Q 1 a $G_X(\theta) = 0.3 + 0.5\theta + 0\theta^2 + 0.2\theta^3 \quad \therefore$

$P(X=0) = 0.3 \quad , \quad P(X=1) = 0.5 \quad , \quad P(X=2) = 0 \quad , \quad P(X=3) = 0.2$

$\cancel{E(X^2}$ $E(Y) = E(X^2+3) = E(X^2) + E(3) = E(X^2) + 3 =$

$\left(\sum_{n=0}^{3} n^2 P(X=n)\right) + 3 =$

$0^2 P(X=0) + 1^2 P(X=1) + 2^2 P(X=2) + 3^2 P(X=3) + 3 =$

$0 \times 0.3 + 1 \times 0.5 + 4 \times 0 + 9 \times 0.2 + 3 = 0.5 + 1.8 + 3 = 5.3$

$P(Y=4) = P(X^2+3=4) = P(X^2=1) = P(X=1) = 0.5$

Q1 b/ $E(Y) = E(2W) = 2E(W)$

$E(W) = G_W^{(1)}(1) = G_W^{(1)}(\theta)\big|_{\theta=1} = \frac{d}{d\theta}\left[G_W(\theta)\right]\big|_{\theta=1} = \frac{d}{d\theta}\left[(2-\theta)^{-1}\right]\big|_{\theta=1} =$

$\left[-1(2-\theta)^{-2}(-1)\right]\big|_{\theta=1} = -1(2-1)^{-2}(-1) = 1 \quad \therefore E(Y) = 2E(W) = 2 \cdot 1 = 2$

$P(Y=4) = P(2W=4) = P(W=2)$

$G_W(\theta) = (2-\theta)^{-1} = (2(1-\frac{1}{2}\theta))^{-1} = 2^{-1}(1-\frac{1}{2}\theta)^{-1} = 0.5\left(1+\left[-\frac{1}{2}\theta\right]\right)^{-1} =$

$0.5\left[1 - 1\left[-\frac{1}{2}\theta\right] + \frac{-1\times(-1-1)}{2!}\left[-\frac{1}{2}\theta\right]^2 + \cdots\right]$ (by binomial expansion formula)

$= 0.5\left(1 + \frac{1}{2}\theta + 1\left[\frac{1}{4}\theta^2\right] + \cdots\right) = 0.5\left(1 + \frac{1}{2}\theta + \frac{1}{4}\theta^2 + \cdots\right) =$

$0.5 + 0.25\theta + 0.125\theta^2 + \cdots \quad \therefore$

$P(W=2) = 0.125 = P(Y=4)$

Q1c $E(Y) = E(W+X) = E(W) + E(X)$

$$E(X) = \sum_{n=0}^{3} nP(X=n) =$$

$$(0)(0.3) + (1)(0.5) + (2)(0) + (3)(0.2) = 1.1$$

$E(W) = 1 \quad \therefore$

$E(Y) = 1 + 1.1 = 2.1$

$G_Y(\theta) = G_{W+X}(\theta) = G_W(\theta)\, G_X(\theta)$ (by independence)

$= (2-\theta)^{-1}(0.3 + 0.5\theta + 0.2\theta^3)$

$$G_W(\theta) = (2-\theta)^{-1} = 0.5\left(1 + \left[-\tfrac{1}{2}\theta\right]\right)^{-1} =$$

$$0.5\left(1 - 1\left[-\tfrac{1}{2}\theta\right] + \frac{-1\times(-1-1)}{2!}\left[-\tfrac{1}{2}\theta\right]^2 + \frac{-1\times(-1-1)\times(-1-2)}{3!}\left[-\tfrac{1}{2}\theta\right]^3 + \frac{-1\times(-1-1)\times(-1-2)\times(-1-3)}{4!}\left[-\tfrac{1}{2}\theta\right]^4 + \dots\right)$$

$$= 0.5\left(1 + \tfrac{1}{2}\theta + 1\left[\tfrac{1}{4}\theta^2\right] - 1\left[-\tfrac{1}{8}\theta^3\right] + 1\left[\tfrac{1}{16}\theta^4\right] + \dots\right) =$$

$$0.5\left(1 + \tfrac{1}{2}\theta + \tfrac{1}{4}\theta^2 + \tfrac{1}{8}\theta^3 + \tfrac{1}{16}\theta^4 + \dots\right) =$$

$$\tfrac{1}{2} + \tfrac{1}{4}\theta + \tfrac{1}{8}\theta^2 + \tfrac{1}{16}\theta^3 + \tfrac{1}{32}\theta^4 + \dots \quad \therefore$$

$$G_Y(\theta) = \left(\tfrac{1}{2} + \tfrac{1}{4}\theta + \tfrac{1}{8}\theta^2 + \tfrac{1}{16}\theta^3 + \tfrac{1}{32}\theta^4 + \dots\right)(0.3 + 0.5\theta + 0.2\theta^3) \quad \therefore$$

$P(Y=4) =$ Coefficient of θ^4 $\therefore$

$$P(Y=4) = \tfrac{1}{32}\times 0.3 + \tfrac{1}{16}\times 0.5 + \tfrac{1}{4}\times 0.2 = \frac{29}{320} = 0.0906 \ (3.S.F.)$$

Q1d $E(Y) = G_Y^{(1)}(1) = G_Y^{(1)}(\theta)\big|_{\theta=1} = \frac{d}{d\theta}\left[G_Y(\theta)\right]\Big|_{\theta=1} =$

$\frac{d}{d\theta}\left[G_X(G_W(\theta))\right]\Big|_{\theta=1} = G_X'(G_W(\theta))G_W'(\theta)\big|_{\theta=1} =$

$G_X'(G_W(1))G_W'(1) = G_X'((2-1)^{-1})G_W'(1) = G_X'(1)G_W'(1) =$

$E(X)E(W) = 1.1 \times 1 = 1.1$

$G_Y(\theta) = G_X(G_W(\theta)) = G_X\left(\frac{1}{2} + \frac{1}{4}\theta + \frac{1}{8}\theta^2 + \frac{1}{16}\theta^3 + \frac{1}{32}\theta^4 + \dots\right) =$

$0.3 + 0.5\left(\frac{1}{2} + \frac{1}{4}\theta + \frac{1}{8}\theta^2 + \frac{1}{16}\theta^3 + \frac{1}{32}\theta^4 + \dots\right) + 0.2\left(\frac{1}{2} + \frac{1}{4}\theta + \frac{1}{8}\theta^2 + \frac{1}{16}\theta^3 + \frac{1}{32}\theta^4 + \dots\right)^3$

$\therefore P(Y=4) =$ Coefficient of θ^4 $\therefore$

$P(Y=4) =$

$0.5 \times \frac{1}{32} + 0.2\left[\frac{1}{32} \times \left(\frac{1}{2}\right)^2 \times \frac{3!}{1!2!} + \frac{1}{16} \times \frac{1}{4} \times \frac{1}{2} \times \frac{3!}{1!1!1!} + \left(\frac{1}{8}\right)^2 \times \frac{1}{2} \times \frac{3!}{2!1!}\right] =$

$\frac{1}{64} + 0.2\left[\frac{3}{128} + \frac{3}{64} + \frac{3}{128}\right] = \frac{1}{64} + \frac{3}{160} = \frac{11}{320} = 0.0344$ (3 s.f.)

Q1e / $G_Y(\theta) = G_{X_1+X_2+X_3}(\theta) = G_{X_1}(\theta)G_{X_2}(\theta)G_{X_3}(\theta)$ (by independence)

$= G_X(\theta)G_X(\theta)G_X(\theta) = (G_X(\theta))^3 \therefore$

$E(Y) = G_Y^{(1)}(1) = G_Y^{(1)}(\theta)\big|_{\theta=1} = \frac{d}{d\theta}\left[G_Y(\theta)\right]\big|_{\theta=1} =$

$\frac{d}{d\theta}\left[(G_X(\theta))^3\right]\big|_{\theta=1} = 3(G_X(\theta))^2 G_X'(\theta)\big|_{\theta=1} =$

$3(0.3+0.5\theta+0.2\theta^3)^2 G_X'(\theta)\big|_{\theta=1} =$

$3(0.3+0.5\times1+0.2\times1^3)^2 G_X'(\theta)\big|_{\theta=1} = 3(1)^2 G_X'(\theta)\big|_{\theta=1} = 3G_X'(\theta)\big|_{\theta=1} =$

$3\frac{d}{d\theta}\left[G_X(\theta)\right]\big|_{\theta=1} = 3\frac{d}{d\theta}\left[0.3+0.5\theta+0.2\theta^3\right]\big|_{\theta=1} =$

$3\left[0.5+0.6\theta^2\right]\big|_{\theta=1} = 3\left[0.5+0.6(1)^2\right] = 3\left[1.1\right] = 3.3$

$G_Y(\theta) = (G_X(\theta))^3 = (0.3+0.5\theta+0.2\theta^3)^3 \therefore$

$P(Y=4) =$ Coefficient of $\theta^4 \therefore$

$P(Y=4) =$

$0.2\times0.5\times0.3\times\frac{3!}{1!1!1!} = \frac{9}{50} = 0.18$

Q 18 $G_Y(\theta) = G_{1+X_1^2+X_2^2+X_3^2+X_4^2}(\theta) =$

$G_1(\theta)G_{X_1^2}(\theta)G_{X_2^2}(\theta)G_{X_3^2}(\theta)G_{X_4^2}(\theta)$ (by independence)

$= G_1(\theta)G_{X^2}(\theta)G_{X^2}(\theta)G_{X^2}(\theta)G_{X^2}(\theta) = G_1(\theta)(G_{X^2}(\theta))^4$

$E(X(X-1)) = E(X^2-X) = E(X^2)-E(X) = G_X^{(2)}(1) = G_X^{(2)}(\theta)|_{\theta=1}$ $\therefore$

$G_X^{(1)}(\theta) = \frac{d}{d\theta}[G_X(\theta)] = \frac{d}{d\theta}[0.3+0.5\theta+0.2\theta^3] = 0.5+3\times0.2\theta^2 = 0.5+0.6\theta^2$

$\therefore G_X^{(2)}(\theta) = \frac{d}{d\theta}[G_X^{(1)}(\theta)] = \frac{d}{d\theta}[0.5+0.6\theta^2] = 2\times0.6\theta = 1.2\theta$ $\therefore$

$E(X) = G_X^{(1)}(1) = G_X^{(1)}(\theta)|_{\theta=1} = [0.5+0.6\theta^2]|_{\theta=1} = 0.5+0.6(1)^2 = 1.1$ $\therefore$

$E(X^2)-E(X) = G_X^{(2)}(\theta)|_{\theta=1} = [1.2\theta]_{\theta=1} = 1.2\times1 = 1.2$ $\therefore$

$E(X^2) = (E(X^2)-E(X))+E(X) = 1.2+1.1 = 2.3$ $\therefore$

$E(X_i^2) = 2.3$, for $i=1,2,3,4$ $\therefore$ $E(X_i^2) = E(X^2)$ $\therefore$

$E(Y) = E(1+X_1^2+X_2^2+X_3^2+X_4^2) = E(1)+E(X_1^2)+E(X_2^2)+E(X_3^2)+E(X_4^2) =$

$1+E(X^2)+E(X^2)+E(X^2)+E(X^2) = 1+4E(X^2) =$

$1+4(2.3) = 10.2$

$P(Y=4) = P((1+X_1^2+X_2^2+X_3^2+X_4^2)=4) = P(X_1^2+X_2^2+X_3^2+X_4^2=3) =$

$P(Y=4) =$ Coefficient of θ^3 $\therefore$

$G_{X_1^2+X_2^2+X_3^2+X_4^2}(\theta) = G_{X_1^2}(\theta)G_{X_2^2}(\theta) * G_{X_3^2}(\theta)G_{X_4^2}(\theta)$ (by independence)

$= G_{X^2}(\theta)G_{X^2}(\theta)G_{X^2}(\theta)G_{X^2}(\theta) = (G_{X^2}(\theta))^4$ $\therefore$

$G_{X^2}(\theta) = E(\theta^{X^2}) = E(\theta^{X\cdot X}) = E((\theta^X)^X) = G_X(\theta^X)$

$P(X=0) = 0.3 = P(X^2=0)$, $P(X=1) = P(X^2=1) = 0.5$,

$P(X=3) = 0.2 = P(X^2=9)$ $\therefore$

$G_{X^2}(\theta) = 0.3+0.5\theta+0.2\theta^9$ $\therefore G_{X_1^2+X_2^2+X_3^2+X_4^2}(\theta) =$

$(G_{X^2}(\theta))^4 = (0.3+0.5\theta+0.2\theta^9)^4$ $\therefore$

$P(Y=4) =$ Coefficient of θ^3 $\therefore$

$P(Y=4) = 0.5^3\times0.3\times\frac{4!}{3!\,1!} = \frac{3}{20} = 0.15$

Q2a $G_X(\theta) = E(\theta^X) = e^{\lambda(\theta^2-1)} = e^{\theta^2\lambda-\lambda} = e^{-\lambda}e^{\theta^2\lambda} =$

$e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\theta^2\lambda)^k}{k!} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{\theta^{2k}\lambda^k}{k!} = \sum_{k=0}^{\infty}\theta^{2k}\frac{\lambda^k}{k!}e^{-\lambda} =$

$a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + \dots = \sum_{k=0}^{\infty}\theta^{2k}P(X=2k) =$

$\cancel{e^{-\lambda} + e^{-\lambda}\frac{\lambda}{k}\theta + e^{-\lambda}\frac{\lambda^2}{2!}\theta^2}$ $e^{-\lambda} + e^{-\lambda}\frac{\lambda^1}{1!}\theta^2 + e^{-\lambda}\frac{\lambda^2}{2!}\theta^4 + \dots =$

$e^{-\lambda} + (0)\theta^1 + e^{-\lambda}\frac{\lambda}{1!}\theta^2 + (0)\theta^3 + e^{-\lambda}\frac{\lambda^2}{2!}\theta^4 + \dots =$

$P(X=0) + (0)\theta + P(X=2)\theta^2 + (0)\theta^3 + P(X=4)\theta^4 + \dots =$

$P(X=0) + P(X=1)\theta + P(X=2)\theta^2 + P(X=3)\theta^3 + P(X=4)\theta^4 + \dots = \sum_{n=0}^{\infty}P(X=n)\theta^n \therefore$

$P(X=2k) = \frac{\lambda^k}{k!}e^{-\lambda} \therefore$

$P(X=n) = a_n$ where a_n is the corresponding coefficients of the taylor series $\therefore$

$a_{2k+1} = 0, \quad a_{2k} = \frac{\lambda^k}{k!}e^{-\lambda} \therefore$

$P(X=2k+1) = 0, \quad P(X=2k) = \frac{\lambda^k}{k!}e^{-\lambda}$ for $n \in \mathbb{Z}_{\geq 0}, \; k \in \mathbb{Z}_{\geq 0} \therefore$

The probability mass function for X is:

$$f_X(k) = P(X=k) = \begin{cases} \frac{\lambda^k}{k!}e^{-\lambda}, & k = 2n \\ 0, & k = 2n+1 \end{cases} \quad \text{for } n \in \mathbb{Z}_{\geq 0}$$

\Q2b/ $S_n \equiv$ number of individuals at generation n

$$G_{T_2}(\theta) = G_{T_x} \circ G_{T_x}(\theta) = G_{T_x}(G_{T_x}(\theta)) = G_{T_x}\left(e^{\lambda(\theta^2-1)}\right)$$

$$G_{T_x}(\theta) = e^{\lambda(\theta^2-1)} = e^{\theta^2\lambda-\lambda} = e^{-\lambda}e^{\theta^2\lambda} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\theta^2\lambda)^k}{k!} =$$

$$e^{-\lambda}\left[\frac{1}{0!}(\theta^2\lambda)^0 + \frac{1}{1!}(\theta^2\lambda)^1 + \frac{1}{2!}(\theta^2\lambda)^2 + \cdots\right] =$$

$$\cancel{e^{-\lambda}\left[1+(\lambda)+1\theta^2\lambda+\frac{1}{2}\theta^4\lambda^2\right] = e^{-\lambda}\left[1+\lambda\theta^2+\frac{1}{2}\lambda^2\theta^4\right]} \neq$$

$$e^{-\lambda}\left[1(1) + 1\theta^2\lambda + \cancel{\tfrac{1}{2}\theta} \cdots\right] = e^{-\lambda}\left[1+\lambda\theta^2+\cdots\right] = e^{-\lambda} + \lambda e^{-\lambda}\theta^2 + \cdots \therefore$$

$$G_{S_2}(\theta) = G_{T_2}(\theta) = G_{T_x}\left(e^{\lambda(\theta^2-1)}\right) = G_{T_x}\left(e^{-\lambda} + \lambda e^{-\lambda}\theta^2 + \cdots\right) =$$

$$e^{\lambda([e^{-\lambda}+\lambda e^{-\lambda}\theta^2+\cdots]^2-1)} = e^{-\lambda}e^{[e^{-\lambda}+\lambda e^{-\lambda}\theta^2+\cdots]^2\lambda} =$$

$$e^{-\lambda}e^{[(e^{-\lambda})^2+2e^{-\lambda}\lambda e^{-\lambda}\theta^2+\cdots]\lambda} = e^{-\lambda}e^{\lambda e^{-2\lambda}+[2\lambda e^{-2\lambda}\theta^2+\cdots]\lambda} =$$

$$e^{-\lambda+\lambda e^{-2\lambda}}e^{2\lambda^2e^{-2\lambda}\theta^2+\cdots} = e^{(-1+e^{-2\lambda})\lambda}\sum_{k=0}^{\infty}\frac{(2\lambda^2e^{-2\lambda}\theta^2+\cdots)^k}{k!} =$$

$$e^{(-1+e^{-2\lambda})\lambda}\left[\frac{1}{0!}(2\lambda^2e^{-2\lambda}\theta^2+\cdots)^0 + \frac{1}{1!}(2\lambda^2e^{-2\lambda}\theta^2+\cdots)^1 + \cdots\right] =$$

$$e^{(-1+e^{-2\lambda})\lambda}\left[1(1) + 2\lambda^2e^{-2\lambda}\theta^2 + \cdots\right] = e^{(-1+e^{-2\lambda})\lambda} + 2\lambda^2e^{(-3+e^{-2\lambda})\lambda}\theta^2 + \cdots \therefore$$

$P(S_2=0) =$ Coefficient of θ^0 $\therefore$

$\cancel{P(S_2)=}$ $P(S_2=0) = e^{(-1+e^{-2\lambda})\lambda}$

$\cancel{P(S_2)}$ $P(S_2=1) =$ Coefficient of θ^1 $\therefore$

$P(S_2=1) = 0$

$P(S_2=2) =$ Coefficient of θ^2 $\therefore$

$P(S_2=2) = 2\lambda^2e^{(-3+e^{-2\lambda})\lambda}$

Q2C / $G_X(\theta) = e^{-\lambda} + \lambda e^{-\lambda}\theta^2 + \ldots$ $\therefore$

$P(X=0) = e^{-\lambda}$

$E(X) = G_X'(1) = G_X'(\theta)\big|_{\theta=1} = \frac{d}{d\theta}\left[G_X(\theta)\right]\big|_{\theta=1} = \frac{d}{d\theta}\left[e^{\lambda(\theta^2-1)}\right]\big|_{\theta=1} =$

$\frac{d}{d\theta}\left[e^{\lambda\theta^2-\lambda}\right]\big|_{\theta=1} = \frac{d}{d\theta}\left[\lambda\theta^2-\lambda\right] e^{\lambda\theta^2-\lambda}\big|_{\theta=1} = 2\lambda\theta e^{\lambda\theta^2-\lambda}\big|_{\theta=1} =$

$2\lambda(1)e^{\lambda(1)^2-\lambda} = 2\lambda e^{\lambda-\lambda} = 2\lambda e^0 = 2\lambda$ $\therefore$

population will ultimately go extinct if $E(X) < 1$ $\therefore$

if $E(X) = 2\lambda < 1$ $\therefore$ $\lambda < \frac{1}{2}$ $\therefore$

ultimate extinction is guaranteed if $\lambda < \frac{1}{2}$ $\therefore$

ultimate extinction is not guaranteed if $\lambda \geq \frac{1}{2}$

$\therefore$ $e = 1$ for $\lambda < \frac{1}{2}$, $e < 1$ for $\lambda \geq \frac{1}{2}$

Q2di For $\lambda = \frac{1}{4}$: $G_x(\theta) = e^{\frac{1}{4}(\theta^2 - 1)}$

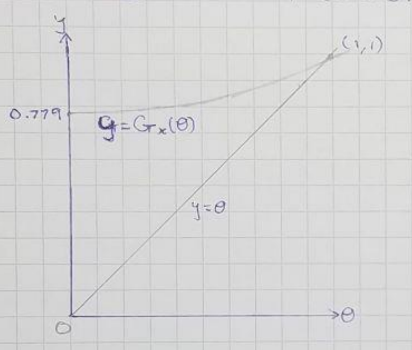
$\lambda = \frac{1}{4} < \frac{1}{2}$ $\therefore$ Ultimate extinction is guaranteed $\therefore e = 1$ $\therefore$

$G_x(\theta) - \theta = 0$ for $\theta = 1$ $\therefore$

$G_x(\theta) - \theta \big|_{\theta=1} = e^{\frac{1}{4}(\theta^2-1)} - \theta \big|_{\theta=1} = e^{\frac{1}{4}(1^2-1)} - 1 = e^0 - 1 = 1 - 1 = 0$ $\therefore$

$\theta = 1$ is a root of $G_x(\theta) - \theta$ $\therefore$

$G_x(0) = e^{\frac{1}{4}(0^2-1)} = e^{-\frac{1}{4}} = 0.779$ (3 s.f.)



Q2dii For $\lambda = \frac{1}{4}$: $G_{T_X}(\theta) = e^{\frac{1}{2}(\theta^2-1)}$

and e^x is an increasing function $\therefore$

$G_{T_X}(\theta) - \theta \big|_{\theta=1} = e^{\frac{1}{2}(1^2-1)} - 1 = e^0 - 1 = 1 - 1 = 0 \quad \therefore$

$G'_{T_X}(\theta) = \frac{d}{d\theta} e^{\frac{1}{2}(\theta^2-1)} = \frac{1}{2}(2\theta) e^{\frac{1}{2}(\theta^2-1)} \quad \therefore$

$G'_{T_X}(\theta) \big|_{\theta=1} = \frac{1}{2}(2 \times 1) e^{\frac{1}{2}(1^2-1)} = \frac{1}{2} \times 2 e^0 = 1 \times 1 = 1$,

$\frac{d}{d\theta}(\theta) \big|_{\theta=1} = 1 \big|_{\theta=1} = 1 \quad \therefore$

$G_{T_X}(\theta) \big|_{\theta=1} = \theta \big|_{\theta=1}$ and $G'_{T_X}(\theta) \big|_{\theta=1} = \frac{d}{d\theta}(\theta) \big|_{\theta=1} \quad \therefore$

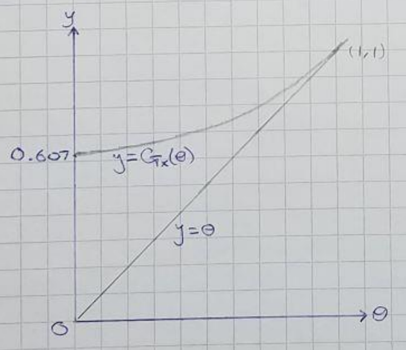
$y = G_{T_X}(\theta)$ intersects and is tangential to $y = \theta$ at $\theta = 1 \quad \therefore$

$y = G_{T_X}(\theta)$ only intersects $y = \theta$ at $\theta = 1 \quad \therefore$

$\theta = 1$ is a root of $G_{T_X}(\theta) - \theta \quad \therefore$

$e = 1 \quad \therefore$ Ultimate extinction is guaranteed.

$\therefore G_{T_X}(0) = e^{\frac{1}{2}(0^2-1)} = e^{-\frac{1}{2}} = 0.607$ (3 s.f.)



Q2diii For $\lambda = 2$: $G_X(\theta) = e^{2(\theta^2-1)}$ $\therefore$

$\lambda = 2 \geq \frac{1}{2}$ $\therefore$ Ultimate extinction is not guaranteed $\therefore e < 1$

but $G_X(\theta) - \theta|_{\theta=1} = e^{2(\theta^2-1)} - \theta|_{\theta=1} = e^{2(1^2-1)} - 1 = e^0 - 1 = 1 - 1 = 0$ $\therefore$

$\theta = 1$ is a root ~~of~~ of $G_X(\theta) - \theta$

and e is also a root of $G_X(\theta) - \theta$

with $P(X=0) > 0$ $\therefore$ $0 < e < 1$ $\therefore$

$G_X(\theta) = e^{2(\theta^2-1)} = e^{2\theta^2-2} = e^{-2}e^{2\theta^2} = e^{-2}\sum_{k=0}^{\infty}\frac{(2\theta^2)^k}{k!} =$

$e^{-2}\left[\frac{(2\theta^2)^0}{0!} + \frac{(2\theta^2)^1}{1!} + \frac{(2\theta^2)^2}{2!} + \frac{(2\theta^2)^3}{3!} + \frac{(2\theta^2)^4}{4!} + \dots\right] =$

$e^{-2}\left[\frac{1}{1} + \frac{2\theta^2}{1} + \frac{4\theta^4}{2} + \frac{8}{6}\theta^6 + \frac{16}{24}\theta^8 + \dots\right] =$

$e^{-2}\left[1 + 2\theta^2 + 2\theta^4 + \frac{4}{3}\theta^6 + \frac{2}{3}\theta^8 + \dots\right] =$

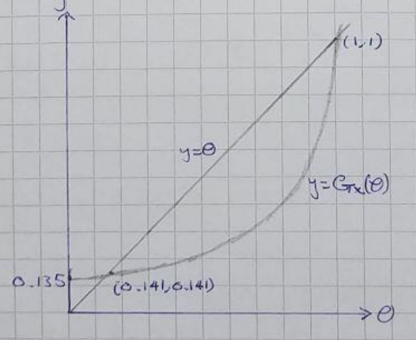
$e^{-2} + 2e^{-2}\theta^2 + 2e^{-2}\theta^4 + \frac{4}{3}e^{-2}\theta^6 + \frac{2}{3}e^{-2}\theta^8 + \dots$ $\therefore$

$G_X(\theta) - \theta = e^{-2} - 1\theta + 2e^{-2}\theta^2 + 2e^{-2}\theta^4 + \frac{4}{3}e^{-2}\theta^6 + \frac{2}{3}e^{-2}\theta^8 + \dots =$

$(\theta - 1)(a_0 + a_1\theta + a_2\theta^2 + a_3\theta^3 + \dots) = 0$ for $a_n \in \mathbb{R}$, $\forall n \in \mathbb{Z}_{\geq 0}$

$\therefore G_X(\theta) - \theta = 0$ for ~~e~~ $\theta = 0.141$ (3 S.f.) $\therefore$

$e = 0.141$, $G_X(0) = e^{2(0^2-1)} = e^{-2} = 0.135$ (3 S f)



Q3a/ Let ~~$A^{(1)}(t)$ the and $A^{(2)}(t)$~~ $A(t)$ be a independent poisson process with rate λ_A, for time t in years $\therefore$

$E(T_A) = 5$ $\therefore$ $\frac{1}{\lambda_A} = E(T_A)$ $\therefore$ $\lambda_A = \frac{1}{5} = 0.2$ per year $\therefore$

Let $S(t) = T_A^{(1)}(t) + T_A^{(2)}(t)$ $\therefore$ Let $G_{A_1, A_2, S}(\theta)$ be the probability generating function of A_1, A_2, S $\therefore$ $G_{T_{A_1}}(\theta) = e^{\lambda_A(\theta - 1)}$, $G_{T_{A_2}}(\theta) = e^{\lambda_A(\theta-1)}$

$\therefore$ by independence: $G_{T_S}(\theta) = G_{T_{A_1}}(\theta) G_{T_{A_2}}(\theta) = e^{\lambda_A(\theta-1)} e^{\lambda_A(\theta-1)} = e^{(\lambda_A + \lambda_A)(\theta - 1)}$ $\therefore$

$S(t)$ is a poisson process with rate $\lambda_S = \lambda_A + \lambda_A = 0.2 + 0.2 = 0.4$ per year.

$\therefore$ Successive events, $S(t)$, are independent and all follow an exponential distribution with rate $\lambda_S = 0.4$ $\therefore$ $f_S = \begin{cases} 0.4e^{-0.4t}, & t \geq 0 \\ 0, & t < 0 \end{cases}$

where $f_S = f_S(t)$ is the probability density function of $S(t)$.

$\therefore$ Let $N(t) \sim Poisson(0.4t)$ $\therefore$

$P(S>1) = P($observe 0 events over time interval $t=1) =$

$P(N(t=1)=0) = P(N(1)=0) = (\lambda_S \times 1)^0 e^{-\lambda_S \times 1}/(0!) = (0.4 \times 1)^0 e^{-0.4 \times 1}/1 =$

$e^{-0.4} = 0.670$ (3 s.f.)

The expected value of S is: $E(S) = \frac{1}{\lambda_S} = \frac{1}{\lambda_A + \lambda_A} = \frac{1}{0.4} = \frac{5}{2} =$

2.5 years. $\because$ $E(S) = \int_0^\infty t f_S(t)\,dt = \int_0^\infty t \lambda_S \exp(-\lambda_S t)\,dt =$

$\left[-t\exp(-\lambda_S t)\right]_0^\infty + \int_0^\infty \exp(-\lambda_S t)\,dt = (0-0) + \left[-\frac{1}{\lambda_S}\exp(-\lambda_S t)\right]_0^\infty = 0 + (0 + \frac{1}{\lambda_S}) = \frac{1}{\lambda_S}$

Q3b/ Let $A(t)$, $B(t)$ be two independent poisson processes with rates λ_A, λ_B respectively, for time t in years $\therefore$

$E[T_A) = 5$, $E(T_B) = 1$ $\therefore$

$\frac{1}{\lambda_A} = E(T_A) \therefore \lambda_A = \frac{1}{5} = 0.2$ per year, $\frac{1}{\lambda_B} = E(T_B) \therefore \lambda_B = \frac{1}{1} = 1$ per year.

Let $V(t) = T_A(t) + T_B(t)$ $\therefore$ Let $G_{T_{A,B,V}}(\theta)$ be the probability generating function of A, B, V $\therefore$ $G_{T_A}(\theta) = e^{\lambda_A(\theta-1)}$, $G_{T_B}(\theta) = e^{\lambda_B(\theta-1)}$ $\therefore$ by independence: $G_{T_V}(\theta) = G_{T_A}(\theta)G_{T_B}(\theta) = e^{\lambda_A(\theta-1)}e^{\lambda_B(\theta-1)} = e^{(\lambda_A+\lambda_B)(\theta-1)}$ $\therefore$

$V(t)$ is a poisson process with rate $\lambda_V = \lambda_A + \lambda_B = 0.2 + 1 = 1.2$ per year.

$\therefore$ Successive events, $V(t)$, are independent and all follow an exponential distribution with rate $\lambda_V = 1.2$ $\therefore$ $f_V = \begin{cases} 1.2e^{-1.2t} & , t \geq 0 \\ 0 & , t < 0 \end{cases}$

where $f_V \doteq f_V(t)$ is the probability density function of $V(t)$.

$\therefore$ The expected value of V is: $E(V) = \frac{1}{\lambda_V} = \frac{1}{\lambda_A + \lambda_B} = \frac{1}{1.2} = \frac{5}{6} =$

0.833 years (3 s.f.) $\because$

$E(V) = \int_0^\infty t f_V(t)\,dt = \int_0^\infty t\lambda_V \exp(-\lambda_V t)\,dt =$

$\left[-t\exp(-\lambda t)\right]_0^\infty + \int_0^\infty \exp(-\lambda_V t)\,dt = (0-0) + \left[-\frac{1}{\lambda_V}\exp(-\lambda t)\right]_0^\infty = 0 + \left(0 + \frac{1}{\lambda_V}\right) = \frac{1}{\lambda_V}$

~~The variance of V is var(V)~~ $E(V^2) = \int_0^\infty t^2 \lambda_V \exp(-\lambda_V t)\,dt =$

$\left[-t^2\exp(-\lambda_V t)\right]_0^\infty + \int_0^\infty 2t\exp(-\lambda_V t)\,dt = (0-0) + \left[-\frac{2}{\lambda_V}t\exp(-\lambda_V t)\right]_0^\infty + \frac{2}{\lambda_V}\int_0^\infty \exp(-\lambda_V t)\,dt =$

$(0-0) + \frac{2}{\lambda_V}\left[-\frac{1}{\lambda_V}\exp(-\lambda_V x)\right]_0^\infty = \frac{2}{\lambda_V^2} = \frac{2}{(1.2)^2} = \frac{25}{18} = 1.39$ (3 s.f.)

$var[V] = E(V^2) - (E(V))^2 = \frac{2}{\lambda_V^2} - \left(\frac{1}{\lambda_V}\right)^2 = \frac{2}{\lambda_V^2} - \frac{1}{\lambda_V^2} = \frac{1}{\lambda_V^2} = \frac{1}{(1.2)^2} = \frac{25}{36} = 0.694$ (3 s.f.)

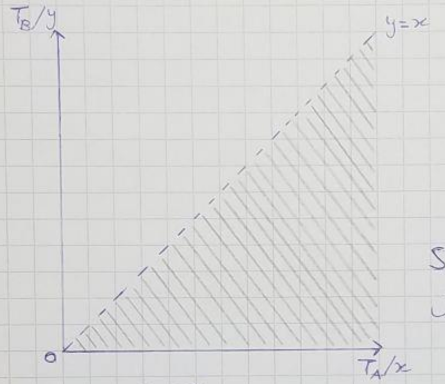
$M_V(t) = E(e^{tT}) = E(\exp(tV)) = \int_{-\infty}^{\infty} \exp(tx) f_V(x)\,dx = \int_0^\infty \exp(tx)\lambda_V \exp(-\lambda_V x)\,dx =$

$\lambda_V \int_0^\infty \exp((t-\lambda_V)x)\,dx = \lambda_V\left[\frac{1}{t-\lambda_V}\exp((t-\lambda_V)x)\right]_0^\infty = \frac{\lambda_V}{\lambda_V - t} = \frac{1.2}{1.2 - t}$

Q3C/ $f_{T_A,T_B}(x,y) = f_{T_A}(x) f_{T_B}(y)$ (by independence)

$= (\lambda_A e^{-\lambda_A x})(\lambda_B e^{-\lambda_B y}) = (0.2e^{-0.2x})(1e^{-1y}) = (0.2e^{-0.2x})(e^{-y})$

For $P(T_A < T_B)$:



Shaded area is the domain being integrated over.

$P(T_A \geq T_B) = \int_0^\infty \int_0^x f_{T_A,T_B}(x,y)\,dy\,dx = \int_0^\infty \int_0^x (0.2e^{-0.2x})(e^{-y})\,dy\,dx =$

$\int_0^\infty \left((0.2e^{-0.2x}) \int_0^x e^{-y}\,dy\right) dx = \int_0^\infty \left((0.2e^{-0.2x})\left[\frac{-1}{1}e^{-y}\right]_0^x\right) dx =$

$\int_0^\infty \left((0.2e^{-0.2x})\left[-e^{-y}\right]_0^x\right) dx = \int_0^\infty \left((0.2e^{-0.2x})\left[-e^{-x} - (-e^{0})\right]\right) dx =$

$\int_0^\infty \left((0.2e^{-0.2x})\left[-e^{-x} - (-1)\right]\right) dx = \int_0^\infty \left((0.2e^{-0.2x})\left[-e^{-x} + 1\right]\right) dx =$

$\int_0^\infty -0.2e^{-1.2x} + 0.2e^{-0.2x}\,dx = \left[\frac{-0.2}{-1.2}e^{-1.2x} + \frac{0.2}{-0.2}e^{-0.2x}\right]_0^\infty =$

$\left[\frac{1}{6}e^{-1.2x} - e^{-0.2x}\right]_0^\infty = \left[\frac{1}{6}(0) - 1(0)\right] - \left[\frac{1}{6}e^{-1.2\times 0} - 1e^{-0.2\times 0}\right] =$

$[0] - \left[\frac{1}{6}e^{0} - 1e^{0}\right] = -\left[\frac{1}{6}\times 1 - 1\times 1\right] = -\left[\frac{1}{6} - 1\right] = -\frac{-5}{6} = \frac{5}{6}$ $\therefore$

$P(T_A < T_B) = 1 - P(T_A \geq T_B) = 1 - \frac{5}{6} = \frac{1}{6}$