Question 1:

$\frac{\partial u}{\partial x} - 3\frac{\partial u}{\partial y} = 0 = u_x - 3u_y = 0 = 1u_x - 3u_y = 0$ ∴

$(1,-3)\cdot(u_x, u_y) = 0 = (1,-3)\cdot(\nabla u) = 0$ ∴

$(dx, dy) \parallel (1,-3)$ so (dx, dy) is parallel to $(1,-3)$ ∴

$\cancel{\frac{dy}{dx} = \frac{-3}{1} = -3 = 1\frac{dy}{dx}}$ ∴ u = constant along $(1,-3)$ ∴

$\cancel{\int 1\frac{dy}{dx}}$ $\frac{dy}{dx} = \frac{-3}{1} = -3 = 1\frac{dy}{dx}$ ∴

$\int 1\frac{dy}{dx}\,dx = \int -3\,dx = \int 1\,dy = y = -3x + C$ where C is an arbitrary constant is the characteristic curves explicit form ∴

$y + 3x = C$ is characteristic curves implicit form ∴

$u = u(x,y) = f(y+3x)$ where f is an arbitrary function is the general solution of the given PDE. ∴ From BC:

$u(x,0) = u(x, y=0) = f(0+3x) = \cancel{\cos 2x}\ f(+3x) = f(3x) = \cos(2x)$ ∴

let $3x = s$ ∴ $x = \frac{1}{3}s$ ∴

$f(s) = \cos(2\cdot\frac{1}{3}s) = \cos(\frac{2}{3}s)$ ∴

$u(x,y) = f(y+3x) = \cos(\frac{2}{3}(y+3x)) = \cos(\frac{2}{3}y + 2x)$

Is the solution to the given PDE that satisfies the BC.

Q1:10/10

Question 2a:

$(1+x^2)\frac{du}{dx} + y\frac{du}{dy} = 0 = (1+x^2)u_x + yu_y = 0$ $\therefore$

$((1+x^2), y)\cdot(u_x, u_y) = 0 = ((1+x^2), y)\cdot \nabla u = 0$ $\therefore$

$(dx, dy) \parallel (1+x^2, y)$ so (dx, dy) is parallel to $(1+x^2, y)$ $\therefore$

$u(x,y) = u = \text{constant}$ along $(1+x^2, y)$ $\therefore$

$\frac{dy}{dx} = \frac{y}{1+x^2}$ $\therefore$

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{1+x^2}$

$\therefore$ The special solution is $y(x) \equiv 0$.

~~is $\frac{1}{1+x^2} = \frac{Az+B}{1+x^2}$~~

let $x = \tan z$ $\therefore$ $\frac{dx}{dz} = \frac{d}{dz}(\tan z) = \sec^2(z) = \frac{1}{\cos^2(z)}$ $\therefore$

$dx = \frac{1}{\cos^2(z)}dz = \sec^2(z)dz$ $\therefore$

$\tan^{-1}(x) = z$ $\therefore$ $1+\tan^2(z) = \frac{1}{\cos^2(z)} = \sec^2(z)$ $\therefore$

$\int \frac{1}{y}\frac{dy}{dx}dx = \int \frac{1}{1+x^2}dx = \int \frac{1}{y}dy = \ln|y| = \int \frac{1}{1+\tan^2(z)}\sec^2(z)\,dz =$

$\int \frac{1}{\sec^2(z)}\sec^2(z)dz = \int 1 dz = z + C = \ln|y| = \tan^{-1}(x) + C$, where C is an arbitrary constant $\therefore$

$|y| = e^{\tan^{-1}(x)+C}$ $\therefore$

$y = \pm e^{\tan^{-1}(x)+C} = Ae^{\tan^{-1}(x)} = Ae^{\arctan(x)}$, where A is an arbitrary constant $\therefore$

$A = \frac{y}{e^{\arctan(x)}} = ye^{-\arctan(x)}$ $\therefore$ ~~is~~ General Solution is:

$u = u(x,y) = f(ye^{-\arctan(x)})$, where f is an arbitrary function

$\therefore$ From boundary condition $u(0,y) = u(x=0,y) = f(ye^{-\arctan(0)}) =$

$f(ye^{-0}) = f(ye^{0}) = f(y\cdot 1) = f(y) = \ln y$ $\therefore$

let $y = s$ $\therefore$ ~~$f(y) = \ln y$~~ $\Leftrightarrow$ $f(s) = \ln s$ $\therefore$

$u(x,y) = f(ye^{-\arctan(x)}) = \ln(ye^{-\arctan(x)}) = \ln(y) + \ln(e^{-\arctan(x)}) =$

$\ln(y) - \arctan(x)$ is the particular solution.

Boundary condition has $y>0$ $\therefore$ $y(x)\equiv 0$ is not a special solution for this boundary condition.

\Question 2b:/ ③

$u(x,y) = f(ye^{-\arctan(x)})$, where f is an arbitrary function

$\therefore$ From boundary Condition $u(x,0) = u(x, y=0) =$
$f((0)e^{-\arctan(x)}) = f(0) = \ln x$ But the function f cannot have different values ($\ln x$) for one and the same value of the argument (0). Because the function cannot have different outputs (multiple outputs) for the same input. So this result cannot be satisfied.

Because the boundary condition is set on $y=0$ which is a Characteristic Curve because $y(x) \equiv 0$ is a special solution. So the solution to the Boundary Condition does not exist. Q2:10/10

Question 3:

$\frac{\partial u}{\partial x} + x^2 y \frac{\partial u}{\partial y} = 0 = u_x + x^2 y u_y = 0 \therefore$

$(1, x^2y) \cdot (u_x, u_y) = 0 = (1, x^2y) \cdot \nabla u \therefore$

$(dx, dy) \parallel (1, x^2y)$ so (dx, dy) is parallel to $(1, x^2y) \therefore$

$u = u(x,y) =$ Constant along $(1, x^2y) \therefore$

$\frac{dy}{dx} = \frac{x^2y}{1} = x^2 y \therefore$

$\frac{1}{y}\frac{dy}{dx} = x^2 \quad \therefore \quad \cancel{y\not\equiv}\ y(x) \equiv 0$ is a special solution $\therefore$

$\int \frac{1}{y}\frac{dy}{dx}\,dx = \int x^2\,dx = \int \frac{1}{y}\,dy = \ln|y| = \frac{1}{3}x^3 + C$, where C is an arbitrary constant $\therefore$

$|y| = e^{\frac{1}{3}x^3 + C} \quad \therefore \quad y = \pm e^{\frac{1}{3}x^3 + C} \quad \therefore$ Characteristic curves is:

$y = Ae^{\frac{1}{3}x^3}$, where A is an arbitrary constant. $\therefore$

$A = ye^{-\frac{1}{3}x^3} \quad \therefore \quad$ General Solution is:

$u = u(x,y) = f(ye^{-\frac{1}{3}x^3})$, where f is an arbitrary function. $\therefore$

is $u_x = \frac{\partial}{\partial x} f(ye^{-\frac{1}{3}x^3}) = \frac{\partial}{\partial x}\left[ye^{-\frac{1}{3}x^3}\right] f'(ye^{-\frac{1}{3}x^3}) = y\frac{\partial}{\partial x}\left[-\frac{1}{3}x^3\right] e^{-\frac{1}{3}x^3} f'(ye^{-\frac{1}{3}x^3})$

$= -yx^2 e^{-\frac{1}{3}x^3} f'(ye^{-\frac{1}{3}x^3})$,

$u_y = \frac{\partial}{\partial y} f(ye^{-\frac{1}{3}x^3}) = \frac{\partial}{\partial y}\left[ye^{-\frac{1}{3}x^3}\right] f'(ye^{-\frac{1}{3}x^3}) = e^{-\frac{1}{3}x^3} f'(ye^{-\frac{1}{3}x^3}) \therefore$

$LHS = \frac{\partial u}{\partial x} + x^2 y \frac{\partial u}{\partial y} = u_x + x^2 y u_y =$

$-yx^2 e^{-\frac{1}{3}x^3} f'(ye^{-\frac{1}{3}x^3}) + x^2 y e^{-\frac{1}{3}x^3} f'(ye^{-\frac{1}{3}x^3}) =$

$(-y+y)x^2 e^{-\frac{1}{3}x^3} f'(ye^{-\frac{1}{3}x^3}) = (0)x^2 e^{-\frac{1}{3}x^3} f'(ye^{-\frac{1}{3}x^3}) = 0 = RHS$

as required $\therefore$

$u(x,y) = f(ye^{-\frac{1}{3}x^3})$ is General Solution.

Q3:10/10

Question 4 (5)

$u_x - 2u_y + u = e^{5x}$ $\therefore$ let $a=1$, $b=-2$, $c=1$, $g(x,y)=e^{5x}$ $\therefore$

$au_x + bu_y + cu = g(x,y)$ $\therefore$ let $\xi = ax+by$, $\eta = bx-ay$ $\therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x = au_\xi + bu_\eta$.

$u_y = u_\xi \xi_y + u_\eta \eta_y = bu_\xi - au_\eta$ $\therefore$ Into PDE:

$a(au_\xi + bu_\eta) + b(bu_\xi - au_\eta) + cu = g(x,y) = \tilde{g}(\xi(x,y), \eta(x,y)) = \tilde{g}(\xi,\eta) =$

$a^2u_\xi + abu_\eta + b^2u_\xi - abu_\eta + cu = (a^2+b^2)u_\xi + cu = \tilde{g}(\xi,\eta) = g(x,y)$ $\therefore$

$u_\xi + \frac{c}{a^2+b^2}u = \tilde{g}(\xi,\eta)\frac{1}{a^2+b^2} = u_\xi + \frac{c}{k}u = \frac{1}{k}\tilde{g}(\xi,\eta)$ for $a^2+b^2=k$ $\therefore$

$h(\xi)u_\xi + h(\xi)\frac{c}{k}u = h(\xi)\frac{1}{k}\tilde{g}(\xi,\eta) = h(\xi)u_\xi + h'(\xi)u = \frac{\partial}{\partial\xi}(h(\xi)u)$ $\therefore$

$h'(\xi) = h(\xi)\frac{c}{k}$ $\therefore$ $h'(\xi)/h(\xi) = \frac{c}{k}$ $\therefore$

$\int \frac{h'(\xi)}{h(\xi)}d\xi = \int \frac{c}{k}d\xi = \ln(h(\xi)) = \frac{c}{k}\xi$ $\therefore$ $h(\xi) = e^{\frac{c}{k}\xi}$ $\therefore$

$\frac{\partial}{\partial\xi}(e^{\frac{c}{k}\xi}u) = e^{\frac{c}{k}\xi}\frac{1}{k}\tilde{g}(\xi,\eta)$ $\therefore$

$e^{\frac{c}{k}\xi}u = \int e^{\frac{c}{k}\xi}\frac{1}{k}\tilde{g}(\xi,\eta)\partial\xi$ $\therefore$ $u(\xi,\eta) = e^{-\frac{c}{k}\xi}\int e^{\frac{c}{k}\xi}\frac{1}{k}\tilde{g}(\xi,\eta)\partial\xi$ $\therefore$

$k = a^2+b^2 = 1^2+(-2)^2 = 1+4 = 5$ $\therefore$ $\frac{c}{k} = \frac{1}{5}$ $\therefore$

$u(\xi,\eta) = e^{-\frac{1}{5}\xi}\int e^{\frac{1}{5}\xi}\frac{1}{5}\tilde{g}(\xi,\eta)\partial\xi$

$ax+by = \xi = x-2y$, $bx-ay = \eta = -2x-y$ $\therefore$

$\Delta = \begin{vmatrix} a & b \\ b & -a \end{vmatrix} = -a^2-b^2$, $\Delta_x = \begin{vmatrix} \xi & b \\ \eta & -a \end{vmatrix} = -a\xi - b\eta$, $\Delta_y = \begin{vmatrix} a & \xi \\ b & \eta \end{vmatrix} = a\eta - b\xi$ $\therefore$

$x = \frac{\Delta_x}{\Delta} = \frac{-a\xi-b\eta}{-a^2-b^2} = \frac{a\xi+b\eta}{a^2+b^2} = \frac{\xi-2\eta}{5} = \frac{1}{5}\xi - \frac{2}{5}\eta$ $\therefore$ $\xi - 2\eta = 5x$,

$y = \frac{\Delta_y}{\Delta} = \frac{-b\xi-a\eta}{-a^2-b^2} = \frac{b\xi-a\eta}{a^2+b^2} = \frac{-2\xi-\eta}{5} = -\frac{2}{5}\xi - \frac{1}{5}\eta$ $\therefore$

$g(x,y) = e^{5x} = e^{5(\frac{1}{5}\xi - \frac{2}{5}\eta)} = e^{\xi-2\eta} = \tilde{g}(\xi,\eta) = e^{\xi}e^{-2\eta}$ $\therefore$

$u(\xi,\eta) = e^{-\frac{1}{5}\xi}\int e^{\frac{1}{5}\xi}\frac{1}{5}e^{\xi}e^{-2\eta}\partial\xi = \frac{1}{5}e^{-\frac{1}{5}\xi}e^{-2\eta}\int e^{\frac{6}{5}\xi}\partial\xi =$

$\frac{1}{5}e^{-\frac{1}{5}\xi}e^{-2\eta}(\frac{5}{6}e^{\frac{6}{5}\xi} + f(\eta)) = \frac{1}{5}e^{-\frac{1}{5}\xi}e^{-2\eta}f(\eta) + \frac{1}{6}e^{\xi}e^{-2\eta} =$

$e^{-\frac{1}{5}\xi}F(\eta) + \frac{1}{6}e^{\xi-2\eta} = u(\xi,\eta) =$

$e^{-\frac{1}{5}(x-2y)}F(-2x-y) + \frac{1}{6}e^{5x} = u(x,y) = e^{-\frac{1}{5}x+\frac{2}{5}y}F(-2x-y) + \frac{1}{6}e^{5x}$,

where f and F are arbitrary functions

Q4:10/10

Question 5:

$\phi = \text{constant}$, $u_{xx} - (1+\phi)u_{xy} + \phi u_{yy} = 0$ $\therefore$

let $u = u(x,y) = f(y+\alpha x)$, f is an arbitrary function, α is a constant $\therefore$

$u_x = \alpha f'(y+\alpha x)$ $\therefore u_{xx} = \alpha^2 f''(y+\alpha x)$,

$u_y = f'(y+\alpha x)$ $\therefore u_{yy} = f''(y+\alpha x)$ $\therefore$

$u_{xy} = \alpha f''(y+\alpha x)$ $\therefore$ Into PDE:

$LHS = \alpha^2 f''(y+\alpha x) - (1+\phi)\alpha f''(y+\alpha x) + \phi f''(y+\alpha x) =$

$[\alpha^2 + (-1-\phi)\alpha + \phi] f''(y+\alpha x) = 0 = RHS$ f is an arbitrary function

$\therefore \exists(x,y)$ such that $f''(y+\alpha x) \neq 0$ $\therefore$

$\alpha^2 + (-1-\phi)\alpha + \phi = 0 = (\alpha-1)(\alpha-\phi) = 0$ $\therefore \alpha = 1$, $\alpha = \phi$ $\therefore$

$\alpha_1 = 1$, $\alpha_2 = \phi$ $\therefore$

$u(x,y) = f_1(y+\alpha_1 x) + f_2(y+\alpha_2 x) = f_1(y+x) + f_2(y+\phi x)$

is a solution, ~~[illegible]~~ where f_1, f_2 are arbitrary functions

Q5:5/10

WHAT IF PHI=1?

Question 6

⑦

$5x^2u_{xx} - y^2u_{yy} + 5xu_x - yu_y = 0$ ∴ use change of variables:

$x = e^t$, $y = e^s$: $(x,y) \mapsto (t,s)$ ∴ $t = \ln x$, $s = \ln y$ ∴

$u_x = u_t t_x = \frac{1}{x}u_t$ ∴

$u_{xx} = \frac{\partial}{\partial x}(\frac{1}{x}u_t) = \frac{\partial}{\partial x}(\frac{1}{x})u_t + \frac{\partial}{\partial x}(u_t)\frac{1}{x} = -\frac{1}{x^2}u_t + \frac{\partial}{\partial x}(u_t)\frac{1}{x} = -\frac{1}{x^2}u_t + \frac{1}{x}\frac{\partial}{\partial x}(u_t) =$

$-\frac{1}{x^2}u_t + \frac{1}{x}\frac{\partial}{\partial t}(u_t)\frac{\partial t}{\partial x} = -\frac{1}{x^2}u_t + \frac{1}{x}u_{tt}\frac{1}{x} = \frac{1}{x^2}u_{tt} - \frac{1}{x^2}u_t$.

$u_y = u_s s_y = \frac{1}{y}u_s$ ∴

$u_{yy} = \frac{\partial}{\partial y}(\frac{1}{y}u_s) = \frac{\partial}{\partial y}(\frac{1}{x})u_s + \frac{\partial}{\partial y}(u_s)\frac{1}{y} = -\frac{1}{y^2}u_s + \frac{\partial}{\partial y}(u_s)\frac{1}{y} = -\frac{1}{y^2}u_s + \frac{1}{y}\frac{\partial}{\partial y}(u_s) =$

$-\frac{1}{y^2}u_s + \frac{1}{y}\frac{\partial}{\partial s}(u_s)\frac{\partial t}{\partial y} = -\frac{1}{y^2}u_s + \frac{1}{y}u_{ss}\frac{1}{y} = \frac{1}{y^2}u_{ss} - \frac{1}{y^2}u_s$ ∴ Into PDE:

$LHS = 5x^2(\frac{1}{x^2}u_{tt} - \frac{1}{x^2}u_t) - y^2(\frac{1}{y^2}u_{ss} - \frac{1}{y^2}u_s) + 5x(\frac{1}{x}u_t) - y(\frac{1}{y}u_s) =$

$5u_{tt} - 5u_t - u_{ss} + u_s + 5u_t - u_s = 5u_{tt} - u_{ss} = 0 = RHS$ ∴

let $u = f(t + \alpha s)$, where f is an arbitrary function, α is constant ∴

$u_t = f'(t+\alpha s)$ ∴ $u_{tt} = f''(t+\alpha s)$,

$u_s = \alpha f'(t+\alpha s)$ ∴ $u_{ss} = \alpha^2 f''(t+\alpha s)$ ∴ Into equation:

$LHS = 5u_{tt} - u_{ss} = 5f''(t+\alpha s) - \alpha^2 f''(t+\alpha s) = (5-\alpha^2)f''(t+\alpha s) = 0 = RHS$

$f = f(t,s)$, f is an arbitrary function ∴ $\exists(t,s)$ such that $f \neq 0$ ∴

$\exists(t,s)$ such that $f''(t+\alpha s) \neq 0$ ∴

$5 - \alpha^2 = 0$ ∴ $5 = \alpha^2$ ∴ $\pm\sqrt{5} = \alpha$ ∴

$\alpha_1 = +\sqrt{5}$, $\alpha_2 = -\sqrt{5}$ ∴

let $\xi = t + \alpha_1 s = t + \sqrt{5}s$, $\eta = t + \alpha_2 s = t - \sqrt{5}s$ ∴

for arbitrary functions h and g:

$u = h(t+\sqrt{5}s) + g(t-\sqrt{5}s) = h(\xi) + g(\eta) =$

$h(\ln(x) + \sqrt{5}\ln(y)) + g(\ln(x) - \sqrt{5}\ln(y)) =$

$h(\ln(x) + \ln(y^{\sqrt{5}})) + g(\ln(x) - \ln(y^{\sqrt{5}}) =$

$h(\ln(y^{\sqrt{5}}x)) + g(-\ln(y^{\sqrt{5}}/x)) =$

$\tilde{F}(y^{\sqrt{5}}x) + \tilde{G}(y^{\sqrt{5}}/x) = u(x,y) = F((y^{\sqrt{5}}x)^{\sqrt{5}}) + G((y^{\sqrt{5}}/x)^{\sqrt{5}}) =$

$F(y^5x^{\sqrt{5}}) + G(y^5/x^{\sqrt{5}}) = u(x,y)$, where $\tilde{F}, \tilde{G}, F, G$ are arbitrary functions

Q6:10/10

\Question 7:/

$u_{xx}+u_{yy}=0 \quad \therefore \quad (x,y)\mapsto(X,Y) \therefore$

$u_x = U_X X_x + U_Y Y_x$

$X = x\cos\theta + y\sin\theta \quad \therefore X_x = \frac{\partial X}{\partial x} = \frac{\partial}{\partial x}(x\cos\theta + y\sin\theta) = \cos\theta$,

$X_y = \frac{\partial}{\partial y}(x\cos\theta + y\sin\theta) = \sin\theta$,

$Y = -x\sin\theta + y\cos\theta \quad \therefore \quad Y_y = \frac{\partial}{\partial y}(-x\sin\theta + y\cos\theta) = \cos\theta$,

$\cancel{Y_{yx}} \quad Y_x = \frac{\partial}{\partial x}(-x\sin\theta + y\cos\theta) = -\sin\theta \quad \therefore$

$X_{xx} = \frac{\partial}{\partial x}(\cos\theta) = 0$, $X_{xy} = X_{yx} = \frac{\partial}{\partial y}(\cos\theta) = 0$, $X_{yy} = \frac{\partial}{\partial y}(\sin\theta) = 0$

$Y_{yy} = \frac{\partial}{\partial y}(\cos\theta) = 0$, $Y_{yx} = Y_{xy} = \frac{\partial}{\partial x}(\cos\theta) = 0$, $Y_{xx} = \frac{\partial}{\partial x}(-\sin\theta) = 0 \quad \therefore$

$u_x = \cos(\theta) U_X \cancel{+} - \sin(\theta) U_Y$,

$u_y = U_X X_y + U_Y Y_y = \sin(\theta) U_X + \cos(\theta) U_Y \quad \therefore$

$u_{xx} = \frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial x}(\cos(\theta)U_X - \sin(\theta)U_Y) = \frac{\partial}{\partial x}[\cos(\theta)U_X] + \frac{\partial}{\partial x}[-\sin(\theta)U_Y] =$

$\cos(\theta)\frac{\partial}{\partial x}[U_X] - \sin(\theta)\frac{\partial}{\partial x}[U_Y] =$

$\cos(\theta)\left(\frac{\partial}{\partial X}[U_X]X_x + \frac{\partial}{\partial Y}[U_X]Y_x\right) - \sin(\theta)\left(\frac{\partial}{\partial X}[U_Y]X_x + \frac{\partial}{\partial Y}[U_Y]Y_x\right) =$

$\cos(\theta)\left(\cos(\theta)U_{XX} - \sin(\theta)U_{XY}\right) - \sin(\theta)\left(\cos(\theta)U_{XY} - \sin(\theta)U_{YY}\right) =$

$\cos^2(\theta)U_{XX} - \sin(\theta)\cos(\theta)U_{XY} - \sin(\theta)\cos(\theta)U_{XY} + \sin^2(\theta)U_{YY} =$

$\cos^2(\theta)U_{XX} - 2\sin(\theta)\cos(\theta)U_{XY} + \sin^2(\theta)U_{YY} =$

$\cos^2(\theta)U_{XX} - \sin(2\theta)U_{XY} + \sin^2(\theta)U_{YY}$ and

$u_{yy} = \frac{\partial}{\partial y}(u_y) = \frac{\partial}{\partial y}(\sin(\theta)U_X + \cos(\theta)U_Y) =$

$\sin(\theta)\frac{\partial}{\partial y}[U_X] + \cos(\theta)\frac{\partial}{\partial y}[U_Y] =$

$\sin(\theta)\left(\frac{\partial}{\partial Y}[U_X]Y_y + \frac{\partial}{\partial X}[U_X]X_y\right) + \cos(\theta)\left(\frac{\partial}{\partial Y}[U_Y]Y_y + \frac{\partial}{\partial X}[U_Y]X_y\right) =$

$\sin(\theta)\left(\cos(\theta)U_{XY} + \sin(\theta)U_{XX}\right) + \cos(\theta)\left(\cos(\theta)U_{YY} + \sin(\theta)U_{XY}\right) =$

$\sin(\theta)\cos(\theta)U_{XY} + \sin^2(\theta)U_{XX} + \cos^2(\theta)U_{YY} + \sin(\theta)\cos(\theta)U_{XY} =$

$\sin^2(\theta)U_{XX} + 2\sin(\theta)\cos(\theta)U_{XY} + \cos^2(\theta)U_{YY} =$

$\sin^2(\theta)U_{XX} + \sin(2\theta)U_{XY} + \cos^2(\theta)U_{YY} \quad \therefore$

V

\Question 7 Continued:/

Sub into PDE:

$LHS = u_{xx} + u_{yy} =$

$\cos^2(\theta)U_{XX} - \sin(2\theta)U_{XY} + \sin^2(\theta)U_{YY} + \sin^2(\theta)U_{XX} + \sin(2\theta)U_{XY} + \cos^2(\theta)U_{YY} =$

$(\cos^2(\theta) + \sin^2(\theta))U_{XX} + (\cos^2(\theta) + \sin^2(\theta))U_{YY} =$

$U_{XX} + U_{YY} = 0 = RHS \therefore$ V

The given PDE invariant under the given rotation.

$\therefore$ Let $U = X^2 - Y^2 \therefore$

$U_X = \frac{\partial}{\partial X}(X^2 - Y^2) = 2X \quad \therefore U_{XX} = \frac{\partial}{\partial X}(2X) = 2$,

$U_Y = \frac{\partial}{\partial Y}(X^2 - Y^2) = -2Y \quad \therefore U_{YY} = \frac{\partial}{\partial Y}(-2Y) = -2 \quad \therefore$

Sub into PDE:

$LHS = U_{XX} + U_{YY} = 2 - 2 = 0 = LHS \therefore$ V

$U(X, Y) = X^2 - Y^2$ is a solution.

And $X^2 = [x\cos(\theta) + y\sin(\theta)]^2 = x^2\cos^2(\theta) + y^2\sin^2(\theta) + 2xy\sin(\theta)\cos(\theta)$,

$Y^2 = (-x\sin(\theta) + y\cos(\theta))^2 = x^2\sin^2(\theta) + y^2\cos^2(\theta) - 2xy\sin(\theta)\cos(\theta) \quad \therefore U(X,Y) =$

$X^2 - Y^2 = x^2\cos^2(\theta) + y^2\sin^2(\theta) + 2xy\sin(\theta)\cos(\theta) - x^2\sin^2(\theta) - y^2\cos^2(\theta) + 2xy\sin(\theta)\cos(\theta)$

$= u(x, y) = (\cos^2(\theta) - \sin^2(\theta))x^2 + (\sin^2(\theta) - \cos^2(\theta))y^2 + 4\sin(\theta)\cos(\theta)xy \quad \therefore$

$u_x = \frac{\partial}{\partial x}[(\cos^2(\theta) - \sin^2(\theta))x^2] + \frac{\partial}{\partial x}[4\sin(\theta)\cos(\theta)xy] =$

$2(\cos^2(\theta) - \sin^2(\theta))x + 4\sin(\theta)\cos(\theta)y \quad \therefore$

$u_{xx} = \frac{\partial}{\partial x}[2(\cos^2(\theta) - \sin^2(\theta))x] = 2(\cos^2(\theta) - \sin^2(\theta)) = 2\cos^2(\theta) - 2\sin^2(\theta)$,

$u_y = \frac{\partial}{\partial y}[(\sin^2(\theta) - \cos^2(\theta))y^2] + \frac{\partial}{\partial y}[4\sin(\theta)\cos(\theta)xy] =$

$2(\sin^2(\theta) - \cos^2(\theta))y + 4\sin(\theta)\cos(\theta)x \quad \therefore$

$u_{yy} = \frac{\partial}{\partial y}[2(\sin^2(\theta) - \cos^2(\theta))y] = 2(\sin^2(\theta) - \cos^2(\theta)) = 2\sin^2(\theta) - 2\cos^2(\theta) \quad \therefore$

Sub into PDE:

$LHS = u_{xx} + u_{yy} = 2\cos^2(\theta) - 2\sin^2(\theta) - 2\sin^2(\theta) - 2\cos^2(\theta) = 0 = RHS \quad \therefore$

$u(x, y) = (\cos^2(\theta) - \sin^2(\theta))x^2 + (\sin^2(\theta) - \cos^2(\theta))y^2 + 4\sin(\theta)\cos(\theta)xy$ is a solution to the PDE

Q7:10/10

Question 8:

$f = f(p) \quad \therefore \quad \frac{\partial f}{\partial p} = \frac{\partial f(p)}{\partial p} = \frac{df(p)}{dp} = f'(p) \quad \therefore$

$x = p + t f(p) \quad \therefore$

$\frac{\partial x}{\partial x} = \frac{\partial}{\partial x}(p + t f(p)) = \frac{\partial p}{\partial x} + \frac{\partial}{\partial x}(t f(p)) = 1 = \frac{\partial p}{\partial x} + f(p)\frac{\partial t}{\partial x} + t\frac{\partial}{\partial x} f(p) =$

$\frac{\partial p}{\partial x} + (0) f(p) + t \frac{\partial}{\partial p}[f(p)] \frac{\partial p}{\partial x} = \frac{\partial p}{\partial x} + t f'(p) \frac{\partial p}{\partial x} = 1 = (1 + t f'(p)) \frac{\partial p}{\partial x} \quad \therefore$

$\frac{\partial p}{\partial x} = \frac{1}{1 + t f'(p)}$ M1

$\frac{\partial}{\partial t}(x) = \frac{\partial}{\partial t}(p + t f(p)) = 0 = \frac{\partial p}{\partial t} + \frac{\partial}{\partial t}(t f(p)) = \frac{\partial p}{\partial t} + f(p)\frac{\partial t}{\partial t} + t \frac{\partial}{\partial t} f(p) =$

$\frac{\partial p}{\partial t} + f(p) + t \frac{\partial}{\partial p}[f(p)] \frac{\partial p}{\partial t} = \frac{\partial p}{\partial t} + f(p) + t f'(p) \frac{\partial p}{\partial t} = (1 + t f'(p)) \frac{\partial p}{\partial t} + f(p) = 0 \quad \therefore$

$(1 + t f'(p)) \frac{\partial p}{\partial t} = -f(p) \quad \therefore \quad \frac{\partial p}{\partial t} = \frac{-f(p)}{1 + t f'(p)}$ M1

$u(x,t) = f(p(x,t)) \quad \therefore \quad u = f(p) \quad \therefore$

$u_t = \frac{\partial}{\partial t}(f(p)) = \frac{\partial}{\partial p}[f(p)] \frac{\partial p}{\partial t} = f'(p) \frac{-f(p)}{1 + t f'(p)}$,

$u_x = \frac{\partial}{\partial x}(f(p)) = \frac{\partial}{\partial p}[f(p)] \frac{\partial p}{\partial x} = f'(p) \frac{1}{1 + t f'(p)} \quad \therefore$ Sub into PDE:

$LHS = u_t + u u_x = f'(p) \frac{-f(p)}{1 + t f'(p)} + f(p) f'(p) \frac{1}{1 + t f'(p)} =$

$(-1+1) \frac{f'(p) f(p)}{1 + t f'(p)} = 0 = RHS$ M3

$u(x,t) = f(p(x,t))$ satisfies the PDE.

~~[illegible]~~ let $u(x,0) = u(x, t=0) = f(p(x, t=0)) = f(p(x)) = f(x)$ is the initial condition at $t=0 \quad \therefore$

$\frac{\partial u(x,0)}{\partial t} = \frac{\partial}{\partial t} f(x) = \cancel{\frac{\partial}{\partial p}[f(x)]} \frac{\partial}{\partial t} f(p(x)) = \frac{\partial}{\partial p}[f(p(x))] \frac{\partial p(x)}{\partial t} =$

$\frac{\partial}{\partial p}[f(p(x))] \frac{\partial p(x,t)}{\partial t}\Big|_{t=0} = f'(p(x)) \frac{-f(p)}{1 + (0) f'(p)} = -f'(p) f(p)\Big|_{t=0} = -f'(x) f(x)$

$\frac{\partial u(x,0)}{\partial x} = \frac{\partial}{\partial x} f(x) = f'(x) \quad \therefore$ ~~[illegible]~~ Sub into PDE:

$LHS = -f'(x) f(x) + f(x) f'(x) = (1-1) f'(x) f(x) = 0 = RHS \quad \therefore$ the solution satisfies the initial condition $u(x,0) = f(x)$ M2

Question 8 continued:

$u(x,0) = f(x)$, $f(x) = A\tanh(x)$ $\therefore$ $u(x,0) = u(x,t=0) = A\tanh(x)$

$\tanh(x) = \frac{1-e^{-2x}}{1+e^{-2x}}$ $\therefore$ $\lim_{x\to\infty} \tanh(x) = \lim_{x\to\infty} \frac{1-e^{-2x}}{1+e^{-2x}} = \frac{1-0}{1+0} = 1$ $\therefore$

$\lim_{x\to\infty} A\tanh(x) = A \lim_{x\to\infty} \tanh(x) = A$

$-1 \leq \tanh(x) < 1$ $\therefore$ $\tanh^2(x) < 1$ $\therefore$ $-1 < -\tanh^2(x)$ $\therefore$ $0 < 1-\tanh^2(x)$ $\therefore$

$0 < A(1-\tanh^2(x))$, $A(1-\tanh^2(x)) \to \infty$ as $A \to \infty$

$\frac{\partial}{\partial x}[u(x,0)] = \frac{\partial}{\partial x}[A\tanh(x)] = A(1-\tanh^2(x))$ $\therefore$

$u_x(x,0) = A(1-\tanh^2(x))$

$-\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x} = \frac{1}{2}\frac{\partial}{\partial x}(u^2)$, $u = f(p)$ $\therefore$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(f(p)) = \frac{\partial}{\partial p}(f(p))\frac{\partial p}{\partial x} = f'(p)\frac{-f(p)}{1+tf'(p)}$ $\therefore$

$\frac{\partial u}{\partial x}(x,0) = \frac{\partial u}{\partial x}(x,t=0) = \left.\frac{-f(p)f'(p)}{1+tf'(p)}\right|_{t=0} = \left.\frac{-f(p)f'(p)}{1+0}\right|_{t=0} = -f(p)f'(p)\Big|_{t=0} = A(1-\tanh^2(x))$

WHEN IS A FRACTION INFINITE?

$x|_{t=0} = p + tf(p)\Big|_{t=0} = p(x,t=0)$ $\therefore$

$-f(x)f'(x) = A(1-\tanh^2(x)) = A\tanh(x)(-f'(x))$ $\therefore$

$u_x(x,t) \to \infty$ as $x \to x_*$ for some $t \to t_*$ $\therefore$

$\lim_{x\to\infty} u_x(x,t) = \lim_{x\to\infty} A(1-\tanh^2(x)) = A\lim_{x\to\infty}(1-\tanh^2(x)) = 0$ $\therefore$

$\lim_{A\to\infty} u_x(x,t) = \lim_{A\to\infty} A(1-\tanh^2(x)) = \infty$ $\therefore$

$u_x(x,t) \to \infty$ as $A \to \infty$ $\therefore$

$\cancel{A}$ $x \to x_*$ and $t \to t_*$ as $A \to \infty$ $\therefore$

$\lim_{A\to\infty} x = x_*$, $\lim_{A\to\infty} t = t_*$ $\therefore$

as $A \to \infty$: $x \to \infty$, $t \to \infty$ $\therefore$

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$x_* = \infty$, $t \to \infty$ $\therefore$

$x_* = A$, $t_* = A$

A IS A CONSTANT

SEEN