MTH3011 Non linear Systems and Control

CW 15th March 20%

- Nonlinear representation A finite dimensional time invariant nonlinear dynamical system can be represented in a compact state space form $\dot{x} = \frac{dx}{dt}$ x is a vec

$$\dot{x}(t) = f(t, x(t), u(t)) \quad , x(t_0) = x_0 \quad t \in [t_0, t_f]$$

time derivative of $x(t)$ — state — control input or forcing term — initial condition — time duration, t_f final time, t_0 initial time

x - state $x(t) \in \mathbb{R}^n$ n-dimensional real space {x is a vector}

$x(t)$ can take any vals from an n-dim realspace (Global) {Global argument}

whereas $x(t) \in D \subset \mathbb{R}^n$ $x(t)$ can take any vals from a set D which is subset of $\mathbb{R}^n$. (Local) {local argument}

$\dot{x}(t) = f(t, x(t), u(t)) \quad , x(0) = x_0$

State $x \in \mathbb{R}^n$ $\quad x[x_1, x_2, \dots, x_n]^T$ $\qquad \dot{x} = [\dot{x}_1, \dot{x}_2, \dots, \dot{x}_n]^T$

Control $u \in \mathbb{R}^m$ $\quad u = [u_1, u_2, \dots, u_m]^T$

$f = [f_1(\cdot) \quad , f_2(\cdot) \quad , \dots , \quad f_n(\cdot)]^T$

$\dot{x}(t) = f(t, x(t), u(t))$

$$\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} f_1(t, x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)) \\ f_2(t, x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)) \\ \vdots \\ f_n(t, x_1(t), \dots, x_n(t), u_1(t), \dots, u_m(t)) \end{bmatrix}$$

$$f(\cdot) : \underset{\text{time}}{\mathbb{R}_+} \times \underset{\text{state}}{\mathbb{R}^n} \times \underset{\text{control}}{\mathbb{R}^m} \longrightarrow \mathbb{R}^n$$

Ex/ $\dot{x}_1 = x_2 \qquad \dot{x}_1 = \frac{dx_1}{dt} \qquad \dot{x}_2 = \frac{dx_2}{dt}$

$\dot{x}_2 = -k_1 x_1 - k_2 x_2 - k x_1^3 + \underbrace{u(t)}_{\text{control}}$ $\qquad x_1(0) = 0.1 \quad x_2(0) = 0.01 \quad k_1, k_2$ are scalar

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -k_1 x_1 - k_2 x_2 - k x_1^3 + u(t) \end{bmatrix}$$

State $x \in \mathbb{R}^2$ (x_1, x_2) $\qquad f_1(x, u) = x_2$

Control $u \in \mathbb{R}^1$ ~~$f_2 x_2$~~ $\qquad f_2(x, u) = -k_1 x_1 - k_2 x_2 - k x_1^3 + u$

- $f(\cdot) : \mathbb{R}^2 \times \mathbb{R} \rightarrow \mathbb{R}^2$

a non linear system with n-states $x \in \mathbb{R}^n$ $x = \{x_1, \dots, x_n\}$

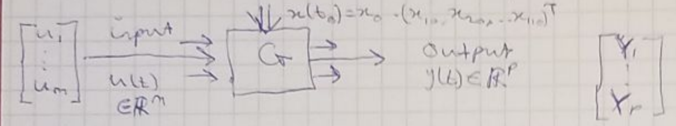
m-inputs $u \in \mathbb{R}^m$ p-outputs $y \in \mathbb{R}^p$ (x_1, x_{1_0})

can be represented as $G \begin{cases} \dot{x}(t) = f(x(t), u(t)), \; x(t_0) = x_0 \\ y(t) = h(x(t), u(t)) \end{cases}$ ← is output eqn

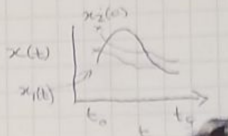
term is not a derivative part like $\dot{x}$

an algebraic func of states & input

i.e. not dynamics



given initial condition

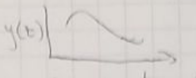


Σ is we know 2 control input structure it

in the State Space representation

$\dot{x}(t) = f(t, x(t), u(t))$ for eg: $\dot{x}_1 = x_2$ $\dot{x}_2 = x_1^2 - x_2$

Suppose time t is not explicitly



t_0 t t_f $t \in [t_0, t_f]$

$\dot{x}_1(t) = x_2(t)$

$\dot{x}_2(t) = x_1(t) - x_2(t)\sin(t)$

$\dot{x} = f(t, x(t), u(t))$

$\dot{x} = f(x, u)$

~~Ex~~

\Ex/ $G \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -k\sin x_1 \\ y = x_1 \end{cases}$ Dynamics ← output eqn

$x \to$ 2 states $x = (x_1, x_2)^T \in \mathbb{R}^2$

Control None

2 states $x = [x_1, x_2]^T \in \mathbb{R}^2$ No control input

one state is measured $\therefore y \in \mathbb{R}^1$ ⟹ Another y could be

$y = (x_1, \mathrm{sign}(x_2))^T$ $y \in \mathbb{R}^2$

obtaining a compact state space representation $\ddot{x}_1 = \frac{d^2 x_1}{dt^2}$

\Ex/ Consider $m_1 \ddot{x}_1 = k_2(x_2 - x_1) - k_1 x_1 + F_1$ is 2nd order ODE coupled

$m_2 \ddot{x} = -k_3 x_2 - k_2(x_2 - x_1) - F_2$

k_1, k_2, k_3 scalar vals

F_1 F_2 external forces control

want to represent in compact state space form eg

i.e. $\dot{x} = f(x, u)$ let us define a new term

$\ddot{x}_1$ ← 2 derivative .. one order less as a new variable

$\dot{x}_1 = \frac{dx_1}{dt} = x_3$ Similarly $\dot{x}_2 = \frac{dx_2}{dt} = x_4$ x_3, x_4 are state variables

we defined where

we have 4 states $(x_1, x_2, \underset{\dot{x}_1}{x_3}, \underset{\dot{x}_2}{x_4})^T \in \mathbb{R}^4$

Eq (1) & (2) as follows $\dot{x}_1 = x_3 \leftarrow f_1(\cdot)$ $\quad \dot{x}_3 = \ddot{x}_1$

$\dot{x}_2 = 4 \leftarrow f_2(\cdot)$

$\frac{dx_3}{dt} = \frac{d\dot{x}_1}{dt} = \ddot{x}_1 \qquad \dot{x}_3 = \ddot{x}_1 = \frac{k_2}{m_1}(x_2 - x_1) - \frac{k_1}{m_1}x_1 + \frac{F_1}{m_1} \leftarrow f_3(\cdot)$

$$\underbrace{\dot{x}_4 = \ddot{x}_2 = -\frac{k_3}{m_2}x_2 - \frac{k_2}{m_2}(x_2 - x_1) - \frac{F_2}{m_2} \leftarrow f_4(\cdot)}_{\dot{x} = f(x,u) \text{ form}}$$

Consider an autonomous nonlinear dynamical system

$\dot{x}(t) = f(x(t)), \quad x \in \mathbb{R}^n$ — (1)

∴ equilibrium pts $x^* \in \mathbb{R}^n$ are the real sol of $f(x(t)) = 0$

know, equilibrium is the sols of simultaneous, possibly nonlinear eqns of states. For autonomous systems, ∵ not explicitly dependent on t the set of equilib holds $\forall t$ (for all time t) $x^* \in \mathbb{R}^n$ can be shifted to origin

$\dot{x} = f(x,u) \qquad y = h(x) \qquad u = g(y)$

$u = g(h(x)) \rightarrow \dot{x} = f(x, g(h(x))) \qquad \dot{x} = f(x)$

$\dot{x}(t) = f(x(t), u(t)) \quad x(t_0) = x_0 \quad t \in [t_0, t_f]$

x - states $\quad$ u = Control

$x \in \mathbb{R}^n \leftarrow$ Global $\qquad x \in D \subset \mathbb{R}^n$ (local) $\qquad u \in \mathbb{R}^m$

Compact state space representation

$\frac{dx}{dt}$ (order 1) $\quad x = [x_1, x_2, \dots, x_n]^T \quad u = [u_1, \dots, u_m]^T$

$\dot{x} = f(x,u) \leftarrow$ dynamics System is Autonomous nonlinear systems

$\rightarrow y = h(x,u)$ is output eqn ∴ $y \in \mathbb{R}^p \quad p \leq n$

$\dot{x}(t) = f(x(t)), \; x(t) \in \mathbb{R}^n$

→0 at steady state

$$\begin{array}{c} 0 \\ 0 \\ \\ 0 \end{array} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} f_1(x_1, \dots, x_n) \\ f_2(x_1, \dots, x_n) \\ \vdots \\ f_n(x_1, \dots, x_n) \end{bmatrix}$$

∴ $f_i(x_1, \dots, x_n) = 0$
$i = 1, \dots, n$

no explicit dependency on $\dot{x}(t) = 0$ for $\dot{x}(t) = f(x(t)), \; x \in \mathbb{R}^n$ by $0 = f(x(t))$

$x^* \in \mathbb{R}^n$ can be shifted to origin by:

Suppose $x^* \neq 0$ (Equilibrium) is a nonzero vector for $x \in \mathbb{R}^n$ $x^* \in \mathbb{R}^n$

$z \in \mathbb{R}^n$ ∴ let $z = x - x^*$ $\leftarrow$ solving $\underset{0}{\dot{x}} = f(x)$

$\frac{dz}{dt} = \dot{z} = \dot{x}$ $\because x^*$ is const $\forall t$ $= f(x)$

from $\circledast$ $\therefore x = z + x^*$ $\therefore \dot{z} = f(z + x^*)$ can be thought as a bias term $\therefore \underbrace{\dot{z} = f(z)}$ which is similar to $\dot{x} = f(x)$

$\Downarrow$ equilibrium origin $z^* = 0$ $\qquad$ $\Downarrow$ equalibrium x^*

$\therefore$ we are studing nonlinear systems ($f(\cdot)$ nonlinear vector functs depending on $x \in \mathbb{R}^n$), it is possible to have multiple equilibriaa or a continuum of equilibriaa, or even no equilibria. it depends on $f(\cdot) = 0$ $\therefore$ Z notion of Isolated equalibrium pt is of interest. in very simple terms, notation it means there are no equilibrium pt other than Z one equilibrium $x^* \in \mathbb{R}^n$ that has been determined in its neighbourhood in Z n dimensional space

say equilibrium pt is $x^* \in \mathbb{R}^n$ any x of n-dim realspace ($x \in \mathbb{R}^n$) that belong to shaded region satisfy Z inequality $\|x - x^*\| < \varepsilon$

$B_\varepsilon(x^*) = \{x \in \mathbb{R}^n \mid \|x - x^*\| < \varepsilon\}$ $\quad$ $\|\cdot\|$ - represents euclidean norm

centre about

set

between x Z x^*

radius $\quad$ $\therefore$ an equilibrium $x^* \in \mathbb{R}^n$ is an isolated equilibrium pt $\exists$ an $\varepsilon > 0$ st $B_\varepsilon(x^*)$ contains no other equilibrium p.

Ex) determine equilibrium pts for $ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$

$\therefore$ define $\theta = x_1$ Z $\dot{\theta} = x_2$ $\therefore \frac{d\theta}{dt} = \frac{dx_1}{dt}$ $\therefore$

$$\left.\begin{aligned} \frac{dx_1}{dt} &= x_2 \\ \frac{dx_2}{dt} &= \ddot{\theta} = \cancel{-mg\sin\theta - kl\dot{\theta}} - \frac{g}{l}\sin\theta - \frac{k}{m}\dot{\theta} \\ &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{aligned}\right\} \longrightarrow \dot{x} = f(x(t)) \quad x \in \mathbb{R}^2$$

$\Downarrow$ equilib

$x(x_1, x_2)^T$

$(\theta, \dot{\theta})^T$

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 \end{aligned}$$

$\therefore$ to determine equilib set $\dot{x}_1 = \dot{x}_2 = 0$ (ie no state or change of state)

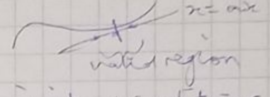
$-\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 = 0$ equili pts $x^* = (x_1^*, x_2^*) = (\pm n\pi, 0)$ $\quad$ $\sin x_1 = 0$

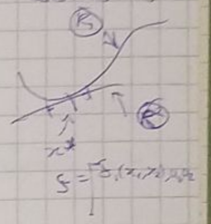
a linearised model is : – an approx of a nonlinear syst

– is valid in a small region around an operating pt

– provides insights about how system behaves in a neighb of EPs especially stability

– at present, design of control using linear design tools is very standard & mature





Consider $\dot{x} = f(x, u)$ $\quad x(0) = [x_1(0), x_2(0)]^T$ ← initial conditions

$x = (x_1, x_2) \in \mathbb{R}^2$ $\quad u = (u_1, u_2) \in \mathbb{R}^2$
states $\qquad$ input

$f = [f_1(x_1, x_2), \ldots]$

Let this system operates along a trajectory $\tilde{x}(t)$ (response of dynamical system)

& input $\tilde{u}(t)$ ensures the evolution of system close to the $\tilde{x}(t)$ $\quad \dot{\tilde{x}}(t) = f(\tilde{x}(t), \tilde{u}(t))$ $\quad$ Let $x(t) = \tilde{x}(t) + \delta x$ & $u(t) = \tilde{u}(t) + \delta u$
small perturbation $\qquad$ small pert

$x(t) = \tilde{x}(t) + \delta x$ ← small perturbation
nominal original trajectory

∴ taking the derivative of the state wrt time gives $\dot{x} = \dot{\tilde{x}} + \delta\dot{x} = f(\tilde{x} + \delta x, \tilde{u} + \delta u)$ $\quad (\because \dot{x} = f(x, u) = \ldots)$ $\quad (\tilde{x}, \tilde{u})$

Let us consider expanding this func $f(\cdot)$ using Taylor expansion $\dot{\tilde{x}} + \delta\dot{x} = f(\tilde{x}, \tilde{u}) + \frac{\partial f}{\partial x}\Big|_{(\tilde{x}, \tilde{u})} \delta x + \frac{\partial f}{\partial u}\Big|_{(u, x)} \delta$

$$\dot{\tilde{x}} + \delta\dot{x} = f(\tilde{x}, \tilde{u}) + \frac{\partial f}{\partial x}\Big|_{(\tilde{x}, \tilde{u})} \delta x + \frac{\partial f}{\partial u}\Big|_{(\tilde{x}, \tilde{u})} \delta u + \text{H.O.T}$$

$\because \dot{\tilde{x}} = f(\tilde{x}, \tilde{u})$ can write $\delta\dot{x} = \frac{\partial f}{\partial x}\Big|_{(\tilde{x}, \tilde{u})} \delta x + \frac{\partial f}{\partial u}\Big|_{(\tilde{x}, \tilde{u})} \delta u + \text{H.O.T}$

$A = \frac{\partial f}{\partial x}\Big|_{(\tilde{x}, \tilde{u})}$ $\qquad \delta\dot{x} = A\,\delta x + B\,\delta u$ $\quad$ considering only first order terms

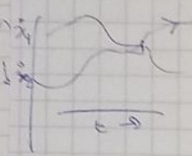
$x \in \mathbb{R}^2$, $f = [f_1(x_1, x_2), f_2(x_1, x_2)]^T$ $\qquad A = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{(\tilde{x}, \tilde{u})} \in \mathbb{R}^{2\times 2}$ ← no. of states

$B = \begin{bmatrix} \partial f_1/\partial u_1 & \partial f_1/\partial u_2 \\ \partial f_2/\partial u_1 & \partial f_2/\partial u_2 \end{bmatrix}\Bigg|_{(\tilde{x}, \tilde{u})} \in \mathbb{R}^{2\times 2}$ $\quad$ is only one I/P your $B \in \mathbb{R}^{2\times 1}$
input matrix

$\dot{x}_1 = a_{11}x_1 + a_{12}x_2 + 0u_2 + 1u_1 \qquad \dot{x}_2 = a_{21}x_1 + a_{22}x_2 + 1u_2 + 0 \cdot u_1$

Linear Algebra Gilbert Strang

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$



$$\dot{x} = Ax + Bu \qquad \delta\dot{x} = A\delta x + B\delta u$$

Second order systems

Autonomous nonlinear system $\dot{x} = f(x)$, $x(0) = x_0$ $t \in [t_0, t_f]$

$x \in \mathbb{R}^2$ $\quad x = (x_1, x_2)^T$ (x_1 state1, x_2 state2)

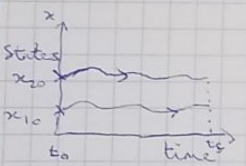
$$\text{'}C_1\text{'} \begin{cases} \dot{x}_1 = f_1(x_1, x_2) \\ \dot{x}_2 = f_2(x_1, x_2) \end{cases} \qquad x_0 = (x_{10}, x_{20})^T$$

assumption: Existance of sol for 'C1' which is <u>unique</u>

Ex/ $\ddot{x} + h(x, \dot{x}) = 0$ ($\ddot{x} = \frac{d^2x}{dt^2}$, h arb non linear func)

$x_1 = x \qquad x_2 = \dot{x} = \frac{dx}{dt} = \left\{ \frac{dx_1}{dt} \right\} = \dot{x}_1$

$\dot{x}_1 = x_2 \qquad \dot{x}_2 = \ddot{x} = -h(x, \dot{x}) = -h(x_1, x_2)$



(not to physics)

∴ Sols x_1, x_2 evolve over time constrained by RHS of dynamics

$\dot{x} = f(x) \quad \dot{x}_1 = x_2 \quad \dot{x}_2 = 2x_1 + x_2$ ($\leftarrow h(x_1, x_2)$)

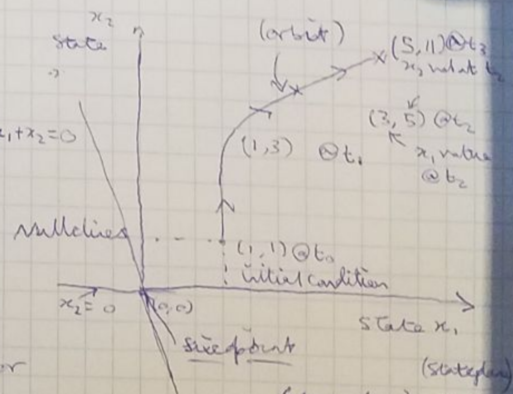
$$\left.\begin{aligned} f_1(\cdot) &= x_2 \\ f_2(\cdot) &= 2x_1 + x_2 \end{aligned}\right] \text{ contrains on evolution of dynamics}$$

$$\begin{array}{c} x_1 \\ x_2 \end{array} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 1 \\ 3 \end{bmatrix} \longrightarrow \begin{bmatrix} 3 \\ 5 \end{bmatrix} \rightarrow \begin{bmatrix} 5 \\ 11 \end{bmatrix}$$

$2 \cdot 1 + 1 \qquad 2 \cdot 1 + 3$

$$f(\cdot) = \begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \end{bmatrix} \qquad x_0 = (1, 1) \quad (x_{10}, x_{20})$$

x_1 - black
x_2 - brown



time is embedded

$f_1(\cdot) = x_2 = 0$
$f_2(\cdot) = 2x_1 + x_2 = 0$
$\dot{x} = f(x)$

$\dot{x}_1 = x_2 = 0$
$\dot{x}_2 = 2x_1 + x_2 = 0$

fixed point or equilib when we solve for x_1, x_2

$\dot{x}_1 = f_1(x_1, x_2) = 0$
$\dot{x}_2 = f_2(x_1, x_2) = 0$

these are called <u>Nullclines</u>

\Ex/ $\dot{x}_1 = x_2(1-x_2^2)$
$\dot{x}_2 = -x_1 - x_2^2$

Nullclines are given by $\dot{x}_1 = \dot{x}_2 = 0$

$\dot{x}_1 = 0 \longrightarrow \underbrace{x_2(1-x_2^2)}_{f_1(\cdot)=0} = 0 \longrightarrow x_2 = 0 \quad \& \; (1-x_2^2) = 0 \quad x_2^2 = 1 \quad x_2 = \pm 1$

$\dot{x}_2 = 0 \longrightarrow \underbrace{-x_1 - x_2^2}_{f_2(\cdot)=0} = 0 \longrightarrow -x_1 = x_2^2 \quad x_2 = \pm\sqrt{-x_1}$

the fixed pts, or equili pts are obtained $\dot{x}_1 = \dot{x}_2 = 0$
& solving for x_1, x_2 as above

$\dot{x}_1 = x_2(1-x_2^2)$
$\dot{x}_2 = -x_1 - x_2^2$
$\dot{x}_1 = 0 \quad \dot{x}_2 = 0$
$x_2 = \pm 1 \quad x_2 = \pm\sqrt{-x_1}$
$\dot{x} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} x$

$\left.\begin{aligned} \dot{x}_1 &= x_2(1-x_2^2) \\ \dot{x}_2 &= -x_1 - x_2^2 \end{aligned}\right\} \dot{x} = f(x)$

\Linearised systems/

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$\dot{x} = \begin{bmatrix} 0 & -2 \\ -1 & -2 \end{bmatrix} x \qquad \dot{x} = \begin{bmatrix} 0 & -2 \\ -1 & 2 \end{bmatrix} x$
at equili

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1-3x_2^2 \\ 1 & -2x_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \text{ (equilibr pts)}$$

$x_1^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad x_2^* = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
$x_3^* = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

$\dot{x} = Ax$

consider 2nd order autonomous system

$\underset{\text{dynamics}}{\dot{x} = f(x)}$, $\underset{\text{initial conds}}{x(t_0) = x_0}$ $\quad \underset{\text{time period}}{t \in J}$

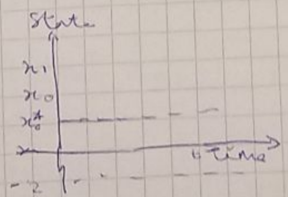
\one-dimensional Ex/ $\dot{x}_1 = \lambda x_1 \quad x_1(0) = x_0 \quad x_1 \in \mathbb{R}$
λ is a scalar $\lambda \in \mathbb{R}$

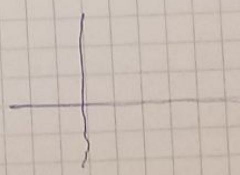
$x_1(t) = e^{\lambda t}$
$\dot{x}_1(t) = \lambda e^{\lambda t}$
$\therefore \dot{x}_1(t) = \lambda x_1(t)$

$\dot{x}_1$ is proportional to state x_1 by λ

Sol: $x_1(t) = \underset{\text{IC}}{x_0}\, e^{\lambda t} \leftarrow$ (IVP) $\quad \because \lambda \in \mathbb{R} \quad , \lambda = 0 \quad \lambda < 0 \quad , \lambda > 0$

When $\lambda = 0 \quad \dot{x}_1 = 0$
x_1 is const over time



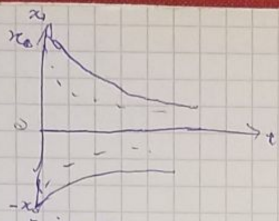


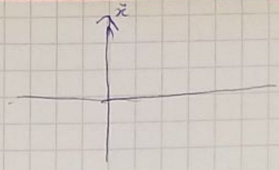
Aim understand dynamical systems
without calcing sols

$\lambda < 0 \quad -\lambda_1$

$\dot{x}_1 = -\lambda_1 x_1$

Sol $x_1 = x_0 e^{-\lambda_1 t}$
(decay)





For any $x<0$ change $\dot{x}$ is +ve

More -ve bigger $\dot{x}$ change
bigger +ve

any $x>0$ change $\dot{x}$ is -ve

System evolves to right & asymptot reach origin
evolves faster to origin

system evolves to left & asympt reach origin

$\lambda > 0 \quad \dot{x}_1 = \lambda_2 x_1 , x_0$

Sol $x_1 = x_0 e^{\lambda_2 t}$

λ_2 +ve val (growth

$\lambda > 0$

$x>0$ chg ~~is~~ is +ve

syst $\dot{x}_1 = \lambda_2 x_1$ evolves to right away from origin

$x<0$ chg is

$\dot{x}_1 = \lambda x_1 \quad x_1 \in \mathbb{R}$

sol is $x_1 = x_0 e^{\lambda t}$ in terms of integral eqn

$x_1(t) = x_0 + \int_0^t \dot{x}_1(\tau)\, d\tau = x_0 + \int_0^t \lambda x_1(\tau)\, dt$

first iteration $x_1(t) = x_0 \leftarrow$ initial cond

$x_{1_1}(t) = x_0 + \int_0^t \lambda x_0\, d\tau = x_0 + \lambda x_0 \int_0^t d\tau = x_0 + \lambda x_0 t = x_0(1+\lambda t)$
iteration index

second iteration $x_{1_2}(t) = x_0 + \int_0^t \lambda x_{1_1}(\tau)\, d\tau = x_0 + \int_0^t \lambda x_0 (1+\lambda \tau)\, d\tau$

$= x_0 + \lambda x_0 t + \frac{\lambda^2 t^2}{2} x_0$

likewise $x_{1_n}(t) = x_0 + \int_0^t \lambda x_{1_{(n-1)}}(\tau)\, d\tau = \underbrace{\left(1 + \frac{\lambda t}{1!} + \frac{\lambda^2 t^2}{2!} + \dots + \frac{\lambda^n t^n}{n!}\right)}_{e^{\lambda t} \text{ taylor expansion}} x_0 \leftarrow$ IC

$x_1(t) = x_0 e^{\lambda t}$ sol in general

For $\dot{x} = Ax$, $x(0) = x_0$, $x \in \mathbb{R}^n$ $A \in \mathbb{R}^{n \times n}$ $x(t) = x_0 + \int_0^t Ax(\tau)\,d\tau$

- $x(t) = x_0 + \int_0^t A\,x(\tau)\,d\tau$

$$x_n(t) = \underbrace{\left[I_{n \times n} + At + \frac{A^2 t^2}{2!} + \dots + \frac{A^n t^n}{n!} \right]}_{\text{Matrix eq}} x_0 \qquad x(t) = x_0 e^{At}$$

Higher order ODE / autonomous compact state-space representation

$\dot{x} = f(x)$, $x \in \mathbb{R}^n$ equil x^* eval Jacobian @ x^*

linear time invariant system $\rightarrow \dot{x} = Ax$, $x(0) = x_0$ $x \in \mathbb{R}^n$ $x(t) = x_0 e^{At}$

$\dot{x} = Ax$, $x \in \mathbb{R}^2$, $A \in \mathbb{R}^{2 \times 2}$ $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

$\det(A - \lambda I_2) = \begin{vmatrix} a_{11} - \lambda & a_{12} \\ a_{21} & a_{22} - \lambda \end{vmatrix} = 0$ $\det(A) = a_{11}a_{22} - a_{21}a_{12}$

$\dot{x} = Ax$ $Tr(A) = \text{Trace}[A] = a_{11} + a_{22}$

$$\lambda^2 - \lambda\underbrace{(a_{11} - a_{22})}_{Tr(A)} + \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{Det(A)} = 0$$

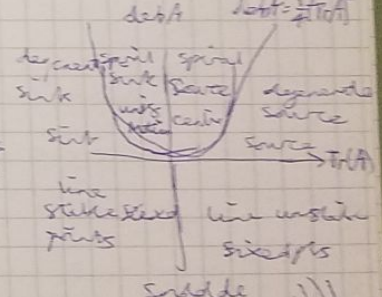
$\lambda^2 - Tr(A)\,\lambda + \det(A) = 0$ (quad in λ)

$$\lambda_{1,2} = \frac{1}{2}\left[Tr(A) \pm \underbrace{\sqrt{Tr(A)^2 - 4\det(A)}}_{\text{discriminant}} \right]$$

$$\lambda_{1,2} = \frac{1}{2}\left[Tr(A) \pm \sqrt{Tr(A)^2 - 4\det(A)} \right]$$

$\hookrightarrow$ $Tr(A)^2 - 4\det(A) \geq 0 \rightarrow \lambda_{1,2} \in \mathbb{R}$ real

$Tr(A)^2 - 4\det(A) < 0$ $\lambda_{1,2} \in \mathbb{C}$ $\lambda_1 = \lambda_2^*$



states

inputs $\rightarrow$ System $\rightarrow$ outputs $\rightarrow$ higher order ODEs (time invariant autonomous)

Determine $\uparrow$

compact state space form

$\dot{x} = f(x, u)$

Lyapunov theory

linearise

$\dot{x} = Ax$

Controller

x^* - equilibrium

$A = \left.\frac{\partial f}{\partial x}\right|_{x^*}$

Limit cycle

Linear System + NL

$$\begin{cases} \dot{x}_1 = 2x_1 + x_2 x_1 \\ \dot{x}_2 = -x_2 + 3x_1 + x_2^2 \end{cases} \quad \therefore \dot{x} = Ax + G(x) \qquad \therefore$$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix}=\begin{bmatrix}2 & 0\\ 3 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}=\begin{bmatrix}x_1x_2\\ x_2^2\end{bmatrix}^{x}$$

$\dot{x}=Ax,\ x\in\mathbb{R}^2 \qquad A\in\mathbb{R}^{2\times2} \qquad A=\begin{bmatrix}a_{11} & a_{12}\\ a_{21} & a_{22}\end{bmatrix}$

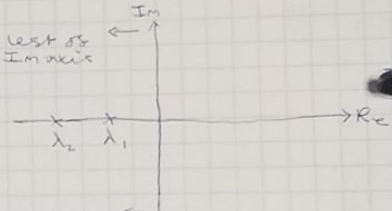
Study stability by looking at eigen values of A

$\det(A-\lambda_2)=\begin{vmatrix}a_{11}-\lambda & a_{12}\\ a_{21} & a_{22}-\lambda\end{vmatrix}=0 \qquad \lambda^2-\lambda\underbrace{(a_{11}+a_{22})}_{Tr\,A}+\underbrace{(a_{11}a_{22}-a_{12}a_{21})}_{\det A}=0$

$\lambda^2-Tr(A)\lambda+\det(A)=0 \qquad \lambda_{1,2}=\frac{1}{2}\left[Tr(A)\pm\sqrt{Tr(A)^2-4\det(A)}\right]$

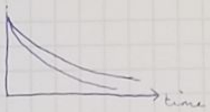
if $Tr(A)^2-4\det(A)\geq0 \quad \lambda_{1,2}\in\mathbb{R}$

$Tr(A)^2-4\det(A)<0 \quad \lambda_{1,2}\in\mathbb{C} \quad \lambda_1=\lambda_2^*$



<u>Second order</u> Systems $\lambda_{1,2}\in\mathbb{R}$

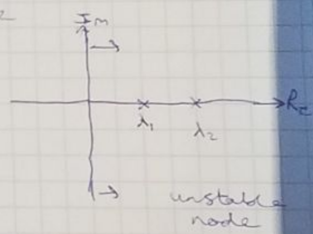
$\dot{x}(t)=A_1x(t), \quad A_1=\begin{bmatrix}-1 & -1\\ -1 & -5\end{bmatrix}$



Eigenvals: $\Lambda=[-5.24 \quad -0.764]$

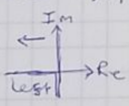
stable node

Eigenvecs: $\begin{bmatrix}0.230 & -0.973\\ 0.973 & 0.230\end{bmatrix}=V$



stable node $\quad A_1V-V\,Diag(\Lambda)=0$

$Ax=\lambda x$



$\lambda_{1,2}\in\mathbb{R}<0$ stable node

$\lambda_{1,2}\in\mathbb{R}>$ Unstable node $\quad \dot{x}(t)=A_2x(t), \quad A_2=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$

Eigenvals: $\Lambda=[1,3]^T$

Both -ve Stable node (sink) $\quad A_1=\begin{bmatrix}-1 & -1\\ -1 & -5\end{bmatrix}$

Both +ve Unstable node (source) $\quad A_2=\begin{bmatrix}2 & 1\\ 1 & 2\end{bmatrix}$

-ve and -ve Saddle point $\quad A_3=\begin{bmatrix}1 & 3\\ 1 & -1\end{bmatrix}$

$\dot{x}(t)=A_3x(t) \qquad$ Eigenvals $\Lambda=[2 \quad -2]^T \qquad A_3=\begin{bmatrix}1 & 3\\ 1 & -1\end{bmatrix}$

Eigenvecs $\quad V=\begin{bmatrix}0.948 & -0.707\\ 0.316 & 0.707\end{bmatrix}$ Saddle node

$\qquad\qquad v_1 \qquad\qquad v_2$

$\dot{x}=Ax+\underbrace{Bu}_{f(x)}$

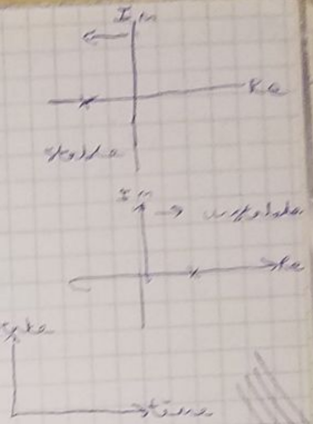
$\lambda_1 = \lambda_2 = \lambda$ -ve Stable Star $A_4 = \begin{bmatrix} -20 & 1 \\ 0 & -20 \end{bmatrix}$

$\lambda_1 = \lambda_2 = \lambda$ +ve Unstable Star $A_5 = \begin{bmatrix} 10 & 1 \\ 0 & 10 \end{bmatrix}$

$Re(\lambda) < 0$ Stable spiral/focus $A_6 = \begin{bmatrix} 0 & 3 \\ -1 & -2 \end{bmatrix}$

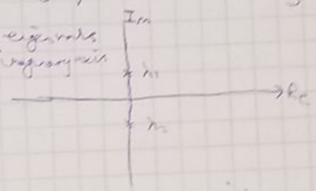
$Re(\lambda) > 0$ Unstable Spiral/focus $A_7 = \begin{bmatrix} 1 & -3 \\ 1 & 2 \end{bmatrix}$

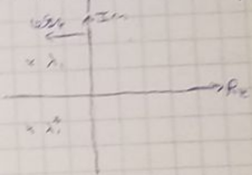
$Re(\lambda) \equiv 0$ Centre $A_8 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$



$\dot{x} = Ax$ $A \in \mathbb{R}^{2\times2}$ eigen vals of $A \to$ classify type of equilib

$\tilde{A} = A + \underbrace{\Delta A}_{\text{small perturbation}}$

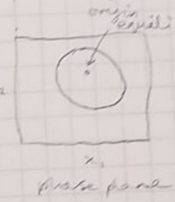




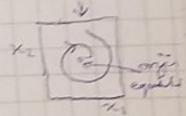
$\dot{x} = Ax$, $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

this gives a centre type

$A + \Delta A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} + \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}$ μ is small scalar val



λ_1, λ_1^* complex eigenvals $Re(\lambda) < 0$ Stable Spiral

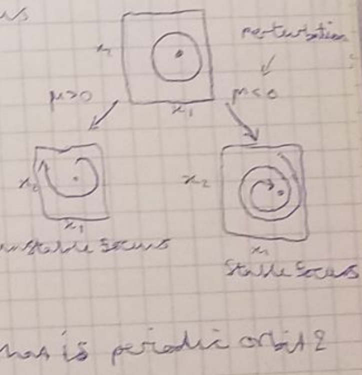


eigen vals for pertubed system

$\dot{x} = \tilde{A}x$ will be $\mu \pm j$

μ is the small scalar val $\mu > 0$ center $\to$ unstable focus

if $\mu < 0$ center $\to$ Stable focus



is NOT structurally Stable!

Periodic orbit / Limit cycle / ① what is periodic orbit? limit cycle? ② how determine a periodic orbit

→ use Poincaré-Bendixson Criterion

→ limit to 2nd order Systems

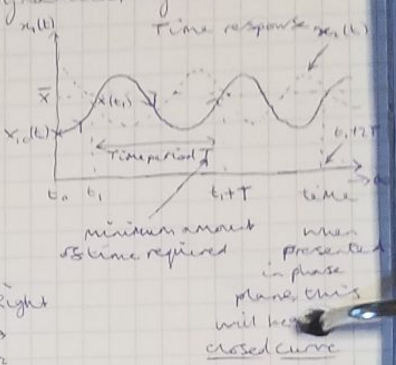
③ how rule out existance of periodic orbit → use Bendixson Criteria

① <u>periodic orbit</u> in general dynamical systems

$\dot{x} = f(t, x(t)) = \dot{x}(t)$, $x(t) \in D \subset \mathbb{R}^n$

if $\exists\ T > 0$ st $f(t, x(t)) = f(t+T, x(t+T))$ $\forall (t, x(t)) \in [t_0, \infty) \times D$

when the dynamical system is a periodic Dynamical System



phase portrait of limit cycle:
near origin: spiraling outwards

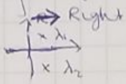
far from origin: attracting towards

a closed curve $t \to \infty$ converge to the orbit

they meet at the periodic orbit?

all trajectories leaving origin asymptotically

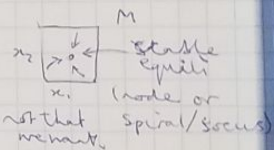
ends up to periodic orbit



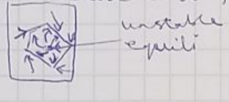
for the existence of limit cycle (2nd order systems)

① A trajectory enters a domain ~~of~~ M at t_1, it remains in domain $M \subset \mathbb{R}^2$ $\forall t > t_1$

② trajectory approaches a periodic orbit (closed orbit M)



we want to have



③ if M ~~contains~~ no stable equili, or no equili:

① holds when M has a periodic orbit

④ if M has one equili but unstable & approach orbit as $t \to \infty$

<u>Limit cycle</u> is a closed trajectory in phase plane st other non-closed trajectories spiral towards it asymptotically from inside or outside the closed curve (as $t \to \infty$

\Existance of Periodic Orbit Limit Cycle/ (Poincaré Bendixson criterion) Consider a $\dot{x} = f(x)$, $x \in \mathbb{R}^2$ & let M set be <u>closed</u>, <u>bounded</u>, <u>subset</u> of phase plane st: M contains no equili pts or contains only one equili pt st the Jacobian Matrix $\left[\frac{\partial f}{\partial x}\right]$ evaluated at equili has eigenvals with +ve real pts
every trajectory starting in M stays in M $\forall$ future time.

then M contains a periodic orbit of limit cycle of $\dot{x} = f(x)$

if M contains only one equili satisfying PB conditions ($Re(\lambda) > 0$) then ~~in~~ its vicinity all trajectories move away from its.

So we can choose a simple closed curve for eg - a circle

$x_1^2 + x_2^2 = C_1$ $\quad x_1, x_2$ rep

- an ellipse $x_1^2 + 3x_1x_2 + x_2^2 = C_2$

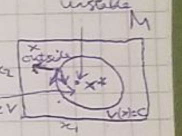
or a polygon around equili

Let that curve be $V(x)$ $\quad (x \in \mathbb{R}^2 \quad x = (x_1, x_2)^T)$ $\quad V(x)$ is continuously differentiable $\quad x \in \mathbb{R}^2 \quad x = (x_1, x_2)^T \quad V(x_1, x_2)$ is identified

$$\nabla V(x_1, x_2) = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}\right]$$

$V(x_1, x_2)$ - a continuously diff func with ~~∇x~~

$\nabla V(x_1, x_2) = \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2}\right]$ $\quad$ recall $\dot{x} = f(x)$ $\quad \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} f_1(x) \\ f_2(x) \end{bmatrix}$



Consider inner product $f(x) \cdot \nabla V(x) = \begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \end{bmatrix} \begin{bmatrix} \frac{\partial V}{\partial x_1} & \frac{\partial V}{\partial x_2} \end{bmatrix} = f_1(\cdot)\frac{\partial V}{\partial x_1} + f_2 \frac{\partial V}{\partial x_2}$

$f(x) \cdot \nabla V(x) < 0$ $\quad f(x)$ pts inwards at x

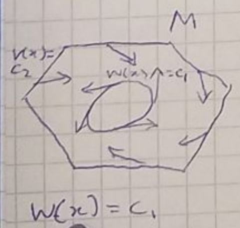
$f(x) \cdot \nabla V(x) = 0$ $\quad$ tangent to curve at x

$f(x) \cdot \nabla V(x) > 0$ $\quad f(x)$ points outwards at x

Trajectories can leave a set only if Z vec field ($f(\cdot)$) pts outwards at some pt on Z boundary i. for a set of Z form

$M = \{x \in \mathbb{R}^2 \mid V(x) \leq C\}$ $\quad C > 0$ trajectories are trapped inside M if

$f(x) \cdot \nabla V(x) \leq 0$ on boundary ie $V(x) = C$



Annular region

$M = \{x \in \mathbb{R}^2 \mid W(x) \geq C_1 \ \& \ V(x) \leq C_2\}$ $\quad C_1, C_2 > 0$

trajectories are trapped inside M if $f(x) \cdot \nabla W(x) \geq 0$

$f(x) \cdot \nabla V(x) \leq 0$ on $V(x) = C_2$

we will solve problem on how to rule out limit cycle

Set $M = \{x \in \mathbb{R}^2 \mid V(x) \leq c\}$ $c>0$ trajectories trapped inside M if

$f(x) \cdot \nabla V(x) \leq 0$

Annular region $M = \{x \in \mathbb{R}^2 \mid W(x) \geq c_1 \;\&\; V(x) \leq c_2\}$ $c_1, c_2 > 0$

trajectories are trapped inside M if $f(x) \cdot \nabla W(x) \geq 0$ on $W(x) = c_2$

$f(x) \cdot \nabla V(x) \leq 0$ on $V(x) = c_2$

Ex

$$\Sigma \begin{cases} \dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2) \leftarrow f_1(x_1, x_2) \\ \dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2) \leftarrow f_2(x_1, x_2) \end{cases}$$

Σ dynamics Σ has a unique equili pt determined by setting $\dot{x}_1 = \dot{x}_2 = 0$ & ~~setting~~ solving $f_1(x_1, x_2) = 0$ & $f_2(x_1, x_2) = 0$ which is $(0,0)^T$ Jacobian $\left.\frac{\partial f}{\partial x}\right|_{(0,0)} =$

$$\begin{bmatrix} 1-3x_1^2 - x_2^2 & 1-2x_1x_2 \\ -2-2x_1x_2 & 1-x_1^2-3x_2^2 \end{bmatrix}_{(0,0)} = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$$ eigvals of $\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$ are $1 \pm j\sqrt{2}$

$Re(\lambda) > 0$

$\therefore$ unstable focus

Consider $V(x) = x_1^2 + x_2^2$ func V

$\dot{x}_1 = f_1(x_1, x_2)$

$\dot{x}_2 = f_2(x_1, x_2)$ $f_1(x_1, x_2)$

$$c = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix}$$

at Z moment not known

$M = \{x \in \mathbb{R}^2 \mid V(x) = x_1^2 + x_2^2 \leq c\}$ $c>0$ $f(x) \cdot \nabla V(x) = 0$

M is a closed and bounded set, contains one equili which is unstable

$V(\cdot) = x_1^2 + x_2^2 = c$ on Z surf $\underset{x_1^2+x_2^2}{V(\cdot)} = c$

$f(x) \cdot \nabla V(x) = f_1(\cdot) \frac{\partial V}{\partial x_1} + f_2(\cdot) \frac{\partial V}{\partial x_2} =$

$$\underbrace{2x_1}_{\frac{\partial V}{\partial x_1}} \cdot \underbrace{\left[x_1 + x_2 - x_1(x_1^2 + x_2^2)\right]}_{f_1(\cdot)} + \underbrace{2x_2}_{\frac{dV}{dx_2}} \underbrace{\left[-2x_1 + x_2 - x_2(x_1^2 + x_2^2)\right]}_{f_2(\cdot)}$$

{on surface its tanget $f(x) \cdot \nabla(x) = 0$}

$= 2(x_1^2 + x_2^2) + 2x_1x_2 - 4x_1x_2 - 2(x_1^2 + x_2^2)^2$

$\{-2x_1^2(x_1^2 + x_2^2) - 2x_2^2(x_1^2 + x_2^2) = -2(x_1^2 + x_2^2)^2\}$

$= +2(x_1^2 + x_2^2) - 2(x_1^2 + x_2^2)^2 - 2x_1x_2$

$\because |2x_1x_2| \leq x_1^2 + x_2^2 \leq 2(x_1^2 + x_2^2) - 2(x_1^2 + x_2^2)^2 + (x_1^2 + x_2^2) = 3c - 2c^2$

$\therefore 3c - 2c^2 = 0$ $\because$ of $V(\cdot) = c$ $\therefore \underbrace{f(x) \cdot \nabla V(x) = 0}_{\text{tangential of x on curve}}$ $\therefore c = 1.5$

$f(x) \cdot \nabla V(x) \leq 0$ holds for $c \geq 1.5$ $\therefore$ P.B cond holds in

Poincaré Bendixson

$M = \{x \in \mathbb{R}^2 \mid V(x_1, x_2) = x_1^2 + x_2^2 \leq 1.5\}$ $\therefore \exists$ a periodic orbit in M

Bendixson Criterion $\dot{x}_1 = f_1(x_1, x_2)$ $\dot{x}_2 = f_2(x_1, x_2)$ region D $\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2}$ is not identically zero, & doesn't chg sign then system has no closed trajectory lying entirely on D

Another equiv representation is by Transfer funcs / obtain by Laplace transform
which converts integral & diff eqn into algebraic eqns
Laplace transform of a signal $f(t)$ is the func

$F(s) = \mathcal{L}(f(t))$ ← time domain func
frequency domain ← Laplace transform operator

$F(s)$ is a complex valued func $F(s) = \int_0^\infty f(t) \cdot e^{-st}\, dt$ $s \in \mathbb{C}$ ← complex domain ← complex frequency variable

eg $f(t) = 1$ (unit step) $F(s) = \int_0^\infty 1 \cdot e^{-st} dt = \left[\frac{1}{s} e^{-st}\right]_0^\infty = \frac{1}{s}$

$f(t) = e^t$ $F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^t \cdot e^{-st} dt = \int_0^\infty e^{(1-s)t} dt = \int_0^\infty e^{-(s-1)t} dt =$

$\left[\frac{e^{-(s-1)t}}{-(s-1)}\right]_0^\infty = \frac{1}{s-a}$ provide $s > a$

eg $\frac{df(\cdot)}{dt}$ $F(s) = \int_0^\infty \left(\frac{df}{dt}\right) e^{-st} dt = \left[f(t) e^{-st}\right]_0^\infty - \int_0^\infty f(t) e^{-st} dt$

$= s\mathcal{L}(f(t)) - f(0)$ ← IC

certain properties: $F(s) = \mathcal{L}(f(t))$ $G(s) = \mathcal{L}(g(t))$ &
α & β are consts

$\therefore$ • linearity $\mathcal{L}(\alpha f(t) + \beta g(t)) = \alpha F(s) + \beta G(s)$

• Differentiation $\mathcal{L}\left(\frac{df(\cdot)}{dt}\right) = s\mathcal{L}(f(\cdot)) - f(0)$ ← IC

• Final val thm $\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s)$ these are useful in studying LTI systems

Laplace transform converts derivs into multiplication by s

$\mathcal{L}\left(\frac{d^2 f}{dt^2}\right) = s\mathcal{L}\left(\frac{df}{dt}\right) - \frac{df}{dt}(0) = s[s\mathcal{L}(f(\cdot)) - f(0)] - \frac{df}{dt}(0) =$

$s^2 \mathcal{L}(f(t)) - \underline{f'(0)} - \underline{sf(0)}$ ICs of first deriv & 2 f itself

Real space $\xrightarrow{\mathcal{L}(\cdot)}$ LT - space 's' domain

Higher order ODE — Algebraic eqn for $Y(s)$

$\downarrow$

Solve & rearrange $Y(s) = \cdots$

Sol $y(t)$ $\xleftarrow{\mathcal{L}^{-1}(s)}$

Ex/

$$\frac{d^2x}{dt^2} + 2\xi\omega_n\frac{dx}{dt} + \omega_n^2 x(t) = \omega_n^2 \delta(t) \qquad ICs \quad x(0) = \dot{x}(0) = 0$$

Taking Laplace Transform

$$\mathcal{L}\left[\frac{d^2x}{dt^2} + 2\xi\omega_n\frac{dx}{dt} + \omega_n^2 x\right] = \mathcal{L}[\omega^2\delta(t)]$$

$$s^2 x(s) + 2\xi\omega_n s x(s) + \omega_n^2(s) = \omega_n^2 F(s)$$

where $X(s) = \mathcal{L}(x(t))$ $\quad F(s) = \mathcal{L}(F(t))$

$$(s^2 + 2\xi\omega_n s + \omega_n^2)\underset{\text{state}}{x(s)} = \omega_n^2 F(s) \leftarrow \text{forcing/input}$$

$$G(s) = \frac{X(s)}{F(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$$ is a ratio of polys in 's'

this is called transfer func of system

$$\begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}_n = \underbrace{\begin{bmatrix} \ddots & \\ & \ddots \end{bmatrix}_{n\times n}}_{A} \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix}_n + \underbrace{\begin{bmatrix} & \\ & \end{bmatrix}_{n\times n}}_{B} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}_{n\times n}$$

$$\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n \quad \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_n = \begin{bmatrix} n\times n \end{bmatrix}_{n\times n} \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}_{n\times 1} + \begin{bmatrix} n\times m \end{bmatrix}_{n\times m} \begin{bmatrix} u_1 \\ \vdots \\ u_m \end{bmatrix}_{m\times 1}$$

Consider a single input single output system

$\underset{u(t)}{\text{input}} \rightarrow \underset{\text{state}}{\text{System}} \rightarrow \underset{y(t)}{\text{output}}$

system is represented as

$$\begin{cases} \dot{x}(t) = Ax(t) + bu(t) \\ y(t) = Cx(t) \end{cases} \quad ①$$

$x \in \mathbb{R}^n$ (states)
$u \in \mathbb{R}^1$ (control input)
$y \in \mathbb{R}^1$ (output)

$A \in \mathbb{R}^{n\times n}$ $\quad b \in \mathbb{R}^{n\times 1}$ $\quad C \in \mathbb{R}^{1\times n}$

assume initial condition $x(0) = 0$

taking Laplace Transform of ① yields

$$s\underset{\mathcal{L}(x(t))}{X}(s) = AX(s) + b\underset{\mathcal{L}(u(t))}{U(s)} \longrightarrow sX(s) - AX(s) = bU(s) \qquad (sI_n - A)X(s) = bU(s)$$

identity matrix n dim

$$\underset{\mathcal{L}(Y(t))}{Y(s)} = CX(s) \qquad X(s) = (sI_n - A)^{-1} b U(s)$$

$$\text{output } \underset{\text{output}}{Y(s)} = CX(s) = \underbrace{C(sI_n - A)^{-1} b}_{\text{System}} \underset{\text{input}}{U(s)}$$

transfer func representation $G(s) = \frac{Y(s)}{U(s)} = C(sI_n - A)^{-1}b$

$u(s) \longrightarrow \boxed{System} \xrightarrow{Y(s)}$

- let us revisit

$\delta(t)$ one input $\rightarrow \boxed{System} \longrightarrow$ one output $Y(t) = x(t)$

Ex/ $\frac{d^2x}{dt^2} + 2\xi\omega_n \frac{dx}{dt} + \omega_n^2 x = \omega_n^2 \delta(t)$

State space form $\dot{x} = Ax + Bu$ $\quad Y = Cx$ $\quad$ triplet (A, B, C)

$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix}$, $C = \begin{bmatrix} 1 & 0 \end{bmatrix}$ $\quad X = [x, \dot{x}]^T$

$\Downarrow$

converted to transfer func

$\frac{Y(s)}{U(s)} = C(sI_n - A)^{-1}B$ $\quad$ use (A, B, C)

$(sI_n - A)^{-1} = \begin{bmatrix} s & 1 \\ -\omega_n^2 & s - 2\xi\omega_n \end{bmatrix}^{-1}$ $\quad \left\{ A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \right\}$

$= \frac{1}{s^2 - 2\xi\omega_n s + \omega_n^2} \begin{bmatrix} -s + 2\xi\omega_n & 1 \\ -\omega_n^2 & s \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix}$ $\quad C = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$\frac{Y(s)}{F(s)} = \begin{bmatrix} 1 & 0 \end{bmatrix} \left(\frac{1}{s^2 + 2\xi\omega_n s + \omega_n^2} \begin{bmatrix} -s + 2\xi\omega_n & 1 \\ -\omega_n^2 & s \end{bmatrix} \right) \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix}$

$G(s) = \frac{Y(s)}{F(s)} = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$

$\frac{dx^2}{dt^2} + 2\xi\omega_n \frac{dx}{dt} + \omega_n^2 x = \omega_n^2 \delta(t)$, $x(0) = \dot{x}(0) = 0$ $\quad$ ODE

$\therefore \mathcal{L}(\cdot) \therefore$ $\quad G(s) = \frac{\omega_n^2}{s^2 + 2\xi\omega_n s + \omega_n^2}$ $\quad \Longleftrightarrow$ equiv representations of linear time invariant systems

state space form $\quad \dot{x} = Ax + Bu$, $\quad Y = Cx$ $\quad x = \left[x \;\; \frac{dx}{dt} \right]^T$

$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix}$ $\quad C = [1, 0]$ $\quad$ input $\rightarrow \boxed{\text{LTI System}} \rightarrow$ output

③ ODEs

by ①: Compact state space

generic $\dot{x} = Ax + BU$

$Y = Cx + DU$

represented as

(A, B, C) triplet

by ②: $\mathcal{L}(\cdot)$ transfer func

$G(s) = C(sI_n - A)^{-1}B$ ratio of poly

i.e. $G(s) = \frac{N(s)}{D(s)}$ charac poly

$D(s) = 0$ (poly) roots of this poly are identified as poles of transfer func (T.F)

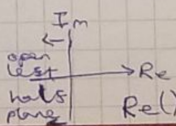
$Re(\text{poles}) < 0$

- stability

can be determined

by looking eigvals of $[A]$

charac eqn, $\det(sI - A) = 0$



$Re(\lambda) < 0$

\Ex/ (A,B,C) $A=\begin{bmatrix}-7 & 12\\ 1 & 0\end{bmatrix}$ $B=\begin{bmatrix}1\\0\end{bmatrix}$ $C=[1\ 2]$

⇒ obtain T.F. $G(s)=\frac{Y(s)}{U(s)}$ $G(s)=C(sI-A)^{-1}B$

$=\frac{s+2}{s^2+7s+12}$ state space:

$\det(\lambda I_n - A)=0$ $\lambda^2+7\lambda+12=0$ $\lambda_{1,2}=-3,-4$ eigvals real neg

T.F. : $\frac{N(s)}{D(s)}$ $D(s)=0$ $s^2+7s+12=0$

poles as -3 & -4 T.F.

\Week 5 linear control design state feedback - pole placement/

Consider LTI system $\dot{x}=Ax+Bu$ ① $y=Cx$ $x\in\mathbb{R}^n$
not known & to be designed

(A,B,C) triple

let $r(t)$ be a reference profile which the LTI system expected to follow. This is tracking problem (ie find a 'k' st system follows $r(t)$) $r(t)\equiv0$, is a special case called regulation problem. our aim is to design control law ② $u=-kx$ states
is state feedback control

∴ with ② into ①: $\dot{x}=Ax-Bkx=(A-Bk)x=A_{CL}x$

is closed loop system 'k' to be designed

suppose $r\equiv0$ we consider regulation problem

$\dot{x}=Ax+Bu$ $y=Cx$ $u=-kx$ to be determined
is identity matrix

$\dot{x}=\underbrace{(A-Bk)}_{A_{CL}}x$ ← is closed loop system our aim is have A_{CL} as a stable matrix ie A_{CL} is Hurwitz (ie $Re(\lambda(A_{CL}))<0$)

the charac poly associated with A_{CL} $\det(sI-(A-Bk))=0$ ③ ← identity matrix

eqn③ treat as poly in 's'

know what the closed loop behaviour should be in terms of its eigvals which are $\mu_1,\mu_2,\dots,\mu_n$ ← desired locations desires specific behaviour for systems

∴ the polyn associated with these eigvals are $\prod_{i=1}^{n}(s-\mu_i)$ ← our desired charac poly

(have assumed for time being 'destinct' eigvals)

$\prod_{i=1}^{n}(s-\mu_i)=(s-\mu_1)(s-\mu_2)\dots(s-\mu_n)$ ~~note~~

\pole placement - design steps / (A,B,C) & desired eigvals locations $\mu_1,\dots,\mu_n$ gives

Step 1: Check 2 Controlability of (A,B)

$M=[B \,\vdots\, AB \,\vdots\, A^2B \,\vdots \cdots \vdots\, A^{n-1}B]$

- rank$(M)=n$ ie full rank is SISO ← single input single out put, det(M) ≠ 0 ← determinant

Step 2: aim to design $u=-kx$

where ~~x∈R~~, $x\in\mathbb{R}^n$ then $k=[k_1\ k_2 \cdots k_n]$

~~vec~~ vec of variabls k_i (here we are considering single input)

\Step 3/ Close loop System $\dot{x}=(A-Bk)x$ charac poly is equated to desired charac polys

$$\det(SI_n-(A-Bk)) = \prod_{i=1}^{n}(s-\mu_i)$$

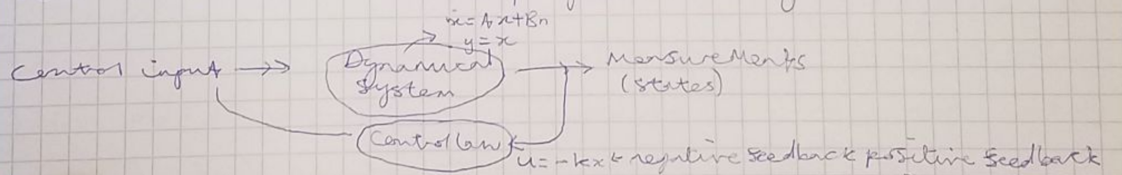
vec of variables k_i

- Step 4: Solve for variables k_i's by matching coeffs of polys

then $u=-kx$ is ~~state~~ feed back control law

\Ex/ $\dot{x}=Ax+Bu$ $y=Cx$ $A=\begin{bmatrix}1&2\\2&1\end{bmatrix}$ $B=\begin{bmatrix}0\\1\end{bmatrix}$ $C=[1\ 0]$

find $u=-kx$ st closed loop System has eignals -4 & -5



- Step 1: Controllability $x\in\mathbb{R}^2$ $M=[B \,\vdots\, AB]=\left[\begin{matrix}0\\1\end{matrix} \,\vdots\, \begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right]=$ $\begin{bmatrix}0&2\\1&1\end{bmatrix}$ rank$(M)=2$ full rank $\det(M)=-2\neq 0$ ∴ (A,B) is controllable

Step 2: $k=[k_1\ k_2]$

Step 3 desired closed loop evals are -4 & -5

$$\prod_{i=1}^{2}(s-\mu_i)=(s+4)(s+5)=s^2+9s+20 \qquad \mu_1=4,\ \mu_2=5$$

Charac poly associated with closed loop system $(A-Bk)$ is

$$A-Bk=\begin{bmatrix}1&2\\2&1\end{bmatrix}-\begin{bmatrix}0\\1\end{bmatrix}[k_1\ k_2]=\begin{bmatrix}1&2\\2-k_1&1-k_1\end{bmatrix}$$

- $\det(SI_2-(A-Bk))=s^2+(k_2-2)s+(-3+2k_1-k_2)$ equating 2 two polys in 's' comparing coeffs $-2+k_2=9$ $-3+2k_1-k_2=20$

$k_2=11$ $k_1=17$ control law $u=-kx=[-17\ -11]x$

towards a General Case $\dot{x} = Ax + Bu$ $x \in \mathbb{R}^n$

$y = Cx$ $u = -Kx$ Suppose our A, B posses a specific structure as follows:

$$A = \begin{bmatrix} 0 & \vdots & 1 & & 0 \\ 0 & \vdots & & 1 & \\ \vdots & \vdots & 0 & & \ddots \\ & \vdots & & & 1 \\ -a_n & \vdots & -a_{n-1} & \cdots & -a_1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

this specific structure is called a Controllable Canonical form

$$\det(sI - A) = s^n + a_1 s^{n-1} + a_2 s^{n-2} + \dots + a_{n-1} s + a_n$$

Closed loop System $\dot{x} = (A - Bx) = A_{CL} x$

$$A_{CL} = \begin{bmatrix} 0_{n-1} & \vdots & \mathbb{I}_{n-1} & & \\ \hline -a_n & -a_{n-1} & -a_{n-2} & \cdots & a_1 \end{bmatrix} - \underbrace{\begin{bmatrix} 0_{n-1} & \vdots & 0_{(n-1)\times(n-1)} & & \\ \hline k_1 & & k_2 & \cdots & k_n \end{bmatrix}}_{B \cdot K} \Big\}\text{only one row}$$

$$A_{CL} = \begin{bmatrix} 0_{n-1} & \vdots & \mathbb{I}_{n-1} & & \\ \hline \underset{\text{known}}{(-a_n - k_1)} & \vdots & (-a_{n-1} - k_2) & \cdots & (-a_1 - k_n) \end{bmatrix} \quad (M_1)$$

only one row

$\therefore$ know $\mu_1, \mu_2, \dots, \mu_n$

destinct desired evals / poles $\prod_{i=1}^{n}(s - \mu_i) = s^n + \alpha_1 s^{n-1} + \dots + \alpha_n$ (M_2)

like earlier associated system matrix will be $\begin{bmatrix} 0_{n-1} & \vdots & \mathbb{I}_{n-1} & \\ \hline -\alpha_n & \vdots & -\alpha_{n-1} \cdots & -\alpha_1 \end{bmatrix}$

Comparing (M_1) & (M_2)

$$+a_n + k_1 = \alpha_n \qquad\longrightarrow\qquad k_1 = \alpha_n - a_n$$

$$a_{n-1} + k_2 = \alpha_{n-1} \qquad\qquad k_2 = \alpha_{n-1} - a_{n-1}$$

$$\vdots \qquad\qquad\qquad \vdots$$

$$\underset{\text{known}}{a_1} + \underset{\text{to be found}}{k_n} = \alpha_n \leftarrow \text{known} \qquad\qquad k_n = \alpha_1 - a_1$$

$\therefore u = -kx$ $k = [k_1, \dots, k_n]$

How to obtain this structure for (A, B) is Controllable Canonical form?

obtain a Controllable Canonical form. Define square invertible matrix T let $\overset{①}{x} = T \overset{\swarrow\text{transformation}}{\tilde{x}}$ $x \in \mathbb{R}^n$ & $\tilde{x} \in \mathbb{R}^n$ represents states & states in transformed coords

taking derivative of ① wrt time $\dot{x} = T\dot{\tilde{x}}$ $\because$ T is invertible:

$$\dot{\tilde{x}} = T^{-1}\dot{x} = T^{-1}(Ax + Bu) = T^{-1}Ax + T^{-1}Bu = T^{-1}AT\tilde{x} + T^{-1}Bu \quad [\because x = T\tilde{x}]$$

$\dot{\tilde{x}} = T^{-1}AT\tilde{x} + T^{-1}Bu$ how can we determine a T st have controllable canonical form $[T]$ is square & invertible

define $T = \overset{\swarrow\text{controllability matrix}}{M}W$ $M = [B \vdots AB \vdots \cdots \vdots A^{n-1}B]$ &

W is chosen as an upper triangular matrix $W = \begin{bmatrix} a_{n-1} & a_{n-2} & a_{n-3} & \cdot & 1 \\ a_{n-2} & a_{n-1} & a_{n-4} & \cdot & 0 \\ \vdots & \vdots & \vdots & \cdot & \vdots \\ a_1 & 1 & 0 & \cdot & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

diagonal is 1 lower triangle part zero

entries a_i are Z coeffs of charac poly associated with A $\det(sI-A)=s^n+\cancel{a_1 Z}\ a_1 s^{n-1}+\dots+a_{n-1}s+a_n$ $\because A\in\mathbb{R}^{n\times n}$

- Z $T=M\cdot W$ will provide us a Controllable Canonical representation, ie Z usesul Structure

Given $\dot{x}=Ax+Bu$ (not necessarily in Controllable canonical form)

$\xrightarrow{\tilde{x}=T^{-1}x}$ {Z $T:=M\cdot W$ M is Controllability Matrix
W is formed from charac poly of A which is upper triangular}

$\rightarrow$ Transformed System: $\dot{\tilde{x}}=(T^{-1}AT)\tilde{x}+(T^{-1}B)u$ $\rightarrow$ design $\rightarrow$

$u=-\tilde{k}\tilde{x}\rightarrow u=\underbrace{-\tilde{k}T^{-1}}_{\bar{k}}x$ $\because x=T\tilde{x}$ ~~$u=\bar{k}x$~~

$u=-k\bar{x}\rightarrow$ Control law for Z original System

where in original coords $\bar{k}=\tilde{k}T^{-1}$

\Ex/ $\dot{x}=Ax+Bu$ find $u=-kx$ st closed loop evals are -4 & -5

$A=\begin{bmatrix}1&2\\2&1\end{bmatrix}$ $B=\begin{bmatrix}0\\1\end{bmatrix}$

<u>Step</u> 1: Check controllability $M=\begin{bmatrix}0&2\\1&1\end{bmatrix}$ M full rank $\det(M)\neq0$

$\therefore$ Controllable

<u>Step</u> 2: Charac poly for $[A]$ $\det(sI-A)=s^2-2s-3$ is of form

$s^2+a_1s+a_2$

<u>Step</u> 3: Obtain square invertible T $T=MW$ where

$W=\begin{bmatrix}a_1&1\\1&0\end{bmatrix}$ upper triangular structure $W=\begin{bmatrix}-2&1\\1&0\end{bmatrix}$

$T\doteq MW=\begin{bmatrix}2&0\\-1&1\end{bmatrix}$

<u>Step</u> 4: desired poles/evals -4 & -5 gives

$\prod_{i=1}^{2}(s-\mu_i)=s^2+9s+20$ $s^2+\overset{9}{\alpha_1}s+\overset{20}{\alpha_2}$ desired form

<u>Step</u> 5: feedback gain matrix $\bar{k}=[(\alpha_2-a_2)\ (\alpha_1-a_1)]T^{-1}=$

$[23\ \ 11]\begin{bmatrix}1&0\\1&2\end{bmatrix}=[34\ \ 22]$ $[17\ \ 11]$ (by scaling 2033)

it is also possible to design control laws using <u>Ackerman's</u> formula $\mathbb{K}=[0^{n-1}\ ;\ 1]M^{-1}\Delta_{des}(A)$ M is Controllability Mat

- $\Delta_{des}(A)$ is desired charac poly evaled at Mat $[A]$.

$u=kx$ [note Z sign change]

\Same Ex example:/ $A=\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}$ $B=\begin{bmatrix}0\\ 1\end{bmatrix}$ controllability mat

$M=\begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}$ $M^{-1}=\text{inv}(M)=\begin{bmatrix}1 & -2\\ -1 & 0\end{bmatrix}$ desired charac poly is

$s^2+9s+20$ (step 4)

following Ackermann's formula $K=\begin{bmatrix}0 & 1\end{bmatrix}\begin{bmatrix}1 & -2\\ -1 & 0\end{bmatrix}\begin{bmatrix}A^2+9A+20\end{bmatrix}$

$s^2+9s+20$
$\uparrow$
$A^2+9A+20$

$K=\begin{bmatrix}-34 & -22\end{bmatrix}=-\begin{bmatrix}17 & 11\end{bmatrix}$ $\therefore$

$u=-\begin{bmatrix}17 & 11\end{bmatrix}x$

let $D\subseteq\mathbb{R}^n$ & $f(\cdot):D\to\mathbb{R}^n$

$\dot{x}=f(x)$
$\mathbb{R}^2 \to \begin{bmatrix}f_1(x_1,x_2)\\ f_2(x_1,x_2)\end{bmatrix}$

\Def:/ $f(\cdot)$ is <u>bounded</u> on D if $\exists\ \alpha>0$ st

$\underbrace{\|f(\cdot)\|}_{\text{norm of } f(\cdot)}\le\alpha \quad \forall x\in D$

\Def:/ $f(\cdot)$ is <u>continuous</u> at $x\in D$, if, for every $\varepsilon>0$, $\exists\ \delta=\delta(\varepsilon,x)>0$ st $\|f(x)-f(y)\|<\varepsilon\ \forall y\in D$ satisfying $\|x-y\|<\delta$

\Def:/ $f(\cdot)$ is <u>uniformly continuous</u> on D if $\forall\ \varepsilon>0$, $\exists\ \delta=\delta(\varepsilon)>0$ st $\|f(x)-f(y)\|<\varepsilon\ \forall x,y\in D$ satisfying $\|x-y\|<\delta$

\Def:/ \Lipschitz continuity/ $f(\cdot)$ is <u>Lipschitz continuous</u> at $x_0\in D$, if $\exists$ a Lipschitz const $L=L(x_0)>0$ & a neighborhood about x_0 $N\subset D$ of x_0 st

$\|f(x)-f(y)\|\le \underbrace{L}_{\text{Lipschitz const}}\|x-y\|$, $x,y\in \underbrace{N(x_0)}_{\text{neighbood centered about } x_0}$

$f(\cdot)$ is Lipschitz continuous on D, if $f(\cdot)$ is Lipschitz cont at every pt in D

\Def:/ $f(\cdot)$ is <u>uniformly Lipschitz</u> continuous on D if $\exists$ $L>0$, st $\|f(x)-f(y)\|\le L\|x-y\|$, $x,y\in D$

\Def:/ $f(\cdot)$ is <u>globally</u> Lipschitz cont if $f(\cdot)$ is uniformly Lipschitz cont on $\mathbb{R}^n$

{useful prerequisit also in week 5}

let $f(\cdot) : [a,b] \times D \to \mathbb{R}^n$ be a cont func for some domain $D \subset \mathbb{R}^n$. Suppose that $\left[\frac{\partial f(\cdot)}{\partial x}\right]$ exists, & is cont on $[a,b] \times D$. If for a convex set $W \subset D$, $\exists$ a const $L \geq 0$ s.t. $\left\|\frac{\partial f(\cdot)}{\partial x}\right\|_\infty \leq L$ on $[a,b] \times W$ then
$\|f(t,x) - f(t,y)\| \leq L\|x-y\|$,
$\forall t \in [a,b]$, $x \in W$ & $y \in W$

$$\begin{cases} \dot{x} = f(x) \\ \dot{x}_1 = -x_1 + x_1 x_2 \\ \dot{x}_2 = x_2 - x_1 x_2 \end{cases}$$

\Ex/ \numerical ex/ $f(x) = \begin{bmatrix} -x_1 + x_1 x_2 \\ x_2 - x_1 x_2 \end{bmatrix}$

Calc Lipschitz const L over Z convex
$W = \{x \in \mathbb{R}^2 \mid |x_1| \leq a_1 \quad |x_2| \leq a_2\}$

\Sol:/ $f(x)$ is continuously diffable on $\mathbb{R}^2$ $\therefore$ Lipschitz on domain D of $\mathbb{R}^2$, or any compact subset of $\mathbb{R}^2$ $\left[\frac{\partial f}{\partial x}\right]$ exists &

$\left[\frac{\partial f}{\partial x}\right] = \begin{bmatrix} -1+x_2 & x_1 \\ -x_2 & 1-x_1 \end{bmatrix}$ $\quad \frac{\partial f}{\partial x}$ is continuous too

using $\|\cdot\|_\infty$ for vecs in $\mathbb{R}^n$; and induce matrix norm for matrices, have $\left\|\frac{\partial f}{\partial x}\right\|_\infty = \max\left\{\underbrace{|-1+x_2| + |x_1|}_{\text{from first row}}, \underbrace{|-x_2| + |1-x_1|}_{\text{2nd row}}\right\}$

$\because |x_1| < a_1$, $|x_2| < a_2$

$|-1+x_2| + |x_1| \leq 1 + a_2 + a_1 = 1 + a_1 + a_2$

$\{$ if $x_2 = a_2$: $|-1+a_2| < |-1| + |a_2|$ $\}$

likewise $|-x_2| + |1-x_1| \leq a_2 + 1 + a_1$ $\therefore$
$\left\|\frac{\partial f}{\partial x}\right\|_\infty \leq 1 + a_1 + a_2$ &
Lipschitz const is $L = 1 + a_1 + a_2$

\Thm/ \Contraction Mapping thm/
let S be a closed subset of a Banach space $\mathcal{X}$ & let $T : S \to S$ suppose that $\|T(x) - T(y)\| \leq \rho\|x-y\|$; $\forall x, y \in S$ & $0 \leq \rho < 1$ $\quad \therefore \exists$ a unique $\bar{x} \in S$ satisfying $\bar{x} = T(\bar{x})$
$\bullet$ $\bar{x}$ can be obtained by Z method of successive approx starting from any initial vec in S

Week 6 Lyapunov Stability theory

Consider $\dot{x} = f(t,x)$, $x \in \mathbb{R}^n$, $x(0)$ initial cond.

Let $x_e \in \mathbb{R}^n$ be equilib, i.e. $f(t, x_e) = 0 \quad \forall t \geq t_0$

Def: equili x_e is stable in the sense of Lyapunov, if

$\forall \varepsilon > 0, \exists \delta > 0$ st if $\|x_0 - x_e\| < \delta$ then

$\|\underbrace{x(t, t_0, t_0)}_{sol} - \underbrace{x_e}_{equilib\ pt}\| < \varepsilon \quad \forall t \geq t_0$ — x_0 near enough to equili x_e

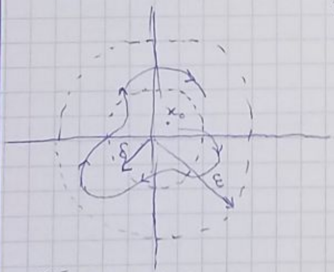
$\Downarrow$

Sols remain in some ball around x_e $\forall$ time t

remark: an equili pt x_e can be shifted to origin for autonomous system via a suitable change of variabs

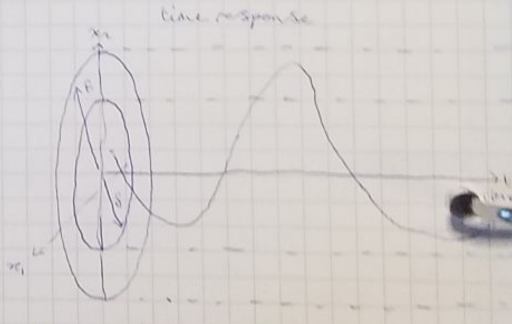
Def: A zero sol. origin, of $\dot{x} = f(x)$ is Lyapunov stable, if $\forall \varepsilon > 0, \exists \delta = \delta(\varepsilon) > 0$ st if $\|x(0)\| < \delta$ then $\|x(t)\| < \varepsilon$, $\forall t \geq 0$

$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \quad \forall t \geq 0 \quad \varepsilon, \delta > 0$ Lyapunov Stability



origin is equili

$\dot{x} = f(x) \quad x \in \mathbb{R}^2$

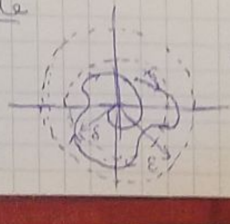


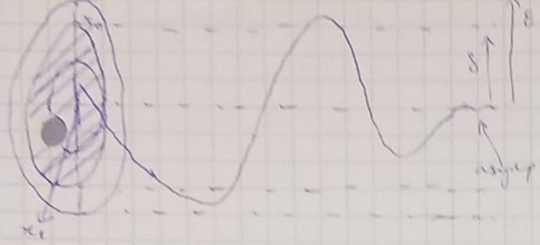
$\forall t > t_0 \quad \|x(t)\| < \varepsilon$

Def: The zero sol of $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ is (locally) asymply stable, if it is Lyapunov stable, & $\exists \delta > 0$ st if $\|x(0)\| < \delta$, then $\lim_{t \to \infty} x(t) = x_e = 0$

$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$ Asymply Stable

(locally) (L.AS)





as time $t \to \infty$

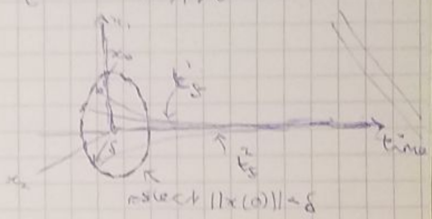
$\lim_{t\to\infty} x(t) - x_e = 0$

\Def:/ Z zero sol origin of $\dot{x} = f(x)$ is (locally) exponentially stable if $\exists$ pos consts $\alpha, \beta, \& \delta$ st

if $\|x(0)\| < \delta$, then $\|x(t)\| \leq \alpha \|x(0)\| \exp^{-\beta t}$, $t \geq 0$

$\|x(0)\| < \delta \Rightarrow \|x(t)\| \leq \alpha \|x(0)\| \exp^{-\beta t}$ $t \geq 0$ $\alpha, \beta, \delta > 0$

exponentially stability



\Def:/ Z origin $\dot{x} = f(x)$ is globally Asymp Stable if it is Lyapunov stable & $\forall x(0) \in \mathbb{R}^n$ $\lim_{t\to\infty} x(t) = 0$

\Def:/ Z origin of $\dot{x} = f(x)$ is globally exponentially stable if $\exists\ \alpha, \beta$ pos scalars st $\|x(t)\| \leq \alpha \|x(0)\| \exp^{-\beta t}$, $t \geq 0$

$\forall x(0) \in \mathbb{R}^n$ - Not $\|x(0)\| < \delta$

in Lyapunov analysis we are intrested in funcs $V(t, x)$ candidate Lyapunov func) & how this $V(\cdot)$ changes along Z vec field $f(\cdot)$ due to Z dynamics $\dot{x} = f(\cdot)$

$\dot{V}(x, t) = \frac{\partial V}{\partial x} \underbrace{f(t, x)}_{\dot{x}} + \underbrace{\frac{\partial V}{\partial t}}_{10}$ is not present for any autonomous systems

Lie derivative of $V(\cdot)$ wrt $f(\cdot)$

when dynamics is autonomous $V(x)$ is sufficient &

$\dot{V}(x) = \frac{\partial V}{\partial x} f(x)$ for $\dot{x} = f(x)$ $x \in \mathbb{R}^n$

$\dot{x} = f(x)$, $x \in \mathbb{R}^n$ $x = [x_1, \ldots, x_n]^T$ $f(\cdot) = [f_1(\cdot), \ldots, f_n(\cdot)]^T$

Consider $V(x) = V(x_1, x_2, \ldots, x_n)$ taking time derivative of

$\dot{V}(x)$ (ie derivative of $V(\cdot)$ along Z trajectories of dynamics $\dot{x} = f(x)$)

$$\frac{dV}{dt} = \frac{\partial V}{\partial x_1} \cdot \underbrace{\frac{dx_1}{dt}}_{\dot{x}_1} + \frac{\partial V}{\partial x_2} \underbrace{\frac{dx_2}{dt}}_{\dot{x}_2} + \dots + \frac{\partial V}{\partial x_n} \underbrace{\frac{dx_n}{dt}}_{\dot{x}_n} = \text{Grad} V \cdot \frac{dx}{dt}$$

$$\text{Grad} V := \left[\frac{\partial V}{\partial x_1}, \frac{\partial V}{\partial x_2} \dots, \frac{\partial V}{\partial x_n} \right] \quad \& \quad \frac{dx}{dt} = \dot{x} = \left[f_1(\cdot) \quad f_2(\cdot) \quad \dots \quad f_n(\cdot) \right]^T$$

idea of Lyapunov theory: Suppose we find some measure of energy of system, for e.g. $V(x) = ||x||^2$, st following conditions hold

- $V(x_e, t) = 0 \quad \forall t > t_0$ ie $V(\cdot)$ ended at equili (origin for autonomous) $V(0)=0 \quad V(x_e)=0$
- $V(x_e, t) > 0$ when $x \neq x_e \quad \forall t > t_0$ ie func $V(\cdot)$ pos for all x other than at equili
- $\dot{V}(x_e, t) \leq 0$ ie energy is non-increasing along all system trajectories $\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial t}$ ommited in auto systems

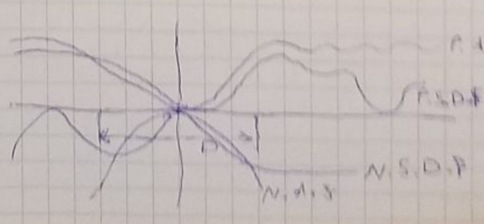
\ posit / negative definiteness / consider domain D for $x \in \mathbb{R}^n$ which contains a neighood of origin (equili pt) $\{x \in \mathbb{R}^n \mid ||x|| < r\}$ where $r > 0$

\Def:/ a func $V(\cdot): D \to \mathbb{R}$ is (locally) positive definite in domain $D \subset \mathbb{R}^n$ if $V(x) > 0 \quad \forall x \in D \setminus \{0\}$ or $\{x_e\}$ in non-zero equili

$V(x) = 0 \Rightarrow x = 0$ ← or x_e for non-zero equili

\Def:/ a func $V(\cdot): D \to \mathbb{R}$ is (locally) negative definite in domain $D \subset \mathbb{R}^n$ if $V(x) < 0 \quad \forall x \in D \setminus \{0\}$ $\quad V(x) = 0 \to x = 0$

\Def:/ a func $V(\cdot): D \to \mathbb{R}$ is (locally) posi semi definite in $D \subset \mathbb{R}^n$ if $V(x) \geq 0 \quad \forall x \in D$



but if you change Z domain D you can change its def

\Lyapunov thm:/ Consider $\dot{x} = f(x), x \in \mathbb{R}^n, x(0) = x_0$ &
assume that there $\exists$ a continuously diffable func $V: D \to \mathbb{R}$,
p.d.f where $D \subset \mathbb{R}^n$ st at origin, assume as equili, $V(0) = 0$
$V(x) > 0, \forall x \in D \setminus \{0\}$ (ie $V(\cdot)$ is strictly posi $\forall x$ in domain D
excluding origin)
$\dot{V}(x) \le 0 \; \forall x \in D$ ← N.S.D.F
then origin is Lyapunov stable this ($\|x(0)\| < \delta \Rightarrow \|x(t)\| < \varepsilon \; \forall t \ge 0$)
in addition if $\dot{V}(x) < 0 \quad \forall x \in D \{0\}$ then Z origin is asymp stable
& if $\exists$ scalars $\alpha, \beta, \varepsilon > 0$ & $p \ge 1$ st $V: D \to \mathbb{R}$ satisfies
$\alpha \|x\|_p \le V(x) \le \beta \|x\|_p \quad x \in D \qquad \dot{V}(x) \le -\varepsilon V(x), \; x \in D \setminus \{0\}$
then Z origin is exponentially stable

\Lyapunov thm/ For $\dot{x} = f(x), x \in \mathbb{R}^n, x(0) = x_0$ suppose that
$\exists$ a continuously diffable $V(x): D \to \mathbb{R}$, where $D \subset \mathbb{R}^n$ st
i) $V(\cdot)$ p.d.f $\quad \dot{V}(\cdot)$ n.s.d.f $\quad$ Lyapunov stable
ii) $V(\cdot)$ p.d.f $\quad \dot{V}(\cdot)$ n.d.f $\quad$ Asymptotically stable
(iii) $V(\cdot)$ p.d.f $\quad \dot{V}(\cdot) \le -\varepsilon V \quad \varepsilon > 0 \quad$ exponentially stable

p.d.f - positive definite func
n.d.f - nega definite func
n.s.d.f nega semi definite func
$\dot{V}(\cdot) = \frac{\partial V}{\partial x} \cdot \dot{x}$

\Quadratic forms/
\Def/: A matrix M is posi definite if $x^T M x > 0, \forall x \neq 0$
(if $M \in \mathbb{R}^{n \times n}, x \in \mathbb{R}^n$) a matrix M is posi semi definite if
$x^T M x \ge 0, \; \forall x$
\Lemma:/ $M = M^T$ is posi definite $\Leftrightarrow \lambda_i(M) > 0 \; \forall i$ eigvals of
Mat M
$M = M^T$ is posi semi definite $\Leftrightarrow \lambda_i(M) \ge 0 \quad \forall i \quad$ if $M = M^T$ & p.d.f.
then $V(\cdot) = x^T M x$ this fulfills $V(0) = 0$ & $V(x) > 0 \quad \forall x \neq 0$
$\therefore V(\cdot)$ is p.d.f {quadratic form $x^T M x \quad M = M^T$ p.d.f}

& $\lambda_{min} ||x||^2 \le x^T M x \le \lambda_{max} ||x||^2$

λ_{min} & λ_{max} are min & max eignals of $[M]$

A func $V: \mathbb{R}^n \to \mathbb{R}$ of Z form $V(x) = x^T M x = \sum_{i,j=1}^{n} M_{ij} x_i x_j$, &

$M = M^T$ p.d.f, is called Quadratic form

Ex $M = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$ M is symmetric eignals of M 0.2984, 6.7016

$\{ \det(\lambda I - M) = 0 \}$ $\therefore \lambda_i(M) > 0$, $i = 1, 2$ $\therefore$

$[M]$ is p.d.f

a quadratic form associated with M for $x \in \mathbb{R}^2$ is

$V(x_1, x_2) = [x_1 \; x_2] \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

2) $V(x_1, x_2) = (x_1 - x_2)^2$ V is posi. square term

V is p.d.f $V(0) = 0$ & $V(x) > 0$ $\forall x \setminus \{0\}$

$\{ x_1 = x_2 = 0$ $V(0) = 0$ $x_1 = x_2$ $V(\cdot) = 0 \}$

$\therefore V(\cdot)$ is not posi definite, posi semi definite $V(x) \ge 0$ $\forall x$

3) $\dot{x}_1 = I_{23} x_2 x_3$ ① $I_{23} = \frac{I_2 - I_3}{I_1}$ $I_{31} = \frac{I_3 - I_1}{I_2}$ $I_{12} = \frac{I_1 - I_2}{I_3}$

$\dot{x}_2 = I_{31} x_3 x_1$ $I_1 > I_2 > I_3$

$\dot{x}_3 = I_{12} x_1 x_2$

$V(x_1, x_2, x_3) = \frac{1}{2}(\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2)$ $\alpha_1, \alpha_2, \alpha_3$ posi

investigate stability of origin

$V(\cdot) = \frac{1}{2} \alpha_i x_i^2$ α_i +ve $\forall i$ x_i^2 +ve $\forall i$ sum of squares

$V(0) = 0$ $V(\cdot)$ is posi $\forall x = (x_1, x_2, x_3)^T / \{0\}$

$V(\cdot)$ is p.d.f.

taking derivative of $V(\cdot)$ along Z dynamics ①

$\dot{V}(\cdot) = \sum_{i=1}^{3} \alpha_i x_i \dot{x}_i$ subing for x_i from eqn ① yields

$\dot{V}(\cdot) = \alpha_1 x_1 (I_{23} x_2 x_3) + \alpha_2 (x_2 (I_{31} x_3 x_1) + \alpha_3 x_3 (I_{12} x_1 x_2) =$

$(\alpha_1 I_{23} + \alpha_2 I_{31} + \alpha_3 I_{12}) x_1 x_2 x_3 = \dot{V}(\cdot) =$

$(\alpha_1 I_{23} + \alpha_2 I_{31} + \alpha_3 I_{12}) \underbrace{x_1 x_2 x_3}_{\text{sign indefinite}}$ $\{ \dot{V}$ n.s.d.f. $\dot{V}(0) = 0$ $\dot{V}(x) \le 0$ $\forall x$, is $\dot{V}$ n.d.f $\dot{V}(0) = 0$ $\dot{V}(x) < 0$ $\forall x \setminus \{0\} \}$

α_i +ve

to be Lyapunov stable is $\alpha_1 I_{23} + \alpha_2 I_{31} + \alpha_3 I_{12} = 0$

then $\dot{V}(\cdot)$ is n.s.d.f. $\therefore$

$I_{31} < 0$ it is always possible to pick a set of $\alpha_1, \alpha_2, \alpha_3$ st

$\alpha_1 I_{23} + \alpha_2 I_{31} + \alpha_3 I_{12} = 0$

$\left\{ \alpha_1 I_{23} - \alpha_2 \underset{\because I_{31} \text{ negn}}{I_{31}} + \alpha_3 I_{12} \right\}$

$V(\cdot)$ is p.d.f. $(\alpha_i > 0)$

$\dot{V}(\cdot) \leq 0 \quad \forall x_1, x_2, x_3$ for Z choice of $\alpha_2 I_{31}$ st

$\alpha_1 I_{23} + \alpha_2 I_{31} + \alpha_3 I_{12} = 0$ with that it is possible to ensure origin Lyapunov stable

\Lyapunov thm - Global Stability/ $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ $\exists$ a contly diffble $V(\cdot): \mathbb{R}^n \to \mathbb{R}$ st

$\left. \begin{array}{l} V(0) \\ V(x) > 0 \quad , x \in \mathbb{R}^n, x \neq 0 \end{array} \right\}$ P.d.f

$\frac{\partial V}{\partial x} \cdot f(x) \to \dot{V}(x) < 0 \quad , x \in \mathbb{R}^n \setminus \{0\}$ $\}$ N.d.f

$V(x) \to \infty$ as $\|x\| \to \infty$ (radially unbounded func) ~~{at origin}~~

then, Z zero sol, $x(t) = 0$ (at origin) to $\dot{x} = f(x)$

is globally asymptotically stable

if $\exists$ $\alpha, \beta, \varepsilon > 0$ & $p \geq 1$

\Def/ A set $M \subset \mathbb{R}^n$ is said to be invariant set with respect to $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ if $x(0) \in M$, implies $x(t) \in M$ $\forall t \in \mathbb{R}$

\Def/ A set $M \subset \mathbb{R}^n$ is said to be positively invariant with respect to $\dot{x} = f(x)$ $x \in \mathbb{R}^n$, if $x(0) \in M$ implies $x(t) \in M$, $\forall t \geq 0$

\Def/ a pt 'p' is Z posi limit pt of $\dot{x} = f(x)$, if $\exists$ a sequence of $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ st $x(t_n) \to p$ as $n \to \infty$. Z set of all posi limit pts of $x(t)$ is called posi limit set of $x(t)$

① Lyapunov stability - LTI syst

② nonlinear / uncertainty bound

③ region of attraction

\Ex/ consider $\dot{x} = Ax$, $x \in \mathbb{R}^n$ LTI syst, $A \in \mathbb{R}^{n \times n}$

if $Re(\lambda_i(A)) < 0 \implies [A]$ is <u>hurwitz</u>

real part of eignvals of $[A]$ & origin of $\dot{x} = Ax$ is <u>asymp stab</u>

consider $V(x) = x^T P x$ where $P = P^T > 0$, $P \in \mathbb{R}^{n \times n}$ is real symmetric p.d.f Mat $\Rightarrow$

$\dot{V}(x)$ is p.d.f $\quad \dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x$ (using $\dot{x} = Ax$)

$= x^T P(Ax) + (Ax)^T P x = x^T P A x + x^T A^T P x \qquad \{(AB)^T = B^T A^T\}$

$= x^T [PA + A^T P] x = x^T (-Q) x$, where Q is symmetric posi definite

$-Q$ is -ve definite

$\dot{V}(x)$ is nega definite

in Z case of linear systs $\dot{x} = Ax$, $x \in \mathbb{R}^n$

suppose start by choosing Q as real symmetric p.d.f Mat

$(Q = Q^T > 0) \qquad PA + A^T P = -Q$ solving for P.

if $PA + A^T P = -Q$ has a p.d.f sol then we can conclude Z origin is Globally asymp stab ($x \in \mathbb{R}^n$)

$PA + A^T P = -Q \quad P = P^T > 0 \quad Q = Q^T > 0$ where $A \in \mathbb{R}^{n \times n}$

$P = Q \in \mathbb{R}^{n \times n}$ is <u>Lyapunov eqn</u>

$V(x) = x^T P x$ - generalised energy

$\dot{V}(\cdot) = -x^T Q x$ - generalised dissipation

For $\dot{x} = Ax$, if $P = P^T > 0$, $Q = Q^T > 0$ then all trajects are bounded & $[A]$ is hurwitz mat &

associated ellipsoidal set

$D = \{x \in \mathbb{R}^n \mid \underbrace{x^T P x}_{V(x)} \leq C\}$ is invariant

$\dot{x} = Ax + Bu$ $\quad \dot{x} = \overbrace{(A - BK)}^{A_{cl}} x \quad u = -Kx$

$PA_{cl} + A_{cl}^T P = -I \quad P(A - BK) + (A - BK)^T P = -I$

$PA + A^T P - PBK - K^T B^T P = -I$

recall Min eigval of [Q] max eigval of Q

$-\lambda_{min}(Q) x^T x \le x^T Q x \le \lambda_{min}(Q) x^T x$

$\lambda_{min}(P) x^T x \le x^T P x \le \lambda_{max}(P) x^T x$ $\quad \left\{ \lambda_{min}(P) x^T x \le x^T P x, \; x^T x \le \frac{x^T P x}{\lambda_{min}(P)} \right\}$

$\dot{V}(\cdot) = -x^T Q x \le -\lambda_{min}(Q) x^T x \le -\lambda_{min}(Q)$

$\le -\lambda_{min}(Q) \frac{x^T P x}{\lambda_{min}(P)}$

$\le -\underbrace{\frac{\lambda_{min}(Q)}{\lambda_{min}(P)}}_{\alpha} x^T P x$

$\dot{V}(\cdot) = -\alpha V(x) \implies$ exponentially stability

~~Ex~~ / for eg = $\dot{V}(\cdot) = -\delta V$ $\quad V$ is $x_1^2 + x_2^2$

$\dot{V} = 2 x_1 x_2 + u$ {for some dynamics}

$= \dot{V} = -\delta x_1^2 - \delta x_2^2 \quad \therefore u = -\delta(x_1^2 + x_2^2) - 2 x_1 x_2$

$\dot{x} = f(x) + u \quad V = x_1^2 + x_2^2$

Converse Lyapunov argument / a Mat [A] is hurwitz, which is associated with $\dot{x} = Ax$, ie $Re(\lambda_i(A)) < 0$ $\forall$ eigvals of A, iff for any given p.d.f symmetric Mat P that satisfy Lyapunov eqn $PA + A^T P = -Q$ for some $Q = Q^T > 0$ & if A is hurwitz, then P is unique sol of $PA + A^T P = -Q$

if [A] is Hurwitz $\exists$ a P st $V = x^T P x$ becomes a Lyapunov func associated with $\dot{x} = Ax$

Nonlinear / perturbation bound

~~Ex~~ Consider $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ $\quad f(\cdot) : D \to \mathbb{R}^n$

$x = 0$ is a domain D, $f(0) = 0$ & equilib pt

$\dot{x} = \underset{\text{linear part}}{Ax} + \underset{\text{nonlinear/perturbed/uncertain}}{g(x)}$

where $A = \frac{\partial f}{\partial x}(0)$ Jacobian evalued at origin

& $|g_i(x)| \le \left\| \frac{\partial f_i(z_i)}{\partial x_1} - \frac{\partial f_i}{\partial x}(0) \right\| \|x\|$ (from Mean value thm) possible to write $\therefore \dfrac{\|g(x)\|}{\|x\|} \to 0$ as $\|x\| \to 0$ (a class of funcs)

suppose in $\dot{x} = Ax + g(x)$

$[A]$ is hurwitz $\Rightarrow Re(\lambda_i(A)) < 0$ by converse Lyapunov argument,

$\exists$ a $V(x) = x^T P x$, $P = P^T > 0$ st $PA + A^T P = -Q$ for some $Q = Q^T > 0$

$\dot{V}(x) = x^T P \dot{x} = \dot{x}^T P x =$

$x^T P[Ax + g(x)] + [Ax + g(x)]^T P x =$

$x^T (PA + A^T P) x + 2x^T P g(x) = \underbrace{-x^T Q x}_{\text{is -ve desinite}} + \underbrace{2 x^T P g(x)}_{\text{sign indesinite}}$

$\therefore \dfrac{\|g(x)\|}{\|x\|} \to 0 \quad \forall \gamma > 0 \ \exists r > 0$ st $\|g(x)\| < \gamma \|x\| \quad \forall \|x\| < r$ ← radius

$\dot{V}(x) \le -x^T Q x + 2\|x\| \|P\| \|g(x)\| \quad \{\|AB\| \le \|A\| \|B\|\}$

$\le -x^T Q x + 2\gamma \|x\| \|x\| \|P\| \quad \forall \|x\| < r$

$\le -\lambda_{min}(Q) \|x\|^2 + 2\gamma \|x\|^2 \|P\| \quad \forall \|x\| < r$

$\le -\underbrace{[\lambda_{min}(Q) - 2\gamma \|P\|]}_{\text{as long as this term is +ve}} \|x\|^2 \quad \forall \|x\| < r$

$\dot{V}(x)$ is N-D-F $\forall \|x\| < r \qquad r < \dfrac{1}{2} \dfrac{\lambda_{min}(Q)}{\|P\|}$ ← Lyapunov Mat

Region of attraction / Let $\dot{x} = f(x)$, $x \in \mathbb{R}^n \quad f: D \to \mathbb{R}^n$

- $\phi(t, x)$ sol
- $x = 0$ is asymp stable

Def:/ Z region of attraction ~~$R_A = \{x \in D \mid \phi(t,x)$~~

$R_A \triangleq \{x \in D \mid \phi(t,x)$ is desined $\forall t \ge 0$, & $\phi(t,x) \to 0$ as $t \to \infty\}$

$x = 0$ asymp stab, R_A is open, connected invariant set

an esti of R_A (not true R_A) is a positively invariant set in Z domain D, where $V: D \to \mathbb{R}$ desined as in Lyapunov thm:

$\Omega_c = \{x \in D \mid V(x) \le c\}$ is simplest compact set $\Omega_c \subset D$ st $\{V(x)$ p.d.f, $\dot{V}(x)$ N.d.f$\}$ every traject in Ω_c stays in Ω_c $\forall$ future time $\therefore$ ~~$\forall$~~ $V(x) = x^T P x$: & $D = \{\|x\|_2 < r\}$

we can ensure that $\Omega_c \subset D$ by choosing

$c < \min_{\|x\|_2 = r} x^T P x = \lambda_{min}(P) r^2$ $\quad \left\{ \lambda_{min}(P) x^T x \le x^T P x \le \lambda_{max}(P) x^T x \right\}$

$\left\{ x^T x = \|x\|^2 = r^2 \right\}$

$\Omega_c = \left\{ x \in \mathbb{R}^n \mid V(x) \le \lambda_{min}(P) r^2 \right\}$

\Ex/ relative degree ex:

\Q1/ $\begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \varepsilon(1-x_1^2)x_2 + u \end{cases}$, $\varepsilon > 0$ scalar, u → control input

2nd order syst

output $y = x_1$ (sensor for measuring state x_1)

determine relative degree

\Sol:/ deriv of y wrt time

there is no explicit u appearing in eqn

$\ddot{y} = \dot{x}_2 = -x_1 + \varepsilon(1-x_1^2)x_2 + u$ → directly appearing or explicit presence of u

$\therefore$ relative degree is 2. for $\mathbb{R}^2$

\part b/ if $y = x_2$ ie a sensor measuring state x_2 is used $\dot{y} = \dot{x}_2 = -x_1 + \varepsilon(1-x_1^2)x_2 + u$

$\therefore$ relative degree is 1 for $\mathbb{R}^2$

\part c/ Suppose output eqn is $y = x_1 + x_2^2 \leftarrow h(x_1, x_2)$

determine relative degree

$\dot{y} = \dot{x}_1 + 2x_2\dot{x}_2$

$\dot{y} = x_2 + 2x_2(-x_1 + \varepsilon(1-x_1^2)x_2 + u)$

coupled with x_2

$x_2 u$ if $x_2 = 0$, u is not present

relative degree is 1 but over a domain where $x_2 \neq 0$

\Week 9 Q3/ $\dot{x}_1 = -2x_1 + ax_2 + \sin x_1$ $\quad \dot{x}_2 = -x_2\cos x_1 + u\cos(2x_1)$

observation: Z syst in Σ is not in $\dot{x} = Ax + B\gamma(x)[u - \alpha(x)]$

$\Sigma \begin{cases} \dot{x}_1 = -2x_1 + ax_2 + \sin x_1 \\ \dot{x}_2 = -x_2\cos x + u\cos(2x_1) \end{cases}$

in Σ, Σ non linearity can not be cancelled by control input u

we make use of $T(x)$ given $z_1 = x_1$ $\quad z_2 = ax_2 + \sin x_1$ $\therefore$ derives:

$\dot{z}_1 = \dot{x}_1 = -2\overset{z_1}{x_1} + \overbrace{a x_2 + \sin x_1}^{z_2}$

$\dot{z}_1 = -2z_1 + z_2$

$\dot{z}_2 = a\dot{x}_2 + \cos x_1 \dot{x}_1 = a(-x_2 \cos x_1 + u\cos(2x_1)) + \cos(x_1)(-2x_1 + a x_2 + \sin x_1)) =$

$= \underset{\sim\sim\sim}{-a x_2 \cos x_1} + a u \cos 2x_1 + \underset{\sim\sim\sim}{a x_2 \cos x_1} - 2x_1 \cos x_1 + \cos x_1 \sin x_1 =$

$-2x_1 \cos x_1 + \cos x_1 \sin x_1 + a u \cos(2x_1)$

$\dot{z}_2 = -2z_1 \cos z_1 + \cos z_1 \sin z_1 + u a \cos(2z_1)$

transformed system is

$z_1 = -2z_1 + z_2$

$\dot{z}_2 = -2z_1 \cos z_1 + \cos z_1 \sin z_1 + u a \cos(2z_1)$

choose $u = \frac{1}{a\cos(2z_1)}\left[v - \cos z_1 \sin z_1 + 2z_1 \cos z_1\right]$

in a bounded ~~do~~ domain D where $\cos(2z_1)$ exits

yields $\dot{z}_1 = -2z_1 + z_2$
$\dot{z}_2 = v$

recall v is a linear statefeedback control law

$v = -k_1 z_1 - k_2 z_2$ & you can design this using pole placement

Final control law ~~transfer~~ back to original coords

\Q1/ $\dot{x}_1 = \underbrace{-\alpha x_1 - \omega x_2}_{\text{linear part}} + \underbrace{(\beta x_1 - \gamma x_2)(x_1^2 + x_2^2)}_{\text{non linear part}}$
$\dot{x}_2 = \omega x_1 - \alpha x_2 + (\gamma x_1 + \beta x_2)\underbrace{(x_1^2 + x_2^2)}_{\|x\|_2^2}$ $\because \|x\|_2 = \sqrt{x_1^2 + x_2^2}$

$\therefore x = [x_1 \quad x_2]^T$

can be written as $\dot{x} = Ax + \|x\|_2^2 Bx$ with $A = \begin{bmatrix} -\alpha & -\omega \\ +\omega & -\alpha \end{bmatrix}$, $B = \begin{bmatrix} \beta & -\gamma \\ \gamma & \beta \end{bmatrix}$

choose $V(x) = x^T x$ which is posi definite

nominal linear syst $A + A^T = -2\alpha I$

$\dot{V}(\cdot) \leq -2\alpha\|x\|^2 + 2\|B\|\|x\|^4 \leq -2\alpha\|x\|^2 + 2\underbrace{\sqrt{\beta^2 + \gamma^2}}_{=\|B\|}\|x\|^4$

for $\|x\| \leq r$: $\dot{V} \leq -2\alpha\|x\|^2 + 2r^2\sqrt{\beta^2 + r^2}\,\|x\|^2 < 0$ $\quad \{(-2\alpha + 2r^2\sqrt{\beta^2 + r^2} = 0\}$

for $\sqrt{\beta^2 + r^2} < \frac{\alpha}{r^2}$ $\Rightarrow$ Calcing deriv of V ~~from~~ Z dynamics (but not considering as linear + NL term)

$\dot{V} = -2\alpha\|x\|^2 + 2\beta\|x\|^4$ when $\beta \leq 0$ $\dot{V} \leq -2\alpha\|x\|^2$ & Z origin is globally exponentially stable $\because$ $V = \|x\|^2$ is posi definite, radially unbounded &

W an upper triangular Mat $W = \begin{bmatrix} a_{n-1} & a_{n-2} & a_{n-3} & \cdots & 1 \\ a_{n-2} & a_{n-3} & a_{n-4} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ a_1 & 1 & 0 & & 0 \\ 1 & 0 & 0 & & 0 \end{bmatrix}$ entries a_i are coeffs of ~~poly~~ charac poly of Mat [A]

$\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n \quad \because A \in \mathbb{R}^{n \times n}$

$T = M \times W$ mat transforming to controllable canonical form

$(A, B) \xrightarrow{T} (T^{-1}AT,\ T^{-1}B)$ in controllable form

start
given syst
LTI $\dot{x} = Ax + Bu$
(not necessarily in controllable canonical form)

$\xrightarrow{\tilde{x} = T^{-1}x}$

2 $T = MW$
$\det(sI-A$

Transformed syst
in controllable canonical form
$\dot{\tilde{x}} = (T^{-1}AT)\tilde{x} + (T^{-1}B)u$

Design $u = -\tilde{k}\tilde{x}$

$\because x = T\tilde{x}$

$u = -\bar{k}x = u = -\tilde{k}(T^{-1}x)$
in original coords: $\bar{k} = \tilde{k}T^{-1}$

control law for syst in original coords

Design steps: $\dot{x} = Ax + Bu$ LTI syst ① check controllability
$M = [B \vdots AB \vdots \cdots \vdots A^{n-1}B]$ rank(M) full

② form charac poly for A $\det(sI - A) = s^n + a_1 s^{n-1} + \cdots + a_{n-1}s + a_n$

③ find transformation mat $T = MW$; W is generated using coeffs of charac poly in step 2. to give a controllable canonical form

④ obtain desired charac poly $\prod_{i=1}^{n}(s - \mu_i) = s^n + \alpha_1 s^{n-1} + \cdots \alpha_n$
μ_i: $i = 1, \ldots, n$ are desired poles

⑤ feed back gain mat $\bar{k} = [(\alpha_n - a_n)\ (\alpha_{n-1} - a_{n-1}) \cdots (\alpha_1 - a_1)] \times T^{-1}$ is now in original coords & can use

Ex: $\dot{x} = Ax + Bu$ $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ find $u = -kx$ st closed loop eigvals are at -4 & -5: step 1: controllable $M = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ full rank

step 2: charac poly of A $\det(sI - A) = s^2 - 2s - 3$ is of form $s^2 + a_1 s + a_2$

step 3: $T = MW$ $W = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}$ $T = MW = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$ square invertible mat (has upper triangular struc)

step 4: desired poles -4 & -5 gives charac poly $(s+4)(s+5) = s^2 + 9s + 20$ (desired one of form $s^2 + \alpha_1 s + \alpha_2$)

step 5: $\therefore$ feedback gain mat $\bar{k} = [(\alpha_2 - a_2)(\alpha_1 - a_1)]T^{-1} = [23\ \ 11]\begin{bmatrix} 1 & 0 \\ 1 & 2 \end{bmatrix} = [34\ \ 22] = [17\ \ 11]$ (we can scale) $u = -kx$ control law $= u = [-17\ \ -11]x$

it is also possible to make use of Ackermann's formula

feedback gain mat $\Rightarrow K = [0 \ \cdots \ 1] \, \mathcal{C}^{-1} \, \Delta_{des}(A)$ (controllability mat) (Ep)

$\Delta_{des}(A)$ is desired charac poly evaluated at mat A, & control law is $u = -Kx$

Ex numerical Ex/ $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $\mathcal{C} = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ $\mathcal{C}^{-1} = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}$

using Ep $K = [0 \ 1] \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix} (A^2 + 9A + 20 I_2)$

{for desired eigvals $-5, -4$ desired charac poly is $s^2 + 9s + 20$ }

this gives $K = [\cdot \ \cdot] = [\cdot \ \cdot]$ control law $u = [\cdot \ \cdot] x$

Lipschitz continuity, contraction mapping/

Let $D \subseteq \mathbb{R}^n$ & $f(\cdot): D \to \mathbb{R}^n$ $\dot{x} = f(x)$

Def/ $f(\cdot)$ is bounded on D if $\exists\, \alpha > 0$ st $\|f(x)\| \le \alpha$, $\forall x \in D$

Def/ $f(\cdot)$ is cont at $x \in D$, if $\forall \varepsilon > 0$, $\exists\, \delta = \delta(\varepsilon, x) > 0$ st $\|f(x) - f(y)\| < \varepsilon$ $\forall y \in D$ satisfying $\|x - y\| < \delta$

Def/ $f(\cdot)$ is uniformly cont on D if $\forall \varepsilon > 0$, $\exists\, \delta = \delta(\varepsilon) > 0$ st $\|f(x) - f(y)\| < \varepsilon$ $\forall x, y \in D$ satisfying $\|x - y\| < \delta$

Def: Lipschitz continuity/ $f(\cdot)$ is Lipschitz cont at $x_0 \in D$, if $\exists$ a Lipschitz const $L = L(x_0) > 0$ & a neighborhood $N \subset D$ of x_0 st $\|f(x) - f(y)\| \le L \|x - y\|$, $x, y \in N(x_0)$

$f(\cdot)$ is Lipschitz cont on D if $f(\cdot)$ is Lipschitz cont at every pt in D

Def/ $f(\cdot)$ is uniformly Lipschitz cont on D if $\exists\, L > 0$ st $\|f(x) - f(y)\| \le L \|x - y\|$ $x, y \in D$ ~~if $\exists L > 0$ st~~

Def/ $f(\cdot)$ is globally Lipschitz cont if $f(\cdot)$ is uniformly Lipschitz cont on $D = \mathbb{R}^n$

Let $f(\cdot): [a, b] \times D \to \mathbb{R}^n$ be a cont func for some domain $D \subset \mathbb{R}^n$ { $\|f(x) - f(y)\| < \varepsilon$ $\forall x, y \in D$ satisfying $\|x - y\| < \delta$ $x \in \mathbb{R}^n$ } Suppose that $\left[\frac{\partial f}{\partial x}\right]$ exists, & is cont on $[a, b] \times D$

$W \subset D$, $\exists$ a const $L \ge 0$ st $\left\|\frac{\partial f}{\partial x}(\cdot)\right\|_{\infty} \le L$ on $[a, b] \times W$ then (Lipschitz) for a convex subset

$\|f(t,x) - f(t,y)\| \le L\|x-y\|$, $\forall\, t \in [a,b]$, $x \in W$ & $y \in W$

Ex numerical/ $f(x) = \begin{bmatrix} -x_1 + x_1x_2 \\ x_2 - x_1x_2 \end{bmatrix} \leftarrow \begin{cases} \dot{x}_1 = -x_1 + x_1x_2 \\ \dot{x}_2 = x_2 - x_1x_2 \end{cases}$ $\quad x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

- calc lipschitz const L over Z convex set

$W = \{ x \in \mathbb{R}^2 \mid |x_1| \le a_1 , |x_2| \le a_2 \}$

Sol:/ $f(x)$ is contly diffable on $\mathbb{R}^2$ $\therefore$ Lipschitz on domain D of $\mathbb{R}^2$, or any ~~Compact~~ compact subset of $\mathbb{R}^2$, $f(\cdot)$ is Lipschitz

$\left[\frac{\partial f}{\partial x}\right]$ exists. & cont too $\left[\frac{\partial f}{\partial x}\right] = \begin{bmatrix} -1+x_2 & x_1 \\ -x_2 & 1-x_1 \end{bmatrix}$ $\quad \begin{bmatrix} |\cdot| \xrightarrow{+} |\cdot| \\ |\cdot| \longrightarrow |\cdot| \end{bmatrix}$ pick max

using $\|\cdot\|_\infty$ for vecs in $\mathbb{R}^n$ Z induced mat norms for mats :

$\left\|\frac{\partial f}{\partial x}\right\|_\infty = \max\{|-1+x_2| + |x_1| , |-x_2| + |1-x_1|\}$ all pts in W satisfy

$|-1+x_2| + |x_1| \le 1 + a_1 + a_2$ $\quad \{|x_1| < a_1 , |x_2| < a_2$ in $W\}$ &

$|x_2| + |1 - x_1| \le a_2 + 1 + a_1$ $\therefore$ $\left\|\frac{\partial f}{\partial x}\right\|_\infty \le 1 + a_1 + a_2$ $\left(\max\{|-1+x_2| + |x_1|, |-x_2| + |1-x_1|\}\right)$

& Z Lipschitz const is $L = 1 + a_1 + a_2$ $\quad \{\because \left\|\frac{\partial f}{\partial x}\right\|_\infty \le L$ on W $\therefore$ $\|f(x) - f(y)\| \le L\|x-y\|$ $\forall x \in W$ & $y \in W\}$

Importance of Lipschitz/ Consider dynamical system $\dot{x} = f(x)$ to predict Z future state of system from its current state x_0 at t_0 , Z initial val probl (IVP) $\dot{x} = f(x)$, $x(t_0) = x_0$ must have a <u>unique</u> sol

- existance & uniqueness can be ensured if $f(x)$ is Lipschitz ie, satisfying $\|f(x) - f(y)\| \le L\|x-y\|$, $\forall x,y$ in some neighbood of x_0

thm :Contraction mapping thm:/ let S be a closed subset of a Banach space X & let $T : S \to S$ suppose $\|T(x) - T(y)\| \le \rho\|x-y\|$ $\forall x,y \in S$ $0 \le \rho < 1$

- $\exists$ a unique $\bar{x} \in S$ satisfying $\bar{x} = T(\bar{x})$
- $\bar{x}$ can be obtained by Z method of successive approx.
  - starting from any initial vec in S
- contraction is special case of Lipschitz with $L < 1$

Week 6 — Lyapunov Stability theory: if total energy is dissipated the system must be stable

consider $\dot{x} = f(t,x)$, $x \in \mathbb{R}^n$, $x(0)$ IC, let $x_e \in \mathbb{R}^n$ - equilib,

ie $f(t, x_e) = 0 \quad \forall t \geq t_0$

Def: equilib x_e is stable in the sense of Lyapunov, if $\forall \varepsilon > 0$,

$\exists \delta > 0$ st if $||x_0 - x_e|| < \delta$ then $||\overset{\text{sol}}{x}(t, x_0, t_0) - x_e|| < \varepsilon \; \forall t \geq t_0$

x_0 near enough to equilib x_e — sols remain in some ball around x_e, $\forall$ time t

remark: an equilib pt can be shifted to origin for autonomous systs via change of variabs

Def: a zero sol, origin, of $\overset{\text{autonomous } x \in \mathbb{R}^n}{\dot{x} = f(x)}$ is Lyapunov stable, if $\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ st if $||x(0)|| < \delta$ then $||x(t)|| < \varepsilon$, $\forall t \geq 0$

$||x(0)|| < \delta \Rightarrow ||x(t)|| < \varepsilon \quad \forall t \geq 0$ Lyapunov stable

any pt in the interior of the circle satisfy $||x(0)|| < \delta$ $\forall$ time $t \geq t_0$

$||x(t)|| < \varepsilon$ — equilib origin — evolution in time (embedded here) is governed by $\dot{x} = f(x)$ $\quad x \in \mathbb{R}^2 \quad \dot{x} = f(x)$, $x \in \mathbb{R}^2$ origin is Lyapunov stable $\quad f(\cdot) \quad ||x(0)|| < \delta \Rightarrow ||x(t)|| < \varepsilon$, $\forall t \geq t_0$

Def: the zero sol (origin) of $\dot{x} = f(\overset{x \in \mathbb{R}^n}{x})$ is (locally) asymp stable, if it is Lyapunov stable & $\exists \delta > 0$ st if $||x(0)|| < \delta$, then $\lim_{t \to \infty} x(t) = x_e = 0$ $\quad ||x(0)|| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$ ⟵ asymp stable

as time $t \to \infty$ $\lim_{t \to \infty} x(t) = x_e = 0$ — equili (x_e) is origin ~~here~~ origin

asymp stability $||\cancel{x}(0)|| < \delta$ $\lim_{t \to \infty} x(t) = x_e = 0$

Def: the zero sol origin of $\dot{x} = f(x)$ is (locally) exponentially stable, if $\exists$ posi consts α, β, δ st if $||x(0)|| < \delta$ then

$||x(t)|| < \alpha ||x(0)|| \exp^{-\beta t}$, $t \geq 0$ $\qquad ||x(0)|| < \delta \Rightarrow$

$||x(t)|| \leq \alpha ||x(0)|| \exp^{-\beta t}$, $t \geq 0$ exponential stability $\alpha, \beta, \delta \in \mathbb{R}^+$

$||x(0)|| < \delta$ evolution of state x due to $\dot{x} = f(x)$

evolution of state is constrained $||x(t)|| \le \alpha ||x(0)|| \exp^{-\beta t}$, $t \ge 0$ $\alpha, \beta, \delta > 0$

- exponentially stable

\Global stability/ \Def:/ Z origin of $\dot{x} = f(x)$ is globally asympt stable if it is Lyapunov stable & $\forall x(0) \in \mathbb{R}^n$ $\lim_{t \to \infty} x(t) = 0$

\Def:/ Z origin of $\dot{x} = f(x)$ is globally exponentially stable if $\exists$ posi consts α, β st $||x(t)|| \le \alpha ||x(0)|| \exp^{-\beta t}$, $t \ge 0$ $\forall x(0) \in \mathbb{R}^n$ {not just $||x(0)|| < \delta$

in Lyapunov analy we interested in funcs $V(t, x)$ (candidate Lyapunov funcs) & how this $V(\cdot)$ changes along Z vec field $f(\cdot)$ due to Z dynamics $\dot{x} = f(\cdot)$ (auton syst dynamics)

\Def:/ Z ie deriv of $V(x, t)$ wrt $f(x, t)$ is $\dot{V}(x, t) = \frac{\partial V}{\partial x} \cdot \underbrace{f(t, x)}_{dx/dt \text{ or } \dot{x}} + \frac{\partial V}{\partial t}$ when dynamics is autono $V(x)$ sufficient & $\dot{V}(x) = \frac{\partial V}{\partial x} f(x)$

$\dot{x} = f(x)$, $x \in \mathbb{R}^n$ $x = [x_1, \ldots, x_n]^T$ $f(\cdot) = [f_1(\cdot), f_2(\cdot), \ldots, f_n(\cdot)]$ (as component state space form) Consider $V(x) = V(x_1, x_2, \ldots, x_n)$ (ie along Z trajects of $\dot{x} = f(x)$) $\frac{dV}{dt} = \frac{\partial V}{\partial x_1} \overbrace{\frac{dx_1}{dt}}^{f_1(\cdot)} + \frac{\partial V}{\partial x_2} \overbrace{\frac{dx_2}{dt}}^{f_2(\cdot)} + \cdots + \frac{\partial V}{\partial x_n} \overbrace{\frac{dx_n}{dt}}^{f_n(\cdot)}$

$= \text{Grad } V \cdot \frac{dx}{dt} = \text{Grad } V \cdot f(\cdot)$

where $\text{Grad } V = [\partial V/\partial x_1 \quad \partial V/\partial x_2 \quad \cdots \quad \partial V/\partial x_n]$ & $\frac{dx}{dt} = \dot{x} = \left[\frac{dx_1}{dt} \quad \frac{dx_2}{dt} \quad \cdots \quad \frac{dx_n}{dt}\right]^T$
$= [f_1(\cdot) \quad f_2(\cdot) \quad \cdots \quad f_n(\cdot)]^T$

\idea of Lyapunov thy/ suppose find some measure of energy of system, for eg. $V(x, t) = ||x||^2$ ← (norm, +ve) st following conds holds:

- $V(x_e, t) = 0$ $\forall (t) > t_0$ ie $V(\cdot)$ centred about equili $V(x_e)$, or $V(0)$ when $x_e = 0$
- $V(x_e, t) > 0$ when $x \ne x_e$, $\forall t > t_0$ (posi everywhere else, other than x_e)
- $\dot{V}(x_e, t) \le 0$ ← $\frac{\partial V}{\partial x} f(t, x) + \frac{\partial V}{\partial t}$ ie energy is non-increasing along all syst trajects

implies x_0 near equili, Z trajec (sol) $x(t, x_0, t_0)$ remains sufficiently near x_e forever

\posi / nega definiteness / Consider a domain D for x which contains a neighborhood of origin or equilib pt $\{x \in \mathbb{R}^n \mid \|x\| < r\}$ where $r > 0$

\Def:/ A func $V : D \to \mathbb{R}$ is (locally) posi definite in $\mathcal{Z}$ domain $D \in \mathbb{R}^n$ if $V(x) > 0$ $\forall x \in D \setminus \{0\}$ $V(x) = 0 \Rightarrow x = 0$

\Def:/ a func $V : D \to \mathbb{R}$ is called (locally) posi semi definite on $D \subset \mathbb{R}^n$ if $V(x) \geq 0$ $\forall x \in D$ if $D \equiv \mathbb{R}^n$ then globally posi definite & Globally posi semi definite

\Class of funcs/ $V(0) = 0$ ← origin or equili pt, $V(x) \geq 0$ $\forall x \in D$ locally posi semi definite $\forall x \in \mathbb{R}^n$ Globally posi semi definite

$V(0) = 0$, $V(x) > 0$ $\forall x \in D \setminus \{0\}$ Locally posi Definite

$\forall x \in \mathbb{R}^n \setminus \{0\}$ Globally posi Definite

$V(0) = 0$, $V(x) \leq 0$ $\forall x \in D$ locally nega semi definite

$\forall x \in \mathbb{R}^n$ Globally nega semidefinite

$V(0) = 0$, $V(x) < 0$ $\forall x \in D \setminus \{0\}$ locally nega definite

$\forall x \in \mathbb{R}^n \setminus \{0\}$ Globally nega definite

"x_e equilib of $\dot{x} = f(x)$ can be shifted to origin"

\Lyapunov thm/ Consider $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, $x(0) = x_0$ & assume $\exists$ a contly differble func $V : D \to \mathbb{R}$ st at origin assume as equili, $V(0) = 0$

$V(x) > 0$, $\forall x \in D \setminus \{0\}$ (ie V is strictly posi in domain D excluding origin)

$\dot{V}(x) \leq 0$ ← $\{$deriv of V wrt time $\frac{\partial V}{\partial x} \cdot f(x)\}$, $\forall x \in D$ (ie energy dissipating/conserved then origin is Lyapunov stable

in addition $\dot{V}(x) < 0$ ← strictly less than, $\forall x \in D \setminus \{0\}$ then origin is asymp stable & finally if $\exists$ scalars $\alpha, \beta, \varepsilon > 0$ & $p \geq 1$ st $V : D \to \mathbb{R}$ satisfies

$\alpha \|x\|_p \leq V(x) \leq \beta \|x\|_p$, $x \in D$ $\{\|x\|_p$: p-norm 1-norm ∞ norm 2-norm$\}$

$\dot{V}(x) \leq -\varepsilon V(x)$, $x \in D \setminus \{0\}$ $\therefore$ origin is Exponentially stable)

For $\dot{x} = f(x)$, $x \in \mathbb{R}^n$, $x(0) = x_0$, Suppose $\exists$ a contly differble $V : D \to \mathbb{R}$ st $V(\cdot)$ p.d.f $\dot{V}$ n.s.d.f Lyapunov stable

Week 1 - Compact State Space representation

continuous time models - typically represented as differential eqns

state space representation

$\dot{x}(t) = f(t, x(t), u(t))$, $x(t_0) = x_0$, $t \in [t_0, t_f]$

time derivative of $x(t)$ — state — control input or forcing term — initial condition — time duration t_f - final time t_0 - initial time

x - state $x(t) \in \mathbb{R}^n$ - n-dimensional real space

$x(t)$ can take any val from an n-dimen real space (Global)

where as $x(t) \in D \subset \mathbb{R}^n$: $x(t)$ can take any vals from a set D which is subset of $\mathbb{R}^n$ (local)

$x \in \mathbb{R}^n$ $u(t) \in \mathbb{R}^m$ - m-dimen control input $x = [x_1, x_2, \ldots, x_n]^T$

$u = [u_1, u_2, \ldots, u_m]^T$ $f = [f_1(\cdot), f_2(\cdot), f_3(\cdot), \ldots, f_n(\cdot)]^T$

$\dot{x}(t) = f(t, x(t), u(t))$ $\begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \\ \vdots \\ \dot{x}_n(t) \end{bmatrix} = \begin{bmatrix} f_1(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)) \\ \vdots \\ \vdots \\ f_n(t, x_1(t), \ldots, x_n(t), u_1(t), \ldots, u_m(t)) \end{bmatrix}$

state space representation

time state control

$f: \mathbb{R}_t \times \mathbb{R}^n \times \mathbb{R}^m$

$\dot{x}(t) = f(t, x(t), u(t))$

Ex

$\dot{x}_1 = x_2$ $\dot{x}_2 = \underbrace{-k_1 x_1 - k_2 x_2}_{\text{linear part}} \underbrace{- k \cdot x_1^3}_{\text{non linearity}} + u(t)$ ← control input forcing term

$x_1(0) = 0.1$ $x_2(0) = 0.01$

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -k_1 x_1 - k_2 x_2 - k_1^3 + u(t) \end{bmatrix}$$

State $x \in \mathbb{R}^2$ (x_1, x_2)

Control input $u \in \mathbb{R}^1$

$f = [f_1(\cdot), f_2(\cdot)]$ $f_1(\cdot) = x_2$ $f_2(\cdot) = -k_1 x_1 - k_2 x_2 - k x_1^3 + u(t)$ $f: \underset{\text{state}}{\mathbb{R}^2} \times \underset{\text{input}}{\mathbb{R}} \to \mathbb{R}^2$

T2DM

Math representation for type 2 diabetes

States $x \in \mathbb{R}^{28}$ $[x_1, \ldots, x_{28}]^T$

$\dot{x} = f(x(t))$ $f(\cdot) = (f_1(\cdot), \ldots, f_{28}(\cdot))$

2 quantities of x can be measured Glucose & insulin vals at tissues

$\dot{x} = f(x(t))$ (omitted $u(t)$ for time being) x_{15}, x_{20}

notion of output - y - output vec $y \in \mathbb{R}^2$ (x_{15}, x_{20})

in Z state space representation $\dot{x}(t) = f(t, x(t), u(t))$ suppose, time t is not explicitly present in dynamics part, then $\dot{x}(t) = f(x(t), u(t))$

this is called <u>autonomous</u> <u>dynamical</u> <u>Systems</u>

$\therefore$ Study such systems

\For eg: Ex/ $\dot{x} = x_2 \quad \dot{x}_2 = x_1^t - x_2$ a non linear system with

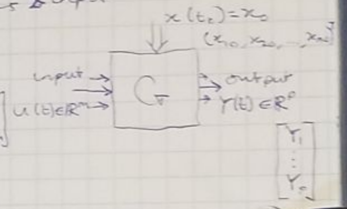
n - States m - inputs p - outputs $x \in \mathbb{R}^n \quad u \in \mathbb{R}^m \quad y \in \mathbb{R}^p$

Can be represented as $G_T \begin{cases} \dot{x}(t) = f(x(t), u(t)), \ x(t_0) = x_0 \\ y(t) = h(x(t), u(t)) \end{cases}$

~~y(t)z~~

this term is not a derivative part like $\dot{x}$

an algebraic func of states & input its not dynamics



\Ex/ $G_T \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -k \sin x_1 \\ y = x_1 \end{cases}$ 2 states $x = [x_1, x_2]^T \in \mathbb{R}^2$

no control input

one state is measurable $\therefore y \in \mathbb{R}^1$ {which is x_1}

another set of y could be $y = (x_1, \mathrm{Sign}(x_2))^T \quad \therefore y \in \mathbb{R}^2$

\Week 1 obtaining a compact state space representation

\Ex 1/ Consider $\overset{①}{m\ddot{x}_1} = k_2(x_2 - x_1) - k_1 x_1 + F_1 \qquad m_2 \ddot{x} = -k_3 x_2 - k_2(x_2 - x_1) + F_2$

2nd order ODEs coupled k_1, k_2, k_3 scalar vals

F_1, F_2 are external forces want to obtain form: $\dot{x} = f(x, u)$

Consider ①: has $\ddot{x}_1$ $\therefore$ lets desire 2 term with one order less as a new variable ie $\dot{x}_1 = \frac{dx_1}{dt} = x_3$

similarly $\dot{x}_2 = \frac{dx_2}{dt} = x_4 \qquad x_3$ & x_4 we have <u>desired</u>

then we have four states in total $x = [x_1, x_2, \overset{\dot{x}_1}{x_3}, \overset{\dot{x}_2}{x_4}]^T \in \mathbb{R}^4 \quad \therefore$ write $\dot{x} = f(x,u)$

$$\begin{aligned} \dot{x}_1 &= x_3 \leftarrow f_1(\cdot) \\ \dot{x}_2 &= x_4 \leftarrow f_2(\cdot) \\ \dot{x}_3 &= \ddot{x}_1 = \frac{k_2}{m_1}(x_2 - x_1) - \frac{k_1}{m_1}x_1 + \frac{F_1}{m_1} \quad \leftarrow f_3(\cdot) \\ \dot{x}_4 &= \ddot{x}_2 = -\frac{k_3}{m_2}x_2 - \frac{k_2}{m_2}(x_2 - x_1) + \frac{F_2}{m_2} \leftarrow f_4(\cdot) \end{aligned}$$

$\leftarrow$ is compact state space representation $\dot{x} = f(x, u) \qquad x \in \mathbb{R}^4$

$u = \in \mathbb{R}^2 \qquad u = \left[\frac{F_1}{m_1}, \frac{F_2}{m_2}\right]^T$

\Ex 2/ Consider following dynamics $\frac{d^3x}{dt^3} - \frac{d^2x}{dt^2}\sin(x) + 2\frac{dx}{dt} - x^2 = 0$

represent this in a <u>State Space</u> form $\leftarrow \dot{x} = f(x(t), u(t))$

3rd order ODE $\therefore$ will have 3 states let us desire new state variables

$x_1 = x \qquad x_2 = \frac{dx}{dt} = \frac{dx_1}{dt} \qquad x_3 = \frac{d^2x}{dt^2} = \frac{dx_2}{dt}$

$\dot{x}_1 = x_2 \leftarrow f_1(\cdot) = x_2$

$\dot{x}_2 = x_3 \leftarrow f_2(\cdot) = x_3$

$\dot{x}_3 = \frac{d^3 x}{dt^3} = \frac{d^2 x}{dt_2} \sin(x) - 2\frac{dx}{dt} + x^2 = x_3 \sin(x_1) - 2x_2 + x_1^2 \leftarrow f_3(\cdot)$

$X = [x_1, x_2, x_3]^T \in \mathbb{R}^3$

& no control input

$\dot{x}(t) = F(X(t))$

$F = [f_1(\cdot), f_2(\cdot), f_3(\cdot)]^T$

\Equalibrium/ consider an autonomous nonlinear dynamical system $\dot{x}(t) = f(x(t))$, $x \in \mathbb{R}^n$ ①

∴ can say equilibrium pts $x^* \in \mathbb{R}^n$ are the real sol of $f(x(t)) = 0$

we know $f(\cdot) = [f_1(\cdot), f_2(\cdot), \dots, f_n(\cdot)]^T$ ∴ equilibrium is Z sols of simultaneous, possibly nonlinear funcs of states. For autonomous systems, ∵ not explicity dependent on t, Z def of equilibrium holds ∀t (for all time t). and $x^* \in \mathbb{R}^n$ can be shifted to origin:

Suppose $x^* \neq 0$ (equilibrium) is non zero vec ∴ let $z = x - x^*$ ⊛

$\frac{dz}{dt} = \dot{z} = \dot{x}$ ∵ x^* is const ∀t

$= f(x)$

now from ⊛, $x = z + x^*$ $\quad \dot{z} = f(z + x^*)$ can be thought as a bias term

~~of~~ $\dot{z} = f(z)$ which is similar to $\dot{x} = f(x)$

$\Downarrow$ equili origin $\qquad \Downarrow$ equili x^*

∵ we are studying nonlinear systems ($f(\cdot)$ nonlinear vec funcs depending on $x \in \mathbb{R}^n$), its possible to have multiple equili, or a continuum of equili, or even ~~no~~ equilibria. it depends on $f(\cdot) = 0$ (Z structure of f). ∴ Z notion of <u>isolated</u> <u>equili</u> pt is of interest. in very simple: it means there are no equili pt other than Z one equili x^* that has been determined, in its neighborhood in Z n-dimen space. Make use of <u>Epsilon</u> <u>open</u> <u>Ball</u> notion

Say equili pt is $x^* \in \mathbb{R}^n$ any x of n-dimen Real Space ($x \in \mathbb{R}^n$) that belong to shaded region satisfy the inequality $||x - x^*|| < \varepsilon$

{$||\cdot||$ represents euclidean norm betw x & x^*} $\quad B_\varepsilon(x^*) = \{x \in \mathbb{R}^n \mid ||x - x^*|| < \varepsilon\}$ (set; radius centered about)

\Def/

∴ an equili $x^* \in \mathbb{R}^n$ is an isolated equili pt if ∃ an $\varepsilon > 0$ st:

- $B_\varepsilon(x^*)$ contains no other equili

\Ex / determine equili pts for $ml\ddot{\theta} = -mg\sin\theta - kl\dot{\theta}$ 2nd order ODE $\Downarrow$ $\dot{x} = f(x(t))$

Define $\theta = x_1$ & ~~$\dot{x}_2$~~ $\dot{\theta} = x_2$

$\frac{dx_1}{dt} = x_2$

$\frac{dx_2}{dt} = \ddot{\theta} = -\frac{g}{l}\sin\theta - \frac{k}{m}\dot{\theta} = \frac{-g}{l}\sin x_1 - \frac{k}{m}x_2$ $\rightarrow \dot{x} = f(x(t))$

$x \in \mathbb{R}^2$ $x = (x_1, x_2)^T$ $(\theta, \dot{\theta})^T$

equili $\Downarrow$ $\dot{x} = f(x(t))$ 0

to determine equili, set $\dot{x}_1 = \dot{x}_2 = 0$

(ie no rate of change of states) solve for $x_2 = 0$,

$-\frac{g}{l}\sin x_1 - \frac{k}{m}x_2 = 0$ equili pts $x^* = (x_1^*, x_2^*) = (\pm n\pi, 0)$

$\because \sin x_1 = 0 = \frac{-g}{l}\sin x_1 - \frac{k}{m}(0)$ $\therefore x_1 = \pm n\pi$

\Linearisation / a linearised model is • an approx of nonlinear system • is valid in a small region around an operating pt • provides key info about how system behaves in a neighbourhood of equili pts, especially stability • at present, design of control using linear design tools is very standard & mature

Consider $\dot{x} = f(x, u)$ $x(0) = [x_1(0), x_2(0)]^T$ ← Initial conds

$x = (x_1, x_2) \in \mathbb{R}^2$ (states) $u = (u_1, u_2) \in \mathbb{R}^2$ (input)

let this system operates along a trajectory $\tilde{x}(t)$ (response of dynamical system) & input $\tilde{u}(t)$ ensures the evolution of system close to the $\tilde{x}(t)$ {the trajectory} $\therefore \dot{\tilde{x}}(t) = f(\tilde{x}(t), \tilde{u}(t))$ $\therefore$ let $x(t) = \tilde{x}(t) + \delta x$ ← small perturbation

& $u(t) = \tilde{u}(t) + \delta u$ ← small perturbation

$x(t) = \underbrace{\tilde{x}(t)}_{\text{nominal original trajectory}} + \underbrace{\delta x}_{\text{small perturbation}}$ taking deriv of state wrt time gives

$\dot{x} = \dot{\tilde{x}} + \delta\dot{x} = f(\tilde{x} + \delta x, \tilde{u} + \delta u)$ $\{\because \dot{x} = f(x, u)\}$

let us consider expanding this func ~~of~~ $f(\cdot)$ using taylor expansion $\therefore$ $\dot{\tilde{x}} + \delta\dot{x} = f(\tilde{x}, \tilde{u}) + \frac{\partial f}{\partial x}\Big|_{(\tilde{x},\tilde{u})}\delta x + \frac{\partial f}{\partial u}\Big|_{(\tilde{x},\tilde{u})}\delta u$ + higher order terms

{taylor series of $h(x)$ at $x = a$ is $h(x) = h(a) + \frac{h'(a)}{1!}(x-a) + \frac{h''(a)}{2!}(x-a)^2 + \cdots$}

$\dot{\tilde{x}} + \delta\dot{x} = f(\tilde{x}, \tilde{u}) + \frac{\partial f}{\partial x}\Big|_{(\tilde{x},\tilde{u})}\delta x + \frac{\partial f}{\partial u}\Big|_{(\tilde{x},\tilde{u})}\delta u + H.O.T.$

$\because \dot{\tilde{x}} = f(\tilde{x}, \tilde{u})$ we can write $\delta\dot{x} = \underbrace{\frac{\partial f}{\partial x}\Big|_{(\tilde{x},\tilde{u})}\delta x + \frac{\partial f}{\partial u}\Big|_{(\tilde{x},\tilde{u})}\delta u}_{\delta\dot{x} = A\delta x + B\delta u} + H.O.T$

Considering only 1st order

terms

$A = \frac{\partial f}{\partial x}\Big|_{(\tilde{x},\tilde{u})}$ $x \in \mathbb{R}^2$, $f = [f_1(x_1,x_2), f_2(x_1,x_2)]^T$

$$A = \begin{bmatrix} \partial f_1/\partial x_1 & \frac{\partial f_1}{\partial x_2} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} \end{bmatrix}\Bigg|_{(\tilde{x},\tilde{u})} \in \mathbb{R}^{2\times 2}$$ no. of states

$$B = \begin{bmatrix} \partial f_1/\partial u_1 & \partial f_1/\partial u_2 \\ \partial f_2/\partial u_1 & \partial f_2/\partial u_2 \end{bmatrix}\Bigg|_{(\tilde{x},\tilde{u})} \in \mathbb{R}^{2\times 2}$$ is only one input (I/P) your

$B \in \mathbb{R}^{2\times 1}$ $\therefore$ $\delta\dot{x} = A\,\delta x + B\,\delta u$ is linearised system

\Ex/ math model of stick balancing problem $\ddot{\theta}(t) = \sin\theta(t) - u(t)\cos\theta(t)$ horizontal force applied

linearize about equili pt ICs are zero

$\theta(t)$ - angular displacement of stick from vertical axis

to obtain state space representation of 2nd order ODE:

$x_1 = \theta(t)$) two states
$x_2 = \dot{\theta}(t)$ } $x = (x_1, x_2) \in \mathbb{R}^2$ taking derivs

$\dot{x}_1 = x_2$ $(\because \dot{\theta})$

$\dot{x}_2 = \ddot{\theta} = \sin x_1 - u(t)\cos x_1$

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix} \overbrace{x_2}^{f_1(\cdot)} \\ \underbrace{\sin x_1 - u\cos x_1}_{f_2(\cdot)}\end{bmatrix} \Big\} \dot{x} = f(x,u)$$

equili pts: $\underbrace{\dot{x}}_{0} = f(x,u)$ $x_2 = 0$

$\sin x_1 - \underbrace{u}_{0}\cos x_1 = 0$ ie $\sin x_1 = 0$ $\because x(0) = 0,\ u(0) = 0$ $\therefore$

$x_1 = 0$ $\therefore$ $x^* = (0,0)$ origin $f_1(\cdot) = x_2$ $f_2(\cdot) = \sin x_1 - u\cos x_1$

we obtain A & B matrices about origin $\dot{x} = f(x,u)$ equili x^*

$\delta\dot{x} = \underbrace{A}_{\frac{\partial f}{\partial x}|_{(x^*,u^*)}}\delta x + \underbrace{B}_{\frac{\partial f}{\partial u}|_{(x^*,u^*)}}\delta u$ $\therefore$ Jacobian: $A = \begin{bmatrix}\partial f_1/\partial x_1 & \partial f/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f/\partial x_2\end{bmatrix}\Bigg|_{x^*, u^*} = \begin{bmatrix} 0 & 1 \\ \cos x_1 + u\sin x_1 & 0\end{bmatrix}\Bigg|_{x^*(0,0),\ u=0}$

$= \begin{bmatrix} 0 & 1 \\ 1 & 0\end{bmatrix}.$ $B = \begin{bmatrix}\partial f_1/\partial u \\ \partial f_2/\partial u\end{bmatrix} = \begin{bmatrix} 0 \\ -\cos x_1\end{bmatrix}\Bigg|_{x(0)=(0,0),\ u=0} = \begin{bmatrix} 0 \\ -1\end{bmatrix}$

using obtained A, B: linearised system is $\dot{x} = Ax + Bu$

$\dot{x} = x_2$
$\dot{x}_2 = x_1 - u$

in original coords $\ddot{\theta} = \theta - u$

\Week 2 Second order systems / Autonomous nonlinear system

$\dot{x} = f(x)$, $x \in \mathbb{R}^n$ $x(0) = x_0$ $t \in [t_0, t_f]$

2nd order system, ie $x \in \mathbb{R}^2$ $x = [x_1, x_2]^T$ $G \begin{cases}\dot{x}_1 = f_1(x_1,x_2) \\ \dot{x}_2 = f_2(x_1,x_2)\end{cases}$

ICs for state x_1, and $x_2 \Rightarrow x_0 = [x_{10}, x_{20}]^T$

Assume: <u>existance</u> of sols for G, which is <u>unique</u>

Ex/ $\ddot{x} + h(x, \dot{x}) = 0$ define $x_1 = x$ & $x_2 = \frac{dx}{dt} = \dot{x}$

$$G \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = \ddot{x} = -h(x, \dot{x}) = -h(x_1, x_2) \end{cases}$$

any arbit func depending on x_1 & x_2

Sols for G evolve over time. Evolution is constrained by RHS of G

{eg vector fields at $[f_1, f_2]^T$}

$f_1(\cdot) = x_2$

$f_2(\cdot) = 2x_1 + x_2$

constrains on evolution of dynamics

$f(\cdot) = \begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \end{bmatrix}$ $x_0 (1, 1)$ x_{10} x_{20} $t_0 = 0$ $t_1 = t_0 + \Delta t$

$t_2 = t_0 + 2\Delta t$ $t_2 = t_1 + \Delta t$

$\begin{matrix} x_1 \\ x_2 \end{matrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \xrightarrow[2 \cdot 1 + 1]{} \begin{bmatrix} 1 \\ 3 \end{bmatrix} \xrightarrow[2 \cdot 1 + 3]{} \begin{bmatrix} 5 \\ 9 \end{bmatrix}$

phase-plane State plane time is embedded

$f_1(\cdot) = x_2 = 0$

$f_2(\cdot) = 2x_1 + x_2 = 0$

$\dot{x}_1 = x_2 = 0$

$\dot{x}_2 = 2x_1 + x_2 = 0$

$\dot{x}_1 = f_1(x_1, x_2) = 0$

$\dot{x}_2 = f_2(x_1, x_2) = 0$

these are called nullclines

$\dot{x} = f(x)$ fixed pt or equili when we solve for x_1, x_2

Ex numerical/2-dimen System

$\dot{x}_1 = x_2(1 - x_2^2)$

$\dot{x}_2 = -x_1 - x_2^2$

nullclines are: $\dot{x}_1 = \dot{x}_2 = 0$ $\dot{x}_1 = 0 \to x_2(1 - x_2^2) = 0 \to$

$x_2 = 0$ and $(1 - x_2^2) = 0$ $\therefore x_2^2 = 1$ $\therefore x_2 = \pm 1$

$\dot{x}_2 \to -x_1 - x_2^2 = 0 \to -x_1 = x_2^2$ $\therefore x_2 = \pm\sqrt{-x_1}$ $\therefore$

2 fixed equili pts. 2 fixed pts, or equili pts are obtained $\dot{x}_1 = \dot{x}_2 = 0$ & solving for x_1, x_2 as above

$x_1^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_2^* = \begin{bmatrix} -1 \\ +1 \end{bmatrix}$ $x_3^* = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$ fixed pts are ~~intersection~~ located at intersection of nullclines

$$\left.\begin{aligned} \dot{x}_1 &= x_2(1 - x_2^2) \\ \dot{x}_2 &= -x_1 - x_2^2 \end{aligned}\right\} \dot{x} = f(x)$$

Linearised System is $\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix}}_{\text{Jacobian}} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

$= \begin{bmatrix} 0 & 1 - 3x_2^2 \\ -1 & -2x_2 \end{bmatrix}_{\text{(equili pts)}} \times \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$ is $\dot{x} = Ax$ form $x_1^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $x_2^* = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ $x_3^* = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$

eg: Higher order ODE $\Rightarrow$ $\dot{x} = f(x)$ find nullclines, determine equili, then linearise to obtain $\dot{x} = Ax$ form about equili

$\dot{x}_1 = x_2(1 - x_2^2)$ $\dot{x}_2 = -x_1 - x_2^2$ has phase portrait

Nullclines by setting $\dot{x}_1 = 0$ $\dot{x}_2 = 0$ $x_2 = 0$ $x_2 = \pm 1$ $x_2 = \pm\sqrt{-x_1}$

intersection of nullclines are fixed pts $[-1, 1]$ $[0, 0]$ $[-1, -1]$ $\therefore$ 3 equilis closed orbits means they're periodic

\Linear systems/ consider 2nd order autonomous system

$\dot{x} = f(x)$, $x(t_0) = x_0$, $t \in T$, $x \in \mathbb{R}^2$

dynamics, IC, time period, states

$\dot{x} = 0$ determine equili pts x^*

determine Jacobian of $f(\cdot)$ ie $\begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = J(f(\cdot))$

evaluate $J(f)$ about x^* yields $\dot{x} = Ax$, $x \in \mathbb{R}^2$, $A \in \mathbb{R}^{2\times2} = J(f)|_{x^*}$

$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ $\therefore$ $\dot{x}_1 = a_{11}x_1 + a_{12}x_2$, $\dot{x}_2 = a_{21}x_1 + a_{22}x_2$ $\therefore \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

variables representing numbers

~~$\dot{x}_1 = \lambda x_1$~~ , ~~$x_1(0) = x_0$~~ λ is a scalar $\lambda \in \mathbb{R}$

\Ex one-dim/ $\dot{x}_1 = \lambda x_1$, $x_1(0) = x_0$, $x_1 \in \mathbb{R}^1$ $\Rightarrow$ $\dot{x}_1$ is proportional to state x_1 by λ

\Sol/ $x_1(t) = x_0 e^{\lambda t}$ (IVP) $x_1(t) = e^{\lambda t}$ $\therefore \dot{x}_1(t) = \lambda e^{\lambda t}$ $\therefore \dot{x}_1(t) = \lambda x_1(t)$

$\because \lambda \in \mathbb{R}$, $\lambda = 0$, $\lambda < 0$, $\lambda > 0$

when $\lambda = 0$: $\dot{x}_1 = 0 \Rightarrow x_1$ is const over time

if system initial condition is x_0 $\because \dot{x} = 0$ system remain at x_0

neither attracts to origin or repell from origin is neutrally stable behaviour

Aim: understand dynamical systems without calcing sols

$\lambda < 0$ $-\lambda_1$ $\dot{x}_1 = -\lambda_1 x_1$ sol $x_1 = x_0 e^{-\lambda_1 t}$ (decay)

$\forall x < 0$ change $\dot{x}$ is +ve system evolves to right & asymptotically reach origin more -ve bigger $\dot{x}$ change bigger +ve evolves faster to origin

any $x > 0$ change $\dot{x}$ is -ve system evolves to left & asymptotically reach origin clearly all trajectories evolve asymptotically to origin (stable fixed pt $\lambda < 0$ - attractor)

$\lambda > 0$ $\dot{x}_1 = \lambda_2 x_1$, x_0 sol $x_1 = x_0 e^{\lambda_2 t}$ λ_2 +ve value (growth)

(repelling behaviour) $\lambda > 0$ $x > 0$ change $\dot{x} > 0$ change $\dot{x}$ is +ve

system $\dot{x}_1 = \lambda_2 x_1$ evolves to right away from origin

$x < 0$ change $\dot{x}$ is -ve system evolves to left

Linear Systems sol/ $\dot{x}_1 = \lambda x_1 \quad x_1 \in \mathbb{R}$ sol is $x_1 = x_0 e^{\lambda t}$ in terms of

integral eqn $x_1(t) = x_0 + \int_0^t \lambda x_1(\tau)\,d\tau$ use $x_1(\tau) = x_0$ first iteration

$\underset{\text{iteration index}}{x_{1_1}(t)} = x_0 + \int_0^t \underbrace{\lambda x_0}_{\text{const}}\,d\tau = x_0 + \lambda x_0 t$ (integrating & applying limits)

$= x_0(1+\lambda t)$ $\qquad \underset{\text{2nd iteration}}{x_{1_2}(t)} = x_0 + \int_0^t \lambda x_{1_1}(\tau)\,d\tau = x_0 + \int_0^t (1 + \lambda^2 t)\,d\tau = \ldots =$

$x_0 + \lambda x_0 t + \frac{\lambda^2 t^2}{2} x_0 = x_0(1 + \lambda t + \frac{\lambda^2}{2} t^2)$ likewise n^{th} iteration

$\underset{n^{th}\text{ iteration}}{x_{1_n}(t)} = x_0 + \int_0^t \lambda x_{1_{n-1}}(\tau)\,d\tau = \underbrace{\left(1 + \frac{\lambda t}{1!} + \frac{\lambda^2 t^2}{2!} + \ldots + \frac{\lambda^n t^n}{n!}\right)}_{e^{\lambda t}\text{ taylor expansion}} x_0$

$x_1(t) = x_0 e^{\lambda t}$ sol in general

following this identically for $\dot{x} = Ax$, $x(0) = x_0$, $x \in \mathbb{R}^2$ (but this x can be of $\mathbb{R}^n$) $\quad x(t) = x_0$ (recall $x \in \mathbb{R}^2$) $\quad x(t) = x_0 + \int_0^t Ax(\tau)\,d\tau \quad x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$

(recall $x \in \mathbb{R}^2$ or $x \in \mathbb{R}^n$) in general, for n-dim $x_n(t) =$

$\left[I_{n \times n} + At + \frac{A^2 t^2}{2!} + \ldots + \frac{A^n t^n}{n!}\right] x_0 = \underbrace{x_0 e^{At}}_{\text{matrix exponential}}$ $\quad$ {$I_{n \times n}$ is identity matrix of dim n}

$x = x_0 e^{At}$
$x \in \mathbb{R}^n$

$\dot{x} = f(x)$ ← higher order ODE
← autonomous compact state-space representation

$\dot{x}_0 = f(x)$, determine equili x^*, evaluate jacobian at x^*

$\dot{x} = Ax$ ← linear time invariant system, $x(0) = x_0$; $x \in \mathbb{R}^n$

phase portrait in 2nd order systems

sol is $x(t) = x_0 e^{At}$

variables for numerical values

linear system $\dot{x} = Ax$, $x \in \mathbb{R}^2$, $A \in \mathbb{R}^{2 \times 2}$ $\qquad A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

for studying stability of linear systems, need to determine eigen vals of A : $\underset{\text{determinant}}{\det}(A - \lambda I) = \begin{vmatrix} a_{11}-\lambda & a_{12} \\ a_{21} & a_{22}-\lambda \end{vmatrix} = 0$ charact polynomial - for system $\dot{x} = Ax$ associated with $[A]$

$\lambda^2 - \lambda\underbrace{(a_{11} + a_{22})}_{Tr(A)} + \underbrace{(a_{11}a_{22} - a_{12}a_{21})}_{Det(A)} = 0$ $\qquad \lambda^2 - Tr(A)\lambda + \det(A) = 0$ (quadratic in λ)

$Tr(A) =$ Trace of $[A]$

$\lambda_{1,2} = \frac{1}{2}\left[Tr(A) \pm \sqrt{\underbrace{Tr(A)^2 - 4\det(A)}_{\text{discriminant}}}\right]$ eigenvals written in terms of $Tr(\cdot)$ & $Det(\cdot)$ of a 2nd order matrix A

$Tr(A) = a_{11} + a_{22}$ real number

$\lambda_{1,2} = \frac{1}{2}\left[Tr(A) \pm (Tr(A)^2 - 4\det(A))^{1/2}\right]$ if $Tr(A)^2 - 4\det(A) \geq 0 \rightarrow \lambda_{1,2} \in \mathbb{R}$ ← real plane

$\underbrace{Tr(A)^2 - 4\det(A)}_{\text{discriminant}} < 0 \rightarrow \lambda_{1,2} \in \mathbb{C}$ ← complex $\quad \therefore \lambda_1 = \lambda_2^*$

Classifications of phase plane of linear systems/

$\dot{x} = Ax$, $x \in \mathbb{R}^2$ specifically to interpret 2 phase plane diagrams will not be asked to plot phase plane diagrams but can give a in an exam a phase plane diagram & from visual ~~diagram~~ observation/ interpretation of what type of stability or what type of linear system they are

$x \in \mathbb{R}^2$ to determine eigvals $\dot{x}(t) = A\,x(t)$ $A \in \mathbb{R}^{2\times 2}$ possible to provide this sort of dynamics

$\dot{x} = Ax$ $x \in \mathbb{R}^2$ A is 2×2 matrix ∴ have 2 eigvals

Both −ve stable node (sink)

Both +ve unstable node (source) $\dot{x} = f(x)$ about equili: $\dot{x} = A_2 x$

+ve & −ve saddle point $\dot{x} = f(x)$

$\begin{cases} \dot{x} = A_1 x \\ \dot{x} = A_2 x \end{cases}$, say $\dot{x} = A_3 x$ have 3 equili

looked at $\lambda_{1,2}$ real & distinct (both +ve, both −ve, & one +v, one −ve)

$\lambda_1 = \lambda_2 = \lambda$ −ve stable star

$\lambda_1 = \lambda_2 = \lambda$ +ve unstable star

remark: $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$ then two eigval at λ } real distinct, real repeated

(real distinct both +ve, both −ve, one +ve, one −ve)
← saddle

(real repeated (both +ve, both −ve)

Case $\lambda_{1,2} \in \mathbb{C}$: $\lambda_1 = \lambda_2^*$

$Re(\lambda) < 0$ stable spiral / focus

$Re(\lambda) > 0$ unstable spiral / focus

$Re(\lambda) \equiv 0$ equili: Center

consider $\dot{x}(t) = A_1 x(t)$, $A_1 = \begin{bmatrix} -1 & -1 \\ -1 & -5 \end{bmatrix}$ eigvals: $\Lambda = [-5.24, -0.764]^T$

$A_1 D - D\,Diag(\Lambda) = 0$, $\lambda_{1,2} \in \mathbb{R} < 0$ ∴

eigvecs $D = \begin{bmatrix} 0.230 & -0.973 \\ 0.973 & 0.230 \end{bmatrix}$ (D_1, D_2)

stable node ∴ (attracting origin) trajectories (orbits) become tangent to slow eigvec as they approach origin & parallel to fast eigvec far from origin

for saddle node: stable eigvec direction & unstable eigvec direction

stable trajectories (orbits) are along stable eigvecs direction D_2 which is of D_1

& unstable along unstable eigvec direction phase planes for degenerate cases $\lambda_{1,2} \in \mathbb{R}$

$\lambda_1 = \lambda_2 = -\lambda$ -ve origin (attracting) there is no slow-fast behaviour observed (cf. to that of a node) stable star

repelling case $\lambda_1 = \lambda_2 = +\lambda$ +ve unstable star

complex pair $\lambda_{1,2} \in \mathbb{C}$

attracting asymptotically reach origin stable spiral/focus $Re(\lambda) < 0$

repelling case unstable spiral/focus $Re(\lambda) > 0$

Center (marginally stable) $Re(\lambda) = 0$ $A = \begin{bmatrix} \cdot & \cdot \\ \cdot & \cdot \end{bmatrix}_{2\times2}$ $\lambda_{1,2} = \pm j\omega$

$\lambda_1 = \lambda_2^*$

$Tr(A)^2 - 4\det(A) \geq 0$ $\lambda_{1,2} \in \mathbb{R}$ is < 0 $\lambda_{1,2} \in \mathbb{C}$

Structurally Stable $\dot{x} = Ax$, $x \in \mathbb{R}^2$ & $\dot{x} = f(x)$ $A = J(f) = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$

perturbations $A \to (A + \Delta A)$ $\Delta A = \begin{bmatrix} -\varepsilon & 0 \\ 0 & -\varepsilon \end{bmatrix}$ $\therefore \begin{bmatrix} a_{11}-\varepsilon & a_{12} \\ a_{21} & a_{22}-\varepsilon \end{bmatrix} = A + \Delta A$

arbitrarily small perturbations

effect of perturbations $A \to (A + \Delta A)$ ΔA is additive perturbation

- effect of perturbations reflect in vals of eigvals, eigvecs, Tr, Det $\therefore$ affect phase plane
- node with distinct eigvals, saddle, & focus is structurally stable $\because$ 2 qualitative behaviours remain same ie not affected under arbit small perturbations ΔA
- node with multiple eigvals (stable star) could become a stable node or a stable focus under arbit small perturbations ΔA
- a center is also not structurally stable $\begin{bmatrix} \mu & 1 \\ -1 & \mu \end{bmatrix} = A + \Delta A$

eigvals are $\mu \pm j$ (μ is a small perturbation param to ~~stable focus~~ about origin)

- $\mu < 0$ (consider as a small perturbation $-\varepsilon$) converts a center to a stable focus
- $\mu > 0$ (small perturb $+\varepsilon$) converts center to unstable focus

$\lambda_1 = \lambda_2 = -\lambda$ is ΔA added eig$(A + \Delta A)$ $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$

is unstable star $\to$ unstable node or unstable focus $\Delta A = \begin{bmatrix} \mu & 0 \\ 0 & \mu \end{bmatrix}$

periodic orbit / limit cycle

1) what is limit cycle ← periodic orbit

2) determine existence of limit cycle (2nd order system) - use of Poincaré-Bendixson criterion

3) rule out existence of limit cycle in a domain - use of Bendixson criterion

a generic dynamical system $\dot{x} = f(t,x)$, $x \in D \subset \mathbb{R}^n$ (subset)

if $\exists\ T>0$ st $f(t,x) = f(t+T, x)\ \forall (t,x) \in [t_0, \infty) \times D$ then Z dynamical system is periodic dynamical system

time response $x_1(t_1) = \bar{x}$ $t_1 \to t_1 + T$ $T>0$ minimum amount of time

$\Rightarrow$ when presented in phaseplane, will be a closed curve

phase portrait: spiralling outwards (unstable focus)

as time $t \to \infty$ all trajectories inside asymptotic converge to the closed curve

at regular intervals state $(\bar{x}_1, \bar{x}_2)$ occurs for $\dot{x} = f(x)$ $x \in \mathbb{R}^2$

$x(t+T) = x(t)$ periodic orbit $D = \{-2 \le x_1 \le 2,\ -2 \le x_2 \le 2\}$ is invariant

all trajectories leave origin asymp ends up to periodic orbit

for existence of limit cycle: 1) a trajectory enters a domain M at t_1, it remains in domain $M \subset \mathbb{R}^2$ $\forall t > t_1$

2) trajectory approach a periodic orbit or an equili

stable is not what we want ∴ want unstable

3) if M contains no equili ① holds when M has a periodic orbit

4) if M has one equili Z trajectories need to move away from that equili & approach orbit as $t \to \infty$

Limit cycle is a closed trajectory in phase plane st other non-closed trajectories spiral towards it asymp from inside & outside (as $t \to \infty$)

Existance of limit cycle (Poincaré-Bendixon Criterion) $x \in \mathbb{R}^2$

Consider $\dot{x} = f(x)$ & let M (set) be closed, bounded subset of plane st.

M contains no equili pts, or contains only one equili pt,

st Z jacobian matrix $[\partial f / \partial x]$ at this pt has eignals with the real

parts. [equili pts is unstable focus or node (unstable)] $\Rightarrow$ every trajectory

starting in M stays in M $\forall$ future time. then M contains a periodic

orbit or limit cycle of $\dot{x} = f(x)$ how we will show that

if M contains only one equili satisfying PB conds (Poincaré-Bendixson)

$Re(\lambda) > 0$ then in its viscinity all trajectories move away from it

$\Rightarrow$ so we can choose a simple closed curve • a circle $x_1^2 + x_2^2 = C_1$

• an ellipse $x_1^2 + 3x_1x_2 + x_2^2 = C_2$ • or a polygon around equili

let that curve be $V(x)$ $V(x)$ is continuously diffable $x \in \mathbb{R}^2$

~~$x = [x_1$~~ $x = [x_1, x_2]^T$ $V(x_1, x_2)$ $\nabla V(x_1, x_2) = [\partial V / \partial x_1, \partial V / \partial x_2]$

$V(x_1, x_2)$ a continuously diffable func with $\nabla V(x_1, x_2) = [\partial V / \partial x_1, \partial V / \partial x_2]$

recall dynamical system $\dot{x} = f(x)$ $f(\cdot) = \begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \end{bmatrix}$ x^* unstable

inside $V(x) = C$ consider inner product $f(x) \cdot \nabla V(x) = \begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \end{bmatrix} [\partial V / \partial x_1 \;\; \partial V / \partial x_2] =$

$f_1(\cdot) \partial V / \partial x_1 + f_2(\cdot) \partial V / \partial x_2$

x and $V(x) = C$ unstable consider about x in figure

$f(x) \cdot \nabla V(x) < 0$ $f(x)$ pts inward

$f(x) \cdot \nabla V(x) = 0$ tangent to curve at x

$f(x) \cdot \nabla V(x) > 0$ pts outward at x

trajectories can leave a set only if Z vec fields $(f(\cdot))$ pts outwards at some pt on boundary $\therefore$

for a set of Z form $M = \{ x \in \mathbb{R}^2 \mid V(x) \le C \}$ $C > 0$

trajectories are trapped inside M if $f(x) \cdot \nabla V(x) \le 0$ on boundary $V(x) = C$

annular region $M = \{ x \in \mathbb{R}^2 \mid W(x) \ge C_1 \;\&\; V(x) \le C_2 \}$ $C_1 > 0$ & $C_2 > 0$

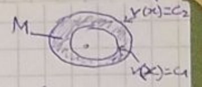
trajectories are trapped inside M if $f(x) \cdot \nabla V(x) \le 0$ on $V(x) = C_2$

$f(x) \cdot \nabla W(x) \ge 0$ on $W(x) = C_2$

\Ex/ $\dot{x}_1 = x_2$
$\dot{x}_2 = -x_1$

$f_1(\cdot) = x_2$
$f_2(\cdot) = -x_1$

Consider annular region



$M = \{x \in \mathbb{R}^2 \mid c_1 \leq V(x) = x_1^2 + x_2^2 = c_2\}$ & $c_2 > c_1 > 0$

Set M is closed & bounded.

equili is origin   M does not contain Z only equili at origin.

if we calc $f \cdot \nabla V(x) = f_1(\cdot)\frac{\partial V}{\partial x} + f_2(\cdot)\frac{\partial V}{\partial x_2} = 0$ $\left\{\frac{\partial V}{\partial x_1} = 2x_1 \quad \frac{\partial V}{\partial x^2} = 2x_2\right\}$

∴ trajectories are trapped within M by PB (Poincaré Bendixon) criterion, we can conclude existance of periodic orbit.

\Ex/ $\dot{x}_1 = x_1 + x_2 - x_1(x_1^2 + x_2^2)$
$\dot{x}_2 = -2x_1 + x_2 - x_2(x_1^2 + x_2^2)$

unique equili at $(0,0)^T$ (Solving $f_1(\cdot)$ & $f_2(\cdot)$ for x_1, x_2 by setting $\dot{x}_1 = \dot{x}_2 = 0$)

Jacobian $\left.\frac{\partial f}{\partial x}\right|_{(0,0)^T} = \left.\begin{bmatrix} 1 - 3x_1^2 - x_2^2 & 1 - 2x_1x_2 \\ -2 - 2x_1x_2 & 1 - x_1^2 - 3x_2^2 \end{bmatrix}\right|_{(0,0)} =$
$\begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix}$ eignvals are $1 \pm i\sqrt{2}$ $Re(\lambda) > 0$ ∴

<u>unstable focus</u> $M = \{x \in \mathbb{R}^2 \mid V(x) = x_1^2 + x_2^2 \leq C\}$, $C > 0$ {in exam V(x) will be given} M is a closed & bounded set & contains one equili which is unstable

on surf $V(x) = C$ $f(x) \cdot \nabla V(x) = f_1(\cdot)\frac{\partial V}{\partial x_1} + f_2\frac{\partial V}{\partial x_2} =$

$2x_1[x_1 + x_2 - x_1(x_1^2 + x_2^2)] + 2x_2[-2x_1 + x_2 - x_2(x_1^2 + x_1^2)]$ $\{\because V(x) = x_1^2 + x_2^2 \therefore f_1(x) = \dot{x}_1,\ f_2(x) = \dot{x}_2\}$

$= 2(x_1^2 + x_2^2) + 2x_1x_2 - 4x_1x_2 - 2(x_1^2 + x_2^2)^2$ $\{-2x_1^2(x_1^2 + x_2^2) - 2x_2^2(x_1^2 + x_2^2) = -2(x_1^2 + x_2^2)^2\}$

$= 2(x_1^2 + x_2^2) - 2(x_1^2 + x_2^2)^2 - 2x_1x_2$ $\{\because |2x_1x_2| \leq x_1^2 + x_2^2\}$

$\leq 2(x_1^2 + x_2^2) - 2(x_1^2 + x_2^2)^2 + (x_1^2 + x_2^2)$

$= 3C - 2C^2$ tangential condition $f(\cdot) \cdot \nabla V(x) = 0$

Setting $3C - 2C^2 = 0$ $C = 1.5$ $f(x) \cdot \nabla V(x) \leq 0$ ∴ $C \geq 1.5$ PB (Poincaré bendixon) condition holds

M using value of C $C \geq 1.5$ by choosing $C \geq 1.5$ it is possible to ensure that all trajectories are trapped inside M

∴ by poincare-Bendixson Criterion, it can be concluded that there is a periodic orbit in $M = \{x \in \mathbb{R}^2 \mid V(x) = x_1^2 + x_2^2 \leq 1.5\}$

(C = 2 aswell since can have any $C \geq 1.5$ for ex)

Week 4 / outline / matrix ~ state space representation of LTI system?

How we can obtain representation - 'transfer function'?

Concept of controllability

Let us consider a 2^{nd} order system represented as ODE

$$\frac{d^2x}{dt^2} = 2\underbrace{\xi}_{\text{damping ratio}}\omega_n \frac{dx}{dt} + \underbrace{\omega_n^2}_{\text{natural frequency}} x = \omega_n^2 f(t) \leftarrow \text{forcing term}$$

Compact state space representation

$z_1 = x \qquad \dot{z}_1 = \dot{x} = z_2$

$z_2 = \dot{x} \qquad \dot{z}_2 = \ddot{x} = -\omega_n^2 x - 2\xi\omega_n \dot{x} + \omega_n^2 f(t) =$

$-\omega_n^2 z_1 - 2\xi\omega_n z_2 + \omega_n^2 f(t)$

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -\omega_n^2 z_1 - 2\xi\omega_n z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n^2 f(t) \end{bmatrix}$$

$$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} \qquad A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \in \mathbb{R}^{2\times 2} \qquad B = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix}$$

in matrix form $\dot{z} = Az + Bu$ where $u \in f(t)$

Suppose we measure only 'x' as the output

ie z_1 only measured $y = Cz + Du$

y is z_1 $\qquad z_1 = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + [0]\,u$ $\qquad$ {Matrix [0] of suitable dim}

C matrix is $[1, 0]$ ∴

$\dot{z} = Az + Bu$
$y = Cz + Du$
is LTI state space form (A, B, C, D)

Linear ODE (with IC) ⇒ Compact state space form (with IC)

in matrix form $\dot{z} = Az + Bu$, $y = Cz + Du$ is equiv representations of Z system

state $z \in \mathbb{R}^n$ $\quad A \in \mathbb{R}^{n\times n}$ $\quad$ control $u \in \mathbb{R}^m$ $\quad B \in \mathbb{R}^{n\times m}$

output $y \in \mathbb{R}^p$ $\quad C \in \mathbb{R}^{p\times n}$ $\quad p \le n$ $\quad D \in \mathbb{R}^{p\times m}$

another equiv representation is: transfer funcs

we obtain them using Lapace transform

Week 4 LTI system, state space form, transfer funcs & controllability

2nd order 'ODE' $\frac{d^2x}{dt^2} + 2\underset{\text{damping ratio}}{\xi}\underset{\text{natural frequency}}{\omega_n}\frac{dx}{dt} + \omega_n^2 x(t) = \omega_n^2 f(t)$

- represent a wide range of systems

Compact form $z_1 = x$ $z_2 = \dot{x}$ $\therefore \dot{z}_1 = \dot{x} = z_2$

$\dot{z}_2 = \ddot{x} = -\omega_n^2 x - 2\xi\omega_n \dot{x} + \omega_n^2 f = -\omega_n^2 z_1 - 2\xi\omega_n z_2 + \omega_n^2 f(t)$

{$\therefore \dot{z}_1$, $\dot{z}_2$ is compact form} in Matrix form $\dot{z} = Az + Bu$, $y = Cz + Du$

$z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ $A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix} \in \mathbb{R}^{2\times 2}$ $B = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} \in \mathbb{R}^{2\times 1}$

$\left\{ \therefore \dot{z} = \begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} z_2 \\ -\omega_n^2 z_1 - 2\xi\omega_n z_2 + \omega_n^2 f(t) \end{bmatrix} = \begin{bmatrix} 0 z_1 + 1 z_2 \\ -\omega_n^2 z_1 - 2\xi\omega_n z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n^2 f(t) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\xi\omega_n \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} f(t) \right\}$

Suppose if we measure only x as the output $\therefore z_1$ is measured $\therefore C = [1, 0]$, D is zero

$\left\{ \therefore y = Cz + Du = C\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + (0)u = \begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = z_1 \right\}$

Linear ODE & IC $\Rightarrow$ Compact form (IC) $\rightarrow$ Mat State space form

z_0 $\dot{z} = Az + Bu$ $y = Cz + Du$ (with the knowledge of what state is measure) $\underset{\text{states}}{z \in \mathbb{R}^n}$ $A \in \mathbb{R}^{n\times n}$ $B \in \mathbb{R}^{n\times m}$ $\underset{\text{control input}}{u \in \mathbb{R}^m}$ $C \in \mathbb{R}^{p\times n}$ $\underset{\text{output measurements}}{y \in \mathbb{R}^p}$

$D \in \mathbb{R}^{p\times m}$ are all equivalent representations of system (linear)

Another one is <u>Transfer funcs</u>: makes use of Laplace transforms

- Laplace transform converts <u>integral</u> & <u>differential</u> eqn into <u>algebraic eqns</u> Laplace transforms of a signal $f(t)$ is the func

$\underset{\text{frequency domain}}{F(s)} = \mathcal{L}(\underset{\text{time domain func}}{f(t)})$ $\mathcal{L}(\cdot)$ represents the Laplace transform

$F(s)$ is a complex valued func of complex number

$F(s) = \int_0^\infty f(t) \cdot e^{-st}\, dt$ $s \in \mathbb{C}$ complex frequency variable sec^{-1}

eg: $f(t) = 1$ (unit step) $F(s) = \int_0^\infty 1 \cdot e^{-st}\, dt = \left[\frac{1}{s}e^{-st}\right]_0^\infty = \frac{1}{s}$

eg: $f(t) = e^t$: $F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^t e^{-st}\, dt = \int_0^\infty e^{(1-s)t}\, dt = \frac{1}{s-1}$

- Certain properties of Laplace transform (LT)

Let α, β be consts $F(s) = \mathcal{L}(f(t))$ $G(s) = \mathcal{L}(g(t))$

- Linearity $\mathcal{L}(\alpha f(t) + \beta g(t)) = \alpha F(s) + \beta G(s)$