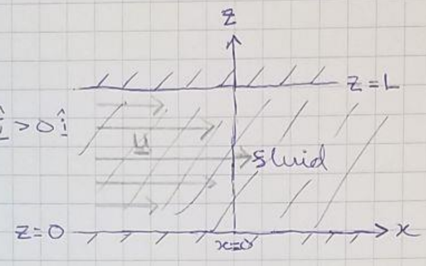
1a/ Incompressible, viscous fluid $\therefore$ $\mu \neq 0$ and $\nabla \cdot \underline{u} = 0$ $\therefore$

$\nabla p = \frac{dp}{dx}\hat{\underline{i}} > 0\hat{\underline{i}}$




$\nabla p = \frac{dp}{dx}\hat{\underline{i}} > 0\hat{\underline{i}}$

1b/ $\nabla p = \frac{dp}{dx}\hat{i}$ and $\frac{dp}{dx} > 0$ $\therefore$ The pressure gradient is not zero along the x-axis but is positive $\therefore$ The pressure is directed in the positive x-direction

but gravity is to be neglected and the boundaries are stationary $\therefore$ no point along the x-axis is different from any other point on the x-axis $\therefore$

we expect the speed of the flow to be independent of $x$ $\therefore$

expect the fluid flow to flow in the positive x-direction and the speed of the flow to depend on $z$ but be independent of $x$

\1c/ gravity is to be neglected ∴

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \mu\nabla^2\underline{u} \quad ∴$$

- steady fluid flow ∴ $\underline{u}$ is independent of time ∴

$$\frac{\partial \underline{u}}{\partial t} = 0$$

$\underline{u}\cdot\nabla\underline{u} = (\underline{u}\cdot\nabla)\underline{u}$ ∴ $\underline{u} = u(z)\hat{\underline{i}}$ ∴

$\underline{u}\cdot\nabla = (u(z)\hat{\underline{i}})\cdot\nabla = u(z)\frac{\partial}{\partial x}$ ∴

$\underline{u}\cdot\nabla\underline{u} = \left(u(z)\frac{\partial}{\partial x}\right)\underline{u} = \left(u(z)\frac{\partial}{\partial x}\right)(u(z)\hat{\underline{i}}) =$

$u(z)\frac{\partial}{\partial x}(u(z)\hat{\underline{i}}) = u(z)\hat{\underline{i}}\frac{\partial}{\partial x}(u(z)) = u(z)\hat{\underline{i}}(0) =$

$0 = \underline{u}\cdot\nabla\underline{u}$ ∴

$\rho(0) = -\nabla p + \mu\nabla^2\underline{u} = 0$ ∴ $\mu\nabla^2\underline{u} = \nabla p$ ∴

$$\mu\nabla^2\underline{u} = \frac{dp}{dx}\hat{\underline{i}} = \mu\nabla^2(u(z)\hat{\underline{i}})$$

from formula sheet: $\nabla^2\underline{u} = \nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{u})$ ∴

$\nabla\cdot\underline{u} = \nabla\cdot(u(z)\hat{\underline{i}}) = \frac{\partial}{\partial x}u(z) = 0$ ∴

$\nabla(\nabla\cdot\underline{u}) = \nabla(0) = 0$

$$\nabla\times\underline{u} = \nabla\times(u(z)\hat{\underline{i}}) = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u(z) & 0 & 0 \end{vmatrix} =$$

$\hat{\underline{i}}\left(\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}(0)\right) - \hat{\underline{j}}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(u(z))\right) + \hat{\underline{k}}\left(\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial y}(u(z))\right) =$

$\hat{\underline{i}}(0) - \hat{\underline{j}}\left(0 - \frac{\partial}{\partial z}(u(z))\right) + \hat{\underline{k}}(0-0) =$

$$\frac{\partial}{\partial z}(u(z))\hat{\underline{j}} = \nabla\times\underline{u} \quad ∴$$

$$\nabla\times(\nabla\times\underline{u}) = \nabla\times\left[\frac{\partial}{\partial z}(u(z))\hat{\underline{j}}\right] = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ 0 & \frac{\partial}{\partial z}u(z) & 0 \end{vmatrix} =$$

$\hat{\underline{i}}\left[\frac{\partial}{\partial y}(0) - \frac{\partial}{\partial z}\left(\frac{\partial}{\partial z}u(z)\right)\right] - \hat{\underline{j}}\left[\frac{\partial}{\partial x}(0) - \frac{\partial}{\partial z}(0)\right] + \hat{\underline{k}}\left[\frac{\partial}{\partial x}\left(\frac{\partial}{\partial z}u(z)\right) - \frac{\partial}{\partial y}(0)\right] =$

$\hat{\underline{i}}\left[0 - \frac{\partial^2}{\partial z^2}u(z)\right] - \hat{\underline{j}}[0-0] + \hat{\underline{k}}\left[\frac{\partial}{\partial z}\left(\frac{\partial}{\partial x}u(z)\right) - 0\right] =$

$-\frac{\partial^2}{\partial z^2}(u(z))\hat{\underline{i}} + \hat{\underline{k}}\left[\frac{\partial}{\partial z}(0)\right] = -\hat{\underline{i}}\frac{\partial^2}{\partial z^2}u(z) = \nabla\times(\nabla\times\underline{u})$ ∴

$\nabla^2\underline{u} = 0 - -\hat{\underline{i}}\frac{\partial^2}{\partial z^2}u(z) = \hat{\underline{i}}\frac{\partial^2}{\partial z^2}u(z) = \nabla^2[u(z)\hat{\underline{i}}]$ ∴

$$\frac{dp}{dx}\hat{\underline{i}} = \mu\hat{\underline{i}}\frac{\partial^2}{\partial z^2}u(z) \quad ∴ \quad \frac{dp}{dx} = \mu\frac{\partial^2}{\partial z^2}u(z)$$

∴ $\frac{dp}{dx}$ is independent of $x$ ∴

1c continued/ $\frac{dp}{dx}$ is either a function of time or constant

$\therefore$ $\frac{dp}{dx} = F(t)$ where F is a function $\therefore$

$$F(t) = \mu \frac{\partial^2}{\partial z^2} U(z) \quad \therefore$$

$$\int F(t)\, dz = \int \mu \frac{\partial^2}{\partial z^2} U(z)\, dz = \mu \int \frac{\partial^2}{\partial z^2} U(z)\, dz = \mu \frac{\partial}{\partial z} U(z) = \cancel{z\,F(t)}$$

$$zF(t) + C_1 = \mu \frac{\partial}{\partial z} U(z) \quad \therefore$$

$$\int zF(t) + C_1\, dz = \int \mu \frac{\partial}{\partial z} U(z) = \mu U(z) = \frac{1}{2} z^2 F(t) + C_1 z + C_2$$

$\therefore$ stationary boundaries $\therefore$

by the no-slip boundary conditions: $U(z=0) = 0$ and $U(z=L) = 0$

$\therefore$ $\cancel{\mu U(z=}$ at $z=0$: $\mu U(z=0) = \frac{1}{2}(0)^2 F(t) + C_1(0) + C_2 = \cancel{0z}$

$$\mu(0) = 0 = 0 + 0 + C_2 = C_2 = 0 \quad \therefore$$

$$\mu U(z) = \frac{1}{2} z^2 F(t) + C_1 z \quad \therefore$$

at $z = L$: $\mu U(z=L) = \mu(0) = 0 = \frac{1}{2} L^2 F(t) + C_1 L \quad \therefore$

$$-\frac{1}{2} L^2 F(t) = C_1 L \quad \therefore$$

$$C_1 = -\frac{1}{2} L F(t) \quad \therefore$$

$$\mu U(z) = \frac{1}{2} z^2 F(t) - \frac{1}{2} L F(t) z \quad \therefore$$

$$U(z) = \frac{1}{2\mu} z^2 F(t) - \frac{1}{2\mu} L F(t) z$$

1d/ $u(z) = \frac{1}{2\mu} F(t) z^2 - \frac{1}{2\mu} L F(t) z = -\frac{1}{2\mu} F(t) z (z-L)$

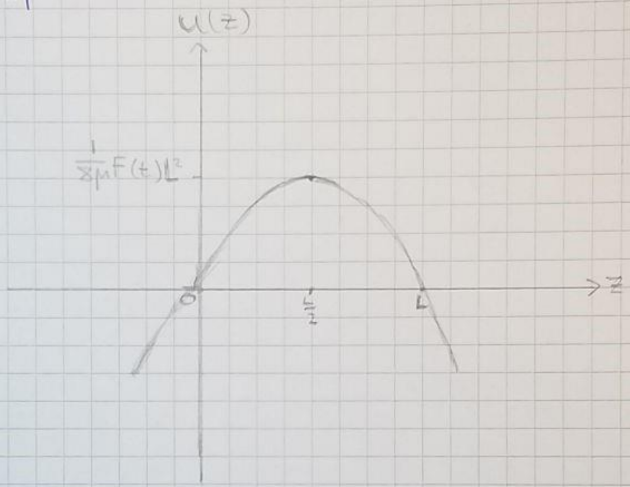
$\therefore$ is $u(z) = 0 = -\frac{1}{2\mu} F(t) z (z-L)$ $\therefore$ $z-L=0$ , $z=0$ $\therefore$

$z=L$ , $z=0$ is its roots and:

$\frac{L+0}{2} = \frac{L}{2}$ $\therefore$ Max $u(z)$ at $z=\frac{L}{2}$ since $u(z)$ is a negative quadratic

$u(z=\frac{L}{2}) = \cancel{-\frac{1}{2\mu} L F t} - \frac{1}{2\mu} F(t) \frac{L}{2}(\frac{L}{2} - L) = -\frac{1}{2\mu} F(t) \frac{L}{2}(-\frac{L}{2}) =$

$-\frac{1}{2\mu} F(t) (-\frac{L^2}{4}) = \frac{1}{8\mu} F(t) L^2$ $\therefore$




$\therefore$ $\underline{u} = u(z)\hat{\underline{i}} = -\frac{1}{2\mu} F(t) z (z-L) \hat{\underline{i}}$ $\therefore$

at $z=0$: $\underline{u} = 0 \hat{\underline{i}}$

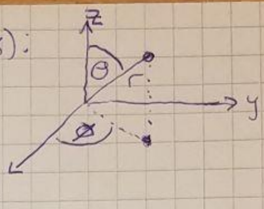
at $z=L$: $\underline{u} = 0 \hat{\underline{i}}$

and Max at $z=\frac{L}{2}$ : $\underline{u} = \frac{1}{8\mu} F(t) L^2 \hat{\underline{i}}$ with $\frac{1}{8\mu} F(t) L^2 > 0$ $\therefore$

with these velocity vectors being independent of $x$

they can be drawn ~~of~~ on the graph:

2a/ In Spherical polar coordinates $(r, \theta, \phi)$:




$\underline{\omega} = \nabla \times \underline{u}$ is vorticity

$$\therefore \quad \nabla \times \underline{u} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\,\hat{\underline{\phi}} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ u_r & r u_\theta & r\sin\theta\, u_\phi \end{vmatrix}$$

$$\therefore \quad \underline{u} = u_r \hat{\underline{r}} + u_\theta \hat{\underline{\theta}} + u_\phi \hat{\underline{\phi}} = r^2\cos\theta\, \hat{\underline{r}} + \frac{1}{r}\hat{\underline{\theta}} + \frac{1}{\sin\theta}\hat{\underline{\phi}} \quad \therefore$$

$$u_r = r^2\cos\theta, \quad u_\theta = \frac{1}{r}, \quad u_\phi = \frac{1}{\sin\theta} \quad \therefore$$

$$\underline{\omega} = \nabla \times \underline{u} = \frac{1}{r^2\sin\theta} \begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\,\hat{\underline{\phi}} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ r^2\cos\theta & r\frac{1}{r} & r\sin\theta\,\frac{1}{\sin\theta} \end{vmatrix} =$$

$$\frac{1}{r^2\sin\theta} \begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\,\hat{\underline{\phi}} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ r^2\cos\theta & 1 & r \end{vmatrix} =$$

$$\frac{1}{r^2\sin\theta}\left[\hat{\underline{r}}\left(\frac{\partial r}{\partial\theta} - \frac{\partial 1}{\partial\phi}\right) - r\hat{\underline{\theta}}\left(\frac{\partial r}{\partial r} - \frac{\partial}{\partial\phi}(r^2\cos\theta)\right) + r\sin\theta\,\hat{\underline{\phi}}\left(\frac{\partial 1}{\partial r} - \frac{\partial(r^2\cos\theta)}{\partial\theta}\right)\right]$$

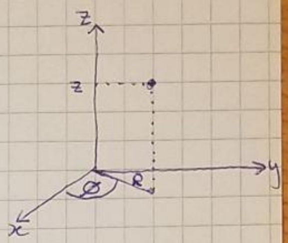
$$= \frac{1}{r^2\sin\theta}\left[\hat{\underline{r}}(0-0) - r\hat{\underline{\theta}}(1-0) + r\sin\theta\,\hat{\underline{\phi}}\left(0 - r^2\frac{\partial}{\partial\theta}\cos\theta\right)\right] =$$

$$\frac{1}{r^2\sin\theta}\left[0\,\hat{\underline{r}} - r\hat{\underline{\theta}} + r\sin\theta\,\hat{\underline{\phi}}(r^2\sin\theta)\right] =$$

$$\frac{1}{r^2\sin\theta}\left[-r\hat{\underline{\theta}} + r^3\sin^2\theta\,\hat{\underline{\phi}}\right] =$$

$$\frac{-1}{r\sin\theta}\hat{\underline{\theta}} + r\sin\theta\,\hat{\underline{\phi}} = \underline{\omega}$$

\2b/ $\underline{u}\cdot\nabla\underline{u} = (\underline{u}\cdot\nabla)\underline{u} \quad \therefore \quad \underline{u} = R\cos\phi\,\hat{\underline{R}} + \sin\phi\,\hat{\underline{\phi}} + 0\,\hat{\underline{z}}$




In cylindrical polar coordinates:

$\underline{u}\cdot\nabla = (R\cos\phi\,\hat{\underline{R}} + \sin\phi\,\hat{\underline{\phi}} + 0\,\hat{\underline{z}})\cdot\nabla =$

$(R\cos\phi\,\hat{\underline{R}} + \sin\phi\,\hat{\underline{\phi}} + 0\,\hat{\underline{z}})\cdot\left(\hat{\underline{R}}\frac{\partial}{\partial R} + \frac{1}{R}\hat{\underline{\phi}}\frac{\partial}{\partial\phi} + \hat{\underline{z}}\frac{\partial}{\partial z}\right) =$

$R\cos\phi\frac{\partial}{\partial R} + \frac{1}{R}\sin\phi\frac{\partial}{\partial\phi} + (0)\frac{\partial}{\partial z} =$

$R\cos\phi\frac{\partial}{\partial R} + \frac{1}{R}\sin\phi\frac{\partial}{\partial\phi} \quad \therefore$

$\underline{u}\cdot\nabla\underline{u} = (\underline{u}\cdot\nabla)\underline{u} = (\underline{u}\cdot\nabla)(R\cos\phi\,\hat{\underline{R}} + \sin\phi\,\hat{\underline{\phi}}) =$

$\left(R\cos\phi\frac{\partial}{\partial R} + \frac{1}{R}\sin\phi\frac{\partial}{\partial\phi}\right)(R\cos\phi\,\hat{\underline{R}} + \sin\phi\,\hat{\underline{\phi}}) =$

$R\cos\phi\frac{\partial}{\partial R}(R\cos\phi\,\hat{\underline{R}}) + \frac{1}{R}\sin\phi\frac{\partial}{\partial\phi}(R\cos\phi\,\hat{\underline{R}}) \cancel{+ R\cos\phi\frac{\partial}{\partial R}(R\cos\phi\,\hat{\underline{R}}}$

$R\cos\phi\frac{\partial}{\partial R}(\sin\phi\,\hat{\underline{\phi}}) + \frac{1}{R}\sin\phi\frac{\partial}{\partial\phi}(\sin\phi\,\hat{\underline{\phi}}) =$

$R\cos\phi\,\hat{\underline{R}}\frac{\partial}{\partial R}(R\cos\phi) + \frac{R\cos\phi}{R}\sin\phi\frac{\partial}{\partial\phi}(\hat{\underline{R}}) + \frac{1}{R}\sin\phi\,\hat{\underline{R}}\frac{\partial}{\partial\phi}(R\cos\phi) +$

$R\cos\phi\,\hat{\underline{\phi}}\frac{\partial}{\partial R}(\sin\phi) + \frac{1}{R}\sin\phi\sin\phi\frac{\partial}{\partial\phi}(\hat{\underline{\phi}}) + \frac{1}{R}\sin\phi\,\hat{\underline{\phi}}\frac{\partial}{\partial\phi}(\sin\phi) =$

$R\cos^2\phi\,\hat{\underline{R}}\frac{\partial R}{\partial R} + \cos\phi\sin\phi\,\hat{\underline{\phi}} + \sin\phi\,\hat{\underline{R}}\frac{\partial}{\partial\phi}(\cos\phi) +$

$R\cos\phi\,\hat{\underline{\phi}}\,(0) + \frac{1}{R}\sin^2\phi(-\hat{\underline{R}}) + \frac{1}{R}\sin\phi\,\hat{\underline{\phi}}\cos\phi =$

$R\cos^2\phi\,\hat{\underline{R}} + \cos\phi\sin\phi\,\hat{\underline{\phi}} + \sin\phi\,\hat{\underline{R}}(-\sin\phi) + \frac{-1}{R}\sin^2\phi\,\hat{\underline{R}} + \frac{1}{R}\sin\phi\cos\phi\,\hat{\underline{\phi}} =$

$R\cos^2\phi\,\hat{\underline{R}} - \sin^2\phi\,\hat{\underline{R}} + \cos\phi\sin\phi\,\hat{\underline{\phi}} - \frac{1}{R}\sin^2\phi\,\hat{\underline{R}} + \frac{1}{R}\sin\phi\cos\phi\,\hat{\underline{\phi}} =$

$\left(R\cos^2\phi - \sin^2\phi - \frac{1}{R}\sin^2\phi\right)\hat{\underline{R}} + \left(\cos\phi\sin\phi + \frac{1}{R}\sin\phi\cos\phi\right)\hat{\underline{\phi}} = \underline{u}\cdot\nabla\underline{u}$

3a/ stagnation point when $\underline{u} = \underline{0}$ $\therefore$

$$\frac{\psi}{R}\hat{\underline{\phi}} = -\frac{1}{R}\frac{R^2}{10}(a^2 - R^2 - z^2)\hat{\underline{\phi}} = -\frac{R}{10}(a^2 - R^2 - z^2)\hat{\underline{\phi}} \quad \therefore$$

$$\underline{u} = \nabla \times \left(\frac{\psi}{R}\hat{\underline{\phi}}\right) = \nabla \times \left(-\frac{R}{10}(a^2 - R^2 - z^2)\hat{\underline{\phi}}\right) =$$

$$\frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial/\partial R & \partial/\partial\phi & \partial/\partial z \\ 0 & -\frac{R^2}{10}(a^2 - R^2 - z^2) & 0 \end{vmatrix} =$$

$$\frac{1}{R}\left[\hat{\underline{R}}\left(0 - \frac{\partial}{\partial z}\left[-\frac{R^2}{10}(a^2 - R^2 - z^2)\right]\right) - R\hat{\underline{\phi}}(0-0) + \hat{\underline{z}}\left(\frac{\partial}{\partial R}\left[-\frac{R^2}{10}(a^2 - R^2 - z^2)\right] - 0\right)\right] =$$

$$\frac{1}{R}\left[\hat{\underline{R}}\left(\frac{1}{10}\frac{\partial}{\partial z}\left[a^2R^2 - R^4 - R^2z^2\right]\right) + \hat{\underline{z}}\left(-\frac{1}{10}\frac{\partial}{\partial R}\left[a^2R^2 - R^4 - R^2z^2\right]\right)\right] =$$

$$\frac{1}{R}\left[\hat{\underline{R}}\left(\frac{1}{10}\left[-2R^2z\right]\right) + \hat{\underline{z}}\left(-\frac{1}{10}\left[2a^2R - 4R^3 - 2Rz^2\right]\right)\right] =$$

$$\frac{-2}{10R}R^2 z\hat{\underline{R}} + \left(\frac{-2a^2R}{10R} + \frac{4R^3}{10R} + \frac{2Rz^2}{10R}\right)\hat{\underline{z}} =$$

$$-\frac{1}{5}Rz\hat{\underline{R}} + \left(-\frac{1}{5}a^2 + \frac{2}{5}R^2 + \frac{1}{5}z^2\right)\hat{\underline{z}} = \underline{u} \quad \therefore$$

for stagnation points of $\underline{u}$:

$$-\frac{1}{5}Rz\hat{\underline{R}} + \left(-\frac{1}{5}a^2 + \frac{2}{5}R^2 + \frac{1}{5}z^2\right)\hat{\underline{z}} = 0\hat{\underline{R}} + 0\hat{\underline{\phi}} + 0\hat{\underline{z}} \quad \therefore$$

$-\frac{1}{5}Rz\hat{\underline{R}} = 0\hat{\underline{R}} \quad \therefore \quad -\frac{1}{5}Rz = 0$ ~~and~~ ① and

$-\frac{1}{5}a^2 + \frac{2}{5}R^2 + \frac{1}{5}z^2 = 0$ ② $\therefore$

from ①: $Rz = 0 \quad \therefore \quad R = 0$ or $z = 0 \quad \therefore$

if $R = 0$: from ②: $-\frac{1}{5}a^2 + \frac{1}{5}z^2 = 0 = -a^2 + z^2 \quad \therefore$

$a^2 = z^2$ and $a \in \mathbb{R}^+ \quad \therefore \pm\sqrt{a^2} = z \quad \therefore$ ~~$z = a$~~ ~~$z$~~ $z = \pm a$

~~$R=0$ and~~ $R = 0$, $z = \pm a$ for all $\phi$ ~~are~~ ~~also~~ solutions for the stagnation point $\therefore (0, \phi, \pm a)$

if $z = 0$: from ②: $-\frac{1}{5}a^2 + \frac{2}{5}R^2 = 0 = -a^2 + 2R^2 \quad \therefore$ but $R \in \mathbb{R}^+ \therefore$

$a^2 = 2R^2 \quad \therefore \quad \frac{a^2}{2} = R^2$ and $a \in \mathbb{R}^+ \quad \therefore +\sqrt{\frac{a^2}{2}} = R \quad \therefore R = \frac{+a}{\sqrt{2}} \quad \therefore$

$R = \frac{+a}{\sqrt{2}}$, $z = 0$ for all $\phi$ is another solution for the stagnation point: $\left(\frac{+a}{\sqrt{2}}, \phi, 0\right)$

3b/ $-\frac{1}{5}Rz\hat{\underline{R}} + \left(-\frac{1}{5}a^2 + \frac{2}{5}R^2 + \frac{1}{5}z^2\right)\hat{\underline{z}} = \underline{u}$ $\therefore$

$$u = |\underline{u}| = \left[\left(-\frac{1}{5}Rz\right)^2 + \left(-\frac{1}{5}a^2 + \frac{2}{5}R^2 + \frac{1}{5}z^2\right)^2\right]^{1/2} =$$

$$\left[\frac{1}{25}R^2z^2 + \left(-\frac{1}{5}a^2 + \frac{2}{5}R^2 + \frac{1}{5}z^2\right)^2\right]^{1/2}$$

but in cylindrical polar coordinates, a sphere of radius $a$ is descried ~~as~~ as: ~~$R^2+z^2 \leq a^2$~~ ?

$0 \leq \phi < 2\pi$, $R^2 + z^2 = a^2$ $\therefore$

subbing $R^2 + z^2 = a^2$ into $\underline{u}$ on the surface of the sphere:

$$\underline{u} = -\frac{1}{5}Rz\hat{\underline{R}} + \left(-\frac{1}{5}a^2 + \frac{1}{5}R^2 + \frac{1}{5}(R^2 + z^2)\right)\hat{\underline{z}} =$$

$$\underline{u} = -\frac{1}{5}Rz\hat{\underline{R}} + \left(-\frac{1}{5}a^2 + \frac{1}{5}R^2 + \frac{1}{5}(a^2)\right)\hat{\underline{z}} = -\frac{1}{5}Rz\hat{\underline{R}} + \frac{1}{5}R^2\hat{\underline{z}} \therefore$$

$$u = |\underline{u}| = \left[\left(-\frac{1}{5}Rz\right)^2 + \left(\frac{1}{5}R^2\right)^2\right]^{1/2} = \left[\frac{1}{25}R^2z^2 + \frac{1}{25}R^4\right]^{1/2} =$$

$$\left[\frac{1}{25}R^2(R^2+z^2)\right]^{1/2} = \left[\frac{1}{25}R^2a^2\right]^{1/2} = \sqrt{\frac{1}{25}R^2a^2} = \frac{1}{5}Ra = u$$

$\therefore a \in \mathbb{R}^+$ and ~~$R z$~~ $R \in \mathbb{R}^+$ $\therefore$ $R^2 + z^2 = a^2$ $\therefore R^2 \leq a^2$ $\therefore$

$R \leq a$ $\therefore$ $\sup(R) = a$ $\therefore$

if $R = a$ : $u = \frac{1}{5}a^2$ $\therefore$

slow speed $u$ is maximal when $R = a$ $\therefore$

$R^2 + z^2 = a^2 + z^2 = a^2$ $\therefore z^2 = 0$ $\therefore z = 0$ $\therefore$

$u_{max}$ is at $(R, \phi, z) = (a, \phi, 0)$ $\therefore$

$R = a$, $z = 0$ for all $\phi$

\3C/ For mass flux use: $\iint_S \underline{F} \cdot \underline{dS}$ where $\underline{dS} = \hat{\underline{n}}\, dS$

where $\hat{n}$ is the unit outward normal to S $\therefore$

the plane $Z = \text{constant}$ in cylindrical polar coordinates

is the $R$-$\phi$ plane $\therefore$

the normal to this plane is $\hat{\underline{z}}$ $\therefore$ $\hat{\underline{n}} = \hat{\underline{z}}$ $\therefore$ $\underline{dS} = \hat{\underline{z}}\, dS$

$\therefore$ mass flow rate $= \iint_S \rho\, \underline{u} \cdot \underline{dS} = \iint_S \rho\, \underline{u} \cdot \hat{\underline{z}}\, dS$ where $\rho$ is

the density of the fluid $\therefore$

$0 \le R \le b$ , $0 \le \phi < 2\pi$ $\therefore$

$$\iint_S \rho\, \underline{u} \cdot \underline{dS} = \iint_S \rho\, \underline{u} \cdot \hat{\underline{z}}\, dS = \rho \int_0^{2\pi}\int_0^b \underline{u} \cdot \hat{\underline{z}}\, dR\, d\phi =$$

$$\rho \int_0^{2\pi}\int_0^b \underline{u} \cdot (0\hat{\underline{R}} + 0\hat{\underline{\phi}} + 1\hat{\underline{z}})\, dR\, d\phi =$$

$$\rho \int_0^{2\pi}\int_0^b \left(-\tfrac{1}{5}RZ\hat{\underline{R}} + 0\hat{\underline{\phi}} + \left(-\tfrac{1}{5}a^2 + \tfrac{2}{5}R^2 + \tfrac{1}{5}Z^2\right)\hat{\underline{z}}\right) \cdot (0\hat{\underline{R}} + 0\hat{\underline{\phi}} + 1\hat{\underline{z}})\, dR\, d\phi$$

$$= \rho \int_0^{2\pi}\int_0^b 0 + 0 + -\tfrac{1}{5}a^2 + \tfrac{2}{5}R^2 + \tfrac{1}{5}Z^2\, dR\, d\phi =$$

$$\rho \int_0^{2\pi}\int_0^b -\tfrac{1}{5}a^2 + \tfrac{2}{5}R^2 + \tfrac{1}{5}Z^2\, dR\, d\phi =$$

$$\rho \int_0^{2\pi} \left( \left[ -\tfrac{1}{5}a^2 R + \tfrac{2}{15}R^3 + \tfrac{1}{5}Z^2 R \right]_{R=0}^{b} \right) d\phi =$$

$$\rho \int_0^{2\pi} \left( \left[ -\tfrac{1}{5}a^2 b + \tfrac{2}{15}b^3 + \tfrac{1}{5}Z^2 b \right] - [0] \right) d\phi =$$

$$\rho \int_0^{2\pi} -\tfrac{1}{5}a^2 b + \tfrac{2}{15}b^3 + \tfrac{1}{5}Z^2 b\, d\phi = \rho\left(-\tfrac{1}{5}a^2 b + \tfrac{2}{15}b^3 + \tfrac{1}{5}Z^2 b\right)\int_0^{2\pi} 1\, d\phi =$$

$$\cancel{\rho\,[\phi]_{\phi=0}^{2\pi}} \quad \rho\left(-\tfrac{1}{5}a^2 b + \tfrac{2}{15}b^3 + \tfrac{1}{5}Z^2 b\right)[\phi]_{\phi=0}^{2\pi} =$$

$$\rho\left(-\tfrac{1}{5}a^2 b + \tfrac{2}{15}b^3 + \tfrac{1}{5}Z^2 b\right)[2\pi - 0] = \tfrac{2}{5}$$

$\tfrac{2}{5}\rho\pi\left(-a^2 b + \tfrac{2}{3}b^3 + Z^2 b\right)$ for $Z = \text{constant}$

is the mass flux through a circular disc of radius $b$

4a/ flow is incompressible is $\nabla \cdot \underline{u} = 0$ $\therefore$

$$\nabla \cdot \underline{u} = \nabla \cdot (Ex, -Ey) = \frac{\partial}{\partial x}(Ex) + \frac{\partial}{\partial y}(-Ey) =$$

$$E - E = 0 = \nabla \cdot \underline{u} \quad \therefore \quad \nabla \cdot \underline{u} = 0 \quad \forall x, \forall y \quad \therefore$$

the flow is incompressible

4b/ Inviscid flow so viscosity is zero $\therefore \mu = 0 \therefore$

the Navier-Stokes equation becomes:

$$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p(x,y)$$

$\underline{u} = (Ex, -Ey) = Ex\,\hat{\underline{i}} - Ey\,\hat{\underline{j}} \quad \therefore$

$\frac{\partial \underline{u}}{\partial t} = \frac{\partial}{\partial t}(Ex\hat{\underline{i}} - Ey\hat{\underline{j}}) = 0\hat{\underline{i}} - 0\hat{\underline{j}} = 0$

$\underline{u}\cdot\nabla\underline{u} = (\underline{u}\cdot\nabla)\underline{u} \quad \therefore$

$\underline{u}\cdot\nabla = (Ex\hat{\underline{i}} - Ey\hat{\underline{j}})\cdot\nabla = Ex\frac{\partial}{\partial x} - Ey\frac{\partial}{\partial y} \quad \therefore$

$(\underline{u}\cdot\nabla)\underline{u} = \left(Ex\frac{\partial}{\partial x} - Ey\frac{\partial}{\partial y}\right)(Ex\hat{\underline{i}} - Ey\hat{\underline{j}}) =$

$Ex\,\hat{\underline{i}}\frac{\partial}{\partial x}(Ex) - Ey\,\hat{\underline{i}}\frac{\partial}{\partial y}(Ex) + Ex\,\hat{\underline{j}}\frac{\partial}{\partial x}(-Ey) - Ey\,\hat{\underline{j}}\frac{\partial}{\partial y}(-Ey) =$

$Ex\,\hat{\underline{i}}E - Ey\,\hat{\underline{i}}(0) + Ex\,\hat{\underline{j}}(0) - Ey\,\hat{\underline{j}}(-E) =$

$E^2x\,\hat{\underline{i}} + E^2y\,\hat{\underline{j}} = (\underline{u}\cdot\nabla\underline{u})$

$\nabla p(x,y) = \hat{\underline{i}}\frac{\partial}{\partial x}p(x,y) + \hat{\underline{j}}\frac{\partial}{\partial y}p(x,y) \quad \therefore$

$\rho(E^2x\,\hat{\underline{i}} + E^2y\,\hat{\underline{j}}) = \hat{\underline{i}}\frac{\partial}{\partial x}p(x,y) + \hat{\underline{j}}\frac{\partial}{\partial y}p(x,y) = \rho E^2x\,\hat{\underline{i}} + \rho E^2y\,\hat{\underline{j}}$

$\therefore$ ~~by consistent~~ by components:

$\hat{\underline{i}}\frac{\partial}{\partial x}p(x,y) = \rho E^2x\,\hat{\underline{i}}$ and $\hat{\underline{j}}\frac{\partial}{\partial y}p(x,y) = \rho E^2y\,\hat{\underline{j}} \quad \therefore$

$\frac{\partial}{\partial x}p(x,y) = \rho E^2x$ ① and $\frac{\partial}{\partial y}p(x,y) = \rho E^2y$ ② $\therefore$

$\therefore$ ~~from ①:~~ since flow is incompressible $\therefore$ density $\rho$ is constant $\therefore$

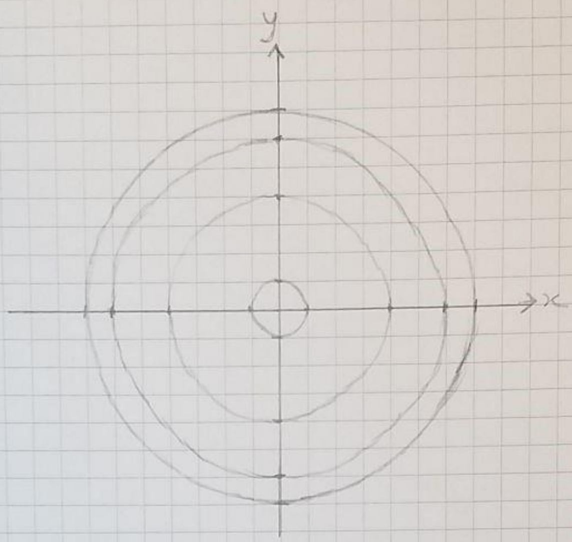
from ①: $p(x,y) = \int\rho E^2x\,dx = \rho E^2\int x\,dx = \frac{1}{2}\rho E^2x^2 + f(y)$

from ②: $p(x,y) = \int\rho E^2y\,dy = \rho E^2\int y\,dy = \frac{1}{2}\rho E^2y^2 + g(x) \quad \therefore$

$\frac{1}{2}\rho E^2y^2 + g(x) = \frac{1}{2}\rho E^2x^2 + f(y) \quad \therefore \quad \frac{1}{2}\rho E^2x^2 = g(x) \quad \therefore$

$p(x,y) = \frac{1}{2}\rho E^2y^2 + \frac{1}{2}\rho E^2x^2 = \frac{1}{2}\rho E^2(x^2+y^2) = p(x,y)$

\4c/



4d $\underline{u} = (Ex, -Ey) = Ex\,\hat{\underline{i}} - Ey\,\hat{\underline{j}} = u\,\hat{\underline{i}} + v\,\hat{\underline{j}}$ $\therefore$

$u = Ex$ , $v = -Ey$ $\therefore$

to find streamlines use formula: $\frac{dx}{u} = \frac{dy}{v}$ assuming

~~since Ex~~ $u, v \neq 0$ $\therefore$ $Ex \neq 0$ and $-Ey \neq 0$ $\therefore$ $x \neq 0$ and $y \neq 0$

$\frac{dx}{Ex} = \frac{dy}{-Ey}$ $\therefore$ $\frac{dx}{x} = \frac{dy}{-y}$ $\therefore$

$\frac{1}{x}dx = -\frac{1}{y}dy$ $\therefore$ $\int \frac{1}{x}dx = \int -\frac{1}{y}dy = -\int \frac{1}{y}dy =$

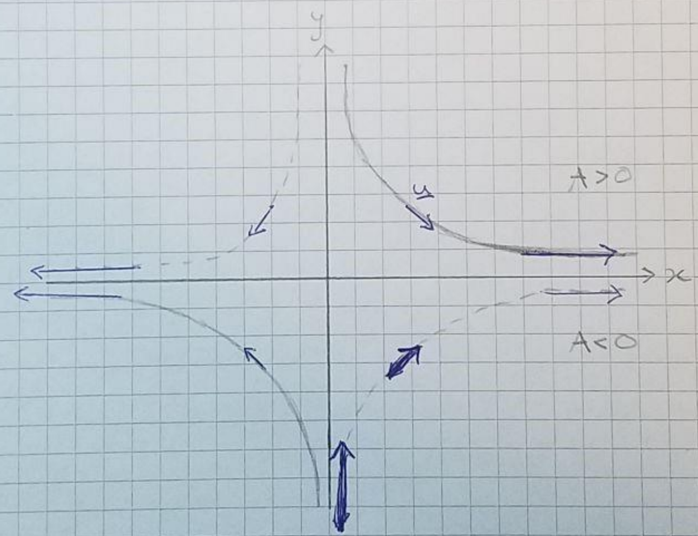
$\ln x = -\ln y + C$ $\therefore$

$e^{\ln x} = e^{-\ln y + C} = x = e^{C}e^{-\ln y} = C_2 e^{\ln(y^{-1})} = C_2 y^{-1} = C_2 \frac{1}{y} = x$ $\therefore$

$C_2 \frac{1}{x} = y$ $C_2 = \text{constant}$

is the equation for the streamlines $\therefore$

$C_2 \frac{1}{x} = y = A\frac{1}{x}$ for $A = C_2$ $\therefore$




$y = A\frac{1}{x}$ $\therefore$

if $A > 0$: if $x$ is large positive $\Rightarrow$ $y$ is small positive

if $x$ is large negative $\Rightarrow$ $y$ is small negative

if $x$ is small positive $\Rightarrow$ $y$ is large ~~negative~~ positive

if $x$ is small negative $\Rightarrow$ $y$ is large negative

this graph is reflected on $y = x$ if $A < 0$ $\therefore$

4e

| $(x, y)$ | $\underline{u} = (Ex, -Ey)$ | $\lvert\underline{u}\rvert = ((Ex)^2 + (-Ey)^2)^{1/2}$ |
|---|---|---|
| $(1, 1)$ | $(E, -E)$ | $(E^2 + (-E)^2)^{1/2} = \sqrt{2}\,E$ |
| $(1, -1)$ | $(E, E)$ | $\sqrt{2}\,E$ |
| $(-1, 1)$ | $(-E, -E)$ | $\sqrt{2}\,E$ |
| $(-1, -1)$ | $(-E, E)$ | $\sqrt{2}\,E$ |
| $(\infty, 0)$ | $(\infty, 0)$ | $\infty$ |
| $(0, \infty)$ | $(0, -\infty)$ | $\infty$ |
| $(2, 1)$ | $(2E, -E)$ | $((2E)^2 + (-E)^2)^{1/2} = \sqrt{5}\,E$ |
| $(+0, -\infty)$ | $(+0, +\infty)$ | $\infty$ |

$\therefore$ velocity vectors on graph:

45/ At $y=0$ the boundary is stationary ∴
using the no-slip boundary condition: we expect the
flow velocity at $y=0$ to be stationary but
$\underline{u} = (Ex, -Ey)$ ∴
$|\underline{u}| = \sqrt{(Ex)^2 + (-Ey)^2}$ and
the equation for the streamlines is: $y = A\frac{1}{x}$ $A$: Constant
∴ $x = A\frac{1}{y}$ ∴
if $y=0 \Rightarrow x = \infty$ for $A>0$ ∴ at $(x, y=0) = (\infty, 0)$
and if $A<0$: $x = -\infty$ ∴ at $(x, y=0) = (-\infty, 0)$ ∴
for either $A>0$ or $A<0$: $|\underline{u}| = ((Ex)^2 + (-Ey)^2)^{1/2} =$
$((\pm\infty)^2 + (0)^2)^{1/2} = \infty \neq 0$ at $y=0$ ∴
at $y=0$ the flow velocity is not zero ∴
the flow velocity does not satisfy the boundary conditions
expected at $y=0$.
This is because the streamlines assumed $x \neq 0$ and $y \neq 0$
∴ $y=0$ breakes this condition
since the streamlines are only asymptotic to $y=0$