\Sol PP2020 \3a Let $\Omega \subset D$ be a compact set that is posivly invariant with $\dot{x} = f(x)$ ∴ let $V_i : D \to \mathbb{R}$ be a contly diffable func ∴ $\dot{V}(x) \le 0$ in Ω ∴ let E be Z set of all pts in Ω where $\dot{V}(x) = 0$. Let M be Z largest invariant set in E, then every sol starting in Ω approaches M as $t \to \infty$

\3bi $\frac{d}{dt} V(x) = \dot{x} P x + x^T P \dot{x} = x^T(A^T P + PA)x + 2x^T P \Phi(x) = -||x(t)||^2 + 2||P||\,||x||\,||\Phi||$ ∴ δ is : $|x| < \delta$ ∴ $\frac{|\Phi(x)|}{x} < \frac{1}{4||P||}$ ∴

$\dot{V} \le -||x||^2 + \frac{1}{2}||x||^2 = -\frac{1}{2}||x||^2$

\3bii $V(x) = x^T P x$, $\dot{V}(x) \le -\frac{1}{2}||x||^2$ provided $||x|| < \delta$ but $\lambda_1 ||x||^2 \le V(x) \le \lambda_n ||x||^2 = ||P||\,||x||^2 = x^T P x$ ∴

$|x| < \delta$ is $\frac{1}{\lambda_1} V(x) = \frac{1}{\lambda_1} x^T P x < \delta^2$ is $x^T P x < \lambda_1 \delta^2$ ∴ $|x| < \delta$ & $\dot{V} < 0$

∴ $D = \{x \mid x^T P x < \lambda_1 \delta^2\}$ is invariant. it is also open as a set of form $\{x \mid V(x) < \beta\}$ ∴ $V(x)$ is posi desi, $\dot{V}(x)$ nega desi, $V(x) < \beta$ level set ∴ in D. ∴ $V(x) > 0$ & $\dot{V} < 0$ ∴ by LaSalle's argument D contained in basin of attraction for $x = 0$

\3biii $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \sqrt{6}\, x_1 x_2 \\ x_1^2 + x_2^2 \end{bmatrix}$ ∴ $A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$ ∴

$A^T P + PA = -I$ lyapunov eqn ∴ $P = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/6 \end{bmatrix}$ ∴

$\lambda_{max} = \frac{3}{12}$, $\lambda_{min} = \frac{2}{12}$ ∴ $\Phi = \begin{bmatrix} \sqrt{6}\, x_1 x_2 \\ x_1^2 + x_2^2 \end{bmatrix}$ ∴

$|\Phi|^2 = 6x_1^2 x_2^2 + (x_1^2 + x_2^2)^2 \le 4(x_1^2 + x_2^2)^2$ ∴ $\frac{|\Phi(x)|}{|x|} \le 2||x||$ ∴

$||P|| = \frac{1}{4}$ ∴ $\frac{1}{4||P||} = 1$ (diagonal of P) ∴ $\frac{|\Phi(x)|}{|x|} < \frac{1}{4||P||} = 1$ is $2||x|| < 1$ ∴

$||x|| < \frac{1}{2}$ ∴ $\delta = \frac{1}{2}$, $\lambda_1 = \frac{1}{6}$ ∴ $D = \{x \mid \frac{1}{12}(3x_1^2 + 2x_2^2) < \frac{1}{4} \cdot \frac{1}{6}\} =$

$\{x \mid 3x_1^2 + 2x_2^2 < \frac{1}{2}\}$

\4a The mapping $f : D \to \mathbb{R}^n$, $g : D \to \mathbb{R}^n$ are vec fields on D deriv wrt time $\dot{y} = \frac{\partial h}{\partial x}[\dot{x}] = \frac{\partial h}{\partial x}[f(x) + g(x)u] = \underbrace{\frac{\partial h}{\partial x} f(x)}_{L_f h(x)} + \underbrace{\frac{\partial h}{\partial x} g(x) u}_{L_g h(x) u}$

are lie deriv of h wrt f along f,

lie deriv of h along $g(x)$ ∴ is.

$L_g h(x) = 0$ then $\dot{y} = L_f h(x)$ indep of u

$y^{(2)} = L_f^2 h(x) + L_g L_f h(x) u \cdots$ etc $\therefore$

the non linear system (1) is said to have relative degree ρ, $1 \le \rho \le n$ in a region $D_0 \subset D$ if

$L_g L_f^{i-1} h(x) = 0$, $i = 1, 2, \ldots, \rho - 1$ $\quad L_g L_f^{\rho-1} h(x) \neq 0 \ \forall x \in D_0$

\4bi/ relative degree 2 $\therefore$ $y = x_1$, $\frac{dy}{dt} = \dot{x}_1$ (no u) so take deriv again following def in (a) to appear u explicity when taking deriv of y occurs when taking 2nd deriv Σ ~~$\mathbb{R}^2$~~ in $\mathbb{R}^2$

\4bii/ $x_2 = \phi(x_1) = -(k+1)x_1$, $k > 0$ with CLF $V(x_1) = \frac{1}{2}x_1^2$, $V(x) \to \infty$ as $x \to \infty$ $\quad \dot{V}(x_1) = x_1 \dot{x}_1 = x_1(x_1 - x_1^3 - (k+1)x_1) = x_1^3 - x_1^4 - kx_1^2 - x_1^2 =$
$-x_1^4 - kx_1^2 < 0 \quad \forall x_1 \in \mathbb{R} \setminus \{0\}$ $\therefore x_2 = \phi(x_1) = -(k+1)x_1$ makes $x_1 = 0$ a globally stable eq

\4biii/ $\dot{x}_1 = x_1 - x_1^3 + z_2 + \phi(x_1) = -kx_1 - x_1^3 + z_2$ $\therefore$

$\dot{z}_2 = \dot{x}_2 - \dot{\phi}(x_1) = u + (k+1)(-kx_1 - x_1^3 - z_2)$ $\therefore$

$V(x_1, z_2) = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$ $\therefore$ $\dot{V} = x_1\dot{x}_1 + z_2\dot{z}_2 =$

$x_1(-kx_1 - x_1^3 + z_2) + z_2(u + (k+1)(-kx_1 - x_1^3 + z_2)) =$

$-kx_1^2 - x_1^4 + \underbrace{z_2(x_1 + u + (k+1)(-kx_1 - x_1^3 + z_2))}_{\text{term to manipulate}}$

requirement $\dot{V} = -kx_1^2 - x_1^4 - cz^2$ (neg definite)

choosing $u = -cz_2 - x_1 - (k+1)(-kx_1 - x_1^3 + z_2)$ $\therefore$ is as required $\therefore$

by control law: $u = -(z_2 - x_1 - (k+1)(-kx_1 - x_1^3 + z_2))$

\1ai/ if $0 < 4a(a+b) < 1$ $\therefore$ $0 < 1 - 4a(a+b) < 1$ $\therefore$

$\sqrt{1-4a(a+b)} \in \mathbb{R}_0$ $\therefore$ $0 < \frac{\sqrt{1-4a(a+b)}}{2} < \frac{1}{2}$ $\therefore$

$-\frac{1}{2} + \frac{\sqrt{1-4a(a+b)}}{2} \in \mathbb{R}_{>0}$ $\therefore$ of $-\frac{1}{2} - \frac{\sqrt{1-4a(a+b)}}{2} \in \mathbb{R}_{<0}$ $\therefore$ $\lambda_{1,2} \in \mathbb{R}_{<0}$

if $0 < 4a(a+b) < 1$ its a stable node

if $4a(a+b) > 1$ $\therefore$ $1 - 4a(a+b) < 0$ $\therefore$ $\pm\sqrt{1-4a(a+b)} \notin \mathbb{R}$ $\therefore$

$\lambda_{1,2} \in \mathbb{C}$ $\therefore$ $Re(\lambda_{1,2}) \in \mathbb{R}_{<0}$ $\therefore$ its a stable focus ,

\PP2020/ $V=0$ for $\omega=(\pm\omega_0,0,0)$ and

$V>0$ for ~~$\omega_x \neq \pm\omega_0$, $\omega_y \neq 0$~~ $\forall \omega \in \mathbb{R}^3 \setminus \{(\pm\omega_0,0,0)\}$ $\therefore$

V is locally positive definite centered at $(\pm\omega_0,0,0)$

$\therefore \dot{V}=0$ $\forall$ points on $(\omega_x, \omega_y, \omega_z)$, in state space is a stable equilibrium $\therefore$

rotation at any constant velocity that is about x-axis alone is stable $\therefore$ ~~rotation~~

rotation about $(\omega_x,0,0)$ for any constant ω_x is stable

\2C/ $\dot{V}=2c\omega_y\dot{\omega}_y+2b\omega_z\dot{\omega}_z+2[2ac\omega_y^2+ab\omega_z^2+bc(\omega_x^2-\omega_0^2)](4ac\omega_y\dot{\omega}_y+2ab\omega_z\dot{\omega}_z+2bc\omega_x\dot{\omega}_x)$

$\therefore$ for $\dot{V}=0$: $2c\omega_y\dot{\omega}_y+2b\omega_z\dot{\omega}_z=0$,

$4ac\omega_y\dot{\omega}_y+2ab\dot{\omega}_z\omega_z+2bc\omega_x\dot{\omega}_x=0$

$\therefore 0=2c\omega_y\dot{\omega}_y+2b\omega_z\dot{\omega}_z=2c\omega_y(-b\omega_z\omega_x)+2b\omega_z(c\omega_x\omega_y)=$

$-2bc\omega_x\omega_y\omega_z+2bc\omega_x\omega_y\omega_z=0$,

$4ac\omega_y\dot{\omega}_y+2ab\omega_z\dot{\omega}_z+2bc\omega_x\dot{\omega}_x=0=$

$4ac\omega_y(-b\omega_z\omega_x)+2ab\omega_z(c\omega_x\omega_y)+2bc\omega_x(a\omega_y\omega_z)=$

$-4abc\omega_x\omega_y\omega_z+2abc\omega_x\omega_y\omega_z+2abc\omega_x\omega_y\omega_z=(-4+2+2)abc\omega_x\omega_y\omega_z=0$

$\therefore \dot{V}=0$ $\forall \omega \in \mathbb{R}^3$,

$V=0$ for $\omega=(\pm\omega_0,0,0)$ and

$V>0$ $\forall \omega \in \mathbb{R}^3 \setminus \{(\pm\omega_0,0,0)\}$ $\therefore$

V is locally positive definite centered at $(\pm\omega_0,0,0)$ $\therefore$

~~$\dot{V}=0$, $V=0$~~ $\forall$ points on ω_x axis in state space is a stable equilibrium

$\therefore$ rotation at any constant velocity that about x-axis alone is stable

\2C/ $\dot{V}(\omega_x,\omega_y,\omega_z)=$

$2c\omega_y\dot{\omega}_y+2b\omega_z\dot{\omega}_z+2[2ac\omega_y^2+ab\omega_z^2+bc(\omega_x^2-\omega_0^2)](4ac\omega_y\dot{\omega}_y+2ab\omega_z\dot{\omega}_z+2bc\omega_x\dot{\omega}_x)=$

$-2c\omega_y b\omega_x\omega_z+2bc\omega_x\omega_y\omega_z+[4ac\omega_y^2+2ab\omega_z^2+2bc(\omega_x^2-\omega_0^2)](4ac\omega_y(-b)\omega_x\omega_z+2abc\omega_x\omega_y\omega_z+2abc\omega_x\omega_y\omega_z)=$

$\omega_x\omega_y\omega_z(-2bc+2bc+[4ac\omega_y^2+2ab\omega_z^2+2bc(\omega_x^2-\omega_0^2)](-4acb+2abc+2abc))=$

$\omega_x\omega_y\omega_z[4ac\omega_y^2+2ab\omega_z^2+2bc(\omega_x^2-\omega_0^2)](0)=0$ $\therefore$

$\dot{V} = 0$

also $V = 0$ iff $\omega = (\pm\omega_0, 0, 0)$ and $V > 0$ wherever $\omega_x \neq \pm\omega_0, \omega_y \neq 0, \omega_z \neq 0$

$\therefore$ V is locally positive definite centered at $(\pm\omega_0, 0, 0)$

$\therefore \dot{V} = 0 \therefore \forall$ points on ω_x axis in state space is a stable equilibrium

$\therefore$ rotation at any const velocity about x-axis alone is stable

3a/ let $\dot{x}(t) = f(x)$, $x \in \mathbb{R}^n$

let $\Omega \subset D \subset \mathbb{R}^n$ be a compact positively invariant set w.r.t $\dot{x} = f(x)$

$\dot{V}(x): D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω

let $E \subset \Omega$ be the set of all points in Ω where $\dot{V}(x) = 0$

let $M \subset E$ be the largest invariant set in E

Then every solution starting in Ω approaches M as $t \to \infty$

3bi/ let $V(x) = x^T P x \therefore \dot{V} = \frac{d}{dt} V(x) = \frac{d}{dt}(x^T P x) = \dot{x}^T P x + x^T P \dot{x} =$

$(Ax + \Phi(x))^T Px + x^T P(Ax + \Phi(x)) = (x^T A^T + \Phi(x)^T) Px + x^T P A x + x^T P \Phi(x) =$

$\cancel{(x^T A^T + \Phi(x)^T) Px + x^T PAx + x^T P \Phi(x)} =$

$x^T A^T P x + \Phi(x)^T P x + x^T P A x + x^T P \Phi(x) =$

$x^T(A^T P x + PAx) + \Phi(x)^T P x + x^T P \Phi(x) = x^T(A^T P + PA)x + \Phi(x)^T P x + x^T P \Phi(x) =$

$x^T(A^T P + PA)x + 2x^T P \Phi(x) =$

$x^T(-I)x + 2x^T P \Phi(x) = -x^T x + 2x^T P \Phi(x) = -x \cdot x + 2x^T P \Phi(x) =$

$-\|x(t)\|^2 + 2x^T P \Phi(x) = -\|x(t)\|^2 + 2\|P\| \|x\| \|\Phi(x)\| = \dot{V}$

$\delta > 0 \because |x| < \delta \therefore \frac{|\Phi(x)|}{|x|} < \frac{1}{4\|P\|} \therefore \frac{1}{\delta} < \frac{1}{|x|} \therefore$

$\frac{|\Phi(x)|}{\delta} < \frac{|\Phi(x)|}{x} < \frac{1}{4\|P\|} \therefore |\Phi(x)| \|P\| < \frac{1}{4}|x| \therefore \|P\| |\Phi(x)| < \frac{1}{4}|x| \therefore$

$\|P\| \|\Phi(x)\| < \frac{1}{4}\|x\|^2 \therefore$

$\dot{V} = -\|x(t)\|^2 + 2\|P\| \|\Phi(x)\| \|x\| \leq -\|x(t)\|^2 + 2\frac{1}{4}\|x\| \|x\| = -\|x\|^2 + \frac{1}{2}\|x\|^2 = -\frac{1}{2}\|x\|^2$

$\therefore \dot{V}(x) \leq -\frac{1}{2}|x|^2$

3bii/ $V = x^T P x$, $\dot{V} \leq -\frac{1}{2}\|x\|^2$ if $\|x\| < \delta \therefore$

$\lambda_1 \|x\|^2 \leq V = x^T P x \leq \lambda_n \|x\|^2 = \|P\| \|x\|^2 \therefore |x| < \delta \therefore |x|^2 < \delta^2 \therefore$

$\frac{1}{\lambda_1} V = \frac{1}{\lambda_1} x^T P x \leq \frac{1}{\lambda_1} \|P\| \|x\|^2 \therefore \frac{1}{\lambda_1} V = \frac{1}{\lambda_1} x^T P x < \delta^2$

if $x^T P x < \lambda_1 \delta^2 \therefore |x| < \delta$, $\dot{V} < 0 \therefore$

PP2020 A set ~~of~~ $D \subset \mathbb{R}^n$ is invariant w.r.t $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ if

$x(0) \in D \Rightarrow x(t) \in D \ \forall t \in \mathbb{R}$

$\therefore$ D is w.r.t $\dot{x} = f(x)$, $x(0) \in D$ $\therefore$ $x(0)^T P x(0) < \lambda_1 \delta^2$

$\therefore$ $x^T P x < \lambda_1 \delta^2$ $\therefore$ $x(t)^T P x(t) < \lambda \delta^2$ $\therefore$ $x(t) \in D \ \forall t \in \mathbb{R}$ $\therefore$

$D = \{x \mid x^T P x < \lambda_1 \delta^2\}$ is invariant

It is also open as a set of form $\{x \mid V(x) < \beta\}$

$V(x)$ p.d.f. $\dot{V}(x)$ n.d.f. $V(x) < \beta$ level set

in D: $V > 0$, $\dot{V} < 0$

$\therefore$ following LaSalle argument. D contained in basin of attraction for $x = 0$

3biii
$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_3 \end{bmatrix} + \begin{bmatrix} \sqrt{6}\, x_1 x_2 \\ x_1^2 + x_2^2 \end{bmatrix} \therefore \quad A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \therefore \quad A^T P + P A = -I$$

is Lyapunov equation $\therefore$ $A^T = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$ $\therefore$

let $P = \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix}$ $\therefore$ $A^T P + P A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_3 & P_4 \end{bmatrix} \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} =$

$$\begin{bmatrix} -2P_1 & -2P_2 \\ -3P_3 & -3P_4 \end{bmatrix} + \begin{bmatrix} -2P_1 & -3P_2 \\ -2P_3 & -3P_4 \end{bmatrix} = \begin{bmatrix} -4P_1 & -5P_2 \\ -5P_3 & -6P_4 \end{bmatrix} = -\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \therefore$$

$-4P_1 = -1$ $\therefore$ $P_1 = \frac{1}{4}$, $-5P_3 = 0 = P_3$, $-5P_2 = 0 = P_2$, $-6P_4 = -1$ $\therefore$ $P_4 = \frac{1}{6}$ $\therefore$

$$P = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/6 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3/2 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \therefore \quad \lambda_{max} = \frac{3}{12}, \ \lambda_{min} = \frac{2}{12} \therefore$$

$$\Phi(x) = \begin{bmatrix} \sqrt{6}\, x_1 x_2 \\ x_1^2 + x_2^2 \end{bmatrix} \therefore \quad \dot{x} = Ax + \Phi(x) \therefore$$

$|\Phi|^2 = \left(\sqrt{(\sqrt{6}\, x_1 x_2)^2 + (x_1^2 + x_2^2)^2}\right)^2 = (\sqrt{6}\, x_1 x_2)^2 + (x_1^2 + x_2^2)^2 = 6x_1^2 x_2^2 + (x_1^2 + x_2^2)^2 =$

$6x_1^2 x_2^2 + x_1^4 + x_2^4 + 2x_1^2 x_2^2 = x_1^4 + 8x_1^2 x_2^2 + x_2^4 \le 4x_1^4 + 4 \times 2x_1^2 x_2^2 + 4x_2^4 =$

$4(x_1^4 + 2x_1^2 x_2^2 + x_2^4) = 4(x_1^2 + x_2^2)^2$ $\therefore$

$\therefore$

$|\Phi|^2 \le 4(x_1^2 + x_2^2)^2 = 4((x_1^2 + x_2^2))^2 = (2(x_1^2 + x_2^2))^2 \le \left(2\left(\left(\sqrt{x_1^2 + x_2^2}\right)^2\right)\right)^2 =$

$(2(\|x\|)^2)^2 = (2\|x\|^2)^2$ $\therefore$

$|\Phi| \le 2\|x\|^2$ $\therefore$

$$\frac{|\Phi|}{|x|} \le 2\frac{\|x\|^2}{|x|} \le 2\|x\| \therefore$$

$\|P\| = \max(\lambda_{max}(P), \lambda_{min}(P)) = \lambda_{max}(P) = \frac{3}{12} = \frac{1}{4}$ $\therefore$ $\frac{1}{4\|P\|} = 1$ $\therefore$

$\cancel{\Phi x}$ $\frac{|\Phi(x)|}{|x|} < \frac{1}{4\|P\|} = 1$ if $\frac{|\Phi|}{|x|} \leq 2\|x\| < 1$ $\therefore$ $\|x\| < \frac{1}{2}$ $\therefore$

$|x| < \delta$ $\therefore$ $\|x\| < \delta$ $\therefore$ $\delta = \frac{1}{2}$ $\therefore$

$\lambda_1 = \lambda_{min}(P) = \frac{2}{12} = \frac{1}{6}$ $\therefore$

$D = \left\{ x \mid x^T P x < \lambda_1 \delta^2 \right\} = \left\{ x \mid x^T \left[\frac{1}{12}\right] \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} x < \frac{1}{6} \times \left(\frac{1}{2}\right)^2 \right\} = \left\{ x \mid \frac{1}{12} \begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} < \frac{1}{6} \times \frac{1}{4} \right\}$

$= \left\{ x \mid \frac{1}{12} \begin{bmatrix} 3x_1 & 2x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} < \frac{1}{24} \right\} = \left\{ x \mid \frac{1}{12}(3x_1^2 + 2x_2^2) < \frac{1}{12} \times \frac{1}{2} \right\} = \left\{ x \mid 3x_1^2 + 2x_2^2 < \frac{1}{2} \right\} = D$

\4a/ The Mapping $f: D \to \mathbb{R}^n$, $g: D \to \mathbb{R}^n$ are vector fields on D

$\therefore$ $\dot{y} = \frac{\partial}{\partial t} y = \frac{\partial}{\partial t}[h(x)] = \frac{\partial h(x)}{\partial x} \frac{\partial x}{\partial t} = \frac{\partial h}{\partial x}[\dot{x}] = \frac{\partial h}{\partial x}[f(x) + g(x)u] =$

$\frac{\partial h}{\partial x} f(x) + \frac{\partial h}{\partial x} g(x) u = L_f h(x) + \frac{\partial h}{\partial x} g(x) u = L_f h(x) + L_g u \frac{\partial h}{\partial x} g(x) =$

$L_f h(x) + u L_g h(x)$ are lie derivatives of h along f and g.

if $L_g h(x) = 0$ $\dot{y} = L_f h(x) + u(0) = L_f h(x)$ is independent of u

$\therefore$ $y^{(2)} = \ddot{y} = \cancel{L_f^2 h(x) + L_f} \frac{\partial}{\partial t}(L_f h(x)) = \frac{\partial L_f h(x)}{\partial x} \dot{x} =$

$\frac{\partial L_f h(x)}{\partial x}(f(x) + g(x)u) = \frac{\partial L_f h(x)}{\partial x} f(x) + u \frac{\partial L_f h(x)}{\partial x} g(x) = L_f L_f h(x) + u L_g L_f h(x) =$

$L_f^2 h(x) + u L_g L_f h(x) = \cancel{\therefore \text{ if }} L_f^2 h(x) \cancel{+ u L_f L_g h(x) L_f^2 h(x) + u L_f(0)} = L_f^2 h(x)$

... etc

$\therefore$ The nonlinear System $\begin{matrix} \dot{x} = f(x) + g(x)u \\ y = h(x) \end{matrix}$ is said to have relative degree

ρ, $1 \leq \rho \leq n$ is a region $D_0 \subset D$ if $L_g L_f^{i-1} h(x) = 0$, $i = 1, 2, \ldots, \rho - 1$

$L_g L_f^{\rho-1} h(x) \neq 0$ $\forall x \in D_0$

\4bi/ let $y = x_1$ $\therefore$ $\dot{y} = \frac{dy}{dt} = \frac{d}{dt} x_1 = \dot{x}_1 = x_1 - x_1^3 + x_2$ independent of u

$\therefore$ $\ddot{y} = \frac{d}{dt} \dot{y} = \frac{d}{dt}(x_1 - x_1^3 + x_2) = \dot{x}_1 - 3x_1^2 \dot{x}_1 + \dot{x}_2 =$

$\dot{x}_1 - 3x_1^2(x_1 - x_1^3 + x_2) + u$ not independent of u $\therefore$

u appears explicitly when taking derivatives of y after second derivative in $\mathbb{R}^2$ $\therefore$ relative degree is 2

\4bii/ let $V(x_1) = \frac{1}{2} x_1^2$ $\therefore$ $V(x) \to \infty$ as $x \to \infty$ $\therefore$ V is p.d. and unbounded

$\therefore$ $\dot{V}(x_1) = x_1 \dot{x}_1 = x_1(x_1 - x_1^3 + x_2) = \cancel{x_1 k} x_1^2 - x_1^4 + x_1 x_2 = x_1^2 - x_1^4 + x_1(-(k+1)x_1) =$

$x_1^2 - x_1^4 - (k+1)x_1^2 = x_1^2 - x_1^4 - kx_1^2 - x_1^2 = -x_1^4 - kx_1^2 = \cancel{-(1+k)x_1^4} - x_1^2(x_1^2 + k) < 0$

$\forall x_1 \in \mathbb{R} \setminus \{0\}$ $\therefore$ $\dot{V}(x_1)$ is negative definite $\therefore$

$x_2 = \phi(x_1) = -(k+1)x_1$ Makes $x_1 = 0$ a globally stable equilibrium.

\PP2020/ \4biii/ $\dot{x}_1 = x_1 - x_1^3 + x_2 = x_1 - x_1^3 + z_2 + \phi(x_1) =$

$x_1 - x_1^3 + z_2 - (k+1)x_1 = -kx_1 - x_1^3 + z_2$

$\therefore \dot{z}_2 = \dot{x}_2 - \dot{\phi}(x_1) = u - \frac{d}{dt}(-(k+1)x_1) = u + (k+1)\dot{x}_1 = u + (k+1)(-kx_1 - x_1^3 + z_2) =$

$\therefore V(x_1, z_2) = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2 \therefore$

$\dot{V}(x_1, z_2) = x_1\dot{x}_1 + z_2\dot{z}_2 = x_1(-kx_1 - x_1^3 + z_2) + z_2(u + (k+1)(-kx_1 - x_1^3 + z_2)) =$

$-kx_1^2 - x_1^4 + z_2x_1 + z_2(u + (k+1)(-kx_1 - x_1^3 + z_2)) =$

$-kx_1^2 - x_1^4 + z_2(x_1 + u + (k+1)(-kx_1 - x_1^3 + z_2)) \Rightarrow -kx_1^2 - x_1^4 - cz_2^2 \therefore$

let $(x_1 + u + (k+1)(-kx_1 - x_1^3 + z_2)) = -cz_2 \Rightarrow$

$x_1 + u - k^2x_1 - kx_1^3 + kz_2 - kx_1 - x_1^3 + z_2 = -cz_2 \therefore$

let $u = -cz_2 - x_1 + k^2x_1 + kx_1^3 - kz_2 + kx_1 + x_1^3 - z_2 \therefore$

$u = -cz_2 - x_1 - (k+1)(-kx_1 - x_1^3 + z_2) = -(cz_2 + x_1 + (k+1)(-kx_1 - x_1^3 + z_2))$

$\therefore$ V_c is positive definite and unbounded

$\dot{V}$ is negative definite $\therefore$ origin is globally asymp stable

\4bii/ $\dot{x}_1 = x_1 - x_1^3 + x_2 = x_1 - x_1^3 - (k+1)x_1 = -kx_1 - x_1^3 = -x_1(k + x_1^2)$

let $V(x_1) = \frac{1}{2}x_1^2$ $\therefore$ V is radially unbounded, positive definite

$\therefore \dot{V} = x_1\dot{x}_1 = x_1(x_1 - x_1^3 + x_2) = x_1^2 - x_1^4 + x_1x_2 = x_1^2 - x_1^4 + x_1x_2 =$

$x_1^2 - x_1^4 + x_1(+(-k-1)x_1) = x_1^2 - x_1^4 - kx_1^2 - x_1^2 = -x_1^4 - kx_1^2 \leq 0$

$\therefore$ $\cancel{V(0)}$ $\dot{V}(x_1 = 0) = 0$, $\dot{V} < 0 \ \forall x_1 < 0 \therefore$

$\dot{V}$ is negative definite $\therefore$

$x_2 = \phi(x_1) = -(k+1)x_1$ Makes $x_1 = 0$ a globally stable equilibrium

\4biii/ $\dot{x}_1 = x_1 - x_1^3 + x_2 = x_1 - x_1^3 + z_2 + \phi(x_1) = x_1 - x_1^3 + \dot{z}_2 - (k+1)x_1 =$

$-kx_1 - x_1^3 + z_2$ $\therefore$

$\cancel{z_2 = x_2}$ $z_2 = x_2 + (k+1)x_1$ $\therefore$

$\dot{z}_2 = \dot{x}_2 + (k+1)\dot{x}_1 = \dot{x}_2 + (k+1)(-kx_1 - x_1^3 + z_2) = u + (k+1)(-kx_1 - x_1^3 + z_2) \therefore$

let $V(x_1, z_1) = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2 \therefore$

$\dot{V} = x_1\dot{x}_1 + z_2\dot{z}_2 = x_1(-kx_1 - x_1^3 + z_2) + z_2(u + (k+1)(-kx_1 - x_1^3 + z_2)) =$

$-kx_1^2 - x_1^4 + z_2x_1 + z_2(u + (k+1)(-kx_1 - x_1^3 + z_2)) = -kx_1^2 - x_1^4 + z_2(x_1 + u + (k+1)(-kx_1 - x_1^3 + z_2))$

$= -kx_1^2 - x_1^4 - cz_2^2$ $\therefore$ $-cz_2 = x_1 + u + (k+1)(-kx_1 - x_1^3 + z_2)$ $\therefore$

$u = -cz_2 - x_1 - (k+1)(-kx_1 - x_1^3 + z_2)$ $\therefore$ $\dot{V}_c \leq 0$ $\therefore$ V_c is p.d.f., $\dot{V}_c$ is n.d.f.

$\therefore$ origin is globally asymp stable

\PP2022/ \1ai/ The ~~values of~~ eigenvalues : $\lambda(A^2) \neq \lambda(A^{-1})$

let v be eigen vector of A $\therefore$ $A^2v = A(Av) = \lambda(Av) = \lambda^2 v$

$\therefore$ $V(A) = V(A^2) = V(A^{-1})$ $\therefore$ eigenvectors are the same $\therefore$

True

\1aii/ $V = x_1^2 + 2x_1x_2 + x_2^2 + 6x_1x_2 = (x_1+x_2)^2 + 6x_1x_2$

$\therefore$ ~~$\|x\|^2$~~ $x_1^2 + x_2^2 = \|x\|$ $\therefore$ $V = \|x\| + 6x_1x_2$ ~~$\therefore V$ is~~

$\therefore$ $V > 0$ $\forall (x_1, x_2) \in \mathbb{R}^2 \setminus \{0\}$ $\therefore$

$V(0,0) = (0+0)^2 + 6(0)(0) = 0$ $\therefore$ V is positive definite $\therefore$

True

\1aiii/ $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$\det(A - \lambda I) = \det\left(\begin{bmatrix} 0-\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix}\right) = \det\left(\begin{bmatrix} -\lambda & 1 \\ -1 & -1-\lambda \end{bmatrix}\right) =$

$-\lambda(-1-\lambda) - 1(-1) = \lambda + \lambda^2 + 1 = 0 = \lambda^2 + \lambda + 1$ $\therefore$

$\lambda_\pm = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \dfrac{-1}{2} \pm \dfrac{\sqrt{-3}}{2} = \dfrac{-1}{2} \pm \dfrac{\sqrt{3}}{2} i$ $\therefore$

$\lambda_\pm \in \mathbb{C}$ $\therefore$ $Re(\lambda_\pm) < 0$ $\therefore$

at $(0,0) = (x_1, x_2)$: $\dot{x}_1 = -(0) = 0$, $\dot{x}_2 = -(0) - 0 = 0$ $\therefore$

$(0,0)$ is an equilibrium and is stable and is a spiral $\therefore$

True

\1aiv/ $\therefore f(y) = y^{1/3}$ $\therefore$ $|f(x) - g(y)| = |x^{1/3} - y^{1/3}| \leq |x^{1/3} + (-y^{1/3})| \leq |x^{1/3}| + |y^{1/3}|$

$= |x|^{1/3} + |y^{1/3}| = |x|^{1/3} + |y|^{1/3}$ ~~$\leq |x-y|$~~ $\therefore$

$|f(x) - f(y)| = |x^{1/3} - y^{1/3}| \leq |x-y|^{1/3} \leq |x-y|$ $\therefore$

$f(x)$ is globally Lipschitz $\therefore$

True

\1av/ $y = x_2$ $\therefore$ $\dot{y} = \dot{x}_2 = -2x_1 - x_2$ is independent of u $\therefore$

$\ddot{y} = -2\dot{x}_1 - \dot{x}_2 = -2(x_2 + u) + 2x_1 + x_2 = -2x_2 - 2u + 2x_1 + x_2 = -x_2 - 2u + 2x_1$ $\therefore$

is not independent of u and has u explicitly $\therefore$

relative degree is 2 $\therefore$

False

\1bi/ $\therefore \ddot{x} = -(2b - g(x))a\dot{x} - a^2x = (g(x)-2b)a\dot{x} - a^2x \therefore$

$\dot{x}_1 = \dot{x} = x_2$, $\dot{x}_2 = \ddot{x} = (g(x)-2b)a\dot{x} - a^2x = (g(x_1)-2b)ax_2 - a^2x_1$

$\therefore \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = (g(x_1)-2b)ax_2 - a^2x_1 \end{cases}$ is new state space representation

\1bii/ $\therefore$ equilibriums : $\dot{x}_1 = 0 = x_2 \therefore$

$\dot{x}_2 = 0 = (g(x_1)-2b)a(0) - a^2x_1 = 0 = -a^2x_1 = 0 = x_1 \therefore$

$(0,0)$ is equilibria $\therefore$

linearisation : $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & 1\\ -a^2 & (g(x_1)-2b)a\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} = A\begin{bmatrix}x_1\\ x_2\end{bmatrix} \therefore$

no periodic ~~sta~~ orbit is A ~~is~~ for A at $(0,0)$:

$A|_{(0,0)} = \begin{bmatrix}0 & 1\\ -a^2 & (g(x_1)-2b)a\end{bmatrix}\Big|_{(0,0)} = \begin{bmatrix}0 & 1\\ -a^2 & (g(0)-2b)a\end{bmatrix} = \begin{bmatrix}0 & 1\\ -a^2 & ak-2ba\end{bmatrix} \therefore$

~~$A = \begin{bmatrix}0 & 1\\ -a^2 & -2ab\end{bmatrix}$~~

$\det(A-\lambda I) = \det\begin{bmatrix}-\lambda & 1\\ -a^2 & ak-2ab-\lambda\end{bmatrix} = \lambda^2 + (-ak+2ab)\lambda + a^2 = 0 \therefore$

$\lambda = \dfrac{+ak-2ab \pm \sqrt{(ak-2ab)^2 - 4a^2}}{2(1)}$

\c/ let $V = x_1^2 + x_2^2 \therefore$ V is positive definite and unbounded

$\therefore V(0) = 0$, $V > 0 \ \forall x \in \mathbb{R}^2 \setminus \{0\}$, $V \to \infty$ as $\|x\| \to \infty$

$\therefore \dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1(-x_1^3 + x_2^4) + 2x_2(-x_2^3 + x_2^4) =$

$-2x_1^4 + 2x_1x_2^4 - 2x_2^4 + 2x_2^5 = -2(x_1^4 + x_2^4) + 2x_2^4(x_1 + x_2)$

$\therefore$ let $x_1 = r\cos\theta$, $x_2 = r\sin\theta \therefore r^2 = x_1^2 + x_2^2 \therefore$

$V = x_1^2 + x_2^2 = r^2 \therefore \dot{V} = 2r\dot{r} \therefore$

\d/ $\|x\|_2 = \sqrt{x_1^2 + x_2^2} \therefore \|x\|_2^2 = (x_1^2 + x_2^2) \therefore$

$\frac{d}{dt}\left(\frac{1}{2}\|x\|_2^2\right) = \frac{1}{2}\frac{d}{dt}(x_1^2 + x_2^2) = \frac{1}{2}(2x_1\dot{x}_1 + 2x_2\dot{x}_2) = x_1\dot{x}_1 + x_2\dot{x}_2 =$

$x_1(-3x_1 + x_1x_2^2) + x_2(-2x_2 - x_1^2x_2 + x_2^2 + u) =$

$-3x_1^2 + x_1^2x_2^2 - 2x_2^2 - x_1^2x_2^2 + x_2^3 + x_2u = -3x_1^2 - 2x_2^2 + x_2^3 + x_2u = -x_1^2 - x_2^2 \therefore$

$x_2u = -x_1^2 - x_2^2 + 3x_1^2 + 2x_2^2 - x_2^3 = 2x_1^2 + x_2^2 - x_2^3 \therefore$

$u = \frac{1}{x_2}(2x_1^2 + x_2^2 - x_2^3) \therefore$

$\frac{1}{2}\frac{d}{dt}(\|x\|_2^2) = -x_1^2 - x_2^2 = -(x_1^2 + x_2^2) = -\|x\|_2^2 \therefore$

$\dfrac{\frac{d}{dt}\|x\|_2^2}{\|x\|_2^2} = -2 \therefore$

\PP2022\ \1d/ $||x_2|| = \sqrt{x_1^2 + x_2^2} \therefore ||x||_2^2 = x_1^2 + x_2^2 \therefore$

$$\frac{d}{dt}\left(\frac{1}{2\sqrt{6}}||x||_2^2\right) = \frac{1}{2\sqrt{6}}\frac{d}{dt}\left(||x||_2^2\right) = \frac{1}{2\sqrt{6}}\frac{d}{dt}\left(x_1^2 + x_2^2\right) = \frac{1}{2\sqrt{6}}(2x_1\dot{x}_1 + 2x_2\dot{x}_2) =$$

$$\frac{1}{\cancel{2}\sqrt{6}}(x_1\dot{x}_1 + x_2\dot{x}_2) = \frac{1}{\sqrt{6}}\left[x_1(-3x_1 + x_1x_2^2) + x_2(-2x_2 - x_1^2x_2 + x_2^2 + u)\right] =$$

$$\frac{1}{\sqrt{6}}\left[-3x_1^2 + x_1^2x_2^2 - 2x_2^2 - x_1^2x_2^2 + x_2^3 + x_2u\right] = -x_1^2 - x_2^2 \doteq -(x_1^2 + x_2^2) = -||x||_2^2 \therefore$$

$$-3x_1^2 + x_1^2x_2^2 - 2x_2^2 - x_1^2x_2^2 + x_2^3 + x_2u = -\sqrt{6}x_1^2 - \sqrt{6}x_2^2 \doteq -3x_1^2 - 2x_2^2 + x_2^3 + x_2u \therefore$$

$$x_2u = -\sqrt{6}x_1^2 - \sqrt{6}x_2^2 + 3x_1^2 \cancel{-x_1^2x_2^2} + 2x_2^2 - x_2^3 = (3-\sqrt{6})x_1^2 + (2-\sqrt{6})x_2^2 - x_2^3 \therefore$$

$$u = x_2^{-1}\left((3-\sqrt{6})x_1^2 + (2-\sqrt{6})x_2^2 - x_2^3\right) \therefore$$

$$\frac{1}{2\sqrt{6}}\frac{d}{dt}\left(||x||_2^2\right) = -||x||_2^2 \therefore \quad \frac{1}{2\sqrt{6}}\frac{d}{dt}\left(||x||^2\right) - ||x||^2 \therefore$$

$$\int \frac{\frac{d}{dt}(||x||^2)}{(||x||^2)} dt = \int -2\sqrt{6}\,dt = \cancel{||x(t)||} \ \ln| ||x||^2 | = -2\sqrt{6}t + C_1$$

$$| ||x||^2 | = e^{-2\sqrt{6}t + C_1} = A_1 e^{-2(\sqrt{6}t)} \therefore$$

$$||x||^2 = ||x(t)||^2 = \pm A_1 e^{-2(\sqrt{6}t)} = e^{-2(\sqrt{6}t)}A_2 \doteq \left(e^{-\sqrt{6}t}\sqrt{A}\right)^2 \therefore$$

$$||x(0)||^2 = e^{-2(\sqrt{6}(0))}A_2 = e^0 A_2 = A_2 = ||x(0)||^2 \therefore$$

$$||x||^2 = e^{-2(\sqrt{6}t)}||x(0)||^2 = \left(e^{-\sqrt{6}t}||x(0)||\right)^2 \therefore$$

$$||x(t)|| = e^{-\sqrt{6}t}||x(0)||$$

\2a/ equilibrium $\therefore$ at $\dot{x}_1 = \dot{x}_2 = 0 \therefore$

$\dot{x}_1 = x_2 = 0 \therefore \dot{x}_2 = 0 = -g(x_1)(x_1 + 0) = -g(x_1)(x_1) \therefore$

g $x_1 \in \mathbb{R} \therefore |g(x_1) - g(y)| \leq L|x_1 - y| \therefore$

$0 = x_1 g(x_1) \therefore x_1 \in \mathbb{R} \therefore g(x_1) \geq 1 \therefore g(x_1) \neq 0 \therefore$

$x_1 = 0 \therefore$

$(0,0) = (x_1, x_2)$ is an equilibria

\2b/ $\cancel{\lim_{x\to\infty}}$ as $||x|| \to \infty \therefore$

$x_2^2 \geq 0$ and is unbounded,

$g(y) \geq 1$ and unbounded $\therefore y \in [0, x_1] \therefore \quad y \geq 0 \therefore$

$yg(y) \geq 0 \therefore \int_0^{x_1} yg(y) \geq 0$ and unbounded $\therefore$

at x_1: $yg(y) = x_1 g(x_1) \geq x_1$ $\therefore$

$x_2^2 + x_1 g(x_1) \geq x_1 x_2 \therefore \quad V \geq 0 \ \forall x \in \mathbb{R}^2 \setminus \{0\} \therefore$

$V(0) = \int_0^0 yg(y)\,dy + 0(0) + 0^2 = 0 + 0 = 0$ and $V \to \infty$ as $||x|| \to \infty \therefore$

V is radially unbounded and positive definite

$\dot{V} = \frac{d}{dt}\int_0^{x_1} y g(y)\,dy + x_1 x_2 + x_2^2 = 2x_2\dot{x}_2 + x_2\dot{x}_1 + x_1\dot{x}_2 + x_1 g(x_1)\dot{x}_1 =$

$2x_2(-g(x_1)(x_1+x_2)) + x_2 x_2 + x_1(-g(x_1)(x_1+x_2)) + x_1 g(x_1) x_2 =$

$(2x_1x_2 + 2x_2^2)(-g(x_1)) + x_2^2 + (x_1^2 + x_1x_2)(-g(x_1)) + x_1x_2 g(x_1) =$

$-2x_1x_2 g(x_1) - 2x_2^2 g(x_1) + x_2^2 - x_1^2 g(x_1) - x_1x_2 g(x_1) + x_1x_2 g(x_1) =$

$-2x_1x_2 g(x_1) - 2x_2^2 g(x_1) - x_1^2 g(x_1) + x_2^2 =$

$-x_1^2 g(x_1) - 2x_1x_2 g(x_1) - x_2^2 g(x_1) + x_2^2 - x_2^2 g(x_1) =$

$-g(x_1)[x_1^2 + 2x_1x_2 + x_2^2] + x_2^2(1 - g(x_1)) =$

$-g(x_1)(x_1+x_2)^2 + x_2^2(1-g(x_1)) \therefore$

$~~g(x)~~ \; x_1 \in \mathbb{R} \therefore g(x_1) \geq 1 \therefore$

$-g(x_1)(x_1+x_2)^2 \leq 0$, $1 - g(x_1) \leq 0 \therefore x_2^2(1-g(x_1)) \leq 0 \therefore$

$\dot{V} \leq 0 \therefore V(0) = -g(0)(0+0)^2 + 0^2(1-g(0)) = 0 \therefore$

$\dot{V}$ is negative definite $\therefore$

The origin is globally asymptotically stable.

\4ai/ equilibria; $\dot{x}_1 = 0 = \dot{x}_2 \therefore \dot{x}_1 = 0 = x_1 - x_1^3 + x_2 \therefore$

$\dot{x}_2 = 3x_1 - x_2 = 0 \therefore x_2 = 3x_1 \therefore 0 = x_1 - x_1^3 + 3x_1 = 4x_1 - x_1^3 = x_1(4 - x_1^2) = 0$

$\therefore x_1 = 0$, $4 - x_1^2 = 0 \therefore x_1^2 = 4 \therefore x_1 = 2, x_1 = -2 \therefore$

$x_2 = 3(0) = 0$, $x_2 = 3(2) = 6$, $x_2 = 3(-2) = -6 \therefore$

$(x_1, x_2) = (0,0), (2,6), (-2,-6)$ are the equilibria

\4aii/ $\therefore \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}\partial f_1/\partial x_1 & \partial f_1/\partial x_2\\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2\end{bmatrix} = \begin{bmatrix}1-3x_1^2 & 1\\ 3 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

$\therefore A|_{(0,0)} = \begin{bmatrix}1-3x_1^2 & 1\\ 3 & -1\end{bmatrix}\Big|_{(0,0)} = \begin{bmatrix}1 & 1\\ 3 & -1\end{bmatrix}$

$A|_{(2,6)} = \begin{bmatrix}1-3(2)^2 & 1\\ 3 & -1\end{bmatrix} = \begin{bmatrix}-11 & 1\\ 3 & -1\end{bmatrix}$ $\quad A|_{(-2,6)} = \begin{bmatrix}1-3(-2)^2 & 1\\ 3 & -1\end{bmatrix} = \begin{bmatrix}-11 & 1\\ 3 & -1\end{bmatrix} \therefore$

$\therefore \det(A|_{(0,0)}) = 1(-1) - 3(1) = -1 - 3 = -4 < 0 \therefore (0,0)$ is unstable $\therefore$

$\det(A|_{(0,0)} - \lambda I) = \begin{bmatrix}1-\lambda & 1\\ 3 & -1-\lambda\end{bmatrix} = -(1-\lambda)(1+\lambda) - 3(1) = -(-\lambda^2+1) - 3 = \lambda^2 - 1 - 3 = \lambda^2 - 4 = 0 \therefore$

$\lambda^2 = 4 \therefore \lambda_1 = +2, \lambda_2 = -2 \therefore \lambda \in \mathbb{R}$, $\lambda_1 < 0$, $\lambda_2 > 0 \therefore$

$(0,0)$ is a saddle point equilibrium

$\det(A|_{(2,6)} - \lambda I) = \begin{bmatrix}-11-\lambda & 1\\ 3 & -1-\lambda\end{bmatrix} = (-11-\lambda)(-1-\lambda) - 3(1) = \lambda^2 + 11 + 12\lambda - 3 = \lambda^2 + 12\lambda + 8 =$

$= 0$ $~~=(\lambda+6)(\lambda+2)~~$ $\therefore \lambda = \frac{-12 \pm \sqrt{12^2 - 4(1)(8)}}{2(1)} = \frac{-12}{2} \pm \frac{4\sqrt{7}}{2} \therefore \lambda_1 = -11.3, \lambda_2 = -0.708$

$\|x\|_2$

\PP 2022/ $\therefore \lambda_{1,2} \in \mathbb{R}$, $\lambda_{1,2} < 0$ $\therefore$

$A|_{(2,6)} = A|_{(-2,-6)}$ $\therefore$

equilibriums $(2,6)$ and $(-2,6)$ are both stable nodes

\4bi/ controllability Matrix $M = \cancel{[B \vdots BA]} \therefore [B \vdots AB]$ $\therefore$

$B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\therefore$ $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ $\therefore$ $g(x) = \begin{bmatrix} -x_1^3 \\ 0 \end{bmatrix}$

$\cancel{AB = \begin{bmatrix} 1 & \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}}$ $\cancel{BA = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & -1 \end{bmatrix} = [\;]}$ $AB = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ $\therefore$

$M = \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \vdots \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ $\therefore$

$\det(M) = 0(-1) - 1(1) = -1 \neq 0$ $\therefore$ ~~M is full~~ $\text{rank}(M) = 2$ $\therefore$

M is full rank $\therefore$ (A, B) is controllable

\4bii/ $A_{cl} = A - BK$ $\therefore$ $K = [k_1 \quad k_2]$ $\therefore$ $BK = \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$ $\therefore$

$A_{cl} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -k_1+3 & -k_2-1 \end{bmatrix}$ $\therefore$

$\det(SI - A_{cl}) = \det\left(\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} 1 & 1 \\ -k_1+3 & -k_2-1 \end{bmatrix} \right) = \det\left(\begin{bmatrix} S-1 & -1 \\ k_1-3 & S+k_2+1 \end{bmatrix} \right) =$

charac eqn

$(S-1)(S+k_2+1) + 1(k_1-3) = S^2 + k_2 S + S - S - k_2 - 1 + k_1 - 3 = S^2 + (k_2)S + (+k_1 - k_2 - 4) =$

$(S+3)(S+5) = S^2 + 3S + 5S + 15 = S^2 + 8S + 15$ $\therefore$

is desired poly of charac eqn

$8 = k_2$ $\therefore$ $15 = k_1 - k_2 - 4 = k_1 - 8 - 4$ $\therefore$ $k_1 = 27$ $\therefore$

$u = -[27 \quad 8]x = [-27 \quad -8] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -27x_1 - 8x_2$

\4biii/ $P = P^T$ $\therefore$ P is symmetric $\lambda_i(P) > 0 \; \forall i$ $\therefore$ P is positive definite

A_{cl} is Hurwitz $\therefore$ $\text{Re}(\lambda_i(A)) < 0$, $i = 1,2$ $\therefore$

The linear part alone $\dot{x} = Ax$ is asymptotically stable origin

and $P = P^T$, $\lambda(P) > 0$ $\therefore$

The Lyapunov function for LTI System $\dot{x} = Ax$ is $V(x) = x^T P x$,

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ $g(x) = \begin{bmatrix} -x_1^3 \\ 0 \end{bmatrix}$

$\therefore \dot{V} = -\|x\|_2^2 + 2x^T P g(x) = -\|x\|_2^2 + 2[x_1 \quad x_2] \begin{bmatrix} 97/240 & 217/240 \\ 217/240 & 37/15 \end{bmatrix} \begin{bmatrix} -x_1^3 \\ 0 \end{bmatrix} =$

$-\|x\|_2^2 + 2[x_1 \quad x_2] \begin{bmatrix} \frac{-97}{240} x_1^3 \\ \frac{-217}{240} x_1^3 \end{bmatrix} = -\|x\|_2^2 + [x_1 \quad x_2] \begin{bmatrix} \frac{-97}{120} x_1^3 \\ \frac{-217}{120} x_1^3 \end{bmatrix} =$

$-\|x\|_2^2 - \frac{97}{120} x_1^4 - \frac{217}{120} x_1^3 x_2 \leq 0$ for $x_1 x_2 \leq 0$ $\therefore$

$\|g(x)\|_2 = \sqrt{(-x_1^3)^2 + 0^2} = \sqrt{x_1^6} = x_1^3$

$\|x\|_2 = \sqrt{x_1^2 + x_2^2}$ ∴

$x_1^3 \leq \beta^2 \sqrt{x_1^2 + x_2^2}$ ∴

$\frac{1}{\beta^2} x_1^3 \leq \sqrt{x_1^2 + x_2^2}$ ∴ $\frac{1}{\beta^4} x_1^6 \leq x_1^2 + x_2^2$. $x_1 x_2 \geq 0$ ∴

~~if~~ $\frac{x_1^3}{\sqrt{x_1^2+x_2^2}} \leq \beta^2$

if $x_1 < 0$: $x_2 < 0$ ∴ $x_1^3 < 0$ ∴ $\sqrt{x_1^2 + x_2^2} > 0$ ∴

$\beta^2 > 0$ ∴ $\frac{x_1^3}{\sqrt{x_1^2+x_2^2}} \leq \beta^2$ true $\forall x_1 < 0$

if $x_1 > 0$: $x_2 > 0$ ∴ $x_1^3 > 0$ ∴ $\frac{x_1^3}{\sqrt{x_1^2+x_2^2}} \leq \beta^2$

∴ $\dot{V} \leq 0$ ∴ $\dot{V}$ is ~~positive~~ negative definite ∴

origin is asymptotically stable

\3ai/ $\dot{V} < 0$ $\frac{\partial V(x)}{\partial t} = \frac{\partial V}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial V}{\partial x}(\dot{x}) = \frac{\partial V}{\partial x}(f(x) + Bk(x))$

∴ $\dot{V} = -W(x)$ ∴ $\dot{V}$ is negative definite

V is positive definite ∴ origin is asympt stable

\3bi/ $V(x)$ is positive definite ∴

$\frac{1}{2}\tilde{\theta}^T \Gamma^{-1} \tilde{\theta} = \frac{1}{2}(\theta - \hat{\theta})^T \Gamma^{-1}(\theta - \hat{\theta})$ ∴

$V_a(x, \tilde{\theta}) = V(x) + \frac{1}{2}(\theta - \hat{\theta})^T \Gamma^{-1}(\theta - \hat{\theta})$ ∴

$\dot{V}_a(x, \tilde{\theta}) = \dot{V} + \frac{1}{2}\left(\frac{\partial}{\partial t}(\theta - \hat{\theta})\right)^T \Gamma^{-1}(\theta - \hat{\theta}) + \frac{1}{2}(\theta - \hat{\theta})^T \Gamma^{-1}\left(\frac{\partial}{\partial t}(\theta - \hat{\theta})\right) =$

$-W(x) + \frac{1}{2}\left(\frac{\partial}{\partial t}(\theta - \hat{\theta})\right)^T \Gamma^{-1}(\theta - \hat{\theta}) + \frac{1}{2}(\theta - \hat{\theta})^T \Gamma^{-1}\left(\frac{\partial}{\partial t}(\theta - \hat{\theta})\right) =$

$-W(x) + (\theta - \hat{\theta})^T \Gamma^{-1} \frac{\partial}{\partial t}(\theta - \hat{\theta})$

$u_a = k(x) - \hat{\theta} = k(x) - \int_0^t \Gamma\left(\frac{\partial V(x)}{\partial x} B\right)^T ds$ ∴

$G(s) = \Gamma\left(\frac{\partial V(s)}{\partial s} B\right) = \Gamma\left(\frac{\partial V(x)}{\partial x} B\right)\Big|_{x=s}$

\1bi/ $\dot{x}_1 = x_2$
$\dot{x}_2 = (g(x_1) - 2b)a x_2 - a^2 x_1$ let $V(x) = x_1^2 + x_2^2$ ∴

$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(x_2) + x_2((g(x_1) - 2b)a x_2 - a^2 x_1) = x_1 x_2 + (g(x_1) - 2b)a x_2^2 - a^2 x_1 x_2 =$

$(1 - a^2)x_1 x_2 + a(g(x_1) - 2b)x_2^2 \leq 0$ ~~for~~ ∴ $a^2 = 1$, $g(x_1) - 2b = 0$ ∴ $b = \begin{cases} 0 & \text{if } |x| > 1 \\ \frac{1}{2}k, & \text{otherwise} \end{cases}$

∴ $f(x) \cdot \nabla(x) = 0$

$\dot{V} = \dot{x}_1 x_2 + a x_2^2 g(x_1) - 2ab x_2^2 - a^2 x_1 x_2$

$=$ ~~let~~ $f(x) \cdot \nabla V = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \cdot \begin{bmatrix} \partial V/\partial x_1 \\ \partial V/\partial x_2 \end{bmatrix} = \dot{x}_1 2x_1 + 2x_2\dot{x}_2 = 2x_1 x_2 + 2x_2((g(x_1) - 2b)a x_2 - a^2 x_1) =$

$2x_1 x_2 + 2x_2(a x_2 g(x_1) - 2ab x_2 - a^2 x_1) = 2x_1 x_2 + 2a x_2^2 g(x_1) - 4ab x_2^2 - 2a^2 x_1 x_2$ ∴

let $a^2 = 1$, $g(x_1) - 2b = 0$ ∴ $b = \frac{1}{2}g(x_1)$ ∴ ~~$b = \frac{1}{2}$~~

\PP2022/ \1bii/ $\dot{x}_1 = x_2$
$\dot{x}_2 = (g(x_1) - 2b)a x_2 - a^2 x_1$ $\therefore$

let $V = x_1^2 + x_2^2$ $\therefore$

$f(x)\cdot\nabla V = \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}\begin{bmatrix}2x_1 & 2x_2\end{bmatrix} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 =$

$2x_1x_2 + 2x_2((g(x_1) - 2b)a x_2 - a^2 x_1) =$

$2x_1x_2 + 2a x_2^2 g(x_1) - 4ab x_2^2 - 2a^2 x_1 x_2 =$

$2(1-a^2)x_1x_2 + 2a x_2^2 (g(x_1) - 2b)$ $\therefore$

let $a^2 < 1$, $a > 0$ $\therefore$ $0 < a < 1$ $\therefore$

if $g(x) - 2b \geq 0$ $\therefore$ $b < \frac{1}{2}g(x)$ $\therefore$ $f(x)\cdot\nabla V > 0$

let $b < \frac{1}{2}\begin{cases}0, & \text{if } |x|>1\\ k, & \text{otherwise}\end{cases}$ $\therefore$ by P.B. criterion, $\nexists$ no periodic ~~about the~~ orbit in the system

\c/ Jacobian: $\frac{\partial f}{\partial x} = \begin{bmatrix}\partial f_1/\partial x_1 & \partial f_1/\partial x_2\\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2\end{bmatrix} = \begin{bmatrix}-3x_1^2 & 4x_2^3\\ 0 & -3x_2^2 + 4x_2^3\end{bmatrix}$

$\therefore \frac{\partial f}{\partial x}\Big|_{(0,0)} = \begin{bmatrix}-3(0)^2 & 4(0)^3\\ 0 & -3(0)^2 + 4(0)\end{bmatrix} = \begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}$

let $x_1 = r\cos\theta$, $x_2 = r\sin\theta$ $\therefore$ $x_1^2 + x_2^2 = r^2$ $\therefore$

~~$\frac{\partial f}{\partial x} = \begin{bmatrix}-3r^2\cos^2\theta\end{bmatrix}$~~ let $V = x_1^2 + x_2^2$ $\therefore$ V is radially unbounded, p.d. $\therefore$

$\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1(-x_1^3 + x_2^4) + 2x_2(-x_2^3 + x_2^4) =$

$-2x_1^4 + 2x_1x_2^4 - 2x_2^4 + 2x_2^5 \leq 0 \ \forall x \notin (0)$

$\dot{V}(0) = 0$ $\therefore$ $\dot{V}$ is negative definite $\therefore$

$(0,0)$ is globally asymp stable

\1d/ let $\|x\|_2 = \sqrt{x_1^2 + x_2^2}$ $\therefore$ $\|x\|_2^2 = x_1^2 + x_2^2$ $\therefore$

$\frac{d}{dt}\left(\frac{1}{2\sqrt{6}}\|x\|_2^2\right) = \frac{1}{2\sqrt{6}}\frac{d}{dt}\left(\|x\|_2^2\right) = \frac{1}{2\sqrt{6}}\frac{d}{dt}(x_1^2 + x_2^2) = \frac{1}{2\sqrt{6}}2x_1\dot{x}_1 + \frac{1}{2\sqrt{6}}2x_2\dot{x}_2 =$

$\frac{1}{\sqrt{6}}x_1(-3x_1 + x_1x_2^2) + \frac{1}{\sqrt{6}}x_2(-2x_2 - x_1^2x_2 + x_2^2 + u) =$

$\frac{1}{\sqrt{6}}(-3x_1^2 + x_1^2x_2^2 - 2x_2^2 - x_1^2x_2^2 + x_2^3 + ux_2) = -\|x\|_2^2 = -(x_1^2 + x_2^2)$ $\therefore$

$-3x_1^2 + x_1^2x_2^2 - 2x_2^2 - x_1^2x_2^2 + x_2^3 + ux_2 = -\sqrt{6}x_1^2 - \sqrt{6}x_2^2$ $\therefore$

let $u = \sqrt{6}x_1^2 + \sqrt{6}x_2^2 + 3x_1^2 - x_1^2x_2^2 + 2x_2^2 + x_1^2x_2^2 - x_2^3 = \left[(\sqrt{6}+3)x_1^2 + (\sqrt{6}+2)x_2^2 - x_2^3\right]\frac{1}{x_2}$ $\therefore$

$\frac{\frac{d}{dt}\|x\|_2^2}{\|x\|_2^2} = -2\sqrt{6}$ $\therefore$ $\frac{\frac{d}{dt}\|x\|^2}{\|x\|^2} = -2\sqrt{6}$ $\therefore$ $\int \frac{\frac{d}{dt}\|x\|^2}{\|x\|^2}dt = \int -2\sqrt{6}\,dt = -2\sqrt{6}t + C_1 =$

$\ln|\|x\|^2| = \ln\|x\|^2$ $\therefore$ $\|x\|^2 = e^{-2\sqrt{6}t + C_1} = A_1 e^{-2\sqrt{6}t} = (\pm e^{-\sqrt{6}t}A_2)^2$ $\therefore$

$\|x(t)\| = \pm e^{-\sqrt{6}t}A_2$ $\therefore$ $\|x(0)\| = \pm e^{-\sqrt{6}(0)}A_2 = \pm e^{0}A_2 = \pm A_2$ $\therefore$ $\|x(t)\| = e^{-\sqrt{6}t}\|x(0)\|$

\4ai/ equilibria is $(\dot{x}_1, \dot{x}_2) = (0,0)$ ∴ $u = 0$ ∴

$\dot{x}_2 = 0 = 3x_1 - x_2$ ∴ $x_2 = 3x_1$ ∴ $\dot{x}_1 = 0 = x_1 - x_1^3 + 3x_1 = 4x_1 - x_1^3 =$

$x_1(4 - x_1^2) = 0$ ∴ $x_1 = 0$, $4 - x_1^2 = 0$ ∴ $x_1^2 = 4$ ∴

$x_1 = 0$, $x_1 = 2$, $x_1 = -2$ ∴

$x_2 = 3(0) = 0$, $x_2 = 3(2) = 6$, $x_2 = 3(-2) = -6$ ∴

equilibria are: $(0,0)$, $(2,6)$, $(-2,-6)$

\4aii/ $\frac{\partial f}{\partial x} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} 1-3x_1^2 & 1 \\ 3 & -1 \end{bmatrix}$ ∴

$\frac{\partial f}{\partial x}\Big|_{(0,0)} = \begin{bmatrix} 1-3(0)^2 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$

$\frac{\partial f}{\partial x}\Big|_{(2,6)} = \begin{bmatrix} 1-3(2)^2 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -11 & 1 \\ 3 & -1 \end{bmatrix}$ $\quad \frac{\partial f}{\partial x}\Big|_{(-2,-6)} = \begin{bmatrix} 1-3(-2)^2 & 1 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} -11 & 1 \\ 3 & -1 \end{bmatrix}$

∴ $\frac{\partial f}{\partial x}(2,6) = \frac{\partial f}{\partial x}\Big|_{(-2,-6)}$

∴ at $(0,0)$: $\det\left(\frac{\partial f}{\partial x}\Big|_{(0,0)} - \lambda I\right) = \begin{bmatrix} 1-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix} = -(1+\lambda)(1-\lambda) - 3(1) =$

$-(-\lambda^2+1) - 3 = \lambda^2 - 1 - 3 = \lambda^2 - 4 = 0$ ∴ $\lambda^2 = 4$ ∴

$\lambda_1 = 2$, $\lambda_2 = -2$ ∴ $\lambda_i \in \mathbb{R}$ ∴ $(0,0)$ is a saddle

$\det\left(\frac{\partial f}{\partial x}\Big|_{(2,6)} - \lambda I\right) = \begin{bmatrix} -11-\lambda & 1 \\ 3 & -1-\lambda \end{bmatrix} = (+11+\lambda)(1+\lambda) - 3 = \lambda^2 + 12\lambda + 11 - 3 = \lambda^2 + 12\lambda + 8 = 0$

∴ $\lambda_{1,2} = \frac{-12 \pm \sqrt{12^2 - 4(1)(8)}}{2(1)} = -6 \pm \frac{4\sqrt{7}}{2} = -6 \pm 2\sqrt{7}$ ∴

$\lambda_1 = -11.3 < 0$, $\lambda_2 = -0.708 < 0$ ∴ $\lambda_{1,2} \in \mathbb{R}_{<0}$ ∴

$(2,6)$ and $(-2,-6)$ are stable nodes.

\4bi/ controllability matrix: $M = [B \vdots AB]$ ∴ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}$ ∴

$AB = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ ∴ $M = \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \vdots \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$ ∴ $\det(M) = -1(0) - 1 = -1 \neq 0$

∴ $\operatorname{rank}(M) = 2$ ∴ full rank ∴ (A,B) is controllable

\4bii/ $K = [k_1 \ k_2]$ ∴ $A_{CL} = A - BK = \begin{bmatrix} 1 & 1 \\ 3 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ -k_1+3 & -k_2-1 \end{bmatrix}$ ∴

$(s+3)(s+5) = s^2 + 8s + 15$ ∴ $\det(sI - A_{CL}) = \det\left(\begin{bmatrix} s-1 & -1 \\ +k_1-3 & s+k_2+1 \end{bmatrix}\right) =$

$(s-1)(s+k_2+1) + (k_1-3) = s^2 + k_2 s - 1 + k_1 - 3 = s^2 + k_2 s + (k_1 - 4) = s^2 + 8s + 15$ ∴

is charac polyn

$8 = k_2$ ∴ $k_1 - 4 = 15 = k_1 - 8 - 4$ ∴ $k_1 = 27$ ∴ $K = [27 \ , 8]$ ∴

$u = -[27 \ \ 8]x = [-27 \ -8]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -27x_1 - 8x_2$

PP2021 / 1ai / $V = x_1^2 + 2x_1x_2 + x_2^2 + x_2^2 - 4x_1x_2 = (x_1+x_2)^2 + x_2^2 - 4x_1x_2 =$
$(x_1+x_2)^2 + x_2(x_2 - 4x_1)$

- $V = x_1^2 - 2x_1x_2 + x_2^2 + x_2^2 = (x_1 - x_2)^2 + x_2^2$

$\therefore (x_1 - x_2)^2 \geq 0$, $x_2^2 \geq 0 \therefore$

$V(0) = (0-0)^2 + 0^2 = 0$

$\therefore V > 0 \ \forall x \in \mathbb{R}^2 \setminus \{0\} \therefore$ V is p.d.f. $\therefore$ True

1aii / $\alpha > 0$ $\therefore$, $\dot{x} = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \therefore$

$A = \begin{bmatrix} \alpha & 1 \\ 0 & \alpha \end{bmatrix} \therefore$ $\det(A - \lambda I) = \begin{vmatrix} \alpha-\lambda & 1 \\ 0 & \alpha-\lambda \end{vmatrix} = (\alpha-\lambda)^2 - 0(1) = (\alpha-\lambda)^2 = 0 \therefore$

$\lambda = \alpha > 0 \therefore$ $\lambda_i > 0 \ \forall i$ $\therefore$ origin is an unstable node $\therefore$
origin is not an ~~the~~ stable node
and at $(x_1 = 0, x_2 = 0)$: $\dot{x}_1 = 0$, $\dot{x}_2 = 0 \therefore$ origin is an equilibria
$\therefore$ False

1aiii / $-x$ is globally lipshitz
$||x| - |y|| \leq |x - y|$ $\therefore 2|x|$ is globally lipshitz
$\therefore f(x)$ is globally lipshitz
$|x|$ is not continuously differentiable $\therefore$
$f(x)$ is not continuously differentiable $\therefore$
False

1aiv / $f(x) = \begin{pmatrix} x_2 \\ 0 \end{pmatrix}$ $g(x) = \begin{pmatrix} 0 \\ x_1 \end{pmatrix}$
$\Rightarrow d_f + d_g = \cancel{\frac{\partial}{\partial}}\ [f(x) + g(x)] \frac{\partial}{\partial x} = \left[\begin{pmatrix} x_2 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ x_1 \end{pmatrix}\right] \frac{\partial}{\partial x} = \begin{pmatrix} x_2 \\ x_1 \end{pmatrix} \frac{\partial}{\partial x}$
$\therefore [f, g](x) = \begin{pmatrix} -x_1 \\ x_2 \end{pmatrix}$ $\therefore$ ~~false~~
$[f, g](x) = \frac{\partial g}{\partial x} f - \frac{\partial f}{\partial x} g = \frac{\partial}{\partial x}\begin{pmatrix} 0 \\ x_1 \end{pmatrix}\begin{pmatrix} x_2 \\ 0 \end{pmatrix} - \frac{\partial}{\partial x}\begin{pmatrix} x_2 \\ 0 \end{pmatrix}\begin{pmatrix} 0 \\ x_1 \end{pmatrix} =$
$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}\begin{pmatrix} x_2 \\ 0 \end{pmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{pmatrix} 0 \\ x_1 \end{pmatrix} = \begin{bmatrix} 0 \\ x_2 \end{bmatrix} - \begin{bmatrix} x_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -x_1 \\ x_2 \end{bmatrix} = (-x_1, x_2)^T \therefore$
True

1bi / $\therefore \dot{x}_1 = 0 = \dot{x}_2 \therefore$ $\dot{x}_2 = 0 = -2(x_2 - x_1) + x_2(1 - x_2^2) \therefore$
$2(x_2 - x_1) = x_2(1 - x_2^2)$ $\therefore$ $\dot{x}_1 = 0 = x_2(1 - x_2^2) + x_1(1 - x_1^2)$
$= x_2 - x_2^3 + x_1 - x_1^3$ $\therefore$ $(0, 0)$ and $(1 - x_2^2 = 0, 1 - x_1^2 = 0) \therefore$
if $1 - x_2^2 = 0 \therefore x_2^2 = 1 \therefore$ $x_2 = +1$, $x_2 = -1 \therefore$

if $x_2 = 1$:

$\dot{x}_1 = 0 = 2(x_2 - x_1) + x_1(1 - x_1^2) = 2x_2 - 2x_1 + x_1 - x_1^3 = 2x_2 - x_1 - x_1^3 \therefore$

$2x_2 = x_1 + x_1^3 \therefore \quad x_2 = \frac{1}{2}x_1 + \frac{1}{2}x_1^3 \therefore$

~~or~~ $\dot{x}_2 = 0 = -2x_2 + 2x_1 + x_2 - x_2^3 =$

$-x_1 - x_1^3 + 2x_1 + \frac{1}{2}x_1 + \frac{1}{2}x_1^3 + \frac{1}{2}x_1 + \frac{1}{2}x_1^3$

$x_2 = \frac{1}{2}x_1 + \frac{1}{2}x_1^3 \quad \therefore$

$x_1 = 0, 1, -1 \quad \therefore \quad x_1 = 0, \; 1 - x_1^2 = 0 \quad \therefore$

if $x_1 = 0$: $x_2 = \frac{1}{2}(0) + \frac{1}{2}(0)^3 = 0$

if $x_1 = 1$: $x_2 = \frac{1}{2}(1) + \frac{1}{2}(1)^3 = \frac{1}{2} + \frac{1}{2} = 1$

if $x_1 = -1$: $x_2 = \frac{1}{2}(-1) + \frac{1}{2}(-1)^3 = -\frac{1}{2} - \frac{1}{2} = -1 \therefore$

equilibriums for $(\dot{x}_1, \dot{x}_2) = (0, 0)$ are:

$(0, 0), (1, 1), (-1, 1)$

\ b ii / let $V(x_1, x_2) = x_1^2 + x_2^2 \therefore f(x) = \dot{x} = \begin{bmatrix} 2x_2 - 2x_1 + x_1 - x_1^3 \\ -2x_2 + 2x_1 + x_2 - x_2^3 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_1 - x_1^3 \\ 2x_1 - x_2 - x_2^3 \end{bmatrix}$

$\therefore \nabla V = \begin{pmatrix} 2x_1 \\ 2x_2 \end{pmatrix} \therefore$

$f(x) \cdot \nabla V = \begin{bmatrix} 2x_2 - 2x_1 + x_1 - x_1^3 \\ -2x_2 + 2x_1 + x_2 - x_2^3 \end{bmatrix} \cdot \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_1 - x_1^3 \\ 2x_1 - x_2 - x_2^3 \end{bmatrix} \cdot \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} =$

$4x_1x_2 - 2x_1^2 - 2x_1^4 + 4x_1x_2 - 2x_2^2 - 2x_2^4 = -2x_1^2 + 8x_1x_2 - 2x_2^2 - 2x_1^4 - 2x_2^4 =$

$-2(x_1^2 - 4x_1x_2 + x_2^2) - 2(x_1^4 + x_2^4) =$

$-2(x_1^2 - 2x_1x_2 + x_2^2) + 4x_1x_2 - 2(x_1^4 + x_2^4) =$

$-2(x_1 - x_2)^2 - 2(x_1^4 - 2x_1x_2 + x_2^4) \cancel{\le -2(x_1 - x_2)^2 - 2(x_1^2 - x_2^2)^2} \le 0$

~~for~~ $= -2(x_1 - x_2)^2 - 2(x_1^4 + 2x_1^2x_2^2 + x_2^2) + 4x_1^2x_2^2 + 4x_1x_2 =$

$-2(x_1 - x_2^2) - 2(x_1^2 + x_2^2)^2 + 4x_1x_2(x_1x_2 + 1) \le 0$ for $x_1x_2 + 1 \le 0 \therefore 1 \le -x_1x_2$

$V = x_1^2 + x_2^2 \therefore$ ~~$x_1x_2 \le$~~ $1 \ge -x_1x_2 \therefore x_1x_2 \ge -1$

all trajectories starting in $M = \{x \in \mathbb{R}^2 \mid x_1x_2 \ge 1\}$ stay in M

for all future time but $x_1x_2 \ge 1$ contains both $(1, 1)$ and $(-1, -1)$

$\therefore$ M doesn't contain only one equilibrium $\therefore$

linearization: $A = \begin{bmatrix} -1 - 3x_1^2 & 2 \\ 2 & -1 - 3x_2^2 \end{bmatrix} \doteq \frac{\partial f}{\partial x} \therefore$

$A|_{(0,0)} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \therefore \det(A_{(0)} - \lambda I) = \begin{vmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = (-1-\lambda)^2 - 2(2) = \lambda^2 + 1 + 2\lambda - 4 = \lambda^2 + 2\lambda - 3 =$

$\cancel{\lambda 2}$ $(\lambda + 3)(\lambda - 1) = 0 \therefore \lambda = -3, \lambda = 1 \quad \therefore (0, 0)$ is saddle

$A|_{(1,1)} = A|_{(-1,-1)} = \begin{bmatrix} -1-3 & 2 \\ 2 & -1-3 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix} \therefore$

\PP2021/ $\det(A|_{(1,1)} - \lambda I) = \det(A|_{(-1,-1)} - \lambda I) = \begin{vmatrix} -4-\lambda & 2 \\ 2 & -4-\lambda \end{vmatrix} =$

$(-4-\lambda)(-4-\lambda) - 2(2) = \lambda^2 + 16 + 8\lambda - 4 = \lambda^2 + 8\lambda + 12 = (\lambda+6)(\lambda+2) = 0$

$\therefore \lambda = -6 \quad , \lambda = -2 \therefore$

both $(1,1)$, $(-1,-1)$ are stable nodes

$\therefore$ by P.B. criterion: no periodic orbit exists

$\therefore$ the system has no limit cycles

\1ci/ controllability matrix $M = [B \vdots AB] = \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \vdots \begin{bmatrix} -3 & 1 \\ 17 & 10 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] =$

$\left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \vdots \begin{bmatrix} 1 \\ 10 \end{bmatrix}\right] = \begin{bmatrix} 0 & 1 \\ 1 & 10 \end{bmatrix} \therefore$

$\det(M) = 0(10) - 1(1) = -1 \neq 0 \quad \therefore \text{rank}(M) = 2 \quad \therefore$ full rank $\therefore$

(A, B) is controllable

\1cii/ let $K = [k_1 \quad k_2] \therefore \quad BK = \begin{bmatrix} 0 \\ 1 \end{bmatrix}[k_1 \quad k_2] = \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} \therefore$

$A_{CL} = A - BK = \begin{bmatrix} -3 & 1 \\ 17 & 10 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} -3 & 1 \\ -k_1+17 & -k_2+10 \end{bmatrix} \therefore$

$\det(SI - A_{CL}) = \det\left(\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ -k_1+17 & -k_2+10 \end{bmatrix}\right) = \det\left(\begin{bmatrix} S+3 & -1 \\ k_1-17 & S+k_2-10 \end{bmatrix}\right) =$

$(S+3)(S+k_2-10) + 1(k_1-17) = S^2 + k_2 S - 10S + 3S + 3k_2 - 30 + k_1 - 17 =$

$S^2 + S(k_2 - 7) + (k_1 + 3k_2 - 47) = 0 =$

$(S+3.618)(S+1.382) = S^2 + 5S + 5.000076 = 0 \therefore$

$k_2 - 7 = 5 \therefore \quad k_2 = 12,$

$k_1 + 3k_2 - 47 = 5.000076 = k_1 + 3(12) - 47 = k_1 - 11 \therefore$

$k_1 = 16.000076 \therefore$

$u = -Kx = -[16.000076 \quad 12]\, x$

\1ciii/ $T = MW \quad \therefore M = \begin{bmatrix} 0 & 1 \\ 1 & 10 \end{bmatrix}, \quad W = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix},$

$\det(SI - A) = \det\left(\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} -3 & 1 \\ 17 & 10 \end{bmatrix}\right) = \det\left(\begin{bmatrix} S+3 & -1 \\ -17 & S-10 \end{bmatrix}\right) =$

$S^2 + a_1 S + a_2 = (S+3)(S-10) + 1(-17) = S^2 - 30 - 7S - 17 = S^2 - 7S - 47 = S^2 + a_1 S + a_2 \therefore$

$a_1 = -7 \therefore \quad \therefore W = \begin{bmatrix} -7 & 1 \\ 1 & 0 \end{bmatrix} \therefore$

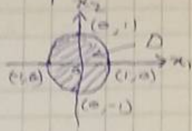
$T = \begin{bmatrix} 0 & 1 \\ 1 & 10 \end{bmatrix}\begin{bmatrix} -7 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ +3 & 1 \end{bmatrix}$

\1d/ let $V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2 \quad \therefore \dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 =$

$x_1(-x_1 + x_1 x_2) + x_2(-x_2) = -x_1^2 + x_1^2 x_2 - x_2^2 = -(x_1^2 - x_1^2 x_2 + x_2^2) = \cancel{x_1^2(x_2} \; x_1^2(-1+x_2) - x_2^2 =$

$-x_1^2(1-x_2) - x_2^2 < 0$ if $1 - x_2 > 0 \quad \therefore \quad \cancel{1 > x_2} \quad x_2 < 1 \therefore \quad x_2^2 < 1 \therefore$

Locally asymptotically stable is $D = \{x_1, x_2 \in \mathbb{R} \mid V = x_1^2 + x_2^2 < 1\}$ ∴



Vle/ $\|x\|_2^2 = (\sqrt{x_1^2 + x_2^2})^2 = x_1^2 + x_2^2$ ∴

$\frac{d}{dt}\left(\frac{1}{2\times 2}\|x\|_2^2\right) = \frac{1}{4}\frac{d}{dt}(\|x\|_2^2) = \frac{1}{4}\frac{d}{dt}(x_1^2 + x_2^2) =$

$\frac{1}{4}(2x_1\dot{x}_1 + \frac{1}{4}(2)x_2\dot{x}_2 = \frac{1}{2}x_1\dot{x}_1 + \frac{1}{2}x_2\dot{x}_2 =$

$\frac{1}{2}x_1(-2x_1 + x_1^2 x_2) + \frac{1}{2}x_2(x_1(2x_2 - x_1^2) + u) =$

$-x_1^2 + \frac{1}{2}x_1^3 x_2 + \frac{1}{2}x_2(2x_1x_2 - x_1^3 + u) = -x_1^2 + \frac{1}{2}x_1^3 x_2 + x_1 x_2^2 - \frac{1}{2}x_1^3 x_2 + \frac{1}{2}x_2 u = \|x\|_2^2 = x_1^2 + x_2^2$

∴ $\frac{1}{2}x_2 u = x_1^2 + x_2^2 + x_1^2 - \frac{1}{2}x_1^3 x_2 - x_1 x_2^2 + \frac{1}{2}x_1^3 x_2 = 2x_1^2 + x_2^2 - x_1 x_2^2$ ∴

$\cancel{2x_1^2 + x_2^2 - \frac{1}{2}x_1^3 x_2}$ $x_2 u = 4x_1^2 + 2x_2^2 - 2x_1 x_2^2$ ∴

$u = x_2^{-1}(4x_1^2 + 2x_2^2 - 2x_1 x_2^2) = 4x_1^2 x_2^{-1} + 2x_2 - 2x_1 x_2$ ∴

$\frac{1}{4}\frac{d}{dt}(\|x\|_2^2) = \|x\|_2^2$ ∴

$\frac{\frac{d}{dt}(\|x\|_2^2)}{\|x\|_2^2} = 4$ ∴ $\int \frac{\frac{d}{dt}(\|x\|_2^2)}{\|x\|_2^2}\,dt = \int 4\,dt = \ln|\|x\|_2^2| = 4t + C_1 = \ln\|x\|_2^2$ ∴

$\|x\|_2^2 = e^{4t + C_1} = C_2 e^{4t}$ ∴

$\|x(0)\|_2^2 = C_2 e^{4(0)} = C_2 e^0 = C_2$ ∴

$\|x\|_2^2 = e^{4t}\|x(0)\|_2^2 = (e^{2t}\|x(0)\|_2)^2$ ∴

$\|x(t)\|_2 = e^{2t}\|x(0)\|_2$ ∴

$\|x(t)\| = e^{2t}\|x(0)\|$

∴ $\frac{1}{4}\frac{d}{dt}(\|x\|^2) = \|x\|^2$ ∴ $\int \frac{\frac{d}{dt}(\|x\|^2)}{\|x\|}\,dt = \int 4\,dt = 4t + C_3 = \ln|\|x\|^2| =$

$\ln\|x\|^2$ ∴ $\|x\|^2 = e^{4t + C_3} = e^{4t}C_4 = e^{2\times 2t}C_4$ ∴

$\|x(0)\|^2 = e^{4(0)}C_4 = e^0 C_4 = C_4$ ∴

$\|x(t)\|^2 = e^{2\times 2t}\|x(0)\|^2 = (e^{2t}\|x(0)\|)^2$ ∴

$\|x(t)\| = e^{2t}\|x(0)\|$

Vle/ $\|x\|_2^2 = x_1^2 + x_2^2$ ∴

$\frac{d}{dt}\left(\frac{1}{2\times 2}\|x\|_2^2\right) = \frac{1}{4}\frac{d}{dt}(x_1^2 + x_2^2) = \frac{1}{4}2x_1\dot{x}_1 + \frac{1}{4}2x_2\dot{x}_2 = \frac{1}{2}x_1\dot{x}_1 + \frac{1}{2}x_2\dot{x}_2 =$

$\frac{1}{2}x_1(-2x_1 + x_1^2 x_2) + \frac{1}{2}x_2(2x_1x_2 - x_1^3 + u) =$

$-x_1^2 + \frac{1}{2}x_1^3 x_2 + x_1 x_2^2 - \frac{1}{2}x_1^3 x_2 + \frac{1}{2}x_2 u = -\|x\|_2^2 = -x_1^2 - x_2^2$ ∴

$\frac{1}{2}x_2 u = -x_1^2 - x_2^2 + x_1^2 - \frac{1}{2}x_1^3 x_2 - x_1 x_2^2 + \frac{1}{2}x_1^3 x_2 = -x_2^2 - \frac{1}{2}x_1^3 x_2$ ∴

$x_2 u = -2x_2^2 - x_1^3 x_2$ ∴ $u = -2x_2 - x_1^3$ ∴

\PP2021/ $\therefore \frac{1}{4}\frac{d}{dt}(\|x\|_2^2) = -\|x\|_2^2 \therefore$

$\cancel{\int \frac{d}{dt}(\|x\|_2^2}$ $\frac{1}{4}\frac{d}{dt}(\|x\|^2) = -\|x\|^2 \therefore$

$\int \frac{\frac{d}{dt}(\|x\|^2)}{\|x\|^2}\,dt = \int -4\,dt = \ln|\|x\|_2^2| = -4t + C_1 = \ln\|x\|^2 \therefore$

$\|x\|^2 = e^{-4t+C_1} = e^{2\times(-2t)}C_2 \therefore$

$\|x(0)\|^2 = e^0 C_2 = C_2 \therefore$

$\|x\|^2 = e^{2\times(-2t)}\|x(0)\|^2 = (e^{-2t}\|x(0)\|)^2 = \|x(t)\|^2 \therefore$

$\|x(t)\| = e^{-2t}\|x(0)\|$

\2a/ let $x_1 = y$, $x_2 = \dot{y} \therefore$ $\ddot{y} = -(a+b\cos y)\dot{y} - c\sin y \therefore$

$\dot{x}_1 = \dot{y} = x_2$, $\dot{x}_2 = \ddot{y} = -(a+b\cos y)\dot{y} - c\sin y = -(a+b\cos x_1)x_2 - c\sin x_1$

$\therefore$ $\dot{x}_1 = x_2$, $\dot{x}_2 = -(a+b\cos x_1)x_2 - c\sin x_1$ is the steady ste space representation

$\therefore$ origin is $(x_1, x_2) = (0,0) \therefore$

~~if~~ at origin: $\dot{x}_1 = 0$, $\dot{x}_2 = -(a+b\cos(0))(0) - c\sin(0) = -c\sin(0) = -c(0) = 0$

$\therefore \dot{x}_1 = 0 = \dot{x}_2$ at origin $\therefore$ origin is an equilibrium

\2b/ let $V = c(1-\cos x_1) + \frac{x_2^2}{2}$, $a \geq 0$, $b \geq 0$ $\therefore V = c - c\cos x_1 + \frac{x_2^2}{2}$

$\therefore \dot{V} = c\sin(x_1)\dot{x}_1 + x_2\dot{x}_2 = cx_2\sin x_1 + x_2(-ax_2 - bx_2\cos x_1 - c\sin x_1) =$

$cx_2\sin x_1 - ax_2^2 - bx_2^2\cos x_1 - cx_2\sin x_1 =$

$-ax_2^2 - bx_2^2\cos x_1$

$\therefore$ $\cos x_1 \geq 0$ for $x_1 \in [-\frac{\pi}{2}, \frac{\pi}{2}] \therefore$

about origin $\dot{V} = -ax_2^2 - bx_2^2\cos x_1 \leq 0 \therefore$

$\dot{V}$ is locally negative definite $\therefore$

by Lyapunov direct method with origin being isolated equilibria the origin is a stable equilibrium

\2c/ $\dot{V} = -ax_2^2 - bx_2^2\cos x_1 = -x_2^2(a+b\cos x_1)$

$\cos x_1 \leq 1 \therefore$ if $a > b \therefore$ $\cancel{a > b\cos x_1}$ $b\cos x_1 \leq b \therefore$

$a > b \geq b\cos x_1 \therefore a - b\cos x_1 > 0 \therefore a + b\cos x_1 > 0 \therefore$

$\cancel{\dot{V} = -x_2^2 < 0} \therefore \dot{V} = -x_2^2(a+b\cos x_1) < 0 \therefore$

$\therefore \dot{V}$ is always negative $\therefore$

the origin is an asymptotically stable equilibrium $\therefore$

$$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ bx_2 \sin x_1 - c\cos x_1 & -a - b\cos x_1 \end{bmatrix} \therefore$$

$$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = A = \begin{bmatrix} 0 & 1 \\ 0 - c\cos(0) & -a - b\cos(0) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -c & -a-b \end{bmatrix} \therefore$$

$$\det(A - \lambda I) = \begin{bmatrix} -\lambda & 1 \\ -c & -a-b-\lambda \end{bmatrix} = -\lambda(-a-b-\lambda) + c(1) = \lambda^2 + (a+b)\lambda + c \therefore$$

$$\lambda = \frac{-a-b \pm \sqrt{(a+b)^2 - 4(1)c}}{2(1)} = \frac{-a-b \pm \sqrt{a^2+2ab-4c+b^2}}{2}$$

\3a/ $y = x_1 \therefore \therefore \dot{y} = \dot{x}_1 = -x_1^3 + x_2$ is independent of explicit u

$\therefore \ddot{y} = -3x_1^2\dot{x}_1 + \dot{x}_2 = -3x_1^2(-x_1^3 + x_2) + u$ is dependent of an explicit $u \therefore$

occured from second derivative of $y \therefore$ relative degree $= 2$

\3bi/ let $V(x_1) = \frac{1}{2}x_1^2 \therefore$ V is positive definite, $V \to \infty$ as $x_1 \to \infty \therefore$

V is radially unbounded $\therefore$

$\dot{V} = x_1\dot{x}_1 = x_1(x_1^2 - x_1x_2) = x_1^3 - x_1^2x_2 = x_1^3 - x_1^2(x_1 + x_1^2) = x_1^3 - x_1^3 - x_1^4 = -x_1^4 \leq 0 \therefore$

$\dot{V}(0) = 0, \quad \dot{V} < 0 \ \forall x_1 \in \mathbb{R}\setminus\{0\} \therefore$

$x_2 = \phi(x_1) = x_1 + x_1^2$ makes $x_1 = 0$ a asymptotically stable equilibrium

\3bii/ $\cancel{\dot{x}_1 = x_1^2 - x_1x_2 = x_1^2 - x_1(\phi(x_1)) - x_1^2 - x_1(x_1 + x_1^2) = x_1^2 - x_1^2 - x_1^3}$

$\therefore$ $x_2 = z_2 + \phi(x_1) \therefore$

$\dot{x}_1 = x_1^2 - x_1x_2 = x_1^2 - x_1(z_2 + \phi(x_1)) = x_1^2 - x_1z_2 - x_1\phi(x_1) = x_1^2 - x_1z_2 - x_1(x_1 + x_1^2) =$

$x_1^2 - x_1z_2 - x_1^2 - x_1^3 = -x_1z_2 - x_1^3$

$\therefore z_2 = x_2 - \phi(x_1) = x_2 - x_1 - x_1^2 \therefore$

$\dot{z}_2 = \dot{x}_2 - \dot{x}_1 - 2x_1\dot{x}_1 = u + x_1z_2 + x_1^3 - 2x_1(-x_1z_2 - x_1^3) =$

$u + x_1z_2 + x_1^3 + 2x_1^2z_2 + 2x_1^4 \therefore$

let $V_c(x_1, z_2) = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2 \therefore$

$\dot{V}_c = x_1\dot{x}_1 + z_2\dot{z}_2 = x_1(-x_1z_2 - x_1^3) + z_2(u + x_1z_2 + x_1^3 + 2x_1^2z_2 + 2x_1^4) =$

$-x_1^2z_2 - x_1^4 + z_2u + x_1z_2^2 + x_1^3z_2 + 2x_1^2z_2^2 + 2x_1^4z_2 =$

$\cancel{-x_1^4 + z_2^2(x_1 + 2x_1^2) + (z}$ $-x_1^4 + (-x_1^2z_2 + z_2u + x_1z_2^2 + x_1^3z_2 + 2x_1^2z_2^2 + 2x_1^4z_2) =$

$-x_1^4 - kz_2^2 \therefore$

$\cancel{-kz_2^2\, z}$ $z_2u = -kz_2^2 + x_1^2z_2 - x_1z_2^2 - x_1^3z_2 - 2x_1^2z_2^2 - 2x_1^4z_2 \therefore$

let $u = -kz_2 + x_1^2 - x_1z_2 - x_1^3 - 2x_1^2z_2 - 2x_1^4 \therefore$

$\dot{V}_c = -x_1^4 - kz_2^2 \leq 0$ for $k \geq 0 \therefore$ V_c is p.d.f, unbounded, $\dot{V}_c$ is n.d.f.

$\therefore$ origin is globally asymptotically stable

\PP2021/ \3a/ $y = x_1 \therefore \quad \dot{y} = \dot{x}_1 = -x_1^3 + x_2$ has not explicit $u \therefore$

$\ddot{y} = -3x_1^2\dot{x}_1 + \dot{x}_2 = -3x_1^2(-x_1^3 + x_2) + u \quad \therefore$ has explicit $u \therefore$

u occures in 2nd derivative of $y \therefore$ relative degree $= 2$

\3bi/ let $V(x_1) = \frac{1}{2}x_1^2 \therefore V$ is positive definite, $V \to \infty$ as $x_1 \to \infty \therefore$

V is unbounded radially $\therefore$

$\dot{V} = x_1\dot{x}_1 = x_1(x_1^2 - x_1x_2) = x_1^3 - x_1^2x_2 = x_1^3 - x_1^2(x_1 + x_1^2) = x_1^3 - x_1^3 - x_1^4 = -x_1^4 \leq 0$

$\therefore \dot{V}(0) = 0, \quad \dot{V} < 0 \ \forall x_1 \in \mathbb{R}\setminus\{0\} \therefore$

$x_2 = \phi(x_1) = x_1 + x_1^2$ makes $x_1 = 0$ a asymptotically stable equilibrium

\3bii/ $x_2 = z_2 + \phi(x_1) \therefore$

$\dot{x}_1 = x_1^2 - x_1x_2 = x_1^2 - x_1(z_2 + \phi(x_1)) = x_1^2 - x_1z_2 - x_1\phi(x_1) = x_1^2 - x_1z_2 - x_1(x_1 + x_1^2) =$

$x_1^2 - x_1z_2 - x_1^2 - x_1^3 = -x_1z_2 - x_1^3 \therefore$

$z_2 = x_2 - \phi(x_1) = x_2 - x_1 - x_1^2 \therefore$

$\dot{z}_2 = \dot{x}_2 - \dot{x}_1 - 2x_1\dot{x}_1 = \cancel{z_2}\ u + x_1z_2 + x_1^3 - 2x_1(-x_1z_2 - x_1^3] = u + x_1z_2 + x_1^3 + 2x_1^2z_2 + 2x_1^4 \therefore$

let $V_c(x_1, z_2) = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2 \therefore$

$\dot{V}_c = x_1\dot{x}_1 + z_2\dot{z}_2 = x_1(-x_1z_2 - x_1^3) + z_2(u + x_1z_2 + x_1^3 + 2x_1^2z_2 + 2x_1^4) =$

$-x_1^2z_2 - x_1^4 + z_2u + x_1z_2^2 + x_1^3z_2 + 2x_1^2z_2^2 + 2x_1^4z_2 = -x_1^4 - kz_2^2 \therefore$

$z_2u = -kz_2^2 + x_1^2z_2 - x_1z_2^2 - x_1^3z_2 - 2x_1^2z_2^2 - 2x_1^4z_2 \therefore$

$u = -kz_2 + x_1^2 - x_1z_2 - x_1^3 - 2x_1^2z_2 - 2x_1^4 \therefore$

$\dot{V}_c = -x_1^4 - kz_2^2 \leq 0$ for $k \geq 0 \therefore$

$\dot{V}_c(0) = 0, \quad \dot{V}_c < 0 \ \forall \mathbb{R}^2\setminus\{0\} \therefore$

$\dot{V}_c$ is negative definite $\therefore$

V_c is positive definite and unbounded $\therefore$

$V_c \to \infty$ as $\|(x_1, z_2)\| \to \infty$, $V_c(0) = 0$, $V_c > 0 \ \forall \mathbb{R}^2\setminus\{0\} \therefore$

origin is globally asymptotically stable

\4a/ $\therefore \det \pi \neq 0$

\4b/ ~~$x = (x_1, x_2) = 0$~~ equilibria is $x = 0 \therefore$

for $x = 0$: $\quad \dot{x} + b(x)|_{x=0} = 0 \therefore$

(0) $b(0) = 0$, at $x = 0$: $\dot{x} = -b(0) \therefore$

let $\dot{x} = 0 \therefore \ 0 + b(x) = 0 = b(x) \quad \therefore x\,b(x) > 0 \ \forall x \neq 0 \therefore$

$b(x) \neq 0 \quad \forall x \neq 0 \therefore$

$b(x)=0$ for $x=0$ $\because$ $b(x)$ is continuous everywhere $\therefore$

$b(0)$ exists $\therefore$ $b(0) \in \mathbb{R}$ $\therefore$ $b(0)=0 \therefore$

$x=0$ is a equilibrium

$xb(x)>0 \therefore$

if $x<0 \therefore b(x)<0$

if $x>0 \therefore b(x)>0$

and $b(0)=0$ $\therefore$ $b(x)$ is a continuous function

let $V(x)=x^2$ $\therefore$ $V(0)=0$, $V>0 \; \forall x \in \mathbb{R}\setminus\{0\} \therefore$ ~~this is~~

$V(x) \to \infty$ as $|x| \to \infty \therefore$ V is radially unbounded, positive definite

$\therefore \dot{V} = 2x\dot{x} = 2x(-b(x)) = -2xb(x) < 0 \; \forall x \neq 0$, $\dot{V}(0)=0 \therefore$

$\dot{V}$ is negative definite $\therefore$

the origin is a globally asymptotically stable equilibrium

\4c/ $V(0) = \frac{1}{2}\dot{x}^2\Big|_{x=0} + \int_0^0 c(s)ds = \frac{1}{2}\dot{x}^2\Big|_{x=0} \therefore$ $\quad \ddot{x} = -b(\dot{x}) - c(x) \therefore$

~~$\dot{V}(x) = 2\dot{x}\ddot{x} + c(x) = 2\dot{x}(-b(\dot{x}) - c(x)) + c(x) =$~~

~~$-2\dot{x}b(\dot{x}) - 2\dot{x}c(x) + c(x) = -2\dot{x}b(\dot{x}) + c(x)(-2\dot{x}+1)$~~

$-2\dot{x}b(\dot{x}) < 0 \therefore$

$\frac{1}{2}(\dot{x})^2 = \dot{x}^2 \geq 0 \therefore$ $\quad xc(x)>0$

$\therefore$ if $x>0 \therefore c(x)>0 \therefore \int_0^x c(s)ds > 0 \therefore$

$V(x) \geq 0$ $\therefore$ $V(x)$ is positive definite and radially unbounded

$\dot{V}(x) = \dot{x}\ddot{x} + c(x)\dot{x} = \dot{x}(-b(\dot{x}) - c(x)) + c(x)\dot{x} = -\dot{x}b(\dot{x}) - \dot{x}c(x) + c(x)\dot{x} =$

$-\dot{x}b(\dot{x}) \leq 0 \therefore$

$V(0) = V(x)\Big|_{x=0} = \frac{1}{2}\dot{x}^2 + \int_0^x c(s)ds \Big|_{\substack{\dot{x}=0 \\ x=0}} = \frac{1}{2}\dot{x}^2\Big|_{\substack{\dot{x}=0 \\ x=0}} + \int_0^0 c(s)ds = \frac{1}{2}\dot{x}^2\Big|_{\substack{\dot{x}=0 \\ x=0}} = 0$

$V(x) \to \infty$ as ~~$|x| \to \infty$~~ $x \to \infty \therefore$

$\dot{V} < 0 \; \forall x \therefore$

$(x, \dot{x}) = (0,0)$ is asymptotically stable

\4d/ need $\dot{V} \leq 0$ so $\dot{V}(0)=0$, $\dot{V}<0 \; \forall x \in \mathbb{R}\setminus\{0\} \therefore$ need $\dot{V}$ to be n.d.f.

$\therefore$ need $\dot{x}b(\dot{x})=0$ at $\dot{x}=0$ for $(x,\dot{x})=(0,0)$ to be globally asymptotically stable

\PP2018/ \1ai/ Controllability matrix $M = [B \vdots AB] = \left[\begin{bmatrix}0\\1\end{bmatrix} \vdots \begin{bmatrix}0 & 1\\0 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right]$

$= \left[\begin{bmatrix}0\\1\end{bmatrix} \vdots \begin{bmatrix}1\\-1\end{bmatrix}\right] = \begin{bmatrix}0 & 1\\1 & -1\end{bmatrix} \therefore$

$\det M = 0(-1) - 1(1) = -1 \neq 0 \therefore$ Rank$(M) = 2$ $\therefore$ full rank $\therefore$

$\therefore$ (A, B) is controllable

\1aii/ $(s+4)(s+5) = s^2 + 4s + 5s + 20 = s^2 + 9s + 20 = 0 \therefore$

let $k = [k_1 \quad k_2] \therefore \quad Bk = \begin{bmatrix}0\\1\end{bmatrix}[k_1 \quad k_2] = \begin{bmatrix}0 & 0\\k_1 & k_2\end{bmatrix} \therefore$

$A_{CL} = A - Bk = \begin{bmatrix}0 & 1\\0 & -1\end{bmatrix} - \begin{bmatrix}0 & 0\\k_1 & k_2\end{bmatrix} = \begin{bmatrix}0 & 1\\-k_1 & -k_2-1\end{bmatrix} \therefore$

$\det(sI - A_{CL}) = \det\left(\begin{bmatrix}s & 0\\0 & s\end{bmatrix} - \begin{bmatrix}0 & 1\\-k_1 & -k_2-1\end{bmatrix}\right) = \det\left(\begin{bmatrix}s & -1\\k_1 & s+k_2+1\end{bmatrix}\right) =$

$(s)(s+k_2+1) + 1(k_1) = s^2 + s(k_2+1) + k_1 = s^2 + 9s + 20 = 0 \therefore$

$k_2 + 1 = 9 \therefore k_2 = 8$, $k_1 = 20 \therefore$

$u = -[20 \quad 8]x$

\aiii/ $T = MW \therefore \quad M = \begin{bmatrix}0 & 1\\1 & -1\end{bmatrix}$,

$W = \begin{bmatrix}a_1 & 1\\1 & 0\end{bmatrix} \therefore$

$\det(sI - A) = \det\left(\begin{bmatrix}s & 0\\0 & s\end{bmatrix} - \begin{bmatrix}0 & 1\\0 & -1\end{bmatrix}\right) = \det\left(\begin{bmatrix}s & -1\\0 & s+1\end{bmatrix}\right) =$

$s(s+1) + 1(0) = s^2 + s + 0 = s^2 + a_1 s + a_2 \therefore$

$a_1 = 1 \quad \therefore \quad W = \begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} \therefore$

FZ transformatrix $T = \begin{bmatrix}0 & 1\\1 & -1\end{bmatrix}\begin{bmatrix}1 & 1\\1 & 0\end{bmatrix} = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}$

\1b/ $f_1 = f_1(x) \quad \therefore \quad |f_1(x) - f_1(y)| \leq L_1|x-y|$,

$|f_2(x) - f_2(y)| \leq L_2|x-y| \therefore$

for $f_1 f_2$: $|f_1 f_2(x) - f_1 f_2(y)| = |f_1(x)f_2(x) - f_1(y)f_2(y)| \leq$

$|f_1(x)f_2(x)| + |-f_1(y)f_2(y)| = |f_1(x)f_2(x)| + |f_1(y)f_2(y)| =$

$|f_1(x)||f_2(x)| + |f_1(y)||f_2(y)| \leq L_1 L_2 |x-y| = L_3|x-y|$

for $L_3 = L_1 L_2 \therefore$ $f_1 f_2$ is locally lipschitz

\1c/ let $V = x_1^2 + x_2^2 \therefore V(0) = 0$, $V > 0 \ \forall x \in \mathbb{R} \setminus \{0\} \therefore$

V is positive definite, $V \to \infty$ as $x \to \infty$ $\therefore$ V is radially unbounded

$\therefore \dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 \therefore$

$\dot{x}_1 = x_1^3 + x_1 x_2^2 - x_1 - x_1^2 x_2 - x_2^3 + x_2$

$\dot{x}_2 = x_1^3 + x_1 x_2^2 - x_1 + x_1^2 x_2 + x_2^3 - x_2 \quad \therefore$

$\dot V = 2x_1(x_1^3+x_1x_2^2-x_1-x_1^2x_2-x_2^3+x_2)+2x_2(x_1^3+x_1x_2^2-x_1+x_1^2x_2+x_2^3-\cancel{x_2}) =$

$2x_1^4+2x_1^2x_2^2-2x_1^2-2x_1^3x_2-2x_1x_2^3+2x_1x_2+2x_1^3x_2+2x_1x_2^3-2x_1x_2+2x_1^2x_2^2+2x_2^4-2x_2^2=$

$2x_1^4+2x_1^2x_2^2-2x_1^2+2x_1^2x_2^2+2x_2^4-2x_2^2=$

$2(x_1^4+x_2^4)+2\cdot 2x_1^2x_2^2-2(x_1^2+x_2^2)=$

$2(x_1^4+2x_1^2x_2^2+x_2^4)-2(x_1^2+x_2^2)=$

$2(x_1^2+x_2^2)^2-2(x_1^2+x_2^2)\le 0$ ∴

$\dot V(0)=0$, $\dot V<0\ \forall x\in\mathbb{R}^2\setminus\{0\}$ ∴ $\dot V$ is negative definite ∴

The origin is locally asymptotically stable

\1 d/ let $V=x_1^2+x_2^2$ ∴ $\nabla V=\begin{bmatrix}2x_1\\2x_2\end{bmatrix}$ ∴

$f(x)\cdot\nabla V=\begin{bmatrix}x_2\\-x_1+2x_2-3x_1^2x_2-2x_2^3\end{bmatrix}\cdot\begin{bmatrix}2x_1\\2x_2\end{bmatrix}=$

$2x_1x_2-2x_1x_2+4x_2^2-6x_1^2x_2^2-4x_2^4=$

$4x_2^2-6x_1^2x_2^2-4x_2^4=-6x_1^2x_2^2-4x_2^2(x_2^2-1)\le 0$

for $x_2^2-1\ge 0$ ∴ $x_2^2\ge 1$ ∴

$x_1^2+x_2^2\ge 1$ ∴ $V\ge 1$ ∴

all trajectories starting in $M=\{x\in\mathbb{R}^2 \mid V\le 1\}$ stay in M $\forall$ future time

equilibriums: $\dot x_1=0=x_2$ ∴ $\dot x_2=0=-x_1+0(2-3x_1^2-2x_2^2)=-x_1=0=x_1$ ∴

M contains only one isolated equilibrium (0,0) ∴

linearisation: $A=\frac{\partial f}{\partial x}=\begin{bmatrix}0&1\\-1-6x_1x_2&2-3x_1^2-6x_2^2\end{bmatrix}$ ∴

$A=\frac{\partial f}{\partial x}\Big|_{(0,0)}=\begin{bmatrix}0&1\\-1&2\end{bmatrix}$ ∴

$\det(A-\lambda I)=\det\begin{bmatrix}-\lambda&1\\-1&2-\lambda\end{bmatrix}=-\lambda(2-\lambda)-1(-1)=\lambda^2-2\lambda+1=(\lambda-1)(\lambda-1)$ ∴

$\lambda=1>0$ ∴ (0,0) is a unstable node ∴

by P.B. criterion $\exists$ a periodic orbit in M ∴

the system has a periodic orbit.

\1 e/ $a,b,c,k,\theta>0$ $y=x_3$ ∴ $\dot y=\dot x_3=\theta x_1x_2$ ∴ no explicit u ∴

$\ddot y=\theta\dot x_1x_2+\theta x_1\dot x_2=\theta(-ax_1+u)+\theta x_1(-bx_2+k-cx_1x_3)=$

$-a\theta x_1+\theta u+\theta x_1(-bx_2+k-cx_1x_3)$ ∴ explicit u ∴ occures at 2nd derivative ∴ system has relative degree = 2

\pp2018/ \5/ $\|x\|_2^2 = x_1^2 + x_2^2 \therefore$

$$\frac{d}{dt}\left(\frac{1}{2}\|x\|_2^2\right) = \frac{1}{2}\frac{d}{dt}\left(\|x\|_2^2\right) = \frac{1}{2}\frac{d}{dx}\left(x_1^2 + x_2^2\right) = \frac{1}{2}\left(2x_1\dot{x}_1 + 2x_2\dot{x}_2\right) =$$

$$x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(-x_1 + x_2 x_1^3) + x_2(x_1^3 + x_1 x_2^2 + u) =$$

$$-x_1^2 + x_1^4 x_2 + x_1^3 x_2 + x_1 x_2^3 + x_2 u = -\|x\|_2^2 = -x_1^2 - x_2^2 \therefore$$

$$x_2 u = -x_1^2 - x_2^2 + x_1^2 - x_1^4 x_2 - x_1^3 x_2 - x_1 x_2^3 = -x_2^2 - x_1^4 x_2 - x_1^3 x_2 - x_1 x_2^3 \therefore$$

let $u = -x_2 - x_1^4 - x_1^3 - x_1 x_2^2 \therefore$

$$\frac{1}{2}\frac{d}{dt}\left(\|x\|_2^2\right) = -\|x\|_2^2 \therefore$$

$$\frac{1}{2}\frac{d}{dt}\left(\|x\|^2\right) = -\|x\|^2 \therefore$$

$$\int \frac{\frac{d}{dt}(\|x\|^2)}{\|x\|^2}\,dt = \int -2\,dt = -2t + C_1 = \ln\left|\|x\|^2\right| = \ln\|x\|^2 \therefore$$

$$\|x\|^2 = e^{-2t + C_1} = e^{-2t}C_2 \therefore$$

$$\|x(0)\|^2 = e^{-2(0)}C_2 = e^0 C_2 = C_2 \therefore$$

$$\|x(t)\|^2 = e^{-2t}\|x(0)\|^2 = \left(e^{-t}\|x(0)\|\right)^2 \therefore$$

$$\|x\| = e^{-t}\|x(0)\|$$

\2a/ let $V(x) = x^T P x \therefore$

$$\dot{V} = \frac{d}{dt}(x^T P x) = \dot{x}^T P x + x^T P \dot{x} =$$

$$(Ax(t) + \phi(x))^T P x + x^T P(Ax + \phi(x)) =$$

$$(x^T A^T + \phi^T) P x + x^T P A x + x^T P \phi =$$

$$x^T A^T P x + \phi^T P x + x^T P A x + x^T P \phi = x^T(A^T P + PA)x + \phi^T P x + x^T P \phi =$$

$$x^T(A^T P + PA)x + 2x^T P \phi =$$

$$x^T(-I)x + 2x^T P \phi = -x^T x + 2x^T P \phi = -x \cdot x + 2x^T P \phi =$$

$$-\|x\|^2 + 2x^T P \phi \leq -\|x\|^2 + 2\|P\|\,\|x\|\,\|\phi\| \therefore$$

$\delta > 0 \therefore \|x\| < \sigma \therefore \frac{\|\phi\|}{\|x\|} < \frac{1}{4\|P\|} \therefore \|P\|\,\|\phi\| < \frac{1}{4}\|x\| \therefore$

$$\dot{V} \leq -\|x\|^2 + 2\|P\|\,\|\phi\|\,\|x\| \leq -\|x\|^2 + 2\frac{1}{4}\|x\|\,\|x\| = -\|x\|^2 + \frac{1}{2}\|x\|^2 = -\frac{1}{2}\|x\|^2 \therefore$$

$$\dot{V} \leq -\frac{1}{2}\|x(t)\|^2$$

\2b/ $V = x^T P x$, $\dot{V} \leq -\frac{1}{2}\|x\|^2$ for $\|x\| < \sigma \therefore$

$$\lambda_{min}\|x\|^2 \leq V = x^T P x \leq \lambda_{max}\|x\|^2 = \|P\|\,\|x\|^2 \therefore$$

$$\|x\| < \sigma \therefore \|x\|^2 < \sigma^2 \therefore$$

$$\frac{1}{\lambda_{min}} V = \frac{1}{\lambda_{min}} x^T P x \leq \frac{1}{\lambda_{min}}\|P\|\,\|x\|^2 \therefore$$

$\frac{1}{\lambda_{min}} V = \frac{1}{\lambda_{min}} x^T P x < \sigma^2$ is $x^T P x < \lambda_{min} \sigma^2$ $\therefore$

$\|x\| < \sigma$, $\dot{V} \leq 0$ $\therefore$

A set $M \subset \mathbb{R}^n$ is invariant w.r.t. $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ if

$x(0) \in D \Rightarrow x(t) \in D \; \forall t \in \mathbb{R}$ $\therefore$

D is w.r.t. $\dot{x} = f(x)$, $x(0) \in D$ $\because x(0)^T P x(0) < \lambda_1 \sigma^2$

$\therefore x^T P x < \lambda_1 \sigma^2 \therefore$ $x^T P x < \lambda \sigma^2 \therefore x \in D \; \forall t \in \mathbb{R} \therefore$

$D = \{x \mid x^T P x < \lambda_{min}(P) \sigma^2$ is invariant

it is also open as it is a set of the form $\{x \mid V(x) < \beta\}$

V is positive definite , $\dot{V}$ is negative definite $\therefore$

$V(x) < \beta$ level set $\therefore$

as in D: $V > 0$, $\dot{V} < 0$ $\therefore$

following Lasalle argument. D contained in basin of attraction $x=0$

\1ai/ controllability matrix $M = [B \vdots AB] = \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \vdots \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] =$

$\left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \vdots \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right] = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \therefore$ $\det M = 0 - 1 = -1 \neq 0 \therefore rank(M) = 2 \therefore$ full rank $\therefore$

(A, B) is controllable

\1aii/ let $K = [k_1 \quad k_2]$ $\therefore$

characteristics: $(s+4)(s+5) = s^2 + 4s + 5s + 20 = s^2 + 9s + 20$ $\therefore$

$A_{cl} = A - BK = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \quad k_2] = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 - 1 \end{bmatrix}$

$\therefore \det(sI - A_{cl}) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ -k_1 & -k_2 - 1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} s & -1 \\ k_1 & s + k_2 + 1 \end{bmatrix}\right) =$

$s(s + k_2 + 1) - (-1)k_1 = \cancel{s^2 + k_2 s + (1 + k_1) = s^2 + 9s + 20 \therefore}$

$\cancel{1 + k_1 = 20 \therefore k_1 = 19, \; k_2 = 9 \therefore}$ $\cancel{K = [19 \quad 9] \therefore}$

$u = \cancel{-Kx = -[19 \quad 9]x = [-19 \quad 9]x}$ $s^2 + (k_2 + 1)s + k_1 = s^2 + 9s + 20$ $\therefore$

$k_2 + 1 = 9 \therefore k_2 = 8$, $k_1 = 20 \therefore$ $\cancel{K = [8, 20]}$ $K = [20, 8]$ $\therefore$

$u = -Kx = -[20, 8]x = [-20 \quad -8]x$

\1aiii/ $T = MW \therefore$ $M = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix}$, $W = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix} \therefore$

$\det(sI - A) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} s & -1 \\ 0 & s+1 \end{bmatrix}\right) =$

$(s)(s+1) - 0 = s^2 + s = s^2 + a_1 s + a_2 \therefore a_1 = 1 \therefore$ $W = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \therefore$

$T = \begin{bmatrix} 0 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

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1b $\cancel{f_1 d}$ $f_1 = f_1(x) \therefore \quad |f_1(x) - f_1(y)| \le L_1|x-y| \; ; \; |f_2(x) - f_2(y)| \le L_2|x-y|$

$\therefore \cancel{|f(x) - f(y)|} = \; f(x) = f_1(x) f_2(x) \therefore$

$|f(x) - f(y)| = |f_1(x) f_2(x) - f_1(y) f_2(y)| =$

$|f_1(x) f_2(x) + f_1(y) f_2(x) - f_1(y) f_2(x) - f_1(y) f_2(xy)| =$

$\cancel{|f_2(x)(f_1(x) + f_1(y)) - f_1(y)(f}$

$|(f_1(x) f_2(x) - f_2(x) f_1(y)) + (f_1(y) f_2(x) - f_1(y) f_2(y))| =$

$|f_2(x)(f_1(x) - f_1(y)) + f_1(y)(f_2(x) - f_2(y))| \le$

$|f_2(x)(f_1(x) - f_1(y))| + |f_1(y)(f_2(x) - f_2(y))| =$

$|f_2(x)||f_1(x) - f_1(y)| + |f_1(y)||f_2(x) - f_2(y)| \le$

$|f_2(x)| L_1 |x-y| + |f_1(y)| L_2 |x-y| =$

$(|f_2(x)| L_1 + |f_1(y)| L_2)|x-y| \le L_3 |x-y|$

$\therefore \because f_i$ are lipschitz bounded on differences $\therefore$

f is lipschitz bounded on differences $\therefore$

$f = f_1 f_2$ is a locally lipschitz function

1c let $V = x_1^2 + x_2^2 \therefore$ $\forall$ ~~x~~ $V \to \infty$ as $|x| \to \infty$;

V is radially unbounded, $V(0) = 0 \therefore V > 0 \; \forall x \in \mathbb{R}^2 \setminus \{0\} \therefore$

V is positive definite $\therefore$

$\cancel{\dot{x}} \quad \cancel{\dot{V} = 2x_1 + 2x_2} \quad \dot{V} = 2x_1 \dot{x}_1 + 2x_2 \dot{x}_2 =$

$2x_1(x_1 - x_2)(x_1^2 + x_2^2 - 1) + 2x_2(x_1 + x_2)(x_1^2 + x_2^2 - 1) =$

$2(x_1^2 + x_2^2 - 1)(x_1^2 - x_1 x_2) + 2(x_1^2 + x_2^2 - 1)(x_1 x_2 + x_2^2) =$

$2(x_1^2 + x_2^2 - 1)[x_1^2 - x_1 x_2 + x_1 x_2 + x_2^2] =$

$2(x_1^2 + x_2^2 - 1)[x_1^2 + x_2^2] \le 0$ for $x_1^2 + x_2^2 - 1 = V - 1 \le 0 \therefore$

$V \le 1 \therefore$ origin is $(x_1 = 0, x_2 = 0) \therefore$ at origin $\dot{V} = 0 \because V = 0 < 1$

linearisation: $\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \left.\begin{bmatrix} (1-0)(x_1^2 + x_2^2 - 1) + (x_1 - x_2)(2x_1 + 0 - 0) & (-1)(x_1^2 + x_2^2 - 1) + (x_1 - x_2)(2x_2) \\ (1+0)(x_1^2 + x_2^2 - 1) + (x_1 + x_2)(2x_1 + 0 - 0) & (1)(x_2^2 + x_1^2 - 1) + (x_1 + x_2)(2x_2) \end{bmatrix}\right|_{(0,0)}$

$\begin{bmatrix} 1(-1) + 0 & (-1)(-1) \\ 1(-1)(0) & (1)(-1) \end{bmatrix} = \begin{bmatrix} -1 & +1 \\ -1 & -1 \end{bmatrix} = A \therefore$

$\det(A - \lambda I) = \det\begin{pmatrix} -1-\lambda & 1 \\ -1 & -1-\lambda \end{pmatrix} = (-1-\lambda)^2 - 1(-1) = \lambda^2 + 2\lambda + 1 + 1 = \lambda^2 + 2\lambda + 2 = 0 \therefore$

$\lambda = \frac{-2 \pm \sqrt{4 - 4(1)(2)}}{2(2)} = \frac{-2}{4} \pm \frac{\sqrt{-4}}{4} = \frac{-1}{2} \pm \frac{2}{4} i = \frac{-1}{2} \pm \frac{1}{2} i \therefore Re(\lambda) = -\frac{1}{2} < 0 \therefore$

$\therefore$ origin is locally asymptotically stable

\1c/ $\dot{x}_1 = x_1^3 + x_1x_2^2 - x_1 - x_2x_1^2 - x_2^3 + x_2$

$\dot{x}_2 = x_1^3 + x_1x_2^2 - x_1 + x_1^2x_2 + x_2^3 - x_2$ ∴

let $V = x_1^2 + x_2^2$. $V \to \infty$ as $|x| \to \infty$ ∴ V is radially unbounded

$V(0) = 0$, $V > 0\ \forall x \in \mathbb{R}^2 \setminus \{0\}$ ∴ V is positive definite ∴

$\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 =$

$2x_1(x_1^3 + x_1x_2^2 - x_1 - x_1^2x_2 - x_2^3 + x_2) + 2x_2(x_1^3 + x_1x_2^2 - x_1 + x_1^2x_2 + x_2^3 - x_2) =$

$2(x_1^4 + x_1^2x_2^2 - x_1^2 - x_1^3x_2 - x_1x_2^3 + x_1x_2 + x_1^3x_2 + x_1x_2^3 - x_1x_2 + x_1^2x_2^2 + x_2^3x_2 - x_2^2) =$

$2(x_1^4 + x_1^2x_2^2 - x_1^2 + x_1^3x_2 + x_1^2x_2^2 - x_2^2) =$

$2(x_1^4 + 2x_1^2x_2^2 - x_1^2 + x_2^4 - x_2^2) =$

$-2(x_1^2 + x_2^2) + 2(x_1^4 + 2x_1^2x_2^2 + x_2^4) =$

$-2(x_1^2 + x_2^2) + 2(x_1^2 + x_2^2)^2 \le 0$ ∴ $\dot{V}(0) = 0$, $\dot{V} < 0\ \forall x \neq 0$ ∴

$\dot{V}$ is negative definite ∴ origin is locally asymp stable

\1c/ let $V = x_1^2 + x_2^2$ ∴ $\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 =$

$2x_1(x_1 - x_2)(x_1^2 + x_2^2 - 1) + 2x_2(x_1 + x_2)(x_1^2 + x_2^2 - 1) =$

$2(x_1^2 + x_2^2 - 1)[x_1^2 - x_1x_2 + x_1x_2 + x_2^2] = 2(x_1^2 + x_2^2 - 1)[x_1^2 + x_2^2] \le 0$

for $x_1^2 + x_2^2 - 1 \le 0$ ∴ $x_1^2 + x_2^2 = V \le 1$ ∴

origin is $(x_1 = 0, x_2 = 0)$ ∴ $V|_{(0,0)} = 0 \le 1$ ∴

$\dot{V}$ is negative semi definite ∴

(0,0) is equilibrium ∴

(0,0) is locally asymptotically stable

\1d/ let $V = x_1^2 + x_2^2$ ∴ $V(0) = 0$, $V > 0\ \forall x \neq 0$, $V \to \infty$ as $|x| \to \infty$ ∴

V is radially unbounded, positive definite ∴ $\dot{x}_2 = -x_1 + 2x_2 - 3x_1^2x_2 - 2x_2^3$ ∴

$\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1x_2 + 2x_2(-x_1 + 2x_2 - 3x_1^2x_2 - 2x_2^3) =$

$2x_1x_2 - 2x_1x_2 + 4x_2^2 - 6x_1^2x_2^2 - 4x_2^4 =$

$4x_2^2 - 6x_1^2x_2^2 - 4x_2^4 = 2x_2^2(2 - 3x_1^2 - 2x_2^2) = 2x_2^2(2 - x_1^2 - 2x_1^2 - 2x_2^2) =$

$2x_2^2(2 - x_1^2) - 4x_2^2(x_1^2 + x_2^2) \le 0$ if $2 - x_1^2 \le 0$ ∴ $2 \le x_1^2$ ∴

$2 \le x_1^2 + x_2^2 = V$ ∴

for $V \le 2$: $M = \{x \mid V \le 2\}$ is domain that all trajectories will remain in for all future time ∴

\PP2018/ $\dot{x}_1 = 0 = x_2$, $\therefore \dot{x}_2 = 0 = -x_1 + 0(2-3x_1^2-2x_2^2) = -x_1 = 0 = x_1 \therefore$
origin is only equilibrium in $V \leq 1 \therefore$

- linearisation: $\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \left.\begin{bmatrix} 0 & 1 \\ -1+x_2(-6x_1) & 2-3x_1^2-2x_2^2+x_2(-4x_2) \end{bmatrix}\right|_{(0,0)} =$

$\begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix} \doteq A \therefore$

$\det(A-\lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{pmatrix} = -\lambda(2-\lambda) - 1(-1) = \lambda^2 - 2\lambda + 1 = (\lambda-1)(\lambda-1) = (\lambda-1)^2 \therefore$
$\lambda = 1 > 0 \therefore$ origin is unstable node
by P-B criterion: $\exists$ a periodic orbit for the system
$\therefore$ periodic orbit in M

\1b/ $\frac{d}{dt}$ let $\|x\|^2 = x_1^2 + x_2^2 \therefore$

$\frac{d}{dt}\left(\frac{1}{2}(x_1^2+x_2^2)\right) = \cancel{x_1^2 + x_2^2} \frac{1}{2}\frac{d}{dt}(x_1^2+x_2^2) =$
$\frac{1}{2}(2x_1\dot{x}_1 + 2x_2\dot{x}_2) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(-x_1+x_2x_1^3) + x_2(x_1^3 + x_1x_2^2 + u) =$
$-x_1^2 + x_1^4x_2 + x_1^3x_2 + x_1x_2^3 + x_2u = -(x_1^2+x_2^2) \doteq -x_1^2 - x_2^2 = -\|x\|^2 = \frac{1}{2}\frac{d}{dt}(\|x\|^2) \therefore$
$x_2u = -x_1^2 - x_2^2 + x_1^2 - x_1^4x_2 - x_1^3x_2 - x_1x_2^3 = -x_2^2 - x_1^4x_2 - x_1^3x_2 - x_1x_2^3 \therefore$
let $u = -x_2 - x_1^4 - x_1^3 - x_1x_2^2 \therefore$
$\frac{1}{2}\frac{d}{dt}(\|x\|^2) = -\|x\| \therefore \frac{\frac{d}{dt}(\|x\|^2)}{\|x\|^2} = -2 \therefore$
$\int \frac{\frac{d}{dt}(\|x\|^2)}{\|x\|^2}dt = \int -2\,dt = \ln|\|x\|^2| = -2t + c_1 \doteq \ln\|x\|^2 \therefore$
$\|x\|^2 = e^{-2t+c_1} = e^{-2t}c_2 \therefore$
$\|x(0)\|^2 = e^{-0}c_2 = c_2 \therefore$
$\|x(t)\|^2 = e^{-2t}\|x(0)\|^2 \doteq (e^{-t}\|x(0)\|)^2 \therefore$
$\|x\| = e^{-t}\|x(0)\|$ ↯

\1e/ $y = x_3 \therefore \dot{y} = \dot{x}_3 = \theta x_1x_2 \therefore$ no explicit $u \therefore$
$\ddot{y} = \theta\dot{x}_1x_2 + \theta x_1\dot{x}_2 = \theta x_2(-ax_1+u) + \theta x_1(-bx_2 + k - cx_1x_3) =$
$-a\theta x_1x_2 + \theta x_2u - b\theta x_1x_2 + k\theta x_1 - c\theta x_1^2x_3 \therefore$ explicit u
appears in 2nd derivative $\therefore$ system relative degree is 2.

\2a/ let $V = x^TPx \therefore \dot{V} = \dot{x}^TPx + x^TP\dot{x} =$
$(Ax+\phi)^TPx + x^TP(Ax+\phi) = (x^TA^T + \phi^T)Px + x^TPAx + x^TP\phi =$
$x^TA^TPx + \phi^TPx + x^TPAx + x^TP\phi = x^T(A^TP+PA)x + \phi^TPx + x^TP\phi =$
$x^T(-I)x + \phi^TPx + x^TP\phi = -x^Tx + 2x^TP\phi = -x\cdot x + 2x^TP\phi = -\|x\|^2 + 2x^TP\phi \leq$

$-\|x\|^2 + 2\|x^T P \phi\| = -\|x\|^2 + 2\|x\|\,\|P\|\,\|\phi\| \therefore \qquad \|P\|\|\phi\| < \frac{1}{4}\|x\| \therefore$

$\dot{V} \leq -\|x\|^2 + 2\|x\|\,\|P\|\,\|\phi\| \leq -\|x\|^2 + 2\|x\|\frac{1}{4}\|x\| = -\|x\|^2 + \frac{1}{2}\|x\|^2 = -\frac{1}{2}\|x\|^2$

\2b/ $V = x^T P x$, $\dot{V} \leq -\frac{1}{2}\|x\|^2$ for $\|x\| < \sigma > 0 \therefore$

$\lambda_{min}\|x\|^2 \leq V = x^T P x \leq \lambda_{max}\|x\|^2 = \|P\|\,\|x\|^2 \qquad \because \|x\|^2 < \sigma^2 \therefore$

$\frac{1}{\lambda_{min}} V = \frac{1}{\lambda_{min}} x^T P x \leq \frac{1}{\lambda_{min}}\|P\|\|x\|^2 \leq \frac{1}{\lambda_{min}}\|P\|\sigma^2 \therefore$

$x^T P x \leq \|P\|\,\|x\|^2 \leq \|P\|\sigma^2$

if $x^T P x < \lambda_{min}\sigma^2 \quad \therefore \|x\| < \sigma \therefore \quad \dot{V} \leq -\frac{1}{2}\|x\|^2 \leq 0 \therefore$

A set $M \subset \mathbb{R}^n$ is invariant w.r.t. $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ if

$x(0) \in D \Rightarrow x(t) \in D \ \forall t \in \mathbb{R} \therefore$

D w.r.t. $\dot{x} = f(x)$, $x(0) \in D \quad \because x(0)^T P x(0) < \lambda_1 \sigma^2$

$\because x^T P x < \lambda_1 \sigma^2 \quad x^T P x < \lambda\sigma^2 \therefore x \in D \ \forall t \in \mathbb{R} \therefore$

$D = \{x \mid x^T P x < \lambda_{min}(P)\sigma^2\}$ is invariant

it is also open as it is a set of the form $\{x \mid V(x) < \beta\} \therefore$

V is positive definite, $\dot{V}$ is negative definite $\therefore$

$V(x) < \beta$ level set as in D: $V > 0$, $\dot{V} < 0 \therefore$

following LaSalle argument, D contained in basin of attraction $x = 0$

\2c/ $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2x_2x_1 \\ x_1^2 + x_2^2 \end{bmatrix} \therefore \quad A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \therefore$

$A^T P + PA = -I$ Lyap equation $\therefore$

$P = P^T = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_4 \end{bmatrix} \therefore \quad A^T = \frac{1}{6}\begin{bmatrix} -3 & 0 \\ 0 & -2 \end{bmatrix} = \begin{bmatrix} -1/2 & 0 \\ 0 & -1/3 \end{bmatrix} \therefore$

$\begin{bmatrix} -1/2 & 0 \\ 0 & -1/3 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2 & P_4 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_4 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2}P_1 & -1/2\, P_2 \\ -\frac{1}{3}P_2 & -\frac{1}{3}P_4 \end{bmatrix} + \begin{bmatrix} -2P_1 & -3P_2 \\ -2P_2 & -3P_4 \end{bmatrix} =$

$\begin{bmatrix} -2.5P_1 & -7/2\, P_2 \\ -7/3\, P_2 & -\frac{10}{3}P_4 \end{bmatrix} = -I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \therefore$

$-\frac{7}{3}P_2 = 0 \therefore P_2 = 0$, $-2.5P_1 = -1 \therefore P_1 = \frac{2}{5}$, $P_4 = \frac{3}{10} \therefore P = \begin{bmatrix} 2/5 & 0 \\ 0 & \frac{3}{10} \end{bmatrix}$

$\cancel{4}$ \2e/ $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 2x_2x_1 \\ x_1^2 + x_2^2 \end{bmatrix} \therefore A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \therefore$

$\det(A) = -2(-3) = 6 \therefore A^T = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} \therefore \quad P = P^T = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_4 \end{bmatrix} \therefore$

$\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2 & P_4 \end{bmatrix} + \begin{bmatrix} P_1 & P_2 \\ P_2 & P_4 \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -2P_1 & -2P_2 \\ -3P_2 & -3P_4 \end{bmatrix} + \begin{bmatrix} -2P_1 & -3P_2 \\ -2P_2 & -3P_4 \end{bmatrix} = \begin{bmatrix} -4P_1 & -5P_2 \\ -5P_2 & -6P_4 \end{bmatrix}$

$= -I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \therefore -1 = -4P_1 \therefore P_1 = \frac{1}{4}$, $0 = -5P_2 \therefore P_2 = 0$, $-1 = -6P_4 \therefore \frac{1}{6} = P_4 \therefore$

\PP2018/ $P = \begin{bmatrix} 1/4 & 0 \\ 0 & \frac{1}{6} \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1/2 & 0 \\ 0 & 1/3 \end{bmatrix} = \frac{1}{6}\begin{bmatrix} 3/2 & 0 \\ 0 & 1 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \therefore$

$\cancel{\lambda_{max} = \frac{3}{12}}$ $\det(P - \lambda I) = \det\begin{pmatrix} 1/4 - \lambda & 0 \\ 0 & \frac{1}{6} - \lambda \end{pmatrix} = (\frac{1}{4} - \lambda)(\frac{1}{6} - \lambda) = 0 \therefore$

$\lambda = \frac{1}{4}, \lambda = \frac{1}{6} \therefore \quad \lambda_{max} = \frac{1}{4}, \lambda_{min} = \frac{1}{6} \therefore$

$\Phi = \begin{bmatrix} 2x_2 x_1 \\ x_1^2 + x_2^2 \end{bmatrix} \therefore$

$|\Phi|^2 = (\sqrt{(2x_2x_1)^2 + (x_1^2 + x_2^2)^2})^2 = (2x_2x_1)^2 + (x_1^2+x_2^2)^2 = 4x_2^2x_1^2 + x_1^4 + 2x_1^2x_2^2 + x_2^4 =$

$x_1^4 + 6x_1^2x_2^2 + x_2^4 \le 2x_1^4 + 2x_2^4 + x_1^4 + 6x_1^2x_2^2 + x_2^4 =$

$3x_1^4 + 6x_1^2x_2^2 + 3x_2^4 = 3(x_1^4 + 2x_1^2x_2^2 + x_2^4) =$

$3(x_1^2 + x_2^2)^2 = 3\|x\|^4 = 3\|x\|^2\|x\|^2 \therefore$

$|\Phi|^2 \le 3|x|^2\|x\|^2 \therefore \quad |\Phi| \le \sqrt{3}|x|\|x\| \therefore$

$\frac{|\Phi|}{|x|} \le \sqrt{3}\|x\| \therefore$

$\cancel{|P| = \det P = \frac{1}{4}}$ $\|P\| = \max(\lambda(P)) = \lambda_{max}(P) = \frac{1}{4} \therefore$

$\frac{1}{4\|P\|} = \frac{1}{4(\frac{1}{4})} = \frac{1}{1} = 1 \therefore$

$\frac{|\Phi(x)|}{|x|} < \frac{1}{4\|P\|} < 1 \quad \text{vs} \quad \sqrt{3}\|x\| < 1 \therefore \quad \|x\| < \frac{1}{\sqrt{3}}$

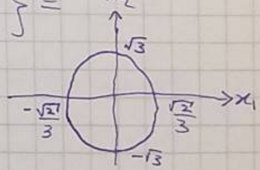
$\sigma = \frac{1}{\sqrt{3}} \therefore \quad \lambda_1 = \lambda_{min} = \frac{1}{6} \therefore \quad \sigma^2 = \frac{1}{3} \therefore$

$D = \{x \mid x^T P x < \lambda_{min}(P)\sigma^2\} = \{x \mid x^T \frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} x < \frac{1}{6} \cdot \frac{1}{3}\} =$

$\{x \mid [x_1 \ \ x_2] \frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} < \frac{1}{18}\} = \{x \mid \frac{1}{12}[x_1 \ \ x_2]\begin{bmatrix} 3x_1 \\ 2x_2 \end{bmatrix} < \frac{1}{18}\} =$

$\{x \mid \frac{1}{12}[3x_1^2 + 2x_2^2] < \frac{1}{18}\} = \{x \mid \frac{1}{2}[3x_1^2 + 2x_2^2] < \frac{1}{3}\} =$

$\{x \mid 3x_1^2 + 2x_2^2 < \frac{2}{3}\}$



\2c/ $P = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} \therefore \lambda_{min} = \frac{1}{6}, \lambda_{max} = \frac{1}{4} \therefore$

$\Phi = \begin{bmatrix} 2x_2x_1 \\ x_1^2 + x_2^2 \end{bmatrix} \therefore \quad \|\Phi\|^2 = (\sqrt{(2x_2x_1)^2 + (x_1^2+x_2^2)^2})^2 = (2x_1x_2)^2 + (x_1^2+x_2^2)^2 =$

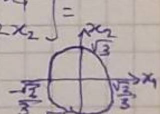
$x_1^4 + 6x_1^2x_2^2 + x_2^4 \le 3(x_1^2+x_2^2)^2 = \sqrt{3}^2(\|x\|^2)^2 \quad \therefore \|\Phi\| < \sqrt{3}\|x\|\|x\| \therefore$

$\frac{\|\Phi\|}{\|x\|} < \sqrt{3}\|x\| < \frac{1}{4\|P\|} = \frac{1}{4(\frac{1}{4})} = \frac{1}{1} = 1 \quad \because \|P\| = \lambda_{max} = \frac{1}{4} \therefore$

$\|x\| < \frac{1}{\sqrt{3}} \therefore \quad \|x\| < \sigma \therefore \frac{1}{\sqrt{3}} = \sigma \therefore \frac{1}{3} = \sigma^2 \therefore \quad \lambda_{min}\sigma^2 = \frac{1}{6} \times \frac{1}{3} = \frac{1}{18} \therefore$

$x^T P x = x^T \frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} x = [x_1 \ \ x_2]\frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{12}[x_1 \ \ x_2]\begin{bmatrix} 3x_1 \\ 2x_2 \end{bmatrix} =$

$\frac{1}{12}[3x_1^2 + 2x_2^2] \therefore \quad D = \{x \mid \frac{1}{12}(3x_1^2 + 2x_2^2) < \frac{1}{18}\} = \{x \mid 3x_1^2 + 2x_2^2 < \frac{2}{3}\}$



2c/ $P = \frac{1}{12}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1/4 & 0 \\ 0 & 1/6 \end{bmatrix} \therefore (-\lambda + \frac{1}{4})(\frac{1}{6} - \lambda) = 0 .$

$\lambda_{min} = \frac{1}{6}, \quad \lambda_{max} = \frac{1}{4} \therefore$

$\Phi = \begin{bmatrix} 2x_1 x_2 \\ x_1^2 + x_2^2 \end{bmatrix} \therefore \quad \|\Phi\|^2 = \left(\left((2x_1x_2)^2 + (x_1^2+x_2^2)^2\right)^{1/2}\right)^2 =$

$(2x_1x_2)^2 + (x_1^2+x_2^2)^2 = 4x_1^2x_2^2 + (x_1^4) + 2x_1^2x_2^2 + x_2^4 \leq$

$3x_1^4 + 6x_1^2x_2^2 + 3x_2^4 = 3(x_1^2+x_2^2)^2 = 3(\|x\|^2)^2 \therefore$

$\|\Phi\| \leq \sqrt{3}\,\|x\|^2 \therefore, \quad \frac{\|\Phi\|}{\|x\|} \leq \sqrt{3}\,\|x\| < \frac{1}{4\|P\|} = \cancel{\frac{1}{4\|P\|}}\ \frac{1}{4(\frac{1}{4})} = \frac{1}{1} = 1$

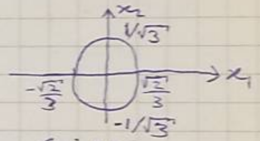
$\therefore$ ~~$\|P\| \geq \lambda_{max} = \frac{1}{4}$~~ and $\|P\| = \left\| \begin{bmatrix} 1/4 & 0 \\ 0 & 1/6 \end{bmatrix} \right\| = Max\left((\frac{1}{4}+0), (\frac{1}{6}+0)\right) =$

$Max(\frac{1}{4}, \frac{1}{6}) = \frac{1}{4} \therefore$

$\|x\| < \frac{1}{\sqrt{3}} = \sigma \therefore \sigma^2 = \frac{1}{3} \therefore \lambda_{min}\sigma^2 = \frac{1}{6} \times \frac{1}{3} = \frac{1}{18} \therefore$

$x^T P x = \frac{1}{12}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 3 & 0 \\ 0 & 2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \frac{1}{12}\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 3x_1 \\ 2x_2 \end{bmatrix} = \frac{1}{12}(3x_1^2 + 2x_2^2) \therefore$

$D = \left\{ x \mid \frac{1}{12}(3x_1^2 + 2x_2^2) < \frac{1}{18} \right\} = \left\{ x \mid 3x_1^2 + 2x_2^2 < \frac{2}{3} \right\} \therefore$



3a/ The mapping $f : D \to R^n$, $g : D \to R^n$ are vector fields on D :

$\dot{y} = \frac{\partial}{\partial t} y = \frac{\partial}{\partial t} h(x) = \frac{\partial h(x)}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial h}{\partial x}\dot{x} = \frac{\partial h}{\partial x}(f(x) + g(x)u) =$

$f(x)\frac{\partial h}{\partial x} + u\,g(x)\frac{\partial h}{\partial x} = \mathcal{L}_f h(x) + u\,\mathcal{L}_g h(x).$

$\therefore$ if $\mathcal{L}_g h(x) = 0 \therefore \dot{y} = \mathcal{L}_f h(x)$ independent of u $\therefore$

$\ddot{y} = y^{(2)} = \frac{\partial}{\partial t}\mathcal{L}_f h(x) = \frac{\partial \mathcal{L}_f h(x)}{\partial x}\frac{\partial x}{\partial t} = \frac{\partial}{\partial x}(\mathcal{L}_f h(x))(f(x) + u g(x)) =$

$f(x)\frac{\partial}{\partial x}\mathcal{L}_f h(x) + u g(x)\frac{\partial}{\partial x}\mathcal{L}_f h(x) = \mathcal{L}_f \mathcal{L}_f h(x) + u\,\mathcal{L}_g \mathcal{L}_f h(x) =$

$\mathcal{L}_f^2 h(x) + u\mathcal{L}_g \mathcal{L}_f h(x) \therefore$

let $\mathcal{L}_g \mathcal{L}_f h(x) = 0 \therefore$

$y^{(2)} = \mathcal{L}_f^2 h(x)$ independent of u --- etc $\therefore$

The nonlinear system is said to have relative degree ρ for $1 \leq \rho \leq n$ in a region $D_0 \in D$ if

$\mathcal{L}_g \mathcal{L}_f^{i-1} h(x) = 0 \quad , i = 1, 2, \ldots, \rho - 1$ and

$\mathcal{L}_g \mathcal{L}_f^{\rho-1} h(x) \neq 0 \quad \forall x \in D_0 \therefore y^{(\rho)} = \mathcal{L}_f^{\rho} h(x) + u\mathcal{L}_g \mathcal{L}_f^{(\rho-1)} h(x) \therefore$

explicit u $\therefore$ relative degree ρ

\PP2018/ \3bi/ $x_2 = -x_1^2$ ∴ $\dot{x}_1 = x_1 x_2 = x_1(-x_1^2) = -x_1^3$

∴ let $V(x_1) = \frac{1}{2}x_1^2$ ∴ $V \to \infty$ $|x_1| \to \infty$ V is radially unbounded

$V(0) = 0$, $V > 0$ $\forall x_1 \neq 0$ ∴ V is positive definite ∴

$\dot{V} = x_1 \dot{x}_1 = x_1(-x_1^3) = -x_1^4 \leq 0$ ∴

$\dot{V}(0) = 0$, $\dot{V}(x_1) < 0$ $\forall x_1 \neq 0$ ∴ $\dot{V}$ is negative definite ∴

∴ origin is globally asymptotically stable

\3bii/ ∴ $z_2 = x_1 - (-x_1^2) = x_1 + x_1^2$ ∴

$\dot{z}_2 = \dot{x}_1 + 2x_1\dot{x}_1 = x_1 x_2 + 2x_1 x_1 x_2 = x_1 x_2 + 2x_1^2 x_2$

∴ $x_2 = z_2 + \phi(x_1) = z_2 - x_1^2$ ∴

$\dot{z}_2 = x_1 x_2 + 2x_1^2 x_2 = x_1(z_2 - x_1^2) + 2x_1^2(z_2 - x_1^2) = x_1 z_2 - x_1^3 + 2x_1^2 z_2 - 2x_1^4$ ∴ let

∴ $V_c = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$ ∴

$\dot{V}_c = x_1\dot{x}_1 + z_2\dot{z}_2 = x_1(x_1 x_2) + z_2(x_1 z_2 - x_1^3 + 2x_1^2 z_2 - 2x_1^4$ ∴

\3bi/ $\dot{x}_1 = x_1 x_2 = x_1(\phi(x_1)) = x_1(-x_1^2) = -x_1^3$ ∴

let $V(x_1) = \frac{1}{2}x_1^2$ ∴

$\dot{V} = x_1\dot{x}_1 = x_1(-x_1^3) = -x_1^4 \leq 0$ $\dot{V}$ is n.d.s ∴ origin is asymp stable

\3bii/ ~~$\dot{x}_1 = x_1 x_2 = 2x_1$~~ $x_2 = z_2 + \phi(x_1)$ ∴

$\dot{x}_1 = x_1 x_2 = x_1(z_2 + \phi(x_1)) = x_1 z_2 + x_1\phi(x_1) = x_1 z_2 + x_1(-x_1^2) = x_1 z_2 - x_1^3$

$z_2 = x_2 - \phi(x_1) = x_2 - (-x_1^2) = x_2 + x_1^2$ ∴

$\dot{z}_2 = \dot{x}_2 + 2x_1\dot{x}_1 = x_1 + u + 2x_1(x_1 z_2 - x_1^3) =$

$x_1 + u + 2x_1^2 z_2 - 2x_1^4$ ∴

let $V_c = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$ ∴

$\dot{V}_c = x_1\dot{x}_1 + z_2\dot{z}_2 = x_1(x_1 z_2 - x_1^3) + z_2(x_1 + u + 2x_1^2 z_2 - 2x_1^4) =$

$x_1^2 z_2 - x_1^4 + x_1 z_2 + z_2 u + 2x_1^2 z_2^2 - 2x_1^4 z_2 =$

$3x_1^3 z_2 - x_1^4 + x_1 z_2 + z_2 u - 2x_1^4 z_2 = -x_1^4 - z_2^2$ ∴

$z_2 u = -x_1^4 - z_2^2 - 3x_1^3 z_2 + x_1^4 - x_1 z_2 + 2x_1^4 z_2 = -z_2^2 - x_1 z_2 - 3x_1^3 z_2 + 2x_1^4 z_2$ ∴

∴ let $u = -z_2 - x_1 - 3x_1^3 + 2x_1^4$ ∴

$\dot{V}_c \leq 0$ ∴ $\dot{V}_c$ is n.d.s ∴ V_c is p.d.s unbounded ∴

origin is globally asymp unbounded

\3bi/ $\dot{x}_1 = x_1 x_2 = x_1 \cdot (-x_1^2) = -x_1^3$ ∴

let $V(x_1) = \frac{1}{2}x_1^2$ ∴ V is radially unbounded, V is p.d.f ∴

$\dot{V}(x_1) = x_1\dot{x}_1 = x_1(-x_1^3) = -x_1^4 \leq 0$ ∴

$\dot{V}$ is n.d.f ∴

origin is globally asymp stable

\3bii/ $x_2 = z_2 + \phi(x_1) = z_2 - x_1^2$ ∴

$\dot{x}_1 = x_1 x_2 = x_1(z_2 - x_1^2) = x_1 z_2 - x_1^3$

$z_2 = x_2 - \phi(x_1) = x_2 - (-x_1^2) = x_2 + x_1^2$ ∴

$\dot{z}_2 = \dot{x}_2 + 2x_1\dot{x}_1 = \cancel{\dot{x}_2 + 2x_1}\ x_1 + u + 2x_1(x_1 z_2 - x_1^3) =$

$x_1 + u + 2x_1^2 z_2 - 2x_1^4$ ∴

let $V_c(x_1, z_2) = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$ ∴

$\dot{V}_c = x_1\dot{x}_1 + z_2\dot{z}_2 = x_1(x_1 z_2 - x_1^3) + z_2(x_1 + u + 2x_1^2 z_2 - 2x_1^4) =$

$x_1^2 z_2 - x_1^4 + x_1 z_2 + z_2 u + 2x_1^2 z_2^2 - 2x_1^4 z_2 = -x_1^4 - z_2^2$ ∴

$\cancel{3x_1^2 z_2}$ $z_2 u = -x_1^4 - z_2^2 - x_1^2 z_2 + x_1^4 - x_1 z_2 - 2x_1^2 z_2^2 + 2x_1^4 z_2 =$

$-z_2^2 - x_1^2 z_2 - x_1 z_2 - 2x_1^2 z_2^2 + 2x_1^4 z_2$ ∴

$u = -z_2 - x_1^2 - x_1 - 2x_1^2 z_2 + 2x_1^4$ ∴

$\dot{V}_c = -x_1^4 - z_2^2$ ∴ $\dot{V}_c$ is n.d.f ∴ origin is globally asymp stable

\3bii/ ∴ $x_2 = z_2 + \phi(x_1) = z_2 + (-x_1^2) = z_2 - x_1^2$ ∴

$\dot{x}_1 = x_1 x_2 = x_1(z_2 - x_1^2) = x_1 z_2 - x_1^3$

$z_2 = x_2 - \phi(x_1) = x_2 - (-x_1^2) = x_2 + x_1^2$ ∴

$\dot{z}_2 = \dot{x}_2 + 2x_1\dot{x}_1 = x_1 + u + 2x_1(x_1 z_2 - x_1^3) = x_1 + u + 2x_1^2 z_2 - 2x_1^4$

∴ let $V_c = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$ ∴ V_c is ~~pd~~ p.d.f, radially unbounded ∴

$\dot{V}_c = x_1\dot{x}_1 + z_2\dot{z}_2 = x_1(x_1 z_2 - x_1^3) + z_2(x_1 + u + 2x_1^2 z_2 - 2x_1^4) =$

$x_1^2 z_2 - x_1^4 + x_1 z_2 + z_2 u + 2x_1^2 z_2^2 - 2x_1^4 z_2 = -x_1^4 - z_2^2$ ∴

$z_2 u = -x_1^4 - z_2^2 - x_1^2 z_2 + x_1^4 - x_1 z_2 - 2x_1^2 z_2^2 + 2x_1^4 z_2 =$

$-z_2^2 - x_1^2 z_2 - x_1 z_2 - 2x_1^2 z_2^2 + 2x_1^4 z_2$ ∴

let $u = -z_2 - x_1^2 - x_1 - 2x_1^2 z_2 + 2x_1^4$ ∴

$\dot{V}_c = -x_1^4 - z_2^2 \leq 0$ ∴ $\dot{V}_c$ is n.d.f ∴ origin is ~~[illegible]~~ globally asymp stable

\PP2018/ ~~\4a/~~ $u(t) = -ky(t)$ ∴ $\begin{pmatrix} \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} = \begin{pmatrix} a & b \\ c & -2 \end{pmatrix}\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} + \begin{pmatrix} 1 \\ 0 \end{pmatrix}(-k)y(t) =$

$$\begin{pmatrix} a & b \\ c & -2 \end{pmatrix}\begin{pmatrix} y(t) \\ z(t) \end{pmatrix} + \begin{pmatrix} -ky(t) \\ 0 \end{pmatrix} = \begin{bmatrix} ay(t)+bz(t) \\ cy(t)-2z(t) \end{bmatrix} + \begin{pmatrix} -ky(t) \\ 0 \end{pmatrix} =$$

$$\begin{bmatrix} ay(t)-ky(t)+bz(t) \\ cy(t)-2z(t) \end{bmatrix} = \begin{bmatrix} (a-k)y(t)+bz(t) \\ cy(t)-2z(t) \end{bmatrix} = \begin{pmatrix} a-k & b \\ c & -2 \end{pmatrix}\begin{pmatrix} y(t) \\ z(t) \end{pmatrix}$$

~~∴ V̇ =~~ \4b/ ∴ let $V(y,z) = \frac{1}{2}y^2 + \frac{1}{2}z^2$ ∴

$\dot{V} = y\dot{y} + z\dot{z}$ ∴ $\dot{y}(t) = (a-k)y(t) + bz(t)$, $\dot{z}(t) = cy(t) - 2z(t)$ ∴

$\dot{V} = y\dot{y} + z\dot{z} = y((a-k)y + bz) + z(cy - 2z) =$

$(a-k)y^2 + bzy + czy - 2z^2 =$

$(a-k)y^2 + (b+c)zy - 2z^2 =$

$(a-k)y^2 + (b+c)zy - z^2 - z^2$

∴ $(b+c)zy - z^2 \leq \left(\frac{b+c}{2}\right)^2 y^2$ ∴

$\dot{V} = (a-k)y^2 + (b+c)zy - z^2 - z^2 \leq (a-k)y^2 + \left(\frac{b-c}{2}\right)^2 y^2 - z^2 = \left(a-k+\left(\frac{b+c}{2}\right)^2\right)y^2 - z^2$

\4c/ let $k \geq k^*$ ∴ $-k^* \geq -k$ ∴ $k^* \leq k$ ∴

$a - k^* \geq a - k$ ∴ $a - k \leq a - k^*$ ∴

$a - k + \left(\frac{b-c}{2}\right)^2 \leq a - k^* + \left(\frac{b-c}{2}\right)^2 \leq -1$ ∴

~~let $a - k + ($~~ $a + \left(\frac{b-c}{2}\right)^2 + 1 \leq k$ ∴ $a + \left(\frac{b-c}{2}\right)^2 + 1 \leq k^*$ ∴

$a + \left(\frac{b-c}{2}\right)^2 + 1 \leq k^* \leq k$ ∴ let $a + \left(\frac{b-c}{2}\right)^2 + 1 = k^*$ ∴

for $k^* \leq k$ ∴ $a + \left(\frac{b-c}{2}\right)^2 + 1 \leq k$ ∴ for large enough k:

$u(t) = -ky(t) \leq -\left(a + \left(\frac{b-c}{2}\right)^2 + 1\right)y(t)$ ∴

$V(y,z) = \frac{1}{2}y^2 + \frac{1}{2}z^2 \geq 0$ ∴ $V \to \infty$ as $\|(y,z)\| \to \infty$ ∴

V is radially unbounded, $V(0) = 0$, $V > 0$ $\forall (y,z) \neq 0$ ∴

V is positive definite ∴

$\dot{V}(0) = 0$, $\dot{V} < 0$ $\forall (x,y) \neq 0$ ∴ $\dot{V}$ is negative definite ∴

for $u(t) = -ky(t)$ ∴ origin is globally asymptotically stable ∴

$u(t) = -ky(t)$ is a stabilizing feedback.

\4d/ k^* ∴ $k \geq k^*$, $\dot{V} \leq -(y^2 + z^2) = -y^2 - z^2$ ∴

∴ let $W = V + \frac{1}{2}(k - k^*)^2$ ∴ $\dot{W} = \dot{V} + (k - k^*)(\dot{k} - \dot{k}^*)$ ∴ $\dot{k}^* = 0$ ∴

$\dot{W} = \dot{V} + (k - k^*)\dot{k} = \dot{V} + (k - k^*)y^2 \leq -y^2 - z^2 + (k - k^*)y^2 \leq -y^2 - z^2 = -(y^2 + z^2)$ ∴

$k - k^* \geq 0 \therefore \quad k \geq k^* \therefore$

$\dot{W} \leq -(y^2 + z^2) \therefore$

$\dot{W} \leq 0 \therefore$ $\exists$ a domain D such that it is an open set

$\therefore$ by LaSalle argument: D contained in basin of attraction

for ~~$x \leq 0$~~ $(y, z) = 0 \therefore$

with ~~the~~ V being; p.d.f, unbounded radially $\therefore$

~~$k(0) \geq 0$~~ $\therefore \frac{1}{2}(k - k^*) \geq 0 \therefore \quad W + V + \frac{1}{2}(k - k^*)^2 \geq 0 \therefore$

W is positive definite,

$\dot{W} \leq 0 \therefore \dot{W}$ is negative definite $\therefore$

$(y = 0, z = 0)$ is globally asymptotically stable $\therefore$

$\lim_{t \to \infty} y(t) = 0, \quad \lim_{t \to \infty} z(t) = 0$

$\therefore$ ~~the~~ $|\lim_{t \to \infty} k(t)| < \infty \therefore k(t)$ converges

$\because \dot{k}(t) = y^2 \therefore$

$\lim_{t \to \infty} y(t) = 0 \therefore \lim_{t \to \infty} y^2(t) = 0 \therefore \lim_{t \to \infty} \dot{k}(t) = 0 \therefore$

$k(t_1) = k(t_2)$ for $t_1 < t_2$ for large enough $t_1 \therefore$

$k(t) < \infty$ and $|\lim_{t \to \infty} k(t)| < \infty \therefore k(t)$ converges

\PP2017/ \1ai/ $Re(\lambda_i) = -\frac{1}{2} < 0$ for $\sqrt{1-4a(a+b)} \notin \mathbb{R}$

equilibrium is stable for $1-4a(a+b) < 0$ ∴

$4a(a+b) > 1$

if $0 < 4a(a+b) < 1$ ∴ $-1 < -4a(a+b) < 0$ ∴

$0 < 1-4a(a+b) < 1$ ∴

$0 < \sqrt{1-4a(a+b)} \leq 1$ ∴ $0 < \frac{\sqrt{1-4a(a+b)}}{2} < \frac{1}{2}$ ∴

$\lambda_i < 0$ ∴ stable node not a stable focus

if $4a(a+b) > 1$ ∴ $-4a(a+b) < -1$ ∴

$1-4a(a+b) < 0$ ∴ $\sqrt{1-4a(a+b)} \notin \mathbb{R}$ ∴

$\lambda_i \notin \mathbb{R}$ ∴ $Re(\lambda_i) = -\frac{1}{2} < 0$ ∴ stable focus not stable node

∴ False

\1aii/ $|x|$ is not continuous ∴

$f(x) = x^2 + |x|$ is not continuous ∴ $f(x) = x^2 + |x|$ is globally lipshitz ∴

False

\1aiii/ $\|A(t)\| \leq a$ for $t \in [t_0, t_1]$ ∴

∴ $A(t)$ is the linearised part of the dynamics

then the system has a unique solution ∴

True

\1aiv/ $V = x_1^2 + 2x_1x_2 + x_2^2 + 2x_2^2 = (x_1+x_2)^2 + 2x_2^2 \geq 0$ ∴

$V(0) = 0$, $V(x) > 0 \; \forall x \neq 0$ ∴ V is positive definite ∴

True

\1b/ $y = x_1$ ∴ $\dot{y} = \dot{x}_1 = x_2$ ∴ no explicit u ∴

$\ddot{y} = \dot{x}_2 = -x_1 + \varepsilon(1-x_1^2)x_2 + u$ ∴ explicit u

appears in 2nd derivative

∴ relative degree is 2 ∴

~~False~~

\1c/ Let $V = \frac{1}{2}kx_1^2 + \frac{1}{2}Mx_2^2$ ∴

$\dot{V} = kx_1\dot{x}_1 + Mx_2\dot{x}_2 = kx_1(x_2) + Mx_2\left(-\frac{k}{M}x_1 - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2|\right) =$

$kx_1x_2 - kx_1x_2 - c_1x_2^2 - c_2x_2^2|x_2| = -x_2^2(c_1 + c_2|x_2|) \leq 0$

∵ $c_1, c_2 > 0$, $|x_2| \geq 0$ ∴ $\dot{V}(0) = 0$, $\dot{V} < 0 \; \forall x \neq 0$ ∴

$\dot{V}$ is negative definite ∴

$V(0)=0$, ~~if~~ $V>0\ \forall x\neq 0$ ∴

V is positive definite ∴

origin is globally asymptotically stable

\|d/ ∴ $\frac{d}{dt}\left(\frac{1}{2}\left[x_1^2+(x_1+x_1^2+x_2)^2\right]\right)=$

$\frac{1}{2}\left[2x_1\dot{x}_1+2(x_1+x_1^2+x_2)(\dot{x}_1+2x_1\dot{x}_1+\dot{x}_2)\right]=$

$x_1(x_1^2+x_1x_2)+(x_1+x_1^2+x_2)(x_1^2+x_1x_2+2x_1(x_1^2+x_1x_2)+x_1^2+u)=$

$x_1^3+x_1^2x_2+(x_1+x_1^2+x_2)(x_1^2+x_1x_2+2x_1^3+2x_1^2x_2+x_1^2+u)=$

$\cancel{x_1^3+x_1^2x_2+(x_1+x_1^2+x}$ $x_1^3+x_1^2x_2+(x_1+x_1^2+x_2)(2x_1^2+x_1x_2+2x_1^3+2x_1^2x_2+u)\ Z_1=$

$\cancel{x_1^3+x_1^2x_2+x_2}$

$x_1^3+x_1^2x_2+2x_1^3+x_1^2x_2+2x_1^4+2x_1^3x_2+x_1u+x_1^3x_2+2x_1^5+2x_1^4x_2+x_1^2u+2x_1^4+2x_1^2x_2+x_1x_2^2+$

$2x_1^3x_2^2+2x_1^2x_2^2+x_2u=$

$3x_1^3+2x_1^2x_2+2x_1^4+4x_1^3x_2+x_1u+x_1^3x_2+2x_1^5+2x_1^4x_2+x_1^2u+2x_1^4+2x_1^2x_2+x_1x_2^2+$

$2x_1^2x_2^2+x_2u=$

$3x_1^3+4x_1^2x_2+4x_1^4+5x_1^3x_2+x_1u+2x_1^5+2x_1^4x_2+x_1^2u+x_1x_2^2+2x_1^2x_2^2+x_2u=$

$-x_1^4-(x_1+x_1^2+x_2)^2=-x_1^4-x_1^2-x_1^4-x_2^2-2x_1^3-2x_1x_2-2x_1^2x_2=$

$-2x_1^4-x_1^2-x_2^2-2x_1^3-2x_1x_2-2x_1^2x_2$ ∴

$x_1u+x_1^2u+x_2u=(x_1+x_2^2+x_2)u=$

$-2x_1^4-x_1^2-x_2^2-2x_1^3-2x_1x_2-2x_1^2x_2-3x_1^3-4x_1^2x_2-4x_1^4-5x_1^3x_2-2x_1^5-2x_1^4x_2-x_1x_2^2-2x_1^2x_2^2$

$=-6x_1^4-x_1^2-x_2^2-5x_1^3-6x_1^2x_2-2x_1x_2-5x_1^3x_2-2x_1^5-2x_1^4x_2-x_1x_2^2-2x_1^2x_2^2$ ∴

let $u=(x_1+x_2^2+x_2)^{-1}(-6x_1^4-x_1^2-x_2^2-5x_1^3-6x_1^2x_2-2x_1x_2-5x_1^3x_2-2x_1^5-2x_1^4x_2-x_1x_2^2-2x_1^2x_2^2)$

\|ei/ let $V(x_1,x_2)=\frac{1}{2}x_1^2+x_2^2$ ∴ $V\to\infty$ as $|x|\to\infty$,

$V(0)=0$, $V(x)>0\ \forall x\neq 0$ ∴ V is positive definite ∴

$\dot{V}=x_1\dot{x}_1+x_2\dot{x}_2=x_1(ax_1-x_1x_2)+x_2(bx_1^2-cx_2)=$

$ax_1^2-x_1^2x_2+bx_2x_1^2-cx_2^2=ax_1^2+(b-1)x_1^2x_2-cx_2^2\leq 0$ for $-x_2\geq 0$ ∴

$0\geq x_2$ ∴ for ~~D = {x|~~ $D=\{x\in\mathbb{R}\mid x_2\geq 0\}$

$\dot{V}(0)=0$, $\dot{V}<0\ \forall x\neq 0$ ∴ $\dot{V}$ is ~~positive~~ negative definite ∴

all trajectories starting in $x_2\geq 0$ stay in D from starting in D

\PP2017/ \1eii/ linearization : $\frac{\partial f}{\partial x}\Big|_{(0,0)} = \begin{bmatrix} a-x_2 & -x_1 \\ 2bx_1 & -c \end{bmatrix}\Big|_{(0,0)} =$

$\begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix} = A \quad \therefore$

$\det(A-\lambda I) = \det\begin{bmatrix} a-\lambda & 0 \\ 0 & -c-\lambda \end{bmatrix} = (a-\lambda)(-\lambda-c) = 0 \therefore$

$a=\lambda, \quad \lambda=-c \therefore \qquad a, c > 0 \therefore$

$\lambda = -c < 0, \quad \lambda = a > 0 \quad \therefore$

$(0,0)$ is a saddle point $\therefore$ ~~not asstable~~ not an unstable node or spiral $\therefore$

by P.B. criterion : ~~not~~ the system has no periodic orbits for $x \in D$

\2a/ $\dot{x} = Ax + Bu \therefore \dot{\tilde{x}} = T^{-1}\dot{x} = T^{-1}(Ax+Bu) =$

$T^{-1}Ax + T^{-1}Bu = (T^{-1}AT)\tilde{x} + (T^{-1}B)u$

for $x = T\tilde{x} \therefore \dot{x} = T\dot{\tilde{x}} \therefore T^{-1}\dot{x} = \dot{\tilde{x}} = T^{-1}(Ax+Bu) =$

$T^{-1}Ax + T^{-1}Bu = T^{-1}AT\tilde{x} + T^{-1}Bu \quad \therefore (T^{-1}AT, \ T^{-1}B, \ CT^{-1})$

for (A, B, C), $T^{-1}AT = \tilde{A}$, $T^{-1}B = \tilde{B}$, $CT^{-1} = \tilde{C}$

$\therefore x = T\tilde{x} \therefore T^{-1}x = \tilde{x} \therefore T^{-1}\dot{x} = \dot{\tilde{x}} \therefore$

$\dot{\tilde{x}} = T^{-1}\dot{x} = T^{-1}(Ax+Bu) = T^{-1}Ax + T^{-1}Bu = T^{-1}AT\tilde{x} + T^{-1}Bu =$

$\tilde{A}\tilde{x} + \tilde{B}u$

\2bi/ $\det(P-\lambda I) = \det\begin{bmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{bmatrix} = (\lambda-2)(\lambda-1) - 1(1) =$

$\lambda^2 + 2 - 3\lambda - 1 = \lambda^2 - 3\lambda - 1$

$\lambda = \frac{3 \pm \sqrt{9-4(1)(-1)}}{2(1)} = \frac{3 \pm \sqrt{13}}{2} \therefore \lambda_{min} = -0.303 < 0, \quad \lambda_{max} = 3.303 > 0$

$\det P = \det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = 2(1) - 1(1) = 2 - 1 = 1 > 0 \quad \therefore P = P^T \therefore$

$\therefore \lambda_{max}(P) = 3.303 > 0 \therefore$ P is positive definite

$BK = \begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} 1, & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} \therefore A - BK = \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ -1 & -2 \end{bmatrix} \therefore$

$(A-BK)^T = \begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix} \quad \therefore \quad (A-BK)^T P + P(A-BK) =$

$\begin{bmatrix} 0 & -1 \\ 0 & -2 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -1 \\ -2 & -2 \end{bmatrix} + \begin{bmatrix} 0 & -1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} -1 & -2 \\ -2 & -3 \end{bmatrix} = -Q = \begin{bmatrix} -Q_1 & -Q_3 \\ -Q_2 & -Q_4 \end{bmatrix}$

$\therefore -1 = -Q_1 \therefore Q = 1, \quad -2 = -Q_3 = -Q_2 \therefore Q_3 = Q_2 = 2, \quad -Q_4 = -3 \therefore Q_4 = 3 \therefore$

$Q = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \quad \therefore$

$\det(Q - \lambda I) = \det\begin{pmatrix} 1-\lambda & 2 \\ 2 & 3-\lambda \end{pmatrix} = (1-\lambda)(3-\lambda) - 2(2) = 3 + \lambda^2 - 3\lambda - \lambda + \cancel{3}\,4$

$\lambda^2 - 4\lambda - 1 \quad \therefore$

$\lambda = \dfrac{4 \pm \sqrt{16 - 4(1)(-1)}}{2(1)} = \dfrac{4 \pm \sqrt{20}}{2} \quad \therefore$

$\lambda_{min} = 2 - \sqrt{5}, \quad \lambda_{max} = 2 + \sqrt{5} \quad \therefore \quad Q = Q^T \quad \therefore$

Q is positive definite

$\therefore$ by linearization the system is stable at the origin

\2bii/ $\dot{V} = \dot{x}^T P x + x^T P \dot{x} =$

$(Ax + (B+x)u)^T P x + x^T P (Ax + (B+x)u) =$

$(x^T A^T + u^T (B+x)^T) P x + x^T P A x + x^T P (B+x) u =$

$x^T A^T P x + u^T (B+x)^T P x + x^T P A x + x^T P (B+x) u =$

$x^T (A^T P + PA) x + u^T (B+x)^T P x + x^T P (B+x) u =$

$-x^T Q x (1 - 2|Kx|) = -x^T \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} (1 - 2 | [1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} |) =$

$-x^T \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 3x_2 \end{bmatrix} (1 - 2|x_1 + x_2|) = -[x_1 \quad x_2] \begin{bmatrix} x_1 + 2x_2 \\ 2x_1 + 3x_2 \end{bmatrix} (1 - 2|x_1 + x_2|) =$

$-(x_1^2 + 2x_1 x_2 + 2x_1 x_2 + 3x_2^2)(1 - 2|x_1 + x_2|) \quad \therefore$

$\dot{V} = x^T (A^T P + PA) x + u^T (B+x)^T P x + x^T P (B+x) u =$

$x^T (A^T P + PA) x + 2x^T P (B+x) u \le -x^T Q x (1 - 2|Kx|)$

\2biii/ P is positive definite $\therefore$

$V = x^T P x \quad \therefore$ quadratic form $\therefore$ V is positive definite

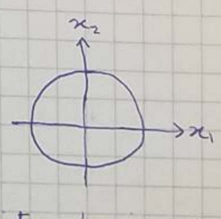
$\therefore \dot{V} \le -x^T Q x (1 - 2|Kx|) \le 0$ for $1 - 2|Kx| \ge 0 \quad \therefore \quad 1 \ge 2|Kx| \quad \therefore$

$|Kx| \le \frac{1}{2} \quad \therefore \quad Kx = [1 \quad 1] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 + x_2 \quad \therefore$

$|x_1 + x_2| \le \frac{1}{2} \quad \therefore \quad \dot{V}$ is negative definite for

$D = \{ x \in \mathbb{R}^2 \mid |x_1 + x_2| \le \frac{1}{2} \} \quad \therefore$

$\Omega = \{ x : x^T P x \le \alpha \} \quad \therefore$



\3a/ Let $\Omega \subset D$ be a compact set that is positively invariant with respect to $\dot{x} = f(x)$. Let $V: D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V} \le 0$ in Ω. Let E be the set of all points in Ω where $\dot{V}(x) = 0$ let M be the largest invariant set in E.

PP2020 Then every sol starting in Ω approaches M as $t \to \infty$

3b $\dot{x} = -b(x)$ ∴ let $V = \frac{1}{2}x^2$ ∴

~~$\dot{V} = x\dot{x}$~~ $V(0)=0$, $V \to \infty$ as $|x| \to \infty$,

$V(x) > 0 \ \forall x \neq 0$ ∴ V is radially unbounded, positive definite

∴ $\dot{V} = x\dot{x} = x(-b(x)) = -bx = -xb \leq 0$ ∵ $bx \geq 0$ ∴

∴ $\dot{V}(0) = 0$, $\dot{V} < 0 \ \forall x \neq 0$ ∴ $\dot{V}$ is negative definite

∴ the origin is an globally asymptotically stable equilibrium point ∴ $\dot{x} = -b(x)$ ~~∴~~

$\ddot{x} = -b(\dot{x}) - c(x)$ ∴

$V(0) = \frac{1}{2}\dot{x}^2 + \int_0^0 c(s)\,ds = \frac{1}{2}\dot{x}^2$

$\dot{V} = \dot{x}\ddot{x} + c(x) = \dot{x}(-b(\dot{x}) - c(x)) + c(x)\dot{x} = -\dot{x}\,b(\dot{x}) - \dot{x}c(x) + \dot{x}c(x)$

$= -\dot{x}\,b(\dot{x}) \leq 0$ ∵ $xb(x) > 0$ ∴ $\dot{x}b(\dot{x}) > 0$ ∴

$\dot{V}(0) = 0$ ∴ $\dot{V}$ is negative definite ∴

$V(0) = 0$, $V > 0 \ \forall x \neq 0$ ∴ V is semi positive definite ∴

the origin $(x, \dot{x}) = (0,0)$ is asymptotically stable ∵

∴ $\ddot{x} = -b(\dot{x}) - c(x)$ ∴ $\ddot{x} = 0 = -b(\dot{x}) - C(x)$ ∴

$b(\dot{x}) = c(x)$ ∴ for $(x, \dot{x}) = (0,0)$ ∴ $\ddot{x} = 0$ ∴

$(x, \dot{x}) = (0,0)$ is an equilibrium

3d globally asymptotically stable is

V is positive definite ∴ $V(0) = 0$ and

$V(x) > 0 \ \forall x \neq 0$ ∴ $\int_0^x c(s)\,ds \geq 0$

4a transfer function: $G(s) = C(sI - A)^{-1}B$ for (A, B, C) ∴

∴ for (TAT^{-1}, TB, CT^{-1}):

$G(s) = CT^{-1}(sI - TAT^{-1})^{-1}TB = C(sI - A)^{-1}B$ ∴

both triples have equal transfer functions

~~4a~~ ∴ $G(s) = CT^{-1}(sI_n - TAT^{-1})^{-1}TB =$

$CT^{-1}(sTT^{-1} - TAT^{-1})^{-1}TB = CT^{-1}((sT - TA)T^{-1})^{-1}TB =$

$CT^{-1}((Ts - TA)T^{-1})^{-1}TB = CT^{-1}(TsI - TA)T^{-1})^{-1}TB =$

$CT^{-1}(T(sI - A)T^{-1})^{-1}TB = CT^{-1}(((T)^{-1})^{-1}(sI - A)^{-1}(T^{-1})^{-1})TB =$

$CT^{-1}(T(SI_n-A)^{-1}T^{-1})TB = CT^{-1}T(SI_n-A)^{-1}T^{-1}TB =$

$C(SI_n-A)^{-1}B = G(s)$ is equal to transfer function of (A,B,C) triple

\4a/ An linear time invariant system is controllable if, $\forall$ $x^*(t)$ and every finite $T>0$ $\exists$ an input function $\mu(t)$, $0<t\le T$ such that the system state goes from $x(0)=0$ to $x(T)=x^*$

System is controllable if the matrix pair (A,B) has controllability matrix $M=[B \vdots AB \vdots A^2B \vdots \cdots \vdots A^{n-1}B]$ is full rank $\therefore$ rank$(M)=n$

\4bi/ $\dot{x}_1 = x_1 - x_1^3 + x_2 = x_1 - x_1^3 - (k+1)x_1 = x_1 - x_1^3 - kx_1 - x_1 = -x_1^3 - kx_1$ $\therefore$

let $V(x_1) = \frac{1}{2}x_1^2$ $\therefore$

$\dot{V} = x_1\dot{x}_1 = x_1(-x_1^3 - kx_1) = -x_1^2(x_1^2+k) \le 0$ $\therefore$

$\because$ $x_1^2 \ge 0$, $k>0$ $\therefore$

$V(0)=0$, $V \to \infty$ as $|x_1| \to \infty$, $V(x_1)>0$ $\forall x_1 \ne 0$ $\therefore$

V is radially unbounded and positive definite $\therefore$

$\dot{V}(0)=0$, $\dot{V}<0$ $\forall x_1 \ne 0$ $\therefore$ $\dot{V}$ is negative definite $\therefore$

origin $x_1=0$ is a globally stable equilibrium

\4bii/ $\therefore x_2 = z_2 + \phi(x_1) = z_2 + (-(k+1)x_1) = z_2 - kx_1 - x_1$ $\therefore$

$\dot{x}_1 = x_1 - x_1^3 + x_2 = x_1 - x_1^3 + z_2 - kx_1 - x_1 = -x_1^3 + z_2 - kx_1$

$z_2 = x_2 - \phi(x_1) = x_2 - (-(k+1)x_1) = x_2 + kx_1 + x_1$ $\therefore$

$\dot{z}_2 = \dot{x}_2 + k\dot{x}_1 + \dot{x}_1 = \cancel{u+k(x_1-x_1^3+x}$

$u + k(-x_1^3 + z_2 - kx_1) + -x_1^3 + z_2 - kx_1 =$

$u - kx_1^3 + kz_2 - k^2x_1 - x_1^3 + z_2 - kx_1 = \cancel{-kx_1^2 - x_1^4 - cz_2^2}$ $\therefore$ let:

$\cancel{u = 2kx_1^2 - x_1^4 - cz_2^2 + kx_1^3 - kz_2 + k^2x_1 + x_1^3 - z_2 + kx_1}$ $\therefore$

let $V_c(x_1, z_2) = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$ $\therefore$ $V_c(0)=0$, $V \to \infty$ as $|(x_1,z_2)| \to \infty$, $V_c > 0$ $\forall (x_1,z_2) \ne 0$ $\therefore$ V_c is p.d.f. $\therefore$

$\dot{V}_c = x_1\dot{x}_1 + z_2\dot{z}_2 = x_1(-x_1^3 + z_2 - kx_1) + z_2(u - kx_1^3 + kz_2 - k^2x_1 - x_1^3 + z_2 - kx_1) =$

$-x_1^4 + x_1z_2 - kx_1^2 + z_2u - kx_1^3z_2 + kz_2^2 - k^2x_1z_2 - x_1^3z_2 + z_2^2 - kx_1z_2 = -kx_1^2 - x_1^4 - cz_2^2$ $\therefore$

let $z_2u = -kx_1^2 - x_1^4 - cz_2^2 + x_1^4 - x_1z_2 + kx_1^2 + kx_1^3z_2 - kz_2^2 + k^2x_1z_2 + x_1^3z_2 - z_2^2 + kx_1z_2 =$

$-cz_2^2 - x_1z_2 + kx_1^3z_2 - kz_2^2 + k^2x_1z_2 + x_1^3z_2 - z_2^2 + kx_1z_2$ $\therefore$ $u = -cz_2 - x_1 + kx_1^3 - kz_2 + k^2x_1 + x_1^3 - z_2 + kx_1$ $\therefore$

$\dot{V}_c \le 0$ $\therefore$ $\dot{V}_c$ is n.d.f. $\therefore$ origin is globally asymp stable

\PP2018/

\2a/ Transfer function is $G(s) = C(sI-A)^{-1}B$ for (A, B, C)

$\therefore$ for (TAT^{-1}, TB, CT^{-1}):

$G(s) = CT^{-1}(sI - TAT^{-1})^{-1}TB =$

$CT^{-1}(TsIT^{-1} - TAT^{-1})^{-1}TB =$

$CT^{-1}(T(sI-A)T^{-1})^{-1}TB = CT^{-1}(T(sI-A)T^{-1})^{-1}TB =$

$C(sI-A)TB = G_{T(A,B,C)}(s)$ is same transfer function for (A,B,C)

\2b/ controllability matrix: $M = [B \vdots AB] = \left[\begin{bmatrix}0\\1\end{bmatrix} \vdots \begin{bmatrix}-3 & 1\\17 & 10\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right] =$

$\left[\begin{bmatrix}0\\1\end{bmatrix} \vdots \begin{bmatrix}1\\10\end{bmatrix}\right] = \begin{bmatrix}0 & 1\\1 & 10\end{bmatrix} \therefore$

$\det(M) = 0(10) - 1(1) = -1 \neq 0 \therefore$ rank$(M) = 2 \therefore$ full rank $\therefore$ pair (A, B) is controllable.

\2bii/ let $K = [k_1 \quad k_2] \therefore$

$A_{cL} = A - BK = \begin{bmatrix}-3 & 1\\17 & 10\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}[k_1 \quad k_2] = \begin{bmatrix}-3 & 1\\17 & 10\end{bmatrix} - \begin{bmatrix}0 & 0\\k_1 & k_2\end{bmatrix} = \begin{bmatrix}-3 & 1\\-k_1+17 & -k_2+10\end{bmatrix}$

$\therefore (s+3.618)(s+1.382) = s^2 + 5.000076 + 5s =$

$\det(sI - A_{cL}) = \det\left(\begin{bmatrix}s & 0\\0 & s\end{bmatrix} - \begin{bmatrix}-3 & 1\\-k_1+17 & -k_2+10\end{bmatrix}\right) = \det\left(\begin{bmatrix}s+3 & -1\\k_1-17 & s+k_2-10\end{bmatrix}\right) =$

$(s+3)(s+k_2-10) - (k_1-17)(-1) =$

$s^2 + (k_2-10)s + 3s + (k_1-17) = s^2 + (k_2-7)s + (k_1-17) \therefore$

$k_1 - 17 = 5.000076 \therefore k_1 = 22.000076$,

$k_2 - 7 = 5 \therefore k_2 = 12 \quad \therefore \quad K = [22.000076 \quad 12] \therefore$

$u = -[22.000076 \quad 12]x = -[22 \quad 12]x = [-22 \quad -12]x$ (3.d.p)

\2biii/ let $V = x^T P x \therefore P = \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} = P^T \therefore P$ is positive definite

$\therefore \lambda(P) = 1 \quad \therefore \quad V = x^T P x$ is in quadratic form $\therefore$

V is positive definite $\therefore$

$\dot V = \dot x^T P x + x^T P \dot x \quad \therefore V = [x_1 \quad x_2]\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} = [x_1 \quad x_2]\begin{bmatrix}x_1\\x_2\end{bmatrix} = x_1^2 + x_2^2 \therefore$

$A - BK = \begin{bmatrix}-3 & 1\\17 & 10\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}[22 \quad 12] = \begin{bmatrix}-3 & 1\\17 & 10\end{bmatrix} - \begin{bmatrix}0 & 0\\22 & 12\end{bmatrix} = \begin{bmatrix}-3 & 1\\-5 & -2\end{bmatrix} = A_{cL} \therefore$

$\dot x = \begin{bmatrix}-3 & 1\\-5 & -2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix} + \begin{bmatrix}x_1 x_2\\0\end{bmatrix} = \begin{bmatrix}-3x_1 + x_2\\-5x_1 - 2x_2\end{bmatrix} + \begin{bmatrix}x_1 x_2\\0\end{bmatrix} \Rightarrow \begin{bmatrix}-3x_1 + x_2 + x_1 x_2\\-5x_1 - 2x_2\end{bmatrix} = \begin{bmatrix}\dot x_1\\\dot x_2\end{bmatrix}$

$\therefore \dot x_1 = -3x_1 + x_2 + x_1 x_2, \quad \dot x_2 = -5x_1 - 2x_2 \quad \therefore$

$\dot V = 2x_1\dot x_1 + 2x_2 \dot x_2 = 2x_1(-3x_1 + x_2 + x_1 x_2) + 2x_2(-5x_1 - 2x_2) =$

~~Sub~~ $2[-3x_1^2 + x_1x_2 + x_1^2x_2 - 5x_1x_2 - 2x_2^2] =$

$2[-3x_1^2 - 4x_1x_2 + x_1^2x_2 - 2x_2^2] = 2[-2x_1^2 - 4x_1x_2 - 2x_2^2 - x_1^2 + x_1^2x_2] =$

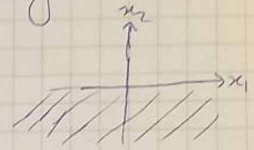
$-4[x_1^2 + 2x_1x_2 + x_2^2] + 2[-x_1^2 + x_1^2x_2] =$

$-4(x_1+x_2)^2 + 2[-x_1^2 + x_1^2x_2] =$

$-4(x_1+x_2)^2 - 2x_1^2 - 2x_1^2(-x_2) \le 0$ for $-x_2 \ge 0$ ∴ $x_2 \le 0$

∴ $\dot V(0) = 0$ ∴ for $x_2 \le 0$: $\dot V < 0$ ∴ $\dot V$ is locally negative definite ∴

origin is ~~locally~~ asymptotically stable



\2iv/ ∴ for $x_2 \le 0$ ∴ $D = \{x \mid x_2 \le 0\}$ is invariant set on which V is positive definite and $\dot V$ is negative definite

\3a/ Asymptot stable: is Lyapunov Stable ∴

$\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ st $\|x(0)\| < \delta$ ∴ $\|x(t)\| < \varepsilon$, $\forall t \ge 0$ ∴

$\|x(0)\| < \delta$ ∴ $\|x(t)\| < \varepsilon$ $\forall t \ge 0$

∴ Asymp Stable: ~~if~~ Lyapunov stable and $\exists \delta > 0$ st if

$\|x(0)\| < \delta$, ∴ $\lim_{t \to \infty} x(t) = x_e = 0$ ∴

$\|x(0)\| < \delta$ ∴ $\lim_{t \to \infty} x(t) = 0$

$\|x(0)\| < \delta$ ∴ $\|x(t)\| \le \alpha \|x(0)\| e^{-\beta t}$, $t \ge 0$

\3bi/ $M(q)\ddot q = u - C(q,\dot q)\dot q - D\dot q - g(q)$ ∴

$\ddot q = \frac{1}{M(q)}u - \frac{1}{M(q)}C(q,\dot q)\dot q - \frac{1}{M(q)}D\dot q - \frac{1}{M(q)}g(q)$ ∴

let $x_1 = q$, $x_2 = \dot q$ ∴ $\dot x_1 = \dot q = x_2$ ∴

$\dot x_2 = \ddot q = \frac{1}{M(x_1)}u - \frac{1}{M(x_1)}C(x_1,x_2)x_2 - \frac{1}{M(x_1)}Dx_2 - \frac{1}{M(x_1)}g(x_1)$

\3bii/ $V = \frac{1}{2}\dot q^T M(q)\dot q + P(q)$ skew symmet: $A^T = -A$ ∴

is p.d. & $\dot q^T M(q)\dot q$ quadratic form   P is p.d.s ∴

$\dot V = \frac{1}{2}\ddot q^T M(q)\dot q + \frac{1}{2}\dot q^T M(q)\ddot q + \frac{\partial P(q)}{\partial q}\dot q + \frac{1}{2}\dot q^T \dot M(q)\dot q$

∴ when $u = 0$, $g(q=0) = g(0) = 0$ ∴

$M(q)\ddot q = -C(q,\dot q) - D\dot q$ ∴ $\ddot q = -\frac{1}{M(q)}C(q,\dot q) - \frac{D}{M}\dot q$

∴ $\frac{\partial P(q)}{\partial q} = g(q)$ ∴ $\dot V = -\dot q^T C$

\PP2016/ \1ai/ Q_1 and Q_3 are stable nodes but Q_2 is a saddle point $\therefore$ is not a stable node $\therefore$ False

\1aii/ $\lambda_1(A) \neq 0 \therefore \lambda_2(A) \neq 0 \therefore$ not all of the systems ~~isolated~~ equilibriums are isolated $\therefore$ the system has some equilibrium subspace $\therefore$ True

\1aiii/ $V = x^T P x$ is in quadratic form $\therefore$ leading principle minors of P are positive $\therefore$ P is positive definite $\therefore$ V is positive definite $\therefore$ True

\1aiv/ $\lim_{x_2 \to \infty} V = \infty$, $\lim_{x_1 \to \infty} V = \infty$ $\therefore V \to \infty$ as $|x| \to \infty \therefore$ V is radially unbounded $\therefore$ True

\1b/ let $x, y \in I \therefore |f(x) - f(y)| = \left|\frac{1}{x^2} - \frac{1}{y^2}\right| \therefore$

$\frac{1}{x^2} - \frac{1}{y^2} = \cancel{\frac{y^2+x^2}{x^2y^2}} = \frac{y^2}{x^2y^2} - \frac{x^2}{x^2y^2} = \frac{y^2-x^2}{x^2y^2} = \frac{(y+x)(y-x)}{x^2y^2} \therefore$

$|f(x) - f(y)| = \left|\frac{(y+x)(y-x)}{x^2y^2}\right| = \frac{1}{|x^2y^2|}|y+x||y-x| \therefore$

$y, x \leq 2 \therefore$ $\cancel{|y+x| \leq 2}$ $y + x \leq 4 \therefore |y+x| \leq 4 \therefore$

$x, y \geq \frac{1}{2} \therefore \frac{1}{x}, \frac{1}{y} \leq 2 \therefore \frac{1}{x^2}, \frac{1}{y^2} \leq 4 \therefore \frac{1}{|x^2|}, \frac{1}{|y^2|} \leq 4 \therefore$

$\cancel{\frac{1}{x^2y^2} \leq 16} \; \frac{1}{|x^2y^2|} = \frac{1}{|x^2||y^2|} = \frac{1}{|x^2|}\frac{1}{|y^2|} \leq 4 \times 4 = 16 \therefore$

$|f(x) - f(y)| = \frac{1}{|x^2||y^2|}|y+x||y-x| \leq \cancel{16 \times 4}\; 16 \times 4|y-x| = 64|y-x| = 64|x-y| \therefore f(x) = \frac{1}{x^2}$ is lipschitz continuous ~~with~~ on I with $L_f = 64 \therefore$

$|f(x) - f(y)| = 64|\cancel{x-y}| \leq 79|x-y| \therefore$ the lipschitz continuous is still valid for $L_f = 79$

\1c/ $\frac{d}{dt}\left(\frac{1}{2}(x_1^2 + x_2^2)\right) = \frac{1}{2}\frac{d}{dt}(\|x\|^2) = \frac{1}{2}(2x_1\dot{x}_1 + 2x_2\dot{x}_2) =$

$x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(-x_1 + x_2x_1^3) + x_2(x_1^3 + x_1x_2^2 + u) =$

$-x_1^2 + x_1^4x_2 + x_1^3x_2 + x_1x_2^3 + x_2u = -2(x_1^2 + x_2^2) = -2\|x\|^2 = -2x_1^2 - 2x_2^2 \therefore$

$x_2u = -2x_1^2 - 2x_2^2 + x_1^2 - x_1^4x_2 - x_1^3x_2 - x_1x_2^3 = -x_1^2 - 2x_2^2 - x_1^4x_2 - x_1^3x_2 - x_1x_2^3 \therefore$

let $u = x_2^{-1}(-x_1^2 - 2x_2^2 - x_1^4x_2 - x_1^3x_2 - x_1x_2^3) = -x_1^2x_2^{-1} - 2x_2 - x_1^4 - x_1^3 - x_1x_2^2 \therefore$

$\frac{\frac{d}{dt}(\|x\|^2)}{\|x\|} = -4 \therefore \int \frac{\frac{d}{dt}(\|x\|^2)}{\|x\|^2}dt = \int -4\,dt = -4t + C_1 = \ln|\|x\|^2| = \ln\|x\|^2 \therefore$

$\|x\|^2 = e^{-4t+C_1} = e^{-4t}C_2 \therefore$

~~$\|x(0)\|^2$~~ $\therefore \|x\| = x_1^2 + x_2^2 = \|V\| \therefore$

$V(0) = e^0 C_2 = C_2 \doteq \|x(0)\|^2 \quad \therefore \|x\|^2 = e^{-4t}\|x(0)\|^2 \quad \therefore \|x\| = e^{-2t}\|x(0)\| \therefore$

$\|V\|^2 = e^{-4t}\|V(0)\|^2 \quad \therefore \|V\| = e^{-2t}\|V(0)\|$

\d/ Let $V(\omega) = \frac{1}{2}\sum_{j=1}^{3} J_i\omega_i^2 = \frac{1}{2}J_1\omega_1^2 + \frac{1}{2}J_2\omega_2^2 + \frac{1}{2}J_3\omega_3^2 \therefore$

$V(0) = 0, \quad V > 0 \ \forall \omega \neq 0 \therefore$

$V \to \infty$ as $|\omega| \to \infty \therefore V$ is radially unbounded, positive definite $\therefore$

$\dot V = J_1\omega_1\dot\omega_1 + J_2\omega_2\dot\omega_2 + J_3\omega_3\dot\omega_3$

$u_1 = -k_1\omega_1, \quad u_2 = -k_2\omega_2, \quad u_3 = -k_3\omega_3 \therefore$

$J_1\dot\omega_1 = (J_2 - J_3)\omega_2\omega_3 - k_1\omega_1$

$J_2\dot\omega_2 = (J_3 - J_1)\omega_3\omega_1 - k_2\omega_2$

$J_3\dot\omega_3 = (J_1 - J_2)\omega_1\omega_2 - k_3\omega_3 \therefore$

$\dot V = \omega_1[(J_2 - J_3)\omega_2\omega_3 - k_1\omega_1] + \omega_2[(J_3 - J_1)\omega_3\omega_1 - k_2\omega_2] + \omega_3[(J_1 - J_2)\omega_1\omega_2 - k_3\omega_3]$

$= [(J_2 - J_3) + (J_3 - J_1) + (J_1 - J_2)](\omega_1\omega_2\omega_3) - k_1\omega_1^2 - k_2\omega_2^2 - k_3\omega_3^2 =$

$-(k_1\omega_1^2 - k_2\omega_2^2 - k_3\omega_3^2) \leq 0 \therefore$

$\dot V(0) = 0, \quad \dot V < 0 \ \forall \omega \neq 0 \therefore \dot V$ is negative definite $\therefore$

The origin is globally asymptotically stable

\e/ Let $V = x_1^2 + x_2^2 \therefore \quad \varepsilon > 0 \therefore \quad V$ is positive definite $\therefore$

$\dot V = 2x_1\dot x_1 + 2x_2\dot x_2 = 2x_1x_2 + 2x_2(-x_1 + \varepsilon x_2(1 - x_1^2 - x_2^2)) =$

$\underbrace{2x_1x_2 - 2x_1x_2} + 2\varepsilon x_2^2(1 - x_1^2 - x_2^2) = 2\varepsilon x_2^2(1 - x_1^2 - x_2^2) \leq 0$ for

$1 - x_1^2 - x_2^2 = 1 - (x_1^2 + x_2^2) = 1 - V \leq 0 \therefore \quad V \geq 1 \therefore$

$M = \{x \mid V \leq 1\}$ such that all trajectories starting in M,

stay in M $\forall$ future time $\therefore M$ has $(0,0)$ is an equilibrium isolated inside

linearization: $\frac{\partial f}{\partial x}\Big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & \varepsilon \end{bmatrix}\Big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & \varepsilon \end{bmatrix} \doteq A$

$\det(A - \lambda I) = \det\begin{pmatrix} -\lambda & 1 \\ -1 & \varepsilon-\lambda \end{pmatrix} = \lambda(\lambda - \varepsilon) - 1(-1) = \lambda^2 - \varepsilon\lambda + 1 \therefore$

$\lambda = \frac{\varepsilon \pm \sqrt{\varepsilon^2 - 4(1)(1)}}{2(1)} \therefore$ if $\lambda \notin \mathbb{R}$: $\mathrm{Re}(\lambda) = \frac{\varepsilon}{2} > 0$,

if $\sqrt{\varepsilon^2 - 4} \in \mathbb{R}$ then $\lambda_1 = \frac{\varepsilon + \sqrt{\varepsilon^2 - 4}}{2} > 0 \quad \therefore$ origin is not a stable equilibria

$\wedge$ by P.B criterion: $\exists$ the system has a periodic orbit in M

\ / $2x_1(-5x_1 + x_1 x_2^2) + 2x_2(-2x_2 + 3x_1) =$

$-10x_1^2 + 2x_1^2 x_2^2 - 4x_2^2 + 6x_2 x_1 =$

$-7x_1^2 + 2x_1^2 x_2^2 - x_2^2 - 3x_1^2 + 6x_1 x_2 - 3x_2^2 =$

$-7x_1^2 + 2x_1^2 x_2^2 - x_2^2 - 3(x_1^2 - 2x_1 x_2 + x_2^2) =$

$-7x_1^2 + 2x_1^2 x_2^2 - x_2^2 - 3(x_1 - x_2)^2 =$

$x_1^2(-7 + 2x_2^2) - x_2^2 - 3(x_1 - x_2)^2 =$

$-x_1^2(7 - 2x_2^2) - x_2^2 - 3(x_1 - x_2)^2 \leq 0$ for $7 - 2x_2^2 \geq 0$ ∴

$7 \geq 2x_2^2$ ∴ $\frac{7}{2} \geq x_2^2$ ∴ $\frac{7}{2} \geq x_1^2 + x_2^2 = V$ ∴ $D = \{x \mid V(x) \leq \frac{7}{2}\}$

is invariant set where V is p.d.f and $\dot{V}$ is negative definite

∴ locally asymp stable

\ ie / let $V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ ∴

$\dot{V} = x_1 \dot{x}_1 + x_2 \dot{x}_2 = x_1(ax_1 - x_1 x_2) + x_2(bx_1^2 - cx_2) = ax_1^2 - x_1^2 x_2 + bx_1^2 x_2 - cx_2^2 =$

$ax_1^2 + (b-1)x_1^2 x_2 - cx_2^2$

$D = \{x \in \mathbb{R}^2 \mid x_2 \geq 0\}$ ∴

on x_1 axis : $\dot{x}_2 = bx_1^2 - c(0) = bx_1^2 \geq 0$ ∴

trajectories starting in D can leave it.

\ ie ii / equilibriums : $\dot{x}_1 = ax_1 - x_1 x_2 = 0 = x_1(a - x_2)$ ∴

$x_1 = 0$ or $a - x_2 = 0$ ∴ $x_2 = a$ ∴

$\dot{x}_2 = 0 = bx_1^2 - cx_2 = b(0)^2 - cx_2 = -cx_2 = 0 = x_2$ ∴

$(0,0)$ or

$\dot{x}_2 = bx_1^2 - cx_2 = bx_1^2 - ca = 0$ ∴ $bx_1^2 = ca$ ∴ $x_1^2 = \frac{ca}{b}$ ∴

$x_1 = \sqrt{\frac{ca}{b}}$, $x_1 = -\sqrt{\frac{ca}{b}}$ ∴ $(a, \sqrt{\frac{ca}{b}})$, $(a, -\sqrt{\frac{ca}{b}})$ ∴

linearisation : $\frac{\partial f}{\partial x} = \begin{bmatrix} a - x_2 & -x_1 \\ 2bx_1 & -c \end{bmatrix} = A$ ∴

$A|_{(0,0)} = \begin{bmatrix} a & 0 \\ 0 & -c \end{bmatrix}$ ∴ $\det(A|_{(0,0)} - \lambda I) = \det\left(\begin{bmatrix} a-\lambda & 0 \\ 0 & -c-\lambda \end{bmatrix}\right) =$

$(a-\lambda)(-c-\lambda) = 0$ ∴ $a - \lambda = 0$ ∴ $\lambda = a > 0$ $-c - \lambda = 0$ ∴ $\lambda = -c < 0$

∴ saddle point

$A|_{(a, \sqrt{\frac{ca}{b}})} = \begin{bmatrix} a - \sqrt{\frac{ca}{b}} & -a \\ 2ba & -c \end{bmatrix}$ ∴ $\det(A|_{(a, \sqrt{\frac{ca}{b}})} - \lambda I) = \det\begin{bmatrix} a - \sqrt{\frac{ca}{b}} - \lambda & -a \\ 2ba & -c-\lambda \end{bmatrix}$

$= (a - \sqrt{\frac{ca}{b}} - \lambda)(-c-\lambda) + a(2ba)$ $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = a - x_2 - c \leq a - c < 0$ $\forall x \in D$

∴ none of the equilibriums in D are unstable nodes or spirals equilib at $y = \frac{Mg}{k}$, $\dot{y} = 0$ $\therefore$

let $x_2 = \dot{y}$ has it at $x_2 = 0$

$\therefore$ for origin: $y - \frac{Mg}{k} = 0 = x_1$ $\therefore$

let $x_1 = y - \frac{Mg}{k}$, $x_2 = \dot{y}$ has $(0,0)$ as equilib

\3bi/ $\dot{x}_1 = x_1^2 + x_2 = x_1^2 + (-x_1^2 - x_1) = x_1^2 - x_1^2 - x_1 = -x_1$ $\therefore$

let $V(x_1) = \frac{1}{2}x_1^2$ $\therefore$ $\dot{V} = x_1\dot{x}_1 = x_1(-x_1) = -x_1^2 \leq 0$ $\therefore$ n.d. $\therefore$ globally asymp stable

\3bii/ $x_2 = z_2 + \phi(x_1) = z_2 + (-x_1^2 - x_1) = z_2 - x_1^2 - x_1$ $\therefore$

$\dot{x}_1 = x_1^2 + x_2 = x_1^2 + z_2 - x_1^2 - x_1 = z_2 - x_1$ $\therefore$

$z_2 = x_2 - \phi(x_1) = x_2 - (-x_1^2 - x_1) = x_2 + x_1^2 + x_1$ $\therefore$

$\dot{z}_2 = \dot{x}_2 + 2x_1\dot{x}_1 + \dot{x}_1 = u + 2x_1(z_2 - x_1) + (z_2 - x_1) =$

$u + 2x_1z_2 - 2x_1^2 + z_2 - x_1$ $\therefore$ let:

$\therefore$ $V_c(x_1, z_2) = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$ $\therefore$

$\dot{V}_c = x_1\dot{x}_1 + z_2\dot{z}_2 = x_1(z_2 - x_1) + z_2(u + 2x_1z_2 - 2x_1^2 + z_2 - x_1) =$

$x_1z_2 - x_1^2 + z_2u + 2x_1z_2^2 - 2x_1^2z_2 + z_2^2 - x_1z_2 =$

$-x_1^2 + z_2u + 2x_1z_2^2 - 2x_1^2z_2 + z_2^2 = -x_1^2 - z_2^2$ $\therefore$

$z_2u = -x_1^2 - z_2^2 + x_1^2 - 2x_1z_2^2 + 2x_1^2z_2 - z_2^2 =$

$V = x^TPx$ $\therefore$ $\dot{V} = \dot{x}^TPx + x^TP\dot{x}$ $\cancel{z}$

$(\cancel{Ax + Bz}$ $\dot{x} = Ax + B[-B^TPx + g] = Ax - BB^TPx + Bg = (A - BB^TP)x + Bg$ $\therefore$

$\dot{V} = [(A - BB^TP)x + Bg]^TPx + x^TP[(A - BB^TP)x + Bg] =$

$[x^T(A - BB^TP)^T + B^Tg]Px + x^TP(A - BB^TP)x + x^TPBg =$

$x^T(A - BB^TP)^T\cancel{Px} + B^TgPx + x^TPBg + x^TP(A - BB^TP)x$

$\cancel{x^T(A^T}$ $x^T[P(A - BB^TP) + (A - BB^TP)^TP]x + 2x^TPBg$

$\therefore$ $Q + 2\alpha P + PBB^TP + \cancel{A^TPx} = -PA - A^TP + 2PBB^TP$ $\therefore$

$P(A - BB^TP) + (A - BB^TP)^TP = PA - PBB^TP + (A^T - P^TBB^T)P =$

$PA - PBB^TP + A^TP - P^TBB^TP = -2PBB^TP + PA + A^TP =$

$-(-PA - A^TP + 2PBB^TP) = -Q - 2\alpha P - PBB^TP$ $\therefore$

$\dot{V} = -x^T(+Q + 2\alpha P + PBB^TP)x + 2x^TP^TBg =$ for $W = B^TPx$:

$-x^T(Q + 2\alpha P)x - PBW + 2W^Tg =$

$-x^TQx - x^T2\alpha Px - PBW + 2W^Tg \leq -2\alpha\lambda_{min}(P)\|x\|^2$

let $x_1 = y$, $x_2 = \dot{y}$ $\therefore$ $\dot{x}_1 = \dot{y} = x_2$,

$\dot{x}_2 = \ddot{y} = -y + \varepsilon \dot{y}(1 - y^2 - \dot{y}^2) = -x_1 + \varepsilon x_2(1 - x_1^2 - x_2^2) \therefore -x_1 + \varepsilon x_2 - \varepsilon x_1^2 x_2 - \varepsilon x_2^3$

$\therefore$ let $V = \frac{1}{2}x_1^2 + \frac{1}{2}x_2^2$ $\therefore$

$\dot{V} = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(x_2) + x_2(-x_1 + \varepsilon x_2 - \varepsilon x_1^2 x_2 - \varepsilon x_2^3) =$

$\cancel{x_1 x_2} - \cancel{x_1 x_2} + \varepsilon x_2^2 - \varepsilon x_1^2 x_2^2 - \varepsilon x_2^4 \leq 0$

$\cancel{\dot{V} = -\varepsilon(x_1^2 + x}$ $\Rightarrow \dot{V} = -\varepsilon x_2^2(1 + x_1^2) - \varepsilon x_2^4 =$

$-\varepsilon x_2^2(-1 + x_1^2 + x_2^2) \leq 0$ for $-1 + x_1^2 + x_2^2 \geq 0 \therefore -1 + V \geq 0 \therefore$

$-1 \geq -V \therefore V \geq 1 \therefore$ all trajectories starting in $M = \{V \leq 1\}$ stay in M

$\dot{V} = \dot{x}^T P x + x^T P \dot{x} = (Ax + \Phi)^T P x + x^T P (Ax + \Phi) =$

$(x^T A^T + \Phi^T) P x + x^T P A x + x^T P \Phi =$

$x^T A^T P x + x^T P A x + \Phi^T P x + x^T P \Phi =$

$x^T(A^T P + P A)x + 2x^T P \Phi = x^T(-I)x + 2x^T P \Phi =$

$-x^T x + 2x^T P \Phi = -x \cdot x + 2x^T P \Phi = -\|x\|^2 + 2x^T P \Phi \leq -\|x\| + 2\|x\|\|P\|\|\Phi\| \therefore$

$\|P\| = \lambda_{max} \therefore \|P\|\|\Phi\| < \frac{\|x\|}{4} \therefore$

$\dot{V} \leq -\|x\|^2 + 2\|x\|\|P\|\|\Phi\| \leq -\|x\|^2 + 2\|x\|\frac{\|x\|}{4} = -\|x\|^2 + \frac{1}{2}\|x\|^2 = -\frac{1}{2}\|x\|^2$

$\cancel{\dot{V} \leq -\|x\|^2}$ $\dot{V} \leq -|x|^2 + 2|x|\|P\||\Phi|$

$|x| < \delta$ $\therefore$ $\cancel{\lambda_1\|x\|^2 \leq V = x^T P x}$ $\lambda_1\|x\|^2 \leq x^T P x \leq \lambda_n\|x\|^2 = \|P\|\|x\|^2$

$\therefore x^T P x = V \therefore \frac{1}{\lambda_1}V \leq \frac{1}{\lambda_1}\|P\|\|x\|^2 \therefore \|x\|^2 < \delta^2 \therefore$

$\cancel{\lambda_1 \leq \|P\| \leq \lambda_n}$ $\lambda_1\|x\|^2 \leq x^T P x \leq \lambda_n\|x\|^2 = \|P\|\|x\|^2$, $\|x\|^2 < \delta^2 \therefore$

$\lambda_1\|x\|^2 \leq x^T P x \leq \|P\|\|x\| \leq \|P\|\delta^2 \therefore$

$\frac{1}{\lambda_1}x^T P x \leq \frac{1}{\lambda_1}\|P\|\delta^2 \leq \delta^2 \therefore$

$x^T P x = V \leq \lambda_1\delta^2$, and $\dot{V} < 0$ $\therefore D = \{x \mid x^T P x < \lambda_1\delta^2\}$ is invariant

$\therefore$ by Lasalles invariance principle, $\therefore$ D is an invariant open set

and V is p.d.f, $\dot{V}$ n.d.f in D, D contains a basin of attraction $x=0$

$$\dot{x} = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \sqrt{8}\, x_1 x_2 \\ x_1^2 + x_2^2 \end{bmatrix} \therefore A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}, \Phi = \begin{bmatrix} \sqrt{8}\, x_1 x_2 \\ x_1^2 + x_2^2 \end{bmatrix}$$

$A^T = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$ $\therefore A^T P + P A = -I$ $\therefore P = P^T = \frac{1}{12}\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & \frac{1}{6} \end{pmatrix} \therefore$

$\lambda_{max} = \frac{1}{4}$, $\lambda_{min} = \frac{1}{6} \therefore$ $\lambda_1\delta^2 = \frac{1}{6}\delta^2$, $x^T P x = \frac{1}{12}(3x_1^2 + 2x_2^2) \therefore \frac{1}{\delta} < \frac{1}{|x|} \therefore$

$\cancel{|x|}$ $|\Phi| < \frac{|x|}{4\|P\|} \therefore |\Phi|^2 < \frac{|x|^2}{16\|P\|^2}$

$$|\Phi|^2 = \left(\left((\sqrt{6}\, x_1 x_2)^2 + (x_1^2 + x_2^2)^2\right)^{1/2}\right)^2 = 6x_1^2 x_2^2 + x_1^4 + x_2^4 + 2x_1^2 x_2^2 =$$

$$x_1^4 + x_2^4 + 8x_1^2 x_2^2 = -3x_1^4 - 3x_2^4 + 4(x_1^4 + 2x_1^2 x_2^2 + x_2^2) =$$

$$-3(x_1^4 + x_2^4) + 4(x_1^2 + x_2^2)^2 \leq 4(x_1^2 + x_2^2)^2 = 4(\|x\|^2)^2 = (2\|x\|^2)^2 \therefore$$

$$|\Phi| \leq 2\|x\|^2 \therefore$$

$$\frac{|\Phi|}{\|x\|} \leq 2\|x\| \leq \frac{1}{4\|p\|} \cancel{= 2\delta \therefore \frac{1}{8\|p\|} = \delta = \frac{1}{8\left(\frac{1}{4}\right)} = \frac{1}{2}}$$

$$\frac{|\Phi|}{|x|} \leq 2\|x\| \leq \frac{1}{4\|p\|} = \frac{1}{4\left(\frac{1}{4}\right)} = \frac{1}{1} = 1 = 2\delta \therefore \quad \delta = \frac{1}{2} \quad \therefore \delta^2 = \frac{1}{4} \therefore$$

$$\therefore \quad \lambda_1 \delta^2 = \frac{1}{6}\,\frac{1}{4} = \frac{1}{24} \therefore \quad D = \left\{x \mid \frac{1}{12}(3x_1^2 + 2x_2^2) < \frac{1}{24}\right\} = \left\{x \mid 3x_1^2 + 2x_2^2 < \frac{1}{2}\right\}$$

$\because u=0 \therefore \quad M(q)\ddot{q} = -C(q,\dot{q})\dot{q} - D\dot{q} - g(q)$

$\dot{q}^T M(q)\dot{q}$ quadratic form, p.d.f $\qquad M^T(q)=M(q)$

$P(q)$ p.d.f $\therefore \frac{\partial P(q)}{\partial q} = g(q) \therefore \qquad \dot{q}^T M(q)\ddot{q} = (\ddot{q}^T M(q)\dot{q})^T$

$\qquad \ddot{q}^T M(q)\dot{q} = (\dot{q}^T M(q)\ddot{q})^T$

let $V(q,\dot{q}) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q) \therefore$

$\dot{V} = \frac{1}{2}\ddot{q}^T M(q)\dot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \frac{1}{2}\dot{q}^T M(q)\ddot{q} + \frac{\partial P(q)}{\partial q}\dot{q} =$

$\frac{1}{2}(\dot{q}^T M(q)\ddot{q})^T + \frac{1}{2}\dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + g(q)\dot{q} =$

$\dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + g(q)\dot{q} =$

$\dot{q}^T[-C(q,\dot{q})\dot{q} - D\dot{q} - g(q)] + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + g^T(q)\dot{q} =$

$-\dot{q}^T C(q,\dot{q})\dot{q} - \dot{q}^T D\dot{q} - \dot{q}^T g(q) + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + g^T(q)\dot{q} =$

$\frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + \frac{1}{2}\dot{q}^T[-2C(q,\dot{q})]\dot{q} - \dot{q}^T D\dot{q} - \dot{q}^T g(q) + g^T(q)\dot{q} =$

$\frac{1}{2}\dot{q}^T[\dot{M}(q) - 2C(q,\dot{q})]\dot{q} - \dot{q}^T D\dot{q} - \dot{q}^T g(q) + g^T(q)\dot{q}$

$\therefore$ at origin : $(q=0, \dot{q}=0) \therefore$

$\dot{M}-2C$ is skew symmet $\therefore \dot{M}(q)-2C=0$ at origin $\therefore$

$\dot{V} = \frac{1}{2}\dot{q}^T(0)\dot{q} - \dot{q}^T D\dot{q} - \dot{q}^T g(q) + g^T(q)\dot{q} =$

$-\dot{q}^T D\dot{q} - \dot{q}^T g(q) + g^T(q)\dot{q} =$

$-\dot{q}^T D\dot{q} - g^T(q)\dot{q} + g^T(q)\dot{q} = -\dot{q}^T D\dot{q} = \dot{V}$ is quadratic form

$\therefore D$ is p. semi defi $\therefore \dot{V}$ is neg. semi. d $\therefore V$ is p.d.f $\therefore$

origin is Lyapunov stable $\therefore$