given by $\frac{d\underline{r}}{ds} = \underline{\omega}(\underline{r},t)$ which can be compared with Z eqn for a streamline $\frac{d\underline{r}}{ds} = \underline{u}(\underline{r},t)$ A vortex tube is Z surface formed by all vortex lines passing though a closed curve C in Z fluid

\Def 6.5/ Z strength of a vortex tube through C is equal to Z flux of vorticity Z open surface S that is bounded by Z closed curve C

note that eqn (6.5) is Z same eqn as Z invisid vorticity eqn (6.4) with $d\underline{r}$ replaced by $\underline{\omega}$ ∴ vorticity vecs are carried & stretched/contracted/rotated precisely as infinitesimal line elements

\thm 6.8 Kelvin's vorticity thm/ for invisicid flow, Z circulation around any closed material contour is const

\Helmholtz laws/ 1. fluid elems having zero vorticity ($\underline{\omega} = \underline{0}$) continue to have zero vorticity follows from $D\underline{\omega}/Dt = \underline{\omega}\cdot\nabla\underline{u}$ if $\underline{\omega}$ is zero at $t=0$ on a fluid elem then $\underline{\omega}=0$ is Z sol at later times

2. vortex lines are material lines have see that vecs $\underline{\omega}$ are transported like material line elems. So if a material curve is a vortex line at $t=0$ (so that $d\underline{r}$ is parallel to $\underline{\omega}$ at each pt) then it remains so

3. Z strength of a vortex tube is const given a vortex tube through a material curve $C(t)$ bounding a surface $S(t)$ its strength is $\int_S \underline{\omega}\cdot d\underline{S} = \int_S \underline{\omega}\cdot\hat{\underline{n}}\, dS = \int_S \nabla\times\underline{u}\cdot\hat{\underline{n}}\, dS = \oint_C \underline{u}\cdot d\underline{r} = \Gamma$ iz penultimate equality being due to Stokes' thm, with Z normal to S & Z direction around C being related by Z right-hand rule) ∴ Z strength is const by Kelvin's thm.

• in certain regimes - magnetic field $\underline{B}$ also obeys $\frac{\partial \underline{B}}{\partial t} + \underline{u}\cdot\nabla\underline{B} = \underline{B}\cdot\nabla\underline{u}$ & ∴ field lines & magnetic flux tubes obey Alfven's laws (like Helmholtz), only now $\underline{u}$ & $\underline{B}$ are not directly lined where as $\underline{\omega} = \nabla\times\underline{u}$)

• As vortex lines & tubes move with Z fluid Z topology of Z velocity field is preserved in time eg, two linked vortex tubes at $t=0$

remain linked (given ($\nu=0$)) there are topological invariants, in particular Z helicity $H=\int_V \underline{u}\cdot\underline{\omega}\,dV$

consider 'strain' flows in which infinitesimal vecs are stretched & ∴ have vorticity intensification

\week 11 notes/ \waves in fluids/ wave motion is one of Z most basic features of all physical phenomena. waves are generated when a system is disturbed & ∃ a restoring force that tries to bring a system back to its undisturbed state, & some kind of inertia that causes Z system to over shoot past its disturbed state. After reviewing some basic properties of waves, we consider Z example of waves that occur at Z free surface of a liquid in which gravity plays Z role of Z restoring force. These are called surface gravity waves.

\wave eqn/ waves travelling only in Z x-direction are described by Z 1 dimens wave eqn $\frac{\partial^2\beta}{\partial t^2}=c^2\frac{\partial^2\beta}{\partial x^2}$

where $\beta(x,t)$ is Z disturbance (eg include Z displacement of Z free surf in a pond when a stone is dropped into it, or Z variation in density in a compressible medium). Z meaning of Z param c will become clear

Z 1 dimens wave eqn has a general sol of Z form (see formulas)

$\beta=f(x-ct)+g(x+ct)$, f & g are arbit funcs

\wave params/ in probls on waves its usual to deal with quantities that are periodic (harmonic) funcs. Consider a wave travelling in Z x-direction. write $\beta=\mathbb{R}[Ae^{ik(x-ct)}]=\mathbb{R}[Ae^{i(kx-\omega t)}]$ on setting $\omega=ck$, (7.2)

, A is Z complex amplitude & $\mathbb{R}$ means 'take Z real part of' ∵

Z wave eqn & Z boundary conditions (see later) are linear, taking Z real part is no trouble & it can be interchanged with Z other operators that enter. eg $\mathbb{R}\left(\frac{\partial\beta}{\partial x}\right)=\frac{\partial}{\partial x}(\mathbb{R}(\beta))$ $\mathbb{R}(\alpha\beta_1+\gamma\beta_2)=\alpha\mathbb{R}(\beta_1)+\gamma\mathbb{R}(\beta_2)$ ~~for real~~

for real α,γ returning to (7.2), if Z amplitude is real, $A=a$, say, have $\beta=a\cos[k(x-ct)]$

\Def 7.3/ • Z amplitude of Z wave is a (note that β varies betw $\pm a$)

• Z wave number is k

• Z wavelength is $\lambda = 2\pi/k$

• Z (circular) frequency is ω

• Z argument $k(x-ct)$ is called Z phase of Z wave & pts of const phase are those where Z wave form has Z same val, say a crest or a trough

• Z phase speed $c = \omega/k$ is Z rate at which Z 'phase' of Z wave (crests & troughs) propagates

• Z period T of Z wave is Z time required for Z wave to travel one wavelength, $T = \frac{\lambda}{c} = \frac{2\pi}{ck} = \frac{2\pi}{\omega}$

Z three-dimens wave eqn reads $\frac{\partial^2 \beta}{\partial t^2} = c^2 \nabla^2 \beta$ (7.3)

for a plane wave in 3 dimens write $\beta = a e^{i(\underline{k}\cdot\underline{r} - \omega t)}$, $\underline{r}$ is Z position vec. note Z real part '$\mathbb{R}$' is omitted but understood

\Def 7.4/ $\underline{k}$ is Z wave vec & $\underline{k} = k\hat{\underline{n}} = \frac{\omega}{c}\hat{\underline{n}}$, $\hat{\underline{n}}$ is a unit vec in Z direction of propagation

\Surface gravity waves/ Consider gravity waves of wavelength λ at Z free surface (unconstrained from above) of a sea of liquid of infinite depth. Z motion is assumed to be generated from rest, by wind action or by dropping a stone.

\Z geometry/ Z geometry of Z probl is 2 dimens waves propagate in Z x-direction (horizontally) only & Z vertical coord is y. gravity is const & acts downwards, $\underline{g} = -g\underline{j}$. denote Z eqn of Z free surface by $y = \eta(x,t)$

\boundary conditions/ BC are to be satisfied at Z free surf. have a kinematic condition. Z kinematic condition is that 'Z surf moves with Z fluid'. Z dynamic condition arises from a condition on Z pressure at Z free surf.

Z difficulty in solving surf-wave probls is due to Z boundary conditions rather than Z differential eqn. Z dynamic condition is non linear & both it & Z kinematic condition must be applied at surf $y=\eta(x,t)$. in real situations on Z surf is not known a priori - it's a quantity that comes out of Z sol itself. fortunately, many features of surf-wave flows are not sensitive to these complicated features & Z difficulties can be overcome by linearising Z probl

/Small amplitude waves/ in order to simplify Z boundary conditions we assume that both Z free surf displacement $\eta(x,t)$ & Z associated fluid velocities u, v are small (in a sense to be made more precise later). Under this assumption we linearise Z probl by rejecting quadratic (& higher order) terms in small quantities.

/Dispersion Relation/ Seek a sinusodial travelling wave sol. Say Z free surf is $\eta = A\cos(kx-\omega t)$, A is Z amplitude of Z surf displacement, ω is Z frequency & k is Z wavenumber. this corres to a wave travelling in Z positive x-direction with speed $c=\omega/k$

Z question we ask: given Z wave amplitude A & Z wavelength $\lambda = 2\pi/k$ what will be Z propagating speed c?. answer determined by solving Laplace's eqn (7.5) together with boundary conditions (7.10) & (7.11) to obtain Z velocity potential.

/Z meaning of small amplitude/ in Z previous section we linearised Z boundary conditions on Z assumption that Z waves are of 'small amplitude'. we wish to answer Z question: small compared to what?

/Dispersion Group velocity/

/Def 7.12/ Z <u>dispersion</u> <u>relation</u> connects Z wave frequency & Z wavenumber & is of Z form $\omega = \omega(k)$

Z <u>group</u> <u>speed</u> is defined as $c_g = \frac{d\omega}{dk}$

Dispersion means that waves of different wavelengths travel at different speeds. if Z phase speed $c=\omega/k$ is indep of number then Z waves are non-dispersive

Week 1 QUIZ

1/ stagnation pt is $\vec{u}=(0,0,0)=(y, z^3-1, x^2-2)$ ∴

$y=0$, $z^3-1=0$, $x^2-2=0$ ∴ $x,y,z \in \mathbb{R}$ ∴

$y=0$, $z^3=1$, $x^2=2$ ∴

$y=0$, $z=1$, $x=\sqrt{2}$ and $y=0$, $z=1$, $x=-\sqrt{2}$ ∴ 2 stagnation pts

2/ material derivative that measures rate of change following the fluid motion is $\frac{\partial}{\partial t}+\vec{u}\cdot\nabla$

3/ $[\vec{u}\times\vec{v}]_i = \varepsilon_{ijk}u_j v_k = \varepsilon_{kij}u_j v_k$

4/ for incompressible fluids the mass continuity eqn is $\nabla\cdot\vec{u}=0$

5/ N-S eqn $\rho\frac{D\vec{u}}{Dt}=\rho\left(\frac{\partial u}{\partial t}+\underline{u}\cdot\nabla\underline{u}\right)=+\mu\nabla^2\underline{u}-\rho g-\nabla p =$

$-\nabla p+\rho\vec{g}+\mu\nabla^2\vec{u}$

6/ pathlines eqns of $\vec{r}(t)$: $\frac{dr}{dt}=\underline{u}(r,t)=\frac{d\vec{r}}{dt}=\vec{u}(\vec{r},t)$

7/ $\varepsilon_{ijk}\varepsilon_{ilm}$ midmid endend − midend endmid ∴

$\varepsilon_{ijk}\varepsilon_{ilm}=\delta_{jl}\delta_{km}-\delta_{jm}\delta_{kl}$

8/ $u_i v_i=\vec{u}\vec{v}$, $\frac{\partial a_j}{\partial x_j}=\frac{\partial}{\partial x_j}a_j=\nabla_j a_j=\nabla\cdot a$ ∴

$u_i v_i+\frac{\partial a_j}{\partial x_j}+3$ in vector notation is $\vec{u}\cdot\vec{v}+\nabla\cdot\vec{a}+3$

9/ $[(\vec{u}\cdot\nabla)\vec{u}]_j = \cancel{\delta}\,[(u_i\nabla_i)\vec{u}]_j=[(u_i\frac{\partial}{\partial x_i})\vec{u}]_j=[\cancel{\delta} u_i\frac{\partial}{\partial x_i}\vec{u}]_j$

$(\vec{u}\cdot\nabla)=u_i\frac{\partial}{\partial x_i}$ ∴ $[(u_i\frac{\partial}{\partial x_i})\vec{u}]_j=[(\vec{u}\cdot\nabla)\vec{u}]_j=u_i\frac{\partial u_j}{\partial x_i}$

10/ $\delta_{ij}=\begin{cases}1 \text{ for } i=j\\ 0 \text{ for } i\neq j\end{cases}$ ∴ $\delta_{ii}=\delta_{ij}$ for $i=j$ ∴ $\delta_{ii}=1$ ✗

$\delta_{ii}=3$ ∵ δ_{ii} for $i=1,2,3$ is $\delta_{ii}=\delta_{11}+\delta_{22}+\delta_{33}=1+1+1=3$

Week 2 QUIZ

1/ no stress boundary condition on free surface is neglect of forces on a surface

2/ $v=\omega r$ ∴ $\vec{u}=(u,v,w)$ ∴ $\vec{u}=\vec{\Omega}\times\vec{r}$

$\int u_\phi R\,d\phi$

3/ $\oint_c \underline{u}\cdot\underline{dr}=\int(u_R\hat{R}+u_\phi\hat{\phi})\cdot(\hat{R}+\hat{\phi})\,dR\,d\phi=\int(u_R+u_\phi)\,dR\,d\phi$ ✗

∵ its radius R

4/ $\nabla\cdot\underline{u}=\nabla\cdot\left(\left(\frac{1}{r^2}\right)\hat{r}+\theta\hat{\theta}+1\hat{\phi}\right)=$

$\frac{1}{r^2}\frac{\partial}{\partial r}(r^2u_r)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\, u_\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(u_\phi)=$

$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{1}{r^2}\right)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\theta)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(1)=\frac{1}{r^2}\frac{\partial}{\partial r}(1)+\frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}(1)=0$

$\cancel{\text{Part}}$

$$\nabla\times\underline{u} = \nabla\times\left(\frac{1}{r^2}\hat r + 0\hat\theta + 1\hat\phi\right) = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat r & r\hat\theta & r\sin\theta\hat\phi \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ \frac{1}{r^2} & r(0) & r\sin\theta \end{vmatrix} =$$

$$\frac{1}{r^2\sin\theta}\left[\left(\frac{\partial}{\partial\theta}(r\sin\theta) - \frac{\partial}{\partial\phi}(0)\right)\hat r - r\hat\theta\left(\frac{\partial}{\partial r}(r\sin\theta) - \frac{\partial}{\partial\phi}\left(\frac{1}{r^2}\right)\right) + r\sin\theta\hat\phi\left(\frac{\partial}{\partial r}(0) - \frac{\partial}{\partial\theta}\left(\frac{1}{r^2}\right)\right)\right]$$

$$= \frac{1}{r^2\sin\theta}\left[\hat r(r\cos\theta) - r\hat\theta(\sin\theta) + r\sin\theta\hat\phi(0)\right] =$$

$$\frac{r\cos\theta}{r^2\sin\theta}\hat r - r\sin\theta\hat\theta = \frac{1}{r\tan\theta}\hat r - r\sin\theta\hat\theta \neq 0 \therefore$$

u is incompressible $\therefore\ \nabla\cdot\underline{u} = 0$

5/ $\underline{u} = \frac{1}{R}\hat R + 0\hat\theta + R\hat\phi \quad \therefore\ \cancel{\underline{u}\cdot\nabla = \left(\frac{1}{R}\hat R + 0\hat\theta + R\hat\phi\right)\cdot\frac{\partial}{\partial R}}$

$\cancel{\left(\frac{1}{R}\hat R + 0\hat\theta + R\hat\phi\right)\cdot\frac{\partial}{\partial r}}$ $\underline{u}\cdot\nabla = \left(\frac{1}{R}\hat R + 0\hat\theta + R\hat\phi\right)\cdot\nabla =$

$\left(\frac{1}{R}\hat R + 0\hat\theta + R\hat\phi\right)\cdot\left(\frac{\partial}{\partial r}\hat R + \hat\theta\frac{\partial}{\partial\theta} + \hat\phi\frac{\partial}{\partial\phi}\right) = \frac{1}{R}\frac{\partial}{\partial R} + R\frac{\partial}{\partial\phi}$ $\cancel{=}$

$\cancel{\underline{u}\cdot\nabla = \left(\frac{1}{R}\hat R\right.}$ $(\underline{u}\cdot\nabla)\underline{u} = \left(\frac{1}{R}\frac{\partial}{\partial R} + R\frac{\partial}{\partial\phi}\right)\left(\frac{1}{R}\hat R + R\hat\phi\right) =$

$\cancel{\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\hat R\right) + R\frac{\partial}{\partial\phi}(R\hat\phi)}$ $\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\hat R + R\hat\phi\right) + R\frac{\partial}{\partial\phi}\left(\frac{1}{R}\hat R + R\hat\phi\right) =$

$\frac{1}{R}\left(\frac{\partial}{\partial R}\left(\frac{1}{R}\hat R\right) + \frac{\partial}{\partial R}(R\hat\phi)\right) + R\left(\frac{\partial}{\partial\phi}\left(\frac{1}{R}\hat R\right) + \frac{\partial}{\partial\phi}(R\hat\phi)\right)$

$\cancel{\frac{1}{R}}$

$\underline{u} = \frac{1}{R}\hat R + R\hat\phi + 0\hat z \quad \therefore\ \underline{u}\cdot\nabla = \left(\frac{1}{R}\hat R + R\hat\phi + 0\hat z\right)\left(\frac{\partial}{\partial R}\hat R + \frac{\partial}{\partial\phi}\hat\phi + \frac{\partial}{\partial z}\hat z\right) =$

$\frac{1}{R}\frac{\partial}{\partial R} + R\frac{\partial}{\partial\phi} \quad \therefore\ (\underline{u}\cdot\nabla)\underline{u} = \left(\frac{1}{R}\frac{\partial}{\partial R} + R\frac{\partial}{\partial\phi}\right)\left(\frac{1}{R}\hat R + R\hat\phi\right) =$

$\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\hat R + R\hat\phi\right) + R\frac{\partial}{\partial\phi}\left(\frac{1}{R}\hat R + R\hat\phi\right) =$

$\frac{1}{R}\left(\frac{\partial}{\partial R}\left(\frac{1}{R}\hat R\right) + \frac{\partial}{\partial R}(R\hat\phi)\right) + R\left(\frac{\partial}{\partial\phi}\left(\frac{1}{R}\hat R\right) + \frac{\partial}{\partial\phi}(R\hat\phi)\right) =$

$\frac{1}{R}\left(\hat R\frac{\partial}{\partial R}\left(\frac{1}{R}\right) + \hat\phi\frac{\partial}{\partial R}(R)\right) + R\left(\frac{1}{R}\hat\phi + \hat R\frac{\partial}{\partial\phi}\left(\frac{1}{R}\right) + R\frac{\partial}{\partial\phi}(\hat\phi)\right) =$

$\frac{1}{R}\left(\hat R(-R^{-2}) + \hat\phi\right) + R\left(\frac{1}{R}\hat\phi + R(-\hat R)\right) = -\frac{1}{R^3}\hat R + \frac{1}{R}\hat\phi + \hat\phi - R^2\hat R = -\left(R^2 - \frac{1}{R^3}\right)\hat R + \left(1 + \frac{1}{R}\right)\hat\phi$

$(\underline{u}\cdot\nabla) = \frac{1}{R}\frac{\partial}{\partial R} + R\frac{\partial}{\partial\phi} \quad \therefore\ (\underline{u}\cdot\nabla)\underline{u} = \left(\frac{1}{R}\frac{\partial}{\partial R} + R\frac{\partial}{\partial\phi}\right)\left(\frac{1}{R}\hat R + R\hat\phi\right) =$

$\left(\frac{1}{R}\right)\frac{\partial}{\partial R}\left(\frac{1}{R}\hat R + R\hat\phi\right) + R\frac{\partial}{\partial\phi}\left(\frac{1}{R}\hat R + R\hat\phi\right) =$

$\frac{1}{R}\left(\frac{\partial}{\partial R}\left(\frac{1}{R}\hat R\right) + \frac{\partial}{\partial R}(R\hat\phi)\right) + R\left(\frac{\partial}{\partial\phi}\left(\frac{1}{R}\hat R\right) + \frac{\partial}{\partial\phi}(R\hat\phi)\right) =$

$\frac{1}{R}\left(\hat R\frac{\partial}{\partial R}\left(\frac{1}{R}\right) + \hat\phi\frac{\partial}{\partial R}(R)\right) + R\left(\frac{1}{R}\frac{\partial}{\partial\phi}(\hat R) + R\frac{\partial}{\partial\phi}(\hat\phi)\right) =$

$\frac{1}{R}\left(-\frac{1}{R^2}\hat R + \hat\phi\right) + R\left(\frac{1}{R}\hat\phi - R\hat R\right) = -\frac{1}{R^3}\hat R + \frac{1}{R}\hat\phi + \hat\phi - R^2\hat R = \left(-R^2 - \frac{1}{R^3}\right)\hat R + \left(1 + \frac{1}{R}\right)\hat\phi$

I got the same answer wrong again so need to be shown the Sol week 2 QUIZ

3 $u = u_R\hat{R} + u_\phi\hat{\phi}$ the circulation around a close circular contour of R is: $\int u_\phi R\,d\phi$ $\therefore$ $\oint_c \underline{u}\cdot\underline{dr}$

5 $u = R^{-1}\hat{R} + R\hat{\phi}$ has nonlinear term $-(R+R^{-3})\hat{R} + 2R^{-1}\hat{\phi}$

week 3 QUIZ

1 $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ $\therefore$ $e_{ij} = e_{ji}$

2 trace is $\left(\frac{\partial u_1}{\partial x_1}, \frac{\partial u_2}{\partial x_2}, \frac{\partial u_3}{\partial x_3}\right) = \nabla\cdot\underline{u}$

3 $\sigma_{ij} = -p\delta_{ij} + d_{ij}$ $\because$ $d_{ij} = 2\mu e_{ij}$ $\therefore$ $\sigma_{ij} = -p\,\delta_{ij} + 2\mu e_{ij}$

4 $R_e = \frac{\rho u L}{\mu}$ $[\rho] = \left[\frac{m}{V}\right] = \frac{M}{L^3}$ $[L] = L$

$[u] = \left[\frac{s}{t}\right] = \frac{L}{T}$ $\mu = \nu\rho$ $\therefore$ $[\nu] = L^2T^{-1}$ $\therefore$

$[\mu] = [\nu\rho] = [\nu][\rho] = \cancel{ML^{-3}}\ L^2T^{-1}ML^{-3} = ML^{-1}T^{-1}$ $\therefore$

$[R_e] = \left[\frac{\rho u L}{\mu}\right] = [\rho][u][L]\left[\frac{1}{\mu}\right] = \left(\frac{M}{L^3}\right)(LT^{-1})(L)(M^{-1}L^{1}T^{1}) = \cancel{L} \; 1$

5 l is indep m is indep g is indep but

$[n] = LT^{-2} = [l][g^{-2}]$ $\therefore$ not indep

$[r] = LT^{-1} = [l][g^{-1}]$ $\therefore$ not indep $\therefore$ 3 params are indep

Week 4 QUIZ

1 $\pi_i = b_i/(a_1^{p_i} \cdots a_k^{r_i})$

2 $\nabla\cdot\underline{u} = 0 = \nabla\cdot(u\underline{i} + v\underline{j} + w\underline{k}) = \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v + \frac{\partial}{\partial z}w = \nabla\cdot(0\underline{i} + v\underline{j} + 0\underline{k}) =$
$\frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}(v) + \frac{\partial}{\partial z}(0) = \frac{\partial}{\partial y}(v) = 0$ $\therefore$ $v = v(x, z)$

3 $[y] = L$ $[t] = T$ $\cancel{[\mu]}$ $[\nu] = L^2T^{-1}$ $\therefore$

$[\nu^{1/2}] = LT^{-1/2}$ $\therefore$ $[\eta] = [yt^q/\nu^{1/2}] = [y][t^q][(\nu^{1/2})^{-1}] =$

$(L)(T^q)(L^{-1}T^{1/2}) = T^{q+\frac{1}{2}} = 1 = T^0$ $\therefore$ $q + \frac{1}{2} = 0$ $\therefore$ $q = -\frac{1}{2}$

4 $[\nu] = L^2T^{-1}$, $[t] = T$ $\therefore$ $[\sqrt{4\nu t}] = [\sqrt{\nu t}] = [(\nu t)^{1/2}] =$
$[\nu^{1/2}t^{1/2}] = [\nu^{1/2}][t^{1/2}] = [\nu]^{1/2}[t]^{1/2} = (LT^{-1/2})(T^{1/2}) = L$

5 $\lim_{x\to\infty} \operatorname{erf}(x) = \lim_{x\to\infty} \frac{2}{\sqrt{\pi}}\int_0^x \exp(-s^2)\,ds = 1$

Week 5 QUIZ

1. $\hat{u}(y)\exp(i\Omega t) = \hat{u}(y)e^{i\Omega t} = \hat{u}(y)[\cos(\Omega t) + i\sin(\Omega t)]$ $\therefore$

$\mathbb{R}[\hat{u}(y)\exp(i\Omega t)] = \hat{u}(y)\cos(\Omega t)$ $\therefore$

$\frac{\partial u}{\partial t} = -\Omega\hat{u}(y)\sin(\Omega t)$ ✓

2. $\sqrt{-i} = (0+(-1)i)^{1/2} = re^{i\theta}$ $\therefore$ $-i = r^2e^{i2\theta}$ $\therefore$ $|-i| = \sqrt{0^2+(-1)^2} = \sqrt{1} = 1$ $\therefore$ $1 = r^2$ $\therefore$ $r = 1$ $\therefore$ $\tan(2\theta) = \frac{y}{x}$ $\therefore$ $2\theta = \frac{\pi}{2}$ $\therefore$ $0 \le 2\theta < 4\pi$ $\therefore$

$2\theta = \frac{\pi}{2}, \frac{5\pi}{2}$ $\therefore$ $\theta = \frac{\pi}{4}, \frac{5\pi}{4}$ $\therefore$

$\sqrt{-i} = 1e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2} = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}} = (1+i)\frac{1}{\sqrt{2}}$

$\therefore$ $\sqrt{-i} = (1-i)\frac{1}{\sqrt{2}}$ ✓

3. $\underline{\omega} = \nabla\times\underline{u}$ $\therefore$ $\frac{D\underline{\omega}}{Dt} = \underline{u}\cdot\nabla\underline{\omega} + \nu\nabla^2\underline{\omega}$ ✗ eqn of vorticity $\underline{\omega} = \nabla\times\underline{u}$ is: $\frac{D\underline{\omega}}{Dt} = \omega\cdot\nabla\underline{u} + \nu\nabla^2\underline{\omega}$

4. $\underline{u} = \nabla\times(\psi\underline{k})$ ✓ curl of NS eqn

5. $[\nu] = L^2T^{-1}$ $[\tau] = T$ $[\delta] = [\sqrt{\nu\tau}] = [\nu^{1/2}][\tau^{1/2}] = [\nu]^{1/2}[\tau]^{1/2}$ $= LT^{-1/2}T^{1/2} = L$ $\therefore$ $[\eta] = \frac{[y]}{[\delta]} = \frac{L}{L} = 1$ $\therefore$ $[F(\eta)] = [\eta] = 1$ $\therefore$

$[\psi] = \frac{[x][\nu]}{[\delta]}[F(\eta)] = \frac{L\,L^2T^{-1}}{L}1 = L^2T^{-1}$ ✓

Week 7 QUIZ

1. $\underline{u} = \nabla\times(\psi\underline{k}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \psi \end{vmatrix} = \underline{i}\frac{\partial\psi}{\partial y} - \frac{\partial\psi}{\partial x}\underline{j} = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right) \doteq (u,v)$ $\therefore$

$u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$ $\therefore$ $\frac{\partial}{\partial y} \sim \delta$ $\frac{\partial}{\partial x} = L$ $\therefore$ $v \sim \frac{\psi}{L}$, $u \sim \frac{\psi}{\delta}$ $\therefore$

$\frac{u}{v} \approx \frac{\psi/\delta}{\psi/L} = \frac{L}{\delta} \gg 1$ $\therefore$ $u \gg v$ $\because$ $\frac{\delta}{L} \ll 1$

2. $\alpha = \frac{\frac{\partial^2}{\partial x^2}u}{\frac{\partial^2}{\partial y^2}u} \approx \frac{\frac{1}{L^2}u}{\frac{1}{\delta^2}u} = \frac{\delta^2}{L^2}$

3. $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{G}{\rho} + \nu\frac{\partial^2 u}{\partial y^2}$ $G = -\frac{dp}{dx}$ $u\frac{\partial u}{\partial x} \approx \frac{G}{\rho} = -\frac{1}{\rho}\frac{\partial p}{\partial x}$

$\therefore$ $u\frac{\partial}{\partial x}(u) = \frac{2u}{2}\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) \approx \frac{\partial}{\partial x}\left(-\frac{p}{\rho}\right)$ $\therefore$ $\frac{u^2}{2} = -\frac{p}{\rho} + c$ $\therefore$

$p = p = -\frac{1}{2}\rho u^2 + c_2$ $\therefore$ $\psi = xy^2 + 2y$ $\therefore$ $\underline{u} = \nabla\times(\psi\underline{k}) = \left|\begin{matrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & xy^2+2y \end{matrix}\right|$

$= \underline{i}\left(\frac{\partial}{\partial y}(xy^2+2y)\right) - \underline{j}\left(\frac{\partial}{\partial x}(xy^2+2y)\right) = \underline{i}(2xy+2) + \underline{j}(-y^2) = u\underline{i} + v\underline{j}$ $\therefore$

$u = 2xy+2$ $\therefore$ $p = -\frac{1}{2}\rho(2xy+2)^2 \doteq -\frac{1}{2}\rho(4x^2y^2+4+8xy)$

$-\frac{dp}{dx} = \rho(2xy+2)(2y) = \rho(4xy^2+4y)$ ✗ 0

4. $\underline{u} = (u,v) = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}\right)$ $\therefore$ $v = -\frac{\partial\psi}{\partial x}$ $\therefore$ at $y = \infty$, $v = U$ $\therefore$ $-\frac{\partial\psi}{\partial x} = U$ ✗ $u = U$ $\therefore$ $\frac{\partial\psi}{\partial y} = U$

5. Drag force from integrating the shear stress ✓

Week 8 QUIZ / 1/ slow flow for $Re \ll 1$

2/ Eliminate pressure by taking the curl of N-S eqn

3/ curl of stokes eqn: $\nabla \times (-\nabla p + \mu \nabla^2 \underline{u}) = 0 = -\nabla \times \nabla p + \mu \nabla \times \nabla^2 \underline{u} =$

$-\nabla \times \nabla p + \mu \nabla \times [\nabla^2 (\nabla \times \Psi \underline{k})] = 0 \qquad \nabla \times \nabla p = 0$

$= \mu \nabla^2 [\nabla \times \nabla \times (\Psi \underline{k})] = 0 = \mu \nabla^2 [\nabla(\nabla \cdot \Psi \underline{k}) - \nabla^2 \Psi \underline{k}] = \mu \nabla^2 [\nabla(\frac{\partial}{\partial z} \Psi) - \nabla^2 \Psi \underline{k}] =$

$\mu \nabla^2 [\nabla(0) - \nabla^2 \Psi \underline{k}] = \mu \nabla^2 [-\nabla^2 \Psi \underline{k}] = -\mu \nabla^2 \nabla^2 (\Psi \underline{k}) = 0 = -\mu \nabla^4 (\Psi \underline{k}) = \nabla^4 (\Psi \underline{k})$

$= \nabla^2 \nabla^2 (\Psi \underline{k}) = \underline{k} \nabla^2 \nabla^2 (\Psi) = \nabla^4 \Psi$

4/ $\nabla \cdot \underline{\omega} = \nabla \cdot \left(-\nabla^2 \left(\frac{\Psi}{r \sin\theta} \hat{\phi}\right)\right) \qquad \nabla \cdot \underline{\omega} = 0$ if $\underline{\omega} = -\nabla^2 \left(\frac{\Psi}{r \sin\theta} \hat{\phi}\right)$

$\nabla \Sigma$ of $\nabla \cdot \underline{\omega} = -\nabla^2 \left(\nabla \cdot \left(\frac{\Psi}{r \sin\theta} \hat{\phi}\right)\right) = -\nabla^2 \left(\frac{\partial}{\partial \phi} \left(\frac{\Psi}{r \sin\theta}\right)\right) = -\nabla^2(0) = 0$

5/ $E^2 \Omega = \frac{\partial^2 \Omega}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial \Omega}{\partial \theta}\right) \therefore$

$E^2 \Psi = E^2 (r \cos^2\theta) = \frac{\partial^2}{\partial r^2} (r \cos^2\theta) + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta} (r \cos^2\theta)\right) =$

$0 + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} \left(-\frac{1}{\sin\theta} 2r \cos\theta \sin\theta\right) = \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta} (-2r \cos\theta) =$

$\frac{\sin\theta}{r^2} 2r \sin\theta = \frac{2 \sin^2\theta}{r}$

Week 9 QUIZ / 1/ flow around a sphere require $E^4 \Psi = 0$

2/ no slip $\therefore u = 0$ on $r = a$ $\Psi(a, \theta) \therefore \frac{\partial \Psi(a,\theta)}{\partial r} = 0 \therefore$ U increases from zero $\therefore$ uniform at $r = \infty$ $\frac{\partial \Psi}{\partial r} = 0$

3/ $\cos\phi \hat{R} - \sin\phi \hat{\phi} = \cos^2\phi \, \underline{i} + \cos\phi \sin\phi \, \underline{j} + \sin^2\phi \, \underline{i} - \sin\phi \cos\phi \, \underline{j} = \underline{i}$

$\therefore$ at $r = \infty$: $\underline{u} = U \underline{k}$

$\underline{r} = x \underline{i} + y \underline{j} + z \underline{k} = r \sin\theta \cos\phi \, \underline{i} + r \sin\theta \cos\phi \, \underline{j} + r \cos\theta \, \underline{k} \qquad \therefore z = r \cos\theta \therefore$

$\sigma_{ij} = -p \delta_{ij} + d_{ij} \qquad v \to -U \sin\theta$

4/ stress tensor $\sigma_{ij} = -p \delta_{ij} + 2\mu e_{ij} \qquad \sigma_{ij} = -p \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$

viscous part of stress tensor σ_{ij} involves $\frac{\partial u_i}{\partial x_j}$

5/ $\hat{\underline{n}} = \hat{\underline{r}} = 1 \hat{\underline{r}} + 0 \hat{\underline{\theta}} + 0 \hat{\underline{\phi}} \therefore \hat{\underline{n}} \cdot \nabla \underline{u} = (\hat{\underline{n}} \cdot \nabla) \underline{u} \quad \therefore (\hat{\underline{n}} \cdot \nabla) =$

$(1 \hat{\underline{r}} + 0 \hat{\underline{\theta}} + 0 \hat{\underline{\phi}}) \cdot \nabla = (1 \hat{\underline{r}} + 0 \hat{\underline{\theta}} + 0 \hat{\underline{\phi}}) \cdot \left(\frac{\partial}{\partial r} \hat{\underline{r}} + \hat{\underline{\theta}} \frac{\partial}{\partial \theta} + \hat{\underline{\phi}} \frac{\partial}{\partial \phi}\right) = \frac{\partial}{\partial r} \therefore$

$(\hat{\underline{n}} \cdot \nabla) \underline{u} = \left(\frac{\partial}{\partial r}\right) (u_r \hat{\underline{r}} + u_\theta \hat{\underline{\theta}}) = \frac{\partial}{\partial r} (u_r \hat{\underline{r}}) + \frac{\partial}{\partial r} (u_\theta \hat{\underline{\theta}}) = \frac{\partial \underline{u}}{\partial r} \therefore$

$2\mu (\hat{\underline{n}} \cdot \nabla \underline{u}) = 2\mu \frac{\partial \underline{u}}{\partial r}$

Week 10 QUIZ / 1/ $\frac{D \underline{\omega}}{Dt} = \underline{\omega} \cdot \nabla \underline{u} + \nu \nabla^2 \underline{\omega}$

2/ at a fixed time t, a vortex line is a curve tangent to Z local vorticity vector, a vortex line has Z vorticity vector as tangents

3/ Kelvins vorticity thm: for inviscid flow, Z circulation around any closed material contour is const, Z line integral of Z velocity around a closed curve moving with Z flow is const for inviscid flow

\4/ $\frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\nabla\underline{u} + \nu\nabla^2\underline{\omega}$ is the vorticity eqn: inviscid case but with the vorticity replaced by the line element $\frac{D}{Dt}(d\underline{r}) = (d\underline{r}\cdot\nabla)\underline{u} = \frac{\partial d\underline{r}}{\partial t} + (\underline{u}\cdot\nabla)d\underline{r}$

\5/ $\underline{u} = U_R\hat{\underline{R}} + U_\phi\hat{\underline{\phi}} + U_Z\hat{\underline{Z}}$ ∴ RZ plane ∴ indep of ϕ ∴ $U_\phi = 0$ ∴ i.e. $\frac{\partial}{\partial\phi}(\underline{u}) = 0$

$$\underline{\omega} = \nabla\times\underline{u} = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{Z}} \\ \partial/\partial R & \partial/\partial\phi & \partial/\partial Z \\ U_R & RU_\phi & U_Z \end{vmatrix} = $$

$$\frac{1}{R}\left[\hat{\underline{R}}\left(\frac{\partial}{\partial\phi}(\cancel{RU_Z}) - \frac{\partial}{\partial Z}(RU_\phi)\right) - R\hat{\underline{\phi}}\left(\frac{\partial}{\partial R}(U_Z) - \frac{\partial}{\partial Z}(U_R)\right) + \hat{\underline{Z}}\left(\frac{\partial}{\partial R}(RU_\phi) - \frac{\partial}{\partial\phi}(U_R)\right)\right] =$$

$$\frac{1}{R}\left[\hat{\underline{R}}(0-0) - R\hat{\underline{\phi}}\left(\frac{\partial}{\partial R}(U_Z) - \frac{\partial}{\partial Z}(U_R)\right) + \hat{\underline{Z}}(0-0)\right] = -R\hat{\underline{\phi}}\left(\frac{\partial}{\partial R}(U_Z) - \frac{\partial}{\partial Z}(U_R)\right)$$

an axisymmetric flow in the Rz plane in cylindrical polar coords has a vorticity that is dependent on space & is Z ϕ direction

\Week 11 QUIZ/ \1/ $f(x-ct)$ ∴ as t increases ∴ if $y = x - ct$ ∴ y decreases, as x increases, y increases ∴ travels in Z positive x-direction with speed $c > 0$

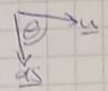
\2/ $T \propto \frac{1}{\omega}$ ∵ period $\propto \frac{1}{\text{frequency}}$

\3/ $\nabla\cdot\underline{u} = 0$, $\nabla\times\underline{u} = 0$ $\quad \underline{u} = \nabla\phi$ satisfies Laplaces eqn $\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$

\4/ For waves of small amplitude we may linearise the problem by neglecting quadratic & higher order terms ∴ linearise the eqn

\5/ $T = \frac{2\pi}{ck}$ ∴ $\frac{1}{T} = \frac{ck}{2\pi}$ the dispersion relation connecting the frequency & Z wavenumber $\omega(k)$ is the relationship between the frequency & Z wavenumber $\omega(k)$

\week 1/



\probl 2.3/ mass inside z volume is $\int_V \rho \, dV$ $\therefore$

z rate of change of mass in V is $\frac{d}{dt}\int_V \rho \, dV = \int_V \frac{\partial \rho}{\partial t} dV$

$\therefore$ mass is conserved this must equal z rate at which mass flows through z boundary (mass flux) $\int_V \frac{\partial \rho}{\partial t} dV = -\int_S \rho \underline{u} \cdot \underline{dS}$

$\{\underline{u} \cdot \underline{dS} = |\underline{u}||\underline{dS}|\cos\theta\}$ with a negative sign insuring that outward flow represents a decrease in mass with respect to time, & an inward flow an increase

__Divergence thm__: $\int_V \nabla \cdot \underline{F} \, dV = \int_S \underline{F} \cdot \underline{dS}$ $\underline{F}$ is a continuously differentiable vec field S is z surf enclosing V $\hat{n}$ is z unit outward normal $\therefore$

$\underline{dS} = \hat{\underline{n}} \, dS$

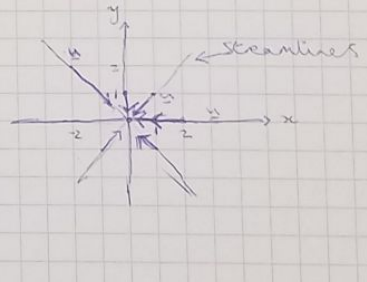
take $\underline{F} = \rho \underline{u}$ $\therefore$ $\int_V \frac{\partial \rho}{\partial t} dV = -\int_S \overbrace{\rho \underline{u}}^{\underline{F}} \cdot \underline{dS} = -\int_V \nabla \cdot (\rho \underline{u}) \, dV$ $\therefore$

$\int_V \frac{\partial \rho}{\partial t} dV + \int_V \nabla \cdot (\rho \underline{u}) \, dV = 0 = \int_V \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) \, dV = 0$ & since V is arbit $\therefore$

$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$ continuity eqn of mass conservation

\probl 2.4/

(x,y)	$\underline{u} = (-x,-y)$	$\lvert\underline{u}\rvert$
(0,0)	(0,0)	0
(1,0)	(-1,0)	1
(0,1)	(0,-1)	1
(1,1)	(-1,-1)	$\sqrt{2}$
(2,0)	(-2,0)	2
(1,-1)	(-1,1)	$\sqrt{2}$
(-1,-1)	(1,1)	$\sqrt{2}$
(-2,2)	(2,-2)	$\sqrt{8}$



streamlines are curves drawn st at any pt along z curve z tangent is z velocity vec

\probl 2.6/ $a_j b_i c_j$ in vec notation $\therefore$ $a_j b_i c_j = \sum_{j=1}^{3} a_j b_i c_j = \sum_{j=1}^{3} a_j c_j b_i =$ $(\underline{a} \cdot \underline{c}) b_i$ $\therefore$ z i^{th} component of z vec $[(\underline{a} \cdot \underline{c})\underline{b}]_i$

\probl 2.7/ $\underline{u} + (\underline{a} \cdot \underline{b})\underline{v} = |\underline{a}|^2 (\underline{b} \cdot \underline{v}) \underline{a}$ in suffix notation $\therefore$

introduce a free suffix i $(i = 1, 2, 3)$ to write eqn in component form

$u_i + (\underline{a} \cdot \underline{b}) v_i = |\underline{a}|^2 (\underline{b} \cdot \underline{v}) a_i$ recall $|\underline{a}|^2 = \left(\sqrt{a_1^2 + a_2^2 + a_3^2}\right)^2 = a_1^2 + a_2^2 + a_3^2 = \underline{a} \cdot \underline{a}$ $\therefore$

$u_i + (\underline{a}\cdot\underline{b})v_i = (\underline{a}\cdot a)(\underline{b}\cdot\underline{v})a_i$

introduce dummy suffix which is repeated & ∴ summer over:

$u_i + a_j b_j v_i = a_j a_j b_k v_k a_i$

\probl 2.8/ $A = \begin{bmatrix} A_{11} & A_{12} & A_{13} \\ A_{21} & A_{22} & A_{23} \\ A_{31} & A_{32} & A_{33} \end{bmatrix}$ ∴ A_{ij} is z entry in z i^{th} row & z j^{th} column

$\mathrm{Trace}(A) = A_{11} + A_{22} + A_{33} = A_{jj} = \sum_{j=1}^{3} A_{jj}$

\probl 2.9/ $\delta_{ij} a_j$ ∴ $\delta_{ij} a_j = \sum_{j=1}^{3} \delta_{ij} a_j = \delta_{i1} a_1 + \delta_{i2} a_2 + \delta_{i3} a_3$

z result depends on z val of i ∴

take $i=1$: $\delta_{i1} = \delta_{11} = 1$ while $\delta_{i2} = \delta_{i3} = 0$ ∴ rhs $= a_1$

$i=2$: a_2

$i=3$: a_3 ∴

$\delta_{ij} a_j = a_i$

$\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$
$\varepsilon_{ijk} = 0$ otherwise
$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = +1$

\probl 2.10/ $(\underline{a}\times\underline{b})_i = \varepsilon_{ijk} a_j b_k$ ∴

take $i=1$: $(\underline{a}\times\underline{b})_1 = \varepsilon_{1jk} a_j b_k = \sum_{j=1}^{3}\sum_{k=1}^{3} \varepsilon_{1jk} a_j b_k$

ε_{ijk} is only non-zero when all 3 of its suffices are different ∴

$j=2\ k=3$ & $j=3\ k=2$ are non-zero in z double sum

$(\underline{a}\times\underline{b})_1 = \varepsilon_{123} a_2 b_3 + \varepsilon_{132} a_3 b_2 = (+1)a_2 b_3 + (-1)a_3 b_2 = a_2 b_3 - a_3 b_2$

$(\underline{a}\times\underline{b})_1 = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \underline{i}\,(a_2 b_3 - a_3 b_2)$ ✓

take $i=2$:

take $i=3$:

\probl 2.11/ $\underline{a}\cdot(\underline{b}\times\underline{c}) = (\underline{a}\times\underline{b})\cdot\underline{c}$ ∴

$\varepsilon_{ijk} = \varepsilon_{kij}$

$\underline{a}\cdot(\underline{b}\times\underline{c}) = a_i(\underline{b}\times\underline{c})_i = a_i(\varepsilon_{ijk} b_j c_k) = \varepsilon_{ijk} a_i b_j c_k = \varepsilon_{kij} a_i b_j c_k$

$= (\varepsilon_{kij} a_i b_j) c_k = (\underline{a}\times\underline{b})_k c_k = (\underline{a}\times\underline{b})\cdot\underline{c}$ $\square$

\probl 2.12/ $\underline{a}\times(\underline{b}\times\underline{c}) = (\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}$ use suffix or no marks ∴

consider z i^{th} component of z LHS: $[\underline{a}\times(\underline{b}\times\underline{c})]_i = \varepsilon_{ijk} a_j (\underline{b}\times\underline{c})_k =$

$\varepsilon_{ijk} a_j \varepsilon_{klm} b_l c_m = \varepsilon_{ijk}\varepsilon_{klm} a_j b_l c_m = \varepsilon_{kij}\varepsilon_{klm} a_j b_l c_m$ ∴

week 1 / know that $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$

(midmid endend − midend endmid)

$\therefore$ have $[\underline{a}\times(\underline{b}\times\underline{c})]_i = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})a_j b_l c_m$

$= \delta_{il}\delta_{jm} a_j b_l c_m - \delta_{im}\delta_{jl} a_j b_l c_m = \delta_{il} b_l \delta_{jm} a_j c_m - \delta_{im} c_m \delta_{jl} b_l a_j$

$= b_i a_m c_m - c_i b_j a_j = b_i(\underline{a}\cdot\underline{c}) - c_i(\underline{b}\cdot\underline{a}) = [(\underline{a}\cdot\underline{c})\underline{b} - (\underline{a}\cdot\underline{b})\underline{c}]_i$ $\square$

prob 2.13 / $\nabla\cdot(\phi\underline{F}) = \phi\nabla\cdot\underline{F} + \underline{F}\cdot\nabla\phi$

$\nabla\cdot\underline{F} = (\nabla)_i(\underline{F})_i = \nabla_i F_i = \frac{\partial}{\partial x_i}F_i$

$\underline{F}\cdot\nabla = F_i\nabla_i = F_i\frac{\partial}{\partial x_i}$

$(\underline{F}\cdot\nabla)\phi = F_i\nabla_i\phi = F_i\frac{\partial\phi}{\partial x_i}$

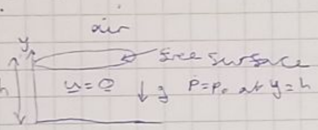
$\nabla\cdot(\phi\underline{F}) = \frac{\partial}{\partial x_i}(\phi\underline{F})_i = \frac{\partial}{\partial x_i}(\phi F_i) = \phi\frac{\partial F_i}{\partial x_i} + F_i\frac{\partial\phi}{\partial x_i}$ (product rule for 2 scalars)

$= \phi\nabla\cdot\underline{F} + (\underline{F}\cdot\nabla)\phi$

{note $(\underline{F}\cdot\nabla)\phi = \underline{F}\cdot\nabla\phi$ $\quad (\underline{F}\cdot\nabla)\underline{G} \neq \underline{F}\cdot\nabla\underline{G}$}

week 2 /

prob 2.14 /



$\underline{g} = -g\underline{j}$ $\qquad$ y comp of N-S eqn: $0 = -\frac{\partial p}{\partial y} - \rho g$ $\therefore \frac{\partial p}{\partial y} = -\rho g$ $\therefore$

$p = -\rho g y + f(x,t)$

x comp of N-S eqn: $0 = -\frac{\partial p}{\partial x}$ $\therefore \frac{\partial f}{\partial x} = 0$ $\therefore$

$p = f(t) - \rho g y$

$\therefore$ at the free surf $y = h$ $p = p_0$ (a const) $\therefore$ $p_0 = f(t) - \rho g h$ $\therefore$

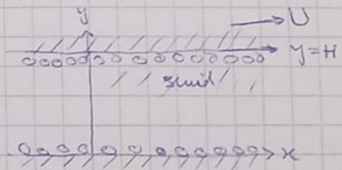
$f(t) = p_0 + \rho g h$ $\therefore$

$p = p_0 + \rho g(h - y)$

$\therefore$ @ ~~$y = p = p_0$~~ $y = h$: $p = p_0$ $\therefore$ as $y \to 0$ $\rho g(h-y)$ increases $\therefore$

pressure increases with depth

prob 2.15 / plane Couette flow /



the motion of the boundary ($y = H$) is driving the flow so the fluid moves in the x-direction

$\underline{u} = u\underline{i}$ $\therefore$

no special location in x. $\therefore$ expect u to be independent of x

const boundary speed U implies const fluid speed u $\therefore$ independent of time

2 boundary conditions at $y=0$ & $y=H$ will make $u=u(y)$ $\therefore \underline{u}=u(y)\underline{i}$

NS eqn: $\rho\{\overbrace{\frac{\partial \underline{u}}{\partial t}}^{0}+\overbrace{\underline{u}\cdot\nabla\underline{u}}^{0}\}=-\nabla p+\mu\nabla^2\underline{u}+\cancel{\rho\underline{g}}$ ← ignore

$\therefore \underline{u}\cdot\nabla = u\underline{i}\cdot(\underline{i}\frac{\partial}{\partial x}+\underline{j}\frac{\partial}{\partial y}+\underline{k}\frac{\partial}{\partial z}) = u\frac{\partial}{\partial x}$ $\therefore$

$(\underline{u}\cdot\nabla)\underline{u} = (\underline{u}\cdot\nabla)(u\underline{i}) = u\frac{\partial}{\partial x}(u\underline{i}) = \underline{i}u\frac{\partial u}{\partial x} = \underline{0}$

x comp: $\frac{\partial p}{\partial x} = \mu\underbrace{\frac{\partial^2 u}{\partial y^2}}_{\text{func of } y \text{ only}}$

$\therefore$ sep of variables $\therefore \frac{\partial p}{\partial x}$ is indep of x $\therefore P=Ax+B$

no imposed pressure gradient $\therefore$ pressure as $x\to-\infty$ must be the same as the pressure as $x\to+\infty$. $\therefore$ $P_L=A(-\infty)+B=P_R=A(\infty)+B$ $\therefore$ $A=0$ $\therefore$

Pressure is not a func of x

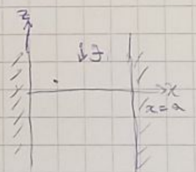
$\therefore$ x comp of NS: $0=\mu\frac{\partial^2 u}{\partial y^2}$ $\therefore u=Cy+D$

apply no slip boundary conditions "flow velocity equals the boundary velocity at the boundary" $\therefore \underline{u}=u\underline{i}$ $\therefore$ $u(y=0)=0$ (stationary boundary) $\therefore D=0$ $\therefore$

$u(y=H)=U$ (moving boundary) $\therefore$ $C=U/H$ $\therefore$

plane Couette flow $u=\frac{U}{H}y$

prob 2.16 poiseuille flow



NS eqn: x comp: $0=-\frac{\partial p}{\partial x}$ $\therefore$ pressure is indep of x

z comp $0=-\frac{\partial p}{\partial z}-\rho g+\mu\frac{\partial^2 u}{\partial x^2}$ $\therefore$ seperation of variable $\therefore \frac{dp}{dz}$ is indep of z

$\therefore$ assume no imposed pressure gradient (eg a long column of fluid open to the atmosphere at each end) to obtain $\partial p/\partial z=0$

$\therefore$ the z-comp of NS eqn $\therefore$ $\mu\frac{d^2u}{dx^2}=\rho g$ $\therefore$ $u=\frac{\rho g x^2}{2\mu}+Ax+B$ $\therefore$

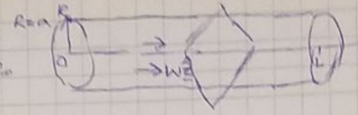
no slip boundary conditions: $u=0$ at $x=0$ $\therefore$ $B=0$

$u=0$ at $x=d$ $\therefore$ $A=-\frac{\rho g d}{2\mu}$ $\therefore$

plane poiseuille flow $u=\frac{\rho g(x-d)x}{2\mu}$

week 2 /

prob 2.18 /

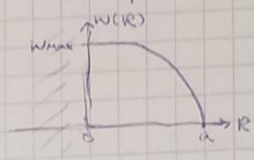


$\underline{u} = W(R)\,\hat{\underline{z}}$ note: $R \to 0$ ∴ $W(R) \to \pm\infty$ depending on z sign of A

an infinite velocity along z centre line is unrealistic, so set $A=0$

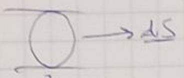
applying z no-slip condition at $R=a$, $W(R) = W(R=a) = 0$ ∴ $B = \frac{Ga^2}{4\mu}$

Hagen-poiseuille flow $W = \frac{G(a^2-R^2)}{4\mu}$ $G = \frac{P_a - P_L}{L}$



$W_{max} = W(R=0) = Ga^2/4\mu$

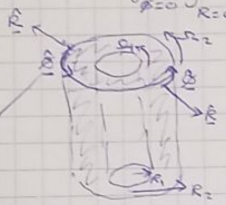
volume flux through a cross-section at const z:



∴ RdR $\underline{dS} = dS\,\hat{\underline{z}} = RdRd\phi\,\hat{\underline{z}}$ ∴

$$Q = \iint \underline{u}\cdot\underline{dS} = \int_{\phi=0}^{2\pi}\int_{R=0}^{a} W(R)\,\hat{\underline{z}}\cdot RdRd\phi\,\hat{\underline{z}} = \frac{\pi G a^4}{8\mu}$$

prob 2.19 /



in general $\underline{u} = u\hat{\underline{R}} + v\hat{\underline{\phi}} + w\hat{\underline{z}}$ ∴

$$\underline{u}\cdot\nabla = \left(u\hat{\underline{R}} + v\hat{\underline{\phi}} + w\hat{\underline{z}}\right)\cdot\left(\hat{\underline{R}}\frac{\partial}{\partial R} + \hat{\underline{\phi}}\frac{1}{R}\frac{\partial}{\partial\phi} + \hat{\underline{z}}\frac{\partial}{\partial z}\right)$$

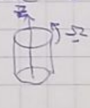
$= u\frac{\partial}{\partial R} + \frac{v}{R}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial z}$ here $u=w=0$ ∴ $\underline{u}\cdot\nabla = \frac{v}{R}\frac{\partial}{\partial\phi}$ ∴

$$(\underline{u}\cdot\nabla)\underline{u} = \frac{v}{R}\frac{\partial}{\partial\phi}(v\hat{\underline{\phi}}) = \big(\underbrace{\tfrac{v}{R}\tfrac{\partial}{\partial\phi}v}_{=0}\big)\hat{\underline{\phi}} + \frac{v}{R}v\frac{\partial\hat{\underline{\phi}}}{\partial\phi} = \frac{v^2}{R}(-\hat{\underline{R}})$$

prob 2.20 / angular velocity is related to z linear velocity by $\underset{\text{linear velocity}}{\underline{u}} = \underset{\text{angular velocity}}{\underline{\Omega}} \times \underset{\text{position vec}}{\underline{r}}$

$$= \begin{vmatrix} \hat{\underline{R}} & \hat{\underline{\phi}} & \hat{\underline{z}} \\ 0 & 0 & \Omega \\ R & 0 & z \end{vmatrix}$$

{taking $\underline{\Omega} = \Omega\hat{\underline{z}}$



$= R\Omega\,\hat{\underline{\phi}} = v\,\hat{\underline{\phi}} = v(R)\hat{\underline{\phi}}$ ∴ $v = R\Omega$ ∴

no-slip condition: R_1, Ω_1 : $v(R_1) = R_1\Omega_1 = AR_1 + \frac{B}{R_1}$

R_2, Ω_2 : $v(R_2) = R_2\Omega_2 = AR_2 + \frac{B}{R_2}$ ∴ Solve for A & B ∴

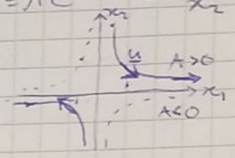
$$A = \frac{R_2^2\Omega_2 - R_1^2\Omega_1}{R_2^2 - R_1^2} \qquad B = \frac{R_1^2R_2^2(\Omega_1 - \Omega_2)}{R_2^2 - R_1^2}$$

\week 3/ \probl 2-2/ $\therefore$ streamlines $\frac{dx_1}{u_1} = \frac{dx_2}{u_2} = \cancel{\frac{dx_3}{u_3}}$ since $u_3 = 0$ $\therefore$

$\frac{dx_1}{u_1} = \frac{dx_2}{u_2}$ $\therefore$ $\frac{dx_1}{x_1} = \frac{dx_2}{-x_2}$ $\therefore \int \frac{dx_1}{x_1} = \int \frac{dx_2}{-x_2}$ $\therefore$

$\ln x_1 = -\ln x_2 + C$ $\therefore e^{\ln x_1} = e^{-\ln x_2 + C} = e^{-\ln x_2} e^{C} = Ae^{-\ln x_2} = Ae^{\ln(x_2^{-1})} = \frac{A}{x_2}$

$\therefore x_1 = A/x_2$ $\therefore x_2 = A/x_1$ $\therefore$ hyperbola $\therefore$



$(x_1, x_2) = (x, y)$	$u = (x_1, -x_2)$	$\lvert u \rvert$
$(1, 1)$	$(1, -1)$	$\sqrt{2}$
$(-1, -1)$	$(-1, 1)$	$\sqrt{2}$
$(\infty, 0)$	$(\infty, 0)$	∞

Deformation Matr $D_{ij} = \frac{\partial u_i}{\partial x_j} = e_{ij} + \xi_{ij} = \overbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}^{e_{ij}} + \overbrace{\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)}^{\xi_{ij}}$

e_{ij} is symmetric since $e_{ji} = e_{ij}$

ξ_{ij} is anti symmetric since $\xi_{ji} = -\xi_{ij}$

$u = (x_1, -x_2, 0)$ $\therefore \frac{\partial u_1}{\partial x_1} = \frac{\partial x_1}{\partial x_1} = 1$ $\frac{\partial u_1}{\partial x_2} = \frac{\partial}{\partial x_2}(x_1) = 0$ $\frac{\partial u_1}{\partial x_3} = 0$

$\frac{\partial u_2}{\partial x_1} = \frac{\partial}{\partial x_1}(-x_2) = 0$ $\frac{\partial u_2}{\partial x_2} = \frac{\partial}{\partial x_2}(-x_2) = -1$ $\therefore$ $e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}(0+0) = 0$

$e_{ij} = \begin{bmatrix} \frac{1}{2}(1+1) = 1 & \frac{1}{2}(0+0) = 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\xi_{ij} = 0 \ \forall ij$ $\therefore \xi_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}(0-0) = 0$

{diagonals of all antisymmetric matrs are zero}

$e_{ii} = \sum_{i=1}^{3} e_{ii} = e_{11} + e_{22} + e_{33} = 1 - 1 + 0 = 0$

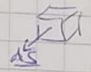
$\therefore \nabla \cdot \underline{u} = \frac{\partial}{\partial x_1}(x_1) + \frac{\partial}{\partial x_2}(-x_2) + \frac{\partial}{\partial x_3}(0) = 1 - 1 + 0 = 0$ $\therefore 0 = 0$ agree ✓

\probl 2.23/ ith component of z momentum balance eqn:

$\int_V \rho \frac{Du_i}{Dt} dV = \int_V \rho\left(\frac{\partial u_i}{\partial t} + \underline{u} \cdot \nabla u_i\right) = \int_V \rho g_i \, dV + \int_S \Sigma_i \, dS$

$= \int_V \rho g_i \, dV + \int_S \sigma_{ij} \hat{n}_j \, dS$

{recall z divergence thm: $\int_V \nabla \cdot \underline{F} \, dV = \int_S \underline{F} \cdot d\underline{S}$



is $\underline{F} = \underline{a}\,\phi(x)$ gives $\int_V \frac{\partial \phi}{\partial x_j} dV = \int_S \phi \hat{n}_j \, dS$ }

$\therefore$ take $\phi = \sigma_{ij}$ $\therefore$ $\int \sigma_{ij} \hat{n}_j \, dS = \int_V \frac{\partial}{\partial x_j} \sigma_{ij} \, dV$ $\therefore$ z momentum balance eqn becomes:

$\int_V \rho \frac{Du_i}{Dt} dV = \int_V \rho g_i \, dV + \int_V \frac{\partial \sigma_{ij}}{\partial x_j} dV$ $\therefore$ since holds for arbit volumes $\therefore$

$\rho \frac{Du_i}{Dt} = \rho\left(\frac{\partial u_i}{\partial t} + \underline{u} \cdot \nabla u_i\right) = \rho g_i + \frac{\partial}{\partial x_j} \sigma_{ij}$

\probl 2.25/ $\sigma_{ij} = -p\delta_{ij} + d_{ij} = -p\delta_{ij} + 2\mu e_{ij} = -p\delta_{ij} + 2\mu \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ $\therefore$

$\frac{\partial \sigma_{ij}}{\partial x_j} = -\frac{\partial}{\partial x_j} = -\frac{\partial}{\partial x_j}(p\delta_{ij}) + \frac{\partial}{\partial x_j}\left\{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right\} =$

\week 3/ $-\delta_{ij}\frac{\partial p}{\partial x_j} + \mu\left\{\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \frac{\partial^2 u_j}{\partial x_j \partial x_i}\right\} =$

$$-\frac{\partial p}{\partial x_i} + \mu\frac{\partial^2 u_i}{\partial x_j \partial x_j} + \mu\frac{\partial}{\partial x_i}\underbrace{\left(\frac{\partial u_j}{\partial x_j}\right)}_{=\nabla\cdot\underline{u}=0 \text{ for incompressible fluid dynamics}}$$

$= -(\nabla p)_i + \mu\nabla^2 u_i$

$\therefore$ momentum balance eqn becomes:

$\rho\left(\frac{\partial u_i}{\partial t} + \underline{u}\cdot\nabla u_i\right) = \rho g_i + \mu\nabla^2 u_i - (\nabla p)_i$ is the i^{th} component of the NS eqn $\therefore$

$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho\underline{g} + \mu\nabla^2\underline{u}$ with $\nabla\cdot\underline{u}=0$ {flow is incompressible}

\prob 3.4/ $\underline{x}' = \frac{x}{L} \therefore \frac{dx'}{dx} = \frac{1}{L} \therefore$ ~~dy~~ $y' = \frac{y}{L} \therefore \frac{dy'}{dy} = \frac{1}{L} \therefore$

$dx' = \frac{1}{L}dx \therefore \frac{d}{dx} = \frac{1}{L}\frac{d}{dx'} \therefore \frac{d}{dy} = \frac{1}{L}\frac{d}{dy'} \therefore \frac{d}{dz} = \frac{1}{L}\frac{d}{dz'} \therefore \nabla z$

$\frac{dt'}{dt} = \frac{1}{L/U} = \frac{U}{L} \therefore dt' = \frac{U}{L}dt \therefore \frac{d}{dt} = \frac{U}{L}\frac{d}{dt'} \therefore \nabla = \frac{1}{L}\nabla' \therefore$

take the NS eqn: $\rho\left\{\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right\} = -\nabla p + \mu\nabla^2\underline{u} \therefore$

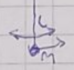
$\rho\left\{\frac{U}{L}\frac{\partial}{\partial t'}(U\underline{u}') + U^2\underline{u}'\cdot\frac{1}{L}\nabla'\underline{u}'\right\} = -\frac{1}{L}\nabla'(\rho U^2 p') + \mu\frac{1}{L^2}(\nabla')^2(U\underline{u}')$

$(\therefore \div \rho U^2/L \therefore)$ $\frac{\partial\underline{u}'}{\partial t'} + \underline{u}'\cdot\nabla\underline{u}' = \frac{L}{\rho U^2}\left\{\left(-\frac{\rho U^2}{L}\right)\nabla' p' + \frac{\mu U}{L^2}(\nabla')^2\underline{u}'\right\} =$

$$-\nabla' p' + \frac{L}{\rho U^2}\mu\frac{U}{L^2}\nabla'^2\underline{u}' = -\nabla' p' + \underbrace{\frac{\mu}{\rho U L}}_{\text{check this ratio is dimensionless}}\nabla'^2\underline{u}'$$

$\therefore$ Dimensionless NS eqn: $\frac{\partial\underline{u}'}{\partial t'} + \underline{u}'\cdot\nabla'\underline{u}' = -\nabla' p' + R_e^{-1}\nabla'^2\underline{u}'$

where $R_e^{-1} = \frac{\mu}{\rho U L}$ $R_e = \frac{\rho U L}{\mu}$ is Reynolds number

\week 4/ \prob 3.6/  Buckingham-Π thm: $y = f(a_1, a_2$

$y = f(a_1, a_2, \ldots a_k, b_1, b_2, \ldots b_m) \therefore$

$\Pi = \Phi(\Pi_1, \Pi_2, \ldots \Pi_m)$

quantity of interest: P

governing params: m, L, g

$[P] = T$ $\underbrace{[m] = M}_{a_1}$ $\underbrace{[L] = L}_{a_2}$ $\underbrace{[g] = LT^{-2}}_{a_3}$

recall $a_1, a_2, \ldots a_k$ have indep dimension $(\therefore k=3)$

$b_1, b_2, \ldots b_m$ other params $\therefore (m=0) \therefore$ (total $n = 3 = m+k$ ✓)

$\Pi = \dfrac{P}{a_1^{\alpha} a_2^{\beta} a_3^{\gamma}}$ $\therefore [P] = T = M^{\alpha} L^{\beta} (LT^{-2})^{\gamma}$

$T: 1 = -2\gamma$ $\therefore$ $\gamma = -1/2$

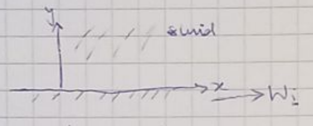
$M: 0 = \alpha$ $\therefore L = 0 = \beta + \gamma = \beta - \frac{1}{2}$ $\therefore \beta = \frac{1}{2}$ $\therefore$

$\Pi = \dfrac{P}{M^0 L^{1/2} g^{-1/2}}$ $\therefore$ $\Pi = \dfrac{P}{L^{1/2} g^{-1/2}}$

Buckingham-Π: $\Pi = \text{const}$ $\therefore \dfrac{P}{L^{1/2} g^{-1/2}} = C$ $\therefore$ $P = c\sqrt{L/g}$

{$c = 2\pi$ but dimensional analy can't tell you}

\prob 3.7/



this is a probl in the xy-plane

plate is infinite and flat & $\therefore$ there is no dependence on x

most generally $\underline{u} = (u(y,t), v(y,t), 0)$

$\nabla \cdot \underline{u} = 0$ $\therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$ $\frac{\partial u}{\partial x} = 0$ $\therefore$ $\frac{\partial v}{\partial y} = 0$ $\therefore v$ is indep of y

no slip boundary condition: on $y = 0$: $\underline{u} = W\underline{i}$ $\therefore u(y=0) = W$, $v(y=0) = 0$ $\therefore$

since v is indep of y & $v(y=0) = 0$, $v = 0$ everywhere $\therefore$

$\underline{u} = (u(y,t), 0, 0)$

\prob 3.8/ N-S eqn: $\rho\left\{\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u}\right\} = -\nabla p + \mu \nabla^2 \underline{u}$

y comp: $\underline{u} \cdot \nabla v = u(y,t)\underline{i} \cdot (\underline{i}\partial_x + \underline{j}\partial_y + \underline{k}\partial_z) v = (\underline{u} \cdot \nabla) v =$

$u \frac{\partial}{\partial x} v = 0$ since v is zero

$-\frac{\partial p}{\partial y} = 0$ $\therefore p = p(x,t)$ only

x comp: $\rho \frac{\partial u}{\partial t} + \underbrace{\rho u \frac{\partial}{\partial x} u}_{\underline{u} \cdot \nabla \, = 0} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u$

$\therefore$

$\therefore$ seperate variables: $\underbrace{\rho \frac{\partial u}{\partial t} - \mu \nabla^2 u}_{\text{not a func of } x} = -\frac{\partial p}{\partial x}$ potentially a func of x

$\therefore \frac{\partial p}{\partial x}$ is indep of x $\therefore$ take $-\frac{\partial p}{\partial x} = G(t)$ $\therefore$ $-p = Gx + F$

$\therefore$ then $p \to p_0$ as $y \to \infty$ $\forall x$ $\therefore$ $F = p_0$ $\therefore G = 0$ $\therefore \frac{\partial p}{\partial x} = 0$

$\therefore \rho \frac{\partial u}{\partial t} = \mu \frac{\partial^2 u}{\partial y^2}$ $\therefore$ $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$ where $\nu = \mu/\rho$

$\therefore$ no slip on the boundary $\therefore$ at $t = 0$: $u(y=0, t=0) = 0$ (fluid & boundary at rest)

for $t > 0$ $u(y=0, t) = W$ $\therefore$ $u(y \to \infty, t \geq 0) \to 0$ "far-field condition" (fluid is at rest at ∞)

\week 4/ \probl 3.9/ we expect that the slow speed u will depend on 4 governing params: y, t, W, ν $[y]=L$ $[t]=T$ $[W]=LT^{-1}$ $[\nu]=L^2T^{-1}$

$\{ \rho = m/V \therefore [\rho]=[m/V]=ML^{-3} \therefore \nu=\mu/\rho \quad [\nu]=L^2T^{-1} \therefore$
$\nu\rho=\mu \therefore [\mu]=[\nu][\rho]=L^2T^{-1}ML^{-3}=ML^{-1}T^{-1} \}$

$\therefore$ take $a_1=y$ $a_2=t$ have indep dimensn $\therefore k=2 \therefore$

$b_1=W$ $b_2=\nu$ $\therefore m=2$ $\therefore b_1$ & b_2 have dependent dimensn

$\therefore n=m+k=2+2=4 \therefore$

Quantity of interest $\Pi = \frac{u}{yt^{-1}} \therefore$

$\Pi_1 = \frac{W}{yt^{-1}}$, $\Pi_2 = \frac{\nu}{y^2t^{-1}}$

Buckingham-Π thm: $\Pi = \Phi(\Pi_1, \Pi_2) \therefore$

$$\frac{u}{yt^{-1}} = \Phi\left(\frac{W}{yt^{-1}}, \frac{\nu}{y^2t^{-1}}\right) = \Phi\left(\frac{Wt}{y}, \frac{\nu t}{y^2}\right) \therefore u = \frac{y}{t}\Phi\left(\frac{Wt}{y}, \frac{\nu t}{y^2}\right)$$

\probl 3.11/ $\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}$ $u(y=0)=W$ $u(y\to\infty)\to 0$

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(WF(\eta)) = W\frac{\partial}{\partial t}(F(\eta)) = W\frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial t} = WF'\frac{\partial \eta}{\partial t}$$

where $\frac{\partial \eta}{\partial t} = y\left(-\frac{1}{2}\right)(4\nu t)^{-3/2}4\nu = \frac{-y4\nu}{2(4\nu t)(4\nu t)^{1/2}} = \frac{-\eta}{2t}$

$$\Rightarrow \frac{\partial u}{\partial t} = WF'\left(\frac{-\eta}{2t}\right)$$

$$\frac{\partial u}{\partial y} = W\frac{\partial}{\partial y}F(\eta) = WF'\frac{\partial \eta}{\partial y} = \frac{WF'}{\sqrt{4\nu t}}$$

$$\frac{\partial^2 u}{\partial y^2} = \frac{W}{\sqrt{4\nu t}}\frac{\partial}{\partial y}(F') = \frac{W}{\sqrt{4\nu t}}F''\frac{1}{\sqrt{4\nu t}} = \frac{WF''}{4\nu t}$$

$\therefore$ sub into the governing eqn: $\cancel{WF'(-\eta)}$ $WF'\left(-\frac{\eta}{2t}\right) = \nu\frac{WF''}{4\nu t} \Rightarrow$

$-\frac{\eta F'}{2} = \frac{F''}{4} \Rightarrow \cancel{F''+q}$ $F''+2\eta F'=0$ with boundary conditions

$u(y=0)=W \Rightarrow WF(\eta=0)=W \Rightarrow F(\eta=0)=1$

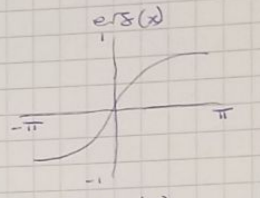
$u(y\to\infty)\to 0 \Rightarrow WF(\eta\to\infty)\to 0 \Rightarrow F(\eta\to\infty)\to 0$

\prob 3.12/ $F'' + 2\eta F' = 0 \Rightarrow \frac{d}{d\eta}(F') = -2\eta F'$

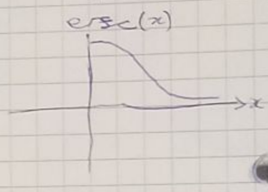
$\Rightarrow \int \frac{dF'}{F'} = -\int 2\eta\, d\eta \Rightarrow \ln F' = -\eta^2 + c \Rightarrow$

$F' = Ae^{-\eta^2}$ $\quad A = e^c = \text{Const}$

$\Rightarrow F(\eta) = A\int_{s=0}^{\eta} e^{-s^2}\, ds + B$



error func $\quad erf(x) = \frac{2}{\sqrt{\pi}}\int_{s=0}^{x} e^{-s^2}\, ds$



complementary error func

$erfc(x) = \frac{2}{\sqrt{\pi}}\int_{x}^{\infty} e^{-s^2}\, ds$

$erf(x) + erfc(x) = 1$

$erf(0) = 0 \quad erfc(\infty) = 0$

$erfc(0) = 1 = 1 - erf(0) = 1$

$erf(\infty) = 1 - erfc(\infty) = 1$

\prob 3.13/ $F(\eta) = A\int_{s=0}^{\eta} e^{-s^2}\, ds + B = C\, erf(\eta) + B$

where $C = \frac{A\sqrt{\pi}}{2}$.

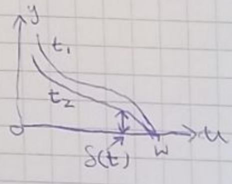
boundary conditions $F(\eta=0) = 1 \Rightarrow C\, erf(0) + B = 1 \Rightarrow B = 1$ since $erf(0) = 0$

$F(\eta \to \infty) = 0 \Rightarrow C\, erf(\infty) + 1 = 0 \Rightarrow C = -1$

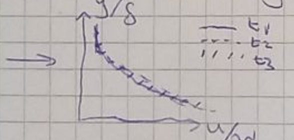
$F(\eta) = -erf(\eta) + 1 = erfc(\eta)$

$u = W F(\eta)$ where $\eta = y/\sqrt{4\nu t} \Rightarrow u = W\, erfc\left(\frac{y}{\sqrt{4\nu t}}\right)$ ~~where $\delta = \sqrt{4\nu t}$~~

$= W\, erfc\left(\frac{y}{\delta}\right)$ where $\delta = \sqrt{4\nu t}$



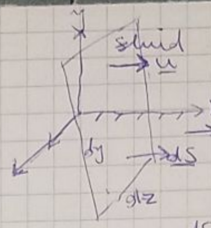
$\delta(t)$ is the boundary layer width



the sols of different times collapse onto a single curve if the flow speed is scaled by W and y is scaled by δ $\therefore$ Self-Similar Sols problem 3.14 / mass flux $= \rho \int_S \underline{u} \cdot d\underline{S}$

$= \rho \int_S u\underline{i} \cdot dy\,dz\,\underline{i} =$

$\rho \int_S u\,dy\,dz$



$dS = dy\,dz$

$\underline{dS} = dS\,\underline{i}$

mass flux unit width in z

$\rho \int_{y=0}^{\infty} u\,dy = \rho \int_{y=0}^{\infty} wF(\eta)\,dy$ $\qquad y = \sqrt{4\nu t}\,\eta \quad dy = \sqrt{4\nu t}\,d\eta$

$= \rho W \int_{\eta=0}^{\infty} F(\mu)\sqrt{4\nu t}\,d\eta = \rho W \sqrt{4\nu t} \int_{\eta=0}^{\infty} F(\eta)\,d\eta$

$= \int_{\eta=0}^{\infty} \overbrace{F(\eta)}^{u}\,\overbrace{d\eta}^{dv}$ $\qquad$ integration by parts $\int u\,dv = [uv] - \int v\,du$

$= [F\eta]_0^{\infty} - \int \eta F'\,d\eta$ $\qquad F(\infty) = \mathrm{erfc}(\infty) = 0$

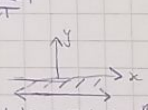
$= [0 - 0] - \int \eta F'\,d\eta$ $\qquad F = \frac{-2}{\sqrt{\pi}} \int e^{-\eta^2} d\eta + 1$

$= -\int_0^{\infty} \eta \left(\frac{-2}{\sqrt{\pi}} e^{-\eta^2}\right) d\eta = \int_0^{\infty} \eta \left(\frac{2}{\sqrt{\pi}} e^{-\eta^2}\right) d\eta = \left[-\frac{1}{\sqrt{\pi}} e^{-\eta^2}\right]_0^{\infty}$

$= \frac{-1}{\sqrt{\pi}} \left\{ \underbrace{e^{-\infty}}_{=0} - e^0 \right\} = \frac{1}{\sqrt{\pi}}$ $\qquad \therefore$ mass flux $= \frac{\rho W \sqrt{4\nu t}}{\sqrt{\pi}}$

week 5 /

problem 4.1 / $\underline{u} = u(y,t)\,\underline{i} \quad \partial_t u = \nu \partial_{yy} u$



$W\cos\Omega t\,\underline{i} = \mathbb{R}[We^{i\Omega t}]\,\underline{i}$

$\therefore u = u(y,t) = \mathbb{R}[\hat{u}(y)e^{i\Omega t}] \therefore$

$\partial_t u = \mathbb{R}[\hat{u}(y)\,i\Omega e^{i\Omega t}]$, $\partial_{yy} u = \mathbb{R}\left[\frac{d^2\hat{u}}{dy^2} e^{i\Omega t}\right] \therefore$

$\partial_t u - \nu \partial_{yy} u = \mathbb{R}\left[\left(\hat{u}\,i\Omega - \nu \frac{d^2\hat{u}}{dy^2}\right)e^{i\Omega t}\right] = 0$ for this to be zero for all times, we must have $\frac{d^2\hat{u}}{dy^2} = \frac{i\Omega}{\nu}\hat{u}(y)$ $\therefore$ sols of form $\hat{u}(y) = e^{ky} \therefore$

$k^2 e^{ky} = \frac{i\Omega}{\nu} e^{ky} \quad \therefore k^2 = \frac{i\Omega}{\nu} \quad \therefore k = \pm\sqrt{\frac{i\Omega}{\nu}} \therefore$

$z = re^{i\theta}$ $\quad$ Im, Re $\quad z = i \therefore z^{1/2} = r^{1/2} e^{i\theta/2} = 1e^{i\pi/4} = \cos(\pi/4) + i\sin\left(\frac{\pi}{4}\right) = \frac{(1+i)}{\sqrt{2}} \therefore$

$k = \pm(1+i)\sqrt{\frac{\Omega}{2\nu}} = \pm\frac{(1+i)}{\delta}$, $\delta = \sqrt{\frac{2\nu}{\Omega}} \therefore$

$\hat{u}(y) = Ae^{(1+i)y/\delta} + Be^{-(1+i)y/\delta}$

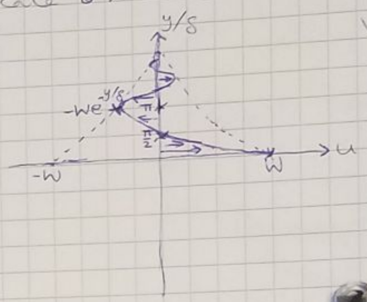
Boundary conditions: as $y \to \infty$ $\hat{u} \to 0 \quad \therefore A = 0$

on $y = 0$: $\hat{u} = W \Rightarrow B = W \therefore \hat{u}(y) = We^{-(1+i)y/\delta} \therefore$

$u = \mathbb{R}[\hat{u}(y)e^{i\Omega t}] = \mathbb{R}[We^{-y/\delta}e^{-iy/\delta}e^{i\Omega t}] = We^{-y/\delta}\mathbb{R}[e^{i(\Omega t - y/\delta)}] =$

$W \underbrace{e^{-y/\delta}}_{\text{decaying exponential}} \underbrace{\cos(\Omega t - y/\delta)}_{\text{wave travelling in } y}$ $\delta = \sqrt{\frac{2\nu}{\Omega}}$

Z cosine represents a wave propogating in Z y-direction, while Z exponential term represents a decay in y ∴ Z sol is damped (decreased) on a lengthscale δ, which is called Z boundary layer width



\prob 4.2/ $u(y) = We^{-y/\delta}\cos(\Omega t - y/\delta)$

$\delta = \sqrt{\frac{2\nu}{\Omega}}$ ∴ take $t=0$:

$u(y) = We^{-y/\delta}\cos(-y/\delta) = We^{-y/\delta}\cos(y/\delta)$

$-1 < \cos y/\delta < 1$, Z envelope is $\pm We^{-y/\delta}$

at $y=0$ $u(0)=W$, at $\frac{y}{\delta} = \frac{\pi}{2}$, $u=0$

at $\frac{y}{\delta} = \pi$: $u = -We^{-\pi}$

\prob 4.3/ {2D incompressible flow ∴} introduce a stream func

$\psi = \psi(x,y)$ $\underline{u} = \nabla \times (\psi \underline{k}) = (\partial_y \psi, -\partial_x \psi, 0)$ ∴

$\nabla \cdot \underline{u} = \nabla \cdot \nabla \times (\psi \underline{k}) \equiv 0$ ∴ flow incompressible $\forall \psi$ ∴

$\underline{\omega} = \nabla \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_y \psi & -\partial_x \psi & 0 \end{vmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & 0 \\ \partial_y \psi & -\partial_x \psi & 0 \end{vmatrix} = (0, 0, -\partial_{xx}\psi - \partial_{yy}\psi) = -\nabla^2 \psi \underline{k}$ ∴

$= -(0, 0, \partial_{xx}\psi + \partial_{yy}\psi)$

$\partial_t \underline{\omega} + \underline{u} \cdot \nabla \underline{\omega} = \underline{\omega} \cdot \nabla \underline{u} + \nu \nabla^2 \underline{\omega}$ {is vorticity eqn}

$\underline{u} \cdot \nabla = u\partial_x + v\partial_y$ ∴ $(\underline{u} \cdot \nabla)\underline{\omega} = (u\partial_x + v\partial_y)\underline{\omega} = (u\partial_x + v\partial_y)(\omega \underline{k})$, where $\omega = -\nabla^2\psi$

$\Rightarrow (\underline{u} \cdot \nabla)\underline{\omega} = (u \cdot \nabla)\omega \underline{k}$

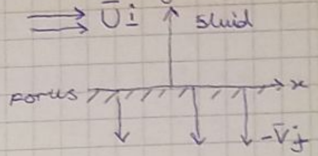
$(\underline{\omega} \cdot \nabla)\underline{u} = (\omega \underline{k} \cdot \nabla)\underline{u} = \omega \partial_z \underline{u} = 0$ ∵ $\underline{u} = \underline{u}(x,y)$ only

$\nabla^2 \underline{\omega} = \nabla^2(\omega \underline{k}) = \nabla^2 \omega \underline{k}$ {∵ $\underline{k}$ is a const vec} ∴

Z vorticity eqn becomes: $\partial_t \omega + (\underline{u} \cdot \nabla)\omega = \nu \nabla^2 \omega \Longrightarrow$

$\frac{D}{Dt}\omega = \nu \nabla^2 \omega$ where $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u} \cdot \nabla$ ∴ Z z-component of Z vorticity ($\underline{\omega} = \omega \underline{k}$) is transported in Z flow ($\frac{D\omega}{Dt}$ is Z material derivative) & vorticity undergoes viscous diffusion ($\nu \nabla^2 \omega$)

probl 4.4 2D vorticity eqn: $\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu \nabla^2 \omega$



steady flow $\therefore \frac{\partial \omega}{\partial t} = 0 \quad \therefore u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu \nabla^2 \omega$

$\underline{\omega} = \nabla \times \underline{u} = \omega \underline{k} = \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\underline{k} \qquad \underline{u} = (u, v, 0)$

but $\underline{u} = (u(y), -\bar{V}, 0) = (u, v, 0) \quad \therefore v = -\bar{V} \quad \therefore \frac{\partial v}{\partial x} = 0$ since $-\bar{V} = $ const

$\therefore \omega = -\frac{\partial u}{\partial y} = -\frac{du}{dy} \quad \therefore \omega = \omega(y)$ only $\quad \therefore \nabla^2 \omega = \frac{d^2 \omega}{dy^2}$,

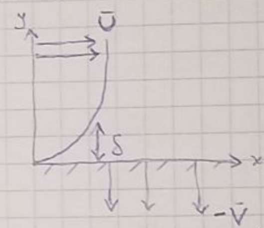
$\frac{\partial \omega}{\partial x} = 0 \quad \therefore$ 2D vorticity $\Rightarrow -\bar{V}\frac{d\omega}{dy} = -\nu \frac{d^2 \omega}{dy^2} \quad \therefore$

Seek sols $\omega \sim e^{py} \Rightarrow -\bar{V}p = \nu p^2 \quad \therefore p = 0$ or $p = -\frac{\bar{V}}{\nu} \quad \therefore$

$\omega = A + Be^{-(\bar{V}/\nu)y} = A + Be^{-y/\delta}, \quad \delta = \nu/\bar{V}$

$\omega = -\frac{du}{dy} \quad \therefore$ integrating wrt y: $u = -Ay + B\delta e^{-y/\delta} + C$

Boundary Conditions: As $y \to \infty$: $u \to \bar{U} \Rightarrow A = 0, \quad C = \bar{U}$

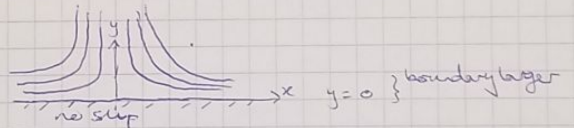


$u(y=0) = 0 \Rightarrow B\delta + \bar{U} = 0 \quad \therefore B\delta = -\bar{U} \quad \therefore$

$u = \bar{U}(1 - e^{-y/\delta})$ & $\omega = -\frac{\bar{U}}{\delta}e^{-y/\delta}, \quad \delta = \nu/\bar{V}$

Boundary layer width δ is controlled by suction $\bar{V}$ & viscosity ν.

probl 4.5



2D incompressible flow $\underline{u} = \nabla \times (\psi \underline{k}) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0\right)$

$\underline{\omega} = \nabla \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z = 0 \\ \partial_y \psi & -\partial_x \psi & 0 \end{vmatrix} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & 0 \\ \partial_y \psi & -\partial_x \psi & 0 \end{vmatrix} = \underline{i}\,0 - \underline{j}\,0 + \underline{k}\left[\partial_x(-\partial_x \psi) - \partial_y(\partial_y \psi)\right] =$

$\underline{k}(-\partial_{xx}\psi - \partial_{yy}\psi) = \underline{k}(-\nabla^2 \psi), \quad \psi = \psi(x, y)$

<u>no normal flow</u>: $v(y=0) = 0 \Rightarrow -\frac{\partial \psi}{\partial x} = 0$ ~~avg~~ on $y = 0 \Rightarrow$

$\psi = $ const on $y = 0 \quad \therefore$ without loss of generality (w.l.o.g.) take $\psi(y=0) = 0$

<u>no slip</u> conditn $u(y=0) = 0 \Rightarrow$ ~~$u(y=0)=0$~~ $\left.\frac{\partial \psi}{\partial y}\right|_{y=0} = 0$

probl 4.6 steady vorticity eqn: $u\frac{\partial}{\partial x}\omega + v\frac{\partial}{\partial y}\omega = \nu \nabla^2 \omega, \quad \omega = -\nabla^2 \omega \quad \therefore$

$u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(\frac{x\nu}{\delta}F(\eta)\right) = \frac{x\nu}{\delta}\frac{\partial F}{\partial y} = \frac{x\nu}{\delta}\frac{\partial F}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{x\nu}{\delta}F'\frac{\partial \eta}{\partial y} = \frac{x\nu}{\delta}F'\frac{1}{\delta} = \frac{x\nu}{\delta^2}F' = \frac{x}{\tau}F'$

$v = -\frac{\partial \psi}{\partial x} = \frac{\partial}{\partial x}\left(\frac{x\nu}{\delta}F(\eta)\right) = \frac{\nu}{\delta}F = -\frac{\delta}{\tau}F$

$\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = 0 - \frac{x}{\tau}\frac{\partial}{\partial y}(F') = -\frac{x}{\tau}\frac{\partial F'}{\partial \eta}\frac{\partial \eta}{\partial y} = -\frac{x}{\tau}F''\frac{\partial \eta}{\partial y} = -\frac{x}{\tau}\frac{F''}{\delta} \quad \therefore$

Sub into vorticity eqn: $\frac{x}{\tau}F'\left(\frac{-F''}{\tau\delta}\right) - \frac{\delta}{\tau}F\left(\frac{-x}{\tau\delta}\frac{\partial}{\partial y}(F'')\right) = \nu\nabla^2\left(\frac{-x}{\tau}\frac{F''}{\delta}\right) \quad \therefore$

$\nabla^2\left(\frac{-xF''}{\tau\delta}\right) = -\frac{1}{\tau\delta}(\partial_{xx}+\partial_{yy})(xF'')$ $\therefore$

$\partial_x(xF'') = F''\partial_x(x) = F''$ $\therefore$ $\partial_{xx}(xF'') = \partial_x(F'') = 0$

$\partial_y(xF'') = x\partial_y(F'') = \frac{xF'''}{\delta}$ $\therefore$ $\partial_{yy}(xF'') = \partial_y\left(\frac{xF'''}{\delta}\right) = \frac{x}{\delta}\partial_y(F''') = \frac{x}{\delta}\frac{F^{iv}}{\delta} = \frac{xF^{iv}}{\delta^2}$

$\therefore$ Sub into eqn: $-F'F''\frac{x}{\tau^2\delta} - \left(\frac{\delta F}{\tau}\right)\left(-\frac{x}{\tau\delta}F'''\frac{1}{\delta}\right) = \nu\left(0 - \frac{1}{\tau\delta}\frac{x}{\delta^2}F^{iv}\right)$ $\therefore$

$-\frac{F'F''x}{\tau^2\delta} + \frac{\delta FF'''x}{\tau^2\delta^2} = -\frac{\nu xF^{iv}}{\tau\delta^3}$ $\therefore$ $\times\frac{\tau^2\delta}{x}$:

$-F'F'' + FF''' = \cancel{\frac{\nu x F^{iv}}{\tau\delta^3}\frac{x}{\tau^2\delta}} = \cancel{-\frac{x^2F^{iv}}{\tau^3\delta^4}}$ $-\frac{\tau\nu F^{(4)}}{\delta^2}$ $= -\frac{\nu F^{iv}\tau}{\delta^2}$ $\therefore$

using $\delta^2 = \nu\tau \Rightarrow -\frac{\nu F^{iv}\tau}{\delta^2} = -\frac{\nu F^{iv}\tau}{\nu\tau} = -F^{iv} = -F'F'' + FF''' \Longrightarrow$

$F'F'' - FF''' = F^{iv} = F^{(4)}$

$\psi = 0$ on $y=0 \Rightarrow \frac{x\nu}{\delta}F(\eta) = 0$ on $y=0$ but $\eta = \frac{y}{\delta}$ $\therefore$

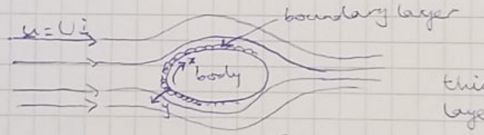
$y=0 \Rightarrow \eta = 0$ & $\frac{x\nu}{\delta}F(\eta=0) = 0$ $\therefore$ $F(\eta=0)=0$

$\frac{\partial\psi}{\partial y} = 0$ on $y=0 \Longrightarrow \frac{x}{\tau}F'(\eta) = 0$ on $\eta = 0$, $\forall x$ $\therefore$ $\frac{x}{\tau}F'(\eta=0) = 0$ $\therefore$ $F'(\eta=0)=0$

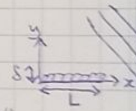
$\therefore F'(0) = 0$

Week 7

probl 4.7



thickness of boundary layer = δ



& lateral scale of boundy $= L$ $\therefore$ $\frac{\delta}{L} \ll 1$ $\therefore$

$\underline{u} = \nabla\times(\psi\underline{k}) = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0\right) = (u, v, 0)$ $\therefore$ $u = \frac{\partial\psi}{\partial y}$, $v = -\frac{\partial\psi}{\partial x}$

$\therefore$ $u = \frac{\partial\psi}{\partial y} \sim \frac{\psi}{\delta}$, $v = -\frac{\partial\psi}{\partial x} \sim \frac{\psi}{L}$ is their typical sizes $\therefore$

$\frac{u}{v} \sim \frac{\psi/\delta}{\psi/L} = \frac{L}{\delta} \gg 1$ $\because$ $\frac{\delta}{L} \ll 1 \Longrightarrow u \gg v$

probl 4.8 x comp: $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{\partial p}{\partial x} - \nu\frac{\partial^2 u}{\partial y^2} + \nu\frac{\partial^2 u}{\partial x^2}$

$u = \frac{\partial\psi}{\partial y} \sim \frac{\psi}{\delta}$, $v = -\frac{\partial\psi}{\partial x} \sim \frac{\psi}{L}$ $(\delta \ll L)$

$u \sim \frac{\psi}{\delta}$, $\frac{\partial}{\partial x} \sim \frac{1}{L}$ $\therefore$ $u\frac{\partial u}{\partial x} = u\frac{\partial}{\partial x}u \sim \frac{\psi}{\delta}\frac{1}{L}\frac{\psi}{\delta}$ ①

$v \sim \frac{\psi}{L}$, $\frac{\partial}{\partial y} \sim \frac{1}{\delta}$, $u \sim \frac{\psi}{\delta}$ $\therefore$ $v\frac{\partial u}{\partial y} = v\frac{\partial}{\partial y}u \sim \frac{\psi}{L}\frac{1}{\delta}\frac{\psi}{\delta}$ ②

$\frac{\partial}{\partial x} \sim \frac{1}{L}$ $\therefore$ $-\frac{1}{\rho}\frac{\partial}{\partial x}P \sim \frac{1}{\rho}P\frac{1}{L}$ ③

$\frac{\partial^2}{\partial x^2} \sim \frac{1}{L^2}$ $u \sim \frac{\psi}{\delta}$ $\therefore$ $\nu\frac{\partial^2 u}{\partial x^2} = \nu\frac{\partial^2}{\partial x^2}u \sim \nu\frac{1}{L^2}\frac{\psi}{\delta}$ ④

$\frac{\partial^2}{\partial y^2} \sim \frac{1}{\delta^2}$, $u \sim \frac{\psi}{\delta}$ $\therefore$ $\nu\frac{\partial^2 u}{\partial y^2} = \nu\frac{\partial^2}{\partial y^2}u \sim \nu\frac{1}{\delta^2}\frac{\psi}{\delta}$ ⑤ $\therefore$

Week 7

$① = \frac{\Psi^2}{L\delta^2}$, $② = \frac{\Psi^2}{L\delta^2}$ $\therefore$ $① \sim ②$

$④ = \nu \frac{\Psi}{\delta}\frac{1}{L^2}$ $⑤ = \nu\frac{\Psi}{\delta}\frac{1}{\delta^2}$ $\therefore$ $④ << ⑤$ $\because$ $\delta << L$ $\therefore \frac{1}{\delta} >> \frac{1}{L}$ $\therefore \frac{1}{\delta^2} >> \frac{1}{L^2}$

$\therefore$ for ③ to be of similar size to ①, ②, ⑤ we require

$\frac{1}{\rho}\frac{P}{L} \sim \frac{\Psi^2}{L\delta^2}$ $\therefore$ $\frac{P}{\rho} \sim \frac{\Psi^2}{\delta^2} \sim ⑤L = \nu\frac{\Psi}{\delta}\frac{1}{\delta^2}L = \frac{\nu\Psi L}{\delta^3}$ $\therefore$ $\frac{P}{\rho} \sim \frac{\nu\Psi L}{\delta^3}$ $(*) \sim \frac{\Psi^2}{\delta^2}$

y comp: $\underset{⑥}{u\frac{\partial v}{\partial x}} + \underset{⑦}{v\frac{\partial v}{\partial y}} = \underset{⑧}{-\frac{1}{\rho}\frac{dp}{dy}} + \underset{⑨}{\nu\frac{\partial^2 v}{\partial x^2}} + \underset{⑩}{\nu\frac{\partial^2 v}{\partial y^2}}$ $\therefore$

$\underbrace{\frac{\Psi}{\delta}\frac{1}{L}\frac{\Psi}{L} \quad \frac{\Psi}{L}\frac{1}{\delta}\frac{\Psi}{L}}_{⑥\sim⑦}$ $\quad \frac{P}{\rho\delta} \quad$ $\underbrace{\frac{\nu}{L^2}\frac{\Psi}{L} \quad \frac{\nu}{\delta^2}\frac{\Psi}{L}}_{⑨ << ⑩}$

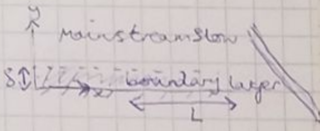
$⑧ : \frac{P}{\rho\delta} \sim \frac{\Psi^2}{\delta^2\delta} \sim ⑥\frac{L^2}{\delta^2} >> ⑥$

$⑩ : \frac{\nu\Psi}{\delta^2 L} = \frac{\nu\Psi}{\delta^2}\frac{1}{L}\frac{L}{L}\frac{\delta}{\delta} \overset{(*)}{\sim} \frac{\Psi^2}{\delta^2}\frac{\delta}{L^2} \sim \frac{\delta^2}{L^2}⑧ << ⑧$ $\therefore$

⑧ is much bigger than all of the other terms & $\therefore$ must have $\frac{\partial p}{\partial y} = 0$ ie p is indep of y. $\therefore$

pressure is constant across the boundary layer & only varies along it (ie wrt x) & so can set $\frac{dp}{dx} = -G(x)$ & $\therefore$ the x-comp of N-S becomes: $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}G + \nu\frac{\partial^2 u}{\partial y^2}$ neglecting $\frac{\partial^2 u}{\partial x^2}$ in comparison with $\frac{\partial^2 u}{\partial y^2}$.

This is known as the (steady) <u>boundary layer eqn</u>



Prob 4.9 $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{G}{\rho} + \nu\frac{\partial^2 u}{\partial y^2}$ $G = -\frac{dp}{dx}$

$\frac{\partial}{\partial y} \sim \frac{1}{\delta}$ $\therefore$ $u\frac{\partial u}{\partial x} \simeq \frac{G}{\rho} = -\frac{1}{\rho}\frac{dp}{dx} \longrightarrow \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) \simeq \frac{\partial}{\partial x}\left(-\frac{P}{\rho}\right) \Rightarrow \frac{u^2}{2} \simeq -\frac{P}{\rho} + c \Longrightarrow$

$p = -\frac{1}{2}\rho u^2 + c'$ $\because$ $\Psi = x^2 y + y^3$ $u = \frac{\partial\Psi}{\partial y} = x^2 + 3y^2$ $\left(v = -\frac{\partial\Psi}{\partial x} = -2xy\right) \Rightarrow$

$p = -\frac{1}{2}\rho(x^2 + 3y^2)^2 + C$ $\therefore$

just outside the boundary layer y is small & $\therefore$ $p \simeq -\frac{1}{2}\rho x^4 + C \Longrightarrow$

$G = -\frac{dp}{dx} = 2\rho x^3$

Prob 4.11 no-slip conditions $\therefore$ $u = 0 \Rightarrow \frac{\partial\Psi}{\partial y} = 0$, $v = 0 \Rightarrow -\frac{\partial\Psi}{\partial x} = 0$

$\Rightarrow \Psi = 0$ w.l.o.g. & far from the plate $u = \frac{\partial\Psi}{\partial y} \to U$

\prob 4.12/ $u\frac{\partial u}{\partial x} \sim \nu \frac{\partial^2 u}{\partial y^2}$

$U\frac{1}{x}U \sim \nu \frac{U}{\delta^2} \Rightarrow U\frac{1}{x}U \sim \frac{\nu U}{\delta^2} \therefore U\frac{1}{x} \sim \frac{\nu}{\delta^2} \Rightarrow$

$\delta(x) = \sqrt{\nu x / U}$

\prob 4.13/ recall: $u = \frac{\partial \psi}{\partial y}$ $\therefore$ at fixed x, can integrat over y to obtain

$\psi = \int u\,dy = \int u\,\delta(x)\,d\eta = \left(\int u\,d\eta\right)\delta(x) = \delta(x)\int U g(\eta)\,d\eta = U\delta(x)f(\eta)$.

$g = \frac{\partial f}{\partial \eta}$

recall: $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \underbrace{\frac{G}{\rho}}_{=0} + \nu\frac{\partial^2 u}{\partial y^2}$ with $u = \frac{\partial \psi}{\partial y}$ & $v = -\frac{\partial \psi}{\partial x}$ & $\therefore$

$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \left(-\frac{\partial \psi}{\partial x}\right)\frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right) = \nu\frac{\partial^2}{\partial y^2}\left(\frac{\partial \psi}{\partial y}\right) \therefore$

$\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu\frac{\partial^3 \psi}{\partial y^3}$

\prob 4.14/ $\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2} = \nu\frac{\partial^3 \psi}{\partial y^3}$ boundary layer eqn

$\psi = U\delta(x)f(\eta)$ $\delta(x) = \sqrt{\nu x / U}$ $g(\eta) = \frac{\partial f}{\partial \eta}$ $\{\eta = y/\delta(x)\}$

$\frac{\partial \psi}{\partial x} = U\left[\delta\frac{\partial f}{\partial x} + f\frac{\partial \delta}{\partial x}\right] = U\left[\delta\frac{\partial f}{\partial \eta}\frac{\partial \eta}{\partial x} + f\frac{\partial \delta}{\partial x}\right]$,

$\frac{\partial \eta}{\partial x} = y\frac{\partial}{\partial x}(\delta^{-1}) = y\left(-\frac{1}{\delta^2}\right)\frac{\partial \delta}{\partial x}$ & $\therefore$

$\frac{\partial \psi}{\partial x} = U\left[-\delta f'\frac{y}{\delta^2}\frac{d\delta}{dx} + f\frac{d\delta}{dx}\right] = U\frac{d\delta}{dx}\left[f - \frac{y}{\delta}f'\right] \Longrightarrow$

$\frac{\partial \psi}{\partial x} = U\frac{d\delta}{dx}\left[f - \eta f'\right] \therefore$

$\frac{\partial^2 \psi}{\partial y \partial x} = \frac{\partial}{\partial y}\left[U\frac{d\delta}{dx}\left[f - \eta f'\right]\right] = U\frac{d\delta}{dx}\frac{d}{dy}\left[f - \eta f'\right] = U\frac{d\delta}{dx}\left[f'\frac{\partial \eta}{\partial y} - \frac{\partial}{\partial \eta}(\eta f')\frac{\partial \eta}{\partial y}\right]$

$= U\frac{d\delta}{dx}\left[\frac{f'}{\delta} - \frac{1}{\delta}(\eta f'' + f')\right] = -U\frac{d\delta}{dx}\frac{\eta f''}{\delta}$

$\frac{\partial \psi}{\partial y} = U\delta(x)\frac{\partial f}{\partial y} = U\delta(x)f'\frac{\partial \eta}{\partial y} = Uf'$

$\frac{\partial^2 \psi}{\partial y^2} = Uf''\frac{\partial \eta}{\partial y} = \frac{Uf''}{\delta}$, $\frac{\partial^3 \psi}{\partial y^3} = \frac{U}{\delta^2}f'''$ $\therefore$ Sub into boundary layer eqn:

$Uf'\left(-U\frac{d\delta}{dx}\frac{\eta F''}{\delta}\right) - U\frac{d\delta}{dx}\left[f - \eta f'\right]\frac{Uf''}{\delta} = \nu\frac{Uf'''}{\delta^2} \Longrightarrow$

$-U\frac{d\delta}{dx}f\frac{Uf''}{\delta} = \frac{\nu Uf'''}{\delta^2} \therefore -U\delta^2\frac{d\delta}{dx}\frac{ff''}{\delta} = \nu f'''$ $\left\{\frac{d\delta^2}{dx} = 2\delta\frac{d\delta}{dx} = \frac{\nu}{U}\right\} \therefore$

$-U\delta\frac{d\delta}{dx}\frac{ff''}{\delta} = \nu f''' \therefore -U\frac{\nu}{2U}ff'' = \nu f''' \therefore -\frac{1}{2}ff'' = f''' \therefore$

$f''' + \frac{1}{2}ff'' = 0 \therefore$

Week 7/ boundary conditions: $\frac{\partial \psi}{\partial y}=0 \Rightarrow U\delta(x)\frac{\partial}{\partial y}f(\eta)=0$ on $y=0 \Rightarrow$
$f'\frac{d\eta}{dy}=0$ on $y=0 \Rightarrow f'(y=0)=0$.

$\psi=0$ on $y=0 \Rightarrow f(0)=0$

far from Z plate $\frac{\partial \psi}{\partial y}\to U \Rightarrow Uf'\to U \Rightarrow f'\to 1$ far from Z plate

probl 4.15/ $\therefore \frac{\partial^2 \psi}{\partial y^2}=U\frac{f''}{\delta} \quad \therefore \tau_0=\mu_0\frac{U}{\delta}f''(0)=\frac{U\rho\nu}{\delta}f''(0)=\rho U^2\frac{\nu}{U\delta}f''(0)$
$=\frac{\rho U^2}{\sqrt{R_e}}f''(0)$, $R_e=\frac{U^2\delta^2}{\nu^2}=\frac{U^2}{\nu^2}\frac{\nu x}{U}=\frac{Ux}{\nu}$ is Z Reynolds number based on Z distance x down Z plate $\therefore \tau_0\sim R_e^{-1/2}\sim x^{-1/2}$

Week 8/ probl 5.1/ Suppose fluid flow of magnitude U on a length scale L, giving a turnover timescale $T=L/U$ $\therefore$
$\underline{u}\sim U \quad \nabla\sim\frac{1}{L} \quad \frac{\partial}{\partial t}\sim\frac{1}{T}=\frac{U}{L}$ $\therefore$
considering Z size of each of Z terms in Z N-S eqn
(1) $\frac{\partial \underline{u}}{\partial t}\sim\frac{U}{L}U=\frac{U^2}{L}$ (2) $\underline{u}\cdot\nabla\underline{u}\sim U\frac{1}{L}U=\frac{U^2}{L}$
(3) $\nu\nabla^2\underline{u}\sim\nu U\frac{1}{L^2}=\frac{\nu U}{L^2}$ $\therefore$
(1)~(2) & $\frac{(1)}{(3)}\sim\frac{U^2}{L}\cdot\frac{L^2}{\nu U}\sim\frac{UL}{\nu}=R_e$ Z Reynolds number R_e gives Z ratio of Z inertia to viscous forces. if R_e is small then may neglect inertia & obtain Z reduced system: $0=-\nabla p+\mu\nabla^2\underline{u}$, $\nabla\cdot\underline{u}=0$, $\nu=\mu/\rho$ known as Stokes eqns (or Z slow flow eqns)

probl 5.2/ $\underline{u}(x,y)=(u,v,0)=\nabla\times(\psi(x,y)\underline{k})$ taking Z curl of Z Stokes eqn: $\underbrace{-\nabla\times\nabla p}_{=0}+\mu\nabla\times[\nabla^2(\nabla\times\psi\underline{k})]=0$ {c d ; $\therefore$ div of curl is 0, curl of grad is 0}
$\therefore \mu\nabla\times\nabla^2(\nabla\times\psi\underline{k})=0 \Rightarrow \mu\nabla^2[\nabla\times\nabla\times(\psi\underline{k})]$ since in Cartesian coords
$\nabla\times\nabla^2\underline{F}=\nabla^2(\nabla\times\underline{F})$ {easiest to prove in suffix, take i^{th} component:
$\varepsilon_{ijk}\partial_j(\nabla^2\underline{F})_k=\varepsilon_{ijk}\partial_j\partial_{mm}F_k=\partial_{mm}(\varepsilon_{ijk}\partial_jF_k)=\partial_{mm}(\nabla\times\underline{F})_i=\nabla^2(\nabla\times\underline{F})_i$ }
$\therefore \mu\nabla^2[\nabla(\nabla\cdot\psi\underline{k})-\nabla^2(\psi\underline{k})]=0$ {$\because \nabla\times\nabla\times\underline{F}=\nabla(\nabla\cdot\underline{F})-\nabla^2\underline{F}$ (see formulas)}
but $\nabla\cdot\psi\underline{k}=\frac{\partial\psi}{\partial z}=0 \quad \therefore -\mu\nabla^2\nabla^2(\psi\underline{k})=0 \Rightarrow -\mu\underline{k}\nabla^2\nabla^2\psi=0 \quad \therefore \nabla^2\nabla^2\psi=0$ $\therefore$
$\nabla^4\psi=\nabla^2(\nabla^2\psi)=0$ with appropriate boundary conditions

prob 5.3/ $\underline{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\hat{\underline{\phi}} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ 0 & 0 & r\sin\theta\frac{\Psi}{r\sin\theta} \end{vmatrix} =$

$\frac{1}{r^2\sin\theta}\left\{\hat{\underline{r}}\frac{\partial\Psi}{\partial\theta} - r\hat{\underline{\theta}}\frac{\partial\Psi}{\partial r} + r\sin\theta\hat{\underline{\phi}}\cdot 0\right\} = -\frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}\hat{\underline{r}} - \frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}\hat{\underline{\theta}}$

$= u\hat{\underline{r}} + v\hat{\underline{\theta}} \Rightarrow u = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta}$ & $v = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}$

prob 5.5/ $\underline{u} = \nabla\times\left(\frac{\Psi}{r\sin\theta}\hat{\underline{\phi}}\right) \therefore \underline{\omega} = \nabla\times\underline{u} = \nabla\times\nabla\times\left(\frac{\Psi}{r\sin\theta}\hat{\underline{\phi}}\right) \therefore$

take $\underline{F} = \frac{\Psi}{r\sin\theta}\hat{\Phi}$ & apply Z identity:

$\underline{\omega} = \nabla\left(\nabla\cdot\left(\frac{\Psi}{r\sin\theta}\hat{\underline{\phi}}\right)\right) - \nabla^2\left(\frac{\Psi}{r\sin\theta}\hat{\underline{\phi}}\right)$ {formula $\nabla\cdot\underline{F} = \ldots$}

$= 0$ {with $F_\phi = \frac{\Psi(r,\theta)}{r\sin\theta} \therefore \frac{\partial F_\phi}{\partial\phi} = 0 \therefore$}

$\underline{\omega} = -\nabla^2\left(\frac{\Psi}{r\sin\theta}\hat{\underline{\phi}}\right)$

prob 5.6/ take Z curl of Z stokes eqn: $\mu\nabla^2\underline{u} - \nabla p = \underline{0} \therefore$

$\mu\nabla\times\nabla^2\underline{u} = \underline{0} \therefore$ curl grad $\equiv 0 \Rightarrow \mu\nabla\times\left[\nabla(\nabla\cdot\underline{u}) - \nabla\times\nabla\times\underline{u}\right] = 0$ {vec identity}

but $\nabla\cdot\underline{u} = 0$ by construction $\underline{u} = \nabla\left(\frac{\Psi}{r\sin\theta}\hat{\underline{\phi}}\right) \therefore$

$\mu\nabla\times\left[-\nabla\times\nabla\times\underline{u}\right] = \underline{0} \therefore \nabla\times(\nabla\times(\nabla\times\underline{u})) = \underline{0} \therefore \nabla\times\nabla\times\underline{\omega} = 0 \therefore$

$\nabla(\nabla\cdot\underline{\omega}) - \nabla^2\underline{\omega} = 0$ (vec identity)

but $\nabla\cdot\underline{\omega} = \nabla\cdot(\nabla\times\underline{u}) = 0$

$\nabla^2\underline{\omega} = 0$ ⊛ $\therefore$ from previous ex: $\underline{\omega} = -\nabla^2\left(\frac{\Psi}{r\sin\theta}\hat{\underline{\phi}}\right) = -\frac{1}{r\sin\theta}(E^2\Psi)\hat{\underline{\phi}}$ ⊕

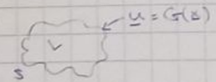
$\therefore$ from ⊛ $-\nabla^2\left(\frac{1}{r\sin\theta}E^2\Psi\hat{\underline{\phi}}\right) = 0$ {E^2 is operator on formula sheet}

& from ⊕ : $\frac{1}{r\sin\theta}E^2(E^2\Psi)\hat{\underline{\phi}} = 0$ $\left\{-\frac{1}{r\sin\theta}(E^2\Psi)\hat{\underline{\phi}} = -\nabla^2\left(\frac{\Psi}{r\sin\theta}\hat{\underline{\phi}}\right)\right.$ &

$+\nabla^2\left(-\frac{1}{r\sin\theta}(E^2\Psi)\hat{\underline{\phi}}\right) = 0 \therefore \nabla^2\left(-\nabla^2\left(\frac{\Psi}{r\sin\theta}\hat{\underline{\phi}}\right)\right) = 0 = \cancel{\frac{1}{r\sin\theta}\nabla^4\Psi} = \nabla^4\left(\frac{\Psi}{r\sin\theta}\hat{\underline{\phi}}\right)$

$\left\{-\nabla^2\left(\frac{1}{r\sin\theta}E^2\Psi\hat{\underline{\phi}}\right) = 0\right.$ & $\left.-\nabla^2\left(\frac{\Psi}{r\sin\theta}\hat{\underline{\phi}}\right) = -\frac{1}{r\sin\theta}(E^2\Psi)\hat{\underline{\phi}}\right\} \therefore$

$\cancel{-\nabla^2\left(\frac{\Psi}{r\sin\theta}\right.}$ $\frac{1}{r\sin\theta}E^2(E^2\Psi)\hat{\underline{\phi}} = 0 \therefore E^4\Psi = E^2(E^2\Psi) = 0$

prob 5.7/ 

know $0 = -\nabla p + \mu\nabla^2\underline{u}$ $\nabla\cdot\underline{u} = 0$ &

$0 = -\nabla\tilde{p} + \mu\nabla^2\tilde{\underline{u}}$ $\therefore$ Subtract: $0 = -\nabla p + \nabla\tilde{p} + \mu\nabla^2(\underline{u} - \tilde{\underline{u}})$, $\nabla\cdot(\underline{u} - \tilde{\underline{u}}) = 0$

$0 = -\nabla q + \mu\nabla^2\underline{w}$, $q = p - \tilde{p}$, $\nabla\cdot\underline{w} = 0$ with Z boundary condition

$\underline{u} = G$ on S & $\tilde{\underline{u}} = G$ on S $\therefore$ $\underline{w} = G - G = 0$ on S

\Week 8/ \probl 5.8/ $\therefore 0 = -\nabla q + \mu \nabla^2 \underline{w}$ $\therefore$ i^{th} component:

$0 = -\frac{\partial q}{\partial x_i} + \mu \frac{\partial^2 w_i}{\partial x_j \partial x_j}$ $\therefore \times w_i$: $0 = -w_i \frac{\partial q}{\partial x_i} + \mu w_i \frac{\partial^2 w_i}{\partial x_j \partial x_j}$ $\therefore$

$$w_i \frac{\partial q}{\partial x_i} = \mu w_i \frac{\partial^2 w_i}{\partial x_j \partial x_j}$$

consider: $\frac{\partial}{\partial x_i}(w_i q) = w_i \frac{\partial q}{\partial x_i} + q \overbrace{\frac{\partial w_i}{\partial x_i}}^{= \nabla \cdot \underline{w} = 0}$ $\left\{ \frac{\partial w_i}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial w_i}{\partial x_i} = \frac{\partial w_1}{\partial x_1} + \frac{\partial w_2}{\partial x_2} + \frac{\partial w_3}{\partial x_3} = \nabla \cdot \underline{w} \right\}$

& $\nabla \cdot \underline{w} = 0$ $\therefore \frac{\partial}{\partial x_i}(w_i q) = w_i \frac{\partial q}{\partial x_i} + q(0) = w_i \frac{\partial q}{\partial x_i}$ $\therefore$

$$\frac{\partial}{\partial x_i}(w_i q) = \mu w_i \frac{\partial^2 w_i}{\partial x_j \partial x_j}$$

now consider: $\frac{\partial}{\partial x_j}\left(w_i \frac{\partial w_i}{\partial x_j}\right) = w_i \frac{\partial}{\partial x_j}\left(\frac{\partial w_i}{\partial x_j}\right) + \frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j} = w_i \frac{\partial^2 w_i}{\partial x_j \partial x_j} + \frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}$

$\therefore \frac{\partial}{\partial x_i}(w_i q) = \mu \left\{ \frac{\partial}{\partial x_j}\left(w_i \frac{\partial w_i}{\partial x_j}\right) - \frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j} \right\}$

\probl 5.9/ $\therefore \frac{\partial}{\partial x_i}(w_i q) = \mu \left\{ \frac{\partial}{\partial x_j}\left(w_i \frac{\partial w_i}{\partial x_j}\right) - \frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j} \right\}$ $\therefore$

$\int \frac{\partial}{\partial x_i}(w_i q)\, dV = \mu \int \left\{ \frac{\partial}{\partial x_j}\left(w_i \frac{\partial w_i}{\partial x_j}\right) - \frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j} \right\} dV$ by Z generalised divergence thm, this becomes {thm used: $\int_V \frac{\partial \phi}{\partial x_i}\, dV = \int_S n_i \phi\, dS$} $\therefore$

$$\int_S n_i w_i q\, dS = \mu \int_S n_j w_i \frac{\partial w_i}{\partial x_j}\, dS - \mu \int_V \frac{\partial w_i}{\partial x_j}\frac{\partial w_i}{\partial x_j}\, dV$$

\probl 5.10/ $\underline{w} = 0$ on S $\therefore$ Z surface integrals vanish B $\therefore$

$\underline{w} = \underline{0}$ on S & $\therefore$ $\mu \int_V \left(\frac{\partial w_i}{\partial x_j}\right)\left(\frac{\partial w_i}{\partial x_j}\right) dV = 0$ $\therefore$ is sum of squares $\therefore$

Z integrand: $\left(\frac{\partial w_1}{\partial x_1}\right)^2 + \left(\frac{\partial w_1}{\partial x_2}\right)^2 + \dots + \left(\frac{\partial w_3}{\partial x_3}\right)^2 \geq 0$

$\left\{ \left(\frac{\partial w_i}{\partial x_j}\right)\left(\frac{\partial w_i}{\partial x_j}\right) = \left(\frac{\partial w_i}{\partial x_j}\right)^2 = \sum_{i=1}^{3}\sum_{j=1}^{3}\left(\frac{\partial w_i}{\partial x_j}\right)^2 = \sum_{i=1}^{3}\left[\left(\frac{\partial w_i}{\partial x_1}\right)^2 + \left(\frac{\partial w_i}{\partial x_2}\right)^2 + \left(\frac{\partial w_i}{\partial x_3}\right)^2\right] \geq 0 \right.$

$\left. \because \left(\frac{\partial w_i}{\partial x_j}\right)^2 \geq 0 \because \right.$ a square is always positive}

& $\therefore$ for this derivative to be zero, every derivative must be zero everywhere is V. $\therefore$ $\underline{w}$ is const in V.

Z boundary condition {$\underline{w} = \underline{0}$ on S} is $\underline{w} = \underline{0}$ on S & $\underline{w}$ is const $\therefore$

$\underline{w} = \underline{0}$ everywhere

$\therefore \underline{w} = \underline{u} - \underline{\tilde{u}} = \underline{0}$ $\therefore \underline{u} = \underline{\tilde{u}}$ $\therefore$ {Z two sols are Z same}

$\therefore$ Z sol is unique

week 9 prob 5.11 $\therefore \underline{u} = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial\theta}\hat{r} - \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\hat{\theta}$

$= u\hat{r} + v\hat{\theta}$ $\therefore \underline{r} = x\underline{i} + y\underline{j} + z\underline{k} = r\sin\theta\cos\phi\,\underline{i} + r\sin\theta\sin\phi\,\underline{j} + r\cos\theta\,\underline{k}$

$\therefore \hat{r} = \sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k}$ &

$\hat{\theta} = \cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\phi\,\underline{j} - \sin\theta\,\underline{k}$ $\therefore$ $\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} =$

$\sin\theta\cos\phi\cos\theta\,\underline{i} - \sin\theta\cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\theta\sin\phi\,\underline{j} - \sin\theta\cos\theta\sin\phi\,\underline{j} +$

$\cos^2\theta\,\underline{k} + \sin^2\theta\,\underline{k} =$

$\cos^2\theta\,\underline{k} + \sin^2\theta\,\underline{k} = \underline{k} = \hat{z}$ $\{= (0,0,1)\}$ $\therefore \cos\theta\,\hat{r} - \sin\theta\,\hat{\theta} = \hat{z}$ $\therefore$

As $r \to \infty$ Z flow approaches Z uniform flow $\underline{u} = U\hat{z} =$

$U(\hat{z}) = U(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) = U\cos\theta\,\hat{r} - U\sin\theta\,\hat{\theta}$ $\therefore$

~~u=r~~ $\underline{u} = u\hat{r} + v\hat{\theta}$ $\therefore$

As $r \to \infty$: $u = U\cos\theta$ & $v = -U\sin\theta$ both valid for large r

prob 5.13 $E^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)$ $\therefore$

$\frac{\partial^2\psi}{\partial r^2} = \frac{d^2 f}{dr^2}\sin^2\theta$, $\frac{\partial\psi}{\partial\theta} = f(r)\,2\sin\theta\cos\theta$ $\therefore$

$\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta} = 2f(r)\cos\theta$ $\therefore \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = -2f(r)\sin\theta$ $\therefore$

$\frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) = \frac{-2f(r)\sin^2\theta}{r^2}$ $\therefore$

$E^2\psi = \frac{d^2 f}{dr^2}\sin^2\theta - 2\frac{f(r)\sin^2\theta}{r^2} = \left(\frac{d^2 f}{dr^2} - \frac{2f}{r^2}\right)\sin^2\theta$

prob 5.14 recall ~~f~~ $\psi = f(r)\sin^2\theta$, $E^2\psi = \sin^2\theta\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)f$ $\therefore$

$E^2\psi = \sin^2\theta\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)r^{\alpha}$ $\{\alpha = \text{const}\}$

$= \sin^2\theta\left\{(\alpha)(\alpha-1)r^{\alpha-2} - 2r^{\alpha-2}\right\} = \sin^2\theta\left[\alpha(\alpha-1) - 2\right]r^{\alpha-2}$ $\therefore$

$E^4\psi = E^2(E^2\psi) = E^2\left\{\sin^2\theta\left[\alpha(\alpha-2) - 2\right]r^{\alpha-2}\right\} = \sin^2\theta\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)\left\{\left[\alpha(\alpha-1) - 2\right]r^{\alpha-2}\right\}$

$= \sin^2\theta\left[\alpha(\alpha-1) - 2\right]\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)r^{\alpha-2} = \sin^2\theta\left[\alpha(\alpha-1) - 2\right]\left\{(\alpha-2)(\alpha-3)r^{\alpha-4} - 2r^{\alpha-4}\right\}$

$\therefore E^4\psi = 0$ requires: $\alpha(\alpha-1) - 2 = 0$ or $(\alpha-2)(\alpha-3) - 2 = 0$ $\therefore$

$\alpha = -1, 2$ or $\alpha = 1, 4$ $\therefore$ Z general sol:

$f(r) = Ar^{-1} + Br^{1} + Cr^{2} + Dr^{4}$

\Week 9/ \probl 5.15/ $f(r) = Ar^{-1} + Br + Cr^2 + Dr^4$ $\quad \psi = f(r)\sin^2\theta$

at large r: , $\psi = \frac{1}{2}Ur^2\sin^2\theta \Rightarrow f(r) = \frac{1}{2}Ur^2$ as $r \to \infty$ $\therefore D = 0$

& $C = \frac{1}{2}U$

on $r = a$: , no flow through the sphere & no-slip gives

$\frac{\partial\psi}{\partial\theta} = \frac{\partial\psi}{\partial r} = 0$ $\quad (u = v = 0$ on $r = a)$

$\frac{\partial\psi}{\partial\theta} = 0 \Rightarrow f(r)2\sin\theta\cos\theta = 0$ on $r = a \Rightarrow f(r = a) = 0$ $\therefore$

$\frac{A}{a} + Ba + \frac{1}{2}Ua^2 + Dr^4 = 0 = \frac{A}{a} + Ba + \frac{1}{2}Ua^2$

$\frac{\partial\psi}{\partial r} = 0$ $\therefore$ $\frac{df}{dr} = 0$ on $r = a$ $\left\{ \psi = f(r)\sin^2\theta \therefore \frac{\partial\psi}{\partial r} = \frac{df}{dr}\sin^2\theta = 0 \therefore \frac{df}{dr} = 0 \right\}$

$\therefore$ $\frac{df}{dr} = -Ar^{-2} + B + 2Cr = 0$ on $r = a$ $\therefore$

$-Aa^{-2} + B + Ua = 0$ $\therefore$ Solve for A & B: $\therefore$

$\psi = \frac{Ua^2}{4}\left[\frac{a}{r} + \frac{3r}{a} + \frac{2r^2}{a^2}\right]\sin^2\theta$

\probl 5.17/ use the Stokes eqn $\nabla p = \mu\nabla^2\underline{u}$ to find the pressure p.

$\nabla p = \mu\nabla^2\underline{u}$ (Stokes eqn) $= \mu[\underbrace{\nabla(\nabla\cdot\underline{u})}_{=0 \text{ by construction}} - \nabla\times(\nabla\times\underline{u})]$ (vec identity) $\quad \underline{u} = \nabla\times\left(\frac{\psi}{r\sin\theta}\hat{\underline{\phi}}\right)$

$-\mu\nabla\times(\nabla\times\underline{u}) = \mu\nabla\times\left(\frac{1}{r\sin\theta}\cdot E^2\psi\hat{\underline{\phi}}\right)$ $\quad$ {earlier notes}

$$= \frac{\mu}{r^2\sin\theta}\begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\hat{\underline{\phi}} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ 0 & 0 & \frac{r\sin\theta E^2\psi}{r\sin\theta} \end{vmatrix} = \frac{\mu}{r^2\sin\theta}\begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\hat{\underline{\phi}} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ 0 & 0 & E^2\psi \end{vmatrix}$$

$\therefore$ r comp: $\frac{\partial p}{\partial r} = \frac{\mu}{r^2\sin\theta}\frac{\partial}{\partial\theta}(E^2\psi)$

θ comp: $\frac{1}{r}\frac{\partial p}{\partial\theta} = \frac{-r\mu}{r^2\sin\theta}\frac{\partial}{\partial r}(E^2\psi) = \frac{-\mu}{r\sin\theta}\frac{\partial}{\partial r}(E^2\psi)$

ϕ comp: $\left\{\frac{1}{r\sin\theta}\frac{\partial p}{\partial\phi} = 0\right\}$ trivial $0 = 0$

$E^2\psi = \left(\frac{d^2f}{dr^2} - \frac{2}{r^2}f\right)\sin^2\theta$ $\quad \{\psi = f(r)\sin^2\theta\}$

$= \sin^2\theta\,\frac{Ua^2}{4}\left(\frac{d^2}{dr^2} - \frac{2}{r^2}\right)\left[\frac{a}{r} - \frac{3r}{a} + \frac{2r^2}{a^2}\right]$

$\therefore \frac{d}{dr}\left[\frac{a}{r} - \frac{3r}{a} + \frac{2r^2}{a}\right] = -\frac{a}{r^2} - \frac{3}{a} + \frac{4r}{a^2}$ $\therefore$ $\frac{d^2}{dr^2}\left[\frac{a}{r} - \frac{3r}{a} + \frac{2r^2}{a}\right] = \frac{2a}{r^3} + \frac{4}{a^2}$

$-\frac{2}{r^2}\left[\frac{a}{r} - \frac{3r}{a} + \frac{2r^2}{a^2}\right] = -\frac{2a}{r^3} + \frac{6r}{r^2a} - \frac{4}{a^2}$ $\therefore E^2\psi = \sin^2\theta\frac{Ua^2}{4}\left\{\frac{2a}{r^3} + \frac{4}{a^2} - \frac{2a}{r^3} + \frac{6}{ar} - \frac{4}{a^2}\right\}$

$= \sin^2\theta\frac{Ua3}{2r} = E^2\psi$ $\therefore$

$\frac{\partial p}{\partial r} = \frac{\mu}{r^2\sin\theta}\frac{\partial}{\partial\theta}(E^2\psi) = \frac{\mu}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left\{\frac{3Ua}{2r}\sin^2\theta\right\} = \frac{\mu}{r^2\sin\theta}\frac{3Ua}{2r}\left\{\sin\theta\cos\theta\right\} =$

$\frac{3Ua}{r^3}\mu\cos\theta \Rightarrow p = \frac{-3Ua\mu}{2r^2}\cos\theta + g(\theta) \Rightarrow$

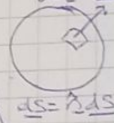
$\frac{\partial p}{\partial \theta} = \frac{3Ua\mu}{2r^2}\sin\theta + g' \overset{*}{=} -\frac{r\mu}{r\sin\theta}\frac{\partial}{\partial r}(E^2\psi) = -\frac{\mu}{\sin\theta}\frac{\partial}{\partial r}\left(\frac{3Ua}{2r}\sin^2\theta\right) =$
$\mu\sin\theta\frac{3Ua}{2r^2}$ ⊕ $\therefore$ comparing $*$ & ⊕ $\Rightarrow g' = 0 \Rightarrow g = \text{constant} = p_\infty$

p_∞ is pressure at $r = \infty$ $\therefore$

$p = p_\infty - \frac{3Ua\mu}{2r^2}\cos\theta$

\prob 5.19/ for a general flow in spherical polar coords (r, θ, ϕ) have $\underline{u} = u_r\hat{\underline{r}} + u_\theta\hat{\underline{\theta}} + u_\phi\hat{\underline{\phi}}$ show $\hat{\underline{n}}\cdot\nabla\underline{u} = \hat{\underline{r}}\frac{\partial}{\partial r}u_r + \hat{\underline{\theta}}\frac{\partial}{\partial r}u_\theta + \hat{\underline{\phi}}\frac{\partial}{\partial r}u_\phi$

for a spherical boundary $\hat{\underline{n}} = \hat{\underline{r}}$



$ds\hat{\underline{n}} = \underline{ds}$

$\hat{\underline{n}}\cdot\nabla = \hat{\underline{r}}\cdot\nabla = \frac{\partial}{\partial r}$

$\Big\{ \hat{\underline{n}}\cdot\nabla = \hat{\underline{r}}\cdot\nabla = (1\hat{\underline{r}} + 0\hat{\underline{\theta}} + 0\hat{\underline{\phi}})\cdot\nabla = (1\hat{\underline{r}} + 0\hat{\underline{\theta}} + 0\hat{\underline{\phi}})\cdot\left(\hat{\underline{r}}\frac{\partial}{\partial r} + \hat{\underline{\theta}}\frac{\partial}{\partial\theta} + \hat{\underline{\phi}}\frac{\partial}{\partial\phi}\right)$ $\quad \underline{ds} = \hat{\underline{n}}\,ds$

$= 1\frac{\partial}{\partial r} + 0\frac{\partial}{\partial\theta} + 0\frac{\partial}{\partial\phi} = \cancel{1\frac{\partial}{\partial r}}\ \cancel{\frac{\partial}{\partial r}} \Big\}$ $\Big\{ \hat{\underline{n}}\cdot\nabla\underline{u} = (\hat{\underline{n}}\cdot\nabla)\underline{u} = \left(\frac{\partial}{\partial r}\right)\underline{u} = \frac{\partial\underline{u}}{\partial r} =$

$= \frac{\partial}{\partial r}(u_r\hat{\underline{r}} + u_\theta\hat{\underline{\theta}} + u_\theta\hat{\underline{\theta}} + u_\phi\hat{\underline{\phi}}) = \hat{\underline{r}}\frac{\partial u_r}{\partial r} + \hat{\underline{\theta}}\frac{\partial u_\theta}{\partial r} + \hat{\underline{\phi}}\frac{\partial u_\phi}{\partial r}$ $\because$ Z basis vecs do not vary with r

\prob 5.21/

for flow around a sphere we have $\underline{u} = u(r,\theta)\hat{\underline{r}} + v(r,\theta)\hat{\underline{\theta}}$ show

$\underline{t} = -p\hat{\underline{n}} + 2\mu\frac{\partial u_r}{\partial r}\hat{\underline{r}} + \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{1}{r}u_\theta + \frac{1}{r}\partial u_\theta + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\hat{\underline{\theta}}$

from Z definition of $\underline{t}$: $\underline{t} = -p\hat{\underline{n}} + 2\mu(\hat{\underline{n}}\cdot\nabla)\underline{u} + \mu\hat{\underline{n}}\times(\nabla\times\underline{u})$, where from earlier $\hat{\underline{n}}\cdot\nabla\underline{u} = \hat{\underline{r}}\frac{\partial u_r}{\partial r} + \hat{\underline{\theta}}\frac{\partial u_\theta}{\partial r} + \hat{\underline{\phi}}\frac{\partial u_\phi}{\partial r}$

& $\hat{\underline{n}}\times(\nabla\times\underline{u}) = -\frac{1}{r^2\sin\theta}\left[r\sin\theta\left(\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right)\hat{\underline{\theta}} + r\left(\frac{\partial}{\partial r}(r\sin\theta u_\phi) - \frac{\partial u_r}{\partial\phi}\right)\hat{\underline{\phi}}\right]$

$\frac{\partial u_\phi}{\partial\phi} = 0 \because u_\phi = 0 \quad u = u(r,\theta),\ v = v(r,\theta) \therefore \frac{\partial u_r}{\partial\phi} = 0 \because u_r = u_r(r,\theta)$ only

$\therefore\ \underline{t} = -p\hat{\underline{r}} + 2\mu\left(\hat{\underline{r}}\frac{\partial u_r}{\partial r} + \hat{\underline{\theta}}\frac{\partial u_\theta}{\partial r}\right) - \frac{\mu}{r^2\sin\theta}\left[r\sin\theta\left(\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right)\right]\hat{\underline{\theta}}$

where $\frac{\partial}{\partial r}(ru_\theta) = r\frac{\partial u_\theta}{\partial r} + u_\theta$ & $\therefore$

$\underline{t} = -p\hat{\underline{r}} + \hat{\underline{r}}\,2\mu\frac{\partial u_r}{\partial r} + \hat{\underline{\theta}}\left\{2\mu\frac{\partial u_\theta}{\partial r} - \frac{\mu r\sin\theta}{r^2\sin\theta}\left(r\frac{\partial u_\theta}{\partial r} + u_\theta\right) + \frac{\mu r\sin\theta}{r^2\sin\theta}\frac{\partial u_r}{\partial\theta}\right\}$

$= -p\hat{\underline{r}} + \hat{\underline{r}}\,2\mu\frac{\partial u_r}{\partial r} + \hat{\underline{\theta}}\left\{\mu\frac{\partial u_\theta}{\partial r} - \frac{\mu}{r}u_\theta + \frac{\mu}{r}\frac{\partial u_r}{\partial\theta}\right\}$

\prob 5.22/ Z total force on Z sphere is $\underline{F} = F\hat{\underline{z}}$ show $F = 6a\pi\mu U$

from earlier $\underline{t} = -p\hat{\underline{r}} + 2\mu\frac{\partial u_r}{\partial r}\hat{\underline{r}} + \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{1}{r}u_\theta + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)\hat{\underline{\theta}}$ this corres to a force $\underline{df} = \underline{t}\,ds = (F_r\hat{\underline{r}} + F_\theta\hat{\underline{\theta}})\,ds$, $F_r = -p + 2\mu\frac{\partial u_r}{\partial r}$ & $F_\theta = \mu\left(\frac{\partial u_\theta}{\partial r} - \frac{1}{r}u_\theta + \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right)$

sub for u_r & u_θ & evaluate on Z surface $r = a$ to obtain

$F_r = -p_\infty + \frac{3\mu U}{2a}\cos\theta \qquad F_\theta = \frac{-3\mu U}{2a}\sin\theta \quad \therefore$

Due to symmetry Z total force is $\underline{F} = F\hat{\underline{z}}$ where

$F = \int_{\text{sphere}} (\underline{F})_z \, dS$ $\qquad F_r \hat{\underline{r}} + F_\theta \hat{\underline{\theta}} = F_r(\sin\theta\cos\phi\,\underline{i} + \sin\theta\sin\phi\,\underline{j} + \cos\theta\,\underline{k}) +$

$F_\theta(\cos\theta\cos\phi\,\underline{i} + \cos\theta\sin\phi\,\underline{j} - \sin\theta\,\underline{k}) \therefore$

$F_z = F_r\cos\theta - F_\theta\sin\theta \quad \therefore \quad F = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(-P_\infty\cos\theta + \frac{3\mu U}{2a}\cos^2\theta + \frac{3\mu U}{2a}\sin^2\theta\right) dS$

with $dS = a^2\sin\theta\,d\theta\,d\phi$ $\qquad$ gives: $F = 6a\pi\mu U$ this is known as Z <u>Stokes</u> <u>drag</u> on a sphere

$$\left\{ F = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(-P_\infty\cos\theta + \frac{3\mu U}{2a}\cos^2\theta + \frac{3\mu U}{2a}\sin^2\theta\right) a^2\sin\theta\,d\theta\,d\phi = \right.$$

$$\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\left(-P_\infty\cos\theta + \frac{3\mu U}{2a}\right)a^2\sin\theta\,d\theta\,d\phi = \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} -a^2P_\infty\cos\theta\sin\theta + \frac{3\mu U a}{2}\sin\theta\,d\theta\,d\phi$$

$$= \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi} -a^2P_\infty\frac{1}{2}2\cos\theta\sin\theta\,d\theta\,d\phi + \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\frac{3\mu U a}{2}\sin\theta\,d\theta\,d\phi =$$

$$-a^2P_\infty\frac{1}{2}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\sin(2\theta)\,d\theta\,d\phi + \frac{3\mu U a}{2}\int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}\sin\theta\,d\theta\,d\phi =$$

$$-a^2P_\infty\frac{1}{2}\int_{\phi=0}^{2\pi}\left[\frac{1}{2}\cos(2\theta)\right]_{\theta=0}^{\pi}d\phi + \frac{3\mu U a}{2}\int_{\phi=0}^{2\pi}\left[-\cos\theta\right]_0^{\pi}d\phi =$$

$$-a^2P_\infty\int_{\phi=0}^{2\pi} -\frac{1}{4}(0)\,d\phi + \frac{3\mu U a}{2}\int_{\phi=0}^{2\pi} +1+1\,d\phi = 0 + 3\mu U a\int_{\phi=0}^{2\pi}d\phi = 3\mu U a 2\pi = 6\mu\pi a =$$

$6a\pi\mu U$ ✓ $\}$ $\qquad Re = (\text{Typical speed} \times \text{length scale} \times \text{fluid density}) / \text{fluid viscosity}$

Probl 6.1 / Show Z eqn for Z vorticity $\underline{\omega}$ reads $\partial_t\underline{\omega} = \nabla\times(\underline{u}\times\underline{\omega}) + \nu\nabla^2\underline{\omega}$

& that for incompressible flows then write $\frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\nabla\underline{u} + \nu\nabla^2\underline{\omega}$

$\therefore$ N-S eqn: $\cancel{\partial_t \underline{u} = \underline{u}\times\underline{\omega}}$ $\{\underline{\omega} = \nabla\times\underline{u}\}$

$\partial_t\underline{u} = \underline{u}\times\underline{\omega} - \nabla(P + \frac{1}{2}|\underline{u}|^2) - \nu\nabla\times\underline{\omega} \quad , \nabla\cdot\underline{u} = 0$

$\because \underline{u}\cdot\nabla\underline{u} = \nabla(\frac{1}{2}|\underline{u}|^2) - \underline{u}\times\underline{\omega} \quad , \nabla^2\underline{u} = \nabla(\underbrace{\nabla\cdot\underline{u}}_{=0}) - \nabla\times\underline{\omega}$ $\qquad$ (formula sheet)

take Z curl: $\nabla\times(\partial_t\underline{u}) = \nabla\times(\underline{u}\times\underline{\omega}) - \underbrace{\nabla\times\nabla(P + \frac{1}{2}|\underline{u}|^2)}_{=0 \text{ since curl grad} = 0} - \nu\nabla\times(\nabla\times\underline{\omega})$

$\partial_t(\nabla\times\underline{u}) = \nabla\times(\underline{u}\times\underline{\omega}) - \nu\nabla\times(\nabla\times\underline{\omega}) =$

$\partial_t\underline{\omega} = \nabla\times(\underline{u}\times\underline{\omega}) - \nu[\nabla(\underbrace{\nabla\cdot\underline{\omega}}_{=0 \because \text{divergence of curl} = 0}) - \nabla^2\underline{\omega}]$ $\qquad$ (formula sheet)

$\Longrightarrow \partial_t\underline{\omega} = \nabla\times(\underline{u}\times\underline{\omega}) + \nu\nabla^2\underline{\omega}$ //

(See formula sheet.) $\nabla\times(\underline{u}\times\underline{\omega}) = \underline{\omega}\cdot\nabla\underline{u} - \underline{u}\cdot\nabla\underline{\omega} + \underline{u}(\overbrace{\nabla\cdot\underline{\omega}}^{=0 \text{ div(curl)} = 0}) - \underline{\omega}(\underbrace{\nabla\cdot\underline{u}}_{=0 \text{ incompressible flows}})$

$\therefore \partial_t\underline{\omega} = \underline{\omega}\cdot\nabla\underline{u} - \underline{u}\cdot\nabla\underline{\omega} + \nu^2\nabla^2\underline{\omega} \quad \therefore$

$\partial_t \underline{\omega} + \underline{u}\cdot\nabla \underline{\omega} = \underline{\omega}\cdot\nabla\underline{u} + \nu\nabla^2\underline{\omega} \implies$

$\frac{D}{Dt}\underline{\omega} = \underline{\omega}\cdot\nabla\underline{u} + \nu\nabla^2\underline{\omega}$ //

$\because \partial_t\underline{\omega} + \underline{u}\cdot\nabla\underline{\omega} = \frac{D\underline{\omega}}{Dt} = \frac{D}{Dt}\underline{\omega}$ $\therefore$ $\partial_t + \underline{u}\cdot\nabla = \frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\nabla$ $\therefore$

$\frac{D}{Dt}\underline{u} = \left(\frac{\partial}{\partial t} + \underline{u}\cdot\nabla\right)\underline{u} = \frac{\partial}{\partial t}\underline{u} + \underline{u}\cdot\nabla\underline{u}$

Prob 6.2

Show for flow in two-dimensions $\{$eg $\underline{u}(x,y,t) = (u,v,0)\}$ Z vorticity equation becomes $\frac{D\underline{\omega}}{Dt} = \nu\nabla^2\underline{\omega}$

general vorticity equation: $\frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\nabla\underline{u} + \nu\nabla^2\underline{\omega}$

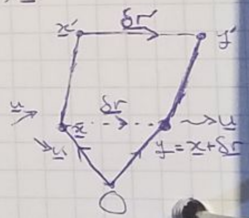
$\underline{\omega} = \nabla\times\underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ u & v & 0 \end{vmatrix} = \left(-\frac{\partial v}{\partial z}, \frac{\partial u}{\partial z}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$ but $\frac{\partial u}{\partial z} = \frac{\partial v}{\partial z} = 0$ $\because$

$\underline{u} = \underline{u}(x,y,t)$ only $\therefore$ $\underline{\omega}\left(=\omega\hat{\underline{z}}\right) = \omega\underline{k}$, where $\omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}$

& $\therefore$ $\underline{\omega}\cdot\nabla\underline{u} = (\omega\underline{k}\cdot\nabla)\underline{u} = \omega\frac{\partial\underline{u}}{\partial z} = \omega\frac{\partial}{\partial z}\underline{u} = \underline{0}$ $\because$ $\underline{u}$ is a func of x,y,t only

$\therefore$ Z vorticity eqn becomes $\frac{D\underline{\omega}}{Dt} = \nu\nabla^2\underline{\omega}$ //

Prob 6.7 / Consider fluid at two near by pts $\underline{x}$ & $\underline{y} = \underline{x} + \delta\underline{r}$ Show Z infinitesimal material element satisfies $\frac{D}{Dt}(d\underline{r}) = (d\underline{r}\cdot\nabla)\underline{u}$



Consider fluid at two nearby pts $\underline{x}$ & $\underline{y} = \underline{x} + \delta\underline{r}$

then in a short time δt, the fluid at $\underline{x}$ has moved to $\underline{x}'$

$\underline{x} \to \underline{x}' = \underline{x} + \underline{u}(\underline{x},t)\delta t$

while $\underline{y} \to \underline{y}' = \underline{y} + \underline{u}(\underline{x}+\delta\underline{r}, t)\delta t$

$\therefore$ $\delta\underline{r}' = \underline{y}' - \underline{x}' = \underline{y} + \underline{u}(\underline{x}+\delta\underline{r}, t)\delta t - \underline{x} - \underline{u}(\underline{x},t)\delta t$

$= \delta\underline{r} + \delta t\left[\underline{u}(\underline{x}+\delta\underline{r}, t) - \underline{u}(\underline{x},t)\right]$ in Z limit $\delta t \to t$

$\frac{\delta\underline{r}' - \delta\underline{r}}{\delta t} = \underline{u}(\underline{x}+\delta\underline{r}, t) - \underline{u}(\underline{x},t)$

$\left\{\text{consider } f(x+h) = f(x) + h\frac{df}{dx} + \frac{h^2}{2}\frac{d^2f}{dx^2} + \dots\right\}$

$\underline{u}(\underline{x}+\delta\underline{r},t) \approx \underline{u}(\underline{x},t) + (\delta\underline{r}\cdot\nabla)\underline{u}(\underline{x},t) = \underline{u}(\underline{x},t) + (\delta\underline{r}\cdot\nabla)\underline{u} + \dots = \underline{u}(x,t) + \delta\underline{r}\cdot\nabla\underline{u} + \dots$ $\therefore$

$\frac{\delta\underline{r}' - \delta\underline{r}}{\delta t} \approx \underline{u}(\underline{x},t) + (\delta\underline{r}\cdot\nabla)\underline{u} + \dots - \underline{u}(\underline{x},t) \approx (\delta\underline{r}\cdot\nabla)\underline{u} + \dots$

& $\therefore$ allowing $\delta\underline{r}$ to tend to zero in length $\frac{D}{Dt}(d\underline{r}) = (d\underline{r}\cdot\nabla)\underline{u}$

$\left\{\frac{D}{Dt}(d\underline{r}) = \frac{\partial}{\partial t}d\underline{r} + \underline{u}\cdot\nabla d\underline{r}\right\}$ as $\delta t \to 0$, $\delta\underline{r} \to 0$: $\frac{\delta\underline{r}' - \delta\underline{r}}{\delta t} = \frac{D}{Dt}(d\underline{r})$

prob 6.9 Kelvin's vorticity thm: for an inviscid flow, Z circulation around any closed material contour is constant.

- By taking Z dot product of Z inviscid N-S eqn with Z line elem, prove Kelvin's vorticity thm:

inviscid flow $\therefore \nu = 0$

circulation: $\Gamma = \oint_C \underline{u} \cdot \underline{dr}$

Material means moving with Z fluid (flow) $\therefore \frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u} \cdot \nabla$

Line elem: $\underline{dr}$

Differentiate following a fluid elem $\therefore$ consider material derivative $\frac{D}{Dt}(\underline{u} \cdot \underline{dr})$ {$\because$ $\underline{u} \cdot \underline{dr}$ appears in Z then (Z circulation when integrate over Z contour} $\therefore$

$$\frac{D}{Dt}(\underline{u} \cdot \underline{dr}) = \frac{D\underline{u}}{Dt} \cdot \underline{dr} + \underline{u} \cdot \frac{D}{Dt}(\underline{dr}) \quad \circledast$$

$\therefore$ consider $\underline{u} \cdot \frac{D}{Dt}(\underline{dr}) = \underline{u} \cdot [(\underline{dr} \cdot \nabla)\underline{u}]$ (previous probl)

$$= u_i [(\underline{dr} \cdot \nabla)\underline{u}]_i = u_i \, dr_j \frac{\partial}{\partial x_j}(u_i) = u_i \left(\frac{\partial u_i}{\partial x_j}\right) dr_j = u_i \left(\frac{\partial u_i}{\partial x_j}\right)(dr_j) =$$

$$u_i \frac{\partial u_i}{\partial x_j} \cdot dr_j = \left(u_i \frac{\partial u_i}{\partial x_j}\right) dr_j = \frac{\partial}{\partial x_j}\left(\tfrac{1}{2} u_i u_i\right) dr_j = \frac{\partial}{\partial x_j}\left(\tfrac{1}{2}|\underline{u}|^2\right) dr_j =$$

$$\frac{\partial}{\partial x_j}\left(\tfrac{1}{2}|\underline{u}|^2\right) \cdot dr_j = \nabla\left(\tfrac{1}{2}|\underline{u}|^2\right) \cdot \underline{dr} \quad \oplus$$

now use Z inviscid vorticity eqn $\frac{D\underline{u}}{Dt} = \frac{\partial}{\partial t}\underline{u} + \underline{u} \cdot \nabla \underline{u} = -\nabla P$ ~~+ ...~~ (inviscid) $(\underline{g} = -\nabla \pi)$

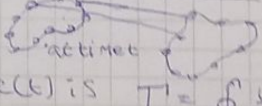
{$\because$ assume other forces are conservative $\therefore$ write in terms of a gradient $-\nabla P$ that could also include gravity eg write $\underline{g} = -\nabla \pi$ $\therefore$ P includes π as well depending if gravity is important eg P could be just fluid pressure or could include other conservative forces} $\Longrightarrow$

$$\frac{D\underline{u}}{Dt} \cdot \underline{dr} = -(\nabla P) \cdot \underline{dr} \quad \oplus\oplus$$

$\therefore$ Sub $\oplus$ & $\oplus\oplus$ into $\circledast$

$$\frac{D}{Dt}(\underline{u} \cdot \underline{dr}) = -\nabla P \cdot \underline{dr} + \nabla\left(\tfrac{1}{2}|\underline{u}|^2\right) \cdot \underline{dr} = \left[\nabla\left(\tfrac{1}{2}|\underline{u}|^2 - P\right)\right] \cdot \underline{dr} \quad \circledast\circledast$$

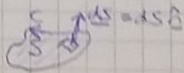
Now suppose that $C(t)$ is a close material contour (so pts on C move with velocity $\underline{u}$) $C(t)$



$\therefore$ Z circulation around $C(t)$ is $\Gamma = \oint_{C(t)} \underline{u} \cdot \underline{dr}$ $\therefore$ $\frac{D\Gamma}{Dt} = \oint_{C(t)} \frac{D}{Dt}(\underline{u} \cdot \underline{dr})$

- $= \oint_{C(t)} \nabla\left(\tfrac{1}{2}|\underline{u}|^2 - P\right) \cdot \underline{dr}$ (using $\circledast\circledast$) let $\underline{F} = \nabla\left(\tfrac{1}{2}|\underline{u}|^2 - P\right)$ (then by

Stokes thm: $\oint_C \underline{F} \cdot \underline{dr} = \int_S (\nabla \times \underline{F}) \cdot \underline{dS}$)



$\therefore$

$$\oint_{C(t)} \nabla\left(\tfrac{1}{2}|\underline{u}|^2 - P\right)\cdot d\underline{r} = \int_S \nabla\times\nabla\left(\tfrac{1}{2}|\underline{u}|^2 - P\right)\cdot d\underline{S} \quad \{\text{curl}(\text{grad}) = 0\}$$

$$= \int_S 0 \cdot d\underline{S} = 0 \quad \{\because \text{curl grad} \equiv 0\}$$

$$\Rightarrow \frac{D\Gamma}{Dt} = 0 \Rightarrow \Gamma = \text{constant following the fluid motion}$$

\prob 6.10\ Consider the 3-d strain flow $\underline{u} = -aR\hat{\underline{R}} + v(R,t)\hat{\underline{\phi}} + 2aZ\hat{\underline{z}}$ where a is a positive const, in cylindrical polar coords, (R,ϕ,z)

Show the only non-trivial component of the vorticity eqn is the z-component which reads $\frac{\partial\omega}{\partial t} - aR\frac{\partial\omega}{\partial R} = 2a\omega + \frac{\nu}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\omega}{\partial R}\right)$

$$\underline{\omega} = \nabla\times\underline{u} = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ -aR & Rv & 2aZ \end{vmatrix} = \frac{1}{R}\hat{\underline{z}}\frac{\partial}{\partial R}(Rv) \;\therefore$$

$\Rightarrow \underline{\omega} = \omega\hat{\underline{z}}$, where $\omega = \frac{1}{R}\frac{\partial}{\partial R}(Rv)$ (is only a func of R & t only)

vorticity eqn: $\frac{\partial\underline{\omega}}{\partial t} + \underline{u}\cdot\nabla\underline{\omega} = \underline{\omega}\cdot\nabla\underline{u} + \nu\nabla^2\underline{\omega}$

$$\underline{u}\cdot\nabla = (-aR\hat{\underline{R}} + v\hat{\underline{\phi}} + 2aZ\hat{\underline{z}})\cdot(\hat{\underline{R}}\partial_R + \tfrac{1}{R}\hat{\underline{\phi}}\partial_\phi + \hat{\underline{z}}\partial_z) =$$

$$-aR\partial_R + \frac{v}{R}\partial_\phi + 2aZ\partial_z \;\therefore$$

$$(\underline{u}\cdot\nabla)\underline{\omega} = \left(-aR\frac{\partial}{\partial R} + \frac{v}{R}\frac{\partial}{\partial\phi} + 2aZ\frac{\partial}{\partial z}\right)(\omega(R,t)\hat{\underline{z}})$$

$= -aR\frac{\partial\omega}{\partial R}\hat{\underline{z}}$ $\{\because \omega$ only depends on R & t & $\hat{\underline{z}}$ is a const vec $\}$

$$\underline{\omega}\cdot\nabla = \omega\hat{\underline{z}}\cdot\nabla = \omega\hat{\underline{z}}\cdot(\hat{\underline{R}}\partial_R - \tfrac{1}{R}\hat{\underline{\phi}}\partial_\phi + \hat{\underline{z}}\partial_z) = \omega\frac{\partial}{\partial z}$$

$$\Rightarrow (\underline{\omega}\cdot\nabla)\underline{u} = \omega\frac{\partial}{\partial z}(\underline{u}) = \omega\frac{\partial}{\partial z}(-aR\hat{\underline{R}} + v(R,t)\hat{\underline{\phi}} + 2aZ\hat{\underline{z}}) = \omega\frac{\partial}{\partial z}(2aZ\hat{\underline{z}}) =$$

$$2a\omega\hat{\underline{z}}$$

$$\nabla^2\underline{\omega} = \nabla(\underbrace{\nabla\cdot\underline{\omega}}_{=0 \because \text{div(curl)}=0}) - \nabla\times(\nabla\times\underline{\omega}) = -\nabla\times(\nabla\times\underline{\omega}) \;\therefore$$

$$\nabla\times\underline{\omega} = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & 0 & \omega \end{vmatrix} = -\frac{R}{R}\hat{\underline{\phi}}\frac{\partial\omega}{\partial R} = -\hat{\underline{\phi}}\frac{\partial\omega}{\partial R} \;\therefore \nabla\times(\nabla\times\underline{\omega}) = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & -R\frac{\partial\omega}{\partial R} & 0 \end{vmatrix} =$$

$$\frac{\hat{\underline{z}}}{R}\frac{\partial}{\partial R}\left(-R\frac{\partial\omega}{\partial R}\right) \;\therefore \nu\nabla^2\underline{\omega} = -\nu\nabla\times(\nabla\times\underline{\omega}) = +\frac{\nu}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\omega}{\partial R}\right)\hat{\underline{z}}$$

$\therefore$ each term in the vorticity eqn only has a component in the z-direction & obtain

$$\underbrace{\frac{\partial\omega}{\partial t} - aR\frac{\partial\omega}{\partial R}}_{\text{material derivative}} = \underbrace{2a\omega}_{\text{vortex stretching term}} + \underbrace{\frac{\nu}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\omega}{\partial R}\right)}_{z\text{ comp of }\nabla^2\underline{\omega}\text{ is viscous term}}$$

\probl 6.11/ verify that sols of Z inviscid vorticity eqn for strain flows take Z form $\omega(R,t)=e^{2at}F(\xi)$, $\xi = Re^{at}$, where F is an arbitrary func of ξ

$\therefore \frac{\partial\omega}{\partial t} - aR\frac{\partial\omega}{\partial R} = 2a\omega + \underbrace{\frac{\nu}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\omega}{\partial R}\right)}_{=0 \text{ inviscid } \nu=0}$ ⊛, $\underline{u} = -aR\hat{\underline{R}} + v(R,t)\hat{\underline{\phi}} + 2az\hat{\underline{z}}$,

($a>0$ const)

$\frac{\partial\xi}{\partial R} = e^{at}$ & $\frac{\partial\xi}{\partial t} = Rae^{at}$

$\therefore \frac{\partial\omega}{\partial t} = e^{2at}\frac{\partial F}{\partial t} + F\frac{\partial}{\partial t}(e^{2at}) = e^{2at}\frac{\partial F}{\partial\xi}\frac{\partial\xi}{\partial t} + F\,2ae^{2at} =$

$e^{2at}F'Rae^{at} + 2aFe^{2at} = aRe^{3at}F' + 2ae^{2at}F$

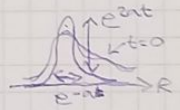
$\frac{\partial\omega}{\partial R} = e^{2at}\frac{\partial F}{\partial R} = e^{2at}\frac{\partial F}{\partial\xi}\frac{\partial\xi}{\partial R} = e^{2at}F'e^{at} = e^{3at}F'$

$\left\{\text{vorticit eqn: } \frac{\partial\omega}{\partial t} - aR\frac{\partial\omega}{\partial R} = +2a\omega \quad ⊛\right\}$

& ∴ rearranging Z vorticity eqn ⊛: $\frac{\partial\omega}{\partial t} - aR\frac{\partial\omega}{\partial R} - 2a\omega = 0 =$

$aRe^{3at}F' + 2ae^{2at}F - aRe^{3at}F' - 2ae^{2at}F = 0$ as required //

Z sol $\omega(R,t) = e^{2at}F(Re^{at})$ ($\forall F$) represents a vorticity distr? (vorticity profile) which <u>contracts in scale</u> as e^{-at} & <u>increases in strength</u> by e^{2at}



consider at $t=0$: $F(Re^{at}) = F(R)$

∴ for what R (R^*, say) is $F(R^*e^{a\tau}) = F(R)$ at time τ later?

need $R = R^*e^{a\tau} \Longrightarrow R^* = Re^{-a\tau} < R$

contracts in scale

\probl 6.12/ show sols of Z steady vorticity eqn for strain flows take Z form $\omega(R) = e^{-R^2/\delta^2}$ where δ is a const that should be found.

were interested in sols to $\underbrace{\frac{\partial\omega}{\partial t}}_{=0 \text{ steady sols}} - aR\frac{\partial\omega}{\partial R} = 2a\omega + \frac{\nu}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\omega}{\partial R}\right)$ ∴

$\frac{d\omega}{dR} = e^{-R^2/\delta^2}\left(\frac{-2R}{\delta^2}\right)$ $\qquad \frac{d}{dR}\left(R\frac{\partial\omega}{\partial R}\right) = \frac{d}{dR}\left(-\frac{2R^2}{\delta^2}e^{-R^2/\delta^2}\right) =$

$-\frac{2R^2}{\delta^2}\frac{d}{dR}\left(e^{-R^2/\delta^2}\right) + e^{-R^2/\delta^2}\frac{d}{dR}\left(\frac{-2R^2}{\delta^2}\right) =$

$\frac{-2R^2}{\delta^2}e^{-R^2/\delta^2}\left(\frac{-2R}{\delta^2}\right) + e^{-R^2/\delta^2}\left(\frac{-4R}{\delta^2}\right) = e^{-R^2/\delta^2}\frac{4R}{\delta^2}\left(\frac{R^2}{\delta^2} - 1\right)$ ∴

Z steady state viscous vorticity eqn is satisfied if

$\left\{-aR\frac{\partial\omega}{\partial R} = 2a\omega + \frac{\nu}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\omega}{\partial R}\right)\right\}$

$\frac{aR^2 2}{\delta^2}e^{-R^2/\delta^2} = 2ae^{-R^2/\delta^2} + \frac{\nu}{R}e^{-R^2/\delta^2}\frac{4R}{\delta}\left(\frac{R^2}{\delta^2} - 1\right)$ ∴

$$\frac{aR^2 2}{\delta^2} = 2a + \frac{\nu}{R}\frac{4R}{\delta^2}\left(\frac{R^2}{\delta^2}-1\right) \therefore$$

$$\frac{2aR^2}{\delta^2} = 2a + 4\frac{\nu R}{R\,\delta^2}\left(\frac{R^2}{\delta^2}-1\right) \Longrightarrow \frac{2aR^2}{\delta^2} = 2a + \frac{4\nu}{\delta^2}\left(\frac{R^2}{\delta^2}-1\right) \Longrightarrow$$

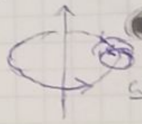
$$R^2\left[\frac{2a}{\delta^2}-\frac{4\nu}{\delta^4}\right] = 2a - \frac{4\nu}{\delta^2} = R^2\frac{1}{\delta^2}\left[2a-\frac{4\nu}{\delta^2}\right] \quad \therefore 2a - \frac{4\nu}{\delta^2} = 0 \therefore$$

if $\delta = \sqrt{\frac{2\nu}{a}} \quad \therefore \delta^2 = \frac{2\nu}{a} \therefore \quad 2a - \frac{4\nu}{\delta^2} = 2a - \frac{4\nu}{2\nu/a} = 2a - 2a = 0 \therefore$

$\delta = \sqrt{\frac{2\nu}{a}}$ // is Z required const

\probl 6.13/ Show Z flow velocity $\underline{u} = u(R,z,t)\hat{\underline{R}} + w(R,z,t)\hat{\underline{z}}$ in cylindrical polar coords (R,ϕ,z), gives rise to 'vortex rings' $\underline{\omega} = \omega(R,z,t)\hat{\underline{\phi}}$ & that Z inviscid vorticity eqn becomes

$$\frac{\partial\omega}{\partial t} + u\frac{\partial\omega}{\partial R} + w\frac{\partial\omega}{\partial z} = \frac{u\omega}{R}$$



$\underline{\omega} = \omega\hat{\underline{\phi}}$

$$\underline{\omega} = \nabla\times\underline{u} = \frac{1}{R}\begin{vmatrix}\hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}}\\ \partial R & \partial\phi & \partial z\\ u & 0 & w\end{vmatrix} = \frac{-1}{R}(R\hat{\underline{\phi}})\left(\frac{\partial w}{\partial R}-\frac{\partial u}{\partial z}\right) = \omega\hat{\underline{\phi}},$$

where $\omega = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial R} \therefore \underline{\omega} = \left(\frac{\partial u}{\partial z}-\frac{\partial w}{\partial R}\right)\hat{\underline{\phi}}$ & $\omega = \omega(R,z,t)$ only

Z vorticity eqn $\frac{\partial\underline{\omega}}{\partial t} + \underline{u}\cdot\nabla\underline{\omega} = \underline{\omega}\cdot\nabla\underline{u} + \underbrace{\nu\nabla^2\underline{\omega}}_{=0 \text{ inviscid}}$

$$\underline{u}\cdot\nabla = (u\hat{\underline{R}} + w\hat{\underline{z}})\cdot\left(\hat{\underline{R}}\partial_R + \frac{1}{R}\hat{\underline{\phi}}\partial_\phi + \hat{\underline{z}}\partial_z\right) = u\frac{\partial}{\partial R} + w\frac{\partial}{\partial z} \therefore$$

$$(\underline{u}\cdot\nabla)\underline{\omega} = \left(u\frac{\partial}{\partial R} + w\frac{\partial}{\partial z}\right)(\omega\hat{\underline{\phi}}) = \left(u\frac{\partial\omega}{\partial R} + w\frac{\partial\omega}{\partial z}\right)\hat{\underline{\phi}}$$

$$\underline{\omega}\cdot\nabla = \omega\hat{\underline{\phi}}\cdot\left(\hat{\underline{R}}\partial_R + \frac{\hat{\underline{\phi}}}{R}\partial_\phi + \hat{\underline{z}}\partial_z\right) = \frac{\omega}{R}\frac{\partial}{\partial\phi} \Longrightarrow$$

$\underline{\omega}\cdot\nabla\underline{u} = \frac{\omega}{R}\frac{\partial}{\partial\phi}(u\hat{\underline{R}} + w\hat{\underline{z}}) = \frac{\omega}{R}u\hat{\underline{\phi}} \because \frac{\partial\hat{\underline{R}}}{\partial\phi} = \hat{\underline{\phi}}$ & u, w are indep of ϕ

$\therefore$ Z only non trivial comp of Z inviscid vorticity eqn is Z ϕ-comp, namely $\frac{\partial\omega}{\partial t} + u\frac{\partial\omega}{\partial R} + w\frac{\partial\omega}{\partial z} = \frac{u\omega}{R}$ //

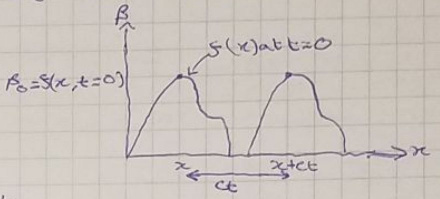
\probl 7.1/ Show $f(x-ct)$ & $g(x+ct)$ represent structures that travel in opposite along Z x-axis with speed c

consider $\beta = f(x-ct)$ when x = const β changes with time, & at any given time β has a different val for different vals of x but, β has Z same val for coord x & time t st Z combination $x - ct$ = const $\therefore x = ct +$ const

[eg if $x = 3$, $t = 0 \Longrightarrow$ const $= 3$ then at time t later $\exists$ a val of

~~wavenumber~~ then Z waves are non-dispersive

x st $x-ct=3$ what is x? $x=3+ct$ ie a distance ct away from $x=3$]



if at time $t=0$, at Z pt x

Z variable β has a certain val

$\beta_0=f(x,t=0)$, then after some time t

Z same val of β is found at a distance ct along Z x-axis

$\therefore$ Z profile moves in Z positive x-direction with speed $c>0$

similarly $g(x+ct)$ represents a wave of arbit shape propagating with speed c in Z negative x-direction $\therefore$ we call $f(x-ct)$ & $g(x+ct)$ travelling waves

{ for $g(x+ct)$: if $\beta=g(x+ct)$ $\therefore x+ct=$ constant $\therefore$

$x=-ct+$ constant $\therefore$

at $t=0$: have $x=x$ $\therefore$ $\beta_0=g(x,t=0)$ then at time t

$g(x',t)=\beta_0$ $\therefore$ $x'=-ct+$ constant $=x-ct$

& at $t=0$: $x=-c(0)+$ constant $=$ constant $\therefore$

$x'-x=-ct+$ const $-$ const $=-ct$ $\therefore$ same val of $\beta_0=(x-ct,t)$ at a distance of $-ct$ along Z x-axis $\therefore$ moves in negative x direction with speed $c>0$ }

\prob 7.5/ consider 2 dimens, incompressible, inviscid water waves with velocity $\underline{u}=(u(x,y,t), v(x,y,t), 0)$ suppose Z fluid motion is irrotational st may introduce Z velocity potential $\phi(x,y,t)$ with $\underline{u}=\nabla\phi$. show ϕ satisfies Laplace's eqn $\frac{\partial^2\phi}{\partial x^2}+\frac{\partial^2\phi}{\partial y^2}=0$

irrotational $\therefore$ ~~ϕ~~ vorticity $\underline{\xi}=\nabla\times\underline{u}=-\left(\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}\right)\underline{k}$

$\therefore$ irrotational $\Longrightarrow \frac{\partial v}{\partial x}=\frac{\partial u}{\partial y}$

introduce Z velocity potential ϕ st $\underline{u}=\nabla\phi$ $\therefore$

$\underline{u}=\left(\frac{\partial\phi}{\partial x},\frac{\partial\phi}{\partial y},0\right)\Longrightarrow\frac{\partial u}{\partial y}=\frac{\partial}{\partial y}\left(\frac{\partial\phi}{\partial x}\right)$ & $\frac{\partial v}{\partial x}=\frac{\partial}{\partial x}\left(\frac{\partial\phi}{\partial y}\right)$

ie Z flow is automatically irrotational

incompressibility $\Longrightarrow\nabla\cdot\underline{u}=0\Longrightarrow\nabla\cdot(\nabla\phi)=0\Longrightarrow\nabla^2\phi=0$ ie

ϕ satisfies Laplace's eqn // $\left\{ \nabla^2\phi = 0 \quad \nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0 \right\}$

$\{ \nabla \cdot (\nabla\phi) = \nabla^2\phi$ ∵ divergence of gradient is Z Laplacian $\nabla^2 g$ $\}$

\prob 7.6/ define $F(x,y,t) = y - \eta(x,t)$ & require that $\frac{DF}{Dt} = 0$ for any particle on Z free surface $y = \eta(x,t)$ show

$\frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} = v$ on $y = \eta(x,t)$

$\frac{D}{Dt}F = 0 \Longrightarrow \frac{\partial F}{\partial t} + \underline{u} \cdot \nabla F = 0$ on $y = \eta(x,t)$

$F = y - \eta(x,t) \Longrightarrow \frac{\partial F}{\partial t} = \frac{\partial}{\partial t}(y - \eta(x,t)) = -\frac{\partial \eta(x,t)}{\partial t} = -\frac{\partial\eta}{\partial t}$

$\underline{u} \cdot \nabla F = u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} = u\left(-\frac{\partial\eta}{\partial x}\right) + v \cdot 1 \; \{ = (\underline{u} \cdot \nabla)F = ((u,v,0) \cdot \nabla)F =$

$((u,v,0) \cdot (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}))F = (u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y} + 0)F = u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} =$

$u\frac{\partial}{\partial x}(y - \eta(x,t)) + v\frac{\partial}{\partial y}(y - \eta(x,t)) = u\frac{\partial}{\partial x}(-\eta(x,t)) + v\frac{\partial}{\partial y}(y) = u\frac{\partial}{\partial x}(-\eta) + v \cdot 1 =$

$-u\frac{\partial}{\partial x}\eta + v = -u\frac{\partial}{\partial x}(\eta(x,t)) + v \}$ & ∴

$-\frac{\partial\eta}{\partial t} + v - u\frac{\partial\eta}{\partial x} = 0 \Longrightarrow \frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} = v$ on $y = \eta(x,t)$ //

\prob 7.7/ write Z inviscid N-S eqn in Z form $\frac{\partial \underline{u}}{\partial t} + \nabla H = \underline{u} \times \underline{\xi}$, ξ is Z vorticity, $H = \frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2 + \Pi$ $\left\{ H = \frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2 + \frac{\Pi}{\rho} \right\}$ is Z <u>Bernoulli func</u> & Π is Z gravitational potential ∴ for inviscid, irrotational flow in Z presence of gravity show

$\frac{\partial\phi}{\partial t} + \frac{1}{2}(u^2 + v^2) + g\eta = 0$ on $y = \eta(x,t)$ (7.9)

N-S: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla\underline{u}\right) = -\nabla p + \rho g$ {inviscid ∴ $\nu = 0$}

{∴ if $g = -\nabla\Pi$} N-S: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla\underline{u}\right) = -\nabla p - \rho\nabla\Pi = -\nabla(p + \rho\Pi)$

when $g = -\nabla\Pi$ $(\Pi = gy)$ $\Longrightarrow \partial_t \underline{u} + \underline{u} \cdot \nabla\underline{u} = -\nabla\left(\frac{p}{\rho} + \Pi\right)$ {capital π}

∴ from formula sheet: $\{ \underline{u} \times (\nabla \times \underline{u}) + \underline{u} \times (\nabla \times \underline{u}) = \nabla(\underline{u} \cdot \underline{u}) - (\underline{u} \cdot \nabla)\underline{u} - (\underline{u} \cdot \nabla)\underline{u} =$

$2\underline{u} \times (\nabla \times \underline{u}) = \nabla(|\underline{u}|^2) - 2(\underline{u} \cdot \nabla)\underline{u}$ ∴ $(\underline{u} \cdot \nabla)\underline{u} = \underline{u} \cdot \nabla\underline{u} = \nabla\left(\frac{|\underline{u}|^2}{2}\right) - \underline{u} \times (\nabla \times \underline{u}) \}$

$\underline{u} \cdot \nabla\underline{u} = \nabla\left(\frac{|\underline{u}|^2}{2}\right) - \underline{u} \times \underline{\xi}$ $\quad \underline{\xi} = \nabla \times \underline{u}$ ∴ have:

$\partial_t \underline{u} + \nabla\left(\frac{|\underline{u}|^2}{2}\right) - \underline{u} \times \underline{\xi} - \nabla\left(\frac{p}{\rho} + \Pi\right) \Longrightarrow$

$\partial_t \underline{u} + \nabla\Big(\underbrace{\tfrac{1}{2}|\underline{u}|^2 + \tfrac{p}{\rho} + \Pi}_{H}\Big) = \underline{u} \times \underline{\xi}$ //

irrotational $\Longrightarrow \nabla \times \underline{u} = \underline{0} \Longrightarrow \underline{\xi} = \underline{0}$ ∴ we can introduce Z velocity potential ϕ with $\underline{u} = \nabla\phi$ & Z N-S eqn becomes $\nabla(\partial_t\phi + H) = 0$ ∴

$\{ \underline{\xi} = 0 \therefore \underline{u} \times \underline{\xi} = 0$ $\therefore \partial_t \underline{u} + \nabla(\frac{1}{2}|\underline{u}|^2 + \frac{p}{\rho} + \Pi) = 0 = \partial_t \underline{u} + \nabla H =$
$\partial_t(\nabla \phi) + \nabla H = \nabla(\partial_t \phi) + \nabla H = \nabla(\partial_t \phi + H) = 0 \} \therefore$

- integrating: $\frac{\partial \phi}{\partial t} + H = f(t) = \frac{\partial \phi}{\partial t} + \frac{1}{2}|\underline{u}|^2 + \frac{p}{\rho} + \Pi$ //

air $y = \eta(x,t)$ liquid

$\therefore$ At Z free surface $y = \eta(x,t)$ we require $p = p_0$ (Z const atmospheric pressure p_0) $\therefore$

choosing $f(t)$ conveniently $\{$take $\frac{p_0}{\rho} = f(t)\}$ $\{\therefore$ can take it into Z $\Pi\}$: $\{\underline{u} = (u, v, 0)\} \therefore$

$\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + v^2 + 0^2) + g\eta = 0 = \frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + v^2) + g\eta = 0$ on $y = \eta(x,t)$ //

$\{\because gy = \Pi$ & $y = \eta\}$

\probl 7.8/ show Z linearised kinematic condition can be approxed as $\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}$ on $y = 0$.

kinematic condition: $\frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = v$ on $y = \eta(x,t)$

waves are of small amplitude (u, v, η are small)

$\therefore$ we neglect Z product $u\frac{\partial \eta}{\partial x}$ to obtain $\frac{\partial \eta}{\partial t} \approx v(x,y,t)$ on $y = \eta(x,t)$

but $\because$ η is small ($\because$ small amplitude definition) we expand Z RHS in a Taylor series $v(x, y=\eta, t) = v(x, y=0, t) + \eta\frac{\partial v}{\partial y}(x,0,t) + \dots$

$\{ f(x+h) \approx f(x) + h\frac{df}{dx} + \frac{h^2}{2}\frac{d^2 f}{dx^2} + \dots \therefore v(x, y=\eta, t) =$
$v(x, y=0, t) + \eta\frac{\partial v}{\partial y}(x,0,t) + \frac{\eta^2}{2}\frac{\partial^2 v}{\partial y^2}(x, y=0, t) + \dots \} \therefore$

- neglecting products of small quantities gives Z approx kinematic condition $\frac{\partial \eta}{\partial t} = v(x, y=0, t)$ but $\underline{u} = \nabla \phi = (\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, 0)$

$\Rightarrow \frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial y}$ on $y = 0$ //

\probl 7.10/ given $\eta = A\cos(kx - \omega t)$ find Z sol for $\phi(x,y,t)$

$\therefore$ derive Z <u>dispersion</u> <u>relation</u> connecting Z frequence & Z wavenumbe, $\omega(k)$

Dynamic condition: $\frac{\partial \phi}{\partial t} = -g\eta$ on $y = 0$ $\frac{\partial \phi}{\partial t} = -gA\cos(kx - \omega t)$ on $y = 0$

kinematic condition $\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}$ on $y = 0$ $\therefore \frac{\partial \phi}{\partial y} = -A\sin(kx - \omega t)(-\omega) =$

- $A\omega\sin(kx - \omega t)$ on $y = 0$ $\{$doesn't depend on y right now $\because$ evaluated on $y = 0\}$

Z boundary conditions suggest seeking ~~a~~ sol for Z velocity potential of Z form $\phi(x,y,t) = f(y)\sin(kx - \omega t)$ $\{\frac{\partial \phi}{\partial y} = A\omega\sin(kx - \omega t) \therefore$

integrating wrt y: $\phi(x,y,t) = \int A\omega \sin(kx-\omega t)\,\partial y = A\omega\sin(kx-\omega t)\int 1\,\partial y =$
$A\omega\sin(kx-\omega t)\,y + B(x,t)$ $\Big\}$ $\therefore \phi(x,y,t) = f(y)\sin(kx-\omega t)$ $\therefore$

Sub into Laplaces eqn for ϕ: $\nabla^2\phi = 0 \Longrightarrow \partial_{xx}\phi + \partial_{yy}\phi = 0 \Longrightarrow$

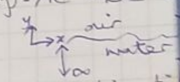
$-k^2 f(y)\sin(kx-\omega t) + f''\sin(kx-\omega t) = 0 = \Big\{ \partial_{xx}\phi + \partial_{yy}\phi = 0 =$

$\frac{\partial^2}{\partial x^2}[f(y)\sin(kx-\omega t)] + \frac{\partial^2}{\partial y^2}[f(y)\sin(kx-\omega t)] = f(y)\frac{\partial^2}{\partial x^2}\sin(kx-\omega t) + \sin(kx-\omega t)\frac{\partial^2 f(y)}{\partial y^2}$

$= -k^2 f(y)\sin(kx-\omega t) + f''(y)\sin(kx-\omega t) = 0 \Big\} \Longrightarrow$

$f'' - k^2 f = 0$ $\quad \{-k^2 f(y) + f''(y) = 0\}$ $\{\therefore$ is $f(y) = e^{py}$: $f'' = p^2 e^{py}$ $\therefore$

$p^2 e^{py} - k^2 e^{py} = 0 = p^2 - k^2 = (p-k)(p+k) = 0$ $\therefore p = k$, $p = -k$ $\therefore f(y) = Ae^{ky} + Be^{-ky}\}$



$\therefore$ Z general sol is $f(y) = Ce^{ky} + De^{-ky}$ //

$\therefore$ take $k > 0$ w.l.o.g. $\{k<0$ just means wave goes in opposite direction$\}$

$\therefore \because$ Z water is of infinite depth, in order that Z velocity remains bounded as $y \to -\infty$ $\{$limit as: $\lim_{y\to-\infty} Ce^{ky} = 0$, $\lim_{y\to-\infty} De^{-ky} = \infty\}$

$\therefore$ must take $D = 0$ & $\therefore$: $\phi = Ce^{ky}\sin(kx-\omega t) = \phi(x,y,t)$ //

Dynamic Cond: $\frac{\partial\phi}{\partial t} \overset{\text{on } y=0}{=} -g\eta \Longrightarrow Ce^{ky}\cos(kx-\omega t)(-\omega) = -\omega Ce^{ky}\cos(kx-\omega t) = -g\eta$
$= -gA\cos(\cancel{kx}-\omega t)$ $\therefore$ $-\omega Ce^{ky} = -gA$ $\{$eval on $y=0\}$ $\therefore$

$-\omega Ce^{k(0)} = -gA = -\omega Ce^0 = -\omega C\cdot 1 = -\omega C = -gA \Longrightarrow$

kinematic condition: on $y=0$: $\frac{\partial\phi}{\partial y} = \frac{\partial\eta}{\partial t} \Longrightarrow Cke^{ky}\sin(kx-\omega t) =$
$-A\sin(kx-\omega t)(-\omega)$ $\therefore$ $Cke^{ky} = +\omega A$ $\therefore$ at $y=0$:

$Cke^{k(0)} = \omega A = Ck\cdot 1 = Ck = \omega A$ $\therefore$ $C = \frac{A\omega}{k}$ $\therefore$

$\phi = \frac{A\omega}{k}e^{ky}\sin(kx-\omega t) = \phi(x,y,t)$ //

Dispersion relation: $\omega = \frac{gA}{C} = \frac{gA}{A\omega/k} = \frac{gk}{\omega} \Longrightarrow \omega^2 = gk$ // $\therefore \omega = \pm\sqrt{gk}$

\prob 7.11/ show we require that A is small compared with λ

kinematic cond: $\frac{\partial\eta}{\partial t} + u\frac{\partial\eta}{\partial x} = v$ on $y = \eta(x,t)$

$\phi = \frac{A\omega}{k}e^{ky}\sin(kx-\omega t)$, $\underline{u} = \nabla\phi$ $\therefore$ $u = \frac{\partial\phi}{\partial x} = \frac{A\omega}{k}e^{ky}\cos(kx-\omega t)k =$
$A\omega e^{ky}\cos(kx-\omega t)$

$v = \frac{\partial\phi}{\partial y} = \frac{A\omega}{k}e^{ky}k\sin(kx-\omega t) = A\omega e^{ky}\sin(kx-\omega t)$ $\therefore$

u & v are of Z same order of magnitude, $A\omega$ & $\therefore$ Z linearised kinematic cond is valid when $\frac{\partial\eta}{\partial x} \ll 1$ $\Big\{\frac{\partial\eta}{\partial x}$ is small$\Big\} \Longrightarrow$

surface displacement is small compared with Z wavelength $\Longrightarrow$

$A << \lambda$ //

● <u>Dynamic</u> <u>cond</u>: $\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2+v^2) + g\eta = 0$ on $y = \eta$

u^2+v^2 is of order $A^2\omega^2 = A^2 gk$ using Z dispersion relation $\therefore$

u^2+v^2 is small compared with $g\eta$ is $A^2 gk \leq gA$ $\therefore$ $Ak << 1$ $\Longrightarrow$

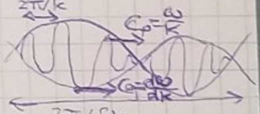
$A << \lambda = 2\pi/k$ //

\prob 7.13/ consider a surf elevation composed of two waves of almost equal wavenumbers $k - \delta k$ & $k + \delta k$, with corres frequencies $\omega - \delta\omega$ & $\omega + \delta\omega$. Show Z resulting disturbance is a product of a long wave & a short wave & determine Z speed of Z individual waves relative to Z wavepacket

$\eta = A\cos[(k-\delta k)x - (\omega - \delta\omega)t] + A\cos[(k+\delta k)x - (\omega+\delta\omega)t]$

$\theta = kx - \omega t$ & $\phi = \delta k x - \delta\omega t$ $\therefore$ $\eta = A\cos[\theta - \phi] + A\cos[\theta + \phi] =$

$2A\cos\theta\cos\phi = 2A\cos(kx - \omega t)\underbrace{\cos(\delta k x - \delta\omega t)}_{\text{long wave}}$, this is a product of a long wave (of wavelength $2\pi/\delta k$) & a short wave (of wavelength $2\pi/k$) & it gives a packet of waves which are almost isolated by a region of calmer water from Z next packet



$\{C_p = \frac{\omega}{k}, C_g = \frac{d\omega}{dk}\}$

δk small

Z individual waves have speeds $\frac{\omega \pm \delta\omega}{k \pm \delta k} \approx \frac{\omega}{k}$

but Z long (or modulating) wave $\cos(\delta k x - \delta\omega t)$ has speed $\frac{\delta\omega}{\delta k}$ which may or may not be ω/k Z individual waves moves relative to Z packet at speed $\frac{\omega}{k} - \frac{d\omega}{dk}$ approx $\{\frac{d\omega}{dk} \approx \frac{\delta\omega}{\delta k}\}$

$\therefore \{C_p = \frac{\omega}{k}, C_g = \frac{d\omega}{dk}$ & $C_p \neq C_g$ can be true $\}$

\prob 7.14/ Calc Z group speed of Z surf gravity waves

Dispersion relation $\omega = \omega(k)$ surface gravity waves $\omega^2 = gk \Longrightarrow$

$\omega = \pm(gk)^{1/2}$ group speed: $\frac{d\omega}{dk} = \pm\frac{1}{2}(gk)^{-1/2} g$. $\{\frac{d\omega}{dk} = \frac{d}{dk}(\pm(gk)^{1/2})\}$

● $= \pm\frac{1}{2}\left(\frac{g}{k}\right)^{1/2} = \pm\frac{1}{2}\sqrt{\frac{g}{k}}$ //

phase speed: $\frac{\omega}{k} = \pm(gk)^{1/2}\frac{1}{k} = \pm\sqrt{\frac{g}{k}}$ $\therefore$ $C_g = \frac{1}{2}C_p$ $\{\therefore C_g \neq C_p\}$ $\therefore$ these are dispersive waves // {group speed C_g, phase speed C_p}

\week 1 sheet/

\1ai/ $\omega = \nabla \times \underline{u}$ is vorticity

$\therefore$ $\underline{u} \times \underline{\omega} = \underline{u} \times (\nabla \times \underline{u})$

and $\underline{u} \times (\nabla \times \underline{u}) + \underline{u} \times (\nabla \times \underline{u}) = \nabla(\underline{u} \cdot \underline{u}) - (\underline{u} \cdot \nabla)\underline{u} - (\underline{u} \cdot \nabla)\underline{u} =$

$2\underline{u} \times (\nabla \times \underline{u}) = \nabla(\underline{u} \cdot \underline{u}) - 2(\underline{u} \cdot \nabla)\underline{u}$ $\therefore$ $\underline{u} \cdot \underline{u} = |\underline{u}|^2 = u^2$ $\therefore$

$\underline{u} \times (\nabla \times \underline{u}) = \underline{u} \times \underline{\omega} = \frac{1}{2}\nabla(\underline{u} \cdot \underline{u}) - (\underline{u} \cdot \nabla)\underline{u} = \frac{1}{2}\nabla(u^2) - (\underline{u} \cdot \nabla)\underline{u}$ $\therefore$

$(\underline{u} \cdot \nabla)\underline{u} = \frac{1}{2}\nabla(u^2) - \underline{u} \times \underline{\omega} = \underline{u} \cdot \nabla \underline{u}$

\sol/ $\underline{\omega} = \nabla \times \underline{u}$ $\therefore$ identity (vii) on the exam formula sheet with $\underline{u} = \underline{v}$

gives $2\underline{u} \times \underline{\omega} = \nabla(\underline{u} \cdot \underline{u}) - 2\underline{u} \cdot \nabla \underline{u}$ $\therefore$

$\underline{u} \cdot \nabla \underline{u} = \frac{1}{2}\nabla u^2 - \underline{u} \times \underline{\omega}$ since $\underline{u} \cdot \underline{u} = |\underline{u}|^2 = u^2$

\1aii/ $(\underline{u} \cdot \nabla)\underline{u} = \frac{1}{2}\nabla(u^2) - \underline{u} \times \underline{\omega} = \frac{1}{2}\nabla(u^2) - \underline{u} \times (\nabla \times \underline{u})$ $\therefore$

$\nabla \times ((\underline{u} \cdot \nabla)\underline{u}) = \nabla \times (\frac{1}{2}\nabla(u^2) - \underline{u} \times \underline{\omega}) =$

$\nabla \times (\frac{1}{2}\nabla(u^2)) + \nabla \times (-\underline{u} \times \underline{\omega}) =$

$\frac{1}{2}\nabla \times (\nabla(u^2)) - \nabla \times (\underline{u} \times \underline{\omega})$

$\therefore (\underline{u} \cdot \nabla)\underline{u} = \underline{u} \cdot \nabla \underline{u} = \frac{1}{2}\nabla(u^2) - \underline{u} \times \underline{\omega}$ $\therefore$

$\nabla \times ((\underline{u} \cdot \nabla)\underline{u}) = \nabla \times (\frac{1}{2}\nabla(u^2) - \underline{u} \times \underline{\omega}) =$

$\nabla \times (\frac{1}{2}\nabla(u^2)) + \nabla \times (-\underline{u} \times \underline{\omega}) =$

$\frac{1}{2}\nabla \times (\nabla(u^2)) - \nabla \times (\underline{u} \times \underline{\omega}) = \frac{1}{2}\nabla \times (\nabla u^2) - \nabla \times (\underline{u} \times \underline{\omega})$

$\therefore \nabla \times ((\underline{u} \cdot \nabla)\underline{u}) - \frac{1}{2}\nabla \times (\nabla u^2) = -\nabla \times (\underline{u} \times \underline{\omega})$

{ $\nabla \times ((\underline{u} \cdot \nabla)\underline{u}) = -\nabla \times (\underline{u} \times \underline{\omega})$ $\therefore$ $-\frac{1}{2}\nabla \times (\nabla u^2) = 0$ $\therefore$

$\nabla \times (\nabla u^2) = 0$ $\therefore$ $\nabla u^2 = \frac{\partial u^2}{\partial x}\hat{i} + \frac{\partial u^2}{\partial y}\hat{j} + \frac{\partial u^2}{\partial z}\hat{k}$ $\therefore$

$\nabla \times (\nabla u^2) = \nabla \times \left(\frac{\partial u^2}{\partial x}\hat{i} + \frac{\partial u^2}{\partial y}\hat{j} + \frac{\partial u^2}{\partial z}\hat{k}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_x u^2 & \partial_y u^2 & \partial_z u^2 \end{vmatrix} =$

$\hat{i}\left(\frac{\partial^2 u^2}{\partial y \partial z} - \frac{\partial^2 u^2}{\partial z \partial y}\right) - \hat{j}\left(\frac{\partial^2 u^2}{\partial x \partial z} - \frac{\partial^2 u^2}{\partial z \partial x}\right) + \hat{k}\left(\frac{\partial^2 u^2}{\partial x \partial y} - \frac{\partial^2 u^2}{\partial y \partial x}\right) = \hat{i}(0) - \hat{j}(0) + \hat{k}(0) = \underline{0} = 0$ }

$\therefore \nabla \times (\nabla u^2) = 0$ $\therefore$ $\nabla \times ((\underline{u} \cdot \nabla)\underline{u}) = -\nabla \times (\underline{u} \times \underline{\omega})$

\sol/ take the curl of the above $\therefore$ $\nabla \times (\underline{u} \cdot \nabla \underline{u}) = \nabla \times (\nabla \frac{u^2}{2}) - \nabla \times (\underline{u} \times \underline{\omega})$ $\therefore$

$\nabla \times (\underline{u} \cdot \nabla \underline{u}) = -\nabla \times (\underline{u} \times \underline{\omega})$ since $\nabla \times \nabla \phi \equiv \underline{0}$ $\forall \phi$

\1aiii/ $\nabla^2 \underline{u} = \nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{u}) \quad \therefore$

$\nabla\times(\nabla^2\underline{u}) = \nabla\times(\nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{u})) =$

$\nabla\times(\nabla(\nabla\cdot\underline{u})) + \nabla\times(-\nabla\times(\nabla\times\underline{u})) = \nabla\times(\nabla(\nabla\cdot\underline{u})) - \nabla\times(\nabla\times(\nabla\times\underline{u}))$

$\nabla^2\underline{\omega} = \nabla(\nabla\cdot\underline{\omega}) - \nabla\times(\nabla\times\underline{\omega}) =$

$\nabla(\nabla\cdot\nabla\times\underline{u}) - \nabla\times(\nabla\times\nabla\times\underline{u}) = \nabla(\nabla\cdot\nabla\times\underline{u}) - \nabla\times(\nabla\times(\nabla\times\underline{u}))$

$\therefore$ is $\nabla\times(\nabla(\nabla\cdot\underline{u})) = \nabla(\nabla\cdot\nabla\times\underline{u})$ then $\nabla\times(\nabla^2\underline{u}) = \nabla^2(\nabla\times\underline{u}) = \nabla^2\underline{\omega}$

\5a/ identity (viii): $\nabla^2\underline{u} = \nabla(\nabla\cdot\underline{u}) - \nabla\times\underline{\omega}$

$\{\nabla^2 u = \nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{u}) \;\therefore\; \nabla\times\underline{\omega} = \nabla\times(\nabla\times\underline{u})\} \quad \therefore$

Take z Curl: $\nabla\times\nabla^2\underline{u} = \nabla\times\nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{\omega}) =$

$-\nabla\times(\nabla\times\underline{\omega})$ $\{$since $\nabla\times\nabla\phi = \underline{0}$ and $\nabla\cdot\underline{u} = \phi\}$ &

$= -\nabla(\nabla\cdot\underline{\omega}) + \nabla^2\underline{\omega}$ using identity (viii) $\{\therefore \nabla\times(\nabla\times\underline{\omega}) =$

$\nabla(\nabla\cdot\underline{\omega}) - \nabla^2\underline{\omega} \quad \therefore \nabla^2\underline{\omega} = \nabla(\nabla\cdot\underline{\omega}) - \nabla\times(\nabla\times\underline{\omega})\} \therefore$

$\cancel{\nabla\times(\nabla\times\underline{\omega}) = \nabla^2\underline{\omega}}$ & $-\nabla\times(\nabla\times\underline{\omega}) = -\nabla(\nabla\cdot\underline{\omega}) + \nabla^2\underline{\omega}$

$= \nabla^2\underline{\omega}$ since $\nabla\cdot\underline{\omega} = \nabla\cdot(\nabla\times\underline{u}) = 0 \;\therefore$ z divergence of a curl is zero

\2a/ for $\nabla\times(\phi\underline{v})$ take i component: $(\nabla\times(\phi v))_i$

for $\phi\nabla\times\underline{v} + \nabla\phi\times\underline{v} = \cancel{(\phi\nabla}$

$\nabla\phi = [\nabla\phi]_i = \nabla_i\phi \quad \therefore$

$[\nabla\phi\times\underline{v}]_i = \varepsilon_{ijk}[\nabla\phi]_j \underline{v}_k = \varepsilon_{ijk}\nabla_j\phi v_k$

$[\phi\nabla]_i = \phi\nabla_i \quad \therefore [\phi\nabla\times\underline{v}]_i = \varepsilon_{ilm}[\phi\nabla]_l \underline{v}_m = \varepsilon_{ilm}\phi\nabla_l v_m$

$[\phi\underline{v}]_i = \phi v_i \quad \therefore [\nabla\times(\phi\underline{v})]_i = \varepsilon_{ijk}\nabla_j[\phi\underline{v}]_k = \varepsilon_{ijk}\nabla_j\phi v_k \quad \therefore$

$[\phi\nabla\times\underline{v} + \nabla\phi\times\underline{v}]_i = [\phi\nabla\times\underline{v}]_i + [\nabla\phi\times\underline{v}]_i = \varepsilon_{ilm}\phi\nabla_l v_m + \varepsilon_{ijk}\nabla_j\phi v_k =$

$\varepsilon_{ijk}\nabla_j\phi v_k = [\nabla\times(\phi\underline{v})]_i$

\3a/ $[\nabla\times(\phi\underline{v})]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\phi\underline{v})_k = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\phi v_k) =$

$\varepsilon_{ijk}\left\{\phi\frac{\partial v_k}{\partial x_j} + v_k\frac{\partial\phi}{\partial x_j}\right\}$ by product rule

$= [\phi\nabla\times\underline{v}]_i + \varepsilon_{ijk}\frac{\partial\phi}{\partial x_j}v_k = [\phi\nabla\times\underline{v}]_i + [\nabla\phi\times\underline{v}]_i$

\2a/ $[\nabla\times(\phi\underline{v})]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\phi\underline{v})_k = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\phi v_k) =$

$\varepsilon_{ijk}\left(\phi\frac{\partial}{\partial x_j}v_k + v_k\frac{\partial\phi}{\partial x_j}\right) = \varepsilon_{ijk}\phi\frac{\partial}{\partial x_j}v_k + \varepsilon_{ijk}\frac{\partial\phi}{\partial x_j}v_k = [\phi\nabla\times\underline{v}]_i + [\nabla\phi\times\underline{v}]_i$

\week1/ \2b/ $[\nabla \times \underline{v}]_i = \varepsilon_{ijk} \frac{\partial}{\partial x_j} v_k$ $\therefore$

$[\nabla \cdot \underline{F}]_i = (\nabla)_i (\underline{F})_i = \nabla_i F_i = \frac{\partial}{\partial x_i} F_i$ $\therefore$

$[\nabla \cdot (\nabla \times v)]_i = \frac{\partial}{\partial x_i} [\nabla \times \underline{v}]_i = \frac{\partial}{\partial x_i} \varepsilon_{ijk} \frac{\partial}{\partial x_j} v_k = \frac{\partial}{\partial x_i} \varepsilon_{kij} \frac{\partial}{\partial x_j} v_k =$

$\varepsilon_{kij} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} v_k = \left[\frac{\partial}{\partial x} \times \frac{\partial}{\partial x}\right]_k v_k = \left[\frac{\partial^2}{\partial x^2}\right]_k v_k$

\sol/ $\nabla \cdot (\nabla \times \underline{v}) = \frac{\partial}{\partial x_i}\left(\varepsilon_{ijk} \frac{\partial}{\partial x_j} v_k\right) = \varepsilon_{ijk} \frac{\partial^2 v_k}{\partial x_i \partial x_j}$ since ε_{ijk} is a const

$= \varepsilon_{ij1} \frac{\partial^2 v_1}{\partial x_i \partial x_j} + \varepsilon_{ij2} \frac{\partial^2 v_2}{\partial x_i \partial x_j} + \varepsilon_{ij3} \frac{\partial v_3}{\partial x_i \partial x_j}$ by z summation convention applied to z repeated k

$\therefore$ now consider z first term:

$\varepsilon_{ij1} \frac{\partial^2 v_1}{\partial x_i \partial x_j} = \varepsilon_{231} \frac{\partial^2 v_1}{\partial x_2 \partial x_3} + \varepsilon_{321} \frac{\partial^2 v_1}{\partial x_3 \partial x_2} = \frac{\partial^2 v_1}{\partial x_2 \partial x_3} - \frac{\partial^2 v_1}{\partial x_3 \partial x_2} = 0$

z second and third terms vanish by similar arguments

$\therefore \nabla \cdot (\nabla \times \underline{v}) = 0$

\2b/ $[\nabla \cdot (\nabla \times \underline{v})]_i = \left(\frac{\partial}{\partial x_i}\right)\left(\varepsilon_{ijk} \frac{\partial}{\partial x_j} v_k\right) = \varepsilon_{ijk} \frac{\partial}{\partial x_i} \frac{\partial}{\partial x_j} v_k = \varepsilon_{ijk} \frac{\partial^2 v_k}{\partial x_i \partial x_j}$

$= \varepsilon_{ij1} \frac{\partial^2 v_1}{\partial x_i \partial x_j} + \varepsilon_{ij2} \frac{\partial^2 v_2}{\partial x_i \partial x_j} + \varepsilon_{ij3} \frac{\partial^2 v_3}{\partial x_i \partial x_j}$ by summation convention for k

$\therefore$ for term: $\varepsilon_{ij1} \frac{\partial^2 v_1}{\partial x_i \partial x_j} = \varepsilon_{231} \frac{\partial^2 v_1}{\partial x_2 \partial x_3} + \varepsilon_{321} \frac{\partial^2 v_1}{\partial x_3 \partial x_2} =$

$(1) \frac{\partial^2 v_1}{\partial x_2 \partial x_3} + (-1) \frac{\partial^2 v_1}{\partial x_3 \partial x_2} = \frac{\partial^2 v_1}{\partial x_2 \partial x_3} - \frac{\partial^2 v_1}{\partial x_2 \partial x_3} = 0$ $\therefore$ by symmetry z 2nd & 3rd terms vanish by similar arguments $\therefore [\nabla \cdot (\nabla \times \underline{v})]_i = 0$

\1b/ NS-eqn: $\rho\left(\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u}\right) = \rho g - \nabla p + \mu \nabla^2 \underline{u}$ $\therefore g = -\nabla \Phi = -\nabla g z$

$\therefore \rho\left(\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u}\right) = \cancel{\rho g} - \rho \nabla g z - \nabla p + \mu \nabla^2 \underline{u}$ $\therefore$

$\nabla \times \left(\rho\left(\frac{\partial \underline{u}}{\partial t} + (\underline{u} \cdot \nabla)\underline{u}\right)\right) = \nabla \times \left(-\rho \nabla g z - \nabla p + \mu \nabla^2 \underline{u}\right)$

$\{\nabla \cdot (\nabla \times \underline{v}) = 0 \quad \nabla \times \nabla \phi \equiv \underline{0}\}$ $\quad \underline{\omega} = \nabla \times \underline{u}$ $\therefore$

$\nabla \times \left(\rho \frac{\partial \underline{u}}{\partial t}\right) + \nabla \times (\rho(\underline{u} \cdot \nabla \underline{u})) = \nabla \times (-\rho \nabla g z) + \nabla \times (-\nabla p) + \nabla \times (\mu \nabla^2 \underline{u}) =$

$\rho \nabla \times \left(\frac{\partial \underline{u}}{\partial t}\right) + \rho \nabla \times (\underline{u} \cdot \nabla \underline{u}) = -\rho \nabla \times (\nabla g z) - \nabla \times (\nabla p) + \nabla \times (\mu \nabla^2 \underline{u})$

\sol/ $\partial_t \underline{u} + \underline{u} \cdot \nabla \underline{u} = -\nabla\left(\frac{p + \rho g z}{\rho}\right) + \nu \nabla^2 \underline{u}$, $\nu = \mu / \rho$

$\therefore$ take z curl: $\partial_t \underline{\omega} + \nabla \times (\underline{u} \cdot \nabla \underline{u}) = 0 + \nu \nabla \times \nabla^2 \underline{u}$ since curl(grad) $\equiv \underline{0}$

$\therefore$ now use part (ii) & (iii) to obtain:

$\partial_t \underline{\omega} = \nabla \times (\underline{u} \times \underline{\omega}) + \nu \nabla^2 \underline{\omega}$ $\quad \{\nabla \times \nabla^2 \underline{u} = \nabla^2 \underline{\omega} = \nabla^2 (\nabla \times \underline{u})$? $\}$

\1b/ NS-eqn: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho \underline{g} + \mu \nabla^2 \underline{u}$ $\therefore$

$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p - \rho \nabla g z + \mu \nabla^2 \underline{u}$ $\therefore$

$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\nabla\left(\frac{p+\rho g z}{\rho}\right) + \nu \nabla^2 \underline{u}$, $\nu = \mu/\rho$ $\therefore$

$\nabla\times\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = \nabla\times\left(-\nabla\left(\frac{p+\rho g z}{\rho}\right)\right) + \nabla\times(\nu\nabla^2\underline{u}) =$

$-\nabla\times\left(\nabla\left(\frac{p+\rho g z}{\rho}\right)\right) + \nabla\times(\nu\nabla^2\underline{u}) = \nabla\times(\nu\nabla^2\underline{u})$ since $\nabla\times\nabla\phi \equiv \underline{0}\ \forall\phi$

$\therefore \nabla\times\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = \nu\nabla\times(\nabla^2\underline{u}) = \frac{\partial}{\partial t}\nabla\times\underline{u} + \nabla\times(\underline{u}\cdot\nabla\underline{u}) =$

$\frac{\partial}{\partial t}\underline{\omega} + \nabla\times(\underline{u}\cdot\nabla\underline{u}) = \cancel{\frac{\partial}{\partial t}}$ $\therefore$ using part (ii):

$\nu\nabla\times(\nabla^2\underline{u}) = \frac{\partial}{\partial t}\underline{\omega} - \nabla\times(\underline{u}\times\underline{\omega})$ $\therefore$

$\frac{\partial \underline{\omega}}{\partial t} = \nabla\times(\underline{u}\times\underline{\omega}) + \nu\nabla\times(\nabla^2\underline{u}) = \nabla\times(\underline{u}\times\underline{\omega}) + \nu\nabla^2(\nabla\times\underline{u}) =$

$\nabla\times(\underline{u}\times\underline{\omega}) + \nu\nabla^2\underline{\omega}$ $\therefore$

$\nabla\times(\nabla^2\underline{u}) = \nabla^2(\nabla\times\underline{u})$ & $\nabla\times\frac{\partial}{\partial t}\underline{u} = \frac{\partial}{\partial t}\nabla\times\underline{u}$

\3/ Divergence thm: $\oiint_S \underline{F}\cdot d\underline{S} = \iiint_V (\nabla\cdot\underline{F})\,dV = \int_V (\nabla\cdot\underline{F})\,dV$

$\underline{F} = \underline{a}\phi$ $\therefore$ $\oiint_S \underline{a}\phi\cdot d\underline{S} = \int_V \nabla\cdot(\underline{a}\phi)\,dV$ $\therefore$ $d\underline{S} = \hat{\underline{n}}\,dS$ $\therefore$

~~for a = i~~: $\int_S \underline{a}\phi\cdot\hat{\underline{n}}\,dS = \int_V \nabla\cdot(\underline{a}\phi)\,dV$ $\therefore$

for $\underline{a} = \underline{i}$: $\int_S \underline{i}\phi\cdot\hat{\underline{n}}\,dS = \int_V \nabla\cdot(\underline{i}\phi)\,dV = \int_V \underline{i}\frac{\partial}{\partial x_1}\phi\,dV$

for $\underline{a} = \underline{j}$: $\int_S \underline{j}\phi\cdot\hat{\underline{n}}\,dS = \int_V \nabla\cdot(\underline{j}\phi)\,dV = \int_V \underline{i}\frac{\partial}{\partial x_2}\phi\,dV$

for $\underline{a} = \underline{k}$: $\int_S \underline{k}\phi\cdot\hat{\underline{n}}\,dS = \int_V \nabla\cdot(\underline{k}\phi)\,dV = \int_V \underline{k}\frac{\partial}{\partial x_3}\phi\,dV$ $\therefore$

$\int_S \phi\,\hat{n}_i\,dS = \int_V \frac{\partial\phi}{\partial x_i}\,dV$ $\therefore$

$\left(\int_S \phi\,\hat{n}_1\,dS, \int_S \phi\,\hat{n}_2\,dS, \int_S \phi\,\hat{n}_3\,dS\right) = \left(\int_V \frac{\partial\phi}{\partial x_1}\,dV, \int_V \frac{\partial\phi}{\partial x_2}\,dV, \int_V \frac{\partial\phi}{\partial x_3}\,dV\right)$

\sol/ the divergence thm: $\int_V \nabla\cdot\underline{F}\,dV = \int_S \underline{F}\cdot d\underline{S}$ where the surf S encloses the volu V &

$d\underline{S} = \hat{\underline{n}}\,dS$, where $\hat{\underline{n}}$ is the unit outward normal to S

take $\underline{F} = \underline{a}\phi(\underline{x})$ where $\underline{a}$ is const $\therefore$ $\int_V \nabla\cdot(\underline{a}\phi)\,dV = \int_S (\underline{a}\phi)\cdot\hat{\underline{n}}\,dS$ $\therefore$

$\int_V \underbrace{\phi\nabla\cdot\underline{a}}_{=0 \text{ since } \underline{a} \text{ is const}} + \underline{a}\cdot\nabla\phi\,dV = \int_S \phi\underline{a}\cdot\hat{\underline{n}}\,dS$ by identity (iii)

$\therefore \int_V \underline{a}\cdot\nabla\phi\,dV = \int_S \phi\underline{a}\cdot\hat{\underline{n}}\,dS$

take $\underline{a} = \underline{i} = (1,0,0)$ obtain: $\int_V \frac{\partial\phi}{\partial x}\,dV = \int \phi\,\hat{n}_x\,dS$, where $\hat{n}_x$ is the x-component of $\hat{\underline{n}}$

take $\underline{a} = \underline{j} = (0,1,0)$: $\int_V \frac{\partial\phi}{\partial y}\,dV = \int \phi\,\hat{n}_y\,dS$

\week 1/ $\underline{a} = \underline{k} = (0,0,1)$: $\int_V \frac{\partial \phi}{\partial z} dV = \int \phi \hat{n}_z dS$ ∴

$\int_V \frac{\partial \phi}{\partial x_i} dV = \int \phi \hat{n}_i dS$ for $i = 1,2,3$ ∴

in vec form: $\int_V \nabla \phi \, dV = \int_S \phi \hat{\underline{n}} dS$

\3/ $\int_V \nabla \cdot (\underline{a} \phi) dV = \int_S \underline{a} \phi \cdot \hat{\underline{n}} dS = \int_V \phi \nabla \cdot \underline{a} + \underline{a} \cdot \nabla \phi \, dV = \int_S \phi \underline{a} \cdot \hat{\underline{n}} dS =$

$\int_V \underline{a} \cdot \nabla \phi \, dV$ since $\nabla \cdot \underline{a} = 0$ since $\underline{a}$ is const ∴ $\phi \nabla \cdot \underline{a} = 0$

∴ $\int_V \underline{a} \cdot \nabla \phi \, dV = \int_S \phi \underline{a} \cdot \hat{\underline{n}} dS$

for $a = \underline{i} = (1,0,0)$: $\int_V \frac{\partial \phi}{\partial x} dV = \int \phi \hat{n}_x dS$ where $\hat{n}_x$ is the x-component of $\hat{\underline{n}}$

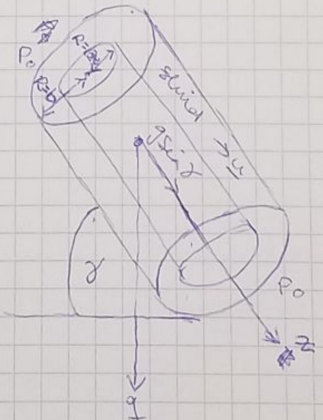
for $\underline{a} = \underline{j} = (0,1,0)$: $\int_V \frac{\partial \phi}{\partial y} dV = \int \phi \hat{n}_y dS$

for $\underline{a} = \underline{k} = (0,0,1)$ ∴ $\int_V \frac{\partial \phi}{\partial z} dV = \int \phi \hat{n}_z dS$ ∴

~~$\int_V \frac{\partial \phi}{\partial x_i} dV = \int_S \phi \hat{n}_i dS$~~ $\int_V \frac{\partial \phi}{\partial x_i} dV = \int \phi \hat{n}_i dS$ for $i = 1,2,3$ ∴

in vec form: $\int_V \nabla \phi \, dV = \int \phi \hat{\underline{n}} dS$

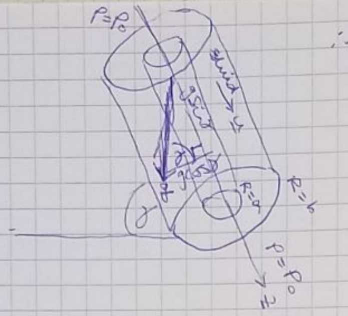
\week 2/ \1a/



NS-eqn: $\rho \left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} \right) + \rho g - \nabla P + \mu \nabla^2 \underline{u}$ ∴ $\underline{u} = (u, v, w)$ ∴

~~z-comp~~ z-comp: $\rho \left(\frac{\partial w}{\partial t} + w \frac{\partial}{\partial z} w \right) = \rho g \sin \gamma + \mu \frac{\partial^2}{\partial z^2} w$

\1a Sol 1/



$\therefore$ $\underline{z}$ - comp of NS eqn is:

$$\rho\big(\underbrace{\partial_t W}_{=0 \text{ since flow is steady}} + [\underline{u}\cdot\nabla\underline{u}]_z\big) =$$

$$-\partial_z p + \rho \sin\gamma + \mu[\nabla^2\underline{u}]_z \quad \therefore$$

$\therefore$ for $\underline{u} = W(R)\hat{\underline{z}}$ should find $[\underline{u}\cdot\nabla\underline{u}]_z = 0$ &

$(\nabla^2\underline{u})_z = \frac{1}{R}\frac{d}{dR}\left(R\frac{dW}{dR}\right)$ (from LN's)

$\underline{u} = W(R)\hat{\underline{z}}$ $\therefore$ $\left(\frac{\partial \underline{u}}{\partial t}\right)_z = \frac{\partial}{\partial t}W(R)$ $\therefore$ $W(R)$ is indep of time $\therefore \frac{\partial W(R)}{\partial t} = 0$

in cylindrical polar coords: $\nabla\Omega = \hat{\underline{R}}\frac{\partial\Omega}{\partial R} + \frac{1}{R}\hat{\underline{\phi}}\frac{\partial\Omega}{\partial\phi} + \hat{\underline{z}}\frac{\partial\Omega}{\partial z}$

$\underline{u} = W(R)\hat{\underline{z}} = W\hat{\underline{z}} = (0,0,W) = 0\hat{\underline{R}} + 0\hat{\underline{\phi}} + W\hat{\underline{z}}$ $\therefore$

$(\underline{u}\cdot\nabla) = (0\hat{\underline{R}} + 0\hat{\underline{\phi}} + W\hat{\underline{z}})\cdot\left(\hat{\underline{R}}\frac{\partial}{\partial R} + \frac{1}{R}\hat{\underline{\phi}}\frac{\partial}{\partial\phi} + \hat{\underline{z}}\frac{\partial}{\partial z}\right) = W\frac{\partial}{\partial z}$ $\therefore$

$(\underline{u}\cdot\nabla)\underline{u} = \left(W\frac{\partial}{\partial z}\right)(0\hat{\underline{R}} + 0\hat{\underline{\phi}} + W(R)\hat{\underline{z}}) = W\frac{\partial}{\partial z}W(R)\,\hat{\underline{z}}$ $\therefore$

$[\underline{u}\cdot\nabla\underline{u}]_z = W\frac{\partial}{\partial z}W(R)$ W is indep of z $\therefore$ $\frac{\partial}{\partial z}W(R) = 0$ $\therefore$

$[\underline{u}\cdot\nabla\underline{u}]_z = 0$ $\therefore$ $\underline{u}\cdot\nabla\underline{u} = 0$

$(\nabla^2\underline{u})_z$: $\therefore$ $\nabla\times(\nabla\times\underline{u}) = \nabla(\nabla\cdot\underline{u}) - \nabla^2\underline{u}$ $\quad \underline{u} = W(R)\hat{\underline{z}}$ $\therefore$

$\nabla\cdot\underline{u} = \nabla\cdot(0\hat{\underline{R}} + 0\hat{\underline{\phi}} + W(R)\hat{\underline{z}}) = \frac{1}{R}\frac{\partial}{\partial R}(Ru_R) + \frac{1}{R}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z} =$

$= \frac{1}{R}\frac{\partial}{\partial R}(0) + \frac{1}{R}\frac{\partial 0}{\partial\phi} + \frac{\partial W(R)}{\partial z} = \frac{\partial W(R)}{\partial z} = 0$

$$\nabla\times\underline{u} = \frac{1}{R}\begin{vmatrix}\hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}}\\ \partial/\partial R & \partial/\partial\phi & \partial/\partial z\\ u_R & Ru_\phi & u_z\end{vmatrix} = \frac{1}{R}\begin{vmatrix}\hat{\underline{R}} & R\hat{\underline{\phi}} & z\\ \partial/\partial R & \partial/\partial\phi & \partial/\partial z\\ 0 & 0 & W(R)\end{vmatrix} =$$

$\frac{1}{R}\hat{\underline{R}}\frac{\partial W(R)}{\partial\phi} - \frac{1}{R}R\hat{\underline{\phi}}\frac{\partial W(R)}{\partial R} + \frac{1}{R}z(0-0) = -\hat{\underline{\phi}}\frac{\partial W(R)}{\partial R}$ $\therefore$

$\nabla\times(\nabla\times\underline{u}) = \nabla\times\left(-\hat{\underline{\phi}}\frac{\partial W(R)}{\partial R}\right) = \nabla\times\left(0\hat{\underline{R}} - \hat{\underline{\phi}}\frac{\partial W(R)}{\partial R} + 0\hat{\underline{z}}\right) =$

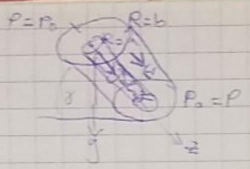
$$\frac{1}{R}\begin{vmatrix}\hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}}\\ \partial R & \partial\phi & \partial z\\ 0 & -R\frac{\partial W(R)}{\partial R} & 0\end{vmatrix} = \frac{1}{R}\hat{\underline{R}}\left(+R\frac{\partial^2 W(R)}{\partial z\,\partial R}\right) - \frac{1}{R}R\hat{\underline{\phi}}(0-0) + \frac{1}{R}\hat{\underline{z}}\left(-\frac{\partial}{\partial R}\left(R\frac{\partial W(R)}{\partial R}\right)\right) =$$

$-\frac{1}{R}\frac{d}{dR}\left(R\frac{dW}{dR}\right)\hat{\underline{z}}$ $\quad \therefore \nabla(\nabla\cdot\underline{u}) = \nabla(0) = 0$ $\therefore$

$-\frac{1}{R}\frac{d}{dR}\left(R\frac{dW}{dR}\right)\hat{\underline{z}} = 0 - \nabla^2\underline{u} = -\nabla^2\underline{u} = 0\hat{\underline{R}} + 0\hat{\underline{\phi}} - \frac{1}{R}\frac{d}{dR}\left(R\frac{dW(R)}{dR}\right)\hat{\underline{z}}$ $\therefore$

$(\nabla^2\underline{u})_z = \frac{1}{R}\frac{d}{dR}\left(R\frac{dW(R)}{dR}\right)$ $\quad \therefore \rho(0+0) = -\partial_z p + \rho g\sin\gamma + \mu\frac{1}{R}\frac{d}{dR}\left(R\frac{dW(R)}{dR}\right) = 0$

week 2 / 1a /



NS: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) =$

$\rho\underline{g} - \nabla p + \mu\nabla^2\underline{u}$ $\quad \underline{u} = (u, v, w(R))$

$\therefore$ z-comp: $\rho\left(\frac{\partial w(R)}{\partial t} + [\underline{u}\cdot\nabla\underline{u}]_z\right) = \rho g \sin\gamma - \frac{\partial}{\partial z}p + \mu[\nabla^2\underline{u}]_z =$

$\rho[\underline{u}\cdot\nabla\underline{u}]_z \quad \therefore$

$\underline{u}\cdot\nabla = w(R)\hat{\underline{z}}\cdot\left(\frac{\partial}{\partial R}\hat{R} + \frac{1}{R}\hat{\underline{\phi}}\frac{\partial}{\partial\phi} + \hat{\underline{z}}\frac{\partial}{\partial z}\right) = w(R)\frac{\partial}{\partial z} \quad \therefore$

$(\underline{u}\cdot\nabla)\underline{u} = w(R)\frac{\partial}{\partial z}(0\hat{\underline{R}} + 0\hat{\underline{\phi}} + w(R)\hat{\underline{z}}) = w(R)\frac{\partial}{\partial z}w(R)\hat{\underline{z}} = 0$

$\therefore [\underline{u}\cdot\nabla\underline{u}]_z = 0 \quad \therefore$

$0 = \rho g \sin\gamma - \frac{\partial}{\partial z}p + \mu[\nabla^2\underline{u}]_z$

$\nabla\times(\nabla\times\underline{u}) = \nabla(\nabla\cdot\underline{u}) - \nabla^2\underline{u} \quad \therefore$

$\nabla\cdot\underline{u} = \nabla\cdot(0\hat{\underline{R}} + 0\hat{\underline{\phi}} + w(R)\hat{\underline{z}}) = \frac{1}{R}\frac{\partial}{\partial R}(R\cdot 0) + \frac{1}{R}\frac{\partial}{\partial\phi}(0) + \frac{\partial}{\partial z}w(R) = \frac{\partial w(R)}{\partial z} = 0$

$\therefore \nabla(\nabla\cdot\underline{u}) = \nabla(0) = 0 \quad \therefore \nabla\times(\nabla\times\underline{u}) = -\nabla^2\underline{u} \quad \therefore$

$\nabla\times\underline{u} = \nabla\times(0\hat{\underline{R}} + 0\hat{\underline{\phi}} + w(R)\hat{\underline{z}}) = -\frac{R}{R}\frac{\partial w(R)}{\partial R}\hat{\underline{\phi}} = -\frac{dw(R)}{dR}\hat{\underline{\phi}} \quad \therefore$

$\nabla\times(\nabla\times\underline{u}) = \nabla\times\left(-\hat{\underline{\phi}}\frac{dw(R)}{dR}\right) = \nabla\times\left(0\hat{\underline{R}} - \hat{\underline{\phi}}\frac{dw(R)}{dR} + 0\hat{\underline{z}}\right) =$

$\frac{1}{R}\hat{\underline{z}}\frac{\partial}{\partial R}\left(-R\frac{dw(R)}{dR}\right) = -\frac{1}{R}\hat{\underline{z}}\frac{d}{dR}\left(R\frac{dw(R)}{dR}\right) = -\nabla^2\underline{u} \quad \therefore$

$\nabla^2\underline{u} = \frac{1}{R}\frac{d}{dR}\left(R\frac{dw(R)}{dR}\right)\hat{\underline{z}} \quad \therefore [\nabla^2\underline{u}]_z = \frac{1}{R}\frac{d}{dR}\left(R\frac{dw(R)}{dR}\right)\hat{\underline{z}} \quad \therefore$

$0 = \rho g \sin\gamma - \frac{\partial}{\partial z}p + \mu\frac{1}{R}\frac{d}{dR}\left(R\frac{dw(R)}{dR}\right)\hat{\underline{z}}$

1b / because the pressure on both ~~sides~~ ends of the cylinder is P_0 so there is no change in pressure going along the cylinder so the pressure gradient along the cylinder is zero

Sol / seperation of variables gives $\frac{\partial p}{\partial t}$ indep of z. Set $\frac{\partial p}{\partial z} = f(R)$ & use conditions $p(z=0) = p(z=L) = p_0$ to find $f = 0 \quad \therefore$

$\frac{\partial}{\partial z}p = \underbrace{\rho g \sin\gamma}_{\text{const}} + \underbrace{\frac{\mu}{R}\frac{d}{dR}\left(R\frac{dw(R)}{dR}\right)}_{\text{func of } R} \quad \therefore \frac{\partial p}{\partial z}$ is indep of z $\therefore$

$\frac{\partial p}{\partial z} = c \quad \therefore \int\frac{\partial p}{\partial z}dz = \int c\,dz = cz + d = p \quad \therefore$ ~~$p(z=0) =$~~

$\therefore p(z=0) = p_0 = c(0) + d = d \quad \therefore p_0 = d \quad \therefore$

$p_0(z=L) = p_0 = cL + p_0 \quad \therefore cL = 0 \quad \therefore c = 0 \quad \therefore p = p_0 \quad \therefore \frac{\partial p}{\partial z} = 0 \quad \therefore$

z pressure gradient along z cylinder is zero

\1b/ $\therefore \frac{\partial p}{\partial z} = \rho g \sin\gamma + \mu \frac{1}{R}\frac{d}{dR}\left(R\frac{dW(R)}{dR}\right) \therefore$

$\rho g \sin\gamma = \text{const}$ & $\mu\frac{1}{R}\frac{d}{dR}\left(R\frac{dW(R)}{dR}\right)$ is a func of R $\therefore$

$\frac{\partial p}{\partial z}$ is indep of z $\therefore$

$\frac{\partial p}{\partial z} = C \therefore p = Cz + d \therefore p(z=0) = P_0 = C(0) + d = d = P_0 \therefore$

$P(z=L) = P_0 = C(L) + P_0 \therefore CL = 0 \therefore C = 0 \therefore \frac{\partial p}{\partial z} = 0 \therefore$

$\therefore$ pressure gradient along z cylinder is zero

\1c/ $\therefore 0 = \rho g \sin\gamma + \mu\frac{1}{R}\frac{d}{dR}\left(R\frac{dW(R)}{dR}\right) \therefore$

$-\frac{\rho g \sin\gamma}{\mu}R = \frac{d}{dR}\left(R\frac{dW(R)}{dR}\right) \therefore A - \frac{\rho g \sin\gamma}{2\mu}R^2 = R\frac{dW(R)}{dR} \therefore$

$-\frac{\rho g \sin\gamma}{2\mu}R + \frac{A}{R} = \frac{dW(R)}{dR} \therefore -\frac{\rho g \sin\gamma}{4\mu}R^2 + A\ln R + B = W(R) \therefore$

\1d/ no slip boundary conditions $\therefore$ at $R=a$ $\underline{u}=0$ and at $R=b$ $\underline{u}=0$

$\therefore W(R=a) = 0$ & $W(R=b) = 0 \therefore$

$W(R=a) = 0 = -\frac{\rho g \sin\gamma}{4\mu}a^2 + A\ln(a) + B$ &

$W(R=b) = 0 = -\frac{\rho g \sin\gamma}{4\mu}b^2 + A\ln b + B \therefore$

$\frac{\rho g \sin\gamma}{4\mu}a^2 - A\ln a = B \therefore 0 = -\frac{\rho g \sin\gamma}{4\mu}b^2 + A\ln b + \frac{\rho g \sin\gamma}{4\mu}a^2 - A\ln a \therefore$

$\frac{\rho g \sin\gamma}{4\mu}(b^2 - a^2) = A(\ln b - \ln a) = A\ln\left(\frac{b}{a}\right) \therefore$

$\frac{\rho g \sin\gamma}{4\mu}(b^2 - a^2)\frac{1}{\ln\left(\frac{b}{a}\right)} = A \therefore$

$W(R) = -\frac{\rho g \sin\gamma}{4\mu}R^2 + \frac{\rho g \sin\gamma}{4\mu\ln\left(\frac{b}{a}\right)}(b^2-a^2)\ln R + \frac{\rho g \sin\gamma}{4\mu}a^2 - \frac{\rho g \sin\gamma(b^2-a^2)}{4\mu\ln\left(\frac{b}{a}\right)}\ln a$

$= -\frac{\rho g \sin\gamma}{4\mu}\left[+R^2 - \frac{(b^2-a^2)\ln R}{\ln\left(\frac{b}{a}\right)} - a^2 + \frac{(b^2-a^2)}{\ln\left(\frac{b}{a}\right)}\ln a\right] =$

$W(R) = -\frac{\rho g \sin\gamma}{4\mu}\left[R^2 - \frac{b^2\ln R}{\ln\left(\frac{b}{a}\right)} + \frac{a^2\ln R}{\ln\left(\frac{b}{a}\right)} - a^2 + \frac{b^2\ln a}{\ln\left(\frac{b}{a}\right)} - \frac{a^2}{\ln\left(\frac{b}{a}\right)}\ln a\right] =$

$-\frac{\rho g \sin\gamma}{4\mu}\left[R^2 + \left(a^2\ln R - b^2\ln R + b^2\ln a - a^2\ln a - a^2\ln\left(\frac{b}{a}\right)\right)/\ln\left(\frac{b}{a}\right)\right] =$

$-\frac{\rho g \sin\gamma}{4\mu}\left[R^2 - \left(+b^2\ln\left(\frac{R}{a}\right) - a^2\ln\left(\frac{R}{b}\right) + a^2\ln a - a^2\ln a\right)/\ln\left(\frac{b}{a}\right)\right] =$

$-\frac{\rho g \sin\gamma}{4\mu}\left[R^2 - \left[\left(b^2\ln\left(\frac{R}{a}\right) - a^2\ln\left(\frac{R}{b}\right)\right]/\ln\left(\frac{b}{a}\right)\right]$ $\square$ QED $\text{for } a < R < b$

\week 2/ \2/ $\therefore \frac{\partial v}{\partial t} = \frac{\mu}{\rho}\left(\frac{\partial^2 v}{\partial R^2} + \frac{1}{R}\frac{\partial v}{\partial R} - \frac{v}{R^2}\right) =$

$\frac{\mu}{\rho}\frac{\partial^2 v}{\partial R^2} + \frac{\mu}{\rho}\frac{1}{R}\frac{\partial v}{\partial R} - \frac{\mu}{\rho}\frac{v}{R^2}$ $\qquad v = f(R)g(t)$

$\therefore \frac{dv}{dt} = \frac{d}{dt}\left(f(R)g(t)\right) = f(R)\frac{d}{dt}g(t)$ & $\therefore \frac{dv}{dR} = g(t)\frac{d}{dR}f(R) \therefore$

$\frac{d^2 v}{dR^2} = \frac{d}{dR}\left(g(t)\frac{d}{dR}f(R)\right) = g(t)\frac{d^2}{dR^2}f(R) \therefore$

$f(R)\frac{d}{dt}g(t) = \nu\left(g(t)\frac{d^2}{dR^2}f(R) + \frac{1}{R}g(t)\frac{d}{dR}f(R) - \frac{f(R)g(t)}{R^2}\right) \therefore$ take $g(t) = e^{-int}$

$\therefore \frac{dg(t)}{dt} = -in e^{-int} = -in g(t) \therefore$

$-in g(t) f(R) = \nu\left(g(t)\frac{d^2}{dR^2}f(R) + \frac{1}{R}g(t)\frac{d}{dR}f(R) - \frac{1}{R^2}f(R)g(t)\right) \therefore$

$-in f(R) = \nu\frac{d^2}{dR^2}f(R) + \nu\frac{1}{R}\frac{d}{dR}f(R) - \nu\frac{1}{R^2}f(R) \therefore$

$0 = \nu f'' + \nu\frac{1}{R}f' + \left(in - \nu\frac{1}{R^2}\right)f \therefore$

$0 = \nu R^2 f'' + \nu R f' + (inR^2 - \nu)f$.

$0 = R^2 f'' + R f' + \left(\frac{in}{\nu}R^2 - 1\right)f = 0 = R^2 f'' + R f' + (\alpha^2 - 1)f$ $\qquad \frac{in}{\nu} = \alpha^2$

$\therefore S = \alpha R \therefore \frac{dS}{dR} = \alpha \therefore dS = \alpha\, dR \therefore \frac{d}{dR} = \alpha\frac{d}{dS} \therefore$

$\frac{d^2}{dR^2} = \alpha^2\frac{d^2}{dS^2} \therefore R^2 f'' = R^2\frac{d^2}{dR^2}f(R) = R^2\alpha^2\frac{d^2}{dS^2}f(R) \therefore$

$Rf' = R\frac{d}{dR}f(R) = R\,\alpha\frac{d}{dS}f(R) \therefore R^2\alpha^2\frac{d^2}{dS^2}f(R) + R\alpha\frac{d}{dS}f(R) + (\alpha^2R^2 - 1)f(R) = 0$

$\therefore S^2\frac{d^2}{dS^2}f(R) + S\frac{d}{dS}f(R) + (S^2 - 1)f(R) = 0$

$\therefore$ Sols $\left.\begin{matrix} f = J_1 \\ f = Y_1 \end{matrix}\right\} \therefore$ general sol: $f = AJ_1 + BY_1$

\2/ $\therefore \frac{dv}{dt} = f\frac{dg}{dt}$, $\frac{dv}{dR} = g f'$ $\therefore \frac{d^2 v}{dR^2} = g f''$ as $g(t) = e^{-int} \therefore$

$\frac{dg}{dt} = -in e^{-int} = -in g \therefore -in g f = \nu\left(g f'' + \frac{1}{R}g f' - \frac{1}{R^2}f g\right) \therefore$

$-in f = \nu f'' + \frac{\nu}{R}f' - \frac{1}{R^2}\nu f \therefore 0 = \nu f'' + \nu\frac{1}{R}f' + \left(in - \nu\frac{1}{R^2}\right)f \therefore$

$0 = \nu R^2 f'' + \nu R f' + (inR^2 - \nu)f \therefore 0 = R^2 f'' + R f' + \left(\frac{in}{\nu}R^2 - 1\right)f \therefore$

$\alpha^2 = \frac{in}{\nu} \therefore 0 = R^2 f'' + R f' + (\alpha^2 R^2 - 1)f \therefore$ as $S = \alpha R \therefore \frac{dS}{dR} = \alpha \therefore$

$\frac{d}{dR} = \alpha\frac{d}{dS} \therefore \frac{d^2}{dR^2} = \alpha^2\frac{d^2}{dS^2} \therefore 0 = R^2\alpha^2\frac{d^2}{dS^2}f + R\alpha\frac{d}{dS}f + (\alpha^2R^2 - 1)f$

$\therefore S^2\frac{d^2 f}{dS^2} + S\frac{df}{dS} + (S^2 - 1)f = 0 \therefore$ Sols: $f = J_1$ & $f = Y_1 \therefore$

general sol: $f = AJ_1 + BY_1$