\3028PP2021/ $\text{constant} + n\ln(\phi) - n\ln(S_x + \phi^2 S_y) =$

$\text{constant} - 2n\ln\left(\left(\frac{S_x + \phi^2 S_y}{2n}\right)^{1/2}\right) + n\ln(\phi) - \frac{S_x + \phi^2 S_y}{2\left(\frac{S_x + \phi^2 S_y}{2n}\right)} =$

$\text{constant} - n\ln\left(\frac{S_x + \phi^2 S_y}{2n}\right) + n\ln(\phi) - n =$

$\text{constant} - n\ln(S_x + \phi^2 S_y) - n(-\ln(2n)) + n\ln(\phi) - n =$

$n\ln(\phi) - n\ln(S_x + \phi^2 S_y) + \text{constant}$

\3b/ mle of ϕ $\therefore$ $l_p'(\phi) = \frac{\partial l_p(\phi)}{\partial \phi} = \frac{n}{\phi} - \frac{2n S_y \phi}{S_x + \phi^2 S_y}$ $\therefore$

Solve $l_p'(\hat{\phi}) = 0$ $\therefore$ $\cancel{\hat{\phi}^2 =}$ $\frac{n}{\hat{\phi}} - \frac{2n S_y \hat{\phi}}{S_x + \hat{\phi}^2 S_y} = 0$ $\therefore$ $\hat{\phi}^2 = \frac{S_x}{S_y}$ $\therefore$

$l_p''(\phi) = -\frac{n}{\phi^2} - \frac{2n S_y}{S_x + \phi^2 S_y} + \frac{4n S_y^2 \phi^2}{(S_x + \phi^2 S_y)^2}$ $\therefore$

$l_p''(\hat{\phi}) = -\frac{n}{\hat{\phi}^2} < 0$ $\therefore$ the likelihood ratio test statistic is

$\Lambda(\phi_0) = \cancel{-2\ln\Lambda}\ \cancel{-2\ln\left[\frac{L(\phi_0)}{L(\hat{\phi})}\right]}\ -2\left[l_p(\hat{\phi}) - l_p(\phi_0)\right] =$

$2n\left(\ln\hat{\phi} - \ln(S_x + \hat{\phi}^2 S_y) - \ln\phi_0 + \ln(S_x + \phi_0^2 S_y)\right) = 2n\ln\left[\frac{\hat{\phi}(S_x + \phi_0^2 S_y)}{\phi_0(S_x + \hat{\phi}^2 S_y)}\right]$

\3c/ an approximate α confidence interval for ϕ is $\{\phi : \Lambda(\phi) < c\}$ where c is the α-quantile of χ_1^2. The inequality $\Lambda(\phi) < c$ is satisfied if and only if $(\hat{\phi} S_y)(\phi^2) - (2 S_x e^{\frac{c}{2n}})(\phi) + (\hat{\phi} S_x) < 0$ which holds if and only if ϕ lies between $\hat{\phi}\left(e^{\frac{c}{2n}} \pm \sqrt{e^{\frac{c}{n}} - 1}\right)$

\4a/ when any of $x_{(2)}, \ldots, x_{(n-1)}$ are omitted, $\hat{\beta}_{-i} = \hat{\beta}$, but when $x_{(1)}$ is omitted, $\hat{\beta}_{-i} = x_{(n)} - x_{(2)}$, and when $x_{(n)}$ is omitted, $\hat{\beta}_{-i} = x_{(n-1)} - x_{(1)}$ $\therefore$ $\hat{\beta}_J = n\hat{\beta} - \frac{n-1}{n}\sum_{i=1}^{n}\hat{\beta}_{-i}$ $\because$

$\hat{\beta}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\beta}_i$, $\hat{\beta}_i = n\hat{\beta} - (n-1)\hat{\beta}_{-i}$ $\therefore$ $\hat{\beta}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\beta}_i =$

$\frac{1}{n}\sum_{i=1}^{n}(n\hat{\beta} - (n-1)\hat{\beta}_{-i}) = \frac{1}{n}\left(\sum_{i=1}^{n} n\hat{\beta}\right) - \frac{1}{n}\sum_{i=1}^{n}(n-1)\hat{\beta}_{-i} = n\hat{\beta} - \frac{n-1}{n}\sum_{i=1}^{n}\hat{\beta}_{-i} = \hat{\beta}_J =$

$\hat{\beta}_J = n\hat{\beta} - \frac{n-1}{n}\sum_{i=1}^{n}\hat{\beta}_{-i} = n\hat{\beta} - \frac{n-1}{n}\left\{(n-2)\hat{\beta} + x_{(n)} - x_{(2)} + x_{(n-1)} - x_{(1)}\right\} =$

$\hat{\beta} + \frac{n-1}{n}\left\{\hat{\beta} - (x_{(n-1)} - x_{(2)})\right\} \geq \hat{\beta}$ $\because$ the term in the braces is non-negative $\because$ $x_{(n)} > x_{(n-1)}$, $x_{(2)} > x_{(1)}$ $\therefore$ $-x_{(1)} > -x_{(2)}$ $\therefore$

$x_{(n)} - x_{(1)} > x_{(n-1)} - x_{(1)} > x_{(n-1)} - x_{(2)} \therefore \quad \hat{\beta} > x_{(n-1)} - x_{(2)} \therefore$

$\hat{\beta} - (x_{(n-1)} - x_{(2)}) > 0 \quad \hat{\beta}_J \geq \hat{\beta}$

\4bi/ The bias corrected estimate is $4.9-(5.1-4.9)=4.7$

$\because \tilde{\beta}^* = \hat{\beta}^* - bias(\hat{\beta}^*)$ & $bias(\hat{\beta}^*) = E(\hat{\beta}^*) - \beta^* \therefore$

~~&~~ $E(\hat{\beta}^*) = \frac{1}{n}\sum_{i=1}^{n} \hat{\beta}_i^* = 5.1$ & $\beta^* = \hat{\beta} = 4.9 \therefore$

$\tilde{\beta}^* = \hat{\beta}^* - (bias(\hat{\beta}^*)) = \cancel{4.9} \; \hat{\beta}^* - (E(\hat{\beta}^*) - \beta^*) = \cancel{\hat{\beta}^*} \; 4.9 - (5.1-4.9) = 4.7$

\4bii/ The estimated standard error is $\sqrt{1.1} = 1.049 \because$

$SE(\beta^*) = \sqrt{var(\hat{\beta}^*)} = \sqrt{1.1} = 1.049$

\4biii/ The basic bootstrap ~~confidence~~ interval is

$(2\hat{\beta} - \hat{\beta}^*_{(195)}, 2\hat{\beta} - \hat{\beta}^*_{(5)}) = (2.6, 6.8)$

$\because \hat{\beta}^*_{(5)} = 3.0, \quad \hat{\beta}^*_{(195)} = 7.4 \therefore \because \hat{\beta}^*_{(5)} < \hat{\beta}^*_{(195)}$

$2(4.9) - 7.4 = 2.4, \quad 2(4.9) - 3.0 = 6.8 \therefore (2.6, 6.8) \because$

$(2\hat{\beta} - \hat{\beta}^*_{((1-\alpha)B)}, 2\hat{\beta} - \hat{\beta}^*_{(\alpha B)}), \quad \alpha = 0.025, B = 200 \therefore$

$(1-\alpha)B = (1-0.025)200 = 195, \quad \alpha B = 0.025(200) = 5 \therefore$

$(2\hat{\beta} - \hat{\beta}^*_{(195)}, \cancel{2\hat{\beta}_{(5)}} \; 2\hat{\beta} - \hat{\beta}^*_{(5)})$

~~\4b~~ \4biv/ The percentile bootstrap interval is $(\hat{\beta}^*_{(5)}, \hat{\beta}^*_{(195)}) = (3.0, 7.2)$

$\because \hat{\beta}^*_{(5)} < \hat{\beta}^*_{(195)} \therefore \quad (\hat{\beta}^*_{(\alpha B)}, \hat{\beta}^*_{((1-\alpha)B)})$

\4c/ When added to the sample, a further data point is equally likely to have any of the $n+1$ possible ranks. The probability that it lies between $x_{(5)}$ and $x_{(95)}$ is $\therefore \frac{(95-5)}{n+1} = 0.9$

$\because$ if $n=2 \therefore$ there are 3 possible ranks $\therefore n+1$ possible ranks

$\therefore \frac{95-5}{99+1} = \frac{90}{100} = 0.9$

\pp2021/ \1a/ $E(X_1)=\int_0^\infty x f(x;\sigma)\,dx=\int_0^\infty x\,2^{1/2}\pi^{-1/2}\sigma^{-1}e^{-x^2/(2\sigma^2)}\,dx$

$=\sigma^{-1}\sqrt{\frac{2}{\pi}}\int_0^\infty x e^{-x^2/(2\sigma^2)}\,dx$

$\therefore$ let: $y=x^2/(2\sigma^2)$ $\therefore$ $\frac{dy}{dx}=\frac{1}{2\sigma^2}2x$ $\therefore$ $\frac{\sigma^2}{x}dy=dx$,

$x=0\rightarrow y=0$, $x=\infty\rightarrow y\rightarrow\infty$ $\therefore$

$E(X_1)=\sigma^{-1}\sqrt{\frac{2}{\pi}}\int_0^\infty x e^{-x^2/(2\sigma^2)}\,dx=\sigma^{-1}\sqrt{\frac{2}{\pi}}\int_0^\infty x e^{-y}\frac{\sigma^2}{x}\,dy=$

$\sigma^{-1}\sigma^2\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-y}\,dy=\sigma\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-y}\,dy=\sigma\sqrt{\frac{2}{\pi}}\left[-e^{-y}\right]_0^\infty=$

$\sigma\sqrt{\frac{2}{\pi}}\left[-0+e^0\right]=\sigma\sqrt{\frac{2}{\pi}}[1]=\sigma\sqrt{\frac{2}{\pi}}$

$\therefore$ let $E(X_1)=\bar{X}$ $\therefore$

Solving $\bar{X}=\hat{\sigma}\sqrt{\frac{2}{\pi}}$ $\therefore$ The method of moments estimator:

$\hat{\sigma}=\bar{X}\sqrt{\frac{\pi}{2}}$

\1a/ $E(X_1)=\int_0^\infty x f(x;\sigma)\,dx=\int_0^\infty x\,2^{1/2}\pi^{-1/2}\sigma^{-1}e^{-x^2/(2\sigma^2)}\,dx=$

$\sigma^{-1}\sqrt{\frac{2}{\pi}}\int_0^\infty x e^{-x^2/(2\sigma^2)}\,dx$ $\therefore$

Let: $y=x^2/(2\sigma^2)$ $\therefore$ $\frac{dy}{dx}=\frac{1}{2\sigma^2}2x=\frac{x}{\sigma^2}$ $\therefore$ $\frac{\sigma^2}{x}dy=dx$,

$x=0$: $y=0$, $x=\infty$ $y=\infty$ $\therefore$

$E(X_1)=\sigma^{-1}\sqrt{\frac{2}{\pi}}\int_0^\infty x e^{-x^2/(2\sigma^2)}\,dx=\sigma^{-1}\sqrt{\frac{2}{\pi}}\int_0^\infty x e^{-y}\frac{\sigma^2}{x}\,dy=$

$\sigma\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-y}\,dy=\sigma\sqrt{\frac{2}{\pi}}\left[-e^{-y}\right]_{y=0}^{\infty}=\sigma\sqrt{\frac{2}{\pi}}\left[-0+e^0\right]=$

$\sigma\sqrt{\frac{2}{\pi}}[1]=\sigma\sqrt{\frac{2}{\pi}}$

$\therefore$ let $E(X_1)=\bar{X}$ $\therefore$

Solving $\bar{X}=\hat{\sigma}\sqrt{\frac{2}{\pi}}$ $\therefore$ The method of moments estimator:

$\hat{\sigma}=\bar{X}\sqrt{\frac{\pi}{2}}$

\1b/ $E(\hat{\sigma})=E\left(\bar{X}\sqrt{\frac{\pi}{2}}\right)=E(\bar{X})\sqrt{\frac{\pi}{2}}=E\left(\frac{1}{n}\sum_{i=1}^n X_i\right)\sqrt{\frac{\pi}{2}}=\sqrt{\frac{\pi}{2}}\frac{1}{n}E\left(\sum_{i=1}^n X_i\right)=$

$\sqrt{\frac{\pi}{2}}\frac{1}{n}\sum_{i=1}^n E(X_i)=\sqrt{\frac{\pi}{2}}\frac{1}{n}\sum_{i=1}^n E(X_1)=\sqrt{\frac{\pi}{2}}\frac{1}{n}n\,E(X_1)=\sqrt{\frac{\pi}{2}}E(X_1)=\sqrt{\frac{\pi}{2}}\sigma\sqrt{\frac{2}{\pi}}=\sigma$

$\therefore$ $\mathrm{Bias}(\hat{\sigma})=E(\hat{\sigma})-\sigma=\sigma-\sigma=0$ $\therefore$ $\hat{\sigma}$ is unbiased

$\mathrm{var}(X_1)=E(X_1^2)-E(X_1)^2=\sigma^2-\left(\sigma\sqrt{\frac{2}{\pi}}\right)^2=\sigma^2-\sigma^2\frac{2}{\pi}=\sigma^2\left(1-\frac{2}{\pi}\right)$ $\therefore$

$\mathrm{var}(\hat{\sigma})=\mathrm{var}\left(\sqrt{\frac{\pi}{2}}\bar{X}\right)=\left(\sqrt{\frac{\pi}{2}}\right)^2\mathrm{var}(\bar{X})=\frac{\pi}{2}\mathrm{var}\left(\frac{1}{n}\sum_{i=1}^n X_i\right)=\frac{\pi}{2n^2}\mathrm{var}\left(\sum_{i=1}^n X_i\right)=$

$\frac{\pi}{2n^2}\sum_{i=1}^n\mathrm{var}(X_i)=\frac{\pi}{2n^2}\sum_{i=1}^n\mathrm{var}(X_1)=\frac{\pi}{2n^2}n\,\mathrm{var}(X_1)=\frac{\pi}{2n}\mathrm{var}(X_1)=\frac{\pi}{2n}\sigma^2\left(1-\frac{2}{\pi}\right)=$

$\left(\frac{\pi}{2n}-\frac{1}{n}\right)\sigma^2=\left(\frac{\pi}{2}-1\right)\frac{\sigma^2}{n}$ $\therefore$

$\mathrm{Bias}(\hat{\sigma})=0\rightarrow 0$, $\mathrm{var}(\hat{\sigma})=\left(\frac{\pi}{2}-1\right)\frac{\sigma^2}{n}\rightarrow 0$ as $n\rightarrow\infty$ $\therefore$

The bias and variance converge to zero as $n \to \infty$ $\therefore$

$\hat{\sigma}$ is consistent

1c The likelihood: $L(\sigma; \underline{x}) = \prod_{i=1}^{n} f(x_i; \sigma)$

$= \prod_{i=1}^{n} 2^{1/2} \pi^{-1/2} \sigma^{-1} e^{-x_i^2/(2\sigma^2)} \propto \prod_{i=1}^{n} \sigma^{-1} e^{-x_i^2/(2\sigma^2)} = \sigma^{-n} e^{-\sum_{i=1}^{n} x_i^2/(2\sigma^2)}$

$\therefore$ The log-likelihood: $l(\sigma, \underline{x}) = \ln L(\sigma; \underline{x}) \propto \ln\left[\sigma^{-n} e^{-\sum_{i=1}^{n} x_i^2/(2\sigma^2)}\right] =$

$-n \ln \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{n} x_i^2$ $\therefore$

$l(\sigma; \underline{x}) = \text{constant} - n \ln \sigma - \frac{1}{2\sigma^2} \sum_{i=1}^{n} x_i^2$ $\therefore$

$l'(\sigma; \underline{x}) = \frac{-n}{\sigma} + \frac{1}{\sigma^3} \sum_{i=1}^{n} x_i^2$ $\therefore$

Solve $l'(\hat{\sigma}; \underline{x}) = 0$: $-\frac{n}{\hat{\sigma}} + \frac{1}{\hat{\sigma}^3} \sum_{i=1}^{n} x_i^2 = 0$ $\therefore$ $\frac{1}{\hat{\sigma}^3} \sum_{i=1}^{n} x_i^2 = \frac{n}{\hat{\sigma}}$ $\therefore$

$\frac{1}{n} \sum_{i=1}^{n} x_i^2 = \hat{\sigma}^2$ $\therefore$ or $n\hat{\sigma}^2 = \sum_{i=1}^{n} x_i^2$ $\therefore$

$\hat{\sigma} = +\left(\frac{1}{n} \sum_{i=1}^{n} x_i^2\right)^{1/2}$ $\because \sigma > 0$ $\therefore$

$l''(\sigma, \underline{x}) = \frac{n}{\sigma^2} - \frac{3}{\sigma^4} \sum_{i=1}^{n} x_i^2$ $\therefore$

$l''(\hat{\sigma}, \underline{x}) = \frac{n}{\hat{\sigma}^2} - \frac{3}{\hat{\sigma}^4} \sum_{i=1}^{n} x_i^2 = \frac{n}{\frac{1}{n}\sum_{i=1}^{n} x_i^2} - \frac{3}{\left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)^2} \sum_{i=1}^{n} x_i^2 = \frac{n}{\hat{\sigma}^2} - \frac{3}{\hat{\sigma}^4} \sum_{i=1}^{n} x_i^2 =$

$\frac{n}{\hat{\sigma}^2} - \frac{3}{\hat{\sigma}^4}(n\hat{\sigma}^2) = \frac{n}{\hat{\sigma}^2} - \frac{3n}{\hat{\sigma}^2} = -\frac{2n}{\hat{\sigma}^2} < 0$ $\because$ $\frac{2n}{\hat{\sigma}^2} > 0$ $\therefore$

$l''(\hat{\sigma}; \underline{x}) < 0$ $\therefore$ $\hat{\sigma}$ is the mle

1d The expected information is $I(\sigma) = -E(l''(\sigma)) =$

$-E\left(\frac{n}{\sigma^2} - \frac{3}{\sigma^4} \sum_{i=1}^{n} x_i^2\right) = -E\left(\frac{n}{\sigma^2}\right) - E\left(-\frac{3}{\sigma^4} \sum_{i=1}^{n} x_i^2\right) = -\frac{n}{\sigma^2} E(1) + \frac{3}{\sigma^4} E\left(\sum_{i=1}^{n} x_i^2\right) =$

$-\frac{n}{\sigma^2} + \frac{3}{\sigma^4} \sum_{i=1}^{n} E(\mathbf{x}_i^2) = -\frac{n}{\sigma^2} + \frac{3}{\sigma^4} \sum_{i=1}^{n} E(X_i^2) = -\frac{n}{\sigma^2} + \frac{3n}{\sigma^4} E(X_i^2) =$

$-\frac{n}{\sigma^2} + \frac{3n}{\sigma^4} \sigma^2 = -\frac{n}{\sigma^2} + \frac{3n}{\sigma^2} = \frac{2n}{\sigma^2}$ $\therefore$

The asymptotic distribution of the mle is

$N\left(\sigma, \frac{1}{I(\sigma)}\right) = N\left(\sigma, \left(\frac{2n}{\sigma^2}\right)^{-1}\right) = N\left(\sigma, \frac{\sigma^2}{2n}\right)$

1e The efficiency of the Method of Moments estimator

is $I(\sigma)^{-1}/\text{var}(\hat{\sigma}) = \frac{1}{I(\sigma)\text{var}(\hat{\sigma})} = \frac{1}{I(\sigma)} \times \frac{1}{\text{var}(\hat{\sigma})} =$

$\frac{\sigma^2}{2n} \times \frac{1}{\text{var}(\hat{\sigma})} = \frac{\sigma^2}{2n}(\text{var}(\hat{\sigma}))^{-1} = \frac{\sigma^2}{2n}\left(\left(\frac{\pi}{2} - 1\right)\frac{\sigma^2}{n}\right)^{-1} = \frac{\sigma^2}{2n}\left(\frac{\pi}{2} - 1\right)^{-1} \frac{n}{\sigma^2} =$

$\frac{1}{2} \frac{1}{\frac{\pi}{2} - 1} = \frac{1}{2\left(\frac{\pi}{2} - 1\right)} = \frac{1}{\pi - 2}$

1f The likelihood ratio is $\Lambda = \frac{L(\sigma_1)}{L(\sigma_0)} = \left(\frac{\sigma_0}{\sigma_1}\right)^n e^{-\frac{1}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right) \sum_{i=1}^{n} x_i^2}$

$= \frac{\sigma_0^n}{\sigma_1^n} e^{\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right) \sum_{i=1}^{n} x_i^2}$ $\because \sigma_1 > \sigma_0$ $\therefore$ $\frac{1}{\sigma_0} > \frac{1}{\sigma_1}$ $\therefore$ $\frac{1}{\sigma_0^2} > \frac{1}{\sigma_1^2}$ $\therefore$ $\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0$ $\therefore$

$\frac{L(\sigma_1)}{L(\sigma_0)}$ increases as ~~the~~ $\sum_{i=1}^{n} x_i^2$ increases $\therefore$

$\frac{L(\sigma_1)}{L(\sigma_0)}$ is large when $\sum_{i=1}^{n} x_i^2$ is large $\therefore$

PP2021 The critical region has the form $\{\underline{x} : \sum_{i=1}^{n} x_i^2 > c\}$ for a critical value c

~~Vol~~

1a $E(X_1) = E(X) = \int_0^\infty x f(x;\sigma)\,dx = \int_0^\infty x 2^{1/2} \pi^{-1/2} \sigma^{-1} e^{-x^2/(2\sigma^2)}\,dx =$

$\sigma^{-1}\sqrt{\frac{2}{\pi}} \int_0^\infty x e^{-x^2/(2\sigma^2)}\,dx \therefore$

let: $y = x^2/(2\sigma^2) \therefore \frac{dy}{dx} = \frac{1}{2\sigma^2} 2x = \frac{x}{\sigma^2} \therefore \frac{\sigma^2}{x} dy = dx$,

$x = 0 : y = 0$, $x = \infty : y = \infty \therefore$

$E(X_1) = \sigma^{-1}\sqrt{\frac{2}{\pi}} \int_0^\infty x e^{-x^2/(2\sigma^2)}\,dx = \sigma^{-1}\sqrt{\frac{2}{\pi}} \int_0^\infty x e^{-y} \frac{\sigma^2}{x}\,dy = \sigma\sqrt{\frac{2}{\pi}} \int_0^\infty e^{-y}\,dy =$

$\sigma\sqrt{\frac{2}{\pi}} \left[-e^{-y}\right]_{y=0}^{\infty} = \sigma\sqrt{\frac{2}{\pi}}\left[-0 + e^0\right] = \sigma\sqrt{\frac{2}{\pi}}[1] = \sigma\sqrt{\frac{2}{\pi}}$

$\therefore \bar{X} \to E(X_1)$ as $n \to \infty \therefore m_1(\hat{\sigma}) = \bar{X} \therefore m_1(\hat{\sigma}) = \hat{\sigma}\sqrt{\frac{2}{\pi}} \therefore$

solving $\bar{X} = \hat{\sigma}\sqrt{\frac{2}{\pi}} \therefore$

The method of moments estimator: $\hat{\sigma} = \bar{X}\sqrt{\frac{\pi}{2}}$

1b $E(\hat{\sigma}) = E\left(\bar{X}\sqrt{\frac{\pi}{2}}\right) = E(\bar{X})\sqrt{\frac{\pi}{2}} = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right)\sqrt{\frac{\pi}{2}} = \sqrt{\frac{\pi}{2}}\frac{1}{n} E\left(\sum_{i=1}^{n} X_i\right) =$

$\sqrt{\frac{\pi}{2}}\frac{1}{n}\sum_{i=1}^{n} E(X_i) = \sqrt{\frac{\pi}{2}}\frac{1}{n}\sum_{i=1}^{n} E(X_1) = \sqrt{\frac{\pi}{2}}\frac{1}{n} n E(X_1) = \sqrt{\frac{\pi}{2}} E(X_1) = \sqrt{\frac{\pi}{2}}\sigma\sqrt{\frac{2}{\pi}} = \sigma$

$\therefore \text{Bias}(\hat{\sigma}) = E(\hat{\sigma}) - \sigma = \sigma - \sigma = 0 \therefore \hat{\sigma}$ is unbiased

$\text{var}(X_1) = E(X_1^2) - E(X_1)^2 = \sigma^2 - \left(\sigma\sqrt{\frac{2}{\pi}}\right)^2 = \sigma^2 - \sigma^2\frac{2}{\pi} = \sigma^2\left(1 - \frac{2}{\pi}\right) \therefore$

$\text{var}(\hat{\sigma}) = \text{var}\left(\sqrt{\frac{\pi}{2}}\bar{X}\right) = \left(\sqrt{\frac{\pi}{2}}\right)^2 \text{var}(\bar{X}) = \frac{\pi}{2}\text{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{\pi}{2n^2}\text{var}\left(\sum_{i=1}^{n} X_i\right) =$

$\frac{\pi}{2n^2}\sum_{i=1}^{n}\text{var}(X_i) = \frac{\pi}{2n^2}\sum_{i=1}^{n}\text{var}(X_1) = \frac{\pi}{2n^2} n\,\text{var}(X_1) = \frac{\pi}{2n}\text{var}(X_1) = \frac{\pi}{2n}\sigma^2\left(1 - \frac{2}{\pi}\right) =$

$\left(\frac{\pi}{2n} - \frac{1}{n}\right)\sigma^2 = \left(\frac{\pi}{2} - 1\right)\frac{\sigma^2}{n} \therefore$

$\text{Bias}(\hat{\sigma}) = 0 \to 0$, $\text{var}(\hat{\sigma}) = \left(\frac{\pi}{2} - 1\right)\frac{\sigma^2}{n} \to 0$ as $n \to \infty \therefore$

The bias and variance converge to zero as $n \to \infty \therefore$

$\hat{\sigma}$ is consistent.

1c The likelihood: $L(\sigma; \underline{x}) = \prod_{i=1}^{n} f(x_i; \sigma) =$

$\prod_{i=1}^{n} 2^{1/2}\pi^{-1/2}\sigma^{-1} e^{-x_i^2/(2\sigma^2)} = 2^{\frac{1}{2}n}\pi^{-\frac{1}{2}n}\sigma^{-n} e^{-\sum_{i=1}^{n} x_i^2/(2\sigma^2)} =$

$2^{n/2}\pi^{-n/2}\sigma^{-n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2} = 2^{n/2}\pi^{-n/2}\sigma^{-n} e^{-\frac{1}{2\sigma^2} n\overline{x^2}} \therefore$

The log-likelihood: $l(\sigma; \underline{x}) = \ln L(\sigma; \underline{x}) = \ln\left(2^{n/2}\pi^{-n/2}\sigma^{-n} e^{-\frac{1}{2\sigma^2} n\overline{x^2}}\right) =$

$\ln(2^{n/2}\pi^{-n/2}) - n\ln\sigma - \frac{1}{2\sigma^2} n\overline{x^2} \therefore$

$l'(\sigma; \underline{x}) = -\frac{n}{\sigma} + \frac{1}{\sigma^3} n\overline{x^2} \therefore$

solve $l'(\hat{\sigma}; \underline{x}) = 0 : -\frac{n}{\hat{\sigma}} + \frac{1}{\hat{\sigma}^3} n\overline{x^2} = 0 \therefore \frac{1}{\hat{\sigma}^3}\overline{x^2} = \frac{1}{\hat{\sigma}} \therefore$

$\overline{x^2} = \hat{\sigma}^2$ ∴

$\hat{\sigma} = +(\overline{x^2})^{1/2}$ ∵ $\sigma > 0$ ∴

$l''(\sigma; \underline{x}) = \frac{n}{\sigma^2} - \frac{3}{\sigma^4}\sum_{i=1}^{n} x_i^2 = \frac{n}{\sigma^2} - \frac{3}{\sigma^4} n\overline{x^2}$ ∴

$l''(\hat{\sigma}; \underline{x}) = \frac{n}{\hat{\sigma}^2} - \frac{3}{\hat{\sigma}^4} n\overline{x^2} = \frac{n}{\overline{x^2}} - \frac{3}{(\overline{x^2})^2} n\overline{x^2} = \frac{n}{\overline{x^2}} - \frac{3n}{\overline{x^2}} = -\frac{2n}{\overline{x^2}} < 0$ ∵

∵ $x^2 \geq 0$ ∴ $\overline{x^2} > 0$ ∴ $\frac{1}{\overline{x^2}} > 0$ ∴ $\frac{2n}{\overline{x^2}} > 0$ ∴

$l''(\hat{\sigma}; \underline{x}) < 0$ ∴ $\hat{\sigma}$ is the mle

\1d/ The expected information is $I(\sigma) = -E(l''(\sigma)) =$

$-E\left(\frac{n}{\sigma^2} - \frac{3}{\sigma^4}\sum_{i=1}^{n} x_i^2\right) = -E\left(\frac{n}{\sigma^2}\right) - E\left(-\frac{3}{\sigma^4}\sum_{i=1}^{n} x_i^2\right) = -\frac{n}{\sigma^2}E(1) + \frac{3}{\sigma^4}E\left(\sum_{i=1}^{n} x_i^2\right) =$

$-\frac{n}{\sigma^2} + \frac{3}{\sigma^4}\sum_{i=1}^{n} E(x_i^2) = -\frac{n}{\sigma^2} + \frac{3}{\sigma^4}\sum_{i=1}^{n} E(X_1^2) = -\frac{n}{\sigma^2} + \frac{3n}{\sigma^4}E(X_1^2) = -\frac{n}{\sigma^2} + \frac{3n}{\sigma^4}\sigma^2 =$

$-\frac{n}{\sigma^2} + \frac{3n}{\sigma^2} = \frac{2n}{\sigma^2}$ ∴

The asymptotic distribution of the mle is: $N(\sigma, \frac{1}{I(\sigma)}) =$

$N(\sigma, (\frac{2n}{\sigma^2})^{-1}) = N(\sigma, \frac{\sigma^2}{2n})$

\1e/ The essiciency of the method of moments estimator is $I(\sigma)^{-1} / var(\hat{\sigma}) = \frac{1}{I(\sigma) \cdot var(\hat{\sigma})} = \frac{1}{I(\sigma)} \times \frac{1}{var(\hat{\sigma})} = \frac{\sigma^2}{2n} \times \frac{1}{var(\hat{\sigma})} =$

$\frac{\sigma^2}{2n}(var(\hat{\sigma}))^{-1} = \frac{\sigma^2}{2n}\left(\left(\frac{\pi}{2} - 1\right)\frac{\sigma^2}{n}\right)^{-1} = \frac{\sigma^2}{2n}\left(\frac{\pi}{2} - 1\right)^{-1}\frac{n}{\sigma^2} = \frac{1}{2}\frac{1}{\frac{\pi}{2} - 1} = \frac{1}{2(\frac{\pi}{2} - 1)} = \frac{1}{\pi - 2}$

\1f/ The whetihood ratio is $\Lambda = \frac{L(\sigma_1)}{L(\sigma_0)} =$

$\left[2^{n/2}\pi^{-n/2}\sigma_1^{-n} e^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^{n} x_i^2}\right] / \left[2^{n/2}\pi^{-n/2}\sigma_0^{-n} e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n} x_i^2}\right] =$

$\frac{\sigma_1^{-n}}{\sigma_0^{-n}} \frac{e^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^{n} x_i^2}}{e^{-\frac{1}{2\sigma_0^2}\sum_{i=1}^{n} x_i^2}} = \left(\frac{\sigma_0}{\sigma_1}\right)^n e^{-\frac{1}{2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)\sum_{i=1}^{n} x_i^2} = \frac{\sigma_0^n}{\sigma_1^n} e^{\frac{1}{2}\left(\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2}\right)\sum_{i=1}^{n} x_i^2}$

∵ $\sigma_1 > \sigma_0$ ∴ $\frac{1}{\sigma_0} > \frac{1}{\sigma_1}$ ∴ $\frac{1}{\sigma_0^2} > \frac{1}{\sigma_1^2}$ ∴ $\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0$ ∴

$\frac{L(\sigma_1)}{L(\sigma_0)}$ increases as $\sum_{i=1}^{n} x_i^2$ increases ∴

$\frac{L(\sigma_1)}{L(\sigma_0)}$ is large when $\sum_{i=1}^{n} x_i^2$ is large ∴

The critical region has the form $\left\{\underline{x} : \sum_{i=1}^{n} x_i > c\right\}$ for a critical value c

\1g/ The wald test statistic is [illegible] $(\hat{\sigma} - \sigma_0)^2 I(\hat{\sigma}) =$

$(\hat{\sigma} - \sigma_0)^2 \frac{2n}{\hat{\sigma}^2} = \left((\hat{\sigma} - \sigma_0)\frac{1}{\hat{\sigma}}\right)^2 2n = \left(1 - \frac{\sigma_0}{\hat{\sigma}}\right)^2 2n$,

The score test statistic is $U(\sigma_0)^2 / I(\sigma_0) = \frac{(l'(\sigma_0))^2}{I(\sigma_0)} =$

$(l'(\sigma_0))^2 \frac{1}{I(\sigma_0)} = (l'(\sigma_0))^2 \frac{\sigma_0^2}{2n} = \left(-\frac{n}{\sigma_0} + \frac{1}{\sigma_0^3}\sum_{i=1}^{n} x_i^2\right)^2 \frac{\sigma_0^2}{2n} =$

$\frac{\sigma_0^2}{2n}\left(-\frac{n}{\sigma_0} + \frac{1}{\sigma_0^3} n\hat{\sigma}^2\right)^2 = \frac{\sigma_0^2}{2n} n^2 \left(-\frac{1}{\sigma_0} + \frac{\hat{\sigma}^2}{\sigma_0^3}\right)^2 = \frac{n\sigma_0^2}{2}\left(\frac{\hat{\sigma}^2}{\sigma_0^3} - \frac{1}{\sigma_0}\right)^2 =$

$\frac{L(\sigma_1)}{L(\sigma_0)}$ is large when $\sum_{i=1}^{n} x_i$

\PP2021/ $\frac{n}{2}\left(\sigma_0\left(\frac{\hat{\sigma}^2}{\sigma_0^3}-\frac{1}{\sigma_0}\right)\right)^2=\frac{n}{2}\left(\frac{\hat{\sigma}^2}{\sigma_0^2}-1\right)^2$

their approximate null distributions is χ_1^2 when n is large

\1g/ The wald test statistic: $(\hat{\sigma}-\sigma_0)^2 I(\hat{\sigma})=(\hat{\sigma}-\sigma_0)^2\frac{2n}{\hat{\sigma}^2}=$
$\left((\hat{\sigma}-\sigma_0)\frac{1}{\hat{\sigma}}\right)^2 2n=\left(1-\frac{\sigma_0}{\hat{\sigma}}\right)^2 2n$

The score test statistic is: $U(\sigma_0)^2/I(\sigma_0)=\frac{(l'(\sigma_0))^2}{I(\sigma_0)}=$
$(l'(\sigma_0))^2\frac{1}{I(\sigma_0)}=(l'(\sigma_0))^2\frac{\sigma_0^2}{2n}=\left(-\frac{n}{\sigma_0}+\frac{1}{\sigma_0^3}\sum_{i=1}^n x_i^2\right)^2\frac{\sigma_0^2}{2n}=$
$\frac{\sigma_0^2}{2n}\left(-\frac{n}{\sigma_0}+\frac{1}{\sigma_0^3}n\hat{\sigma}^2\right)^2=\frac{\sigma_0^2}{2n}n^2\left(-\frac{1}{\sigma_0}+\frac{\hat{\sigma}^2}{\sigma_0^3}\right)^2=\frac{n\sigma_0^2}{2}\left(\frac{\hat{\sigma}^2}{\sigma_0^3}-\frac{1}{\sigma_0}\right)^2=$
$\frac{n}{2}\left(\sigma_0\left(\frac{\hat{\sigma}^2}{\sigma_0^3}-\frac{1}{\sigma_0}\right)\right)^2=\frac{n}{2}\left(\frac{\hat{\sigma}^2}{\sigma_0^2}-1\right)^2$

their ~~anti Zeto~~ approximate null distributions is χ_1^2 when n is large

\2ai/ we know $\frac{X_i}{\sigma}$ are independent $N(0,1)$ random variables
~~Y~~ $\because X_i\sim N(0,\sigma^2)$ $\therefore \frac{X_i-\mu}{\sigma}\sim N(0,1)$ $\therefore \frac{X_i-0}{\sigma}=\frac{X_i}{\sigma}\sim N(0,1)$ $\therefore$
$Y\sim\chi_n^2$ $\because \frac{X_i}{\sigma}\sim N(0,1)$ $\therefore \frac{X_i}{\sigma}=Z_i\sim N(0,1)$ $\therefore$
$Y=\sum_{i=1}^n\frac{X_i^2}{\sigma^2}=\sum_{i=1}^n\left(\frac{X_i}{\sigma}\right)^2=\sum_{i=1}^n Z_i^2\sim\chi_n^2$ $\therefore Y\sim\chi_n^2$ $\therefore$
The χ_n^2 distribution has no unknown parameters $\therefore$ Y is a pivot

\2ai Sol/ know $\frac{X_i}{\sigma}$ are independent $N(0,1)$ ~~rv~~ r.v.s $\therefore$
$Y\sim\chi_n^2$ $\therefore$ The χ_n^2 distribution has no ~~parameters~~ unknown parameters $\therefore$ Y is a pivot

\2ai/ $X_i\sim N(0,\sigma^2)$ are independent $\therefore$
$\frac{X_i-\mu}{\sigma}=\frac{X_i-0}{\sigma}=\frac{X_i}{\sigma}\sim N(0,1)$ are independent,
$Z_i\sim N(0,1)$ $\therefore$ $\frac{X_i}{\sigma}=Z_i$ $\therefore$
$Y=\sum_{i=1}^n\frac{X_i^2}{\sigma^2}=\sum_{i=1}^n\left(\frac{X_i}{\sigma}\right)^2=\sum_{i=1}^n Z_i^2\sim\chi_n^2$ $\therefore Y\sim\chi_n^2$ $\therefore$
The χ_n^2 distribution has no unknown parameters $\therefore$ Y is a pivot

\2aii/ $0.9=Pr(q_{0.05}<Y<q_{0.95})=Pr\left(q_{0.05}<\sum_{i=1}^n\frac{X_i^2}{\sigma^2}<q_{0.95}\right)=$
$Pr\left(q_{0.05}<\frac{1}{\sigma^2}\sum_{i=1}^n X_i^2<q_{0.95}\right)=Pr\left(\frac{1}{q_{0.95}}\sum_{i=1}^n X_i^2<\sigma^2<\frac{1}{q_{0.05}}\sum_{i=1}^n X_i^2\right)=$
$Pr\left(\sum_{i=1}^n X_i^2/q_{0.95}<\sigma^2<\sum_{i=1}^n X_i^2/q_{0.05}\right)$
a 90% confidence interval for σ^2 is $\left(\sum_{i=1}^n X_i^2/q_{0.95},\ \sum_{i=1}^n X_i^2/q_{0.05}\right)$

\2aii/ let q_p denote the p-quantile of χ^2_n $\therefore$

$0.9 = Pr(q_{0.05} < Y < q_{0.95}) = Pr\left(q_{0.05} < \sum_{i=1}^{n} \frac{X_i^2}{\sigma^2} < q_{0.95}\right) =$

$Pr\left(\frac{1}{q_{0.95}} \sum_{i=1}^{n} X_i^2 < \sigma^2 < \frac{1}{q_{0.05}} \sum_{i=1}^{n} X_i^2\right) = Pr\left(\sum_{i=1}^{n} \frac{X_i^2}{q_{0.95}} < \sigma^2 < \sum_{i=1}^{n} \frac{X_i^2}{q_{0.05}}\right)$ $\therefore$

a 90% confidence interval for σ^2 is $\left(\sum_{i=1}^{n} \frac{X_i^2}{q_{0.95}}, \sum_{i=1}^{n} \frac{X_i^2}{q_{0.05}}\right)$

\2bi/ The ancillary statistic $T = \frac{X_0^2}{W} = \frac{X_0^2}{\sum_{i=1}^{n} X_i^2/n} = \frac{X_0^2}{\frac{1}{n}\sum_{i=1}^{n} X_i^2}$,
where $W = \sum_{i=1}^{n} X_i^2/n$

$\therefore \frac{X_i}{\sigma} = Z_i \therefore \frac{X_0}{\sigma} = Z_0 \therefore$

$\frac{Z_0^2}{\cancel{Z^2}} = \frac{Z_0}{\frac{1}{n}\sum_{i=1}^{n} Z_i^2} = \frac{\left(\frac{X_0}{\sigma}\right)^2}{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\sigma}\right)^2} = \frac{\left(\frac{X_0^2}{\sigma^2}\right)}{\frac{1}{n}\sum_{i=1}^{n}\frac{X_i^2}{\sigma^2}} = \frac{\left(\frac{X_0^2}{\sigma^2}\right)}{\frac{1}{n}\frac{1}{\sigma^2}\sum_{i=1}^{n} X_i^2} = \frac{X_0^2}{\frac{1}{n}\sum_{i=1}^{n} X_i^2} = T$

$\therefore T$ is ancillary and has a $F_{1,n}$ distribution

$\frac{Y_1/1}{Y_2/n}$ has a $F_{1,n}$ distribution $\therefore Y_1 \sim \chi^2_1$, $Y_2 \sim \chi^2_n$

$\sum_{i=0}^{0} Z_0^2 = Z_0^2 \sim \chi^2_1$, $\sum_{i=1}^{n} Z_i^2 \sim \chi^2_n \therefore \frac{Z_0^2/1}{\frac{1}{n}\sum_{i=1}^{n} Z_i^2} \sim F_{1,n}$

\2bi/ The ancillary statistic $T = \frac{X_0^2}{\sum_{i=1}^{n} X_i^2/n} = \frac{X_0^2}{\frac{1}{n}\sum_{i=1}^{n} X_i^2} \therefore$

$\frac{X_i}{\sigma} = Z_i \therefore \frac{X_0}{\sigma} = Z_0 \therefore$

$\frac{Z_0}{\frac{1}{n}\sum_{i=1}^{n} Z_i^2} = \frac{\left(\frac{X_0}{\sigma}\right)}{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\sigma}\right)^2} = \frac{\left(\frac{X_0^2}{\sigma^2}\right)}{\frac{1}{n}\sum_{i=1}^{n}\frac{X_i^2}{\sigma^2}} = \frac{\left(\frac{X_0^2}{\sigma^2}\right)}{\frac{1}{n}\frac{1}{\sigma^2}\sum_{i=1}^{n} X_i} = \frac{X_0^2}{\frac{1}{n}\sum_{i=1}^{n} X_i^2} = T \therefore$

$\frac{Y_1/1}{Y_2/n}$ has a $F_{1,n}$ distribution $\therefore Y_1 \sim \chi^2_1$, $Y_2 \sim \chi^2_n \therefore$

$\sum_{i=0}^{0} Z_i^2 = Z_0^2 \sim \chi^2_1$, $\sum_{i=1}^{n} Z_i^2 \sim \chi^2_n \therefore \frac{Z_0^2/1}{\frac{1}{n}\sum_{i=1}^{n} Z_i^2} \sim F_{1,n} \therefore$

T is ancillary and has an $F_{1,n}$ distribution

\2bi/ let $Y_1 = \sum_{i=0}^{0} Z_i^2 = Z_0^2 \sim \chi^2_1$, $Y_2 = \sum_{i=1}^{n} Z_i^2 \sim \chi^2_n \therefore \frac{X_i}{\sigma} = Z_i \therefore \frac{X_0}{\sigma} = Z_0 \therefore$

$\frac{Y_1/1}{Y_2/n} = \frac{Y_1/1}{\frac{1}{n}Y_2} = \frac{Z_0^2/1}{\frac{1}{n}\sum_{i=1}^{n} Z_i^2} = \frac{\left(\frac{X_0}{\sigma}\right)^2}{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\sigma}\right)^2} = \frac{(X_0^2/\sigma^2)}{\frac{1}{n}\sum_{i=1}^{n}\frac{X_i^2}{\sigma^2}} = \frac{\frac{1}{\sigma^2}X_0^2}{\frac{1}{n}\frac{1}{\sigma^2}\sum_{i=1}^{n} X_i^2} = \frac{X_0^2}{\frac{1}{n}\sum_{i=1}^{n} X_i^2} = T$

$\therefore T$ is ~~as~~ ancillary and has an $F_{1,n}$ distribution

\2bii/ let c^2 denote the 0.9-quantile of $F_{1,n}$ $\therefore$

$0.9 = Pr(T < c^2) = Pr\left(\frac{X_0^2}{\frac{1}{n}\sum_{i=1}^{n} X_i^2} < c^2\right) = Pr\left(-c < \left(\frac{X_0^2}{\frac{1}{n}\sum_{i=1}^{n} X_i^2}\right)^{1/2} < c\right) =$

PP 2021 $\Pr\left(-c < \frac{X_0}{\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right)^{1/2}} < c\right) = \Pr\left(-c\sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2} < X_0 < c\sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2}\right)$

$\therefore$ a 90% prediction interval for X_0 is $\left(-c\sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2}, c\sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2}\right)$

2bii Let c^2 denote the 0.9-quantile of $F_{1,n}$ $\therefore$

$0.9 = \Pr(T < c^2) = \Pr\left(\frac{X_0^2}{\frac{1}{n}\sum_{i=1}^{n} X_i^2} < c^2\right) = \Pr\left(-c < \left(\frac{X_0^2}{\frac{1}{n}\sum_{i=1}^{n} X_i^2}\right)^{1/2} < c\right) =$

$\Pr\left(-c < \frac{X_0}{\left(\frac{1}{n}\sum_{i=1}^{n} X_i^2\right)^{1/2}} < c\right) = \Pr\left(-c\sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2} < X_0 < c\sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2}\right) \therefore$

a 90% prediction interval for X_0 is $\left(-c\sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2}, c\sqrt{\frac{1}{n}\sum_{i=1}^{n} X_i^2}\right)$

3a The log-likelihood $l(\sigma, \phi; \underline{x}, \underline{y}) = \ln L(\sigma, \phi; \underline{x}, \underline{y}) =$

$\ln((2\pi)^{-n}) - 2n\ln\sigma + n\ln\phi - \frac{S_x + \phi^2 S_y}{2\sigma^2} \therefore$

$\frac{\partial l}{\partial \sigma} = l_\sigma = \frac{-2n}{\sigma} + \frac{S_x + \phi^2 S_y}{\sigma^3} \therefore$

Solve $l_\sigma = 0$ for σ:

$-\frac{2n}{\hat{\sigma}} + \frac{S_x + \phi^2 S_y}{\hat{\sigma}^3} = 0 \therefore S_x + \phi^2 S_y = 2n\hat{\sigma}^2 \therefore \frac{S_x + \phi^2 S_y}{2n} = \hat{\sigma}(\phi)^2 \therefore$

$l_{\sigma\sigma} = \frac{2n}{\sigma^2} - \frac{3(S_x + \phi^2 S_y)}{\sigma^4} \therefore l_{\sigma\sigma}(\hat{\sigma}(\phi)) = \frac{2n}{\hat{\sigma}(\phi)^2} - \frac{3(S_x + \phi^2 S_y)}{(\hat{\sigma}(\phi)^2)^2} =$

$\frac{2n}{\hat{\sigma}(\phi)^2} - \frac{3(S_x + \phi^2 S_y)}{\left(\frac{S_x + \phi^2 S_y}{2n}\right)^2} = \frac{2n}{\hat{\sigma}(\phi)^2} - 3 \cdot 2n \frac{2n \cdot 1}{S_x + \phi^2 S_y} = \frac{2n}{\hat{\sigma}(\phi)^2} - 6n \frac{1}{\hat{\sigma}(\phi)^2} = -4n \frac{1}{\hat{\sigma}(\phi)^2} < 0$

$\therefore$ The profile log likelihood is $\therefore$

$l_p(\phi; \underline{x}, \underline{y}) = l(\hat{\sigma}(\phi), \phi, \underline{x}, \underline{y}) = \text{Constant} - 2n\ln(\hat{\sigma}(\phi)) + n\ln(\phi) - \frac{S_x + \phi^2 S_y}{2\hat{\sigma}(\phi)^2}$

$= \text{const} + n\ln(\phi) - 2n\ln\left(\left(\frac{S_x + \phi^2 S_y}{2n}\right)^{1/2}\right) + \cancel{n\ln(S_x + \phi^2 S_y)} - \frac{S_x + \phi^2 S_y}{2\left(\frac{S_x + \phi^2 S_y}{2n}\right)}$

$= \text{const} + n\ln\phi - n\ln\left(\frac{S_x + \phi^2 S_y}{2n}\right) - \frac{2n}{2} =$

$\text{const} + n\ln\phi - n\ln(S_x + \phi^2 S_y) + n\ln(2n) - n =$

$n\ln\phi - n\ln(S_x + \phi^2 S_y) + \text{Constant}$

3b $-2\ln\Lambda = 2\left[l(\hat{\theta}) - l(\theta_0)\right] = -2\ln\frac{L(\theta_0)}{\sup_\theta L(\theta)} = -2\ln\frac{L(\theta_0)}{L(\hat{\theta})} \therefore$

$\therefore$ MLE of ϕ: $l_p'(\phi) = \frac{\partial}{\partial\phi} l_p(\phi) = \frac{n}{\phi} - n\frac{2\phi S_y}{S_x + \phi^2 S_y} \therefore$

$l_p'(\hat{\phi}) = 0 = \frac{n}{\hat{\phi}} - \frac{2n\hat{\phi} S_y}{S_x + S_y\hat{\phi}^2} \therefore \frac{2n\hat{\phi} S_y}{S_x + S_y\hat{\phi}^2} = \frac{n}{\hat{\phi}} \therefore$

$2n\hat{\phi}^2 S_y = nS_x + nS_y\hat{\phi}^2 \therefore 2n\hat{\phi}^2 S_y - nS_y\hat{\phi}^2 = nS_x = (2nS_y - nS_y)\hat{\phi}^2 = nS_y \therefore$

$\hat{\phi}^2 = \frac{S_x}{S_y} \therefore$

$$l_p''(\phi) = -\frac{n}{\phi^2} - \frac{2nS_y}{S_x+\phi^2S_y} - \frac{2nS_y\phi\,(-1)}{(S_x+\phi^2S_y)^2}(2\phi S_y) = -\frac{n}{\phi^2} - \frac{2nS_y}{S_x+\phi^2S_y} + \frac{4nS_y^2\phi^2}{(S_x+\phi^2S_y)^2}$$

$$\therefore l_p''(\hat\phi) = -\frac{nS_y}{S_x} - \frac{2nS_y}{S_x+\frac{S_x}{S_y}S_y} + \frac{4nS_y^2\frac{S_x}{S_y}}{(S_x+\frac{S_x}{S_y}S_y)^2} = -\frac{S_y}{S_x} - \frac{2nS_y}{S_x+S_x} + \frac{4nS_yS_x}{(S_x+S_x)^2} =$$

$$-\frac{nS_y}{S_x} - \frac{2nS_y}{2S_x} + \frac{4nS_yS_x}{(2S_x)^2} = -\frac{nS_y}{S_x} - \frac{nS_y}{S_x} + \frac{4nS_yS_x}{4S_x^2} = \frac{-nS_y}{S_x} - \frac{nS_y}{S_x} + \frac{nS_y}{S_x} = -\frac{nS_y}{S_x} = -\frac{n}{\hat\phi^2} < 0$$

$\therefore$ likelihood ratio test statistic is

$$\Lambda(\phi_0) = \cancel{-2\ln} -2\ln\left(\frac{L(\phi_0)}{\sup L(\phi)}\right) = -2\ln\left(\frac{L(\phi_0)}{L(\hat\phi)}\right) = -2\left[l(\phi_0) - l(\hat\phi)\right] =$$

$$2\left[l(\hat\phi) - l(\phi_0)\right] = 2\left[l_p(\hat\phi) - l_p(\phi_0)\right] =$$

$$2\left[n\ln(\hat\phi) - n\ln(S_x+\hat\phi^2S_y) + \text{const} - n\ln(\phi_0) + n\ln(S_x+\phi_0^2S_y) - \text{const}\right] =$$

$$2\left[n\ln\left(\frac{\hat\phi}{\phi_0}\right) + n\ln\left(\frac{S_x+\phi_0^2S_y}{S_x+\hat\phi^2S_y}\right)\right] = 2n\ln\left(\frac{\hat\phi(S_x+\phi_0^2S_y)}{\phi_0(S_x+\hat\phi^2S_y)}\right)$$

3c an approximate α confidence interval for ϕ is $\{\phi : \Lambda(\phi) < c\}$ where c is the α-quantile of χ_1^2

The inequality $\Lambda(\phi) < c$ is satisfied if and only if $\Lambda(\phi) < c$ $\therefore$

$$2n\ln\left(\frac{\hat\phi(S_x+\phi^2S_y)}{\phi(S_x+\hat\phi^2S_y)}\right) < c \quad \therefore \quad \cancel{e^{\frac{c}{2n}}} \quad \frac{\hat\phi(S_x+\phi^2S_y)}{\phi(S_x+\hat\phi^2S_y)} < e^{\frac{c}{2n}} \quad \therefore$$

$$\hat\phi(S_x+\phi^2S_y) < e^{\frac{c}{2n}}(S_x+\hat\phi^2S_y)\phi \quad \hat\phi S_y(\phi^2) + (\hat\phi S_x) \quad \therefore$$

$$(\hat\phi S_y)(\phi^2) + \left(-(S_x+\hat\phi^2S_y)e^{\frac{c}{2n}}\right)(\phi) + (\hat\phi S_x) < 0 \quad \therefore$$

$$\phi = \left((S_x+\hat\phi^2S_y)e^{\frac{c}{2n}} \pm \sqrt{(S_x+\hat\phi^2S_y)^2e^{\frac{c}{n}} - 4\hat\phi S_y\hat\phi S_x}\right) \Big/ (2\hat\phi S_y)$$

$$\left(\left((S_x+\hat\phi^2S_y)e^{\frac{c}{2n}} - \sqrt{(S_x+\hat\phi^2S_y)^2e^{\frac{c}{n}} - 4\hat\phi S_y\hat\phi S_x}\right)\Big/(2\hat\phi S_y),\ \left((S_x+\hat\phi^2S_y)e^{\frac{c}{2n}} \pm \left((S_x+\hat\phi^2S_y)^2e^{\frac{c}{n}} - 4\hat\phi S_y\hat\phi S_x\right)^{1/2}\right)\Big/(2\hat\phi S_y)\right)$$

is the α confidence interval for ϕ $\therefore$

ϕ lies between $\hat\phi\left(e^{\frac{c}{2n}} \pm \sqrt{e^{\frac{c}{n}} - 1}\right)$

4a when any of $x_{(2)}, \dots, x_{(n-1)}$ are omitted, $\hat\beta_{-i} = \hat\beta$

when $x_{(1)}$ is omitted $\hat\beta_{-i} = x_{(n)} - x_{(2)}$,

when $x_{(n)}$ is omitted $\hat\beta_{-i} = x_{(n-1)} - x_{(1)}$ $\therefore$

$$\cancel{\hat\beta_J = n\hat\beta} \quad \hat\beta_J = \frac{1}{n}\sum_{i=1}^{n}\hat\beta_i = \frac{1}{n}\sum_{i=1}^{n}\left[n\hat\beta - (n-1)\hat\beta_{-i}\right] = \frac{1}{n}\sum_{i=1}^{n}(n\hat\beta) - \frac{1}{n}\sum_{i=1}^{n}(n-1)\hat\beta_{-i} =$$

$$\sum_{i=1}^{n}\hat\beta - \frac{n-1}{n}\sum_{i=1}^{n}\hat\beta_{-i} = n\hat\beta - \frac{n-1}{n}\sum_{i=1}^{n}\hat\beta_{-i}$$

$$\hat\beta_J = \frac{1}{n}\sum_{i=1}^{n}\hat\beta_i \qquad \hat\beta_i = n\hat\beta - (n-1)\hat\beta_{-i}$$

$$\therefore \hat\beta_J = \frac{1}{n}\sum_{i=1}^{n}\hat\beta_i = \frac{1}{n}\sum_{i=1}^{n}(n\hat\beta - (n-1)\hat\beta_{-i}) = \sum_{i=1}^{n}\hat\beta - \frac{n-1}{n}\sum_{i=1}^{n}\hat\beta_{-i} = n\hat\beta - \frac{n-1}{n}\sum_{i=1}^{n}\hat\beta_{-i} =$$

$$n\hat\beta - \frac{n-1}{n}\left\{x_{(n)} - x_{(2)} + \sum_{i=2}^{n-1}\hat\beta_{-i} + x_{(n-1)} - x_{(1)}\right\} = n\hat\beta - \frac{n-1}{n}\left\{(n-2)\hat\beta + x_{(n)} - x_{(2)} + x_{(n-1)} - x_{(n)}\right\} =$$

$$n\hat\beta + \frac{n-1}{n}\left\{(n-2)\hat\beta + \hat\beta + (x_{(n-1)} - x_{(2)})\right\} = n\hat\beta + \frac{n-1}{n}(n-2)\hat\beta + \frac{n-1}{n}\left\{\hat\beta - (x_{(n-1)} - x_{(2)})\right\} =$$

$L(\theta_0)$ $i=1$

\PP2021/ $\frac{n^2\hat{\beta}}{n} + \frac{n^2+2-3n}{n} + \frac{n-1}{n}\left\{\hat{\beta} - (x_{(n-1)} - x_{(2)})\right\} =$

$n\hat{\beta} + \frac{n-1}{n}\left\{(n-2)\hat{\beta} + \hat{\beta} + (x_{(n-1)} - x_{(2)})\right\} =$

$n\hat{\beta} + \frac{n-1}{n}\left\{(n-2)\hat{\beta} + \hat{\beta} - n\hat{\beta} + n\hat{\beta} + (x_{(n-1)} - x_{(2)})\right\} =$

$n\hat{\beta} + \frac{n-1}{n}(-n\hat{\beta}) + \frac{n-1}{n}\left\{(n-2)\hat{\beta} + \hat{\beta} + n\hat{\beta} + (x_{(n-1)} - x_{(2)})\right\} =$

$n\hat{\beta} + \cancel{(n+1)}\hat{\beta} + \frac{n-1}{n}\left\{(n-2)\hat{\beta} + (n+1)\hat{\beta} + (x_{(n-1)} - x_{(2)})\right\} =$

$\hat{\beta} + \frac{n-1}{n}\left\{(2n-1)\hat{\beta} + (x_{(n-1)} - x_{(2)})\right\} \cancel{\geq \hat{\beta}} \cancel{= \hat{\beta}} \geq \hat{\beta} \therefore \hat{\beta}_J = \hat{\beta} + \frac{n-1}{n}\left\{\hat{\beta} - (x_{(n-1)} - x_{(2)})\right\} \geq \hat{\beta} \therefore$

$\because$ The term is the braces is non-negative $\because x_{(n)} > x_{(n-1)}$,

$x_{(2)} > x_{(1)} \quad \therefore -x_{(1)} > -x_{(2)} \therefore \quad x_{(n)} - x_{(1)} > x_{(n-1)} - x_{(2)} \therefore$

$\hat{\beta} > x_{(n-1)} - x_{(2)} \therefore$

$\hat{\beta} - (x_{(n-1)} - x_{(2)}) > 0 \qquad \hat{\beta}_J \geq \hat{\beta}$

\1bi/ The bias corrected estimate is $\tilde{\beta} = \hat{\beta} - \frac{\epsilon}{\epsilon}(E(\hat{\beta}) - \beta) \therefore$

$\tilde{\beta} = \hat{\beta} - (E(\hat{\beta}) - \beta) \therefore$

$\hat{\beta} = 4.9$, $\therefore \cancel{\hat{\beta} =} \therefore \tilde{\beta}^* = \hat{\beta} - \text{bias}(\hat{\beta}^*) = \hat{\beta} - (E(\hat{\beta}^*) - \beta^*)$

$\therefore E(\hat{\beta}^*) = \frac{1}{n}\sum_{i=1}^{n}\hat{\beta}_i^* = 5.1$, $\beta^* = \hat{\beta}_n^* - \hat{\beta}_1^* = \hat{\beta}_{200}^* - \hat{\beta}_1^* = 7.9 - 2.7 = 5.2 \therefore$

Let $\beta^* = 4.9 \therefore \qquad \tilde{\beta}^* = 4.9 - (5.1 - 4.9) = 4.7$

\4bii/ The ~~bes~~ estimated Standard error is $\sqrt{1.1} = 1.049 \therefore$

$SE(\beta^*) = \sqrt{var(\hat{\beta}^*)} = \sqrt{1.1} = 1.049$

\4biii/ $\hat{\beta}_{(5)}^* = 3$, $\hat{\beta}_{(195)}^* = 7.4 \therefore \hat{\beta}_{(5)}^* < \hat{\beta}_{(195)}^* \therefore$

$\alpha = 0.025, \quad B = 200 \therefore$

$(1-\alpha)B = (1-0.025)200 = 195$, $\alpha B = 0.025(200) = 5 \therefore$

$(2\hat{\beta} - \hat{\beta}_{(195)}^*, 2\hat{\beta} - \hat{\beta}_{(5)}^*) = (2 \times 4.9 - 7.4, 2 \times 4.9 - 3) = (2.6, 6.8)$

\4biv/ $\hat{\beta}_{(5)}^* < \hat{\beta}_{(195)}^* \therefore (\hat{\beta}_{(\alpha B)}^*, \hat{\beta}_{((1-\alpha)B)}^*) = (\hat{\beta}_{(5)}^*, \hat{\beta}_{(195)}^*) = (3, 7.2)$

is the percentile bootstrap interval

\4c/ when added to the sample, a further data point is equally likely to have any of the $n+1$ possible ranks.

$\therefore$ The probability that it lies between $x_{(5)}$ and $x_{(95)}$ is

$\therefore \frac{95-5}{n+1} = \frac{95-5}{99+1} = \frac{90}{100} = 0.9$

$\because$ if $n=2$ there are 3 possible ranks $\therefore n+1$ possible ranks

\PP2019/ \1a/ Likelihood func: $L(\theta; \underline{x}) = \prod_{i=1}^{n} f(x_i;\theta) =$

$\prod_{i=1}^{n} \frac{1}{2\theta} e^{-|x_i|/\theta} = \frac{1}{2^n \theta^n} e^{-\sum_{i=1}^{n}|x_i| \frac{1}{\theta}} = \cancel{\frac{1}{2^n\theta}}\ 2^{-n}\theta^{-n} e^{-\frac{1}{\theta}\sum_{i=1}^{n}|x_i|}$ ∴

$l(\theta;\underline{x}) = \ln(L(\theta;\underline{x})) = -n\ln(2) - n\ln(\theta) - \frac{1}{\theta}\sum_{i=1}^{n}|x_i| =$

constant $-n\ln(\theta) - \theta^{-1}\sum_{i=1}^{n}|x_i|$ ∴

$l'(\theta;\underline{x}) = -\frac{n}{\theta} - 2\theta^{-2}\sum_{i=1}^{n}|x_i|$ ∴

$-\frac{n}{\theta} - \frac{2}{\theta^2}\sum_{i=1}^{n}|x_i| = 0$ ∴ $-2\sum_{i=1}^{n}|x_i| = n\hat{\theta}$ ∴

$\hat{\theta} = -\frac{2}{n}\sum_{i=1}^{n}|x_i|$ ∴

$l''(\theta;\underline{x}) = \frac{2n}{\theta^2} + 4\theta^{-3}\sum_{i=1}^{n}|x_i|$ ∴

$l''(\hat{\theta};\underline{x}) = \cancel{2n\frac{1}{(}}\ 2n\frac{1}{\left(-\frac{2}{n}\sum_{i=1}^{n}|x_i|\right)^2} + 4\frac{\sum_{i=1}^{n}|x_i|}{\left(-\frac{2}{n}\sum_{i=1}^{n}|x_i|\right)^3} =$

$2n\left(\frac{4}{n^2}\right)\frac{1}{\left(\sum_{i=1}^{n}|x_i|\right)^2} + 4\left(-\frac{8}{n^3}\right)\frac{1}{\left(\sum_{i=1}^{n}|x_i|\right)^2} =$

$\left(\frac{8}{n} - \frac{32}{n^3}\right)\frac{1}{\left(\sum_{i=1}^{n}|x_i|\right)^2} < 0$ ∴

$\hat{\theta} = -\frac{2}{n}\sum_{i=1}^{n}|x_i|$ is mle

$I(\theta) = \cancel{-E(J(\theta,x))} = -E(l''(\theta,\underline{x})) = -E\left(\frac{2n}{\theta^2} + \frac{4}{\theta^3}\sum_{i=1}^{n}|x_i|\right) =$

$-\frac{2n}{\theta^3} - \frac{4}{\theta^3}E\left(\sum_{i=1}^{n}|x_i|\right) = -\frac{2n}{\theta^3} - \frac{4}{\theta^3}\sum_{i=1}^{n}E(|x_i|)$ $\quad E(|x_i|) = \int_{-\infty}^{\infty}|x_i|\frac{1}{2\theta}e^{-|x|/\theta}dx$

\1a Sol/ By independence, the likelihood is:

$L(\theta;\underline{x}) = \prod_{i=1}^{n} f(x_i;\theta) \propto \theta^{-n} e^{-\sum_{i=1}^{n}|x_i|/\theta}$ ∴

$l(\theta;\underline{x}) =$ constant $-n\ln\theta - \theta^{-1}\sum_{i=1}^{n}|x_i|$ ∴

$l'(\theta;\underline{x}) = -n\theta^{-1} + \theta^{-2}\sum_{i=1}^{n}|x_i|$ ∴

Sol: $l'(\hat{\theta};\underline{x}) = 0$: $\hat{\theta} = n^{-1}\sum_{i=1}^{n}|x_i|$ ∴

$l''(\hat{\theta}) = n\hat{\theta}^{-2} - 2\hat{\theta}^{-3}\sum_{i=1}^{n}|x_i| = -n\hat{\theta}^{-2} < 0$ ∴ $\hat{\theta}$ is mle ∴

$I(\theta) = -E(J(\theta)) = -E(l''(\theta)) = -n\theta^{-2} + 2\theta^{-3}\sum_{i=1}^{n}E(|X_i|)$

$f(x;\theta)$ is is an even function ∴ let $y = x/\theta$ ∴

$E(|X_i|) = \int_{-\infty}^{\infty}|x| f(x;\theta)dx = 2\int_0^{\infty}\frac{x}{2\theta}e^{-x/\theta}dx = \theta\int_0^{\infty} y e^{-y}dy = \theta\int_0^{\infty} y^1 e^{-y}dy = \theta(1!) = \theta$

∴

$I(\theta) = -n/\theta^2 + 2n\theta/\theta^3 = n/\theta^2$ ∴

asymptotic distribution of the mle is $N(\theta\ \theta^2/n)$

1a/S01/ likelihood : $L(\theta;\underline{x}) = f(\underline{x};\theta) = \prod_{i=1}^{n} f(x_i;\theta) =$
$\prod_{i=1}^{n} \frac{1}{2\theta} e^{-|x_i|/\theta} \propto \frac{1}{\theta^n} e^{-\frac{1}{\theta}\sum_{i=1}^{n}|x_i|} = \theta^{-n} e^{-\frac{1}{\theta}\sum_{i=1}^{n}|x_i|}$ ∴

$l(\theta;\underline{x}) = l(\theta) = \ln(L(\theta)) = -n\ln(\theta) - \frac{1}{\theta}\sum_{i=1}^{n}|x_i| + \text{const}$
$l(\theta;\underline{x}) = \text{constant} - n\ln(\theta) - \theta^{-1}\sum_{i=1}^{n}|x_i|$ ∴

$l'(\theta;\underline{x}) = -\frac{n}{\theta} + \theta^{-2}\sum_{i=1}^{n}|x_i|$ ∴
for $l'(\hat\theta;\underline{x}) = 0$:
$-\frac{n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n}|x_i| = 0$ ∴ $\frac{1}{n}\sum_{i=1}^{n}|x_i| = \hat\theta$ ∴

$l''(\theta;\underline{x}) = l''(\theta) = \frac{n}{\theta^2} - 2\theta^{-3}\sum_{i=1}^{n}|x_i|$ ∴
$l''(\hat\theta) = n\hat\theta^{-2} - 2\hat\theta^{-3}\sum_{i=1}^{n}|x_i| = n\frac{1}{n^{-2}}\left(\sum_{i=1}^{n}|x_i|\right)^{-2} - 2\frac{1}{n^{-3}}$

$l''(\theta) = n\hat\theta^{-2} - 2\hat\theta^{-3}\sum_{i=1}^{n}|x_i|$ ∴ $\hat\theta = n^{-1}\sum_{i=1}^{n}|x_i|$ ∴

$\hat\theta^{-2} = n^2\left(\sum_{i=1}^{n}|x_i|\right)^{-2}$, $\hat\theta^{-3} = n^3\left(\sum_{i=1}^{n}|x_i|\right)^{-3}$ ∴

$l''(\theta) = n^3\left(\sum_{i=1}^{n}|x_i|\right)^{-2} - 2n^3\left(\sum_{i=1}^{n}|x_i|\right)^{-2} = -n^3\left(\sum_{i=1}^{n}|x_i|\right)^{-2} =$

$-n\left(n\left(\sum_{i=1}^{n}|x_i|\right)^{-1}\right)^2 = -n\left(\hat\theta^{-1}\right)^2 = -n\hat\theta^{-2} < 0$ $\hat\theta$ is mle ∴

expected information is: $I(\theta) = -E(l''(\theta)) = -E\left(n\theta^{-2} - 2\theta^{-3}\sum_{i=1}^{n}|x_i|\right)$
$= -n\theta^{-2} + 2\theta^{-3}E\left(\sum_{i=1}^{n}|x_i|\right) = -n\theta^{-2} + 2\theta^{-3}\sum_{i=1}^{n}E(|x_i|) =$
$-n\theta^{-2} + 2\theta^{-3}\sum_{i=1}^{n}E(|X|) = -n\theta^{-2} + 2n\theta^{-3}E(|X|)$ ∴
$f(x;\theta)$ is an even ~~function~~ ∴
$E(|X|) = \int_{-\infty}^{\infty}|x| f(x;\theta)\,dx = 2\int_0^{\infty} x f(x;\theta)\,dx = 2\int_0^{\infty}\frac{x}{2\theta}e^{-x/\theta}\,dx$
let $y = x/\theta$ ∴ $dy = \frac{1}{\theta}dx$ ∴ $\theta dy = dx$, ∴ $x=0 \to y=0$, $x=\infty \to y=\infty$.
$E(|X|) = \int_0^{\infty}\frac{x}{\theta}e^{-x/\theta}\,dx = \int_0^{\infty} y e^{-y}\theta\,dy = \theta\int_0^{\infty} y^1 e^{-y}\,dy =$
$\theta(1!) = \theta$ ∴

PP2019 $I(\theta) = -n\theta^{-2} + 2n\theta^{-3}\theta = -n\theta^{-2} + 2n\theta^{-2} = n\theta^{-2} = \frac{n}{\theta^2}$ ∴

asymptotic distribution: $\hat{\theta} \sim N(\theta, I(\theta)^{-1}) = N(\theta, \frac{\theta^2}{n})$ ✓

a $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} \frac{1}{2\theta} e^{-|x_i|/\theta} =$

$2^{-n}\theta^{-n} e^{-\frac{1}{\theta}\sum_{i=1}^{n}|x_i|}$ ∴

$l(\theta) = \ln(L(\theta)) = -n\ln 2 - n\ln(\theta) - \frac{1}{\theta}\sum_{i=1}^{n}|x_i| =$

constant $-n\ln(\theta) - \theta^{-1}\sum_{i=1}^{n}|x_i|$ ∴

$l'(\theta) = -\frac{n}{\theta} + \theta^{-2}\sum_{i=1}^{n}|x_i|$ ∴

$l'(\hat{\theta}) = 0 = -\frac{n}{\hat{\theta}} + \hat{\theta}^{-2}\sum_{i=1}^{n}|x_i|$ ∴

$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}|x_i|$ ∴

$l''(\theta) = n\theta^{-2} - 2\theta^{-3}\sum_{i=1}^{n}|x_i|$ ∴

$l''(\hat{\theta}) = n\,n^2 \frac{1}{(\sum_{i=1}^{n}|x_i|)^2} - 2n^3 \frac{\sum_{i=1}^{n}|x_i|}{(\sum_{i=1}^{n}|x_i|)^3} = (n^3 - 2n^3)\frac{1}{(\sum_{i=1}^{n}|x_i|)^2} =$

$\cancel{-n^3}$ $-n\left(\frac{n}{\sum_{i=1}^{n}|x_i|}\right)^2 = -n\left(\frac{1}{\hat{\theta}}\right)^2 = -n(\hat{\theta}^{-1})^2 = -n\theta^{-2} < 0$ ∴

$-n\left(\frac{1}{\hat{\theta}}\right)^2 < 0$ ∴

$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n}|x_i|$ is MLE ∴

$I(\theta) = -E(J(\theta)) = -E(l''(\theta)) = -E\left(n\theta^{-2} - 2\theta^{-3}\sum_{i=1}^{n}|x_i|\right) =$

$-n\theta^{-2} + 2\theta^{-3}E\left(\sum_{i=1}^{n}|x_i|\right) = -n\theta^{-2} + 2\theta^{-3}\sum_{i=1}^{n}E(|x_i|) = -n\theta^{-2} + 2\theta^{-3}\sum_{i=1}^{n}E(|X|)$ ∴

$f(x;\theta)$ is even ∴

$E(|X|) = \int_{-\infty}^{\infty}|x| f(x;\theta)\,dx = 2\int_0^{\infty} x f(x;\theta)\,dx = 2\int_0^{\infty}\frac{x}{2\theta}e^{-x/\theta}\,dx$ ∴

let $y = \frac{x}{\theta}$ ∴ $\theta dy = dx$ $x=0: y=0$, $x=\infty: y=\infty$ ∴

$E(|X|) = 2\int_0^{\infty}\frac{1}{2}ye^{-y}\theta\,dy = \theta\int_0^{\infty}y^1e^{-y}\,dy = \theta(1!) = \theta$ ∴

∴ $I(\theta) = -n\theta^{-2} + 2\theta^{-3}\sum_{i=1}^{n}\theta = -n\theta^{-2} + 2\theta^{-3}n\theta = -n\theta^{-2} + 2n\theta^{-2} = n\theta^{-2} = \frac{n}{\theta^2}$

∴ $\hat{\theta} \sim N(\theta, I(\theta)^{-1}) = N(\theta, \frac{\theta^2}{n})$

\1b/ an estimator is unbiased if the bias$(\theta) = 0$,

bias$(\theta) = E(\hat{\theta}) - \theta$,

they're efficient if $\frac{I(\theta)^{-1}}{var(\hat{\theta})} = 1$

$\therefore$ estimator is unbiased and efficient if ~~$\hat{\theta}$~~ the score can be written in the form ~~$\hat{\theta} \Rightarrow I(\theta) =$~~ $U(\theta) = b(\hat{\theta} - \theta)$, $b = I(\theta)$

\1b/ an estimator, $\hat{\theta}$, for a parameter, θ, is unbiased if $E(\hat{\theta}) = \theta$ and is efficient if $var(\hat{\theta}) = I(\theta)^{-1}$

\1b/ ~~score~~ an estimator $\hat{\theta}$ is unbiased if $E(\hat{\theta}) - \theta = 0$

its efficient if $\frac{I(\theta)^{-1}}{var(\hat{\theta})} = 1$

\1c/ score is $U(\theta) = l'(\theta)$ $\therefore$

$U(\theta) = l'(\theta) = -\frac{n}{\theta} + \theta^{-2} \sum_{i=1}^{n} |x_i| = -\frac{n}{\theta} + \theta^{-2} n \frac{1}{n} \sum_{i=1}^{n} |x_i| = -\frac{n}{\theta} + \frac{1}{\theta^2} n \hat{\theta} =$

$\frac{n}{\theta^2}\hat{\theta} - \frac{n}{\theta^2}\theta = \frac{n}{\theta^2}(\hat{\theta} - \theta)$ $\therefore$ $\hat{\theta}$ is unbiased and efficient

\1c/ score: $U(\theta) = n\theta^{-2}(\hat{\theta} - \theta)$ $\therefore$ by thm 1.2, $\hat{\theta}$ is unbiased and efficient.

\1d/ $W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta})$

$S = U(\theta_0)^2 / I(\theta_0)$

$-2\ln\Lambda = 2(l(\hat{\theta}) - l(\theta_0))$ $\therefore$

$\therefore W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta}) = (\hat{\theta} - \theta_0)^2 \frac{n}{\hat{\theta}^2} = n\left((\hat{\theta} - \theta_0)\frac{1}{\hat{\theta}}\right)^2 = n\left(1 - \frac{\theta_0}{\hat{\theta}}\right)^2$,

$S = U(\theta_0)^2 / I(\theta_0) = (l'(\theta_0))^2 / I(\theta_0) = \left(\frac{n}{\theta_0^2}(\hat{\theta} - \theta_0)\right)^2 \frac{1}{I(\theta_0)} =$

$\left(\frac{n^2}{\theta_0^4}\right)(\hat{\theta} - \theta_0)^2 \frac{\theta_0^2}{n} = \frac{n}{\theta_0^2}(\hat{\theta} - \theta_0)^2 = n\left(\frac{1}{\theta_0}(\hat{\theta} - \theta_0)\right)^2 = n\left(\frac{\hat{\theta}}{\theta_0} - 1\right)^2$

likelihood ratio: $-2\ln\Lambda = 2(l(\hat{\theta}) - l(\theta_0)) =$

$2\left(\text{const} - n\ln(\hat{\theta}) - \frac{1}{\hat{\theta}} \sum_{i=1}^{n} |x_i| - \text{const} + n\ln(\theta_0) + \frac{1}{\theta_0} \sum_{i=1}^{n} |x_i|\right) =$

$2\left(n(\ln(\theta_0) - \ln(\hat{\theta})) + \left(\frac{1}{\theta_0} - \frac{1}{\hat{\theta}}\right) \sum_{i=1}^{n} |x_i|\right) =$

$2n\ln\left(\frac{\theta_0}{\hat{\theta}}\right) + 2\left(\frac{1}{\theta_0} - \frac{1}{\hat{\theta}}\right) \sum_{i=1}^{n} |x_i| = 2n\ln\left(\frac{\theta_0}{\hat{\theta}}\right) + 2\left(\frac{1}{\theta_0} - \frac{1}{\hat{\theta}}\right) n\hat{\theta} =$

$2n\ln(\theta_0/\hat{\theta}) + 2n(\hat{\theta}/\theta_0 - 1)$

When n is large they all have a approx null distri of ~~the~~ χ^2 distribution

\PP2019/ ~~\1d sow~~ The wald test statistic is $(\hat\theta-\theta_0)^2 I(\hat\theta) = n(1-\theta_0/\hat\theta)^2$

the score test statistic is $U(\theta_0)^2 I(\theta_0)^{-1} = n(1-\hat\theta/\theta_0)^2$

the likelihood ratio test statistic is

$2(l(\hat\theta)-l(\theta_0)) = 2n(\ln(\theta_0/\hat\theta) + \hat\theta/\theta_0 - 1)$

their null distris are all approx χ^2_1 for large n

\1d/ $W = (\hat\theta-\theta_0)^2 I(\hat\theta) = (\hat\theta-\theta_0)^2 \frac{n}{\hat\theta^2} = \left(\frac{\hat\theta}{\hat\theta} - \frac{\theta_0}{\hat\theta}\right)^2 n = n\left(1-\frac{\theta_0}{\hat\theta}\right)^2$

$S = U(\theta_0)^2 I(\theta_0)^{-1} = \left(\frac{n}{\theta_0^2}(\hat\theta-\theta_0)\right)^2 \frac{\theta_0^2}{n} = \frac{n^2}{\theta_0^4}(\hat\theta-\theta_0)^2 \frac{\theta_0^2}{n} =$

$\frac{n}{\theta_0^2}(\hat\theta-\theta_0)^2 = n\left(\frac{\hat\theta}{\theta_0} - \frac{\theta_0}{\theta_0}\right)^2 = n\left(\frac{\hat\theta}{\theta_0}-1\right)^2 = n(-1)^2\left(\frac{\hat\theta}{\theta_0}-1\right)^2 =$

$n\left(-1\left(\frac{\hat\theta}{\theta_0}-1\right)\right)^2 = n\left(1-\hat\theta/\theta_0\right)^2$

likelihood ratio: $-2\ln\Lambda = 2(l(\hat\theta) - l(\theta_0)) =$

$2\left(\text{constant} - n\ln(\hat\theta) - \frac{1}{\hat\theta}\sum_{i=1}^n |x_i| - \text{constant} + n\ln(\theta_0) + \frac{1}{\theta_0}\sum_{i=1}^n |x_i|\right) =$

$2\left(\cancel{\cos}\ n(\ln(\theta_0)-\ln(\hat\theta)) + \left(\frac{1}{\theta_0} - \frac{1}{\hat\theta}\right)\sum_{i=1}^n |x_i|\right) =$

$2\left(n\ln(\theta_0/\hat\theta) + \left(\frac{1}{\theta_0} - \frac{1}{\hat\theta}\right)n\hat\theta\right) = 2\left(n\ln(\theta_0/\hat\theta) + n\left(\frac{\hat\theta}{\theta_0}-1\right)\right) =$

$\cancel{2(n\ln(\theta_0/\hat\theta)}$ $2n\left(\ln(\theta_0/\hat\theta) + \frac{\hat\theta}{\theta_0} - 1\right)$

their null distris are all approx χ^2_1 for large n

\1e/ let c be the a-quantile of the χ^2_1 distribution

the critical region is $\{\underline{x} : W \geq c\} = \{\underline{x} : n(1-\theta_0/\hat\theta)^2 \geq c\}$ ∴ the α confidence interval is $\{\theta : n(1-\theta/\hat\theta)^2 < c\}$ this inequality is satisfied iff $(1-\theta/\hat\theta)^2 < \frac{c}{n}$ ∴

$-\sqrt{\frac{c}{n}} < 1 - \frac{\theta}{\hat\theta} < \sqrt{\frac{c}{n}}$ ∴ $1-\sqrt{\frac{c}{n}} < \frac{\theta}{\hat\theta} < 1+\sqrt{\frac{c}{n}}$ ∴

$\hat\theta\left(1-\sqrt{\frac{c}{n}}\right) < \theta < \hat\theta\left(1+\sqrt{\frac{c}{n}}\right)$ ∴ $\left(\hat\theta\left(1-\sqrt{\frac{c}{n}}\right), \hat\theta\left(1+\sqrt{\frac{c}{n}}\right)\right)$ is the α confidence interval for θ

\1e/ ~~critical~~ let d be the α quantile of the χ^2_1 distri ∴

the CR is $\{\underline{x} : W \geq d\} = \{\underline{x} : n(1-\theta_0/\hat\theta)^2 \geq d\}$ ∴

α CI is: $\{\theta : n(1-\theta/\hat\theta)^2 \leq d\}$ ∴ $(1-\theta/\hat\theta)^2 < \frac{d}{n}$ ∴

$-\sqrt{\frac{d}{n}} < 1 - \frac{\theta}{\hat\theta} < \sqrt{\frac{d}{n}}$ ∴ $\hat\theta\left(1-\sqrt{\frac{d}{n}}\right) < \theta < \hat\theta\left(1+\sqrt{\frac{d}{n}}\right)$ ∴ $\left(\hat\theta\left(1-\sqrt{\frac{d}{n}}\right), \hat\theta\left(1+\sqrt{\frac{d}{n}}\right)\right)$ is the α CI for θ

\1f/ $Pr(X_1 \le x) = \int_{-\infty}^{x} f(t;\theta)dt = \int_{-\infty}^{x} \frac{1}{2\theta} e^{-|t|/\theta} dt$, $x \le 0 \therefore t \le 0 \therefore$

$Pr(X_1 \le x) = \int_{-\infty}^{x} f(x;\theta)dx = \int_{-\infty}^{x} \frac{1}{2\theta} e^{-|x|/\theta} dx = \frac{1}{2\theta}\int_{-\infty}^{x} e^{x/\theta} dx =$

\1f/ $Pr(X_1 \le x) = \int_{-\infty}^{x} f(x;\theta)dx = \frac{1}{2\theta}\int_{-\infty}^{x} e^{x/\theta} dx = \frac{1}{2}\left[e^{x/\theta}\right]_{-\infty}^{x} = \frac{1}{2}e^{x/\theta}$

$\therefore$ solving $Pr(X_1 \le x_p) = p \therefore$ p-quantile: $x_p = \theta\ln(2p)$

\1f/ $Pr(X_1 \le x) = \int_{-\infty}^{x} f(x;\theta)dx = \int_{-\infty}^{x} \frac{1}{2\theta} e^{-|x|/\theta} dx \therefore x \le 0 \therefore$

$Pr(X_1 \le x) = \int_{-\infty}^{x} \frac{1}{2\theta} e^{-(-x)/\theta} dx = \frac{1}{2\theta}\int_{-\infty}^{x} e^{x/\theta} dx =$

$\frac{1}{2\theta}\frac{1}{(1/\theta)}\left[e^{x/\theta}\right]_{-\infty}^{x} = \frac{1}{2}\left[e^{x/\theta} - 0\right] = \frac{1}{2}e^{x/\theta} \therefore$

solving $Pr(X_1 \le x_p) = p \therefore$

$Pr(X_1 \le x_p) = \frac{1}{2}e^{x_p/\theta} = p \therefore$

$e^{x_p/\theta} = 2p \therefore x_p/\theta = \ln(2p) \therefore x_p = \theta\ln(2p)$ is the p-quantile

\1g sol/ The plugin estimate of the p-quantile is $\hat{x}_p = \hat{\theta}\ln(2p)$ for $p \le \frac{1}{2}$. As the distri is symmetric about $x=0$, the $(1-p)$-quantile is $\hat{x}_{1-p} = -\hat{x}_p$ and an equal tailed $(1-2\alpha)$ PI is $(\hat{x}_\alpha, \hat{x}_{1-\alpha})$

\1g/ The plugin esti of p-quantile is $\hat{x}_p = \hat{\theta}\ln(2p)$ for $p \le \frac{1}{2}$ $\therefore \because$ distri is symmetric about $x=0 \therefore$ the $(1-p)$-quantile is $\hat{x}_{1-p} = -\hat{x}_p \therefore$ an equaltailed $(1-2\alpha)$ PI is $(\hat{x}_\alpha, \hat{x}_{1-\alpha})$

\2a/ The quasi-score function is $G = \mu_\theta^T \Sigma^{-1}(\underline{X} - \mu)$ where μ_θ is the gradient of μ.

The quasi-score is unbiased $\because E(G) = E(\mu_\theta^T \Sigma^{-1}(\underline{X} - \mu)) =$
$\mu_\theta^T E(\Sigma^{-1}(\underline{X} - \mu)) = \mu_\theta^T \Sigma^{-1}(E(\underline{X} - \mu)) = \mu_\theta^T \Sigma^{-1}(E(\underline{X}) - \mu) = \mu_\theta^T \Sigma^{-1}(\mu - \mu) =$
$\mu_\theta^T \Sigma^{-1}(0) = 0$

\2a/ The quasi-score function is $G = \mu_\theta^T \Sigma^{-1}(\underline{X} - \mu) \therefore$
The quasiscore is unbiased $\because E(G) = E(\mu_\theta^T \Sigma^{-1}(\underline{X} - \mu)) =$
$\mu_\theta^T E(\Sigma^{-1}(\underline{X} - \mu)) = \mu_\theta^T \Sigma^{-1}(E(\underline{X} - \mu)) = \mu_\theta^T \Sigma^{-1}(E(\underline{X}) - \mu) = \mu_\theta^T \Sigma^{-1}(\mu - \mu) = \mu_\theta^T \Sigma^{-1}(0) = 0$

\PP2019/ $\therefore \mu = (\theta, \ldots, \theta)^T$ and $\Sigma = \phi\theta(1-\theta)I$ $\therefore$

$\mu_\theta = (1, \ldots, 1)^T$, $\Sigma^{-1} = (\phi\theta(1-\theta))^{-1} I$ and

$$G = \frac{n(\bar{X}-\theta)}{\phi\theta(1-\theta)}$$

$\therefore$ Solving $G = 0 = \frac{n(\bar{X}-\theta)}{\phi\theta(1-\theta)}$ $\therefore$ $n(\bar{X}-\theta) = 0$ $\therefore$ $\bar{X}-\theta = 0$

yields quasi-likelihood estimator $\hat{\theta} = \bar{X}$

The asymptotic distribution is $N(\theta, K^{-1})$, where

$K = \mu_\theta^T \Sigma^{-1} \mu_\theta = \frac{n}{\phi\theta(1-\theta)}$

\2b/ $\mu = (\theta, \ldots, \theta)^T$ and $\Sigma = \phi\theta(1-\theta)I$ $\therefore$

$\mu_\theta = (1, \ldots, 1)^T$ $\Sigma^{-1} = (\phi\theta(1-\theta))^{-1} I$ and

$$G = \mu_\theta^T \Sigma^{-1}(\underline{X}-\mu) = (1, \ldots, 1)^T (\phi\theta(1-\theta))^{-1} I (\bar{X}-\mu) =$$

$$(1, \ldots, 1)^T (\phi\theta(1-\theta))^{-1} I (\bar{X} - (\theta, \ldots, \theta)^T) =$$

$$\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} \frac{1}{\phi\theta(1-\theta)} \begin{bmatrix} 1 & & 0 \\ & \ddots & \\ 0 & & 1 \end{bmatrix} (\bar{X} - (\theta, \ldots, \theta)^T) = \frac{n(\bar{X}-\theta)}{\phi\theta(1-\theta)}$$

$\therefore G = 0$ $\therefore$ $\frac{n(\bar{X}-\theta)}{\phi\theta(1-\theta)} = 0$ $\therefore$ $\bar{X}-\theta = 0$ $\therefore$

The quasi-likelihood estimator $\hat{\theta} = \bar{X}$.

The asymptotic distribution is $N(\theta, K^{-1})$, where

$K = \mu_\theta^T \Sigma^{-1} \mu_\theta = \frac{n}{\phi\theta(1-\theta)}$

\2b/ $\mu = (\theta, \ldots, \theta)^T$, $\Sigma = \phi\theta(1-\theta)I$ $\therefore$

$\mu_\theta = (1, \ldots, 1)^T$, $\Sigma^{-1} = (\phi\theta(1-\theta))^{-1} I = \frac{1}{(\phi\theta(1-\theta))} I$ $\therefore$

$$G = \mu_\theta^T \Sigma^{-1}(\underline{X}-\mu) = (1, \ldots, 1)^T \frac{1}{(\phi\theta(1-\theta))} I (\bar{X}-\mu) =$$

$$(1, \ldots, 1)^T \frac{1}{(\phi\theta(1-\theta))} I (\bar{X} - (\theta, \ldots, \theta)^T) = \frac{n(\bar{X}-\theta)}{\phi\theta(1-\theta)}$$

$\therefore G = 0$ $\therefore$ $\frac{n(\bar{X}-\theta)}{\phi\theta(1-\theta)} = 0$ $\therefore$ $\bar{X}-\theta = 0$ $\therefore$

The quasi-likelihood estimator is: $\hat{\theta} = \bar{X}$ $\therefore$

The asymptotic distribution is: $\hat{\theta} \sim N(\theta, K(\theta)^{-1}) = N(\theta, K^{-1})$,

where $K(\theta) = \mathrm{var}(G) = \mu_\theta^T \Sigma^{-1} \mu_\theta = (1, \ldots, 1) \frac{1}{\phi\theta(1-\theta)} I (1, \ldots, 1)^T = \frac{n}{\phi\theta(1-\theta)}$

\2c/ let $\Sigma_0 = \theta(1-\theta)I$ then an estimator for ϕ is

$$\frac{1}{n-1}(\underline{X}-\mu)^T \Sigma_0^{-1}(\underline{X}-\mu)\big|_{\theta=\hat{\theta}} = \frac{\sum_{i=1}^{n}(X_i-\bar{X})^2}{(n-1)\hat{\theta}(1-\hat{\theta})}$$

\2c/ $\Sigma(\theta) = \phi\Sigma_0(\theta) = \phi\theta(1-\theta)I$ $\therefore$ $\theta(1-\theta)I = \Sigma_0(\theta) = \Sigma_0$.

$\hat{\phi} = \frac{1}{n-1}(\underline{x}-\mu)^T\Sigma_0^{-1}(\underline{x}-\mu)\Big|_{\theta=\hat\theta} = \frac{1}{n-1}(\underline{x}-\mu)^T\left(\frac{1}{\theta(1-\theta)}\right)I(\underline{x}-\mu)\Big|_{\theta=\hat\theta} =$

$\frac{1}{(n-1)\hat\theta(1-\hat\theta)}\sum_{i=1}^{n}(x_i-\bar{x})^2$

\3a/ Consider testing $H_0: \theta=\theta_0$ against $H_1: \theta=\theta_1$ $\therefore$

The most powerful test of a given size has a critical region of the form $\{\underline{x}: L(\theta_1;\underline{x})/L(\theta_0;\underline{x}) \geq c\}$

\3a/

The most powerful test of a given size has a critical region of the form $\{\underline{x}: L(\theta_1;\underline{x})/L(\theta_0;\underline{x}) \geq c\}$

\3b/ subtracting $P_r(\underline{X} \in C \cap C';\theta_1)$ from both probabilities, the left hand side of the inequality becomes

$P_r(\underline{X} \in C \cap \overline{C'};\theta_1) - P_r(\underline{X} \in C' \cap \overline{C};\theta_1)$

let $f(\underline{x},\theta)$ denote the density of $\underline{X}$. By the def of C, $f(\underline{x};\theta_1) \geq cf(\underline{x};\theta_0)$ for $\underline{x} \in C$, and $f(\underline{x};\theta_1) \leq cf(\underline{x};\theta_0)$ for $\underline{x} \in \overline{C}$ $\therefore$, $P_r(\underline{X} \in C \cap \overline{C'}) \geq cP_r(\underline{X} \in C \cap \overline{C'};\theta_0)$

and $P_r(\underline{X} \in C' \cap \overline{C};\theta_1) \leq cP_r(\underline{X} \in C' \cap \overline{C};\theta_0)$ $\therefore$

$P_r(\underline{X} \in C,\theta_1) - P_r(\underline{X} \in C',\theta_1) \geq$

$c[P_r(\underline{X} \in C \cap \overline{C'};\theta_0) - P_r(\underline{X} \in C' \cap \overline{C},\theta_0)] =$

$c[P_r(\underline{X} \in C,\theta_0) - P_r(\underline{X} \in C';\theta_0)] = 0$

$\therefore$ $c[P_r(\underline{X} \in C \cap \overline{C'};\theta_0) - P_r(\underline{X} \in C' \cap \overline{C},\theta_0)] =$

$c[(P_r(\underline{X} \in C \cap \overline{C'};\theta_0) + P_r(\underline{X} \in C \cap C',\theta_0)) - (P_r(\underline{X} \in C' \cap \overline{C};\theta_0) + P_r(\underline{X} \in C \cap C',\theta_0))]$

$= c[P_r(\underline{X} \in C;\theta_0) - P_r(\underline{X} \in C';\theta_0)] = 0$ $\therefore P_r(\underline{X} \in C;\theta_0) = P_r(\underline{X} \in C';\theta_0)$

\3b/ $P_r(\underline{X} \in C;\theta_1) - P_r(\underline{X} \in C \cap C';\theta_1) = P_r(\underline{X} \in C \cap \overline{C'};\theta_1)$

$P_r(\underline{X} \in C';\theta_1) - P_r(\underline{X} \in C \cap C';\theta_1) = P_r(\underline{X} \in C' \cap \overline{C};\theta_1)$

$\therefore$ $P_r(\underline{X} \in C;\theta_1) - P_r(\underline{X} \in C';\theta_1) =$

$[P_r(\underline{X} \in C;\theta_1) - P_r(\underline{X} \in C \cap C';\theta_1)] - [P_r(\underline{X} \in C';\theta_1) - P_r(\underline{X} \in C \cap C';\theta_1)]$

$= P_r(\underline{X} \in C \cap \overline{C'};\theta_1) - P_r(\underline{X} \in C' \cap \overline{C};\theta_1)$

let $f(\underline{x};\theta)$ denote the density of $\underline{X}$. by definition of C,

$f(\underline{x};\theta_1) \geq cf(\underline{x},\theta_0)$ $\therefore$ $\frac{f(\underline{x};\theta_1)}{f(\underline{x};\theta_0)} \geq C$ for $\underline{x} \in C$, $\therefore$

\PP2019/ $f(\underline{x};\theta_1) \leq c f(\underline{x};\theta_0)$ for $x \in \overline{C}$ $\therefore$

$Pr(\underline{X} \in C \cap \overline{C'};\theta_1) \geq c Pr(\underline{X} \in C \cap \overline{C'};\theta_0)$.

$Pr(\underline{X} \in C' \cap \overline{C};\theta_1) \leq c Pr(\underline{X} \in C' \cap \overline{C};\theta_0)$ $\therefore$

$Pr(\underline{X} \in C;\theta_1) - Pr(\underline{X} \in C';\theta_1) =$

$Pr(\underline{X} \in C \cap \overline{C'};\theta_1) - Pr(\underline{X} \in C' \cap \overline{C};\theta_1) \geq$

$c Pr(\underline{X} \in C \cap \overline{C'};\theta_0) - Pr(\underline{X} \in C' \cap \overline{C};\theta_1) \geq$

$c Pr(\underline{X} \in C \cap \overline{C'};\theta_0) - c Pr(\underline{X} \in C' \cap \overline{C};\theta_0) =$

$c(Pr(\underline{X} \in C \cap \overline{C'};\theta_0) - Pr(\underline{X} \in C' \cap \overline{C};\theta_0)) =$

$c([Pr(\underline{X} \in C \cap C';\theta_0) + Pr(\underline{X} \in C \cap \overline{C'};\theta_0)] - [Pr(\underline{X} \in C' \cap \overline{C};\theta_0) + Pr(\underline{X} \in C \cap C';\theta_0)]) =$

$c[Pr(\underline{X} \in C;\theta_0) - Pr(\underline{X} \in C';\theta_0)] = 0$ $\therefore$

$\cancel{Pr(\underline{X} \in C \cap C}$ $Pr(\underline{X} \in C;\theta_0) = Pr(\underline{X} \in C';\theta_0)$

\3C/ A test is uniformly most powerful if its power is at least as large as the power of any other test of the same size for all distributions specified by the alternative hypothesis. The Neyman Pearson theorem can be used to construct the critical region for the most powerful test for any simple hypothesis contained within the composite alternative. If this critical region is the same for all simple hypotheses contained with the composite alternative then it also defines the uniformly most powerful test. If the critical region changes for some simple alternatives then no uniformly most powerful test exists

\3C/ A test is unifly most powerful if its power is at least as large as the power of any other test of the same size for all distributions specified by the alternative hypothesis. The Neyman Pearson theorem can be used to construct the CR for the most powerful test for any simple hypothesis contained within the composite alternative.

If this CR is the same for all simple hypotheses contained within the composite alternative then ~~also~~ it also defines the uniformly most powerful test.
If the CR changes for some simple alternatives then no uniformly most powerful test exists.

\4a/ Compute t for the original data then simulate a large number of samples of size n from the ~~dis~~ distribution with parameter $\theta=\theta_0$ and compute the test statistic for each sample. The p-value is the proportion ~~of~~ of simulated test statistics exceeding t

\4a/ compute t for the original data then simulate a large number of samples of size n from the distr with param $\theta=\theta_0$ and compute the test stat for each sample. The p-value is the ~~perop~~ proportion of simulated test statistics exceeding t.

\4b/ The jackknife version of $\hat\theta$ is: $\hat\theta_J=\frac{1}{n}\sum_{i=1}^{n}\left\{n\hat\theta-(n-1)\hat\theta_{-i}\right\}=$

$$\frac{1}{n}\sum_{i=1}^{n}\left\{\frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2-\cancel{\ }(n-1)\left(\frac{1}{n-1}\right)^2\left(\sum_{j\neq i}x_j\right)^2\right\}=$$

$$\frac{1}{n}\sum_{i=1}^{n}\left\{\frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2-\frac{1}{n-1}\left(\sum_{j\neq i}x_j\right)^2\right\}=\frac{1}{n(n-1)}\sum_{i=1}^{n}\left\{\frac{n-1}{n}\left(\sum_{i=1}^{n}x_i\right)^2-\left(\sum_{j=1}^{n}x_j-x_i\right)^2\right\}=$$

$$\frac{1}{n(n-1)}\left\{(n-1)\left(\sum_{i=1}^{n}x_i\right)^2-n\left(\sum_{j=1}^{n}x_j\right)^2+2\left(\sum_{i=1}^{n}x_i\right)^2-\sum_{i=1}^{n}x_i^2\right\}=$$

$$\frac{1}{n(n-1)}\left\{(n-1)\left(\sum_{i=1}^{n}x_i\right)^2-n\left(\sum_{i=1}^{n}x_i\right)^2+2\left(\sum_{i=1}^{n}x_i\right)^2-\sum_{i=1}^{n}x_i^2\right\}=\frac{1}{n(n-1)}\left\{\left(\sum_{i=1}^{n}x_i\right)^2-\sum_{i=1}^{n}x_i^2\right\}$$

\4b/ $\hat\theta_{-i}=\left(\frac{1}{n-1}\sum_{j\neq i}x_j\right)^2=\frac{1}{(n-1)^2}\left(\sum_{j\neq i}x_j\right)^2=\frac{1}{(n-1)^2}\left(\sum_{j=1}^{n}x_j-x_i\right)^2$ $\therefore$

$$\hat\theta_J=\frac{1}{n}\sum_{i=1}^{n}\hat\theta_i=\frac{1}{n}\sum_{i=1}^{n}\left\{n\hat\theta-(n-1)\hat\theta_{-i}\right\}=\frac{1}{n}\sum_{i=1}^{n}\left\{n\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2-(n-1)\frac{1}{(n-1)^2}\left(\sum_{j=1}^{n}x_j-x_i\right)^2\right\}$$

$$\cancel{\frac{1}{n}\sum_{i=1}^{n}\left\{\frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2-\ (n-1)\right\}}=\frac{1}{n}\sum_{i=1}^{n}\left\{\frac{1}{n}\left(\sum_{i=1}^{n}x_i\right)^2-\frac{1}{n-1}\left(\sum_{j=1}^{n}x_j-x_i\right)^2\right\}=$$

$$\frac{1}{n(n-1)}\sum_{i=1}^{n}\left\{\frac{n-1}{n}\left(\sum_{i=1}^{n}x_i\right)^2-\left(\sum_{j=1}^{n}x_j-x_i\right)^2\right\}=\frac{1}{n(n-1)}\left\{n\,\frac{n-1}{n}\left(\sum_{i=1}^{n}x_i\right)^2-\sum_{i=1}^{n}\left(\sum_{j=1}^{n}x_j-x_i\right)^2\right\}=$$

$$\frac{1}{n(n-1)}\left\{(n-1)\left(\sum_{i=1}^{n}x_i\right)^2-\sum_{i=1}^{n}\left[\left(\sum_{j=1}^{n}x_j\right)^2-2x_i\sum_{j=1}^{n}x_j+(x_i)^2\right]\right\}=$$

\PP2019/ $\frac{1}{n(n-1)}\left\{(n-1)\left(\sum_{i=1}^{n}x_i\right)^2-\sum_{i=1}^{n}\left(\sum_{j=1}^{n}x_j\right)^2+2\sum_{i=1}^{n}x_i\sum_{j=1}^{n}x_j-\cancel{2}\sum_{i=1}^{n}(x_i)^2\right\}=$

$\frac{1}{n(n-1)}\left\{(n-1)\left(\sum_{i=1}^{n}x_i\right)^2-n\left(\sum_{j=1}^{n}x_j\right)^2+2\left(\sum_{j=1}^{n}x_j\right)\sum_{i=1}^{n}x_i-\cancel{2}\sum_{i=1}^{n}(x_i)^2\right\}=$

$\frac{1}{n(n-1)}\left\{(n-1)\left(\sum_{i=1}^{n}x_i\right)^2-n\left(\sum_{i=1}^{n}x_i\right)^2+2\left(\sum_{i=1}^{n}x_i\right)\sum_{i=1}^{n}x_i-\cancel{2}\sum_{i=1}^{n}(x_i)^2\right\}=$

$\frac{1}{n(n-1)}\left\{(n-1)\left(\sum_{i=1}^{n}x_i\right)^2-n\left(\sum_{i=1}^{n}x_i\right)^2+2\left(\sum_{i=1}^{n}x_i\right)^2-\cancel{2}\sum_{i=1}^{n}(x_i)^2\right\}=$

$\frac{1}{n(n-1)}\left\{(n-1-n+2)\left(\sum_{i=1}^{n}x_i\right)^2-\sum_{i=1}^{n}(x_i)^2\right\}=\frac{1}{n(n-1)}\left\{\left(\sum_{i=1}^{n}x_i\right)^2-\sum_{i=1}^{n}x_i^2\right\}$

\4c/ $\hat{\theta}^*=\frac{1}{n^2}\left(\sum_{i=1}^{n}x_i^*\right)^2=\frac{1}{n^2}\left(\sum_{i=1}^{n}x_i^{*2}+\sum_{i\neq j}x_i^*x_j^*\right)$

where $x_1^*,\dots,x_n^*$ are independent with $E(x_i^*)=\sqrt{\hat{\theta}}$ and

$E(x_i^{*2})=var(x_i^*)+E(x_i^*)^2=\hat{\theta}+\hat{\theta}=2\hat{\theta}$. have:

$E(\hat{\theta}^*)=\frac{1}{n^2}\left\{nE(x_i^{*2})+n(n-1)E(x_i^*)^2\right\}=\frac{1}{n^2}\left\{2n\hat{\theta}+n(n-1)\hat{\theta}\right\}=\frac{n+1}{n}\hat{\theta}$

$\therefore$ The bias-corrected estimator is $2\hat{\theta}-E(\hat{\theta}^*)=\left(1-\frac{1}{n}\right)\hat{\theta}$

\4e/ $\hat{\theta}^*=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^*\right)^2=\frac{1}{n^2}\left(\sum_{i=1}^{n}x_i^*\right)^2=\frac{1}{n^2}\left(\sum_{i=1}^{n}x_i^{*2}+\sum_{i\neq j}x_i^*x_j^*\right)$

where $x_1^*,\dots,x_n^*$ are independent with $E(x_i^*)=\sqrt{\hat{\theta}}$

and $E(x_i^{*2})=var(x_i^*)+E(x_i^*)^2=\hat{\theta}+(\sqrt{\hat{\theta}})^2=\hat{\theta}+\hat{\theta}=2\hat{\theta}$ $\therefore$

$E(\hat{\theta}^*)=E\left(\frac{1}{n^2}\left(\sum_{i=1}^{n}x_i^{*2}+\sum_{i\neq j}x_i^*x_j^*\right)\right)=\frac{1}{n^2}\left(E\left(\sum_{i=1}^{n}x_i^{*2}\right)+E\left(\sum_{i\neq j}x_i^*x_j^*\right)\right)=$

$\frac{1}{n^2}\left\{\sum_{i=1}^{n}E(x_i^{*2})+\sum_{i\neq j}E(x_i^*x_j^*)\right\}=\frac{1}{n^2}\left\{\sum_{i=1}^{n}E(x_i^{*2})+\sum_{i\neq j}E(x_i^*)E(x_j^*)\right\}=$

$\frac{1}{n^2}\left\{nE(x_1^{*2})+\sum_{i\neq j}E(x_1^*)E(x_1^*)\right\}=$

$\frac{1}{n^2}\left\{nE(x_1^{*2})+\sum_{i\neq j}E(x_1^*)^2\right\}=\frac{1}{n^2}\left\{nE(x_1^{*2})+n(n-1)E(x_1^*)^2\right\}=$

$\frac{1}{n^2}\left\{n2\hat{\theta}+n(n-1)(\sqrt{\hat{\theta}})^2\right\}=\frac{1}{n^2}\left\{2n\hat{\theta}+n(n-1)\hat{\theta}\right\}=$

$\frac{1}{n^2}\left\{2n\hat{\theta}+n^2\hat{\theta}-n\hat{\theta}\right\}=\frac{1}{n^2}\left\{n^2\hat{\theta}-n\hat{\theta}\right\}=\frac{1}{n^2}\left((n^2-n)\hat{\theta}\right)=\frac{n+1}{n}\hat{\theta}$

$\therefore$ The bias-corrected estimator is $2\hat{\theta}-E(\hat{\theta}^*)=2\hat{\theta}-\frac{n+1}{n}\hat{\theta}=$

$\frac{2n}{n}\hat{\theta}+\frac{-n-1}{n}\hat{\theta}=\frac{2n-n-1}{n}\hat{\theta}=\frac{n-1}{n}\hat{\theta}=\hat{\theta}-\frac{1}{n}\hat{\theta}=\left(1-\frac{1}{n}\right)\hat{\theta}$

\3028 PP2020/ \1a/ likelihood $L(\theta;\underline{x}) = \prod_{i=1}^{n} \frac{x_i}{\theta^2} e^{-x_i/\theta} =$

$\theta^{-2n} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i \quad \therefore$

loglikelihood $= l(\theta) = \ln(L(\theta)) = \ln\left[\theta^{-2n} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} (x_i)\right] =$

$-2n\ln(\theta) - \frac{1}{\theta}\left(\sum_{i=1}^{n} x_i\right) + \ln\left(\prod_{i=1}^{n} x_i\right) = -2n\ln(\theta) - \frac{1}{\theta}\left(\sum_{i=1}^{n} x_i\right) + \sum_{i=1}^{n} \ln x_i$

\1b/ $\therefore l'(\theta) = -2n\frac{1}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i \quad \therefore$

$l'(\hat{\theta}) = 0 \quad \therefore \quad -2n\frac{1}{\hat{\theta}} + \frac{1}{\hat{\theta}^2}\sum_{i=1}^{n} x_i = 0 \quad \therefore \frac{1}{\hat{\theta}^2}\sum_{i=1}^{n} x_i = 2n\frac{1}{\hat{\theta}} \quad \therefore$

$\sum_{i=1}^{n} x_i = 2n\hat{\theta} \quad \therefore \quad \frac{1}{2}\frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{2}\bar{x} = \hat{\theta} \quad \therefore$

$l''(\theta) = 2n\frac{1}{\theta^2} - 2\frac{1}{\theta^3}\sum_{i=1}^{n} x_i \quad \therefore$

$l''(\hat{\theta}) = 2n\frac{1}{\hat{\theta}^2} - 2\frac{1}{\hat{\theta}^3}2n\hat{\theta} = \frac{2n}{\hat{\theta}^2} - \frac{4n}{\hat{\theta}^2} = -\frac{4n}{\hat{\theta}^2} < 0 \quad \because \frac{4n}{\hat{\theta}^2} > 0 \quad \therefore$

$l''(\hat{\theta}) < 0 \quad \therefore \quad \hat{\theta} = \frac{1}{2}\bar{x}$ is the ~~B~~ MLE.

\1c/ $E(\hat{\theta}) = E\left(\frac{1}{2}\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{2n}E\left(\sum_{i=1}^{n} x_i\right) = \frac{1}{2n}\sum_{i=1}^{n} E(x_i) = \frac{1}{2n}\sum_{i=1}^{n} E(X) =$

$\frac{1}{2n} n E(X) = \frac{1}{2}E(X)$

$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta = \frac{1}{2}E(X) - \theta$

$var(\hat{\theta}) = var\left(\frac{1}{2}\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{4}\frac{1}{n^2} var\left(\sum_{i=1}^{n} x_i\right) = \frac{1}{4n^2}\sum_{i=1}^{n} var(x_i) = \frac{1}{4n^2}\sum_{i=1}^{n} var(X) =$

$\frac{1}{4n} var(X) = \frac{1}{4n}\left[E(X^2) - E(X)^2\right]$

$\therefore E(X) = \int_0^\infty x f(x;\theta) = \int_0^\infty x \frac{x}{\theta^2} e^{-x/\theta} dx = \int_0^\infty \left(\frac{x}{\theta}\right)^2 e^{-x/\theta} dx \quad \therefore$

let $y = \frac{x}{\theta} \quad \therefore \frac{dy}{dx} = \frac{1}{\theta} \quad \therefore \theta dy = dx \quad x=0 \to y=0, \; x=\infty \to y=\infty \quad \therefore$

$E(X) = \int_0^\infty y^2 e^{-y} \theta dy = \theta\int_0^\infty y^2 e^{-y} dy = \theta\left[-y^2 e^{-y}\right]_0^\infty - \theta\int_0^\infty -2y e^{-y} dy =$

$\theta[-0+0] + 2\theta\left[-y e^{-y}\right]_0^\infty - 2\theta\int_0^\infty -e^{-y} dy = 0 + 2\theta[-0+0] + 2\theta\left[-e^{-y}\right]_0^\infty =$

$0 + 2\theta[-0+1] = 2\theta = \theta 2! = \theta\Gamma(3) = \theta\int_0^\infty y^{3-1} e^{-y} dy = 2\theta = E(X) \quad \therefore$

$E(X^2) = \int_0^\infty x^2 \frac{x}{\theta^2} e^{-x/\theta} dx = \theta\int_0^\infty \left(\frac{x}{\theta}\right)^3 e^{-x/\theta} dx = \theta\int_0^\infty y^3 e^{-y} \theta dy =$

$\theta^2\int_0^\infty y^3 e^{-y} dy = \theta^2\int_0^\infty y^{4-1} e^{-y} dy = \theta^2\Gamma(4) = \theta^2(3!) = 6\theta^2 = E(X^2) \quad \therefore$

$Bias(\hat{\theta}) = \frac{1}{2}(2\theta) - \theta = \theta - \theta = 0$,

$var(\hat{\theta}) = \frac{1}{4n}\left[6\theta^2 - (2\theta)^2\right] = \frac{1}{4n}\left[6\theta^2 - 4\theta^2\right] = \frac{1}{4n}\left[2\theta^2\right] = \frac{\theta^2}{2n}$

1d/ score θ: $U(\theta) = l'(\theta) = -\frac{2n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i$,

$\hat{\theta} = \frac{1}{2n}\sum_{i=1}^{n} x_i$, $\sum_{i=1}^{n} x_i = 2n\hat{\theta}$ $\therefore$

$U(\theta) = -\frac{2n}{\theta} + \frac{1}{\theta^2}2n\hat{\theta} = 2n\left[-\frac{1}{\theta} + \frac{\hat{\theta}}{\theta^2}\right]$ $\therefore$

$U(\theta) = b(\theta + \hat{\theta})$ $\therefore$

$l''(\theta) = 2n\frac{1}{\theta^2} - 2\frac{1}{\theta^3}\sum_{i=1}^{n} x_i$ $\therefore$

$I(\theta) = -E(l''(\theta)) = -E\left(2n\frac{1}{\theta^2} - 2\frac{1}{\theta^3}\sum_{i=1}^{n} x_i\right) =$

$-2n\frac{1}{\theta^2}E(1) + 2\frac{1}{\theta^3}E\left(\sum_{i=1}^{n} x_i\right) = -2n\frac{1}{\theta^2} + 2\frac{1}{\theta^3}\sum_{i=1}^{n} E(x_i) =$

$-\frac{2n}{\theta^2} + 2\frac{1}{\theta^3}\sum_{i=1}^{n} E(X) = -\frac{2n}{\theta^2} + 2n\frac{1}{\theta^3}2\theta = -\frac{2n}{\theta^2} + \frac{4n}{\theta^2} = \frac{2n}{\theta^2}$ $\therefore$

$\frac{2n}{\theta^2}(\theta - \hat{\theta}) = \frac{2n}{\theta^2}\theta - \frac{2n}{\theta^2}\hat{\theta} = \frac{2n}{\theta} - \frac{1}{\theta^2}\sum_{i=1}^{n} x_i$ $\therefore$

$U(\theta) = -\frac{2n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i = \cancel{-2n\left(\frac{1}{\theta}\right.}$ $-\frac{2n}{\theta} + \frac{1}{\theta^2}2n\hat{\theta} = -2n\left[\frac{1}{\theta} - \frac{1}{\theta^2}\hat{\theta}\right] =$

$-\frac{2n}{\theta^2}(\theta - \hat{\theta})$ $\therefore$ $\frac{2n}{\theta^2}(\hat{\theta} - \theta) = b(\hat{\theta} - \theta)$ b is constant

$\therefore$ $\hat{\theta}$ is unbiased and efficient.

1e/ let $E(X) = \bar{X}$ $\therefore$ $\hat{\theta} = \frac{1}{2}\bar{X} = \frac{1}{2}E(X) = mle(\hat{\theta})$

1f/ $\theta_1 > \theta_0$ Neyman-Pearson theorem is that the power of a test is ~~A=k~~ found from the likelihood ratio

$\Lambda = \frac{L(\theta_1)}{L(\theta_0)}$ $\therefore$

$$\Lambda(\underline{x}) = \frac{L(\theta_1)}{L(\theta_0)} = \frac{\theta_1^{-2n} e^{-\frac{1}{\theta_1}\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i}{\theta_0^{-2n} e^{-\frac{1}{\theta_0}\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i} = \left(\frac{\theta_0}{\theta_1}\right)^{2n} e^{\left(-\frac{1}{\theta_1} + \frac{1}{\theta_0}\right)\sum_{i=1}^{n} x_i} \therefore$$

$\theta_1 > \theta_0$ $\therefore$ $\frac{1}{\theta_1} < \frac{1}{\theta_0}$ $\therefore$ $-\frac{1}{\theta_1} > -\frac{1}{\theta_0}$ $\therefore$ $-\frac{1}{\theta_1} + \frac{1}{\theta_0} > 0$ $\therefore$

Λ increases as $\sum_{i=1}^{n} x_i$ increases $\therefore$ ~~the~~ $\sum_{i=1}^{n} x_i$ is the critical region $\therefore$ $\left\{\underline{x} : \sum_{i=1}^{n} x_i > c\right\}$, $\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \geq d$ $\therefore$ $\{\underline{x} : \bar{x} \geq d\}$

$\therefore$ reject H_0 if $\bar{x} \geq d$, d is constant

\3028PP2020/ $I(\theta) = \frac{2n}{\theta^2}$ $\therefore$ $W = (\theta - \hat{\theta})^2 I(\theta) =$

$(\theta - \hat{\theta})^2 \frac{2n}{\theta^2} = 2n\left((\theta - \hat{\theta})\frac{1}{\theta}\right)^2 = 2n\left(1 - \frac{\hat{\theta}}{\theta}\right)^2 = 2n\left(1 - \frac{1}{2}\frac{\bar{x}}{\theta}\right)^2$ $\therefore$

when H_0 is true and n is large : W follows a χ^2_1 distribution $\therefore$

critical region is $Pr(W) \leq \alpha$

\ii/ $\therefore$ $\left(\left(1 - \frac{\theta_0}{\hat{\theta}}\right)^2 > c_\alpha\right)$ $\therefore$ $\left(1 - \frac{\theta_0}{\hat{\theta}} < -\sqrt{c_\alpha}, 1 - \frac{\theta_0}{\hat{\theta}} > +\sqrt{c_\alpha}\right)$ $\therefore$

$(\hat{\theta} - \theta_0 < -\hat{\theta}\sqrt{c_\alpha}, \hat{\theta} - \theta_0 > +\hat{\theta}\sqrt{c_\alpha}) = (\hat{\theta} + \hat{\theta}\sqrt{c_\alpha} < \theta_0, \hat{\theta} - \hat{\theta}\sqrt{c_\alpha} > \theta_0)$

$= (\hat{\theta} + \hat{\theta}\sqrt{c_\alpha} < \theta_0 < \hat{\theta} - \hat{\theta}\sqrt{c_\alpha})$ $\therefore$ $(\hat{\theta} + \hat{\theta}\sqrt{c_\alpha}, \hat{\theta} - \hat{\theta}\sqrt{c_\alpha})$

\3a/ bootstrap resampling ~~is~~ when you use the sampled data to find its estimated parameters and then use this sample to ~~start~~ create new samples of equal size and estimate the parameters and distribution of the new samples ~~then~~ and also do tests on them.

parametric means you calculate the parameters of the observed sample and estimate its distribution and ~~appro~~ generate the new bootstrap samples from this distribution.

non parametric means to not do this but instead to only use the given data to ~~predict~~ ~~or~~ generate randomly the new data which will take the value of one of the old data values.

\3bi/ you create a basic bootstrap confidence interval by first bootstrapping the data then using the formula for the basic CI and calculating the parameters of the new data then plugging it all into the formula for the interval.

\3bii/ similar to the basic but instead you order the new data in accending order and say the q_p which is the p-quantile of the new data is which ever value the new data takes on that quantile $\therefore$ find the q_α and $q_{1-\alpha}$ quantile so the quantile interval: $(q_\alpha, q_{1-\alpha})$

\3c/ take the bootstrap algorithem and preform it 10000s of times to generate 1000s of confidence intervals for the parameter you are testing ~~and~~ for both the basic and quantile method.
the calculate the estimated value of the parameter ~~from~~ the original ~~test~~ data then test to see what ~~so~~ proportion of this parameter is contained within the 10000s of intervals of each test then which ever proportion of the two is higher, has the better coverage

\1d/ $\hat{\theta} = \frac{1}{\bar{x}}$, $\hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i$, $\hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$,

$$\hat{\theta}_{-i} = \frac{1}{\frac{1}{n-1}\sum_{j\neq i} x_j} = \frac{1}{\frac{1}{n-1}\left[\left(\sum_{i=1}^{n} x_i\right) - x_i\right]} \quad \therefore \quad n=5, \ \bar{x} = \frac{1.55}{5} = 0.31 \therefore \hat{\theta} = \frac{100}{31}$$

i	1	2	3	4	5
$\frac{1}{\frac{1}{n-1}\sum_{j\neq i} x_j} = \hat{\theta}_{-i}$	$\frac{50}{13}$	$\frac{100}{29}$	$\frac{400}{119}$	$\frac{400}{129}$	$\frac{50}{19}$
$5\left(\frac{100}{31}\right) - 4\hat{\theta}_{-i} = \hat{\theta}_i$	$\frac{800}{403}$	$\frac{2100}{899}$	2.684	3.726	$\frac{3300}{589}$

$$\therefore \hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i = \frac{1}{5}\sum_{i=1}^{n}\hat{\theta}_i = \frac{1}{5}(15.093) = 3.019 \ (4 \text{ s.f.})$$

\4a/ $\therefore Z_0 = \frac{X_0}{\theta}$ $\therefore$ $T = \frac{X_0}{X_{(n)}} = \frac{X_0/\theta}{X_{(n)}/\theta} = \frac{Z_i}{Z_{(n)}}$ $\therefore$ let $X_{(n)} = X_n$ wlog
if Z_i is a pivot: T is ancillary $\therefore$

$Z_i = \frac{X_i}{\theta}$; $P(Z_i \le x)$ $Pr(X_i/\theta \le x) = Pr(X_i \le \theta x) = e^{-\theta/(\theta x)} = e^{-\frac{\theta}{\theta x}} = e^{-\frac{1}{x}}$

$\therefore P(Z_i \le x) = e^{-\frac{1}{x}}$ $\therefore$ $\frac{d}{dx}e^{-\frac{1}{x}} = f(Z_i) = e^{-\frac{1}{x}}\left(\frac{d}{dx}(-x^{-1})\right) = x^{-2}e^{-\frac{1}{x}}$

which is independent on any unknown parameters $\therefore$ Z_i is a pivot $\therefore$ T is ancillary

\3023PP2020/ \4a/ let t_5 ~~and~~ t_{95} be the 5% and 95% quantile of T $\therefore$ $0.9 = (t_5, t_{95}) = (t_5 \le T \le t_{95})$ $\therefore$

$$Pr(T \le t_5) = \frac{n}{n + \frac{1}{t_5}} \quad , \quad Pr(T \le t_{95}) = \frac{n}{n + \frac{1}{t_{95}}}$$

\4b/ $F(X_{(n)} \le x) = P(X_{(n)} \le x) = P(X_i \le x) = P(X_1 \le x)^n = (e^{-\theta/x})^n = e^{-n\theta/x}$

$\therefore$ $\cancel{\int_\theta^\infty Pr(X_i \le x_i; \theta)}$ $\therefore$ $f(x_{(n)}) = \frac{d}{dx}(e^{-n\theta/x}) = +n\theta x^{-2} e^{-n\theta/x}$

\1a/ $L(\theta) = L(\theta; \underline{x}) = \prod_{i=1}^{n} f(x_i; \theta) = \prod_{i=1}^{n} \frac{x_i}{\theta^2} e^{-x_i \frac{1}{\theta}} = \theta^{-2n} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i$ $\therefore$

$l(\theta) = \ln L(\theta) = \ln\left(\theta^{-2n} e^{-\frac{1}{\theta}\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i\right) = -2n\ln(\theta) - \frac{1}{\theta}\sum_{i=1}^{n} x_i + \ln\left(\prod_{i=1}^{n} x_i\right) =$

$-2n\ln(\theta) - \frac{1}{\theta}\sum_{i=1}^{n} x_i + \sum_{i=1}^{n} \ln(x_i)$

~~\1b/~~ \1a sol/ $L(\theta; \underline{x}) = \theta^{-2n}\left(\prod_{i=1}^{n} x_i\right) e^{-\sum_{i=1}^{n} x_i/\theta}$

$l(\theta; \underline{x}) = -2n\ln(\theta) + \sum_{i=1}^{n} \ln(x_i) - \frac{1}{\theta}\sum_{i=1}^{n} x_i$

\1b/ $E(x) = \int_0^\infty x f(x; \theta)\, dx = \int_0^\infty x \frac{x}{\theta^2} e^{-x/\theta}\, dx = \int_0^\infty \left(\frac{x}{\theta}\right)^2 e^{-x/\theta}\, dx$

$\therefore$ let $x/\theta = y$ $\therefore$ $\frac{dy}{dx} = \frac{1}{\theta}$ $\therefore$ $\theta\, dy = dx$. $x = 0$ $\therefore$ $y = \frac{0}{\theta} = 0$

$x = \infty$: $y = \frac{\infty}{\theta} = \infty$ $\therefore$ $E(x) = \int_0^\infty y^2 e^{-y} \theta\, dy = \theta\int_0^\infty y^2 e^{-y}\, dy =$

$\theta\left[y^2(-1)e^{-y}\right]_0^\infty - \theta\int_0^\infty -2y e^{-y}\, dy = \theta\left[-y^2 e^{-y}\right]_0^\infty + 2\theta\int_0^\infty y e^{-y}\, dy =$

$\theta[-0 + 0] + 2\theta\left[y(-1)e^{-y}\right]_0^\infty - 2\theta\int_0^\infty -e^{-y}\, dy =$

$2\theta\left[-ye^{-y}\right]_0^\infty + 2\theta\int_0^\infty e^{-y}\, dy = 2\theta[-0+0] + 2\theta\left[-e^{-y}\right]_0^\infty =$

$2\theta[-0 + e^0] = 2\theta[1] = 2\theta$ $\therefore$

$E(x) = \theta\int_0^\infty y^2 e^{-y}\, dy = \theta\Gamma(3) = \theta\Gamma(2+1) = \theta\Gamma(3) = \theta(3-1)! = \theta(2!) = \theta(2) = 2\theta$

$\because$ $\int_0^\infty y^k e^{-y}\, dy = k!$ $\therefore$ $E(x) = \theta\int_0^\infty y^2 e^{-y}\, dy = \theta(2!) = \theta(2) = 2\theta$

$\therefore$ mle by: $l'(\theta) = -2n\frac{1}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i + 0 = -\frac{2n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n} x_i$ $\therefore$

$l'(\hat\theta) = 0 = -\frac{2n}{\hat\theta} + \frac{1}{\hat\theta^2}\sum_{i=1}^{n} x_i$ $\therefore$ $\frac{2n}{\hat\theta} = \frac{1}{\hat\theta^2}\sum_{i=1}^{n} x_i$ $\therefore$ $\hat\theta(2n) = \sum_{i=1}^{n} x_i$ $\therefore$

$\hat\theta = \frac{1}{2n}\sum_{i=1}^{n} x_i$ $\therefore$

$l''(\theta) = \frac{2n}{\theta^2} - 2\frac{1}{\theta^3}\sum_{i=1}^{n} x_i$ $\therefore$

$l''(\hat\theta) = \frac{2n}{\left(\frac{1}{2n}\sum_{i=1}^{n} x_i\right)^2} - \frac{2}{\left(\frac{1}{2n}\sum_{i=1}^{n} x_i\right)^3}\sum_{i=1}^{n} x_i = \frac{2n}{\frac{1}{4n^2}\left(\sum_{i=1}^{n} x_i\right)^2} - \frac{2}{\frac{1}{8n^3}\left(\sum_{i=1}^{n} x_i\right)^2} = \frac{8n^3}{\left(\sum_{i=1}^{n} x_i\right)^2} - \frac{16n^3}{\left(\sum_{i=1}^{n} x_i\right)^2} =$

$-8\frac{n^3}{(\sum_{i=1}^{n}x_i)^2} < 0$ ∴

$\hat\theta = \frac{1}{2n}\sum_{i=1}^{n}x_i = \frac{1}{2n}n\bar{x} = \frac{1}{2}\bar{x}$ is mle ∴

~~Method of moments: $E(x) = 2\theta$~~

\1c/ $E(x) = \int_0^\infty x f(x;\theta)\,dx = \int_0^\infty x\frac{x}{\theta^2}e^{-x/\theta}dx = \int_0^\infty (\frac{x}{\theta})^2 e^{-x/\theta}\,dx$ ∴

let $\frac{x}{\theta} = y$ ∴ $\frac{dy}{dx} = \frac{1}{\theta}$ ∴ $\theta\,dy = dx$ ∴ $x=0: y=\frac{0}{\theta}=0$, $x=\infty: y=\frac{\infty}{\theta}=\infty$

∴ $E(x) = \int_0^\infty y^2 e^{-y}\theta\,dy = \theta\int_0^\infty y^2 e^{-y}dy = \theta[2!] = \theta(2) = 2\theta$

$E(x^2) = \int_0^\infty x^2\frac{x}{\theta^2}e^{-x/\theta}dx = \theta\int_0^\infty \frac{x^2}{\theta}\frac{x}{\theta^2}e^{-x/\theta}dx = \theta\int_0^\infty (\frac{x}{\theta})^3 e^{-x/\theta}dx =$

$\theta\int_0^\infty (y)^3 e^{-y}\theta\,dy = \theta^2\int_0^\infty y^3 e^{-y}dy = \theta^2(3!) = 6\theta^2$ ∴

$var(x) = E(x^2) - E(x)^2 = 6\theta^2 - 4\theta^2 = 2\theta^2$ ∴

$bias(\hat\theta) = E(\hat\theta) - \theta$

∴ $E(\hat\theta) = E(\frac{1}{2}\bar{x}) = \frac{1}{2}E(\bar{x}) = \frac{1}{2}E(\frac{1}{n}\sum_{i=1}^{n}x_i) = \frac{1}{2n}E(\sum_{i=1}^{n}x_i) = \frac{1}{2n}\sum_{i=1}^{n}E(x_i) =$

$\frac{1}{2n}\sum_{i=1}^{n}E(X) = \frac{n}{2n}E(x) = \frac{1}{2}(2\theta) = \theta$ ∴

$bias(\hat\theta) = \theta - \theta = 0$

$var(\hat\theta) = var(\frac{1}{2}\bar{x}) = \frac{1}{4}var(\bar{x}) = \frac{1}{4}var(\frac{1}{n}\sum_{i=1}^{n}x_i) = \frac{1}{4n^2}var(\sum_{i=1}^{n}x_i) =$

$\frac{1}{4n^2}\sum_{i=1}^{n}var(x_i) = \frac{1}{4n^2}\sum_{i=1}^{n}var(X) = \frac{n}{4n^2}var(X) = \frac{1}{4n}var(X) = \frac{1}{4n}(2\theta^2) = \frac{\theta^2}{2n}$

\1c sol/ $E(X_1) = \int_0^\infty x f(x;\theta)\,dx = \theta\int_0^\infty y^2 e^{-y}dy = 2\theta$

$E(X_1^2) = \int_0^\infty x^2 f(x;\theta)\,dx = \theta^2\int_0^\infty y^3 e^{-y}dy = 6\theta^2$ ∴

$var(X) = E(X^2) - E(X)^2 = 2\theta^2$ ∴

bias of $\hat\theta$: $E(\hat\theta) - \theta = \frac{E(X_1)}{2} - \theta = \frac{2\theta}{2} - \theta = \theta - \theta = 0$

$var(\hat\theta) = \frac{1}{4n^2}\sum_{i=1}^{n}var(X_1) = \frac{var X_1}{4n} = \frac{2\theta^2}{4n} = \frac{\theta^2}{2n}$

\1d/ score function $U(\theta) = l'(\theta)$ if ~~$var(\hat\theta)$~~ $\frac{I(\theta)^{-1}}{var(\hat\theta)} = 1$ then efficient

$I(\theta) = E(-l''(\theta))$

if $U(\theta) = b(\hat\theta - \theta)$ then $\hat\theta$ is unbiased and efficient ∴

$U(\theta) = l'(\theta) = -\frac{2n}{\theta} + \frac{1}{\theta^2}\sum_{i=1}^{n}x_i = -\frac{2n}{\theta} + \frac{1}{\theta^2}2n\frac{1}{2n}\sum_{i=1}^{n}x_i = -\frac{2n}{\theta} + \frac{2n}{\theta^2}\hat\theta =$

$-2n(\frac{1}{\theta} - \frac{1}{\theta^2}\hat\theta) = 2n(\frac{1}{\theta^2}\hat\theta - \frac{1}{\theta}) = \frac{2n}{\theta}(\frac{1}{\theta}\hat\theta - 1) = \frac{2n}{\theta^2}(\hat\theta - \theta) = b(\hat\theta - \theta)$ ∴

$\hat\theta$ is unbiased and efficient ∵ $b = \frac{2n}{\theta^2}$ = const

\PP2020/ \1d sol/ The score func: $U(\theta) = l'(\theta) = \frac{2n}{\theta^2}\left(\frac{\bar{x}}{2} - \theta\right) = \frac{2n}{\theta^2}(\hat{\theta} - \theta)$

which is of the form $U = b(\hat{\theta} - \theta)$ $\therefore \hat{\theta}$ is efficient

\1e/ mle: $\hat{\theta} = \frac{\bar{x}}{2}$ $\quad \therefore 2\hat{\theta} = \bar{x} \therefore$

$\therefore$ Method of Moments $E(x) = \bar{x} = 2\hat{\theta} \therefore$

$E(x) = 2\hat{\theta} = E(x) = 2\theta \therefore$

$\hat{\theta} = \theta$

\1e sol/ MOM estimator obtained by equating the expectation with sample mean $\therefore E(x) = \bar{x} \quad \therefore$

$\hat{\theta} = \frac{\bar{x}}{2} \therefore 2\hat{\theta} = \bar{x} \therefore E(x) = \bar{x} = 2\hat{\theta}$

$\therefore E(x) = 2\theta \therefore E(x) = 2\theta = \bar{x} \therefore$

$\hat{\theta} = \frac{\bar{x}}{2}$ is the MOM estimator is identical to the mle

\1f/ The most powerful test of size α for the null hypothesis $H_0: \theta = \theta_0$ against the alternative $H_1: \theta = \theta_1$ has a critical region of the form $C = \{x; \Lambda(x) \geq c\}$, $\Lambda(x) = \frac{L(\theta_1; x)}{L(\theta_0; x)}$

$$\therefore \Lambda(x) = \frac{L(\theta_1)}{L(\theta_0)} = \frac{\theta_1^{-2n} e^{-\frac{1}{\theta_1}\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i}{\theta_0^{-2n} e^{-\frac{1}{\theta_0}\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i} = \left(\frac{\theta_1}{\theta_0}\right)^{-2n} \frac{e^{-\frac{1}{\theta_1}\sum_{i=1}^{n} x_i}}{e^{-\frac{1}{\theta_0}\sum_{i=1}^{n} x_i}} = \cancel{\left(\frac{\theta_0}{\theta_1}\right)^{2n}}$$

$$\left(\frac{\theta_0}{\theta_1}\right)^{2n} e^{-\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)\sum_{i=1}^{n} x_i} = \left(\frac{\theta_0}{\theta_1}\right)^{2n} e^{\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)\sum_{i=1}^{n} x_i}$$

$\therefore \theta_1 > \theta_0 \therefore \frac{1}{\theta_0} > \frac{1}{\theta_1} \therefore \frac{1}{\theta_0} - \frac{1}{\theta_1} > 0 \therefore$

$\Lambda(x)$ increases as $\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right)$ increases $\therefore$

$\Lambda(x)$ increases as $\sum_{i=1}^{n} x_i$ increases $\therefore$

Λ is large when $\sum_{i=1}^{n} x_i$ is large $\therefore \frac{1}{n}\sum_{i=1}^{n} x_i$ is large

critical region has the form $\left\{x : \frac{1}{n}\sum_{i=1}^{n} x_i \geq d\right\}$ for critical value d

$\therefore \{x : \bar{x} \geq d\}$

\1f sol/ Neyman-Pearson theorem: The most powerful test of size α for the simple null hypothesis $H_0: \theta = \theta_0$ against the simple alternative hypothesis $H_1: \theta = \theta_1$ has a critical region of the form $C = \{x : \Lambda(x) \geq c\}$ where $\Lambda(x) = L(\theta_1; x) / L(\theta_0; x)$ is the likelihood ratio $\therefore$

$\Lambda(x) = \left(\frac{\theta_1}{\theta_0}\right)^{-2n} e^{-n\bar{x}\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)}$ $\therefore \theta_1 > \theta_0 \therefore$

likelihood ratio ~~prop~~ increases is $\bar{x}$ increases $\therefore$

reject H_0 is $\Lambda(x) \geq c$ $\therefore$ reject H_0 is $\bar{x} \geq d$; d = constant

\1g/ they both approximately follow a χ_1^2 distribution when n is large $\therefore W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta})$

$S = U(\theta_0)^2 / I(\theta_0)$

$\therefore U(\theta) = \frac{2n}{\theta^2}(\hat{\theta} - \theta) = I(\theta)^{-1}(\hat{\theta} - \theta) \therefore$ $I(\theta)^{-1} = \frac{2n}{\theta^2} \therefore \frac{\theta^2}{2n} = I(\theta) \therefore$

wald test statistic: $W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta}) = (\hat{\theta} - \theta_0)^2 \frac{\hat{\theta}^2}{2n} =$

$((\hat{\theta} - \theta_0)\hat{\theta})^2 \frac{1}{2n} = \frac{1}{2n}(\hat{\theta}^2 - \theta_0\hat{\theta})^2$

score test statistic: $S = U(\theta_0)^2 / I(\theta_0) = \left(\frac{2n}{\theta_0^2}(\hat{\theta} - \theta_0)\right)^2 \frac{2n}{\theta_0^2}$ ~~Z~~

\1g/ $U(\theta) = \frac{2n}{\theta^2}(\hat{\theta} - \theta) = b(\hat{\theta} - \theta) \therefore$

θ is unbiased and efficient $\therefore I(\theta) = b = \frac{2n}{\theta^2} \therefore$

wald test statistic: $W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta}) = (\hat{\theta} - \theta_0)^2 \frac{2n}{\hat{\theta}^2} =$

$((\hat{\theta} - \theta_0)\frac{1}{\hat{\theta}})^2 2n = 2n\left(1 - \frac{\theta_0}{\hat{\theta}}\right)^2$ ~~Z~~

score test statistic: $S = U(\theta_0)/I(\theta_0) = \frac{2n}{\theta_0^2}(\hat{\theta} - \theta_0) I(\theta_0)^{-1} =$

$\frac{2n}{\theta_0^2}(\hat{\theta} - \theta_0)\frac{\theta_0^2}{2n} = (\hat{\theta} - \theta_0)$

they both approximately follow a χ_1^2 distribution when n is large

\1g/ $W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta})$ ~~$= (\hat{\theta} - \theta_0)$~~ $\therefore$

$U(\theta) = \frac{2n}{\theta^2}(\hat{\theta} - \theta) = I(\theta)(\hat{\theta} - \theta) \therefore I(\theta) = \frac{2n}{\theta^2} \therefore$

$W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta}) = (\hat{\theta} - \theta_0)^2 \frac{2n}{\hat{\theta}^2} = ((\hat{\theta} - \theta_0)\frac{1}{\hat{\theta}})^2 2n = 2n\left(1 - \frac{\theta_0}{\hat{\theta}}\right)^2$

it approximately follows a χ_1^2 distribution when n is large

$\therefore$ reject H_0 is W is large $\therefore$ critical region for test size α is $\{x : W \geq d\}$ d is constant

\1g sol/ The expected information is given by $I(\theta) = E(-l''(\theta)) =$

$E\left(-\frac{2n}{\theta^2} + 2\frac{1}{\theta^3}\sum_{i=1}^{n} x_i\right) = E\left(-\frac{2n}{\theta^2}\right) + E\left(2\frac{1}{\theta^3}\sum_{i=1}^{n} x_i\right) = -\frac{2n}{\theta^2} + 2\frac{1}{\theta^3}E\left(\sum_{i=1}^{n} x_i\right) =$

$-\frac{2n}{\theta^2} + \frac{2}{\theta^3}\sum_{i=1}^{n} E(x_i) = -\frac{2n}{\theta^2} + \frac{2}{\theta^3}\sum_{i=1}^{n} E(x) = -\frac{2n}{\theta^2} + \frac{2}{\theta^3} n E(x) = -\frac{2n}{\theta^2} + \frac{2n}{\theta^3} 2\theta =$

$-\frac{2n}{\theta^2} + \frac{4n}{\theta^2} = \frac{2n}{\theta^2}$

$\therefore$ using MLE $\hat{\theta} = \frac{\bar{x}}{2}$, the wald test stat: $W = (\hat{\theta} - \theta_0)^T I(\hat{\theta})(\hat{\theta} - \theta_0) =$

\PP2020/ $(\hat\theta-\theta_0)^2 I(\hat\theta) = (\hat\theta-\theta_0)^2 \frac{2n}{\hat\theta^2} = 2n\left((\hat\theta-\theta_0)\frac{1}{\hat\theta}\right)^2 = 2n\left(1-\frac{\theta_0}{\hat\theta}\right)^2$

if H_0 is true, and for large n, W has a χ^2_1 distribution

The critical region for a test of size α is given by all data sets $\underline{x}$ for which W exceeds the $(1-\alpha)$ quantile $\chi_{(1-\alpha)}$ of the χ^2_1 distribution

$C(\theta_0) = \{x : W \geq \chi_{(1-\alpha)}\}$

\1h/ $\left(1-\frac{\theta_0}{\hat\theta}\right)^2 > C_\alpha \therefore \cancel{\frac{\theta_0}{\hat\theta}}$ $-\left(1-\frac{\theta_0}{\hat\theta}\right) < \sqrt{C_\alpha} < \left(1-\frac{\theta_0}{\hat\theta}\right) \therefore$

$-(\hat\theta-\theta_0) < \hat\theta\sqrt{C_\alpha} < \hat\theta - \theta_0$

$-\hat\theta < \theta_0 + \hat\theta\sqrt{C_\alpha} < \hat\theta \therefore$

$-\hat\theta - \hat\theta\sqrt{C_\alpha} < \theta_0 < \hat\theta - \hat\theta\sqrt{C_\alpha} \therefore$

$-\hat\theta(1+\sqrt{C_\alpha}) < \theta_0 < \hat\theta(1-\sqrt{C_\alpha}) \therefore$

$1-\alpha$ confidence interval for θ is $(-\hat\theta(1+\sqrt{C_\alpha}), \hat\theta(1-\sqrt{C_\alpha}))$

\1g sol/ Inverting the hypothesis test means to include in the confidence set those values of θ which are not rejected by the test $\therefore S(\underline{x}) = \{\theta : (1-\frac{\theta}{\hat\theta})^2 \leq C_\alpha\}$

$\therefore (1-\frac{\theta}{\hat\theta})^2 \leq C_\alpha \therefore$

$-\sqrt{C_\alpha} \leq 1 - \frac{\theta}{\hat\theta} \leq \sqrt{C_\alpha} \therefore -\sqrt{C_\alpha}\,\hat\theta \leq \hat\theta - \theta \leq \sqrt{C_\alpha}\,\hat\theta \therefore$

$-\sqrt{C_\alpha}\,\hat\theta - \hat\theta \leq -\theta \leq \sqrt{C_\alpha}\,\hat\theta - \hat\theta \therefore$

$-\sqrt{C_\alpha}\,\hat\theta + \hat\theta \leq \theta \leq \sqrt{C_\alpha}\,\hat\theta + \hat\theta \therefore \hat\theta(1-\sqrt{C_\alpha}) \leq \theta \leq \hat\theta(1+\sqrt{C_\alpha})$

$\therefore$ a $(1-\alpha)$ confidence interval for θ is given by

$[\hat\theta(1-\sqrt{C_\alpha}), \hat\theta(1+\sqrt{C_\alpha})]$

\2a/ $h(\hat\theta) \approx h(\theta) + (\hat\theta-\theta)h'(\theta) \therefore$

$\hat\phi = h(\hat\theta) \approx h(\theta) + (\hat\theta-\theta)h'(\theta) = \phi + (\hat\theta-\theta)h'(\theta) \therefore$

$E(\hat\phi) = E(h(\hat\theta)) \approx E(\phi + (\hat\theta-\theta)h'(\theta)) \approx$

$var(\hat\phi) = var(h(\hat\theta)) \approx var(h(\theta) + (\hat\theta-\theta)h'(\theta)] = var(\hat\theta h'(\theta)) = (h'(\theta))^2 var(\hat\theta)$

$E(\hat\phi) = E(h(\hat\theta)) \approx E(h(\theta) + (\hat\theta-\theta)h'(\theta)) = h(\theta) - \theta h'(\theta) + h'(\theta)E(\hat\theta) =$

$\phi + (E(\hat\theta)-\theta)h'(\theta) \approx \phi + (0)h'(\theta) = \phi$

$h(\hat\theta) \approx h(\theta) + (\hat\theta-\theta)h'(\theta) + \frac{1}{2}(\hat\theta-\theta)^2 h''(\theta) \therefore$

$E(\hat{\phi}) = E(h(\hat{\theta})) \approx E(h(\theta) + (\hat{\theta}-\theta)h'(\theta) + \frac{1}{2}(\hat{\theta}-\theta)^2 h''(\theta)) =$

$h(\theta) + (E(\hat{\theta})-\theta)h'(\theta) + \frac{1}{2}E((\hat{\theta}-\theta)^2)h''(\theta) =$

$h(\theta) + \frac{1}{2}E((\hat{\theta}-\theta)^2)h''(\theta) = h(\theta) + \frac{1}{2}(\text{var}(\hat{\theta}))h''(\theta) =$

$\phi + \frac{1}{2}\text{var}(\hat{\theta})h''(\theta)$

$\therefore \text{var}(\hat{\phi}) \approx (h'(\theta))^2 \text{var}(\hat{\theta})$

$E(\hat{\phi}) \approx \phi + \frac{1}{2}\text{var}(\hat{\theta})h''(\theta) \therefore$

$\text{Bias}(\hat{\phi}) = E(\hat{\phi}) - \phi = \phi + \frac{1}{2}\text{var}(\hat{\theta})h''(\theta) - \phi = \frac{1}{2}\text{var}(\hat{\theta})h''(\theta)$

\2a Sol/ $h(\hat{\theta}) \approx h(\theta) + (\hat{\theta}-\theta)h'(\theta) + \frac{1}{2}(\hat{\theta}-\theta)^2 h''(\theta)$

$\therefore \text{var}(\hat{\phi}) = \text{var}(h(\hat{\theta})) \approx \text{var}(h(\theta) + (\hat{\theta}-\theta)h'(\theta)) =$

$\text{var}((\hat{\theta}-\theta)h'(\theta)) = \text{var}(\hat{\theta}\, h'(\theta)) = (h'(\hat{\theta}))^2 \text{var}(\hat{\theta})$

~~E(h(θ~~ $E(\hat{\phi}) = E(h(\hat{\theta})) \approx E(h(\theta) + (\hat{\theta}-\theta)h'(\theta) + \frac{1}{2}(\hat{\theta}-\theta)^2 h''(\theta)) =$

$h(\theta) + (E(\hat{\theta})-\theta)h'(\theta) + \frac{1}{2}E((\hat{\theta}-\theta)^2)h''(\theta) =$

$\phi + (0)h'(\theta) + \frac{1}{2}\cancel{E}\text{var}(\hat{\theta})h''(\theta) = \phi + \frac{1}{2}\text{var}(\hat{\theta})h''(\theta) \therefore$

approx of the bias: $\text{Bias}(\hat{\phi}) = E(\hat{\phi}) - \phi \approx \phi + \frac{1}{2}\text{var}(\hat{\theta})h''(\theta) - \phi =$
$\frac{1}{2}\text{var}(\hat{\theta})h''(\theta)$

\2bi/ $\therefore E(x_i) = \theta$, $\text{var}(x_i) = \theta^2$ $\quad \hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} x_i$

$\text{var}(\hat{\theta}) = \text{var}(\frac{1}{n}\sum_{i=1}^{n} x_i) = \frac{1}{n^2}\text{var}(\sum_{i=1}^{n} x_i) = \frac{1}{n^2}\sum_{i=1}^{n}\text{var}(x_i) = \frac{1}{n^2}\sum_{i=1}^{n}\text{var}(x) =$
$\frac{n}{n^2}\text{var}(x) = \frac{1}{n}\text{var}(x) = \frac{1}{n}\text{var}(x_i) = \frac{\theta^2}{n}$ ✓

\2bi sol/ $\text{var}(\hat{\theta}) = \frac{1}{n^2}\sum_{i=1}^{n}\text{var}(x_i) = \frac{1}{n}\text{var}(x_1) = \frac{\theta^2}{n}$

\2bii/ $\text{var}(\hat{\phi}) = \text{var}(\sqrt{\hat{\theta}}) = \text{var}(\hat{\theta}^{1/2})$

$\text{var}(\hat{\phi}) = E(\hat{\phi}^2) - E(\hat{\phi})^2 \therefore$

$\phi = h(\theta) = \sqrt{\theta} = \theta^{1/2} \therefore \hat{\phi} = h(\hat{\theta}) = \sqrt{\hat{\theta}} = \hat{\theta}^{1/2} \therefore h'(\hat{\theta}) = \frac{1}{2}\hat{\theta}^{-1/2}$

$\therefore \text{var}(\hat{\phi}) = \text{var}(h(\hat{\theta})) \approx \text{var}(h(\theta) + (\hat{\theta}-\theta)h'(\theta) + \frac{1}{2}(\hat{\theta}-\theta)^2 h''(\theta)) \approx$

$\text{var}(h(\theta) + (\hat{\theta}-\theta)h'(\theta)) = \text{var}(\hat{\theta}-\theta)(h'(\theta))^2 = \text{var}(\hat{\theta})(h'(\hat{\theta}))^2 = (h'(\hat{\theta}))^2 \text{var}(\hat{\theta})$

$\therefore \text{var}(\hat{\phi}) \approx (h'(\hat{\theta}))^2 \text{var}(\hat{\theta}) = (\frac{1}{2}\hat{\theta}^{-1/2})^2 \text{var}(\frac{1}{n}\sum_{i=1}^{n} x_i) =$

$\frac{1}{4}\hat{\theta}^{-1}\text{var}(\hat{\theta}) = \frac{1}{4}\hat{\theta}^{-1}\frac{\hat{\theta}^2}{n} = \frac{\hat{\theta}}{4n}$

\2bii/ let $h(\theta) = \theta^{1/2} \therefore h'(\theta) = \frac{1}{2}\theta^{-1/2} = \frac{1}{2\sqrt{\theta}} \therefore$

$\text{var}(\hat{\phi}) \approx [h'(\theta)]^2 \text{var}(\hat{\theta}) = [h'(\theta)]^2 \frac{\theta^2}{n} = (\frac{1}{2}\theta^{-1/2})^2 \frac{\theta^2}{n} = \frac{1}{4}\theta^{-1}\frac{\theta^2}{n} = \frac{\theta}{4n}$

\PP2020/ \2biii/ $\text{Bias}(\hat{\phi}) \approx E(\hat{\phi}) - \phi \approx$

$E(h(\hat{\theta})) - \phi \approx E\left(h(\theta) + (\hat{\theta}-\theta)h'(\theta) + \frac{1}{2}(\hat{\theta}-\theta)^2 h''(\theta)\right) - \phi =$

$h(\theta) + h''(\theta)E((\hat{\theta}-\theta)^2) - \phi = h(\theta) + \frac{1}{2}h''(\theta)\text{var}(\hat{\theta}) - \phi =$

$\phi \approx \frac{1}{2}h''(\theta)\text{var}(\hat{\theta}) \doteq \frac{1}{2}h''(\theta)\frac{\theta^2}{n}$

$h(\theta) = \sqrt{\theta} = \theta^{1/2}$ $\therefore$ $h'(\theta) = \frac{1}{2}\theta^{-1/2}$ $\therefore$ $h''(\theta) = -\frac{1}{4}\theta^{-3/2}$ $\therefore$

$\text{Bias}(\hat{\phi}) \approx \frac{1}{2}h''(\theta)\frac{\theta^2}{n} = \frac{1}{2}\left(-\frac{1}{4}\theta^{-3/2}\right)\frac{\theta^2}{n} = -\frac{1}{8}\theta^{1/2}\frac{1}{n} = \frac{-\theta^{1/2}}{8n}$

Bias corrected estimator: $\tilde{\phi}$

$\cancel{\hat{\phi} - \text{Bias}(\hat{\phi})} = \tilde{\phi} = \hat{\phi} - \text{Bias}(\hat{\phi}) = h(\hat{\theta}) + \frac{\theta^{1/2}}{8n} = \sqrt{\theta} + \frac{\theta^{1/2}}{8n}$

\2biii/ $h''(\theta) = \frac{-1}{4\sqrt{\theta^3}} = \frac{-\theta^{-3/2}}{4}$ $\therefore$

$\text{Bias}(\hat{\phi}) = E(\hat{\phi}) - \phi \approx -\frac{\theta^2}{n}\frac{1}{8\sqrt{\theta^3}} = -\frac{\sqrt{\theta}}{8n} = -\frac{\theta^{1/2}}{8n}$

$\therefore$ biased corrected estimator: $\tilde{\phi} = \hat{\phi} - \left(\frac{-\hat{\theta}^{1/2}}{8n}\right) = h(\hat{\theta}) + \frac{\hat{\theta}^{1/2}}{8n} =$

$\hat{\theta}^{1/2} + \frac{\hat{\theta}^{1/2}}{8n} = \hat{\theta}^{1/2}\left(1 + \frac{1}{8n}\right) = \hat{\phi}\left(1 + \frac{1}{8n}\right)$

\2C/ consistent $\therefore$ $\forall \varepsilon > 0$: $\lim_{n\to\infty} \Pr(|\hat{\theta} - \theta| > \varepsilon) = 0$

$\therefore$ mean squared error$(\hat{\theta}) \to 0$ as $n \to \infty$

$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta \neq 0$ $\therefore$

ϕ $E(\hat{\phi}) = E(h(\hat{\theta})) \approx E\left(h(\theta) + (\hat{\theta}-\theta)h'(\theta) + \frac{1}{2}(\hat{\theta}-\theta)^2 h''(\theta)\right) =$

$h(\theta) + (E(\hat{\theta}) - \theta)h'(\theta) + \frac{1}{2}E((\hat{\theta}-\theta)^2)h''(\theta) =$

$\phi + (E(\hat{\theta}) - \theta)h'(\theta) + \frac{1}{2}\text{var}(\hat{\theta})h''(\theta)$

$\text{Bias}(\hat{\phi}) = \cancel{\phi} \; E(\hat{\phi}) - \phi =$

$\phi + (E(\hat{\theta}) - \theta)h'(\theta) + \frac{1}{2}\text{var}(\hat{\theta})h''(\theta) - \phi =$

$\text{Bias}(\hat{\theta})h'(\theta) + \frac{1}{2}\text{var}(\hat{\theta})h''(\theta) =$

$\text{Bias}(\hat{\theta})\frac{1}{2}\theta^{-1/2} + \frac{1}{2}\text{var}(\hat{\theta})\left(-\frac{1}{4}\theta^{-3/2}\right) =$

$\frac{1}{2}\theta^{-1/2}\text{Bias}(\hat{\theta}) - \frac{1}{8}\theta^{-3/2}\text{var}(\hat{\theta}) =$

$\frac{1}{2}\theta^{-1/2}\text{Bias}(\hat{\theta}) - \frac{1}{8}\theta^{-3/2}\frac{\theta^2}{n} = \frac{1}{2}\theta^{-1/2}\text{Bias}(\hat{\theta}) - \frac{\theta^{1/2}}{8n} = \frac{1}{2\sqrt{\theta}}\text{Bias}(\hat{\theta}) - \frac{\sqrt{\theta}}{8n}$

$= \frac{1}{2\sqrt{\theta}}(E(\hat{\theta}) - \theta) - \frac{\sqrt{\theta}}{8n} = \frac{1}{2\sqrt{\theta}}\left(E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)\right) - \frac{\sqrt{\theta}}{8n} =$

$\frac{1}{2\sqrt{\theta}}E(x_1) - \frac{\sqrt{\theta}}{8n} = \frac{1}{2\sqrt{\theta}}\theta - \frac{\sqrt{\theta}}{8n} = \frac{\theta^{1/2}}{2} - \frac{\theta^{1/2}}{8n} = \sqrt{\theta}\left(\frac{1}{2} - \frac{1}{8n}\right)$

2c sol / $E(\hat{\phi}) = E(h(\hat{\theta})) \approx E\left(h(\theta) + (\hat{\theta}-\theta)h'(\theta) + \frac{1}{2}(\hat{\theta}-\theta)^2 h''(\theta)\right) =$

$h(\theta) + (E(\hat{\theta})-\theta)h'(\theta) + \frac{1}{2}E((\hat{\theta}-\theta)^2)h''(\theta) = \phi + (E(\hat{\theta})-\theta)h'(\theta) + \frac{1}{2}E((\hat{\theta}-\theta)^2)h''(\theta)$ ∴

$\text{Bias}(\hat{\phi}) = E(\hat{\phi}) - \phi = \phi + (E(\hat{\theta})-\theta)h'(\theta) + \frac{1}{2}E((\hat{\theta}-\theta)^2)h''(\theta) - \phi =$

$(E(\hat{\theta})-\theta)h'(\theta) + \frac{1}{2}E((\hat{\theta}-\theta)^2)h''(\theta) =$

$h'(\theta)\text{Bias}(\hat{\theta}) + \frac{1}{2}h''(\theta)\text{mse}(\hat{\theta}) =$

$h'(\theta)\text{bias}(\hat{\theta}) + \frac{1}{2}h''(\theta)(\text{var}(\hat{\theta}) + \text{bias}^2(\hat{\theta}))$

3a / parametric means using known parameters of the underlying data   non parametric means only using what can be calculated from the given data
bootstrap resampling means using the observed data to generate many more sets of data
∴ it will give a parametric or non parametric biases and variances

3b sol / Bootstrap resampling creates new samples of size n by simulating values from the Exp(θ) distribution (parametric) or by drawing randomly, with replacement from the original data x (non parametric).
if the estimator is evaluated ~~from the original~~ for each of many such samples then the bias of $\hat{\theta}$ may be estimated as the difference between the mean of the bootstrap estimates and $\hat{\theta}$, and the variance may be estimated as the sample variance of the bootstrap estimates
eg $\text{bias}(\hat{\theta}) = \text{Mean}(\hat{\theta}^*_i) - \hat{\theta} = \overline{\hat{\theta}^*_B} - \hat{\theta}$, $\text{var}(\theta) = \text{var}(\hat{\theta}^*_I)$

3b / Basic Bootstrap interval is $(2\hat{\theta} - \hat{\theta}^*_{((1-\alpha)B)}, 2\hat{\theta} - \hat{\theta}^*_{(\alpha B)})$
∴ $\hat{\theta}$ is measured from the original data then bootstraping new data find the $\hat{\theta}^*$ of each sample and order them and find the $(1-\alpha)B$ and αB quantiles of the $\hat{\theta}^*$ where

\PP2020/ α is for a $(1-2\alpha)$ confidence interval
and B is the number of bootstrapped samples

~~\3b~~ \3bii/ percentile boostrap interval is $(\hat{\theta}^*_{(\alpha B)}, \hat{\theta}^*_{((1-\alpha)B)})$

\3b sol/ let $\bar{x}^*_r$ denote the mean of the r^{th} bootstrap sample and define $\hat{\theta}^*_r = 1/\bar{x}^*_r$. let $\hat{\theta}^*_{(pR)}$ denote the sample p-quantile of $\hat{\theta}_1, \dots, \hat{\theta}^*_R$ where $R=1000$ or a similarly large number.

A basic bootstrap, equaltailed $(1-2\alpha)$-CI for θ is
$(2\hat{\theta} - \hat{\theta}^*_{((1-\alpha)R)}, 2\hat{\theta} - \hat{\theta}^*_{(\alpha R)})$.

A percentile bootstrap equaltailed $(1-2\alpha)$ CI for θ is $(\hat{\theta}^*_{(\alpha R)}, \hat{\theta}^*_{((1-\alpha)R)})$

\3c/ use the original data to generate a large number of new $\hat{\theta}^*$ estimates and check what proportionat of the estimates do fall into the calculated intervals and whichever one has the most fall into it ~~of~~ has the better converage.

\3c sol/ Sample replicate data $x^* = (x^*_1, \dots, x^*_n)^T$ from $\mathrm{Exp}(\hat{\theta})$ and construct $1-\alpha$ confidence intervals using the two methods. Over many replications count how ~~so~~ often the two types of CI cover the true value $\hat{\theta}$. The type of CI whose converage frequently ~~is close~~ is close to $1-\alpha$ and has a higher proportion of coverages from the replications has better converage and is $\therefore$ preferable.

\3d/

$\hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i \qquad \hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i} = 5\left(\frac{100}{31}\right) - 4\hat{\theta}_{-i}$

$\hat{\theta} = \frac{1}{\bar{x}} \quad \therefore \quad \bar{x} = \frac{1}{5}(0.51 + 0.39 + 0.36 + 0.26 + 0.03) = 0.31 \therefore \hat{\theta} = \frac{1}{0.31} = \frac{100}{31}$

x_i	0.51	0.39	0.36	0.26	0.03
$\bar{x}_{-i}$	0.26	0.29	0.2975	0.3225	0.38
$\hat{\theta}_{-i}$	$\frac{50}{13}$ ✓	$\frac{100}{29}$ ✓	$\frac{400}{199}$ ✗	$400/129$ ✓	$50/19$ ✓
θ_i	$\frac{300}{403}$	$\frac{2100}{899}$	$\frac{49900}{6169}$	3.7259	$\frac{3300}{589}$

$\therefore \hat{\theta}_J = \frac{1}{5}\left(\frac{300}{403} + \frac{2100}{899} + \frac{49900}{6169} + 3.7259 + \frac{3300}{589}\right) = 4.10 \; (3 s.f.)$ ✗ 3.02

3d sol / define $\hat{\theta}_{-i} = \frac{n-1}{n\bar{x} - x_i} = \frac{1}{\left(\frac{n\bar{x} - x_i}{n-1}\right)} = \frac{1}{\frac{1}{n-1}(n\bar{x} - x_i)} =$

$\frac{1}{\frac{1}{n-1}\left(n\frac{1}{n}\sum_{i=1}^{n} x_i - x_i\right)} = \frac{1}{\frac{1}{n-1}\left(\sum_{i=1}^{n} x_i - x_i\right)} = \frac{1}{\frac{1}{n-1}\left(\sum_{j\neq i} x_j\right)}$ $\therefore$

$\hat{\theta}_J = n\hat{\theta} - \frac{n-1}{n}\sum_{i=1}^{n}\hat{\theta}_{-i} = n\hat{\theta} - (n-1)\hat{\theta}_i$, ~~$\hat{\theta}_i = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i$~~

$\therefore$ $\hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$ $\therefore$

$\hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i = \frac{1}{n}\sum_{i=1}^{n}(n\hat{\theta} - (n-1)\hat{\theta}_{-i}) = \frac{1}{n} n \sum_{i=1}^{n}\hat{\theta} - \frac{(n-1)}{n}\sum_{i=1}^{n}\hat{\theta}_{-i} = n\hat{\theta} - \frac{n-1}{n}\sum_{i=1}^{n}\hat{\theta}_{-i}$

$\therefore n = 5$, $\hat{\theta} = 3.23$,

$\hat{\theta}_{-1} = 3.85$, $\hat{\theta}_{-2} = 3.45$ $\hat{\theta}_{-3} = 3.36$ $\hat{\theta}_{-4} = 3.10$ $\hat{\theta}_{-5} = 2.63$ $\therefore$

$\hat{\theta}_i$: 0.729 2.33 2.69 3.73 5.61 $\therefore$

$\hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i = \frac{1}{5}(0.729 + 2.33 + 2.69 + 3.73 + 5.61) = 3.02$

3a / $P(Z_i \le x; \theta) = P(\frac{X_i}{\theta} \le x; \theta) = P(X_i \le \theta x; \theta) = e^{-\frac{\theta}{(\theta x)}} = e^{-\frac{1}{x}}$ $\therefore$

~~θ~~ $P(Z_i \le x) = e^{-\frac{1}{x}}$ is independent of any parameters $\therefore$

Z_i is a pivot

$T = \frac{X_0}{X_{(n)}} = \frac{Z_0\theta}{Z_n\theta} = \frac{Z_0}{Z_n}$ $\therefore$ Z_0, Z_n are pivots $\therefore$

$\frac{Z_0}{Z_n} = T$ is independent of any parameters $\therefore$ T is ancillary

4a sol / The distribution of Z_i is $Pr(Z_i \le t; \theta) = Pr(\frac{X_i}{\theta} \le t; \theta) =$

$Pr(X_i \le \theta t; \theta) = e^{-\frac{\theta}{(\theta t)}} = e^{-\frac{1}{t}}$ The distribution of Z_i is

independent of θ and $\therefore$ Z is a pivot

$\therefore$ ~~$T = \theta Z_0$~~ $X_{(n)} = \text{Max}\{X_1, \dots, X_n\}$ $\therefore$ $X_i = \theta Z_i$ $\therefore$ $X_0 = \theta Z_0$ $\therefore$

$X_{(n)} = \text{Max}\{X_1, \dots, X_n\} = \text{Max}\{\theta Z_1, \dots, \theta Z_n\}$ $\therefore$

$T = \frac{X_0}{X_{(n)}} = \frac{\theta Z_0}{\text{Max}\{\theta Z_1, \dots, \theta Z_n\}} = \frac{\theta Z_0}{\theta\,\text{Max}\{Z_1, \dots, Z_n\}} = \frac{Z_0}{\text{Max}\{Z_1, \dots, Z_n\}}$

which is independent of θ. $\therefore$ the distribution of $Z_0, \dots, Z_n$ are independent of θ, the distribution of T is independent of θ, and $\therefore$ T is an ancillary statistic.

4b / $X_{(n)} = \text{Max}\{X_1, \dots, X_n\}$ $\therefore$ $F(x_{(n)}; \theta) = Pr(X_{(n)} \le x; \theta) =$

$Pr(\text{Max}\{X_1, \dots, X_n\} \le x; \theta) = Pr(X_1 \cap X_2 \cap \dots \cap X_n \le x; \theta) =$

$Pr(X_1 \le x, X_2 \le x, \dots, X_n \le x; \theta) = P(X_1 \le x)P(X_2 \le x) \cdots P(X_n \le x) =$

\PP2020/ $(Pr(X_n \le x))^n = (e^{-\theta/x})^n = e^{-n\theta/x}$ $\therefore$

$f_{X_{(n)}}(x;\theta) = \frac{d}{dx}(Pr(X_{(n)} \le x;\theta)) = \frac{d}{dx}(e^{-n\theta/x}) = e^{-n\theta/x}\frac{d}{dx}(-n\theta x^{-1}) =$
$e^{-n\theta/x}(n\theta x^{-2}) = \frac{n\theta}{x^2}e^{-n\theta/x}$ $\therefore$

$T = \frac{X_0}{X_{(n)}} = \frac{e^{-\theta/x}}{\frac{n\theta}{x^2}e^{-n\theta/x}} = \frac{x^2}{n\theta}e^{-\theta/x}e^{n\theta/x} = \frac{x^2}{n\theta}e^{-\frac{\theta}{x}+n\frac{\theta}{x}} = \frac{x^2}{n\theta}e^{\frac{\theta}{x}(-1+n)}$

\4b SA/ By independence of $X_1, \dots, X_n$: $Pr(X_{(n)} \le x) =$
$Pr(\max\{X_1, \dots, X_n\} \le x) = Pr(X_1, \dots, X_n \le x) = Pr(X_1 \le x, \dots, X_n \le x) =$
by independence $P(X_1 \le x) \dots P(X_n \le x) = (P(X \le x))^n = F(x;\theta)^n = \cancel{e^{-\theta/x}}\, e^{-n\theta/x}$
$\therefore$ The density of $X_{(n)}$ is given by $g(x) = \frac{d}{dx}Pr(X_{(n)} \le x) =$
$\frac{d}{dx}(e^{-n\theta/x}) = n\theta x^{-2}e^{-n\theta/x}$ $\therefore$

~~Pr~~ \4b/ ~~PR~~ $g(x) = \frac{d}{dx}(Pr(X_{(n)} \le x)) = n\theta^{-2}e^{-n\theta/x}$
$\therefore Pr(T \le t) = Pr\left(\frac{X_0}{X_{(n)}} \le t\right) = Pr(X_0 \le tX_{(n)}) = \int_0^\infty Pr(X_0 \le tX_{(n)} \mid X_{(n)} = x)\, P(X_{(n)} = x)\, dx =$
$\int_0^\infty P(X_0 \le tx)\, P(X_{(n)} = x)\, dx = \int_0^\infty P(X_0 \le tx)\, g(x)\, dx = \int_0^\infty e^{-\frac{\theta}{tx}} n\theta x^{-2} e^{-\frac{n\theta}{x}} dx =$
$n\theta\int_0^\infty x^{-2}e^{-\frac{1}{t}\frac{\theta}{x} - n\frac{\theta}{x}}dx = n\theta\int_0^\infty x^{-2}e^{-(n+\frac{1}{t})\frac{\theta}{x}} = n\theta\int_0^\infty x^{-2}e^{-\theta(n+\frac{1}{t})/x}dx$ $\therefore$

let $y = \theta(n+\frac{1}{t})/x = \theta(n+\frac{1}{t})x^{-1}$ $\therefore \frac{dy}{dx} = -\theta(n+\frac{1}{t})x^{-2}$ $\therefore$
$\frac{1}{-\theta(n+\frac{1}{t})}dy = x^{-2}dx$ $\therefore$ $x = \infty : y = \theta(n+\frac{1}{t})\frac{1}{\infty} = 0$, $x = 0 : y = \theta(n+\frac{1}{t})\frac{1}{0} = \infty$

$\therefore Pr(T \le t) = n\theta\int_0^\infty e^{-\theta(n+\frac{1}{t})\frac{1}{x}}x^{-2}dx = n\theta\int_\infty^0 e^{-y}\frac{-1}{\theta(n+\frac{1}{t})}dy =$
$\frac{n\theta}{\theta(n+\frac{1}{t})}\int_0^\infty e^{-y}dy = \frac{n}{n+\frac{1}{t}}\left[-e^{-y}\right]_{y=0}^{\infty} = \frac{n}{n+\frac{1}{t}}\left[-(0)+e^0\right] = \frac{n}{n+\frac{1}{t}}[1] =$
$\frac{n}{n+\frac{1}{t}} = Pr(T \le t)$

\4c/ $Pr(T \le t) = \frac{n}{n+\frac{1}{t}}$ $\therefore$ $Pr(T \le t_{0.95}) = 0.95 = \frac{n}{n+\frac{1}{t_{0.95}}}$ $\therefore$
$0.95\left(n+\frac{1}{t_{0.95}}\right) = n = 0.95n + \frac{0.95}{t_{0.95}} = n$ $\therefore$ $\frac{0.95}{t_{0.95}} = 0.05n$ $\therefore$ $n = \frac{19}{t_{0.95}}$

$Pr(T \le t_{0.05}) = 0.05 = \frac{n}{n+\frac{1}{t_{0.05}}}$ $\therefore$ $0.05\left(n+\frac{1}{t_{0.05}}\right) = n = 0.05n + \frac{0.05}{t_{0.05}} = n$ $\therefore$
$\frac{0.05}{t_{0.05}} = 0.95n$ $\therefore$ $n = \frac{1}{19t_{0.05}}$ $\therefore$

t_α is the α-quantile of the T distribution $\therefore$
90% PI : $(t_{0.05} \le T \le t_{0.95}) = (t_{0.05}, t_{0.95}) = \left(\frac{n}{n+\frac{1}{t_{0.05}}}, \frac{n}{n+\frac{1}{t_{0.95}}}\right)$.

4C Sol / let $P_r(T \le t_p) = p$ for p-quantile $\therefore$

$P_r(T \le t_p) = p = \frac{n}{n + \frac{1}{t_p}}$ $\therefore$ $p(n + \frac{1}{t_p}) = n = pn + \frac{p}{t_p}$ $\therefore$ $n(1-p) = \frac{p}{t_p}$ $\therefore$

$t_p = \frac{p}{n(1-p)}$ $\therefore$ $t_{0.05} = \frac{0.05}{n(1-0.05)} = \frac{1}{19n}$, $t_{0.95} = \frac{19}{n}$ $\therefore$

$0.9 = P_r(t_{0.05} < T < t_{0.95}) = P_r(\frac{1}{19n} < T < \frac{19}{n}) = P_r(\frac{1}{19n} < \frac{X_0}{X_{(n)}} < \frac{19}{n}) =$

$P_r(\frac{X_{(n)}}{19n} < X_0 < \frac{19X_{(n)}}{n})$ $\therefore$ $(\frac{X_{(n)}}{19n}, \frac{19X_{(n)}}{n})$ is a 90% PI for X_0.

PP2022 / 1a / $E(X_1) = E(X) = \int_0^\infty x f(x;\theta)\,dx = \int_0^\infty x \sqrt{4\theta}^{-1} x^{-1/2} e^{-(x/\theta)^{1/2}}\,dx$

$= \frac{1}{\sqrt{4}} \int_0^\infty \theta^{-1/2} x^{1/2} e^{-(x/\theta)^{1/2}}\,dx = \frac{1}{\sqrt{4}} \int_0^\infty \left(\frac{x}{\theta}\right)^{1/2} e^{-(x/\theta)^{1/2}}\,dx$

$\therefore$ let $y = \left(\frac{x}{\theta}\right)^{1/2} = \frac{1}{\theta^{1/2}} x^{1/2}$ $\therefore$ $\frac{dy}{dx} = \frac{1}{2\theta^{1/2}} x^{-1/2}$ $\therefore$ $y = \frac{1}{\theta^{1/2}} x^{1/2}$

$2\theta^{1/2} dy = x^{-1/2} dx$ $\therefore$ $2\theta^{1/2} x^{1/2} dy = dx$ $\therefore$ $\theta^{1/2} y = x^{1/2}$

$\therefore$ $2\theta^{1/2} x^{1/2} dy = 2\theta^{1/2} \theta^{1/2} y = 2\theta y\,dy = dx$ $\therefore$

$x = 0$: $y = \left(\frac{0}{\theta}\right)^{1/2} = 0$, $x \to \infty$: $y = \left(\frac{\infty}{\theta}\right)^{1/2} = \infty$ $\therefore$

$E(X_1) = \frac{1}{\sqrt{4}} \int_0^\infty y e^{-y}\, 2\theta y\,dy = \frac{2\theta}{\sqrt{4}} \int_0^\infty y^2 e^{-y}\,dy = \frac{2\theta}{\sqrt{4}} (2!) = \frac{2\theta \cdot 2}{\sqrt{4}} = \frac{4\theta}{\sqrt{4}} = \sqrt{4}\theta$

$\therefore$ MOM: $\bar{x} = E(X_1) = \sqrt{4}\hat{\theta}$ $\therefore$ $\frac{\bar{x}}{2} = \hat{\theta} = \frac{\bar{x}}{\sqrt{4}}$ is MOM estimator

$\therefore E(x) = 2\theta$

1b / $\text{bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$,

$\therefore E(\hat{\theta}) = E\left(\frac{1}{\sqrt{4}}\bar{X}\right) = \frac{1}{\sqrt{4}} E(\bar{X}) = \frac{1}{\sqrt{4}} E\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \frac{1}{\sqrt{4}\,n} \sum_{i=1}^n E(X_i) = \frac{1}{\sqrt{4}} \sum_{i=1}^n E(X) =$

$\frac{1}{\sqrt{4}} n E(X) = \frac{1}{n\sqrt{4}} n \sqrt{4}\theta = n\theta$

$\therefore \text{bias}(\hat{\theta}) = E(\hat{\theta}) - \theta = \frac{n\theta}{n} - \theta = \theta - \theta = 0$

$\text{var}(\hat{\theta}) = \text{var}\left(\frac{1}{\sqrt{4}}\bar{X}\right) = \frac{1}{4}\text{var}(\bar{X}) = \frac{1}{4}\text{var}\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \frac{1}{4n^2}\text{var}\left(\sum_{i=1}^n X_i\right) =$

$\frac{1}{4n^2}\sum_{i=1}^n \text{var}(X_i) = \frac{1}{4n^2}\sum_{i=1}^n \text{var}(X) = \frac{1}{4n^2} n\,\text{var}(X) = \frac{1}{4n}\text{var}(X)$ $\therefore$

$\text{var}(X) = E(X^2) - E(X^2)$ $\therefore$

$E(X^2) = \int_0^\infty x^2 f(x,\theta)\,dx = \int_0^\infty x^2 \frac{1}{\sqrt{4}} (x^{-1/2}) \theta^{-1/2} e^{-(x/\theta)^{1/2}}\,dx =$

$\frac{1}{\sqrt{4}} \int_0^\infty x^{3/2} \theta^{-1/2} e^{-(x/\theta)^{1/2}}\,dx = \frac{1}{\sqrt{4}} \int_0^\infty x \left(\frac{x}{\theta}\right)^{1/2} e^{-(x/\theta)^{1/2}}\,dx$ $\therefore$

$y = \left(\frac{x}{\theta}\right)^{1/2}$ $\therefore$ $x = 0$: $y = 0$, $x = \infty$: $y = \infty$,

$\theta^{1/2} y = x^{1/2}$ $\therefore$ $x = \theta y^2$, $2\theta y\,dy = dx$ $\therefore$

$E(X^2) = \frac{1}{\sqrt{4}} \int_0^\infty \theta y^2\, y e^{-y}\, 2\theta y\,dy = \frac{1}{\sqrt{4}} \int_0^\infty 2\theta^2 y^4 e^{-y}\,dy = \frac{2\theta^2}{\sqrt{4}} \int_0^\infty y^4 e^{-y}\,dy =$

$\frac{2\theta^2}{\sqrt{4}} 4! = \frac{2\theta^2}{\sqrt{4}} 24 = 24\theta^2$ $\therefore$

$\therefore \text{var}(X) = 24\theta^2 - (2\theta)^2 = 24\theta^2 - 4\theta^2 = 20\theta^2$ $\therefore$

$\text{var}(\hat{\theta}) = \frac{1}{4n} 20\theta^2 = \frac{5\theta^2}{n}$ $\therefore$

$\text{var}(\hat{\theta}) \to 0$ as $n \to \infty$ $\therefore$ $\text{mse}(\hat{\theta}) \to 0$ as $n \to \infty$ $\therefore$

$\hat{\theta}$ is consistent.

\1c/ $L(\theta;\underline{x}) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} 4^{-1/2}\theta^{-1/2}x_i^{-1/2}e^{-\frac{1}{\theta^{1/2}}x_i^{1/2}} =$

$4^{-n/2}\theta^{-n/2}\left(\prod_{i=1}^{n} x_i^{-1/2}\right)e^{-\frac{1}{\theta^{1/2}}\sum_{i=1}^{n} x_i^{1/2}}$ $\therefore$

$l(\theta;\underline{x}) = \ln(4^{-n/2}) + \ln(\theta^{-n/2}) + \ln\left(\prod_{i=1}^{n} x_i^{-1/2}\right) - \frac{1}{\theta^{1/2}}\sum_{i=1}^{n} x_i^{1/2} =$

$-\frac{n}{2}\ln\theta - \theta^{-1/2}\sum_{i=1}^{n} x_i^{1/2} + \left(-\frac{n}{2}\ln(4) + \ln\left(\prod_{i=1}^{n} x_i^{-1/2}\right)\right) =$

$-\frac{n}{2}\ln\theta - \theta^{-1/2}\sum_{i=1}^{n} x_i^{1/2}$ + constant

$\therefore$ $l'(\theta) = -\frac{n}{2}\frac{1}{\theta} + \frac{1}{2}\theta^{-3/2}\sum_{i=1}^{n} x_i^{1/2}$ $\therefore$

$l'(\hat\theta) = 0 = -\frac{n}{2}\frac{1}{\hat\theta} + \frac{1}{2}\frac{1}{\hat\theta^{3/2}}\sum_{i=1}^{n} x_i^{1/2}$ $\therefore$

$\frac{n}{2}\frac{1}{\hat\theta} = \frac{1}{2}\frac{1}{\hat\theta^{3/2}}\sum_{i=1}^{n} x_i^{1/2}$ $\therefore$ $\frac{n}{2}\hat\theta^{1/2} = \frac{1}{2}\sum_{i=1}^{n} x_i^{1/2}$ $\therefore$ $\hat\theta^{1/2} = \frac{1}{n}\sum_{i=1}^{n} x_i^{1/2}$ $\therefore$

$\hat\theta = \frac{1}{n^2}\left(\sum_{i=1}^{n} x_i^{1/2}\right)^2$ $\therefore$

$l''(\theta) = \frac{n}{2}\frac{1}{\theta^2} - \frac{3}{4}\theta^{-5/2}\sum_{i=1}^{n} x_i^{1/2}$ $\therefore$

$l''(\hat\theta) = \frac{n}{2}\frac{1}{\frac{1}{n^2}\left(\sum_{i=1}^{n} x_i^{1/2}\right)^2} - \frac{3}{4}\frac{1}{\frac{1}{n^5}\left(\sum_{i=1}^{n} x_i^{1/2}\right)^5}\sum_{i=1}^{n} x_i^{1/2} = \frac{1}{2n\left(\sum_{i=1}^{n} x_i^{1/2}\right)^2} - \frac{3}{4n\;n^4\left(\sum_{i=1}^{n} x_i^{1/2}\right)^4} =$

$\frac{1}{2n\left(\sum_{i=1}^{n} x_i^{1/2}\right)^2}\left(1 - \frac{3}{2n^4\left(\sum_{i=1}^{n} x_i^{1/2}\right)^2}\right) < 0$ $\therefore$

$\hat\theta = \frac{1}{n^2}\left(\sum_{i=1}^{n} x_i^{1/2}\right)^2$ is mle

\1d/ $U(\theta) = l'(\theta) = -\frac{n}{2}\frac{1}{\theta} + \frac{1}{2}\frac{1}{\theta^{3/2}}\sum_{i=1}^{n} x_i^{1/2}$.

$I(\theta) = -E(l''(\theta)) = -E\left(\frac{n}{2\theta^2} - \frac{3}{4}\frac{1}{\theta^{5/2}}\sum_{i=1}^{n} x_i^{1/2}\right) =$

$-E\left(\frac{n}{2\theta^2}\right) - E\left(-\frac{3}{4}\frac{1}{\theta^{5/2}}\sum_{i=1}^{n} x_i^{1/2}\right) = \frac{-n}{2\theta^2} + \frac{3}{4}\frac{1}{\theta^{5/2}}E\left(\sum_{i=1}^{n} x_i^{1/2}\right) =$

$-\frac{n}{2\theta^2} + \frac{3}{4}\frac{1}{\theta^{5/2}}\sum_{i=1}^{n} E(x_i^{1/2}) = -\frac{n}{2\theta^2} + \frac{1}{\theta^{5/2}}\sum_{i=1}^{n} E(x^{1/2}) = \frac{-n}{2\theta^2} + \frac{n}{\theta^{5/2}}E(X^{1/2})$

$\therefore$ is $U(\theta) = b(\hat\theta - \theta) = b\left(\frac{1}{n^2}\left(\sum_{i=1}^{n} x_i^{1/2}\right)^2 - \theta\right)$

$\therefore$ bias$(\hat\theta) = E(\hat\theta) - \theta = E\left(\frac{1}{n^2}\left(\sum_{i=1}^{n} x_i^{1/2}\right)^2\right) - \theta = \frac{1}{n^2}E\left(\left(\sum_{i=1}^{n} x_i^{1/2}\right)^2\right) - \theta \neq 0$ $\therefore$

$\therefore$ bias$(\hat\theta) \neq 0$ $\therefore$ $\hat\theta$ is unbiased

and $U(\theta) \neq b(\hat\theta - \theta)$ where b is constant $\therefore$

$\hat\theta$ is both unbiased and not efficient and an estimator)

for θ that is both unbiased and efficient does not exist

\PP2022/\1e/ $\theta_1 > \theta_0 \therefore$ $$\frac{L(\theta_1)}{L(\theta_0)} = \frac{4^{-n/2}\theta_1^{-n/2}\left(\prod_{i=1}^{n} x_i^{-1/2}\right)e^{-\frac{1}{\theta_1^{1/2}}\sum_{i=1}^{n} x_i^{1/2}}}{4^{-n/2}\theta_0^{-n/2}\left(\prod_{i=1}^{n} x_i^{-1/2}\right)e^{-\frac{1}{\theta_0^{1/2}}\sum_{i=1}^{n} x_i^{1/2}}} =$$

$$\left(\frac{\theta_1}{\theta_0}\right)^{-n/2} e^{\left(\frac{1}{\theta_0^{1/2}} - \frac{1}{\theta_1^{1/2}}\right)\sum_{i=1}^{n} x_i^{1/2}} = \left(\frac{\theta_0}{\theta_1}\right)^{n/2} e^{\left(\frac{1}{\theta_0^{1/2}} - \frac{1}{\theta_1^{1/2}}\right)\sum_{i=1}^{n} x_i^{1/2}} \therefore$$

$\theta_1 > \theta_0 \therefore \theta^{1/2} > \theta_0^{1/2} \therefore \frac{1}{\theta_0^{1/2}} > \frac{1}{\theta_1^{1/2}} \therefore \frac{1}{\theta_0^{1/2}} - \frac{1}{\theta_1^{1/2}} > 0 \therefore$

likelihood ratio increases if $\sum_{i=1}^{n} x_i^{1/2}$ increases $\therefore$

reject H_0 if $\sum_{i=1}^{n} x_i^{1/2} \geq d$, d = constant is critical value $\therefore$

critical region: $\{x : \sum_{i=1}^{n} x_i^{1/2} \geq d\}$

\1f/ $I(\theta) \therefore$

wald test statistic: $W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta}) = (\hat{\theta} - \theta_0)^2 \frac{n}{\hat{\theta}^2} = n\left((\hat{\theta} - \theta_0)\frac{1}{\hat{\theta}}\right)^2 =$

$n\left(1 - \frac{\theta_0}{\hat{\theta}}\right)^2$

and it approximately follows a χ_1^2 distribution when n is large if H_0 is true

\1g/ $\therefore$ for a critical region of test size α is given by all data sets $\underline{x}$ for which W exceeds the $(1-\alpha)$ quantile $\chi_{(1-\alpha)}$ of the χ_1^2 distribution $\therefore$

$\{\underline{x} : W \geq \chi_{(1-\alpha)}\}$ $\quad W = \left(1 - \frac{\theta_0}{\hat{\theta}}\right)^2 n > d_\alpha \therefore$

inverting the hypothesis test means including the CS of θ not rejected by the test $S(\underline{X}) = \left\{\theta : \left(1 - \frac{\theta}{\hat{\theta}}\right)^2 n \leq d_\alpha\right\} \therefore$

$\left(1 - \frac{\theta}{\hat{\theta}}\right)^2 n \leq d_\alpha \therefore \left(1 - \frac{\theta}{\hat{\theta}}\right)^2 \leq \frac{d_\alpha}{n} \therefore$

$-\sqrt{d_\alpha/n} \leq \left(1 - \frac{\theta}{\hat{\theta}}\right) \leq \sqrt{d_\alpha/n} \therefore -\sqrt{d_\alpha/n}\,\hat{\theta} \leq \hat{\theta} - \theta \leq \sqrt{d_\alpha/n}\,\hat{\theta} \therefore$

$-\sqrt{d_\alpha/n}\,\hat{\theta} \leq -\hat{\theta} + \theta \leq \sqrt{d_\alpha/n}\,\hat{\theta} \therefore \hat{\theta} - \sqrt{d_\alpha/n}\,\hat{\theta} \leq \theta \leq \hat{\theta} + \sqrt{d_\alpha/n}\,\hat{\theta} \therefore$

$(1 - \sqrt{d_\alpha/n})\hat{\theta} \leq \theta \leq (1 + \sqrt{d_\alpha/n})\hat{\theta} \therefore$

$((1 - \sqrt{d_\alpha/n})\hat{\theta}, (1 + \sqrt{d_\alpha/n})\hat{\theta})$ is a $1-\alpha$ CI for θ

\2a/ let $X = Z\theta$: $F(x,\theta) = F(z\theta;\theta) = 1 - e^{-z\theta/\theta} = 1 - e^{-z}$

and z is independent of θ $\therefore$ $1 - e^{-z}$ is independent of $\theta \therefore$

F is a scale model $\therefore$ let $X_i = \theta Z_i \therefore X_{(1)} = \min\{\theta Z_1, \ldots, \theta Z_n\} \therefore$

$$T = \frac{X_n}{X_{(1)}} = \frac{X_n}{\min\{X_1, \ldots, X_n\}} = \frac{Z_n\theta}{\min\{Z_1\theta, \ldots, Z_n\theta\}} = \frac{Z_n\theta}{\theta\min\{Z_1, \ldots, Z_n\}} = \frac{Z_n}{\min\{Z_1, \ldots, Z_n\}}$$

which is independent of θ $\therefore$ the distribution of T is independent of θ $\therefore$ T is an ancillary statistic

$\therefore$ let $Z_i = \frac{X_i}{\theta}$ $\therefore$ $F(z_i, \theta) = F\left(\frac{X}{\theta} \leq \theta\right) = Pr\left(\frac{X}{\theta} \leq x, \theta\right) = Pr(X \leq x\theta, \theta)$

$= 1 - e^{-\frac{x\theta}{\theta}} = 1 - e^{-x}$ is independent of θ $\therefore$ Z is a pivot

F is a Scale model $\therefore$

distri of $Z_0, \dots, Z_n$ are indep of θ

\2b/ $Pr(X_{(1)} > x) = \cancel{Pr(X_{(1)}} = Pr(\min\{X_1 \dots X_n\} > x) =$

$Pr(X_1, \dots, X_n > x) = Pr(X_1 > x, \dots, X_n > x) =$ by independence

$P(X_1 > x)P(X_2 > x) \cdots P(X_n > x) =$

$(1 - P(X_1 \leq x))(1 - P(X_2 \leq x)) \cdots (1 - P(X_n \leq x)) =$

$(1 - P(X_i \leq x))^n = (1 - F(x; \theta))^n = (1 - (1 - e^{-x/\theta}))^n = (e^{-x/\theta})^n = e^{-nx/\theta}$ $\therefore$

$Pr(T \leq t) = Pr\left(\frac{X_0}{X_{(1)}} \leq t\right) = Pr(X_0 \leq tX_{(1)}) = \int_0^\infty Pr(X_0 \leq tX_{(1)} \mid X_{(1)} = x) Pr(X_{(1)} = x) dx$

$\therefore$ $Pr(X_{(1)} = x) = g(x) = \frac{d}{dx} P(X_{(1)} \leq x) = \frac{d}{dx}(1 - P(X_{(1)} > x)) = \frac{d}{dx}(1 - e^{-nx/\theta}) =$

$e^{-nx/\theta}$ $\therefore$

$Pr(T \leq t) = \int_0^\infty Pr(X_0 \leq tX_{(1)} \mid X_{(1)} = x) g(x) dx = \int_0^\infty P(X_0 \leq tx) g(x) dx =$

$\int_0^\infty (1 - e^{-tx/\theta}) e^{-nx/\theta} dx = \int_0^\infty e^{-nx/\theta} - e^{-t\frac{x}{\theta} - n\frac{x}{\theta}} dx =$

$\int_0^\infty e^{-\frac{n}{\theta}x} - e^{\left(\frac{-t-n}{\theta}\right)x} dx = \left[-\frac{\theta}{n} e^{-\frac{n}{\theta}x}\right]_0^\infty - \left[\frac{\theta}{-t-n} e^{\frac{-t-n}{\theta}x}\right]_0^\infty =$

$\left[-\frac{\theta}{n}(0) + \frac{\theta}{n} e^0\right] - \left[\frac{\theta}{-t-n}(0) - \frac{\theta}{-t-n} e^0\right] =$

$\frac{\theta}{n} + \frac{\theta}{-t-n} = \frac{\theta(-t-n)}{n(-t-n)} + \frac{\theta n}{n(-t-n)} = \frac{-\theta t - n\theta + n\theta}{-nt - n^2} = \frac{-\theta t}{-n(t+n)} = \frac{\theta t}{n(t+n)}$ $\frac{t}{t+n}$

\2c/ let t_p be the p-quantile of the T distribution $\therefore$

$Pr(t_{0.1} \leq T \leq t_{0.9}) = 0.8$ $\therefore$

$Pr(T \leq t_p) = \frac{t_p}{t_p + n} = p$ $\therefore$ $p(t_p + n) = pt_p + np = t_p$ $\therefore$ $np = t_p - pt_p = (1-p)t_p$ $\therefore$

$\frac{np}{1-p} = t_p$ $\therefore$ $t_{0.9} = \frac{n(0.9)}{1-0.9} = 9n$, $t_{0.1} = \frac{n}{9}$ $\therefore$

$0.8 = \left(\frac{n}{9} \leq T \leq 9n\right)$ $\therefore$ ~~$\left(\frac{n}{9}, 9n\right)$ is a~~ $\therefore 0.8 = \left(\frac{n}{9} \leq \frac{X_0}{X_{(1)}} \leq 9n\right)$ $\therefore$

$0.8 = \left(\frac{nX_{(1)}}{9} \leq X_0 \leq 9nX_{(1)}\right)$ $\therefore$ $\left(\frac{nX_{(1)}}{9}, 9nX_{(1)}\right)$ is the 80% PI for X_0

\pp2020/ $var(X) = E(X^2) - E(X)^2 \therefore$

$$E(X_i^2) = \int_0^\infty x^2 f(x;\theta)\, d\theta = \int_0^\infty x^2 \frac{1}{\sqrt{4}} x^{-1/2} \theta^{-1/2} e^{-\left(\frac{x}{\theta}\right)^{1/2}} dx =$$

$$\frac{1}{2}\int x^{3/2} \theta^{-1/2} e^{-\left(\frac{x}{\theta}\right)^{1/2}} dx = \frac{1}{2}\int_0^\infty x \left(\frac{x}{\theta}\right)^{1/2} e^{-\left(\frac{x}{\theta}\right)^{1/2}} dx$$

$\therefore y = \left(\frac{x}{\theta}\right)^{1/2} \therefore \quad x=0: y=0, \quad x=\infty: y=\infty, \quad 2\theta y\,dy = dx, \ x = \theta y^2 \therefore$

$$E(X_i^2) = \frac{1}{2}\int_0^\infty \theta y^2 (y) e^{-y} 2\theta y\, dy = \theta^2 \int_0^\infty y^4 e^{-y}\, dy = \theta^2 (4!) = 24\theta^2$$

$$\therefore var(X) = 24\theta^2 - (2\theta)^2 = 24\theta^2 - 4\theta^2 = 20\theta^2 \therefore$$

$$var(\hat\theta) = var\left(\frac{1}{2}\bar{X}\right) = \frac{1}{4} var(\bar{X}) = \frac{1}{4} var\left(\frac{1}{n}\sum_{i=1}^n X_i\right) = \frac{1}{4n^2} var\left(\sum_{i=1}^n X_i\right) =$$

$$\frac{1}{4n^2}\sum_{i=1}^n var(X_i) = \frac{1}{4n^2}\sum_{i=1}^n var(X) = \frac{n}{4n^2} var(X) = \frac{1}{4n} var(X) = \frac{20\theta^2}{4n} = \frac{5\theta^2}{n} \therefore$$

$\therefore var(\hat\theta) \to 0$ as $n \to \infty$, bias$(\hat\theta) = 0 \therefore$

mse$(\hat\theta) \to 0$ as $n\to\infty \therefore \hat\theta$ is consistent

\1c/ $L(\theta; \underline{x}) = L(\theta) = \prod_{i=1}^n f(x_i;\theta) = \prod_{i=1}^n 4^{-1/2} \theta^{-1/2} x_i^{-1/2} e^{-\left(\frac{x_i}{\theta}\right)^{1/2}} =$

$$\prod_{i=1}^n 2^{-1} \theta^{-1/2} x_i^{-1/2} e^{-\theta^{-1/2} x_i^{1/2}} = 2^{-n}\theta^{-n/2}\left(\prod_{i=1}^n x_i^{-1/2}\right) e^{-\theta^{-1/2}\sum_{i=1}^n x_i^{1/2}} \therefore$$

$$l(\theta;\underline{x}) = \ln L(\theta) = \ln(2^{-n}) + \ln(\theta^{-n/2}) + \ln\left(\prod_{i=1}^n x_i^{-1/2}\right) - \theta^{-1/2}\sum_{i=1}^n x_i^{1/2} =$$

$$-\frac{n}{2}\ln(\theta) + \ln(2^{-n}) + \ln\left(\prod_{i=1}^n x_i^{-1/2}\right) - \theta^{-1/2}\sum_{i=1}^n x_i^{1/2} =$$

$$-\frac{n}{2}\ln\theta - \theta^{-1/2}\sum_{i=1}^n x_i^{1/2} + \text{Constant} \doteq l(\theta) \therefore$$

$$l'(\theta) = -\frac{n}{2}\frac{1}{\theta} + \frac{1}{2}\theta^{-3/2}\sum_{i=1}^n x_i^{1/2} \therefore$$

$$-\frac{n}{2}\frac{1}{\hat\theta} + \frac{1}{2}\hat\theta^{-3/2}\sum_{i=1}^n x_i^{1/2} = 0 \therefore \quad \frac{1}{2}\frac{1}{\hat\theta^{3/2}}\sum_{i=1}^n x_i^{1/2} = \frac{n}{2}\frac{1}{\hat\theta} \therefore$$

$$\frac{1}{2}\sum_{i=1}^n x_i^{1/2} = \frac{n}{2}\hat\theta^{1/2} \therefore \quad \frac{1}{n}\sum_{i=1}^n x_i^{1/2} = \hat\theta^{1/2} \therefore \quad \hat\theta = \left(\frac{1}{n}\sum_{i=1}^n x_i^{1/2}\right)^2 = \frac{1}{n^2}\left(\sum_{i=1}^n x_i^{1/2}\right)^2$$

$$\therefore l''(\theta) = \frac{n}{2}\frac{1}{\theta^2} - \frac{3}{4}\theta^{-5/2}\sum_{i=1}^n x_i^{1/2} \therefore$$

$$l''(\hat\theta) = \frac{n}{2\hat\theta^2} - \frac{3}{4}\hat\theta^{-5/2}\sum_{i=1}^n x_i^{1/2} = \frac{n}{2}\frac{n^4}{\left(\sum_{i=1}^n x_i^{1/2}\right)^4} - \frac{3}{4}\left(\frac{n^2}{\left(\sum_{i=1}^n x_i^{1/2}\right)^2}\right)^{5/2}\sum_{i=1}^n x_i^{1/2} =$$

$$\frac{n^5}{2\left(\sum_{i=1}^n x_i^{1/2}\right)^4} - \frac{3}{4}\frac{n^5}{\left(\sum_{i=1}^n x_i^{1/2}\right)^5}\sum_{i=1}^n x_i^{1/2} = \frac{1}{2}\frac{n^5}{\left(\sum_{i=1}^n x_i^{1/2}\right)^4} - \frac{3}{4}\frac{n^5}{\left(\sum_{i=1}^n x_i^{1/2}\right)^4} = -\frac{1}{4}\frac{n^5}{\left(\sum_{i=1}^n x_i^{1/2}\right)^4} < 0$$

$\therefore \hat\theta = \frac{1}{n^2}\left(\sum_{i=1}^n x_i^{1/2}\right)^2$ is the mle

\1d/ $U(\theta) = l'(\theta) = -\frac{n}{2}\frac{1}{\theta} + \frac{1}{2}\frac{1}{\theta^{3/2}}\sum_{i=1}^n x_i^{1/2} \therefore$

$$\text{bias}(\theta) = E(\hat\theta) - \theta = E\left(\frac{1}{n^2}\left(\sum_{i=1}^n X_i^{1/2}\right)^2\right) - \theta = \frac{1}{n^2}E\left(\left(\sum_{i=1}^n X_i^{1/2}\right)^2\right) - \theta \neq 0$$

$$-E(l''(\theta)) = I(\theta) = -E\left(\frac{n}{2\theta^2} - \frac{3}{4}\theta^{-5/2}\sum_{i=1}^n X_i^{1/2}\right) = -E\left(\frac{n}{2\theta^2}\right) - E\left(-\frac{3}{4}\frac{1}{\theta^{5/2}}\sum_{i=1}^n X_i^{1/2}\right)$$

$$= -\frac{n}{2\theta^2} + \frac{3}{4}\frac{1}{\theta^{5/2}} E\left(\sum_{i=1}^n X_i^{1/2}\right) = -\frac{n}{2\theta^2} + \frac{3}{4}\frac{1}{\theta^{5/2}}\sum_{i=1}^n E(X_i^{1/2}) =$$

$-\frac{n}{2\theta^2}+\frac{3}{4\theta^{5/2}}\sum_{i=1}^{n}E(x^{1/2}) = -\frac{n}{2\theta^2}+\frac{3n}{4\theta^{5/2}}E(x^{1/2})$ ∴

an estimator for θ is unbiased and efficient

if bias$(\hat\theta)=0$ and $U(\theta)=b(\hat\theta-\theta)=l'(\theta)$, $b=$ constant

∴ ~~for~~ $\forall\hat\theta$: $U(\theta)\neq b(\hat\theta-\theta)$ ∴

and $\hat\theta$ is unbiased and not efficient and an estimator for θ that is both unbiased and efficient does not exist

\1e/ $\theta_1>\theta_0$ ∴ $\frac{L(\theta_1)}{L(\theta_0)}=\frac{4^{-n/2}\theta_1^{-n/2}\left(\prod_{i=1}^{n}x_i^{1/2}\right)e^{-\frac{1}{\theta_1^{1/2}}\sum_{i=1}^{n}x_i^{1/2}}}{4^{-n/2}\theta_0^{-n/2}\left(\prod_{i=1}^{n}x_i^{1/2}\right)e^{-\frac{1}{\theta_0^{1/2}}\sum_{i=1}^{n}x_i^{1/2}}}=$

$\left(\frac{\theta_1}{\theta_0}\right)^{-n/2}e^{-\frac{1}{\theta_1^{1/2}}\cdot\sum_{i=1}^{n}x_i^{1/2}+\frac{1}{\theta_0^{1/2}}\sum_{i=1}^{n}x_i^{1/2}}$

$=\left(\frac{\theta_0}{\theta_1}\right)^{n/2}e^{\left(\frac{1}{\theta_0^{1/2}}-\frac{1}{\theta_1^{1/2}}\right)\sum_{i=1}^{n}x_i^{1/2}}$ ∴

$\theta_1>\theta_0$ ∴ $\frac{1}{\theta_0}>\frac{1}{\theta_1}$ ∴ $\cancel{\frac{1}{\theta_0}\ \frac{1}{\theta_1}}$ $\frac{1}{\theta_0^{1/2}}>\frac{1}{\theta_1^{1/2}}$ ∴ $\frac{1}{\theta_0^{1/2}}-\frac{1}{\theta_1^{1/2}}>0$ ∴

likelihood ratio increases if $\sum_{i=1}^{n}x_i^{1/2}$ increases ∴

reject H_0 if $\sum_{i=1}^{n}x_i^{1/2}\geq d$, $d=$ constant is critical value ∴

critical region: $\left\{x:\sum_{i=1}^{n}x_i^{1/2}\geq d\right\}$

\/ $\frac{L(\theta_1)}{L(\theta_0)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $\frac{L(\theta_1)}{L(\theta_0)}$

\1f/ $W=(\hat\theta-\theta_0)^2I(\hat\theta)$ $W=(\hat\theta-\theta_0)^2I(\hat\theta)$ $(\hat\theta-\theta_0)^2I(\hat\theta)$ $(\hat\theta-\theta_0)^2I(\hat\theta)$ $W=(\hat\theta-\theta_0)^2I(\hat\theta)$
$W=(\hat\theta-\theta_0)^2I(\hat\theta)$ $W=(\hat\theta-\theta_0)^2I(\hat\theta)$ $(\hat\theta-\theta_0)^2I(\hat\theta)$ $W=(\hat\theta-\theta_0)^2I(\hat\theta)$ $W=(\hat\theta-\theta_0)^2I(\hat\theta)$ $W=(\hat\theta-\theta_0)^2I(\hat\theta)$

\1f/ wald test statistic: $W=(\hat\theta-\theta_0)^2I(\hat\theta)=(\hat\theta-\theta_0)^2\frac{n}{\hat\theta^2}=\left((\hat\theta-\theta_0)\frac{1}{\hat\theta}\right)^2n=$

$\left(1-\frac{\theta_0}{\hat\theta}\right)^2n$

which has an approximate χ_1^2 distribution for large n if H_0 is true

\1g/ for a critical region of test size α is given by all data sets $\underline{x}$ for which W exceeds the $(1-\alpha)$ quantile $\chi_{(1-\alpha)}$ of the χ_1^2 distribution ∴ $\{x: W\geq\chi_{(1-\alpha)}\}$ ∴

$W=\left(1-\frac{\theta_0}{\hat\theta}\right)^2n>d_\alpha$ ∴ inverting the hypothesis test means including the CS of θ not rejected by the test $S(\underline{x})=\left\{\theta:\left(1-\frac{\theta}{\hat\theta}\right)^2n\leq d_\alpha\right\}$

∴ $\left(1-\frac{\theta}{\hat\theta}\right)^2n\leq d_\alpha$ ∴ $\left(1-\frac{\theta}{\hat\theta}\right)^2n\leq\frac{d_\alpha}{n}$ for d_α is critical value for a test size α ∴ $-\sqrt{\frac{d_\alpha}{n}}\leq\left(1-\frac{\theta}{\hat\theta}\right)\leq\sqrt{\frac{d_\alpha}{n}}$ ∴ $-\sqrt{d_\alpha/n}\,\hat\theta\leq\hat\theta-\theta\leq\sqrt{d_\alpha/n}\,\hat\theta$ ∴

$\hat\theta-\sqrt{d_\alpha/n}\,\hat\theta\leq\theta\leq\hat\theta+\sqrt{d_\alpha/n}\,\hat\theta$ ∴ $(1-\sqrt{d_\alpha/n})\hat\theta\leq\theta\leq(1+\sqrt{d_\alpha/n})\hat\theta$ ∴

$((1-\sqrt{d_\alpha/n})\theta,(1+\sqrt{d_\alpha/n})\hat\theta)$ is ~~a~~ α CI for θ

\PP2022/\2a/ let $x = z\theta$ ∴ $F(x;\theta) = F(z\theta;\theta) = 1 - e^{-z\theta/\theta} = 1 - e^{-z}$

and z is independent of θ ∴ $1-e^{-z}$ is independent of θ ∴

F is a scale model ∴

let $x_i = \theta z$ ∴ $X_{(1)} = \min(\theta z_1, \dots, \theta z_n\}$ ∴

$$T = \frac{X_0}{X_{(1)}} = \frac{X_0}{\min\{x_1, \dots, x_n\}} = \frac{z_0\theta}{\min\{z_1\theta, \dots, \theta z_n\}} = \frac{z_0\theta}{\theta \min\{z_1, \dots, z_n\}} = \frac{\theta}{\min\{z_1, \dots, z_n\}}$$

which is independent of θ ∴ the distribution of T is independent of θ ∴ T is an ancillary statistic.

let $z_i = \frac{X_i}{\theta}$ ∴ $F(z_i;\theta) = F(\frac{X_i}{\theta};\theta) = P_r(\frac{X_i}{\theta} \le x, \theta) = P_r(X_i \le x\theta;\theta) =$

$1 - e^{-\frac{x\theta}{\theta}} = 1 - e^{-x}$ is independent of θ ∴ z_i is a pivot ∴

F is a scale model ∴

distri of $z_0, \dots, z_n$ are indep of θ

\2b/ $P_r(X_{(1)} > x) = P_r(\min\{x_1, \dots, x_n\} > x) =$

$P(X_1, \dots, X_n > x) = P(X_1 > x, \dots, X_n > x) =$ by indep

$P(X_1 > x)P(X_2 > x) \cdots P(X_n > x) = (1 - P(X_1 \le x))(1 - P(X_n \le x)) \cdots (1 - P(X_n \le x)) =$

$(1 - P(X \le x))^n = (1 - F(x;\theta))^n = (1 - (1 - e^{-x/\theta}))^n = (e^{-x/\theta})^n = e^{-nx/\theta}$

∴ $P_r(T \le t) = P_r(\frac{X_0}{X_{(1)}} \le t) = P_r(X_0 \le tX_{(1)}) = \int_0^\infty P_r(X_0 \le tX_{(1)} \mid X_{(1)} = x) P(X_{(1)} = x)\,dx$

$= \cancel{\int_0^\infty P_r}$ ∴ $\cancel{P_r(X_{(1)} = x)}$ $= \int_0^\infty P_r(X_0 \le tX_{(1)} \mid X_{(1)} = x) f_{X_{(1)}}(x;\theta)$.

∴ $f_{X_{(1)}}(x;\theta) = \frac{d}{dx} P(X_{(1)} \le x) = \frac{d}{dx}(1 - P(X_{(1)} > x)) = \frac{d}{dx}(1 - e^{-nx/\theta}) =$

$\cancel{e^{z}}$ $\frac{n}{\theta} e^{-nx/\theta}$ ∴

$$P_r(T \le t) = \int_0^\infty P_r(X_0 \le tX_{(1)} \mid X_{(1)} = x) f_{X_{(1)}}(x;\theta)\,dx = \int_0^\infty P_r(X_0 \le tx) f_{X_{(1)}}(x;\theta)\,dx$$

$$= \int_0^\infty (1 - e^{-tx/\theta}) \frac{n}{\theta} e^{-nx/\theta}\,dx = \frac{n}{\theta}\int_0^\infty (1 - e^{-tx/\theta}) e^{-nx/\theta}\,dx = \frac{n}{\theta}\int_0^\infty e^{-nx/\theta} - e^{-\frac{tx}{\theta}} e^{-\frac{nx}{\theta}}\,dx$$

$$= \frac{n}{\theta}\int_0^\infty e^{-\frac{n}{\theta}x} - e^{-\frac{1}{\theta}(t+n)x}\,dx = \frac{n}{\theta}\left[\frac{1}{(-n/\theta)} e^{-\frac{n}{\theta}x} - \frac{1}{(\frac{-1}{\theta}(t+n))} e^{-\frac{1}{\theta}(t+n)x}\right]_0^\infty$$

$$= \frac{n}{\theta}\left[\frac{1}{(-n/\theta)}(0) - \frac{1}{(-n/\theta)} e^0 - \frac{1}{(\frac{-1}{\theta}(t+n))}\left[0 - e^0\right]\right] =$$

$$\frac{n}{\theta}\left(\frac{\theta}{n} + \frac{\theta}{t+n}[-1]\right) = \cancel{\frac{n}{\theta}}\ \frac{n}{\theta}\frac{\theta}{n} - \frac{n}{\theta}\frac{\theta}{t+n} = 1 - \frac{n}{t+n} = \frac{t+n}{t+n} - \frac{n}{t+n} = \frac{t}{t+n}$$

\2c/ let t_p be the p-quantile of the T distribution ∴

$P_r(t_{0.1} \le T \le t_{0.9}) = 0.8$ ∴ $P_r(T \le t_p) = \frac{t_p}{t_p + n} = p$ ∴ $p(t_p + n) = pt_p + np = t_p$ ∴

$np = t_p - pt_p = (1-p)t_p$ ∴ $\frac{np}{1-p} = t_p$ ∴ $t_{0.9} = \frac{n(0.9)}{1-0.9} = 9n$, $t_{0.1} = \frac{n}{9}$ ∴

$0.8 = P(\frac{n}{9} \le T \le 9n) = P(\frac{n}{9} \le \frac{X_0}{X_{(1)}} \le 9n) = (\frac{nX_{(1)}}{9} \le X_0 \le 9nX_{(1)})$ ∴ $(\frac{nX_{(1)}}{9}, 9nX_{(1)})$ is the 80% PI for X_0

\3a/ let $\underline{z} = (x_1, \dots, x_m, y_1, \dots, y_n)$ and $\theta = (\lambda, \gamma) \therefore \theta^T = \begin{bmatrix} \lambda \\ \gamma \end{bmatrix} \therefore$

$\mu = (\lambda, \dots, \lambda, \gamma\lambda, \dots, \gamma\lambda)$ and $\Sigma = \mathrm{diag}(\lambda, \dots, \lambda, \gamma\lambda, \dots, \gamma\lambda) \therefore$

$$\mu_\theta^T = \begin{bmatrix} \partial\mu_1/\partial\lambda & \cdots & \partial\mu_m/\partial\lambda & \partial\mu_{m+1}/\partial\lambda & \cdots & \partial\mu_{m+n}/\partial\lambda \\ \partial\mu_1/\partial\gamma & & \partial\mu_m/\partial\gamma & \partial\mu_{m+1}/\partial\gamma & \cdots & \partial\mu_{m+n}/\partial\gamma \end{bmatrix}^T = \begin{bmatrix} 1 & \cdots & 1 & \gamma & \cdots & \gamma \\ 0 & \cdots & 0 & \lambda & \cdots & \lambda \end{bmatrix}^T$$

$\therefore \Sigma^{-1} = \mathrm{diag}\left(\frac{1}{\lambda}, \dots, \frac{1}{\lambda}, \frac{1}{\gamma\lambda}, \dots, \frac{1}{\gamma\lambda}\right) \therefore$

$$G(\theta; z) = \mu_\theta^{\cancel{T}} \Sigma^{-1}(z-\mu) = \begin{bmatrix} 1 & \cdots & 1 & \gamma & \cdots & \gamma \\ 0 & \cdots & 0 & \lambda & \cdots & \lambda \end{bmatrix} \mathrm{diag}\left(\frac{1}{\lambda} \cdots \frac{1}{\lambda}\ \frac{1}{\gamma\lambda} \cdots \frac{1}{\gamma\lambda}\right)(z-\mu) =$$

$$\begin{bmatrix} \frac{1}{\lambda} & \cdots & \frac{1}{\lambda} & \frac{\gamma}{\gamma\lambda} & \cdots & \frac{\gamma}{\gamma\lambda} \\ \frac{0}{\lambda} & \cdots & \frac{0}{\lambda} & \frac{\lambda}{\gamma\lambda} & \cdots & \frac{\lambda}{\gamma\lambda} \end{bmatrix}^{\cancel{T}} (z-\mu) = \begin{bmatrix} \frac{1}{\lambda} & \cdots & \frac{1}{\lambda} & \frac{1}{\lambda} & \cdots & \frac{1}{\lambda} \\ 0 & \cdots & 0 & \frac{1}{\gamma} & \cdots & \frac{1}{\gamma} \end{bmatrix}^T (z-\mu) =$$

$$\begin{bmatrix} \frac{1}{\lambda} & \cdots & \frac{1}{\lambda} & \frac{1}{\lambda} & \cdots & \frac{1}{\lambda} \\ 0 & \cdots & 0 & \frac{1}{\gamma} & \cdots & \frac{1}{\gamma} \end{bmatrix}^{\cancel{T}} (x_1, \dots, x_m\ y_1 \dots y_n) - (\lambda \dots \lambda\ \gamma\lambda \dots \gamma\lambda) =$$

$$\begin{bmatrix} \frac{1}{\lambda} & \cdots & \frac{1}{\lambda} & \frac{1}{\lambda} & \cdots & \frac{1}{\lambda} \\ 0 & \cdots & 0 & \frac{1}{\gamma} & \cdots & \frac{1}{\gamma} \end{bmatrix} (x_1-\lambda \cdots x_m-\lambda\ y_1-\gamma\lambda \cdots y_n-\gamma\lambda) =$$

$$\begin{bmatrix} \sum_{i=1}^m (x_i-\lambda)\frac{1}{\lambda} + \sum_{i=1}^n (y_i-\gamma\lambda)\frac{1}{\lambda} \\ 0 + \sum_{i=1}^n (y_i-\gamma\lambda)\frac{1}{\gamma} \end{bmatrix} = \begin{bmatrix} \frac{1}{\lambda}m\bar{x} - m + \frac{1}{\lambda}n\bar{y} - n\gamma \\ \frac{1}{\gamma}n\bar{y} - n\lambda \end{bmatrix} = 0 \therefore$$

$\frac{1}{\lambda}m\bar{x} - m + \lambda n\bar{y} - n\gamma = 0, \quad \frac{1}{\gamma}n\bar{y} - n\lambda = 0 \therefore$

$n\lambda = \frac{1}{\gamma}n\bar{y} \therefore \lambda = \frac{1}{\gamma}\bar{y} \therefore \lambda\gamma = \bar{y} \therefore$

$\frac{1}{\lambda}m\bar{x} = m - \lambda n\bar{y} + n\gamma = m - \lambda n \lambda\gamma + n\gamma = \frac{1}{\lambda}m\bar{x} = m - \lambda^2\gamma n + n\gamma \therefore$

$m\lambda + n\lambda^3\gamma + n\lambda\gamma = m\bar{x}$ X

$\frac{1}{\gamma}n\bar{y} - n\lambda = 0 \therefore \frac{1}{\gamma}n\bar{y} = n\lambda \therefore$

$\frac{1}{\gamma}\bar{y} = \lambda \therefore \frac{1}{\lambda} = \frac{\gamma}{\bar{y}} \therefore$

$\frac{1}{\lambda}m\bar{x} - m + \lambda n\bar{y} - n\gamma = \frac{\gamma}{\bar{y}}m\bar{x} - m + \frac{1}{\gamma}\bar{y}n\bar{y} - n\gamma = \gamma(\cancel{m\bar{y}\bar{x}})$. X

$\frac{m}{\lambda}\bar{x} - m + \frac{n}{\lambda}\bar{y} - \gamma n = 0, \quad \frac{n}{\gamma}\bar{y} - \lambda n = 0 \therefore \quad \lambda = \frac{1}{\gamma}\bar{y}, \quad \frac{1}{\lambda} = \frac{1}{\bar{y}}\gamma$

$\therefore \frac{1}{\lambda}m\bar{x} - m + n\frac{1}{\lambda}\bar{y} - \gamma n = 0 = \frac{1}{\lambda}(m\bar{x} + n\bar{y}) - m - \gamma n \therefore$

$\frac{1}{\bar{y}}\gamma(m\bar{x} + n\bar{y}) - m - (\gamma n) = \gamma\left[-n + \frac{m\bar{x}}{\bar{y}} + n\right] - m = 0 \therefore$

$\gamma\left[-n + \frac{m\bar{x}}{\bar{y}} + n\right] = m = \gamma\left(\frac{m\bar{x}}{\bar{y}}\right] \therefore \quad 1 = \gamma\frac{\bar{x}}{\bar{y}} \therefore \frac{\bar{y}}{\bar{x}} = \hat{\gamma},$

$\hat{\lambda} = \frac{1}{\gamma}\bar{y} = \frac{\bar{x}}{\bar{y}}\bar{y} = \bar{x}$

\3b/ $P_r(Y_i = y) = (\gamma\lambda)^y e^{-\gamma\lambda}\frac{1}{y!} \therefore$

$L(\lambda, \gamma) = \prod_{i=1}^m P_r(X_i = x)\prod_{i=1}^n P_r(Y_i = y) = \prod_{i=1}^m \lambda^{x_i}\frac{1}{x_i!}e^{-\lambda}\prod_{i=1}^n (\gamma\lambda)^{y_i}e^{-\gamma\lambda}\frac{1}{y_i!} =$

$e^{-m\lambda}e^{-n\gamma\lambda}\prod_{i=1}^m e^{\ln\lambda^{x_i}}\frac{1}{x_i!}\prod_{i=1}^n e^{\ln(\gamma\lambda)^{y_i}}\frac{1}{y_i!} = e^{-m\lambda}e^{-n\gamma\lambda}\prod_{i=1}^m e^{x_i\ln(\lambda)}\frac{1}{x_i!}\prod_{i=1}^n e^{y_i\ln(\gamma\lambda)}\frac{1}{y_i!} =$

PP2022 / 2c / let $Pr(T \le t_p) = p$ for p-quantile $\therefore$

$Pr(T \le t_p) = p = \frac{t_p}{t_p + n} \therefore \quad p(t_p + n) = t_p = pt_p + np \quad \therefore t_p - pt_p = (1-p)t_p = np \therefore$

$t_p = \frac{np}{1-p} \therefore$

$t_{0.1} = \frac{n(0.1)}{1-0.1} = \frac{n}{9}, \quad t_{0.9} = \frac{n(0.9)}{1-0.9} = 9n \therefore$

$0.8 = Pr(t_{0.1} < T < t_{0.9}) = Pr\left(\frac{n}{9} < T < 9n\right) = Pr\left(\frac{n}{9} < \frac{X_0}{X_{(1)}} < 9n\right) =$

$Pr\left(\frac{nX_{(1)}}{9} < X_0 < 9nX_{(1)}\right) \therefore \left(\frac{n}{9}X_{(1)}, 9nX_{(1)}\right)$ is a 80% PI for X_0

3a / let $z = (x_1, \dots, x_m, y_1, \dots, y_n)$ and $\theta = (\lambda, \gamma) \therefore \theta^T = \begin{bmatrix} \lambda \\ \gamma \end{bmatrix}$

$\mu = (\lambda, \dots, \lambda, \gamma\lambda, \dots, \gamma\lambda)$ and

$\Sigma = diag(\lambda, \dots, \lambda, \gamma\lambda, \dots, \gamma\lambda) \therefore$

$$\mu_\theta^T = \begin{bmatrix} \partial\mu_1/\partial\lambda & \cdots & \partial\mu_{m+n}/\partial\lambda \\ \partial\mu_1/\partial\gamma & \cdots & \partial\mu_{m+n}/\partial\gamma \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 1 & \gamma & \cdots & \gamma \\ 0 & \cdots & 0 & \lambda & \cdots & \lambda \end{bmatrix}$$

$\Sigma^{-1} = diag\left(\frac{1}{\lambda}, \dots, \frac{1}{\lambda}, \frac{1}{\gamma\lambda}, \dots, \frac{1}{\gamma\lambda}\right) \therefore$

$$G(\theta; z) = \mu_\theta^T \Sigma^{-1}(z - \mu) = \begin{bmatrix} 1 & \cdots & 1 & \gamma & \cdots & \gamma \\ 0 & \cdots & 0 & \lambda & \cdots & \lambda \end{bmatrix} diag\left(\frac{1}{\lambda}, \dots, \frac{1}{\lambda}, \frac{1}{\gamma\lambda}, \dots, \frac{1}{\gamma\lambda}\right)(z - \mu) =$$

$$\begin{bmatrix} \frac{1}{\lambda} & \cdots & \frac{1}{\lambda} & \frac{1}{\lambda} & \cdots & \frac{1}{\lambda} \\ 0 & \cdots & 0 & \frac{1}{\gamma} & & \frac{1}{\gamma} \end{bmatrix}(x_1 - \lambda, \dots, x_m - \lambda, y_1 - \gamma\lambda, \dots, y_n - \gamma\lambda) =$$

$$\begin{bmatrix} \frac{1}{\lambda}(x_1 - \lambda) + \frac{1}{\lambda}(x_2 - \lambda) + \dots + \frac{1}{\lambda}(x_m - \lambda) + \frac{1}{\lambda}(y_1 - \gamma\lambda) + \dots + \frac{1}{\lambda}(y_n - \gamma\lambda) \\ \frac{1}{\gamma}(y_1 - \gamma\lambda) + \dots + \frac{1}{\gamma}(y_n - \gamma\lambda) \end{bmatrix} =$$

$$\begin{bmatrix} \frac{1}{\lambda}\sum_{i=1}^{m} x_i + \sum_{i=1}^{m}(-1) + \frac{1}{\lambda}\sum_{i=1}^{n} y_i - \gamma\sum_{i=1}^{n} 1 \\ \frac{1}{\gamma}\sum_{i=1}^{n} y_i - \lambda\sum_{i=1}^{n} 1 \end{bmatrix} = \begin{bmatrix} \frac{m}{\lambda}\bar{x} - m + \frac{n}{\lambda}\bar{y} - \gamma n \\ \frac{n}{\gamma}\bar{y} - \lambda n \end{bmatrix} = 0 \therefore$$

$\frac{m}{\lambda}\bar{x} - m + \frac{n}{\lambda}\bar{y} - \gamma n = 0 = \frac{1}{\lambda}(m\bar{x} + n\bar{y}) - m - \gamma n$

$\frac{n}{\gamma}\bar{y} - \lambda n = 0 \therefore \quad \lambda n = \frac{n}{\gamma}\bar{y} \quad \therefore \lambda = \frac{1}{\gamma}\bar{y} \quad \therefore \quad \frac{1}{\lambda} = \frac{1}{\bar{y}}\gamma \therefore$

$\frac{1}{\bar{y}}\gamma(m\bar{x} + n\bar{y}) - m - \gamma n = 0 = \gamma\left(\frac{m\bar{x}}{\bar{y}} + n - n\right) - m = \gamma\left(\frac{m\bar{x}}{\bar{y}}\right) - m = 0 \therefore$

$\gamma\left(\frac{m\bar{x}}{\bar{y}}\right) = m \therefore \quad \gamma = \frac{m\bar{y}}{m\bar{x}} = \frac{\bar{y}}{\bar{x}} = \hat{\gamma} \therefore$

$\hat{\lambda} = \frac{1}{\hat{\gamma}}\bar{y} = \frac{\bar{x}}{\bar{y}}\bar{y} = \bar{x}$

3b / $\therefore$ ~~LP~~ $Pr(Y_1 = y) = \frac{(\gamma\lambda)^y}{y!}e^{-\gamma\lambda} \therefore$

$L(\lambda, \gamma) = \prod_{i=1}^{m} Pr(X_i = x) \prod_{i=1}^{n} Pr(Y_i = y) = \prod_{i=1}^{m} \lambda^{x_i}\frac{1}{x_i!}e^{-\lambda} \prod_{i=1}^{n} (\gamma\lambda)^{y_i}\frac{1}{y_i!}e^{-\gamma\lambda} =$

$e^{-m\lambda}\prod_{i=1}^{m}\left(\frac{1}{x_i!}\right)\prod_{i=1}^{m} e^{x_i \ln\lambda} e^{-n\gamma\lambda}\prod_{i=1}^{n}\left(\frac{1}{y_i!}\right)\prod_{i=1}^{n} e^{y_i \ln\gamma\lambda} =$

$\prod_{i=1}^{m}\left(\frac{1}{x_i!}\right)\prod_{i=1}^{n}\left(\frac{1}{y_i!}\right)e^{-m\lambda}e^{-n\gamma\lambda}e^{\ln(\lambda)\sum_{i=1}^{m} x_i} e^{\ln(\gamma\lambda)\sum_{i=1}^{n} y_i} \therefore$

$l(\lambda, \gamma) = \ln L(\lambda, \gamma) = -m\lambda - n\gamma\lambda + \ln(\lambda)\sum_{i=1}^{m} x_i + \ln(\gamma\lambda)\sum_{i=1}^{n} y_i$ + Constant $\therefore$

$$l_\lambda'(\lambda, \gamma) = -m - n\gamma + \frac{1}{\lambda}\sum_{i=1}^{m} x_i + \frac{1}{\lambda}\sum_{i=1}^{n} y_i = 0 \quad \therefore$$

$$\lambda(-m - n\gamma) + \sum_{i=1}^{m} x_i + \sum_{i=1}^{n} y_i = 0 \quad \therefore$$

$$\lambda(m + n\gamma) = \sum_{i=1}^{m} x_i + \sum_{i=1}^{n} y_i \quad \therefore$$

$$\hat{\lambda}(\gamma) = \frac{1}{m+n\gamma}\left(\sum_{i=1}^{m} x_i + \sum_{i=1}^{n} y_i\right) = \frac{1}{m+n\gamma}(m\bar{x} + n\bar{y}) = \frac{m\bar{x}+n\bar{y}}{m+n\gamma} \quad \therefore$$

$$l_p(\gamma) = l(\hat{\lambda}(\gamma), \gamma) = -m\hat{\lambda}(\gamma) - n\gamma\hat{\lambda}(\gamma) + \ln(\hat{\lambda}(\gamma))\sum_{i=1}^{n} x_i + \ln(\gamma\hat{\lambda}(\gamma))\sum_{i=1}^{n} y_i + \text{constant}$$

$$= -m\frac{m\bar{x}+n\bar{y}}{m+n\gamma} - n\gamma\frac{m\bar{x}+n\bar{y}}{m+n\gamma} + \ln\left(\frac{m\bar{x}+n\bar{y}}{m+n\gamma}\right)\sum_{i=1}^{n} x_i + \ln\left(\gamma\frac{m\bar{x}+n\bar{y}}{m+n\gamma}\right)\sum_{i=1}^{n} y_i + \text{constant}$$

\4a/ $\hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i$, $\quad \hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i} \quad \therefore$

$$\hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i = \frac{1}{n}\sum_{i=1}^{n}(n\hat{\theta} - (n-1)\hat{\theta}_{-i}) = \frac{1}{n}\sum_{i=1}^{n} n\hat{\theta} - \frac{1}{n}\sum_{i=1}^{n}(n-1)\hat{\theta}_{-i} =$$

$$\sum_{i=1}^{n}\hat{\theta} - \frac{n-1}{n}\sum_{i=1}^{n}\hat{\theta}_{-i} = n\hat{\theta} - \frac{n-1}{n}\sum_{i=1}^{n}\hat{\theta}_{-i} \quad \therefore$$

$$\hat{\theta}_{-i} = \prod_{j\neq i} X_j^{1/n} = \frac{\prod_{i=1}^{n} X_i^{1/n}}{X_i^{1/n}} = \frac{\hat{\theta}}{X_i^{1/n}} = \hat{\theta}X_i^{-1/n} \quad \therefore$$

$$\hat{\theta}_J = n\hat{\theta} - \frac{n-1}{n}\sum_{i=1}^{n}\hat{\theta}X_i^{-1/n} = n\hat{\theta} - \frac{n-1}{n}\hat{\theta}\sum_{i=1}^{n}X_i^{-1/n} = \hat{\theta}\left(n - \frac{n-1}{n}\sum_{i=1}^{n}X_i^{-1/n}\right)$$

\4b/ $\hat{\theta}^* = \prod_{i=1}^{n} X_i^{*\,1/n}$ where $X_1^*, \dots, X_n^*$ are independent with common ~~mass function~~ distribution function for $i = 1, \dots, n \quad \therefore$

$$E(\hat{\theta}^*) = E\left(\prod_{i=1}^{n} X_i^{*\,1/n}\right) = E(X_1^{*\,1/n})\cdots E(X_n^{*\,1/n}) \text{ by independence}$$

$$= E(e^{\ln X_1^{*\,1/n}})\cdots E(e^{\ln X_n^{*\,1/n}})$$

\1a/ $E(X_i) = E(X) = \int_0^\infty x f(x;\theta)\,dx = \int_0^\infty x(4\theta x)^{-1/2} e^{-(\frac{x}{\theta})^{1/2}}\,dx =$

$$\int_0^\infty 4^{-1/2} x\theta^{-1/2}x^{-1/2}e^{-(\frac{x}{\theta})^{1/2}}\,dx = 4^{-1/2}\int_0^\infty x^{1/2}\theta^{-1/2}e^{-(\frac{x}{\theta})^{1/2}}\,dx = 0.5\int_0^\infty \left(\frac{x}{\theta}\right)^{1/2}e^{-(\frac{x}{\theta})^{1/2}}\,dx$$

$\therefore$ let $y = \left(\frac{x}{\theta}\right)^{1/2} = \frac{1}{\theta^{1/2}}x^{1/2} \quad \therefore \quad \frac{dy}{dx} = \frac{1}{2\theta^{1/2}}x^{-1/2} \quad \therefore \quad \theta^{1/2}y = x^{1/2} \quad \therefore$

$$2\theta^{1/2}x^{1/2}\,dy = dx = 2\theta^{1/2}\theta^{1/2}y\,dy = 2\theta y\,dy = dx \quad \therefore$$

$$x = 0:\ y = \frac{0^{1/2}}{\theta^{1/2}} = 0, \quad x = \infty:\ y = \frac{\infty^{1/2}}{\theta^{1/2}} = \infty \quad \therefore$$

$$E(X) = \frac{1}{2}\int_0^\infty y e^{-y}\, 2\theta y\,dy = \theta\int_0^\infty y^2 e^{-y}\,dy = \theta(2!) = 2\theta \quad \therefore$$

MOM: $\bar{x} = E(X) = 2\hat{\theta} \quad \therefore \quad \hat{\theta} = \frac{\bar{x}}{2}$ is MOM estimator

\1b/ $E(\hat{\theta}) = E(\frac{1}{2}\bar{x}) = \frac{1}{2}E(\bar{x}) = \frac{1}{2}E\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right) = \frac{1}{2n}E\left(\sum_{i=1}^{n}x_i\right) = \frac{1}{2n}\sum_{i=1}^{n}E(x_i) =$

$$\frac{1}{2n}\sum_{i=1}^{n}E(x) = \frac{1}{2n}nE(X) = \frac{1}{2}E(X) = \frac{1}{2}2\theta = \theta \quad \therefore$$

$$\text{Bias}(\hat{\theta}) = E(\hat{\theta}) - \theta = \theta - \theta = 0$$

\PP2022/ $e^{-m\lambda}e^{-n\gamma\lambda}e^{\ln(\lambda)\sum_{i=1}^{m}x_i}e^{\ln(\gamma\lambda)\sum_{i=1}^{n}y_i}\left(\prod_{i=1}^{m}\frac{1}{x_i!}\right)\left(\prod_{i=1}^{n}\frac{1}{y_i!}\right)$ $\therefore$

$$l(\lambda,\gamma)=\ln L(\lambda,\gamma)=-m\lambda-n\gamma\lambda+\ln(\lambda)\sum_{i=1}^{m}x_i+\ln(\gamma\lambda)\sum_{i=1}^{n}y_i+\ln\left[\left(\prod_{i=1}^{m}\frac{1}{x_i!}\right)\left(\prod_{i=1}^{n}\frac{1}{y_i!}\right)\right]$$

$$=-m\lambda-n\gamma\lambda+\ln(\lambda)\sum_{i=1}^{m}x_i+\ln(\gamma\lambda)\sum_{i=1}^{n}y_i+\text{Constant}\ \therefore$$

$$l_\lambda(\lambda,\gamma)=\frac{\partial}{\partial\lambda}l(\lambda,\gamma)=-m-n\gamma+\frac{1}{\lambda}\sum_{i=1}^{m}x_i+\frac{1}{\lambda}\sum_{i=1}^{n}y_i\ \therefore$$

$$-m-n\gamma+\cancel{\frac{n}{\hat\lambda(\gamma)}}\frac{1}{\hat\lambda(\gamma)}\left(\sum_{i=1}^{m}x_i+\sum_{i=1}^{n}y_i\right)=0\ \therefore$$

$$\frac{1}{\hat\lambda(\gamma)}\left(\sum_{i=1}^{m}x_i+\sum_{i=1}^{n}y_i\right)=m+n\gamma\ \therefore\ \frac{1}{m+n\gamma}\left(\sum_{i=1}^{m}x_i+\sum_{i=1}^{n}y_i\right)=\hat\lambda(\gamma)=\frac{m\bar{x}+n\bar{y}}{m+n\gamma}$$

$$\therefore\ l(\hat\lambda(\gamma),\gamma)=-m\frac{m\bar{x}+n\bar{y}}{m+n\gamma}-n\gamma\frac{m\bar{x}+n\bar{y}}{m+n\gamma}+\ln\left(\frac{m\bar{x}+n\bar{y}}{m+n\gamma}\right)\left[\sum_{i=1}^{m}x_i+\sum_{i=1}^{n}y_i\right]+\ln(\gamma)\sum_{i=1}^{n}y_i+\text{Constant}$$

\4a/ $\hat\theta_J=\frac{1}{n}\sum_{i=1}^{n}\hat\theta_i$ $\quad\hat\theta_i=n\hat\theta-(n-1)\hat\theta_{-i}$ $\quad\hat\theta_{-i}=\prod_{j\neq i}X_j^{1/n}$ $\therefore$

$$\hat\theta_J=\frac{1}{n}\sum_{i=1}^{n}\hat\theta_i=\frac{1}{n}\sum_{i=1}^{n}\left(n\hat\theta-(n-1)\hat\theta_{-i}\right)=\frac{1}{n}\sum_{i=1}^{n}n\hat\theta-\frac{1}{n}\sum_{i=1}^{n}(n-1)\hat\theta_{-i}=\sum_{i=1}^{n}\hat\theta-\frac{n-1}{n}\sum_{i=1}^{n}\hat\theta_{-i}=$$
$$n\hat\theta-\frac{n-1}{n}\sum_{i=1}^{n}\hat\theta_{-i}\ \therefore$$

$$\hat\theta_{-i}=\prod_{j=1}X_j^{1/n}=\frac{\prod_{j=1}^{n}X_j^{1/n}}{X_i^{1/n}}=\frac{\hat\theta}{X_i^{1/n}}=\hat\theta X_i^{-1/n}\ \therefore$$

$$\hat\theta_J=n\hat\theta-\frac{n-1}{n}\sum_{i=1}^{n}\hat\theta X_i^{-1/n}=n\hat\theta-\frac{n-1}{n}\hat\theta\sum_{i=1}^{n}X_i^{-1/n}=n\hat\theta-\frac{n-1}{n}\hat\theta\sum_{i=1}^{n}e^{\ln X_i^{-1/n}}=$$
$$n\hat\theta-\frac{n-1}{n}\hat\theta\sum_{i=1}^{n}e^{-\frac{1}{n}\ln X_i}=n\hat\theta-\frac{n-1}{n}\hat\theta\left(\sum_{i=1}^{n}e^{-\frac{1}{n}\sum_{i=1}^{n}\ln X_i}\right)=\hat\theta\left(n-\frac{n-1}{n}e^{-\frac{1}{n}\sum_{i=1}^{n}\ln X_i}\right)$$
$$=\hat\theta\left(n-\frac{n-1}{n}e^{-\frac{1}{n}\ln\prod_{i=1}^{n}X_i}\right)=\hat\theta\left(n-\frac{n-1}{n}e^{\ln\left(\prod_{i=1}^{n}X_i\right)^{-\frac{1}{n}}}\right)=\hat\theta\left(n-\frac{n-1}{n}\left(\prod_{i=1}^{n}X_i\right)^{-\frac{1}{n}}\right)$$

\4b/ $\hat\theta^*=\prod_{i=1}^{n}X_i^{*\,1/n}$ where $X_1^*,\dots,X_n^*$ are independent with common distribution function for $i=1,\dots,n$ $\therefore$

~~$E(\hat\theta^*)=E$~~ $\quad E(X_i^*)=\sum_{i=1}^{n}x_i\Pr(X_i^*=x_i)$

$\therefore\ \because\ \Pr(X_i^*=x_i)=\frac{1}{n}$ for $i=1,\dots,n$:

$$E(X_i^*)=\sum_{i=1}^{n}x_i\Pr(X_i^*=x_i)=\frac{1}{n}\sum_{i=1}^{n}x_i=\bar{x}$$

estimate expectation of $\hat\theta$ from $\hat\theta^*$ $\therefore$ $E(\hat\theta^*)=E\left(\prod_{i=1}^{n}X_i^{1/n}\right)=$

$E(X_1^{*\,1/n})\cdots E(X_n^{*\,1/n})$ by independence $=E(X_1^{*\,\frac{1}{n}})\cdots E(X_1^{*\,\frac{1}{n}})=\left(E(X_1^{*\,\frac{1}{n}})\right)^n$

$$\therefore\ E(X_i^{*\,\frac{1}{n}})=\sum_{i=1}^{n}x_i^{1/n}\Pr(X_i^*=x_i)=\sum_{i=1}^{n}x_i^{1/n}\frac{1}{n}=\frac{1}{n}\sum_{i=1}^{n}x_i^{1/n}\ \therefore$$

$$\left(E(X_1^{*\,1/n})\right)^n=E(\hat\theta^*)=\left(\frac{1}{n}\sum_{i=1}^{n}x_i^{1/n}\right)^n$$

$$\text{var}(X_i^*)=E\left[(X_i^*-E(X_i^*))^2\right]=\sum_{i=1}^{n}E\left[(X_i^*-\bar{x})^2\right]=\sum_{i=1}^{n}(x_i-\bar{x})^2\Pr(X_i^*=x_i)=$$
$$\sum_{i=1}^{n}(x_i-\bar{x})^2\frac{1}{n}=\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\ \therefore\ \text{by Symmetry: }\text{var}(X_i^{*\,\frac{1}{n}})=\frac{1}{n}\sum_{i=1}^{n}\left(x_i^{1/n}-\frac{1}{n}\sum_{i=1}^{n}x_i^{1/n}\right)^2\ \therefore$$

$\text{var}(\hat\theta^*)=\text{var}\left(\prod_{i=1}^{n}X_i^{1/n}\right)$ ~~$=\text{var}(\prod_{i=1}^{n}X_i$~~

\4ci/ interval is : $(2\hat{\theta} - \hat{\theta}^*_{((1-\alpha)B)}, 2\hat{\theta} - \hat{\theta}^*_{(\alpha B)})$ X

let $t^*_{(1)} \leq \ldots \leq t^*_{(B)}$ are order statistics $\therefore$

$\hat{q}_p = t^*_{(pB)}$ is an p-quantile estimate

$(1-2\alpha)$ CI : $(\hat{\theta} - \hat{q}_{1-\alpha}, \hat{\theta} - \hat{q}_{\alpha})$ $\therefore$

for 99% CI $\therefore$ $\frac{1-0.99}{2} = 0.005$ $\therefore$ $B = 1000$ $\therefore$

$CI = \cancel{(\hat{\theta} - \hat{q}_{\alpha}, \hat{\theta} - \hat{q}_{1-\alpha}) = (\hat{\theta} - t^*_{(\alpha B)}, \hat{\theta} -}$

$(\hat{\theta} - \hat{q}_{1-\alpha}, \hat{\theta} - \hat{q}_{\alpha}) = (\hat{\theta} - t^*_{((1-\alpha)B)}, \hat{\theta} - t^*_{(\alpha B)}) = (\hat{\theta} - t^*_{((1-0.005)1000)}, \hat{\theta} - t^*_{(0.005\times 1000)})$

$= (\hat{\theta} - t^*_{(0.995\times 1000)}, \hat{\theta} - t^*_{(0.005\times 1000)}) = (\hat{\theta} - t^*_{995}, \hat{\theta} - t^*_{5}) = (\hat{\theta} - 2.97, \hat{\theta} - 0.65)$

$= (1.2 - 2.97, 1.2 - 0.65) = (-1.77, 0.55)$ is the basic bootstrap interval

\4cii/ percentile bootstrap interval : $(\hat{\theta}^*_{(\alpha B)}, \hat{\theta}^*_{((1-\alpha)B)}) =$

$(\hat{\theta}^*_{(0.005\times 1000)}, \hat{\theta}^*_{((1-0.005)1000)}) = (\hat{\theta}^*_{(5)}, \hat{\theta}^*_{((0.995)1000)}) = (\hat{\theta}^*_{(5)}, \hat{\theta}^*_{(995)}) =$

$(0.65, 2.97)$

\PP2018/ \1a/ $\therefore \int_0^\infty y^k e^{-y}\,dy = k! \therefore E(X_i) = E(X) = \int_0^\infty x f(x;\theta)\,dx =$
$\int_0^\infty x \frac{\theta^k}{\Gamma(k)} x^{-k-1} e^{-\theta/x}\,dx = \frac{1}{\Gamma(k)}\int_0^\infty \theta^k x^{-k} e^{-\theta/x}\,dx = \frac{1}{\Gamma(k)}\int_0^\infty \left(\frac{\theta}{x}\right)^k e^{-\theta/x}\,dx$

$\therefore$ let $y = \theta/x = \theta x^{-1} \therefore \frac{dy}{dx} = -\theta x^{-2} \therefore y^2 = \theta^2 x^{-2} \therefore \frac{1}{\theta} y^2 = \theta x^{-2} \therefore$
$\frac{dy}{dx} = -\frac{1}{\theta} y^2 \therefore -\theta y^{-2}\,dy = dx \quad x = 0 \Rightarrow y = \frac{\theta}{0} = \infty \quad x = \infty \Rightarrow y = \frac{\theta}{\infty} = 0 \therefore$
$E(X_i) = \frac{1}{\Gamma(k)}\int_0^\infty \left(\frac{\theta}{x}\right)^k e^{-\theta/x}\,dx = \frac{1}{\Gamma(k)}\int_\infty^0 y^k e^{-y}(-\theta) y^{-2}\,dy = \frac{-\theta}{\Gamma(k)}\int_\infty^0 y^{k-2} e^{-y}\,dy$
$= \frac{-\theta}{\Gamma(k)}\int_0^\infty - y^{k-2} e^{-y}\,dy = \frac{\theta}{\Gamma(k)}\int_0^\infty y^{k-2} e^{-y}\,dy = \frac{\theta}{\Gamma(k)}(k-2)! = \frac{\theta}{(k-1)!}(k-2)! =$
$\frac{\theta}{(k-1)(k-2)!}(k-2)! = \frac{\theta}{k-1}$ for $k > 1 \therefore$
MOM: $\bar{x} = E(X) = \frac{\hat{\theta}}{k-1} \therefore (k-1)\bar{x} = \hat{\theta}$

\1b/ $L(\theta;\underline{x}) = L(\theta) = \prod_{i=1}^n f(x_i;\theta) = \prod_{i=1}^n \frac{\theta^k}{\Gamma(k)} x_i^{-k-1} e^{-\theta/x_i} =$
$\Gamma(k)^{-n} \theta^{kn} e^{-\theta \sum_{i=1}^n \frac{1}{x_i}} \prod_{i=1}^n x_i^{-k-1} \therefore$

$l(\theta;\underline{x}) = \ln L(\theta) = \ln(\Gamma(k)^{-n}) + \ln(\theta^{kn}) - \theta\sum_{i=1}^n \frac{1}{x_i} + \ln\left(\prod_{i=1}^n x_i^{-k-1}\right) =$
$nk\ln(\theta) - \theta\sum_{i=1}^n x_i^{-1}$ + constant, constant $= \ln(\Gamma(k)^{-n}) + \ln\left(\prod_{i=1}^n x_i^{-k-1}\right) \therefore$

$l'(\theta) = l'(\theta;\underline{x}) = l_\theta(\theta) = \frac{\partial}{\partial\theta} l(\theta) = nk\frac{1}{\theta} - \sum_{i=1}^n x_i^{-1} \therefore$
$nk\frac{1}{\hat{\theta}} - \sum_{i=1}^n x_i^{-1} = 0 \therefore nk\frac{1}{\hat{\theta}} = \sum_{i=1}^n x_i^{-1} \therefore \hat{\theta} = nk\frac{1}{\sum_{i=1}^n x_i^{-1}} \therefore$

$l''(\theta) = -nk\frac{1}{\theta^2} < 0 \therefore$
$\hat{\theta} = nk\frac{1}{\sum_{i=1}^n x_i^{-1}}$ is the mle $\therefore$

$I(\theta) = -E(l''(\theta)) = -E\left(-nk\frac{1}{\theta^2}\right) = -(-nk)\frac{1}{\theta^2}E(1) = nk\frac{1}{\theta^2} \therefore$
$\hat{\theta} \sim N(\theta, I(\theta)^{-1}) = N\left(\theta, \left(nk\frac{1}{\theta^2}\right)^{-1}\right) = N\left(\theta, \frac{\theta^2}{nk}\right)$

\1c/ $\theta_1 < \theta_0 \therefore \Lambda(\underline{x}) = \frac{L(\theta_1)}{L(\theta_0)} = \frac{\Gamma(k)^{-n}\theta_1^{kn} e^{-\theta_1\sum_{i=1}^n \frac{1}{x_i}} \prod_{i=1}^n x_i^{-k-1}}{\Gamma(k)^{-n}\theta_0^{kn} e^{-\theta_0\sum_{i=1}^n \frac{1}{x_i}} \prod_{i=1}^n x_i^{-k-1}} =$
$\left(\frac{\theta_1}{\theta_0}\right)^{kn} e^{-\theta_1\sum_{i=1}^n \frac{1}{x_i} + \theta_0\sum_{i=1}^n \frac{1}{x_i}} = \left(\frac{\theta_1}{\theta_0}\right)^{kn} e^{(\theta_0-\theta_1)\sum_{i=1}^n \frac{1}{x_i}} \therefore \theta_0 - \theta_1 > 0 \therefore$
which $\Lambda(\underline{x})$ increases as $\sum_{i=1}^n \frac{1}{x_i}$ increases $\therefore$
$\Lambda(\underline{x})$ is large when $\sum_{i=1}^n \frac{1}{x_i}$ is large $\therefore$
The critical region has the form $\{\underline{x} : \sum_{i=1}^n \frac{1}{x_i} > c\}$ for a critical value c
\1d/ The most powerful test of size α for the simple ~~null~~ hypothesis has a critical region of the form $C = \{\underline{x} : \Lambda(x) \geq c\}$, $\Lambda(x) = \frac{L(\theta_1)}{L(\theta_0)}$ is likelihood ratio

\1d/ for the test of size α has a critical region of the form
$C=\{\underline{x} : \Lambda(\underline{x}) \geq c\}$: $\Lambda(\underline{x})=\frac{L(\theta_1)}{L(\theta_0)}$ is the likelihood ratio.

\1d/ the most powerful test of size α for a simple null and alternative hypothesis has critical region of the form
$C=\{\underline{x} : \Lambda(\underline{x}) \geq c\}$: $\Lambda(\underline{x})=\frac{L(\theta_1)}{L(\theta_0)}$ is the likelihood ration

\1e/ Score test statistic : $\cancel{U(\theta_0)}=$ $S=U(\theta_0)^2 I(\theta_0)^{-1}$ ∴

$I(\theta_0)=nk\frac{1}{\theta_0^2}$, $U(\theta_0)=l'(\theta_0)=nk\frac{1}{\theta_0}-\sum_{i=1}^{n} x_i^{-1}=nk\frac{1}{\theta_0}-nk\frac{1}{\hat{\theta}}=nk\left(\frac{1}{\theta_0}-\frac{1}{\hat{\theta}}\right)$

$S=\left(nk\left(\frac{1}{\theta_0}-\frac{1}{\hat{\theta}}\right)\right)^2\frac{\theta_0^2}{nk}=(nk)^2\frac{1}{nk}\left(\left(\frac{1}{\theta_0}-\frac{1}{\hat{\theta}}\right)\theta_0\right)^2=\cancel{\frac{1}{nk}}\,nk\left(1-\frac{\theta_0}{\hat{\theta}}\right)^2$

∴ if H_0 is true and n is large: S has an approximate χ^2_1 distribution

\1f/ let d^2 be the α-quantile of the χ^2_1 distribution :
the critical region is $\{\underline{x} : S \geq d^2\}=\{x : nk(1-\frac{\theta_0}{\hat{\theta}})^2 \geq d^2\}$ ∴
α CI is $\{\theta : kn(1-\frac{\theta_0}{\hat{\theta}})^2 < d^2\}$ ∴ $nk(1-\frac{\theta_0}{\hat{\theta}})^2 < d^2$ ∴
$(1-\frac{\theta_0}{\hat{\theta}})^2 < \frac{d^2}{nk}$ ∴ $-\frac{d}{\sqrt{nk}} < 1-\frac{\theta_0}{\hat{\theta}} < \frac{d}{\sqrt{nk}}$ ∴
$-\frac{d}{\sqrt{nk}}\hat{\theta} < \hat{\theta}-\theta_0 < \frac{d}{\sqrt{nk}}\hat{\theta}$ ∴ $\hat{\theta}-\frac{d}{\sqrt{nk}}\hat{\theta}=\hat{\theta}\left(1-\frac{d}{\sqrt{nk}}\right) < \theta_0 < \hat{\theta}+\frac{d}{\sqrt{nk}}\hat{\theta}=\hat{\theta}\left(1+\frac{d}{\sqrt{nk}}\right)$
∴ $\left(\hat{\theta}\left(1-\frac{d}{\sqrt{nk}}\right), \hat{\theta}\left(1+\frac{d}{\sqrt{nk}}\right)\right)$ is the α CI for θ

\1gi/ for $k=1$: $f(x;\theta)=\frac{\theta^1}{\Gamma(1)}x^{-1-1}e^{-\theta/x}=\frac{\theta^1}{0!}x^{-2}e^{-\theta/x}=\theta x^{-2}e^{-\theta/x}$
for $x>0$
∴ $P(X_1 \leq x)=\int_0^x \theta x^{-2}e^{-\theta/x}\,dx$ ∴
let $y=\frac{\theta}{x}=\theta x^{-1}$ ∴ $\frac{dy}{dx}=-\theta x^{-2}$ ∴ $-dy=\theta x^{-2}dx$ ∴
$x=x \rightarrow y=\frac{\theta}{x}$, $x=0 \rightarrow y=\frac{\theta}{0}=\infty$ ∴
$P(X_1 \leq x)=\int_0^x e^{-\theta/x}\theta x^{-2}dx=\int_\infty^{\theta/x} e^{-y}(-1)dy=-\int_{\theta/x}^{\infty}-e^{-y}dy=\int_{\theta/x}^{\infty}e^{-y}dy=$
$\left[-e^{-y}\right]_{y=\frac{\theta}{x}}^{\infty}=-e^{-\infty}+e^{-\theta/x}=-0+e^{-\theta/x}=e^{-\theta/x}$

∴ let q_p be the p-quantile of X_1 ∴
$Pr(X_1 \leq q_p)=e^{-\theta/q_p}=p$ ∴ $\ln(e^{-\theta/q_p})=-\theta/q_p=\ln p$ ∴
$q_p=-\frac{\theta}{\ln p}$ $q_{\alpha-1}=-\frac{\theta}{\ln(\alpha-1)}$

\1gii/ ∴ $q_\alpha=-\frac{\theta}{\ln\alpha}$ ∴ $P(q_\alpha < X_0 < q_{\alpha-1})=1-2\alpha=P\left(-\frac{\theta}{\ln\alpha} < X_0 < -\frac{\theta}{\ln(\alpha-1)}\right)$
∴ $\left(-\frac{\theta}{\ln\alpha}, -\frac{\theta}{\ln(\alpha-1)}\right)$ is a $1-2\alpha$ PI for X_0

\PP2018/ \2a/ let Z have a distribution independent of θ ∴

let $\frac{X}{\theta} = Z$ ∴ $X = Z\theta$ ∴ $F(x;\theta) = P(X \le x) = e^{-\theta/x}$ ∴

$\cancel{P(X \le x)} = P(Z \le x) = P(\frac{X}{\theta} \le x) = P(X \le x\theta)$ ∴

$P(X \le x) = P(Z\theta \le x) = P(Z \le x/\theta)$

$\rightarrow$

$F(x) = F(Z\theta)$

$P(Z \le z) = P(\frac{X}{\theta} \le z) = P(X \le z\theta) = e^{-\frac{\theta}{z\theta}} = e^{-\frac{1}{z}}$ which is indep of θ ∴

∴ $F(x;\theta)$ is a scale model ∴ $Z_i\theta = X_i$ ∴

$X_{(n)} = \text{Max}\{X_1, \dots, X_n\} = \text{Max}\{Z_1\theta, \dots, Z_n\theta\} = \theta\text{Max}\{Z_1, \dots, Z_n\}$ ∴

$T = \frac{X_0}{X_{(n)}} = \frac{Z_0\theta}{\theta\text{Max}\{Z_1, \dots, Z_n\}} = \frac{Z_0}{\text{Max}\{Z_1, \dots, Z_n\}}$ which is indep of θ

∴ The distribution of T is indep of θ ∴ T is an ancillary statistic

\2b/ let $P_r(X_{(n)} \le x) = P_r(\text{Max}\{X_1, \dots, X_n\} \le x) = P_r(X_1 \cap \dots \cap X_n \le x) =$

$P(X_1 \le x \cap \dots \cap X_n \le x) = P(X_1 \le x) \cdots P(X_n \le x) = e^{-\theta/x} \cdots e^{-\theta/x} =$

$(e^{-\theta/x})^n = e^{-n\theta/x}$ ∴

$f_{X_{(n)}}(x;\theta) = \frac{d}{dx} P_r(X_n \le x) = \frac{d}{dx}(e^{-n\theta x^{-1}}) = -n\theta(-1)x^{-2}e^{-n\theta x^{-1}} = n\theta x^{-2}e^{-n\theta x^{-1}}$

∴ $P_r(T \le t) = P_r(\frac{X_0}{X_{(n)}} \le t) = P_r(X_0 \le tX_{(n)}) = \int_0^\infty P_r(X_0 \le tX_{(n)} | X_{(n)} = x) f_{X_{(n)}}(x)dx$

$= \int_0^\infty P_r(X_0 \le tx) f_{X_{(n)}}(x)dx = \int_0^\infty P_r(X_0 \le tx) n\theta x^{-2} e^{-n\theta x^{-1}} dx =$

$\int_0^\infty e^{-\frac{\theta}{tx}} n\theta x^{-2} e^{-n\theta x^{-1}} dx = \int_0^\infty n\theta x^{-2} e^{-\frac{\theta}{t}\frac{1}{x} - n\theta\frac{1}{x}} dx = n\theta \int_0^\infty x^{-2} e^{-(\frac{\theta}{t} + n\theta)x^{-1}} dx$

$= n\theta \int_0^\infty x^{-2} e^{-\theta(\frac{1}{t} + n)x^{-1}} dx$ ∴

let $y = +\theta(\frac{1}{t} + n)x^{-1}$ ∴ $\frac{dy}{dx} = -\theta(\frac{1}{t} + n)x^{-2}$ ∴

$\frac{-1}{\theta(\frac{1}{t}+n)} dy = x^{-2}dx$ ∴ $x = 0 \rightarrow y = +\theta(\frac{1}{t}+n)\frac{1}{0} = +\infty$, $x = \infty \rightarrow y = \theta(\frac{1}{t}+n)\frac{1}{\infty} = 0$

∴ $P_r(T \le t) = n\theta \int_0^\infty e^{-\theta(\frac{1}{t}+n)x^{-1}} x^{-2} dx = n\theta \int_\infty^0 e^{-y} \frac{-1}{\theta(\frac{1}{t}+n)} dy =$

$\frac{n\theta}{\theta(\frac{1}{t}+n)} \int_\infty^0 -e^{-y} dy = \frac{n\theta}{\theta(\frac{1}{t}+n)} \int_0^\infty e^{-y} dy = \frac{n\theta}{\theta(\frac{1}{t}+n)} \int_0^\infty y^0 e^{-y} dy =$

$\frac{n\theta}{\theta(\frac{1}{t}+n)}(0!) = \frac{n}{\frac{1}{t}+n}(1) = \frac{n}{\frac{1}{t}+n}$ ∴

$0.8 = P_r(t_{0.1} \le T \le t_{0.9})$ for t_α is the α-quantile of T ∴

$P_r(T \le t_\alpha) = \frac{n}{\frac{1}{t_\alpha}+n} = \alpha$ ∴ $\alpha(\frac{1}{t_\alpha}+n) = n = \alpha\frac{1}{t_\alpha} + \alpha n$ ∴ $\alpha\frac{1}{t_\alpha} = n - n\alpha = n(1-\alpha)$ ∴

$\alpha = n(1-\alpha)t_\alpha$ ∴ $t_\alpha = \frac{\alpha}{n(1-\alpha)}$ ∴ $t_{0.1} = \frac{0.1}{n(1-0.1)} = \frac{1}{9n}$, $t_{0.9} = \frac{0.9}{n(1-0.9)} = \frac{9}{n}$ ∴

$Pr(t_{0.1} \le T \le t_{0.9}) = 0.8 = Pr\left(t_{0.1} \le \frac{X_0}{X_{(n)}} \le t_{0.9}\right) = Pr(t_{0.1} X_{(n)} \le X_0 \le t_{0.9} X_{(n)})$

$= Pr\left(\frac{1}{9n} X_{(n)} \le X_0 \le \frac{9}{n} X_{(n)}\right) \therefore$

$\left[\frac{1}{9n} X_{(n)}, \frac{9}{n} X_{(n)}\right]$ is the 80% PI for X_0

\4a/ $\therefore \theta = (1-\phi)/\phi \therefore \theta\phi = 1 - \phi \therefore \theta\phi + \phi = 1 = \phi(\theta+1) \therefore$

$\phi = \frac{1}{\theta+1} = h(\theta) \therefore \hat{\phi} = h(\hat{\theta}) = \frac{1}{\hat{\theta}+1} \therefore$

$h(\hat{\theta}) \approx h(\theta) + (\hat{\theta}-\theta)h'(\theta) + \frac{1}{2}(\hat{\theta}-\theta)^2 h''(\theta) \therefore$

$E(\hat{\phi}) = E(h(\hat{\theta})) \approx E\left(h(\theta) + (\hat{\theta}-\theta)h'(\theta) + \frac{1}{2}(\hat{\theta}-\theta)^2 h''(\theta)\right) =$

$E(h(\theta)) + E((\hat{\theta}-\theta)h'(\theta)) + \frac{1}{2}E(\hat{\theta}-\theta)^2 h''(\theta) =$

$h(\theta) + (E(\hat{\theta})-\theta)h'(\theta) + \frac{1}{2}var(\hat{\theta})h''(\theta) =$

$h(\theta) + (\theta-\theta)h'(\theta) + \frac{1}{2}var(\hat{\theta})h''(\theta) = \phi + \frac{1}{2}var(\hat{\theta})h''(\theta) \therefore$

$bias(\hat{\phi}) = E(\hat{\phi}) - \phi = \phi + \frac{1}{2}var(\hat{\theta})h''(\theta) - \phi = \frac{1}{2}var(\hat{\theta})h''(\theta)$

$var(\hat{\phi}) = var(h(\hat{\theta})) \approx var(h(\theta) + (\hat{\theta}-\theta)h'(\theta)) =$

$var(h(\theta)) + var((\hat{\theta}-\theta)h'(\theta)) = var(\hat{\theta}-\theta)[h'(\theta)]^2 =$

$(var(\hat{\theta}) + var(\theta))[h'(\theta)]^2 = var(\hat{\theta})[h'(\theta)]^2 \therefore$

$h(\theta) = (\theta+1)^{-1} \therefore h'(\theta) = -(\theta+1)^{-2} \therefore h''(\theta) = 2(\theta+1)^{-3} \therefore$

$bias(\hat{\phi}) \approx \frac{1}{2}var(\hat{\theta})h''(\theta) = \frac{1}{2}2(\theta+1)^{-3}var(\hat{\theta}) = (\theta+1)^{-3}var(\hat{\theta})$

$var(\hat{\phi}) \approx var(\hat{\theta})[h'(\theta)]^2 = var(\hat{\theta})[2(\theta+1)^{-3}]^2 = 4(\theta+1)^{-6}var(\hat{\theta})$

\4b/ bootstrap resampling creates new samples of size n by simulating values from the Geo($\hat{\phi}$) distribution for parametric $\therefore$

evaluates $\hat{\phi}$ for each of the many samples the bias($\hat{\phi}$) may be estimated as the difference between the mean of the bootstrap estimates and $\hat{\phi}$.

The variance estimated as the sample variance of the bootstrap estimates

\4ei/ $\hat{\phi}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\phi}_i$, $\hat{\phi}_i = n\hat{\phi} - (n-1)\hat{\phi}_{-i} \therefore n = 5 \therefore$

$\hat{\phi}_J = \frac{1}{n}\sum_{i=1}^{n}(n\hat{\phi} - (n-1)\hat{\phi}_{-i}) = \frac{1}{n}\sum_{i=1}^{n} n\hat{\phi} - \frac{1}{n}(n-1)\sum_{i=1}^{n}\hat{\phi}_{-i} = \sum_{i=1}^{n}\hat{\phi} - \frac{n-1}{n}\sum_{i=1}^{n}\hat{\phi}_{-i} =$

$n\hat{\phi} - \frac{n-1}{n}\sum_{i=1}^{n}\hat{\phi}_{-i} \therefore \hat{\phi} = \frac{1}{1+\hat{\theta}} = \frac{1}{1+\bar{x}} \therefore \bar{x} = \frac{1}{5}(2+2+0+12+5) = 4.2 \therefore \hat{\phi} = \frac{1}{1+4.2} = \frac{5}{26}$

\PP 2018/ $\hat{\theta}_{-i} = \frac{1}{n-1}\sum_{j\neq i} x_j = \frac{1}{n-1}\left[\sum_{j=1}^{n} x_j - x_i\right]$ $\therefore \sum_{j=1}^{n} x_j = 2+2+0+12+5 = 21$ $\therefore$

$\theta_{-1} = \frac{1}{4}(21-2) = \frac{19}{4}$ $\theta_{-2} = \frac{19}{4}$, $\theta_{-3} = \frac{21}{4}$, $\theta_{-4} = \frac{9}{4}$ $\theta_{-5} = 4$ $\therefore$

$\hat{\phi}_{-1} = \frac{1}{1+\frac{19}{4}} = \frac{4}{23}$ $\hat{\phi}_{-2} = \frac{4}{23}$, $\hat{\phi}_{-3} = \frac{4}{25}$ $\hat{\phi}_{-4} = \frac{4}{13}$ $\hat{\phi}_{-5} = \frac{1}{5}$ $\therefore$

$\hat{\phi}_J = 5 \times \frac{5}{26} - \frac{4}{5}\left(\frac{4}{23} + \frac{4}{23} + \frac{4}{25} + \frac{4}{13} + \frac{1}{5}\right) = 0.149$

\1c ii/ $\hat{V}_J = \frac{1}{n(n-1)}\sum_{i=1}^{n}(\hat{\phi}_i - \hat{\phi}_J)^2 = \frac{1}{20}\sum_{i=1}^{5}(\hat{\phi}_i - \hat{\phi}_J)^2$

$\because \hat{\phi}_i = n\hat{\phi} - (n-1)\hat{\phi}_{-i} = 5\hat{\phi} - 4\hat{\phi}_{-i} = \frac{25}{26} - 4\hat{\phi}_{-i}$ $\therefore$

$\hat{\phi}_1 = \frac{159}{598}$ $\hat{\phi}_2 = \frac{159}{598}$ $\hat{\phi}_3 = \frac{209}{650}$ $\hat{\phi}_4 = -\frac{7}{26}$ $\hat{\phi}_5 = \frac{21}{130}$ $\therefore$

$(\hat{\phi}_1 - \hat{\phi}_J)^2 = 0.013633$, $(\hat{\phi}_2 - \hat{\phi}_J) = 0.013633$ $(\phi_3 - \hat{\phi}_J)^2 = 0.029727$

$(\phi_4 - \phi_J)^2 = 0.17502$ $(\hat{\phi}_5 - \phi_J)^2 = 0.00015412S$ $\therefore$

$\hat{V}_J = \frac{1}{20}\sum_{i=1}^{5}(\phi_i - \hat{\phi}_J)^2 = 0.0116$ (3 s.f.)

\3a/ $l(\theta_0) \approx l(\hat{\theta}) + (\theta_0 - \hat{\theta})U(\hat{\theta}) - \frac{1}{2}(\theta_0 - \hat{\theta})^2 J(\hat{\theta}) =$

$l(\hat{\theta}) - \frac{1}{2}(\theta_0 - \hat{\theta})^2 J(\hat{\theta})$ $\because U(\hat{\theta}) = 0$ for large n $\therefore$

$-2\ln\Lambda = -2\ln\frac{L(\theta_0)}{L(\hat{\theta})} = +2\left(l(\hat{\theta}) - l(\theta_0)\right) \approx$

$(\theta_0 - \hat{\theta})^2 J(\hat{\theta}) = (\theta_0 - \hat{\theta})^2\left(-\frac{\partial^2 l}{\partial\theta^2}\Big|_{\hat{\theta}}\right) \approx (\theta_0 - \hat{\theta})^2 I(\hat{\theta}) = (\hat{\theta} - \theta_0)^2 I(\hat{\theta}) = W$

$\therefore$ The likelihood ratio test $\therefore$ rejects H_0 at level a is $-2\ln\Lambda$ exceeds the $(1-\alpha)$-quantile of the χ_p^2 distribution

$\therefore L(\alpha, \beta; \underline{x}) = L(\alpha, \beta) = \prod_{i=1}^{n} f(x_i; \alpha, \beta) = \prod_{i=1}^{n}(2\pi)^{-1/2} e^{-\frac{1}{2}(x_i - \alpha - \beta z_i)^2} =$

$(2\pi)^{-n/2}\prod_{i=1}^{n} e^{-\frac{1}{2}(x_i^2 + \alpha^2 + \beta^2 z_i^2 - 2\alpha x_i - 2\beta z_i x_i + 2\alpha\beta z_i)} =$

$(2\pi)^{-n/2} e^{-\frac{1}{2}\sum_{i=1}^{n}(x_i^2 + \alpha^2 + \beta^2 z_i^2 - 2\alpha x_i - 2\beta z_i x_i + 2\alpha\beta z_i)}$ $\therefore$

$l(\alpha, \beta) = \ln L(\alpha, \beta) = \ln\left((2\pi)^{-n/2}\right) - \frac{1}{2}\sum_{i=1}^{n} x_i^2 + \alpha^2 + \beta^2 z_i^2 - 2\alpha x_i - 2\beta z_i x_i + 2\alpha\beta z_i =$

Constant $-\frac{1}{2}n\alpha^2 - \frac{1}{2}\beta^2\sum_{i=1}^{n} z_i^2 + \alpha\sum_{i=1}^{n} x_i + \beta\sum_{i=1}^{n} z_i x_i - \alpha\beta\sum_{i=1}^{n} z_i =$

Constant $-\frac{1}{2}n\alpha^2 - \frac{n}{2}\beta^2\overline{z^2} + n\alpha\bar{x} + n\beta\overline{zx} - \alpha\beta n\bar{z}$ $\therefore$

$l_\alpha(\alpha, \beta) = -n\alpha + n\bar{x} - \beta n\bar{z}$ $\therefore$

$-n\hat{\alpha} + n\bar{x} - \beta n\bar{z} = 0$ $\therefore n(\bar{x} - \beta\bar{z}) = n\hat{\alpha}$ $\therefore \hat{\alpha} = \bar{x} - \beta\bar{z} = \hat{\alpha}(\beta)$ $\therefore$

$l_p(\hat{\alpha}(\beta), \beta) = l(\hat{\alpha}(\beta), \beta) = -\frac{1}{2}n(\hat{\alpha}(\beta))^2 + \hat{\alpha}(\beta)(n\bar{x} - \beta n\bar{z}) - \frac{n}{2}\beta^2\overline{z^2} + n\beta\overline{zx} =$

Constant $-\frac{1}{2}n(\bar{x} - \beta\bar{z})^2 + (\bar{x} - \beta\bar{z})(n\bar{x} - \beta n\bar{z}) - \frac{n}{2}\overline{z^2}\beta^2 + n\overline{zx}\beta$

\3bii/ $-2\ln\Lambda = -2\ln\frac{L(\beta_0)}{L(\hat{\beta})} = -2[\ln(L(\beta_0)) - \ln L(\hat{\beta})] = -2[l(\beta_0) - l\hat{\beta}]$

$\therefore l_p'(\beta) = -\frac{1}{2}n2(\bar{x}-\beta\bar{z})(-\bar{z}) + (-\bar{z})(n\bar{x}-\beta n\bar{z}) + (\bar{x}-\beta\bar{z})(-n\bar{z}) - n\overline{z^2}\beta + n\overline{zx}$

$\therefore +\frac{\bar{z}}{2}n(\bar{x}-\hat{\beta}\bar{z}) - \bar{z}(n\bar{x}-\hat{\beta}n\bar{z}) + (\bar{x}-\hat{\beta}\bar{z})(-n\bar{z}) - n\overline{z^2}\hat{\beta} + n\overline{zx} = 0$

$l_p''(\hat{\beta}) < 0 \therefore$

$\Lambda(\beta_0) = 2(l_p(\hat{\beta}) - l_p(\beta_0))$

PP2017 / 3ai / $L(\theta,\gamma,\underline{x},\underline{y}) = \prod_{i=1}^{m} f(x_i;\theta) \prod_{i=1}^{n} f(y_i;\theta,\gamma) =$

$\prod_{i=1}^{m} \theta x_i^{-\theta-1} \prod_{i=1}^{n} \gamma\theta y_i^{-\gamma\theta-1} = \theta^m \prod_{i=1}^{m} e^{(-\theta-1)\ln x_i} \gamma^n \theta^n \prod_{i=1}^{n} e^{(-\gamma\theta-1)\ln y_i}$

$= \theta^{m+n}\gamma^n e^{(-\theta-1)\sum_{i=1}^{m}\ln x_i} e^{(-\gamma\theta-1)\sum_{i=1}^{n}\ln y_i}$

3aii / $\therefore \ell(\theta,\gamma) = \ln L(\theta,\gamma) =$

$(m+n)\ln\theta + n\ln\gamma + (-\theta-1)\sum_{i=1}^{m}\ln x_i + (-\gamma\theta-1)\sum_{i=1}^{n}\ln y_i$

$\ell_\theta(\theta,\gamma) = \frac{m+n}{\theta} - \sum_{i=1}^{m}\ln x_i - \gamma\sum_{i=1}^{n}\ln y_i \therefore$

$\ell_\theta(\hat\theta,\gamma) = 0 = \frac{m+n}{\hat\theta} + \sum_{i=1}^{m}\ln x_i - \gamma\sum_{i=1}^{n}\ln y_i \therefore$

$\left(\gamma\sum_{i=1}^{n}\ln y_i - \sum_{i=1}^{m}\ln x_i\right)^{-1}(m+n) = \hat\theta = \hat\theta(\gamma) = (\gamma\overline{\ln y} - \overline{\ln x})^{-1}(m+n)$

$\ell_p(\gamma) = \ell(\hat\theta(\gamma),\gamma) =$

$(m+n)\ln\hat\theta + n\ln\gamma + (\hat\theta-1)\sum_{i=1}^{m}\ln x_i + (-\gamma\hat\theta-1)\sum_{i=1}^{n}\ln y_i$

$= (m+n)\ln\left((m+n)(\gamma\overline{\ln y} - \overline{\ln x})^{-1}\right) + n\ln\gamma + \left((m+n)(\gamma\overline{\ln y}-\overline{\ln x})^{-1}-1\right)\overline{\ln x} + (\gamma\hat\theta-1)\overline{\ln y}$

$\therefore -2\ln\Lambda = -2\ln\frac{L(\gamma_0)}{L(\hat\gamma)} = -2[\ell(\gamma_0)-\ell(\hat\gamma)] = 2(\ell_p(\hat\gamma) - \ell_p(\gamma_0))$

$= 2(\ell_p(\hat\gamma) - \ell_p(1))$

$\therefore \ell_p'(\hat\gamma) = 0,\ \ell_p''(\hat\gamma) < 0$

3b / $L(\theta,\gamma,\phi) = \prod_{i=1}^{m} f(x_i;\theta,\phi) \prod_{i=1}^{n} f(y_i;\theta,\gamma,\phi) =$

$\prod_{i=1}^{m} \theta\phi^\theta x_i^{-\theta-1} \prod_{i=1}^{n} \gamma\theta\phi^{\gamma\theta} y_i^{-\gamma\theta-1} =$

$\theta^m\phi^{m\theta} e^{(-\theta-1)\overline{\ln x}} \gamma^n\theta^n \phi^{n\gamma\theta} e^{(-\gamma\theta-1)\overline{\ln y}} = \theta^{m+n}\gamma^n\phi^{m\theta+n\gamma\theta} e^{(-\theta-1)\overline{\ln x}+(-\gamma\theta-1)\overline{\ln y}}$

$\ell(\theta,\gamma,\phi) = (m+n)\ln\theta + n\ln\gamma + (m\theta+n\gamma\theta)\ln\phi + (-\theta-1)\overline{\ln x} + (-\gamma\theta-1)\overline{\ln y}$

$U(\theta) = I(\theta)(\hat\theta-\theta)$ $h(\hat\theta) \approx h(\theta) + (\hat\theta-\theta)h'(\theta)$

$\ell_\phi(\theta,\gamma,\phi) = (m\theta+n\gamma\theta)\frac{1}{\phi} \therefore$ $+\frac{1}{2}(\hat\theta-\theta)^2 h''(\theta)$ $I(\theta) = -E(\ell''(\theta))$ $N(\theta, I(\theta)^{-1})$

$(m\theta+n\gamma\theta)\frac{1}{\phi} = 0$ $\frac{I(\theta)^{-1}}{\mathrm{var}(\hat\theta)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $W = (\hat\theta-\theta_0)^2 I(\hat\theta)$ $S = U(\theta_0)^2 I(\theta_0)^{-1}$ $-2\ln\frac{L(\theta_0)}{L(\hat\theta)}$

$G = \mu_\theta^T\Sigma^{-1}(\underline{x}-\mu)$ $E(G) = 0$ $\hat\theta \sim N(\theta, k^{-1})$ $k = \mu_\theta^T\Sigma^{-1}\mu_\theta$ $\hat\theta_J = \frac{1}{n}\sum_{i=1}^{n}\hat\theta_i$

$\ell_\theta(\hat\theta,\gamma,\phi) = \frac{m+n}{\hat\theta} + (m+n\gamma)\ln\phi + (-1)\overline{\ln x} + (-\gamma)\overline{\ln y} = 0$

$\hat\theta_i = n\hat\theta - (n-1)\hat\theta_{-i}$ $\hat V_J = \frac{1}{n(n-1)}\sum_{i=1}^{n}(\hat\theta_i - \hat\theta_J)$ $U(\theta) = I(\theta)(\hat\theta-\theta)$

$\ell_\gamma(\theta,\hat\gamma,\phi)$

$h(\hat\theta) \approx h(\theta) + (\hat\theta-\theta)h'(\theta) + \frac{1}{2}(\hat\theta-\theta)^2 h''(\theta)$

4a / Let Z be indep of $\mu \therefore X = Z + \mu \therefore$

$F(x;\mu) = P(X \le x) = P(Z+\mu \le x) = \cancel{P(Z}$

$F(z;\mu) = P(Z \le z) = P(X-\mu \le z) = P(X \le z+\mu) = (1+e^{\mu-z-\mu})^{-1} =$

$(1+e^{-z})^{-1}$ is indep of $\mu \therefore F$ is a location model $\therefore X_i = Z_i + \mu$

4b / $T = X_0 - \bar X = X_0 - \frac{1}{n}\sum_{i=1}^{n} X_i = Z_0 + \mu - \frac{1}{n}\sum_{i=1}^{n}(Z_i+\mu) = Z_0 + \mu - \mu - \frac{1}{n}\sum_{i=1}^{n} Z_i =$

$Z_0 - \frac{1}{n}\sum_{i=1}^{n} Z_i$ is indep of $\mu \therefore$ distn of T indep of $\mu \therefore$ ancillary

$h(\hat\theta) \approx h(\theta) + (\hat\theta-\theta)h'(\theta) + \frac{1}{2}(\hat\theta-\theta)^2 h''(\theta)$ $\mathrm{var}(X) = E((X-E(X))^2)$

\4c/ would simulate samples of equal size n from the ~~X~~ distribution and for each sample calculate the T distribution

Simulate with $\hat{\mu}_0$ let $T^* = T(\underline{X}^*, X_0^*)$ $\underline{X}^* = (X_1^*, \dots, X_n^*)$ have indep distn $F(x; \hat{\theta})$

Simulate x_{b0}^* & $\underline{x}_b^* = (x_{b1}^*, \dots, x_{bn}^*)$ from $F(x; \hat{\theta})$

calc $t^* = T(x_b^*, x_{b0}^*)$

\1di/ ~~(2θ~~ $(2\hat{\mu} - \hat{\mu}^*_{(1-\alpha)B}, 2\hat{\mu} - \hat{\mu}^*_{\alpha B})$ $B = 1000$

$1 - 2\alpha = 0.99 \therefore 2\alpha = 0.01 \quad \therefore \alpha = 0.005 \therefore (1-\alpha)B = 995, \ \alpha B = 5 \therefore$

$\hat{\mu} = (-1.4 - 1 - 0.5 + 0.3 + 2.3)/5 = -0.06 \therefore$

$(2(-0.06) - 6.1, \ 2(-0.006) - -5.7) = (-6.22, 5.58)$

\1dii/ $\theta = e^{\mu}$, $\hat{\theta} = e^{\bar{x}} = e^{\frac{1}{n}\sum_{i=1}^{n} x_i} \therefore$

$n = 5 \quad \frac{1}{n}\sum_{i=1}^{n} x_i = \frac{1}{5}(-0.3) = -0.06$

$\hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n} \hat{\theta}_i = \frac{1}{n}\sum_{i=1}^{n}(n\hat{\theta} - (n-1)\hat{\theta}_{-i}) = n\hat{\theta} - \frac{n-1}{n}\sum_{i=1}^{n}\hat{\theta}_{-i} = 5\hat{\theta} - \frac{4}{5}\sum_{i=1}^{5}\hat{\theta}_{-i} \therefore$

$\therefore \hat{\theta} = e^{-0.06}$

$\hat{\theta}_{-i} = e^{\frac{1}{n-1}\sum_{j \neq i} x_j} = e^{\frac{1}{4}\sum_{j \neq i} x_j} = e^{\frac{1}{4}(\sum_{j=1}^{n} x_j - x_i)}$

$\hat{\theta}_{-1} = e^{\frac{27}{20}} \quad \hat{\theta}_{-2} = e^{19/20} \quad \hat{\theta}_{-3} = e^{9/20} \quad \hat{\theta}_{-4} = e^{-7/20} \quad \hat{\theta}_{-5} = e^{-47/20}$

$\therefore \hat{\theta}_J = 5 \times e^{-0.06} - \frac{4}{5}(e^{27/20} + e^{19/20} + e^{9/20} + e^{-7/20} + e^{-45/20}) =$
-2.35

\\/ $I(\theta) = -E(l''(\theta))$ $N(\theta, I(\theta)^{-1})$ $\frac{I(\theta)^{-1}}{var(\hat{\theta})}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $w = (\hat{\theta} - \theta_0)^2 I(\hat{\theta})$ $S = U(\theta_0)^2 I(\theta_0)^{-1}$
$-2\ln\frac{L(\theta_0)}{L(\hat{\theta})}$ $(2\hat{\theta} - \hat{\theta}^*_{(1-\alpha)B}, 2\hat{\theta} - \hat{\theta}^*_{\alpha B})$ $(\hat{\theta}^*_{\alpha B}, \hat{\theta}^*_{(1-\alpha)B})$ $(\hat{\theta} - \hat{q}_{(1-\alpha)}S, \hat{\theta} - \hat{q}_{\alpha}S)$ $\int_0^\infty y^k e^{-y} dy = k!$
$G = \mu_\theta^T \Sigma^{-1}(\underline{x} - \mu)$ $E(G) = 0$ $\hat{\theta} \sim N(\theta, k^{-1})$ $k = \mu_\theta^T \Sigma^{-1} \mu_\theta$ $h(\hat{\theta}) \approx h(\theta) + (\hat{\theta} - \theta)h'(\theta) + \frac{1}{2}(\hat{\theta} - \theta)^2 h''(\theta)$
$I(\theta) = -E(l''(\theta))$ $N(\theta, I(\theta)^{-1})$ $\frac{I(\theta)^{-1}}{var(\hat{\theta})}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $w = (\hat{\theta} - \theta_0)^2 I(\hat{\theta})$ $S = U(\theta_0)^2 I(\theta_0)^{-1}$ $-2\ln\frac{L(\theta_0)}{L(\hat{\theta})}$
$(2\hat{\theta} - \hat{\theta}^*_{(1-\alpha)B}, 2\hat{\theta} - \hat{\theta}^*_{\alpha B})$ $(\hat{\theta}^*_{\alpha B}, \hat{\theta}^*_{(1-\alpha)B})$ $(\hat{\theta} - \hat{q}_{(1-\alpha)}S, \hat{\theta} - \hat{q}_{\alpha}S)$ $\int_0^\infty y^k e^{-y} dy = k!$ $G = \mu_\theta^T \Sigma^{-1}(\underline{x} - \mu)$
$E(G) = 0$ $\hat{\theta} \sim N(\theta, k^{-1})$ $k = \mu_\theta^T \Sigma^{-1} \mu_\theta$ $h(\hat{\theta}) \approx h(\theta) + (\hat{\theta} - \theta)h'(\theta) + \frac{1}{2}(\hat{\theta} - \theta)^2 h''(\theta)$ $\hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i$
$\hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$ $\hat{V}_J = \frac{1}{n(n-1)}\sum_{i=1}^{n}(\hat{\theta}_i - \hat{\theta}_J)^2$ 1.65 1.96 $U(\theta) = I(\theta)(\hat{\theta} - \theta)$ $I(\theta) = -E(l''(\theta))$
$N(\theta, I(\theta)^{-1})$ $\frac{I(\theta)^{-1}}{var(\hat{\theta})}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $w(\hat{\theta} - \theta_0)^2 I(\hat{\theta})$ $S = U(\theta_0)^2 I(\theta_0)^{-1}$ $2\ln\frac{L(\theta_0)}{L(\hat{\theta})}$ $(2\hat{\theta} - \hat{\theta}^*_{(1-\alpha)B}, 2\hat{\theta} - \hat{\theta}^*_{\alpha B})$
$(\hat{\theta}^*_{\alpha B}, \hat{\theta}^*_{(1-\alpha)B})$ $(\hat{\theta} - \hat{q}_{(1-\alpha)}S, \hat{\theta} - \hat{q}_{\alpha}S)$ $\int_0^\infty y^k e^{-y} dy = k!$ $G = \mu_\theta^T \Sigma^{-1}(\underline{x} - \mu)$ $E(G) = 0$ $\hat{\theta} \sim N(\theta, k^{-1})$
$k = \mu_\theta^T \Sigma^{-1} \mu_\theta$ $\hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i$ $\hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$ $\hat{V}_J = \frac{1}{n(n-1)}\sum_{i=1}^{n}(\hat{\theta}_i - \hat{\theta}_J)^2$ 1.65 1.96

$I(\theta) = -E(l''(\theta))$ $N(\theta, I(\theta)^{-1})$ $\frac{I(\theta)^{-1}}{var(\hat{\theta})}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta})$

$S = U(\theta_0)^2 I^{-1}(\theta_0)$ $S = U(\theta_0)^2 I(\theta_0)^{-1}$ $-2\ln\frac{L(\theta_0)}{L(\hat{\theta})}$ $(\hat{\theta} - \hat{q}_{1-\alpha}, \hat{\theta} - \hat{q}_{\alpha})$

$(\hat{\theta}^*_{(\alpha B)}, \hat{\theta}^*_{((1-\alpha)B)})$ $(\hat{\theta} - \hat{q}_{1-\alpha} S, \hat{\theta} - \hat{q}_{\alpha} S)$ $U = I(\theta)(\hat{\theta} - \theta)$

$I(\theta) = -E(l''(\theta))$ $N(\theta, I(\theta)^{-1})$ $\frac{I(\theta)^{-1}}{var(\hat{\theta})}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta})$

$S = U(\theta_0)^2 I(\theta_0)^{-1}$ $-2\ln\frac{L(\theta_0)}{L(\hat{\theta})}$ $(\hat{\theta} - \hat{q}_{1-\alpha}, \hat{\theta} - \hat{q}_{\alpha})$ $(\hat{\theta}^*_{(\alpha B)}, \hat{\theta}^*_{((1-\alpha)B)})$

$(\hat{\theta} - \hat{q}_{1-\alpha} S, \hat{\theta} - \hat{q}_{\alpha} S)$ $U = I(\theta)(\hat{\theta} - \theta)$ $\int_0^\infty y^k e^{-y} dy = k!$ $\int_0^\infty y^k e^{-y} dy = k!$

$G(\theta; \underline{X}) = \mu_\theta^T \Sigma^{-1}(\underline{X} - \mu)$ $E(G) = 0$ $I(\hat{\theta}) = -E(l''(\theta))$ $N(\theta, I(\theta)^{-1})$ $\frac{I(\theta)^{-1}}{var(\hat{\theta})}$ $\frac{L(\theta_1)}{L(\theta_0)}$

$W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta})$ $S = U(\theta_0)^2 I(\theta_0)^{-1}$ $-2\ln\frac{L(\theta_0)}{L(\hat{\theta})}$ $(\hat{\theta} - \hat{q}_{(1-\alpha)B}, \hat{\theta} - \hat{q}_{\alpha B})$ $(\hat{\theta}^*_{(\alpha B)}, \hat{\theta}^*_{((1-\alpha)B)})$

$(\hat{\theta} - \hat{q}_{1-\alpha} S, \hat{\theta} - \hat{q}_{\alpha} S)$ $U = I(\theta)(\hat{\theta} - \theta)$ $\int_0^\infty y^k e^{-y} dy = k!$ $G = \mu_\theta^T \Sigma^{-1}(\underline{X} - \mu)$ $E(G) = 0$

$I(\theta) = -E(l''(\theta))$ $N(\theta, I^{-1}(\theta))$ $\frac{I(\theta)^{-1}}{var(\hat{\theta})}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta})$ $S = U(\theta_0)^2 I(\theta_0)^{-1}$

$-2\ln\frac{L(\theta_0)}{L(\hat{\theta})}$ $(\hat{\theta} - \hat{q}_{1-\alpha}, \hat{\theta} - \hat{q}_{\alpha})$ $(\hat{\theta}^*_{\alpha B}, \hat{\theta}^*_{(1-\alpha)B})$ $(\hat{\theta} - \hat{q}_{1-\alpha} S, \hat{\theta} - \hat{q}_{\alpha} S)$

$U = I(\theta)(\hat{\theta} - \theta)$ $\int_0^\infty y^k e^{-y} dy = k!$ $G = \mu_\theta^T \Sigma^{-1}(\underline{X} - \mu)$ $E(G) = 0$

$I(\theta) = -E(l''(\theta))$ $N(\theta, I^{-1}(\theta))$ $\frac{I^{-1}(\theta)}{var(\hat{\theta})}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta})$ $S = U(\theta_0)^2 I(\theta_0)^{-1}$

$-2\ln\frac{L(\theta_0)}{L(\hat{\theta})}$ $(\hat{\theta} - \hat{q}_{1-\alpha}, \hat{\theta} - \hat{q}_{\alpha})$ $(\hat{\theta}^*_{\alpha B}, \hat{\theta}^*_{(1-\alpha)B})$ $(\hat{\theta} - \hat{q}_{1-\alpha} S, \hat{\theta} - \hat{q}_{\alpha} S)$

$U = I(\theta)(\hat{\theta} - \theta)$ $\int_0^\infty y^k e^{-y} dy = k!$ $G = \mu_\theta^T \Sigma^{-1}(\underline{X} - \mu)$ $E_\theta(G) = 0$

$I(\theta) = -E(l''(\theta))$ $N(\theta, I^{-1}(\theta))$ $\frac{I^{-1}(\theta)}{var(\hat{\theta})}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta})$ $S = U(\theta_0)^2 I(\theta_0)^{-1}$

$-2\ln\frac{L(\theta_0)}{L(\hat{\theta})}$ $(\hat{\theta} - \hat{q}_{1-\alpha}, \hat{\theta} - \hat{q}_{\alpha})$ $(\hat{\theta}^*_{\alpha B}, \hat{\theta}^*_{(1-\alpha)B})$ $(\hat{\theta} - \hat{q}_{1-\alpha} S, \hat{\theta} - \hat{q}_{\alpha} S)$

$U = I(\theta)(\hat{\theta} - \theta)$ $\int_0^\infty y^k e^{-y} dy = k!$ $G = \mu_\theta^T \Sigma^{-1}(\underline{X} - \mu)$ $E(G) = 0$

$I(\theta) = -E(l''(\theta))$ $N(\theta, I^{-1}(\theta))$ $\frac{I^{-1}(\theta)}{var(\hat{\theta})}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta})$ $S = U(\theta_0)^2 I(\theta_0)^{-1}$

$-2\ln\frac{L(\theta_0)}{L(\hat{\theta})}$ $(\hat{\theta} - \hat{q}_{1-\alpha}, \hat{\theta} - \hat{q}_{\alpha})$ $(\hat{\theta}^*_{\alpha B}, \hat{\theta}^*_{(1-\alpha)B})$ $(\hat{\theta} - \hat{q}_{1-\alpha} S, \hat{\theta} - \hat{q}_{\alpha} S)$ $U = I(\theta)(\hat{\theta} - \theta)$

$\int_0^\infty y^k e^{-y} dy = k!$ $G = \mu_\theta^T \Sigma^{-1}(\underline{X} - \mu)$ $E(G) = 0$

$I(\theta) = -E(l''(\theta))$ $N(\theta, I(\theta)^{-1})$ $\frac{I(\theta)^{-1}}{var(\hat{\theta})}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta})$ $S = U(\theta_0)^2 I(\theta_0)^{-1}$

$-2\ln\frac{L(\theta_0)}{L(\hat{\theta})}$ $(\hat{\theta} - \hat{q}_{(1-\alpha)}, \hat{\theta} - \hat{q}_{\alpha})$ $(\hat{\theta}^*_{\alpha B}, \hat{\theta}^*_{(1-\alpha)B})$ $(\hat{\theta} - \hat{q}_{1-\alpha} S, \hat{\theta} - \hat{q}_{\alpha} S)$ $U(\theta) = I(\theta)(\hat{\theta} - \theta)$

$\int_0^\infty y^k e^{-y} dy = k!$ $G = \mu_\theta^T \Sigma^{-1}(\underline{X} - \mu)$ $E(G) = 0$ $\hat{\theta}_J = \frac{1}{n}\sum_{i=1}^n \hat{\theta}_i$ $\hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$

$\hat{v}_J = \frac{1}{n(n-1)}\sum_{i=1}^n (\hat{\theta}_i - \hat{\theta}_J)^2$ 1.65 1.96 $h(\hat{\theta}) \approx h(\theta) + (\hat{\theta} - \theta)h'(\theta) + \frac{1}{2}(\hat{\theta} - \theta)^2 h''(\theta)$

$var(x) = E((x - E(x))^2)$ $cov(x, y) = E((x - E(x))(y - E(y)))$ $(2\hat{\theta} - \hat{\theta}^*_{(1-\alpha)B}, 2\hat{\theta} - \hat{\theta}^*_{\alpha B})$ $(\hat{\theta}^*_{\alpha B}, \hat{\theta}^*_{(1-\alpha)B})$

$I(\theta) = -E(l''(\theta))$ $N(\theta, I(\theta)^{-1})$ $\frac{I(\theta)^{-1}}{var(\hat{\theta})}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta})$ $S = U(\theta_0)^2 I(\theta_0)^{-1}$

$-2\ln\frac{L(\theta_0)}{L(\hat{\theta})}$ $(2\hat{\theta} - \hat{\theta}^*_{(1-\alpha)B}, 2\hat{\theta} - \hat{\theta}^*_{\alpha B})$ $(\hat{\theta}^*_{\alpha B}, \hat{\theta}^*_{(1-\alpha)B})$ $(\hat{\theta} - \hat{q}_{1-\alpha} S, \hat{\theta} - \hat{q}_{\alpha} S)$

$\int_0^\infty y^k e^{-y} dy = k!$ $G = \mu_\theta^T \Sigma^{-1}(\underline{X} - \mu)$ $E(G) = 0$ $\hat{\theta}_J = \frac{1}{n}\sum_{i=1}^n \hat{\theta}_i$ $\hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$ $\hat{v}_J = \frac{1}{n(n-1)}\sum_{i=1}^n (\hat{\theta}_i - \hat{\theta}_J)^2$

1.65 1.96 $h(\hat{\theta}) \approx h(\theta) + (\hat{\theta} - \theta)h'(\theta) + \frac{1}{2}(\hat{\theta} - \theta)^2 h''(\theta)$ $var(x) = E((x - E(x))^2)$

$\hat\theta \sim N(\theta, var(G)^{-1})$ $G=\mu_\theta^T \Sigma^{-1}(\underline{x}-\mu)$ $E(G)=0$ $\hat\theta\sim N(\theta,k^{-1})$ $k^{-1}=var(G)=\mu_\theta^T\Sigma^{-1}\mu_\theta$

$\frac{1}{n-1}(\underline{x}-\mu)^T\Sigma_0^{-1}(\underline{x}-\mu)|_\theta$ $I(\theta)=-E(l''(\theta))$ $N(\theta, I(\theta)^{-1})$ $\frac{I(\theta)^{-1}}{var(\hat\theta)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $w=(\hat\theta-\theta_0)^2 I(\hat\theta)$

$S=U(\theta_0)^2 I(\theta_0)^{-1}$ $-2\ln\frac{L(\theta_0)}{L(\hat\theta)}$ $(2\hat\theta-\hat\theta^*_{(1-\alpha)B}, 2\hat\theta-\hat\theta^*_{\alpha B})$ $(\hat\theta^*_{\alpha B},\hat\theta^*_{(1-\alpha)B})$ $(\hat\theta-\hat q_{1-\alpha}S, \hat\theta-\hat q_\alpha S)$

$\int_0^\infty y^k e^{-y}dy=k!$ $G=\mu_\theta^T\Sigma^{-1}(\underline{x}-\mu)$ $E(G)=0$ $\hat\theta_J=\frac1n\sum_{i=1}^n\hat\theta_i$ $\hat\theta_i=n\hat\theta-(n-1)\hat\theta_{-i}$ $\hat V_J=\frac{1}{(n-1)n}\sum_{i=1}^n(\hat\theta_i-\hat\theta_J)^2$

1.65 1.96 $h(\hat\theta)\approx h(\theta)+(\hat\theta-\theta)h'(\theta)+\frac12(\hat\theta-\theta)^2h''(\theta)$ $var(x)=E((x-E(x))^2)$ $I(\theta)=-E(l''(\theta))$

$N(\theta, I(\theta)^{-1})$ $\frac{I(\theta)^{-1}}{var(\hat\theta)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $w=(\hat\theta-\theta_0)^2 I(\hat\theta)$ $S=U(\theta_0)^2I(\theta_0)^{-1}$ $-2\ln\frac{L(\theta_0)}{L(\hat\theta)}$

$(2\hat\theta-\hat\theta^*_{(1-\alpha)B}, 2\hat\theta-\hat\theta^*_{\alpha B})$ $(\hat\theta^*_{\alpha B},\hat\theta^*_{(1-\alpha)B})$ $(\hat\theta-\hat q_{(1-\alpha)}S,\hat\theta-\hat q_\alpha S)$ $\int_0^\infty y^ke^{-y}dy=k!$

$G=\mu_\theta^T\Sigma^{-1}(\underline{x}-\mu)$ $E(G)=0$ $\hat\theta\sim N(\theta,k^{-1})$ $k=\mu_\theta^T\Sigma^{-1}\mu_\theta$ $\hat\theta_J=\sum_{i=1}^n\hat\theta_i$

$\hat\theta_i=n\hat\theta-(n-1)\hat\theta_{-i}$ $\hat V_J=\frac{1}{n(n-1)}\sum_{i=1}^n(\hat\theta_i-\hat\theta_J)^2$ 1.65 1.96 $h(\hat\theta)\approx h(\theta)+(\hat\theta-\theta)h'(\theta)+\frac12(\hat\theta-\theta)^2h''(\theta)$

$var(X)=E((X-E(X))^2)$ $I(\theta)=-E(l''(\theta))$ $N(\theta,I(\theta)^{-1})$ $\frac{I(\theta)^{-1}}{var(\hat\theta)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $w=(\hat\theta-\theta_0)^2I(\hat\theta)$

$S=U(\theta_0)^2I(\theta_0)^{-1}$ $-2\ln\frac{L(\theta_0)}{L(\hat\theta)}$ $(2\hat\theta-\hat\theta^*_{(1-\alpha)B},2\hat\theta-\hat\theta^*_{\alpha B})$ $(\hat\theta^*_{\alpha B},\hat\theta^*_{(1-\alpha)B})$ $(\hat\theta-\hat q_{(1-\alpha)}S,\hat\theta-\hat q_\alpha S)$

$\int_0^\infty y^ke^{-y}dy=k!$ $G=\mu_\theta^T\Sigma^{-1}(\underline x-\mu)$ $E(G)=0$ $\hat\theta\sim N(\theta,k^{-1})$ $k=\mu_\theta^T\Sigma^{-1}\mu_\theta$ $\hat\theta_J=\sum_{i=1}^n\hat\theta_i$ $\hat\theta_i=n\hat\theta-(n-1)\hat\theta_{-i}$

$\hat V_J=\frac{1}{n(n-1)}\sum_{i=1}^n(\hat\theta_i-\hat\theta_J)^2$ 1.65 1.96 $h(\hat\theta)\approx h(\theta)+(\hat\theta-\theta)h'(\theta)+\frac12(\hat\theta-\theta)^2h''(\theta)$ $U(\theta)=I(\theta)(\hat\theta-\theta)$

$I(\theta)=-E(l''(\theta))$ $N(\theta,I(\theta)^{-1})$ $\frac{I(\theta)^{-1}}{var(\hat\theta)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $w=(\hat\theta-\theta_0)^2I(\hat\theta)$ $S=U(\theta_0)^2I(\theta_0)^{-1}$ $-2\ln\frac{L(\theta_0)}{L(\hat\theta)}$

$(2\hat\theta-\hat\theta^*_{(1-\alpha)B},2\hat\theta-\hat\theta^*_{\alpha B})$ $(\hat\theta^*_{\alpha B},\hat\theta^*_{(1-\alpha)B})$ $(\hat\theta-\hat q_{(1-\alpha)}S,\hat\theta-\hat q_\alpha S)$ $\int_0^\infty y^ke^{-y}dy=k!$ $G=\mu_\theta^T\Sigma^{-1}(\underline x-\mu)$

$E(G)=0$ $\hat\theta\sim N(\theta,k^{-1})$ $k=\mu_\theta^T\Sigma^{-1}\mu_\theta$ $\hat\theta_J=\frac1n\sum_{i=1}^n\hat\theta_i$ $\hat\theta_i=n\hat\theta-(n-1)\hat\theta_{-i}$ $\hat V_J=\frac{1}{n(n-1)}\sum_{i=1}^n(\hat\theta_i-\hat\theta_J)^2$ 1.65 1.96

$h(\hat\theta)\approx h(\theta)+(\hat\theta-\theta)h'(\theta)+\frac12(\hat\theta-\theta)^2h''(\theta)$ $U(\theta)=I(\theta)(\hat\theta-\theta)$ $var(x)=E((x-E(x))^2)$ $I(\theta)=-E(l''(\theta))$

$N(\theta,I(\theta)^{-1})$ $\frac{I(\theta)^{-1}}{var(\hat\theta)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $w=(\hat\theta-\theta_0)^2I(\hat\theta)$ $S=U(\theta_0)^2I(\theta_0)^{-1}$ $-2\ln\frac{L(\theta_0)}{L(\hat\theta)}$ $(2\hat\theta-\hat\theta^*_{(1-\alpha)B},2\hat\theta-\hat\theta^*_{\alpha B})$

$(\hat\theta^*_{\alpha B},\hat\theta^*_{(1-\alpha)B})$ $(\hat\theta-\hat q_{(1-\alpha)}S,\hat\theta-\hat q_\alpha S)$ $\int_0^\infty y^ke^{-y}dy=k!$ $G=\mu_\theta^T\Sigma^{-1}(\underline x-\mu)$ $E(G)=0$ $\hat\theta\sim N(\theta,k^{-1})$

$k=\mu_\theta^T\Sigma^{-1}\mu_\theta$ $\hat\theta_J=\frac1n\sum_{i=1}^n\hat\theta_i$ $\hat\theta_i=n\hat\theta-(n-1)\hat\theta_{-i}$ $\hat V_J=\frac{1}{n(n-1)}\sum_{i=1}^n(\hat\theta_i-\hat\theta_J)^2$ 1.65 1.96 $U(\theta)=I(\theta)(\hat\theta-\theta)$

$h(\hat\theta)\approx h(\theta)+(\hat\theta-\theta)h'(\theta)+\frac12(\hat\theta-\theta)^2h''(\theta)$ $var(x)=E((x-E(x))^2)$ $I(\theta)=-E(l''(\theta))$ $N(\theta,I(\theta)^{-1})$

$\frac{I(\theta)^{-1}}{var(\hat\theta)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $w=(\hat\theta-\theta_0)^2I(\hat\theta)$ $S=U(\theta_0)^2I(\theta_0)^{-1}$ $2\ln\frac{L(\theta_0)}{L(\hat\theta)}$ $(2\hat\theta-\hat\theta^*_{(1-\alpha)B},2\hat\theta-\hat\theta^*_{\alpha B})$

$(\hat\theta^*_{\alpha B},\hat\theta^*_{(1-\alpha)B})$ $(\hat\theta-\hat q_{(1-\alpha)}S,\hat\theta-\hat q_\alpha S)$ $\int_0^\infty y^ke^{-y}dy=k!$ $G=\mu_\theta^T\Sigma^{-1}(\underline x-\mu)$ $E(G)=0$ $\hat\theta\sim N(\theta,k)$

$k=\mu_\theta^T\Sigma^{-1}\mu_\theta$ $\theta_J=\frac1n\sum_{i=1}^n\hat\theta_i$ $\hat\theta_i=n\hat\theta-(n-1)\hat\theta_{-i}$ $\hat V_J=\frac{1}{n(n-1)}\sum_{i=1}^n(\hat\theta_i-\hat\theta_J)^2$ 1.65 1.96

$U(\theta)=I(\theta)(\hat\theta-\theta)$ $h(\hat\theta)\approx h(\theta)+(\hat\theta-\theta)h'(\theta)+\frac12(\hat\theta-\theta)^2h''(\theta)$ $I(\theta)=-E(l''(\theta))$

$N(\theta,I(\theta)^{-1})$ $\frac{I(\theta)^{-1}}{var(\hat\theta)}$ $\frac{L(\theta_1)}{L(\theta_0)}$ $w=(\hat\theta-\theta_0)^2I(\hat\theta)$ $S=U(\theta_0)^2I(\theta_0)^{-1}$ $-2\ln\frac{L(\theta_0)}{L(\hat\theta)}$

$(2\hat\theta-\hat\theta^*_{(1-\alpha)B},2\hat\theta-\hat\theta^*_{\alpha B})$ $(\hat\theta^*_{\alpha B},\hat\theta^*_{(1-\alpha)B})$ $(\hat\theta-\hat q_{(1-\alpha)}S,\hat\theta-\hat q_\alpha S)$ $\int_0^\infty y^ke^{-y}dy=k!$

$G=\mu_\theta^T\Sigma^{-1}(\underline x-\mu)$ $E(G)=0$ $\hat\theta\sim N(\theta,k)$ $k=\mu_\theta^T\Sigma^{-1}\mu_\theta$ $\theta_J=\frac1n\sum_{i=1}^n\theta_i$ $\hat\theta_i=n\hat\theta-(n-1)\hat\theta_{-i}$

2019: 36

$\hat V_J=\frac{1}{n(n-1)}\sum_{i=1}^n(\hat\theta_i-\hat\theta_J)^2$ 1.65 1.96 $U(\theta)=I(\theta)(\hat\theta-\theta)$ $h(\hat\theta)\approx h(\theta)+(\hat\theta-\theta)h'(\theta)+\frac12(\hat\theta-\theta)^2h''(\theta)$