MTH3008 Partial Differential equations

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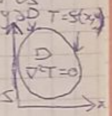
CW1 set Week2 Due week5 10%

CW2 set week6 Due week9 10%

ODE's unknown funcs depend on one variable, which $\therefore$ have ordinary derivatives

\Ex 1.1/ Z pde of a func $u(x,y)$ $\frac{\partial u}{\partial x} = 0$ states that u does not depend on x. Its general sol is $\therefore$ $u = w(y)$ where w is an arbit func of x



PDE's usually solved in Z presence of boundary conditions

It is a nontrivial task to determine how much boundary info is approp for a given PDE

Linear PDEs occur most frequently

Modern approaches seek methods applicable to nonlinear PDEs as well as linear ones

existence & uniqueness results & those concerning Z regularity results

\Ex 1.2/ find all $u(x,y)$ satisfying $\frac{\partial^2 u}{\partial x^2} = 0$

$\left\{ \frac{\partial u}{\partial x} = w(y) \quad \therefore u = x\,w(y) + g(y) \right\}$

\Sol/ let $\frac{\partial u}{\partial x} = v = v(x,y)$ $\quad \frac{\partial^2 u}{\partial x^2} = \frac{\partial v}{\partial x} = 0 \quad \forall (x,y) \quad \therefore$

$v(x,y) = f(y)$ "Direct integration"

$\frac{\partial v}{\partial x} = 0 \qquad \int \frac{\partial v}{\partial x} \partial x = \int 0 \, \partial x = f(y)$ $\leftarrow$ "constant of integration as far as x is concerned"

non standard notion! "partial integration"

next step: $v = \frac{\partial u}{\partial x} = f(y)$

"Direct integration": $\int \frac{\partial u}{\partial x} \partial x = \int \underbrace{f(y)}_{\text{const "as far as } x \text{ is concerned"}} \partial x = f(y)x + g(y)$

answer $u(x,y) = f(y)x + g(y)$, where f, g are arbit funcs

\Ex 1.3/ all $u(x,y)$ for $\frac{\partial^2 u}{\partial x \partial y} = 0$

\Sol/ Direct integration $\frac{\partial u}{\partial x} = v(x,y) \quad \therefore \frac{\partial v}{\partial y} = \frac{\partial^2 u}{\partial x \partial y} = 0$

$\int \frac{\partial v}{\partial y} \partial y = \int 0 \, \partial y = f(x)$ so $\frac{\partial u}{\partial x} = f(x) \quad \therefore$

Direct integration again $u = \int \frac{\partial u}{\partial x} \partial x = \int f(x)\, \partial x = F(x) + g(y)$

$F(x)$ where $F'(x) = f(x)$ arbit func

$F(x)+g(y)$ is Gen sol. F, g arb funcs

introduction to business analytics weeks 1-4 write time but less maths

\Ex 1.4/ $a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0 = au_x + bu_y$

$(u_x \equiv \frac{\partial u}{\partial x}\ ;\ u_y \equiv \frac{\partial u}{\partial y})$ a, b = const $a \neq 0$ or $b \neq 0$

\Sol/ change of variables $x, y \to \xi, \eta$ (ξ, η)

$\xi = ax + by$; $\eta = bx - ay$ ∴ $u(x,y) = v(\xi, \eta)$ ∴

$u_x = v_\xi \overset{=a}{\xi_x} + v_\eta \overset{=b}{\eta_x} = av_\xi + bv_\eta$

$\left\{ \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} \right\}$

$u_y = v_\xi \overset{=b}{\xi_y} + v_\eta \overset{=-a}{\eta_y}$ ∴ Sub:

$au_x + bu_y = a(av_\xi + bv_\eta) + b(bv_\xi - av_\eta)$

$\underbrace{(a^2+b^2)}_{\neq 0} v_\xi + \underbrace{(ab - ba)}_{\neq 0} v_\eta = 0$ ∴

$v_\xi = 0$ ∴ $v = f(\eta)$ ∴

$u(x,y) = v(\xi(x,y), \eta(x,y)) = f(\eta(x,y))$ ∴

$u(x,y) = f(bx - ay)$

Method ② similarly: if $\xi = x$, $\eta = bx - ay$ $(a \neq 0)$

(abuse of notation:) $u(x,y) = v(\xi, \eta)$ $(u(\xi, \eta))$ "physics style"

$u_x = u_\xi \overset{=1}{\xi_x} + u_\eta \overset{=b}{\eta_x} = u_\xi + bu_\eta$ ∴

$u_y = u_\xi \overset{=0}{\xi_y} + u_\eta \overset{=-a}{\eta_y} = -au_\eta$ ∴

$au_x + bu_y = a(u_\xi + bu_\eta) - abu_\eta = 0$

$au_\xi + \underbrace{(ab - ab)}_{=0} u_\eta = 0$ $| u_\xi = 0$, $u = f(\eta)$ Same as before

Method ③: how to invent $\eta = bx - ay$? we "know the answer"

$u(x,y) = f(\xi)$ where ξ is some lin comb of x & y eg

let $\xi = \alpha x + y$, for α to be found & F is arbit func

Try $u_x = F'(\xi)\xi_x = \alpha F(\xi)$ $u_y = F'(\xi)\xi_y = F'(\xi)$

$au_x + bu_y = \alpha a F(\xi) + bF'(\xi) = (\alpha a + b)F'(\xi) = 0$

F arbit func $\Longrightarrow$ $\alpha a + b = 0$ $\alpha = -b/a$; $\xi = -\frac{b}{a}x + y$ ∴

Ex 1.4 / $a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0 \therefore \xi = ax+by, \ \eta = bx - ay \therefore$

$u(x,y) = f(bx - ay) \therefore$

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = a\frac{\partial u}{\partial \xi} + b\frac{\partial u}{\partial \eta}$$

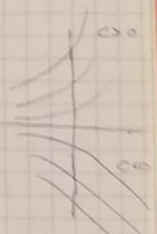
Ex 1.6 / PDE $\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0 \quad \therefore u_x + y u_y = 0$

Sol / the PDE says the geometric meaning $(1,y)\cdot(u_x, u_y) = 0 \therefore$ maybe can expect $u = \text{const}$ along lines s.t. the slope is parallel to $(1,y)$ $\therefore \frac{dy}{dx} = \frac{y}{1}$ $(dx, dy) \parallel (1,y)$ { $\parallel$ means parallel two vectors are to each other } $\therefore$

$\frac{dx}{1} = \frac{dy}{y}$ $\quad \therefore \frac{dy}{dx} = \frac{y}{1}$ is an ODE $\Delta \therefore$ is a separable eqn $\therefore$

$\int \frac{dy}{y} = \int dx$ $\quad \left\{ \frac{1}{y}\frac{dy}{dx} = 1 \therefore \int \frac{1}{y}\frac{dy}{dx} dx = \int 1\, dx = \int \frac{1}{y} dy = \int dx \right\}$

$\therefore \ln|y| = x + A$ | $y = 0$ is specific sol $\therefore$



$y = \pm e^A e^x = Ce^x$ are the characteristic curves

$y = Ce^x \leftarrow$ "explicit"

characteristic curve for every point (x,y)

Resolve charac curve wrt C: $C = ye^{-x} \therefore \leftarrow$ "implicit"

General sol: $u(x,y) = f(c) = f(ye^{-x})$

verify / $u_x = f'(ye^{-x})\frac{\partial}{\partial x}(ye^{-x}) = -ye^{-x} f'(ye^{-x})$

$u_y = f'(ye^{-x})\frac{\partial}{\partial y}(ye^{-x}) = e^{-x} f'(ye^{-x})$ $\quad$ sub: $u_x + y u_y = \cancel{f'(ye^{-x})}$

$f'(ye^{-x})\underbrace{[-ye^{-x} + ye^{-x}]}_{=0} = 0 \ \forall f$

to make sure this is the general sol?

By change of variables $\xi = ye^{-x}$; $\eta = x \therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x = -ye^{-x}u_\xi + u_\eta$

$u_y = u_\xi \xi_y + u_\eta \eta_y = e^{-x}u_\xi \therefore$

$$u_x + y u_y = \underbrace{-ye^{-x}u_\xi + u_\eta}_{u_x} + \underbrace{ye^{-x}u_\xi}_{yu_y}$$

$= u_\eta = 0 = \text{RHS} \implies u(\xi,\eta) = f(\xi) \iff u(x,y) = f(ye^{-x})$ $\square$

$u(x,y) = F(-\frac{b}{a}x + y) = f(bx - ay)$ equivalent

equivalence $F(\underbrace{-\frac{b}{a}x + y}_{\xi}) = F(-\frac{1}{a}(\underbrace{-ay + bx}_{\eta}))$

$= F(-\frac{1}{a}(\eta)) = f(\eta)$ ie $f(\eta) = F(-\frac{1}{a}\eta)$

NB $\quad au_x + bu_y = \underbrace{(a,b) \cdot \nabla u}_{\text{directional derivative}} = 0$

u is const along lines in the direction (a,b)

$\eta = bx - ay = \text{const}$ are such lines

$\xi = y - \frac{b}{a}x = \text{const}$ also such lines

Ex 1.5/ $4\frac{\partial u}{\partial x} - 3\frac{\partial u}{\partial y} = 0$ with BCs $u(0,y) = y^3$

$u(x, -3x/4) = 1 \quad u(x, -3x/4) = x^3$

Sols/ GS: $u = f(bx - ay) \quad a = 4 \quad b = -3 \quad \therefore$

$= f(-3x - 4y) = g(3x + 4y) \quad f, g$ arbit funcs are same

(a) $u(0,y) = g(3 \cdot 0 + 4y) = g(4y) = y^3 \quad$ let $4y = s \quad y = s/4$

$\therefore g(s) = (s/4)^3 = \frac{1}{64}s^3 \quad \therefore \quad u(x,y) = g(3x + 4y) = \frac{1}{64}(3x + 4y)^3$

unique answer

(b) $u(x, -\frac{3x}{4}) = 1 = g(3x + 4(-\frac{3x}{4})) = g(3x - 3x) = g(0) = 1 \quad \therefore$

$u(x,y) = g(3x - 4y) \quad \forall g$ st $g(0) = 1$

(c) $u(x, -\frac{3x}{4}) = x^3 = g(3x + 4(-\frac{3x}{4})) = g(0) = x^3 \quad \therefore$

impossible (contradicts def of a func) $\therefore$ no sol

say $y = Ce^x$ is explicit eqn of a charac curve

- $C = ye^{-x}$ is implicit eqn of a charac curve

implicit form provides it

Ex 1.7/ $\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial x} = 0$ satisfying auxiliary eqn $u(0,y) = y^3$

Sol/ GS $u(x,y) = f(ye^{-x})$

BC: $x = 0$; $u(0,y) = f(y) = y^3 \implies u(x,y) = (ye^{-x})^3 = y^3e^{-3x}$

Ex 1.8/ $u_x + 2xy^2 u_y = 0$

Sol/ Charac eqn (ODE) $(dx, dy) \parallel (1, 2xy^2) \quad \therefore$

$\frac{dy}{dx} = \frac{2xy^2}{1}$ (separable) $y=0$ special sol

$\int \frac{dy}{y^2} = \int 2x\,dy = c - \frac{1}{y} = x^2 \quad \therefore \quad c = \frac{1}{y} + x^2$ (implicit)

$\longrightarrow u(x,y) = f\left(\frac{1}{y} + x^2\right)$

explicit $y = \frac{1}{c - x^2}$ (or $y=0$)

\Ex 1.9/ $\frac{\partial u}{\partial x} - a\frac{\partial u}{\partial y} - bu = 0 \quad a \,\&\, b \neq 0$

\Sol/ $u = A(\xi)e^{bx} = A(y+ax)e^{bx}$

\Sol/ consider $b=0$ (or drop $-bu$ term) {eg $bu=0$}

$\tilde{u}_x - a\tilde{u}_y = 0 \Longrightarrow \tilde{u}(x,y) = f(y+ax)$

let change of variables $x, y \mapsto \xi, \eta$,

$\xi = y + ax$; $\eta = x$, to Z orig PDE:

$u_x = u_\xi \xi_x + u_\eta \eta_x = au_\xi + u_\eta$,

$u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi$ $\therefore$ sub $LHS = u_x - au_y - bu = 0$

$au_\xi + u_\eta - au_\xi - bu = u_\eta - bu = 0 = RHS \therefore$

$u_\eta - bu = 0$ $\therefore$ depend on ξ as a param!

$\frac{\partial u}{\partial \eta} - bu = 0 \quad \therefore \quad \frac{\partial u}{\partial \eta} = bu \quad \therefore \quad u = Ae^{b\eta}$

Const as far as η is concerned!

$u = A(y+ax)e^{bx}$

are equivalent

\Alternatively/ $\xi = y + ax$; $\eta = y$

$\therefore u = B(y+ax)e^{-\frac{by}{a}}$ | $B(\xi)e^{-\frac{b\eta}{a}}$

\Ex 1.10/ $au_x + bu_y + cu = g(x,y)$

\Sol/ find charac coord for $a\tilde{u}_x + b\tilde{u}_y = 0$

say $\xi = ax + by$ $\eta = bx - ay$ $\therefore$ $u_x = au_\xi + bu_\eta$

$\therefore LHS = au_x + bu_y + cu = a(au_\xi + bu_\eta) + b(bu_\xi - au_\eta) + u$

$= (a^2+b^2)u_\xi + u = g(x,y) = \tilde{g}(\xi, \eta)$

ODE for $u(\xi)$ depend on η as param

$\rightarrow$ linear ODE 1st order, solve by IF

$\frac{\partial u}{\partial \xi} + \frac{1}{a^2+b^2}u = \frac{\tilde{g}(\xi,\eta)}{a^2+b^2} \qquad h = \exp\left[\frac{\xi}{a^2+b^2}\right] \therefore$

$e^{\frac{\xi}{a^2+b^2}} \frac{\partial u}{\partial \xi} + \frac{e^{\frac{\xi}{a^2+b^2}}}{a^2+b^2} u = \frac{\tilde{g}\,e^{\frac{\xi}{a^2+b^2}}}{a^2+b^2} = \frac{\partial}{\partial \xi}\left[e^{\frac{\xi}{a^2+b^2}} u\right] \therefore$

$$e^{\frac{\xi}{a^2+b^2}} u = \int \frac{\tilde{g}(\xi,\eta)\, e^{\frac{\xi}{a^2+b^2}}}{a^2+b^2}\, \partial\xi \quad \therefore$$

$$u(\xi,\eta) = e^{-\frac{\xi}{a^2+b^2}} \int \frac{\tilde{g}(\xi,\eta)\, e^{\frac{\xi}{a^2+b^2}}}{a^2+b^2}\, \partial\xi$$

note: if $g(x,y) = e^{(a^2+b^2)x}$ $\therefore \tilde{g}(\xi,\eta) = ?$ $\therefore$

need to resolve wrt x, y : $\begin{cases} ax+by = \xi \\ bx-ay = \eta \end{cases}$

Cramer's rule is: $\Delta = \begin{vmatrix} a & b \\ -b & a \end{vmatrix} = a^2+b^2$ $\therefore \begin{cases} ax+by = \xi \\ bx-ay = \eta \end{cases}$

$\Delta_x = \begin{vmatrix} \xi & b \\ \eta & a \end{vmatrix} = a\xi - b\eta \qquad \Delta_y = \begin{vmatrix} a & \xi \\ b & \eta \end{vmatrix} = a\eta + b\xi \quad \therefore$

$$x = \frac{\Delta x}{\Delta} = \frac{a\xi - b\eta}{a^2+b^2} \qquad y = \frac{\Delta y}{\Delta} = \frac{b\xi + b\eta}{a^2+b^2}$$

$$\therefore g = e^{(a^2+b^2)x} = g(x,y) = e^{a\xi - b\eta} = \tilde{g}(\xi,\eta)$$

$$\therefore u(\xi,\eta) = e^{-\frac{\xi}{a^2+b^2}} \int \frac{e^{a\xi - b\eta}}{a^2+b^2}\, e^{\frac{\xi}{a^2+b^2}}\, \partial\xi$$

$$= e^{-\frac{\xi}{a^2+b^2}} \cdot e^{-\frac{b\eta}{a^2+b^2}} \int e^{k\xi}\, \partial\xi \qquad k = \frac{1}{a^2+b^2} + a$$

$$u = \frac{1}{a^2+b^2} e^{-\frac{\xi}{a^2+b^2} - b\eta} \left(\frac{e^{k\xi}}{k} + A(\eta) \right)$$ where $A(\ )$ is arbit func

Second order PDE

Ex 3.1 $2u_{xx} - u_{xy} - 3u_{yy} = 0$

Sol look for $u(x,y) = f(y+\alpha x)$; α = ? f : arb

$\therefore u_x = \alpha f'(y+\alpha x) \qquad \therefore u_{xx} = \alpha^2 f''(y+\alpha x)$

$u_y = f'(y+\alpha x) \qquad u_{yy} = f''(y+\alpha x) \quad \therefore u_{xy} = \alpha f''(y+\alpha x)$

$\therefore 2u_{xx} - u_{xy} - 3u_{yy} = 2\alpha^2 f''(y+\alpha x) - \alpha f''(y+\alpha x) - 3f''(y+\alpha x) =$

$\underbrace{(2\alpha^2-\alpha-3)}_{=0} \underbrace{f''(y+\alpha x)}_{\text{arbit}} = 0 \quad \therefore (2\alpha^2-\alpha-3) = 0 = (2\alpha-3)(\alpha+1) = 0$

$\Rightarrow \alpha = -1,\ \alpha = \frac{3}{2}$

$\Rightarrow u(x,y) = f_1(y-x) + f_2(y+\frac{3}{2}x)$ is a sol for arb f_1, f_2

to obtain GS, do change of variables (chg of var):

$\xi = 2(y+\frac{3}{2}x) = 2y+3x = 3x+2y \qquad \eta = -(y-x) = x-y \quad \therefore$

$u_x = 3u_\xi + u_\eta \quad \Big| \quad \partial_x = 3\partial_\xi + \partial_\eta$

$u_y = 2u_\xi + - u_\eta \quad \Big| \quad \partial_y = 2\partial_\xi - \partial_\eta \quad \therefore$

$u_{xx} = (3\partial_\xi + \partial_\eta) = (3\partial_\xi + \partial_\eta) u = (9\partial_\xi^2 + 6\partial_\xi\partial_\eta + \partial_\eta^2) u$

$= 9u_{\xi\xi} + 6u_{\xi\eta} + u_{\eta\eta}$

$u_{xy} = (3\partial_\xi + \partial_\eta)(2\partial_\xi - \partial_\eta) u = (6\partial_\xi^2 - \partial_\xi\partial_\eta - \partial_\eta^2) u =$

$6u_{\xi\xi} - u_{\xi\eta} - u_{\eta\eta}$ & similarly

$u_{yy} = 4u_{\xi\xi} - 4u_{\xi\eta} + u_{\eta\eta} \quad \therefore$

Sub: $2u_{xx} - u_{xy} - 3u_{yy} = 2(9u_{\xi\xi} + 6u_{\xi\eta} + u_{\eta\eta}) -$

$(6u_{\xi\xi} - u_{\xi\eta} - u_{\eta\eta}) - 3(4u_{\xi\xi} - 4u_{\xi\eta} + u_{\eta\eta})$

$= \underbrace{(18 - 6 - 12)}_{=0} u_{\xi\xi} + \underbrace{(12 + 1 + 12)}_{=25} u_{\xi\eta} + \underbrace{(2 + 1 - 3)}_{=0} u_{\eta\eta} =$

$25u_{\xi\eta} = 0 = u_{\xi\eta} \implies$

$u = g(\xi) + h(\eta) = g(3x + 2y) + h(x - y) \quad \square$

Ex 1.12 ~~$2\frac{\partial^2 u}{\partial x^2} - \frac{\partial^2 u}{\partial x \partial y} - 3\frac{\partial^2 u}{\partial y^2} = 0$~~ $\quad u_{xx} - 2u_{xy} + u_{yy} = 0$

Sol $u = f(y + \alpha x)$ (reuse previous formulas)

$\underbrace{(\alpha^2 - 2\alpha + 1)}_{=0} \underbrace{f''(y + \alpha x)}_{\text{arb}} = 0 \qquad (\alpha - 1)^2 = 0 \quad \therefore \alpha = 1$ double root

Let us choose $\xi = x + y$ & $\eta = x - y$ (say) $\therefore$

$u_x = u_\xi + u_\eta \qquad \partial_x = \partial_\xi + \partial_\eta \qquad \partial_y = \partial_\xi - \partial_\eta$

$u_y = u_\xi - u_\eta \qquad u_{xx} = (\partial_\xi + \partial_\eta)^2 u = u_{\xi\xi} + 2u_{\xi\eta} + u_\eta u_\eta$

$u_{xy} = (\partial_\xi + \partial_\eta)(\partial_\xi - \partial_\eta) = u_{\xi\xi} - u_{\eta\eta}$

$u_{yy} = (\partial_\xi - \partial_\eta)^2 = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} \quad \therefore$

LHS $= u_{xx} - 2u_{xy} + u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} - 2(u_{\xi\xi} - u_{\eta\eta}) + u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$

$= \underbrace{(1 - 2 + 1)}_{=0} u_{\xi\xi} + \underbrace{(2 - 2)}_{=0} u_{\xi\eta} + (1 + 2 + 1) u_{\eta\eta} = 4u_{\eta\eta} = 0 = u_{\eta\eta} =$ RHS $\therefore$

$u = A(\xi)\eta + B(\xi) \quad \therefore u = A(x + y)(x - y) + B(x + y)$ what is $\eta = x$ or $\eta = y$?

\Ex 1.13/ $x^2u_{xx} - y^2u_{yy} + xu_x - yu_y = 0$ $(x>0,\ y>0)$

\Sol/ $x = e^t,\ y = e^s$ $(x,y)\to(t,s)$ $t = \ln x,\ s = \ln y$

$u_x = u_t t_x = \frac{1}{x}u_t$

$u_{xx} = \partial_x(\frac{1}{x})u_t + \frac{1}{x}\partial_x(u_t) = \frac{1}{x}[\frac{1}{x}\frac{\partial}{\partial t}(u_t)] - \frac{1}{x^2}u_t =$

$\frac{1}{x^2}u_{tt} - \frac{1}{x^2}u_t$ $\quad u_{yy} = \frac{1}{y^2}u_{ss} - \frac{1}{y^2}u_s$ $\quad u_y = \frac{1}{y}u_s$

$\therefore x^2[\frac{1}{x^2}u_{tt} - \frac{1}{x^2}u_t] - y^2[\frac{1}{y^2}u_{ss} - \frac{1}{y^2}u_s] + x\frac{1}{x}u_t - y\frac{1}{y}u_s =$

$u_{tt} - u_t - u_{ss} + u_s + u_t - u_s = u_{tt} - u_{ss} = 0 = RHS$ $\therefore$

So look $u = f(t+\alpha s)$ $\therefore$

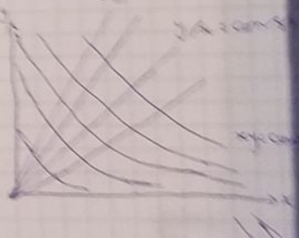
$\underbrace{(1-\alpha^2)}_{=0}\underbrace{f''(t+\alpha s)}_{\neq 0} = 0$ $\therefore \alpha = 1$ or $\alpha = -1$

$\therefore$ say ~~$\xi = x+y$~~ ~~$\eta = x-y$~~ $\xi = t+s$, $\eta = t-s$

$\therefore u = f(t+s) + g(t-s) = f(\xi) + g(\eta)$ $\quad \{t = \ln x,\ s = \ln y\}$

$= f(\ln x + \ln y) + g(\ln x - \ln y) =$

$f(\ln(xy)) + g(-\ln(y/x)) = F(xy) + G(y/x) = u$



\Def 1.14/

we say that a partial differential eqn containing partial derivatives of the unknown func & all its derivat

\Def 1.15/ say PDE is linear if it is linear w.r.t the unknown func & all its derivatives

indep vars & RHS, free term, either zero eqn homogeneous

\Def 1.16/ PDE is quasilinear if its linear w.r.t all the highest order derivatives of the unknown func, the coefs of the linear combs as well as the free term

\Def 1.17/ Semilinear is its linear wrt all Z highest order derivs of Z unknown func

) eg $a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} = f(x,y,u,u_x,u_y)$

no universally accepted def of nonlinear PDEs

\Ex 1.18/ $u_t + uu_x + u_{xxx} = 0$ not linear $\because$ uu_x term but is semilinear and quasilinear $\because$ its linear wrt Z only third order deriv u_{xxx} (its 3rd order)

\Ex 1.19/ $(1+u_x^2)u_{yy} + (1+u_y^2)u_{xx} - 2u_x u_y u_{xy} = 0$
is 2nd order. its not linear, nor semilinear

\Ex 1.20/ $u_t + \frac{1}{2}(u_x^2 + u_y^2 + u_z^2) = 0$
1st order is "fully nonlinear" ie not even quasilinear $\because$ nonlinear dependence on Z 1st order deriv

\Def 1.21/ let $(x,y) \longmapsto (\xi,\eta)$ be diffable transformation of Z (x,y) plane ie $\xi(x,y)$, $\eta(x,y)$ are diffable funcs

$\therefore$ Jacobian Matrix

$$\underline{J} = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{bmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{bmatrix}$$

Jacobian determinant $J = \det \underline{J} = \det\left[\frac{\partial(\xi,\eta)}{\partial(x,y)}\right] = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x\eta_y - \xi_y\eta_x$

is called Z Jacobian determinant

consider $a(x,y)u_{xx} + 2b(x,y)u_{xy} + c(x,y)u_{yy} = f(x,y,u,u_x,u_y)$

$(x,y) \longmapsto (\xi,\eta)$ $\qquad \xi(x,y) = \text{const} \qquad \eta(x,y) = \text{const}$

$$\nabla\xi \nparallel \nabla\eta \iff \frac{\xi_x}{\xi_y} \neq \frac{\eta_x}{\eta_y} \iff \det\left[\frac{\partial(\xi,\eta)}{\partial(x,y)}\right] \neq 0$$

$(x,y) \to (\xi,\eta)$

chain rule: $u_x = u_\xi \xi_x + u_\eta \eta_x$

$u_y = u_\xi \xi_y + u_\eta \eta_y$

$u_{xx} = \partial_x [u_\xi \xi_x + u_\eta \eta_x] = \partial_x [u_\xi] \xi_x + u_\xi \xi_{xx} + \partial_x [u_\eta] \eta_x + u_\eta \eta_{xx} =$

$[u_{\xi\xi} \xi_x + u_{\xi\eta} \eta_x] \xi_x + u_\xi \xi_{xx} + [u_{\eta\xi} \xi_x + u_{\eta\eta} \eta_x] \eta_x + u_\eta \eta_{xx} =$

$\xi_x^2 u_{\xi\xi} + 2 \xi_x \eta_x u_{\xi\eta} + \eta_x^2 u_{\eta\eta} + \xi_{xx} u_\xi + \eta_{xx} u_\eta$

$\xi = \xi(x,y)$ $\quad \eta = \eta(x,y)$ $\quad u_x = u_\xi \xi_x + u_\eta \eta_x$

$u_y = u_\xi \xi_y + u_\eta \eta_y$

$u_{xx} = \xi_x^2 u_{\xi\xi} + 2 \xi_x \eta_x u_{\xi\eta} + \eta_x^2 u_{\eta\eta} + \xi_{xx} u_\xi + \eta_{xx} u_\eta$ lower-order terms

semi linear: L.O.T. $\quad a(x,y) u_{xx} + 2b(x,y) u_{xy} + c(x,y) u_{yy} =$
$f(x,y,u,u_x,u_y)$

similarly $u_{yy} = \xi_y^2 u_{\xi\xi} + 2 \xi_y \eta_y u_{\xi\eta} + \eta_y^2 u_{\eta\eta}$ + LOT

& (similarly u_{xx})

$u_{xy} = \xi_x \xi_y u_{\xi\xi} + (\xi_x \eta_y + \xi_y \eta_x) u_{\xi\eta} + \eta_x$

sub into ODE: LHS $= a u_{xx} + 2b u_{xy} + c u_{yy} =$

$a[\xi_x^2 u_{\xi\xi} + 2 \xi_x \eta_x u_{\xi\eta} + \eta_x^2 u_{\eta\eta}] + 2b[\xi_x \xi_y u_{\xi\xi} + (\xi_x \eta_y + \xi_y \eta_x) u_{\xi\eta} +$
$\eta_x \eta_y u_{\eta\eta}] + c[\xi_y^2 u_{\xi\xi} + 2 \xi_y \eta_y u_{\xi\eta} + \eta_y^2 u_{\eta\eta}]$ + LOT = RHS = f

collecting similar terms $\alpha u_{\xi\xi} + 2\beta u_{\xi\eta} + \gamma u_{\eta\eta} = g$

where $\alpha = a \xi_x^2 + 2b \xi_x \xi_y + c \xi_y^2$

$\beta = a \xi_x \eta_x + b(\xi_x \eta_y + \xi_y \eta_x) + c \xi_y \eta_y$

~~$\gamma = a\eta$~~ $\gamma = a \eta_x^2 + 2b \eta_x \eta_y + c \eta_y^2$

$g = f$ − L.O.T (from the RHS)

$a = a(x,y)$ etc, so need to transform $(x,y) \mapsto (\xi,\eta)$.

so $\alpha = \alpha(\xi,\eta)$, $\beta = \beta(\xi,\eta)$, $\gamma = \gamma(\xi,\eta)$

\Def 1.28/ ODE $a(x,y)\left(\frac{dy}{dx}\right)^2 - 2b(x,y)\frac{dy}{dx} + c(x,y) = 0$

is characteristic eqn any sols are characteristic curves or

) just characteristics

\Ex/ \Ex 1.23/ $x^2 u_{xx} - y^2 u_{yy} + x u_x - y u_y = 0$ $x>0, y>0$

\Sol/ $a u_{xx} + 2b u_{xy} + c u_{yy} = 0$

$a = x^2$: $b = 0$: $c = -y^2$

Charact ODE $a(y')^2 - 2by' + c = 0$

$\therefore x^2\left(\frac{dy}{dx}\right)^2 - y^2 = 0$ $\therefore \frac{dy}{dx} = \frac{y}{x}$ or $\frac{dy}{dx} = -\frac{y}{x}$

~~separations~~ vars $\int \frac{dy}{y} = \int \frac{dx}{x}$ $\int \frac{dy}{y} = -\int \frac{dx}{x}$

$\ln y = \ln x + c$ $\ln y = -\ln x + c$

explicit $c = \ln y - \ln x$ $c = \ln y + \ln x$

or $D = y/x$ $D = yx$

So we can take $\xi = \ln y - \ln x$: $\eta = \ln y + \ln x$

or $\xi = y/x$ $\eta = yx$

this is what we used : $\ln x = s$, $\ln y = t$

$\xi = s - t$; $\eta = s + t$

$\Rightarrow$ either way, $u_{\xi\eta} = 0$ $(g \equiv 0$

so $u = f_1(\xi) + f_2(\eta) = f(\ln y - \ln x) + g(\ln y + \ln x)$

$= F(y/x) + G(xy)$

\Def 1.24/ 2nd order PDE

- hyperbolic if $(b^2 - ac) > 0$
- parabolic if $(b^2 - ac) = 0$
- elliptic if $(b^2 - ac) < 0$

\Ex 1.25/ $u_{xx} - u_{tt} = 0$ with $a = 1$, $b = 0$, $c = -1$ $f = 0$ $\therefore$

$b^2 - ac = 1 > 0$ $\therefore$ hyperbolic

\Ex 1.26/ $u_t = u_{xx}$ $a = 1$, $b = c = 0$ $f = u_t$ $\therefore b^2 - ac = 0$ $\therefore$ parabolic

\Ex 1.27/ $u_{xx} + u_{yy} = 0$ $a = 1$, $b = 0$ $b^2 - ac < 0$ elliptic

\Ex 1.28/ $u_{xx} - 5u_{xy} = 0$ $\therefore a = 1$ $b = -\frac{5}{2}$ $c = 0$ $\therefore$

$b^2 - ac = \frac{25}{4} - 1(0) = \frac{25}{4}$ $\therefore$ hyperbolic

\Ex 1.29/ $4u_{xx} + 6u_{xy} + 9u_{yy} = 0$ $a = 4$, $b = 3$, $c = 9$

$b^2 - ac = 3^2 - 4 \cdot 9 = 9 - 36 = -27 < 0$ $\therefore$ elliptic

\Ex 1.30/ $x^2 u_{xx} - y^2 u_{yy} + x u_x - y u_y = 0$

$a = x^2$, $b = 0$, $c = -y^2$ $\therefore$ $b^2 - ac = x^2 y^2$

$\therefore$ hyperbolic

\thm 1.31/ (1.3) transforms as $u(x,y) = v(\xi, \eta)$, $\xi = \xi(x,y)$ $\eta = \eta(x,y)$

$\therefore \alpha v_{\xi\xi} + 2\beta v_{\xi\eta} + \gamma v_{\eta\eta} = g(\xi, \eta, u, v_\xi, v_\eta)$ $\therefore$

$\beta^2 - \alpha\gamma = (b^2 - ac) J^2$, $J = \det\left(\frac{\partial(\xi,\eta)}{\partial(x,y)}\right) = \xi_x \eta_y - \xi_y \eta_x$

if $J \neq 0$: $\operatorname{sgn}(\beta^2 - \alpha\gamma) = \operatorname{sgn}(b^2 - ac)$

\proof/ \thm 1.31:/ $\beta^2 - \alpha\gamma = (b^2 - ac) J^2 \Rightarrow \operatorname{sgn}(\beta^2 - \alpha\gamma) = \operatorname{sgn}(b^2 - ac)$

$\therefore$ ① compute J^2; $J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x \eta_y - \xi_y \eta_x$

$\cancel{J^2 = \xi_x^2}$ $J^2 = \xi_x^2 \eta_y^2 - 2\xi_x \xi_y \eta_x \eta_y + \xi_y^2 \eta_x^2$ (>0)

② $\beta^2 - \alpha\gamma =$

$$[a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y]^2 - [a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2][a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2] =$$

$$\underbrace{a^2[\xi_x^2\eta_x^2 - \xi_x^2\eta_x^2]}_{0} + \underbrace{c^2[\xi_y^2\eta_y^2 - \xi_y^2\eta_y^2]}_{0} + b^2[\xi_x^2\eta_y^2 + 2\xi_x\xi_y\eta_x\eta_y + \xi_y^2\eta_x^2 - 4\xi_x\xi_y\eta_x\eta_y] +$$

$$\underbrace{ab[2\xi_x^2\eta_x\eta_y + 2\xi_x\xi_y\eta_x^2 - 2\xi_x^2\eta_x\eta_y - 2\xi_x\xi_y\eta_x^2]}_{0} +$$

$$\underbrace{bc[2\xi_y^2\eta_y\eta_x + 2\xi_y\xi_x\eta_y^2 - 2\xi_y^2\eta_y\eta_x - 2\xi_y\xi_x\eta_y^2]}_{0} +$$

$$ac[2\xi_x\xi_y\eta_x\eta_y - \xi_x^2\eta_y^2 - \xi_y^2\eta_x^2] =$$

$$b^2(\xi_x^2\eta_y^2 - 2\xi_x\xi_y\eta_x\eta_y + \xi_y^2\eta_x^2 + \xi_y^2\eta_x^2) - ac(\xi_x^2\eta_y^2 - 2\xi_x\xi_y\eta_x\eta_y + \xi_y^2\eta_x^2) =$$

$(b^2 - ac) J^2$ $\square$

Ex 1.32/ $x^2 u_{xx} - y^2 u_{yy} + x u_x - y u_y = 0$ has coeffs

$a = x^2$, $b = 0$, $c = -y^2$ $\therefore b^2 - ac = x^2 y^2$ $\therefore$

transformation $\xi = y/x$, $\eta = yx$ $\therefore$

$\xi_x = -y/x^2$, $\xi_y = 1/x$ $\eta_x = y$, $\eta_y = x$ $\therefore$

$J = \xi_x \eta_y - \xi_y \eta_x = -2y/x$ the transformed eqn in that exercise has

$-4y^2 u_{\xi\eta} = 0$ that is $\alpha = 0$, $\beta = -2y^2$, $\gamma = 0$ $\therefore$

$\beta^2 - \alpha\gamma = (-2y^2)^2 - 0 = 4y^4 = x^2 y^2 (-2y/x)^2 = (b^2 - ac) J^2$

in accordance with thm 1.31

BCs: Dirichlet BC: specified at boundary

or derivative $\partial u/\partial n$ to the boundary

or combination, $\partial u/\partial n + a(x, t) u$ specified

if set at same boundary they corres to Cauchy BC

if one indep variable is time & BC at $t = t_0$ then called IC

good PDE enjoys: • Existence (of at least one u & initial & BCs)

• Uniqueness: only one such sol

• Stability (small changes of initial or BCs produce small changes in sol

Chapter 2 Hyperbolic eqns/

$a u_{xx} + 2b u_{xy} + c u_{yy} = f(x, y, u, u_x, u_y)$ a, b & c are funcs of x & y

$\alpha u_{\xi\xi} + 2\beta u_{\xi\eta} + \gamma u_{\eta\eta} = g$ $\beta^2 - \alpha\gamma = (b^2 - ac)(\xi_x \eta_y - \xi_x$

is hyperbolic if $(b^2 - ac) > 0 \iff (\beta^2 - \alpha\gamma) > 0$

Def 2.1/ the (transformed) hyperbolic semilinear PDE is in the

canonical form if $\alpha \equiv 0$, $\gamma \equiv 0$ $\therefore \frac{\partial^2 u}{\partial \xi \partial \eta} = \tilde{g}(\xi, \eta, u, u_\xi, u_\eta)$

Ex 2.2/ $u_{xx} - u_{yy} = f(x, y)$ f is a func by bringing to the canonical

Sol/ $a u_{xx} + 2b u_{xy} + c u_{yy} = u_{xx} - u_{yy}$ $a = 1$, $b = 0$, $c = -1$

$b^2 - ac = 0 - 1(-1) = 1 > 0$ $\therefore$ hyperbolic

with charac eqn:

$a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0$

$\left(\frac{dy}{dx}\right)^2 - 1 = 0$ $\qquad \frac{dy}{dx} = 1 \qquad \frac{dy}{dx} = -1$

charac curves: $y = x - C_1 \qquad y = -x + C_2$

$C_1 = x - y \qquad C_2 = x + y$

charac coords: $\xi = x - y \qquad \eta = x + y$

$x = \frac{1}{2}(\xi + \eta)$; $y = \frac{1}{2}(-\xi + \eta)$ $\therefore$

by chainrule: $\xi_x = 1$; $\xi_y = -1 \qquad \eta_x = 1 \quad \eta_y = 1$

$u_x = u_\xi + u_\eta$; $u_y = -u_\xi + u_\eta$

$u_{xx} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \qquad \partial_x = \partial_\xi + \partial_\eta \qquad \partial_y = -\partial_\xi + \partial_\eta$

$u_{yy} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$

$LHS = u_{xx} - u_{yy} = 4u_{\xi\eta} = RHS = f(x,y) = f\left(\frac{\xi+\eta}{2}, \frac{-\xi+\eta}{2}\right) = g(\xi,\eta) = 4u_{\xi\eta}$

$\therefore$ Direct integration twice: $u(\xi,\eta) = \frac{1}{4}\iint g(\xi,\eta)\,\partial\xi\,\partial\eta + F(\eta) + G(\xi)$

$\xi = x - y \quad \eta = x + y$

Ex 2.3 $\quad u_{tt} - C^2 u_{xx} = 0$; $C = \text{const} > 0$ $\qquad$ 1 Dimen wave eqn

Sol $\quad a\left(\frac{dx}{dt}\right)^2 - 2b\left(\frac{dx}{dt}\right) + c = 0 \qquad a = 1$; $b = 0 \qquad c = -C^2$

$\left(\frac{dx}{dt}\right)^2 - C^2 = 0 \qquad \frac{dx}{dt} = \pm C \qquad x \pm Ct = \text{Const}$

$\xi = x - Ct$; $\eta = x + Ct$; $\cdots$ (DIY) $\quad \{f = 0 \Rightarrow g = 0\}$

$\frac{\partial^2 u}{\partial\xi\partial\eta} = 0 \qquad u = f_1(\xi) + f_2(\eta) = f_1(x - Ct) + f_2(x + Ct)$

D'Alembert's representation

$u(x,t) = f_1(x + Ct) + f_2(x - Ct)$ $\qquad$ solve Cauchy problem for it

$u_{t=0} = \phi(x) \qquad \left(\frac{\partial u}{\partial t}\right)_{t=0} = \psi(x)$ $\quad \therefore$ from IC: $\phi(x) = f_1(x) + f_2(x)$,

$\psi(x) = Cf_1'(x) - Cf_2'(x)$ $\quad \therefore$ integrate wrt x: $\int_{x_0}^{x} \psi(s)\,ds =$

$C(f_1(x) - f_1(x_0)) - C(f_2(x) - f_2(x))$ $\qquad x_0$ is arb const

$\therefore$ System of simultaneous eqns $\quad f_1(x) - f_2(x) = \frac{1}{C}\int_{x_0}^{x} \cancel{\psi(s)\,ds + f_1(x_0) + f_2(x_0)}$

$\frac{1}{C}\int_{x_0}^{x} \psi(s)\,ds + f_1(x_0) + f_2(x_0) \qquad f_1(x) + f_2(x) = \phi(x)$ $\quad \therefore$

$f_1(x) = \frac{\phi(x)}{2} + \frac{1}{2C}\int_{x_0}^{x} \psi(s)\,ds + \frac{1}{2}\left[f_1(x_0) - f_2(x_0)\right]$

$f_2(x) = \frac{\phi(x)}{2} - \frac{1}{2c}\int_{x_0}^{x} \psi(s)\,ds - \frac{1}{2}[f_1(x_0) - f_2(x_0)]$ $\therefore$ Sol satisfying ICs:

$$u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$$

known as d'Alembert's formula (hint $\int_{x-ct}^{x_0} + \int_{x_0}^{x+ct} = \int_{x-ct}^{x+ct}$)

\Ex 2.4/ $\phi(x) = 0$: $\psi(x) = \cos x$ is ICs to wave eqn

\Sol/ $u = \frac{1}{2}[\underbrace{\phi(x+ct)}_{=0} + \underbrace{\phi(x-ct)}_{=0}] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds$

$= \frac{1}{2c}\int_{x-ct}^{x+ct} \cos(s)\,ds = \frac{1}{2c}\left[\sin(s)\right]_{s=x-ct}^{x=c+ct} =$

$\frac{1}{2c}\left[\sin(x+ct) - \sin(x-ct)\right] = \frac{1}{c}\cos(x)\sin(ct)$

effect of initial position $\phi(x)$ is a pair of waves travelling in either direction at speed c & at half 2 original amplitude

if ask $u_t(x,t)$: $u_t(x,t) = \frac{c}{2}[\phi'(x+ct) - \phi'(x-ct)] +$
$\frac{1}{2}[\psi(x+ct) - \psi(x-ct)]$
depends only on $\psi(x+ct)$ & $\psi(x-ct)$

~~Canon~~ Canonical form of hyperbolic eqn: $u_{\xi\eta} = f(\xi, \eta, \ldots)$

\§2.2.3 Cauchy problem on a charactistic/

PDE: $u_{\xi\eta} = f(\xi, \eta)$

BC 1: $u(\xi, 0) = \phi(\xi)$ BC 2: $u_\eta(\xi, 0) = \psi(\xi)$

BC 2 is true for $\forall \xi$, $\therefore \frac{\partial}{\partial \xi}$: $u_{\xi\eta}(\xi, 0) = \psi'(\xi)$ compared

$\therefore$ by continuity PDE gives $u_{\xi\eta}(\xi, 0) = f(\xi, 0)$

① if $\psi'(\xi) \neq f(\xi, 0)$ anywhere $\to$ Contradiction $\therefore$ no sols

② $\psi'(\xi) = f(\xi, 0)$ $\forall \xi$ $\Rightarrow$ BC 2 is redundant, 2 Sol depends on 1 arb func

consider s/s on whole real line $x \in \mathbb{R} = (-\infty, \infty)$ have eqn half line; $x \in (0, \infty)$ BCs at $x = 0$ $\therefore$ $v_{tt} - c^2 v_{xx} = 0$, $x \in (0, \infty)$ $t \in (0, \infty)$

$v(x, 0) = \phi(x)$, $v_t(x, 0) = \psi(x)$ $v(0, t) = 0$

Z odd extension of both ICs to Z whole line

$$\phi_{odd}(x) = \begin{cases} \phi(x) & x>0 \\ -\phi(-x), & x<0 \\ 0, & x=0 \end{cases} \quad \text{\& similarly} \quad \psi_{odd}(x) = \begin{cases} \psi(x), & x>0 \\ -\psi(-x), & x<0 \\ 0, & x=0 \end{cases}$$

Let $u(x,t)$ be Z sol of $u_{tt} - cu_{xx} = 0$ $\quad x \in (-\infty, \infty)$, $t \in (0, \infty)$

ICs $\quad u(x,0) = \phi_{odd}(x) \quad u_t(x,0) = \psi_{odd}(x)$

\Lemma 2.5/ if Z initial data satisfy $\phi_{odd}(-x) = -\phi_{odd}(x)$

$\therefore u(-x,t) = -u(x,t) \ \forall x,t$ $\quad$ for $\psi_{odd}(-x) = -\psi_{odd}(x)$ as well

$\therefore$ PDE $\quad c^2u_{xx} - u_{tt} = 0 \qquad$ IC1 $\ u(x,0) = \phi(x)$

IC2 $\quad u_t(x,0) = \psi(x) \qquad x \in \mathbb{R}, \ t \in \mathbb{R}_+$

\Sol/ (Sol is unique $\because$ D'Alembert's formula)

S1: $\phi(-x) \equiv -\phi(x) \ \forall x \qquad$ S2 $\ \psi(-x) \equiv -\psi(x) \ \forall x$

$\therefore u(-x,t) \equiv -u(x,t) \quad \forall x,t$

\proof/ change of variables. $\ u(x,t) = -v(\xi, \tau)$ ⊛

$x = -\xi$, $\ t = \tau$ $\therefore$

$u = -v \quad \therefore u_t = v_\tau \tau_t = -v_\tau$

$u_{tt} = -v_{\tau\tau}$

$u_x = -v_\xi \underbrace{\xi_x}_{-1} = v_\xi \qquad u_{xx} = v_{\xi\xi} \underbrace{\xi_x}_{-1} = -v_{\xi\xi} \quad \therefore$ into PDE:

$c^2u_{xx} - u_{tt} = 0 \qquad c^2(-v_{\xi\xi}) - (-v_{\tau\tau}) = 0 \qquad c^2v_{\xi\xi} - v_{\tau\tau} = 0$

$v(\xi, \tau)$ satisfies Z same PDE as $u(x,t)$ $\therefore$ PDE is invariant wrt chng of variabs

IC1 $\ u(x,0) = \phi(x) = u(x,t)\big|_{t=0} \qquad \{t = \tau \quad t=0 \Leftrightarrow \tau = 0\}$

$= -v(\xi, \tau)\big|_{\tau=0} = -v(\xi, 0) = \phi(-\xi) \overset{S1}{=} -\phi(\xi)$

$\therefore v(\xi, 0) = \phi(\xi)$ $\therefore$ $v(\xi, \tau)$ satisfies Z same IC1 as $u(x,t)$

IC2: $v(\xi, \tau)$ satisfies Z same IC2 as $u(x,t)$

(use S2 $\ \psi(-\xi) = -\psi(\xi)$)

So $v(\xi, \tau)$ satisfies Z same IVP as $u(x,t)$, sol to IVP is unique $\Rightarrow u: (x,t) \mapsto u(x,t) \qquad v: (\xi, \tau) \mapsto v(\xi, \tau)$ are one and the same func $\therefore$

they should give 2 same result given 2 same argument, ie

$v(\xi,\tau) = u(\xi,\tau)$ $\quad\quad$ $v(\xi,\tau) = u(\xi,\tau)$

$\| \quad\quad\quad\quad\quad\quad\quad\quad \therefore$ by ⊛ $\quad \| \quad\quad \|$ by ⊛

$-u(x,t)$ (by ⊛) $\quad\quad\quad\quad -u(x,t) = u(-x,t)$ $\quad \square$

Reflection of Waves $\quad v_{tt} - c^2 v_{xx} = 0 \quad$ IC1 $\; v(x,0) = \phi(x)$

IC2 $\; v_t(x,0) = \psi(x) \quad\quad$ BC $\; v(0,t) = 0$

$$\hat{\phi}(x) = \begin{cases} \phi(x), & x \geq 0 \\ -\phi(-x), & x < 0 \end{cases} \quad\quad \hat{\psi}(x) = \begin{cases} \psi(x), & x \geq 0 \\ -\psi(-x), & x < 0 \end{cases}$$

$\hat{\phi}, \hat{\psi}$ are odd funcs

"odd extension of BVP" $\; u_{tt} - c^2 u_{xx} = 0 \quad$ IC1 $\; u(x,0) = \hat{\phi}(x)$

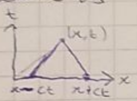
IC2: $\; u_t(x,0) = \hat{\psi}(x)$

$u(x,t)$ - by D'Alemberts

$v(x,t) = u(x,t) \quad | x \geq 0$

by D'alemberts $\quad u(x,t) = \frac{1}{2}\left[\hat{\phi}(x+ct) + \hat{\phi}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \hat{\psi}(s)\,ds$

Case 1 :



$x \in [x-ct, x+ct] \quad x > 0$

only $\phi(x), \psi(x) \quad x - ct \geq 0$

$x \geq t \quad\quad \therefore ct \leq x$

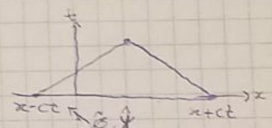
$x \geq ct, \; t \leq x/c$

$\hat{\phi}(x-ct) = \phi(x-ct)$

$\hat{\psi}(s) = \psi(s)$,

$v(x,t) = u(x,t) = \frac{1}{2}\left[\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi(s)\,ds\right] = v_1(x,t)$

Case 2:



$x - ct < 0 \quad\quad x < ct$

$ct > x$

$x < ct : \; t > x/c$

$\phi(x-ct) = -\phi(ct-x)$

$s \in [x-ct, x+ct] = [x-ct, 0] \cup [0, x+ct]$

$$I = \int_{x-ct}^{x+ct} \hat{\psi}(s)\,ds = \underbrace{\int_{x-ct}^{0} \hat{\psi}(s)\,ds}_{I_1} + \underbrace{\int_{0}^{x+ct} \hat{\psi}(s)\,ds}_{I_2}$$

$$I_2 = \int_0^{x+ct} \hat{\psi}(s)\,ds = \int_0^{x+ct} \psi(s)\,ds$$

$$I_1 = \int_{x-ct}^{0} \hat{\psi}(s)\,ds \quad \Big| \; s = -s', \; ds = -ds'$$

$$= \int_{ct-x}^{0} \hat{\psi}(-s')(-ds') \quad \Big|_{s' \to s}$$

$$= \int_0^{ct-x} \hat{\psi}(-s)\,ds = -\int_0^{ct-x} \hat{\psi}(s)\,ds = I_1$$

$$I = I_1 + I_2 = \int_0^{x+ct} \psi(s)\,ds - \int_0^{ct-x} \psi(s)\,ds =$$

$$\int_{ct-x}^{ct+x} \psi(s)\,ds$$

So D'Alembert's gives:

$$v(x,t) = \frac{1}{2}\left[\phi(ct+x) - \phi(ct-x)\right] + \frac{1}{2c}\int_{ct-x}^{ct-x} \psi(s)\,ds =$$

$v_2(x,t)$

$$v(x,t) = \begin{cases} v_1(x,t), & x-ct \geq 0 \\ v_2(x,t), & x-ct < 0 \end{cases}$$

$$\begin{cases} \delta(x) = 0 & x \neq 0 \\ \int_{-\infty}^{\infty} \delta(x)\,dx = 1 \end{cases}$$

given $f(x)$ $\int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = \int_{-\infty}^{\infty} \cancel{(f(x)-f(x_0))\,\delta(x-x_0)\,dx}$

$$\int_{-\infty}^{\infty} \underbrace{(f(x)-f(x_0))}_{\text{zero at } x=x_0} \underbrace{\delta(x-x_0)}_{\substack{\text{zero} \\ \text{for } x\neq x_0}}\,dx + \int_{-\infty}^{\infty} f(x_0)\,\delta(x-x_0)\,dx$$

whole first integral is zero

$$= f(x_0)\int_{-\infty}^{\infty} \delta(x-x_0)\,dx = f(x_0)\underbrace{\int_{-\infty}^{\infty} \delta(z)\,dz}_{1} \qquad \Big|\; z = x - x_0$$

$= f(x_0)$

$$\Rightarrow \int_{-\infty}^{\infty} f(x)\,\delta(x-x_0)\,dx = f(x_0)$$

Ex 2.6 / PDE & BCs & ICs

1st seperate variables & apply BCs

2) form a sum of these 3) apply ICs

$u_{tt} = c^2 u_{xx}$ ∴ Put $u = X(x)T(t)$ & seperate to give

$X T'' = c^2 X'' T$ or $\dfrac{T''}{c^2 T} = \dfrac{X''}{X} = -\lambda = \text{constant}$ ∴

$X'' + \lambda X = 0$, $T'' + \lambda c^2 T = 0$ $X(0) = 0$, $X(l) = 0$ BCs

Cases $\lambda < 0$, $\lambda = 0$, $\lambda > 0$ ∴

$\underline{\lambda < 0}$: Let $\lambda = -a^2$, $X'' = a^2 X \Rightarrow X = Ae^{ax} + Be^{-ax}$ ∴

BCs $\Rightarrow A + B = 0$, $Ae^{al} + Be^{-al} = 0 \Rightarrow \ldots A = B = 0$

$\underline{\lambda = 0}$: $X'' = 0 \Rightarrow X = A + Bx$ BCs $\Rightarrow A = 0$, $A + Bl = 0 \Rightarrow A = B = 0$

$\lambda > 0$, let $\lambda = a^2$ $\therefore X = A\cos ax + B\sin ax$ $\therefore$

BCs $\Rightarrow A = 0$ & $A\cos aL + B\sin aL = 0 \Rightarrow B = 0$ or $\sin aL = 0 \Rightarrow$

$aL = n\pi \qquad n = 1, 2, 3, \ldots$

$\Rightarrow X(x) = \sin\frac{n\pi x}{L} \qquad \lambda = a^2 = \left(\frac{n\pi}{L}\right)^2$

$\Rightarrow T'' + c^2\lambda T = 0 \Rightarrow T'' + c^2 a^2 T = 0$

$T = A\sin(act) + B\cos(act) \leftarrow$ different from previous A, B

$= A\sin\left(\frac{n\pi ct}{L}\right) + B\cos\left(\frac{n\pi ct}{L}\right)$

our seperated sol is:

$u(x,t) = \left(A\sin\left(\frac{n\pi ct}{L}\right) + B\cos\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$

General sol is:

$$u(x,t) = \sum_{n=1}^{\infty}\left(A_n\sin\left(\frac{n\pi ct}{L}\right) + B_n\cos\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$$

now apply ICs: $u(x,0) = 0$, ~~$\frac{\partial u}{\partial t}(x,$~~ $\frac{\partial u}{\partial t}(x,0) = I\,\delta(x - x_0)$

at $t = 0$: $0 = u(x,0) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{L}\right)$ must hold $\forall x \Rightarrow$

$B_n = 0 \;\forall n$

look at $\frac{\partial u}{\partial t}\Big|_{t=0} = \frac{\partial u}{\partial t}(x,0) =$

$$\sum_{n=1}^{\infty} \frac{n\pi c}{L} A_n \sin\left(\frac{n\pi x}{L}\right) = I\,\delta(x - x_0) \quad (*)$$

now:

$$\int_0^L \sin\left(\frac{m\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx = \begin{cases} 0 & m \neq n \\ L/2 & m = n \end{cases}$$

Multiply $(*)$ by $\sin\left(\frac{m\pi x}{L}\right)$ for some m & integrate from 0 to L, this pulls out $n = m$ only & gives:

$$\frac{m\pi c}{L} A_m \frac{L}{2} = \int_0^L I \sin\left(\frac{m\pi x}{L}\right)\delta(x - x_0)\,dx$$

recall $\int_{-\infty}^{\infty} f(x)\,\delta(x - x_0)\,dx = f(x_0)$ here

$$\frac{m\pi c}{L} A_m \frac{L}{2} = I\sin\left(\frac{m\pi x_0}{L}\right) \Rightarrow A_m = \frac{2I}{m\pi c}\sin\left(\frac{m\pi x_0}{L}\right) \quad \text{\& so}$$

$$u(x,t) = \sum_{n=1}^{\infty} \frac{2I}{n\pi c}\sin\left(\frac{n\pi x_0}{L}\right)\sin\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right)$$

Ex 2.7 $\frac{\partial^2 u}{\partial t^2} - c^2 u_{xx} = 0 \qquad 0 \le x \le L \qquad u_x(0,t) = 0$

$u(L,t) + h u_x(L,t) = 0 \qquad u(x,0) = 0 \qquad u_t(x,0) = I\,\delta(x_0 - x_0)$

$u_x(0,t) = 0$ is Neumann

$u(L,t) + h\, u_x(L,t) = 0$ "Robin"

... \Sol/ ... $X'' + \lambda X = 0$ $\quad X'(0) = 0$ $\quad , X(L) + hX'(L) = 0$

Look at $\underline{\lambda > 0}$ $\therefore$ $\lambda = a^2$ $\quad X'' = a^2 X = 0$ $\quad \therefore X(x) = A\cos(ax) + B\sin(ax)$

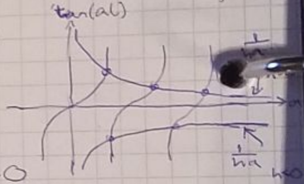
$X'(x) = -Aa\sin(ax) + Ba\cos(ax)$ $\therefore$

$X'(0) = 0 \Rightarrow B = 0$ & $X = A\cos(ax)$

Now look at $X(L) + hX'(L) = 0 \Longrightarrow A\cos(aL) - hAa\sin(aL) = 0 \Longrightarrow$

$A = 0$ or $\tan(aL) = \frac{1}{ha}$

Look for sols for 'a' given h, L {find where $\tan(aL)$ and $\frac{1}{ha}$ intersect on a graph $\because$ transsendental eqn can't be solved}



\Ex2.7/ $u_{tt} = C^2 u_{xx}$ $\quad \frac{\partial}{\partial x} u(0,t) = 0$ Neumann

$u(L,t) + h\frac{\partial u}{\partial x}(L,t) = 0$ "Robin"

$\frac{\partial u}{\partial x}(x,t) = I\delta(x - x_0)$ $\quad , u(x,0) = 0$

seperating variab $X'' + \lambda X = 0$ $\quad T'' + C^2 \lambda T = 0$

$\underline{\lambda > 0}$: , $\lambda = a^2$ $\quad X = A\cos ax$ $\quad \tan(La) = \frac{1}{ha}$ infinetly sols for both $h > 0$ and $h < 0$ $\quad$ infinetly many sols label $a_1, a_2, a_3, \ldots$

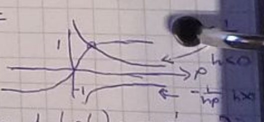
$\therefore X = A\cos(a_n x)$ $\quad T = A\cos(a_n Ct) + B\sin(a_n Ct)$ $\therefore$

$u = (A_n\cos(a_n Ct) + B_n\sin(a_n Ct))\cos(a_n x)$

$\underline{\lambda = 0}$: doesn't give a sol

$\underline{\lambda < 0}$: $\lambda = -p^2$ $\quad X'' - p^2 X = 0$, $\quad X = Ae^{px} + Be^{-px} =$

$C\cosh(px) + D\sinh(px)$



apply $X'(0) = 0$ $\quad X(L) + (h)(X'(L)) = 0 \Longrightarrow D = 0$ & $\tanh(pL) = -\frac{1}{hp}$??

call this sol $p = p_*$ for $h < 0$ or no sol for $h > 0$

$X(x) = C\cosh(p_* x)$ $\quad T(t) = A\cosh(p_* Ct) + B\sinh(p_* Ct)$

from $X'' - p^2 X = 0 \Rightarrow p_*$

$T'' - p^2 C^2 T = 0 \rightarrow$

$\therefore$ My exceptional sol is ~~$u = C$~~

$u = (A_*\cosh(Cp_* t) + B_*\sinh(Cp_* t))\cosh(p_* x)$

General sol is $u = \{(A_*\cosh(Cp_* t) + B_*\sinh(Cp_* t))\cosh(p_* x)\} +$

$\sum_{n=1}^{\infty}(A_n\cos(Ca_n t) + B_n\sin(Ca_n t))\cos(a_n x)$ with

$\{\dots\}$ eg $\{(A_* \cosh(Cp_* t) + B_* \sinh(Cp_* t))\cosh(p_* x)\}$ only present for $h<0$

we need to apply $u(x,0)=0$ & $\frac{\partial u}{\partial t}(x,0) = I\,\delta(x-x_0)$

set $A_*, A_n = 0$ $\forall n$ to satisfy $u(x,0)=0$ $\Rightarrow$ differentiate wrt. to time t & set $t=0$ to find:

$$\frac{\partial u}{\partial t} = \{Cp_* B_* \cosh(p_* x)\} + \sum_{n=1}^{\infty} Ca_n B_n \cos(a_n x) =$$

$$\sum_n Ca_n B_n \phi_n(x) \qquad \phi_n(x) = \cos(a_n x), \quad \phi_*(x) = \cosh(p_* x)$$

sum over n includes $*$ if present

$$\left.\frac{\partial u}{\partial t}\right|_{t=0} = \sum_n Ca_n B_n \phi_n(x) = I\,\delta(x-x_0) \qquad \circledast$$

although a_n, a_* are "messy" we have orthogonality (SL {Sturm Liouville} theory) $\int_0^L \phi_n(x)\phi_m(x)\,dx = \begin{cases} 0 & n \neq m \\ M_n^{-1} & n=m \end{cases}$

take $\circledast$ multiply by ϕ_m (or ϕ_*) & integ $Ca_m B_m M_m^{-1}$

$$= \int I\,\phi_m(x)\,\delta(x-x_0)\,dx$$

$$= I\,\phi_m(x_0)$$

$$\Rightarrow B_m = \frac{I}{Ca_m M_m^{-1}}\phi_m(x_0) = \frac{I M_m}{Ca_m}\phi_m(x_0)$$

$$B_* = \frac{I}{Ca_* M_*^{-1}}\phi_*(x) \quad \text{starred case} \quad \frac{I M_*}{Cp_*}\phi_*(x_0)$$

finally $u = \frac{I}{C}\left\{\frac{M_*}{p_*}\cosh(p_* x_0)\sinh(p_* Ct)\cosh(p_* x)\right\} +$

$$\sum_{n=1}^{\infty} \frac{I}{C}\left(\frac{M_n}{a_n}\cos(a_n x_0)\sin(Ca_n t)\cos(a_n x)\right)$$

after some work: $M_* = \frac{2}{L - \frac{h}{1-h^2p_*^2}}$ $\qquad M_n = \frac{2}{L - \frac{h}{1+h^2a_n^2}}$

Thm 2.3 Let $u(x,t)$ be a sol of 2 above problem (with any of 2 four possible combinations of 2 B(s)) &

$E(t) = \int_0^L \left(\frac{1}{2}u_t^2 + \frac{C^2}{2}u_x^2\right)dx$ $\therefore dE/dt = 0$, $\therefore E(t) = \text{const}$

have eqn in 1-d $u_{tt} = C^2 u_{xx}$ what about in 3-d?

$$\frac{\partial^2 u}{\partial t^2} = C^2\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = C^2\nabla^2 u \quad \text{"vector calculus"}$$

Conservation of energy:

$E(t) = \int_{\mathbb{R}^3} (\frac{1}{2} u_t^2 + \frac{1}{2} c^2 |\nabla u|^2 \, dV$ is constant $\therefore \frac{dE}{dt} = 0$ under some conditions...

\vector calculus:/ $\nabla = (\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}) = \underline{i} \frac{\partial}{\partial x} + \underline{j} \frac{\partial}{\partial y} + \underline{k} \frac{\partial}{\partial z}$

$\text{grad} f = \nabla f = (\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z})$

$\underline{A} = (A_x, A_y, A_z)$

$\text{div} \underline{A} = \nabla \cdot \underline{A} = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$

$\nabla \cdot (f \underline{A}) = f \nabla \cdot \underline{A} + \underline{A} \cdot \nabla f$ vector calc identities

\Gauss ~~thm~~ thm / divergence thm / ostrgradsky thm:

$\int_V \nabla \cdot \underline{A} \, dV = \int_S \underline{A} \cdot \underline{n} \, dS$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$

$\nabla \cdot (f \vec{A}) = f \nabla \cdot \vec{A} + \nabla f \cdot \vec{A}$

divergence thm: $\int_V \nabla \cdot \vec{A} \, dV = \int_S \vec{A} \cdot \vec{n} \, dS$

\1 dim wave eqn to conservation of energy proof/

\thm 2.8/ $u(x,t)$ is a sol of $u_{tt} - c^2 u_{xx} = 0$

$x \in (0, l)$; $t \geq 0$; subject to BC BC1: $u(0,t) = 0$; or $u_x(0,t) = 0$

BC2: $u(l,t) = 0$, or $u_x(l,t) = 0$ $\therefore$ 4 different combinations

let $E(t) = \int_0^l (\frac{1}{2} u_t^2 + \frac{1}{2} c^2 u_x^2) dx$ $\therefore E(t) \geq 0$ $\therefore$

$\frac{dE}{dt} = 0$, ie $E(t) = \text{const}$

\proof/ $\frac{dE}{dt} = \frac{d}{dt} \int_0^l (\frac{1}{2} u_t^2 + \frac{1}{2} c^2 u_x^2) dx = \int_0^l \frac{\partial}{\partial t} (\frac{1}{2} u_t^2 + \frac{1}{2} c^2 u_x^2) dx =$

$\int_0^l u_t u_{tt} + c^2 u_x u_{xt}) dx = I_1 + I_2 = \int_0^l u_t u_{tt} dx + I_2 =$

$\cancel{I_1 + c^2}$ $I_1 + c^2 \int_0^l u_x u_{xt} dx = I_1 + I_2$

$I_2 = c^2 \int_0^l u_x u_{xt} dx = c^2 \int_0^l u_x d[u_t] =$

$c^2 [u_x u_t]_0^l - c^2 \int_0^l u_t u_{xx} dx$ $\therefore$

$\frac{dE}{dt} = c^2 [u_x u_t]_0^l + \int_0^l u_t \underbrace{(u_{tt} - c^2 u_{xx})}_{= 0 \text{ by PDE}} dx = c^2 [u_x u_t]_0^l + \int_0^l 0 dx = c^2 [u_x u_t]_0^l =$

$n=1$

$\cancel{c^2[u_x(l)u_t(l) - u_x(0)u_t(0)]}$ $c^2[u_x(l,t)u_t(l,t) - u_x(0,t)u_t(0,t)]$

$\therefore u_x(l) = 0$ for neumann

) $u_x(0) = 0$ for neumann

also: of say $u(0,t) = 0 \ \forall t \Rightarrow u_t(0,t) = 0$, similarly

$u(L,t) = 0 \Rightarrow u_t(L,0) = 0 \ \therefore$

$\frac{dE}{dt} = 0$ for any of 2 four BC $\square$

\3 dim wave eqn/ $u_{tt} - c^2(u_{xx} + u_{yy} + u_{zz}) = 0$

$\therefore u_{tt} - c^2\nabla^2 u = 0$ IC $u(\vec{x},0) = 0$ $u_t(\vec{x},0) = \psi(\vec{x})$

asymp conditions (BCs at infinity) $|\nabla u| \to 0$, $|\vec{x}| \to \infty$

\thm 2.9/ let $u(\vec{x},t)$ be a sol eq $u(\vec{x},t)$ satisfying

$|u_t(\vec{x},t)| \le A|\vec{x}|^{-p}$, $|\nabla u(\vec{x},t)| \le B|\vec{x}|^{-q}$ $\forall \vec{x}, t$ where $A>0, B>0$,

$p>0$, $q>0$ are some consts & $p+q>2$ & let

$E(t) = \int_{R^2} (\frac{1}{2}u_t^2 + \frac{1}{2}c^2|\nabla u|^2)\,dV \quad \therefore \frac{dE}{dt} = 0$

\thm 2.9/ $u_{tt} - c^2\nabla^2 u = 0$ PDE BC $|u_t| \le A|\underline{x}|^{-p}$ $(\underline{x} \equiv \vec{x})$

$\nabla u| \le B|\underline{x}|^{-q}$ $A, B, p, q > 0$; $p+q>2$

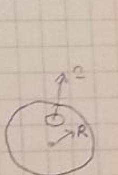
$E(t) = \int_{R^3} (\frac{1}{2}u_t^2 + \frac{1}{2}c^2(\nabla u)^2)\,dV \Rightarrow \frac{dE}{dt} = 0$

\proof/ first consider finite domain V

$V = \{\underline{x} \mid |\underline{x}| \le R\}$ $\cancel{E_V = \iiint_V (\frac{u_t^2}{2}}$ $E_V = \iiint_V (\frac{u_t^2}{2} + \frac{c^2}{2}(\nabla u)^2)\,dV \ \therefore$

$\frac{dE_V}{dt} = \iiint_V (u_t \overbrace{u_{tt}}^{c^2\nabla^2 u \text{ by PDE}} + c^2\nabla u \cdot \nabla u_t)\,dV = c^2\iiint_V (u_t\nabla^2 u + \nabla u \cdot \nabla u_t)\,dV$

$\{\nabla\cdot(\xi\underline{A}) = \xi\nabla\cdot\underline{A} + \underline{A}\cdot\nabla\xi \quad \therefore \underline{A} = \nabla u,\ \xi = u_t\}$



$= c^2\iiint_V \nabla\cdot(u_t\nabla u)\,dV$ ($\therefore$ by divergence thm:)

$= c^2\iint_S u_t \underbrace{\nabla u\cdot\bar{n}}_{\partial u/\partial r}\,dS \quad \therefore \frac{dE_V}{dt} = c^2\iint_S u_t \overbrace{u_r}^{\nabla u\cdot\bar{n}}\,dS$

$\{|\frac{dE_V}{dt}| = c^2\iint_S |u_t\nabla u\cdot\bar{n}|\,dS \ge c^2\iint_S \underbrace{|u_t|}_{\le A|x|^{-p}}\underbrace{|\nabla u\cdot n|}_{\le B|x|^{-q}}\,dS$ $S: \{\underline{x} \mid |\underline{x}| = R\}$ $\}$

$\therefore \frac{dE_V}{dt} = c^2 A B R^{-p-q} \underbrace{\iint_S dS}_{4\pi R^2}$ $\{4\pi R^2$ is ~~[illegible]~~ area of sphere$\}$

$= c^2 AB \cdot 4\pi \; R^{2-p-q} \xrightarrow[R\to\infty]{} 0 \Rightarrow \left|\frac{dE}{dt}\right| = 0$ $\square$

Kirchoffs formula / D'Alemberts 1DWE 3DWE ⊛ $u_{tt} - c^2\nabla^2 u = 0$ $\vec{x} \in \mathbb{R}^3$

$\underline{x} = (x,y,z)$ $u(\underline{x},0) = \phi(x) = 0$ $u_t(\underline{x},0) = \psi(\underline{x})$ WLOG solve at $\underline{x} = \underline{0} = (0,0,0)$

method of spherical means: $V = \{\underline{x} \mid |\underline{x}| \le R\}$ $S = \partial V = \{\underline{x} \mid |\underline{x}| = R\}$ integrate ⊛ over $\underline{x} \in V$

S = sphere

$\underbrace{\iiint_V u_{tt}\, dV}_{LHS} = \underbrace{c^2 \iiint_V \nabla^2 u}_{RHS}$ $\quad \underline{n} = \frac{\underline{x}}{R}$

by divergence thm $RHS = c^2 \iiint_V \nabla(\nabla u)\, dV = c^2 \oiint \underbrace{\nabla u \cdot \underline{n}}_{\partial u/\partial r} dS$

Spherical coords $(x,y,z) \mapsto (r,\theta,\phi)$

$dV = r^2 \underline{\sin\theta\, d\theta\, d\phi}\, dr$

$dS = R^2 \underbrace{\sin\theta\, d\theta\, d\phi}_{= d\Omega}$ "element of solid angle"

$\therefore dV = r^2 d\Omega\, dr$; $dS = R^2 d\Omega$

$(V: r \le R)$

in this notation: $LHS = \int_0^R \oiint_\Omega u_{tt}(r,\theta,\phi,t)\, r^2\, dr\, d\Omega$

$= RHS = c^2 \oiint_\Omega u_r(R,\theta,\phi,t) R^2\, d\Omega$

define spherical means: $\bar{u}(\rho,t) = \frac{1}{4\pi\rho^2} \oiint_{|x|=\rho} u(\underline{x},t)\, dS =$

$\frac{1}{4\pi\rho^2} \oiint u(r,\phi,\theta,t)\, \rho^2\, d\Omega = \frac{1}{4\pi} \oiint u(\rho,\theta,\phi,t)\, d\Omega$ use

this in the integral form

$LHS = \frac{\partial^2}{\partial t^2} \int_0^R \Big[\underbrace{\oiint u\, d\Omega}_{4\pi\bar{u}(r,t)}\Big] r^2 dr = 4\pi \frac{\partial^2}{\partial t^2} \int_0^R \bar{u}(r,t)\, r^2\, dr$

$RHS = c^2 R^2 \frac{\partial}{\partial r} \underbrace{\oiint u\, d\Omega}_{\leftarrow 4\pi\bar{u}(r,t)} \Big|_{r=R^2}$

$= 4\pi c^2 R^2 \left(\frac{\partial\bar{u}}{\partial r}\right)\Big|_{r=R}$ $\therefore$ $LHS = RHS$ $\quad \int_0^R \bar{u}_{tt}(r,t)\, r^2\, dr = c^2 R^2 \bar{u}_r(R,t)$

integro-differential eqn $\to$ PDE by $\frac{\partial}{\partial R}$:

$\bar{u}_{tt}(R,t) R^2 = c^2 (R^2 \bar{u}_{rr}(R,t) + 2R\bar{u}_r(R,t))$ $\quad (R \to r)$

$r^2 \bar{u}_{tt}(r,t) = c^2 \left[r^2 \bar{u}_{rr}(r,t) + 2r\bar{u}_r(r,t) \right]$ PDE for $\bar{u}(r,t)$

dropping args $\quad r\bar{u}_{tt} = c^2 (r\bar{u}_{rr} + 2\bar{u}_r)$ ⊛ $\therefore$

let $r\bar{u}(r,t) = v(r,t)$ $\therefore$ $\bar{u}(r,t) = \frac{1}{r} v(r,t) \Longrightarrow$

$v_{tt} = c^2 v_{rr}$ (1WE)

this PDE defined $r > 0$

1) need BC at $r = 0$ $\quad v(0,t) = ?$

reminder $v(r,t) = r\bar{u}(r,t)$ $\quad \bar{u}(r,t)$ is bounded $\Rightarrow v(0,t) = 0$

~~summary $2x > ct$ $v(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-c$~~

d'Alembert's formula: $v(r,t) =$

$\frac{1}{2}\left[\tilde{\phi}(ct+r) - \tilde{\phi}(ct-r)\right] + \frac{1}{2c}\int_{ct-r}^{ct+r} \tilde{\Psi}(s)\,ds$ where $\tilde{\phi}(r) = v(r,0) =$

$r\underbrace{\bar{u}(r,0)}_{\frac{1}{4\pi r^2}\oiint \underbrace{\phi(\underline{x})}_{=0} d\underline{x}} = 0$

$\tilde{\Psi}(r) = v_t(r,0) = r\bar{u}_t(r,0) = 0 = r\bar{\Psi}(r)$

$\bar{\Psi}(r) = \frac{1}{4\pi r^2}\oiint_{|\underline{x}|=r} \Psi(\underline{x})\,dS \Rightarrow \bar{u}(r,t) = \frac{1}{r}v(r,t) =$

$\frac{1}{2cr}\int_{ct-r}^{ct+r} s\bar{\Psi}(s)\,ds = \frac{\text{Numerator}(r)}{\text{Denominator}(r)}$ $\quad \{\bar{u}(0,t) = ?\}$

$\text{Numerator}(r) = \int_{ct-r}^{ct+r} s\bar{\Psi}(s)\,ds$ $\quad \text{Denominator}(r) = 2cr$ $\therefore$

by continuity: $\bar{u}(0,t) = \lim_{r\to 0}\frac{\text{Num}(r)}{\text{Den}(r)} = \lim_{r\to 0}\frac{\text{Num}'(r)}{\text{Den}'(r)}$ $\quad$ {L'opitals rule}

$\text{Den}'(r) = 2c$ $\quad$ ~~$\text{Num}'(r) = s\bar{\Psi}(s)\big|_{s=ct+r}\frac{\partial}{\partial r}(ct+r)$~~

$\text{Num}'(r) = s\bar{\Psi}(s)\big|_{s=ct+r}\frac{d}{dr}(ct+r) - s\bar{\Psi}(s)\big|_{s=ct-r}\frac{d}{dr}(ct-r) =$

$(ct+r)\bar{\Psi}(ct+r) + (ct-r)\bar{\Psi}(ct-r)$

$\lim_{r\to 0}\text{Num}'(r) = 2ct\Psi(ct)$ $\quad \therefore \bar{u}(0,t) = \frac{2ct\bar{\Psi}(ct)}{2c} = t\bar{\Psi}(ct) =$

$t\frac{1}{4\pi c^2 t^2}\oiint_{|\underline{x}|=ct} \Psi(\underline{x})\,dS = \bar{u}(0,t) = u(\underline{0},t)$

$u(0,t) = \frac{1}{4\pi c^2 t}\oiint_{|\underline{x}|=ct} \overbrace{u_t(\underline{x},0)}^{\Psi}\,dS$

2) to find sol at any pt $\underline{x}_*$ $\quad$ let $\underline{x} = (x_1, x_2, x_3)$ $\quad \tilde{\underline{x}}(\tilde{x}_1, \tilde{x}_2, \tilde{x}_3)$

$\tilde{\underline{x}} = \underline{x} - \underline{x}_*$ $\quad \left|\begin{array}{l}\underline{x} = \underline{x}_* \\ \tilde{\underline{x}} = \underline{0}\end{array}\right.$ $\quad \therefore \nabla_{\underline{x}} = \nabla_{\tilde{\underline{x}}}$ $\quad u(\underline{x},t) = \tilde{u}(\tilde{\underline{x}},t)$

$\tilde{u}_{tt} = c^2\nabla^2_{\tilde{\underline{x}}}\tilde{u}$ $\quad \tilde{u}(\underline{0},t) = \frac{1}{4\pi c^2 t}\oiint_{|\tilde{\underline{x}}|=ct} \tilde{u}_t(\tilde{\underline{x}},0)\,dS \Longrightarrow$

$u(\underline{x}_*, t) = \frac{1}{4\pi c^2 t} \oiint_{|\underline{x}-x_*|=ct} u_t(\underline{x}, 0)\, ds$ is Kirchhoff's formula

\chapter 3/ \parabolic eqns/

\uniqueness/

\thm 3.1/ PDE $u_t - Du_{xx} = f(x,t)$ in Z domain $\underline{x} \in (0, L)$, $t \in (0, \infty)$ subject to $u(x,0) = \phi(x)$ &

$u(0,t) = \psi_0(t)$, $u(L,t) = \psi_L(t)$, $D > 0$, $L > 0$ & $f(x,t)$, $\phi(x)$

$\psi_0(x)$ & $\psi_L(x)$ are given funcs, has a unique sol

\thm 3.1/ $u_t - Du_{xx} = f(x,t)$ PDE $D > 0$ $\partial_t = \frac{\partial}{\partial t}$

$x \in (0, L)$ $t \in (0, \infty)$ IC: $u(x,0) = \phi(x)$

BC: $u(0,t) = \psi_0(t)$; $u(L,t) = \psi_L(t) \Rightarrow$

sol is unique

\proof/ suppose u_1, u_2 is sols

PDE $\partial_t u_1 - D\partial_x^2 u_1 = f(x,t)$ PDE $\partial_t u_2 - D\partial_x^2 u_2 = f(x,t)$

IC $u_1(x,0) = \phi(x) = u_2(x,0)$

BC1 $u_1(0,t) = \psi_0(t) = u_2(0,t)$

BC2 $u_2(L,t) = \psi_L(t) = u_2(L,t)$

consider $w(x,t) = u_1(x,t) - u_2(x,t)$ need $w(x,t) \equiv 0$ $\forall (x,t)$

$\Leftrightarrow (u_1 \equiv u_2)$

PDE1 − PDE2: $w_t - Dw_{xx} = 0$

IC $w(x,0) = 0$ BC: $w(0,t) = 0$; $w(L,t) = 0$ need to show

$w(x,t) \equiv 0$

consider $E(t) = \int_0^L \frac{1}{2} w^2(x,t)\, dx$ then $\frac{dE}{dt} = \frac{1}{2}\int_0^L \frac{\partial}{\partial t}(w^2(x,t))\, dx =$

$\int_0^L w w_t\, dx$ (product rule) $= D\int_0^L w w_{xx}\, dx$ (by PDE) $=$

$D[w w_x]_0^L - D\int_0^L w_x^2\, dx$ (by parts) $=$

$D[\underbrace{w(L,t)}_{=0} w_x(L,t) - \underbrace{w(0,t)}_{=0} w_x(0,t)]$ (by BC)

$= -D\int_0^L w_x^2\, dx$ $\therefore \frac{dE}{dt} = -D\int_0^L w_x^2\, dx \leq 0$ $(D > 0)$ (*)

$\{$ note $E(t) \geq 0$ $\because \frac{dE}{dt} \leq 0$ $E[0] = \frac{1}{2}\int_0^L \underbrace{w^2(x,0)}_{=0 \text{ by IC}}\, dx \Rightarrow E(t) = 0 \}$ $\therefore$

$\frac{1}{2}\int_0^l w^2(x,t)\,dx = 0 \implies w(x,t) \equiv 0 \quad \forall (x,t)$

$\implies u_1 = u_2$ $\square$

thm 3.4 / wrt norm $L^2([0,l])$ $u_1(x,t)$ & $u_2(x,t)$ are sols

$\int_0^l [u_1(x,t) - u_2(x,t)]^2\,dx \le \int_0^l [\phi_1(x) - \phi_2(x)]^2\,dx \quad \forall t > 0$

thm 3.4 / u_1, u_2 : sols

same PDE, BC, different IC:

$u_1(x,0) = \phi_1(x)$: $u_2(x,0) = \phi_2(x) \implies$

$\int_0^l (u_1(x,t) - u_2(x,t))^2\,dx \le \int_0^l (\phi_1(x) - \phi_2(x))^2\,dx$

proof / same until (*) $E = \frac{1}{2}\int_0^l (u_1 - u_2)^2\,dx = \frac{1}{2}\int_0^l w^2\,dx$

$\frac{dE}{dt} = -D\int_0^l w_x^2 \le 0$ (*) $E(0) = \frac{1}{2}\int_0^l (u_1(x,0) - u_2(x,0))^2\,dx =$

$\frac{1}{2}\int_0^l (\phi_1(x) - \phi_2(x))^2\,dx$ $\quad \frac{dE}{dt} \le 0 \ \forall t \implies E(t) \le E(0) \quad (t \ge 0)$

that is $\frac{1}{2}\int_0^l (u_1(x,t) - u_2(x,t))^2\,dt \le \frac{1}{2}\int_0^l (\phi_1(x) - \phi_2(x))^2\,dx$ $\square$

Ex 3.5 / $u_t = Du_{xx}$ $x \in (-l, l)$, $t > 0$ $u(x,0) = \phi(x)$,

$u(-l,t) = u(l,t)$ $\quad u_x(-l,t) = u_x(l,t)$

Ex 3.5 / $u_t = Du_{xx}$ $x \in (-l, l)$; $t \ge 0$ $u(-l,t) = u(l,t)$

$u_x(-l,t) = u_x(l,t)$, $u(x,0) = \phi(x)$

Sol / sep of variabs ① $u = X(x)T(t)$, sat PDE + BC

② LC there of ③ choose const of LC to sat Z IC

①: $u = X(x)T(t)$ $\quad X(x)T'(t) = DX''(x)T(x)$ $\quad \therefore \times \frac{1}{DXT}$:

$\frac{T'(t)}{DT(t)} = \frac{X''(t)}{X(x)} = -\lambda \implies$

$X''(x) + \lambda X(x) = 0$ $\quad X(-l) = X(l)$; $X'(-l) = X'(l)$

$T'(t) = -\lambda D T(t)$

$X'' + \lambda X = 0$; $\quad \left| X(-l) = X(l) \quad X'(-l) = X'(l) \right.$

look for sols $X(x) = e^{mx}$

$(m^2 + \lambda)X(x) = 0$; $X \ne 0$

$m^2 + \lambda = 0$: $\implies m_{1,2} = \pm\sqrt{-\lambda}$: ① $-\lambda > 0$: $\lambda < 0$: $\sqrt{-\lambda} = p > 0$

$\Longrightarrow$ GS $X(x) = C_1 e^{px} + C_2 e^{-px} = A\cosh(px) + B\sinh(px)$

BC1: $X(-L) = X(L)$

$A\underbrace{\cosh}_{\text{even}}(-pL) + B\underbrace{\sinh}_{\text{odd}}(-pL) = A\cosh(pL) + B\sinh(pL)$

$\Longleftrightarrow 2B\underbrace{\sinh(pL)}_{>0} = 0 \quad \Longrightarrow B = 0$

so $X(x) = A\cosh(px)$

$X'(x) = Ap\sinh(px)$

BC2: $X'(-L) = X'(L)$

$Ap\underbrace{\sinh}_{\text{odd}}(-pL) = Ap\sinh(pL)$

$\Longleftrightarrow 2Ap\sinh(pL) = 0 \Longrightarrow A = 0$

$\Longrightarrow X(x) \equiv 0 \qquad (A = B = 0)$

② $-\lambda = 0$; $\sqrt{-\lambda} = 0$; $m_{1,2} = 0$

$X(x) = A + Bx$

BC1: $X(-L) = X(L) \qquad \therefore A - BL = A + BL \quad \therefore -BL = BL \therefore$

$2BL = 0 \Longrightarrow B = 0 \quad \therefore$

$X(x) = A \qquad , X'(x) = 0$ | $\lambda = 0$ is eval

BC2: $X'(-L) = X'(L)$ | $X(x) = 1$ is efuncs

③ $-\lambda < 0$: $\lambda > 0$: $\sqrt{-\lambda} = i\sqrt{\lambda} = iq$:

$q = \sqrt{\lambda} > 0$; $m_{1,2} = \pm iq \qquad \lambda = q^2$

GS: $X(x) = A\cos(qx) + B\sin(qx)$

BC1: $X(-L) = X(L) \qquad A\cos(qL) + B\sin(qL) = A\cos(qL) + B\sin(qL)$

$\therefore A\cos(qL) - B\sin(qL) = A\cos(qL) + B\sin(qL) \quad \therefore$

$2B\sin(qL) = 0 \quad \therefore$

$B \neq 0$ possible is $\sin(qL) = 0 \quad \therefore$

$qL = n\pi$: $n \in \mathbb{Z}$

$q > 0 \Longrightarrow q = \frac{n\pi}{L}$; $n \in \mathbb{Z}_+ = \{1, 2, 3, \dots\}$

BC2: $X'(x) = qA\sin(qx) - qB\cos(qx)$

$X'(-L) = X'(L) \quad \therefore$

$qA\sin(-qL) - qB\cos(-qL) = qA\sin(qL) - qB\cos(+qL) \quad \therefore$

$-qA\sin(ql) - qB\cos(ql) = qA\sin(ql) - qB\cos(ql) \iff$

$2qA\sin(ql) = 0$

$A \neq 0$ possible is $\sin(ql) = 0$;

$q = \frac{n\pi}{l}$, $n \in \mathbb{Z}_+$ $\{\therefore B \neq 0$ & $A \neq 0\}$

So for $q = q_n = \frac{n\pi}{l}$ (evals)

$X(x) = A\cos(q_n x) + B\sin(q_n x)$

So 2 efuncs for every q_n (double eval)

$X_n^{(1)}(x) = \cos(q_n x)$; $X_n^{(2)}(x) = \sin(q_n x)$

& remainder $q_0 = 0$

$X_0(x) = 1$

$\lambda_0 = 0$, $\lambda_n = (n\pi/l)^2$

For $T(t)$: $T(t) = e^{-\lambda D t}$

For $\lambda = \lambda_n$ $T_n(t) = \exp(-\lambda_n D t) = \exp\left(-\frac{n^2\pi^2 D t}{l^2}\right)$

(2) Step 2: (Linear Combination:) LC:

$u(x,t) = A_0 + \sum_{n=1}^{\infty}(A_n\cos(q_n x) + B_n\sin(q_n x))\exp(-q_n^2 D t)$ $\quad | \; q_n = \frac{n\pi}{l}$

solves PDE + BC1 + BC2

(3) Choose A_0, A_n, B_n to satisfy IC

$u(x,0) = \phi(x)$

$A_0 + \sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{n\pi x}{l}\right) + B_n\sin\left(\frac{n\pi x}{l}\right)\right) = \phi(x)$

- Fourier series;

$A_0 = \frac{1}{2l}\int_{-l}^{l}\phi(x)\,dx$ {thought of as average val of the func}

$A_n = \frac{1}{l}\int_{-l}^{l}\phi(x)\cos\left(\frac{n\pi x}{l}\right)dx$ {similar but multiply by corres efuncs}

$B_n = \frac{1}{l}\int_{-l}^{l}\phi(x)\sin\left(\frac{n\pi x}{l}\right)dx$

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[\int_{-\infty}^{\infty} f(s)e^{-iks}\,ds\right]e^{ikx}\,dk$ write:

$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}F(k)e^{ikx}\,dx$ is Fourier inversion formula or Fourier integral & $F(k) = \int_{-\infty}^{\infty} f(s)e^{-iks}\,ds$, where $F(k)$ is called the Fourier transform of $f(x)$

Solving 1D HE by FT §3.4.2 #4

$u_t = D u_{xx};\ x \in \mathbb{R},\ t \geq 0 \qquad u(x,0) = \phi(x) \qquad D > 0 \qquad$ FT&Z Sol

Sol $u(x,t) = \frac{1}{2\pi}\int V(k,t) e^{ikx}\, dk$ 1)

$V(k,t) = \int u(x,t)\, e^{-ikx}\, dx \qquad \left(\int \equiv \int_{-\infty}^{\infty} \right)$

Sub into PDE:

LHS $u_t = \frac{1}{2\pi}\int V_t(k,t) e^{ikx}\, dk$

RHS $= D u_{xx} = \frac{D}{2\pi}\int V(k,t) \frac{\partial^2}{\partial x^2}(e^{ikx})\, dx =$

$\frac{1}{2\pi}\int (-Dk^2)\, V(k,t)\, e^{ikx}\, dk$

LHS $-$ RHS $= 0 \qquad \frac{1}{2\pi}\int (V_t + Dk^2 V) e^{ikx}\, dk = 0$

$= \frac{1}{2\pi}\int Z(k,t) e^{ikx}\, dk = 0 \ \forall x, \forall t$

$\Rightarrow Z(k,t) = 0 \quad \forall k, t$

$V_t(k,t) + Dk^2 V(k,t) = 0$

$V_t = Dk^2 V$ | dependent on k as a param $\Rightarrow$ ODE

Separable ODE 1st order $\frac{dU}{U} - Dk^2\, dt$

$\ln|U| = -Dk^2 t + A \quad ; A = A(k)$

$U = Be^{-Dk^2 t} \qquad B = B(k)$

$U(k,t) = B(k)e^{-Dk^2 t} \qquad$ at $t=0 \quad U(k,0) = B(k)$

$\Rightarrow U(k,t) = U(k,0)\, e^{-Dk^2 t} \qquad U(k,0)$ found from $u(x,0) = \phi(x)$

$U(k,0) = \int u(x,0) e^{ikx}\, dx = \int \phi(x) e^{-ikx}\, dx = \int \phi(s) e^{-iks}\, ds$

So $U(k,t) = \left[\int \phi(s) e^{-iks}\, ds\right] e^{-Dk^2 t}$

$u(x,t) = \frac{1}{2\pi}\int U(k,t) e^{ikx}\, dk = \frac{1}{2\pi}\int \left\{ \left[\int \phi(s) e^{-iks}\, ds\right] e^{-Dk^2 t} \right\} e^{ikx}\, dx$

$= \frac{1}{2\pi}\iint \phi(s) e^{-iks} e^{ikx} e^{-Dk^2 t}\, dk\, ds =$

$\int \phi(s) \underbrace{\left[\frac{1}{2\pi}\int e^{i(x-s)k} e^{-Dk^2 t}\, dk \right]}_{K(x-s,t)} ds$

$= \int \phi(s) K(x-s,t)\, ds$

$K(x-s,t) = \frac{1}{2\pi} J(\beta, Q)$, $\beta = x - s$, $Q = Dt$

$J(\beta, Q) = \int_{-\infty}^{\infty} e^{i\beta k} e^{-Qk^2} dk$

(can be also done with Cauchy residue thm)

consider Q = fixed: J as func of β,

$\frac{\partial J}{\partial \beta} = \int (ik) e^{i\beta k} e^{-Qk^2} dk = \frac{i}{2} \int e^{i\beta k} e^{-Q(k^2)} d(k^2) =$ (by parts)

$\frac{-i}{2Q} \int e^{i\beta k}\, d\left[e^{-Qk^2}\right] = \cancel{\frac{i}{2Q}\int}$

$\underbrace{\left[-\frac{i}{2Q} e^{i\beta k} e^{-Qk^2}\right]_{k=-\infty}^{\infty}}_{=0} + \frac{i}{2Q} \int e^{-Qk^2} d\left[e^{i\beta k}\right] = \frac{i}{2Q} \int e^{-Qk^2} i\beta e^{i\beta k} dk$

$= \frac{-\beta}{2Q} \underbrace{\int e^{-Qk^2} e^{i\beta k} dk}_{J(\beta, Q)}$

So $\frac{\partial J(\beta, Q)}{\partial \beta} = -\frac{\beta}{2Q} J(\beta, Q)$ | ODE for $J = J(\beta)$

Seperable 1st order ODE, $\frac{dJ}{J} = -\frac{\beta}{2Q} d\beta$

$\ln|J| = -\frac{\beta^2}{4Q} + A$ $(A = A(Q))$

$J = B e^{-\beta^2/4Q}$ $B = B(Q)$ $B(Q) = ?$

From $J(\beta, Q)$ at $\beta = 0$

$J(\beta, Q) = J(0, Q) e^{-\beta^2/4Q}$ but

$J(0, Q) = \int_{-\infty}^{\infty} e^{-Qk^2} dk \Big|_{k = \frac{z}{\sqrt{Q}}} = \frac{1}{\sqrt{Q}} \underbrace{\int_{-\infty}^{\infty} e^{-z^2} dz}$ < "Gaussian integral" $= \sqrt{\pi}$

$\therefore J(0, Q) = \sqrt{\pi/Q}$ so

$J(\beta, Q) = \sqrt{\pi/Q}\, \exp\left(-\frac{\beta^2}{4Q}\right)$

$k(x-s, t) = \frac{1}{2\pi} J(x-s, Dt) = \frac{1}{2\pi} \sqrt{\frac{\pi}{Q_{\leftarrow Dt}}} \exp\left(-\frac{(x-s)^2}{4Dt}\right)$

& finally $u(x,t) = \frac{1}{2\sqrt{\pi D t}} \int_{-\infty}^{\infty} \phi(s) e^{-(x-s)^2/4Dt} ds$

\Ex 4.1/ charac coords elliptic eqn

$x u_{xx} + u_{yy} + \frac{1}{2} u_x = 0 \quad x > 0$ to canonical form 1)

\Ex 4.1/ bring to canonical form

$x u_{xx} + u_{yy} + \frac{1}{2} u_x = 0 \quad (x > 0)$

\Sol/ $a u_{xx} + 2b u_{xy} + c u_{yy} = f$ ∴

$a = x, \ b = 0, \ c = 1$ ∴

$b^2 - ac = -x < 0 \quad (x > 0)$ ∴ elliptic

Charac eqn: $a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0$ ∴

$x\left(\frac{dy}{dx}\right)^2 + 1 = 0$ ∴ $\frac{dy}{dx} = \pm i x^{-1/2}$ ∴

explicit: $y = \pm 2 i x^{1/2} + C_1$

implies $A = \underbrace{y + 2 i x^{1/2}}_{\xi}$, $B = \underbrace{y - 2 i x^{1/2}}_{\eta}$

∴ take $\mu = \mathrm{Re}(\xi) = \frac{1}{2}(\xi + \eta) = y$

$\nu = \mathrm{Im}(\xi) = \frac{1}{2i}(\xi - \eta) = 2x^{1/2}$

$(x, y) \mapsto (\mu, \nu)$

$\frac{\partial}{\partial y} \equiv \frac{\partial}{\partial \mu}$; $u_x = u_\nu \nu_x \quad (+ u_\mu \mu_x = 0)$

$= x^{-1/2} u_\nu$

~~So~~ $u_{xx} = (x^{-1/2})^2 u_{\nu\nu} - \frac{1}{2} x^{-3/2} u_\nu$

$u_{yy} = u_{\mu\mu}$

Sub into PDE $x u_{xx} + u_{yy} + \frac{1}{2} u_x = u_{\nu\nu} - \frac{1}{2} x^{-1/2} u_\nu + u_{\mu\mu} + \frac{1}{2} x^{-1/2} u_\nu$

$= 0$ ∴ $u_{\nu\nu} + u_{\mu\mu} = 0$ is canonical form

\Thm 4.2/ u satisfies poissons eqn $u_{xx} + u_{yy} = f(x, y)$ in a bounded domain D with piecewise smooth boundary ∂D & Z normal $\vec{n}$, subject to Z boundary ~~eqn~~ cond $A(x,y) u + B(x,y) u_n = g(x,y)$ for $(x,y) \in \partial D$ st $A > 0, B \geq 0$ everywhere on ∂D then is a sol exists, it is unique thm A.2 / $\nabla^2 u = f(\underline{r})$: $\underline{r} \in D$ ∂D : pw smooth,

$A(\underline{r}) + u + B(\underline{r}) \quad (\bar{n}\cdot\nabla)u = g(\underline{r})$, $r \in \partial D$

& $A > 0$; $B \geq 0$; $\forall r \in \partial D$ $\Rightarrow$ then $u(\underline{r})$ is unique

proof / let $u_1(\underline{r})$, $u_2(\underline{r})$ be sols, let

$\omega(\underline{r}) = u_1(\underline{r}) - u_2(\underline{r})$ need to prove $\omega(\underline{r}) \equiv 0 \quad \forall r \in D$

linearity of PDE + BC $\Rightarrow$

$\nabla^2 \omega = 0$; $r \in D$

$A\omega + B(n\cdot\nabla)\omega = 0$ $r \in \partial D$

$\therefore$ by divergence thm: $\int_D \nabla\cdot\underline{F}\, d\underline{r} = \oint_{\partial D} \underline{F}\cdot\underline{n}\, ds$

"vector product rule" $\nabla\cdot(\underline{\Psi}\,\underline{F}) = (\nabla\underline{\Psi})\cdot\underline{F} + \underline{\Psi}(\nabla\cdot\underline{F})$

$\nabla^2\omega = 0$ $\nabla\cdot(\nabla\omega) = 0$

let vector field $\underline{F} = \omega\nabla\omega$

divergence thm : LHS $= \int_D \nabla\cdot(\omega\nabla\omega)\, d\underline{r} = \oint_{\partial D} \omega\nabla\omega\cdot\underline{n}\, ds =$ RHS

for "vector product rule" $\Psi = \omega$ $\underline{F} = \nabla\omega$

LHS $= \int_D \nabla\cdot(\omega\nabla\omega)\, d\underline{r} = \int_D \nabla\omega\cdot\nabla\omega\, dr + \int_D \omega\underbrace{(\nabla\cdot\nabla\omega)}_{=0} dr$ $= \nabla^2\omega$ by

PDE

$= \int_D (\nabla\omega)^2\, dr \geq 0$

RHS $= \int_D \omega\nabla\omega\cdot\underline{n}\, ds$

| $A > 0$ $B \geq 0$

$A\omega + B(n\cdot\nabla)\omega = 0$ $\circledast$

$A \neq 0$ $\omega = -\frac{B}{A}(n\cdot\nabla)\omega$

$\therefore$ so RHS $= -\int_D \frac{B(r)}{A(r)}(n\nabla\omega)^2\, dS \leq 0$ $\frac{B}{A} \geq 0$!

$\therefore$ LHS ≥ 0, RHS ≤ 0 $\therefore$ ~~RP~~ LHS = RHS $\Rightarrow$

LHS = RHS = 0 $\therefore$ LHS $= \int_D (\nabla\omega)^2\, dr = 0 \Rightarrow \nabla\omega \equiv 0$ $\therefore$

then $\circledast$ $\omega(\underline{r}) = 0$, $r \in \partial D$ $\Rightarrow$ $\omega(\underline{r}) = 0$, $r \in D$ $\square$

§4.2.2 \nonuniqueness/ \for Neumann problem/

\Corollary 4.3/ if instead $A=0,\ B\neq 0$ $\quad n\cdot \nabla(\omega)=0 \mid \underline{r}\in\partial D$

$RHS=\oint_{\partial D} \omega \underbrace{(n\cdot\nabla)}_{=0}\omega\, ds$

$LHS\geq 0$, $RHS=0$ $\quad LHS=RHS$ $\quad\therefore LHS=RHS=0$ $\therefore$

$\int_D (\nabla\omega)^2\, d\underline{r}=0 \Rightarrow \nabla\omega(\underline{r})\equiv\underline{0}$ $\quad r\in D$

$\omega(\underline{r})=\text{const}=C \Rightarrow \nabla\omega=0$ $\quad r\in\partial D$

verify: $\omega(\underline{r})=C$ $\quad \nabla^2\omega=0$ $\quad (D)$

$(\bar{n}\cdot\nabla\omega)=0$ $\quad(\partial D)$ $\quad$ is a sol

So in Z original problem $\quad \nabla^2 u=f$ $\quad (\underline{r}\in D$

$\nabla u\cdot\underline{n}=g$ $\quad(\underline{r}\in\partial D)$

if $u=u_1(\underline{r})$ is a sol $\Rightarrow u=u_2(\underline{r})=u_1(\underline{r})+C$

is also a sol $\forall C\in\mathbb{R}$

"unique up to additive const"

\Corollary/ when $f=0$ & D is a <u>connected</u> set, then Z BC is $\partial w/\partial n=0$. still have $\iint_D |\nabla w|^2\,dx\,dy=0$ & consequently $\nabla w=0$ & $w=$ const but Z conclusion $w=0$ on ∂D doesn't follow $\therefore$ (4.1) is no longer in force $\therefore$ for this BC, Z sol u, if exists, is not unique, but only unique up to an additive const

\Def 4.4/ Func $u(\vec{r})$ is called harmonic in a domain D if $\nabla^2 u=0$ in D ~~we RtP~~

\thm 4.5/ \Mean val property of harmonic funcs/

Suppose that $u(x,t)$ is harmonic in a disk of radius $r>0$ & center at (x_0,y_0) that is $D_{(x_0,y_0),r}=\{(x,y) \mid (x-x_0^2)+(y-y_0^2)\leq r^2$ $\therefore$ $u(x_0,y_0)=\frac{1}{2\pi r}\int_{\partial D} u(x,y)\,ds$ that is Z val of u at Z centre of Z disk equals its average over Z boundary of Z disk

Corollary 4.6 / if $\nabla^2 u(x,y) = 0$ within a disk, then the val at the center of the disk

Thm 4.5 / $D: (x-x_0)^2 + (y-y_0)^2 \leq R$

$\nabla^2 u = 0 \quad \forall \underline{x} \in D \quad \underline{x} = (x, y)$

$\Rightarrow u(\underline{x}_0) = \frac{1}{2\pi R} \int_{\partial D} u(\underline{x})\, dl$

Corollary 4.6 / $u(\underline{x}_0) = \frac{1}{\pi R^2} \int_D u(\underline{x})\, d^2\underline{x}$

proof / WLOG $\underline{x}_0 = (0,0)$

(otherwise, $\underline{x}_0 = (x_0, y_0)$ then $\tilde{x} = x - x_0$; $\tilde{y} = y - y_0$)

then $u(0) \stackrel{?}{=} \frac{1}{\pi R^2} \iint_{x^2+y^2 \leq R} u(x,y)\, dx\, dy$ (polar coords $x = r\cos\theta$, $y = r\sin\theta$)

[illegible] $dx\,dy$

$dx\,dy = r\,dr\,d\theta$

$u(0,0) \stackrel{?}{=} \frac{1}{\pi R^2} \int_{\theta=0}^{2\pi} \int_{r=0}^{R} u(r,\theta)\, r\, dr\, d\theta =$

$\frac{1}{\pi R^2} \int_{r=0}^{R} \underbrace{\left[\int_0^{2\pi} u(r,\theta)\, d\theta\right]}_{\text{by mean value theorem}} r\, dr = \frac{2\pi u(0,0)}{\pi R^2} \underbrace{\int_{0}^{R} r\, dr}_{= R^2/2}$

$\frac{2\pi u(0,0)}{\pi R^2} \cdot \frac{R^2}{2} = u(0,0)$ ✓ □

Thm 4.7 / "maximum principle" / D connected domain (bounded)

$\nabla^2 u = 0$, $\underline{x} \in D$ & let maximum val $M = \max_{\underline{x} \in D \cup \partial D} u(\underline{x})$

then either $u(\underline{x}) < M \ \forall \underline{x} \in D \setminus \partial D$ or $u(\underline{x}) = M \ \forall \underline{x} \in D \cup \partial D$

proof / ① suppose $u(\underline{x}_0) = M \quad \underline{x}_0 \in D \setminus \partial D$

then $\exists$ disk D_r, of radius r centr at $\underline{x} = \underline{x}_0$

by assumption ④ $u(\underline{x}) \leq M = u(\underline{x}_0) \ \forall \underline{x} \in D_r$

② by Corollary, $u(\underline{x}_0) = \frac{1}{\pi r^2} \int_{D_r} u(\underline{x})\, d^2\underline{x} \Rightarrow$

$u(\underline{x}) = M \ \forall \underline{x} \in D_r$

③ [illegible] [illegible] $\Rightarrow u(\underline{x}) = [illegible] = M$

$\forall \underline{x} \in D$ □

in other words $u(x,y) < \operatorname{Max} u(x',y')$ $(\underline{x} \in D)$ $(\underline{x}' \in \partial D)$

Corollary 4.8 / $u(x,y)\big|_D > \operatorname{Min}_{\partial D} u(x',y')$ "minimum principle"

proof / consider thm 4.7 for $-u(x,y)$

thm 4.9 / $\nabla^2 u = f(x,y)$ $(x,y) = \underline{x} \in D$

u_1, u_2 : sols,

$u_1(\underline{x}) = g_1(\underline{x})\big|_{\underline{x} = \partial D}$ $\quad u_2(\underline{x}) = g_2(\underline{x})\big|_{\underline{x} = \partial D}$ $\Longrightarrow$

$|u_1(\underline{x}) - u_2(\underline{x})| \leq \max_{x' \in \partial D} |g_1(\underline{x}') - g_2(x')|$ $\quad \forall \underline{x} \in D$

"stability wrt boundary data"

proof / $w = u_1 - u_2$; $\nabla^2 w = 0$; by thm 4.7, corollary 4.8:

$w(\underline{x}) < \operatorname{Max}_{x' \in \partial D} (w(x'))$ $\quad (\underline{x} \in D)$

$\operatorname{Min}_{x' \in \partial D} (w(x')) \leq w(\underline{x}) \leq \operatorname{Max}_{x' \in \partial D} (w(x'))$

$|w(\underline{x})|_{\underline{x} \in D} \leq \operatorname{Max}_{x' \in \partial D} (|w(x')|)$

that is $|u_1 - u_2| \leq \operatorname{Max}_{\underline{x}' \in D} (|u_1(\underline{x}') - u_2(\underline{x}')|)$ $\quad (\underline{x} \in D)$

$= \operatorname{Max}_{x' \in D} (|g_1(x') - g_2'(x')|)$

Ex 4.1 / Solve by fourier series 2 2D Laplace eq

(PDE) $u_{xx} + u_{yy} = 0$ $\quad a \leq x \leq b$, $0 \leq y \leq L$, two variants (a & b)

of BCs (BC 1-3) $u(a,y) = u(b,y) = 0$, $u(x,0) = f(x)$,

(BC 4a) & either $u(x,L) = 0$

(BC 4b) or $u_y(x,0) = 0$

Ex 4.10 / PDE: $u_{xx} + u_{yy} = 0$ $\quad x \in (a,b)$; $y \in (0,L)$

BC1 $u(a,y) = 0$, BC2 $u(b,y) = 0$ $\quad$ BC3: $u(x,0) = f(x)$ (given)

BC4a $u(x,L) = 0$

Sol / 1°) separable sols PDE + BC1 + BC2

$u(x,y) = X(x)\,Y(y)$ sub into PDE:

$X''(x)Y(y) = -X(x)Y''(y)$ $\therefore$

$$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda = \text{const}$$

(A) $X''(x) + \lambda X(x) = 0$; $X(a) = X(b) = 0$

(B) $Y''(y) - \lambda Y(y) = 0$: (no BCs at this stage yet)

(A): Reduce to a previously seen problem: $\xi = x - a$; $x = a + \xi$;

$x \in (a,b)$ $\xi \in (0, \tilde{L}) = (0, b-a)$: $\frac{dx}{d\xi} = 1$: $X(x) = \tilde{X}(\xi)$ $\therefore$ $\frac{d\xi}{dx} = 1$ $\therefore$

$$\frac{dX}{dx} = \frac{d\tilde{X}}{d\xi}\frac{d\xi}{dx} = \frac{d\tilde{X}}{d\xi} \qquad \frac{d^2X}{dx^2} = \frac{d^2\tilde{X}}{d\xi^2} \qquad \frac{d^2\tilde{X}}{d\xi} + \lambda\tilde{X} = 0 \; ; \; \tilde{X}(0) = \tilde{X}(\tilde{L}) = 0$$

{Let $\cdot$ $t'' + \lambda t = 0$ $t(0) = t(L) = 0$ seen before}

$\phi_n = \sin\left(\frac{n\pi x}{L}\right)$ $\lambda_n = \left(\frac{n\pi}{L}\right)^2$ $\Rightarrow \tilde{X}_n(\xi) = \sin\left(\frac{n\pi\xi}{\tilde{L}}\right)$ $\lambda_n = \left(\frac{n\pi}{\tilde{L}}\right)^2$

$X_n(x) = \sin\left(\frac{n\pi}{\tilde{L}}(x-a)\right)$

Let $k_n = \frac{n\pi}{\tilde{L}} = \frac{n\pi}{b-a}$

$X_n(x) = \sin(k_n(x-a))$: $\lambda_n = k_n^2$ $n \in \mathbb{Z}_+ = \{1, 2, 3, \dots\}$

(B) $Y''(y) - k_n^2 Y(y) = 0$

G.S.: $Y(y) = C_1 e^{k_n y} + C_2 e^{-k_n y} = A\cosh(k_n y) + B\sinh(k_n y) = Y_n(y)$

$u(x,y) = X_n(x)Y_n(y)$ solves PDE + BC1 + BC2

(2) $u(x,y) = \sum_{n=1}^{\infty} X_n(x) Y_n(y)$ also solves that

$$= \sum_{n=1}^{\infty} \sin(k_n(x-a))\left[C_{1,n}e^{k_n y} + C_{2,n}e^{-k_n y}\right]$$ (this is common for variant (a) and variant (b))

For variant (a), (BC4a) $u(x,L) = 0$

$Y_n(y) = C_{1,n}e^{k_n y} + C_{2,n}e^{k_n y}$ Let us choose C st $Y_n(L) = 0$

$C_{1,n}e^{k_n L} + C_{2,n}e^{-k_n L} = 0$ $C_{1,n} = -\frac{1}{2}\tilde{A}_n e^{-k_n L}$ $C_{2,n} = \frac{1}{2}\tilde{A}_n e^{+k_n L}$

So $Y_n(y) = -\frac{1}{2}\tilde{A}_n e^{-k_n L}e^{k_n y} + \frac{1}{2}\tilde{A}_n e^{k_n L}e^{-k_n y} =$

$\frac{\tilde{A}_n}{2}\left[e^{k_n(L-y)} - e^{k_n(y-L)}\right] = \tilde{A}_n \sinh(k_n(L-y))$ So:

$$u = \sum_{n=1}^{\infty} \tilde{A}_n \sinh(k_n(L-y))\sinh(k_n(x-a)) \; \therefore$$

PDE + BC1 + BC2 + BC4a satisfied

(3) Satisfying the unknown BC3: $u(x,0) = f(x)$ =

$$u(x,0) = \sum_{n=1}^{\infty} \tilde{A}_n \sinh(k_n L) \sin(k_n(x-a)) = f(x) \quad \forall x \in (a,b)$$

$$(x-a=\xi): \quad \sum_{n=1}^{\infty} \overbrace{\tilde{A}_n \sinh(k_n L)}^{\tilde{c}_n} \sin\left(\frac{n\pi}{\tilde{L}}\xi\right) = f(a+\xi) \quad ; \xi \in (0, \tilde{L})$$

$$= \sum_{n=1}^{\infty} \tilde{c}_n \sin\left(\frac{n\pi}{L}\xi\right) = f(a+\xi) \qquad \text{(Fourier Sine-series)}$$

$$\tilde{c}_n = \frac{2}{\tilde{L}} \int_0^{\tilde{L}} f(a+\xi) \sin\left(\frac{n\pi\xi}{\tilde{L}}\right) d\xi = \frac{2}{b-a} \int_a^b f(x) \sin(k_n(x-a))\, dx$$

$\because$ odd and periodic

Alternatively may use orthogonality: $\phi_n(x)$ orthog fns on $[a,b]$

$$y(x) = \sum c_n \phi_n(x) \Rightarrow$$

$$c_n = \frac{\int_a^b \phi_n y(x) w(x)\, dx}{\int_a^b \phi_n^2(x) w(x)\, dx} \qquad \phi_n(x) = \sin(k_n(x-a)), \ w(x) = 1$$

$$\Rightarrow \int_a^b \phi_n^2(x)\, dx = \frac{b-a}{2}$$

So the answer is: ... $F_n = \frac{2}{b-a} \int_a^b f(x) \sin(k_n(x-a))\, dx$

$$u(x,y) = \sum_{n=1}^{\infty} U_n(y) \sin(k_n(x-a)), \text{ where } U_n(y) = F_n \frac{\sinh(k_n(L-y))}{\sinh(k_n L)}$$

Note that $y \in (0,L)$ so $|U_n(y)| \leq |F_n|$

Compare with predictions of max principle, thm 4.9

we can show $\|u(x,y)\|_{L^2} \leq \|u(x,0)\|_{L^2}$ (as func of x)

<u>Variant (b)</u> reuse sol: PD+BC1+BC2:

$$u(x,t) = \sum_{n=1}^{\infty} \sin(k_n(x-a)) Y_n(y) \qquad Y_n(y) = C_{1n} e^{k_n y} + C_{2n} e^{-k_n y}$$

$$= A_n \cosh(k_n y) + B_n \sinh(k_n y).$$

$k_n = \frac{\pi n}{b-a}$ BC3

BC4b:

if $Y_n(y) = 0$ at $y=0 \Rightarrow$ BC4b satisfied

So $Y_n'(0) = 0$ $\quad Y_n'(y) = k_n A_n \sinh(k_n y) + k_n B_n \cosh(k_n y)$

$Y_n'(0) = k_n B_n = 0 \Rightarrow B_n = 0$

So $u(x,y) = \sum_{n=1}^{\infty} A_n \cosh(k_n y) \sin(k_n(x-a))$ | satisfy PDE+BC1+BC2+BC4b

So for BC3: $u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin(k_n(x-a)) \Rightarrow$

$$A_n = \frac{2}{b-a} \int_a^b f(x) \sin(k_n(x-a))\, dx \quad (= F_n)$$

does this series converge?

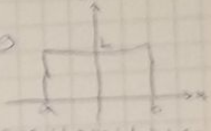
NB $k_n = \frac{\pi}{b-a} n$ as $n \to \infty$, A_n ↘; $\cosh(k_n y)$ ↗

For convergence $A_n < e^{-k_n y}$ $\forall y$ $(n \to \infty)$

remember, Fourier coeffs of function of finite number of continuous derivatives $A_n \sim n^{-k}$

$\Rightarrow A_n e^{k_n y} \sim n^{-k} e^{y k_n} \to \infty$ Cauchy ill posed

§4.3.2 Dirichlet for 2D Lap in half plane



use Fourier transform $u_{xx} + u_{yy} = 0$ $x \in \mathbb{R}$ $y \geq 0$

$u(x,0) = f(x)$; $u(\pm\infty, y) = 0$; $|u(x,+\infty)| < \infty$

$a \to -\infty$; $b \to +\infty$

Sol/ look for $u(x,y) = \frac{1}{2\pi}\int U(k,y) e^{ikx} dk$

$U(k,y) = \int u(s,y) e^{-iks} ds$ $\left(\int = \int_{-\infty}^{\infty}\right)$

Sub into PDE $u_{xx} = \frac{1}{2\pi}\int U(k,y)(-k^2) e^{ikx} dk$

$u_{yy} = \frac{1}{2\pi}\int U_{yy}(k,y) e^{ikx} dk$

$u_{xx} + u_{yy} = \frac{1}{2\pi}\int \underbrace{\left[U_{yy}(k,y) - k^2 U(k,y)\right]}_{Z(k)} e^{ikx} dk = 0$ $\forall x, y$

$\Rightarrow Z(k) \equiv 0$ $\forall k, y$ $\left\{ \text{NB } u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} U(k,y) e^{ikx} dk \right\}$

need to solve $U_{yy} - k^2 U = 0$ $(\forall k, y)$

$\Rightarrow$ de facto ODE wrt $U(y)$ with k params

$\Rightarrow U(k,y) = C_+ e^{ky} + C_- e^{-ky}$ $(k \neq 0)$

for $y \to +\infty$ we need $U(k,+\infty) \to 0$, or at least $|U(k,+\infty)| < \infty$

so we need $C_+(k) = 0$, $k > 0$ $C_-(k) = 0$; $k < 0$

$u(x,y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \left(C_+(k) e^{ky} + C_-(k) e^{-ky}\right] e^{ikx} =$

$\frac{1}{2\pi}\int_{-\infty}^{0} C_+(k) e^{ky} e^{ikx} dk + \frac{1}{2\pi}\int_{0}^{+\infty} C_-(k) e^{-ky} e^{ikx} dk$

$= \frac{1}{2\pi}\left\{ \int_{-\infty}^{0} C_+(k) e^{-y|k|} e^{ikx} dk + \int_{0}^{\infty} C_-(k) e^{-y|k|} e^{ikx} dk \right\} = u(x,y)$

$\frac{1}{2\pi}\int_{-\infty}^{+\infty} C(k) e^{-y|k|} e^{ikx} dk$, where $C(k) = \begin{cases} C_+(k), & k < 0 \\ C_-(k), & k > 0 \end{cases}$

∴ solves PDE + "homog - BCs"

\Ex 4.11/ $u_{xx}+u_{yy}=0$; $y\geq 0$ $u(x,0)=f(x)$;

$u(\pm\infty, y)=0$; $|u(x,+\infty)|<\infty$

\Sol/ by FT ... $u(x,y)=\frac{1}{2\pi}\int_{-\infty}^{\infty} C(k)e^{-|k|y}e^{ikx}\,dk$

$C(k)$ arbit func ∴ this eqn satisfies PDE + homog BC

need to choose $C(k)$ to satisfy non homog BC

$f(x)=u(x,0)=\frac{1}{2\pi}\int_{-\infty}^{\infty} C(k)e^{ikx}\,dx$ by inverse FT

$C(k)=\int_{-\infty}^{\infty} f(s)e^{-iks}\,ds$ sub into answer

$u(x,y)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{2\pi}e^{-y|k|}e^{ikx}e^{-iks}f(s)\,ds\,dk =$

$\int_{-\infty}^{\infty} f(s)\underbrace{\left[\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-s)}e^{-y|k|}\,dk\right]}_{K(x-s,y)}ds$

$=\int_{-\infty}^{\infty} f(s)K(x-s,y)\,ds$ $\{x-s=\beta\}$

$\{e^{i\alpha}=\cos\alpha+i\sin\alpha\}$

$2\pi K(\beta,y)=\int_{-\infty}^{\infty}e^{i\beta k}e^{-y|k|}\,dk$

$=\int_{-\infty}^{\infty}\underbrace{\underbrace{\cos(\beta k)}_{\text{even}}\underbrace{e^{-y|k|}}_{\text{even}}}_{\text{even}}dk+i\underbrace{\int_{-\infty}^{\infty}\underbrace{\sin(\beta k)}_{\text{odd}}\underbrace{e^{-y|k|}}_{\text{even}}dk}_{\text{odd func } \therefore -\infty \text{ to } \infty}\rightarrow = 0$

$=2\int_0^{\infty}\cos(\beta k)e^{-y|k|}\,dk=2\int_0^{\infty}\cos(\beta k)e^{-yk}\,dk=2J$

$J=\int_0^{\infty}\cos(\beta k)e^{-yk}\,dk$ {parts parts twice}

$=\frac{1}{y}-\frac{\beta^2}{y^2}\int_0^{\infty}e^{-yk}\cos(\beta k)\,dk=\frac{1}{y}-\frac{\beta^2}{y^2}J$

$J=\frac{1}{y}-\frac{\beta^2}{y^2}J \Rightarrow J=\frac{y}{y^2+\beta^2} \Rightarrow K(\beta,y)=\frac{y}{\pi}\frac{1}{y^2+\beta^2}$ $\{\beta=x-s\}$

So finally $u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(s)}{y^2+(x-s)^2}\,ds$ $\{u(x,0)=f(s)\}$

$u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{u(s,0)}{y^2+(x-s)^2}\,ds$ poisson's formula for half plane

\Ex 4.12/ 2D Laplace eqn in a disk: $u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0 \quad r\leq a$

subject to $u(a,\theta)=g(\theta)$

\Ex 4.12/ $u(r,\theta)$? PDE: $u_{rr}+\frac{1}{r}u_r+\frac{1}{r^2}u_{\theta\theta}=0$, $r\in(0,a]$ $\theta\in\mathbb{R}$

$|u(0,\theta)|<\infty \qquad u(a,\theta)=g(\theta)$ given

$u(r,\theta+2\pi)=u(r,\theta)$

\Sol/ try separ of variables

① $u(r,\theta)=R(r)\,Q(\theta)$ ∴ into PDE

$R''(r)Q(\theta)+\frac{1}{r}R'(r)Q(\theta)+\frac{1}{r^2}R(r)Q''(\theta)=0 \quad ∴ \quad \times\frac{r^2}{RQ}$:

$$\frac{r^2R''(r)+rR'(r)}{R(r)}=-\frac{Q''(\theta)}{Q(\theta)}=\lambda=\text{const} \Rightarrow$$

$r^2R''(r)+rR'(r)-\lambda R(r)=0 \quad , \quad Q''(\theta)+\lambda Q(\theta)=0$

First solve $Q(\theta)$ periodic BC

$Q(\theta)\equiv Q(\theta+2\pi)$, $\theta\in\mathbb{R} \iff \theta\in[-\pi,\pi]$, $Q(-\pi)=Q(\pi)$

$Q'(-\pi)=Q'(\pi)$

$Q''(\theta)+\lambda Q(\theta)=0$ reminder for $(-L,L)$ now $L=\pi$

- $\lambda=0$; $Q_0(\theta)=1$
- $\lambda=k^2$; $Q_n^{(1)}(\theta)=\cos(n\theta) \quad Q_n^{(2)}(\theta)=\sin(n\theta) \quad \}\, n\in\mathbb{Z}_+$

For $R(r)$ $\quad r^2R''(r)+rR'(r)-k^2R(r)=0$

$R(r)=\cancel{e^{mr}}$

reminder euler Cauchy 2nd order LODE

$a_2x^2y''+a_1xy'+a_0y=0$

we try $R(r)=r^m$ then $R'=mr^{m-1}$, $R''=m(m-1)r^{m-2}$ ∴

$r^2m(m-1)r^{m-2}+rmr^{m-1}-n^2r^m=0$

$r^m\left[m(m-1)+m-n^2\right]=0 \quad \forall n \Rightarrow m^2-m+m-n^2=0 \quad ∴ \quad m^2-n=0$

$\Rightarrow m=n$ or $m=-n$ ∴ So

$R(r)=Ar^n+Br^{-n} \quad n\in\mathbb{Z}_+$

But as $r\to 0$ $|R(0)|<\infty \Rightarrow B=0$

So $R(r)=r^n$ (or any multiple)

For $\lambda=0$; $r^2R''+rR'=0$ let $R'=S$ ∴

$r\frac{dS}{dr} + S = 0 \quad \therefore \frac{dS}{S} = -\frac{dr}{r} \quad \therefore$

$\ln|S| = C - \ln|r| \quad \therefore S = Dr^{-1} \qquad D = \pm e^{c}$ or $D = 0$ {special sol}

$S = Dr^{-1} = \frac{dR}{dr}$

$R = D\int r^{-1}\,dr = D\ln r + E$

$|R(0)| < \infty \Rightarrow R(r) = 1$ (or multiple)

② $u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right]$

③ BC at $r = a$ $\quad u(a,\theta) = g(\theta) = A_0 + \sum_{n=1}^{\infty} a^n\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right]$

= standard Fourier series for 2π periodic func

$u(r,\theta) = A_0 + \sum_{n=0}^{\infty} r^n\left[A_n\cos(n\theta) + B_n\sin(n\theta)\right]$ $\quad$ at $r = a$ $\qquad \theta \to \xi$

$u(a,\theta) = g(\theta) \quad A_0 = \frac{1}{2\pi}\int_0^{2\pi} g(\theta)\,d\theta \qquad A_n = \frac{1}{a^n}\frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos(n\theta)\,d\theta$

mean value property

$B_n = \frac{1}{a^n}\frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin(n\theta)\,d\theta$

Sub A_0, A_n, B_n into sol $\quad u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} g(\xi)\,d\xi\ +$

$\frac{1}{\pi}\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left[\cos(n\theta)\int_0^{2\pi} g(\xi)\cos(n\xi)\,d\xi + \sin(n\theta)\int_0^{2\pi} g(\xi)\sin(n\xi)\,d\xi\right]$

Swap order of integration & summation

$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} g(\xi)\left[1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left(\cos(n\theta)\cos(n\xi) + \sin(n\theta)\sin(n\xi)\right)\right]d\xi =$

$\frac{1}{2\pi}\int_0^{2\pi} g(\xi)\,K(r,\theta,\xi)\,d\xi$

$K(r,\theta,\xi) = 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\left(\cos(n\theta)\cos(n\xi) + \sin(n\theta)\sin(n\xi)\right)$

$\{\cos(\alpha+\beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta \qquad \cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta\}$

$= 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos(n\theta - n\xi) \qquad \{\phi = \theta + \xi\}$

$= 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n\cos(n\phi) \qquad \{e^{i\alpha} = \cos\alpha + i\sin\alpha$

$e^{-i\alpha} = \cos$

$K(r,$

$1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n$

$= 1 + \sum_{n=1}^{\infty} q^n$

{Geometric

$\therefore K$

$= 1 + \frac{re^{i\theta}}{1 - re}$

where $D =$

$a^2 - 2ar\cos$

$N = a^2 - 2$

$a^2 - 2ar\cos$

So $K =$

So $u(r,$

$\frac{a^2 - r^2}{2\pi}\int_0$

$u(r,\theta) =$

$e^{-ix} = \cos x - i\sin x \therefore \cos x = \frac{1}{2}(e^{ix} + e^{-ix}) \quad \therefore x = n\phi \therefore \}$

- $K(r,\theta,\xi) = 1 + 2\sum_{n=1}^{\infty}\left(\frac{r}{a}\right)\frac{e^{in\phi} + e^{-in\phi}}{2} =$

$1 + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n \underbrace{e^{in\phi}}_{(e^{i\phi})^n} + \sum_{n=1}^{\infty}\left(\frac{r}{a}\right)^n e^{-in\phi}$

$= 1 + \sum_{n=1}^{\infty} q^n + \sum_{n=1}^{\infty}\overline{q^n}$ where $q = \frac{r}{a}e^{i\phi}$

$\{$Geometric progression: $\sum_{n=1}^{\infty} q^n = q + q^2 + q^3 + \ldots = \frac{q}{1-q} \quad |q|<1 \}$

$\therefore K = 1 + \frac{q}{1-q} + \frac{\bar{q}}{1-\bar{q}} \qquad \{q = \frac{r}{a}e^{i\phi}\}$

$= 1 + \frac{re^{i\phi}/a}{1 - re^{i\phi}/a} + \frac{re^{-i\phi}/a}{1 - re^{-i\phi}/a} = 1 + \frac{re^{i\phi}}{a - re^{i\phi}} + \frac{re^{-i\phi}}{a - re^{-i\phi}} = \frac{N}{D}$

where $D = (a - re^{i\phi})(a - re^{-i\phi}) = a^2 - are^{i\phi} - are^{-i\phi} + r^2 =$

$a^2 - 2ar\cos\phi + r^2$

$N = a^2 - 2ar\cos\phi + r^2 + re^{i\phi}(a - re^{-i\phi}) + re^{-i\phi}(a - re^{i\phi}) =$

$a^2 - 2ar\cos\phi + r^2 \underbrace{+ are^{i\phi} - r^2 + are^{-i\phi}}_{2ar\cos\phi} - r^2 = a^2 - r^2$

So $K = \frac{N}{D} = \frac{a^2 - r^2}{a^2 - 2ar\cos(\theta - \xi) + r^2}$

So $u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} g(\xi)\frac{a^2 - r^2}{a^2 - 2ar\cos(\theta-\xi) + r^2}d\xi =$

$\frac{a^2 - r^2}{2\pi}\int_0^{2\pi} g(\xi)\frac{d\xi}{a^2 - 2ar\cos(\theta - \xi) + r^2} \qquad \{g(\xi) = u(a,\xi)\}.$

$u(r,\theta) = \frac{a^2 + r^2}{2\pi}\int_0^{2\pi}\frac{u(a,\xi)}{a^2 - 2ar\cos(\theta - \xi) + r^2}$ is poisson's formula for Z disk

- $\nabla^2 u = \mu u$ $\{$ Helmholtz's eqn

Thm 5.1 ∃ sol for that BVP can expressed as

$y(x) = \int_a^b G(x;\xi) R(\xi)\, d\xi$ where $G(x;\xi)$ satisfies

$$\frac{d^2 G}{dx^2} + P(x)\frac{dG}{dx} + Q(x) G(x) = \delta(x-\xi), \quad a \le x \le b, \quad a \le \xi \le b$$

Subject to $\alpha_1 G(a;\xi) + \beta_1 G_x(a;\xi) = 0$

$\alpha_2 G(b;\xi) + \beta_2 G_x(b;\xi) = 0$

Func $G(x;\xi)$ is called Greens func

Thm 5.1 Sol of BVP expressed $y(x) = \int_a^b G(x;\xi) R(\xi)\, d\xi$,

$G(x;\xi)$ satisfies $\frac{d^2 G}{dx^2} + P(x)\frac{dG}{dx} + Q(x) G(x) = \delta(x-\xi)$

let $G(x;\xi)$ ODE $G_{xx} + P(x) G_x + Q(x) G = \delta(x-\xi)$

BC1 $\alpha_1 G(a;\xi) + \beta_1 G_x(a;\xi) = 0$

BC2 $\alpha_2 G(b;\xi) + \beta_2 G_x(b;\xi) = 0$ $\quad \{ \forall \xi \in (a,b) \; x \in (a,b)$

$\therefore y(x) = \int_a^b G(x;\xi) R(\xi)\, d\xi$

Satisfies ODE $y'' + P(x) y' + Q(x) y$

BC1y: $\alpha_1 y(a) + \beta_1 y'(a) = 0$ $\quad \alpha_2 y(b) + \beta_2 y'(b) = 0$

proof: by sub ODE y LHS $= y'' + P y' + Q y =$

$$\frac{d^2}{dx^2}\int_a^b G(x;\xi) R(\xi)\, d\xi + P(x)\frac{d}{dx}\int_a^b G(x;\xi) R(\xi)\, d\xi + Q(x)\int_a^b G(x;\xi) R(\xi)\, d\xi$$

$$= \int_a^b \left[G_{xx}(x;\xi) + P(x) G_x(x;\xi) + Q(x) G(x;\xi) \right] R(\xi)\, d\xi$$

$$= \int_a^b \delta(x-\xi) R(\xi)\, d\xi = R(x) = RHS$$

BC1y $\quad 0 = \alpha_1 y(a) + \beta_1 y'(a) =$

$$\alpha_1 \int_a^b G(a;\xi) R(\xi)\, d\xi + \beta_1 \int_a^b G_x(a;\xi) R(\xi)\, d\xi$$

$$= \int_a^b \underbrace{\left[\alpha_1 G(a;\xi) + \beta_1 G_x(a;\xi) \right]}_{=0 \text{ by BC1G}} R(\xi)\, d\xi = 0$$

& similar for BC2y ($x=a \to x=b$) $\therefore y(x)$ satis ODE + BC1 + BC2

□

what does it mean that $\underbrace{G'' + P(x)G' + Q(x)G}_{L(G)} = \delta(x-\xi)$?

1) $L(G) = 0 \quad \forall x \neq \xi$

so $G = C_1 y_1(x) + C_2 y_2(x)$ for either interval (a,ξ) & (ξ, b)

2) $1 = \int_{\xi-\varepsilon}^{\xi+\varepsilon} \delta(x-\xi)\,dx \qquad \left\{\lim_{\varepsilon\to 0^+}\right\}$

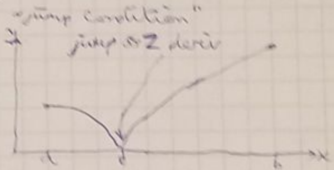
$= \int_{\xi-\varepsilon}^{\xi+\varepsilon} G''(x;\xi)\,dx + \int_{\xi-\varepsilon}^{\xi+\varepsilon} P(x)G'(x;\xi)\,dx + \int_{\xi-\varepsilon}^{\xi+\varepsilon} Q(x)G(x;\xi)\,dx =$

$G'(\xi+\varepsilon) - G'(\xi-\varepsilon) + \left[P(x)G(x)\right]_{\xi-\varepsilon}^{\xi+\varepsilon} - \int_{\xi-\varepsilon}^{\xi+\varepsilon} G(x)P'(x)\,dx + \int_{\xi-\varepsilon}^{\xi+\varepsilon} Q(x)G(x)\,dx \quad \{\varepsilon\to 0\}$

note $P(x)$, $Q(x)$ are bounded, $P(x)$ is smooth, &

assume G is bounded in $x \in (\xi-\varepsilon, \xi+\varepsilon)$

then $\lim_{\varepsilon\to 0}\left[P(x)G(x)\right]_{\xi-\varepsilon}^{\xi+\varepsilon} = 0$



$\lim_{\varepsilon\to 0}\int_{\xi-\varepsilon}^{\xi+\varepsilon} P'(x)G(x)\,dx = 0$

$\lim_{\varepsilon\to 0}\int_{\xi-\varepsilon}^{\xi+\varepsilon} G(x)G(x)\,dx = 0 \quad \therefore \quad \lim_{\varepsilon\to 0}\left[G'(\xi+\varepsilon) - G'(\xi-\varepsilon)\right] = 1$

$G'(\xi+0) - G'(\xi-0) = 1$

$\lim_{a\to 0^+}\int_{-\infty}^{\infty} f(x)\,\frac{e^{-x^2/a^2}}{a\sqrt{\pi}}\,dx = f(0) \quad \forall f$ ~~dot this for~~

to this for $f(x)$ Lipschitz cont (LC)

Def: $f(x)$ is LC is $\exists K = 0$ st $\forall x_1, x_2 \in \text{domain}(f)$

$|f(x_1) - f(x_2)| \leq K|x_1 - x_2|$

$f = \sqrt[3]{x}$ is cont but not LC. on an interval containing $x = 0$

$\therefore |f(x_1) - f(x_2)| < K|x_1 - x_2| \qquad$ ① $\int_{-\infty}^{\infty}\frac{e^{-x^2/a^2}}{a\sqrt{\pi}}\,dx\Big|_{x=az}$

$\leq \frac{1}{\sqrt{\pi}}\underbrace{\int_{-\infty}^{\infty} e^{-z^2}\,dz}_{\text{Gaussian integral} = \sqrt{\pi}} = 1$

② need to prove $\lim \int f(x)\,\frac{e^{-x^2/a^2}}{a\sqrt{\pi}}\,dx = f(0)\underbrace{\int_{-\infty}^{\infty}\frac{e^{-x^2/a^2}}{a\sqrt{\pi}}\,dx}_{1}$

$$\Leftrightarrow \lim_{a\to 0}\int_{-\infty}^{\infty}(f(x)-f(0))\frac{e^{-x^2/a^2}}{a\sqrt{\pi}}\,dx=0$$

$|f(x_1)-f(x_2)|\le K|x_1-x_2| \quad \forall x_1, x_2$ ii)

$\int_{-\infty}^{\infty}\frac{e^{-x^2/a^2}}{a\sqrt{\pi}}\,dx=1 \;;\; \forall a>0$ need to prove

$$0=\lim_{a\to 0}\underbrace{\int_{-\infty}^{\infty}(f(x)-f(0))\frac{e^{-x^2/a^2}}{a\sqrt{\pi}}\,dx}_{=I(a)}$$

$\therefore$ have : $|I(a)|=\left|\int_{-\infty}^{\infty}(f(x)-f(0))\frac{e^{-x^2/a^2}}{a\sqrt{\pi}}\,dx\right|$

$$\le\int_{-\infty}^{\infty}|f(x)-f(0)|\frac{e^{-x^2/a^2}}{a\sqrt{\pi}}\,dx$$

$$\le\int_{-\infty}^{\infty}\underbrace{K|x|\frac{e^{-x^2/a^2}}{a\sqrt{\pi}}}_{\text{even}}\,dx=\frac{2K}{a\sqrt{\pi}}\int_{0}^{\infty}xe^{-x^2/a^2}\,dx\Big|_{x=az}$$

$$=\frac{2K}{\cancel{a}\sqrt{\pi}}\cancel{a}^{2}\underbrace{\int_{0}^{\infty}ze^{-z^2}\,dz}_{=\text{const}}=\frac{2K}{\sqrt{\pi}}a\int_{0}^{\infty}ze^{-z^2}\,dz\to 0 \text{ as } a\to 0 \;\therefore$$

by sandwich thm: $|I(a)|\to 0$ as $a\to 0$ $\therefore$

$\lim_{a\to 0}I(a)=0$

consider Z 2nd order PDE $u_t-Du_{xx}=f(x,t)$ $0\le x\le L$, $0\le t\le T$

for given $D>0$, $L>0$, $T>0$ & $f(x,t)$ subject to

$u_x(0,t)=u_x(L,t)=0 \qquad u(x,0)=0$

following Z ODE ex in Z 1st section, would like to express

u in Z form $u(x,t)=\int_0^T\int_0^L f(\xi,\tau)G(x,t;\xi,\tau)\,d\xi\,d\tau$

where $G(x,t;\xi,\tau)$ is Z Green's func satisfying

$G_x(0,t;\xi,\tau)=G_x(L,t;\xi,\tau)=0$

$G(x,t;\xi,\tau)=0 \;\forall\, t<\tau \qquad G_t-DG_{xx}=\delta(x-\xi)\delta(t-\tau)$

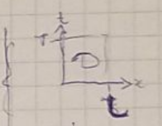
~~theory~~ let $G(x,t;\xi,\tau)$ PDE $G_t - DG_{xx} = \delta(x-\xi)\delta(t-\tau)$

(as ξ, τ of x, t)

BC$_G$ $G_x(0,t) = G_x(L,t) = 0$ $\forall \xi \in (0,L)$, $\forall \tau \in (0,T)$ they are params

IC$_G$ $G(x,0) = 0$

& $u(x,t) = \int_0^T \int_0^L G(x,t;\xi,\tau) f(\xi,\tau)\, d\xi\, d\tau$



then u satisfies PDE$_u$ $u_t - Du_{xx} = f(x,t)$

BC$_u$ ~~$u_t - Du_{xx} = f(x,t)$~~ $u_x(0,t) = u_x(L,t) = 0$

IC$_u$ $u(x,0) = 0$

$\int_0^T \int_0^L \cdots d\xi\, d\tau = \int_D$

~~proof~~ verify by substitution:

PDE$_u$ LHS $= u_t - Du_{xx} = \frac{\partial}{\partial t} \iint_D G f\, d\xi\, d\tau - D \frac{\partial^2}{\partial x^2} \iint_D G f\, d\xi\, d\tau$

{diff. integral by params}

$= \iint_D (G_t - DG_{xx}) f\, d\xi\, d\tau$

$= \iint_D \overbrace{[G_t(x,t;\xi,\tau) - DG_{xx}(x,t;\xi,\tau)]}^{\text{by PDE}} f(\xi,\tau)\, d\xi\, d\tau$

$= \iint_D \delta(x-\xi)\delta(t-\tau) f(\xi,\tau)\, d\xi\, d\tau = f(x,t) =$ RHS

BC$_u$ $u_x(x,t) = \frac{\partial}{\partial x} \iint_D G f\, d\xi\, d\tau = \iint_D G_x(x,t;\xi,\tau)\, d\xi\, d\tau$

$0 = u_x(0,t) = \iint_D \overbrace{G_x(0,t;\xi,\tau)}^{\text{by BC}_G}\, d\xi\, d\tau = \iint_D 0\, d\xi\, d\tau = 0$

similarly $u_x(L,t) = 0$ $(x=0 \Rightarrow x=L)$

IC$_u$: $u(x,0) = \iint_D \overbrace{G(x,0;\xi,\tau)}^{\text{by IC}_G} f(\xi,\tau)\, d\xi\, d\tau = \iint_D 0\, d\xi\, d\tau = 0$ $\square$

$G(x;t;\xi,\tau) = G(x,t)$

~~BC~~ $G_t - G_{xx} = \delta(x-\xi)\delta(t-\tau)$

BC: $G_x(0,t)$; $G_x(L,t) = 0$, $G(x,0) = 0$ $\xi \in (0,L)$ $\tau \in (0,T)$

Consider proof $D = D_1 \cup D_2 \cup D_3$

$D_1 = [0,L] \times [0, \tau - \varepsilon]$ $D_2 = [0,L] \times [\tau - \varepsilon, \tau + \varepsilon]$

$D_3 = [0,L] \times [\tau + \varepsilon, T]$

Plan: ① Solve in D_1, F.C. of D_1 = IC of D_2

② Solve in D_2, FC of D_2 = IC of D_3

③ Solve in D_3 ④ $\varepsilon \to 0^+$

①: IC $G(x,0)=0$, in D_1 $G_x(a,t)=G_x(b,t)=0$

$G_t - G_{xx} = 0 \Rightarrow G(x,t)=0$

F.C. $G(x,\tau-\varepsilon)=0$

②: D_2: IC $G(x,\tau-\varepsilon)=0$ $G_x(a,t)=G_x(b,t)=0$

$G_t - G_{xx} = \delta(x-\xi)\,\delta(t-\tau)$

only need $G(x,\tau+\varepsilon)$

Naive approaches: $G(x,\tau+\varepsilon) = \underbrace{G(x,\tau-\varepsilon)}_{=0} + \int_{\tau-\varepsilon}^{\tau+\varepsilon} G_t(x,t)\,dt$ by PDE

main thm of Calculus)

$= \delta(x-\xi)\underbrace{\int_{\tau-\varepsilon}^{\tau+\varepsilon}\delta(t-\tau)\,dt}_{=1} + \underbrace{D\int_{\tau-\varepsilon}^{\tau+\varepsilon} G_{xx}\,dt}_{\varepsilon\to 0 \;\to 0}$ {not correct argument}

$\xrightarrow[\varepsilon\to 0^+]{} \delta(x-\xi)$ correct answer

More pedantically: $G(x,\tau+\varepsilon) = \phi(x) = \phi(x;\varepsilon)$ want to show

$\lim_{\varepsilon\to 0^+}\phi(x) = \delta(x-\xi)$ in a weak sense

namely $\lim_{\varepsilon\to 0}\left[\int_a^b \phi(x)\,dx\right] = \int_a^b \delta(x-\xi)\,dx = \begin{cases} 1, & \xi\in(a,b) \\ 0, & \xi<a \text{ or } \xi>b \end{cases}$

note $G(t,x)$ is smooth at $t=\tau+\varepsilon$

indeed $\int_a^b \phi(x)\,dx = \int_a^b G(x,\tau+\varepsilon)\,dx$ (by Main thm of calc)

$= \int_a^b\int_{\tau-\varepsilon}^{\tau+\varepsilon} G_t(x,t)\,dx\,dt =$ by PDE *

$\underbrace{\int_a^b\int_{\tau-\varepsilon}^{\tau+\varepsilon} D\,G_{xx}(x,t)\,dx\,dt}_{=I(\varepsilon)} + \underbrace{\int_a^b\int_{\tau-\varepsilon}^{\tau+\varepsilon}\delta(x-\xi)\,\delta(t-\tau)\,d\xi\,dt}_{=J(\varepsilon)}$

where $J(\varepsilon) = \begin{cases} 1 & \xi\in(a,b) \\ 0 & \text{otherwise.} \end{cases}$

and $I(\varepsilon) = D\int_a^b\int_{\tau-\varepsilon}^{\tau+\varepsilon} G_{xx}(x,t)\,dt\,dx$

$= D\int_{\tau-\varepsilon}^{\tau+\varepsilon}\left[\int_a^b G_{xx}(x,t)\,dx\right]dt =$ {by main theory of calc}

$$D\int_{\tau-\varepsilon}^{\tau+\varepsilon}\left[G_{\tau x}(b,t)-G_{\tau x}(a,t)\right]dt$$

$\xrightarrow[\varepsilon\to 0^+]{} 0$ as $G_{\tau x}(x,t)$ is smooth at $x=a$ or $x=b$
so bounded for $t\in(\tau-\varepsilon,\tau+\varepsilon)$

so $\lim_{\varepsilon\to 0}\phi(x;\varepsilon)=\delta(x-\xi)$ (weak)

as in the naive approach

③: in D_3 $G_{\tau t}-DG_{\tau xx}=0$; $G_{\tau x}(0,t)=G_{\tau x}(L,t)=0$

$G_\tau(x,\tau+0)=\lim_{\varepsilon\to 0^+}G_\tau(x,\tau+\varepsilon)=\delta(x-\xi)$

Solve by Fourier series after sep of vars,

1° ~~G_τ~~ $G_\tau(x,t)=X(x)T(t)$ $\quad \dfrac{X''(x)}{X(x)}=\dfrac{T'(t)}{DT(t)}=-\lambda$

$$\begin{cases} X''(x)+\lambda X(x)=0\\ X'(0)=0\ ;\ X'(L)=0\end{cases}$$

standard eigenvalue probl, $\lambda_n=\left(\frac{n\pi}{L}\right)^2$; $n\in\mathbb{Z}_*\in\{0,1,2,\dots\}$

$n=0$ $\quad X_n=1$

$n\geq 1$ $\quad X_n=\cos(k_n x)$, $k_n=\frac{n\pi}{L}$ $\quad \lambda_n=k_n^2$

$T'(t)=-\lambda D\,T(t)$; $\lambda=\lambda_n$

$T(t)=e^{-\lambda_n Dt}=\exp\left[-Dk_n^2 t\right]$

2° $\quad G_\tau(x,t)=\sum_{n=0}^{\infty}\cos(k_n x)\,e^{-k_n^2 Dt}A_n$

3° $\quad G_\tau(x,\tau)=\phi(x)=\delta(x-\xi)=\sum_{n=0}^{\infty}\left[A_n\cos(k_n x)e^{-k_n^2 D\tau}\right]$

$$=\sum_{n=0}^{\infty}\tilde{A}_n\cos\left(\frac{n\pi x}{L}\right)=\phi(x) \qquad \text{cos series}$$

$\tilde{A}_0=\frac{1}{L}\int_0^L\phi(x)\,dx$

$\tilde{A}_n=\frac{2}{L}\int_0^L\phi(x)\cos\left(\frac{n\pi x}{L}\right)dx$

$\Big\{$ vs $\delta(x)=\sum_{n=1}^{\infty}C_n\phi_n(x)\Rightarrow C_n=\dfrac{\langle\phi_n,\delta\rangle}{\langle\phi_n,\phi_n\rangle}=\dfrac{\int_0^L\phi_n(x)\,\delta(x)\,dx}{\int_0^L\phi_n^2(x)\,dx}$

$\phi_n(x)=\cos\left(\frac{n\pi x}{L}\right)$ $\qquad \int_0^L\cos^2\left(\frac{n\pi x}{L}\right)dx=\frac{L}{2}$ $\Big\}$

Summarise: $\tilde{A}_n=\dfrac{2}{Q_n L}\int_0^L\phi(x)\cos\left(\frac{n\pi x}{L}\right)dx$ $\qquad Q_n=\begin{cases}2, & n=0\\ 1, & n\geq 1\end{cases}$

For $\phi(x)=\delta(x,\xi)$

$\tilde{A}_n=\dfrac{2}{LQ_n}\int_0^L\delta(x-\xi)\cos\left(\frac{n\pi x}{L}\right)dx=\dfrac{2}{LQ_n}\cos\left(\frac{n\pi\xi}{L}\right)$ where

Z singularity in B is circumvented by using Z divergence thm

$$\iiint_V \nabla\cdot\vec{A}\, d^3\vec{r} = \oiint_{\partial V} \vec{A}\cdot\vec{n}\, dS$$

$\nabla\cdot\vec{A} = \nabla^2 \frac{1}{|\vec{r}|}$; $\vec{A} = \nabla\left(\frac{1}{|\vec{r}|}\right)$ $\rightleftharpoons \nabla\cdot\left(\nabla \frac{1}{|\vec{r}|}\right)$

$$I = \oiint_{\partial B} \nabla\left(\frac{1}{|\vec{r}|}\right)\cdot\vec{n}\, dS$$

note ∂B; sphere of radius μ

$\vec{n} = \frac{\vec{r}}{|\vec{r}|}\Big|_{|\vec{r}|=\mu}$ in spherical coords (r,θ,ϕ)

$|\vec{r}| = r$

$\nabla\left(\frac{1}{|\vec{r}|}\cdot\vec{n}\right)$

$\nabla\left(\frac{1}{r}\right) = \left(\frac{\partial}{\partial r}\left(\frac{1}{r}\right), 0, 0\right) = \left(-\frac{1}{r^2}, 0, 0\right)$

$\vec{n}\cdot\nabla\left(\frac{1}{r}\right) = -\frac{1}{r^2}\Big|_{r=\mu} = -\frac{1}{\mu^2}$ $\therefore$

$$I = \oiint_{\partial B}\left(-\frac{1}{\mu^2}\right)dS = -\frac{1}{\mu^2}\underbrace{\oiint_{\partial B} dS}_{\text{area of sphere } 4\pi\mu^2} = -\frac{1}{\mu^2}(4\pi\mu^2) = -4\pi \quad \square$$

\inviscid burgers/ $u_t + uu_x = 0$ $\qquad x = X(t)$

$u_t = u_t + u_x \frac{dX}{dt} = -u u_x + u_x \frac{dX}{dt} = u_x\left(\frac{dX}{dt} - u\right) = 0$

characteristics are straight lines

$u_t + uu_x = 0$ sol by charnes

\Conservation laws/ aka first integrals

$\frac{\partial u}{\partial t} = -u\frac{\partial u}{\partial x}$

For any sol desire $M(t) = \int_{\mathbb{R}} u(x,t)\,dx$ "Momentum"

$\frac{dM}{dt} = \frac{d}{dt}\int_{\mathbb{R}} u(x,t)\,dx = \int_{\mathbb{R}} u_t(x,t)\,dx = -\int_{\mathbb{R}} uu_x = -\left[\frac{u^2}{2}\right]_{-\infty}^{\infty}$

if eg $u(\pm\infty; t) = 0$ then $\frac{dM}{dt} = 0$; $M = \text{Const}$ "total Momentum is Conserved"

Now consider $E(t) = \int_{\mathbb{R}}\left(\frac{u^2}{2}\right)dx = \int_{\mathbb{R}} uu_t\, dx = -\int_{\mathbb{R}} u\cdot uu_x\, dx = -\int_{\mathbb{R}} u^2 u_x\, dx =$

$-\left[\frac{u^3}{3}\right]_{-\infty}^{\infty} = 0$ under same assumptions

"total energy is Conserved"

Week 1 / 1 / Change of variables $x, y \mapsto \xi, \eta$ ∴ if $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$ ∴)

$u_x + 2u_y = 0$ ∴ $\xi = x + \cancel{2}y$, $\eta = x - 2y$ ∴

$u_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$

$u_y = u_\xi \xi_y + u_\eta \eta_y = 2u_\xi - 2u_\eta$ ∴

$u_\xi + u_\eta + 2(2u_\xi - 2u_\eta) = 0 = u_\xi + u_\eta + 4u_\xi - 4u_\eta = 0 = 5u_\xi - 3u_\eta = 0$ ∴

$u_\xi = 0$, $u_\eta = 0$ ∴

$\frac{\partial u}{\partial \xi} = 0$, $\frac{\partial u}{\partial \eta} = 0$ ∴ $\Rightarrow$ ~~tz~~

$\cancel{u = \eta}$ $u = f(\eta)$, $u = g(\xi)$ ∴

$\cancel{u = \xi}$ $u = g(x + 2y)$

1 sol / $\nabla u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}$ ∴ $(1,2) \cdot \nabla u = (1,2) \cdot \left(\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y}\right) =$

$\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$ the given eqn is equiv to $(1,2) \cdot \nabla u = 0$. So characteristic curves satisfy $(dx, dy) \parallel (1,2)$ or $\frac{dy}{dx} = 2$ ∴ so the characteristic curves have the form $y = 2x + C$

1 retry / $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0$ ∴ if $a = 1$, $b = 2$: $a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0$

∴ if $\xi = ax + by$, $\eta = bx - ay$ ∴

$u_x = u_\xi \xi_x + u_\eta \eta_x = au_\xi + bu_\eta$ $u_y = u_\xi \xi_y + u_\eta \eta_y = bu_\xi - au_\eta$ ∴

$au_x + bu_y = 0 = \cancel{au_\xi + bu_\eta}$

$a(au_\xi + bu_\eta) + b(bu_\xi - au_\eta) = 0 = a^2u_\xi + abu_\eta + b^2u_\xi - abu_\eta =$

$a^2u_\xi + b^2u_\xi = (a^2 + b^2)u_\xi = 0$ $a = 1$, $b = 2$ ∴ $a^2 + b^2 = 1^2 + 2^2 = 1 + 4 = 5 \neq 0$ ∴

$\frac{\partial u}{\partial \xi} = 0$ ∴ $\cancel{u = \eta}$ $u = f(\eta) = f(bx - ay) = u(x,y) = f(2x - y)$

1 retry / $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0 = u_x + 2u_y = 0$ ∴ if $a = 1$, $b = 2$: $au_x + bu_y = 0$ ∴

if $\xi = y$, $\eta = bx - ay$: $u_x = u_\xi \xi_x + u_\eta \eta_x = 0u_\xi + bu_\eta = bu_\eta$

$u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi - au_\eta$ ∴

$a\frac{\partial u}{\partial x} + b\frac{\partial u}{\partial y} = 0 = abu_\eta + b(u_\xi - au_\eta) = abu_\eta + bu_\xi - abu_\eta = bu_\xi = 0 = u_\xi$ ∴

$u = f(\eta) = f(bx - ay) = u(x,y) = f(2x - y)$

if $u(x,y) = F(\xi)$, $\xi = \alpha x + y$ ∴ $u_x = \alpha F'(\alpha x + y)$, $u_y = F'(\alpha x + y)$,

∴ $au_x + bu_y = a\alpha F'(\alpha x + y) + bF'(\alpha x + y) = (a\alpha + b)F'(\alpha x + y) = 0$ is satisfied for arbitrary differentiable F for $\alpha = -\frac{b}{a}$ ∴ $\xi = -\frac{b}{a}x + y = \alpha x + y$

$\{\therefore$ is $\xi = -\frac{b}{a}x + y$ & is $\eta = bx - ay \Rightarrow \{-\frac{1}{a}\eta = -\frac{1}{a}(bx-ay) = -\frac{b}{a}x + y = \alpha x + y = \xi$

$\xi = \alpha x + y = -\frac{b}{a}x + y = -\frac{1}{a}(bx - ay) = -\frac{1}{a}\eta \therefore \xi$ is a scaled version of η

sols represented via scaled coords are equiv shown by:

$f(\eta) = f(bx-ay) = f(-a(y - \frac{b}{a}x)) = F(y - \frac{b}{a}x) = F(\xi)$

$u(x,y) = f(bx - ay) = f(2x - y) \therefore$

the curves (straight lines) $bx - ay = C$, for various values of C, are characteristic curves. The sol has the same value $f(C)$ $\forall$ pts on a characteristic curve. The variable $\eta = bx - ay$ or any of its scaled version is characteristic coord $\therefore$

$bx - ay = C \therefore bx - C = ay \therefore y = \frac{b}{a}x - \frac{C}{a} = y(x, C) = \frac{2}{1}x - \frac{C}{1} = 2x - C$

~~[illegible]~~ is the characteristic curves $\therefore$

$\therefore f(x,y) = C = f(bx - ay) \equiv u(x,y) = f(2x-y)$

$u(x,0) = f(2x - 0) = f(2x) = e^x \therefore$ let $2x = s \therefore x = \frac{s}{2} \therefore$

$f(s) = e^{s/2} = (e^s)^{1/2} = (e^{1/2})^s = (\sqrt{e})^s \therefore u(x,y) = f(2x-y) = (\sqrt{e})^{2x-y}$
$= (e^{\frac{1}{2}})^{2x-y} = e^{x - \frac{1}{2}y}$

\\ retry / $\frac{\partial u}{\partial x} + 2\frac{\partial u}{\partial y} = 0 = u_x + 2u_y = 0 \therefore (1,2)\cdot(u_x, u_y) = 0 \therefore$

expect $u = $ const along $(1,2) \therefore \frac{dy}{dx} = \frac{2}{1} \therefore (dx, dy) \parallel (1,2) \therefore$

$\frac{dx}{1} = \frac{dy}{2} \therefore \frac{dy}{dx} = 2 \therefore \int \frac{dy}{dx}dx = \int 2\,dx = \int dy = $ ~~[illegible]~~

$y = 2x + C \therefore C = y - 2x \therefore u = f(y - 2x)$ is general sol of PDE with charact curves $y = 2x + C \therefore$

when $y = 0 \therefore u = f(0 - 2x) = f(-2x) = e^x = \exp(-\frac{1}{2}(-2x)) = e^{-\frac{1}{2}(-2x)} \therefore$

$f(s) = e^{-\frac{1}{2}(s)} = e^{-\frac{1}{2}s} = \exp(-\frac{1}{2}s)$ &

~~[illegible]~~ $u = f(y - 2x) = e^{-s/2}\Big|_{s = y-2x} = e^{x - y/2}$

~~[illegible]~~ \\ ✓ $\frac{du}{dx} + 2\frac{du}{dy} = 0 \therefore (1,2)\cdot(u_x, u_y) = 0 \therefore$

$u = $ constant along $(1,2) \therefore \frac{dy}{dx} = \frac{2}{1} = 2 \therefore \int \frac{dy}{dx}dx = \int 2\,dx = \int dy =$

$y = 2x + C \therefore y - 2x = C \therefore u = u(x,y) = f(y - 2x) \therefore$

$u(x,0) = f(0 - 2x) = f(-2x) = e^x \therefore$ let $-2x = s \therefore x = -\frac{1}{2}s \therefore$

$f(s) = e^{-\frac{1}{2}s} \therefore u(x,y) = f(y - 2x) = e^{-\frac{1}{2}(y-2x)} = e^{x - \frac{1}{2}y}$

\Week 1/ \2a/ $(\cos^2 x)u_x + y u_y = 0 = (\cos^2 x, y)\cdot(u_x, u_y) = 0$

$\therefore u$ = constant along $(\cos^2 x, y)$ $\therefore \frac{dy}{dx} = \frac{y}{\cos^2 x}$ $\therefore$

- $\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos^2 x}$ $\therefore$ forgets the special sol $\therefore$ $\int \frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{1}{\cos^2 x}\,dx = \int \frac{1}{y}\,dy = \ln|y|$

$\left\{ \frac{d}{dx}\tan x = \frac{d}{dx}\frac{\sin x}{\cos x} = \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \right\}$

$\ln|y| = \int \frac{1}{\cos^2 x}\,dx = \tan x + C$ $\therefore$ $|y| = e^{\tan x + C} = y = Ae^{\tan x}$ $\therefore$

$A = \frac{y}{e^{\tan x}}$ $\therefore u = u(x,y) = f\left(\frac{y}{e^{\tan x}}\right)$ $\therefore$

$u(0,y) = f\left(\frac{y}{e^{\tan 0}}\right) = y^2 = f\left(\frac{y}{e^0}\right) = f\left(\frac{y}{1}\right) = f(y)$ $\therefore$ let $y = s$ $\therefore$

$f(s) = s^2$ $\therefore$ $u(x,y) = f\left(\frac{y}{e^{\tan x}}\right) = \left(\frac{y}{e^{\tan x}}\right)^2 = \frac{y^2}{(e^{\tan x})^2} = \frac{y^2}{e^{2\tan x}}$

\2b/ $u(x,0) = f\left(\frac{0}{e^{\tan x}}\right) = x^2 = f(0)$ $\therefore$ let $0 = s$ $\therefore$

$f(0) = 0^2 = 0$ $\therefore u(x,y) = f\left(\frac{y}{e^{\tan x}}\right) = 0$ but for $u(x,0) = x^2$ the solution doesn't exist $\because$ the boundary cond is not a function of x or y eg $f(0) = x^2$ does not make sense $\because x$ is undefined

\2a Sol/ $(\cos^2 x, y)\cdot\nabla u = 0$ $\therefore$ charac curves $(dx, dy) \parallel (\cos^2 x, y)$

or $\frac{dy}{dx} = \frac{y}{\cos^2 x}$ $\therefore$ $\int \frac{dy}{y} = \int \frac{dx}{\cos^2 x} \rightarrow \ln|y| = \tan x + \text{const}$ $\therefore$

- $y = \pm\exp(A + \tan x) = C\exp(\tan x)$ A is arbit $C = \pm e^A$

special sol $y(x) \equiv 0$ is covered by the same formula with $C = 0$

$\therefore C = y\exp(-\tan x)$ $\therefore$ CFS $u = f(y\exp(-\tan x))$

when $x = 0$. $u(0,y) = f(y) = y^2$ $\therefore u(x,y) = y^2\exp(-2\tan x)$

\2b Sol/ when $y = 0$: $u(x,y) = f(0) = x^2$ which cannot be satisfied $\because$ func f cannot have different vals (x^2) for one & the same val of the argument (0) {function cannot have different outputs (multiple outputs) for the same input} $\therefore$ the sol doesn't exist

- $\because$ the BC is set on $y = 0$ which is a charac curve

{ $y(x) \equiv 0$ is the special sol $\therefore$ maybe this counts as a charac curve? }

3 $u_x + xy^3 u_y = 0 \therefore (1, xy^3)\cdot(u_x, u_y) = 0 \therefore$

u is constant along $(1, xy^3) \therefore \frac{dy}{dx} = \frac{xy^3}{1} = xy^3 \therefore$

$\cancel{\therefore}$ $y^{-3}\frac{dy}{dx} = x \therefore \int y^{-3}\frac{dy}{dx}\,dx = \int x\,dx = \int y^{-3}\,dy = \frac{1}{-2}y^{-2} =$

$\frac{1}{2}x^2 + C_2 \therefore -y^{-2} = x^2 + C_3 \therefore y^{-2} = -x^2 + C_4 \therefore \frac{1}{y^2} = -x^2 + C \therefore$

$\frac{1}{-x^2+C} = y^2 \therefore \left(\frac{1}{-x^2+C}\right)^{1/2} = y$ is charac curves

$y = 0$ is a special sol

$\frac{1}{y^2} + x^2 = C = f(x,y) = u = u(x,y) = f\left(\frac{1}{y^2} + x^2\right)$

$\therefore$ is $u_x = \frac{\partial}{\partial x} f\left(\frac{1}{y^2} + x^2\right) = 2x f'\left(\frac{1}{y^2} + x^2\right)$

$u_y = \frac{\partial}{\partial y} f(\cancel{y}y^{-2} + x^2) = -2y^{-3} f'\left(\frac{1}{y^2} + x^2\right) = -2y^{-3} f'(y^{-2} + x^2) \therefore$

$u_x + xy^3 u_y = LHS = 2x f'(y^{-2} + x^2) + xy^3(-2)y^{-3} f'(y^{-2} + x^2) =$

$2x f'(y^{-2}+x^2) - 2x f'(y^{-2}+x^2) = (2x - 2x) f'(y^{-2} + x^2) = (0) f'(y^{-2}+x^2) =$

$0 = RHS \therefore u = f(y^{-2} + x^2)$ is GS

3 Soln rewrite eqn: $(1, xy^3)\cdot\nabla u \therefore$ charac curves: $(dx, dy) \parallel (1, xy^3)$

or $\frac{dy}{dx} = xy^3 \therefore \int \frac{dy}{y^3} = \int x\,dx \therefore -\frac{1}{2y^2} = \frac{1}{2}x^2 + A = \frac{x^2 - C}{2} \therefore$

$y^2 = \frac{1}{C - x^2} \therefore y = \pm\frac{1}{\sqrt{C - x^2}}$ A is arb const & $C = -2A \therefore$

$C = x^2 + \frac{1}{y^2} \therefore$ sol: $u = f\left(x^2 + \frac{1}{y^2}\right)$ for arb func f

Check: $\frac{\partial u}{\partial y} = (-2/y^3) f' \quad \frac{\partial u}{\partial x} = (2x) f' \therefore$

$\frac{\partial u}{\partial x} + xy^3\frac{\partial u}{\partial y} = (2x + xy^3(-2/y^3)) f' = (2x - 2x) f' = 0 = RHS$

4 is $a = 1, b = 3, c = 10, 20e^{10x} = g(x,y) \therefore$

$au_x + bu_y + cu = g(x,y) \therefore$ is $\xi = ax + by,\ \eta = bx - ay \therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x = au_\xi + bu_\eta \quad$ $u_y = u_\xi \xi_y + u_\eta \eta_y = bu_\xi - au_\eta \therefore$

$a(au_\xi + bu_\eta) + b(bu_\xi - au_\eta) + cu = g(x,y) = \tilde{g}(\xi,\eta) =$

$a^2 u_\xi + abu_\eta + b^2 u_\xi - abu_\eta + cu = (a^2 + b^2)u_\xi + cu = \tilde{g}(\xi,\eta) =$

$\tilde{g}(\xi(x,y), \eta(x,y)) = g(x,y) \therefore u_\xi + \frac{c}{a^2+b^2}u = \frac{\tilde{g}(\xi,\eta)}{a^2+b^2} \therefore$

$h(\xi)u_\xi + h(\xi)\frac{c}{a^2+b^2}u = h(\xi)\frac{\tilde{g}(\xi,\eta)}{a^2+b^2} = h(\xi)u_\xi + h'(\xi)u = \frac{\partial}{\partial\xi}(h(\xi)u) \therefore$

Week 1/

$$h'(\xi) = \frac{d}{d\xi}h(\xi) = h(\xi)\frac{c}{a^2+b^2} \therefore$$

$$\frac{h'(\xi)}{h(\xi)} = \frac{c}{a^2+b^2} \therefore \int \frac{h'(\xi)}{h(\xi)}d\xi = \int \frac{c}{a^2+b^2}d\xi = \ln(h(\xi)) = \frac{c\xi}{a^2+b^2} \therefore$$

$$h(\xi) = \exp\left[\frac{c\xi}{a^2+b^2}\right] \therefore$$

$$\frac{\partial}{\partial\xi}\left[e^{\frac{c\xi}{a^2+b^2}}u\right] = e^{\frac{c\xi}{a^2+b^2}}\frac{\tilde{g}(\xi,\eta)}{a^2+b^2} \therefore$$

$$e^{\frac{c\xi}{a^2+b^2}}u = \int \frac{\tilde{g}(\xi,\eta)}{a^2+b^2}e^{\frac{c\xi}{a^2+b^2}}\,\partial\xi \therefore$$

$$u(\xi,\eta) = e^{-\frac{c\xi}{a^2+b^2}}\int \frac{\tilde{g}(\xi,\eta)\,e^{\frac{c\xi}{a^2+b^2}}}{a^2+b^2}\,\partial\xi$$

$\therefore$ $\xi = ax+by$, $\eta = bx-ay$: $\quad \begin{matrix} ax+by=\xi \\ bx-ay=\eta \end{matrix} \therefore$

$$\Delta = \begin{vmatrix} a & b \\ -b & a \end{vmatrix} = a^2+b^2 \quad \therefore \begin{cases} ax+by=\xi \\ bx-ay=\eta \end{cases} \therefore$$

$$\Delta_x = \begin{vmatrix} \xi & b \\ \eta & a \end{vmatrix} = a\xi - b\eta \qquad \Delta_y = \begin{vmatrix} a & \xi \\ b & \eta \end{vmatrix} = a\eta \mp b\xi \therefore$$

$$x = \frac{\Delta_x}{\Delta} = \frac{a\xi-b\eta}{a^2+b^2} \qquad y = \frac{\Delta_y}{\Delta} = \frac{a\eta \mp b\xi}{a^2+b^2}$$

$$g = e^{(a^2+b^2)x} = g(x,y) = e^{a\xi-b\eta} = \tilde{g}(\xi,\eta)$$

$$\cancel{u(\xi,\eta) = e^{-\frac{c\xi}{a^2+b^2}}\int \frac{e^{a\xi-b\eta}}{\ldots}} \quad u(\xi,\eta) = e^{\frac{-c\xi}{a^2+b^2}}\int \frac{e^{a\xi-b\eta}}{a^2+b^2}e^{\frac{c\xi}{a^2+b^2}}\,\partial\xi =$$

$$e^{-\frac{c\xi}{a^2+b^2}}\int \frac{1}{a^2+b^2}e^{a\xi}e^{-b\eta}e^{\frac{c}{a^2+b^2}\xi}\,\partial\xi = \frac{1}{a^2+b^2}e^{\frac{-c\xi}{a^2+b^2}}e^{-b\eta}\int e^{\left(a+\frac{c}{a^2+b^2}\right)\xi}\,\partial\xi$$

$$= \frac{1}{a^2+b^2}e^{-\frac{c\xi}{a^2+b^2}-b\eta}\left(\frac{1}{\left(a+\frac{c}{a^2+b^2}\right)}e^{\left(a+\frac{c}{a^2+b^2}\right)\xi} + A(\eta)\right)$$ A is arbitrary func. $\therefore$

$$u(\xi,\eta) = \frac{1}{1^2+3^2}e^{-\frac{10\xi}{1^2+3^2}-3\eta}\left(\frac{1}{\left(1+\frac{10}{1^2+3^2}\right)}e^{\left(1+\frac{10}{1^2+3^2}\right)\xi} + A(\eta)\right) =$$

$$\frac{1}{10}e^{-\frac{10\xi}{10}-3\eta}\left(\frac{1}{\left(1+\frac{10}{10}\right)}e^{\left(1+\frac{10}{10}\right)\xi} + A(\eta)\right) =$$

$$\frac{1}{10}e^{-\xi-3\eta}\left(\frac{1}{2}e^{2\xi} + A(\eta)\right) = u(\xi,\eta)$$

4Sol/ Change of Coords $\xi = x+3y$ $\quad \eta = 3x-y$

$$\frac{\partial u}{\partial x} = u_x = \xi_x u_\xi + \eta_x u_\eta = u_\xi + 3u_\eta$$

$u_y = \xi_y u_\xi + \eta_y u_\eta = 3u_\xi - u_\eta$ then $u_x + 3u_y = u_\xi + 3u_\eta + 3(3u_\xi - u_\eta) =$

$10u_\xi$ notice also: $x = (\xi + 3\eta)/10$ then Z PDE becomes:

$10u_\xi + 10u = 20e^{\xi+3\eta}$ or just $u_\xi + u = 2e^{\xi+3\eta}$

∵ there are no differentiation wrt η, to solve this eqn hold η as a const param, Z integrate this as a first-order linear ODE wrt Z indep variable ξ. this can be done eg: using integrating factor, which is e^ξ giving $e^\xi u_\xi + e^\xi u = \frac{\partial}{\partial \xi}(e^\xi u) = 2e^{2\xi+3\eta} \Rightarrow$

$e^\xi u = e^{2\xi+3\eta} + f(\eta) \Rightarrow$ ~~$(f = e^{\xi+3\eta}$~~ $u = e^{\xi+3\eta} + e^{-\xi} f(\eta)$

alternatively: solve using Z method of undetermined coeffs.

Complementary func: $u_\xi + u = 0 \Rightarrow u = e^{-\xi} f(\eta) = u_{CF}$

for particular integral try $u = Ce^\xi$, then

$LHS = u_\xi + u = 2Ce^\xi = RHS = 2e^{\xi+3\eta}$, $C = e^{3\eta}$, $u = e^{\xi+3\eta} = u_{PI}$

GS: ~~the~~ $u = u_{PI} + u_{CF} = e^{-\xi} f(\eta) + e^{\xi+3\eta}$ as before

∴ in original coords: $u = e^{10x} + e^{-x-3y} f(3x - y)$

\4 retry $u_x + 3u_y + 10u = 20e^{10x}$ is $a=1$, $b=3$, $c=10$, $20e^{10x} = g(x,y)$:

$au_x + bu_y + cu = g(x,y)$ ∴ if $\xi = ax + by$, $\eta = bx - ay$:

$u_x = u_\xi \xi_x + u_\eta \eta_x = au_\xi + bu_\eta$ $u_y = u_\xi \xi_y + u_\eta \eta_y = bu_\xi - au_\eta$ ∴

$a(au_\xi + bu_\eta) + b(bu_\xi - au_\eta) + cu = g(x,y) = \tilde{g}(\xi, \eta) =$

$a^2 u_\xi + ab u_\eta + b^2 u_\xi - ab u_\eta + cu = (a^2+b^2) u_\xi + cu = \tilde{g}(\xi,\eta) = \tilde{g}(\xi(x,y), \eta(x,y)) = g(x,y)$

∴ $u_\xi + \frac{c}{a^2+b^2} u = \tilde{g}(\xi,\eta) \frac{1}{a^2+b^2} = u_\xi + \frac{c}{k} u = \frac{1}{k} \tilde{g}(\xi,\eta)$ for $a^2+b^2 = k$ ∴

$h(\xi) u_\xi + h(\xi) \frac{c}{k} u = h(\xi) \frac{1}{k} \tilde{g}(\xi,\eta) = h(\xi) u_\xi + h'(\xi) u = \frac{\partial}{\partial \xi}(h(\xi) u)$ ∴

$h'(\xi) = h(\xi) \frac{c}{k}$ ∴ $\int \frac{h'(\xi)}{h(\xi)} d\xi = \int \frac{c}{k} d\xi = \ln(h(\xi)) = \frac{c}{k} \xi$ ∴ $h(\xi) = e^{\frac{c}{k}\xi}$ ∴

$\frac{\partial}{\partial \xi}(e^{\frac{c}{k}\xi} u) = e^{\frac{c}{k}\xi} \frac{1}{k} \tilde{g}(\xi,\eta)$ ∴

$e^{\frac{c}{k}\xi} u = \int e^{\frac{c}{k}\xi} \frac{1}{k} \tilde{g}(\xi,\eta) \partial \xi$ ∴ $u(\xi,\eta) = e^{-\frac{c}{k}\xi} \int e^{\frac{c}{k}\xi} \frac{1}{k} \tilde{g}(\xi,\eta) \partial \xi$ ∴

$k = a^2 + b^2 = 1^2 + 3^2 = 10$ ∴ ~~$u(\xi,\eta) = e$~~ ∴ $\frac{c}{k} = \frac{10}{10} = 1$ ∴

$u(\xi,\eta) = e^{-\xi} \int e^\xi \frac{1}{10} \tilde{g}(\xi,\eta) \partial \xi$ ∴ $ax + by = \xi = x + 3y$, $bx - ay = \eta = 3x - y$

$\Delta = \begin{vmatrix} a & b \\ b & -a \end{vmatrix} = -a^2 - b^2$ $\Delta_x = \begin{vmatrix} \xi & b \\ \eta & -a \end{vmatrix} = -a\xi - b\eta$ ∴ $x = \frac{\Delta_x}{\Delta} = \frac{-a\xi - b\eta}{-a^2 - b^2} = \frac{a\xi + b\eta}{a^2 + b^2}$

$= \frac{\xi + 3\eta}{10} = \frac{1}{10}\xi + \frac{3}{10}\eta$

Week 1/ $\Delta_y = \begin{vmatrix} a & \xi \\ b & \eta \end{vmatrix} = a\eta - b\xi$ $\therefore y = \frac{\Delta_y}{\Delta} = \frac{-b\xi + a\eta}{-a^2 - b^2} = \frac{b\xi - a\eta}{a^2+b^2} =$

$\frac{3\xi - \eta}{10} = \frac{3}{10}\xi - \frac{1}{10}\eta$ $\therefore$

$g(x,y) = 20e^{10x} = 20e^{10(\frac{1}{10}\xi + \frac{3}{10}\eta)} = 20e^{\xi + 3\eta} = \tilde{g}(\xi,\eta) \equiv 20e^{\xi}e^{3\eta}$

$u(\xi,\eta) = e^{-\xi}\int e^{\xi}\frac{1}{10}20e^{\xi}e^{3\eta}\,\partial\xi =$

$2e^{-\xi}e^{3\eta}\int e^{2\xi}\,\partial\xi = 2e^{-\xi}e^{3\eta}\left(\frac{1}{2}e^{2\xi} + f(\eta)\right) =$

$2e^{-\xi}e^{3\eta}f(\eta) + e^{\xi}e^{3\eta} = e^{-\xi}F(\eta) + e^{\xi+3\eta} = u(\xi,\eta)$

$= e^{-x-3y}F(\eta) + e^{10x} = u(x,y) = e^{-x-3y}F(3x-y) + e^{10x}$ F arb

5a/ $2u_{xx} + 5u_{xy} - 3u_{yy} = 0$

$\therefore$ if $u(x,y) = f(y+\alpha x)$ f as arbit function $\therefore$

$u_x = \alpha f'(y+\alpha x)$ $\therefore u_{xx} = \alpha^2 f''(y+\alpha x)$

$u_y = f'(y+\alpha x)$ $\therefore u_{yy} = f''(y+\alpha x)$ $\therefore$ $u_{xy} = \alpha f''(y+\alpha x)$ $\therefore$

~~$2\alpha^2$~~ $LHS = 2\alpha^2 f''(y+\alpha x) + 5\alpha f''(y+\alpha x) - 3f''(y+\alpha x) = 0 = RHS =$

$(2\alpha^2 + 5\alpha - 3)f''(y+\alpha x) = 0$ $\cancel{f''(y+\alpha x)}$ f is arbit funct $\therefore$

$f''(y+\alpha x) \neq 0$ $\forall x,y$ $\therefore 2\alpha^2 + 5\alpha - 3 = 0$

$(2\alpha-1)(\alpha+3) = 2\alpha^2 + 6\alpha - \alpha - 3 = 2\alpha^2 + 5\alpha - 3 = 0$ $\therefore 2\alpha - 1 = 0$, $\alpha + 3 = 0$ $\therefore$

$\alpha = \frac{1}{2}$, $\alpha = -3$ $\therefore$ $\alpha_1 = \frac{1}{2}$, $\alpha_2 = -3$ $\therefore$

$u(x,y) = f_1(y + \alpha_1 x) + f_2(y + \alpha_2 x) = f_1(y + \frac{1}{2}x) + f_2(y - 3x)$

is a Sol for arb f_1, f_2

5a Sol/ try $u = \phi(y + \alpha x)$ $\therefore$ PDE: $2\alpha^2\phi'' + 5\alpha\phi'' - 3\phi'' = 0 \Rightarrow$

$2\alpha^2 + 5\alpha - 3 = 0 \Rightarrow (2\alpha - 1)(\alpha + 3) = 0 \Rightarrow \alpha = \frac{1}{2}$ or $\alpha = -3 \Rightarrow$

$u = f(y + x/2) + g(y - 3x)$ for arbit funcs f & g

5b/ $u_x - 4u_y - 3u = 0$ if $a = -4$, $b = +3$ $\therefore$ ~~$u_x + au_y + bu = 0$~~

~~if $b = 0$ $\tilde{u}_x$~~ $u_x - au_y - bu = 0$ $\therefore$ if $b = 0$:

$\tilde{u}_x + a\tilde{u}_y = 0 = (1, -a)\cdot(\tilde{u}_x, \tilde{u}_y)$ $\therefore$ $\frac{dy}{dx} = \frac{-a}{1}$ $\therefore$ $y = -ax + C$

$\therefore C = y + ax$ $\therefore$ $\tilde{u}(x,y) = f(y + ax)$ f arb func $\therefore$

Change of variables $x,y \mapsto \xi,\eta$: $\xi = y + ax$, $\eta = x$ $\therefore$ into original PDE:

$u_x = u_\xi\xi_x + u_\eta\eta_x = au_\xi + u_\eta$, $u_y = u_\xi\xi_y + u_\eta\eta_y = u_\xi$ $\therefore$

LHS $= u_x - a u_y - bu = a u_\xi + u_\eta - a u_\xi - bu = u_\eta - bu = 0 =$ RHS $\therefore$

$u_\eta - bu = 0 \therefore u_\eta = bu \therefore \frac{\left(\frac{\partial u}{\partial \eta}\right)}{u} = b \therefore \int \frac{\frac{\partial u}{\partial \eta}}{u} d\eta = \int b \, d\eta = b\eta + c = \ln|u|$

~~$b\eta = \ln|u|$~~ $\therefore$ u depend on ξ as a param $\therefore$

$|u| = e^{b\eta + c} = e^c e^{b\eta} \therefore u = \pm e^c e^{b\eta} = Ae^{b\eta}$ A is arb const

as far as η is concerned $\therefore$ A is a function of ξ $\therefore$

$u(x,y) = A(\xi)e^{b\eta} = A(y+ax)e^{bx}$

$\Big\{ \int \frac{\frac{\partial u}{\partial \eta}}{u} \partial\eta = \int b \, \partial\eta = \ln|u| = b\eta + f(\xi) \therefore |u| = e^{b\eta + f(\xi)} = e^{f(\xi)}e^{b\eta} \therefore$

$u = \pm e^{f(\xi)}e^{b\eta} = A(\xi)e^{b\eta}$ $A(\xi)$ is arb func of ξ $\therefore$

$u(\xi,\eta) = A(\xi)e^{b\eta} = A(y+ax)e^{bx} = u(x,y) \Big\}$

~~5b Sol 2~~ $\therefore u(x,y) = A(y+4x)e^{3x}$

\5b Sol/ 2 charac coord be found by trying $u = f(\alpha x + y)$ for Z

homogenous version of Z eqn giving $(4-\alpha) f'(\alpha x + y) = 0 \therefore$

$\alpha = 4 \therefore$ take $\xi = 4x + y$ & $\eta = x \therefore$ ~~$u_x = 4u_\xi$~~ $u_x = 4u_\xi + u_\eta$, $u_y = u_\xi$

$\therefore u_x - 4u_y - 3u = (4u_\xi + u_\eta) - 4(u_\xi) - 3u = u_\eta - 3u = 0$

Sol to this is $u(\xi,\eta) = e^{3\eta} f(\xi)$ with arb func f in Z

the original coords gives: $u = e^{3x} f(4x+y)$

alternative equiv forms: $u = f(4x+y)e^{-3y/4}$, $u = f(4x+y)e^{3(x-4y)/17}$

\5c/ $x^2 u_{xx} - 4y^2 u_{yy} + x u_x - 4y u_y = 0$

$\therefore$ change of variables: $x = e^t$, $y = e^s$ $(x,y) \mapsto (t,s) \therefore t = \ln x, s = \ln y$

$\therefore u_x = u_t t_x = \frac{1}{x} u_t \therefore u_{xx} = \frac{\partial}{\partial x}\left(\frac{1}{x} u_t\right) = \frac{\partial}{\partial x}\left(\frac{1}{x}\right) u_t + \frac{\partial}{\partial x}(u_t)\frac{1}{x} =$

$-\frac{1}{x^2} u_t + \frac{\partial}{\partial x}(u_t)\frac{1}{x} = -\frac{1}{x^2}u_t + \frac{1}{x}\frac{\partial}{\partial x}(u_t) = -\frac{1}{x^2}u_t + \frac{1}{x}\frac{\partial}{\partial t}(u_t)\frac{\partial t}{\partial x} = -\frac{1}{x^2}u_t + \frac{1}{x}u_{tt}\frac{1}{x} =$

$\frac{1}{x^2}u_{tt} - \frac{1}{x^2}u_t$ $u_y = u_s s_y = \frac{1}{y} u_s$ $u_{yy} = \frac{1}{y^2}u_{ss} - \frac{1}{y^2}u_s \therefore$

LHS $= x^2\left(\frac{1}{x^2}u_{tt} - \frac{1}{x^2}u_t\right) - 4y^2\left(\frac{1}{y^2}u_{ss} - \frac{1}{y^2}u_s\right) + x\left(\frac{1}{x}u_t\right) - 4y\left(\frac{1}{y}u_s\right) =$

$u_{tt} - u_t - 4u_{ss} + 4u_s + u_t - 4u_s = u_{tt} - 4u_{ss} = 0 =$ RHS $\therefore$

if $u = f(t + \alpha s) \therefore u_{tt} = f''(t+\alpha s)$ $u_{ss} = \alpha^2 f''(t+\alpha s) \therefore$

$f''(t+\alpha s) - 4\alpha^2 f''(t+\alpha s) = (1-4\alpha^2) f''(t+\alpha s) = 0$ f is arb func $\therefore f \neq 0$

$\therefore f''(t+\alpha s) \neq 0 \; \forall t,s \therefore 1 - 4\alpha^2 = 0 \therefore \frac{1}{4} = \alpha^2 \therefore \alpha_1 = \frac{1}{2}, \alpha_2 = -\frac{1}{2}$

Week 1 $\therefore \xi = t + \frac{1}{2}s \quad , \eta = t - \frac{1}{2}s \quad \therefore$

$u = f(t + \frac{1}{2}s) + g(t - \frac{1}{2}s) = f(\xi) + g(\eta) =$

$f(\ln x + \frac{1}{2}\ln y) + g(\ln x - \frac{1}{2}\ln y) =$

$f(\ln x + \ln(y^{1/2})) + g(\ln x - \ln(y^{1/2})) = \cancel{txt}$

$\cancel{f(\ln(y^{1/2} x)) + g(\ln}$ $f(\ln(y^{1/2} x)) + g(-\ln(y^{1/2}/x)) =$

$\tilde{F}(y^{1/2} x) + \tilde{G}(y^{1/2}/x) = u(x,y) = F((y^{1/2}x)^2) + G((y^{1/2}/x)^2) =$

$F(yx^2) + G(y/x^2) = u(x,y)$

5c sol put $x = e^s \quad , y = e^t \quad , s = \ln x \quad t = \ln y$

$u_x = \frac{1}{x} u_s \quad u_{xx} = \frac{1}{x^2}(u_{ss} - u_s) \quad u_y = \frac{1}{y} u_t \quad , u_{yy} = \frac{1}{y^2}(u_{tt} - u_t) \quad \therefore$

PDE becomes: $u_{ss} - 4u_{tt} = 0 \therefore$ let $u = f(t + \alpha s) \Rightarrow \alpha^2 - 4 = 0 \Rightarrow$

$\alpha = \pm 2 \quad \therefore$ sol is: $u = f(t + 2s) + g(t - 2s) =$

$f(\ln y + 2\ln x) + g(\ln y - 2\ln x) = F(x^2 y) + G(y/x^2)$

6 $u_{xx} - u_{tt} = 0 \qquad \xi = Ax + Bt \qquad \tau = Cx + Dt$

$\therefore$ is $u = f(x + \alpha t) \quad \therefore u_{xx} = f''(x + \alpha t) \quad , u_{tt} = \alpha^2 f''(x + \alpha t) \quad \therefore$

$LHS = f''(x + \alpha t) - \alpha^2 f''(x + \alpha t) = (1 - \alpha^2) f''(x + \alpha t) = 0$

f is arb func $\therefore \exists f''(x + \alpha t) \neq 0 \quad \therefore \quad 1 - \alpha^2 = 0 = (1 + \alpha)(1 - \alpha)$

$\therefore \alpha = -1 \quad , \alpha = 1 \qquad \therefore \alpha_1 = 1 \quad , \alpha_2 = -1 \quad \therefore$

$\xi = x + 1t \quad , \tau = x - 1t \quad \therefore$

$\xi = Ax + Bt = 1x + 1t \quad , \tau = Cx + Dt = 1x - 1t \quad \therefore$

$A = 1 \quad , B = 1 \quad , C = 1 \quad , D = -1$ coeffs are real, constant, independent

$\cosh(0) = 1 = A = \cosh(\phi) \quad \therefore \phi = 0 \quad \therefore \quad B = C = \cosh\phi$.

$D = -1 = -1(1) = -\cosh(\phi)$

6 sol use formulas ~~for~~ Z generic transformation of Z coeffs of Z quatilinear 2nd order PDE $\alpha = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2$

$\beta = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y \qquad \gamma = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2$

where need to change $y \mapsto t, \eta \mapsto \tau$ giving:

$\alpha = a\xi_x^2 + 2b\xi_x\xi_t + c\xi_t^2 \qquad \beta = a\xi_x\tau_x + b(\xi_x\tau_t + \xi_t\tau_x) + c\xi_t\tau_t$

$\gamma = a\tau_x^2 + 2b\tau_x\tau_t + c\tau_t^2$ for Z requested linear transformation, Identifying $\xi_x = A$, $\xi_t = B$, $\tau_x = C$, $\tau_t = D$ further, ii 2 gives eqn 8

its transformation form we have $a = \alpha = 1$, $b = \beta = 0$, $c = \gamma = -1$

$\therefore$ subbing into further transformation formulas:

$A^2 - B^2 = 1 \qquad AC - BD = 0 \qquad C^2 - D^2 = -1$

from $A^2 - B^2 = 1$ & given $B > 0$: $B = \sinh\phi$

similarly from $D^2 - C^2 = 1$, $C > 0$ & $D > 0$: $D = \cosh\psi$, $C = \sinh\psi$

for some $\psi > 0$ $\therefore$ reformulating $AC - BD = 0$ as $B/A = C/D$

see: $\tanh\phi = \tanh\psi \therefore \psi = \phi \therefore A = D = \cosh\phi$, $B = C = \sinh\phi$

~~Stating:~~ $(x, t) \mapsto (\xi, \tau) \therefore u_x = u_\xi \xi_x + u_\tau \tau_x \therefore$

$u_{xx} = \frac{\partial}{\partial x}\left[u_\xi \xi_x + u_\tau \tau_x\right] = \frac{\partial}{\partial x}\left[u_\xi \xi_x\right] + \frac{\partial}{\partial x}\left[u_\tau \tau_x\right] =$

$\frac{\partial}{\partial x}\left[u_\xi\right]\xi_x + u_\xi \frac{\partial}{\partial x}\left[\xi_x\right] + \frac{\partial}{\partial x}(u_\tau)\tau_x + u_\tau \frac{\partial}{\partial x}(\tau_x) =$

$\left[\frac{\partial}{\partial x}(u_\xi)\right]\xi_x + u_\xi \xi_{xx} + \left[\frac{\partial}{\partial x}(u_\tau)\right]\tau_x + u_\tau \tau_{xx} =$

$\left[\frac{\partial}{\partial \xi}(u_\xi)\xi_x + \frac{\partial}{\partial \tau}(u_\xi)\tau_x\right]\xi_x + u_\xi \xi_{xx} + \left[\frac{\partial}{\partial \xi}(u_\tau)\xi_x + \frac{\partial}{\partial \tau}(u_\tau)\tau_x\right]\tau_x + u_\tau \tau_{xx} =$

$\left[u_{\xi\xi}\xi_x + u_{\xi\tau}\tau_x\right]\xi_x + u_\xi \xi_{xx} + \left[u_{\xi\tau}\xi_x + u_{\tau\tau}\tau_x\right]\tau_x + u_\tau \tau_{xx} =$

$u_{\xi\xi}\xi_x\xi_x + u_{\xi\tau}\xi_x\tau_x + u_\xi \xi_{xx} + u_{\xi\tau}\xi_x\tau_x + u_{\tau\tau}\tau_x\tau_x + u_\tau \tau_{xx} =$

$u_{\xi\xi}\xi_x^2 + u_{\xi\tau}\xi_x\tau_x + u_\xi \xi_{xx} + u_{\xi\tau}\xi_x\tau_x + u_{\tau\tau}\tau_x^2 + u_\tau \tau_{xx} =$

$u_{\xi\xi}\xi_x^2 + u_\xi \xi_{xx} + 2u_{\xi\tau}\xi_x\tau_x + u_{\tau\tau}\tau_x^2 + u_\tau \tau_{xx}$ and

$u_t = u_\xi \xi_t + u_\tau \tau_t \therefore$

$u_{tt} = \frac{\partial}{\partial t}\left[u_\xi \xi_t + u_\tau \tau_t\right] = \frac{\partial}{\partial t}\left[u_\xi \xi_t\right] + \frac{\partial}{\partial t}\left[u_\tau \tau_t\right] =$

$\frac{\partial}{\partial t}\left[u_\xi\right]\xi_t + u_\xi \frac{\partial}{\partial t}\left[\xi_t\right] + \frac{\partial}{\partial t}(u_\tau)\tau_t + u_\tau \frac{\partial}{\partial t}(\tau_t) =$

$\left[\frac{\partial}{\partial t}(u_\xi)\right]\xi_t + u_\xi \xi_{tt} + \left[\frac{\partial}{\partial t}(u_\tau)\right]\tau_t + u_\tau \tau_{tt} =$

$\left[\frac{\partial}{\partial \xi}(u_\xi)\xi_t + \frac{\partial}{\partial \tau}(u_\xi)\tau_t\right]\xi_t + u_\xi \xi_{tt} + \left[\frac{\partial}{\partial \xi}(u_\tau)\xi_t + \frac{\partial}{\partial \tau}(u_\tau)\tau_t\right]\tau_t + u_\tau \tau_{tt} =$

$\left[u_{\xi\xi}\xi_t + u_{\xi\tau}\tau_t\right]\xi_t + u_\xi \xi_{tt} + \left[u_{\xi\tau}\xi_t + u_{\tau\tau}\tau_t\right]\tau_t + u_\tau \tau_{tt} =$

$u_{\xi\xi}\xi_t\xi_t + u_{\xi\tau}\xi_t\tau_t + u_\xi \xi_{tt} + u_{\xi\tau}\xi_t\tau_t + u_{\tau\tau}\tau_t\tau_t + u_\tau \tau_{tt} =$

$u_{\xi\xi}\xi_t^2 + u_\xi \xi_{tt} + 2u_{\xi\tau}\xi_t\tau_t + u_{\tau\tau}\tau_t^2 + u_\tau \tau_{tt}$ $\therefore$ Sub into ODE:

$LHS = u_{xx} - u_{tt} =$