\Week 2 Sheet/ \3a/ $\dot{x}_1 = x_2$ $\dot{x}_2 = -x_1 - 2\tan^{-1}(x_1+x_2)$ to determine equili

$\therefore \dot{x} = f(x)$ form so $\dot{x} = 0$ ie $x_2 = 0$ $-x_1 - 2\tan^{-1}(x_1+x_2) = 0$ $\therefore x_1 = 2\tan^{-1}(x_1)$ $\therefore \tan\left(\frac{x_1}{2}\right) = x_1$

$\therefore x_1 \approx \pm 2.33$ $\therefore$ syst has 3 equili pts at $(x_1^*, x_2^*)^T$ $\therefore (0,0)^T, (2.33,0)^T, (-2.33,0)^T$

Jacobian Mat $\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ 1 - \frac{2}{1+(x_1+x_2)^2} & -\frac{2}{1+(x_1+x_2)^2} \end{bmatrix}$ $\frac{\partial f}{\partial x}\Big|_{(0,0)^T} = \begin{bmatrix} 0 & 1 \\ -1 & -2 \end{bmatrix}$ $\therefore \lambda_{1,2} = -1, -1$

multiple eignvals at -1 $\therefore$ stable star $\therefore$ $\frac{\partial f}{\partial x}\Big|_{(2.33,0)^T} = \begin{bmatrix} 0 & 1 \\ 0.6892 & -0.3108 \end{bmatrix}$ $\therefore \lambda_{1,2} = 0.6892, -1$

$\therefore (2.33, 0)^T$ is saddle as one eignval on LH plane & saddle other RH plane

similarly $(-2.33, 0)^T$ is also a saddle $\therefore$ in Z given diag

\3b/ $0 = x_2$ $0 = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2)$ $\therefore x_2 = 0$ & $x_1 = 0$ $\exists$ a unique equili pt at Z origin

$\frac{\partial f}{\partial x}\Big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix}$ $\therefore \lambda_{1,2} = \frac{1}{2} \pm i\frac{\sqrt{3}}{2}$ $\therefore (0,0)^T$ is unstable focus $\therefore \exists$ a periodic orbit

Z origin from Z diag (periodic orbit & existence)

\4/ $\dot{x}_1 = -x_1^2 + x_1 + \sin x_2 \leftarrow f_1$ $\dot{x}_2 = \cos x_2 - x_1^3 - 5x_2 \leftarrow f_2$ $\therefore$ at equili $x_0 = (1, 0)^T$

Jacobian $\frac{\partial f}{\partial x} = \begin{bmatrix} -2x_1+1 & \cos x_2 \\ -3x_1^2 & -\sin x_2 - 5 \end{bmatrix}\Big|_{(1,0)} = \begin{bmatrix} -1 & 1 \\ -3 & -5 \end{bmatrix}$ eignvals are -2 & -4 $\therefore$ both eignval are

real nega $(1, 0)^T$ is a stable node

\5a/ False $\therefore$ diag Q2 is saddle

\5b true

\5c/ False $\therefore$ $\dot{x}_1 = x_2$ $\dot{x}_2 = -h(x_1, x_2)\sin x_2^{t^2} - 5x_1x_2$ has explicit appearence of 't' on

RHS of dynamics $\therefore$ not auto syst . non auto-time varying syst

\5d/ false set equili space corres to one or both eignvals are zero . $\pm i$ p corres to centre

Week 2 Sheet

$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 1 \\ 0.1-2x_1-0.3x_1^2 & -2 \end{bmatrix}$ $\therefore$ $\frac{\partial f}{\partial x}\Big|_{(0,0)^T} = \begin{bmatrix} -1 & 1 \\ 0.1 & -2 \end{bmatrix}$ $\therefore$ $\lambda_{1,2} = -2.07, -0.9$

$\therefore (0,0)$ is stable node, $\therefore$ $\frac{\partial f}{\partial x}\Big|_{(-2.76,-2.76)^T} = \begin{bmatrix} -1 & 1 \\ 0.1+2(2.76)-2.76^2(0.3) & -2 \end{bmatrix}$ $\frac{\partial f}{\partial x}\Big|_{(0,0)^T} = \begin{bmatrix} -1 & 1 \\ \cancel{0.1} & -2 \end{bmatrix}$ $\therefore$

$\lambda_{1,2} = -3.39, 0.393$ $\therefore (-2.76, -2.76)^T$ is saddle

$\frac{\partial f}{\partial x}\Big|_{(-7.23,-7.23)^T} = \begin{bmatrix} -1 & 1 \\ -1.168 & -2 \end{bmatrix}$ & $\lambda_{1,2} = -1.5 \pm 0.958i$ $\therefore Re(\lambda) < 0$ $\therefore (-7.23, -7.23)$ is stable focus

1b

$0 = x_1(1+x_2)$ ①

$0 = -x_2 + x_2^2 + x_1x_2 - x_1^3$ ② $\therefore$ ①: $x_1 = 0$ or $x_2 = -1$ $\therefore$ $x_1 = 0$ $\therefore 0 = -x_2 + x_2^2$ $\therefore x_2 = 0$ or $x_2 = 1$

$\therefore x_2 = -1$ $\therefore 0 = 2 - x_1 - x_1^3$ $\therefore x_1 = 1$ $\therefore$ have 3 equili pts $(0,0)^T$, $(0,1)^T$, $(1,-1)^T$ Z

Jacobian of $\frac{\partial f}{\partial x} = \begin{bmatrix} 1+x_2 & x_1 \\ x_2-3x_1^2 & -1+2x_2+x_1 \end{bmatrix}$ $\frac{\partial f}{\partial x}\Big|_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ $\therefore \lambda_{1,2} = 1, -1$ $\therefore (0,0)^T$ is $\therefore$ saddle

(+ve & -ve real eigvals)

$\frac{\partial f}{\partial x}\Big|_{(0,1)^T} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ $\therefore$ $\lambda_{1,2} = 2, 1$ 2 +eigvals $\therefore (0,1)^T$ is unstable node

$\frac{\partial f}{\partial x}\Big|_{(1,-1)^T} = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix}$ $\therefore \lambda_{1,2} = -1 \pm i\sqrt{2}$ $\therefore Re(\lambda) < 0$ $\therefore (1,-1)^T$ is stable focus

1c

$0 = (x_1 - x_2)(1 - x_1^2 - x_2^2)$ $0 = (x_1 + x_2)(1 - x_1^2 - x_2^2)$ $x_1^2 + x_2^2 = 1$ is an equili set

$(0,0)$ is an isolated equili pt $\frac{\partial f}{\partial x}\Big|_{(0,0)^T} = \begin{bmatrix} 1-3x_1^2-x_2^2+2x_1x_2 & -2x_1x_2-1+1x_1^2+3x_2^2 \\ 1-3x_1^2-x_2^2-2x_1x_2 & -2x_1x_2+1-x_1^2-3x_2^2 \end{bmatrix}_{(0,0)^T} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ $\therefore$

$(0,0)^T$ is an unstable focus

1d

$0 = (1-x_1)x_1 - \frac{2x_1x_2}{1+x_1}$ $0 = \left(2 - \frac{x_2}{1+x_1}\right)x_2$ $\therefore x_2 = 0$, $x_2 = 2(1+x_1)$ $\therefore$ $x_2 = 0$ $\therefore x_1 = 0, x_1 = 1$ $\therefore$

$x_2 = 2(1+x_1)$ $\therefore 0 = (x_1+3)x_1$ $\therefore x_1 = 0, x_1 = -3$ $\therefore$ 4 equili pts $\therefore (0,0)^T$, $(0,2)^T$, $(1,0)^T$, $(-3,-4)^T$

Jacobian: $\frac{\partial f}{\partial x} = \begin{bmatrix} 1-2x_1-\frac{2x_2}{(1+x_1)^2} & \frac{-2x_1}{1+x_1} \\ x_2^2/(1+x_1)^2 & 2-\frac{2x_2}{1+x_1} \end{bmatrix}$ $\therefore$ $\frac{\partial f}{\partial x}\Big|_{(0,0)^T} = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ $\lambda_{1,2} = 1, 2$ $\therefore$ 2 real +ve $\therefore$

$(0,0)$ is unstable node

$\frac{\partial f}{\partial x}\Big|_{(1,0)^T} = \begin{bmatrix} -1 & -1 \\ 0 & 2 \end{bmatrix}$ $\lambda_{1,2} = -1, 2$ $\therefore$ -ve, +ve $\therefore (1,0)^T$ is saddle

$\frac{\partial f}{\partial x}\Big|_{(0,2)^T} = \begin{bmatrix} -3 & 0 \\ 4 & -2 \end{bmatrix}$ $\lambda_{1,2} = -3, -2$ 2 real -ve $\therefore (0,2)^T$ stable node

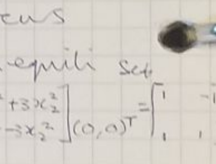
$\frac{\partial f}{\partial x}\Big|_{(-3,-4)^T} = \begin{bmatrix} 9 & -3 \\ 4 & -2 \end{bmatrix}$ $\lambda_{1,2} = 7.772, -0.772$ $\therefore$ +ve, -ve $\therefore (-3,-4)^T$ is saddle

2a

$\dot{x}_1 = x_1 - x_2$ $\dot{x}_2 = x_1^2 + x_2^2 - 2$ $\therefore 0 = x_1 - x_2$ $\therefore x_1 = x_2$, $0 = x_1^2 + x_2^2 - 2$ $\therefore x_1 = \pm 1$ $\therefore$ 2 equili

pts $(1,1)^T$ & $(-1,-1)^T$ $\frac{\partial f}{\partial x} = \begin{bmatrix} 1 & -1 \\ 2x_1 & 2x_2 \end{bmatrix}$ $\frac{\partial f}{\partial x}\Big|_{(1,1)} = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$ $\therefore \lambda_{1,2} = \frac{3}{2} \pm \frac{\sqrt{7}}{2}i$ $\because Re(\lambda) > 0$ $\therefore$

& complex, $(1,1)^T$ is unstable focus/spiral

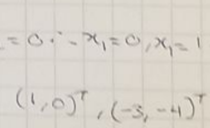


$\frac{\partial f}{\partial x}\Big|_{(-1,-1)^T} = \begin{bmatrix} 1 & -1 \\ -2 & -2 \end{bmatrix}$ $\therefore$ $\lambda_{1,2} = -\frac{1}{2} \pm \frac{\sqrt{17}}{2}$ $\therefore$ $\lambda_1 = 1.56$, $\lambda_2 = -2.56$

real eigvals +ve & -ve $\therefore (-1,-1)^T$ is saddle

2b

$\dot{x}_1 = x_1 - x_1x_2$ $\dot{x}_2 = x_2 + 2x_1x_2$ $0 = x_1 - x_1x_2$ $\therefore x_1(1-x_2) = 0$ $\therefore x_1 = 0$ or $x_1 = 1$ $\therefore$



$0 = x_2 + 2x_1x_2$ $\therefore x_2(1+2x_1) = 0$ $\therefore x_2 = 0$ or $x_1 = -\frac{1}{2}$ $\therefore$ equili pts are $(0,0)^T$ & $(-\frac{1}{2}, 1)^T$

$\frac{\partial f}{\partial x} = \begin{bmatrix} 1-x_2 & -x_1 \\ 2x_2 & 1+2x_1 \end{bmatrix}$ $\frac{\partial f}{\partial x}\Big|_{(0,0)^T} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ $\therefore \lambda_{1,2} = 1$ $\therefore$ 2 repeated eigvals

which are +ve $\therefore$ unstable star $\therefore$ for NL consider as unstable node $\therefore$ $\frac{\partial f}{\partial x}\Big|_{(-\frac{1}{2},1)^T} = \begin{bmatrix} 0 & \frac{1}{2} \\ 2 & 0 \end{bmatrix}$ $\therefore \lambda_{1,2} = 1, -1$

$\therefore (-\frac{1}{2}, 1)^T$ is saddle pt. $\therefore$

Week 8 / 1 / $\dot{x} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x_1^2 x_2 \end{bmatrix}$ $\therefore A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$

$\therefore \det(A - \lambda I) = \det\begin{bmatrix} -\lambda & -1 \\ 1 & -1-\lambda \end{bmatrix} = \lambda(1+\lambda) + 1(1) = \lambda^2 + \lambda + 1 \doteq 0 \therefore$

$\lambda = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1}{2} \pm \frac{\sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \therefore$

$\cancel{\lambda_{1,2}}$ $Re(\lambda_{1,2}(A)) = -\frac{1}{2} < 0$, $\lambda_{1,2}(A) \notin \mathbb{R} \therefore$ A is hurwitz

system $\dot{x} = Ax$ is ~~stable~~ asymp stable origin

$\therefore$ by converse Lyapunov argument $\therefore$

$P = P^T > 0 \quad \forall Q = Q^T > 0$ satisfying $\therefore$

$A^T P + PA = -Q \quad \therefore$ let $Q = -I \therefore \quad A^T P + PA = -I \therefore$

$P = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} \therefore \quad A^T = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \therefore \quad A^T P = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix} = \begin{bmatrix} P_2 & P_3 \\ -P_1 - P_2 & -P_2 - P_3 \end{bmatrix} \therefore$

$PA = \begin{bmatrix} P_1 & P_2 \\ P_2 & P_3 \end{bmatrix}\begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} P_2 & -P_1 - P_2 \\ P_3 & -P_2 - P_3 \end{bmatrix} \therefore$

$A^T P + PA = \begin{bmatrix} 2P_2 & -P_1 - P_2 + P_3 \\ -P_1 - P_2 + P_3 & -2P_2 - 2P_3 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \therefore$

$2P_2 = -1 \therefore P_2 = -\frac{1}{2} \therefore \quad -2(-\frac{1}{2}) - 2P_3 = -1 \therefore -2P_3 = -2 \therefore P_3 = 1 \therefore$

$-P_1 - P_2 + P_3 = 0 = -P_1 + \frac{1}{2} + 1 = -P_1 + \frac{3}{2} = 0 \therefore P_1 = \frac{3}{2} \therefore \quad P = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1 \end{bmatrix}$

$\therefore \det(P - \lambda I) = \det\begin{bmatrix} 3/2 - \lambda & -1/2 \\ -1/2 & 1-\lambda \end{bmatrix} = (\frac{3}{2} - \lambda)(1-\lambda) + \frac{1}{2}(-\frac{1}{2}) =$

$\frac{3}{2} - \frac{3}{2}\lambda - \lambda + \lambda^2 - \frac{1}{4} = \lambda^2 - \frac{5}{2}\lambda + \frac{5}{4} = 0 \therefore$

$\lambda = \frac{\frac{5}{2} \pm \sqrt{(-\frac{5}{2})^2 - 4(1)(\frac{5}{4})}}{2(1)} = \frac{5}{4} \pm \frac{\sqrt{\frac{5}{4}}}{2} = \frac{5}{4} \pm \frac{\sqrt{5}}{4} \therefore \quad \lambda_1 = 0.691, \ \lambda_2 = 1.8 \therefore$

$\lambda_{min} = 0.691 = \lambda_{min}(P) \therefore$

let $V = x^T P x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} \frac{3}{2}x_1 - \frac{1}{2}x_2 \\ -\frac{1}{2}x_1 + x_2 \end{bmatrix} = \frac{3}{2}x_1^2 - \frac{1}{2}x_1 x_2 - \frac{1}{2}x_1 x_2 + x_2^2 =$

$\frac{3}{2}x_1^2 - x_1 x_2 + x_2^2 \quad \therefore \dot{x}_1 = -x_2, \ \dot{x}_2 = x_1 + x_1^2 x_2 - x_2 \therefore$

$\dot{V} = 3x_1\dot{x}_1 - x_2\dot{x}_1 - x_1\dot{x}_2 + 2x_2\dot{x}_2 =$

$3x_1(-x_2) - x_2(-x_2) - x_1(x_1 + x_1^2 x_2 - x_2) + 2x_2(x_1 + x_1^2 x_2 - x_2) =$

$-3x_1 x_2 + x_2^2 - x_1^2 - x_1^3 x_2 + x_1 x_2 + 2x_1 x_2 + 2x_1^2 x_2^2 - 2x_2^2 =$

$x_2^2 - x_1^2 - x_1^3 x_2 + 2x_1^2 x_2^2 - 2x_2^2 = -x_2^2 - x_1^2 - x_1^3 x_2 + 2x_1^2 x_2^2 = -(x_1^2 + x_2^2) - x_1^2 x_2(x_1 + 2x_2) =$

$-\|x\|_2^2 - x_1^2 x_2(x_1 + 2x_2)$ ~~$\leq -\|x\|_2^2 \neq 0$~~ $= -\|x\|_2^2 - x_1(x_1 x_2)(x_1 + 2x_2) \leq$

$-\|x\|^2 - \|x_1\| \|x_1 x_2\| \|x_1 + 2x_2\| \leq -\|x\|^2 - \|x_1\| \|x\|^2 \frac{1}{2} \|x_1 + 2x_2\| =$

$-\|x\|^2 - \frac{1}{2}\|x_1\| \|x\|^2 \|x_1 + 2x_2\| \leq -\|x\|^2 - \frac{1}{2}\|x_1\| \|x\|^2 \sqrt{1^2 + (2)^2} \|x\| = \cancel{-\|x\|}$

$-\|x\|^2 - \frac{\sqrt{5}}{2}\|x_1\| \|x\|^2 \|x\| \leq -\|x\|^2 - \frac{\sqrt{5}}{2}\|x\| \|x\|^2 \|x\| = -\|x\|^2 - \frac{\sqrt{5}}{2}\|x\|^4 < 0 \quad \forall x \neq 0 \therefore$

$\therefore \dot{V}$ is n.d.f $\therefore$ origin is globally asymp stable is $\dot{V} \leq -\|x\|^2\left(1 - \frac{\sqrt{5}}{2}\|x\|^2\right) \leq 0$ if $1 - \frac{\sqrt{5}}{2}\|x\|^2 \geq 0$ $\therefore$

$\frac{2}{\sqrt{5}} \geq \|x\|^2 \geq 0$ $\quad\therefore\quad$ let $r^2 = \frac{2}{\sqrt{5}}$ $\therefore$

$\Omega : \{x \in \mathbb{R}^2 \mid V < \lambda_{min}(P) \times r^2\} = \{x \in \mathbb{R}^2 \mid V < \frac{0.691}{\sqrt{5}} \times 2\} = \{x \in \mathbb{R}^2 \mid V < 0.618\}$

is region of attraction stability

$V(\cdot)$ p.d.f $\dot{V}$ n.d.f asymp stable

$V(\cdot)$ p.d.f $\dot{V} \le -\varepsilon V$, $\varepsilon > 0$ exponentially stable $\left\{ \dot{V} = \frac{\partial V}{\partial x} \cdot f(x) \right\}$

- p.d.f is posi definite

n.s.d.f is nega semi definite

n.d.f is nega definite

when $\dot{V}(x) < 0$ $\quad \overbrace{\frac{\partial V}{\partial x}}^{\text{grad}V} \cdot \overbrace{f(x)}^{\text{vec field}} < 0 \implies$ Angle subtended betw $\frac{\partial V}{\partial x}$ & $f(x)$ is greater than $90°$ Z deriv $\dot{V}(\cdot)$ along a phase trajec is everywhere nega in domain D then Z trajec $x(t)$ tends to Z origin, ie syst is Stable. Otherwise, $\dot{V}$ is +ve, traject moves away from origin, repel off origin, ie syst is unstable for that choice of V

$\dot{V}(x)$ p.d.f $\frac{\partial V}{\partial x} \cdot f(x) < 0$ $\quad \left|\frac{\partial V}{\partial x}\right|$ Gradient vec (Grad V) normal to level surf it pts in Z direct of greatest rate of increase in $V(\cdot)$

Quadratic forms

$$[x_1 \; x_2]\begin{bmatrix} M_{11} & M_{12} \\ M_{21} & M_{22} \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Def: a mat M is posi definite, if $x^T M x > 0$, $\forall x \neq 0$ x is vec $x \in \mathbb{R}^n$

a mat M is posi semi definite if $x^T M x \ge 0$ $\forall x$

Lemma: $M = M^T$ is posi definite $\iff \lambda_i(M) > 0$, $\forall i$ (eignvals of M)

posi semi definite $\iff \lambda_i(M) \ge 0$, $\forall i$

{if $M = M^T$ p.d.f.} then $V(x) = x^T M x$ fulfills $V(0) = 0$ & $V(x) > 0$ $\forall x \neq 0$ P.D.F & $\lambda_{min} \|x\|^2 \le x^T M x \le \lambda_{max} \|x\|^2$ {λ_{min} & λ_{max} are min, max eignvals of M}

a func $V: \mathbb{R}^n \to \mathbb{R}$ of Z form $V(x) = x^T M x = \sum_{i,j=1}^{n} M_{ij} x_i x_j$ is called a quadratic form

remark: $M \in \mathbb{R}^{n \times n}$ is symmetric, ie $M = M^T$ eignvals of M are real

thm Lyapunov thm - Global Stability thm

- consider $\dot{x} = f(x)$, $x \in \mathbb{R}^n$; $\exists$ a contly diffable $V(\cdot): \mathbb{R}^n \to \mathbb{R}$

st $\left.\begin{array}{l} V(0) = 0 \\ V(x) > 0, x \in \mathbb{R}^n, x \neq 0 \end{array}\right\}$ p.d.f

$\frac{\partial V}{\partial x} \cdot f(x) \longrightarrow \dot V(x) < 0, \; x \in \mathbb{R}^n \setminus \{0\}$ ← N.d.f

$V(x) \to \infty$ as $\|x\| \to \infty$ (radially unbounded func) ∴ Z zero sol $x(t) = 0$ or origin, to $\dot x = f(x)$ is globally asymp stab

if $\exists\, \alpha, \beta, \varepsilon > 0$ & $p \geq 1$ st $V(\cdot): \mathbb{R}^n \to \mathbb{R}$ satisfies $\alpha\|x\|^p \leq V(x) \leq \beta\|x\|^p$, $x \in \mathbb{R}^n$ $\dot V(x) = \frac{\partial V}{\partial x} f(x) \leq -\varepsilon V(x)$, $x \in \mathbb{R}^n$ ∴ Zero sol to $\dot x = f(x)$ is globally exponentially stab

{① $V(\cdot)$ radially unbounded ② $V(\cdot): \mathbb{R}^n \to \mathbb{R}$ (not D) conditions need to satisfy in $\mathbb{R}^n$ not just a D}

Ex/consider $\Sigma: \begin{cases} \dot x_1 = x_2 \\ \dot x_2 = -a\sin x_1 - b x_2 \end{cases}$ $\{a, b > 0\}$ ∴ candidate func:

$V(x) = \underbrace{a(1-\cos x_1)}_{\text{bounded func} \geq 0} + \frac{1}{2}x_2^2$ ← square term a, b are +ve $V(x)$ is p.d.f ∵ $V(0,0) = 0$

∵ $V(x) > 0 \; \forall x \in \mathbb{R}^2 \setminus \{0,0\}$ taking time deriv: $\dot V(x) = (a\sin x_1)\dot x_1 + x_2 \dot x_2$

~~$x = (x_1, x_2)$~~ $= -b x_2^2$ {$x = (x_1, x_2)^T$ (V N.D.F $\Rightarrow V(0,0) = 0$

$\dot V(x) < 0 \; \forall x \in D \setminus \{0\}$ } $\dot V(x)$ is nega semi definite

$\dot V(x) = 0$ for $x_2 = 0$, & no matter what Z val of x_1

origin is stab in Z sense of Lyapunov

{Failure of a candidate func to satisfy conds for stability, or asymp stability, does not mean that equilib is not stab, or not asymp stab. Note: Lyapunov candidate conds are sufficiency conds}

∴ one can modify existing $V(\cdot)$ $V_1(x) = \frac{1}{2}x^T P x + a(1-\cos x_1)$

{presently $\frac{1}{2}x_2^2 \Rightarrow \frac{1}{2}x^T \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} x$ a more generic P to be considered $P = P^T$ present in $V(\cdot)$ $x = [x_1, x_2]$ $P = \begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}$ }

$V_1(x) = \overbrace{\frac{1}{2}[x_1 \; x_2] \underbrace{\begin{bmatrix} p_{11} & p_{12} \\ p_{12} & p_{22} \end{bmatrix}}_{P} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}^{+ve} + a(1-\cos x_1)$ P is posit definite $P = P^T > 0$

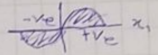
(quadratic form) $p_{11} > 0$ ∵ P is 2 dimen mat

$p_{11}p_{22} - p_{12}^2$ all Z leading principle minors of Z mat posi ensures Z posi definiteness of P mat   {taking deriv of $V_1(\cdot)$ along Z trajects of dynamics} $\dot V_1(x) = (p_{11}x_1 + p_{12}x_2 + a\sin x_1)x_2 + (p_{12}x_1 + p_{22}x_2)(-a\sin x_1 - b x_2) =$

$a(1-P_{22})x_2 \sin x_1$ $- a P_{12} x_1 \sin x_1$ $+ (P_{11} - P_{12} b) x_1 x_2$ $+ (P_{12} - P_{22} b) x_2^2$

0 no struc & can't claim definiteness So $1 - P_{22} = 0$ is ideal $P_{22} = 1$

+ve term; becomes −ve; +ve



this term sign indefinite ∴ best to have $P_{11} - P_{12} b = 0$ ∴ $P_{12} = \frac{b}{2}$ can choose

−ve term; can select $P_{22} = 1$

$\dot{V}_1(x) = -\frac{1}{2} a b x_1 \sin x_1 - \frac{1}{2} b x_2^2$ $\quad V_1(x)$ pdf

$\dot{V}_1(x)$ NDF in $D = \{x \in \mathbb{R}^2 \mid |x_1| < \pi\}$ ∴ origin is asymp stab locally in set D

made use of quadratic form with a known struc to obtain a Lyapunov form

choice of $P = \begin{bmatrix} P_{11} & P_{21} \\ P_{12} & P_{22} \end{bmatrix}$ $\quad \cdot P_{11} = b P_{12} \quad \cdot P_{12} = \frac{b}{2} \quad \cdot P_{22} = 1$

Ex/ consider dynamics $\Sigma : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a \sin x_1 - b x_2 \end{cases}$ $\quad \{\dot{V}(x) = -b x_2^2 \text{ for } \Sigma$

choice of $V(x)$ $\begin{bmatrix} f_1(\cdot) \\ f_2(\cdot) \end{bmatrix}$ }

if $x_2 = 0 \Rightarrow \dot{x}_2 = 0$ ∴ $0 = -a \sin x_1 - b x_2$ ∴ $-a \sin x_1 = 0$ ∴ $x_1 = 0$ $\quad -\pi \quad \pi$ only at origin

From struc we make sure confinement of $x_1 = 0$ ∴ on segment $-\pi < x_1 < \pi$ of line $x_2 = 0$, Σ syst can maintain $\dot{V}(x) = 0$ condition only at $x_1 = 0$ & $x_2 = 0$ ie $(0,0)$. $V(\cdot)$ p.d.f, $\dot{V}(\cdot)$ N.S.D.F, by above claim origin of syst Σ is locally asymp stab

Def/ a set $M \subset \mathbb{R}^n$ is said to be invariant set wrt $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ if $x(0) \in M$, implies $x(t) \in M$, $\forall t \in \mathbb{R}$

Def:/ a set $M \subset \mathbb{R}^n$ is said to be positively invariant set wrt $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ if $x(0) \in M$ implies $x(t) \in M$, $\forall t \geq 0$

Def:/ a pt 'p' is the posi limit pt of $\dot{x} = f(x)$, if $\exists$ a sequence of $\{t_n\}$, with $t_n \to \infty$, as $n \to \infty$ st $x(t_n) \to p$ as $n \to \infty$. The set of all posi limit pts of $x(t)$ is called posi limit set of $x(t)$

La Salle's Invariance principle/ let dynamics be $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ let $\Omega \subset D \subset \mathbb{R}^n$ be a compact positively invariant set $\{x(0) \in \Omega \Rightarrow x(t) \in \Omega \ \forall t \geq 0\}$ wrt $\dot{x} = f(x)$

Let $V(x) : D \to \mathbb{R}$ be a contly diffable func st $\dot{V}(x) \leq 0$ in Ω {negat semi definite} let $E \subset \Omega$ be a set of all pts in Ω where $\dot{V}(x) \equiv 0$

define $M \subset E$ be Z largest invariant set E. then $\forall$ sols starting within Ω approaches M as $t \to \infty$

set in which $\dot{V}(x) \equiv 0$ — Domain where $V(\cdot)$ is defined $V(\cdot): D \to \mathbb{R}$

$$M \subset E \subset \Omega \subset D \subset \mathbb{R}^n \longleftrightarrow \text{n dimen state space } \dot{x} = f(x),\ x \in \mathbb{R}^n$$

largest invariant set in E — Ω is +vely invariant compact set in which $\dot{V}(x) \leq 0$, ie nega semi definiteness of $V(x)$

all trajects $x(t)$ are bounded & approach a posi limit set $L^+ \leq M$ as $t \to \infty$. asymp stab equili posi limit pt M our interest is to show $x(t) \to 0$ as $t \to \infty$ establish largest invariant set in M is origin, ie $M = \{0\}$

\Corollary: LaSalle's Corollary/ thms of Barbashin & Krasovskii

let $x = 0$ be equili pt for $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ let $V(\cdot): D \to \mathbb{R}$ be contly diffable posi definite func on domain D containing Z origin $x = 0$, st $\dot{V}(x) \leq 0$ in D.

~~$\dot{V}(x) \equiv 0$~~ let $S = \{x \in D \mid \dot{V}(x) \equiv 0\}$ & suppose no sols can ~~be~~ identically stay in S, other than Z trivial sol $x(t) \equiv 0$ then origin asymp stab (locally)

\Week 8/ \Lyapunov Stability - LTI syst/

\Ex/ $\dot{x} = Ax$, $x \in \mathbb{R}^n$ LTI syst $A \in \mathbb{R}^{n \times n}$

if $Re(\lambda_i(A)) < 0$ $[A]$ is Hurwitz

λ_i → eignvals of A

& origin of $\dot{x} = Ax$ is asymp stab

\Ex/ $V(x) = x^T P x$, where $P = P^T > 0$ $P \in \mathbb{R}^{n \times n}$ is real symmetric posi definite matrix. {recall $(AB^T = B^T A^T)$}

$\dot{V}$ nega definite

$$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T P(Ax) + (Ax)^T P x = x^T [PA + A^T P] x = -x^T Q x,$$

where Q is symmetric posi defin, $-Q$ is nega defin

in Z case of linear systs $\dot{x} = Ax$, $x \in \mathbb{R}^n$, suppose start by choosing Q as real symmetric posi defin mat $(Q = Q^T > 0)$ solving for $PA + A^T P = -Q$, for P. if $PA + A^T P = -Q$ has a posi definite sol then we can conclude Z origin is globally asymp stable (ie $x \in \mathbb{R}^n$)

known linear systMat

$PA + A^T P = -Q$ ← can be our choice $P = P^T > 0$ $Q = Q^T > 0$ $A \in \mathbb{R}^{n \times n}$ $Q, P \in \mathbb{R}^{n \times n}$

is Lyapunov eqn physics $\begin{cases} V(x) = x^T P x & \text{- generalised energy} \\ \dot V(\cdot) = -x^T Q x & \text{- generalised dissipation} \end{cases}$

for $\dot x = Ax$, if $P = P^T > 0$, $Q = Q^T > 0$ then all <u>trajectories</u> <u>are bounded</u> & $[A]$ is <u>hurwitz</u> mat ← $Re(\lambda_i(A)) < 0 \quad i = 1, \dots, n$

& <u>associated</u> <u>ellipsoidal</u> <u>set</u> $D = \{x \in \mathbb{R}^n \mid \underbrace{x^T P x}_{V(x)} \le C\}$ is <u>invariant</u>

recall $\lambda_{min}(Q) x^T x \le x^T Q x \le \lambda_{max}(Q) x^T x$ (min eignval of Q mat; max eignval of Q mat) $Q = Q^T \quad P = P^T > 0$

& $\lambda_{min}(P) x^T x \le x^T P x \le \lambda_{max}(P) x^T x$ $\therefore -x^T Q x \le -\lambda_{min}(Q) x^T x$

$\therefore \dot V(x) \le -\lambda_{min}(Q) x^T x \le -\lambda_{min}(Q) \left[\frac{x^T P x}{\lambda_{min}(P)}\right]$ $\left\{ \lambda_{min}(P) x^T x \le x^T P x \quad x^T x \le \frac{x^T P x}{\lambda_{min}(P)} \right\}$

$\le -\frac{\lambda_{min}(Q)}{\lambda_{min}(P)} (V(x)) = -\alpha V(x)$ $\alpha = \frac{\lambda_{min}(Q)}{\lambda_{min}(P)}$

$\therefore$ exponentially stable

\ Converse Lyapunov argument thm / A mat $[A]$ is hurwitz, (associate: $\dot x = Ax$) that is $Re(\lambda_i(A)) < 0$ $\forall$ eignvals of A, iff for any given posi defi symmetric mat P that satisfies the Lyapunov eqn $PA + A^T P = -Q$ for some $Q = Q^T > 0$ & if A is hurwitz, then P is the unique sol of $PA + A^T P = -Q$.

if $[A]$ is hurwitz, $\exists$ a P st $V = x^T P x$ becomes the Lyapunov func associated with $\dot x = Ax$

\ nonlinear / perturbation bound /

\ Ex / $\dot x = f(x)$, $x \in \mathbb{R}^n$ $f : D \to \mathbb{R}^n$ $x = 0$ is in Domain D, $f(0) = 0$

& it is an equili pt. it is possible to write in certain cases

$\dot x = \overbrace{Ax}^{\text{linear part}} + g(x)$ ← nonlinear/perturbed part wrt linear syst $x \in \mathbb{R}^n$

where $A = \frac{\partial f}{\partial x}(0)$ Jacobian evaluated at origin. &

$|g_i(x)| \le \| \frac{\partial f_i}{\partial x}(z_i) - \frac{\partial f_i}{\partial x}(0) \| \, \|x\|$ (from mean val thm)

possible to write: $\frac{\|g(x)\|}{\|x\|} \to 0$ as $\|x\| \to 0$ (a class of nonlinearity)

Suppose in $\dot x = Ax + g(x)$ $[A]$ to be hurwitz matrix $\therefore Re(\lambda_i(A)) < 0$

$\dot x = Ax$ is stable LTI syst origin is equili. By converse Lyapunov argument $\exists$ a $V(x) = x^T P x$, $P = P^T > 0$ st $PA + A^T P = -Q$ for a $Q = Q^T > 0$

$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T P[Ax + g(x)] + [Ax + g(x)]^T P x$ (subbing for $\dot{x}$)

$= x^T(PA + A^T P)x + 2x^T P g(x) = -x^T Q x + 2x^T P g(x)$ ← this is sign indefinite term

a neg definite term $Q = Q^T > 0$ ∴ yes P is posi defin

∵ $\frac{\|g(x)\|}{\|x\|} \to 0$ as $\|x\| \to 0$ (class of nonlinearity satisfies this condition)

for any $r > 0$ $\exists\, \gamma > 0$ st $\|g(x)\| < \gamma\|x\|$ $\forall \|x\| < r$ (bound on Z nonlinearity)

∴ $\dot{V}(x) \le -x^T Q x + 2\|x\|\,\|P\|\,\|g(x)\| \le -x^T Q x + 2\gamma\|x\|\,\|x\|\,\|P\|$, $\forall \|x\| < r$

$\le -\lambda_{min}(Q)\|x\|^2 + 2\gamma\|x\|^2\|P\|$, $\forall \|x\| < r$ $\quad \|AB\| \le \|A\|\,\|B\|$

$\le -[\lambda_{min}(Q) - 2\gamma\|P\|]\|x\|^2$, $\forall \|x\| < r$

as long as this term is posi $\quad \dot{V}$ is neg defin in $\|x\| < r$

$\lambda_{min}(Q) = 2\gamma\|P\| = 0 \quad \lambda_{min}(Q) = 2\gamma\|P\| \quad \frac{\lambda_{min}(Q)}{2\|P\|} = \gamma$

for that choose $\gamma < \frac{1}{2}\frac{\lambda_{min}(Q)}{\|P\|}$

\region of attraction/ Let $\dot{x} = f(x)$, $x \in \mathbb{R}^n$ $\quad f: D \to \mathbb{R}^n$

* $\phi(t, x)$ sol $\quad$ * $x = 0$ is asymp stable

\Def/ Z region of attraction

$R_A = \{x \in D \mid \phi(x,t)$ is defined $\forall t \ge 0$, & $\phi(x,t) \to 0$ as $t \to \infty\}$

$x = 0$ asymp stable, R_A is open, connected invariant set, but hard to find

an esti of R_A (not true R_A) is a compact posi invariant sub-set in the domain D, where $V: D \to \mathbb{R}$ (Lyapunov func) defined as in Lyapunov thm

$\Omega_c = \{x \in \mathbb{R}^n \mid V(x) \le c\}$ is simplest compact set $\Omega_c \subset D$ st every traject in Ω_c stays in Ω_c $\forall$ future time ∵ we have $V(x) = x^T P x$ &

$D = \{\|x\|_2 < r\}$ we ensure $\Omega_c \subset D$ by choosing

$c < \min_{\|x\|_2 = r} x^T P x = \lambda_{min}(P) \underset{\uparrow \|x\|^2}{r^2} \qquad \Omega_c = \{x \in \mathbb{R}^n \mid V(x) \le \lambda_{min}(P) r^2\}$

\week 9/

\Ex/ $\dot{x} = f(x) + g(x)u$ (a class of nonlinear systs)

$\underset{\leftarrow \text{state}}{x \in \mathbb{R}^n} \quad \underset{\leftarrow \text{control}}{u \in \mathbb{R}}$ $\quad f(\cdot)$ & $g(\cdot)$ Lipschitz in domain D.

where $\exists$ a state feedback control $u = \alpha(x) + \beta(x)v$ & chg of variabs $z = T(x)$ that ∴ nonlinear syst to equivalent linear syst st closed loop syst can be made stable designing a v using linear control techniques

Consider Stability

Ex/ stability of origin $\Sigma : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -a[\sin(x_1+\delta) - \sin\delta] - bx_2 + cu \end{cases}$

- in Σ a control $u(\cdot)$ to be designed

A choice of $u := \frac{a}{c}[\sin(x_1+\delta) - \sin\delta] + \frac{\nu}{c}$ $(*)$ (of Z form $\alpha(x) + \underbrace{\beta(x)}_{1/c}\nu$)

ensures the non linear term $a[\sin(x_1+\delta) - \sin\delta]$ in Σ is cancelled.

Sub u from $(*)$ in Σ yields $\Sigma_l : \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -bx_2 + \nu \end{cases}$ is a linear probl

$$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & -b \end{bmatrix}}_{A} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \nu \qquad \dot{x} = Ax + B\nu$$

in Σ_l we need to determine ν to stabilize origin. do by linear (pole placement) design techniques learned earlier $\therefore$

$\nu = -k_1x_1 - k_2x_2$ (of Z form $-kx$) are able to follow a pole placement design Z closed loop is $\begin{matrix} \dot{x}_1 = x_2 \\ \dot{x}_2 = -k_1x_1 - (k_2+b)x_2 \end{matrix}$ $(A-Bk)$ having desired eignals

one can choose arbitrary desired eignals locations on Z LHP of complex domain & design $k = [k_1 \ \ k_2]$

overall Z state feedback control law for original syst Σ becomes

$u = \left(\frac{a}{c}\right)[\sin(x_1+\delta) - \sin\delta] - \frac{1}{c}(k_1x_1 + x_2x_2)$ is $\alpha(x) + \beta(x)\nu$ form

this is a nonlinear part of control law (which originally allowed us to have a linear equiv form)

linear part of control law designed using pole placement ideas ensuring certain performance through Z choice of desired eignals

presence of 'δ' term in Σ & control accounts Z regulation probl

- $\theta \to \delta$ a specific angle of interest.

is it possible to cancel nonlinearity always? in Z Ex we exploited certain structural property of syst, which allowed to cancel non linearity. in Σ, Z $\dot{x}_2$ eqn, 'sin(·)' in terms & 'u' are aligned.

channel $\dot{x}_2 = -\underbrace{a[\sin(x_1-\delta) - \sin\delta]}_{\leftarrow \text{nonlinearity}} - bx_2 + cu$ ← control was in same channel ← in Z same state channel

this need not be Z case with every non linear syst.

To cancel a nonlinear term $(\alpha(x))$ by subtraction/addition Z control 'u' & Z nonlinearity $\alpha(x)$ must appear in Z same channel.

- To cancel $\gamma(x)$, a nonlinear term, Z control 'u' & non linearity $\gamma(x)$ need to appear as product eg $\gamma(x)u$

($cu \to \gamma(x)$ which was a const) & $\gamma(x)$ must be non singular in Z domain of interest. then it can be cancelled by $u=\beta(x)\nu$; $\beta(x)=\gamma(x)^{-1}$

Z ability to use feedback to convert a nonlinear state eqn to a <u>controllable</u> <u>linear</u> <u>state</u> eqn by cancelling nonlinearities

requires following structure for Z sys

$\circledast$ $\quad \dot{x} = Ax + B\gamma(x)[u - \alpha(x)] \qquad x \in \mathbb{R}^n, \; u \in \mathbb{R}^p$ ← generic multiple inputs

$A \in \mathbb{R}^{n\times n} \quad B \in \mathbb{R}^{n\times p} \quad (A, B)$ controllable

$\alpha(x): \mathbb{R}^n \to \mathbb{R}^p \quad \gamma(x): \mathbb{R}^n \to \mathbb{R}^{p\times p}$ non singular in domain $D \subset \mathbb{R}^n$

matched in same channel as u containing origin

then can linearize Z syst via feedback $u = \alpha(x) + \beta(x)\nu$;

$\beta(x) = \gamma(x)^{-1}$ yields $\dot{x} = Ax + B\nu$ (is linear form (A,B) controllable)

now design $\nu = -kx$ st $(A - Bk)$ is hurwitz & possesses Z eignvals at desired locations. overall nonlinear control law

$u = \alpha(x) - \beta(x)kx$

what if Z nonlinear syst doesn't have structure in $\circledast$ $\dot{x} = Ax + B\gamma(x)[u - \alpha(x)]$

\Ex/ $\begin{cases} \dot{x}_1 = a\sin x_2 \\ \dot{x}_2 = -x_1^2 + u \end{cases}$ clearly this is not in Z form of $\dot{x} = Ax + B\gamma(x)[u - \alpha(x)]$

define new variabs chng of coords if we pick $z_1 = x_1$ &

$z_2 = a\sin x_2 = \dot{x}_1 \qquad (x_1, x_2) \xrightarrow{T(x)} (z_1, z_2)$

$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} z_1 \\ \sin^{-1}(z_2/a) \end{bmatrix} \qquad \begin{aligned} x_1 &= z_1 \\ x_2 &= \sin^{-1}(z_2/a) \end{aligned}$ which is well defined for $-a < z_2 < a$

Z transformed eqns then becomes $\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= a\cos x_2(-x_1^2 + u) \end{aligned}$

recall $\begin{aligned} z_1 &= x_1 \\ z_2 &= a\sin x_2 \end{aligned} \qquad \therefore \; \begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \underbrace{a\cos\left(\sin^{-1}\left(\tfrac{z_2}{a}\right)\right)}_{\gamma(\cdot)} \, (\underbrace{-z_1^2}_{\alpha(\cdot)} + u) \end{aligned}$

now have recovered $\dot{x} = Ax + B\gamma(x)[u - \alpha(x)]$ type structure

achieved via $z = T(x)$ state transformation $x \xrightarrow[T]{} z$

$T(x)$ is invertible, ie $x = T^{-1}(z) \; \forall z \in T(D)$ Z map $T(\cdot)$ is invertible $\therefore$ $\exists$ an inverse map $T(\cdot)^{-1}$ st $x = T^{-1}(z) \; \forall z \in T(D)$ where D is domain of T

(valid) $\therefore$ derivatives of z & x should be cont $T(\cdot)$ & ~~$T/$~~ $T^{-1}(\cdot)$ to be contly diffable

\Def/ a contly diffable map with a contly diffable inverse is known as a <u>diffeomorphism</u>. if $\left[\frac{\partial T}{\partial x}\right]$ is non singular at $x_0 \in D$ then from <u>inverse</u> <u>func thm</u> that $\exists$ a neighhood N of x_0 st T

restricted to N is a diffeomorphism of N.

A map T is said to global diffeomorphism if it is a diffeomorphism

- on $\mathbb{R}$ & $T(\mathbb{R}^n) = \mathbb{R}^n$ & $\lim_{|x|\to\infty} \|T(x)\| \to \infty$

a non linear syst $\dot{x} = f(x) + g(x)u$ $f: D \to \mathbb{R}^n$ $g(\cdot): D \to \mathbb{R}^{n \times p}$ sufficiently smooth / Lipschitz on domain $D \subset \mathbb{R}^n$ is said to be feedback linearizable (or input to state linearizable) if $\exists$ a diffeomorphism $T: D \to \mathbb{R}^n$ st $D_z = T(D)$ contains Z origin & Z chng of variab $z = T(x)$ transforms $\dot{x} = f(x) + g(x)u \xrightarrow{z = T(x)} \dot{z} = Az + B\gamma(\cdot)[u - \alpha(\cdot)]$ with (A, B) controllable & $\gamma(\cdot)$ nonsingular in domain D.

{Ex/ $\dot{x}_1 = a \sin x_2$, $\dot{x}_2 = -x_1^2 + u$ (not in any required form) $\xrightarrow{z = T(x)}$ $z_1 = x_1$, $z_2 = a \sin x_2$

$z = T(x)$ yields $\left.\begin{aligned} \dot{z}_1 &= z_2 \\ \dot{z}_2 &= \underbrace{a \cos x_2}_{\gamma(x)} (\underbrace{-x_1^2}_{\alpha(x)} + u) \end{aligned}\right\} (**)$ (of Z form $\dot{z} = Az + B\gamma[u - \alpha]$

$\therefore u = \alpha(x) + \beta(x)v$ & $\beta(x) = \gamma^{-1}(x)$ $u = +x_1^2 + \frac{1}{a \cos x_2} v$ (for convenience of representation x_1 & x_2 where maintained here) Subing u from above in $(**)$ yields $\dot{z}_1 = z_2$, $\dot{z}_2 = v$ $v = -Kx$ $\therefore u = x_1^2 + \frac{1}{a \cos x_2}[-Kx]$

relative degree: consider a single input - single output syst

$\dot{x} = f(x) + g(x)u$ $y = h(x)$ $x \in \mathbb{R}^n$, $u \in \mathbb{R}$ $y \in \mathbb{R}$ output

recall $f(\cdot): D \to \mathbb{R}^n$ $g(\cdot): D \to \mathbb{R}^n$ are vec fields

taking deriv of y wrt time t $\frac{dy}{dt} = \dot{y} = \frac{\partial h}{\partial x}(\dot{x})$

$$\dot{y} = \frac{\partial h}{\partial x}[f(x) + g(x)u] = \frac{\partial h}{\partial x} \cdot f(x) + \frac{\partial h}{\partial x} \cdot g(x)u$$ ($f(\cdot)$, $g(\cdot)$ are vec fields)

$$= L_f h(x) + L_g h(x) u$$

$L_f h(x) = \frac{\partial h(\cdot)}{\partial x} \cdot f(x)$ is Z Lie derivative of $h(\cdot)$ wrt $f(\cdot)$ along $f(\cdot)$

$L_g h(x) = \frac{\partial h}{\partial x} g(x)$ is Z Lie deriv of $h(\cdot)$ wrt $g(\cdot)$ along $f(\cdot)$

concept of Lie deriv is convenient when Z calculation of deriv is done repeatedly

- Like in normal deriv write $L_g(L_f h(x)) = \frac{\partial(L_f h(x))}{\partial x} \cdot g(x)$

$\underbrace{L_f^2 h(x)}_{L_f(L_f h(x))} = \frac{\partial(L_f h(x))}{\partial x} \cdot f(x)$ (ie $L_f L_f h(x)$)

$L_f^k h(x) = L_f(L_f^{k-1} h(x)) = \frac{\partial(L^{k-1} h(x))}{\partial x} f(x)$. $\dot{y} = L_f h(x) + L_g h(x) u$ suppose

$L_g h(x) = 0$ in above eqn of $\dot{y}$ ie $\dot{y} = L_f h(x)$ & is indep of term u then continue calcing second deriv $\ddot{y} = \frac{\partial L_f h(x)}{\partial x}[f(x)+g(x)u] =$

$\frac{\partial L_f h(x)}{\partial x} f(x) + \frac{\partial L_f h(x)}{\partial x} g(x)u = L_f^2 f(x) + L_g L_f h(x) u$

Suppose $L_g L_f h(x)$ also zero, then $\ddot{y} = L_f^2 f(x)$ in that case

we repeat Z process until it is possible to observe if $h(x)$ satisfies

$L_g L_f^{i-1} h(x) = 0 \quad i = 1, 2, \dots, \rho-1$

$L_g L_f^{\rho-1} h(x) \neq 0$ (ie presence of u appears) then u does not appear in eqns of $y, \dot{y}, \ddot{y}, \dots, \frac{d^{\rho-1} y}{dt^{\rho-1}}$ & explicitly u appears in Z eqn of $y^{(\rho)} \leftarrow \left(\frac{d^{\rho} y}{dt^{\rho}}\right)$ with a non-zero coeffs $y^{(\rho)} = L_f^{(\rho)} h(x) + \underbrace{L_g L_f^{(\rho-1)} h(x)}_{\text{not zero}} u$

then it becomes evident that Z syst is input-output linearisable with a control law $u = \frac{1}{L_g L_f^{(\rho-1)} h(x)}\left[-L_f^{(\rho)} h(x) + v\right]$ numerical control design

$L_g L_f^{(\rho-1)} h(x) \neq 0$ in a domain D (so inverse exist) 'ρ' is relative degree

Z use of this 'u' reduces Z input-output map to $y^{(\rho)} = v$, which is a chain of 'ρ' integrators

\Def/ (relative degree) Z syst $\begin{array}{ll} \dot{x} = f(x) + g(x)u & \leftarrow \text{state eqn} \\ y = h(x) & \leftarrow \text{output eqn} \end{array}$

has a relative degree ρ, $1 \leq \rho \leq n$ in $D_0 \subset D$ if $\forall x \in D_0$

$L_g L_f^{i-1} h(x) = 0, \ i = 1, 2, \dots, \rho-1 \qquad L_g L_f^{\rho-1} h(x) \neq 0$

(remark:) Z relative degree corres to Z choice of output measurement available.

\Ex/ $\begin{array}{l} \dot{x}_1 = x_2 \\ \dot{x}_2 = -x_1 + \varepsilon(1 - x_1^2)x_2 + u \end{array}$, $\varepsilon > 0$ $\quad y = x_1$ determine relative degree of this configuration.

calc $\dot{y} = \dot{x}_1 = x_2$ (here there is no explicit appearence of (u)) ∴ take next time deriv $\ddot{y} = \dot{x}_2 \quad \ddot{y} = -x_1 + \varepsilon(1 - x_1^2)x_2 + u$ now explicit appearence of u ∴ relative degree is 2 in $\mathbb{R}^2$ (no domain ~~restriction~~ restriction)

\Week 10/ \Ex/ a nonlinear syst subsystem $\dot{\eta} = f(\eta) + g(\eta)\xi$ ①

subsyst $\dot{\xi} = u \leftarrow$ control input ② $\quad [\eta^T \ \ \xi]^T \in \mathbb{R}^{n+1}$ represents states $\eta \in \mathbb{R}^n$

$u \in \mathbb{R}$ control inputs $f(\cdot): D \to \mathbb{R}^n \quad g(\cdot): D \to \mathbb{R}^n$ smooth funcs in domain $D \subseteq \mathbb{R}^n$ D contains $\eta = 0$ the origin & $f(0) = 0$

objectives: Design state feedback control law 'u' to stabilize the nonlinear syst ① - ② about origin ($\eta^T = 0_n$, $\xi = 0$)

assumption: $f(\cdot)$ is known $g(\cdot)$ is known

all states are available for design

step 1: in ① $\dot{\eta} = f(\eta) + g(\eta)\xi$ ①

treat ξ as "virtual input"

Let ① be stabilised by a smooth state feedback control $\xi = \phi(\eta)$ ← states of ①

& $\phi(0) = 0$ i.e. $\dot{\eta} = f(\eta) + g(\eta)\phi(\eta)$ is asymp stable

suppose have a smooth posi definite Lyapunov func $V(\eta)$ that satisfy

derivative of V along the dynamics ① with $\xi = \phi(\eta)$

$$\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] \leq -W(\eta), \quad \forall \eta \in D$$ where $W(\eta)$ is a posi definite func

here $W(\eta)$ can introduce certain level of performance in design

Step 2: let us rewrite $\dot{\eta} = f(\eta) + g(\eta)\xi$ ← $\xi = \phi(\eta)$; $\dot{\xi} = u$ as

$$\dot{\eta} = f(\eta) + g(\eta)\xi + \underbrace{g(\eta)\phi(\eta)}_{\text{virtual input}} - g(\eta)\phi(\eta) \quad ④ \qquad \dot{\xi} = u \quad ⑤$$

define $z = \xi - \phi(\eta)$ change of variables

$\dot{z} = \dot{\xi} - \dot{\phi}(\eta)$; $\dot{\phi}(\eta) = \frac{\partial \phi}{\partial \eta}\dot{\eta}$ | $g(\eta)\xi - g(\eta)\phi(\eta)$ $g(\eta)[\xi - \phi(\eta)]$

∴ the dynamics can be: ⑥ $\dot{\eta} = [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z$ ← using def of z, $z = \xi - \phi(\cdot)$

⑦ $\dot{z} = \dot{\xi} - \dot{\phi}(\cdot) = u - \dot{\phi}(\cdot)$; $\dot{\phi} = \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi]$

This step is viewed as 'backstepping' the ∴ the name.

suppose $\nu = u - \dot{\phi}(\cdot)$ ← chosen virtual input

⑧ $\dot{\eta} = [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z$ ⑨ $\dot{z} = \nu$

⑧ - ⑨ dynamics has similar structure of ① - ②

i.e. $\dot{\eta} = \underbrace{f(\eta) + g(\eta)\phi(\eta)}_{\text{this has stable origin}} + g(\eta)z$

step 3: desired a composite Lyapunov func which is posi definite

$V_c(\eta, z) = V(\eta) + \frac{1}{2}z^2$ ← this was the one used earlier

taking derivative of $V_c(\cdot)$ along the trajecs of dynamics ⑧ - ⑨

yields $\dot{V}_c(\cdot) = \frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] + \frac{\partial V}{\partial \eta}g(\eta)z + z\nu$ ← from $\dot{z}$

from $\dot{\eta}$

$\dot{V}_c(\cdot) = \underbrace{\frac{\partial V}{\partial \eta}\left[f(\eta) + g(\eta)\phi(\eta)\right]}_{-W(\eta)} + \frac{\partial V}{\partial \eta} g(\eta) z + z\nu$

$\leq -W(\eta) + \frac{\partial V}{\partial \eta} g(\eta) z + z\nu$ ← design freedom

→ presence of z

$W(\eta)$ is posi desin

$\frac{\partial V}{\partial \eta} g(\eta) z$ sign indesin

ν is design freedom one can pick ν st $\dot{V}_c$ is nega desin

choose a ν appropriately

$\dot{V}_c(\cdot) \leq -W(\cdot) + \frac{\partial V}{\partial \eta} g(\eta) z + z\nu$ ← design freedom

choose $\nu = -\frac{\partial V}{\partial \eta} g(\eta) - kz$; k is a scalar

with this ν $\quad \dot{V}_c(\cdot) \leq -W(\eta) - kz^2$ is nega desin

$\therefore$ Z origin $(\eta^T = 0_n, z = 0)$ ← corres to ⑧-⑨ is asymp stable

Sub V, z & $\phi(\cdot)$ $\quad u = \frac{\partial \phi}{\partial \eta}\left[f(\eta) + g(\eta)\xi\right] - \frac{\partial V}{\partial \eta} g(\eta) - k\left[\xi - \phi(\cdot)\right]$

is backstepping control law (original coords or states of ①-②)

if all assumps hold globally $V(\eta)$ & $(V_c(\eta, z)$ are radially unbounded,

then Z origin is globally asymp stable

\Ex/ $\dot{x}_1 = \underbrace{x_1^2}_{f} \underbrace{- x_1^3}_{} + \overset{g(\cdot)=1}{x_2}$ ← virtual input ⊛ $\qquad \dot{x}_2 = u$ ⊛⊛

x_1 is like η $\quad x_2$ is like ξ (connection to what we discussed in ① ② of theory

$f(\eta) = x_1^2 - x_1^3$ $\quad g(\eta) = 1$ $\quad$ Consider $\dot{x}_1 = x_1^2 - x_1^3 + x_2$ (Subsyst 1)

treat 'x_2' as "virtual control" need to design $x_2 = \phi(x_1)$

Let $x_2 = \overbrace{-x_1^2 - x_1}^{\phi(x_1)}$ choice will be given. Above choice cancels Z

nonlinear term x_1^2 $\quad \dot{x}_1 = -x_1 - x_1^3$ $\quad (\dot{x}_1 = x_1^2 - x_1^3 + x_2 = -x_1^3 - x_1)$

$V(x_1) = \frac{1}{2}x_1^2$ $\quad \dot{V}(x_1) = x_1\dot{x}_1 = -x_1^2 - x_1^4$ $\quad (-\|x_1\|^2 - \|x_1\|^2\|x_1\|^2 \leq \|x_1\|^2 - \|x\|^2)$

for $\dot{V}(x_1) \leq -W(x_1)$ say $\dot{V}(x) = -\underbrace{x_1^2}_{2(\frac{1}{2}x_1^2)} - x_1^4 \leq -2V$ so origin is stable exponentially

globally. To do backstepping $z_2 = x_2 - \phi(x_1)$ (like $\xi - \phi(\eta)$ term)

$z_2 = x_2 + x_1 + x_1^2$ $\quad \dot{z}_2 = \dot{x}_2 + \dot{x}_1 + 2x_1\dot{x}_1$

$\dot{x}_1 = -x_1 - x_1^3 + z_2$ $\quad \dot{z}_2 = \dot{x}_2 + \dot{x}_1 + 2x_1\dot{x}_1 = u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)$ desiring

Composite Lyapunov func $V_c(x_1, z_2) = V(x_1) + \frac{1}{2}z_2^2$ $\qquad \dot{V}_c = x_1\dot{x}_1 + z_2\dot{z}_2 =$

$x_1(-x_1 - x_1^3 + z_2) + z_2(u + (1 + 2x_1)(-x_1 - x_1^3 + z_2)) =$

$\underbrace{-x_1^2 - x_1^4}_{\leq W(\cdot)} + z_2\left[x_1 + (1 + 2x_1)(-x_1 - x_1^3 + z_2) + \boxed{u}\right]$ ← unknown design $\quad u = -x_1 - (1 + 2x_1)(-x_1 - x_1^3 - z_2) - z_2$

(if asked to be in original coords replace z_2) $\quad$ yields

$\dot{V}_c = -x_1^2 - x_1^4 - z_2^2$ which is globally asymp stable

\PP2019/

\1ai/ False Q_1 stable , Q_2 saddle, Q_3 saddle

\1ai/ Q_1 is stable , Q_2 is saddle , Q_3 is saddle $\therefore$ False.

\1aii/ False. $M = M^T$ positive desinite matrix $\lambda_i(M) > 0 \ \forall i$,
$\lambda_i(M) \geq 0$ positive semidesinite matrix

\1aii/ M is symmetric $\therefore M = M^T$.
positive desinite matrix $\therefore \lambda_i(M) > 0$ but
is $\lambda_i(M) \geq 0$: positive semi desinite $\therefore$ False

\1aiii sol/ True $\because V(x) > 0 \ \forall x \in \mathbb{R}^2 \setminus \{0\}$, $V(0) = 0$ at $x = (0,0)$ $\therefore$
$V(x) \to \infty$ as $\|x\| \to \infty$ $\therefore V(\cdot) : \mathbb{R}^2 \to \mathbb{R}$ is radially unbounded

\1aiii/ $V(x) > 0 \ \forall x \in \mathbb{R}^2 \setminus \{0\}$, $V(0) = 0$ at $x = (0,0)$ $\therefore$
positive desinite , $V(x) \to \infty$ as $\|x\| \to \infty$ $\therefore V : \mathbb{R}^2 \to \mathbb{R}$ is
radially unbounded $\therefore$ True

\1aiv/ False. By desinition, said to be critically min phase
is its zero dynamics is lyapunov stable, min phase is its
zero dynamics is asymptotically stable

\1aiv/ System is at critically min phase is its zero
dynamics is lyapunov stable, min phase is its zero
dynamics is asymptotically ~~min phase is its zero~~ stable
$\therefore$ False

\1aiv/ a system is at min phase is its ~~dynamics~~
zero dynamics is asymptotically stable.
a system is at critically min phase ~~is~~ its zero dynamics
is lyapunov stable $\therefore$ False.

\1ai/ Q_1 is stable , Q_2 is saddle , Q_3 saddle $\therefore$ False.

\1aii/ M symmetric $\therefore M^T = M$, M posi desinite $\therefore$
$\lambda_i(M) > 0 \ \forall i$ but is $\lambda_i(M) \geq 0 \ \forall i$ then posi semi desi $\therefore$ False

\1aiii/ $V(x) > 0 \ \forall x \in \mathbb{R}^2 \setminus \{0\}$, $V(x) = 0$ for $x = 0$ $\therefore$ posidesi.
$V \to \infty$ as $\|x\| \to \infty$ $\therefore$ $V: \mathbb{R}^2 \to \mathbb{R}$ is radially unbounded $\therefore$ True

\1aiv/ a system is at min phase is its zero dynamics are asymptotically ~~lyapunov~~ stable. is at critically min phase is its zero dynamics are lyapunov stable $\therefore$ False

\1bi/ controllability matrix $M = [B \vdots AB] = \left[\begin{bmatrix}0\\1\end{bmatrix} \vdots \begin{bmatrix}0 & 1\\2 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right] = \left[\begin{bmatrix}0\\1\end{bmatrix} \vdots \begin{bmatrix}1\\-1\end{bmatrix}\right] = \begin{bmatrix}0 & 1\\1 & -1\end{bmatrix}$ $\therefore$ System is controllable is controllability Matrix has full rank $\therefore$ rank$(M) = 2$, $\det(M) = -1 \neq 0$

\1bi/ controllability matrix $M = [B \vdots AB] = \left[\begin{bmatrix}0\\1\end{bmatrix} \vdots \begin{bmatrix}0 & 1\\2 & -1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right] = \left[\begin{bmatrix}0\\1\end{bmatrix} \vdots \begin{bmatrix}1\\-1\end{bmatrix}\right] = \begin{bmatrix}0 & 1\\1 & -1\end{bmatrix}$ $\therefore$ $\det(M) = (0)(-1) - 1(1) = 0 - 1 = -1 \neq 0$ $\therefore$
rank$(M) = 2$ $\therefore$ full rank Matrix $\therefore$
System is controllable

\1bii/ Desired eigenvalues at -1 and -2 gives characteristic equation $(s+1)(s+2) = s^2 + 3s + 2$

closed loop system characteristic polynomial

$\det(sI - A^{+}_{cl}) = \det(sI - (A - BK)) = \det\left(\begin{bmatrix}s & 0\\0 & s\end{bmatrix} - \left(\begin{bmatrix}0 & 1\\2 & -1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}[k_1 \ k_2]\right)\right) =$

$\det\left(\begin{bmatrix}s & 0\\0 & s\end{bmatrix} - \left(\begin{bmatrix}0 & 1\\2 & -1\end{bmatrix} - \begin{bmatrix}0 & 0\\k_1 & k_2\end{bmatrix}\right)\right) = \det\left(\begin{bmatrix}s & 0\\0 & s\end{bmatrix} - \left(\begin{bmatrix}0 & 1\\2-k_1 & -1-k_2\end{bmatrix}\right)\right) =$

$\det\left(\begin{bmatrix}s & -1\\k_1 - 2 & s + k_2 + 1\end{bmatrix}\right) = s(s + k_2 + 1) - (-1)(k_1 - 2) =$

$s^2 + \cancel{s k_2}\, k_2 s + s + k_1 - 2 = s^2 + (k_2 + 1)s + (k_1 - 2) = s^2 + 3s + 2$ $\therefore$

$k_2 + 1 = 3$ $\therefore$ $k_2 = 2$

$k_1 - 2 = 2$ $\therefore$ $k_1 = 4$ $\therefore$ $K = [4 \ \ 2]$ $\therefore$

$u = -[4 \ \ 2]x$

\1bii/ Desired eigenvalues at -1 and -2 gives characteristic equation $(s+1)(s+2) = s^2 + 2s + s + 2 = s^2 + 3s + 2$ $\therefore$ $K = [k_1 \ k_2]$ $\therefore$

$A^{+}_{cl} = A - BK = \begin{bmatrix}0 & 1\\2 & -1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}[k_1 \ k_2] = \begin{bmatrix}0 & 1\\2 & -1\end{bmatrix} - \begin{bmatrix}0 & 0\\k_1 & k_2\end{bmatrix} = \begin{bmatrix}0 & 1\\2-k_1 & -1-k_2\end{bmatrix} = \begin{bmatrix}0 & 1\\-k_1+2 & -k_2-1\end{bmatrix}$

$\therefore$ closed loop system characteristic polynomial

$\det(sI - A^{+}_{cl}) = \det(sI - (A - BK)) = \det\left(\begin{bmatrix}s & 0\\0 & s\end{bmatrix} - \begin{bmatrix}0 & 1\\-k_1+2 & -k_2-1\end{bmatrix}\right) =$

$\det\left(\begin{bmatrix}s & -1\\k_1 - 2 & s + k_2 + 1\end{bmatrix}\right) = s(s + k_2 + 1) - (-1)(k_1 - 2) = s^2 + k_2 s + s + k_1 - 2 =$

\PP2019/ $s^2+(k_2+1)s+(k_1-2)=s^2+3s+2$ ∴

$k_2+1=3$ ∴ $k_2=2$, $k_1-2=2$ ∴ $k_1=4$ ∴

$K=[4 \quad 2]$ ∴ $u=-[4 \quad 2]x$

\1c/ $f(x)=\begin{bmatrix}-x_1^2+x_1x_2 \\ x_1+\frac{3}{2}x_1^2+\frac{1}{2}x_2^2\end{bmatrix}$ ∴ $\frac{\partial f}{\partial x}=\begin{bmatrix}\partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2\end{bmatrix}=$

$\begin{bmatrix}-2x_1+x_2 & x_1 \\ 1+3x_1 & x_2\end{bmatrix}$ ∴

$\left\|\frac{\partial f}{\partial x}\right\|_\infty=\max\{|-2x_1+x_2|+|x_1| , |1+3x_1|+|x_2|\}$ ∴

all points in W satisfy:

$|-2x_1+x_2|+|x_1|\le 2a+b+a\le 3a+b$,

$|1+3x_1|+|x_2|\le 1+3a+b$ ∴ $1+3a+b>3a+b$ ∴

$\left\|\frac{\partial f}{\partial x}\right\|_\infty\le 1+3a+b$ ∴ is lipschitz constant bound $1+3a+b$

\1c/ $f(x)=\begin{bmatrix}-x_1^2+x_1x_2 \\ x_1+\frac{3}{2}x_1^2+\frac{1}{2}x_2^2\end{bmatrix}$ ∴ $\frac{\partial f}{\partial x}=\begin{bmatrix}\partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2\end{bmatrix}=\begin{bmatrix}-2x_1+x_2 & x_1 \\ 1+3x_1 & x_2\end{bmatrix}$

∴ $\left\|\frac{\partial f}{\partial x}\right\|_\infty=\max\{|-2x_1+x_2|+|x_1| , |1+3x_1|+|x_2|\}$ ∴

all points in W satisfy $|x_1|\le a$, $|x_2|\le b$ ∴

~~2a+b+2a~~ $|-2x_1+x_2|+|x_1|\le 2|x_1|+|x_2|+|x_1|\le 2a+b+a=3a+b$,

$|1+3x_1|+|x_2|\le |1|+3|x_1|+|x_2|\le 1+3a+b$ ∴ $1+3a+b>3a+b$ ∴

the lipschitz constant over the convex set is $1+3a+b$

\1d/ $\dot{x}_1=-5x_1+x_1x_2^2$, $\dot{x}_2=3x_1-2x_2$ ∴ $V=x_1^2+x_2^2$ is posi definite ∴

~~V=2x1+2x2~~ $\dot{V}=2x_1\dot{x}_1+2x_2\dot{x}_2=2x_1(-5x_1+x_1x_2^2)+2x_2(-2x_2+3x_1)=$

$-10x_1^2+2x_1^2x_2^2-4x_2^2+2(3x_1)x_2=$

$-x_1^2-9x_1^2+2x_1^2x_2^2-3x_2^2-x_2^2+2(3x_1)x_2=$

$-x_1^2+2x_1^2x_2^2-3x_2^2+(-9x_1^2+2(3x_1)x_2-x_2^2)=$

$-x_1^2-3x_2^2+2x_1^2x_2^2-(9x_1^2-2(3x_1)x_2+x_2^2)$

$-x_1^2-3x_2^2+2x_1^2x_2^2-(3x_1-x)^2\le -x_1^2-3x_2^2+2x_1^2x_2^2=$

$-x_1^2(1-2x_2^2)-3x_2^2$ or $-x_1^2-x_2^2(3-2x_1^2)$

∴ for $\dot{V}\le -x_1^2(1-2x_2^2)-3x_2^2<0$ if $1-2x_2^2>0$ ∴ $1>2x_2^2$ ∴

$\frac{1}{2}>x_2^2$ ∴ $\frac{1}{2}>x_1^2+x_2^2$ ∴ $x_1^2+x_2^2=V<\frac{1}{2}$ ∴ $D_1=\{x \mid V(x)<\frac{1}{2}\}$

~~for~~ $\dot{V} \le -x_1^2 - x_2^2(3-2x_1^2) < 0$ is ~~x_1^2~~ $3-2x_1^2>0$ ∴ $3>2x_1^2$ ∴

$\frac{3}{2}>x_1^2$ ∴ $\frac{3}{2}>x_1^2+x_2^2$ ∴ ~~$D_2=\{x\}$~~ ∴ $x_1^2+x_2^2=V<\frac{3}{2}$ ∴

$D_2=\{x \mid V(x)<\frac{3}{2}\}$

are the ~~sets where~~ invariant sets which V is posi desi

and $\dot{V}$ is nega desi ∴

origin is locally asymp stable

\1d/ V is posi desi ∴

$\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1(-5x_1 + x_1x_2^2) + 2x_2(3x_1 - 2x_2) =$

$-10x_1^2 + 2x_1^2x_2^2 + 2(3x_1)x_2 - 4x_2^2 =$

$-x_1^2 + 2x_1^2x_2^2 - 3x_2^2 - 9x_1^2 + 2(3x_1)x_2 - x_2^2 =$

$-x_1^2 + 2x_1^2x_2^2 - 3x_2^2 - (9x_1^2 - 2(3x_1)x_2 + x_2^2) =$

$-x_1^2 + 2x_1^2x_2^2 - 3x_2^2 - (3x_1 - x_2)^2 \le -x_1^2 + 2x_1^2x_2^2 - 3x_2^2$

$\le -x_1^2 - x_2^2(-2x_1^2+3)$ ∴

$\dot{V}<0$ is $-x_1^2 - x_2^2(-2x_1^2+3) < 0$ ∴

$-2x_1^2+3>0$ ∴ $\frac{3}{2}>x_1^2$ ∴ $\frac{3}{2}>x_1^2+x_2^2=V$ ∴

invariant set which V is posi desi and $\dot{V}$ nega desi

$D=\{x \mid V(x)<\frac{3}{2}\}$ ∴ origin is locally asymp stable

\1ei/ $D=\{x \in \mathbb{R}^2 \mid x_2 \ge 0\}$ the value of $\dot{x}_2$ on the x_1-axis is

$\dot{x}_2 = bx_1^2 \ge 0$ ∴ the trajectories starting in D cant leave it

\1ei/ $D=\{x \in \mathbb{R}^2 \mid x_2 \ge 0\}$, $\dot{x}_2 = bx_1^2 - cx_2$ ∴ the value of $\dot{x}_2$ on the

x_1-axis is $\dot{x}_2 = bx_1^2$, $a, b, c > 0$ ∴ $\dot{x}_2 = bx_1^2 > 0$ ∴

$x_2 \ge 0$ ∴ ~~$x_2(t)$~~ $\dot{x}_2 > 0$ ∴ ~~x_2~~ $x_2(t_1) \le x_2(t_2)$ $\forall t_1 \le t_2$ ∴

$x_2(0) \ge 0$, $x_2(t) \ge x_2(0) \ge 0$ $\forall t \ge 0$ ∴ $x_2 \ge 0 \; \forall t$ ∴

$x_2 \ge 0$ ∴ $D=\{x \in \mathbb{R}^2 \mid x_2 \ge 0\}$ ∴ the trajectories starting in D cant

leave it

\1ei/ the value of $\dot{x}_2$ on the x_1-axis is $\dot{x}_2 = bx_1^2 > 0$ ∴

$x_2 \ge 0 \; \forall t$, $x_2(0) \ge 0$ ∴ the trajectories starting in D cant

leave it

\PP2019/ \1eii/ $\frac{\partial f_1}{\partial x_1} = a - x_2$, $\frac{\partial f_2}{\partial x_2} = -c$ $\therefore$

$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = a - x_2 - c$ $\because$ $x_2 > 0$ $\therefore$ $-x_2 < 0$ $\therefore$

$a - x_2 - c = a - c - x_2 = -(-a+c) - x_2 = -(c-a) - x_2 \leq -(c-a) < 0$

$\because$ $c > a$ $\therefore$ $c - a > 0$ $\therefore$ $-(c-a) < 0$ $\forall x \in D$ $\therefore$

by Bendixsons criterion there can be no closed orbits entirely on D. $\therefore$ trajectories starting in D cannot leave it, a closed orbit through anypoint in D must lie entirely in D $\therefore$ $\exists$ no closed orbits through anypoint in D

\eii/ $\frac{\partial f_1}{\partial x_1} = a - x_2$, $\frac{\partial f_2}{\partial x_2} = -c$ $\therefore$

$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = a - x_2 - c = -(c-a) - x_2 \leq -(c-a) < 0 \; \forall x \in D$ ~~$\forall x_2 \leq 0$~~ $\because$ $-x_2 \leq 0$

$\therefore$ By Bendixons criterion, $\exists$ no closed orbits enterly on D.

$\therefore$ trajectories starting in D must lie entirely in D $\therefore$

$\exists$ no closed orbits through anypoint in D

\1eii/ $\frac{\partial f_1}{\partial x_1} = a - x_2$, $\frac{\partial f_2}{\partial x_2} = -c$ $\therefore$

$\frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = a - x_2 - c = -(c-a) - x_2 \leq -(c-a) < 0 \quad \forall x \in D$ $\because$ $-x_2 \leq 0$ &

~~By Ben trajectories starting in D must lie entirely on D~~

$\therefore$ trajectories starting in D cannot leave it, a closed orbit through anypoint in D must lie entirely in D $\therefore$

By Bendixons criterion, $\exists$ no closed orbits entirely on D.

$\therefore$ $\exists$ no closed orbits through anypoint in D

\2a/ La Salle's thm: Let $\Omega \subseteq D$ be a compact set that is positively invariant wrt $\dot{x} = f(x)$

Let $V: D \to R$ be a contly diffable func st $\dot{V}(x) \leq 0$ in Ω.

Let E be the set of all pts in Ω where $\dot{V}(x) = 0$. Let M be the largest invariant set in E. Then every sol starting in Ω approaches M as $t \to \infty$

\2a/ Let $\Omega \subset D$ be a compact set that is positively invariant wrt $\dot{x} = f(x)$

Let $V: D \to R$ be a contly diffable func st $\dot{V}(x) \leq 0$ in Ω. Let E be set of all pts in Ω where $\dot{V}(x) = 0$, Let M be the largest invariant set in E then every sol starting in Ω approaches M as $t \to \infty$

\2bi/ The system has equili pt at $y = Mg/k$ and $\dot{y} = 0$

\2bi/ equili is $\dot{y} = 0$, $\ddot{y} = 0$ ∴

$M\ddot{y} = 0 = Mg - ky - c_1(0) - c_2(0)|0| = Mg - ky = 0$ ∴ $Mg = ky$ ∴ $Mg/k = y$))

∴ equili pt at $y = Mg/k$ and $\dot{y} = 0$

\2bii/ let $x_1 = y - Mg/k$, $x_2 = \dot{y} - 0 = \dot{y}$ ∴ equili is shifted to $(0,0)$ ∴ $x_1 + \frac{Mg}{k} = y$ ∴

$\dot{x}_1 = \frac{d}{dt}\left(y - \frac{Mg}{k}\right) = \dot{y} = x_2$,

$\dot{x}_2 = \ddot{y} = g - \frac{k}{M}y - \frac{c_1}{M}\dot{y} - \frac{c_2}{M}\dot{y}|\dot{y}| = g - \frac{k}{M}\left(x_1 + \frac{Mg}{k}\right) - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2| =$
$g - \frac{k}{M}x_1 - g - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2| = -\frac{k}{M}x_1 - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2|$

\2bii/ let $x_1 = y - \frac{Mg}{k}$, $x_2 = \dot{y} - 0 = \dot{y}$ ∴

equili is shifted to origin $(0,0)$ ∴

$x_1 + \frac{Mg}{k} = y$ ∴ $\dot{x}_1 = \frac{d}{dt}\left(y - \frac{Mg}{k}\right) = \dot{y} = x_2$,

$\dot{x}_2 = \ddot{y} = g - \frac{k}{M}y - \frac{c_1}{M}\dot{y} - \frac{c_2}{M}\dot{y}|\dot{y}| = g - \frac{k}{M}\left(x_1 + \frac{Mg}{k}\right) - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2| =$
$g - \frac{k}{M}x_1 - g - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2| = -\frac{k}{M}x_1 - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2|$

\2biii/ $V(x) = ax_1^2 + bx_2^2$ $a, b > 0$ $V(x) > 0\ \forall x \in \mathbb{R}^2 \setminus \{0\}$, $V(x) \to \infty$ as $\|x\| \to \infty$, $V(0) = 0$ ∴ V is posi desi

$\dot{V}(x) = 2\left(a - \frac{bk}{M}\right)x_1x_2 - \frac{2bc_1}{M}x_2^2 - \frac{2bc_2}{M}x_2^2|x_2|$ ∴

taking $a = \frac{k}{2}$, $b = \frac{M}{2}$:

$\dot{V} = -c_1x_2^2 - c_2x_2^2|x_2| \leq 0$ ∴ $\dot{V} \leq 0\ \forall x$

if $\dot{V} = 0$: $x_2 = 0$ ∴ $x_1 = 0$ ∴ nega desi ∴

using Lasalle's principle, origin is asymp stable

\2biii/ $V(x) > 0\ \forall x \in \mathbb{R}^2 \setminus \{0\}$, $V(x) \to \infty$ as $\|x\| \to \infty$, $V(0) = 0$ ∴

V is posi desi ∴

$\dot{V} = 2ax_1\dot{x}_1 + 2bx_2\dot{x}_2 =$

$2ax_1x_2 + 2bx_2\left(-\frac{k}{M}x_1 - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2|\right) =$

$2\left(a - \frac{bk}{M}\right)x_1x_2 - \frac{2bc_1}{M}x_2^2 - \frac{2bc_2}{M}x_2^2|x_2|$ ∴

let $a = \frac{k}{2}$, $b = \frac{M}{2}$ ∴ $\dot{V} = -c_1x_2^2 - c_2x_2^2|x_2| \leq 0$ ∴ $\dot{V} \leq 0\ \forall x$, $\dot{V}(0) = 0$ ∴

nega desi ∴ by Lasalle's principle, origin is asymp stable

\PP2019\ \3a\ $\dot{x}=f(x)+g(x)u \quad y=h(x)$

f, g, h are sufficiently smooth in $D \subset \mathbb{R}^n$.

$f: D \to \mathbb{R}^n$, $g\ D \to \mathbb{R}^n$ are vector fields on D

$\therefore$ $\cancel{\dot{y}=\frac{\partial h}{\partial x}}$ $\quad \dot{y}=\frac{\partial y}{\partial t}=\frac{\partial}{\partial t}[h(x)]=\cancel{\frac{\partial h(x)}{\partial x}}\ \frac{\partial h(x)}{\partial x}\frac{\partial x}{\partial t}=$

$\frac{\partial h(x)}{\partial x}\dot{x}=\frac{\partial h(x)}{\partial x}[f(x)+g(x)u]=\frac{\partial h(x)}{\partial x}f(x)+\frac{\partial h(x)}{\partial x}g(x)u=$

$L_f h(x)+L_g h(x)u$ $\therefore$

$L_f h(x)=\frac{\partial h}{\partial x}f(x)$, $L_g h(x)=\frac{\partial h}{\partial x}g(x)$

Lie derivative of h w.r.t f, or along f.

$L_g L_f h(x)=\frac{\partial (L_f h)}{\partial x}g(x)$ and so on.

If $L_g h(x)=0$, then $\dot{y}=L_f h(x)$, independent of u

If we continue to calculate the second derivative

$y^{(2)}=\cancel{\ }\ \frac{\partial (L_f h)}{\partial x}[f(x)+g(x)u]=$

$L_f^2 h(x)+L_g L_f h(x)u$

if $L_g L_f h(x)=0$, $y^{(2)}=L_f^{(2)} h(x)$ independent of u,

then continuing, $y^{(\rho)}=L_f^{\rho} h(x)+L_g L_f^{(\rho-1)} h(x)u$

(n nonzero) Then L system is said to be of

relative degree ρ, $1\le \rho \le n$ in a region

$D_0 \subset D$ if $L_g L_f^{i-1} h(x)=0$, $i=1,2,\ldots,\rho-1$

$L_g L_f^{\rho-1} h(x)\neq 0 \quad \forall x \in D_0$

\3a\ $f: D \to \mathbb{R}^n$, $g\ D \to \mathbb{R}^n$ are vector fields on D.

$\dot{y}=\frac{\partial y}{\partial t}=\frac{\partial}{\partial t}[h(x)]=\frac{\partial h(x)}{\partial x}\frac{\partial x}{\partial t}=\frac{\partial h(x)}{\partial x}\dot{x}=$

$\frac{\partial h}{\partial x}[f(x)+g(x)u]=\frac{\partial h}{\partial x}f(x)+\frac{\partial h}{\partial x}g(x)u$

$=L_f h(x)+L_g h(x)u$ $\therefore$

$L_f h(x)=\frac{\partial h}{\partial x}f(x)$, $L_g h(x)=\frac{\partial h}{\partial x}g(x)$

Lie derivative of h w.r.t f, or along f.

$L_g L_f h(x)=\frac{\partial (L_f h)}{\partial x}g(x)$ and so on $\therefore$

If $L_g h(x)=0$, then $\dot{y}=L_f h(x)$, independent of u

If we continue to calculate the Second derivative

$\cancel{y^{(2)} = \frac{\partial(L_f h)}{\partial x}}$ $y^{(2)} = \frac{\partial \dot{y}}{\partial t} = \frac{\partial}{\partial t}[L_f h(x)] = \frac{\partial}{\partial x}[L_f h(x)]\dot{x} =$

$\frac{\partial(L_f h)}{\partial x}[f(x)+g(x)u] = \frac{\partial(L_f h)}{\partial x} f(x) + \frac{\partial(L_f h)}{\partial x} g(x)u =$

$L_f^2 h(x) + L_g L_f h(x) u$

if $L_g L_f h(x) = 0$, $y^{(2)} = L_f^{(2)} h(x)$ independent of u,

then continuing, $y^{(p)} = L_f^p h(x) + L_g L_f^{(p-1)} h(x) u$

(n-nonzero) Then L system is said to be of relative degree p, $1 \le p \le n$, in a region

$D_0 \subset D$ if $L_g L_f^{i-1} h(x) = 0$, $i = 1, 2, \dots, p-1$

$L_g L_f^{p-1} h(x) \neq 0 \quad \forall x \in D_0$

3bi/ $\dot{x}_1 = x_1^2 + x_2$, $\dot{x}_2 = u$ ∴

start with $\dot{x}_1 = x_1^2 + x_2$, choosing $x_2 = \phi(x_1) = -x_1^2 - x_1$ as input ∴

$\dot{x}_1 = x_1^2 + x_2 = x_1^2 + (-x_1^2 - x_1) = -x_1$ ∴

choosing Lyapunov function $V(x) = \frac{1}{2} x_1^2$

∴ V is positive definite ∴

$\dot{V} = x_1 \dot{x}_1 = x_1(-x_1) = -x_1^2 \le 0$ ∴

$\dot{V}$ is negative definite ∴

$\dot{x}_1 = -x_1$ is globally asymptotically stable ∴

$x_1 = 0$ is a globally stable equili of $\dot{x}_1 = x_1^2 + x_2$

3bi/ ~~other~~ let $x_2 = \phi(x_1) = -x_1^2 - x_1$ as input ∴

$\dot{x}_1 = x_1^2 + x_2 = x_1^2 - x_1^2 - x_1 = -x_1$ ∴

let Lyapunov function be $V(x) = \frac{1}{2} x_1^2$ ∴

V is posi defi ∴

$\dot{V} = x_1 \dot{x}_1 = x_1(-x_1) = -x_1^2 \le 0$ ∴

$\dot{V}$ is nega defi ∴

$\dot{x}_1 = -x_1$ is globally asymp stable ∴

$x_1 = 0$ is a globally stable equili of $\dot{x}_1 = x_1^2 + x_2$

\pp2019/ \3b ii/ $z_2 = x_2 - \phi(x_1) = x_2 - (-x_1^2 - x_1) = x_2 + x_1^2 + x_1$

$V_c(x_1, z_2) = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2 = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1^2 + x_1) = \frac{1}{2}x_2 + x_1^2 + \frac{1}{2}x_1$

$\dot{V}_c = x_1\dot{x}_1 + z_2\dot{z}_2$

~~$x_1 t$~~ $\dot{x}_1 = x_1^2 + x_2 = -x_1 + (x_2 + x_1^2 + x_1) = -x_1 + z_2$,

$\dot{z}_2 = \frac{d}{dt}(x_2 + x_1 + x_1^2) = \dot{x}_2 + \dot{x}_1 + 2x_1\dot{x}_1 = u + -x_1 + z_2 + 2x_1(-x_1 + z_2) =$

$u + (1+2x_1)(-x_1 + z_2)$ ∴

$\dot{V}_c = x_1\dot{x}_1 + z_2\dot{z}_2 =$

$x_1(-x_1 + z_2) + z_2[u + (1+2x_1)(-x_1 + z_2)] =$

$-x_1^2 + z_2[x_1 + (1+2x_1)(-x_1 + z_2) + u]$ ∴

let $u = -x_1 - (1+2x_1)(-x_1 + z_2) - z_2 \triangleq \psi(x_1, x_2) = \psi(x_1, z_2 + \phi(x_1)) = \psi(x_1, z_2)$

$\dot{V}_c = -x_1^2 + z_2[x_1 + (1+2x_1)(-x_1 + z_2) - x_1 - (1+2x_1)(-x_1 + z_2) - z_2] =$

$-x_1^2 + z_2[-z_2] = -x_1^2 - z_2^2$ ∴

$u = -x_1 - (1+2x_1)(-x_1 + z_2) - z_2 =$

$-x_1 - (1+2x_1)(-x_1 + x_2 + x_1^2 + x_1) - x_2 - x_1^2 - x_1 =$

~~$-2(x_1 + x_2 + x_1x_2 + x_1^2 + x_1^3)$~~ $-x_1 - (1+2x_1)(x_2 + x_1^2) - x_2 - x_1 - x_1^2 =$

$-2x_1 - [x_2 + x_1^2 + 2x_1x_2 + 2x_1^3] - x_2 - x_1^2 =$

$-2x_1 - 2x_2 - 2x_1^2 - 2x_1x_2 - 2x_1^3 = -2(x_1 + x_2 + x_1x_2 + x_1^2 + x_1^3)$

choosing such 'u' yields $\dot{V}_c = -2V_c$

∴ $z_2^2 = (x_2 + x_1^2 + x_1)^2$ & ∴

$\dot{V}_c = -x_1^2 - z_2^2 = -(x_1^2 + z_2^2) = -2 \times \frac{1}{2}(x_1^2 + z_2^2) = -2V_c \leq 0$

V_c is posi desi ∴

$\dot{V}_c$ is nega desi ∴

origin is globally asymp stable

~~\4a/ suppose $\bar{x} \in D$ is an equili pt~~

\3b ii/ $z_2 = x_2 - \phi(x_1) = x_2 - (x_1^2 - x_1) = x_2 + x_1^2 + x_1$ ∴

$V_c(x_1, z_2) = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2 = \frac{1}{2}x_1^2 + \frac{1}{2}(x_2 + x_1^2 + x_1) = \frac{1}{2}x_2 + x_1^2 + \frac{1}{2}x_1$ ∴

~~$\dot{V}_c = 2x_1$~~ $\dot{x}_1 = x_1^2 + x_2 = -x_1 + (x_2 + x_1^2 + x_1) = -x_1 + z_2$,

$\dot{z}_2 = \frac{d}{dt}(x_2 + x_1 + x_1^2) = \dot{x}_2 + \dot{x}_1 + 2x_1\dot{x}_1 = u + -x_1 + z_2 + 2x_1(-x_1 + z_2) = u + (1+2x_1)(-x_1 + z_2)$ ∴

$\dot{V}_c = x_1\dot{x}_1 + z_2\dot{z}_2 = x_1(-x_1 + z_2) + z_2[u + (1+2x_1)(-x_1 + z_2)] =$

$-x_1^2 + z_2[x_1 + (1+2x_1)(-x_1+z_2) + u]$ ∴

let $u = -x_1 - (1+2x_1)(-x_1+z_2) - z_2$ ∴

$\dot{V}_c = -x_1^2 + z_2[x_1 + (1+2x_1)(-x_1+z_2) + -x_1 - (1+2x_1)(-x_1+z_2) - z_2] =$

$-x_1^2 + z_2[-z_2] = -x_1^2 - z_2^2 = -(x_1^2 + z_2^2) = -2 \times \frac{1}{2}(x_1^2 + z_2^2) = -2V_c \leq 0$

∴ ~~orgi~~ V_c is posi desi ∴ $\dot{V}_c$ is nega desi ∴

origin is globally asymp stable

\4a/ suppose $\bar{x} \in D$ is an equili pt, ie: $f(\bar{x}) = 0$.

The equili pt $\bar{x}$ of $\dot{x} = f(x)$ is stable is for each $\varepsilon > 0$; $\exists\, \delta = \delta(\varepsilon) > 0$

st $\|x(0)\| < \delta \rightarrow \|x(t)\| < \varepsilon \; \forall t \geq 0$

asymp stable is its stable and δ can be chosen st

$\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} \|x(t)\| = \bar{x}$ is locally exponentially stable,

is $\exists$ +ve α, β and δ st $\|x(0)\| < \delta$ then

$\|x\| \leq \delta \|x(0)\| \exp(-\beta t)$, $t \geq 0$

$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \alpha \|x(0)\| \exp(-\beta t)$, $\forall t \geq 0$

\4a/ suppose $\bar{x} \in D$ is an equili pt, ie $f(\bar{x}) = 0$

The equili pt $\bar{x}$ of $\dot{x} = f(x)$ is stable is for each $\varepsilon > 0$; $\exists\, \delta =$

$\delta(\varepsilon) > 0$ st $\|x(0)\| < \delta \rightarrow \|x(t)\| < \varepsilon \quad \forall t \geq 0$

its asymp stable, is its stable and δ can be chosen

st $\|x(0)\| < \delta \Rightarrow \lim_{t\to\infty} x(t) = \bar{x}$.

it is locally exponentially stable, is $\exists$ +ve α, β, δ st $\|x(0)\| < \delta$,

then $\|x\| \leq \delta \|x(0)\| e^{-\beta t}$, $t \geq 0$.

$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \alpha \|(x)\| e^{-\beta t} \quad \forall t \geq 0$

\4bi/ with $u = -B^T P x$: The closed loop dynamics is:

$\dot{x} = (A - BB^T P)x + Bg(t,x)$ ∴

~~$\dot{x} = \frac{d}{dt}$~~ $\dot{x} = Ax + B[u + g(t,x)] = Ax + Bu + Bg(t,x) =$

$Ax + B(-B^T P x) + Bg(t,x) = (A - BB^T P)x + Bg(t,x)$

\4bi/ with $u = -B^T P x$ The closed loop dynamics is: $\dot{x} =$

~~$(A - BB$~~ $Ax + Bu + Bg(t,x) = Ax + B(-B^T P x) + Bg(t,x) = (A - BB^T P)x + Bg(t,x)$

\PP2019/ \4bii/ let $V(x)=x^T P x$ ∴ taking ~~trajectories~~ along derivatives along the trajectories of the closed loop system:

$\dot{V}(x) = x^T[P(A-BB^TP)+(A-BB^TP)^TP]x + 2x^TPBg(t,x)$

which may be written in terms of the Riccati equation ∴

$\dot{V}(x) = -x^T[Q + PBB^TP + 2\alpha P]x + 2x^TPBg(x,t)$ ∴

using the given bounds and $w = B^TPx$:

$\dot{V}(x) \leq -k^2\|x\|^2 - \|w\|^2 - 2\alpha\lambda_{min}(P)\|x\|^2 + 2k\|w\|\|x\| =$

$-[k\|x\| - \|w\|]^2 - 2\alpha\lambda_{min}(P)\|x\|^2 \leq -2\alpha\lambda_{min}(P)\|x\|^2$

\4bii/ let $V(x) = x^TPx$ ∴ taking derivatives along the trajectories of the closed loop system:

$\dot{V} = \frac{d}{dt}[V(x)] = \frac{d}{dt}[x^TPx] = \dot{x}^TPx + x^TP\dot{x}$

$\dot{x}^T = [(A-BB^TP)x + Bg(t,x)]^T = x^T(A-BB^TP)^T + B^Tg(t,x)$

$\dot{V} = [x^T(A-BB^TP)^T + B^Tg(t,x)]Px + x^TP[(A-BB^TP)x + Bg(t,x)] =$

$x^T(A-BB^TP)^TPx + B^Tg(t,x)Px + x^TP(A-BB^TP)x + x^TPBg(t,x) =$

$x^T(A-BB^TP)^TPx + x^TP(A-BB^TP)x + [x^TPBg(t,x)]^T + x^TPBg(t,x) =$

$x^T[P(A-BB^TP) + (A-BB^TP)^TP]x + 2x^TPBg(t,x)$.

$Q + 2\alpha P + PBB^TP = 2PBB^TP - PA - A^TP$ ∴

$P(A-BB^TP) + (A-BB^TP)^TP = PA - PBB^TP + (A^T - P^TBB^T)P =$

$PA - PBB^TP + A^TP - P^TBB^TP = -(2PBB^TP - PA - A^TP) =$

$-Q - 2\alpha P - PBB^TP$ ∴

$\dot{V} = -x^T(Q + 2\alpha P + PBB^TP)x + 2x^TPBg(t,x) \leq$

$-k^2\|x\|^2 - \|w\|^2 - 2\alpha\lambda_{min}(P)\|x\|^2 + 2k\|w\|\|x\|$ for $w = B^TPx$

∴ $\dot{V} = -x^T(Q + 2\alpha P + PBwx^{-1})x + 2x^TPBg(t,x) =$

$(-x^TQ - 2x^T\alpha P - x^TPBwx^{-1})x + 2x^TPBg(t,x) =$

$-x^TQx - 2x^T\alpha Px - x^TPBw + 2x^TPBg(t,x)$ ∴ for $w = B^TPx$:

$\dot{V} \leq -k^2\|x\|^2 - \|w\|^2 - 2\alpha\lambda_{min}(P)\|x\|^2 + 2k\|w\|\|x\| =$

$-[k\|x\| - \|w\|]^2 - 2\alpha\lambda_{min}(P)\|x\|^2 \leq -2\alpha\lambda_{min}(P)\|x\|^2$

4b iii $\because \dot{V}(t,x) \leq -2\alpha \lambda_{min}(P)\|x\|^2$, $V(x) = (x^T P x) \geq 0$ $\therefore$

can conclude the origin is globally exponentially stable:

$\lambda_{min}\|x\|^2 \leq x^T P x \leq \lambda_{max}\|x\|^2$, $\dot{V} \leq \varepsilon V$, $x \in \mathbb{R}^n$

\ CW sols / \ 1a / equilibria: $\dot{x}_1 = 0 = 2(x_2 - x_1) + x_1(1 - x_1^2)$ ①

$\dot{x}_2 = 0 = -2(x_2 - x_1) + x_2(1 - x_2^2) + 0$ ② $\because u = 0 \therefore$

) ① − ②: $2(x_2 - x_1) + x_1(1 - x_1^2) + 2(x_2 - x_1) - x_2(1 - x_2^2) = 0 =$

$4(x_2 - x_1) + x_1 - x_1^3 - x_2 + x_2^3 = (x_2 - x_1)(3 + x_2^2 + x_1^2 + x_1 x_2) = 0 \therefore$

$x_1 = x_2$ $\because$ $3 + x_2^2 + x_1^2 + x_1 x_2 = 0$ has no real solutions $\therefore x_1 = x_2 \therefore$

into ①: $\cancel{x_2}(x_1 - x_1) + x_1(1 - x_1^2) = x_1(1 - x_1^2) = 0 \therefore$

$1 - x_1^2 = 0 \therefore x_1^2 = 1 \therefore x_1 = 1, x_1 = -1, x_1 = 0 \therefore$

equilibria are: $(0,0)$, $(1,1)$, $(-1,-1)$

\ 1b / To linearise system, determine Jacobian

$$D_f = \begin{bmatrix} -1-3x_1^2 & 2 \\ 2 & -1-3x_2^2 \end{bmatrix}$$

at $(0,0)$: $D_f|_{(0,0)} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ $\cancel{f}$

$\therefore$ characteristic eqn: $\mathrm{Det}(D_f - \lambda I) = \begin{bmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} = (\lambda + 1)^2 - 4 = 0$

$\therefore$ eigvals: $\lambda = -3, 1 \therefore$

equili $(0,0)$ is saddle

at $(1,1)$ and $(-1,-1)$:

$D_f|_{(1,1)} = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix}$ eigvals are $-2, -6$ stable node

$D_f|_{(-1,-1)} = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix}$ eigvals are $-2, -6$, stable node

\ 1c / To find out about limit cycles, use Bendixson's theorem, to see if any closed orbits exist,

$$\nabla f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -2 + 1 - 3x_1^2 - 2 + 1 - 3x_2^2 = -2 - 3(x_1^2 + x_2^2) < 0$$

$\therefore \because$ the divergence is always negative, it can never have a sign change, or be equivalently zero. $\therefore$, no closed orbits can exist in this system so neither can have a limit cycle.

\ / a / equilibria: $\dot{x} = 0 = 2(x_2 - x_1) + x_1(1 - x_1^2)$ ①

$\dot{x}_2 = 0 = -2(x_2 - x_1) + x_2(1 - x_2^2) + 0$ ② $\because u = 0 \therefore$

) ① − ②: $2(x_2 - x_1) + x_1(1 - x_1^2) + 2(x_2 - x_1) - x_2(1 - x_2^2) = 0 =$

$4(x_2 - x_1) + x_1 - x_1^3 - x_2 + x_2^3 = (x_2 - x_1)(3 + x_2^2 + x_1^2 + x_1 x_2) = 0 \therefore$

$3 + x_2^2 + x_1^2 + x_1 x_2 \neq 0 \therefore$ if $3 + x_2^2 + x_1^2 + x_1 x_2 = 0$ then

$x_2^2 + \cancel{x_1 x_2} + x_1^2 = -3$ but $(x_1 + x_2)^2 - x_1 x_2$ but $|x_1 x_2| \leq |x_1^2 + x_2^2| \therefore$

$3+x_1^2+x_1x_2+x_2^2 \geq 0 \therefore \quad 3+x_1^2+x_1x_2+x_2^2 \neq 0 \therefore$

$(x_2-x_1)=0 \therefore \quad x_2=x_1 \therefore$

$0=2(x_1-x_1)+x_1(1-x_1^2)=x_1(1-x_1^2)=0 \therefore$

$x_1=0 \quad , x_1^2=1 \therefore \quad x_1=1 \, , x_1=-1 \therefore$

equilibria: $(0,0)$, $(1,1)$, $(-1,-1)$

\1b/ $\frac{\partial f}{\partial x} = D_f = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} \partial_{x_1}(2x_2-x_1-x_1^3) & \partial_{x_2}(2x_2-x_1-x_1^3) \\ \partial_{x_1}(-x_2+2x_1-x_2^3) & \partial_{x_2}(-x_2+2x_1-x_2^3) \end{bmatrix} =$

$\begin{bmatrix} -1-3x_1^2 & 2 \\ 2 & -1-3x_2^2 \end{bmatrix}$

at $(0,0)$: $D_f|_{(0,0)} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \therefore$

$\text{Det}(D_f(0,0)-\lambda I) = \begin{bmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} = (\lambda+1)^2-4=0 \therefore$

eigvals: $\lambda=-3, 1 \therefore$

$(0,0)$ equili is saddle

at $(1,1)$, $(-1,-1)$: $D_f = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix} \therefore$

eigvals: $-2, -6 \therefore$ stable node

\1c/ by Bendixsons thm: closed orbits exist ~~iff~~:

$\nabla f = \frac{\partial f_1}{\partial x_1} + \frac{\partial f_2}{\partial x_2} = -2+1-3x_1^2-2+1-3x_2^2 = -2-3(x_1^2+x_2^2) < 0 \therefore$

$\therefore$ the divergence is always negative, it can never have a sign change, or be equivalently zero $\therefore$ no closed orbits can exist in this system $\therefore$ neither can have a limit cycle

\1d/ $\dot{x} = Ax+Bu$ about origin: $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2x_2-x_1-x_1^3 \\ -x_2+2x_1-x_2^3 \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix} =$

$\begin{bmatrix} -1-x_1^2 & 2 \\ 2 & -1-x_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \therefore$

~~A= at origin~~: $\cancel{\dot{x} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}}$ $A = \begin{bmatrix} -1-x_1^2 & 2 \\ 2 & -1-x_2^2 \end{bmatrix} \therefore$

at origin $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

\1e/ controllability matrix: $M = [B \,\vdots\, AB] = \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \vdots \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]$

$= \begin{bmatrix} 0 & \vdots & 2 \\ 1 & \vdots & -1 \end{bmatrix} \therefore \det(M) = -2 \neq 0 \therefore$ full rank $\therefore$

rank $M=2 \therefore$ pair (A,B) is controllable

\CW sols/ \1d/ $\dot{x} = Ax + Bu \therefore \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_1 - x_1^3 \\ -x_2 + 2x_1 - x_2^3 \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix} =$

$\begin{bmatrix} -1-x_1^2 & 2 \\ 2 & -1-x_2^2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u \therefore$ ~~A=~~

$$A = \begin{bmatrix} -1-x_1^2 & 2 \\ 2 & -1-x_2^2 \end{bmatrix} \therefore$$

~~At~~ at origin $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ $\quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

\1e/ controllability matrix: $M = [B \vdots AB] = \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \vdots \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] =$

$\left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \vdots \begin{bmatrix} 2 \\ -1 \end{bmatrix} \right] = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \therefore$

$\det(M) = -2 \neq 0 \therefore$ full rank ~~2~~ $\therefore \operatorname{rank}(M) = 2 \therefore$

pair (A, B) is controllable

\1f/ Desired poles are -5 and $-8 \therefore$

associated desired characteristic polynomial is:

$(s+5)(s+8) =$ ~~$s^2 + 13s + 40$~~ $s^2 + 8s + 5s + 40 = s^2 + 13s + 40 = 0 \therefore$

let $K = [k_1 \;\; k_2]$

and the closed loop stable space equation is $\dot{x} = (A - BK)x$,

$A - BK = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \;\; k_2] = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2-k_1 & -k_2-1 \end{bmatrix}$.

The desired polynomial equals to the characteristic equation associated with $\det(sI_2 - (A - BK)) = 0 = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ -k_1+2 & -k_2-1 \end{bmatrix} \right) =$

$$\det\left(\begin{bmatrix} s+1 & -2 \\ k_1-2 & s+k_2+1 \end{bmatrix} \right) = (s+1)(s+k_2+1) + 2(k_1-2) =$$

$s^2 + k_2 s + s + s + k_2 + 1 + 2k_1 - 4 = s^2 + (k_2+2)s + (k_2 + 2k_1 - 3) = 0 \therefore$

Comparing with $s^2 + 13s + 40 = 0 \therefore$

$s^2 + (k_2+2)s + (k_2 + 2k_1 - 3) = s^2 + 13s + 40 \therefore$

$k_2 + 2 = 13 \therefore k_2 = 11$, $k_2 + 2k_1 - 3 = 40 = 11 + 2k_1 - 3 \therefore$

$2k_1 = 40 - 11 + 3 = 32 \therefore k_1 = 16 \therefore \quad K = [16 \;\; 11] \therefore$

control law is $u = -Kx = -[16 \;\; 11]x =$ ~~$-16x$~~ $[-16 \;\; -11] \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

$-16x_1 - 11x_2$

\1f/ Desired poles are -5 and $-8 \therefore$ associated desired characteristic polynomial is: $(s+5)(s+8) =$

$s^2 + 8s + 5s + 40 = s^2 + 13s + 40 = 0 \therefore$

let $K=[k_1 \;\; k_2]$ ∴

and the closed loop state space equation is $\dot{x}=(A-B$

$\dot{x} = A_{cl}^{+} x = (A-BK)x$ ∴

$A_{cl}^{+} = A - BK = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}[k_1 \;\; k_2] = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ -k_1+2 & -k_2-1 \end{bmatrix}$ ∴

The desired polynomial equals the characteristic equation associated with: $\det(sI_2 - A_{cl}^{+}) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ -k_1+2 & -k_2-1 \end{bmatrix}\right) =$

$\det\left(\begin{bmatrix} s+1 & -2 \\ k_1-2 & s+k_2+1 \end{bmatrix}\right) = (s+1)(s+k_2+1)+2(k_1-2) =$

$s^2 + k_2 s + s + s + k_2 + 1 + 2k_1 - 4 = s^2 + (k_2+2) + (2k_1+k_2-3) = 0$ ∴

Comparing with $s^2+13s+40=0$:

$s^2+(k_2+2)+(2k_1+k_2-3) = s^2+13s+40$ ∴

$k_2+2=13$ ∴ $k_2=11$ ∴

$2k_1+k_2-3 = 2k_1+11-3 = 40$ ∴ $2k_1 = 32$ ∴

$k_1 = 16$ ∴

$K=[16 \;\; 11]$ ∴

Control law is: $u=-Kx=-[16 \;\; 11]x = [-16 \;\; -11]\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = -16x_1 - 11x_2$

\1g/ The transformation matrix $T=MW$, where M is controllability matrix and $W=\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}$, where to obtain a_1, determine charactistic polynomial associated with A. ∴

$\det(sI-A) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} s+1 & -2 \\ -2 & s+1 \end{bmatrix}\right) =$

$(s+1)(s+1)+2(-2) = \cancel{s^2+2s+1-4=s^2} \; s^2+s+s+1-4 = s^2+2s-3 = 0$ ∴

$s^2+a_1 s + a_2 = 0$ ∴

$s^2+2s-3 = s^2+a_1 s + a_2$ ∴ $a_1 = 2$ ∴

$W=\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ ∴

$T=\begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

\1g/ The transformation matrix $T=MW$, where M is controllability matrix $M=\begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$, and $W=\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}$, where to obtain a_1, determine characteristic polynomial

\ CW Sols / with A. $\therefore$ $\det(sI - A) = s^2 + a_1 s + a_2$ $\therefore$

$\det(sI - A) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}\right) = \det\left(\begin{bmatrix} s+1 & -2 \\ -2 & s+1 \end{bmatrix}\right) =$

$(s+1)(s+1) + 2(-2) = s^2 + s + s + 1 - 4 = s^2 + 2s - 3 = 0$ $\therefore$

$s^2 + a_1 s + a_2 = 0$ $\therefore$

$s^2 + 2s - 3 = s^2 + a_1 s + a_2$ $\therefore$ $a_1 = 2$ $\therefore$ $W = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ $\therefore$

$T = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

\ 2 / $\dot{x}_1 = x_2$, $\dot{x}_2 = -x_1 + x_2(2 - 3x_1^2 - 2x_2^2)$ $\therefore$ let $V(x_1, x_2) = x_1^2 + x_2^2$ $\therefore$

$f(x) = \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$ $\therefore$

$f(x) \cdot \nabla V(x_1, x_2) = f(x) \cdot \begin{bmatrix} \partial_{x_1}(x_1^2 + x_2^2) \\ \partial_{x_2}(x_1^2 + x_2^2) \end{bmatrix} = f(x) \cdot \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} =$

$\begin{bmatrix} x_2 \\ -x_1 + 2x_2 - 3x_1^2 x_2 - 2x_2^3 \end{bmatrix} \cdot \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = 2x_1x_2 - 2x_1x_2 + 2x_2^2(2 - 3x_1^2 - 2x_2^2) =$

$2x_2^2(2 - 3x_1^2 - 2x_2^2) = 2x_2^2(2 - 2x_1^2 - 2x_2^2 - x_1^2) =$

$(2)x_2^2(2 - 2x_1^2 - 2x_2^2) - 2x_1^2 x_2^2 = 4x_2^2(1 - x_1^2 - x_2^2) - 2x_1^2 x_2^2 \leq$

$4x_2^2(1 - x_1^2 - x_2^2)$ $\therefore$

$f(x) \cdot \nabla V(x) \leq 0$ for $1 - x_1^2 - x_2^2 \leq 0$ $\therefore$ $V = x_1^2 + x_2^2 \geq 1$ $\therefore$ $V \geq 1$

$\therefore$ In particular, all trajectories starting in

$M = \{x \in \mathbb{R}^2 \mid V(x) \leq 1\}$ stay in M for all future time.

$\therefore$ M contains only one equilibrium point, which is the origin. And linearization at the origin yields us

$A = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ and its eigvals are 1 and 1.

$\therefore$ f analytic in x, the origin is unstable node.

$\therefore$ By ~~PB~~ P.B. criterion, there is a periodic orbit in M.

\ 2 / $f(x) = \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$ $\therefore$ $f(x) \cdot \nabla V(x_1, x_2) = f(x) \cdot \nabla(x_1^2 + x_2^2)$

$\therefore$ let $V(x_1, x_2) = x_1^2 + x_2^2$ $\therefore$

$f(x) \cdot \nabla V = f(x) \cdot \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + x_2(2 - 3x_1^2 - 2x_2^2) \end{bmatrix} \cdot \begin{bmatrix} 2x_1 \\ 2x_2 \end{bmatrix} =$

$2x_1x_2 - 2x_1x_2 + 2x_2^2(2 - 3x_1^2 - 2x_2^2) =$

$2x_2^2(2 - 2x_1^2 - 2x_2^2 - x_1^2) = 2x_2^2(2 - 2x_1^2 - 2x_2^2) - 2x_1^2 x_2^2 =$

$4x_2^2(1 - x_1^2 - x_2^2) - 2x_1^2 x_2^2 \leq 4x_2^2(1 - x_1^2 - x_2^2) \leq 0$ for $1 - x_1^2 - x_2^2 \leq 0$ $\therefore$

$f(x) \cdot \nabla V \leq 0$ $x_1^2 + x_2^2 \geq 1$ $\therefore$ $V(x) \geq 1$ $\therefore$

4/ let $V(x, x_2) = x_1^2 + x_2^2$ $\therefore$ V is positive desinite, and unbounded $\therefore$

$\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1(-3x_1 + x_1x_2) + 2x_2(-2x_2 + x_1) =$

$-6x_1^2 - 4x_2^2 + 2x_1^2x_2 + 2x_1x_2 = (-5x_1^2 - x_1^2) + 2x_1^2x_2 + (-3x_2^2 - x_2^2) + 2x_1x_2 =$

$\cancel{(-5+2x)}\ (-5x_1^2 + 2x_1^2x_2 - 3x_1^2) + (-x_1^2 + 2x_1x_2 - x_2^2) =$

$(-5x_1^2 + 2x_1^2x_2 - 3x_1^2) - (x_1^2 - 2x_1x_2 + x_2^2) =$

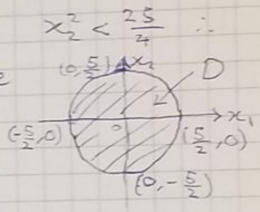
$(-5x_1^2 + 2x_1^2x_2 - 3x_1^2) - (x_1 - x_2)^2 \leq -5x_1^2 + 2x_1^2x_2 - 3x_2^2 =$

$-x_1^2(5 - 2x_2) - 3x_2^2$ $\therefore$

$\dot{V} < 0$ is $5 - 2x_2 > 0$ $\therefore$ $x_2 < \frac{5}{2}$ $\therefore$ $x_2^2 < \frac{25}{4}$ $\therefore$

negative desinite $\therefore$ locally asymp stable

$D = \{x_1, x_2 \in \mathbb{R} \mid x_1^2 + x_2^2 < \frac{25}{4}\}$



4/ let $V(x_1, x_2) = x_1^2 + x_2^2$ $\therefore$ V is positive desinite and unbounded $\therefore$

$\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1(-3x_1 + x_1x_2) + 2x_2(-2x_2 + x_1) =$

$\cancel{(-3x}\ -6x_1^2 + 2x_1^2x_2 + 2x_1x_2 - 4x_2^2 = (-5x_1^2 - x_1^2) + 2x_1^2x_2 + (-3x_2^2 - x_2^2) + 2x_1x_2 =$

$(-5x_1^2 + 2x_1^2x_2 - 3x_2^2) + (-x_1^2 + 2x_1x_2 - x_2^2) =$

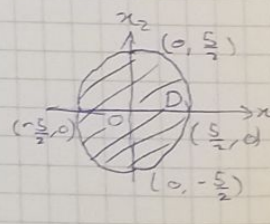
$(-5x_1^2 + 2x_1^2x_2 - 3x_2^2) - (x_1^2 - 2x_1x_2 + x_2^2) = (-5x_1^2 + 2x_1^2x_2 - 3x_1^2) - (x_1 - x_2)^2 \leq$

$-5x_1^2 + 2x_1^2x_2 - 3x_2^2 = -x_1^2(5 - 2x_2) - 3x_2^2$ $\therefore$

$\dot{V} < 0$ is $5 - 2x_2 > 0$ $\therefore$ $x_2 < \frac{5}{2}$ $\therefore$ negative desinite $\therefore$

locally asymp stable $\therefore$ $x_2^2 < \frac{25}{4}$ $\therefore$

$D = \{x_1, x_2 \in \mathbb{R} \mid x_1^2 + x_2^2 < \frac{25}{4}\}$ $\therefore$



5a/ $x_1 = y - \frac{Mg}{k}$, $x_2 = \dot{y}$ $\therefore$

$\dot{x}_1 = \frac{d}{dt}(x_1) = \frac{d}{dt}\left(y - \frac{Mg}{k}\right) = \dot{y} = x_2$ $\therefore$

$\dot{x}_2 = \frac{d}{dt}(x_2) = \frac{d}{dt}(\dot{y}) = \ddot{y} = g - \frac{c_1}{M}\dot{y} - \frac{c_2}{M}\dot{y}|\dot{y}| - \frac{k}{M}y =$

$g - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2| - \frac{k}{M}\left(x_1 + \frac{Mg}{k}\right) = g - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2| - \frac{k}{M}x_1 - g =$

$\dot{x}_2 = -\frac{k}{M}x_1 - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2|$ is the steady space reprecentation

\CW sols/ with x_1, x_2

\5a/ ~~$\ddot{y} = \frac{-C_1}{M}\dot{y} + \frac{-C_2}{M}\dot{y}|\dot{y}| + g - \frac{k}{M}y$~~ $M\ddot{y} = Mg - C_1\dot{y} - C_2\dot{y}|\dot{y}| - ky$ ∴

$\ddot{y} = g - \frac{C_1}{M}\dot{y} - \frac{C_2}{M}\dot{y}|\dot{y}| - \frac{k}{M}y$ ∴

$x_1 = y - \frac{Mg}{k}$, $x_2 = \dot{y}$ ∴ $x_1 + \frac{Mg}{k} = y$ ∴

$\dot{x}_1 = \frac{d}{dt}(x_1) = \frac{d}{dt}\left(y - \frac{Mg}{k}\right) = \dot{y} = x_2$ ∴ $\dot{x}_1 = x_2$, ∴

$\dot{x}_2 = \frac{d}{dt}(x_2) = \frac{d}{dt}(\dot{y}) = \ddot{y} = g - \frac{C_1}{M}\dot{y} - \frac{C_2}{M}\dot{y}|\dot{y}| - \frac{k}{M}y =$

$g - \frac{C_1}{M}x_2 - \frac{C_2}{M}x_2|x_2| - \frac{k}{M}\left(x_1 + \frac{Mg}{k}\right) =$

$g - \frac{C_1}{M}x_2 - \frac{C_2}{M}x_2|x_2| - \frac{k}{M}x_1 - g =$

$\dot{x}_2 = -\frac{k}{M}x_1 - \frac{C_1}{M}x_2 - \frac{C_2}{M}x_2|x_2|$, $\dot{x}_1 = x_2$ ~~State~~ is the steady space representation with x_1, x_2

\5b/ let $V = ax_1^2 + bx_2^2$, $a, b > 0$ ∴ V is positive definite and radially unbounded ∴

$\dot{V} = 2ax_1\dot{x}_1 + 2bx_2\dot{x}_2 =$

$2ax_1x_2 + 2bx_2\left(-\frac{k}{M}x_1 - \frac{C_1}{M}x_2 - \frac{C_2}{M}x_2|x_2|\right) =$

$2ax_1x_2 - \frac{2bk}{M}x_1x_2 - \frac{2bC_1}{M}x_2^2 - \frac{2bC_2}{M}x_2^2|x_2| =$

$2\left(a - \frac{bk}{M}\right)x_1x_2 - \frac{2bC_1}{M}x_2^2 - \frac{2bC_2}{M}x_2^2|x_2|$ ∴

taking $b = \frac{M}{2}$:

$\dot{V} = 2\left(a - \frac{(M/2)k}{M}\right)x_1x_2 - C_1x_2^2 - C_2x_2^2|x_2| =$

$\dot{V} = 2\left(a - \frac{k}{2}\right)x_1x_2 - C_1x_2^2 - C_2x_2^2|x_2|$ ∴

taking $a = \frac{k}{2}$:

$\dot{V} = 2\left(\frac{k}{2} - \frac{k}{2}\right)x_1x_2 - C_1x_2^2 - C_2x_2^2|x_2| = -C_1x_2^2 - C_2x_2^2|x_2| \le 0 \quad \forall x$

∴ independent of x_1 ∴ negative semidefinite only ∴

and $\dot{V} = 0$ for $x_2 = 0$ ∴ $x_1 = 0$ ∴

using LaSalle's theorem, conclude the origin is the trivial solution about which $\dot{V} = 0$, and for all other x, $\dot{V}$ is negative definite and ∴ the origin is globally asymptotically stable.

5b/ 6b $V = ax_1^2 + bx_2^2$; $a, b > 0$ ∴

V is positive definite and radially unbounded

$\dot{V} = 2ax_1\dot{x}_1 + 2bx_2\dot{x}_2 =$

$2ax_1x_2 + 2bx_2\left(-\frac{k}{M}x_1 - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2|\right) =$

$2ax_1x_2 - 2\frac{bk}{M}x_1x_2 - 2\frac{bc_1}{M}x_2^2 - 2\frac{bc_2}{M}x_2^2|x_2| =$

$2\left(a - \frac{bk}{M}\right)x_1x_2 - 2\frac{bc_1}{M}x_2^2 - 2\frac{bc_2}{M}x_2^2|x_2|$ ∴

taking $b = \frac{M}{2}$ ∴

$\dot{V} = 2\left(a - \frac{\left(\frac{M}{2}\right)k}{M}\right)x_1x_2 - 2\frac{\left(\frac{M}{2}\right)c_1}{M}x_2^2 - 2\frac{\left(\frac{M}{2}\right)c_2}{M}x_2^2|x_2| =$

$2\left(a - \frac{k}{2}\right)x_1x_2 - c_1x_2^2 - c_2x_2^2|x_2|$ ∴

taking $a = \frac{k}{2}$ ∴

$\dot{V} = 2\left(\frac{k}{2} - \frac{k}{2}\right)x_1x_2 - c_1x_2^2 - c_2x_2^2|x_2| = -c_1x_2^2 - c_2x_2^2|x_2| \leq 0 \; \forall x$

is negative semidefinite ∵ independent of x_1,

but for $\dot{V} = 0$ ∴ $x_2 = 0 \implies x_1 = 0$ ∴

negative definite ∴

using LaSalle's theorem, conclude that the origin is the trivial solution about which $\dot{V} = 0$, and for all other x, $\dot{V}$ is negative definite ∴

the origin is globally asymptotically stable.

\3011 PP2020/

\1ai/ if $0<4a(a+b)<1$ ∴ $0<1-4a(a+b)<1$ ∴

$\sqrt{1-4a(a+b)} \in \mathbb{R}_{>0}$ ∴ $0<\frac{\sqrt{1-4a(a+b)}}{2}<\frac{1}{2}$ ∴

$-\frac{1}{2}+\frac{\sqrt{1-4a(a+b)}}{2} \in \mathbb{R}_{<0}$, $-\frac{1}{2}-\frac{\sqrt{1-4a(a+b)}}{2} \in \mathbb{R}_{<0}$ ∴ $\lambda_{1,2} \in \mathbb{R}_{<0}$

if $0<4a(a+b)<1$ its a stable ~~focus~~, node.

if $4a(a+b)>1$ ∴ $1-4a(a+b)<0$ ∴

$\pm\sqrt{1-4a(a+b)} \notin \mathbb{R}$ ∴ $\lambda_{1,2} \in \mathbb{C}$, $Re(\lambda_{1,2}) \in \mathbb{R}_{<0}$ ∴

if $4a(a+b)>1$ its a stable focus

if $a(a+b)<0$ ∴ $-4a(a+b)>0$ ∴ $1-4a(a+b)>1$ ∴

$+\sqrt{1-4a(a+b)} > 1$ ∴ $1-\sqrt{1-4a(a+b)}<0$ ∴

$\lambda_1 \in \mathbb{R}_{>0}$, $\lambda_2 \in \mathbb{R}_{<0}$ ∴ point is saddle ∴

statement is true.

\1aii/ $-x$ is ~~[illegible]~~ lipschitz globally ∵ $|-x-(-y)|=|-x+y|=|x-y|$

$\leq |x-y|$, $||x|-|y|| \leq |x-y|$ ∴ $|x|$ is global lipshitz ∴

$|x|$ is not continuously differentiable ∴ the function means

false.

\1aiii/ $x^T P x$ is in quadratic form ∴ $x^T x$ is positive

definite and P is not negative definite ∴

False.

\1aiv/ False ∵ there cannot be a periodic ~~orbit~~

enclosing both a stable focus and saddle point

∵ no domain exists that entraps all trajectories in that domain

\1bi/ controllability matrix $M = [B \vdots AB] = \left[\begin{bmatrix}0\\1\end{bmatrix} \vdots \begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right] =$

$\left[\begin{bmatrix}0\\1\end{bmatrix} \vdots \begin{bmatrix}2\\1\end{bmatrix}\right] = \begin{bmatrix}0&2\\1&1\end{bmatrix}$ ∴

$eig(M) = \lambda_{1,2} = \lambda$ ∴ $det(M-\lambda I) = \begin{vmatrix}-\lambda & 2\\ 1 & 1-\lambda\end{vmatrix} = -\lambda(1-\lambda)-2(1) = -\lambda+\lambda^2-2 =$

$\lambda^2-\lambda-2=(\lambda-2)(\lambda+1)=0$ ∴ $\lambda_1=2$, $\lambda_2=-1$ ∴

~~\1bii/~~ $\lambda_2<0$, $\lambda_{1,2} \in \mathbb{R}$ ∴ M is controllable.

\1bi/ let $K = \begin{bmatrix} k_1 & k_2 \end{bmatrix}$ $\therefore$ system equation: $(s+4)(s+5) =$

$s^2 + 4s + 5s + 20 = s^2 + 9s + 20$

$BK = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} k_1 & k_2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}$ $\therefore$ $\cancel{\det}(A - BK) = \cancel{\det}$

$\cancel{\det}\left(\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} \right) = \cancel{\det}\left(\begin{bmatrix} 1 & 2 \\ 2-k_1 & 2-k_2 \end{bmatrix} \right)$ $\therefore$

$\det(sI - (A - BK)) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2-k_1 & 2-k_2 \end{bmatrix} \right) = \det\left(\begin{bmatrix} s-1 & -2 \\ k_1-2 & s+k_2-2 \end{bmatrix} \right) =$

$(s-1)(s+k_2-2) - (-2)(k_1-2) =$

$s^2 + k_2 s - 2s - s - k_2 + 2 + 2k_1 - 4 = s^2 + s(k_2 - 3) + (2k_1 - k_2 - 2) =$

$s^2 + 9s + 20$ $\therefore$

$k_2 - 3 = 9$ $\therefore$ $k_2 = 12$,

$2k_1 - k_2 - 2 = 20 = 2k_1 - (12) - 2 = 20$ $\therefore$ $2k_1 = 34$ $\therefore$ $k_1 = 17$ $\therefore$

$K = \begin{bmatrix} 17 & 12 \end{bmatrix}$ $\therefore$ $u = -Kx = -\begin{bmatrix} 17 & 12 \end{bmatrix} x = \begin{bmatrix} -17 & -12 \end{bmatrix} x$ ✗

\1biii/ $T = WM$, $W = \begin{bmatrix} a_2 & 1 \\ 1 & 0 \end{bmatrix}$, $s + a_1 s + a_2$ $\therefore$

$\det(sI - A) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \right) = \det\left(\begin{bmatrix} s-1 & -2 \\ -2 & s-1 \end{bmatrix} \right) =$

$(s-1)(s-1) - (-2)(-2) = s^2 - 2s + 1 - 4 = s^2 - 2s - 3 = s + a_1 s + a_2$ $\therefore$

$a_2 = -3$ $\therefore$ $W = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix}$ $\therefore$

$T = WM = \begin{bmatrix} -3 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -3(2)+1(1) \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -5 \\ 0 & 2 \end{bmatrix}$ ✗

\1c/ $V(0,0) = \frac{1}{2}(0^2 + 0^2) = 0$, $x_1^2, x_2^2 \geq 0$ $\therefore$

$V(x_1, x_2) > 0 \quad \forall (x_1, x_2) \in \mathbb{R}^2 \setminus \{0, 0\}$ $\therefore$ V is positive definite

$\therefore$ taking the derivatives along the trajectories of the system:

$\dot{V} = \frac{dV}{dt} = \frac{1}{2}(2x_1\dot{x}_1 + 2x_2\dot{x}_2) = x_1\dot{x}_1 + x_2\dot{x}_2 =$

$x_1(-x_1 + g(x_2)) + x_2(-x_2 + h(x_1)) = -x_1^2 + g(x_2)x_1 - x_2^2 + h(x_1)x_2 =$

$-(x_1^2 + x_2^2) + g(x_2)x_1 + h(x_1)x_2 = -2\left[\frac{1}{2}(x_1^2 + x_2^2)\right] + g(x_2)x_1 + h(x_1)x_2 =$

$-2V + g(x_2)x_1 + h(x_1)x_2 = \dot{V}$ $\therefore$ $V \geq 0$ $\therefore$ $-2V \leq 0$ $\therefore$

$\dot{V} \leq g(x_2)x_1 + h(x_1)x_2$ $\therefore$ $|g(x_2)| \leq |x_2|/2$ $\therefore$ $|g(x_2)|x_1 \leq |x_2|x_1/2$ $\therefore$

$g(x_2)x_1 \leq x_2 x_1/2$, $|h(x_1)| \leq |x_1|/2$ $\therefore$ ~~$|h(x_1)| \leq x$~~ $|h(x_1)|x_2 \leq |x_1|x_2/2$ $\therefore$

$h(x_1)x_2 \leq x_1 x_2/2$ $\therefore$ $g(x_2)x_1 + h(x_1)x_2 \leq x_2 x_1/2 + x_1 x_2/2 = x_1 x_2$ $\therefore$

$\dot{V} \leq -(x_1^2 + x_2^2) + x_1 x_2 \leq \cancel{z}$ $\therefore$

\3011PP2020/ $x_1^2+x_2^2 \ge x_1x_2$ ∴ $-(x_1^2+x_2^2)+x_1x_2 \le 0$ ∴

$\dot V \le 0$ ∴ $\dot V$ is negative semidefinite ∴

the origin is globally asymptotically stable

\1d/ $x=\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ∴ $\dot x=\begin{bmatrix} \dot x_1 \\ \dot x_2 \end{bmatrix}=\begin{bmatrix} -x_1+x_2x_1^3 \\ x_1^3+x_1x_2^2+u \end{bmatrix}=\begin{bmatrix} -x_1 \\ 0 \end{bmatrix}+\begin{bmatrix} x_2x_1^3 \\ x_1^3+x_1x_2^2+u \end{bmatrix}=$

$Ax+Bu=\begin{bmatrix} -x+x_2x_1^3 \\ x_1^3+x_1x_2^2 \end{bmatrix}+\begin{bmatrix} 0 \\ 1 \end{bmatrix}u$ ∴ $B=\begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $A=\begin{bmatrix} -x+x_2x_1^3 \\ x_1^3+x_1x_2^2 \end{bmatrix}$ ∴

$\|x\|=|x_1x_2|$

\1e/ ~~$V(0,0)=$~~ let $V(x_1,x_2)=x_1^2+x_2^2$ ∴ $V(0,0)=0$,

$V>0\ \forall(x_1,x_2)\in\mathbb{R}^2 \setminus \{(0,0)\}$ ∴ V is positive definite ∴

Let $x_1=y$ $x_2=\dot y$ ∴ $\dot x_1=\dot y=x_2$, $\ddot y=-y+\varepsilon\dot y(1-y^2-\dot y^2)$ ∴

$\dot x_2=\ddot y=-y+\varepsilon\dot y(1-y^2-\dot y^2)=-x_1+\varepsilon x_2(1-x_1^2-x_2^2)$ ∴

taking derivite of V with time:

$\dot V=2x_1\dot x_1+2x_2\dot x_2=2x_1(x_2)+2x_2(-x_1+\varepsilon x_2(1-x_1^2-x_2^2))=$

$2x_1x_2-2x_1x_2+2\varepsilon x_2^2(1-x_1^2-x_2^2)=2\varepsilon x_2^2(1-x_1^2-x_2^2)=\dot V$ ∴

$2\varepsilon x_2^2\ge 0$ ∴ $\dot V\le 0$ if $(1-x_1^2-x_2^2)<0$ ∴

$x_1^2+x_2^2=V>1$ ∴ by Poincaré Bendixson criterion,

the system has a periodic orbit for all trajectories in the domain $D=\{(x_1,x_2)\in\mathbb{R}^2 \mid x_1^2+x_2^2=V<1\}$

\2a/ equilibria when $\dot\omega_x=0$, $\dot\omega_y=0$, $\dot\omega_z=0$ ∴

$a\omega_y\omega_z=0$, $-b\omega_z\omega_x=0$, $c\omega_x\omega_y=0$ ∴ $a,b,c>0$ ∴

ω_y or $\omega_z=0$ ∴ when $\omega_y=0$: $c\omega_x(0)=0$ ∴ ω_x arbitrary ∴

$\omega_z=0$ ∴ $(\omega_x,0,0)$

when $\omega_z=0$: ω_x arbitrary ∴ $\omega_y=0$ ∴ $(\omega_x,0,0)$, ∴

when $\omega_x=0$: $+c(0)(\omega_y)=0$ ∴ ω_y arbitrary ∴ $\omega_z=0$ ∴

$(0,\omega_y,0)$, when $\omega_x=\omega_y=\omega_z=0$ $\dot\omega_x=\dot\omega_y=\dot\omega_z=0$ ∴ $(0,0,0)$ is equilibria

when $\omega_z=0$: $-b(0)\omega_x=0$ ∴ ω_x arbitrary ∴ $\omega_y=0$ ∴

when $\omega_y=0$: ω_z arbitrary ∴ $-b\omega_x(\omega_z)=0$ ∴ $\omega_x=0$ ∴ $(0,0,\omega_z)$ ∴

equilibria are: $(\omega_x,0,0)$, $(0,\omega_y,0)$, $(0,0,\omega_z)$, $(0,0,0)$

\2b/ $\omega_x^2, \omega_y^2, \omega_z^2 \geq 0$ ∴ $a, b, c > 0$ ∴

if $p > 0$: $ap > 0$?

if $p, \cancel{p} > 0$: $ap + cr > 0$ ∴ $-bq < 0$ ∴ $q > 0$,

$p, r, q >$ is true ∴

$V(0,0,0) = 0$, $V(\omega_x, \omega_y, \omega_z) > 0 \ \forall (\omega_x, \omega_y, \omega_z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ ∴

V is positive definite ∴

$\dot{V} = 2p\omega_x\dot{\omega}_x + 2q\omega_y\dot{\omega}_y + 2r\omega_z\dot{\omega}_z =$

$2p\omega_x(a\omega_y\omega_z) + 2q(\omega_y)(-b\omega_z\omega_x) + 2r\omega_z(c\omega_x\omega_y) =$

$2ap\,\omega_x\omega_y\omega_z - 2bq\,\omega_x\omega_y\omega_z + 2cr\,\omega_x\omega_y\omega_z =$

$2(ap - bq + cr)(\omega_x\omega_y\omega_z) = 2(0)(\omega_x\omega_y\omega_z) = 0$ ∴

V is positive definite , $\dot{V} = 0$ and the origin is an equilibria ∴ it is Lyapunov stable.

\2c/ ω_0 is constant ∴ let $2ac\omega_y^2 + 4ab\omega_z^2 + bc(\omega_x^2 - \omega_0^2) = F(\omega_x, \omega_y, \omega_z)$ ∴

$\dot{V} = 2c\omega_y\dot{\omega}_y + 2b\omega_z\dot{\omega}_z + 2F(\omega_x,\omega_y,\omega_z)4ac\omega_y\dot{\omega}_y + 2(F)2ab\omega_z\dot{\omega}_z + 2(F)2bc\omega_x\dot{\omega}_x$

$= 2c\omega_y\dot{\omega}_y + 2b\omega_z\dot{\omega}_z + F(\omega_x,\omega_y,\omega_z)\left[8ac\omega_y\dot{\omega}_y + 4ab\omega_z\dot{\omega}_z + 4bc\omega_x\dot{\omega}_x\right] =$

$2c\omega_y\dot{\omega}_y + 2b\omega_z\dot{\omega}_z + F(\omega_x,\omega_y,\omega_z)\left[8ac\omega_y(-b\omega_z\omega_x) + 4ab\omega_z(c\omega_x\omega_y) + 4bc\omega_x(a\omega_y\omega_z)\right] =$

$2c\omega_y\dot{\omega}_y + 2b\omega_z\dot{\omega}_z + F(\omega_x,\omega_y,\omega_z)\left[\omega_x\omega_y\omega_z\right]\underbrace{\left[-8abc + 4abc + 4abc\right]}_{=0} =$

$2c\omega_y\dot{\omega}_y + 2b\omega_z\dot{\omega}_z + 0 = 2c\omega_y\dot{\omega}_y + 2b\omega_z\dot{\omega}_z =$

$2c\omega_y(-b\omega_z\omega_x) + 2b\omega_z(c\omega_x\omega_y) =$

$-2bc\omega_x\omega_y\omega_z + 2bc\omega_y\omega_x\omega_z = 0 = \dot{V}$ ∴

$V = c\omega_y^2 + b\omega_z^2 + (F(\omega_x, \omega_y, \omega_z))^2$, $F(0,0,0) = 0$ ∴

$V(0,0,0) = 0$, $V > 0 \ \forall (\omega_x, \omega_y, \omega_z) \in \mathbb{R}^3 \setminus \{(0,0,0)\}$ ∴

V is positive definite, $\dot{V} = 0$ ∴

The non-zero rotational Motion is stable

\3a/ if the origin is asymptotically lipschitz, then the x is locally lipschitz

\3bi/ $eig(A) = \lambda_{1,2}(A)$ ∴ $Re(\lambda_{1,2}(A)) < 0$ ∴ A is hurwitz ∴

$A^T P + PA = -I$, $P, P^T > 0$ ∴

\3aii pt 2020 / $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$ ∴ $P = P^T$ ∴

$\dot{V} = \dot{x}^T P x + x^T P \dot{x}$ ∴ $\dot{x}^T = (Ax + \Phi(x))^T$ ∴

$\dot{V} = (Ax + \Phi(x))^T Px + x^T P (Ax + \Phi(x)) =$

$((Ax)^T + \Phi(x)^T) Px + x^T P(Ax) + x^T P \Phi(x) =$

$(Ax)^T Px + \Phi(x)^T Px + x^T P(Ax) + xP\Phi(x) =$

$x^T A^T P x + x^T P A x + \Phi(x)^T P x + xP\Phi(x) = \dot{V}$ and

$(xP\Phi(x))^T = \Phi(x)^T (xP)^T = \Phi(x)^T P^T x^T = \Phi(x)^T P x^T$ ∴

$\Phi(x)^T Px + xP\Phi(x) = 2xP\Phi(x)$,

$(x^T P A x)^T = ((x^T P)(Ax))^T = (Ax)^T (x^T P)^T = x^T A^T P^T x = x^T A^T P x$ ∴

$x^T A^T P x + x^T P A x = 2x^T P A x$ ∴ is in quadratic form

$\dot{V} = 2x^T P A x + 2xP\Phi(x)$ ∴

$|\Phi(x)| < \frac{1}{4\|P\|}|x|$ ∴ $P|\Phi| < \frac{1}{4\|P\|} P|x|$ ∴ $xP|\Phi| < \frac{1}{4\|P\|} xP|x|$ ∴

$\dot{V}(x) \leq -\frac{1}{2}|x|^2$

\3bii / $\delta > 0$, $\frac{|\Phi(x)|}{|x|} < \frac{1}{4\|P\|}$, $\forall |x| < \delta$ ∴

$x^T P x$ is in quadratic form & $P > 0$ ∴ $x^T P x$ is positive definite ∴ $\lambda_1 < \lambda_2$ ∴ $\Omega = \{x \mid x^T P x < \lambda_1 \delta^2\}$

\3biii/ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ∴ $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + \sqrt{6}\, x_1 x_2 \\ -3x_2 + x_1^2 + x_2^2 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ -3x_2 \end{bmatrix} + \begin{bmatrix} \sqrt{6}\, x_1 x_2 \\ x_1^2 + x_2^2 \end{bmatrix} =$

$\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \sqrt{6}\, x_1 x_2 \\ x_1^2 + x_2^2 \end{bmatrix}$ ∴

$\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} = E$ ∴ $\det(E) = \det\begin{pmatrix} -2-\lambda & 0 \\ 0 & -3-\lambda \end{pmatrix} = (-2-\lambda)(-3-\lambda) = 6 + \lambda^2 + 5\lambda$ ∴

$-2-\lambda = 0$ ∴ $\lambda_1 = -3$, $-3-\lambda = 0$ ∴ $\lambda_2 = -2$ ∴ $\lambda_2 > \lambda_1$ ∴

\3biii/ $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} -2x_1 + \sqrt{6}\, x_1 x_2 \\ -3x_2 + x_1^2 + x_2^2 \end{bmatrix} = \begin{bmatrix} -2x_1 \\ -3x_2 \end{bmatrix} + \begin{bmatrix} \sqrt{6}\, x_1 x_2 \\ x_1^2 + x_2^2 \end{bmatrix} =$

$\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} \sqrt{6}\, x_1 x_2 \\ x_1^2 + x_2^2 \end{bmatrix}$ ∴ $A = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}$ ∴ $\det(A - \lambda I) = \begin{bmatrix} -2-\lambda & 0 \\ 0 & -3-\lambda \end{bmatrix} =$

$(-2-\lambda)(-3-\lambda) = 0$ ∴ $\lambda_1 = -3$, $\lambda_2 = -2$ ∴ $\lambda_i < 0$, $i = 1, 2$ ∴ A is hurwitz ∴ $PA + A^T P = -Q$ ∴ let $Q = I$ ∴ $P = P^T$, $P, P^T > 0$ ∴

$PA + A^T P = -I$ ∴ $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$ ∴

$A^T = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} = A$ ∴ $A^T P = AP$ ∴ $PA = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} -2P_{11} & -3P_{12} \\ -2P_{12} & -3P_{22} \end{bmatrix}$

∴ $AP = \begin{bmatrix} -2 & 0 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} = \begin{bmatrix} -2P_{11} & -2P_{12} \\ -3P_{12} & -3P_{22} \end{bmatrix}$ ∴

$PA + A^T P = -I = \begin{bmatrix} -2P_{11} & -3P_{12} \\ -2P_{12} & -3P_{22} \end{bmatrix} + \begin{bmatrix} -2P_{11} & -2P_{12} \\ -3P_{12} & -3P_{22} \end{bmatrix} = \begin{bmatrix} -4P_{11} & -5P_{12} \\ -5P_{12} & -6P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ ∴

$-5P_{12} = 0$ ∴ $P_{12} = 0$, $-6P_{22} = -1$ ∴ $P_{22} = \frac{1}{6}$, $-4P_{11} = -1$ ∴ $P_{11} = \frac{1}{4}$ ∴

$P = \begin{bmatrix} 1/4 & 0 \\ 0 & \frac{1}{6} \end{bmatrix}$ ∴ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ∴ $x^T = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$ ∴ $x^T P x =$

$\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 1/4 & 0 \\ 0 & 1/6 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} \frac{1}{4}x_1 \\ \frac{1}{6}x_2 \end{bmatrix} = \frac{1}{4}x_1^2 + \frac{1}{6}x_2^2$ ∴

$D = \left\{ x \mid \frac{1}{4}x_1^2 + \frac{1}{6}x_2^2 < \lambda_1 \delta^2 \right\}$

\4a/ The relative degree of the system is ~~[illegible]~~ to ~~what~~ how many of ~~either~~ ~~[illegible]~~ derivatives of ~~x~~ either $\dot{x}$ or y must be take to ~~ast~~ have a explicit u appear.

\4bi/ ~~$\dot{x}_1 = x_1 - x_1^3 + x_2$~~ $\ddot{x}_1 = \dot{x}_1 - 3x_1^2\dot{x}_1 + \dot{x}_2 =$

$x_1 - x_1^3 + x_2 - 3x_1^2(x_1 - x_1^3 + x_2) + u$ ∴ delative degree is one.

\4bii/ $k > 0$ ∴ $k+1 > 0$ ∴ $-(k+1) < 0$ ∴

at ~~$x = 0$~~ ~~$\dot{x}_1 = 0$~~ $\dot{x}_1 = x_1 - x_1^3 + \phi(x_1) = \dot{x}_1 = x_1 - x_1^3 - (k+1)x_1$

$= x_1 - x_1^3 - kx_1 - x_1 = -x_1^3 - kx_1 = -(x_1^3 + kx_1) = -x_1(x_1^2 + k)$ ~~$\neq$~~ ∴

~~x_1~~ f is $x_1 = 0$; $\dot{x}_1 = 0 \;\forall x_2$ ∴ $x_1 = 0$ is a globally stable equilibrium

\4biii/ $\dot{V}_c = \frac{d}{dt}V_c = \frac{1}{2}(2x_1\dot{x}_1 + 2z_2\dot{z}_2) = x_1\dot{x}_1 + z_2\dot{z}_2$ ~~z~~

~~$\dot{z}_2 = z_2 = x_2 - \phi(x_1) = -(k+1)x$~~ $\dot{z}_2 = \dot{x}_2 - \dot{\phi}(x_1) = u - \dot{\phi}(x_1)$ ∴

$\dot{V}_c = x_1(x_1 - x_1^3 + x_2) + z_2(u - \dot{\phi}(x_1)) =$

$x_1^2 - x_1^4 + x_1x_2 + (x_2 - \phi(x_1))(u - \dot{\phi}(x_1))$ ∴

$\dot{V}_c(x_1, z_2) = -k_1^2 - x_1^4 - cz_2^2 \leq 0$ ∵ $k_1^2 \geq 0$, $x_1^4 \geq 0$, $z_2^2 \geq 0$

∴ $\dot{V}_c \leq 0$ for $c > 0$ ∴ $\dot{V}_c$ is negative semi definite,

$V_c(0,0) = 0$, $V_c > 0 \;\forall (x_1, z_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$ ∴ V is positive definite ∴

the origin is globally asymptotically stable.

\3011PP2020/

\1ai sol/ $\lambda = \frac{-1 \pm \sqrt{1-4a(a+b)}}{2}$ equilibria (0,0) is stable node if

1) $0 < 4a(a+b) < 1$ $\quad \lambda_1 \neq \lambda_2 \neq 0$ & $\lambda_i > 0$

Stable focus if $4a(a+b) > 1$ $\quad \lambda_{1,2} = \alpha \pm j\beta$ complex eignal $\alpha < 0$

Saddle if $a(a+b) < 0$ $\quad \lambda_2 < 0 < \lambda_1$ one stable one unstable

$\therefore$ True

\1aii/ $f(x) = -x + 2|x|$ $\quad -x+2|x|$ is not contly differentiable but it is globally lipshitz $\because$ x & $|x|$ are globally lipshitz $\therefore$ False

\1aiii/ $P = P^T \in \mathbb{R}^{n \times n}$ $\quad [P]$ is nega definite iff all leading principle minors of $[P]$ are of alternate in sign $\therefore$ quadratic form $\therefore V(x) = x^T P x < 0 \; \forall x \in \mathbb{R}^n$ $\therefore P$ is nega defi $\quad$ True

\1aiv/ 2 only combination of equili pts that can be encircled by a periodic orbit is a single focus. Other possibility of periodic orbit encircling both are ruled out $\quad$ False.

\1bi/ Controllability mat $M = [B \vdots AB] = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ $\therefore$

$\det(M) \neq 0$ $\quad \therefore \text{rank}(M) = 2 \therefore$ full rank $\quad \therefore$ controllable

\1bii/ desired eignals -4 & -5 $\therefore (s+4)(s+5) = s^2 + 9s + 20 \therefore$

$u = -Kx$ closed loop syst, charac polyn: $\det(sI - A_{cl}) = 0$

$A_{cl} = [A - BK] = \begin{bmatrix} 1 & 2 \\ 2-k_1 & 1-k_2 \end{bmatrix}$ $\quad \therefore s^2 + (k_2 - 2)s - 3 + 2k_1 - k_2 = 0 \therefore$

coeffs: $-2 + k_2 = 9 \therefore k_2 = 11$, $\quad -3 + 2k_1 - k_2 = 20 \quad \therefore k_1 = 17$

\1biii/ $\det(sI - A) = s^2 - 2s - 3 = s^2 + a_1 s + a_2 \therefore$ $\quad T = MW \therefore$

$W = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}$ $\quad \therefore T = MW = \begin{bmatrix} 2 & 0 \\ -1 & 1 \end{bmatrix}$ is of $x = T\tilde{x}$

\1c/ $V(x) > 0 \; \forall x \in \mathbb{R}^2 \setminus \{0\}$, $V(0) = 0$ $\quad V(x) \to \infty$ as $\|x\| \to \infty \therefore$ radially unbounded, $\dot{V}(x) = x_1 \dot{x}_1 + x_2 \dot{x}_2 = x_1(-x_1 + g(x_2)) + x_2(-x_2 + h(x_1)) =$

$-x_1^2 - x_2^2 + x_1 g(x_2) + x_2 h(x_1)$ $\quad \because |g(x_2)| \leq \frac{|x_2|}{2}$, $|h(x_1)| \leq \frac{|x_1|}{2} \therefore$

$\dot{V}(x) = -x_1^2 - x_2^2 + g(x_2)x_1 + x_2 h(x_1) \leq -x_1^2 - x_2^2 + |x_1 x_2|$

$\because |x_1 x_2| \leq \frac{1}{2}(x_1^2 + x_2^2)$ $\quad \therefore \dot{V}(x) \leq -\frac{1}{2}(x_1^2 + x_2^2) = -V \therefore \dot{V}$ is nega defi

$\therefore$ syst is globally exponentially stable (GES)

\1d/ $\frac{d}{dt}\left(\frac{1}{2}(x_1^2+x_2^2)\right) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(-x_1+x_2x_1^3) + x_2(x_1^3+x_1x_2^2+u) = -x_1^2-x_2^2$

iff $x_1^4x_2 + x_2x_1^3 + x_1x_2^3 + x_2u = -x_2^2$ iff

$u = -x_2 - x_1x_2^2 - x_2x_1^3 - x_1^4$ ∴ $\frac{d}{dt}(\|x\|^2) = -2\|x\|^2$ ∴

$\|x(t)\|^2 = e^{-2t}\|x(0)\|^2$ ∴ $\|x(t)\| = e^{-t}\|x(0)\|$

\1e/ let $x_1 = y$, $x_2 = \dot{y}$, $V(x) = x_1^2 + x_2^2$ ∴ $\dot{x}_1 = x_2$;

$\dot{x}_2 = -x_1 + \varepsilon x_2(1 - x_1^2 - x_2^2)$ ∴ $f(x)\cdot\nabla V(x) = 2\varepsilon x_2^2(1-x_1^2-x_2^2) = 2\varepsilon x_2^2(1-V)$ ∴

$f(x)\cdot\nabla V(x) \le 0$ for $V(x) \ge 1$ ∴ all trajecs starting in $M = \{V(x) \le 1\}$

stay in M ∀ future time. M contains only one equili pt at

origin ∴ at origin: linearization at origin: $\begin{bmatrix} 0 & 1 \\ -1 & \varepsilon \end{bmatrix}$ ∴ origin

is unstable node or unstable focus ∴ by P.B criterion:

∃ a periodic orbit in M

\2a/ equili pts $\dot{\omega}_x = 0$ ∴ $\omega_y = 0$ or $\omega_z = 0$ ie at least two of

$\omega_x, \omega_y, \omega_z$ must be zero for $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z$ ∴ every pt in

state space lying on Z ω_x axis, ω_y axis, or ω_z axis is

equili pt

\2b/ show equili at $\omega = 0$ ∴ $V = p\omega_x^2 + q\omega_y^2 + r\omega_z^2$ ∴ V is posi

defi iff $p, q, r > 0$, $\forall \omega_x, \omega_y, \omega_z \setminus \{0\}$ ∴

$\dot{V} = 2p\omega_x\dot{\omega}_x + 2q\omega_y\dot{\omega}_y + 2r\omega_z\dot{\omega}_z = 2(pa - qb + rc)\omega_x\omega_y\omega_z$ ∴ choosing

p, q, r ∴ $p>0$, $q>0$, $r>0$ ∴ $pa - qb + rc = 0$ always possible

∴ $q = \frac{pa+rc}{b}$ is posi ∀ choosen $p, r > 0$ ∴ $\dot{V} = 0$ ∴ $\omega = 0$ is a

Lyapunov stable equili pt by Lyapunovs direct method.

\2c/ ∴ $\dot{V} = 2c\omega_y\dot{\omega}_y + 2b\omega_z\dot{\omega}_z + 2[2ac\omega_y^2 + ab\omega_z^2 + bc(\omega_x^2 - \omega_0^2)]($

$4ac\omega_y\dot{\omega}_y + 2ab\omega_z\dot{\omega}_z + 2bc\,\omega_x\dot{\omega}_x)$ ∴

$2c\omega_y\dot{\omega}_y + 2b\omega_z\dot{\omega}_z = 0$, $4ac\omega_y\dot{\omega}_y + 2ab\omega_z\dot{\omega}_z + 2bc\,\omega_x\dot{\omega}_x = 0$

$\dot{V} = 0$, & $V = 0$ iff $\omega = (\pm\omega_0, 0, 0)$ ∴ $V > 0$ if $\omega_x \ne \pm\omega_0$, $\omega_y \ne 0$, $\omega_z \ne 0$

∴ V is locally posi defi centered at equili $(\pm\omega_0, 0, 0)$ ∴

$\dot{V} = 0$ ∴ ∀ pt on ω_x axis in state space is a stable equili

∴ rotation at any const velocity that about x-axis alone is

stable

PP2020 is $a(a+b)<0$ ∴ $-4a(a+b)>0$ ∴ $1-4a(a+b)>1$ ∴

$+\sqrt{1-4a(a+b)}>1$ ∴ $1-\sqrt{1-4a(a+b)}<0$ ∴

$\lambda_1 \in \mathbb{R}_{>0}$, $\lambda_2 \in \mathbb{R}_{<0}$ ∴ point is saddle ∴

True.

1ai $\lambda = \frac{-1 \pm \sqrt{1-4a(a+b)}}{2}$ equilibrium is $(0,0)$ ∴

is $0<4a(a+b)<1$ ∴ $-1<-4a(a+b)<0$ ∴

$0<1-4a(a+b)<1$ ∴ $0<\sqrt{1-4a(a+b)}<1$

~~$\sqrt{1-4a(a+b)} \in \mathbb{R}$~~ , $0<\frac{\sqrt{1-4a(a+b)}}{2}<\frac{1}{2}$ ∴

~~$-\frac{1}{2}+\frac{\sqrt{1-4a(a+b)}}{2} \in \mathbb{R}_{<0}$~~ , $-\frac{1}{2} \pm \frac{\sqrt{1-4a(a+b)}}{2}<0$ ∴

$\lambda_{1,2}<0$ ∴ its a stable node

is $4a(a+b)>1$ ∴ $\Leftrightarrow -4a(a+b)<-1$ ∴

$1-4a(a+b)<0$ ∴ $\pm\sqrt{1-4a(a+b)} \notin \mathbb{R}$ ∴

$\lambda_{1,2} \in \mathbb{C}$ ∴ $Re(\lambda_{1,2}) \in \mathbb{R}_{<0}$ ∴ its a stable focus

is $a(a+b)<0$ ∴ $-4a(a+b)>0$ ∴

$1-4a(a+b)>1$ ∴ $\sqrt{1-4a(a+b)}>1$ ∴

$\frac{1+\sqrt{1-4a(a+b)}}{2}>0$, $\frac{1-\sqrt{1-4a(a+b)}}{2}<0$ ∴ $\lambda_1>0$, $\lambda_2<0$ ∴

point is saddle ∴

True

1ai at equilibrium $(0,0)$:

is $0<4a(a+b)<1$: $\lambda_1 \neq \lambda_2 \neq 0$, $\lambda_i<0$ ∴ stable node

is $4a(a+b)>1$: $\lambda_i = \alpha + i\beta$ ∴ complex eigenvalues , $\alpha<0$ ∴

stable focus

is $a(a+b)<0$: $\lambda_2<0<\lambda_1$ ∴ one stable, one unstable ∴ saddle ∴

True.

1aii $-x$ is globally lipschitz ∴ $|-x-(-y)|=|-x+y|=|x-y| \leq |x-y|$,

1aii $|-x-(-y)|=|-x+y|=|x-y| \leq |x-y|$ ∴ $-x$ is globally lipschitz ,

$||x|-|y|| \leq |x-y|$ ∴ $|x|$ is globally lipschitz ,

$-x$ is continuously differentiable ,

$|x|$ is not continuously differentiable ∴

$f(x)$ is not continuously differentiable $\therefore$

False

\1aii/ x, $|x|$ are globally lipschitz.

x is continuously differentiable but $|x|$ is not $\therefore$

$f(x)$ is globally lipschitz but not continuously differentiable

$\therefore$ False

\1aiii/ $\because P = P^T \in \mathbb{R}^{n \times n}$

quadratic form $\therefore$

all leading principle minors of $[P]$ are of alternate sign $\therefore$

$[P]$ is negative definite $\therefore$

$V(x) = x^T P x < 0 \quad \forall x \in \mathbb{R}^n \therefore$ negative definite $\therefore$

True.

\1aiii/ $\because P = P^T \in \mathbb{R}^{n \times n} \therefore$

all leading principle minors of $[P]$ are of alternate sign $\therefore$

$[P]$ is negative definite $\therefore$

quadratic form $V = x^T P x < 0 \quad \forall x \in \mathbb{R}^n \therefore$ negative definite

$\therefore$ True

\1aiv/ $\dot{x}_1 = -x_1 + x_1 x_2$, $\dot{x}_2 = x_1 + x_2 - 2x_1 x_2$

For this given $(0,0)$ saddle , $(1,1)$ stable focus equilibrium. The only combination of equilibrium points that can be encircled by a periodic orbit is a single focus. other possibility of periodic orbit encircling both are ruled out. $\therefore$ False.

\1aiv/ The only combination of equilibrium points that can be encircled by a periodic orbit is a single focus. other possibility of periodic orbit encircling both are ruled out. $\therefore$ False.

1)

\1bi/ controllability matrix $M = [B \vdots AB] = \left[\begin{bmatrix}0\\1\end{bmatrix} \vdots \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right] = \left[\begin{bmatrix}0\\1\end{bmatrix} \vdots \begin{bmatrix}2\\1\end{bmatrix}\right] = \begin{bmatrix}0 & 2\\1 & 1\end{bmatrix}$

$\therefore \det(M) = 0 - 2 = -2 \neq 0 \therefore \text{rank}(M) = 2 \therefore$ full rank $\therefore$ controllable

\pp2020/ \1bi/ controllability matrix : $M = [B \vdots AB] = \left[\begin{bmatrix}0\\1\end{bmatrix} \vdots \begin{bmatrix}1&2\\2&1\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right] = \left[\begin{bmatrix}0\\1\end{bmatrix} \vdots \begin{bmatrix}2\\1\end{bmatrix}\right]$
$= \begin{bmatrix}0&2\\1&1\end{bmatrix}$ ∴

$\det(M) = 0 - 2 = -2 \neq 0$ ∴ ~~Su~~ $\mathrm{rank}(M) = 2$ ∴ full rank ∴ controllable

\1bii/ desired eigen values : $-4, -5$ ∴ $(s+4)(s+5) = s^2 + 5s + 4s + 20 =$
$s^2 + 9s + 20$ ∴ let $K = [k_1 \;\; k_2]$ ∴

$u = -Kx$ closed loop system : characteristic polynomial :

$\det(sI - A_{cl}) = 0$: $A_{cl} = [A - BK] = \left[\begin{bmatrix}1&2\\2&1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}[k_1 \;\; k_2]\right] =$
$\left[\begin{bmatrix}1&2\\2&1\end{bmatrix} - \begin{bmatrix}0&0\\k_1&k_2\end{bmatrix}\right] = \begin{bmatrix}1&2\\2-k_1&1-k_2\end{bmatrix} = \begin{bmatrix}1&2\\-k_1+2&-k_2+1\end{bmatrix}$

$\det(sI - A_{cl}) = \det\left(\begin{bmatrix}s&0\\0&s\end{bmatrix} - \begin{bmatrix}1&2\\-k_1+2&-k_2+1\end{bmatrix}\right) = \det\left(\begin{bmatrix}s-1&-2\\k_1-2&s+k_2-1\end{bmatrix}\right) =$

$(s-1)(s+k_2-1) - (k_1-2)(-2) = s^2 + k_2 s - s - s - k_2 + 1 + 2k_1 - 4 =$

$s^2 + (k_2 - 2)s + (-k_2 + 2k_1 - 3) = 0$ ∴

$s^2 + 9s + 20 = s^2 + (k_2 - 2)s + (-k_2 + 2k_1 - 3)$ ∴

$9 = k_2 - 2$ ∴ $k_2 = 11$, $20 = -k_2 + 2k_1 - 3 = -11 + 2k_1 - 3 = -14 + 2k_1$ ∴

$34 = 2k_1$ ∴ $k_1 = 17$ ∴

$u = -Kx = -[17 \;\; 11]x = [-17 \;\; -11]x = [-17 \;\; -11]\begin{bmatrix}x_1\\x_2\end{bmatrix} = -17x_1 - 11x_2$

\1bii/ desired eigen values : $-4, -5$ ∴ $(s+4)(s+4) = s^2 + 5s + 4s + 20 =$
$s^2 + 9s + 20$ ∴

let $K = [k_1 \;\; k_2]$ ∴

$u = -Kx$ closed loop system : characteristic polynomial :

$\det(sI - A_{cl}) = \det(sI - (A - BK)) = \det\left(sI - \left(\begin{bmatrix}1&2\\2&1\end{bmatrix} - \begin{bmatrix}0\\1\end{bmatrix}[k_1 \;\; k_2]\right)\right) =$

$\det\left(sI - \left(\begin{bmatrix}1&2\\2&1\end{bmatrix} - \begin{bmatrix}0&0\\k_1&k_2\end{bmatrix}\right)\right) = \det\left(sI - \begin{bmatrix}1&2\\-k_1+2&-k_2+1\end{bmatrix}\right) = \det\left(\begin{bmatrix}s&0\\0&s\end{bmatrix} + \begin{bmatrix}-1&-2\\k_1-2&k_2-1\end{bmatrix}\right)$

$\det\left(\begin{bmatrix}s-1&-2\\k_1-2&s+k_2-1\end{bmatrix}\right) = (s-1)(s+k_2-1) + 2(k_1-2) =$

$s^2 + k_2 s - s - s - k_2 + 1 + 2k_1 - 4 = s^2 + (k_2 - 2)s + (2k_1 - k_2 - 3) = s^2 + 9s + 20$ ∴

$9 = k_2 - 2$ ∴ $k_2 = 11$, $20 = 2k_1 - k_2 - 3 = 2k_1 - 11 - 3 = 2k_1 - 14$ ∴ $2k_1 = 34$ ∴

$k_1 = 17$ ∴ $u = -Kx = -[17 \;\; 11]x = [-17 \;\; -11]x$

\1biii/ ~~$T = MW = \begin{bmatrix}2&0\\2&1\end{bmatrix}$~~ $T = MW$ ∴ $M = \begin{bmatrix}0&2\\1&1\end{bmatrix}$,
$W = \begin{bmatrix}\alpha_1&1\\1&0\end{bmatrix}$ ∴ $\det(sI - A) = \det\left(\begin{bmatrix}s&0\\0&s\end{bmatrix} - \begin{bmatrix}1&2\\2&1\end{bmatrix}\right) = \det\left(\begin{bmatrix}s-1&-2\\-2&s-1\end{bmatrix}\right) =$

$(s-1)(s-1)+2(-2) = s^2-s-s+1-4 = s^2-2s-3 = s^2+a_1s+a_2$ $\therefore$

$a_1=-2$ $\therefore$ $W=\begin{bmatrix}-2 & 1\\ 1 & 0\end{bmatrix}$ $\therefore$

~~$W = \begin{bmatrix}-2 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix} = \begin{bmatrix}1 & -3\\ 0 & 2\end{bmatrix}$~~ $T=\begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}-2 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}2 & 0\\ -1 & 1\end{bmatrix}$ is of $x = T\tilde{x}$

\1biii/ $T=MW$ $\therefore$ $M=\begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}$ $\therefore$

$W=\begin{bmatrix}a_1 & 1\\ 1 & 0\end{bmatrix}$ $\therefore$

$\det(sI-A) = \det\left(\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix}-\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\right) = \det\left(\begin{bmatrix}s-1 & -2\\ -2 & s-1\end{bmatrix}\right)=$

~~det~~ $(s-1)(s-1)+2(-2) = s^2-s-s+1-4 = s^2-2s-3 = s^2+a_1s+a_2s$ $\therefore$

$a_1=-2$ $\therefore$ $W=\begin{bmatrix}-2 & 1\\ 1 & 0\end{bmatrix}$ $\therefore$

$T=\begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}-2 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}2 & 0\\ -1 & 1\end{bmatrix}$ is of $x=T\tilde{x}$ is transformation T

\1biii/ Transformation Matrix: $T=MW$, $M=\begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}$ $\therefore$

$W=\begin{bmatrix}a_1 & 1\\ 1 & 0\end{bmatrix}$

$\det(sI-A) = \det\left(\begin{bmatrix}s & 0\\ 0 & s\end{bmatrix}-\begin{bmatrix}1 & 2\\ 2 & 1\end{bmatrix}\right) = \det\left(\begin{bmatrix}s-1 & -2\\ -2 & s-1\end{bmatrix}\right)=$

$(s-1)(s-1)+2(-2) = s^2-s-s+1-4 = s^2-2s-3 = s^2+a_1s+a_2s$ $\therefore$

$a_1=-2$ $\therefore$ $W=\begin{bmatrix}-2 & 1\\ 1 & 0\end{bmatrix}$

$T=MW=\begin{bmatrix}0 & 2\\ 1 & 1\end{bmatrix}\begin{bmatrix}-2 & 1\\ 1 & 0\end{bmatrix}=\begin{bmatrix}2 & 0\\ -1 & 1\end{bmatrix}$

\1c/ let $V=\frac{1}{2}(x_1^2+x_2^2)$ $\therefore$ $V(x)>0\ \forall x\in\mathbb{R}^2\setminus\{0\}$, $V(0)=0$, $V(x)\to\infty$ as $\|x\|\to\infty$

$\therefore$ V is positive definite and radially unbounded

$\dot{V}=x_1\dot{x}_1+x_2\dot{x}_2 = x_1(-x_1+g(x_2))+x_2(-x_2+h(x_1)) = -x_1^2+x_1g(x_2)-x_2^2+x_2h(x_1)=$

$-x_1^2-x_2^2+x_1g(x_2)+x_2h(x_1)$ $\therefore$ $|g(x_2)|\le\frac{|x_2|}{2}$, $|h(x_1)|\le\frac{|x_1|}{2}$ $\therefore$

$\dot{V}=-x_1^2-x_2^2+x_1g(x_2)+x_2h(x_1)\le -x_1^2-x_2^2+x_1\frac{|x_2|}{2}+x_2\frac{|x_1|}{2}=$

$-x_1^2-x_2^2+\frac{1}{2}(x_1|x_2|+x_2|x_1|)\le -x_1^2-x_2^2+\frac{1}{2}(|x_1x_2|+|x_1x_2|) = -x_1^2-x_2^2+|x_1x_2|$

$\therefore$ $|x_1x_2|\le\frac{1}{2}(x_1^2+x_2^2)$ $\therefore$

$\dot{V}\le -x_1^2-x_2^2+|x_1x_2|\le -x_1^2-x_2^2+\frac{1}{2}(x_1^2+x_2^2) = -1(x_1^2+x_2^2)+\frac{1}{2}(x_1^2+x_2^2) = -\frac{1}{2}(x_1^2+x_2^2)=-V$

$\therefore$ $\dot{V}\le -V$ $\therefore$ $\dot{V}$ is negative definite $\therefore$

system is globally exponentially stable $\therefore$

origin is globally asymptotically stable

\PP2020/ \1c/ let $V=\frac{1}{2}(x_1^2+x_2^2)$ ∴ $V(x)>0\ \forall x\in\mathbb{R}^2\setminus\{0\}$, $V(0)=0$, $V(x)\to\infty$ as $\|x\|\to\infty$

∴ V is positive definite and radially unbounded ∴

$\dot V = x_1\dot x_1 + x_2\dot x_2 = x_1(-x_1+g(x_2)) + x_2(-x_2+h(x_1)) = -x_1^2+x_1g(x_2)-x_2^2+x_2h(x_1) =$

$-x_1^2-x_2^2+x_1g(x_2)+x_2h(x_1)$ ∵ $|g(x_2)|\le \frac{|x_2|}{2}$, $|h(x_1)|\le\frac{|x_1|}{2}$ ∴

$\dot V = -x_1^2-x_2^2+x_1g(x_2)+x_2h(x_1) \le -x_1^2-x_2^2+x_1\frac{|x_2|}{2}+x_2\frac{|x_1|}{2} = -(x_1^2+x_2^2)+\frac{1}{2}(x_1|x_2|+|x_1|x_2) \le$

$-(x_1^2+x_2^2)+\frac{1}{2}(|x_1x_2|+|x_1x_2|) = -(x_1^2+x_2^2)+|x_1x_2|$

∵ $|x_1x_2|\le\frac{1}{2}(x_1^2+x_2^2)$ ∴

$\dot V\le -(x_1^2+x_2^2)+|x_1x_2| \le -(x_1^2+x_2^2)+\frac{1}{2}(x_1^2+x_2^2) = -\frac{1}{2}(x_1^2+x_2^2) = -V$ ∴

$\dot V\le -V$ ∴ $\dot V$ is negative definite ∴

System is globally exponentially stable ∴

∴ origin is globally asymptotically stable

\1d/ let $\begin{bmatrix}\dot x_1\\ \dot x_2\end{bmatrix} = \begin{bmatrix}x_1\\ x_2\end{bmatrix}$ ∴ $x=x(t)=\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ ∴ $\|x(t)\|_2=\|x\|_2=(x_1^2+x_2^2)^{1/2}$ ∴

$\|x\|_2^2 = x_1^2+x_2^2$ ∴

$\cancel{\frac{d}{dt}(\frac{1}{2}(x_1^2+x_2^2))=}$

$\frac{d}{dt}(\frac{1}{2}\|x\|_2^2) = \frac{d}{dt}(\frac{1}{2}(x_1^2+x_2^2)) = x_1\dot x_1+x_2\dot x_2 =$

$x_1(-x_1+x_2x_1^3)+x_2(x_1^3+x_1x_2^2+u) = -x_1^2+x_1^4x_2+x_1^3x_2+x_1x_2^3+x_2u =$

$-x_1^2-(-x_1^4x_2-x_1^3x_2-x_1x_2^3-x_2u)$

∴ let $-x_1^4x_2-x_1^3x_2-x_1x_2^3-x_2u = x_2^2 = x_2(-x_1^4-x_1^3-x_1x_2^2-u)$ ∴

$\cancel{x_2 \Rightarrow}$ $x_2 = -x_1^4-x_1^3-x_1x_2^2-u$ ∴

$u=-x_1^4-x_1^3-x_1x_2^2-x_2$ ∴

$\frac{d}{dt}(\frac{1}{2}\|x\|_2^2) = -x_1^2-(x_2^2) = -(x_1^2+x_2^2) = -\|x\|_2^2$ ∴

$\frac{d}{dt}(\frac{1}{2}\|x\|_2^2) = \frac{1}{2}\frac{d}{dt}(\|x\|_2^2) = -\|x\|_2^2$ ∴ $\frac{d}{dt}(\|x\|_2^2) = -2\|x\|_2^2$ ∴

$\frac{d}{dt}(\|x\|^2) = -2\|x\|^2$ ∴

$\frac{\frac{d}{dt}(\|x\|^2)}{\|x\|^2} = -2$ ∴ $\int -2\,dt = \int\frac{\frac{d}{dt}(\|x\|^2)}{\|x\|^2}dt = \ln|\|x\|^2| = -2t+c$ ∴

$e^{\ln|\|x\|^2|} = \|x\|^2 = e^{-2t+c} = e^ce^{-2t} = Ae^{-2t} = \|x(t)\|^2$ ∴

∵ $\|x(0)\|^2 = Ae^{-2(0)} = \|x(0)\|^2 = Ae^0 = A(1) = A$ ∴

$\|x(t)\|^2 = \|x(0)\|^2e^{-2t} = \|x(0)\|^2(e^{-t})^2 = (\|x(0)\|(e^{-t}))^2$ ∴

$\|x(t)\| = \|x(0)\|(e^{-t}) = e^{-t}\|x(0)\|$

\1d/ let $\cancel{x = 4t}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ∴ let $\|x\|_2 = (x_1^2 + x_2^2)^{1/2}$ ∴ $\|x\|_2^2 = x_1^2 + x_2^2$ ∴

$\frac{d}{dt}\left(\frac{1}{2}(x_1^2 + x_2^2)\right) = \frac{1}{2}(2x_1\dot{x}_1 + 2x_2\dot{x}_2) = x_1\dot{x}_1 + x_2\dot{x}_2 = x_1(-x_1 + x_2x_1^3) + x_2(x_1^3 + x_1x_2^2 + u) =$

$-x_1^2 + (x_1^4x_2 + x_1^3x_2 + x_1x_2^3 + x_2u) = \frac{d}{dt}\left(\frac{1}{2}\|x\|_2^2\right)$ ∴

let $x_1^4x_2 + x_1^3x_2 + x_1x_2^3 + x_2u = -x_2^2$ ∴

$x_2u = \cancel{-x_2^2} - x_1^4x_2 \cancel{+} - x_1^3x_2 - x_1x_2^3 - x_2^2 = +x_2(-x_1^4 - x_1^3 - x_1x_2^2 - x_2)$ ∴

$u = -x_1^4 - x_1^3 - x_1x_2^2 - x_2$ ∴

$\frac{d}{dt}\left(\frac{1}{2}\|x\|_2^2\right) = -x_1^2 - x_2^2 = -(x_1^2 + x_2^2) = -\|x\|_2^2$ ∴

$\cancel{\frac{d}{dt}}$ $\cancel{\frac{d}{dt}(\|x\|^2) = 2\|x\|^2}$ $\frac{d}{dt}\left(\frac{1}{2}\|x\|^2\right) = -\|x\|^2$ ∴

$\frac{d}{dt}(\|x\|^2) = \cancel{2} -2\|x\|^2$ ∴

$\frac{\frac{d}{dt}(\|x\|^2)}{\|x\|^2} = -2$ ∴ $\int \frac{\frac{d}{dt}(\|x\|^2)}{\|x\|^2}dt = \int -2\,dt = \ln|\|x\|^2| = \ln\|x\|^2 = -2t + C_1$ ∴

$e^{\ln\|x\|^2} = \|x\|^2 = e^{-2t+C_1} = e^{C_1}e^{-2t} = C_2e^{-2t} = \|x(t)\|^2$ ∴

$\|x(0)\|^2 = C_2e^{-2(0)} = C_2e^0 = C_2(1) = C_2$ ∴

$\|x(t)\|^2 = \|x(0)\|^2e^{-2t} = (\|x(0)\|e^{-t})^2$ ∴

$\|x(t)\| = \|x(0)\|e^{-t}$

\1e/ let $x_1 = y$, $x_2 = \dot{y}$ ∴ $\dot{x}_1 = \dot{y} = x_2$,

$\dot{x}_2 = \ddot{y} = \varepsilon\dot{y}(1 - y^2 - \dot{y}^2) - y = \varepsilon x_2(1 - x_1^2 - x_2^2) - x_1$ ∴

let $V = x_1^2 + x_2^2$ ∴ $\nabla V = \left[\frac{\partial}{\partial x_1}(x_1^2 + x_2^2), \frac{\partial}{\partial x_2}(x_1^2 + x_2^2)\right] = [2x_1, 2x_2]$ ∴

$f(x)\cdot\nabla V = [\dot{x}_1 \quad \dot{x}_2]\cdot[2x_1 \quad 2x_2] = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2(x_1\dot{x}_1 + x_2\dot{x}_2) =$

$2(x_1x_2 + x_2(\varepsilon x_2(1 - x_1^2 - x_2^2) - x_1)) =$

$2(x_1x_2 + \varepsilon x_2^2(1 - x_1^2 - x_2^2) - x_1x_2) = 2(\varepsilon x_2^2(1 - x_1^2 - x_2^2)) = 2\varepsilon x_2^2(1 - x_1^2 - x_2^2) =$

$2\varepsilon x_2^2(1 - (x_1^2 + x_2^2)) = 2\varepsilon x_2^2(1 - V) \le 0$ for $1 - V \le 0$ ∴ $V(x) \ge 1$ ∴

All trajectories starting in $M = \{V(x) \le 1\}$ stay in M for all ~~further~~ future time. $\dot{x}_1 = x_2 = 0$ ∴ $\dot{x}_2 = \varepsilon(0)(1 - x_1^2 - x_2^2) - x_1 = -x_1 = 0 = x_1$ ∴

only equilibrium point is $(0,0)$ ∴ M only contains one equilibrium point at origin $(0,0)$.

∴ Linearization at the origin yields the Matrix $\begin{bmatrix} 0 & 1 \\ -1 & \varepsilon \end{bmatrix}$ ∴

$\begin{vmatrix} 0 & 1 \\ -1 & \varepsilon \end{vmatrix} = 0 - 1(-1) = 1 > 0$ ∴ origin is unstable node or unstable focus ∴ By P.B. criterion, $\exists$ a periodic orbit in M

\PP2020/1e/ let $x_1 = y$, $x_2 = \dot{y}$ $\therefore$

$\dot{x}_1 = \dot{y} = x_2$

$\therefore$ $\dot{x}_2 = \ddot{y} = -y + \varepsilon\dot{y}(1 - y^2 - \dot{y}^2) = -x_1 + \varepsilon x_2(1 - x_1^2 - x_2^2)$

$\therefore$ let $V = x_1^2 + x_2^2$ $\therefore$ $\nabla V = \left[\frac{\partial}{\partial x_1}(x_1^2 + x_2^2) \quad \frac{\partial}{\partial x_2}(x_1^2 + x_2^2)\right] = \left[2x_1 \quad 2x_2\right]$ $\therefore$

$f(x) = [\dot{x}_1 \quad \dot{x}_2]^T = \dot{x} = \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix}$ $\therefore$

$f(x)\nabla V = \begin{bmatrix}\dot{x}_1 \\ \dot{x}_2\end{bmatrix}[2x_1 \quad 2x_2] = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1(x_2) + 2x_2(-x_1 + \varepsilon x_2(1 - x_1^2 - x_2^2)) =$

$2(x_1x_2 - x_1x_2 + \varepsilon x_2^2(1 - x_1^2 - x_2^2)) = 2(\varepsilon x_2^2(1 - x_1^2 - x_2^2)) = 2\varepsilon x_2^2(1 - x_1^2 - x_2^2) =$

$2\varepsilon x_2^2(1 - (x_1^2 + x_2^2)) = 2\varepsilon x_2^2(1 - V) \leq 0$ for $1 - V \leq 0$ $\therefore$ $V \geq 1$ $\therefore$

All trajectories starting in $M = \{V(x) = x_1^2 + x_2^2 \leq 1\}$ stay in M for all future time $\therefore$

$\therefore$ $\dot{x}_1 = x_2 = 0$ $\therefore$ $\dot{x}_2 = -x_1 + \varepsilon(0)(1 - x_1^2 - 0^2) = -x_1 = 0 = x_1$ $\therefore$

only equilibria is $(0,0)$ $\therefore$

M only contains one equilibrium point at origin $(0,0)$.

$\therefore$ $\dot{x} = \begin{bmatrix}x_2 \\ -x_1 + \varepsilon x_2 - \varepsilon x_1^2 x_2 - \varepsilon x_2^3\end{bmatrix} = \begin{bmatrix}0x_1 + x_2 \\ -x_1 + \varepsilon x_2\end{bmatrix} + \begin{bmatrix}0 \\ -\varepsilon x_1^2 x_2 - \varepsilon x_2^3\end{bmatrix} = \begin{bmatrix}0 & 1 \\ -1 & \varepsilon\end{bmatrix}\begin{bmatrix}x_1 \\ x_2\end{bmatrix} + \begin{bmatrix}0 \\ -\varepsilon x_1^2 x_2 - \varepsilon x_2^3\end{bmatrix}$

$\therefore$ $A = \begin{bmatrix}0 & 1 \\ -1 & \varepsilon\end{bmatrix}$ $\therefore$

Linearization at the origin yields the matrix $\begin{bmatrix}0 & 1 \\ -1 & \varepsilon\end{bmatrix}$ $\therefore$

$\begin{vmatrix}0 & 1 \\ -1 & \varepsilon\end{vmatrix} = 0(\varepsilon) - 1(-1) = 1 > 0$ $\therefore$ origin in unstable node or spiral.

By P.B. criterion, $\exists$ a periodic orbit in M

\2a/ equilibrium points: $\dot{\omega}_x = 0$ $\therefore$ $\omega_y = 0$ or $\omega_z = 0$ $\therefore$

at least two of $\omega_x, \omega_y, \omega_z$ must be zero for $\dot{\omega}_x = \dot{\omega}_y = \dot{\omega}_z = 0$ $\therefore$

every point in state space ~~lyi~~ lying on the ω_x axis, ω_y axis, ω_z axis is equilibrium point

\2a/ equilibrium points: $\dot{\omega}_x = 0$ $\therefore$ $\omega_y = 0$ or $\omega_z = 0$ $\therefore$

if $\omega_y = 0$: $\dot{\omega}_z = c\omega_x(0) = 0$, $\dot{\omega}_y = -b\omega_z\omega_x = 0$ $\therefore$ $\omega_z = 0$ or $\omega_x = 0$

$\therefore$ $(\omega_x, 0, 0)$ and $(0, 0, \omega_z)$ are equilibrium points

$\therefore$ if $\omega_z = 0$: $\dot{\omega}_y = -b(0)\omega_x = 0$, $\dot{\omega}_z = c\omega_x\omega_y = 0$ $\therefore$ $\omega_x = 0$ or $\omega_y = 0$

$\therefore$ $(0, \omega_y, 0)$ is an equilibria $\therefore$

equilibrium points are: $(\omega_x, 0, 0)$, $(0, \omega_y, 0)$, $(0, 0, \omega_z)$.

\2a/ for $\dot{\omega}_x = a\omega_y\omega_z = 0$ ∴ $\omega_y = 0$ or $\omega_z = 0$ ∴

if $\omega_y = 0$: $\dot{\omega}_z = c\omega_x(0) = 0$ ∴ $\dot{\omega}_y = -b\omega_z\omega_x = 0$ for $\omega_z = 0$ or $\omega_x = 0$ ∴

$(0, 0, \omega_z)$, $(\omega_x, 0, 0)$ are equilibria

if $\omega_z = 0$: $\dot{\omega}_y = -b(0)\omega_x = 0$ ∴ $\dot{\omega}_z = c\omega_x\omega_y = 0$ for $\omega_x = 0$ or $\omega_y = 0$ ∴

$(0, \omega_y, 0)$ is also a equilibria ∴

The equilibria are: $(\omega_x, 0, 0)$, $(0, \omega_y, 0)$, $(0, 0, \omega_x)$

\2b/ To show stability of equilibrium at $\omega = 0$:

let $V = p\omega_x^2 + q\omega_y^2 + r\omega_z^2$

V is positive definite if p, q, r are all positive $\forall \omega_x, \omega_y, \omega_z \setminus \{0\}$

∴ $\dot{V} = 2p\omega_x\dot{\omega}_x + 2q\omega_y\dot{\omega}_y + 2r\omega_z\dot{\omega}_z = 2(p\omega_x\dot{\omega}_x + q\omega_y\dot{\omega}_y + r\omega_z\dot{\omega}_z) =$

$2(p\omega_x(a\omega_y\omega_z) + q\omega_y(-b\omega_z\omega_x) + r\omega_z(c\omega_x\omega_y)) =$

~~2(ap~~ $2(pa\omega_x\omega_y\omega_z - qb\,\omega_y\omega_z\omega_x + rc\,\omega_x\omega_y\omega_z) =$

$2(pa - qb + rc)\omega_x\omega_y\omega_z = 0$ for $pa - qb + rc = 0$ ∴

$pa + rc = qb$ ∴ $\frac{pa + rc}{b} = q$ ∴ let $p, r > 0$ ∴ $p, q, r > 0$ ∴

$\dot{V} = 0$ ∴ $\omega = 0$ is a Lyapunov stable equilibrium point by Lyapunovs direct method ∴

The equilibrium $\omega = 0$ is Lyapunov stable

\2b/ To show stability of equilibrium at $\omega = 0$:

let $V = p\omega_x^2 + q\omega_y^2 + r\omega_z^2$ ∴

V is positive definite if p, q, r are all positive $\forall \omega_x, \omega_y, \omega_z \setminus \{0\}$

∴ $\dot{V} = 2p\omega_x\dot{\omega}_x + 2q\omega_y\dot{\omega}_y + 2r\omega_z\dot{\omega}_z = 2(p\omega_x\dot{\omega}_x + q\omega_y\dot{\omega}_y + r\omega_z\dot{\omega}_z) =$

$2(p\omega_x(a\omega_y\omega_z) + q\omega_y(-b\omega_z\omega_x) + r\omega_z(c\omega_x\omega_y)) =$

$2(pa - qb + rc)(\omega_x\omega_y\omega_z = 0$ for $pa - qb + rc = 0$ ∴

$pa + rc = qb$ ∴ $\frac{pa + rc}{b} = q$ ∴ let $p, r > 0$ ∴ $p, q, r > 0$ ∴

$\dot{V} = 0$ ∴ $\omega = 0$ is a Lyapunov stable equilibrium by Lyapunovs direct method

\2c/ $\dot{V} = 2c\omega_y\dot{\omega}_y + 2b\omega_z\dot{\omega}_z + 2[2ac\omega_y^2 + ab\omega_z^2 + bc(\omega_x^2 - \omega_0^2)](4ac\omega_y\dot{\omega}_y + 2b\omega_z\dot{\omega}_z + 2bc\omega_x\dot{\omega}_x)$

∴ for $\dot{V} = 0$: $2c\omega_y\dot{\omega}_y + 2b\omega_z\dot{\omega}_z = 0$,

$4ac\omega_y\dot{\omega}_y + 2ab\omega_z\dot{\omega}_z + 2bc\omega_x\dot{\omega}_x = 0$ ∵ $2ac\omega_y^2 + ab\omega_z^2 + bc(\omega_x^2 - \omega_0^2) > 0$ ∴