix. There are all kinds of non-linear terms on the LHS proportional to $gf^{abc}A^2$, but you can show that they do not contribute and you end up with just getting a Maxwell-type equation, which should have an inverse-square solution:

$$\partial_\mu F^{a\mu t} = g(t^a)_{11}\delta^{(3)}(\vec{x}) \quad F^{art} \sim \frac{1}{r^2} \quad (5.46)$$

This suggests that a quark should make a color electric field that falls off as $1/r^2$.

Physically, however, this is completely incorrect. In real life the gluons are not massless, and it is impossible to even *find* an isolated quark, let alone observe that it creates a color electric field that goes like $1/r^2$. Note the difference with EM: there we get more or less the right answer in the full quantum theory from just looking at the classical equations of motion. For QCD we get completely the wrong answer: apparently the classical theory is a terrible approximation to the quantum theory.

5.3 Quantizing non-Abelian gauge theories

Thus motivated, we turn now to the quantum theory. We will temporarily throw out the quarks and work out the Feynman rules for pure Yang-Mills theory, i.e.

$$S[A] = \int d^4x \left(-\frac{1}{4} F_{\mu\nu}^a F^{a\mu\nu} \right) \quad (5.47)$$

We turn back to a general gauge group G .

We first recall that since we have

$$F_{\mu\nu}^a = (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a) + g A_\mu^b A_\nu^c f^{abc} \quad (5.48)$$

this theory is no longer quadratic, and thus even before we add any matter it is *already* interacting. Again, the interaction structure is completely fixed by gauge-invariance. Expanding out the action we see that we have

$$S[A] = \int d^4x \left(-\frac{1}{4} (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g A_\mu^b A_\nu^c f^{abc}) (\partial_\mu A_\nu^a - \partial_\nu A_\mu^a + g A_\mu^d A_\nu^e f^{ade}) \right) \quad (5.49)$$

Thus there will be an interaction vertex that couples three A 's and one that couples four of them.

We can work out the vertices through the usual techniques, which I will not go through in detail on the board as it is just bookkeeping. Note that this the first example we have seen of an interaction term that involves a derivative of the field, and this means that the Feynman rule will explicitly involve the momentum in the line connected to the vertex. I encourage you to work through the details at home (see, in particular, Exercise 2 from Week 4 for a simpler example with the same idea).

This structure is highly constrained – if you just wrote out the most general interaction there would be many other possible terms. In particular, note that it all depends on a single parameter g .

Now that we have all the vertices worked out, we should perform manipulations similar to those that we did for the Abelian case to derive the *propagator* for the gauge field. We will do precisely the same thing as we did there: we will define a gauge-fixing function

$$G_\omega^a(A) \equiv \partial_\mu A^{a\mu} - \omega^a(x) \quad (5.50)$$

Note that this is actually one gauge fixing function for each gluon. We further define a general gauge transformation by $\exp(i\alpha^a(x)t^a)$ and denote the gauge transform of A by α by A^α , whose components are:

$$(A^\alpha)_\mu^a t^a \equiv e^{i\alpha^a(x)t^a} \left(A_\mu^a(x) t^a + \frac{i}{g} \partial_\mu \right) e^{-i\alpha^a(x)t^a} \quad (5.51)$$

It is helpful to work out the infinitesimal version of this transformation, which is

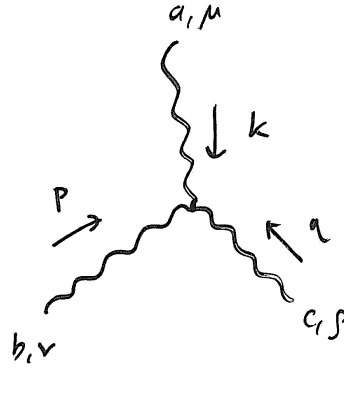
$$(A^\alpha)_\mu^a = A_\mu^a + \frac{1}{g} \partial_\mu \alpha^a + f^{abc} A_\mu^b \alpha^c = A_\mu^a + \frac{1}{g} D_\mu \alpha^a \quad (5.52)$$

where in the last term we used the formula previously derived for the covariant derivative of a field in the adjoint (the field in this case is the gauge-transformation parameter α^a). Note the difference from the Abelian case is in the term with the f^{abc} (relatedly, the fact that the gauge-transformation parameter α^a itself has a Lie algebra index). Just as before, we write:

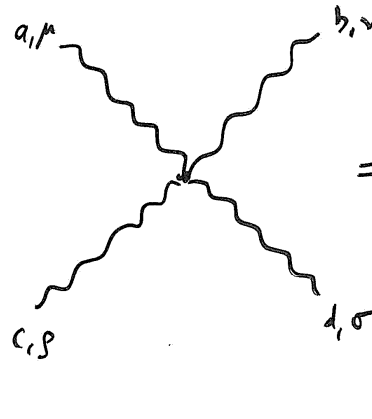
$$1 = \int [\mathcal{D}\alpha] \delta[G_\omega(A^\alpha)] \det \left(\frac{\delta G_\omega(A^\alpha)}{\delta \alpha} \right) \quad (5.53)$$

and plug this whole thing into the path integral to find:

$$Z = \int [\mathcal{D}A \mathcal{D}\alpha] \delta[G_\omega(A^\alpha)] \det \left(\frac{\delta G_\omega(A^\alpha)}{\delta \alpha} \right) \exp(iS[A]) \quad (5.54)$$



$$= g f^{abc} \left[\eta^{\mu\nu} (k-p)^\rho + \eta^{\nu\rho} (p-q)^\mu + \eta^{\rho\mu} (q-k)^\nu \right]$$



$$= -ig^2 \left[f^{abe} f^{cde} (\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho}) + f^{ace} f^{bde} (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\sigma} \eta^{\nu\rho}) + f^{ade} f^{bce} (\eta^{\mu\nu} \eta^{\rho\sigma} - \eta^{\mu\rho} \eta^{\nu\sigma}) \right]$$

Figure 2: Interaction vertices for non-Abelian gauge fields

Now we will encounter a difference. Consider the operator in the functional determinant:

$$\frac{\delta G_\omega(A^\alpha)}{\delta \alpha} = \frac{\delta}{\delta \alpha} (\partial_\mu (A^\alpha)^\mu - \omega) \quad (5.55)$$

$$= \frac{\delta}{\delta \alpha} \left(\partial_\mu \left(A^\mu + \frac{1}{g} D^\mu \alpha \right) - \omega \right) \quad (5.56)$$

$$= \frac{1}{g} \partial_\mu D^\mu \quad (5.57)$$

Recall that D_μ is the covariant derivative of a field acting on the adjoint – thus there is an A in it! In other words, this time the determinant depends on A_μ , and so we cannot take it outside the functional integral.

However everything else goes on exactly as in the Abelian case – we integrate over all ω with a Gaussian weighting, etc. etc. Thus we find at the end of the day

$$Z = \int [\mathcal{D}A] \det \left(\frac{1}{g} \partial_\mu D^\mu \right) \exp \left(iS[A] - i \int d^4x \frac{1}{2\xi} (\partial_\mu A^\mu)^2 \right) \quad (5.58)$$

What about the determinant? We would like to have a useful way to represent it. Think back to two lectures ago when we first studied Grassman integrals – then we learned that if we do an integral over a Grassman variable, we get a determinant in the numerator. In other words, we are free to write

$$\det (\partial_\mu D^\mu) = \int [dc d\bar{c}] \exp \left(i \int d^4x \bar{c} (-\partial_\mu D^\mu) c \right) \quad (5.59)$$

where c^a is a Grassman field that is a scalar in spacetime but is in the *adjoint* with respect to gauge transformations. Note that its Grassman (so its anticommuting) but it is *not* a spinor. Such excitations are called *Faddeev-Popov ghosts*. I will say a lot more about them in a second.

But first let's recap: our final functional integral takes the form

$$Z = \int [\mathcal{D}A \mathcal{D}c \mathcal{D}\bar{c}] \exp \left(i \int d^4x \left(\frac{1}{2} \text{tr}(F^2) - \frac{1}{2\xi} (\partial_\mu A^\mu)^2 - \bar{c} (\partial_\mu D^\mu) c \right) \right) \quad (5.60)$$

where the action is completely gauge fixed, at the costs of introducing mysterious new particles called *ghosts*.

So first the good news – the quadratic part of the action is just like $d(G)$ free Maxwell actions, and the new gauge-fixing term is diagonal in this space. So we can simply repeat the previous analysis to find the non-Abelian gauge field propagator:

$$D_{\mu\nu}^{ab}(p) = -\frac{i}{p^2} \left(\eta_{\mu\nu} - (1-\xi) \frac{p_\mu p_\nu}{p^2} \right) \delta^{ab} \quad (5.61)$$

exactly the same as the Abelian one, except with an extra δ^{ab} making it diagonal in color space. In this case the choice of gauge parameter $\xi = 1$ is called Feynman-t'Hooft gauge.

Now we turn to the ghosts. Writing out their Lagrangian explicitly we find

$$S_{\text{ghost}}[c, \bar{c}] = \int d^4x \bar{c}^a (-\partial^2 \delta^{ac} - g \partial^\mu f^{abc} A_\mu^b) c^c \quad (5.62)$$

This is a completely normal Lagrangian – we can easily work out the ghost propagator and the interaction vertex. The Feynman rules are shown in Figure 4.

The ghost propagator is

$$D_{\text{ghost}}^{ab}(p) = \frac{i}{p^2} \delta^{ab} \quad (5.63)$$

$$\begin{array}{c} a \\ \mu \end{array} \text{---} \text{wavy line} \text{---} \begin{array}{c} b \\ \nu \end{array} = -\frac{i}{p^2} \left(\eta^{\mu\nu} - (1-\xi) \frac{p^\mu p^\nu}{p^2} \right) \delta^{ab}$$

Figure 3: Propagator for non-Abelian gauge fields

$$\begin{array}{c} a \text{---} \text{dashed line} \text{---} b \\ \text{---} \text{wavy line} \text{---} \begin{array}{c} b, \mu \\ \text{---} \text{dashed line} \text{---} c \\ \text{---} \text{dashed line} \text{---} a \\ \text{---} \text{dashed line} \text{---} p \end{array} \end{array} = \frac{i \delta^{ab}}{p^2} = g f^{abc} p^\mu$$

Figure 4: Feynman rules for the ghosts

$$i \longrightarrow j = \frac{i}{\not{k} - m} \delta_{ij}$$

Figure 5: Propagator for fermion transforming in fundamental

Apparently to calculate things we have to include these ghosts. Let me stress that they actually quantitatively change things: for example, the vacuum polarization diagram for a gluon receives a contribution from a ghost loop, which is calculable **Figure**.

Having said that, this is the *complete set of Feynman rules*. Now you can calculate anything you like in perturbation theory: gluon-gluon scattering, etc. Unfortunately the course is almost over, so we won't really get to, though I hope to say some qualitative things.

Now let's return to these ghosts. Here are some questions you may have:

1. *Q. Can I make them in a particle physics experiment?*

A. No, they are just fictitious objects that we made up to represent a functional determinant. Of course they run in loops and can affect Feynman diagrams, but the construction does not permit an "external ghost line", so you need never worry about making a ghost at the LHC.

2. *Q. Shouldn't I be worried that they are scalar particles that are anticommuting? You mumbled something about spin and statistics earlier.*

A. Indeed, they violate the spin-statistics theorem. This is okay, because as I said earlier, they can never be external particles.

3. *Q. What are they the ghosts of?*

A. Interestingly enough, this question has an answer. Recall that the gauge field has 4 components but only 2 actual physical polarizations, meaning that 2 polarizations are spurious. Now imagine a loop – naively these 2 spurious components want to run in the loop, but we shouldn't let them (remember, they are unphysical). One way to interpret the ghosts is that due to the anticommuting minus signs, they *cancel* the two spurious polarizations. This is the reason the ghosts exist.

Finally, we can put back the quarks. If we do this the quark part of the action

$$S_{\text{quark}}[\bar{q}, q, A] = \int d^4x \bar{q}^i (i\gamma^\mu (\partial_\mu - igA_\mu^a t^a)_{ij} q^j) \quad (5.64)$$

where I have temporarily put back the color indices $i \in 1, \dots, N$ on the quarks. From here, we can read off the quark propagator and the interaction of the quark with the gauge field:

- The propagator is the usual one, but is now diagonal in color space: $D_F^{ij}(p) = \frac{i\delta^{ij}}{\not{p} - m}$
- The interaction vertex is $-ig\gamma^\mu t_{ij}^a$. Note the representation matrix plays the role of tying together all of the indices.

From these we can now calculate anything that we like.

5.4 Qualitative discussion: Non-Abelian gauge theory at long distances

These are the Feynman rules – we can now use them to calculate things. In practice, we are almost out of time, so we will not do much quantitatively, but I will try to vaguely argue for what can happen. This ends the precise part of the course, from now on I will be hand-wavy.

Let's remember when we first calculated the one-loop correction to the amplitude for 2-2 scattering in $\lambda\phi^4$ theory. We calculated the diagrams in Section ?? and found that at high energies the amplitude as a function of s (and t, u). At large $s \gg m$ it went like

$$i\mathcal{M} = -i\lambda - i\frac{\lambda^2}{32\pi^2} \left(\log\left(\frac{s}{s_0}\right) + \dots \right) \quad (5.65)$$

where s_0 was a renormalization point and I've not written down the dependence on t and u (but it is similar). Now from staring at this, you can conclude that the interaction λ gets *stronger* at high energies, as increasing s acts in the same direction as increasing λ . Also note this depends crucially on the *sign* of the prefactor of the one-loop term: if it had the other sign, then increasing s would have decreased the interaction strength. You can do a similar calculation in QED and arrive at a similar answer, QED also gets strongly interacting in the UV.

Now let's imagine doing a similar calculation in non-Abelian gauge theory: the goal is to understand whether it gets stronger or weaker as we go to high energies. We need to pick some interaction process: there are many different processes that we could pick, they should all give the same result because they are all controlled by the same number g . So we pick the quark-quark-gluon coupling and calculate. We look at the diagrams. There is only one at tree level, but there are several at one loop.

So now we calculate them all (see Chapter 16.5 of Peskin), then we get a result that looks like (very very schematically)

$$i\mathcal{M} = -ig + i\frac{g^3}{(4\pi)^2} \left(\frac{11}{3}N - \frac{2}{3}n_q \right) \left(\log\left(\frac{E}{E_0}\right) + \dots \right) \quad (5.66)$$

where n_q is the number of quark fields, and where E is the energy flowing into the vertex. I say this is schematic because I am being deliberately vague about what I mean by “energy flowing into the vertex”: it is clear that there are several different energies you could pick. So actually the correct way to parametrize this energy is to *not* do what I described, but to instead compute the *beta function* of the coupling g (again, to be done in the course on renormalization). However this gives you a reasonable idea of what is happening.

The crucial thing to note here is the *sign*! For a small enough number of quark fields n_q , it looks like this thing has the opposite sign as the ϕ^4 theory, and so this means that the theory becomes *weakly coupled* – in fact, a free theory – in the UV. This property is called *asymptotic freedom*: in the context of QCD, it was first discovered by Gross, Politzer and Wilczek, for which they won the Nobel Prize in 2006. To appreciate its importance, we should think not in the UV, but in the IR – if the interaction is weak in the UV, then it seems to be getting stronger and stronger at low energies. So what happens?

There are a few possibilities – one thing that could happen is that the theory ends up getting Higgsed. This happens for the $SU(2) \times U(1)$ part of the Standard Model; you have studied this in that course, so I will discuss it no further here. I stress that you can often still understand that within perturbation theory – you just expand around the new vacuum.

But let us assume this does not happen, and the interaction just gets stronger and stronger. Now you have to take my work on what happens – from a renormalization group analysis it turns out that the strength of the interaction diverges and goes to infinity at a scale that is

$$\Lambda_{QCD} = E_0 \exp\left(-\frac{8\pi^2}{b_0 g^2}\right) \quad (5.67)$$

where E_0 is the renormalization point (recall our definition of the renormalized coupling g depends on this choice of renormalization point), and $b_0 = (\frac{11}{3}N - \frac{2}{3}n_q)$. This is the *strong-coupling scale* of non-Abelian gauge theory: at this scale we lose all hope of doing perturbation theory. This is why our classical intuition does not work well for non-Abelian gauge theory. It is thus scale at which bound states are expected to form – a bound state is a very non-perturbative thing.

So now we have a picture of what is happening for QCD. In the UV, we have a theory of quarks and gluons. They interact and run around in a manner that can be understood in perturbation theory. However, if you go to long distances, the theory becomes more and more strongly interacting. Eventually we don't understand perturbation theory any more, but it *looks* like all the quarks and gluons are bound up into bound states that are all singlets under the gauge transformations. (If they were not singlets, they would interact with the gauge field, which is strongly interacting, and that would be bad – therefore all bound states are singlets). This phenomenon – where in the IR you have only gauge-singlet bound states, and no massless gauge bosons – is called *confinement*.

How do we make gauge-singlets out of quarks? Schematically, there are two ways to combine fundamental degrees of freedom into gauge singlets. We can imagine having $\bar{q}q$ – these are called *mesons*, and examples are the pions. Or we could have $\epsilon^{ijk}q_iq_jq_k$, which are gauge singlets since the determinant of an $SU(3)$ matrix is 1. These are called *baryons*, and two examples that we know and love are the proton and neutron. I stress that these expressions are schematic things designed to capture the quantum numbers – the fields that create mesons are not *actually* simple products of the quark fields, but rather products of the quark fields surrounded by a gluon cloud.

Note that we expect all the masses of these bound states to be around Λ_{QCD} , which is roughly 200 MeV. Indeed the proton and neutron have masses around 938 MeV, while the pions have masses of around 135 MeV (this relative low-ness has to do with chiral symmetry breaking, which I will not discuss).

5.4.1 What are the gauge fields doing in a confined phase?

Now I will go even more qualitative. I would like to try to explain what the gauge fields are doing in the confined phase.

Let's first get oriented by going back to normal $U(1)$ gauge theory, i.e. electromagnetism. Imagine having an electron and a positron separated by a distance R . The positron makes electric field lines that come out and go into the electron – because EM is weakly coupled, we can solve for their shape, and it is an elementary problem in EM to show that the Coulomb energy from the interaction between the two particles goes like

$$E(R) \sim \frac{e^2}{R} \quad (5.68)$$

Now this is *not* what happens in non-Abelian gauge theory. Imagine pinning down a quark and an anti-quark separated by a distance R . They make *color* electric fields – but what we *expect* to happen is that the color electric fields do not spread out in all directions like they do in free EM. Instead they remain focused into a tight tube called a *flux* tube that attaches the quark to the antiquark. This tube actually has a tension, and so the energy is

$$E(R) \sim \alpha' R \quad (5.69)$$

where I have called the tension of the flux tube α' for some reason. But now note that you cannot separate two quarks – if you try to pull them apart, then you pull on this flux tube and this costs infinite energy. It is like the two quarks are attached by a string. This is the idea of confinement. In this picture you can imagine that a meson is a quark and an antiquark attached by a bit of string – the different vibrational modes of the bound state are the different mesons that exist in real life.

This is pretty much as far as we will go. There are a few things you could try to do now: one promising idea was to try to take the flux tube as the starting point: in other words, try to build a theory out of quantizing the dynamics of flux tubes. The problem with this approach is that it *always* resulted in a spin-2 massless particle, which is really annoying because if you look at the QCD spectrum there is no such particle.

Of course eventually it was realized that this theory of strings was really solving an entirely different problem, and this is how string theory as a theory of quantum gravity was discovered. This spin-2 particle is the graviton.

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