—Q1a:—

$u_{xx} - 4u_{xy} + 3u_{yy} - u_x - u_y - 2u = 0$ ∴

$u_{xx} - 4u_{xy} + 3u_{yy} = u_x + u_y - 2u = au_{xx} + 2bu_{xy} + cu_{yy}$ ∴

$a = 1$, $b = -2$, $c = 3$ ∴ The discriminant is

$b^2 - ac = (-2)^2 - 1\times 3 = 4 - 3 = 1 > 0$ ∴ the equation is hyperbolic.

The characteristic equation is $a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0 =$

$1\left(\frac{dy}{dx}\right)^2 - 2(-2)\frac{dy}{dx} + 3 = \left(\frac{dy}{dx}\right)^2 + 4\frac{dy}{dx} + 3 = \left(\frac{dy}{dx} + 1\right)\left(\frac{dy}{dx} + 3\right) = 0$ ∴

$\frac{dy}{dx} = -1$, $\frac{dy}{dx} = -3$ ∴

$\int \frac{dy}{dx}\,dx = \int -1\,dx = \int 1\,dy = -x + C_1 = y$,

$\int \frac{dy}{dx}\,dx = \int -3\,dx = \int 1\,dy = -3x + C_2 = y$ ∴

$C_1 = y + x$, $C_2 = y + 3x$ ∴

The characteristic coordinates can be chosen as:

$\xi = x + y$, $\eta = 3x + y$.

— Q1b: —

$\xi = x+y$, $\eta = 3x+y$ $\therefore$

$\eta - 3x = y$ $\therefore$ $\xi = x + (\eta - 3x) = x + \eta - 3x = -2x + \eta$ $\therefore$ $2x = -\xi + \eta$ $\therefore$

$x = -\frac{1}{2}\xi + \frac{1}{2}\eta$ $\therefore$

$\eta - 3(-\frac{1}{2}\xi + \frac{1}{2}\eta) = y = \eta + \frac{3}{2}\xi - \frac{3}{2}\eta = \frac{3}{2}\xi - \frac{1}{2}\eta = y$ $\therefore$

$x_\xi = \frac{\partial}{\partial\xi}(-\frac{1}{2}\xi + \frac{1}{2}\eta) = -\frac{1}{2}$, $x_\eta = \frac{\partial}{\partial\eta}(-\frac{1}{2}\xi + \frac{1}{2}\eta) = \frac{1}{2}$,

$y_\xi = \frac{\partial}{\partial\xi}(\frac{3}{2}\xi - \frac{1}{2}\eta) = \frac{3}{2}$, $y_\eta = \frac{\partial}{\partial\eta}(\frac{3}{2}\xi - \frac{1}{2}\eta) = -\frac{1}{2}$ $\therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x$ $\therefore$

$\xi_x = \frac{\partial}{\partial x}(x+y) = 1$, $\xi_y = \frac{\partial}{\partial y}(x+y) = 1$,

$\eta_x = \frac{\partial}{\partial x}(3x+y) = 3$, $\eta_y = \frac{\partial}{\partial y}(3x+y) = 1$ $\therefore$

$u_x = 1u_\xi + 3u_\eta$, $u_y = u_\xi \xi_y + u_\eta \eta_y = 1u_\xi + 1u_\eta = u_\xi + u_\eta$ $\therefore$

$u_{xx} = \frac{\partial}{\partial x}(u_x) = \frac{\partial}{\partial x}(u_\xi + 3u_\eta) = \partial_x(u_\xi) + 3\partial_x(u_\eta) =$

$u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + 3u_{\xi\eta}\xi_x + 3u_{\eta\eta}\eta_x =$

$1u_{\xi\xi} + 3u_{\xi\eta} + 1\times 3u_{\xi\eta} + 3\times 3u_{\eta\eta} = u_{\xi\xi} + 6u_{\xi\eta} + 9u_{\eta\eta} = u_{xx}$

$u_{xy} = \frac{\partial}{\partial x}(u_y) = \frac{\partial}{\partial x}(u_\xi + u_\eta) = \partial_x(u_\xi) + \partial_x(u_\eta) =$

$u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x = 1u_{\xi\xi} + 3u_{\xi\eta} + 1u_{\xi\eta} + 3u_{\eta\eta} =$

$u_{\xi\xi} + 4u_{\xi\eta} + 3u_{\eta\eta}$

$u_{yy} = \partial_y(u_y) = \partial_y(u_\xi + u_\eta) = \partial_y(u_\xi) + \partial_y(u_\eta) =$

$u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y + u_{\xi\eta}\xi_y + u_{\eta\eta}\eta_y = 1u_{\xi\xi} + 1u_{\xi\eta} + 1u_{\xi\eta} + 1u_{\eta\eta} =$

$u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$ $\therefore$ Sub them all into PDE:

$LHS = u_{xx} - 4u_{xy} + 3u_{yy} - u_x - u_y - 2u =$

$u_{\xi\xi} + 6u_{\xi\eta} + 9u_{\eta\eta} - 4(u_{\xi\xi} + 4u_{\xi\eta} + 3u_{\eta\eta}) + 3(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) - u_x - u_y - 2u =$

$u_{\xi\xi} + 6u_{\xi\eta} + 9u_{\eta\eta} - 4u_{\xi\xi} - 16u_{\xi\eta} - 12u_{\eta\eta} + 3u_{\xi\xi} + 6u_{\xi\eta} + 3u_{\eta\eta} - u_x - u_y - 2u =$

$(1-4+3)u_{\xi\xi} + (6-16+6)u_{\xi\eta} + (9-12+3)u_{\eta\eta} - u_x - u_y - 2u =$

$0u_{\xi\xi} - 4u_{\xi\eta} + 0u_{\eta\eta} - (u_\xi + 3u_\eta) - (u_\xi + u_\eta) - 2u =$

$-4u_{\xi\eta} - u_\xi - 3u_\eta - u_\xi - u_\eta - 2u =$

$-4u_{\xi\eta} - 2u_\xi - 4u_\eta - 2u = 0 = RHS$ Is the canonical form of the equation

Q1C:

③

$u = e^{\lambda\xi+\mu\eta}v$ ∴

$u_\xi = \partial_\xi(e^{\lambda\xi+\mu\eta}v) = \lambda e^{\lambda\xi+\mu\eta}v + e^{\lambda\xi+\mu\eta}v_\xi$,

$u_\eta = \partial_\eta(e^{\lambda\xi+\mu\eta}v) = \mu e^{\lambda\xi+\mu\eta}v + e^{\lambda\xi+\mu\eta}v_\eta$ ∴

$u_{\xi\eta} = \partial_\xi(u_\eta) = \partial_\xi(\mu e^{\lambda\xi+\mu\eta}v) + \partial_\xi(e^{\lambda\xi+\mu\eta}v_\eta) =$

$\lambda\mu e^{\lambda\xi+\mu\eta}v + \mu e^{\lambda\xi+\mu\eta}v_\xi + \lambda e^{\lambda\xi+\mu\eta}v_\eta + e^{\lambda\xi+\mu\eta}v_{\xi\eta}$ ∴ Subbing into canonical form of the equation:

$LHS = -4u_{\xi\eta} - 2u_\xi - 4u_\eta - 2u =$

$-4(\lambda\mu v e^{\lambda\xi+\mu\eta} + \mu v_\xi e^{\lambda\xi+\mu\eta} + \lambda v_\eta e^{\lambda\xi+\mu\eta} + v_{\xi\eta}e^{\lambda\xi+\mu\eta}) - 2u_\xi - 4u_\eta - 2ve^{\lambda\xi+\mu\eta} =$

$e^{\lambda\xi+\mu\eta}[-4\lambda\mu v - 2v - 4\mu v_\xi - 4\lambda v_\eta - 4v_{\xi\eta}] - 2u_\xi - 4u_\eta =$

$e^{\lambda\xi+\mu\eta}[-4\lambda\mu v - 2v - 4\mu v_\xi - 4\lambda v_\eta - 4v_{\xi\eta}] - 2(\lambda v e^{\lambda\xi+\mu\eta} + v_\xi e^{\lambda\xi+\mu\eta}) - 4u_\eta =$

$e^{\lambda\xi+\mu\eta}[-4\lambda\mu v - 2v - 4\mu v_\xi - 4\lambda v_\eta - 4v_{\xi\eta} - 2\lambda v - 2v_\xi] - 4u_\eta =$

$e^{\lambda\xi+\mu\eta}[(-4\lambda\mu - 2 - 2\lambda)v + (-4\mu - 2)v_\xi - 4\lambda v_\eta - 4v_{\xi\eta}] - 4(\mu v e^{\lambda\xi+\mu\eta} + v_\eta e^{\lambda\xi+\mu\eta}) =$

$e^{\lambda\xi+\mu\eta}[(-4\lambda\mu - 2 - 2\lambda)v + (-4\mu - 2)v_\xi - 4\lambda v_\eta - 4v_{\xi\eta} - 4\mu v - 4v_\eta] =$

$e^{\lambda\xi+\mu\eta}[(-4\lambda\mu - 2 - 2\lambda - 4\mu)v + (-4\mu - 2)v_\xi + (-4\lambda - 4)v_\eta - 4v_{\xi\eta}] = 0 = RHS$

∴ Let $-4\mu - 2 = 0$, $-4\lambda - 4 = 0$ ∴ $-2 = 4\mu$, $-4 = 4\lambda$ ∴

$\mu = -\frac{1}{2}$, $-1 = \lambda$ ∴ $-4\lambda\mu - 2 - 2\lambda - 4\mu = -4(-1)(\frac{-1}{2}) - 2 - 2(-1) - 4(-\frac{1}{2}) = 0$ ∴

Sub back into equation ∴ $LHS = e^{-\xi-\frac{1}{2}\eta}[0v + 0v_\xi + 0v_\eta - 4v_{\xi\eta}] =$

$-4v_{\xi\eta}e^{-\xi-\frac{1}{2}\eta} = -4e^{-\xi-\frac{1}{2}\eta}v_{\xi\eta} = 0 = RHS$ ∴

$e^{-\xi-\frac{1}{2}\eta}v_{\xi\eta} = 0$ and $e^{-\xi-\frac{1}{2}\eta} \neq 0$ ∴

$v_{\xi\eta} = 0$ ∴

$\frac{\partial^2 v}{\partial\xi\partial\eta} = 0$ is the simplist possible form of the equation. for $\mu = -\frac{1}{2}$, $\lambda = -1$

— Q2a: —

$u_{xx} - 4u_{xy} + 4u_{yy} - u_x - u_y + u = 0 = au_{xx} + 2bu_{xy} + cu_{yy} - u_x - u_y + u$

$\therefore a = 1,\ b = -2,\ c = 4$ $\therefore$ The discriminant is

$b^2 - ac = (-2)^2 - 1 \times 4 = 4 - 4 = 0$ $\therefore$ The equation is parabolic. $\therefore$

The characteristic equation is: $a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0 =$

$1\left(\frac{dy}{dx}\right) - 2(-2)\left(\frac{dy}{dx}\right) + 4 = \left(\frac{dy}{dx}\right)^2 + 4\frac{dy}{dx} + 4 = \left(\frac{dy}{dx} + 2\right)\left(\frac{dy}{dx} + 2\right) = \left(\frac{dy}{dx} + 2\right)^2 = 0$

$\therefore$ double repeated root of $\frac{dy}{dx} = -2$ $\therefore$

$\int \frac{dy}{dx}\,dx = \int -2\,dx = \int 1\,dy = -2x + C = y$ $\therefore$

$C = y + 2x$ $\therefore$

$\xi = 2x + y$ $\therefore$ The second characteristic coordinates can be chosed arbitrarily $\therefore$ ~~Let $\eta = y$~~ as long as it is independent of ξ $\therefore$ Let $\eta = y$ $\therefore$

The characteristic coordinates can be chosen as:

$\xi = 2x + y,\ \eta = y$

— Q2b: —

$\xi = 2x + y$, $\eta = y$ $\therefore$

$\xi_x = 2$, $\xi_y = 1$, $\eta_x = 0$, $\eta_y = 1$ $\therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x = 2u_\xi + 0u_\eta = 2u_\xi$,

$u_y = u_\xi \xi_y + u_\eta \eta_y = 1u_\xi + 1u_\eta = u_\xi + u_\eta$,

$u_{xx} = \partial_x(u_x) = \partial_x(2u_\xi) = 2\,\partial_x(u_\xi) = 2u_{\xi\xi}\xi_x + 2u_{\xi\eta}\eta_x =$

$2 \times 2u_{\xi\xi} + 2 \times 0\, u_{\xi\eta} = 4u_{\xi\xi}$,

$u_{xy} = \partial_x(u_y) = \partial_y(u_x) = \partial_y(2u_\xi) = 2\partial_y(u_\xi) = 2u_{\xi\xi}\xi_y + 2u_{\xi\eta}\eta_y =$

$2 \times 1u_{\xi\xi} + 2 \times 1u_{\xi\eta} = 2u_{\xi\xi} + 2u_{\xi\eta}$,

$u_{yy} = \partial_y(u_y) = \partial_y(u_\xi + u_\eta) = \partial_y(u_\xi) + \partial_y(u_\eta) =$

$u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y + u_{\xi\eta}\xi_y + u_{\eta\eta}\eta_y = 1u_{\xi\xi} + 1u_{\xi\eta} + 1u_{\eta\eta} + 1u_{\eta\eta} =$

$\cancel{u_{\xi\eta} + 2}$ $u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$ $\therefore$ Sub them all into PDE:

$LHS = u_{xx} - 4u_{xy} + 4u_{yy} - u_x - u_y + u =$

$4u_{\xi\xi} - 4(2u_{\xi\xi} + 2u_{\xi\eta}) + 4(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) - 2u_\xi - (u_\xi + u_\eta) + u =$

$4u_{\xi\xi} - 8u_{\xi\xi} - 8u_{\xi\eta} + 4u_{\xi\xi} + 8u_{\xi\eta} + 4u_{\eta\eta} - 2u_\xi - u_\xi - u_\eta + u =$

$(4 - 8 + 4)u_{\xi\xi} + (-8 + 8)u_{\xi\eta} + (4)u_{\eta\eta} - 3u_\xi - u_\eta + u =$

$(0)u_{\xi\xi} + (0)u_{\xi\eta} + 4u_{\eta\eta} - 3u_\xi - u_\eta + u =$

$4u_{\eta\eta} - 3u_\xi - u_\eta + u = 0 = RHS$ is the canonical form of the equation

— Q2C: —

$u = e^{\lambda\xi+\mu\eta}v$ $\therefore$

$u_\xi = \partial_\xi(e^{\lambda\xi+\mu\eta}v) = \lambda e^{\lambda\xi+\mu\eta}v + e^{\lambda\xi+\mu\eta}v_\xi = e^{\lambda\xi+\mu\eta}(\lambda v + v_\xi)$

$u_\eta = \partial_\eta(e^{\lambda\xi+\mu\eta}v) = \mu e^{\lambda\xi+\mu\eta}v + e^{\lambda\xi+\mu\eta}v_\eta = e^{\lambda\xi+\mu\eta}(\mu v + v_\eta)$ $\therefore$

$u_{\eta\eta} = \partial_\eta\left[e^{\lambda\xi+\mu\eta}(\mu v + v_\eta)\right] = (\mu v + v_\eta)\frac{\partial}{\partial\eta}(e^{\lambda\xi+\mu\eta}) + e^{\lambda\xi+\mu\eta}\frac{\partial}{\partial\eta}(\mu v + v_\eta) =$

$(\mu v + v_\eta)\mu e^{\lambda\xi+\mu\eta} + e^{\lambda\xi+\mu\eta}(\mu v_\eta + v_{\eta\eta}) = e^{\lambda\xi+\mu\eta}(\mu^2 v + \mu v_\eta + \mu v_\eta + v_{\eta\eta}) =$

$e^{\lambda\xi+\mu\eta}(\mu^2 v + 2\mu v_\eta + v_{\eta\eta})$ $\therefore$ Subbing into Canonical form of the equation: LHS $= 4u_{\eta\eta} - 3u_\xi - u_\eta + u =$

$4e^{\lambda\xi+\mu\eta}(\mu^2 v + 2\mu v_\eta + v_{\eta\eta}) - 3e^{\lambda\xi+\mu\eta}(\lambda v + v_\xi) - e^{\lambda\xi+\mu\eta}(\mu v + v_\eta) + e^{\lambda\xi+\mu\eta}v$

$= e^{\lambda\xi+\mu\eta}(4\mu^2 v + 8\mu v_\eta + 4v_{\eta\eta} - 3\lambda v - 3v_\xi - \mu v - v_\eta + v) =$

$e^{\lambda\xi+\mu\eta}\left[(4\mu^2 - 3\lambda - \mu + 1)v + (8\mu - 1)v_\eta + (-3)v_\xi + 4v_{\eta\eta}\right] = 0 = \text{RHS}$ $\therefore$

Let $8\mu - 1 = 0$, $4\mu^2 - 3\lambda - \mu + 1 = 0$ $\therefore$

$8\mu = 1 \therefore \mu = \frac{1}{8} \therefore 4\left(\frac{1}{8}\right)^2 - 3\lambda - \frac{1}{8} + 1 = 0 = \frac{1}{16} - 3\lambda + \frac{7}{8} = \frac{15}{16} - 3\lambda \therefore$

$3\lambda = \frac{15}{16}$ $\therefore$ $\lambda = \frac{5}{16}$ $\therefore$ Sub back into equation:

$\text{LHS} = e^{\frac{5}{16}\xi + \frac{1}{8}\eta}\left[(0)v + (0)v_\eta - 3v_\xi + 4v_{\eta\eta}\right] =$

$e^{\frac{5}{16}\xi + \frac{1}{8}\eta}\left[-3v_\xi + 4v_{\eta\eta}\right] = 0 = \text{RHS}$ $\therefore$

$e^{\frac{5}{16}\xi + \frac{1}{8}\eta} \neq 0$ $\therefore$

$4v_{\eta\eta} - 3v_\xi = 0$ is the simplist possible form of the equation, for $\mu = \frac{1}{8}$, $\lambda = \frac{5}{16}$.

— Q3: —

$u_t - u_{xx} = f(x,t)$ ∴

if u_1, u_2 are solutions to the problem ∴

$u_{1_t} - u_{1_{xx}} = f(x,t)$, $u_1(x,0) = \phi(x)$, $u_1(-l,t) = u_1(l,t)$,

$u_{1_x}(-l,t) = u_{1_x}(l,t)$, $-l \leq x \leq l$, $t \geq 0$.

$u_{2_t} - u_{2_{xx}} = f(x,t)$, $u_2(x,0) = \phi(x)$, $u_2(-l,t) = u_2(l,t)$,

$u_{2_x}(-l,t) = u_{2_x}(l,t)$, $-l \leq x \leq l$, $t \geq 0$ ∴

Let $w(x,t) = W = u_1(x,t) - u_2(x,t) = u_1 - u_2$ ∴

$W_t - W_{xx} = f(x,t) - f(x,t) = \partial_t(W) - \partial_{xx}(W) =$

$\partial_t(u_1 - u_2) - \partial_{xx}(u_1 - u_2) = u_{1_t} - u_{2_t} - u_{1_{xx}} - u_{2_{xx}} =$

$(u_{1_t} - u_{1_{xx}}) - (u_{2_t} - u_{2_{xx}}) = (f(x,t)) - (f(x,t)) = 0$, $-l \leq x \leq l$, $t \geq 0$,

$w(x,0) = u_1(x,0) - u_2(x,0) = \phi(x) - \phi(x) = 0$,

$w(-l,t) = u_1(-l,t) - u_2(-l,t) = u_1(l,t) - u_2(l,t) = w(l,t)$,

$W_x(-l,t) = u_{1_x}(-l,t) - u_{2_x}(-l,t) = u_{1_x}(l,t) - u_{2_x}(l,t) = W_x(l,t)$ ∴

$W_t - W_{xx} = 0$ ∴ $W(W_t - W_{xx}) = W \times 0 = WW_t - WW_{xx} = 0$ and

$\frac{1}{2}\frac{\partial}{\partial t}(W^2) = \frac{1}{2} \times 2W\partial_t(W) = WW_t$,

$\frac{\partial}{\partial x}(WW_x) = W_x\frac{\partial}{\partial x}(W) + W\frac{\partial}{\partial x}(W_x) = W_xW_x + WW_{xx}$ ∴

$WW_{xx} = \frac{\partial}{\partial x}(WW_x) - (\frac{\partial}{\partial x}W)^2$ ∴

$\frac{1}{2}\frac{\partial}{\partial t}(W^2) - \frac{\partial}{\partial x}(WW_x) + (\frac{\partial}{\partial x}W)^2 = 0$ ∴ $\int_{-l}^{l} WW_t - WW_{xx}\,dx =$

$\int_{-l}^{l} \frac{1}{2}\frac{\partial}{\partial t}(W^2) - \frac{\partial}{\partial x}(WW_x) + (\frac{\partial}{\partial x}W)^2 dx = \int_{-l}^{l} 0\,dx = 0 =$

$\frac{1}{2}\int_{-l}^{l}\frac{\partial}{\partial t}(W^2)\,dx - \int_{-l}^{l} \cancel{WW_x}\frac{\partial}{\partial x}(WW_x)\,dx + \int_{-l}^{l}(\frac{\partial}{\partial x}W)^2 dx =$

$\frac{1}{2}\frac{\partial}{\partial t}\int_{-l}^{l} W^2\,dx - \left[WW_x\right]_{-l}^{l} + \int_{-l}^{l}(\frac{\partial}{\partial x}W)^2 dx$ ∴

$-\left[WW_x\right]_{-l}^{l} = -\left[W(l,t)W_x(l,t) - W(-l,t)W_x(-l,t)\right] =$

$-\left[W(l,t)W_x(l,t) - W(l,t)W_x(l,t)\right] = -0 = 0$ ∴

$\frac{1}{2}\int_{-l}^{l}\frac{\partial}{\partial t}(W^2)\,dx - 0 + \int_{-l}^{l}(\frac{\partial}{\partial x}W)^2 dx = 0 = \frac{1}{2}\frac{\partial}{\partial t}\int_{-l}^{l} W^2 dx + \int_{-l}^{l}(\frac{\partial}{\partial x}W)^2 dx$ ∴

— Q3 Continued: —

Let $E(t) = \int_{-l}^{l} \frac{1}{2} W^2(x,t)\,dx = \frac{1}{2}\int_{-l}^{l} (W(x,t))^2\,dx = \frac{1}{2}\int_{-l}^{l} W^2\,dx$ ∴

$W^2 \geq 0$ ∴ $E(t) = \frac{1}{2}\int_{-l}^{l} W^2\,dx \geq 0$ ∴

$\frac{dE(t)}{dt} = \frac{\partial E(t)}{\partial t} = \frac{\partial}{\partial t}\left[\frac{1}{2}\int_{-l}^{l} W^2\,dx\right] = \frac{1}{2}\int_{-l}^{l} \frac{\partial}{\partial t}(W^2)\,dx = \frac{1}{2}\int_{-l}^{l} 2WW_t\,dx =$

$\int_{-l}^{l} WW_t\,dx$, $W_t = W_{xx}$ ∴

$\frac{dE}{dt} = \frac{dE(t)}{dt} = \int_{-l}^{l} WW_t\,dx = \int_{-l}^{l} WW_{xx}\,dx = \left[WW_x\right]_{-l}^{l} - \int_{-l}^{l} W_xW_x\,dx =$

$0 - \int_{-l}^{l} (W_x)^2\,dx = -\int_{-l}^{l} (W_x)^2\,dx$ ∴ $(W_x)^2 \geq 0$ ∴ $\int_{-l}^{l} (W_x)^2 \geq 0$ ∴

$\frac{dE}{dt} = -\int_{-l}^{l} (W_x)^2\,dx \leq 0$ ∴ as t increases, E decreases ∴

$E(t_2) \geq E(t_1)$ for $t_1 \leq t_2$ ∴

~~$E(t=0)$~~ $E(0) \geq E(t)$ ∴

$E(0) = \int_{-l}^{l} \frac{1}{2} W^2(x,0)\,dx \geq \int_{-l}^{l} \frac{1}{2} W^2(x,t)\,dx = E(t)$ ∴ $\int_{-l}^{l} W^2(x,0)\,dx \geq \int_{-l}^{l} W^2(x,t)\,dx$

and $W(x,0) = 0$ ∴ $W^2(x,0) = 0$ ∴ $\int_{-l}^{l} 0\,dx \geq \int_{-l}^{l} W^2(x,t)\,dx \geq 0$ ∴

$0 \geq \int_{-l}^{l} W^2(x,t)\,dx \geq 0$ ∴ $\int_{-l}^{l} W^2(x,t)\,dx = 0$ ∴

$\int_{-l}^{l} W(x,t)\,dx = 0$

$\int_{-l}^{l} W^2(x,t)\,dx = \int_{-l}^{l} \left[U_1(x,t) - U_2(x,t)\right]^2\,dx$,

$\int_{-l}^{l} W(x,t)\,dx = \int_{-l}^{l} U_1(x,t) - U_2(x,t)\,dx = \int_{-l}^{l} U_1(x,t)\,dx - \int_{-l}^{l} U_2(x,t)\,dx = 0$ ∴

$\int_{-l}^{l} U_1(x,t)\,dx = \int_{-l}^{l} U_2(x,t)\,dx$ ∴ $W = 0 = W(x,t) = U_1 - U_2 = U_1(x,t) - U_2(x,t)$

∴ $U_1 = U_2 = U_1(x,t) = U_2(x,t)$ ∴ The solution to the problem is unique.

Q4:

Let f, ϕ be p-periodic in x for $p>0$ ∴

$f(x+p,t)=f(x,t)$, $\phi(x+p)=\phi(x)$ ∴

$u_t - u_{xx} = f(x,t) = f(x+p,t) = u_t(x,t) - u_{xx}(x,t)$ ∴

$u_t(x+p,t) - u_{xx}(x+p,t) = f(x+p,t) = f(x,t) = u_t(x,t) - u_{xx}(x,t)$

$u(x,0)=\phi(x)=\phi(x+p)$ ∴

$u(x+p,0) = \phi(x+p) = \phi(x) = u(x,0)$ and $u(x,t)-u(x+p,t) = u(x,t)-u(x+p,t)$ ∴

$u_t(x,t)-u_t(x+p,t) = u_{xx}(x,t) - u_{xx}(x+p,t)$ ∴

$\frac{\partial}{\partial t}\left[u(x,t)-u(x+p,t)\right] = \partial_{xx}\left[u(x,t)-u(x+p,t)\right]$ ∴

$$\int \partial_t\left[u(x,t)-u(x+p,t)\right]\partial t = \iint \partial_{xx}\left[u(x,t)-u(x+p,t)\right]\partial x\,\partial x$$ ∴

$\left[u(x,t)-u(x+p,t)\right]t + C_2 = \left[u(x,t)-u(x+p,t)\right]x^2$ ∴

$\left[t-x^2\right]\left[u(x,t)-u(x+p,t)\right] = -C_2 = C_1$; C_2, C_1 : constants ∴

at $t=0$: $\left[0-x^2\right]\left[u(x,0)-u(x+p,0)\right] = C_1 = -x^2\left[\phi(x)-\phi(x)\right] =$

$-x^2\left[0\right] = 0 = C_1$ ∴

$\left[t-x^2\right]\left[u(x,t)-u(x+p,t)\right]=0$ $\forall x\in\mathbb{R}, \forall t\ge 0$ ∴

$u(x,t)-u(x+p,t)=0$ ∴

$u(x,t)=u(x+p,t)$ for $p>0$, $\forall x\in\mathbb{R}$, $\forall t \ge 0$ ∴

The solution is p-periodic in x with period $p>0$.

— Q5a: —

$w_t - w_{xx} = 0$ ∴

Let $w = u(x,t) + x$ ∴ $w - x = u$ ∴

$u_t = w_t$, $u_x = w_x - 1$ ∴ $u_{xx} = w_{xx}$ ∴

$u_t - u_{xx} = w_t - w_{xx} = 0$ $\quad -\pi \le x \le \pi$,

$u(x,0) = w(x,0) - x = 0 - x = -x$,

~~$u(\pi,t) = u(-\pi,t) = w(\pi,t) - \pi = w(-\pi,t) - x$~~

$u(\pi,t) = w(\pi,t) - \pi$, $u(-\pi,t) = w(-\pi,t) - (-\pi) = w(-\pi,t) + \pi$ ∴

$u(\pi,t) - u(-\pi,t) = w(\pi,t) - \pi - w(-\pi,t) - \pi = 2\pi - 2\pi = 0$,

$u_x(\pi,t) = w_x(\pi,t) - 1$, $u_x(-\pi,t) = w_x(-\pi,t) - 1$ ∴

$u_x(\pi,t) - u_x(-\pi,t) = w_x(\pi,t) - 1 - w_x(-\pi,t) + 1 = w_x(\pi,t) - w_x(-\pi,t) = 0$ ∴

Let $u = X(x)T(t)$ ∴ $u_t = X(x)T'(t)$, $u_x = X'(x)T(t)$ ∴

$u_{xx} = X''(x)T(t)$ ∴ $X(x)T'(t) - X''(x)T(t) = 0$ ∴ $X(x)T'(t) = X''(x)T(t)$ ∴

$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} = -\lambda =$ constant ∴ $X''(x) = -\lambda X(x)$, $T'(t) = -\lambda T(t)$ ∴

$X''(x) + \lambda X(x) = 0$, $T'(t) + \lambda T(t) = 0$ ∴

For $X'' + \lambda X = 0$: ODE for $X(x)$ has BC $X(\pi) - X(-\pi) = 0$,

$X'(\pi) - X'(-\pi) = 0$ ∴ $X'(\pi) = X'(-\pi)$ ∴

Case 1: $\lambda < 0$ ∴ $\lambda = -a^2 < 0$ for $a \in \mathbb{R}$ ∴ $X'' - a^2 X = 0$ ∴

$X = X(x) = Ae^{ax} + Be^{-ax}$ ∴ from BCs:

$X(\pi) = Ae^{a\pi} + Be^{-a\pi}$, $X(-\pi) = Ae^{-a\pi} + Be^{a\pi}$ ∴

$X(\pi) - X(-\pi) = Ae^{a\pi} + Be^{-a\pi} - Ae^{-a\pi} - Be^{a\pi} = 0 = (A - B)e^{a\pi} + (-A + B)e^{-a\pi}$

$X'(x) = Aae^{ax} - Bae^{-ax}$ ∴ $X'(\pi) = Aae^{a\pi} - Bae^{-a\pi}$,

$X'(-\pi) = Aae^{-a\pi} - Bae^{a\pi}$ ∴ $Aae^{a\pi} - Bae^{-a\pi} - Aae^{-a\pi} + Bae^{a\pi} = 0 =$

$(A + B)e^{a\pi} - (A + B)e^{-a\pi} = A(e^{a\pi} - e^{-a\pi}) + B(e^{a\pi} - e^{-a\pi}) = 0$ and

$e^x > 0$ ∴ $e^{a\pi} \neq e^{-a\pi}$ ∴ $A + B = 0$ ∴ $B = -A$ ∴

$0 = (A + A)e^{a\pi} + (-A - A)e^{-a\pi} = 2Ae^{a\pi} - 2Ae^{-a\pi} = 0 = Ae^{a\pi} - Ae^{-a\pi} =$

$A(e^{a\pi} - e^{-a\pi}) = 0 = A$ ∴ $B = 0$

Case 2: $\lambda = 0$ ∴ $X'' + 0X = 0 = X''$ ∴ $X' = B$ ∴ $X = A + Bx = X(x)$ ∴

$X(\pi) = A + B\pi$, $X(-\pi) = A - B\pi$ ∴ $X(\pi) - X(-\pi) = A + B\pi - A + B\pi = 0 = 2\pi B$ ∴

$B = 0$ ∴ $X(x) = A$ ∴ $X'(x) = 1$ ∴ A is arbitrary .

— Q5a Continued: —

Case 3: $\lambda > 0$ $\therefore$ $\lambda = a^2 > 0$ $a \in \mathbb{R}$ $\therefore$ $X'' + a^2 X = 0$ $\therefore$

- $X(x) = A\cos ax + B\sin ax$ $\therefore$ $X'(x) = -Aa\sin ax + Ba\cos ax$ $\therefore$

$X'(\pi) = -Aa\sin a\pi + Ba\cos a\pi =$; $X'(-\pi) = -Aa\sin(-a\pi) + Ba\cos(-a\pi) =$

$Aa\sin(a\pi) + Ba\cos a\pi$ $\therefore$ $X'(\pi) - X'(-\pi) =$

$-Aa\sin a\pi + Ba\cos a\pi - Aa\sin(a\pi) - Ba\cos a\pi = -2Aa\sin(a\pi) = 0$ $\therefore$

$A\sin(a\pi) = 0$ $\therefore$ $X(\pi) = A\cos a\pi + B\sin a\pi$,

$X(-\pi) = A\cos(-a\pi) + B\sin(-a\pi) = A\cos a\pi - B\sin a\pi$ $\therefore$ $X(\pi) - X(-\pi) =$

$A\cos a\pi + B\sin a\pi - A\cos a\pi + B\sin a\pi = 2B\sin a\pi = 0 = B\sin a\pi$ $\therefore$

$B = 0$ or $a = n$, $n = 1, 2, 3, \cdots$ $\therefore$ $n \in \mathbb{N}$ $\therefore$

- $\lambda = n^2$, $n \in \mathbb{N}$ $\therefore$ $X_n(x) = \sin(nx)$, $n \in \mathbb{N}$ $\therefore$

$T'(t) + n^2 T(t) = 0$ $\therefore$ $\frac{T'(t)}{T(t)} = -n^2$ $\therefore$ $\int \frac{T'(t)}{T(t)} dt = \int -n^2 dt = \ln|T(t)| =$

$-n^2 t + C_3$ $\therefore$ $|T(t)| = e^{-n^2 t + C_3} = e^{C_3} e^{-n^2 t} = C_2 e^{-n^2 t}$ $\therefore$

$T(t) = Ce^{-n^2 t} = T_n(t)$ C is ~~constant~~ arbitrary constant

and IC: $w(x,0) = 0$ $\therefore$ $u(x,0) = w(x,0) - x = 0 - x = -x$ $\therefore$

$u(x,t) = X(x)T(t) = A_n \sin(nx) e^{-n^2 t}$ $\therefore$

From Forier Series: $l = \pi$ $\therefore$ $A_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin(nx)dx = \frac{1}{\pi}\int_{-\pi}^{\pi} -x\sin(nx)dx$

$\therefore$ $-x$ is an odd function, so is $\sin(nx)$ $\therefore$

- $A_n = \frac{2}{\pi}\int_0^{\pi} -x\sin(nx)dx = \frac{2}{\pi}\left[-x\left(-\frac{\cos(nx)}{n}\right)\right]_0^{\pi} - \frac{2}{\pi}\int_0^{\pi} -\left(-\frac{\cos(nx)}{n}\right)dx =$

$\frac{2}{\pi}\left[+\frac{\pi\cos(n\pi)}{n}\right] - \frac{2}{\pi}\int_0^{\pi}\frac{1}{n}\cos(nx)\,dx = \frac{2}{n}\cos(n\pi) - \frac{2}{\pi}\left[\frac{1}{n^2}\sin(nx)\right]_0^{\pi} =$

$\frac{2}{n}\cos(n\pi) - \frac{2}{\pi n^2}\left[\sin(n\pi) - \sin 0\right] = \frac{2}{n}\cos(n\pi) - \frac{2}{\pi n^2}[0] =$

$\frac{2}{n}\cos(n\pi) = \frac{2}{n}(-1)^n$ $\therefore$

$u(x,t) = \sum_{n=1}^{\infty} \frac{2}{n}(-1)^n \sin(nx) e^{-n^2 t}$ $\therefore$

$w(x,t) = x + u(x,t) = x + \sum_{n=1}^{\infty}\frac{2}{n}(-1)^n \sin(nx) e^{-n^2 t}$

Q5b:

$$w(x,t) = x + \sum_{n=1}^{\infty} \frac{2}{n}(-1)^n \sin(nx) e^{-n^2 t} \therefore$$ approximate

Fist term Solution is $w(x,t) \approx x \therefore$

approximate three terms Solution is

$$w(x,t) \approx x + \sum_{n=1}^{2} \frac{2}{n}(-1)^n \sin(nx) e^{-n^2 t} =$$

$$x + \frac{2}{1}(-1)^1 \sin(x) e^{-1^2 t} + \frac{2}{2}(-1)^2 \sin(2x) e^{-2^2 t} =$$

$$x - 2\sin(x) e^{-t} + \sin(2x) e^{-4t}$$

— Q6: —

$u_t - Du_{xx} = 0$ ∴

Let $w(x,t)$ be the solution of $w_t - Dw_{xx} = 0$, $-\infty < x < \infty$

ICs: $w(x,0) = \phi(x)$, BCs: $w(0,t) = 0$ ∴

$u(x,t) = w(x,t)$, $0 \le x < \infty$ ∴ by ~~d'Alembert's~~ formula:

~~$w(x,t) = \frac{1}{2}[\phi(x+\sqrt{D}t) + \phi(x-\sqrt{D}t)]$~~ and derivation of the solution,

the solution to the new problem is:

$$w(x,t) = \frac{1}{\sqrt{4D\pi t}} \int_{-\infty}^{\infty} w(s,0) e^{-(x-s)^2/(4Dt)} \, ds =$$

$$\frac{1}{2\sqrt{D\pi t}} \int_{-\infty}^{\infty} \phi(s) e^{-(x-s)^2/(4Dt)} \, ds$$ by IC $w(s,0) = \phi(s)$ ∴

$$w(x,t) = \frac{1}{2\sqrt{D\pi t}} \int_{-\infty}^{\infty} \phi(s) e^{-(x-s)^2/(4Dt)} \, ds =$$

$$\frac{1}{2\sqrt{D\pi t}} \left[\int_{-\infty}^{a} \phi(s) e^{-(x-s)^2/(4Dt)} \, ds + \int_{a}^{b} \phi(s) e^{-(x-s)^2/(4Dt)} \, ds + \int_{b}^{\infty} \phi(s) e^{-(x-s)^2/(4Dt)} \, ds \right]$$

∴ $\phi(s) = \begin{cases} 1, & a \le s \le b, \\ 0 & \text{otherwise} \end{cases}$ ∴

for $-\infty < s < a$: $\phi(s) = 0$, for $a \le s \le b$: $\phi(s) = 1$,

for $b < s < \infty$: $\phi(s) = 0$ ∴

$$w(x,t) = \frac{1}{2\sqrt{D\pi t}} \left[\int_{-\infty}^{a} 0 e^{-(x-s)^2/(4Dt)} \, ds + \int_{a}^{b} 1 e^{-(x-s)^2/(4Dt)} \, ds + \int_{b}^{\infty} 0 e^{-(x-s)^2/(4Dt)} \, ds \right]$$

$$= \frac{1}{2\sqrt{D\pi t}} \left[\int_{-\infty}^{a} 0 \, ds + \int_{a}^{b} 1 e^{-(x-s)^2/(4Dt)} \, ds + \int_{b}^{\infty} 0 \, ds \right] = \frac{1}{2\sqrt{D\pi t}} \left[0 + \int_{a}^{b} e^{-(x-s)^2/(4Dt)} \, ds + 0 \right]$$

$= \frac{1}{2\sqrt{D\pi t}} \int_{a}^{b} e^{-(x-s)^2/(4Dt)} \, ds$ ∴ if $x - s = \sqrt{4Dt}\, z = 2\sqrt{Dt}\, z$ ∴ $x - 2\sqrt{Dt}\, z = s$

∴ $\frac{ds}{dz} = -2\sqrt{Dt}$ ∴ $ds = -2\sqrt{Dt}\, dz$ and when $s = a$: $\frac{x-s}{2\sqrt{Dt}} = z$ ∴

$\frac{x-a}{2\sqrt{Dt}} = z$, when $s = b$: $\frac{x-b}{2\sqrt{Dt}} = z$ ∴

$$w(x,t) = \int_{a}^{b} \frac{1}{2\sqrt{D\pi t}} e^{-(x-s)^2/(4Dt)} \, ds =$$

Q6 Continued:

$$\int_{\frac{x-a}{2\sqrt{Dt}}}^{\frac{x-b}{2\sqrt{Dt}}} \frac{1}{2\sqrt{Dt}\sqrt{\pi}} e^{-(2\sqrt{Dt}Z)^2/(4Dt)} (-2\sqrt{Dt})\, dZ =$$

$$-\int_{\frac{x-a}{2\sqrt{Dt}}}^{(x-b)/(2\sqrt{Dt})} \frac{1}{\sqrt{\pi}} e^{-4DtZ^2/(4Dt)}\, dZ =$$

$$-\frac{1}{\sqrt{\pi}} \int_{(x-a)/(2\sqrt{Dt})}^{(x-b)/(2\sqrt{Dt})} e^{-z^2}\, dz =$$

$$-\frac{1}{\sqrt{\pi}} \left[\int_0^{(x-b)/(2\sqrt{Dt})} e^{-z^2}\, dz - \int_0^{(x-a)/(2\sqrt{Dt})} e^{-z^2}\, dz \right] =$$

$$-\frac{1}{\sqrt{\pi}} \left[\frac{\sqrt{\pi}}{2} \operatorname{erf}\left((x-b)/(2\sqrt{Dt})\right) - \frac{\sqrt{\pi}}{2} \operatorname{erf}\left(\frac{(x-a)}{2\sqrt{Dt}}\right) \right] = w(x,t) =$$

$$\frac{1}{2}\left[-\operatorname{erf}\left(\frac{x-b}{2\sqrt{Dt}}\right) + \operatorname{erf}\left(\frac{x-a}{2\sqrt{Dt}}\right) \right] = u(x,t) \text{ for } 0 \le x < \infty$$

— Q7: —

$u_t - Du_{xx} + ktu = 0$ $\therefore$

- Let $u(x,t) = V(x,t)e^{-kt^2/2}$, $x \in \mathbb{R}$ $\therefore$ $u = Ve^{-kt^2/2}$ $\therefore$

$u(x,0) = u(x,t=0) = V(x,0)e^{-k(0)^2/2} = V(x,0)e^0 = V(x,0)\,1 = V(x,0) = \phi(x)$

$\therefore u_t = \partial_t(V(x,t)e^{-kt^2/2}) = e^{-kt^2/2}V_t - ktVe^{-kt^2/2} = e^{-kt^2/2}(V_t - ktV)$,

$u_x = \partial_x(Ve^{-kt^2/2}) = e^{-kt^2/2}V_x$ $\therefore$ $u_{xx} = \partial_x(e^{-kt^2/2}V_x) = e^{-kt^2/2}V_{xx}$ $\therefore$

Sub into equation: $RHS = u_t - Du_{xx} + ktu =$

$e^{-kt^2/2}(V_t - ktV) - De^{-kt^2/2}V_{xx} + ktVe^{-kt^2/2} = 0 = RHS$ $\therefore$

$e^t > 0$ $\therefore$ $e^{-kt^2/2} \neq 0$ $\therefore$

$V_t - ktV - DV_{xx} + ktV = 0 = V_t - DV_{xx} = 0$, $x \in \mathbb{R}$ $\therefore$

- by Formula:

$$V(x,t) = \frac{1}{\sqrt{4D\pi t}} \int_{-\infty}^{\infty} V(s,0)e^{-(x-s)^2/(4Dt)}\,ds$$

and $V(x,0) = \phi(x)$ $\therefore$ $V(s,0) = \phi(s)$ $\therefore$

$$V(x,t) = \frac{1}{\sqrt{4D\pi t}} \int_{-\infty}^{\infty} \phi(s)e^{-(x-s)^2/(4Dt)}\,ds \quad \text{and}$$

$u(x,t) = V(x,t)e^{-kt^2/2} =$

$$\frac{1}{\sqrt{4D\pi t}} e^{-kt^2/2} \int_{-\infty}^{\infty} \phi(s)e^{-(x-s)^2/(4Dt)}\,ds = u(x,t) \quad , x \in \mathbb{R} \quad \therefore$$

$$u(x,t) = \frac{1}{\sqrt{4D\pi t}} \int_{-\infty}^{\infty} \phi(s)e^{\frac{-(x-s)^2}{4Dt} - \frac{kt^2}{2}}\,ds \quad , x \in \mathbb{R} \quad \therefore$$

$$u(x,t) = \frac{1}{\sqrt{4D\pi t}} \int_{-\infty}^{\infty} \phi(s)e^{\frac{-(x-s)^2 - 2Dkt^3}{4Dt}}\,ds \quad , x \in \mathbb{R}$$

— Q8: —

$u_t = D(u_{rr} + \frac{2}{r}u_r)$ ∴

Let $u(r,t) = \frac{1}{r}v(r,t) = u = \frac{1}{r}v$. $0 \le r \le R$ ∴

$u_t = \frac{1}{r}v_t$, $u_r = \frac{1}{r}v_r - \frac{1}{r^2}v$, $u_{rr} = \frac{1}{r}v_{rr} - \frac{2}{r^2}v_r + \frac{2}{r^3}v$ ∴

Sub into PDE: LHS $= u_t = \frac{1}{r}v_t = D(u_{rr} + \frac{2}{r}u_r) =$

$D(\frac{1}{r}v_{rr} - \frac{2}{r^2}v_r + \frac{2}{r^3}v + \frac{2}{r}[\frac{1}{r}v_r - \frac{1}{r^2}v]) =$

$D(\frac{1}{r}v_{rr} - \frac{2}{r^2}v_r + \frac{2}{r^3}v + \frac{2}{r^2}v_r - \frac{2}{r^3}v) = D(\frac{1}{r}v_{rr} + 0v_r + 0v) =$

$D(\frac{1}{r}v_{rr}) = \frac{1}{r}Dv_{rr}$ = RHS ∴

$v_t = Dv_{rr}$ $\quad 0 \le r \le R$ ∴

$v = ru$ ∴ $|u(0,t)| = |u(r=0,t)| < \infty$ ∴ $u(0,t) \in \mathbb{R}$ ∴

$v(0,t) = v(r=0,t) = [r(u(r,t)]|_{r=0} = (0)u(r,0) = 0$,

$v_r = ru_r + u$ ∴ $u_r = \frac{1}{r}v_r - \frac{1}{r^2}v$ ∴ $r^2u_r = rv_r - v$ ∴

at $r=R$: $R^2u_r(R,t) = Rv_r(R,t) - v(R,t) = R^2(0) = 0$ ∴

Let $v = v(r,t) = X(r)T(t)$ ∴

$v_t = X(r)T'(t)$, $v_r = X'(r)T(t)$ ∴ $v_{rr} = X''(r)T(t)$ ∴

$X(r)T'(t) = DX''(r)T(t)$ ∴ $\frac{T'(t)}{DT(t)} = \frac{X''(r)}{X(r)} = -\lambda$ = Constant ∴

$T'(t) = -\lambda DT(t)$, $X''(r) = -\lambda X(r)$ ∴

For $X'' + \lambda X = 0$ ∴ ODE for $X(r)$ has BC:

$X(0) = X(r=0) = 0$, $RX'(R) - X(R) = 0$ ∴

Case 1: $\lambda < 0$ ∴ $\lambda = -a^2 < 0$ for $a \in \mathbb{R}$ ∴ $X'' - a^2X = 0$ ∴

$X = X(r) = Ae^{ar} + Be^{-ar}$ ∴ $X' = X'(r) = Aae^{ar} - Bae^{-ar}$ ∴

$X(0) = Ae^{a(0)} + Be^{-a(0)} = A + B = 0$ ∴ $B = -A$ ∴ $X(r) = Ae^{ar} - Ae^{-ar}$ ∴

$X'(r) = Aae^{ar} + Aae^{-ar}$ ∴

$X'(R) = Aae^{aR} + Aae^{-aR}$, $X(R) = Ae^{aR} - Ae^{-aR}$ ∴

$RX'(R) - X(R) = AaRe^{aR} + AaRe^{-aR} - Ae^{aR} + Ae^{-aR} = 0 =$

$(ARe^{aR} + ARe^{-aR})a - A(e^{aR} - e^{-aR}) = 0 = AaR(e^{aR} + e^{-aR}) - A(e^{aR} - e^{-aR})$

∴ $AaRe^{aR} + AaRe^{-aR} + Ae^{-aR} = Ae^{aR}$ ∴

$e^{aR} > 0$, $e^{-aR} > 0$ ∴ $AaRe^{aR} + AaRe^{-aR} + Ae^{-aR} > Ae^{aR}$ if $A \neq 0$

∴ $A = 0$ ∴ $-A = B = 0$

— Q 8 continued: —

Case 2: $\lambda = 0 \therefore X'' + 0X = 0 = X'' \therefore X' = B \therefore X = A + Br = X(r)$

$\therefore X(0) = A + B(0) = A = 0 \therefore X(r) = Br \therefore$

$X'(R) = B, \quad X(R) = BR \therefore RX'(R) = RB \therefore$

$RX'(R) - X(R) = RB - BR = B(R-R) = B(0) = 0 = 0 \therefore$

B is arbitrary.

Case 3: $\lambda > 0 \therefore \lambda = a^2 > 0 \quad a \in \mathbb{R} \therefore X'' + a^2 X = 0 \therefore$

$X(r) = A\cos(ar) + B\sin(ar) \therefore X'(r) = -Aa\sin(ar) + Ba\cos(ar) \therefore$

$X(0) = A\cos(a \times 0) + B\sin(a \times 0) = A \times 1 + B \times 0 = A = 0 \therefore$

$X(r) = B\sin(ar), \quad X'(r) = Ba\cos(ar) \therefore$

$X'(R) = Ba\cos(aR), \quad X(R) = B\sin(aR) \therefore RX'(R) = BaR\cos(aR) \therefore$

$RX'(R) - X(R) = BaR\cos(aR) - B\sin(aR) =$

$BaR\cos(aR) - B\frac{\cos(aR)}{\cos(aR)}\sin(aR) = BaR\cos(aR) - B\cos(aR)\tan(aR) =$

$B\cos(aR)[aR - \tan(aR)] = 0 \therefore aR - \tan(aR) = 0 \therefore$

$aR = \tan(aR) \therefore aR = q \therefore a = \frac{q}{R} \therefore a^2 = \frac{q^2}{R^2} \therefore$

$\lambda = \frac{q^2}{R^2} \therefore$

$T'(t) = -q^2 \frac{1}{R^2} D T(t) \therefore \frac{T'(t)}{T(t)} = -q^2 \frac{1}{R^2} D \therefore \int \frac{T(t)}{T(t)} dt = \int -q^2 \frac{1}{R^2} D dt =$

$\ln|T(t)| = -q^2 \frac{1}{R^2} Dt + C_3 \therefore |T(t)| = e^{-q^2 \frac{1}{R^2} Dt + C_3} = e^{C_3} e^{-q^2 \frac{1}{R^2} Dt} =$

$C_2 e^{-q^2 \frac{1}{R^2} Dt} \therefore T(t) = Ce^{-q^2 \frac{1}{R^2} Dt}$ C is arbitrary constant

$\therefore$ IC : $u(r,0) = 2 + \frac{3}{r}\sin(qr/R) = \frac{1}{r}V(r,0) \therefore$

$2r + 3\sin(qr/R) = V(r,0) \therefore$ for $n \in \mathbb{N}$:

$V(r,t) = X(r)T(t) = A_0 r + \sum_{n=1}^{\infty} B_n \sin(a_n r) e^{-a_n^2 Dt}$ where

a_n is solutions to $a_n R = \tan(a_n R) \therefore$

$u(r,t) = \frac{1}{r}V(r,t) = A_0 + \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin(a_n r) e^{-a_n^2 Dt} \therefore$ by IC :

$u(r,0) = A_0 + \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin(a_n r) e^{-a_n^2 D(0)} = A_0 + \frac{1}{r}\sum_{n=1}^{\infty} B_n \sin(a_n r) = 2 + \frac{3}{r}\sin\left(\frac{qr}{R}\right)$

$\therefore A_0 = 2 \therefore$ Let $u(r,0) = 2 + \frac{B}{r}\sin(ar) = 2 + \frac{B}{r}\sin\left(\frac{qr}{R}\right) = 2 + \frac{3}{r}\sin\left(\frac{qr}{R}\right) \therefore B\sin(qr/R) = 3\sin(qr/R) \therefore B = 3 \therefore$

$u(r,t) = 2 + 3\frac{1}{r}\sin\left(q\frac{1}{R}r\right)e^{-q^2 \frac{1}{R^2} Dt}$

(18)

Q 9a:

Dirichlet Boundary Condition : $u(\vec{x},t)=0$ for $\vec{x}\in\partial V$ $\therefore$

$$\frac{dE}{dt}=\frac{d}{dt}(E(t))=\frac{d}{dt}\left(\frac{1}{2}\int_V u^2(\vec{x},t)dV\right)=\frac{d}{dt}\left(\frac{1}{2}\int_V u^2 dV\right)=$$

$$\frac{1}{2}\int_V \frac{\partial}{\partial t}(u^2)dV=\frac{1}{2}\int_V 2u\frac{\partial u}{\partial t}dV=\int_V uu_t\,dV=\int_V u\frac{\partial u}{\partial t}dV=\int_V u\nabla^2 u\,dV \quad \therefore$$

Divergence theorem is: $\int_S F\cdot\vec{n}\,dS=\int_V \nabla\cdot F\,dV$ $\therefore$

$\nabla u=\partial_x u+\partial_y u+\partial_z u=u_x+u_y+u_z$ $\therefore$

$\nabla\cdot(u\nabla u)=\nabla\cdot(u[u_x+u_y+u_z])=\nabla\cdot(uu_x+uu_y+uu_z)=$

$(uu_x)_x+(uu_y)_y+(uu_z)_z=$

$u_xu_x+uu_{xx}+u_yu_y+uu_{yy}+u_zu_z+uu_{zz}=$

$u_x^2+uu_{xx}+u_y^2+uu_{yy}+u_z^2+uu_{zz}$ and

$u\nabla^2u=u(u_{xx}+u_{yy}+u_{zz})=uu_{xx}+uu_{yy}+uu_{zz}$ $\therefore$

$\nabla\cdot(u\nabla u)=u\nabla^2u+[(u_x)^2+(u_y)^2+(u_z)^2]$ and

$S=\partial V$ $\therefore$

$$\int_{\partial V}u\nabla u\cdot\vec{n}\,d(\partial V)=\int_S u\nabla u\cdot\vec{n}\,dS=\int_V\nabla\cdot(u\nabla u)dV=$$

$$\int_V u\nabla^2u+(u_x)^2+(u_y)^2+(u_z)^2\,dV=\int_V u\nabla^2u\,dV+\int_V(u_x)^2+(u_y)^2+(u_z)^2\,dV$$

$$\therefore\frac{dE}{dt}=\int_V u\nabla^2u\,dV=\int_{\partial V}u\nabla u\cdot\vec{n}\,d(\partial V)-\int_V(u_x)^2+(u_y)^2+(u_z)^2\,dV=\frac{dE}{dt}$$

and $(u_x)^2\geq 0$, $(u_y)^2\geq 0$, $(u_z)^2\geq 0$ $\therefore\int_V(u_x)^2+(u_y)^2+(u_z)^2dV\geq 0$

$\therefore-\int_V(u_x)^2+(u_y)^2+(u_z)^2dV\leq 0$ $\therefore$

For BC: $u=0$ for $\vec{x}\in\partial V$ $\therefore$

$$\int_{\partial V}u\nabla u\cdot\vec{n}\,d(\partial V)=\int_{\partial V}(0)\nabla u\cdot\vec{n}\,d(\partial V)=\int_{\partial V}0\,d(\partial V)=0\quad\therefore$$

$$\frac{dE}{dt}=0-\int_V(u_x)^2+(u_y)^2+(u_z)^2dV=-\int_V(u_x)^2+(u_y)^2+(u_z)^2dV\leq 0\quad\therefore$$

$\frac{dE}{dt}\leq 0$ for this boundary condition can be asserted.

—— Q9b: ——

Neumann boundary condition: $\vec{n} \cdot \nabla u(x,t) = \nabla u \cdot \vec{n} = 0$ for $\vec{x} \in \partial V$

$\therefore \frac{dE}{dt} = \int_{\partial V} u \nabla u \cdot \vec{n} \, d(\partial V) - \int_V (u_x)^2 + (u_y)^2 + (u_z)^2 \, dV$

and $-\int_V (u_x)^2 + (u_y)^2 + (u_z)^2 \, dV \leq 0 \quad \therefore$

For BC: $\nabla u \cdot \vec{n} = 0$ for $\vec{x} \in \partial V \therefore$

$\int_{\partial V} u \nabla u \cdot \vec{n} \, d(\partial V) = \int_{\partial V} u (\nabla u \cdot \vec{n}) \, d(\partial V) = \int_{\partial V} u(0) \, d(\partial V) = \int_{\partial V} 0 \, d(\partial V) = 0 \quad \therefore$

$\frac{dE}{dt} = 0 - \int_V (u_x)^2 + (u_y)^2 + (u_z)^2 \, dV = -\int_V (u_x)^2 + (u_y)^2 + (u_z)^2 \leq 0 \therefore$

$\frac{dE}{dt} \leq 0$ for this boundary condition can be asserted.