$-\nu\sqrt{\nu Q}\,(F')^2 - \sqrt{\nu Q}\,\nu F F''\eta + \eta\nu\sqrt{\nu Q}\,F'F'' = -Q^2\frac{\nu^{3/2}}{Q^{3/2}} - \nu\sqrt{\nu Q}\,F''' \therefore$

$-(F')^2 - FF''\eta + \eta F'F'' = -1 - F''' \therefore$

$F''' - (F')^2 + 1 = 0$

\1e/ on $y=0$ $u=0 \therefore F'(0)=0 \therefore v=0 \therefore \eta=0 \therefore$

$u = -\frac{\sqrt{\nu Q}}{\delta} F'(\eta\to\infty) \to -\frac{Q}{x} \therefore -\frac{\nu^{1/2}Q^{1/2}}{x\nu^{1/2}} Q^{1/2} F'(\eta\to\infty) \to -\frac{Q}{x} \therefore$

$F'(\eta\to\infty)\to 1 \quad \therefore u \approx U = -\frac{Q}{x} \quad \therefore \partial_y t = 0 \therefore F''(\eta\to\infty) = 0$

\1f/ $G = F' \therefore G'' - G^2 + 1 = 0 \quad \therefore G'G'' - G'G^2 + G' = 0 \therefore$

$\partial_\eta \frac{(G')^2}{2} - \partial_\eta\left(\frac{G^3}{3}\right) + G' = 0 \therefore \frac{(G')^2}{2} - \frac{G^3}{3} + G = \text{const} \therefore$

$F'(\infty) = 1 \therefore G(\infty) = 1 \quad \therefore F''(\infty) = 0 \therefore G'(\infty) = 0 \therefore$

$0 - \frac{1}{3} + 1 = \text{const} \quad \therefore \frac{(G')^2}{2} - \frac{G^3}{3} + G = \frac{2}{3} \therefore$

$(G')^2 = \frac{2}{3}(2 - 3G + G^3) \quad \therefore (G')^2 = \frac{2}{3}(1-G)^2(2+G)$

\1g/ $H^2 = G + 2 \quad \therefore 2HH' = G' \quad \therefore (G')^2 = 4H^2(H')^2 \therefore$

$4\cancel{H^2}(H')^2 = \frac{2}{3}(1-(H^2-2))^2\,\cancel{H^2} \qquad (H')^2 = \frac{1}{6}(-H^2+3)^2 \therefore$

$H' = \pm\frac{1}{\sqrt{6}}(3-H^2) \therefore H = \sqrt{3}\tanh(I) \therefore H' = \sqrt{3}(1-\tanh^2 I)I' \therefore$

$\sqrt{3}\,(1-\tanh^2 I)I' = \pm\frac{1}{\sqrt{6}}|3 - 3\tanh^2 I|$

$\therefore \sqrt{3}\,I' = \pm\frac{3}{\sqrt{6}} \quad \therefore I' = \pm 1/\sqrt{2} \therefore$

$I = \pm\frac{1}{\sqrt{2}}\eta + C \therefore H = \sqrt{3}\tanh\left[\pm\frac{\eta}{\sqrt{2}} + C\right] \therefore$

$F' = G = H^2 - 2 = 3\tanh^2\left[C \pm \frac{\eta}{\sqrt{2}}\right] - 2$

\week8/

\1b/ $\underline{u} = 0,\ R = a \quad u = \frac{1}{R}\partial_\theta\psi = 0,\ R = a$

$v = -\partial_R\psi = 0,\ R = a \quad \therefore \underline{u} = \nabla\times(\psi\hat{\underline{z}}),\ \psi = \psi(R,\theta),$

$R\to\infty: \underline{u} \to U\underline{i} = U(\cos\theta\hat{\underline{R}} - \sin\theta\hat{\underline{\theta}}) \therefore$

$\frac{1}{R}\frac{\partial\psi}{\partial\theta} \quad \therefore \frac{1}{R}\frac{\partial\psi}{\partial\theta} \approx U\cos\theta,\ \partial_R\psi = U\sin\theta \therefore$

$\psi = UR\sin\theta + g(\theta) \therefore \partial_\theta\psi = UR\cos\theta + \frac{\partial g}{\partial\theta} \therefore \partial_\theta g = 0 \therefore$

$\psi = UR\sin\theta,\ R\to\infty$

\1c/ $\psi = R^m\sin\theta \therefore \nabla^2\psi = (m^2-1)R^{m-2}\sin\theta \therefore$

$\nabla^4\psi = (m^2-1)(m^2-4m+3)R^{m-4}\sin\theta,$

$\nabla^4\psi = 0 \therefore m = \pm 1,\ m = 1,3 \therefore F(R) = \frac{A}{R} + BR + CR\ln R + DR^3 \therefore$

\1d/ $f(R) \approx UR,\ R\to\infty \therefore B = U,\ C = D = 0 \therefore$

\week 8/ $\partial_\theta \psi = 0$, $R = a$ ∴ $f(a) = \frac{A}{a} + Ua = 0$

$\partial_R \psi = 0$, $R = a$ ∴ $f'(a) = -\frac{A}{a^2} + U = 0$

$\underline{\Omega} = \Omega \hat{\underline{z}}$, $u = w \hat{\underline{\phi}}$ ∴ $\underline{u} = \underline{\Omega} \times \hat{\underline{r}}$ ∴

~~$\hat{\underline{r}} = (\Omega, 0, \Omega)$~~, $\underline{r} = (1, 0, 0)$

~~$\underline{\Omega} = (\underline{\Omega}$~~ ∴ $\underline{\Omega} \times \hat{\underline{r}}$

$$\underline{\Omega} \times \underline{r} = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\hat{\underline{\phi}} \\ \cancel{\theta} & 0 & \Omega \\ 1 & 0 & 0 \end{vmatrix}$$

$= \frac{-r}{r^2 \sin\theta} \hat{\underline{\phi}} (-\Omega) = \hat{\underline{\theta}}$

\week 8/ $\nabla^4 \psi = \nabla^2(\nabla^2 \psi) = 0$, $\underline{u} = \nabla \times (\psi \underline{k}) = (\partial_y \psi, -\partial_x \psi, 0)$

$\psi(x, y) = f(x) e^{\lambda y}$ ∴ $\nabla^4 \psi = (f'''' + \lambda^2 f'' + \lambda^2 f'' + \lambda^4 f) e^{\lambda y}$

∴ $p^4 + 2p^2 \lambda^2 + \lambda^4 = 0$ ∴ $p = \pm i\lambda$ ∴

$f(x) = A \sin\lambda x + B \cos\lambda x + Cx \sin\lambda x + Dx \cosh\lambda x$

∴ $A = D = 0$ ∴ $\lambda + \sin\lambda = 0$ ∴

$\sin\lambda - \lambda = 0$

$u_R = \frac{1}{R} \partial_\phi \psi$ $u_\phi = -\partial_R \psi$

∴ $\nabla^4 \psi = \frac{1}{R^4}(4f'' + f'''')$

$f = A\cos 2\phi + B\sin 2\phi + C + D\phi$

$u_R = \frac{1}{R} \partial_\phi f = -\frac{C}{R}(\cos 2\phi - \cos 2\alpha)$

\week 9/ $\underline{u} = -\alpha + nU\cos[n(x - \alpha y t)]$

$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}$

$\frac{\partial f}{\partial t} = -\nu m^2 (1 + \alpha^2 t^2) F$

$F(t) = A\exp[-\nu m^2 (t + \alpha^2 t^3/3)]$ ∴ $A = 1$

$u_R = \frac{3U}{2} R z a^3 (R^2 + z^2)^{-5/2}$

$\omega = \frac{15UR}{2a^2}$ ∴ $\omega = 0$

\week 11/ $\frac{\partial^2 \beta}{\partial \xi \partial \eta} = 0$ $\partial_\xi = F(\xi)$ $\beta = F(\xi) + g(\eta) = f(x - ct) + g(x + ct)$

$\frac{\partial \beta}{\partial t} = -cf' + cg'$

∴ $\beta = a e^{i(\underline{k} \cdot \underline{r} - \omega t)}$ ∴ $\frac{\partial \beta}{\partial t} = -i\omega\beta$, $\beta_{tt} = (-i\omega)^2 \beta$,

$\beta_{xx} = -k_1^2 \beta$ ∴ $\nabla^2 \beta = -k^2 \beta$ $k = |\underline{k}|$ ∴

$-\omega^2 = -c^2 k^2$ ∴ $\omega = \pm ck$

$\partial_y \phi = \partial_t \eta \quad , \partial_t \phi + g\eta = 0 \quad \partial_y \phi = 0 \quad \therefore \quad \eta = A\cos(kx - \omega t)$

$\phi = f(y)\sin(kx - \omega t) \quad \therefore f'' - k^2 f = 0 \therefore f = Ce^{ky} + De^{-ky} \therefore$

$\partial_y \phi = 0 \therefore Cke^{-kH} - Dke^{kH} = 0 \therefore$

$\partial_y \phi = \partial_t \eta \therefore Cke^{0} - Dke^{0} = A\omega \quad \therefore$

$\omega^2 = gk\tanh(kH) \therefore$

\pp2018/ \1ar/ $\delta_{ij} = \begin{cases} 1, & i=j, \\ 0, & i \neq j. \end{cases}$ $\varepsilon_{ijk} = \begin{cases} 0, & i=j, i=k, j=k, \\ 1, & (i,j,k) = (1,2,3), (3,1,2), (2,3,1), \\ -1, & (i,j,k) = (1,3,2), (2,1,3), (3,2,1). \end{cases}$

\1ai/ $[\nabla \cdot \nabla \times \underline{V}]_i = \nabla \cdot \varepsilon_{ijk} \nabla_j V_k = \nabla \cdot \varepsilon_{ijk} \partial_j V_k =$

$\partial_m \varepsilon_{ijk} \partial_j V_k =$

$\varepsilon_{ijk} \partial_m \partial_j V_k$

$[\nabla \cdot \nabla \times \underline{V}]_i = \nabla_i \cdot (\nabla \times \underline{V})_i = \partial_i \varepsilon_{ijk} \nabla_j V_k = \partial_i \varepsilon_{ijk} \partial_j V_k = \varepsilon_{ijk} \partial_j \partial_i V_k =$

$\varepsilon_{kij} \partial_j \partial_i V_k = V_k \varepsilon_{ijk} \partial_i \partial_j$

$\cancel{\nabla_i}$ $[\nabla \cdot \nabla \times V]_i = \nabla_i \cdot (\nabla \times V)_i = \partial_i \varepsilon_{ijk} \partial_j V_k = \cancel{\partial_i \varepsilon_j} \; \cancel{-\partial_i \varepsilon_j}$

$\cancel{-\varepsilon}$ $-\partial_i \varepsilon_{jik} \partial_j V_k = \cancel{-\partial_j \varepsilon_{jik} \partial_j V}$

$-\partial_j \varepsilon_{jik} \partial_i V_k = -\partial_j (\nabla \times V)_j = -[\nabla \cdot \nabla \times V]_j \therefore$

by symmetry $[\nabla \cdot \nabla \times \underline{V}]_j = -[\nabla \cdot \nabla \times \underline{V}]_k$ ~~or~~ and

$[\nabla \cdot \nabla \times \underline{V}]_i = -[\nabla \cdot \nabla \times \underline{V}]_k \therefore$

$[\nabla \cdot \nabla \times \underline{V}]_i = -[\nabla \cdot \nabla \times \underline{V}]_j = [\nabla \cdot \nabla \times \underline{V}]_k = -[\nabla \cdot \nabla \times \underline{V}]_i$

$[\nabla \cdot \nabla \times \underline{V}]_i = 0 \therefore \quad \nabla \cdot \nabla \times \underline{V} = 0$

\1aii/ $[\nabla \times (\nabla \times u)]_i = \varepsilon_{ijk} \nabla_j (\nabla \times u)_k = \varepsilon_{ijk} \partial_j \varepsilon_{klm} \nabla_l u_m =$

$\varepsilon_{kij} \partial_j \varepsilon_{klm} \partial_l u_m = \varepsilon_{kij} \varepsilon_{klm} \partial_j \partial_l u_m =$

$(\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \partial_j \partial_l u_m =$

$\delta_{il} \delta_{jm} \partial_j \partial_l u_m - \delta_{im} \delta_{jl} \partial_j \partial_l u_m =$

$\partial_j \partial_i u_j - \partial_j \partial_j u_i = \partial_i (\partial_j u_j) - (\nabla \cdot \nabla)_j u_i =$

$\partial_i (\nabla \cdot u)_j - (\nabla \cdot \nabla)_j u_i = \partial_i (\nabla \cdot u)_i - (\nabla^2)_i u_i =$

$\nabla_i (\nabla \cdot u)_i - (\nabla^2)_i u_i = [\nabla (\nabla \cdot \underline{u}) - \nabla^2 \underline{u}]_i$

$\partial_i (\partial_j u_j) - (\nabla \cdot \nabla)_j u_i = \partial_i (\nabla u)_j - (\nabla^2)_j u_i = \partial_i (\nabla u)_i - (\nabla^2)_i u_i =$

$[\nabla (\nabla \underline{u}) - \nabla^2 \underline{u}]_i$

\1ai/ $[\nabla \cdot \nabla \times \underline{V}]_i = [\nabla \cdot (\nabla \times \underline{V})]_i = \partial_i (\nabla \times \underline{V})_i = \partial_i \varepsilon_{ijk} \partial_j V_k =$

$\varepsilon_{ijk} \partial_i \partial_j V_k = \varepsilon_{ijk} \frac{\partial^2 V_k}{\partial x_i \partial x_j} = \varepsilon_{ij1} \frac{\partial^2 V_1}{\partial x_i \partial x_j} + \varepsilon_{ij2} \frac{\partial^2 V_2}{\partial x_i \partial x_j} + \varepsilon_{ij3} \frac{\partial^2 V_3}{\partial x_i \partial x_j}$

$\therefore \varepsilon_{ij1} \frac{\partial^2 V_1}{\partial x_i \partial x_j} = \varepsilon_{231} \frac{\partial^2 V_1}{\partial x_2 \partial x_3} + \varepsilon_{321} \frac{\partial^2 V_1}{\partial x_3 \partial x_2} = \frac{\partial^2 V_1}{\partial x_2 \partial x_3} - \frac{\partial^2 V_1}{\partial x_3 \partial x_2} = 0$

$\therefore$ by symmetry $\varepsilon_{ij2} \frac{\partial^2 V_2}{\partial x_i \partial x_j} = \varepsilon_{ij3} \frac{\partial^2 V_3}{\partial x_i \partial x_j} = 0 \quad \therefore [\nabla \cdot \nabla \times \underline{V}]_i = 0 \therefore \nabla \cdot \nabla \times \underline{V} = 0$

\1bi/ $\frac{d}{dr}\left(r\frac{dw}{dr}\right) = \frac{-G}{\mu}r \therefore \quad r\left(\frac{dw}{dr}\right) = -\frac{G}{2\mu}r^2 + A \therefore$

$-\frac{G}{2\mu}r + \frac{A}{r} = \frac{dw}{dr} \therefore$

$w = w(r) = \frac{-G}{4\mu}r^2 + A\ln|r| + B = -\frac{G}{4\mu}r^2 + A\ln r + B \quad \because r>0$

\1bii/ at $r=a$: $w(r=a) = w(a) = 0 \quad \because$ no slip $\therefore$

$w(a) = 0 = -\frac{G}{4\mu}(a)^2 + A\ln a + B \therefore \quad B = \frac{G a^2}{4\mu} - A\ln a \therefore$

$\cancel{w(r) = \frac{G a^2}{4\mu}} \quad w(r) = -\frac{G}{4\mu}r^2 + A\ln r + \frac{G a^2}{4\mu} - A\ln a =$

$\frac{G}{4\mu}(a^2 - r^2) + A\ln\left(\frac{r}{a}\right)$

$w'(r) = -\frac{G}{2\mu}r + \frac{A}{r} \therefore \quad$ max $w(r)$ at $r=0 \therefore$

$\ln(0)$ is undefined $\therefore \quad \lim_{r\to 0}\ln(r) = -\infty \therefore$

$|w(r=0)| < \infty \therefore$ let $A=0$ such that is not infinite at $r=0$

$\therefore w(r) = \frac{G}{4\mu}(a^2 - r^2)$

\1biii/ $M = \rho\int_0^a w(r)2\pi r\,dr = \rho\int_0^a \frac{G}{4\mu}(a^2 - r^2)2\pi r\,dr = \frac{2\pi\rho G}{4\mu}\int_0^a a^2 r - r^3\,dr =$

$\frac{\pi\rho G}{2\mu}\left[\frac{1}{2}a^2r^2 - \frac{1}{4}r^4\right]_0^a = \frac{\pi\rho G}{2\mu}\left[\frac{1}{2}a^2(a^2-0^2) - \frac{1}{4}(a^2-0^2)\right] =$

$\frac{\pi\rho G}{2\mu}\left[\frac{1}{2}a^4 - \frac{1}{4}a^4\right] = \frac{\pi\rho G}{8\mu}a^2$

\1ci/ NS: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho g + \mu\nabla^2\underline{u} \therefore$

$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\nabla\frac{p}{\rho} + g + \nu\nabla^2\underline{u} \quad \because \quad \frac{\mu}{\rho} = \nu \quad , \nabla\cdot\underline{u} = 0 \therefore$

$\cancel{u\times\omega = \underline{u}\times\nabla\times\underline{u}\times(\nabla\times\underline{u}) = \nabla(\underline{u}\cdot\underline{u})}$ absence of body forces $\therefore g = 0 \therefore$

$2\,\underline{u}\times(\nabla\times\underline{u}) = \nabla(\underline{u}\cdot\underline{u}) - (\underline{u}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)\underline{u} = \nabla|\underline{u}|^2 - 2(\underline{u}\cdot\nabla)\underline{u} \therefore$

$\underline{u}\times(\nabla\times\underline{u}) = \frac{1}{2}\nabla|\underline{u}|^2 - (\underline{u}\cdot\nabla)\underline{u} = \underline{u}\times\underline{\omega} \therefore$

$\underline{u}\cdot\nabla\underline{u} = (\underline{u}\cdot\nabla)\underline{u} = \frac{1}{2}\nabla|\underline{u}|^2 - \underline{u}\times\underline{\omega}$,

$\cancel{\nu\nabla^2\underline{u} = \nu\nabla\cdot\nabla\times\underline{u}} \quad \nabla\times\underline{\omega} = \nabla\times(\nabla\times\underline{u}) = \nabla(\nabla\cdot\underline{u}) - \nabla^2\underline{u} = \nabla(0) - \nabla^2\underline{u} =$

$-\nabla^2\underline{u} \therefore \quad \nu\nabla^2\underline{u} = -\nu\nabla\times\underline{\omega} \therefore$

$\frac{\partial \underline{u}}{\partial t} + \frac{1}{2}\nabla|\underline{u}|^2 - \underline{u}\times\underline{\omega} = -\nabla\frac{p}{\rho} - \nu\nabla\times\underline{\omega} \therefore$

$\frac{\partial \underline{u}}{\partial t} - \underline{u}\times\omega = -\nabla\frac{p}{\rho} - \nabla\frac{1}{2}|\underline{u}|^2 - \nu\nabla\times\underline{\omega} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2\right) - \nu\nabla\times\underline{\omega}$

\PP 2018/ $\because$ constant $= \rho \therefore \frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \underline{u}\cdot\nabla\rho = 0 \therefore$

$$\frac{\partial \rho}{\partial t} + \nabla\cdot(\underline{u}\,\rho) = 0 \therefore$$

$$\frac{\partial \rho}{\partial t} + \underline{u}\cdot\nabla\rho + \rho\nabla\cdot\underline{u} = 0 = \rho\nabla\cdot\underline{u} = 0 = \nabla\cdot\underline{u}$$

\1Cii/ $\frac{\partial \underline{u}}{\partial t} - \underline{u}\times\underline{\omega} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2\right) - \nu\nabla\times\underline{\omega}$, $\nabla\cdot\underline{u} = 0 \therefore$

taking curls : $\nabla\times\frac{\partial \underline{u}}{\partial t} = \frac{\partial}{\partial t}\nabla\times\underline{u} = \frac{\partial}{\partial t}\underline{\omega} = \frac{\partial \underline{\omega}}{\partial t}$

$$-\nabla\times\nabla\left(\frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2\right) = -\nabla\times\nabla\cancel{\phi} = -0 = 0$$

$$\nabla\times(\nabla\times\underline{\omega}) = \nabla(\nabla\cdot\underline{\omega}) - \nabla^2\underline{\omega} = \nabla(\nabla\cdot\nabla\times\underline{u}) - \nabla^2\underline{\omega} =$$

$$\nabla(0) - \nabla^2\underline{\omega} = -\nabla^2\underline{\omega} \therefore$$

$$-\nu\nabla\times(\nabla\times\underline{\omega}) = -\nu(-\nabla^2\underline{\omega}) = \nu\nabla^2\underline{\omega}$$

$$-\nabla\times(\underline{u}\times\underline{\omega}) = -\underline{u}(\nabla\cdot\underline{\omega}) + \underline{\omega}(\nabla\cdot\underline{u}) \cancel{+} -(\underline{\omega}\cdot\nabla)\underline{u} + (\underline{u}\cdot\nabla)\underline{\omega} =$$

$$\underline{u}(0) - \underline{\omega}(0) \cancel{+} -(\underline{\omega}\cdot\nabla)\underline{u} + (\underline{u}\cdot\nabla)\underline{\omega} = -(\underline{\omega}\cdot\nabla)\underline{u} + (\underline{u}\cdot\nabla)\underline{\omega} \therefore$$

$$\frac{\partial \underline{\omega}}{\partial t} - (\underline{\omega}\cdot\nabla)\underline{u} + (\underline{u}\cdot\nabla)\underline{\omega} = \nu\nabla^2\underline{\omega} \therefore$$

$$\frac{\partial \underline{\omega}}{\partial t} + (\underline{u}\cdot\nabla)\underline{\omega} = (\underline{\omega}\cdot\nabla)\underline{u} + \nu\nabla^2\underline{\omega} \therefore$$

$$\nabla\cdot\underline{\omega} = \nabla\cdot(\nabla\times\underline{u}) = 0$$

\1Ciii/

$$\underline{u} = \underline{u}(x, y, t) = (u, v, w) \therefore$$

$$\underline{\omega} = \nabla\times\underline{u} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \partial_x & \partial_y & \partial_z \\ u & v & w \end{vmatrix} = \hat{\underline{i}}\left(\frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}\right) - \hat{\underline{j}}\left(\frac{\partial w}{\partial x} - \frac{\partial u}{\partial z}\right) + \hat{\underline{k}}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$$= \hat{\underline{i}}\frac{\partial w}{\partial y} - \hat{\underline{j}}\frac{\partial w}{\partial x} + \hat{\underline{k}}\left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \left(\frac{\partial w}{\partial y}, -\frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)$$

$\therefore$ NS : $\frac{\partial \underline{u}}{\partial t} - \underline{u}\times\underline{\omega} = -\nabla\left(\frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2\right) - \nu\nabla^2\underline{\omega} \therefore$

z component : $\frac{\partial w}{\partial t}$,

NS : $\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\nabla\frac{p}{\rho} + \nu\nabla^2\underline{u} \therefore$

$$\cancel{\underline{u}\cdot\nabla\underline{u} = (\underline{u}\cdot\nabla)\left(\frac{\partial w}{\partial y}, -\frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)} \therefore$$

$$\cancel{\underline{u}\cdot\nabla = \left(\frac{\partial w}{\partial y}, -\frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\cdot\nabla = \frac{\partial w}{\partial y}\frac{\partial}{\partial x} + -\frac{\partial w}{\partial x}\frac{\partial}{\partial y} + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right)\frac{\partial}{\partial z}}$$

$$\therefore \cancel{\frac{\partial w}{\partial z} = 0 \therefore [\underline{u}\cdot\nabla\underline{u}]_z = \left(\frac{\partial w}{\partial y}\frac{\partial}{\partial x} - \frac{\partial w}{\partial x}\frac{\partial}{\partial y}\right)(w)}$$

$$\cancel{\frac{\partial w}{\partial y}\frac{\partial w}{\partial x} - \frac{\partial w}{\partial x}\frac{\partial w}{\partial y}} \qquad \frac{\partial w}{\partial z} = 0 \therefore$$

for z component: $\underline{u}\cdot\nabla(w) = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)w = u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} \therefore$

$$\nabla^2(u, v, w) = \nabla^2\underline{u} = (\nabla^2 u, \nabla^2 v, \nabla^2 w)$$

$$\frac{\partial \underline{u}}{\partial t} = \frac{\partial}{\partial t}(u, v, w) = \left(\frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial w}{\partial t}\right) \therefore$$

$\nearrow$ z comp = 0

z component of $\nabla\left(\frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2\right)$ is zero $\therefore$

$$\left(\frac{\partial}{\partial z}\left(\frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2\right)\right) = 0 \quad \therefore$$

~~$\frac{\partial w}{\partial t}$~~ z component:

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} = \nu\nabla^2 w$$

$$\frac{\partial \underline{\omega}}{\partial t} + (\underline{u}\cdot\nabla)\underline{\omega} = (\underline{\omega}\cdot\nabla)\underline{u} + \nu\nabla^2\underline{\omega} \therefore$$

$$\underline{\omega} = \left(\frac{\partial w}{\partial y}, -\frac{\partial w}{\partial x}, \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) = \left(\frac{\partial w}{\partial y} - \frac{\partial w}{\partial x}, \omega\right) \therefore$$

z component: $\frac{\partial \underline{\omega}}{\partial t} \rightarrow \frac{\partial \omega}{\partial t}$

$$(\underline{u}\cdot\nabla)\underline{\omega} \rightarrow (\underline{u}\cdot\nabla)\omega = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)\omega = u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y},$$

$$\nu\nabla^2\underline{\omega} \rightarrow \nu\nabla^2\omega \therefore$$

$$\frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \nu\nabla^2\omega$$

\1di/ $\frac{\partial W}{\partial t} + \underline{u}\cdot\nabla W = \nu\nabla^2 W \therefore \quad \underline{u} = V\hat{\underline{\theta}} + W\hat{\underline{z}}$

$$\therefore \underline{u}\cdot\nabla = (V\hat{\underline{\theta}} + W\hat{\underline{z}})\cdot\left(\frac{1}{R}\frac{\partial}{\partial\phi}\hat{\underline{\phi}} + \frac{\partial}{\partial z}\hat{\underline{z}}\right) = V\frac{1}{R}\frac{\partial}{\partial\phi} + W\frac{\partial}{\partial z} \therefore$$

$$\left(V\frac{1}{R}\frac{\partial}{\partial\phi} + W\frac{\partial}{\partial z}\right)W = \underline{u}\cdot\nabla W = V\frac{1}{R}\frac{\partial W}{\partial\phi} + W\frac{\partial W}{\partial z} = V\frac{1}{R}\frac{\partial W}{\partial\phi} + W(0) = V\frac{1}{R}\frac{\partial W}{\partial\phi}$$

$$\nabla^2 W = \left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial W}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 W}{\partial\phi^2} + \frac{\partial^2 W}{\partial z^2}\right] = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial W}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 W}{\partial\phi^2}$$

$$\therefore \frac{\partial W}{\partial t} + \frac{V}{R}\frac{\partial W}{\partial\phi} = \nu\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial W}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 W}{\partial\phi^2}\right]$$

\1dii/ let $V = V(R) = \frac{\gamma}{R} \therefore$

$$\frac{\partial W}{\partial t} + \frac{\gamma}{R}\frac{1}{R}\frac{\partial W}{\partial\phi} = \nu\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial W}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 W}{\partial\phi^2}\right] =$$

$$\frac{\partial W}{\partial t} + \frac{\gamma}{R^2}\frac{\partial W}{\partial\phi} = \nu\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial W}{\partial R}\right) + \nu\frac{1}{R^2}\frac{\partial^2 W}{\partial\phi^2} \therefore$$

let $W = \mathrm{Re}[F(R)e^{i\phi}] \therefore$ ~~$\frac{\partial W}{\partial t} = \mathrm{Re}[\phi F(r)e^{i\phi}]$~~, $\frac{\partial W}{\partial t} = 0$,

$W_\phi = \mathrm{Re}(\phi F(R)e^{i\phi}) \quad W_{\phi\phi} = \mathrm{Re}[F(R)e^{i\phi} + \phi^2 F(R)e^{i\phi}]$,

$W_R = \mathrm{Re}(F'(R)e^{i\phi}) \therefore RW_R = R\,\mathrm{Re}[F'(R)e^{i\phi}] = \mathrm{Re}[RF'(R)e^{i\phi}]$

$$\therefore \frac{\partial}{\partial R}\left(R\frac{\partial W}{\partial R}\right) = \mathrm{Re}[F'(R)e^{i\phi} + RF''(R)e^{i\phi}] \therefore$$

$$\frac{\gamma}{R^2}\mathrm{Re}[\phi F(R)e^{i\phi}] = \nu\frac{1}{R}\mathrm{Re}[F'(R)e^{i\phi} + RF''(R)e^{i\phi}] + \frac{1}{R^2}\mathrm{Re}[F(R)e^{i\phi} + \phi^2 F(R)e^{i\phi}]$$

$$\therefore \frac{\partial W}{\partial t} + \frac{\gamma}{R^2}\frac{\partial W}{\partial\phi} - \nu\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial W}{\partial R}\right) - \nu\frac{1}{R^2}\frac{\partial^2 W}{\partial\phi^2} = 0 \therefore$$

\pp2018/

$$\cancel{\frac{\gamma}{R^2}} \quad Re\left[\frac{\gamma}{R^2}\phi F(R)e^{i\phi} - \nu\frac{1}{R}F'(R)e^{i\phi} - R\frac{\nu}{R}F''(R)e^{i\phi} - \frac{1}{R^2}F(R)e^{i\phi} - \frac{\phi^2}{R^2}F(R)e^{i\phi}\right]$$

$$= Re\left[F(R)\left(\frac{\gamma\phi}{R^2}e^{i\phi} - \frac{1}{R^2}e^{i\phi} - \frac{\phi^2}{R^2}e^{i\phi}\right) - \frac{\nu}{R}F'(R)e^{i\phi} - \nu F''(R)e^{i\phi}\right]$$

$= \cancel{F(R)}$

$$\left[-\nu F''(R) - \frac{\nu}{R}F'(R) + \left(\frac{\gamma\phi}{R^2} - \frac{1}{R^2} - \frac{\phi^2}{R^2}\right)\right]Re\left(e^{i\phi}\right) = 0$$

let $F(R) = R^{\alpha}$ ∴

$\alpha R^{\alpha-1} = F'$, $\alpha(\alpha-1)R^{\alpha-2} = F'' = (\alpha^2-\alpha)R^{\alpha-2}$ ∴

$$(\alpha^2-\alpha)R^{\alpha-2} + \frac{1}{R}\alpha R^{\alpha-1} - \frac{1}{R^2}\left(1 - \frac{i\gamma}{\nu}\right)R^{\alpha} =$$

$$R^{\alpha-2}\left[\alpha^2 - \alpha + \alpha - 1 + \frac{i\gamma}{\nu}\right] = R^{\alpha-2}\left[\alpha^2 - 1 + \frac{i\gamma}{\nu}\right] = 0$$

∴ $\alpha^2 - 1 + \frac{i\gamma}{\nu} = 0$ ∴ $\alpha^2 = 1 - \frac{i\gamma}{\nu}$ ∴ $\alpha = \pm\left(1 - \frac{i\gamma}{\nu}\right)^{1/2}$

\2a/ $\underline{u}x\,(U(x), 0, 0) = \left(-\frac{Q}{2\pi x}, 0, 0\right) = (u, 0, 0)$ ∴

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{-Q}{2\pi}x^{-1}\right) = \frac{Q}{2\pi}x^{-2} \therefore$$

$$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(\frac{-Q}{2\pi}x^{-1}\right) = 0, \quad \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}(0) = 0 \therefore$$

$$u\frac{\partial u}{\partial x} = -\frac{Q}{2\pi x}\frac{Q}{2\pi}x^{-2} = -\frac{Q^2}{(2\pi)^2}\frac{1}{x^3} \therefore$$

$$-\frac{Q^2}{(2\pi)^2}\frac{1}{x^3} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \therefore \qquad \frac{\partial p}{\partial x} = \rho\left(\frac{Q}{2\pi}\right)^2\frac{1}{x^3}$$

\2b/ $\eta = \frac{y}{\delta} = \frac{y}{\left(\frac{2\pi\nu}{Q}\right)^{1/2}x} = \frac{1}{\left(\frac{2\pi\nu}{Q}\right)^{1/2}}yx^{-1}$ ∴

$$u = -\frac{Q}{2\pi x}F(\eta) = -\frac{Q}{2\pi}x^{-1}F\left(\frac{1}{\left(\frac{2\pi\nu}{Q}\right)^{1/2}}yx^{-1}\right),$$

$$v = -\frac{Q}{2\pi x^2}F(\eta) = -\frac{Q}{2\pi}x^{-2}F\left(\frac{1}{\left(\frac{2\pi\nu}{Q}\right)^{1/2}}yx^{-1}\right) \therefore$$

$$\nabla\cdot\underline{u} = \frac{\partial}{\partial x}u + \frac{\partial}{\partial y}v = \frac{\partial}{\partial x}\left(-\frac{Q}{2\pi}x^{-1}F\left(\frac{1}{\left(\frac{2\pi\nu}{Q}\right)^{1/2}}yx^{-1}\right)\right) + \frac{\partial}{\partial y}\left(-\frac{Q}{2\pi}x^{-2}F\left(\frac{1}{\left(\frac{2\pi\nu}{Q}\right)^{1/2}}yx^{-1}\right)\right)$$

$$= \frac{Q}{2\pi}x^{-2}F\left(\frac{1}{\left(\frac{2\pi\nu}{Q}\right)^{1/2}}yx^{-1}\right) - \frac{Q}{2\pi}x^{-1}F'(\eta)(-1)\frac{yx^{-2}}{\left(\frac{2\pi\nu}{Q}\right)^{1/2}} - \frac{Qx^{-2}}{2\pi}F(\eta)\frac{1}{\left(\frac{2\pi\nu}{Q}\right)^{1/2}}x^{-1}$$

\2c/ when $F^2 = 1 + F''$ ∴

3a/ NS: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho g + \mu\nabla^2\underline{u}$ ∴

~~for steady flow~~

$\underline{u} \sim U$, ~~t~~ $x \sim L$ ∴ $t \sim T$ ∴ $\frac{\partial}{\partial t} \sim \frac{1}{T}$,

$\nabla \sim \frac{1}{L}$ ∴ $\frac{L}{T} = U$ ∴ $\frac{1}{T} \simeq \frac{U}{L} \sim \frac{\partial}{\partial t}$ ∴

$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p + g + \nu\nabla^2\underline{u}$ ∴ $\nu = \frac{\mu}{\rho}$ ∴

$\frac{\partial \underline{u}}{\partial t} \sim \frac{U}{L}U = \frac{U^2}{L}$, $\underline{u}\cdot\nabla\underline{u} \simeq U\frac{1}{L}U = \frac{U^2}{L} \sim \frac{\partial \underline{u}}{\partial t}$

$\nu\nabla^2\underline{u} \sim \nu\frac{1}{L^2}U = \frac{\nu U}{L^2}$ ∴

$\frac{\frac{\partial \underline{u}}{\partial t}}{\nu\nabla^2\underline{u}} \sim \frac{\underline{u}\cdot\nabla\underline{u}}{\nu\nabla^2\underline{u}} \sim \frac{U^2/L}{\nu U/L^2} = \frac{UL}{\nu} = R_e << 1$ for small R_e ∴

$\frac{\partial \underline{u}}{\partial t}, \underline{u}\cdot\nabla\underline{u} << \nu\nabla^2\underline{u}$ ∴ can neglect $\frac{\partial \underline{u}}{\partial t}$, $\underline{u}\cdot\nabla\underline{u}$ for $R_e << 1$

∴ $\rho(0+0) = 0 = -\nabla p + \mu\nabla^2\underline{u}$ ∵ neglect gravity

3b/ $\nabla\times(\nabla\times(\nabla\times(\psi\hat{\underline{z}}))) = \nabla\times(\nabla\times\underline{u}) = \nabla(\nabla\cdot\underline{u}) - \nabla^2\underline{u} =$

$\nabla(0) - \nabla^2\underline{u} = -\nabla^2\underline{u}$ ∴

$\nabla\times(\nabla\times(\times(\nabla\times(\psi\hat{\underline{z}})))) = \nabla\times(-\nabla^2\underline{u}) = \cancel{\nabla\times(-\nabla^2(\psi\hat{\underline{z}}))} =$

$\nabla\times(-\nabla^2(\nabla\times(\psi\hat{\underline{z}}))) = -\nabla^2(\nabla\times(\nabla\times(\psi\hat{\underline{z}}))) =$

$-\nabla^2(\nabla(\nabla\cdot\psi\hat{\underline{z}})) + \nabla^2\nabla^2(\psi\hat{\underline{z}}) = -\nabla^2(\nabla(\frac{\partial}{\partial z}\psi)) + \nabla^2\nabla^2(\psi\hat{\underline{z}}) =$

$-\nabla^2(\nabla(0)) + \nabla^2\nabla^2(\psi\hat{\underline{z}}) = \nabla^2\nabla^2(\psi\hat{\underline{z}}) = \frac{\partial^2}{\partial z^2}\psi = 0$

∴ $\nabla^2\nabla^2\psi = \nabla^2\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi =$

$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)\psi = \left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2}\right)^2\psi = 0$

$\psi_\theta = r^\lambda f'$, $\psi_r = \lambda r^{\lambda-1} f$, $\psi_{rr} = \lambda(\lambda-1)r^{\lambda-2}f = (\lambda^2-\lambda)r^{\lambda-2}f$

\PP2021/ \1ai/ N-S: $\rho\left(\frac{\partial \underline{u}}{\partial t}+\underline{u}\cdot\nabla\underline{u}\right)=-\nabla p+\rho \underline{g}+\mu\nabla^2\underline{u}$ Q3 a

\1aii/ $\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\underline{u})=0$ ∴ incompressible ∴ $\frac{\partial\rho}{\partial t}=0$ ∴ $\frac{D\rho}{Dt}=\frac{\partial\rho}{\partial t}+u\cdot\nabla\rho=0$

⇒ $0=\nabla\cdot(\rho\underline{u})=\rho\nabla\cdot\underline{u}+\underline{u}\cdot\nabla\rho=0$ ∴ $\nabla\rho=0$ ∴

$u\cdot\nabla\rho=0$ ∴ $\rho\nabla\cdot\underline{u}=0=\nabla\cdot\underline{u}$ $\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\underline{u})=\frac{\partial\rho}{\partial t}+\underline{u}\cdot\nabla\rho+\rho\nabla\cdot\underline{u}=\rho\nabla\cdot\underline{u}=0$ ∴ $\rho\nabla\cdot\underline{u}=0=\nabla\cdot\underline{u}$

incompressibility means the material derivative of density is zero ∴ $\frac{D\rho}{Dt}=0=\frac{\partial\rho}{\partial t}+u\cdot\nabla\rho=0$ ∴ $\frac{\partial\rho}{\partial t}=0$, $u\cdot\nabla\rho=0$

∴ continuity mass equation: $\frac{\partial\rho}{\partial t}+\nabla\cdot(\rho\underline{u})=0$ ∴

$\frac{\partial\rho}{\partial t}+\rho\nabla\cdot\underline{u}+\underline{u}\cdot\nabla\rho=0$ ∴ $\rho\nabla\cdot\underline{u}=0=\nabla\cdot\underline{u}$

\1bi/ incompressible is $\nabla\cdot\underline{u}=0$ ∴ $\nabla\cdot\underline{u}=\nabla\cdot\left(0\hat{\underline{r}}+\frac{-r}{\sin\theta}\hat{\underline{\theta}}+0\hat{\underline{\phi}}\right)$

$=0+\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\left(-\frac{r}{\sin\theta}\right)\right)+0=\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(-r)=0=\nabla\cdot\underline{u}$ ∴

flow incompressible

\1bii/ $\underline{\omega}=\nabla\times\underline{u}=\nabla\times\left(0\hat{\underline{r}}+\frac{-r}{\sin\theta}\hat{\underline{\theta}}+0\hat{\underline{\phi}}\right)=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\hat{\underline{\phi}}\\ \partial_r & \partial_\theta & \partial_\phi\\ 0 & \frac{-r^2}{\sin\theta} & 0\end{vmatrix}$

$\frac{1}{r^2\sin\theta}\left(\hat{\underline{r}}(0)-r\hat{\underline{\theta}}(0)+r\sin\theta\hat{\underline{\phi}}\left(\frac{\partial}{\partial r}\left(\frac{-r^2}{\sin\theta}\right)\right)\right)=$

$\frac{1}{r^2\sin\theta}\left(r\sin\theta\hat{\underline{\phi}}\left(\frac{-1}{\sin\theta}2r\right)\right)=\frac{1}{r^2\sin\theta}(-2r)\hat{\underline{\phi}}=\frac{-2}{r\sin\theta}\hat{\underline{\phi}}$

\1biii/ $\underline{u}\cdot\nabla\underline{u}=(\underline{u}\cdot\nabla)\underline{u}$ ∴ $(\underline{u}\cdot\nabla)=\left(0\hat{\underline{r}}+\frac{-r}{\sin\theta}\hat{\underline{\theta}}+0\hat{\underline{\phi}}\right)\cdot\nabla=$

$\frac{-r}{\sin\theta}\frac{1}{r}\frac{\partial}{\partial\theta}=\frac{-1}{\sin\theta}\frac{\partial}{\partial\theta}$ ∴

$\underline{u}\cdot\nabla\underline{u}=\frac{-1}{\sin\theta}\frac{\partial}{\partial\theta}(\underline{u}))=\frac{-1}{\sin\theta}\frac{\partial}{\partial\theta}\left(0\hat{\underline{r}}+\frac{-r}{\sin\theta}\hat{\underline{\theta}}+0\hat{\underline{\phi}}\right)=$

$\frac{-1}{\sin\theta}\frac{\partial}{\partial\theta}\left(-\frac{r}{\sin\theta}\hat{\underline{\theta}}\right)=\frac{+r}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\hat{\underline{\theta}}\right)=\frac{r}{\sin\theta}\frac{\partial}{\partial\theta}\left(((\sin\theta))^{-1}\hat{\underline{\theta}}\right)=$

$\frac{r}{\sin\theta}(-1)(\sin\theta)^{-2}(\cos\theta)\hat{\underline{\theta}}+\frac{r}{\sin\theta}\frac{1}{\sin\theta}(-\hat{\underline{r}})=$

$\frac{-r\cos\theta}{(\sin\theta)^3}\hat{\underline{\theta}}+\frac{-r}{\sin^2\theta}\hat{\underline{r}}$

\1biv/ $\underline{u}=\nabla\times\left(\frac{\psi}{r\sin\theta}\hat{\underline{\phi}}\right)=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\hat{\underline{\phi}}\\ \partial_r & \partial_\theta & \partial_\phi\\ 0 & 0 & \psi(r,\theta)\end{vmatrix}=$

$\frac{1}{r^2\sin\theta}\left(\hat{\underline{r}}\left(\frac{\partial\psi(r,\theta)}{\partial\theta}\right)-r\hat{\underline{\theta}}\left(\frac{\partial\psi(r,\theta)}{\partial r}\right)+r\sin\theta\hat{\underline{\phi}}(0)\right)=$

$\frac{+1}{r^2\sin\theta}\frac{\partial\psi(r,\theta)}{\partial\theta}\hat{\underline{r}}+\frac{-1}{r}\frac{\partial\psi(r,\theta)}{\partial\theta}\hat{\underline{\theta}}=\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi(r,\theta)}{\partial\theta},\frac{-1}{r}\frac{\partial\psi(r,\theta)}{\partial\theta},0\right)$

\1bi/ $\underline{u} = \frac{-r}{\sin\theta}\hat{\underline{\phi}} \therefore \nabla\cdot\underline{u} = 0$

\1bii/ $\therefore \underline{\omega} = \nabla\times\underline{u} = \frac{-2}{\sin\theta}\hat{\underline{\phi}}$

\1biii/ $\underline{u}\cdot\nabla = \frac{-1}{\sin\theta}\frac{\partial}{\partial\theta} \therefore \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\sin\theta}\frac{\partial u}{\partial\theta} =$

$\frac{-r\cos\theta}{\sin^3\theta}\hat{\underline{\theta}} - \frac{r}{\sin^2\theta}\hat{\underline{r}}$

\1biv/ $\underline{u} = u_r\hat{\underline{r}} + u_\theta\hat{\underline{\theta}} = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\hat{\underline{r}} + \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r}\hat{\underline{\theta}} \therefore$

$\therefore u_r = 0 = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} \therefore \frac{\partial\psi}{\partial\theta} = 0 \therefore \psi = f(r)$,

$\frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -\frac{r}{\sin\theta} \therefore \frac{-1}{r}\frac{\partial\psi}{\partial r} = -r \therefore \frac{\partial\psi}{\partial r} = r^2$

$\therefore \psi = \frac{1}{3}r^3 + g(r) = \psi(r) \therefore \frac{\partial\psi}{\partial r} = \frac{\partial f(r)}{\partial r} \therefore$

$\frac{-1}{r\sin\theta}\frac{df(r)}{dr} = \frac{-r}{\sin\theta} \therefore \frac{df(r)}{dr} = r^2 \therefore f(r) = \frac{1}{3}r^3 + C \therefore$

$\psi = \frac{1}{3}r^3 + C$ ✓ C = constant

\1ci/ $e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) = e_{21}$, $\therefore$

$e_{13} = e_{31}$, $e_{23} = e_{32} \therefore$

e_{ij} has 6 ✓ independent components

\1cii/ $-e_{12} = e_{21}$, $-e_{13} = e_{31}$, $-e_{23} = e_{32}$,

~~eij~~ $e_{11} = e_{22} = e_{33} = 0 \therefore$ 4 independent components ✗

\1cii/ 3 $\because \xi_{ij} = -\xi_{ji}$, 0 diagonal $\therefore$ 3 independent components

\1ciii/ because $\epsilon_{ijk} \therefore \nabla\times\nabla\phi \equiv 0 \therefore$ is $\nabla\phi = \underline{u} \therefore$

$\underline{\omega} = \nabla\times\underline{u} = \nabla\times\nabla\phi = 0 \therefore$ shows vorticity is zero

$u = \nabla\phi$ guarantees $\nabla\times\underline{u} = 0 \therefore$

$[u]_i = [\nabla\phi]_i = \nabla_i\phi = \partial_i\phi \therefore [\underline{u}]_j = \partial_j\phi$

$\xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial}{\partial x_i}u_j\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_j}(\partial_i\phi) - \frac{\partial}{\partial x_i}(\partial_j\phi)\right) =$

$\frac{1}{2}\left(\frac{\partial^2}{\partial x_j\partial x_i}\phi - \frac{\partial}{\partial x_j\partial x_i}\phi\right) = 0$

\1ciii sol/ $\underline{u} = \nabla\phi$ guarantees $\nabla\times\underline{u} = 0 \therefore \underline{u} \to [\underline{u}]_i = [\nabla\phi]_i = \nabla_i\phi =$

$\partial_i\phi \therefore [\underline{u}]_j = \partial_j\phi \therefore \xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_j}(\partial_i\phi) - \frac{\partial}{\partial x_i}(\partial_j\phi)\right) =$

$\frac{1}{2}\left(\frac{\partial^2}{\partial x_j\partial x_i}\phi - \frac{\partial}{\partial x_j\partial x_i}\phi\right) = 0$

\1civ/ incompressible $\therefore$ ~~u·∇~~ $\nabla\cdot\underline{u} = 0 \therefore \frac{\partial u_i}{\partial x_i} + \frac{\partial u_j}{\partial x_j} + \frac{\partial u_k}{\partial x_k} = 0 = \frac{\partial u_i}{\partial x_i} \therefore$

\PP2021/ $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ ∴ normally 6 independent components ∴

$e_{ij} = e_{ji}$ for $j \neq i$ but now $\frac{\partial u_i}{\partial x_i} = 0$ ∴

only 3 ~~independent~~ components when $\nabla \cdot \underline{u} = 0$

\Ic iv/ incompressible ∴ $\nabla \cdot \underline{u} = 0$ ∴ $\frac{\partial u_i}{\partial x_i} = 0 = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$ ∴

$\frac{\partial u_1}{\partial x_1} = -\frac{\partial u_2}{\partial x_2} - \frac{\partial u_3}{\partial x_3}$ ∴ $\frac{\partial u_1}{\partial x_1}$ not independent ∴ 5 independent components

\Ic v/ $\underline{u} = (u_1, u_2, u_3)$

$\underline{u} = \underline{u}(x_1, x_2, x_3)$ ∴ $\underline{\omega} = \nabla \times \underline{u} = \nabla \times (u_1, u_2, u_3) =$ $\begin{vmatrix} \hat{i} & \hat{j} & \hat{z} \\ \partial x_1 & \partial x_2 & \partial x_3 \\ u_1 & u_2 & u_3 \end{vmatrix} =$

$\hat{i}\left(\frac{\partial}{\partial x_2}(u_3) - \frac{\partial}{\partial x_3} u_2\right) - \hat{j}\left(\frac{\partial}{\partial x_1}u_3 - \frac{\partial u_1}{\partial x_3}\right) + \hat{z}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) =$

$= \hat{i}\, 2\xi_{32} - \hat{j}\, 2\xi_{31} + \hat{z}\, 2\xi_{21} = 2\xi_{32}\hat{i} + 2\xi_{13}\hat{j} + 2\xi_{21}\hat{z} = \omega_1 \hat{i} + \omega_2 \hat{j} + \omega_3 \hat{z}$

$\underline{\omega} = 2(\xi_{32}, \xi_{13}, \xi_{21})$ ∴ $\omega_k = 2\varepsilon_{kij}\xi_{ji}$ ∴ $\frac{1}{2}\omega_k = \varepsilon_{kji}\xi_{ji}$ ∴ $\xi_{ij} = -\frac{1}{2}\varepsilon_{ijk}\omega_k = -\frac{1}{2}(-1)\varepsilon_{kji}\omega_k = \frac{1}{2}\varepsilon_{kji}\omega_k$

\Ic vi/ $\underline{\omega} = \hat{i}\, 2\xi_{32} + 2\xi_{13}\hat{j} + 2\xi_{21}\hat{z}$ ∴ ~~$\omega = 2\varepsilon_{kji}\xi$~~ ∴ $\underline{\omega} = 2(\xi_{32}, \xi_{13}, \xi_{21})$

~~$\frac{1}{2}\omega =$~~ $\omega_k = 2\varepsilon_{kji}\xi_{ji}$ ∴ $\frac{1}{2}\omega_k = \varepsilon_{kji}\xi_{ji}$ ∴ $\omega_k = 2\varepsilon_{kji}\xi_{ji}$ ∴

~~$\xi_{ij} = \frac{1}{2}\varepsilon$~~ $\xi_{ij} = -\frac{1}{2}\varepsilon_{ijk}\omega_k = \frac{1}{2}(-1)\varepsilon_{kji}\omega_k = \frac{1}{2}\varepsilon_{kji}\omega_k$ ∴ $\frac{1}{2}\omega_k = \varepsilon_{kji}\xi_{ji}$ ∴

$2\xi_{ij} = \varepsilon_{kji}\omega_k$ ∴ ~~$2\xi_{ij} = \varepsilon_{kji}$~~ ∴ $-2\xi_{ij} = 2\xi_{ji} = \varepsilon_{kji}\omega_k$ ∴ $\frac{1}{2}\varepsilon_{kji}\omega_k = \varepsilon_{kji}\varepsilon_{kji}\xi_{ji}$

$-2\xi_{ij} = 2\xi_{ji} = \varepsilon_{kji}\omega_k$ ∴ $\frac{1}{2}\varepsilon_{kji}\omega_k = \xi_{ji}$ ∴

∴ $-\xi_{ij} = \xi_{ji}$ ∴ $2\xi_{ji} = -2\xi_{ij} = \cancel{2}\varepsilon_{kji}\omega_k = -\cancel{2}\varepsilon_{ijk}\omega_k$ ∴ $-\frac{1}{2}\varepsilon_{kij}\omega_k = \xi_{ij}$ ∴

$\xi_{ji} = -\frac{1}{2}\varepsilon_{ijk}\omega_k$ $\xi_{ji} = \frac{1}{2}\varepsilon_{kji}\omega_k = -\frac{1}{2}\varepsilon_{ijk}\omega_k$ $-\frac{1}{2}\varepsilon_{kji}\omega_k = \xi_{ij}$ ∴

$\underline{\omega} = 2(\xi_{32}, \xi_{12}, \xi_{21}) = (\omega_1, \omega_2, \omega_3)$ ~~$-\frac{1}{2}\varepsilon_{ikj}$~~

\Id i/ $\underline{\Omega} \times \underline{u}$ ∴ $\underline{u} = (u, v, w)$ ∴

$(0, 0, \Omega) \times (u, v, w) = \underline{\Omega} \times \underline{u} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ 0 & 0 & \Omega \\ u & v & w \end{vmatrix} =$

$\hat{x}(0 - \Omega v) - \hat{y}(0 - \Omega u) + \hat{z}(0 - 0) = -\Omega v \hat{x} + \Omega u \hat{y} + 0\hat{z}$ ∴

$-\frac{1}{\rho}\nabla p = -\frac{1}{\rho}\frac{\partial p}{\partial x}\hat{x} + -\frac{1}{\rho}\frac{\partial p}{\partial y}\hat{y} - \frac{1}{\rho}\frac{\partial p}{\partial z}\hat{z}$ ∴

$\frac{\partial \underline{u}}{\partial t} = \frac{\partial u}{\partial t}\hat{x} + \frac{\partial v}{\partial t}\hat{y} + \frac{\partial w}{\partial t}\hat{z}$ ∴

$2\underline{\Omega} \times \underline{u} = -2\Omega v \hat{x} + 2\Omega u \hat{y} + 0\hat{z}$ ∴

$\frac{\partial u}{\partial t} - 2\Omega v = -\frac{1}{\rho}\frac{\partial p}{\partial x}$ // $\frac{\partial v}{\partial t} + 2\Omega u = -\frac{1}{\rho}\frac{\partial p}{\partial y}$; $\frac{\partial w}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial z}$ are $\hat{x}, \hat{y}, \hat{z}$ components ∴

$2\underline{\Omega} \times \underline{u} = 2\Omega(-v\hat{x} + u\hat{y}) = 2\Omega(-v, u)$ $\frac{d}{dy}$

$\nabla \cdot \left(\frac{\partial \underline{u}}{\partial t}\right) + 2\nabla \cdot (\underline{\Omega} \times \underline{u}) = -\frac{1}{\rho}\nabla \cdot \nabla p$ ∴ $\nabla \cdot \frac{\partial \underline{u}}{\partial t} = \frac{\partial}{\partial t}\nabla \cdot \underline{u} = \frac{\partial}{\partial t}(0) = 0$ ∴

∴ $+2\nabla \cdot (\underline{\Omega} \times \underline{u}) = -\frac{1}{\rho}\nabla \cdot \nabla p$ ∴ $2\nabla \cdot (\underline{\Omega} \times \underline{u}) = 2\Omega \nabla \cdot (-v\hat{x} + u\hat{y}) =$

$-2\Omega\frac{\partial v}{\partial x} + 2\Omega\frac{\partial u}{\partial y}$, $-\frac{1}{\rho}\nabla \cdot \nabla p = -\frac{1}{\rho}\nabla^2 p$ ∴ $2\Omega\left(\frac{\partial v}{\partial x} + \frac{\partial u}{\partial y}\right) = -\frac{1}{\rho}\nabla^2 p$ ∴

$= 2\Omega\frac{\partial u}{\partial y} - 2\Omega\frac{\partial v}{\partial x} = 2\Omega\left(\frac{\partial u}{\partial y} - \frac{\partial v}{\partial x}\right)$

$\frac{\partial}{\partial y}(x\text{component}) - \frac{\partial}{\partial x}(y\text{component}) =$ $\quad \frac{\partial u}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v$

$\frac{\partial v}{\partial t} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u \therefore$

$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t} - 2\Omega v\right) - \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t} + 2\Omega u\right) =$ $\quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right) - \frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right) = \frac{\partial}{\partial t}(u_y - v_x) =$

$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) - 2\Omega\frac{\partial v}{\partial y} - \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) - 2\Omega\frac{\partial u}{\partial x} = \frac{\partial}{\partial y}\left(-\frac{1}{\rho}\right.$

$\frac{\partial}{\partial y}\left(-\frac{1}{\rho}\frac{\partial p}{\partial x}\right) - \frac{\partial}{\partial x}\left(-\frac{1}{\rho}\frac{\partial p}{\partial y}\right) = -\frac{1}{\rho}\frac{\partial^2 p}{\partial y \partial x} + \frac{1}{\rho}\frac{\partial^2 p}{\partial y \partial x} = 0 \therefore$

$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y}\right) - \frac{\partial}{\partial t}\left(\frac{\partial v}{\partial x}\right) = 2\Omega\frac{\partial v}{\partial y} + 2\Omega\frac{\partial u}{\partial x} \therefore$

$\frac{\partial}{\partial t}\left(\frac{\partial u}{\partial y} - v_x\right) = 2\Omega(u_x + v_y)$

$\partial_{tt} = \partial_{tt}(-2\rho\Omega(u_y - v_x)) = \partial^2\Omega(u_{y}-v_{x}\ldots)$

\1dii/ $-\rho 2\Omega\left(\frac{\partial u}{\partial y} - v_x\right) = \nabla^2 p = -2\rho\Omega(u_y - v_x) \therefore$

$\frac{\partial^2}{\partial t^2}\{-2\rho\Omega\left(\frac{\partial^2}{\partial t^2}u_y - \frac{\partial^2}{\partial t^2}v_x\right) = -2\rho\Omega\left(\frac{\partial^3}{\partial t^2 \partial y}u - \frac{\partial^3}{\partial t^2 \partial x}v\right) = \frac{\partial^2}{\partial t^2}\nabla^2 p$

$2\Omega\left(\frac{\partial^3 u}{\partial t^2 \partial y} - \frac{\partial^3 v}{\partial t^2 \partial x}\right) = -\cancel{\rho}\frac{\partial^2}{\partial t^2}\nabla^2 p \quad \therefore 2\Omega(u_{tty} - v_{ttx}) = -\frac{1}{\rho}\partial_{tt}\nabla^2 p$

$\cancel{\frac{\partial}{\partial t}}\ \frac{\partial^2}{\partial t^2}\left(-\frac{1}{\rho}\nabla^2 p\right) = -\frac{1}{\rho}\frac{\partial^2}{\partial t^2}\nabla^2 p = \frac{\partial^2}{\partial t^2}\left(2\Omega(u_y - v_x)\right) =$

$\frac{\partial}{\partial t}\left(2\Omega\frac{\partial}{\partial t}(u_y - v_x)\right) = 2\Omega\frac{\partial}{\partial t}\left[\frac{\partial}{\partial t}(u_y - v_x)\right] = \cancel{2\Omega\frac{\partial}{\partial t}(u_x+v_y)}$

$= 2\Omega\frac{\partial}{\partial t}(2\Omega(u_x + v_y))$

$2\Omega\frac{\partial}{\partial t}\left[2\Omega(u_x + v_y)\right] = 4\Omega^2\frac{\partial}{\partial t}(u_x + v_y) = -\frac{1}{\rho}\frac{\partial^2}{\partial t^2}(\nabla^2 p) = 4\Omega^2\frac{\partial}{\partial t}(u_x + v_y)$

$= 2\Omega\frac{\partial}{\partial t}(2\Omega(u_x + v_y)) = -\frac{1}{\rho}\frac{\partial^2}{\partial t^2}(\nabla^2 p) =$ $\quad = 4\Omega^2\frac{\partial}{\partial t}(-w_z) =$

$4\Omega^2\frac{\partial}{\partial t}(u_x + v_y) \therefore \quad \nabla\cdot\underline{u} = u_x + v_y + w_z = 0 \therefore u_x + v_y = -w_z \therefore$ $\quad -4\Omega^2\frac{\partial}{\partial z}\left(\frac{\partial w}{\partial t}\right) =$

$-4\Omega^2\partial_z\left(-\frac{1}{\rho}\frac{\partial p}{\partial z}\right)$

$\nabla\cdot\underline{u} = 0 = u_x + v_y + w_z \quad \therefore u_x + v_y = -w_z \therefore$ $\quad = \frac{4\Omega^2}{\rho}\frac{\partial^2 p}{\partial z^2}$

$4\Omega^2\frac{\partial}{\partial t}(-w_z) = -\frac{1}{\rho}\frac{\partial^2}{\partial t^2}(\nabla^2 p) \therefore$ $\quad = -\frac{1}{\rho}\partial_{tt}\nabla^2 p \therefore$

$4\Omega^2\frac{\partial}{\partial t}(w_z) = \frac{1}{\rho}\frac{\partial^2}{\partial t^2}(\nabla^2 p) \therefore$ $\quad \partial_{tt}\nabla^2 p = -4\Omega^2\frac{\partial^2 p}{\partial z^2}$

$4\Omega^2\frac{\partial}{\partial t}(\rho w_z) = \frac{\partial^2}{\partial t^2}(\nabla^2 p) = 4\Omega^2\frac{\partial}{\partial z}\left(\rho\frac{\partial}{\partial t}w\right) \therefore$

$\rho w_t = -\frac{\partial p}{\partial z} \therefore \quad 4\Omega^2\frac{\partial}{\partial z}\left(-\frac{\partial p}{\partial z}\right) = \cancel{4\Omega^2}\frac{\partial^2}{\partial t^2}(\nabla^2 p) \therefore$

$\frac{\partial^2}{\partial t^2} = -4\Omega^2\frac{\partial^2 p}{\partial z^2}$

\1diii/ $p = Ae^{i(\underline{k}\cdot\underline{x} - \omega t)}$ $\quad \underline{x} = (x_1, x_2, x_3) \therefore \underline{k}\cdot\underline{x} = (k_x, k_y, k_z)\cdot(x_1, x_2, x_3) =$

$k_x x_1 + k_y x_2 + k_z x_3 \therefore$ dispersion relation is $\omega = \omega(k) \therefore$

\1div/ $\underline{c}_g = \frac{\partial\omega}{\partial k} = \frac{\partial}{\partial \underline{k}}(\omega) = \frac{\partial}{\partial(k_x, k_y, k_z)}\omega = \frac{\partial\omega}{\partial k_x}\hat{i} + \frac{\partial\omega}{\partial k_y}\hat{j} + \frac{\partial\omega}{\partial k_z}\hat{k}$

$= (\omega_{k_x}, \omega_{k_y}, \omega_{k_z})$

\PPZ 021/ $\underline{k} \cdot \underline{x} = k_x x_1 + k_y x_2 + k_z x_3$ ∴

$p = A e^{i(k_x x_1 + k_y x_2 + k_z x_3 - \omega t)} = A e^{i(\underline{k} \cdot \underline{x} - \omega t)} = A e^{i(\underline{k} \cdot \underline{x}) - \omega i t} =$

$A e^{i(\underline{k} \cdot \underline{x})} e^{-\omega i t}$ $\cancel{e^{\omega}}$ $\cancel{e^{it}}$ $\cancel{e^{ti\omega}}$ $e^{it\omega} = \frac{1}{p} A e^{i(\underline{k} \cdot \underline{x})}$ ∴

$\ln(e^{it\omega}) = it\omega = \ln\left(\frac{1}{p} A e^{i(\underline{k} \cdot \underline{x})}\right) = \ln\left(\frac{A}{p}\right) + i(\underline{k} \cdot \underline{x})$ ∴

$t\omega = \frac{1}{i} \ln\left(\frac{A}{p}\right) + \underline{k} \cdot \underline{x} = -i \ln\left(\frac{A}{p}\right) + \underline{k} \cdot \underline{x}$ ∴

$\omega = -i \frac{1}{t} \ln\left(\frac{A}{p}\right) + \frac{1}{t} \underline{k} \cdot \underline{x} = \omega(k)$

\1diii sol/ $\underline{x} = (x, y, z)$ ∴ $p = A e^{i(k_x x + k_y y + k_z z - \omega t)}$ ∴

$\frac{\partial p}{\partial x} = i k_x p$, $\frac{\partial p}{\partial y} = i k_y p$, $\frac{\partial p}{\partial z} = i k_z p$ ∴

$\frac{\partial^2 p}{\partial x^2} = (i k_x)^2 p = -k_x^2 p$, $\frac{\partial^2 p}{\partial y^2} = -k_y^2 p$, $\frac{\partial^2 p}{\partial z^2} = -k_z^2 p$ ∴

$\nabla^2 p = -k_x^2 p - k_y^2 p - k_z^2 p = -(k_x^2 + k_y^2 + k_z^2) p = -(\underline{k} \cdot \underline{k}) p = -k^2 p$ ∴

∴ $\omega^2 = \frac{4\Omega^2 k_z^2}{k^2}$ ∴ $\frac{\partial^2}{\partial t^2}(\nabla^2 p) = -4\Omega^2 \frac{\partial^2 p}{\partial z^2}$ ∴

$\frac{\partial p}{\partial t} = \frac{\partial}{\partial t}(A e^{i(\underline{k} \cdot \underline{x} - \omega t)}) = -\omega A e^{i(\underline{k} \cdot \underline{x} - \omega t)} = -\omega p$ ∴

$\frac{\partial^2 p}{\partial t^2} = -\omega(-\omega p) = \omega^2 p$ ∴

$\nabla^2 p = -k^2 p$, ∴ $\frac{\partial^2}{\partial t^2}(\nabla^2 p) = \frac{\partial^2}{\partial t^2}(-k^2 p) = -k^2 \frac{\partial}{\partial t^2}(p) = -k^2 \omega^2 p =$

$-4\Omega^2 \frac{\partial^2 p}{\partial z^2}$ ∴ $\frac{\partial^2 p}{\partial z^2} = -k_z^2 p$, ∴ $-k^2 \omega^2 p = -4\Omega^2(-k_z^2 p) = \cancel{\omega}$

$4\Omega^2 k_z^2 p$ ∴ $-k^2\omega^2 = 4\Omega^2 k_z^2$ ∴ $\omega^2 = \frac{-4\Omega^2 k_z^2}{k^2}$

\1div/ $\underline{k}$ is the wave vector and $\underline{k} = k\hat{n} = \frac{\omega}{c}\hat{n}$

∴ $\underline{k} = (k_x, k_y, k_z)$ ∴, $\underline{k} \cdot \underline{c}_g$ $(k_x, k_y, k_z) \cdot \left(\frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_y}, \frac{\partial \omega}{\partial k_z}\right) =$

$k_x \frac{\partial \omega}{\partial k_x} + k_y \frac{\partial \omega}{\partial k_y} + k_z \frac{\partial \omega}{\partial k_z}$

$c_g = \left(\frac{\partial \omega}{\partial k_x}, \frac{\partial \omega}{\partial k_y}, \frac{\partial \omega}{\partial k_z}\right)$ ∴

$\frac{\partial \hat{x}}{\partial x} = 0$

$\cancel{\omega = k}$ $\omega = \sqrt{-4\Omega^2 k_z^2 k^{-2}}$ ∴ $\cancel{\omega_{k_x}}$ $\frac{\partial \omega}{\partial k_x} =$

$\nabla = \hat{\underline{R}} \frac{\partial}{\partial R} + \hat{\underline{\phi}} \frac{1}{R} \frac{\partial}{\partial \phi} + \hat{\underline{z}} \frac{\partial}{\partial z}$ $\underline{u} \cdot \nabla \underline{u}$

$= \sqrt{-4}\,\Omega\, k_z \sqrt{k^{-2}} = 2i\Omega k_z \sqrt{k^{-2}} = 2i\Omega k_z (k_x^2 + k_y^2 + k_z^2)^{-1/2}$ ∴

$\omega_{k_x} = 2i\Omega k_z \left(-\frac{1}{2}\right)(k^2)^{-3/2} 2k_x = -2i\Omega k_z (k^2)^{-3/2} k_x$,

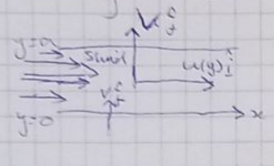
$\omega_{k_y} = 2i\Omega k_z \left(-\frac{1}{2}\right)(k^2)^{-3/2} 2k_y = -2i\Omega k_z (k^2)^{-3/2} k_y$,

$\omega_{k_z} = 2i\Omega (k^2)^{-1/2} + 2i\Omega k_z \left(-\frac{1}{2}\right)(k^2)^{-3/2} 2k_z = 2i\Omega\left((k^2)^{-1/2} - k_z^2 (k^2)^{-3/2}\right)$ ∴

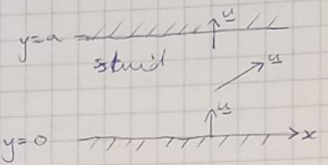
$\underline{k} \cdot \underline{c}_g = -2i\Omega k_z (k^2)^{-3/2} k_x^2 - 2i\Omega k_z (k^2)^{-3/2} k_y^2 + 2i\Omega k_z \left(-\frac{1}{2}\right)(k^2)^{-3/2} k_z^2 + 2i\Omega (k^2)^{-1/2} k_z$

$\omega^2 = \frac{4\Omega^2 k_z^2}{k^2}$ ∴ $\omega = \left(\frac{4\Omega^2 k_z^2}{k^2}\right)^{1/2}$ ∴ $c_g \cdot \underline{k} = \left(\frac{\partial}{\partial k_x}\left[\left(\frac{4\Omega^2 k_z^2}{k^2}\right)^{1/2}\right], \frac{\partial}{\partial k_y}\left[\left(\frac{4\Omega^2 k_z^2}{k^2}\right)^{1/2}\right], \frac{\partial}{\partial k_z}\left[\left(\frac{4\Omega^2 k_z^2}{k^2}\right)^{1/2}\right]\right) \cdot \underline{k} = 0$

\2a/ steady $\therefore \frac{\partial \underline{u}}{\partial t} = 0$, incompressible $\therefore \nabla \cdot \underline{u} = 0$



$\frac{\partial \rho}{\partial t} = 0$ $\quad \frac{dp}{dx} \neq 0$ $\quad v > 0$



\2b/ N-S $\therefore \rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho \underline{g} + \mu \nabla^2 \underline{u}$ $\therefore \underline{g} = 0 \therefore$

$\frac{\partial \underline{u}}{\partial t} = 0$ $\quad \frac{\partial \rho}{\partial t} + \underline{u}\cdot\nabla\rho$ $\quad \nabla\cdot(\rho\underline{u}) = \rho\nabla\cdot\underline{u} + \underline{u}\cdot\nabla\rho = \rho(0) + \underline{u}\cdot\nabla\rho =$

$\underline{u}\cdot\nabla\rho \therefore \nabla\rho = 0 \therefore \nabla\cdot(\rho\underline{u}) = 0 \therefore$

$\frac{D\rho}{Dt} = \frac{\partial \rho}{\partial t} + \underline{u}\cdot\nabla\rho = 0 \therefore \nabla\cdot\underline{u} = 0 \therefore$

$\rho(\underline{u}\cdot\nabla\underline{u}) = -\nabla p + \mu\nabla^2\underline{u} \therefore \nabla p = \frac{\partial p}{\partial x}\hat{\underline{i}} + \frac{\partial p}{\partial y}\hat{\underline{j}} \therefore \frac{\partial p}{\partial y} \neq 0 \therefore$

$\underline{u}\cdot\nabla \therefore \underline{u} = (u, v) = (u(y), v) \therefore \underline{u}\cdot\nabla = (u(y), v)\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right) = u(y)\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}$

$\therefore \underline{u}\cdot\nabla\underline{u} = \left(u(y)\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(u(y), v) = \left(u(y)\frac{\partial u(y)}{\partial x} + v\frac{\partial u(y)}{\partial y}, u(y)\frac{\partial v}{\partial x} + v\frac{\partial v(y)}{\partial y}\right) =$

$\left(0 + v\frac{\partial u(y)}{\partial y}, 0 + \cancel{\frac{\partial u(y)}{\partial y}}\right) = \left(v\frac{\partial u(y)}{\partial y}\right)\hat{\underline{i}} + 0\hat{\underline{j}} \therefore \nabla^2\underline{u} = 0$

$\rho v \frac{\partial u(y)}{\partial y}\hat{\underline{i}} = -\frac{\partial p}{\partial x}\hat{\underline{i}} \cancel{+\mu\partial} \therefore$

y component: $0 = 0$, x component: $\rho v \frac{\partial u(y)}{\partial y} = -\frac{\partial p}{\partial x} = -\frac{dp}{dx}$

$\rho v \frac{\partial u(y)}{\partial y}$ is constant $\therefore \rho v \frac{\partial u(y)}{\partial y} = G$, $\frac{dp}{dx} = -G$

\2b Sol/ flow is steady $\therefore \frac{\partial \underline{u}}{\partial t} = 0$, gravity neglected $\therefore \rho\underline{g} = 0$

$\underline{u}\cdot\nabla\underline{u} = v\frac{du(y)}{dy}\hat{\underline{i}}$ $\quad \underline{u}\cdot\nabla\underline{u} = \left(u\frac{\partial}{\partial x} + v\frac{\partial}{\partial y}\right)(u, v) = \left(v\frac{\partial u}{\partial y}, 0\right)$

$\therefore \nabla^2\underline{u} = \nabla^2(u(y), v) = (\partial_{yy}u, 0)$

y-comp: $\frac{\partial p}{\partial y} = 0$, x-comp: $v\frac{du}{dy}\rho = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u(y)}{dy^2} \therefore$

$\therefore \nabla^2\underline{u} = \nabla^2(u(y), v) = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u(y), v) = \frac{\partial^2 u(y)}{\partial x^2}\hat{\underline{i}} \therefore$

x-comp: $v\frac{du}{dy}\rho = -\frac{\partial p}{\partial x} + \mu\frac{d^2 u(y)}{dy^2} \therefore$

let $\mu\frac{d^2u(y)}{dy^2} = \tilde{f}(y)$, $-v\frac{du}{dy}\rho = \hat{f}(y) \therefore \frac{\partial p}{\partial x} = -v\rho\frac{\partial u}{\partial y} + \mu^2\frac{\partial^2 u(y)}{dy^2} = \hat{f}(y) + \tilde{f}(y)$

$\therefore \frac{\partial p}{\partial x} = f(y)$, $\frac{\partial p}{\partial y} = 0 \therefore P = \text{Constant}$ $\quad \frac{\partial p}{\partial x} = \mu\frac{d^2u}{dy^2} + v\frac{du}{dy}$

$p = x f(y) + f^*(y) \therefore \frac{\partial p}{\partial y} = 0 = x f'(y) + f^{*\prime}(y) = 0 \therefore$

$p = h(x) \therefore \hat{f}(y) = 0$, $f^*(y) = 0 \therefore$ ~~P = Constant~~ $\therefore \frac{\partial p}{\partial x} = f(y)$, $\frac{\partial p}{\partial y} = 0$

$\therefore$ ~~P = Constant~~ $\therefore$

pp2021 $\frac{\partial p}{\partial x} = f(y)$, $\frac{\partial p}{\partial y} = 0$ ∴ p indep of y, ∴ p only func of x, ∴ $p = \hat{f}(x)$ ∴ ∴

$\frac{\partial p}{\partial y} = 0$ ∴ $p = \hat{f}(x)$ ∴ $\frac{\partial p}{\partial x} = \hat{f}'(x) = \tilde{f}(x) = f(y) = \text{const}$

$\frac{\partial p}{\partial x} = \frac{\partial}{\partial x}\hat{f}(x) = f(y)$ ∴ $f(y) =$ constant ∴ $\frac{\partial p}{\partial x} =$ constant ∴

$\frac{dp}{dx} \equiv -G$

2c $[a] = L$ $[V] = LT^{-1}$ $[\rho] = ML^{-3}$ $\nu = \frac{\mu}{\rho}$

$[\nu] = LT^{-2}$ $\left[\frac{\partial u}{\partial t}\right] = [\nu \nabla^2 u] = [\nu] L^{-2} L T^{-1} = L^{-1}T^{-1}$

of $\left[\frac{\partial u}{\partial t}\right] = [\nu \nabla^2 u] = [\nu][\nabla^2 u]$ $= LT^{-1}T^{-1} = LT^{-2}$ ∴

$\nu = \frac{\mu}{\rho}$ ∴ $[\nu][\rho] = [\mu] = [\nu] ML^{-3}$ $[\nu] = L^2 T^{-1}$

$\left[\frac{\partial u}{\partial t}\right] = LT^{-2}$, $[\nabla^2 u] = L^{-2} L T^{-1} = L^{-1} T^{-1}$ ∴

$L^2 T^{-1} = [\nu] = [\mu]\left[\frac{1}{\rho}\right] = [\mu] M^{-1} L^3$ ∴

$[\mu] = ML^{-1}T^{-1}$ $[G] = \left[\frac{dp}{dx}\right] = L^{-1}[p] = L^{-1} M L T^{-2} L^{-2} = ML^{-2}T^{-2}$

$[G] = \left[\frac{dp}{dx}\right] = [\rho \underline{u} \cdot \nabla \underline{u}] = ML^{-3} L T^{-1} L^{-1} L T^{-1} = ML^{-2}T^{-2}$ ∴

2d a, V, ρ, μ, G a has independent dimension, V has independent, ρ has independent dimension ∴ 3 independent dimension.

∴ 2 = m dependent dimension: μ, G ∴

$\Pi_1 = \frac{\mu}{a^\alpha V^\beta \rho^\gamma}$ ∴ $[\mu] = ML^{-1}T^{-1} = [a]^\alpha [V]^\beta [\rho]^\gamma =$

$[L]^\alpha [LT^{-1}]^\beta [ML^{-3}]^\gamma = M^\gamma L^\alpha L^\beta L^{-3\gamma} T^{-\beta} = M^\gamma L^{\alpha+\beta-3\gamma} T^{-\beta} = M^1 L^{-1} T^{-1}$

∴ $1 = \gamma$, $-1 = -\beta$ ∴ $\beta = 1$ ∴ $\alpha + \beta - 3\gamma = -1 = \alpha + 1 - 3 = \alpha - 2$ ∴ $\alpha = 1$ ∴

$\Pi_1 = \frac{\mu}{aV\rho}$,

$\Pi_2 = \frac{G}{a^\alpha V^\beta \rho^\gamma}$ ∴ $[G] = M^1 L^{-2} T^{-2} = [a]^\alpha [V]^\beta [\rho]^\gamma = M^\gamma L^{\alpha+\beta-3\gamma} T^{-\beta}$

∴ $\gamma = 1$ $-2 = -\beta$ ∴ $\beta = 2$ ∴ $\alpha + \beta - 3\gamma = -2 = \alpha + 2 - 3 = \alpha - 1$ ∴ $-1 = \alpha$ ∴

$\Pi_2 = \frac{G}{a^{-1} V^2 \rho} = \frac{Ga}{V^2 \rho}$ $\frac{\mu}{\rho} u'' - Vu' =$

$Vu' = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu u'' = -\frac{G}{\rho} + \nu u''$

2e ∴ let $u'' - \frac{V\rho}{\mu} u' = -\frac{G}{\mu}$ ∴ for $u'' - \frac{V\rho}{\mu} u' = 0$ ∴

$u = e^{qy}$ ∴ $q^2 - \frac{V\rho}{\mu} q = 0 = q\left(q - \frac{V\rho}{\mu}\right) = 0$ ∴ $q = 0$, $q = \frac{V\rho}{\mu}$ ∴

$Vu'\rho = V\rho u'$ ∴ $V\rho u' = +G + \mu u''$ ∴

$\mu u'' - V\rho u' = -G$ ∴ $u'' - \frac{V\rho}{\mu} u' = -\frac{G}{\mu}$ ∴ $q\left(q - \frac{V\rho}{\mu}\right) = 0$ ∴ $q = \frac{V\rho}{\mu}$,

$q = 0$ ∴ $u = e^{qy}$ ∴ $u(y) = A_1 + B_1 e^{\frac{V\rho}{\mu} y}$,

let $u = \alpha y^2 + \beta y + \gamma$ ∴ $u' = 2\alpha y + \beta$ ∴ $u'' = 2\alpha$ ∴

$$2\alpha - \frac{V\rho}{\mu}(2\alpha y + \beta) = -\frac{G_T}{\mu} = 2\alpha - 2\alpha\frac{V\rho}{\mu}y - \beta\frac{V\rho}{\mu} \therefore$$

$$-2\alpha\frac{V\rho}{\mu} = 0 \quad \therefore \quad \alpha = 0 \quad \therefore \quad 2\alpha - \beta\frac{V\rho}{\mu} = -\frac{G_T}{\mu} = -\beta\frac{V\rho}{\mu} \therefore$$

$$G_T = \beta V\rho \therefore \quad \frac{G_T}{V\rho} = \beta \therefore \quad u_{IF} = \frac{G_T}{V\rho}y + \gamma \quad , \gamma \text{ arbitrary} \therefore$$

$$u_{GS}(y) = A_1 + B_1 e^{\frac{V\rho}{\mu}y} + \frac{G_T}{V\rho}y + \gamma = A_2 + B_1 e^{\frac{V\rho}{\mu}y} + \frac{G_T}{V\rho}y \therefore$$

$u(0) = 0$, $u(a) = 0 \therefore$

$u(0) = 0 = A_2 + B_1 \quad \therefore \quad A_2 = -B_1 \therefore$

$$u(y) = \cancel{-B} \; B_1\left(-1 + e^{\frac{V\rho}{\mu}y}\right) + \frac{G_T}{V\rho}y \therefore$$

$$u(a) = 0 = B_1\left(-1 + e^{\frac{V\rho}{\mu}a}\right) + \frac{G_T}{V\rho}a \therefore$$

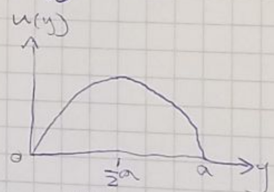
$$-\frac{G_T a}{V\rho} = B_1\left(-1 + e^{\frac{V\rho}{\mu}a}\right) \therefore$$

$$\frac{-G_T a}{V\rho\left(-1 + e^{\frac{V\rho a}{\mu}}\right)} = B_1 \therefore$$

$$u(y) = \frac{-G_T a}{V\rho\left(-1 + e^{\frac{V\rho a}{\mu}}\right)}\left(-1 + e^{\frac{V\rho}{\mu}y}\right) + \frac{G_T}{V\rho}y =$$

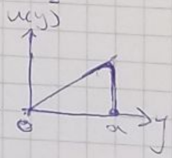
$$\frac{G_T}{V\rho}\left(y - \frac{a e^{\frac{V\rho}{\mu}y}}{-1 + \exp\left(\frac{V\rho a}{\mu}\right)}\right) \therefore \cancel{Re} \quad Re = \frac{V\rho a}{\mu} \; , \; \frac{G_T}{V\rho} = C$$

$A(y) = y$, $\quad a e^{\frac{V\rho}{\mu}} = B(y)$



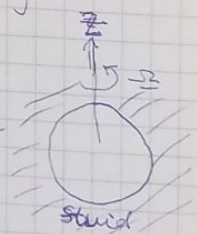
~~5~~ $\therefore Re = \frac{10^{-4} \times 10^{3} \times 1}{10^{-3}} = 100$ $\therefore$

at $\frac{1}{2}a = y$: $u(\frac{1}{2}a) = \frac{G_T}{V\rho}\left(\frac{1}{2}a - \frac{a e^{\frac{V\rho}{\mu}\frac{1}{2}a}}{-1 + \exp\left(\frac{V\rho a}{\mu}\right)}\right)$



$Re = 100 \gg 1 \therefore u(y) \simeq C(y - 0) = Cy = \frac{G_T}{V\rho}y \therefore$

$\underline{\Omega} = \Omega\hat{z} = \Omega(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta}) = \Omega\cos\theta\,\hat{r} - \Omega\sin\theta\,\hat{\theta}$



~~3a~~ as $r \to \infty$ $\quad \underline{u} \to 0 \therefore \omega \to 0$

no slip BC $\therefore \underline{u} = \underline{r} \times \underline{\Omega} = \omega\hat{\phi} = \underline{r} \times \Omega\hat{z} = \omega(r,\theta)\hat{\phi}$

$\therefore \omega = \Omega \quad \underline{u} = \underline{\Omega} \times \underline{r} = (\Omega\cos\theta\,\hat{r} - \Omega\sin\theta\,\hat{\theta})(a, 0, 0)$

$w \to 0$ as $r \to \infty$

no slip BC at $r = a$: $\underline{u} = \underline{\Omega} \times \underline{r} \therefore \forall \theta$

$\underline{r} = r\sin\theta\cos\phi\,\hat{i} + r\sin\theta\sin\phi\,\hat{j} + r\cos\theta\,\hat{k} \therefore$

$r\cos\theta = 0$, $\quad \underline{\Omega} = 0\hat{r} + 0\hat{\theta} + \Omega\hat{z} \therefore$

let ~~$\hat{\underline{r}} = (r_r \hat{r})_{r}$~~ let $\underline{r} = a\hat{\underline{r}} + 0\hat{\underline{\theta}} + 0\hat{\underline{\phi}}$ ∴

$\underline{u} = \omega(r,\theta)\hat{\underline{\phi}} = \underline{u} = \underline{\Omega} \times \underline{r} = (0\hat{\underline{r}} + 0\hat{\underline{\theta}} + 0\hat{\underline{\phi}}) \times$ ~~$(a\hat{\underline{r}} + 0\hat{\underline{\theta}}$~~ $\times \underline{\Omega}$

$\underline{\Omega} = \Omega\hat{\underline{k}}$ ∴ $\underline{r} = a\hat{\underline{r}} + 0\hat{\underline{\theta}} + 0\hat{\underline{\phi}}$

$\underline{u} = \underline{\Omega} \times \underline{r}$ gives $\underline{u}$ ∴ $w(a,\theta) = a\Omega\sin\theta$

~~$\hat{r}$~~ $\underline{r} = a\sin\theta\cos\phi\,\underline{i} + r\sin\theta\sin\phi\,\underline{j} + a\cos\theta\,\underline{k}$ ∴

$$\underline{\Omega} \times \underline{r} = \begin{vmatrix} \hat{\underline{r}} & \hat{\underline{\theta}} & \hat{\underline{\phi}} \\ & -a\Omega\sin\theta & \\ a & 0 & 0 \end{vmatrix} \qquad \underline{\Omega} = \Omega_r\hat{\underline{r}} - \Omega\sin\theta\,\hat{\underline{\theta}} + \Omega_\phi\hat{\underline{\phi}}$$ ∴

$\underline{\Omega} = 0\hat{\underline{x}} + 0\hat{\underline{y}} + \Omega\hat{\underline{z}} =$ $\qquad z = r\cos\theta$

$\underline{\Omega} = \Omega_r\hat{\underline{r}} - a\Omega\sin\theta\,\hat{\underline{\theta}} + \Omega_\phi\hat{\underline{\phi}}$ ∴ $\underline{r} = a\hat{\underline{r}} + 0\hat{\underline{\theta}} + 0\hat{\underline{\phi}}$ ∴

$$\underline{\Omega} \times \underline{r} = \begin{vmatrix} \hat{\underline{r}} & \hat{\underline{\theta}} & \hat{\underline{\phi}} \\ \Omega_r & -\Omega\sin\theta & \Omega_\phi \\ a & 0 & 0 \end{vmatrix} = \hat{\underline{r}}(0) - \hat{\underline{\theta}}(-\Omega_\phi a) + \hat{\underline{\phi}}(-a(-\Omega)\sin\theta) =$$

$\hat{\underline{\theta}}\,\Omega_\phi a + \hat{\underline{\phi}}\,a\Omega\sin\theta = \underline{u} = w(a,\theta)\hat{\underline{\phi}} = 0\hat{\underline{r}} + 0\hat{\underline{\theta}} + w(a,\theta)\hat{\underline{\phi}}$ ∴

$\Omega_\phi a = 0$ ∴ $\Omega_\phi = 0$ ∴

$\underline{\Omega} = \Omega_r\hat{\underline{r}} - a\Omega\sin\theta\,\hat{\underline{\theta}} + 0\hat{\underline{\phi}}$ ∴ let $\Omega_r = a\cos\theta$

3b/ $\nabla \cdot \underline{u} = 0$ ∴ incompressible fluid, $\underline{g} = 0$ ∴

N-S: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \mu\nabla^2\underline{u} = 0$

∴ $\frac{\partial u}{\partial t} = 0$, $\underline{u}\cdot\nabla\underline{u} = 0$, ∴ $\nabla p = \mu\nabla^2\underline{u}$

Re ≪ 1 ∴ let ~~∇~~ let $\underline{u} \sim U$ ∴

$\nabla \sim \frac{1}{L}$ ∴ $\nabla^2 \sim \frac{1}{L^2}$ ∴

$\underline{u}\cdot\nabla\underline{u} \sim U\frac{1}{L}U = \frac{U^2}{L}$ ∴ $\mu\nabla^2\underline{u} \sim \mu\frac{U}{L^2}$ ∴

$\frac{U^2}{L} \leq \mu\frac{U^2}{L}$ the inertial forces are much smaller then the viscous ∴ inertia ≪ viscous forces

3c/ p = constant ∴ $\nabla p = 0$ ∴ $\nabla^2\underline{u} = 0$

p = constant ∴ $\nabla p = 0$ ∴ $\mu\nabla^2\underline{u} = 0 = \nabla^2\underline{u}$ ∴ $\nabla\cdot\underline{u} = 0$ ∴

∴ $\nabla^2\underline{u} = \nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{u}) = 0 = \nabla(0) - \nabla\times(\nabla\times\underline{u}) = -\nabla\times(\nabla\times\underline{u}) = 0$

∴ $\nabla\times(\nabla\times\underline{u}) = 0$ ∴

$w(a,\theta) = a\Omega\sin\theta$ ∴ $w(r,\theta) = a\Omega\sin\theta$ ∴

$\nabla\times(\underline{u}) = \nabla\times(w(r,\theta)\hat{\underline{\phi}}) = \nabla\times(0\hat{\underline{r}} + 0\hat{\underline{\theta}} + w(r,\theta)\hat{\underline{\phi}}) =$

$$\frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\,\hat{\underline{\phi}} \\ \partial_r & \partial_\theta & \partial_\phi \\ 0 & 0 & r\sin\theta W(r,\theta) \end{vmatrix} = \frac{1}{r^2\sin\theta}\left(\hat{\underline{r}}\left(\frac{\partial}{\partial\theta}(\sin\theta W(r,\theta))r\right) - r\hat{\underline{\theta}}\left(\sin\theta\frac{\partial}{\partial r}(rW(r,\theta))\right)\right)$$

$$= \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta W)\hat{\underline{r}} - \cancel{\frac{1}{r\sin\theta}}\frac{1}{r}\frac{\partial}{\partial r}(rW)\hat{\underline{\theta}} \doteq \nabla\times\underline{u} \therefore$$

$$\nabla\times(\nabla\times\underline{u}) = 0 = \nabla\times\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta W)\hat{\underline{r}} - \frac{1}{r}\frac{\partial}{\partial r}(rW)\hat{\underline{\theta}}\right) =$$

$$\frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\,\hat{\underline{\phi}} \\ \partial_r & \partial_\theta & \partial_\phi \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta W) & -\frac{\partial}{\partial r}(rW) & 0 \end{vmatrix} =$$

$$\frac{1}{r^2\sin\theta}\left[r\sin\theta\,\hat{\underline{\phi}}\left(-\frac{\partial^2}{\partial r^2}(rW) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(W\sin\theta)\cancel{\frac{r}{r}}\right)\right) + \dots\right] = 0$$

$\cancel{z}$ $\therefore$ $\hat{\underline{\phi}}$ component $= 0 \therefore$

$$-\frac{\partial^2}{\partial r^2}(rW) - \frac{1}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(W\sin\theta)\right) = 0 \therefore$$

$$(rW)_{rr} + \frac{1}{r}\partial_\theta\left(\frac{1}{\sin\theta}\partial_\theta(W\sin\theta)\right) = 0$$

\3d/ let $W = f(r)g(\theta) \therefore \frac{\partial}{\partial r}((rW)_r) = \frac{\partial}{\partial r}(W + rW_r) = \cancel{\frac{\partial}{\partial r}(W}$

$W_r + W_r + rW_{rr} = 2W_r + rW_{rr} = 2f'(r)g(\theta) + rf''(r)g(\theta)$,

$\frac{\partial}{\partial\theta}(W\sin\theta) = W\cos\theta \quad \therefore \frac{1}{\sin\theta}\partial_\theta(W\sin\theta) = \frac{1}{\sin\theta}W\cos\theta =$

$\frac{W}{\tan\theta} \therefore \frac{\partial}{\partial\theta}\left(W\frac{\cos\theta}{\sin\theta}\right) = W\partial_\theta\left(\frac{\cos\theta}{\sin\theta}\right) = \frac{\sin\theta(-\sin\theta) - \cos\theta(\cos\theta)}{\sin^2\theta}W =$

$-\left(\frac{\sin^2\theta + \cos^2\theta}{\sin^2\theta}\right)W = \frac{-W}{\sin^2\theta} \therefore$

$$rf''(r)g(\theta) + \frac{1}{r}\frac{-1}{\sin^2\theta}f(r)g(\theta) \cancel{\theta} + 2f'(r)g(\theta) = 0$$

\3d sol/ $\frac{\partial^2(rW)}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(W\sin\theta)\right] = 0$, $W = f(r)g(\theta) \therefore$

$\partial_{rr}(rf(r)g(\theta)) = g(\theta)\partial_{rr}(rf(r))$,

$\therefore g(\theta)\partial_{rr}(rf(r)) + f(r)\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(g(\theta)\sin\theta)\right] = 0 \therefore$

$\cancel{\partial_r}\ g(\theta)\partial_{rr}(rf(r)) = -f(r)\frac{1}{r}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(g(\theta)\sin\theta)\right] \therefore$

$\frac{r}{f(r)}\frac{\partial^2}{\partial r^2}(rf(r)) = -\frac{1}{g(\theta)}\frac{\partial}{\partial\theta}\left[\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(g(\theta)\sin\theta)\right] = \text{constant}$

\3e/ let $g(\theta) = \sin\theta \therefore f(\infty) = 0$, $\cancel{f(a) = }$ $f(a) = a\Omega \therefore$

let $\frac{r}{f(r)}\frac{\partial^2}{\partial r^2}(rf(r)) = -\lambda = \text{constant} \therefore$

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$$-\frac{1}{g}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(g\sin\theta)\right) = \text{constant} = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\sin\theta)\right) =$$

$$-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta)\right) = -\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}2\sin\theta\cos\theta\right) =$$

$$-\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(2\cos\theta) = \frac{-2}{\sin\theta}(-\sin\theta) = 2 \therefore$$

$$\frac{r}{f(r)}\frac{\partial^2}{\partial r^2}(r f(r)) = 2 \therefore$$

$$r\frac{\partial}{\partial r}(f(r) + r f'(r)) = 2f(r) \therefore$$

$$r(f'(r) + f'(r) + r f''(r)) = 2f(r) \therefore$$

$$r^2 f''(r) + 2r f'(r) - 2f(r) = 0 \therefore$$

$f(r) = r^m \therefore \quad f' = m r^{m-1}, \quad f'' = m(m-1)r^{m-2} = (m^2-m)r^{m-2} \therefore$

$\therefore m^2 - m + 2m - 2 = 0 = m^2 + m - 2 = (m+2)(m-1) \therefore$

$m = -2, \quad m = 1 \therefore \quad f(r) = A\frac{1}{r^2} + Br \therefore$

as $r \to \infty$ $f(r) = 0 \therefore A = 0 \quad \therefore f(r) = Br \therefore$

as $f(r=a) = f(a) = a\Omega \therefore \quad B = \Omega \therefore$

$f(r) = \Omega r$

\3f/ work out ϕ component of all the known terms on the RHS of the expression for t, given the normal to a spherical surface is the radial direction $\hat{r}$

\3f/ $\therefore \hat{n} = \hat{r} \quad \therefore$ let $\hat{r} = (1, 0, 0) \therefore$

$$\hat{n}\cdot\nabla = \hat{r}\cdot\nabla = (1,0,0)\cdot\nabla = (1,0,0)\cdot\left(\frac{\partial}{\partial r}, \frac{1}{r}\frac{\partial}{\partial\theta}, \frac{1}{r\sin\theta}\frac{\partial}{\partial\phi}\right) =$$

$\frac{\partial}{\partial r} \qquad \underline{u} = w(r,\theta)\hat{\underline{\phi}} \quad \therefore$

$$(\hat{\underline{n}}\cdot\nabla)\underline{u} = \left(\frac{\partial}{\partial r}\right)w(r,\theta)\hat{\underline{\phi}} = \frac{\partial(w(r,\theta)\hat{\underline{\phi}})}{\partial r} = \hat{\underline{\phi}}\frac{\partial w}{\partial r}$$

$\therefore 2\mu(\hat{\underline{n}}\cdot\nabla)\underline{u} = 2\mu\hat{\underline{\phi}}\frac{\partial w}{\partial r} = 2\mu\frac{\partial w}{\partial r}\hat{\underline{\phi}}$

$$\nabla\times\underline{u} = \nabla\times w(r,\theta)\hat{\underline{\phi}} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\hat{\underline{\phi}} \\ \partial r & \partial\theta & \partial\phi \\ 0 & 0 & r\sin\theta w(r,\theta)\end{vmatrix} =$$

$$\frac{1}{r^2\sin\theta}\left(\hat{\underline{r}}\, r\frac{\partial}{\partial\theta}(\sin\theta w) - r\hat{\underline{\theta}}\sin\theta\frac{\partial}{\partial r}(r w)\right) =$$

$$\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta w)\hat{\underline{r}} - \frac{1}{r}\frac{\partial}{\partial r}(r w)\hat{\underline{\theta}} \therefore$$

$$\hat{\underline{n}}\times(\nabla\times\underline{u}) = \hat{\underline{r}}\times(\nabla\times\underline{u}) = (1,0,0)\times\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta w), -\frac{1}{r}\frac{\partial}{\partial r}(r w), 0\right) =$$

$$= \begin{vmatrix}\hat{\underline{r}} & \hat{\underline{\theta}} & \hat{\underline{\phi}} \\ 1 & 0 & 0 \\ \frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta w) & -\frac{1}{r}\frac{\partial}{\partial r}(r w) & 0\end{vmatrix} =$$

$\hat{\underline{r}}(0) - \hat{\underline{\theta}}(0) + \hat{\underline{\phi}}\left(-\frac{1}{r}\frac{\partial}{\partial r}(rw)\right) = -\frac{1}{r}\frac{\partial}{\partial r}(rw)\hat{\underline{\phi}}$

$\therefore -p\hat{\underline{n}} = \cancel{-p(\underline{r},0)}\ -p\hat{\underline{r}} = -p(1,0,0) = -p\hat{\underline{r}}\ \therefore$

$\mu\hat{\underline{n}} \times (\nabla \times \underline{u}) = -\frac{1}{\mu} r\frac{\partial}{\partial r}(rw)\hat{\underline{\phi}}\ \therefore$

$t = t_r\hat{\underline{r}} + t_\theta\hat{\underline{\theta}} + t_\phi\hat{\underline{\phi}}\ \therefore\ \hat{\underline{\phi}}$ component:

$\cancel{t_\phi = \frac{1}{2\mu} r \frac{\partial}{\partial}}\quad t_\phi = 2\mu\frac{\partial w}{\partial r} - \frac{1}{\mu} r\frac{\partial}{\partial r}(rw)\ \therefore$

$\frac{\partial}{\partial r}\left(\frac{w}{r}\right) = \frac{\partial}{\partial r}(wr^{-1}) = r^{-1}\frac{\partial w}{\partial r} + w\frac{\partial}{\partial r}(r^{-1}) = \frac{1}{r}\frac{\partial w}{\partial r} - \frac{1}{r^2}w\ \therefore$

$t_\phi = \mu r\frac{\partial}{\partial r}\left(\frac{w}{r}\right) = \mu r\left(\frac{1}{r}\frac{\partial w}{\partial r} - \frac{1}{r^2}w\right) = \mu\frac{\partial w}{\partial r} - \mu\frac{1}{r}w$

$\therefore r\frac{\partial}{\partial r}(rw) = r\left(r\left(\frac{\partial w}{\partial r}\right) + \frac{\partial w}{\partial r}\right) = r^2\frac{\partial w}{\partial r} + r\frac{\partial w}{\partial r}$

$\therefore t_\phi = \mu r\frac{\partial}{\partial r}\left(\frac{w}{r}\right)$

\4a/ $\therefore \nabla \times \left(\frac{\psi}{R}\hat{\underline{\phi}}\right) = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & \psi & 0 \end{vmatrix} =$

$\frac{1}{R}\left(\hat{\underline{R}}\left(-\frac{\partial\psi}{\partial z}\right) - R\hat{\underline{\phi}}(0) + \hat{\underline{z}}\left(\frac{\partial\psi}{\partial r}\right)\right) = -\frac{1}{R}\frac{\partial\psi}{\partial z}\hat{\underline{R}} + \frac{1}{R}\frac{\partial\psi}{\partial R}\hat{\underline{z}} = u\hat{\underline{R}} + w\hat{\underline{z}}\ \therefore$

$u = -\frac{1}{R}\frac{\partial\psi}{\partial z}$ ✓ , $w = \frac{1}{R}\frac{\partial\psi}{\partial R}$ ✓

\4b/ $\therefore$ streamlines: $\frac{dx}{u} = \frac{dy}{w} =$

$\frac{dR}{\left(-\frac{1}{R}\frac{\partial\psi}{\partial z}\right)} = \frac{\partial z}{\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)}\ \therefore -R\frac{dR}{\partial\psi/\partial z} = R\frac{\partial z}{\partial\psi/\partial R}$

$\therefore \cancel{-R}\ -\frac{dR}{\partial\psi/\partial z} = \frac{\partial z}{\partial\psi/\partial R}\ \therefore\ -\frac{\partial R\,\partial z}{\partial\psi} = \frac{\partial R\,\partial z}{\partial\psi}$

\4b/ $\therefore \frac{dR}{u} = \frac{dz}{w}\ \therefore\ \frac{dR}{-\frac{1}{R}\frac{\partial\psi}{\partial z}} = \frac{\partial z}{\frac{1}{R}\frac{\partial\psi}{\partial R}}\ \therefore$

$\frac{1}{R}\frac{\partial\psi}{\partial R}dR = -\frac{1}{R}\frac{\partial\psi}{\partial z}dz = \cancel{\psi}\ \therefore$

$\int\frac{1}{R}\partial\psi = \int -\frac{1}{R}\partial\psi\ \therefore\ 2\int\frac{1}{R}d\psi = 0 = \frac{1}{R}\int 1\,d\psi\ \therefore\ \int 1\,d\psi = 0\ \therefore$

$\psi + C_1 = 0\ \therefore\ \psi = C_2 = $ Constant

\4c/ $\underline{\omega} = \nabla \times \underline{u} = \nabla \times \left(u\hat{\underline{R}} + 0\hat{\underline{\phi}} + w\hat{\underline{z}}\right) =$

$\nabla \times \left(-\frac{1}{R}\frac{\partial\psi}{\partial z}, 0, \frac{1}{R}\frac{\partial\psi}{\partial R}\right) = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ -\frac{1}{R}\frac{\partial\psi}{\partial z} & 0 & \frac{1}{R}\psi_R \end{vmatrix} =$

$\frac{1}{R}\left[\hat{\underline{R}}\left(\frac{\partial}{\partial\phi}\left(\frac{1}{R}\psi_R\right)\right) - R\hat{\underline{\phi}}\left(\frac{\partial}{\partial R}\left(\frac{1}{R}\psi_R\right) - \frac{\partial}{\partial z}\left(-\frac{1}{R}\psi_z\right)\right) + \hat{\underline{z}}\left(\frac{\partial}{\partial\phi}\left(-\frac{1}{R}\psi_z\right)\right)\right] =$

$\frac{1}{R}\left[0\hat{\underline{R}} - R\hat{\underline{\phi}}\left(-\frac{1}{R^2}\psi_R + \frac{1}{R}\psi_{RR} + \frac{1}{R}\psi_{zz}\right) + \hat{\underline{z}}(0)\right] = \left(\frac{1}{R^2}\frac{\partial\psi}{\partial R} - \frac{1}{R}\frac{\partial^2\psi}{\partial R^2} - \frac{1}{R}\frac{\partial^2\psi}{\partial z^2}\right)\hat{\underline{\phi}}$

\PP2021/ $= \omega \underline{\hat{e}} \therefore \omega = \frac{1}{R^2}\frac{\partial \psi}{\partial R} - \frac{1}{R}\frac{\partial^2 \psi}{\partial R^2} \cancel{+} \frac{1}{R}\frac{\partial^2 \psi}{\partial z^2}$

\2d/ invisid $\therefore \mu = 0$, incompressible $\therefore \nabla \cdot \underline{u} = 0 \therefore$

$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\nabla \therefore$ N-S: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho g$

$\nabla \cdot \underline{u} = 0 \therefore \frac{D\rho}{Dt} = 0 \therefore \frac{\partial \rho}{\partial t} + u\cdot\nabla\rho = 0 \therefore$

$\nabla \times \left(\rho \frac{\partial \underline{u}}{\partial t}\right) + \nabla \times (\underline{u}\cdot\nabla\underline{u}) = \nabla \times (-\nabla p) + \nabla \times (\rho g)$

$\frac{D\rho}{Dt} = 0 = \frac{\partial \rho}{\partial t} + \underline{u}\cdot\nabla\rho = 0 \therefore \frac{\partial \rho}{\partial t} = 0, u\cdot\nabla\rho = 0,$

$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{u}) = 0$ continuitity mass equation

$\therefore \frac{\partial \rho}{\partial t} + \underline{u}\cdot\nabla\rho + \rho\nabla\cdot\underline{u} = 0 \therefore \frac{\partial \rho}{\partial t} = 0, \nabla\rho = 0 \therefore \underline{u}\cdot\nabla\rho = 0 \therefore$

$\rho\nabla\cdot\underline{u} = 0 = \nabla\cdot\underline{u} \therefore$

$\therefore$ ~~look~~ $\nabla \times (-\nabla p) = -\nabla \times \nabla p = -0 = 0 \therefore$

$\rho\nabla \times \left(\frac{\partial \underline{u}}{\partial t}\right) + \nabla \times (\underline{u}\cdot\nabla\underline{u}) = \rho\nabla \times (g) \therefore$

$\therefore$ N-S: ~~$\rho\left(\frac{\partial \underline{u}}{\partial t}\right) = -\nabla p$~~ $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p$.

$\nabla \times \left(\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right)\right) = \nabla \times (-\nabla p) - \nabla \times (\nabla p) = -0 = 0 \therefore$

$\rho\nabla \times \left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = 0 = \nabla \times \left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) =$

$\nabla \times \left(\frac{\partial \underline{u}}{\partial t}\right) + \nabla \times (\underline{u}\cdot\nabla\underline{u}) = 0 \therefore$

$\nabla \times \left(\frac{\partial \underline{u}}{\partial t}\right) = \frac{\partial}{\partial t}\nabla \times \underline{u} = \frac{\partial}{\partial t}\underline{\omega} = -\nabla \times (\underline{u}\cdot\nabla\underline{u}) = -\nabla \times ((\underline{u}\cdot\nabla)\underline{u}) =$

$-(\underline{u}\cdot\nabla)\nabla \times \underline{u} = -(\underline{u}\cdot\nabla)\underline{\omega} =$

$\nabla \times (\underline{u} \times \underline{\omega}) - \underline{u}(\nabla\cdot\underline{\omega}) + \underline{\omega}(\nabla\cdot\underline{u}) - (\underline{\omega}\cdot\nabla)\underline{u} \therefore$

$\nabla\cdot\underline{u} = 0 \therefore \underline{\omega}(\nabla\cdot\underline{u}) = 0,$

$\nabla\cdot\underline{\omega} = \nabla\cdot\nabla \times \underline{u} = 0 \therefore \underline{u}(\nabla\cdot\underline{\omega}) = 0 \therefore$

$\frac{\partial}{\partial t}\underline{\omega} = \nabla \times (\underline{u} \times \underline{\omega}) - (\underline{\omega}\cdot\nabla)\underline{u} \therefore$

~~$\nabla \times (\underline{u} \times \underline{\omega}$~~ N-S $\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} = -\nabla p \therefore \nabla\cdot\underline{u} = 0 \therefore$

$\nabla \times \left(\frac{\partial \underline{u}}{\partial t}\right) + \nabla \times ((\underline{u}\cdot\nabla)\underline{u}) = \nabla \times (-\nabla p) = 0 = \frac{\partial}{\partial t}\underline{\omega} + \nabla \times ((\underline{u}\cdot\nabla)\underline{u})$

~~$\frac{\partial}{\partial t}\underline{\omega} + (\underline{u}\cdot\nabla)\nabla \times \underline{u}$~~ is $\nabla \times ((\underline{u}\cdot\nabla)\underline{u}) = (\underline{u}\cdot\nabla)\nabla \times \underline{u} = (\underline{u}\cdot\nabla)\underline{\omega}$

$= \underline{u}(\nabla\cdot\underline{\omega}) - \underline{\omega}(\nabla\cdot\underline{u}) + (\underline{\omega}\cdot\nabla)\underline{u} - \nabla \times (\underline{u} \times \underline{\omega}) = (\underline{\omega}\cdot\nabla)\underline{u} - \nabla \times (\underline{u} \times \underline{\omega})$

$\therefore \frac{\partial}{\partial t}\underline{\omega} + (\underline{\omega}\cdot\nabla)\underline{u} - \nabla \times (\underline{u} \times \underline{\omega}) = 0 \therefore$

vorticity eqn: $\frac{\partial \underline{\omega}}{\partial t} = \nabla \times (\underline{u} \times \underline{\omega}) + \nu \nabla^2 \underline{\omega}$ $\quad \frac{\partial \underline{\omega}}{\partial t} = \nabla \times (\underline{u} \times \underline{\omega}) + \nu \nabla^2 \underline{\omega}$

$\frac{\partial \underline{\omega}}{\partial t} = \nabla \times (\underline{u} \times \underline{\omega}) + \nu \nabla^2 \underline{\omega}$ $\quad \frac{\partial \underline{\omega}}{\partial t} = \nabla \times (\underline{u} \times \underline{\omega}) + \nu \nabla^2 \underline{\omega}$

\4d/ vorticity equation: $\frac{\partial \underline{\omega}}{\partial t} = \nabla \times (\underline{u} \times \underline{\omega}) + \nu \nabla^2 \underline{\omega}$ ∴

$\mu = 0$ ∴ $\nu = \frac{\mu}{\rho} = \frac{0}{\rho} = 0$ ∴ $\frac{\partial \underline{\omega}}{\partial t} = \nabla \times (\underline{u} \times \underline{\omega})$ ∴

$\frac{D}{Dt}\left(\frac{\omega}{R}\right) = \frac{\partial}{\partial t}\left(\frac{\omega}{R}\right) + \underline{u} \cdot \nabla\left(\frac{\omega}{R}\right)$ ∴ $\nabla \cdot \underline{\omega} = 0$, $\nabla \cdot \underline{u} = 0$ ∴

$\nabla \times (\underline{u} \times \underline{\omega}) = \underline{u}(\nabla \cdot \underline{\omega}) - \underline{\omega}(\nabla \cdot \underline{u}) + (\underline{\omega} \cdot \nabla)\underline{u} - (\underline{u} \cdot \nabla)\underline{\omega} =$

$(\underline{\omega} \cdot \nabla)\underline{u} - (\underline{u} \cdot \nabla)\underline{\omega} = \frac{\partial \underline{\omega}}{\partial t}$ ∴

$(\underline{u} \cdot \nabla)\underline{\omega}$ $\cancel{=}$, $(\underline{\omega} \cdot \nabla)\underline{u}$ ∴ $(\underline{\omega} \cdot \nabla) = \omega \hat{\underline{\phi}} \cdot \nabla = \omega \frac{1}{R}\frac{\partial}{\partial \phi}$ ∴

$(\underline{\omega} \cdot \nabla)\underline{u} = \omega \frac{1}{R}\frac{\partial}{\partial \phi}\underline{u}$ $\cancel{= \omega \frac{1}{R}\frac{\partial}{\partial \phi}\left(\frac{\psi}{R}\hat{\underline{\phi}}\right) = \omega \frac{1}{R}\frac{\psi}{R}\frac{\partial}{\partial \phi}(\hat{\underline{\phi}}) = \omega \frac{1}{R}\frac{\psi}{R}\hat{\underline{R}}}$ ∴

$\cancel{\omega \cdot \nabla} - \cancel{u \cdot \nabla}$ ∴

$(\underline{\omega} \cdot \nabla)\underline{u} = \omega \frac{1}{R}\frac{\partial}{\partial \phi}\underline{u} = \omega \frac{1}{R}\frac{\partial}{\partial \phi}\left(-\frac{1}{R}\frac{\partial \psi}{\partial z}\hat{\underline{R}} + \frac{1}{R}\frac{\partial \psi}{\partial R}\hat{\underline{z}}\right) =$

$\omega \frac{1}{R}\left(-\frac{1}{R}\right)\frac{\partial \psi}{\partial z}\frac{\partial \hat{\underline{R}}}{\partial \phi} = -\frac{\omega}{R^2}\frac{\partial \psi}{\partial z}\hat{\underline{\phi}}$;

$\underline{u} \cdot \nabla = \left(-\frac{1}{R}\frac{\partial \psi}{\partial z}\hat{\underline{R}} + \frac{1}{R}\frac{\partial \psi}{\partial R}\hat{\underline{z}}\right) \cdot \left(\frac{\partial}{\partial R}\hat{\underline{R}} + \frac{\partial}{\partial z}\hat{\underline{z}}\right) =$

$-\frac{1}{R}\frac{\partial \psi}{\partial z}\frac{\partial}{\partial R} + \frac{1}{R}\frac{\partial \psi}{\partial R}\frac{\partial}{\partial z}$ ∴

$(\underline{u} \cdot \nabla)\underline{\omega} = (\underline{u} \cdot \nabla)\omega \hat{\underline{\phi}} = \hat{\underline{\phi}}\left(-\frac{1}{R}\frac{\partial \psi}{\partial z}\frac{\partial \omega}{\partial R} + \frac{1}{R}\frac{\partial \psi}{\partial R}\frac{\partial \omega}{\partial z}\right)$

$\frac{D}{Dt}\left(\frac{\omega}{R}\right) = \frac{\partial}{\partial t}\left(\frac{\omega}{R}\right) + \underline{u} \cdot \nabla\left(\frac{\omega}{R}\right)$ ∴

$\frac{\partial \underline{\omega}}{\partial t} = -\frac{\omega}{R^2}\frac{\partial \psi}{\partial z}\hat{\underline{\phi}} + \frac{1}{R}\frac{\partial \psi}{\partial z}\frac{\partial \omega}{\partial R}\hat{\underline{\phi}} - \frac{1}{R}\frac{\partial \psi}{\partial R}\frac{\partial \omega}{\partial z}\hat{\underline{\phi}}$ ∴ $\frac{D}{Dt}\left(\frac{\omega}{R}\right) = 0$

\4e/ $\frac{D}{Dt}\left(\frac{\omega}{R}\right) = 0$ ∴ is R changes.

\4f/ $\frac{D}{Dt}\left(\frac{\omega}{R}\right) = 0$ ∴ steady flow ∴ $\frac{\partial}{\partial t}\underline{\omega} = 0$ ∴ $\frac{D}{Dt} = u \cdot \nabla\left(\frac{\omega}{R}\right) = 0$

$\underline{u} \cdot \nabla\left(\frac{\omega}{R}\right) = \nabla \times \left(\frac{\psi}{R}\hat{\underline{\phi}}\right) \cdot \nabla\left(\frac{\omega}{R}\right) = 0 = \left(-\frac{1}{R}\frac{\partial \psi}{\partial z}\hat{\underline{R}} + \frac{1}{R}\frac{\partial \psi}{\partial R}\hat{\underline{z}}\right) \cdot \nabla\left(\frac{\omega}{R}\right)$ ∴

$\nabla \psi \cdot \nabla\left(\frac{\omega}{R}\right) = 0$

\4gi/ stagnation points when $\underline{u} = 0$ ∴ $\underline{u} = \nabla \times \left(\frac{\psi}{R}\hat{\underline{\phi}}\right)$ ∴

$$\underline{u} = \nabla \times \left(-\frac{R}{10}(a^2 - R^2 - z^2)\hat{\underline{\phi}}\right) = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & -\frac{R^2}{10}(a^2 - R^2 - z^2) & 0 \end{vmatrix} =$$

$\frac{1}{R}\left(\hat{\underline{R}}\left(-\frac{2R^2}{10}z\right) + \hat{\underline{z}}\left(-\frac{2R}{10}(a^2 - R^2 - z^2) + 4\frac{R^3}{10}\right)\right) = \underline{u} = (0, 0, 0)$ ∴

$\hat{\underline{R}}\left(-2\frac{R}{10}z\right) + \hat{\underline{z}}\left(-\frac{2}{10}(a^2 - R^2 - z^2) + 4\frac{R^2}{10}\right)$ ∴

\PP2021/ $-\frac{2R}{10}z=0 \therefore z=0,\ R=0 \therefore$

$\cancel{-\frac{2}{10}(a^2)}$ $\therefore$ for $R=0$:

$-\frac{2}{10}(a^2-0^2-z^2)+\frac{4(0)^2}{10}=-\frac{2}{10}a^2+\frac{2}{10}z^2=0=a^2-z^2 \therefore$

$z^2=a^2 \therefore z=\pm a \therefore$

$(0, \phi, \pm a)$

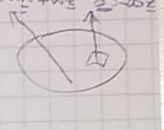
if $z=0$: $-\frac{2}{10}a^2+\frac{2}{10}R^2+\frac{4R^2}{10}=0 \therefore$

$\frac{2}{10}a^2=\frac{6R^2}{10} \therefore 2a^2=6R^2 \therefore \frac{2}{6}a^2=R^2 \therefore$

$R>0 \therefore R=+\sqrt{\frac{2}{6}}a \therefore (+\sqrt{\frac{2}{6}}a, \phi, 0)$

\4gii/ flow speed $=|\underline{u}| \therefore R^2+z^2=a^2$ describes the surface of the sphere $\therefore$ max flow speed at $R=a$

$\underline{u}=u\hat{R}+w\hat{z}$ $\quad \underline{dS}=dS\hat{z}$



\4giii/ volume flux $=\int \underline{u}\cdot\underline{dS}=\dots=\frac{2\pi b^2}{10}(b^2+z^2-a^2)$

\PP2019/ \1ai/ N-S: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla P + \rho \underline{g} + \mu\nabla^2\underline{u}$

\1aii/ $\frac{D}{Dt}\rho = 0 = \frac{\partial \rho}{\partial t} + \underline{u}\cdot\nabla\rho$

\1aiii/ $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\nabla$

\1bi/ ~~[∇×~~ $\nabla\times(\nabla^2\underline{u}) \rightarrow [\nabla\times(\nabla^2\underline{u})]_i = \varepsilon_{ijk}\nabla_j(\nabla^2\underline{u})_k$

~~∇~~ $\underline{\omega} = \nabla\times\underline{u}$ ∴ $\nabla^2\underline{\omega} \rightarrow [\nabla^2\underline{\omega}]_i = \nabla_i^2\omega_i$

\1bii/ ~~[∇×u~~ $\nabla\times\underline{u} \rightarrow [\nabla\times\underline{u}]_i = \varepsilon_{ijk}\nabla_j u_k = \varepsilon_{ijk}\partial_{x_j} u_k =$

\1ci/ $\nabla\times(\psi\underline{k}) = \begin{vmatrix}\hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \psi\end{vmatrix} = \underline{i}(\partial_y\psi - \partial_z(0)) - \hat{\underline{j}}(\partial_x\psi - \partial_z(0)) + \hat{\underline{k}}(0-0) =$

$\frac{\partial}{\partial y}(\psi)\hat{\underline{i}} - \frac{\partial}{\partial x}(\psi)\hat{\underline{j}}$ ∴ $\underline{u} = \frac{\partial}{\partial y}(\psi)\hat{\underline{i}} - \frac{\partial}{\partial x}(\psi)\hat{\underline{j}} + W\underline{k} = U\hat{\underline{i}} + V\hat{\underline{j}} + W\hat{\underline{k}}$

∴ $U = \frac{\partial\psi}{\partial y}$, $V = -\frac{\partial\psi}{\partial x}$

\1cii/ $\underline{\omega} = \nabla\times\underline{u} = \nabla\times\left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, W\underline{k}\right) = \begin{vmatrix}\underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{\partial\psi}{\partial y} & -\frac{\partial\psi}{\partial x} & W\end{vmatrix} =$

$\left(\partial_y W + \frac{\partial^2}{\partial z\partial x}\psi, -(\partial_x W - \frac{\partial^2}{\partial z\partial y}\psi), -\frac{\partial^2}{\partial x^2}\psi - \frac{\partial^2}{\partial y^2}\psi\right) =$

$\left(\frac{\partial W}{\partial y}, -\frac{\partial W}{\partial x}, -\nabla^2\psi\right)$

\1ciii/ $\partial_z \partial_y$ $\nabla\cdot\underline{u} = (\partial_x, \partial_y, \partial_z)\cdot\left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, W\underline{k}\right) =$

$\frac{\partial\psi}{\partial x\partial y} - \frac{\partial\psi}{\partial x\partial y} + \frac{\partial W}{\partial z} = \frac{\partial W}{\partial z} = \frac{\partial}{\partial z}W(x,y) = 0$

$\nabla\cdot\underline{\omega} = \nabla\cdot(\nabla\times\underline{u}) \equiv 0$ ∵ $\nabla\cdot\nabla\times\underline{V} \equiv 0$

\1civ/ ∴ $U = \frac{\partial\psi}{\partial y} = x$, $V = -\frac{\partial\psi}{\partial x} = -y$,

$\frac{dx}{U} = \frac{dy}{V} = \frac{dz}{W} = \frac{dx}{x} = \frac{dy}{-y}$

\1di/ Solid pipe boundary ∴ no slip condition ∴

boundary unmoving stationary ∴ $\underline{u} = 0\hat{R} + 0\hat{\theta} + \omega\hat{z}$ ∴

no slip ∴ $\omega(R=a) = 0$ ∴ $\omega = \omega(R)$ ∴ $\underline{u} = \omega(R)\hat{\underline{z}}$

\diii/ N-S: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho \underline{g} + \mu\nabla^2\underline{u}$ ∴

∴ $\frac{\partial \underline{u}}{\partial t} = 0$, $\nabla\cdot\underline{u} = 0$, $\underline{g} = 0$ ∴ $\underline{u}\cdot\nabla\underline{u} = 0$ ∴

$\nabla\times\underline{u} = \underline{\omega} = 0$ ∴ $\cancel{\nabla^2 u = 0}$ ∴ $0 = -\nabla p + \mu\nabla^2\underline{u}$ ∴

$\nabla p = \mu\nabla^2\underline{u}$

\idiv/ ∴ $\frac{dp}{dz} = \mu\frac{d^2}{dR^2}\omega(R)$ ∴ $\frac{1}{\mu}\frac{dp}{dz} = \frac{d^2}{dR^2}(\omega(R))$ ∴

$\frac{1}{\mu}\frac{dp}{dz}R + \cancel{A} = \frac{d}{dR}(\omega(R))$ ∴ $\frac{1}{2\mu}\frac{dp}{dz}R^2 + AR + B = \omega(R)$

$\omega = \frac{1}{4\mu}\frac{dp}{dz}R^2 + A\log R + B$ ∴ $\frac{d\omega}{dR} = \frac{1}{2\mu}\frac{dp}{dz}R + \frac{A}{R}$,

$\frac{d^2}{dR^2}\omega(R) = \frac{1}{2\mu}\frac{dp}{dz} - A\frac{1}{R^2}$

is $\omega(R=a) = 0$, $\omega(R=a) = 0 = \frac{1}{4\mu}\frac{dp}{dz}a^2 + A\ln(a) + B = 0$

∴ $B = -\frac{1}{4\mu}\frac{dp}{dz}a^2 - A\ln(a)$

\3 aii B Sol/ inviscid means $\nu = 0$ ∴ $\omega = e^{2at}F(\xi)$, $\xi = Re^{at}$

$\frac{\partial\omega}{\partial R} = e^{2at}\frac{\partial F}{\partial R} = e^{2at}\frac{\partial F}{\partial\xi}\frac{\partial\xi}{\partial R} = e^{2at}F'e^{at}$,

$\frac{\partial\omega}{\partial t} = 2ae^{2at}F + e^{2at}\frac{\partial F}{\partial t} = 2ae^{2at} + e^{2at}(F')Re^{at}$ ∴

$\frac{\partial\omega}{\partial t} - aR\frac{\partial\omega}{\partial R} = 2a\omega$

the strength = amplitude = e^{2at} ∴ as t increases, it and ∴ the strength exponentially increases

the scale = $\xi^{-1} = (Re^{at})^{-1}$ ∴ as t increases, it and ∴ the scale exponentially decreases.

\3biA/ $\underline{\omega}\cdot\nabla \rightarrow [\underline{\omega}\cdot\nabla]_i = \omega_i\nabla_i = \omega_i\partial_i$

∴ $(\underline{\omega}\cdot\nabla)\underline{u} \rightarrow [(\underline{\omega}\cdot\nabla)u]_i = \varepsilon_{ijk}(\underline{\omega}\cdot\nabla)_j u_k = \varepsilon_{ijk}\omega_j\partial_j u_k =$

$\omega_j\varepsilon_{ijk}\partial_j u_k = \omega_j(\nabla\times\underline{u})_i$

$\underline{u}\cdot(\underline{\omega}\cdot\nabla)\underline{u} \Longrightarrow [\underline{u}\cdot(\underline{\omega}\cdot\nabla)\underline{u}]_i = u_i((\underline{\omega}\cdot\nabla)\underline{u})_i =$

$u_i\varepsilon_{ijk}(\underline{\omega}\cdot\nabla)_j u_k = u_i\varepsilon_{ijk}\omega_j\partial_j u_k = \omega_j u_i\varepsilon_{ijk}\partial_j u_k =$

$\omega_j u_i[\nabla\times\underline{u}]_i = \omega_j u_i[\nabla\times\underline{u}]_j = \omega_j u_i\varepsilon_{jki}\nabla_k u_i =$

$\omega_j\varepsilon_{jki}\partial_k u_i^2 = \omega_j(\nabla u_i^2)_j \Longrightarrow [\underline{\omega}\cdot\nabla(\frac{u^2}{2})]_i \rightarrow \underline{\omega}\cdot\nabla(\frac{u^2}{2})$

\pp2019/ $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho g + \mu\nabla^2\underline{u}$ $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho g + \mu\nabla^2\underline{u}$

$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{u}) = 0$ ∴ $\frac{\partial \rho}{\partial t} + \rho\nabla\cdot\underline{u} + \underline{u}\cdot\nabla\rho = 0 = \left(\frac{\partial \rho}{\partial t} + \underline{u}\cdot\nabla\rho\right) + \rho\nabla\cdot\underline{u} = 0$

$) = \frac{D\rho}{Dt} + \rho\nabla\cdot\underline{u} = 0$ ∴ ~~in~~compressible ∴ $\frac{D\rho}{Dt} = 0$ ∴ $\rho\nabla\cdot\underline{u} = 0$ ∴

$\nabla\cdot\underline{u} = 0$

$\frac{\partial \rho}{\partial t} + \nabla\cdot(\rho\underline{u}) = 0$

\ai/ write the whole navier stokes equation

\aii/ very simplified equation of continuity equation ∵ $\frac{D\rho}{Dt} = 0$ for incompressible fluid

\aiii/ material derivative acting on nothing

\bi/ i^{th} component of curl of vec ∴ first ~~find~~ k^{th} component of grad squared u_i and shows by using grad squared is a scalar operator ∴ ~~so~~ ith comp of curl of grad squared of u is elpison delta, grad squared = epsilon delta introduce subscript m

\bi/ $\nabla\times\underline{f} \rightarrow \cancel{[\nabla\times(\nabla^2\underline{u})]_i}$ $[\nabla\times\underline{f}]_i = \varepsilon_{ijk}\nabla_j f_k = \varepsilon_{ijk}\partial_j f_k$,

here $\underline{f} = \nabla^2\underline{u}$ ∴ ⇒ $f_k = (\nabla^2\underline{u})_k = \nabla^2 u_k$ ∵ $\nabla^2 = \partial_{xx} + \partial_{yy} + \partial_{zz}$ ∴ is a scalar operator

∴ $\nabla\times(\nabla^2\underline{u}) \rightarrow [\nabla\times(\nabla^2\underline{u})]_i = \varepsilon_{ijk}\nabla_j(\nabla^2\underline{u})_k = \varepsilon_{ijk}\partial_j(\nabla^2\underline{u})_k =$

$\varepsilon_{ijk}\partial_j(\nabla^2 u_k) = \varepsilon_{ijk}\partial_j(\partial_{mm}u_k) = \partial_{mm}(\varepsilon_{ijk}\partial_j u_k) = \partial_{mm}[\nabla\times\underline{u}]_i = \partial_{mm}[\underline{\omega}]_i =$

$\partial_{mm}[\omega_i] = \partial_{mm}\omega_i = [\nabla^2\omega_i) = [\nabla\underline{\omega}]_i \rightarrow \nabla^2\underline{\omega}$

\bii/ $\underline{u} = (2ax_1, a^2x_2, x_3)$ ∴ $e_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \frac{1}{2}(2)\frac{\partial u_1}{\partial x_2} = \frac{\partial u_1}{\partial x_2} = \frac{\partial}{\partial x_1}(2ax_1) = 2a$

$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_2}(2ax_1) + \frac{\partial}{\partial x_1}(a^2x_2)\right) = \frac{1}{2}(0+0) = 0$,

$e_{22} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right) = \frac{\partial u_2}{\partial x_2} = \frac{\partial}{\partial x_2}(a^2x_2) = a^2$,

$e_{33} = \frac{\partial u_3}{\partial x_3} = \frac{\partial}{\partial x_3}(x_3) = 1$, $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial j} + \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}(0+0) = 0$ for $i \neq j$ ∴

diagonal matrix e_{ij} with entries $2a, a^2, 1$ on diagonal, 0 elsewhere $\begin{bmatrix} 2a & 0 & 0 \\ 0 & a^2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

∴ find a for which u is incompressible means use e_{ij} ∴

$\nabla\cdot\underline{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = e_{ii} = e_{11} + e_{22} + e_{33} = 2a + a^2 + 1 = 0 = (a+1)^2$ ∴

$a = -1$ ∴

$\underline{u} = (-2x_1, x_2, x_3)$ ∴ $\nabla\cdot\underline{u} = \frac{\partial}{\partial x_1}(-2x_1) + \frac{\partial}{\partial x_2}(x_2) + \cancel{\partial_3}\,\partial_{x_3}(x_3) =$

$-2\cancel{x}+1+1 = 0 = \nabla\cdot\underline{u}$ ∴ $a = -1$ is correct

\1biiB/ $\nabla \times \underline{u} = \nabla \times (u_1, u_2, u_3) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial x_1 & \partial x_2 & \partial x_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = \cancel{\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x}}$

$\underline{i}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) - \underline{j}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) + \underline{k}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = \underline{i}\,2\xi_{32} - \underline{j}\,2\xi_{31} + \underline{k}\,2\xi_{21}$

$= \underline{i}\,2\xi_{32} + \underline{j}\,2(-\xi_{31}) + \underline{k}\,2\xi_{21} = \underline{i}\,2 - \underline{j} + 2\xi_{21}\underline{k} = 2\underline{i} - \underline{j} + 0\underline{k}$

\1ci/ $\nabla \times (\psi \underline{k})$ $\therefore U = \frac{\partial \psi}{\partial y}, V = -\frac{\partial \psi}{\partial x}$ $\therefore \nabla \times (\psi \underline{k}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \psi \end{vmatrix} =$

$\underline{i}\left(\frac{\partial}{\partial y}\psi - \frac{\partial}{\partial z}(0)\right) - \underline{j}\left(\frac{\partial}{\partial x}\psi - \frac{\partial}{\partial z}(0)\right) + \underline{k}\left(\partial_x(0) - \partial_y(0)\right) = \underline{i}\frac{\partial \psi}{\partial y} - \frac{\partial \psi}{\partial x}\underline{j} = U\underline{i} + V\underline{j}$

$\therefore U = \frac{\partial \psi}{\partial y}, \quad -\frac{\partial \psi}{\partial x} = V$

\1cii/ $\underline{\omega} = \nabla \times \underline{u} = \nabla \times \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, W\right) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{\partial \psi}{\partial y} & -\frac{\partial \psi}{\partial x} & W \end{vmatrix} =$

$\left[\frac{\partial}{\partial y}(W) - \frac{\partial}{\partial z}\left(-\frac{\partial \psi}{\partial x}\right)\right]\underline{i} - \underline{j}\left[\frac{\partial}{\partial x}(W) - \frac{\partial}{\partial z}\left(\frac{\partial \psi}{\partial y}\right)\right] + \underline{k}\left[\frac{\partial}{\partial x}\left(-\frac{\partial \psi}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)\right] =$

$\left(\frac{\partial W}{\partial y} + \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial z}\right)\right)\underline{i} - \underline{j}\left[\frac{\partial W}{\partial x} - \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial z}\right)\right] + \underline{k}\left(-\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial y \partial y}\right)\right) =$

$\left(\frac{\partial W}{\partial y} + 0\right)\underline{i} - \underline{j}\left(\frac{\partial W}{\partial x} - 0\right) + -\underline{k}\left(\nabla^2 \psi\right) = \frac{\partial W}{\partial y}\underline{i} - \frac{\partial W}{\partial x}\underline{j} - \nabla^2 \psi\, \underline{k} =$

$\left(\frac{\partial W}{\partial y}, -\frac{\partial W}{\partial x}, -\nabla^2 \psi\right) = \left(\frac{\partial W}{\partial y}, -\frac{\partial W}{\partial x}, \omega\right) = \underline{\Omega}$

\1ciii/ $\nabla \cdot \underline{u} = \nabla \cdot (U, V, W) = \frac{\partial U}{\partial x} + \frac{\partial}{\partial y}V + \frac{\partial}{\partial z}W = \frac{\partial}{\partial x}\left(\frac{\partial \psi}{\partial y}\right) + \frac{\partial}{\partial y}\left(-\frac{\partial \psi}{\partial x}\right) + \frac{\partial W}{\partial z}$

$\therefore$ W indep of z $\therefore \quad = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial x \partial y} + 0 = 0$

$\overset{\frown}{\text{Edy}}$ vortex lines have $\underline{\omega}$ as tangents just like streamlines have $\underline{u}$ as tangents $\therefore$ circles

$\nabla \cdot \underline{\Omega} = \nabla \cdot (\nabla \times \underline{u}) = \nabla \cdot \left(\frac{\partial W}{\partial y}, -\frac{\partial W}{\partial x}, \omega\right) = \cancel{\nabla Z}\frac{\partial}{\partial x}\frac{\partial W}{\partial y} - \frac{\partial^2 W}{\partial y \partial x} + \frac{\partial}{\partial z}\omega =$

$\frac{\partial^2 W}{\partial x \partial y} - \frac{\partial^2 W}{\partial x \partial y} + \frac{\partial}{\partial z}\left(-\nabla^2 \psi\right) = 0 + -\frac{\partial}{\partial z}\left(\nabla^2 \psi(x,y)\right) = -\nabla^2 \frac{\partial}{\partial z}\psi(x,y) = -\nabla^2(0) = 0$

\1civ/ vortex lines have $\underline{\omega}$ as tangents $\therefore$

$\underline{\omega} = \left(\frac{\partial W}{\partial y}, -\frac{\partial W}{\partial x}, \omega\right) = \left(\frac{\partial}{\partial y}(x^2+y^2), -\frac{\partial}{\partial x}(x^2+y^2), -\nabla^2(xy)\right) =$

$\left(2y, -2x, -\left(\frac{\partial^2}{\partial x \partial x}(xy) + \frac{\partial^2}{\partial y \partial y}(xy)\right)\right) = (2y, -2x, 0) \quad \therefore$

is $\frac{dx}{2y} = \frac{dy}{-2x} \quad \therefore \quad -2x\,dx = 2y\,dy \quad \therefore \int -2x\,dx = \int 2y\,dy = -x^2 + C = y^2 \quad \therefore$

$C = y^2 + x^2 = r^2 \quad \therefore$ vortex lines form the shapes of circles

\1di/ The flow is driven by the difference in pressure along the z-direction, so $\underline{u} = W\hat{\underline{z}}$ now need to explain why ~~W~~ $W = W(R)$ only

$\therefore$ BCs at $R = a$ cause $W = W(R)$, W indep of z, ϕ, t, only dependent of stationary non slip boundary.

\PP2019/ \1dii/ ∴ show $\nabla \cdot \underline{u} = 0$ ∴ $\underline{u} = W(R)\hat{\underline{z}} \doteq 0\hat{\underline{R}} + 0\hat{\underline{\phi}} + W(R)\hat{\underline{z}}$

∴ $\nabla \cdot \underline{u} = \nabla \cdot (0\hat{\underline{R}} + 0\hat{\underline{\phi}} + W(R)\hat{\underline{z}}) = \frac{1}{R}\frac{\partial}{\partial R}(R(0)) + \frac{1}{R}\frac{\partial}{\partial \phi}(0) + \frac{\partial}{\partial z}(W(R)) =$

$\frac{1}{R}(0) + \frac{1}{R}(0) + 0 = 0$

\1diii/ N-S: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla P + \rho \underline{g} + \mu \nabla^2 \underline{u}$ ∴

$\underline{u} = W(R)\hat{\underline{z}}$ ∴ $\frac{\partial \underline{u}}{\partial t} = 0$, $\nabla = \hat{\underline{R}}\frac{\partial}{\partial R} + \hat{\underline{\phi}}\frac{1}{R}\frac{\partial}{\partial \phi} + \hat{\underline{z}}\frac{\partial}{\partial z}$ ∴

$\underline{u}\cdot\nabla = W(R)\frac{\partial}{\partial z}$ ∴ $\underline{u}\cdot\nabla\underline{u} = (\underline{u}\cdot\nabla)W(R)\hat{\underline{z}} = W(R)\frac{\partial}{\partial z}(W(R)\hat{\underline{z}}) =$

$\hat{\underline{z}}W(R)\frac{\partial}{\partial z}(W(R)) = \hat{\underline{z}}W(R)(0) = 0$ ∴

$\rho(0+0) = 0 = -\nabla P + \rho\underline{g} + \mu\nabla^2\underline{u}$, $\underline{g} = 0$ ∴ $0 = -\nabla P + \mu\nabla^2\underline{u}$ ∴

$\nabla P = \mu\nabla^2\underline{u}$

\1div/ $\nabla P = \mu\nabla^2\underline{u}$ ∴ z-component: $\frac{\partial P}{\partial z} = (\nabla^2\underline{u})_z$ ∴

not cartesian coords ∴ curvilinear coords: identity (viii):

$\nabla^2\underline{u} = \nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{u})$ ∴ $\nabla\times\underline{u} = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & 0 & W(R) \end{vmatrix} =$

$\frac{-1}{R}\hat{\underline{\phi}}\left(\frac{\partial W(R)}{\partial R}\right) = \frac{-1}{R}\frac{\partial}{\partial R}(W(R))\hat{\underline{\phi}} \doteq \frac{-1}{R}\frac{\partial W}{\partial R}\hat{\underline{\phi}}$ ∴

$\nabla\times(\nabla\times\underline{u}) = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ \cancel{\frac{1}{R}\frac{\partial W}{\partial R}} & -\frac{\partial W}{\partial R} & 0 \end{vmatrix} = \frac{1}{R}\hat{\underline{z}}\left(\frac{\partial}{\partial R}\left(-\frac{\partial W}{\partial R}\right)\right)$ X

$\nabla^2\underline{u} = \nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{u})$ ∴

$\nabla\cdot\underline{u} = 0$ ∴ $\nabla(\nabla\cdot\underline{u}) = \nabla(0) = 0$ ∴

$\nabla\times\underline{u} = \nabla\times(W(R)\hat{\underline{z}}) = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & 0 & W(R) \end{vmatrix} = \frac{1}{R}R\hat{\underline{\phi}}\left(\frac{\partial W}{\partial R}\right) = -\hat{\underline{\phi}}\frac{\partial W}{\partial R} = -\frac{\partial W}{\partial R}\hat{\underline{\phi}}$

∴ $\nabla\times(\nabla\times\underline{u}) = \nabla\times\left(-\frac{\partial W}{\partial R}\hat{\underline{\phi}}\right) = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & -R\frac{\partial W}{\partial R} & 0 \end{vmatrix} = \frac{1}{R}\hat{\underline{z}}\left(\frac{\partial}{\partial R}\left(-R\frac{\partial W}{\partial R}\right)\right) =$

$\frac{-1}{R}\hat{\underline{z}}\left(\frac{\partial}{\partial R}\left(R\frac{\partial W}{\partial R}\right)\right) = \frac{-1}{R}\hat{\underline{z}}\left(\frac{\partial W}{\partial R} + R\frac{\partial^2 W}{\partial R^2}\right) = \left(-\frac{1}{R}\frac{\partial W}{\partial R} - \frac{\partial^2 W}{\partial R^2}\right)\hat{\underline{z}}$ ∴

$\nabla^2\underline{u} = \left(\frac{1}{R}\frac{\partial W}{\partial R} + \frac{\partial^2 W}{\partial R^2}\right)\hat{\underline{z}} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial W}{\partial R}\right)\hat{\underline{z}}$ ∴

$\frac{\partial P}{\partial z} = \mu\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial W}{\partial R}\right)$ ∴

$R\frac{\partial P}{\partial z}\frac{1}{\mu} = \frac{\partial}{\partial R}\left(R\frac{\partial W}{\partial R}\right)$ ∴ $\frac{1}{2\mu}\frac{\partial P}{\partial z}R^2 + A = R\frac{\partial W}{\partial R}$ ∴

$\frac{1}{2\mu}\frac{\partial P}{\partial z}R + AR^{-1} = \frac{\partial W}{\partial R}$ ∴ $\frac{1}{4\mu}\frac{\partial P}{\partial z}R^2 = A\ln(R) + B = W = W(R)$

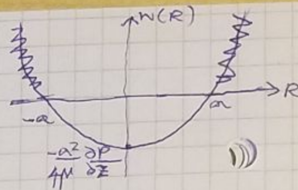
let $\left.\frac{\partial W}{\partial R}\right|_{R=0} = 0$ ∴ $\cancel{0 =}$ $\frac{1}{2\mu}\frac{\partial P}{\partial z}R^2 + A = R\frac{\partial W}{\partial R}$ ∴ at $R=0$:

$\frac{1}{2\mu}\frac{\partial P}{\partial z}(0)^2 + A = (0)\frac{\partial W}{\partial R} = 0 = 0 + A = A = 0$ ∴ $W = \frac{1}{4\mu}\frac{\partial P}{\partial z}R^2 + B$ ∴

let $W(R=a) = 0$ ∴ $0 = \frac{1}{4\mu}\frac{\partial P}{\partial z}a^2 + B$ ∴ $B = -\frac{1}{4\mu}\frac{\partial P}{\partial z}a^2 = \frac{-a^2}{4\mu}\frac{\partial P}{\partial z}$ ∴

$W(R) = \frac{1}{4\mu}\frac{\partial P}{\partial z}R^2 - \frac{a^2}{4\mu}\frac{\partial P}{\partial z}$

\1dV/ $w(R) = \frac{1}{4\mu}\frac{\partial p}{\partial z}R^2 - \frac{a^2}{4\mu}\frac{\partial p}{\partial z}$ ∴



if $w(R)=0$: $\frac{a^2}{4\mu}\frac{\partial p}{\partial z} = \frac{1}{4\mu}\frac{\partial p}{\partial z}R^2$ ∴ $a^2=R^2$

∴ $R = \pm a$

$R=0$ $\quad w(0) = -\frac{a^2}{4\mu}\frac{\partial p}{\partial z}$ ∴

Max speed at $R=0$ is $\frac{-a^2}{4\mu}\frac{\partial p}{\partial z} = w(R)$

$\underline{dS} = \hat{\underline{n}}\,dS$ ∴ $\hat{\underline{n}} = \hat{\underline{k}}$ ∴

Q_z if $Q = \int \underline{u}\cdot d\underline{S} = \int w(R)\hat{\underline{z}}\cdot d\underline{S} = \int \left(\frac{1}{4\mu}\frac{\partial p}{\partial z}R^2 - \frac{a^2}{4\mu}\frac{\partial p}{\partial z}\right)\hat{\underline{z}}\cdot d\underline{S}$

$= \frac{1}{4\mu}\frac{\partial p}{\partial z}\int (R^2 - a^2)\hat{\underline{z}}\cdot d\underline{S}$ ∴ if $d\underline{S} = R\hat{\underline{z}}\,dR\,d\phi$ ∴

$\frac{1}{4\mu}\frac{\partial p}{\partial z}\int_0^{2\pi}\int_0^a (R^2-a^2)\hat{\underline{z}}\cdot(R\hat{\underline{z}}\,dR\,d\phi) = \frac{1}{4\mu}\frac{\partial p}{\partial z}\int_0^{2\pi}\int_0^a R^3 - a^2R\,dR\,d\phi =$

$\frac{1}{4\mu}\frac{\partial p}{\partial z}\int_0^{2\pi}\left[\frac14R^4 - \frac12a^2R^2\right]_0^a d\phi = \frac{1}{4\mu}\frac{\partial p}{\partial z}\int_0^{2\pi}\frac14[a^4] - \frac12a^2[a^2]\,d\phi =$

$\frac{1}{4\mu}\frac{\partial p}{\partial z}\int_0^{2\pi} -\frac14a^4\,d\phi = \frac{1}{4\mu}\frac{\partial p}{\partial z}\left(-\frac14a^4\right)[\phi]_0^{2\pi} = \frac{-a^4}{16\mu}\frac{\partial p}{\partial z}2\pi = \frac{-a^4\pi}{8\mu}\frac{\partial p}{\partial z}$

\1ei/ $\phi = ae^{i(kx-\omega t)}$ ∴ $\phi_t = -\omega i a e^{i(kx-\omega t)}$ ∴ $\phi_{tt} = \omega^2(i)^2ae^{i(kx-\omega t)} =$
$-\omega^2ae^{i(kx-\omega t)}$, $\phi_x = kiae^{i(kx-\omega t)}$ ∴ $\phi_{xx} = k^2(i)^2ae^{i(kx-\omega t)} =$
$-k^2ae^{i(kx-\omega t)}$ ∴ into PDE: $-\omega^2ae^{i(kx-\omega t)} = -c^2k^2ae^{i(kx-\omega t)}$ ∴
$\omega^2 = c^2k^2$ ∴ $\omega = \pm\sqrt{c^2k^2} = \pm ck$ ✓

\1eii/ $C_p = \omega/k$ $\quad C_g = \frac{d\omega}{dk}$. C_p is phase speed ∴

∴ $\omega = \pm\sqrt{gk} = \pm\sqrt{g}\,k^{1/2}$ ∴ $\omega/k = C_p = \pm\sqrt{g}\,k^{1/2}/k = \pm\sqrt{g}\frac{1}{\sqrt{k}} = \pm\sqrt{\frac{g}{k}}$

group speed is $C_g = \frac{d\omega}{dk}$ ∴ $\frac1g\omega^2 = k$ ∴ $\frac{dk}{d\omega} = \frac1g 2\omega$ ∴

$\frac{d\omega}{dk} = C_g = 1/\left(\frac{dk}{d\omega}\right) = \frac{g}{2\omega}$,

$\frac{d}{dk}((\omega)^2) = \frac{d}{dk}(gk) = g = 2\omega\frac{d\omega}{dk}$ ∴ $\frac{d\omega}{dk} = \frac{g}{2\omega}$

∴ waves are dispersive if waves of different wavelength λ (and ∴ wave number k) travel at different phase speeds

∴ $C_p = \pm\sqrt{\frac{g}{k}}$ ∴ C_p is a function of k ∴ waves are dispersive

\2a/ N-S: $\rho\left(\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho g + \mu\nabla^2\underline{u}$, $\nabla\cdot\underline{u} = 0$ ∴

∴ change of variables: $x' = \frac{x}{L}$, $t' = \frac{t}{T}$, $u' = \frac{u}{U}$ ∴

\PP2019/ $\frac{dx'}{dx} = \frac{1}{L} \therefore dx' = \frac{1}{L}dx$, $\frac{\partial t'}{\partial t} = \frac{1}{T} \therefore \partial t' = \frac{1}{T}\partial t$, $\therefore \frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x'} \therefore$

$\therefore \nabla = \frac{1}{L}\nabla'$, $\frac{\partial}{\partial t} = \frac{1}{T}\frac{\partial}{\partial t'} \therefore R_e = \frac{UL}{\nu} \therefore \frac{\mu}{\rho} = \nu \therefore$

$\therefore$ N-S: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \mu\nabla^2\underline{u} \therefore$

$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}) = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\underline{u} \therefore$

$U = \frac{L}{T} \therefore U = \frac{u}{u'} = \frac{x/x'}{t/t'} = \frac{xt'}{x't} = \frac{u}{u'} \therefore \frac{u't'}{x'} = \frac{ut}{x} \therefore$

$\underline{u} \sim U$, $x \sim L$, $\cancel{t \sim T} \therefore x' = \frac{x}{L} \therefore \frac{\partial x'}{\partial x} = \frac{1}{L} \therefore \frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x'} \therefore$

$\cancel{\frac{\partial u}{\partial t}}$ $\nabla = \frac{1}{L}\nabla'$ $\therefore \nabla \sim \frac{1}{L}$, $T = \frac{L}{U} \therefore \frac{\partial}{\partial t} \sim \frac{1}{T} = \frac{U}{L} \therefore$

$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}(\underline{u}) \approx \frac{U}{L}U = \frac{U^2}{L}$, $\therefore \underline{u}\cdot\nabla \approx U\frac{1}{L} \therefore \underline{u}\cdot\nabla\underline{u} = U\frac{1}{L}U = \frac{U^2}{L}$

$\cancel{\nabla^2\underline{u} = \nabla\cdot\nabla\times}$ $\nabla\cdot\nabla\omega = \nabla^2\omega$ $\quad \nabla^2\underline{u} = \nabla(\nabla\underline{u})$ n $\therefore$

$\nabla^2\underline{u} \sim \frac{1}{L}\frac{1}{L}U = \frac{U}{L^2} \therefore$ for $R_e << 1 \therefore$

$\frac{\mu}{\rho}\nabla^2\underline{u} = \nu\nabla^2\underline{u} \sim \nu\frac{U}{L^2} \therefore$

$\frac{U^2}{L} \sim \frac{U^2}{L} \therefore \frac{\partial u}{\partial t} \sim \underline{u}\cdot\nabla\underline{u} \therefore \frac{(\partial \underline{u}/\partial t)}{\nu\nabla^2\underline{u}} \sim \frac{U^2/L}{\nu U/L^2} = \frac{UL}{\nu} = R_e \therefore$

for $R_e << 1$: $\frac{UL}{\nu} << 1 \therefore$

$\frac{\partial u}{\partial t}, \underline{u}\cdot\nabla\underline{u} << \nu\nabla^2\underline{u} = \frac{\mu}{\rho}\nabla^2\underline{u} \therefore$

may neglect $\frac{\partial u}{\partial t}$, $\underline{u}\cdot\nabla\underline{u} \therefore 0 = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\underline{u} \therefore$

$0 = -\nabla p + \mu\nabla^2\underline{u}$, $\nabla\cdot\underline{u} = 0$

\2b/ $\nabla^4\psi = \nabla^2(\nabla^2\psi) \therefore \cancel{\nabla\times}$ $0 = -\nabla p + \mu\nabla^2\underline{u} \therefore$

$\nabla\times(0) = 0 = \nabla\times(-\nabla p + \mu\nabla^2\underline{u}) \therefore$

is $\cancel{\quad}$ $\nabla^2(\nabla\times\underline{u}) = 0$; $\underline{u} = \nabla\times(\psi\hat{\underline{z}}) \therefore$

$\nabla^2(\nabla\times\underline{u}) = 0 = \nabla^2(\nabla\times(\nabla\times(\psi\hat{\underline{z}}))) \therefore$

$\nabla\times(\nabla\times(\psi\hat{\underline{z}})) = \nabla(\nabla\cdot\underline{u}) - \nabla^2\underline{u} \therefore \nabla\cdot\underline{u} = 0 \therefore$

$\nabla\times(\nabla\times(\psi\hat{\underline{z}})) = \nabla(0) - \nabla^2\underline{u} = -\nabla^2\underline{u} \therefore$

$0 = \nabla^2(-\nabla^2\underline{u}) = -\nabla^2(\nabla^2\underline{u}) = 0 \therefore \nabla^2(\nabla^2\underline{u}) = 0 = \nabla^4\underline{u}$

$\nabla^2(\nabla\times\underline{u}) = \nabla(\nabla\cdot(\nabla\times\underline{u})) - \nabla\times(\nabla\times(\nabla\times\underline{u}))$

\2b/ $0 = -\nabla p + \mu\nabla^2\underline{u}$, $\nabla\cdot\underline{u} = 0 \therefore \nabla p = \mu\nabla^2\underline{u} \therefore$ curl

$\cancel{\nabla\times(0) = 0 = \nabla\times(-\nabla p + \mu}$ $\nabla\times(\nabla p) = 0 = \nabla\times(\mu\nabla^2\underline{u}) = \mu\nabla\times(\nabla^2\underline{u})\therefore$

$\therefore 0 = \mu \nabla \times (\nabla^2 \underline{u}) \therefore \nabla \times (\nabla^2 \underline{u}) = 0 \therefore$

$\nabla^2 (\nabla \times \underline{u}) = 0$ $\because \nabla^2$ is a linear operator $\therefore$

$\underline{u} = \nabla \times (\psi \hat{\underline{z}}) \therefore$

$\nabla \times \underline{u} = \nabla \times (\underline{u}) = \nabla \times (\nabla \times (\psi \hat{\underline{z}})) = \nabla(\nabla \cdot (\nabla \times (\psi \hat{\underline{z}}))) - \nabla^2 (\nabla \times (\psi \hat{\underline{z}})) =$

$\nabla(\nabla \cdot \underline{u}) - \nabla^2(\nabla \times (\psi \hat{\underline{z}})) = -\nabla^2(\nabla \times (\psi \hat{\underline{z}})) = -\nabla^2 \underline{u}$

$\nabla^2(\nabla \times \underline{u}) = \nabla^2[-\nabla^2 \underline{u}] = 0 = -\nabla^2(\nabla^2 \underline{u}) = 0 = \nabla^2(\nabla^2 \underline{u})$

\2b/ $\cancel{\nabla p}$ $0 = -\nabla p + \mu \nabla^2 \underline{u} \therefore \nabla p + \mu \nabla^2 \underline{u} \therefore$

$\nabla \times (\nabla p) = 0 = \nabla \times (\mu \nabla^2 \underline{u}) = \mu \nabla \times (\nabla^2 \underline{u}) = 0 = \nabla \times (\nabla^2 \underline{u}) = \nabla^2 (\nabla \times \underline{u}) = 0$

$= \nabla^2 [\nabla \times (\nabla \times (\psi \hat{\underline{z}}))] = \cancel{\nabla^2 [\hat{z}(\nabla \cdot (\nabla \times (\psi \hat{z}))}$

$\nabla^2 [\nabla(\nabla \cdot (\psi \hat{\underline{z}})) - \nabla^2 (\psi \hat{\underline{z}})] = -\nabla^2[\nabla^2(\psi)] = 0 = \nabla^2(\nabla^2(\psi)) = \nabla^4 \psi$

\2c/ $\underline{u} = \nabla \times (\psi \hat{\underline{z}}) = \underline{u} = \nabla \times (f(\phi)\hat{\underline{z}}) = \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\phi} & \hat{z} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & 0 & f(\phi) \end{vmatrix} =$

$\frac{1}{R}\hat{\underline{R}}\left(\frac{\partial}{\partial \phi}(f(\phi))\right) = \frac{1}{R}\hat{\underline{R}} f'(\phi) = \frac{f'(\phi)}{R}\hat{\underline{R}} = \underline{u} = u(R,\phi)\hat{\underline{R}}$ for $u(R,\phi) = \frac{f'(\phi)}{R}$

$\therefore \underline{u}$ only has a component along $\hat{\underline{R}}$ $\therefore$ it is radial flow.

\2d/ $\nabla^4 \psi = \nabla^2(\nabla^2 \psi) = \nabla^2(\nabla^2 f(\phi)) = \nabla^2\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}f(\phi)\right) + \frac{1}{R^2}\frac{\partial^2}{\partial \phi^2}f(\phi) + \frac{\partial^2 f(\phi)}{\partial z^2}\right]$

$= \nabla^2\left[\frac{1}{R}\frac{\partial}{\partial R}(R \times 0) + \frac{1}{R^2}f''(\phi) + 0\right] = \nabla^2\left[\frac{1}{R^2}f''(\phi)\right]$

$\frac{1}{R}\frac{\partial}{\partial R}\left(R \times \frac{\partial}{\partial R}\left(\frac{1}{R^2}f''(\phi)\right)\right) + \frac{1}{R^2}\frac{\partial^2}{\partial \phi^2}\left(\frac{1}{R^2}f''(\phi)\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{R}f''(\phi)\right) =$

$\cancel{\frac{1}{R}\frac{\partial}{\partial R}} \frac{1}{R}\frac{\partial}{\partial R}\left(R(-2)\frac{1}{R^{3}}f''(\phi)\right) + \frac{1}{R^4}f''''(\phi) + 0 =$

$\frac{-2}{R}f''(\phi)\frac{\partial}{\partial R}\left(\frac{1}{R^2}\right) + \frac{1}{R^4}f''''(\phi) =$

$\frac{-2}{R}f''(\phi)(-2)\frac{1}{R^3} + \frac{1}{R^4}f''''(\phi) =$

$\frac{4}{R^4}f''(\phi) + \frac{1}{R^4}f''''(\phi) = 0 = f''''(\phi) + 4f'' = f'''' + 4f''$

\2e/ let $f(\phi) = e^{m\phi} \therefore$ let $f'' = g \therefore f^{(4)} = g'' \therefore$

$g'' + 4g = 0 \therefore q^2 + 4 = 0 \therefore q^2 = -4 \therefore q = \pm 2i \therefore$

$g(\phi) = E\sin(2\phi) + F\cos(2\phi) = f''(\phi) \therefore$

$\cancel{g'}$ $f'(\phi) = -\frac{1}{2}E\cos(2\phi) + \frac{1}{2}F\sin(2\phi) + B \therefore$

$f(\phi) = -\frac{1}{4}E\sin(2\phi) + \frac{1}{4}F\cos(2\phi) + B\phi + A =$

$A + B\phi + C\sin(2\phi) + D\sin(2\phi) = f = \psi \therefore f' = B + 2C\sin(2\phi) + 2D\cos(2\phi)$

$u = u(R,\phi) = \frac{f'(\phi)}{R} = B\frac{1}{R} + 2C\frac{1}{R}\sin(2\phi) + 2D\frac{1}{R}\cos(2\phi) \therefore$

$R\,u(R,\phi) = B + 2C\sin(2\phi) + 2D\cos(2\phi) \therefore$

\pp2019/ $\therefore$ at $\phi = \alpha$, $u = 0 = R(0) = 0 = B + 2C\sin(2\alpha) + 2D\cos(2\alpha)$

$\therefore$ at $\phi = -\alpha$; $u = 0 = R(0) = 0 = B + 2C\sin(2(-\alpha)) + 2D\cos(2(-\alpha)) =$

$\cancel{B \neq}$ $B - 2C\sin(2\alpha) + 2D\cos(2\alpha) = 0$

$\therefore 2D\cos(2\alpha) = -B + 2C\sin(2\alpha) \therefore$

$\cancel{D\cos(2\alpha)} = -\frac{1}{2}B\frac{1}{\cos(2\alpha)} + C\cdot\frac{\sin(2\alpha)}{\cos(2\alpha)} \therefore$

$0 = B + 2C\sin(2\alpha) + (-B + 2C\sin(2\alpha)) = 4C\sin(2\alpha) = 0 \therefore C = 0 \therefore$

$\therefore Ru = B + 2D\cos(2\phi) \therefore$

\2e/ $Ru(R, \phi) = B + 2C\sin(2\phi) + 2D\cos(2\phi) \therefore$

at $\phi = \alpha$: $0 = B - 2C\sin(2\alpha) + 2D\cos(2\alpha)$,

$0 = B + 2C\sin(2\alpha) + 2D\cos(2\alpha) \therefore$

$\sin(2\alpha)2C = B + 2D\cos(2\alpha) \therefore$

$\cancel{0 = B + 2C\sin(2\alpha}$ $0 = B + B + 2D\cos(2\alpha) + 2D\cos(2\alpha) =$

$2B + 4D\cos(2\alpha) = 0 = B + 2D\cos(2\alpha)$

$2D\cos(2\alpha) = -B + 2C\sin(2\alpha) \therefore$

$0 = B + -B + 2C\sin(2\alpha) = 0 = 2C\sin(2\alpha) = C = 0 \therefore$

$Ru = B + 2D\cos(2\phi) \therefore$

at $\phi = -\alpha$: $u = 0$: $0 = B + 2D\cos(-2\alpha) = B + 2D\cos(2\alpha) \therefore$

$B = -2D\cos(2\alpha) \therefore$

$\cancel{u = Ct}$ $Ru = C\cos(2\phi) - \cos(2\alpha) \therefore$

$Ru = -2D\cos(2\alpha) + 2D\cos(2\phi) = 2D(\cos(2\phi) - \cos(2\alpha)) =$

$C(\cos(2\phi) - \cos(2\alpha)) \therefore$

$u = CR^{-1}(\cos(2\phi) - \cos(2\alpha))$

\2f/ radially inwards flow has $u < 0$ $\therefore \underline{u} = u\hat{\underline{R}} = (-v_e)\hat{\underline{R}} \therefore$

$u = C(\cos 2\phi - \cos 2\alpha)\frac{1}{R} \therefore C < 0 \therefore$ for $u < 0$:

$\cos 2\phi - \cos 2\alpha > 0 \therefore \cos 2\phi > \cos 2\alpha \therefore$

$\cancel{\cos\ \text{let}}$ $\arccos(\cos 2\phi) > 2\alpha \therefore 2\phi > 2\alpha \therefore \cancel{\phi >}$ $\alpha < \phi$

for $\phi \in [0, 2\pi)$

\3ai/ $\therefore \nabla\cdot\underline{u}=0$

N-S: $\rho\left(\frac{\partial \underline{u}}{\partial t}+\underline{u}\cdot\nabla\underline{u}\right)=-\nabla P+\rho \underline{g}+\mu\nabla^2\underline{u}$

$\frac{\partial \rho}{\partial t}+\nabla\cdot(\rho\underline{u})=0 \quad \rho=\text{constant} \therefore$

$\frac{\partial \rho}{\partial t}=0 \therefore \nabla\cdot(\rho\underline{u})=0=\rho\nabla\cdot\underline{u}+\underline{u}\cdot\nabla\rho \therefore$

$\nabla\rho=0 \therefore \underline{u}\cdot\nabla\rho=\nabla\cdot(0)=0 \therefore \rho\nabla\cdot\underline{u}=0=\nabla\cdot\underline{u}$

$\therefore \frac{\partial \underline{u}}{\partial t}+\underline{u}\cdot\nabla\underline{u}=-\frac{1}{\rho}\nabla P+\underline{g}+\frac{\mu}{\rho}\nabla^2\underline{u}=-\frac{1}{\rho}\nabla P-\nabla\Pi+\nu\nabla^2\underline{u} \therefore$

$=\nabla\left(-\frac{1}{\rho}P\right)-\nabla\Pi+\nu\nabla^2\underline{u} \therefore$

$\nabla\times\left(\frac{\partial \underline{u}}{\partial t}+\underline{u}\cdot\nabla\underline{u}\right)=\nabla\times\left[\nabla\left(-\frac{1}{\rho}P\right)\right]+\nabla\times\left[-\nabla\Pi\right]+\nabla\times\left[\nu\nabla^2\underline{u}\right]=$

$0-\nabla\times\left[\nabla\Pi\right]+\nu\nabla\times\left[\nabla^2\underline{u}\right]=-0+\nu\nabla\times\nabla^2\underline{u}=\nu\nabla\times\nabla^2\underline{u}=$

$\nabla\times\left(\frac{\partial \underline{u}}{\partial t}\right)+\nabla\times(\underline{u}\cdot\nabla\underline{u})=\frac{\partial}{\partial t}(\nabla\times\underline{u})+\nabla\times(\underline{u}\cdot\nabla\underline{u})=\nu\nabla\times\nabla^2\underline{u}$

$=\nu\nabla^2(\nabla\times\underline{u}) \quad \therefore$

$\underline{\omega}=\nabla\times\underline{u} \therefore \nu\nabla^2(\underline{\omega})=\frac{\partial}{\partial t}(\underline{\omega})+\nabla\times(\underline{u}\cdot\nabla\underline{u}) \therefore$

$\frac{\partial \underline{\omega}}{\partial t}=-\nabla\times(\underline{u}\cdot\nabla\underline{u})+\nu\nabla^2\underline{\omega}$

$\therefore \underline{u}\cdot\nabla\underline{u}=(\underline{u}\cdot\nabla)\underline{u} \therefore$

$\nabla\times(\underline{u}\cdot\nabla\underline{u})=\nabla\times\left[(\underline{u}\cdot\nabla)\underline{u}\right]=(\underline{u}\cdot\nabla)(\nabla\times\underline{u})\,\overline{=}\,(\nabla\times\underline{u}\cdot\nabla)\underline{u}$

$=(\underline{u}\cdot\nabla)\nabla\times\underline{u}+\nabla(\underline{u}\cdot\nabla)\times\underline{u}=(\underline{u}\cdot\nabla)\underline{\omega}+\nabla(\underline{u}\cdot\nabla)\underline{u}=(\underline{u}\cdot\nabla)\underline{\omega}\,\overline{=}\,(\underline{\omega}\cdot\nabla)\underline{u}$

$=(\underline{u}\cdot\nabla)\underline{\omega}-(\nabla\times\underline{u}\cdot\nabla)\underline{u} \therefore$

$\nabla(\underline{u}\cdot\nabla)\underline{u}=-(\nabla\times\underline{u}\cdot\nabla)\underline{u}$

\3ai/ N-S: $\frac{\partial \underline{u}}{\partial t}+\underline{u}\cdot\nabla\underline{u}=-\nabla\left(\frac{1}{\rho}P\right)+\underline{g}+\nu\nabla^2\underline{u}\;\overline{=}\;-\nabla\left(\frac{1}{\rho}P\right)-\nabla\Pi+\nu\nabla^2\underline{u}$

$\therefore$ taking the curl of the N-S equation: $\nabla\times\underline{u}=\underline{\omega} \therefore$

$\nabla\times\left(\frac{\partial \underline{u}}{\partial t}\right)=\frac{\partial}{\partial t}\nabla\times\underline{u}=\frac{\partial}{\partial t}\underline{\omega}$,

$\nabla\times\left(-\nabla\left(\frac{1}{\rho}P\right)\right)=-\nabla\times\left[\nabla\left(\frac{1}{\rho}P\right)\right]=0$,

$\nabla\times\left[-\nabla\Pi\right]=-\nabla\times\left[\nabla\Pi\right]=0$,

$\nabla\times\left[\nu\nabla^2\underline{u}\right]=\nu\nabla\times\left[\nabla^2\underline{u}\right]=\nu\nabla^2(\nabla\times\underline{u})=\nu\nabla^2\underline{\omega}$

$\therefore \frac{\partial \underline{\omega}}{\partial t}+\nabla\times(\underline{u}\cdot\nabla\underline{u})=\nu\nabla^2\underline{\omega} \therefore$

$\nabla\times(\underline{u}\cdot\nabla\underline{u})=(\underline{u}\cdot\nabla)\underline{\omega}-(\underline{\omega}\cdot\nabla)\underline{u}=(\underline{u}\cdot\nabla)(\nabla\times\underline{u})-((\nabla\times\underline{u})\cdot\nabla)\underline{u}$

$=\nabla\times((\underline{u}\cdot\nabla)\underline{u})=(\underline{u}\cdot\nabla)\nabla\times\underline{u}+\nabla(\underline{u}\cdot\nabla)\times\underline{u} \therefore$

$\nabla(\underline{u}\cdot\nabla)\times\underline{u}=-((\nabla\times\underline{u})\cdot\nabla)\underline{u}$

\PP2079/ $\nabla^2 \underline{u} = \nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{u}) = \nabla(\nabla\cdot\underline{u}) - \nabla\times\underline{\omega} \quad \therefore$

$\nabla\times\nabla^2\underline{u} = \nabla\times\left(\nabla(\nabla\cdot\underline{u}) - \nabla\times\underline{\omega}\right) = \nabla\times\nabla(\nabla\cdot\underline{u}) - \nabla\times\nabla\times\underline{\omega} =$

$\nabla\times\nabla(\varnothing) - \nabla\times\nabla\times\underline{\omega} = -\nabla\times\nabla\times\underline{\omega} \qquad \because \nabla\times\nabla\varnothing \equiv 0$

$= -\nabla\times(\nabla\times\underline{\omega}) = -\nabla(\nabla\cdot\underline{\omega}) + \nabla^2\underline{\omega} = -\nabla(\nabla\cdot\nabla\times\underline{u}) + \nabla^2\underline{\omega} =$

$-0 + \nabla^2\underline{\omega} \qquad \because \nabla\cdot\nabla\times \equiv 0$

$= \nabla^2\underline{\omega} \quad \therefore$

$\frac{\partial\underline{\omega}}{\partial t} + \nabla\times(\underline{u}\cdot\nabla\underline{u}) = \nu\nabla^2\underline{\omega}$

$\underline{u}\times(\nabla\times\underline{u}) + \underline{u}\times(\nabla\times\underline{u}) = \nabla(\underline{u}\cdot\underline{u}) - (\underline{u}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)\underline{u} =$

$2\underline{u}\times(\nabla\times\underline{u}) = \nabla(u^2) - 2(\underline{u}\cdot\nabla)\underline{u} = 2\underline{u}\times\underline{\omega} \quad \because \underline{u}\cdot\underline{u} = |\underline{u}|^2 = u^2 \quad \therefore$

$\underline{u}\times\underline{\omega} = \frac{1}{2}\nabla(u^2) - (\underline{u}\cdot\nabla)\underline{u} \quad \therefore$

$(\underline{u}\cdot\nabla)\underline{u} = \underline{u}\cdot\nabla\underline{u} = \frac{1}{2}\nabla(u^2) - \underline{u}\times\underline{\omega} = \frac{1}{2}\nabla u^2 - \underline{u}\times\underline{\omega} \quad \therefore$

$\nabla\times(\underline{u}\cdot\nabla\underline{u}) = \nabla\times\left(\frac{1}{2}\nabla u^2 - \underline{u}\times\underline{\omega}\right) = \frac{1}{2}\nabla\times\nabla u^2 - \nabla\times(\underline{u}\times\underline{\omega})$

$= \frac{1}{2}\nabla\times\nabla\theta - \nabla\times(\underline{u}\times\underline{\omega}) \cancel{= -\nabla\times(\underline{u}\times\underline{\omega})}$

$= \frac{1}{2}(0) - \nabla\times(\underline{u}\times\underline{\omega}) \qquad \because \nabla\times\nabla\theta \equiv 0$

$= -\nabla\times(\underline{u}\times\underline{\omega}) \quad \therefore$

$\frac{\partial\underline{\omega}}{\partial t} - \nabla\times(\underline{u}\times\underline{\omega}) = \nu\nabla^2\underline{\omega}$

\3ai/ NS: $\rho\left(\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho g + \mu\nabla^2\underline{u} \quad \therefore \frac{\mu}{\rho} = \nu, \quad \nabla\times\underline{u} = \underline{\omega},$

$\nabla\cdot\underline{u} = 0 \quad \therefore \quad g = -\nabla\Pi \quad \therefore \quad \frac{\rho g}{\rho} = g = -\nabla\Pi \quad \therefore$

$\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p - \nabla\Pi + \nu\nabla^2\underline{u} \quad \therefore$

$\nabla\times\left(\frac{\partial\underline{u}}{\partial t}\right) = \frac{\partial}{\partial t}\nabla\times\underline{u} = \frac{\partial\underline{\omega}}{\partial t},$

$-\nabla\times\nabla\Pi = -0 = 0, \quad -\frac{1}{\rho}\nabla\times\nabla p = -\frac{1}{\rho}(0) = 0 \quad \therefore$

$\frac{\partial\underline{\omega}}{\partial t} + \nabla\times(\underline{u}\cdot\nabla\underline{u}) = \nu\nabla\times\nabla^2\underline{u} \quad \therefore$

$\nabla^2\underline{u} = \nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{u}) = \nabla(\nabla\cdot\underline{u}) - \nabla\times\underline{\omega} \quad \therefore$

$\nabla\times\nabla^2\underline{u} = \nabla\times\nabla(\nabla\cdot\underline{u}) - \nabla\times\nabla\times\underline{\omega} = \nabla\times\nabla(\varnothing) - \nabla\times\nabla\times\underline{\omega} = -\nabla\times\nabla\times\underline{\omega}$

$= -\nabla\times(\nabla\times\underline{\omega}) = -\nabla(\nabla\cdot\underline{\omega}) + \nabla^2\underline{\omega} = -\nabla(\nabla\cdot\nabla\times\underline{u}) + \nabla^2\underline{\omega} = -0 + \nabla^2\underline{\omega} = \nabla^2\underline{\omega} \quad \therefore$

$\frac{\partial\underline{\omega}}{\partial t} + \nabla\times(\underline{u}\cdot\nabla\underline{u}) = \nu\nabla^2\underline{\omega}$

$\underline{u}\times(\nabla\times\underline{u}) + \underline{u}\times(\nabla\times\underline{u}) = \nabla(\underline{u}\cdot\underline{u}) - (\underline{u}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)\underline{u} =$

$2\underline{u}\times(\nabla\times\underline{u}) = \nabla(u^2) - 2(\underline{u}\cdot\nabla)\underline{u} = 2\underline{u}\times\underline{\omega} \quad \therefore$

$\underline{u} \times \underline{\omega} = \frac{1}{2}\nabla u^2 - \underline{u} \cdot \nabla \underline{u}$ $\therefore$

$\underline{u} \cdot \nabla \underline{u} = \frac{1}{2}\nabla u^2 - \underline{u} \times \underline{\omega}$ $\therefore$

$\nabla \times (\underline{u} \cdot \nabla \underline{u}) = \frac{1}{2}\nabla \times \nabla u^2 - \nabla \times (\underline{u} \times \underline{\omega}) = \frac{1}{2}\nabla \times \nabla \theta - \nabla \times (\underline{u} \times \underline{\omega}) =$

$\frac{1}{2}(0) - \nabla \times (\underline{u} \times \underline{\omega}) = -\nabla \times (\underline{u} \times \underline{\omega})$ $\therefore$

$\frac{\partial \underline{\omega}}{\partial t} - \nabla \times (\underline{u} \times \underline{\omega}) = \nu \nabla^2 \underline{\omega}$ $\therefore$

$\frac{\partial \underline{\omega}}{\partial t} = \nabla \times (\underline{u} \times \underline{\omega}) + \nu \nabla^2 \underline{\omega}$

$\therefore \nabla \cdot \underline{u} = 0$, $\nabla \times (\underline{u} \times \underline{\omega}) = \underline{u}(\nabla \cdot \underline{\omega}) - \underline{\omega}(\nabla \cdot \underline{u}) + (\underline{\omega} \cdot \nabla)\underline{u} - (\underline{u} \cdot \nabla)\underline{\omega}$

$\therefore \underline{\omega}(\nabla \cdot \underline{u}) = \underline{\omega}(0) = 0$ $\therefore$

$\nabla \times (\underline{u} \times \underline{\omega}) = \underline{u}(\nabla \cdot \underline{\omega}) + (\underline{\omega} \cdot \nabla)\underline{u} - (\underline{u} \cdot \nabla)\underline{\omega}$

$\therefore \underline{u}(\nabla \cdot \underline{\omega}) = \underline{u}(\nabla \cdot \nabla \times \underline{u}) = \underline{u}(0) = 0$ $\therefore$

$\nabla \times (\underline{u} \times \underline{\omega}) = (\underline{\omega} \cdot \nabla)\underline{u} - (\underline{u} \cdot \nabla)\underline{\omega}$ $\therefore$

$\frac{\partial \underline{\omega}}{\partial t} = (\underline{\omega} \cdot \nabla)\underline{u} - (\underline{u} \cdot \nabla)\underline{\omega} + \nu \nabla^2 \underline{\omega} \equiv$

$\frac{\partial \underline{\omega}}{\partial t} = -(\underline{u} \cdot \nabla)\underline{\omega} + (\underline{\omega} \cdot \nabla)\underline{u} + \nu \nabla^2 \underline{\omega}$

\3aii) $\underline{\omega} = \nabla \times \underline{u} = \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\theta} & \hat{z} \\ \partial_R & \partial_\theta & \partial_z \\ -\alpha R & Rv & 2\alpha z \end{vmatrix} =$

$\frac{1}{R}\left(\hat{R}(0) - R\hat{\theta}(0) + \hat{z}\left[\frac{\partial}{\partial R}(Rv) - \frac{\partial}{\partial \theta}(-\alpha R)\right]\right) = \frac{1}{R}\hat{z}\left(v + R\frac{\partial v}{\partial R}\right) =$

$\left(\frac{1}{R}v + R\frac{\partial v}{\partial R}\right)\hat{z} = \omega\hat{z}$ $\therefore$ $\frac{\partial \underline{\omega}}{\partial t} = \frac{\partial \omega}{\partial t}\hat{z}$ $\therefore$

$\underline{\omega} \cdot \nabla = \omega\hat{z} \cdot \nabla = \omega\hat{z} \cdot \left(\hat{R}\frac{\partial}{\partial R} + \hat{\theta}\frac{1}{R}\frac{\partial}{\partial \theta} + \hat{z}\frac{\partial}{\partial z}\right) = \omega\frac{\partial}{\partial z}$ $\therefore$

$\underline{\omega} \cdot \nabla \underline{u} = (\underline{\omega} \cdot \nabla)\underline{u} = \omega\frac{\partial}{\partial z}\underline{u} = \omega\frac{\partial}{\partial z}(-\alpha R\hat{R} + v\hat{\theta} + 2\alpha z\hat{z}) =$

$\omega\frac{\partial}{\partial z}(-\alpha R)\hat{R} + \omega\frac{\partial}{\partial z}(v)\hat{\theta} - \omega\frac{\partial}{\partial z}(2\alpha z)\hat{z} = 0\hat{R} + 0\hat{\theta} + \omega 2\alpha\hat{z} = 2\alpha\omega\hat{z}$

$(\underline{u} \cdot \nabla)\underline{\omega} = (\underline{u} \cdot \nabla)\omega\hat{z} = (\underline{u} \cdot \nabla)(0\hat{R} + 0\hat{\theta} + \omega\hat{z}) =$

$\left(-\alpha R\frac{\partial}{\partial R} + \frac{v}{R}\frac{\partial}{\partial \theta} + 2\alpha z\frac{\partial}{\partial z}\right)\left(\left(\frac{1}{R}v + R\frac{\partial v}{\partial R}\right)\hat{z}\right) =$

$-\alpha R\frac{\partial}{\partial R}\left[\frac{1}{R}v + R\frac{\partial v}{\partial R}\right]\hat{z} = -\alpha R\left[-R^{-2}v + \frac{1}{R}\frac{\partial v}{\partial R} + \frac{\partial v}{\partial R} + R\frac{\partial^2 v}{\partial R^2}\right]\hat{z} = -\alpha R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}(Rv)\right)\hat{z} =$

$-\alpha R\frac{\partial}{\partial R}\omega\hat{z} = \left(-\alpha R\frac{\partial \omega}{\partial R}\right)\hat{z}$ $\therefore$

$\nu\nabla^2\underline{\omega}$ $\therefore$ $\nabla^2\underline{\omega} = \nabla(\nabla \cdot \underline{\omega}) - \nabla \times (\nabla \times \underline{\omega}) = \nabla(\nabla \cdot (\nabla \times \underline{u})) - \nabla \times (\nabla \times \underline{\omega})$

$\nabla \cdot \nabla \times \underline{u} = 0$ $\therefore \nabla(\nabla \cdot (\nabla \times \underline{u})) = 0 = \nabla(\nabla \cdot \underline{\omega})$ $\therefore$

$\nabla^2\underline{\omega} = -\nabla \times (\nabla \times \underline{\omega}) = -\nabla \times (\nabla \times (\omega\hat{z}))$ $\therefore$ $\nabla \times (\omega\hat{z}) = \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\theta} & \hat{z} \\ \partial_R & \partial_\theta & \partial_z \\ 0 & 0 & \omega \end{vmatrix} =$

$\frac{1}{R}\left(-R\hat{\theta}\frac{\partial}{\partial R}(\omega)\right) = -\hat{\theta}\frac{\partial \omega}{\partial R}$

$\nabla \times (\nabla \times (\omega\hat{z})) = \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\theta} & \hat{z} \\ \partial_R & \partial_\theta & \partial_z \\ 0 & -R\frac{\partial \omega}{\partial R} & 0 \end{vmatrix} = \frac{1}{R}\left(\hat{z}\left(\frac{\partial}{\partial R}\left(-R\frac{\partial \omega}{\partial R}\right)\right)\right) = \frac{1}{R}\frac{\partial}{\partial R}\left(-R\frac{\partial \omega}{\partial R}\right)\hat{z}$ $\therefore$

The vorticity equation only has z-components

\PP2019/ $\nabla = \hat{\underline{R}}\frac{\partial}{\partial R} + \hat{\underline{\phi}}\frac{1}{R}\frac{\partial}{\partial \phi} + \hat{\underline{z}}\frac{\partial}{\partial z}$ ∴ $\underline{u} = u\hat{\underline{R}} + v\hat{\underline{\phi}} + w\hat{\underline{z}}$ ∴

\3aii ~~A~~ A: let $\underline{\omega} = \nabla \times \underline{u} = \omega\hat{\underline{z}}$ ∴

) $\frac{\partial}{\partial t}\underline{\omega} = \frac{\partial}{\partial t}(\omega\hat{\underline{z}}) = -(\underline{u}\cdot\nabla)\underline{\omega} + (\omega\cdot\nabla)\underline{u} + \nu\nabla^2\underline{\omega} =$

$-(\underline{u}\cdot\nabla)\omega\hat{\underline{z}} + (\omega\hat{\underline{z}}\cdot\nabla)\underline{u} + \nu\nabla^2\omega\hat{\underline{z}} = \frac{\partial\omega}{\partial t}\hat{\underline{z}} =$

$-(\underline{u}\cdot\nabla)\omega\hat{\underline{z}} + \hat{\underline{z}}(\omega\cdot\nabla)\underline{u} + (\nu\nabla^2\omega)\hat{\underline{z}} =$

$-\left(u\frac{\partial}{\partial R} + v\frac{1}{R}\frac{\partial}{\partial\phi} + w\frac{\partial}{\partial z}\right)\omega\hat{\underline{z}} + \hat{\underline{z}}\left(\omega\frac{\partial}{\partial R} + \omega\frac{1}{R}\frac{\partial}{\partial\phi} + \omega\frac{\partial}{\partial z}\right)\underline{u} + (\nu\nabla^2\omega)\hat{\underline{z}}$

$= \left(-u\frac{\partial\omega}{\partial R} + v\frac{1}{R}\frac{\partial\omega}{\partial\phi} + w\frac{\partial\omega}{\partial z}\right)\hat{\underline{z}} + \left(\omega\frac{\partial W}{\partial R} + \omega\frac{1}{R}\frac{\partial W}{\partial\phi} + \omega\frac{\partial W}{\partial z}\right) + (\nu\nabla^2\omega)\hat{\underline{z}}$

∴ ∵ $\underline{\omega} = \omega\hat{\underline{z}}$ ∴ $\underline{\omega}$ only has a z component ∴
vorticity equation only has a z component ∴

$$\underline{\omega} = \nabla\times\underline{u} = \nabla\times\left(-aR\hat{\underline{R}} + v(R,t)\hat{\underline{\phi}} + 2az\hat{\underline{z}}\right) = \frac{1}{R}\begin{vmatrix}\hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}}\\ \partial_R & \partial_\phi & \partial_z\\ -aR & Rv(R,t) & 2az\end{vmatrix}$$

$= \frac{1}{R}\hat{\underline{z}}\left(\frac{\partial}{\partial R}(Rv(R,t)) - \frac{\partial}{\partial\phi}(-aR)\right) = \frac{1}{R}\hat{\underline{z}}\left(v(R,t) + R\frac{\partial}{\partial R}(v(R,t))\right) =$

$= \left(\frac{1}{R}v(R,t) + \frac{\partial}{\partial R}v(R,t)\right)\hat{\underline{z}} = \omega\hat{\underline{z}}$

\3aii A) first calculate $\underline{\omega} = \nabla\times\underline{u} = \frac{1}{R}\begin{vmatrix}\hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}}\\ \partial_R & \partial_\phi & \partial_z\\ -aR & Rv(R,t) & 2az\end{vmatrix} =$

$\frac{1}{R}\left(\hat{\underline{R}}(0) - R\hat{\underline{\phi}}(0) + \hat{\underline{z}}\left(\frac{\partial}{\partial R}(Rv(R,t)) - \frac{\partial}{\partial\phi}(-aR)\right)\right) =$

$\frac{1}{R}\hat{\underline{z}}\left(v(R,t) + R\frac{\partial}{\partial R}v(R,t)\right) = \left(\frac{1}{R}v(R,t) + R\frac{\partial}{\partial R}v(R,t)\right)\hat{\underline{z}} = \omega\hat{\underline{z}}$ ∴

$\underline{\omega}\cdot\nabla = \omega\hat{\underline{z}}\cdot\nabla = \omega\hat{\underline{z}}\cdot\left(\hat{\underline{R}}\frac{\partial}{\partial R} + \hat{\underline{\phi}}\frac{1}{R}\frac{\partial}{\partial\phi} + \hat{\underline{z}}\frac{\partial}{\partial z}\right) = \omega\frac{\partial}{\partial z}$ ∴

$\underline{\omega}\cdot\nabla\underline{u} = (\underline{\omega}\cdot\nabla)\underline{u} = \omega\frac{\partial}{\partial z}\underline{u} = \omega\frac{\partial}{\partial z}\left(-aR\hat{\underline{R}} + v(R,t)\hat{\underline{\phi}} + 2az\hat{\underline{z}}\right) =$

$= \omega\frac{\partial}{\partial z}(-aR)\hat{\underline{R}} + \omega\frac{\partial}{\partial z}(v(R,t))\hat{\underline{\phi}} + \omega\frac{\partial}{\partial z}(2az)\hat{\underline{z}} =$

$0\hat{\underline{R}} + 0\hat{\underline{\phi}} + \omega 2a\hat{\underline{z}} = 2a\omega\hat{\underline{z}}$,

$(\underline{u}\cdot\nabla)\underline{\omega} = (\underline{u}\cdot\nabla)\omega\hat{\underline{z}} = (\underline{u}\cdot\nabla)(0\hat{\underline{R}} + 0\hat{\underline{\phi}} + \omega\hat{\underline{z}}) =$

$\left(-aR\frac{\partial}{\partial R} + v(R,t)\frac{1}{R}\frac{\partial}{\partial\phi} + 2az\frac{\partial}{\partial z}\right)\left(\left(\frac{1}{R}v(R,t) + R\frac{\partial v(R,t)}{\partial R}\right)\hat{\underline{z}}\right) =$

$-aR\frac{\partial}{\partial R}\left[\frac{1}{R}v(R,t) + R\frac{\partial v(R,t)}{\partial R}\right]\hat{\underline{z}} =$

$-aR\left[-R^{-2}v(R,t) + \frac{1}{R}\frac{\partial v(R,t)}{\partial R} + \frac{\partial v(R,t)}{\partial R} + R\frac{\partial^2 v(R,t)}{\partial R^2}\right]\hat{\underline{z}} = (-aR)\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}(Rv)\right)\hat{\underline{z}}$

$= (-aR)\frac{\partial}{\partial R}\omega\hat{\underline{z}} = -aR\frac{\partial\omega}{\partial R}$ ∴ e.g.

) $\nu\nabla^2\underline{\omega}$ ∵ $\nabla^2\underline{\omega} = \nabla(\nabla\cdot\underline{\omega}) - \nabla\times(\nabla\times\underline{\omega}) = \nabla(\nabla\cdot(\nabla\times\underline{u})) - \nabla\times(\nabla\times\underline{\omega})$

∴ $\nabla\cdot\nabla\times\underline{u} = 0$ ∴ $\nabla(\nabla\cdot(\nabla\times\underline{u})) = 0 = \nabla(\nabla\cdot\underline{\omega})$ ∴

$\nabla^2\underline{\omega} = -\nabla\times(\nabla\times\underline{\omega}) = -\nabla\times(\nabla\times(\omega\hat{\underline{z}}))$ ∴ $\nabla\times(\underline{\omega}\hat{\underline{z}}) = \frac{1}{R}\begin{vmatrix}\hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}}\\ \partial_R & \partial_\phi & \partial_z\\ 0 & 0 & \omega(R,t)\end{vmatrix} =$

$$\frac{1}{R}\left(-R\,\hat{\underline{\phi}}\,\frac{\partial}{\partial R}\left(\omega(R,t)\right)\right) = -\hat{\underline{\phi}}\left(\frac{\partial \omega(R,t)}{\partial R}\right) \therefore$$

$$\nabla \times (\nabla \times (\omega \hat{\underline{z}})) = \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & -R\frac{\partial \omega(R,t)}{\partial R} & 0 \end{vmatrix} =$$

$$\frac{1}{R}\left(\hat{\underline{z}}\left(\frac{\partial}{\partial R}\left(-R\frac{\partial \omega(R,t)}{\partial R}\right)\right)\right) = \frac{1}{R}\frac{\partial}{\partial R}\left(-R\frac{\partial \omega(R,t)}{\partial R}\right)\hat{\underline{z}} \therefore$$

the vorticity equation only has z-components

\3aiiB/ inviscid case $\therefore \nu = 0 \because \mu = 0 \therefore$

$$\frac{\partial \underline{\omega}}{\partial t} = -(\underline{u}\cdot\nabla)\underline{\omega} + (\underline{\omega}\cdot\nabla)\underline{u} \therefore$$

$$\underline{\omega} = \omega\hat{\underline{z}} = \omega(R,t)\hat{\underline{z}} = e^{2at}F(\xi)\hat{\underline{z}} = e^{2at}F(Re^{at})\hat{\underline{z}} \therefore$$

$$\frac{\partial \underline{\omega}}{\partial t} = \frac{\partial}{\partial t}\left(e^{2at}F(Re^{at})\hat{\underline{z}}\right) \therefore$$

$$\frac{\partial \omega}{\partial t} = -aR\frac{\partial \omega}{\partial R} = 2a\omega + \frac{\nu}{R}\frac{\partial}{\partial R}\left(\frac{\partial \omega}{\partial R}\right)$$

\3aii A/ $\frac{\partial \underline{\omega}}{\partial t} = \frac{\partial \omega}{\partial t}\hat{\underline{z}}$, $\cancel{((\underline{u}\cdot\nabla)\underline{\omega})\underline{z}}$ $((\underline{\omega}\cdot\nabla)\underline{u}) = 2a\omega\hat{\underline{z}}$,

$(\underline{u}\cdot\nabla)\underline{\omega} = -aR\frac{\partial \omega}{\partial R}\hat{\underline{z}}$, $\nabla^2\underline{\omega} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \omega}{\partial R}\right)\hat{\underline{z}} \therefore$

$$\nu\nabla^2\underline{\omega} = \frac{\nu}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \omega}{\partial R}\right)\hat{\underline{z}} \therefore$$

$$\cancel{2a\omega\hat{\underline{z}} = aR\frac{\partial \omega}{\partial R}\hat{\underline{z}} + \frac{\nu}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \omega}{\partial R}\right)\hat{\underline{z}} - aR\frac{\partial \omega}{\partial R}\hat{\underline{z}}} \therefore$$

$$\frac{\partial \omega}{\partial t}\hat{\underline{z}} = +aR\frac{\partial \omega}{\partial R}\hat{\underline{z}} + 2a\omega\hat{\underline{z}} + \frac{\nu}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \omega}{\partial R}\right)\hat{\underline{z}} \therefore$$

$$\frac{\partial \omega}{\partial t} - aR\frac{\partial \omega}{\partial R} = 2a\omega + \frac{\nu}{R}\frac{\partial}{\partial R}\left(R\frac{\partial \omega}{\partial R}\right)$$

\3aiiB/ inviscid means $\nu = 0 \therefore \frac{\partial \omega}{\partial t} - aR\frac{\partial \omega}{\partial R} = 2a\omega$

$$\omega = e^{2at}F(\xi) = e^{2at}F(Re^{at}) \therefore$$

$$\frac{\partial \omega}{\partial t} = 2ae^{2at}F(Re^{at}) + e^{2at}F'(Re^{at})\,Rae^{at},$$

$$\frac{\partial \omega}{\partial R} = e^{2at}F'(Re^{at})e^{at}$$

$$\frac{\partial \omega}{\partial t} - aR\frac{\partial \omega}{\partial R} - 2a\omega = \cancel{e^{2at}F'(Re^{at})Rae^{at} - aRe^{2at}F'(Re^{at})e^{at} -}$$

$$\cancel{2ae^{2at}F(Re^{at})} =$$

$$\cancel{F'(Re^{at})}\left[e^{2at}Rae^{at}\right.$$

$$2ae^{2at}F(Re^{at}) + e^{2at}F'(Re^{at})Rae^{at} - aRe^{2at}F'(Re^{at})e^{at} - 2ae^{2at}F(Re^{at})$$

$$= F(Re^{at})\left(2ae^{2at} - 2ae^{2at}\right) + F'(Re^{at})\left(e^{2at}Rae^{at} - aRe^{2at}e^{at}\right)$$

$$= F(Re^{at})(0) + F'(Re^{at})(0) = 0 + 0 = 0$$ as required $\therefore$

$\underline{\omega} = \omega\hat{\underline{z}} = \omega(R,t) = e^{2at}F(\xi) \therefore \underline{\omega}$ exponentially increases as t increases for $a > 0$ and vice verse for $a < 0$

\PP2019/ \4c/ fluid moves with the boundary on the boundary and that far away from the boundary, the fluid doesn't know about the boundary motion

\4d/
let $u' = \frac{u}{U}$ ∴ $\frac{\partial u'}{\partial t} = \frac{1}{U}\frac{\partial u}{\partial t}$, $\frac{\partial u'}{\partial y} = \frac{1}{U}\frac{\partial u}{\partial y}$ ∴

$u'_{yy} = \frac{1}{U} u_{yy}$ ∴ $\frac{1}{U} u_t = \nu \frac{1}{U} u_{yy}$ ∴

$u'_t = \nu u'_{yy}$

$u'(0) = 1$ ∵ $u(y=0, t>0) = U$ ∴ $u'(y=0, t>0) = \frac{u(y=0, t>0)}{U} = \frac{U}{U} = 1$

$u'(y \to \infty) \to 0$ ∵ $u'(y \to \infty) = \frac{u(y \to \infty)}{U} \to \frac{0}{U} = 0$

\4e/ $[y] = L$, $[t] = T$, $[\frac{\partial u}{\partial t}] = LT^{-1}T^{-1} = LT^{-2}$

$[\nu \nabla^2 u] = [\nu] L^{-2} LT^{-1} = L^{-1}T^{-1}$ ∴

$[\nu] = LT^{-2} LT = L^2 T^{-1}$ ∴

y, t independent, ν is dependent ∴

The quantity of interest is u'

3 parameters: $[y] = L$, $[t] = T$, $[\nu] = L^2T^{-1}$

y, t independent, ν dependent ∴

$n=3$, $k=2$, $m=1$ ∴ $a_1 = y$, $a_2 = t$ ∴

$\Pi_1 = \frac{\nu}{a_1^{\alpha} a_2^{\beta}} = \frac{\nu}{y^{\alpha} t^{\beta}}$ ∴ $[\nu] = L^2T^{-1} = [y]^{\alpha}[t]^{\beta} = L^{\alpha}T^{\beta}$ ∴

$\alpha = 2$, $\beta = -1$ ∴ $\Pi_1 = \frac{\nu}{y^2 t^{-1}}$ ∴

$\Pi = \Phi(\Pi_1) = \Phi\left(\frac{\nu}{y^2 t^{-1}}\right)$

where Π is the quantity of interest made dimensionless

Π_1 is the dependent parameters made dimensionless using the parameters with independent ∴

$\Pi = u'$, $\Pi_1 = \frac{\nu}{y^2 t^{-1}}$ ∴ $u' = \Phi(\eta) = \Phi\left(\frac{y}{\sqrt{4\nu t}}\right)$ ∴

$u' = \Phi\left(\frac{\nu}{y^2 t^{-1}}\right)$ ∴ $\Phi\left(\left(\frac{\nu}{y^2 t^{-1}}\right)^{-\frac{1}{2}}\right) = \Phi\left(\left(\frac{y^2 t^{-1}}{\nu}\right)^{\frac{1}{2}}\right) = \Phi\left(\frac{y^2}{\nu t}\right)^{\frac{1}{2}} =$

$\Phi\left(\sqrt{\frac{y^2}{\nu t}}\right) = \Phi\left(\frac{y}{\sqrt{\nu t}}\right)$ ∴ $u' = \Phi\left(\frac{y}{\sqrt{\nu t}}\right)$ ∴

if $\Phi(x) = \frac{1}{2} x$ ∴ $u' = \Phi\left(\frac{y(0.5)}{\sqrt{\nu t}}\right) = \Phi\left(\frac{1}{2}\frac{y}{\sqrt{\nu t}}\right) = \Phi\left(\frac{y}{\sqrt{4}\sqrt{\nu t}}\right) = \Phi\left(\frac{y}{\sqrt{4\nu t}}\right)$

\4h/ ∴ the part multiplied by viscosity μ ∴ vector n normal to surface is $\hat{j}$ ∴

$\Sigma_1^{viscous}\Big|_{y=0} = -U\left(\frac{\rho\mu}{t\pi}\right)^{\frac{1}{2}}$

\48/ substitute $u' = \Phi(\eta)$, $\eta = \frac{y}{\sqrt{4\nu t}}$ into $u_t = \nu u_{yy}$, $\nu = \frac{\mu}{\rho}$

and the BCs: $u'(y=0, t>0) = 1$,

$u(y=0, t>0) = U$, $u(y\to\infty, t>0)\to 0$, $u'(y\to\infty, t>0)\to 0$, ∴

$\cancel{u_y = u_\eta \frac{y}{t}}$ $u'_y = u'_\eta \eta_y = \Phi_\eta \frac{1}{\sqrt{4\nu t}}$, ∴

$u'_{yy} = \frac{\partial}{\partial y}\left(\frac{1}{\sqrt{4\nu t}}\Phi_\eta\right) = \frac{1}{\sqrt{4\nu t}}\Phi_{\eta\eta}\eta_y = \frac{1}{4\nu t}\Phi_{\eta\eta} = \frac{1}{4\nu t}\Phi''$,

$\cancel{u_t}$ $u'_t = \Phi_\eta \eta_t = \frac{\partial}{\partial t}\left(\frac{y}{\sqrt{4\nu}} t^{-1/2}\right)\Phi = -\frac{1}{2}t^{-3/2}\frac{y}{\sqrt{4\nu}}\Phi'$ ∴ $u'_t = \nu u'_{yy}$ ∴

$-\frac{1}{2}t^{-3/2}\frac{y}{\sqrt{4\nu}}\Phi' = \nu\frac{1}{4\nu t}\Phi'' = \frac{1}{4t}\Phi''$ ∴

$-\frac{1}{2}t^{-1/2}\frac{y}{\sqrt{4\nu}}\Phi' = \frac{1}{4}\Phi''$ ∴ $-2\frac{y}{\sqrt{4\nu t}}\Phi' = \Phi''$ ∴

$\Phi'' + 2\frac{y}{\sqrt{4\nu t}}\Phi' = 0 = \Phi'' + 2\eta\Phi' = 0$,

$u'(y=0, t>0) = 1$ ∴ $u'(y=0, t>0) = \Phi\left(\frac{0}{\sqrt{4\nu t}}\right) = \Phi(0) = 1$,

$u'(y\to\infty, t>0) = \Phi(\eta\to\infty) = 0$ ∴

\49/ $\Phi'' + 2\eta\Phi' = 0$ ∴ ∴ $\frac{d}{d\eta}\Phi' = -2\eta\Phi'$ ∴

$\frac{1}{\Phi'}\frac{d\Phi'}{d\eta} = -2\eta$ ∴ $\int\frac{1}{\Phi'}d\Phi' = \int -2\eta\, d\eta = \ln|\Phi'| = -2\eta + A_1$ ∴

$|\Phi'| = e^{-2\eta + A_1} = A_2 e^{-2\eta}$ ∴ $\Phi' = A_3 e^{-2\eta}$ ∴

$\Phi = \frac{A_3}{-2}e^{-2\eta} + B = Ae^{-2\eta} + B$ ∴

$\Phi(0) = 1 = Ae^0 + B = A + B$ ∴ $\cancel{1-B=A}$ ∴ $\cancel{B}$ $1 - B = A$, $1 - A = B$ ∴

$\Phi = Ae^{-2\eta} + 1 - A = A(e^{-2\eta} - 1) + 1$

$\Phi(\eta\to\infty) = 0 = \lim_{\eta\to\infty}(A(e^{-2\eta} - 1)) + 1 = -A + 1$ ∴ $A = 1$, ∴

let $B = -1$ ∴ $\Phi = e^{-2\eta} - 1$

\49 sol/ $\Phi'' + 2\eta\Phi' = 0$ ∴ $\cancel{\int\frac{1}{\Phi'}\frac{d\Phi}{d}}$ $\frac{d}{d\eta}\Phi' = -2\eta\Phi$ ∴ $\int\frac{d\Phi'}{\Phi'} = -2\int\eta\, d\eta$

$\ln|\Phi'| = -2\eta + c$ ∴ $\Phi' = A_1 e^{-2\eta}$ ∴

$\Phi(\eta) = A_2 e^{-2\eta} + B_1$ ∴ $\Phi(\eta) = A\frac{2}{\sqrt{\pi}}\int_0^\eta e^{-s^2}ds + B$

$\Phi(0) = 1 = A(0) + B = 1 = B$,

$\Phi(\eta\to\infty) = A(1) \cancel{\frac{2}{\sqrt{\pi}}} + 1 = 0 = A + 1$ ∴ $A = -1$ ∴ $\Phi(\eta) = \frac{-2}{\sqrt{\pi}}\int_0^\eta e^{-s^2}ds + 1$

\~~4~~h/ force on area element dS with unit normal $\underline{n}$ is $dF_i = \sigma_{ij}n_j\, dS$

$= \Sigma_i\, dS$, σ_{ij} is the stress tensor given ∴ want viscous part ∴

\PP2019/ \1ai/ $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \rho g + \mu\nabla^2\underline{u}$

\1aii/ $\nabla\cdot\underline{u} = 0 \therefore \frac{\partial\rho}{\partial t} + \underline{u}\cdot\nabla = 0$ ✗

$\cancel{\partial\rho}$ $\nabla\cdot\underline{u} = 0 \quad \therefore \rho\nabla\cdot\underline{u} = 0 \therefore$

$\nabla\cdot(\rho\underline{u}) = \rho\nabla\cdot\underline{u} + \underline{u}\cdot\nabla\rho = \underline{u}\cdot\nabla\rho \therefore$

$\cancel{\frac{D\rho}{Dt} =}\ \cancel{u}\ \partial_t\rho + \underline{u}\cdot\nabla\rho = 0$

\1ai/ $\rho(\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \rho g + \mu\nabla^2\underline{u} = \rho(\partial_t\underline{u} + \underline{u}\cdot\nabla\underline{u})$

$\cancel{\frac{D}{Dt}}$ \1aii/ $\frac{D}{Dt}\rho = 0 = \frac{\partial\rho}{\partial t} + \underline{u}\cdot\nabla\rho$

\1aii/ $\cancel{\partial t}\ \cancel{u}\ \nabla\cdot\underline{u} = 0 \therefore$

$0 = \frac{D}{Dt}\rho = \partial_t\rho + \nabla\cdot(\rho\underline{u})$

$\nabla\cdot\underline{u} \quad \therefore \cancel{\partial t}\ \partial_t\rho + \nabla\cdot(\rho\underline{u}) = \partial_t\rho + \rho\nabla\cdot\underline{u} + \underline{u}\cdot\nabla\rho =$

$\partial_t\rho + \rho(0) + \underline{u}\cdot\nabla\rho = \partial_t\rho + \underline{u}\cdot\nabla\rho = 0 = \frac{D\rho}{Dt}$

\1aii/ $\nabla\cdot\underline{u} = 0 \therefore \partial_t\underline{u} + \nabla\cdot(\rho\underline{u}) = \partial_t\underline{u} + \rho\nabla\cdot\underline{u} + \underline{u}\cdot\nabla\rho =$

$\partial_t\underline{u} + \rho(0) + \underline{u}\cdot\nabla\rho = \partial_t\underline{u} + \underline{u}\cdot\nabla\rho = 0 = \frac{D\rho}{Dt} \therefore$

$\partial_t\underline{u} + \underline{u}\cdot\nabla\rho = 0$

\1aiii/ $\frac{D}{Dt} = \partial_t + \underline{u}\cdot\nabla$

\1bi/ $\nabla\times(\nabla^2\underline{u}) \rightarrow [\nabla\times(\nabla^2\underline{u})]_i = \varepsilon_{ijk}\nabla_j(\nabla^2\underline{u})_k$

\1ai/ $\rho(\partial_t\underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \rho g + \mu\nabla^2\underline{u}$

\1aii/ $\nabla\cdot\underline{u} = 0 \therefore \cancel{\frac{D\rho}{Dt} =}\ \cancel{\partial\rho}\ \partial_t\rho + \nabla\cdot(\rho\underline{u}) = 0 =$

$\partial_t\rho + \rho\nabla\cdot\underline{u} + \underline{u}\cdot\nabla\rho = \partial_t\rho + \rho(0) + \underline{u}\cdot\nabla\rho = \partial_t\rho + \underline{u}\cdot\nabla\rho = 0 = \frac{D\rho}{Dt}$

\1aiii/ $\frac{D}{Dt} = \partial_t + \underline{u}\cdot\nabla$

\1bi/ $\nabla\times(\nabla^2\underline{u}) \rightarrow [\nabla\times(\nabla^2\underline{u})]_i = \varepsilon_{ijk}\nabla_j(\nabla^2\underline{u})_k = \varepsilon_{ijk}\partial_j(\nabla^2\underline{u})_k =$

$\varepsilon_{ijk}\partial_j(\nabla^2 u_k)$

\1bi/ $\nabla^2 = \partial_{xx} + \partial_{yy} + \partial_{zz}$ in cartesian coords $\therefore$ is scalar $\therefore$

$[\nabla\times(\nabla^2\underline{u})]_i = \varepsilon_{ijk}\nabla_j(\nabla^2\underline{u})_k = \varepsilon_{ijk}\partial_j(\nabla^2 u_k) = \varepsilon_{ijk}\partial_j(\partial_{mm}u_k) =$

$\partial_{mm}\varepsilon_{ijk}\partial_j(u_k) = \partial_{mm}(\nabla\times\underline{u})_i = \partial_{mm}\omega_i = [\nabla^2\underline{\omega}]_i$

\1bi/ $\nabla^2 = \partial_{xx} + \partial_{yy} + \partial_{zz}$ in cartesian coords $\therefore$ is scalar $\therefore$

$[\nabla\times(\nabla^2\underline{u})]_i = \varepsilon_{ijk}(\nabla)_j(\nabla^2\underline{u})_k = \varepsilon_{ijk}\partial_j(\nabla^2 u_k) =$

$\varepsilon_{ijk}\partial_j(\partial_{mm}u_k) = \partial_{mm}\varepsilon_{ijk}\partial_j(u_k) = \partial_{mm}\varepsilon_{ijk}(\nabla)_j(u_k) = \partial_{mm}(\nabla\times\underline{u})_i =$

$\partial_{mm}\boxed{\omega_i} = [\nabla^2\underline{\omega}]_i$

\1bii A/ $u=(2ax_1, a^2x_2, x_3)$ $\therefore$ $e_{11}=\frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}+\frac{\partial u_1}{\partial x_1}\right)=\frac{1}{2}(2)\frac{\partial u_1}{\partial x_1}=\frac{\partial u_1}{\partial x_1}=$
$\partial_{x_1}(2ax_1)=2a$, $e_{12}=\frac{1}{2}(\partial_{x_2}u_1+\partial_{x_1}u_2)=\frac{1}{2}(\partial_{x_2}(2ax_1)+\partial_{x_1}(a^2x_2))=$
$\frac{1}{2}(0+0)=0$ $e_{22}=\frac{1}{2}(\partial_{x_2}u_2+\partial_{x_2}u_2)=\partial_{x_2}u_2=\partial_{x_2}(a^2x_2)=a^2$
$e_{33}=\partial_{x_3}u_3=\partial_{x_3}(x_3)=1$, $e_{ij}=\frac{1}{2}(\partial_{x_j}u_i+\partial_{x_i}u_j)=\frac{1}{2}(0+0)=0$ for $i\neq j$ $\therefore$
diagonal matrix e_{ij} with entries $2a, a^2, 1$ on diagonal,
0 otherwise $\begin{bmatrix}2a & 0 & 0\\ 0 & a^2 & 0\\ 0 & 0 & 1\end{bmatrix}$ $\therefore$ find a for which $\underline{u}$ is incompressible
means use e_{ij} $\therefore$ $\nabla\cdot\underline{u}=\partial_{x_1}u_1+\partial_{x_2}u_2+\partial_{x_3}u_3=e_{ii}=e_{11}+e_{22}+e_{33}=$
$a^2+2a+1=0=(a+1)^2$ $\therefore$ $a=-1$

\1bii A/ $e_{11}=\frac{1}{2}(\partial_{x_1}u_1+\partial_{x_1}u_1)=\partial_{x_1}u_1=\partial_{x_1}(2ax_1)=2a$,
$e_{12}=\frac{1}{2}(\partial_{x_2}u_1+\partial_{x_1}u_2)=\frac{1}{2}(\partial_{x_2}(2ax_1)+\partial_{x_1}(a^2x_2))=0$,
$e_{22}=\frac{1}{2}(\partial_{x_2}u_2+\partial_{x_2}u_2)=\partial_{x_2}u_2=\partial_{x_2}(a^2x_2)=a^2$
$e_{33}=\partial_{x_3}u_3=\partial_{x_3}(x_3)=1$ $\therefore$
$e_{ij}=\frac{1}{2}(\partial_{x_j}u_i+\partial_{x_i}u_j)=\frac{1}{2}(0+0)=0$ for $i\neq j$ ~~0 otherwise $\therefore e_{ij}$~~ $\therefore$
diagonal matrix e_{ij} with entries $2a, a^2, 1$ on diagonal
0 otherwise $\begin{bmatrix}2a & 0 & 0\\ 0 & a^2 & 0\\ 0 & 0 & 1\end{bmatrix}$ $\therefore$ find a for which $\underline{u}$ is incompressible
means use e_{ij} $\therefore$ $\nabla\cdot\underline{u}=\partial_{x_1}u_1+\partial_{x_2}u_2+\partial_{x_3}u_3=e_{ii}=e_{11}+e_{22}+e_{33}$
$2a+a^2+1=(a+1)^2=0$ $\therefore$ $a=-1$

\1bii B/ $\nabla\times\underline{u}=\nabla\times(u_1,u_2,u_3)=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \partial x_1 & \partial x_2 & \partial x_3\\ u_1 & u_2 & u_3\end{vmatrix}=$
$\underline{i}(\partial_{x_2}u_3-\partial_{x_3}u_2)-\underline{j}(\partial_{x_1}u_3-\partial_{x_3}u_1)+\underline{k}(\partial_{x_1}u_2-\partial_{x_2}u_1)=$
$\underline{i}\,2\xi_{32}-\underline{j}\,2\xi_{31}+\underline{k}\,2\xi_{21}=2\xi_{32}\underline{i}+2(-\xi_{31})\underline{j}+2\xi_{21}\underline{k}=2(1)\underline{i}+2(-\frac{1}{2})\underline{j}+2\xi_{21}\underline{k}=$
$2\underline{i}-\underline{j}+2(0)\underline{k}=2\underline{i}-\underline{j}+0\underline{k}$

\1bii B/ $\nabla\times\underline{u}=\nabla\times(u_1,u_2,u_3)=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \partial x_1 & \partial x_2 & \partial x_3\\ u_1 & u_2 & u_3\end{vmatrix}=$
$\underline{i}(\partial_{x_2}u_3-\partial_{x_3}u_2)-\underline{j}(\partial_{x_1}u_3-\partial_{x_3}u_1)+\underline{k}(\partial_{x_1}u_2-\partial_{x_2}u_1)=$
$\underline{i}\,2\xi_{32}-\underline{j}\,2\xi_{31}+\underline{k}\,2\xi_{21}=2(1)\underline{i}+2(-\frac{1}{2})\underline{j}+2(0)\underline{k}=2\underline{i}-\underline{j}+0\underline{k}$

\1ci/ $\nabla\times(\psi\underline{k})=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \partial_x & \partial_y & \partial_z\\ 0 & 0 & \psi\end{vmatrix}=\underline{i}(\partial_y\psi-\partial_z(0))-\underline{j}(\partial_x\psi-\partial_z(0))+\underline{k}(\partial_x(0)-\partial_y(0))=$
$\partial_y\psi\,\underline{i}-\partial_x\psi\,\underline{j}=U\underline{i}+V\underline{j}$ $\therefore$ $U=\partial_y\psi$, $-\partial_x\psi=V$

\1ci/ $(U,V)=\nabla\times(\psi\underline{k})=U\underline{i}+V\underline{j}=\begin{vmatrix}\underline{i} & \underline{j} & \underline{k}\\ \partial_x & \partial_y & \partial_z\\ 0 & 0 & \psi\end{vmatrix}=\underline{i}(\partial_y\psi-\partial_z(0))-\underline{j}(\partial_x\psi-\partial_z(0))+\underline{k}(\partial_x(0)-\partial_y(0))$
$=\underline{i}\,\partial_y\psi-\underline{j}\,\partial_x\psi=U\underline{i}+V\underline{j}$ $\therefore$ $U=\partial_y\psi$, $V=-\partial_x\psi$

PP2019 c ii / $\underline{\omega} = \nabla \times \underline{u} = \nabla \times (\partial_y \psi, -\partial_x \psi, W) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_y \psi & -\partial_x \psi & W \end{vmatrix} =$

$[\partial_y W - \partial_z(-\partial_x \psi)]\underline{i} - \underline{j}[\partial_x(W) - \partial_z(\partial_y \psi)] + \underline{k}[\partial_x(-\partial_x \psi) - \partial_y(\partial_y \psi)] =$

$(\partial_y W + \partial_x(\partial_z \psi))\underline{i} - \underline{j}[\partial_x W - \partial_y(\partial_z(\psi))] + \underline{k}(-\partial_{xx}\psi \mp \partial_{yy}\psi) =$

$(\partial_y W + 0)\underline{i} - \underline{j}(\partial_x W - 0) - \underline{k}(\nabla^2 \psi) = \partial_y W \underline{i} - \partial_x W \underline{j} - \nabla^2 \psi \underline{k} =$

$(\partial_y W, -\partial_x W, -\nabla^2 \psi) = (\partial_y W, -\partial_x W, \omega) = \underline{\omega}$

c ii / $\underline{\omega} = \nabla \times \underline{u} = \nabla \times (\partial_y \psi, -\partial_x \psi, W) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_y \psi & -\partial_x \psi & W \end{vmatrix} =$

$[\partial_y W - \partial_z(-\partial_x \psi)]\underline{i} - [\partial_x W - \partial_z(\partial_y \psi)]\underline{j} + [\partial_x(-\partial_x \psi) - \partial_y(\partial_y \psi)]\underline{k} =$

$[\partial_y W + \partial_x(\partial_z \psi)]\underline{i} - [\partial_x W - \partial_y(\partial_z \psi)]\underline{j} + (-(\partial_{xx}\psi + \partial_{yy}\psi))\underline{k} =$

$\partial_y W \underline{i} - \partial_x W \underline{j} - \nabla^2 \psi \underline{k} = (\partial_y W, -\partial_x W, -\nabla^2 \psi) = (\partial_y W, -\partial_x W, \omega) = \underline{\omega}$

c iii / $\nabla \cdot \underline{u} = \nabla \cdot (U, V, W) = \partial_x U + \partial_y V + \partial_z W =$

$\partial_x(\partial_y \psi) - \partial_y(-\partial_x \psi) + \partial_z W \therefore$ W indep of z $\therefore$

$= \partial_{xy}\psi - \partial_{xy}\psi + 0 = 0 + 0 = 0$

edj vortex lines have $\underline{\omega}$ as tangents just like streamlines have $\underline{u}$ as tangents $\therefore$ circles

$\nabla \cdot \underline{\omega} = \nabla \cdot (\nabla \times \underline{u}) = \nabla \cdot (\partial_y W, -\partial_x W, \omega) = \partial_{xy} W - \partial_{yx} W + \partial_z \omega =$

$\partial_{xy} W - \partial_{xy} W + (\partial_z(-\nabla^2 \psi)) = 0 + -\partial_z(\nabla^2 \psi(x,y)) = -\nabla^2 \partial_z \psi(x,y) = -\nabla^2(0) = 0$

c ii / $\nabla \cdot \underline{u} = \nabla \cdot (U, V, W) = \partial_x U + \partial_y V + \partial_z W =$

$\partial_x(\partial_y \psi) + \partial_y(-\partial_x \psi) + \partial_z \psi(x,y) = \partial_{xy}\psi - \partial_{xy}\psi + 0 = 0$

vortex lines $\underline{\omega}$ as tangents just like streamlines have $\underline{u}$ as tangents $\therefore$ circles $\therefore$ edj $\therefore$

$\nabla \cdot \underline{\omega} = \nabla \cdot (\nabla \times \underline{u}) = \nabla \cdot (\partial_y W, -\partial_x W, \omega) = \partial_{xy} W - \partial_{yx} W + \partial_z \omega =$

$\partial_{xy} W - \partial_{xy} W + (\partial_z(-\nabla^2 \psi)) = 0 - \partial_z(\nabla^2 \psi(x,y)) = -\nabla^2 \partial_z \psi(x,y) = -\nabla^2(0) = 0$

$\therefore$ edj $\because \nabla \cdot \nabla \times = 0 \therefore \nabla \cdot \underline{\omega} = \nabla \cdot (\nabla \times \underline{u}) \equiv 0$

c iv / vortex lines have $\underline{\omega}$ as tangents $\therefore$

$\underline{\omega} = (\partial_y W, -\partial_x W, \omega) = (\partial_y(x^2+y^2), -\partial_x(x^2+y^2), -\nabla^2(xy)) =$

$(2y, -2x, -\partial_{xx}(xy) - \partial_{yy}(xy)) = (2y, -2x, 0) \therefore$

$\therefore \frac{dx}{2y} = \frac{dy}{-2x} \therefore -2x\,dx = 2y\,dy \therefore \int -2x\,dx = \int 2y\,dy = -x^2 + C = y^2 \therefore$

$C = y^2 + x^2 = r^2 \therefore$ vortex lines form the shapes of circles

\1ciV/ $\underline{\omega} = (\partial_y W, -\partial_x W, 0) = (\partial_y(x^2+y^2), -\partial_x(x^2+y^2), -\nabla^2\chi) =$
$(2y, -2x, -\nabla^2(xy)) = (2y, -2x, - \partial_{xx}(xy) - \partial_{yy}(xy)) =$
$(2y, -2x, 0) = \underline{\omega}$ ∴
vortex lines have ω as tangents ∴
$\frac{dx}{+2y} = \frac{dy}{-2x}$ ∴ $-2x\frac{dx}{dy} = -2y$ ∴
$\int -2x \frac{dx}{dy} dy = \int +2y\, dy = \int -2x\, dx = -x^2 + C = y^2$ ∴
$C = x^2 + y^2 = r^2$ ∴
vortex lines form the shapes of circles

\1di/ The flow is driven by the difference in pressure along the z-direction, so $\underline{u} = W\hat{\underline{z}}$ now need to explain ~~that~~ $W = W(R)$ only ∴ BCs at $R=a$ cause ~~that~~ $W = W(R)$, W independent of $z, \varnothing, t$, only dependent of stationary non slip boundary.

\1di/ The flow is driven by the difference is pressure along the z-direction ∴ $\underline{u} = W\hat{\underline{z}}$
and ∴ BCs at $R=a$ cause $W = W(R)$, and W independent of $z, \varnothing, t$, only dependent on stationary non slip boundary

\1dii/ incompressible ∴ $\nabla \cdot \underline{u} = 0$ ∴
$\underline{u} = W(R)\hat{\underline{z}} = 0\hat{\underline{R}} + 0\hat{\underline{\varnothing}} + W(R)\hat{\underline{z}}$ ∴
$\nabla \cdot \underline{u} = \nabla \cdot (0\hat{\underline{R}} + 0\hat{\underline{\varnothing}} + W(R)\hat{\underline{z}}) = \frac{1}{R}\partial_R(R \times 0) + \frac{1}{R}\partial_\varnothing(0) + \partial_z(W(R)) =$
$\frac{1}{R}\partial_R(0) + 0 + 0 = 0 + 0 = 0$ ∴ flow satisfies the incompressible continuity equation

\1dii/ incompressible ∴ $\nabla \cdot \underline{u} = 0$ ∴
$\underline{u} = W(R)\hat{\underline{z}} = 0\hat{\underline{R}} + 0\hat{\underline{\varnothing}} + W(R)\hat{\underline{z}}$ ∴
$\nabla \cdot \underline{u} = \nabla \cdot (0\hat{\underline{R}} + 0\hat{\underline{\varnothing}} + W(R)\hat{\underline{z}}) = \frac{1}{R}\partial_R(R \times 0) + \frac{1}{R}\partial_\varnothing(0) + \partial_z(W(R)) =$
$\frac{1}{R}\partial_R(0) + 0 + 0 = 0 + 0 = 0$ ∴ flow satisfies the incompressible continuity equation

\PP2019/ \1diii/ N-S: $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla P + \rho \underline{g} + \mu\nabla^2\underline{u}$ ∴

$\underline{u} = W(R)\hat{\underline{z}}$ ∴ $\partial_t \underline{u} = \partial_t(W(R)\hat{\underline{z}}) = 0$,

$\nabla = \hat{\underline{R}}\,\partial_R + \hat{\underline{\phi}}\,\frac{1}{R}\partial_\phi + \hat{\underline{z}}\,\partial_z$ ∴

$\underline{u}\cdot\nabla = W(R)\,\partial_z$ ∴

$\underline{u}\cdot\nabla\underline{u} = W(R)\,\partial_z(W(R)\hat{\underline{z}}) = W(R)\,\partial_z(W(R))\hat{\underline{z}} = W(R)(0)\hat{\underline{z}} = 0$ ∴

$\rho(0+0) = -\nabla P + \rho\underline{g} + \mu\nabla^2\underline{u} \doteq 0$ ∴

$\underline{g} = 0$ ∴

$-\nabla P + \mu\nabla^2\underline{u} = 0$ ∴ $\nabla P = \mu\nabla^2\underline{u}$

\1diii/ N-S: $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla P + \rho\underline{g} + \mu\nabla^2\underline{u}$ ∴

ignore gravity ∴ $\underline{g} = 0$ ∴

$\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla P + \mu\nabla^2\underline{u}$ ∴

$\partial_t\underline{u} = \partial_t(W(R)\hat{\underline{z}}) = \partial_t(W(R))\hat{\underline{z}} = 0\hat{\underline{z}} = 0$ ∴

$\rho(\underline{u}\cdot\nabla\underline{u}) = -\nabla P + \mu\nabla^2\underline{u}$ ∴

$\nabla = \hat{\underline{R}}\,\partial_R + \hat{\underline{\phi}}\,\frac{1}{R}\partial_\phi + \hat{\underline{z}}\,\partial_z$ ∴

$\underline{u}\cdot\nabla = (W(R)\hat{\underline{z}})\cdot\nabla = W(R)\,\partial_z$ ∴

$\underline{u}\cdot\nabla\underline{u} = W(R)\,\partial_z(W(R)\hat{\underline{z}}) = W(R)\,\partial_z(W(R))\hat{\underline{z}} = W(R)(0)\hat{\underline{z}} = 0$ ∴

$\rho(0) = 0 = -\nabla P + \mu\nabla^2\underline{u}$ ∴ $\nabla P = \mu\nabla^2\underline{u}$

\div/ $\nabla P = \mu\nabla^2\underline{u}$ ∴

z-component: $\partial_z P = \mu(\nabla^2\underline{u})_z$ ∴

$\nabla^2\underline{u} = \nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{u})$ ∴

$\nabla\cdot\underline{u} = 0$ ∴ $\nabla(\nabla\cdot\underline{u}) = \nabla(0) = 0$ ∴

$\nabla^2\underline{u} = -\nabla\times(\nabla\times\underline{u})$ ∴ in cylindrical coords:

$$\nabla\times\underline{u} = \nabla\times(W(R)\hat{\underline{z}}) = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & 0 & W(R) \end{vmatrix} = \frac{1}{R}\left[\hat{\underline{R}}(0-0) - R\hat{\underline{\phi}}(\partial_R W(R) - 0) + \hat{\underline{z}}(0-0)\right] =$$

$$\frac{1}{R}(-R)\hat{\underline{\phi}}\,\partial_R W(R) = -\hat{\underline{\phi}}\,\partial_R W(R) \doteq -\partial_R W(R)\,\hat{\underline{\phi}}$$

$$-\nabla\times(\nabla\times\underline{u}) = -\nabla\times(-\partial_R W(R)\hat{\underline{\phi}}) = -\frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & -R\partial_R W(R) & 0 \end{vmatrix} =$$

$$-\frac{1}{R}\left[\hat{\underline{R}}(0-0) - R\hat{\underline{\phi}}(0-0) + \hat{\underline{z}}(\partial_R(-R\partial_R W) - 0)\right] =$$

$$-\frac{1}{R}\partial_R(-R\partial_R W)\hat{\underline{z}} = -\left(-\frac{1}{R}\partial_R(R\partial_R W)\hat{\underline{z}}\right) = \frac{1}{R}\partial_R(R\partial_R W)\hat{\underline{z}}$$ ∴

$\partial_z P = \mu\frac{1}{R}\partial_R(R\partial_R W)$ ∴

$\frac{1}{\mu}\partial_z(P)R = \partial_R(R\partial_R W)$ ∴

$\frac{1}{\mu}\partial_z(P)\frac{1}{2}R^2 + A = R\partial_R W = \frac{1}{2\mu}\partial_z(P)R^2 + A \therefore$

$\frac{1}{2\mu}\partial_z(P)R + A\frac{1}{R} = \partial_R W \therefore$

$W = \frac{1}{2\mu}\partial_z(P)\frac{1}{2}R^2 + A\ln R + B = \frac{1}{4\mu}\partial_z(P)R^2 + A\ln R + B$

\ div / $\nabla P = \mu\nabla^2 \underline{u} \therefore$

z-component: $\partial_z P = \mu[\nabla^2 \underline{u}]_z \therefore$

$\nabla^2 \underline{u} = \nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{u}) \therefore$

$\nabla\cdot\underline{u} = 0 \therefore \nabla(\nabla\cdot\underline{u}) = \nabla(0) = 0 \therefore$

$\nabla^2 \underline{u} = -\nabla\times(\nabla\times\underline{u}) \therefore$ in cylindrical coords:

$\nabla\times\underline{u} = \nabla\times(W(R)\hat{z}) = \frac{1}{R}\begin{vmatrix}\hat{R} & R\hat{\phi} & \hat{z}\\ \partial_R & \partial_\phi & \partial_z\\ 0 & 0 & W(R)\end{vmatrix} =$

$\frac{1}{R}[\hat{R}(0-0) - R\hat{\phi}(\partial_R W(R) - 0) + \hat{z}(0-0)] =$

$\frac{1}{R}[-R\hat{\phi}(\partial_R W(R))] = -\partial_R W(R)\hat{\phi} \therefore$

$-\nabla\times(\nabla\times\underline{u}) = -\nabla\times(-\partial_R W(R)\hat{\phi}) = -\frac{1}{R}\begin{vmatrix}\hat{R} & R\hat{\phi} & \hat{z}\\ \partial_R & \partial_\phi & \partial_z\\ 0 & -R\partial_R W(R) & 0\end{vmatrix} =$

$-\frac{1}{R}[\hat{R}(0-0) - R\hat{\phi}(0-0) + \hat{z}(\partial_R(-R\partial_R W(R)) - 0)] =$

$-\frac{1}{R}[\hat{z}(\partial_R(-R\partial_R W(R)))] = -\frac{1}{R}\partial_R(-R\partial_R W(R))\hat{z} = \frac{1}{R}\partial_R(R\partial_R W(R))\hat{z} \therefore$

$[\nabla^2 \underline{u}]_z = \frac{1}{R}\partial_R(R\partial_R W(R)) \therefore$

$\partial_z P = \mu\frac{1}{R}\partial_R(R\partial_R W(R)) \therefore$

$\frac{1}{\mu}\partial_z(P)R = \partial_R(R\partial_R W(R)) \therefore$

$\frac{1}{\mu}\partial_z(P)\frac{1}{2}R^2 + A = R\partial_R W(R) = \frac{1}{2\mu}\partial_z(P)R^2 + A \therefore$

$\frac{1}{2\mu}\partial_z(P)R + A\frac{1}{R} = \partial_R W(R) \therefore$

$W(R) = \frac{1}{2\mu}\partial_z(P)\frac{1}{2}R^2 + A\ln(R) + B = \frac{1}{4\mu}\partial_z(P)R^2 + A\ln(R) + B \therefore$

let $\frac{\partial W}{\partial R}\Big|_{R=0} = 0 \therefore$

$\cancel{\partial_R W(R=0) = \frac{1}{2\mu}\partial_z(P)(0)} - \frac{1}{2\mu}\partial_z(P)(0)^2 + A = (0)\partial_R W(R=0) = 0 = 0 + A =$

$A = 0 \therefore$

$W(R) = \frac{1}{4\mu}\partial_z(P)R^2 + B \therefore$

let $W(R=a) = 0 \because$ of noslip BC $\therefore$

$W(R=a) = 0 = \frac{1}{4\mu}\partial_z(P)a^2 + B \therefore B = -\frac{1}{4\mu}\partial_z(P)a^2 = -\frac{a^2}{4\mu}\partial_z(P) \therefore$

$W(R) = \frac{1}{4\mu}\partial_z(P)R^2 - \frac{a^2}{4\mu}\partial_z(P)$

\pp2019/ \div/ $w = \frac{1}{4\mu}\partial_z(P)R^2 + A\ln(R) + B$ ∴

let $\partial_R(w(R))|_{R=0} = 0$ ∴

$R\partial_R w(R) = \frac{1}{4\mu}\partial_z(P)R^2 + A$ ∴

~~(o)~~ at $R=0$ ∴ $(0)\partial_R(w(R=0)) = \frac{1}{2\mu}\partial_z(P)(0)^2 + A = 0 = 0 + A = A = 0$ ∴

$w = \frac{1}{4\mu}\partial_z(P)R^2 + B$ ∴

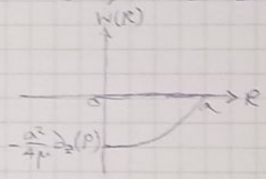
$w(R=a) = 0 = \frac{1}{4\mu}\partial_z(P)a^2 + B$ ∴

$B = -\frac{a^2}{4\mu}\partial_z(P)$, $A = 0$.

\ldv/ $w(R) = \frac{1}{4\mu}\partial_z(P)R^2 - \frac{a^2}{4\mu}\partial_z(P) = \frac{1}{4\mu}\partial_z(P)(R^2 - a^2)$, ~~$a < R \le \infty$~~

$R \ge 0$, $R \le a$ ∴ $0 \le R \le a$ ∴

$W(R=a) = 0$,



~~$\because$ at $R \ge 0$~~ ∴ $W(0) = -\frac{a^2}{4\mu}\partial_z(P)$ ∴

∴ Max speed is at $R=0$ and is: $W(R) = \frac{a^2}{4\mu}\partial_z(P)$

$\underline{dS} = \hat{\underline{n}}\, dS$ ∴ ~~$\hat{\underline{n}} = \hat{\underline{R}}$~~ ∴

$Q = \int \underline{u} \cdot \underline{dS} = \int w(R)\hat{\underline{z}} \cdot \underline{dS} = \int \left(\frac{1}{4\mu}\partial_z(P)(R^2 - a^2)\hat{\underline{z}}\right) \cdot \underline{dS} = \frac{1}{4\mu}\partial_z(P)\int (R^2 - a^2)\hat{\underline{z}} \cdot \underline{dS}$

∴ $\hat{\underline{z}} = \hat{\underline{n}}$ ∴ $\underline{dS} = \hat{\underline{n}} \cdot dS = \hat{\underline{n}} \cdot R\,dR\,d\theta = \hat{\underline{z}} \cdot R\,dR\,d\theta = R\,\hat{\underline{z}}\,dR\,d\theta$ ∴

$Q = \frac{1}{4\mu}\partial_z(P)\int (R^2 - a^2)\hat{\underline{z}} \cdot \underline{dS} = \frac{1}{4\mu}\partial_z(P)\int (R^2 - a^2)\hat{\underline{z}} \cdot R\,\hat{\underline{z}}\,dR\,d\theta =$

$\frac{1}{4\mu}\partial_z(P)\int (R^2 - a^2)R\,dR\,d\theta = \frac{1}{4\mu}\partial_z(P)\int_0^{2\pi}\int_0^a (R^2 - a^2)R\,dR\,d\theta =$

$\frac{1}{4\mu}\partial_z(P)\,2\pi\int_0^a (R^2 - a^2)R\,dR = \frac{\pi}{2\mu}\partial_z(P)\int_0^a R^3 - a^2R\,dR =$

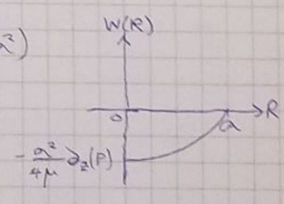
$\frac{\pi}{2\mu}\partial_z(P)\left[\frac{1}{4}R^4 - \frac{1}{2}a^2R^2\right]_{R=0}^{a} = \frac{\pi}{2\mu}\partial_z(P)\left[\frac{1}{4}a^4 - \frac{1}{2}a^2a^2\right] =$

$\frac{\pi}{2\mu}\partial_z(P)\left[\frac{1}{4}a^4 - \frac{1}{2}a^4\right] = \frac{\pi}{2\mu}\partial_z(P)\left(-\frac{1}{4}\right)a^4 = \frac{-a^4\pi}{8\mu}\partial_z(P)$

\ldv/ $w(R) = \frac{1}{4\mu}\partial_z(P)R^2 - \frac{a^2}{4\mu}\partial_z(P) = \frac{1}{4\mu}\partial_z(P)(R^2 - a^2)$

$R \ge 0$, $R \le a$ ∴ $0 \le R \le a$ ∴

$w(R=a) = 0$, ∴



∴ Max speed at $R=0$ ∴

$w(0) = -\frac{a^2}{4\mu}\partial_z(P)$ ∴

$|W(0)| = \frac{a^2}{4\mu}\partial_z(P)$ is Max speed ∴

$\underline{dS} = \hat{\underline{n}}\, dS$ ∴ $dS = R\, dR\, d\phi$ ∴ $\boxed{D} \rightarrow \hat{\underline{z}} = \hat{\underline{n}}$ ∴ $\hat{\underline{z}} = \hat{\underline{n}}$ ∴

$\underline{dS} = \hat{\underline{z}} \cdot R\, dR\, d\phi = R\, \hat{\underline{z}}\, dR\, d\phi$ ∴

$Q = \int \underline{u} \cdot \underline{dS} = \int (w(R)\, \hat{\underline{z}}) \cdot R\, \hat{\underline{z}}\, dR\, d\phi = \int R\, w(R)\, dR\, d\phi =$

$\int_0^{2\pi} \int_0^a R\, w(R)\, dR\, d\phi = 2\pi \int_0^a R\, w(R)\, dR =$

$2\pi \int_0^a R \times \frac{1}{4\mu} \partial_z(p)(R^2 - a^2)\, dR = \frac{\pi}{2\mu} \partial_z(p) \int_0^a R^3 - a^2 R\, dR =$

$\frac{\pi}{2\mu} \partial_z(p) \left[\frac{1}{4} R^4 - \frac{a^2}{2} R^2\right]_{R=0}^{a} = \frac{\pi}{2\mu} \partial_z(p) \left[\frac{1}{4} a^4 - \frac{a^2}{2} a^2\right] =$

$\frac{\pi}{2\mu} \partial_z(p) \left(-\frac{1}{4}\right) a^4 = \frac{-\pi a^4}{8\mu} \partial_z(p)$

\e i/ $\phi = a e^{i(kx - \omega t)}$ ∴ $\phi_t = -\omega i a e^{i(kx - \omega t)}$, ∴

$\phi_{tt} = -\omega i(-\omega i) a e^{i(kx - \omega t)} = -\omega^2 a e^{i(kx - \omega t)}$,

$\phi_x = k i a e^{i(kx - \omega t)}$ ∴ $\phi_{xx} = k i (k i) a e^{i(kx - \omega t)} = -k^2 a e^{i(kx - \omega t)}$ ∴

into PDE: $-\omega^2 a e^{i(kx - \omega t)} = -c^2 k^2 a e^{i(kx - \omega t)}$ ∴

$-\omega^2 = -c^2 k^2$ ∴ $\omega^2 = c^2 k^2$ ∴

$\omega = \pm\sqrt{c^2 k^2} = \pm ck$

\e i/ $\phi = a e^{i(kx - \omega t)}$ ∴ $\phi_t = -\omega i a e^{i(kx - \omega t)}$ ∴

$\phi_{tt} = -\omega i(-\omega i) a e^{i(kx - \omega t)} = -\omega^2 a e^{i(kx - \omega t)}$,

$\phi_x = k i a e^{i(kx - \omega t)}$ ∴ $\phi_{xx} = k i (k i) a e^{i(kx - \omega t)} = -k^2 a e^{i(kx - \omega t)}$ ∴

into PDE: $\omega^2 (-a e^{i(kx - \omega t)}) = c^2 k^2 (-a e^{i(kx - \omega t)})$ ∴

$\omega^2 = c^2 k^2$ ∴ $\omega = \pm\sqrt{c^2 k^2} = ck$

\e ii/ $c_p = \frac{\omega}{k}$, $c_g = \frac{d\omega}{dk}$ ∴ c_p is phase speed ∴

$\omega = \pm\sqrt{gk} = \pm\sqrt{g}\, k^{1/2}$ ∴ $c_p = \frac{\omega}{k} = \pm\sqrt{g}\, k^{1/2} k^{-1} = \pm\sqrt{g} \frac{1}{\sqrt{k}} = \pm\sqrt{\frac{g}{k}}$

is phase speed,

group speed is: $c_g = \frac{d\omega}{dk}$ ∴ $\cancel{\frac{1}{g}\omega^2 = k}$ ∴

$\cancel{c_g = \frac{d\omega}{dk}} = \frac{d}{dk}(\omega^2) = 2\omega \frac{d\omega}{dk} = \frac{d}{dk}(gk) = g$ ∴

$c_g = \frac{d\omega}{dk} = \frac{g}{2\omega}$ ∴

waves of different wavelengths λ and ∴ different wave numbers k travel at different phase speeds $c_p = \pm\sqrt{\frac{g}{k}}$ ∴ c_p is a function of k ∴ the waves are dispersive

\pp2019/ \1eii/ $c_p = \frac{\omega}{k}$, $c_g = \frac{d\omega}{dk}$, c_p is phase speed, c_g is group speed $\therefore$

$\omega = \pm\sqrt{gk} = \pm\sqrt{g}\, k^{1/2}$ $\therefore$

phase ~~spheed~~ speed is: $c_p = \frac{\omega}{k} = \omega k^{-1} = \pm\sqrt{g}\, k^{1/2} k^{-1} = \pm\sqrt{g}\, k^{-1/2} = \pm\sqrt{\frac{g}{k}}$

$\frac{d(\omega^2)}{dk} = 2\omega \frac{d\omega}{dk} = \frac{d}{dk}(gk) = g$ $\therefore$

group ~~speed~~ speed: $c_g = \frac{d\omega}{dk} = \frac{g}{2\omega}$ $\therefore$

waves of different wavelengths λ and $\therefore$ ~~wave numbers~~ k ~~travel~~ different wavenumbers k travel at different ~~spho~~ phase speeds $c_p = \pm\sqrt{\frac{g}{k}}$ $\because$ c_p is a function of k $\therefore$ The waves are dispersive.

\2a/ N-S: $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \rho \underline{g} + \mu \nabla \underline{u}$, $\nabla\cdot\underline{u} = 0$ $\therefore$

$\underline{u} \sim U$, $x \sim L$, $\nabla \sim \frac{1}{L}$, $T = \frac{L}{U}$ $\therefore$ $\frac{\partial}{\partial t} \sim \frac{1}{T} = \frac{U}{L}$ $\therefore$

$\partial_t \underline{u} = \frac{\partial}{\partial t}(\underline{u}) \sim \frac{U}{L} U = \frac{U^2}{L}$,

$\underline{u}\cdot\nabla \sim U\frac{1}{L}$ $\therefore$ $\underline{u}\cdot\nabla\underline{u} \sim U\frac{1}{L}U = \frac{U^2}{L}$,

$\nabla^2\underline{u} = \nabla(\nabla\underline{u}) \sim \frac{1}{L}(\frac{1}{L}U) = \frac{U}{L^2}$ $\therefore$

N-S: $(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\frac{1}{\rho}\nabla p + \underline{g} + \frac{\mu}{\rho}\nabla^2\underline{u}$

$\therefore$ when $R_e \ll 1$: $\frac{\mu}{\rho}\nabla^2\underline{u} = \nu\nabla^2\underline{u} \sim \nu\frac{U}{L^2}$ $\therefore$

$\frac{U^2}{L} \sim \frac{U^2}{L}$ $\therefore$ $\partial_t\underline{u} \sim \underline{u}\cdot\nabla\underline{u}$ $\therefore$

$\frac{(\partial_t \underline{u})}{\nu\nabla^2\underline{u}} \sim \frac{U^2/L}{\nu U/L^2} = \frac{UL}{\nu} = R_e$ $\therefore$

for $R_e \ll 1$ $\therefore$ $\frac{UL}{\nu} \ll 1$ $\therefore$ $(U^2/L)/(\nu U/L^2) \ll 1$ $\therefore$

$\frac{U^2}{L} \ll \nu\frac{U}{L^2}$ $\therefore$

$\partial_t\underline{u}$, $\underline{u}\cdot\nabla\underline{u} \ll \nu\nabla^2\underline{u} = \frac{\mu}{\rho}\nabla^2\underline{u}$ $\therefore$

may neglect $\partial_t\underline{u}$, $\underline{u}\cdot\nabla\underline{u}$ $\therefore$

~~$0 = -\nabla p$~~ $0 = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\underline{u}$ $\therefore$

$-\nabla p + \mu\nabla^2\underline{u} = 0$ and incompressible $\therefore$ $\nabla\cdot\underline{u} = 0$

\2a/ N-S: $\rho(\partial_t\underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \rho \underline{g} + \mu\nabla^2\underline{u}$,

incompressible $\therefore$ $\nabla\cdot\underline{u} = 0$ $\therefore$

$\underline{u} \sim U$, $x \sim L$ $\therefore$ $\nabla \sim \frac{1}{L}$, $T = \frac{L}{U}$ $\therefore$ $\partial_t \sim \frac{1}{T} = \frac{U}{L}$ $\therefore$

$\partial_t \underline{u} \sim \frac{U}{L} U = \frac{U^2}{L}$,

$\underline{u}\cdot\nabla = U\frac{1}{L} \therefore \underline{u}\cdot\nabla\underline{u} = U\frac{1}{L}U = \frac{U^2}{L}$

$\partial_t\underline{u} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p + g + \frac{\mu}{\rho}\nabla^2\underline{u} \therefore$

$\frac{\mu}{\rho}\nabla^2\underline{u} = \nu\nabla^2\underline{u} = \nu\nabla(\nabla\underline{u}) = \nu\frac{1}{L}\left(\frac{1}{L}U\right) = \nu\frac{U}{L^2} \therefore$

$\cancel{R_e = \frac{UL}{\nu} = U^2}$ $\frac{(\partial_t \underline{u})}{\nu\nabla^2\underline{u}} \sim \frac{U^2/L}{\nu U/L^2} = \frac{UL}{\nu} = R_e \therefore$

for $R_e \ll 1 \therefore \frac{UL}{\nu} \ll 1 \therefore \frac{U^2/L}{\nu U/L^2} \ll 1 \therefore \frac{U^2}{L} \ll \nu\frac{U}{L^2} \doteq \frac{\mu}{\rho}\frac{U}{L^2} \therefore$

$\partial_t \underline{u} \ll \frac{\mu}{\rho}\nabla^2\underline{u}, \underline{u}\cdot\nabla\underline{u} \ll \frac{\mu}{\rho}\nabla^2\underline{u} \therefore$

may neglect $\partial_t\underline{u}$, $\underline{u}\cdot\nabla\underline{u} \therefore$

$\rho(0+0) = 0 = -\nabla p + \mu\nabla^2\underline{u}$ $\qquad \nabla\cdot\underline{u} = 0$

\2b/ $\nabla^4\psi = \nabla^2(\nabla^2\psi) \therefore$ $\cancel{0 = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\underline{u}}$ $\quad 0 = -\nabla p + \mu\nabla^2\underline{u} \therefore$

$\cancel{\nabla\times(0) = 0 =}$ $\quad \nabla p = \mu\nabla^2\underline{u} \therefore$

$\cancel{\nabla}$ cdg $\quad \nabla\times(\nabla p) = 0 = \nabla\times(\mu\nabla^2\underline{u}) = \mu\nabla\times(\nabla^2\underline{u}) = 0 \therefore$

$\nabla\times(\nabla^2\underline{u}) = 0 = \nabla^2(\nabla\times\underline{u}) = \nabla^2[\nabla\times(\nabla\times(\psi\hat{\underline{z}}))] =$

$\nabla^2[\nabla(\nabla\cdot(\psi\hat{\underline{z}})) - \nabla^2(\psi\hat{\underline{z}})] = 0 \therefore$

cdg $\therefore \nabla(\nabla\cdot(\psi\hat{\underline{z}})) = \nabla(\nabla\cdot(\psi(R,\phi)\hat{\underline{z}}))$ $\therefore$

$\nabla\cdot(\psi(R,\phi)\hat{\underline{z}}) = \partial_z\psi(R,\phi) = 0 \therefore$

$\nabla(\nabla\cdot(\psi\hat{\underline{z}})) = \nabla(0) = 0 \therefore$

$\nabla^2[0 - \nabla^2(\psi\hat{\underline{z}})] = 0 = \nabla^2[-\nabla^2(\psi\hat{\underline{z}})] = \cancel{-\nabla^4\psi}$

$-\nabla^2[\nabla^2(\psi)\hat{\underline{z}}] = -\nabla^2[\nabla^2(\psi)]\hat{\underline{z}} = 0 \therefore$

$\nabla^2[\nabla^2(\psi)] = \nabla^4\psi = 0$

\2b/ $\nabla^4\psi = \nabla^2(\nabla^2\psi) \therefore \qquad 0 = -\nabla p + \mu\nabla^2\underline{u} \therefore$

$\nabla p = \mu\nabla^2\underline{u} \therefore \qquad$ cdg

$\nabla\times(\nabla p) = 0 = \nabla\times(\mu\nabla^2\underline{u}) = \mu\nabla\times(\nabla^2\underline{u}) = 0 \therefore$

$\nabla\times(\nabla^2\underline{u}) = 0 = \nabla^2[\nabla\times(\nabla\times(\psi\hat{\underline{z}}))] =$

$\nabla^2[\nabla(\nabla\cdot(\psi\hat{\underline{z}})) - \nabla^2(\psi\hat{\underline{z}})] = 0 \therefore$

$\nabla(\nabla\cdot(\psi\hat{\underline{z}})) = \nabla(\nabla\cdot(\psi(R,\phi)\hat{\underline{z}})) = \nabla(\partial_z(\psi(R,\phi))) =$

$\nabla(0) = 0 \therefore$

$\nabla^2[0 - \nabla^2(\psi\hat{\underline{z}})] = 0 = \nabla^2[-\nabla^2(\psi)\hat{\underline{z}}] = -\nabla^2[\nabla^2(\psi)]\hat{\underline{z}} = 0 \therefore$

$\nabla^2[\nabla^2(\psi)] = \nabla^4(\psi) = 0$

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\2C/ $\underline{u}=\nabla\times(\psi\hat{\underline{z}})=\underline{u}=\nabla\times(f(\phi)\hat{\underline{z}})=\frac{1}{R}\begin{vmatrix}\hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}}\\ \partial_R & \partial_\phi & \partial_z \\ 0 & 0 & f(\phi)\end{vmatrix}=\frac{1}{R}\hat{\underline{R}}\left(\frac{\partial}{\partial\phi}(f(\phi))\right)=\frac{1}{R}\hat{\underline{R}}f'(\phi)=$

$\frac{f'(\phi)}{R}\hat{\underline{R}}=\underline{u}=u(R,\phi)\hat{\underline{R}}$, for $u(R,\phi)=\frac{f'(\phi)}{R}$

$\therefore$ $\underline{u}$ only has a component along $\hat{\underline{R}}$ $\therefore$ it is a radial flow

\2C/ $\underline{u}=\nabla\times(\psi\hat{\underline{z}})=\underline{u}=\nabla\times(f(\phi)\hat{\underline{z}})=\frac{1}{R}\begin{vmatrix}\hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}}\\ \partial_R & \partial_\phi & \partial_z \\ 0 & 0 & f(\phi)\end{vmatrix}=\frac{1}{R}\hat{\underline{R}}\left(\frac{\partial}{\partial\phi}(f(\phi))\right)=\frac{1}{R}\hat{\underline{R}}f'(\phi)=$

$\frac{f'(\phi)}{R}\hat{\underline{R}}=\underline{u}=u(R,\phi)\hat{\underline{R}}$, for $u(R,\phi)=\frac{f'(\phi)}{R}$ $\therefore$

u only has a component along $\hat{\underline{R}}$ $\therefore$ it is a radial flow

\2d✓ $\nabla^4\psi=\nabla^2(\nabla^2\psi)=\nabla^2(\nabla^2 f(\phi))=\nabla^2\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}f(\phi)\right)+\frac{1}{R^2}\frac{\partial^2}{\partial\phi^2}f(\phi)+\frac{\partial^2 f(\phi)}{\partial z^2}\right]$

$=\nabla^2\left[\frac{1}{R}\frac{\partial}{\partial R}(R\times 0)+\frac{1}{R^2}f''(\phi)+0\right]=\nabla^2\left[\frac{1}{R^2}f''(\phi)\right]=\cancel{\frac{1}{R}\frac{\partial}{\partial R}\left(R\times\frac{\partial}{\partial R}\left(\frac{1}{R^2}f''(\phi)\right)\right)}$

$\frac{1}{R}\frac{\partial}{\partial R}\left(R\times\frac{\partial}{\partial R}\left(\frac{1}{R^2}f''(\phi)\right)\right)+\frac{1}{R^2}\frac{\partial}{\partial\phi^2}\left(\frac{1}{R^2}f''(\phi)\right)+\frac{\partial^2}{\partial z^2}\left(\frac{1}{R}f''(\phi)\right)=$

$\frac{1}{R}\frac{\partial}{\partial R}\left(R(-2)\frac{1}{R^3}f''(\phi)\right)+\frac{1}{R^4}\frac{\partial}{\partial\phi^2}(f''(\phi))+0=$

$\frac{-2}{R}f''(\phi)\frac{\partial}{\partial R}\left(\frac{1}{R^2}\right)+\frac{1}{R^4}f^{(4)}(\phi)=\frac{-2}{R}f''(\phi)(-2)\frac{1}{R^3}+\frac{1}{R^4}f''''(\phi)=$

$\frac{4}{R^4}f''(\phi)+\frac{1}{R^4}f''''(\phi)=0=f''''(\phi)+4f''(\phi)=f''''+4f''=0$

\2d/ $0=\nabla^4\psi=\nabla^2(\nabla^2\psi)=\nabla^2(\nabla^2 f(\phi))=\nabla^2\left[\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}f(\phi)\right)+\frac{1}{R^2}\frac{\partial}{\partial\phi^2}f(\phi)+\frac{\partial^2 f(\phi)}{\partial z^2}\right]$

$=\nabla^2\left[\frac{1}{R}\frac{\partial}{\partial R}(R\times 0)+\frac{1}{R^2}f''(\phi)+0\right]=\nabla^2\left[\frac{1}{R^2}f''(\phi)\right]=$

$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\left(\frac{1}{R^2}f''(\phi)\right)\right)+\frac{1}{R^2}\frac{\partial^2}{\partial\phi^2}\left(\frac{1}{R^2}f''(\phi)\right)+\frac{\partial^2}{\partial z^2}\left(\frac{1}{R^2}f''(\phi)\right)=$

$\frac{1}{R}\frac{\partial}{\partial R}\left(Rf''(\phi)\frac{\partial}{\partial R}\left(\frac{1}{R^2}\right)\right)+\frac{1}{R^2}\frac{1}{R^2}\frac{\partial^2}{\partial\phi^2}(f''(\phi))+\frac{1}{R^2}f''(\phi)\frac{\partial}{\partial z^2}(0)=$

$\frac{1}{R}\frac{\partial}{\partial R}\left(Rf''(\phi)(-2)\frac{1}{R^3}\right)+\frac{1}{R^4}f''''(\phi)+0=$

$\frac{-2}{R}f''(\phi)\frac{\partial}{\partial R}\left(\frac{1}{R^2}\right)+\frac{1}{R^4}f''''(\phi)=\frac{-2}{R}f''(\phi)(-2)\frac{1}{R^3}+\frac{1}{R^4}f''''(\phi)=$

$\frac{4}{R^4}f''(\phi)+\frac{1}{R^4}f''''(\phi)=0=4f''(\phi)+f''''(\phi)=f''''+4f''=0$

\2e/ let $g=f''$ $\therefore$ $g''=f''''$ $\therefore$

$g''+4g=0$ $\therefore$ $q^2+4=0$ $\therefore$ $q^2=-4$ $\therefore$ $q=\pm\sqrt{-4}=\pm\sqrt{4}\,i=\pm 2i$ $\therefore$

$g=E\sin(2\phi)+F\cos(2\phi)=f''$ $\therefore$

$\cancel{f'=E\sin(2\phi)+F\cos(2\phi)}$ $f'=-\frac{1}{2}E\cos(2\phi)+\frac{1}{2}F\sin(2\phi)+B$ $\therefore$

$f(\phi)=-\frac{1}{4}E\sin(2\phi)+\frac{-1}{4}F\cos(2\phi)+B\phi+A=$

$A+B\phi+C\sin(2\phi)+D\cos(2\phi)=f=\psi$ $\therefore$ $f'=B+2C\cos(2\phi)-2D\sin(2\phi)$ $\therefore$

$u = u(R,\phi) = \frac{f'(\phi)}{R} = B\frac{1}{R} + 2C\frac{1}{R}\cos(2\phi) \mp 2D\frac{1}{R}\sin(2\phi)$

$Ru(R,\phi) = B + 2C\cos(2\phi) - 2D\sin(2\phi)$

$\therefore$ at $\phi = \alpha$: $u = 0 = u(R,\alpha)$ $\therefore$

$R(0) = B + 2C\cos(2\alpha) - 2D\sin(2\alpha) = 0$. $\therefore$

$2D\sin(2\alpha) = B + 2C\cos(2\alpha)$ $\therefore$

$D = \frac{B}{2\sin(2\alpha)} + \frac{2C\cos(2\alpha)}{2\sin(2\alpha)} = \frac{B}{2\sin(2\alpha)} + \frac{C\cos(2\alpha)}{\sin(2\alpha)}$ $\therefore$

$\cancel{u(R,\phi) = B\frac{1}{R} + 2C\frac{1}{R}}$

$Ru(R,\phi) = B + 2C\cos(2\phi) - \frac{2\sin(2\phi)B}{2\sin(2\alpha)} - \frac{2\sin(2\phi)C\cos(2\alpha)}{\sin(2\alpha)}$ $\therefore$

$u(R,-\alpha) = 0$ $\therefore$

at $\phi = -\alpha$:

$R(0) = 0 = B + 2C\cos(-2\alpha) - \frac{2\sin(-2\alpha)B}{2\sin(2\alpha)} - \frac{2\sin(-2\alpha)C\cos(2\alpha)}{\sin(2\alpha)} =$

$B + 2C\cos(2\alpha) - \frac{2(-1)\sin(2\alpha)B}{2\sin(2\alpha)} - \frac{2(-1)\sin(2\alpha)C\cos(2\alpha)}{\sin(2\alpha)} =$

$B + 2C\cos(2\alpha) + B + 2C\cos(2\alpha) = \cancel{4R}\ 2B + 4C\cos(2\alpha) = 0 =$

$B + 2C\cos(2\alpha) = 0$ $\therefore$ $B = -2C\cos(2\alpha)$ $\therefore$

$Ru(R,\phi) = -2C\cos(2\alpha) + 2C\cos(2\phi) - \frac{2\sin(2\phi)(-2)C\cos(2\alpha)}{2\sin(2\alpha)} - \frac{2\sin(2\phi)C\cos(2\alpha)}{\sin(2\alpha)}$

$= 2C\cos(2\phi) - 2C\cos(2\alpha) + \frac{2\sin(2\phi)C\cos(2\alpha)}{\cancel{2}\sin(2\alpha)} - \frac{2\sin(2\phi)C\cos(2\alpha)}{\sin(2\alpha)}$

$= 2C\cos(2\phi) - 2C\cos(2\alpha) = 2C(\cos(2\phi) - \cos(2\alpha)) = Ru(R,\phi) =$

$C(\cos(2\phi) - \cos(2\alpha))$ $\therefore$

$u(R,\phi) = CR^{-1}(\cos 2\phi - \cos 2\alpha)$

2e $f''''(\phi) + 4f''(\phi) = 0$ $\therefore$ let $g = f'' = g(\phi)$ $\therefore$ $g''(\phi) = f''''$ $\therefore$

$g''(\phi) + 4g(\phi) = 0$ $\therefore$ $q^2 + 4 = 0$ $\therefore$ $q = \pm 2i$ $\therefore$

$g(\phi) = D_1\sin(2\phi) + E_1\cos(2\phi) = f''(\phi)$ $\therefore$

$f'(\phi) = -\frac{1}{2}D_1\cos(2\phi) + \frac{1}{2}E_1\sin(2\phi) + F = A\cos(2\phi) + B\sin(2\phi) + F$ $\therefore$

$u = u(R,\phi) = \frac{f'(\phi)}{R}$ $\therefore$ $Ru(R,\phi) = f'(\phi) = A\cos(2\phi) + B\sin(2\phi) + F$

$\therefore$ ~~at~~ at $\phi = -\alpha$: $u(R,\phi=-\alpha) = u(R,-\alpha) = 0$ $\therefore$

$R(0) = A\cos(-2\alpha) + B\sin(-2\alpha) + F = A\cos(2\alpha) - B\sin(2\alpha) + F = 0$ $\therefore$