Q1 a:

$u = 0 \therefore$

$\dot{x}_1 = 2x_2 - 2x_1 + x_1 - x_1^3$, $\dot{x}_2 = -2x_2 + 2x_1 + x_2 - x_2^3 \therefore$

equilibria when $(\dot{x}_1, \dot{x}_2) = (0, 0) \therefore$ let:

$\dot{x}_1 = 2x_2 - x_1 - x_1^3 = 0 \quad \therefore \frac{1}{2}x_1^3 + \frac{1}{2}x_1 = x_2 \quad \therefore$

$\dot{x}_2 = -x_2 + 2x_1 - x_2^3 = 0 = -(\frac{1}{2}x_1^3 + \frac{1}{2}x_1) + 2x_1 - (\frac{1}{2}x_1^3 + \frac{1}{2}x_1)^3 \quad \therefore$

if $x_1 = 0$: $-(\frac{1}{2}(0)^3 + \frac{1}{2} \times 0) + 2(0) - (\frac{1}{2}(0)^3 + \frac{1}{2}(0))^3 = 0 = \dot{x}_2 \quad \therefore \frac{1}{2}(0)^3 + \frac{1}{2}(0) = 0 = x_2$

$\therefore (x_1 = 0, x_2 = 0) = (0, 0)$ is an equilibrium point.

$\because 2(0) - 0 - (0)^3 = \dot{x}_1 = 0$.

if $x_1 = 1$: $-(\frac{1}{2}(1)^3 + \frac{1}{2}(1)) + 2(1) - (\frac{1}{2}(1)^3 + \frac{1}{2}(1))^3 =$

$-1 + 2 - 1 = 0 = \dot{x}_2 \quad \therefore \quad \frac{1}{2}(1)^3 + \frac{1}{2}(1) = x_2 = 1 \quad \therefore$

$(x_1 = 1, x_2 = 1) = (1, 1)$ is an equilibrium point.

$\because 2(1) - 1 - (1)^3 = \dot{x}_1 = 0$

if $x_1 = -1$: $\frac{1}{2}(-1)^3 + \frac{1}{2}(-1) = x_2 = -1 \quad \therefore$

$2(-1) - (-1) - (-1)^3 = \dot{x}_1 = 0$,

$-(\frac{1}{2}(-1)^3 + \frac{1}{2}(-1)) + 2(-1) - (\frac{1}{2}(-1)^3 + \frac{1}{2}(-1))^3 =$

$1 - 2 + 1 = \dot{x}_2 = 0 \quad \therefore$

$(x_1 = -1, x_2 = -1) = (-1, -1)$ is an equilibrium point. $\therefore$

the equilibria are: $(0, 0)$, $(1, 1)$, $(-1, -1)$.

Q1b

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 2x_2 - x_1 - x_1^3 \\ -x_2 + 2x_1 - x_2^3 \end{bmatrix} \therefore$$

$f_1 = 2x_2 - x_1 - x_1^3$, $f_2 = -x_2 + 2x_1 - x_2^3$ $\therefore$

$\frac{\partial f_1}{\partial x_1} = -1 - 3x_1^2$, $\frac{\partial f_2}{\partial x_1} = 2$,

$\frac{\partial f_1}{\partial x_2} = 2$, $\frac{\partial f_2}{\partial x_2} = -1 - 3x_2^2$ $\therefore$

$$\frac{\partial f}{\partial x} = \frac{\partial}{\partial x}\begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} -1-3x_1^2 & 2 \\ 2 & -1-3x_2^2 \end{bmatrix} \therefore$$

for equilibria $(0,0)$: $\frac{\partial f}{\partial x}\Big|_{(0,0)} = \begin{bmatrix} -1-3(0)^2 & 2 \\ 2 & -1-3(0)^2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \therefore$

$\frac{\partial f}{\partial x}\Big|_{(0,0)} - \lambda I = \begin{bmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{bmatrix} \therefore$

$\det\left(\frac{\partial f}{\partial x}\Big|_{(0,0)} - \lambda I\right) = \begin{vmatrix} -1-\lambda & 2 \\ 2 & -1-\lambda \end{vmatrix} = (-1-\lambda)(-1-\lambda) - 2\times 2 = 1 + \lambda^2 + 2\lambda - 4 =$

$\lambda^2 + 2\lambda - 3 = (\lambda+3)(\lambda-1) = 0$ $\therefore$

$\lambda_1 = -3$, $\lambda_2 = 1$ $\therefore$ $\lambda_1 \in \mathbb{R}_{<0}$, $\lambda_2 \in \mathbb{R}_{>0}$ $\therefore$

the equilibrium $(0,0)$ is a saddle point.

for equilibria $(1,1)$: $\frac{\partial f}{\partial x}\Big|_{(1,1)} = \begin{bmatrix} -1-3x_1^2 & 2 \\ 2 & -1-3x_2^2 \end{bmatrix}\Bigg|_{(1,1)} = \begin{bmatrix} -1-3(1)^2 & 2 \\ 2 & -1-3(1)^2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix}$

$\therefore$ for its eigenvalues: $\det\left(\frac{\partial f}{\partial x}\Big|_{(1,1)} - \lambda I\right) = \begin{vmatrix} -4-\lambda & 2 \\ 2 & -4-\lambda \end{vmatrix} =$

$(-4-\lambda)(-4-\lambda) - 2\times 2 = 16 + \lambda^2 - 8\lambda - 4 = \lambda^2 + 8\lambda + 12 = (\lambda+6)(\lambda+2) = 0$ $\therefore$

$\lambda_1 = -6$, $\lambda_2 = -2$ $\therefore$ $\lambda_1, \lambda_2 \in \mathbb{R}_{<0}$ $\therefore$

the equilibria $(1,1)$ is a stable node.

for equilibria $(-1,-1)$: $\frac{\partial f}{\partial x}\Big|_{(-1,-1)} = \begin{bmatrix} -1-3x_1^2 & 2 \\ 2 & -1-3x_2^2 \end{bmatrix}\Bigg|_{(-1,-1)} = \begin{bmatrix} -1-3(-1)^2 & 2 \\ 2 & -1-3(-1)^2 \end{bmatrix} = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix} \therefore$

same as before: for its eigenvalues: $\det\left(\frac{\partial f}{\partial x}\Big|_{(-1,-1)} - \lambda I\right) =$

$\begin{vmatrix} -4-\lambda & 2 \\ 2 & -4-\lambda \end{vmatrix} = (\lambda+6)(\lambda+2) = 0$ $\therefore$ $\lambda_1 = -6$, $\lambda_2 = -2$ $\therefore$ $\lambda_1, \lambda_2 \in \mathbb{R}_{<0}$ $\therefore$

the equilibria $(-1,-1)$ is a stable node.

\Question 1C:/

The equilibria (0,0) has eigen values : $\lambda_1 = -3$, $\lambda_2 = 1$.
The equilibria (1,1) has eigen values $\lambda_1 = -6$, $\lambda_2 = -2$.
The equilibria (-1,-1) has eigen values $\lambda_1 = -6$, $\lambda_2 = -2$ $\therefore$
all the equilibrium points have atleast one eigen value that is negative $\therefore$ they all have atleast one eigen value that is not positive $\therefore$ none of the equilibria are unstable focus or unstable nodes.

$\therefore$ By the Poincare-Bendixson Criterion; the Criterion for the existence of a limit cycle is not fulfilled.

$\therefore$ The System does not have limit cycles.

Question 1d:

$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \therefore$$

$$\dot{x} = \frac{d}{dt} x = \frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = Ax + Bu =$$

$$\begin{bmatrix} 2(x_2 - x_1) + x_1(1 - x_1^2) \\ -2(x_2 - x_1) + x_2(1 - x_2^2) + u \end{bmatrix} = \begin{bmatrix} 2(x_2 - x_1) + x_1(1 - x_1^2) \\ -2(x_2 - x_1) + x_2(1 - x_2^2) \end{bmatrix} + \begin{bmatrix} 0 \\ u \end{bmatrix} =$$

$$\begin{bmatrix} 2x_2 - x_1 - x_1^3 \\ -x_2 + 2x_1 - x_2^3 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = \begin{bmatrix} (-1 - x_1^2)x_1 + 2(x_2) \\ (2)x_1 + (-1 - x_2^2)x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u =$$

$$\begin{bmatrix} -1 - x_1^2 & 2 \\ 2 & -1 - x_2^2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u = A\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + Bu = Ax + Bu \quad \therefore$$

$$A = \begin{bmatrix} -1 - x_1^2 & 2 \\ 2 & -1 - x_2^2 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$\therefore$ about the origin is $(x_1, x_2) = (0, 0)$ $\therefore$

$$A = \begin{bmatrix} -1 - 0^2 & 2 \\ 2 & -1 - 0^2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Question 1e:

Controllability matrix is: $M = [B \ \vdots \ AB] =$

$$\left[\begin{bmatrix}0\\1\end{bmatrix} \vdots \begin{bmatrix}-1-x_1^2 & 2\\ 2 & -1-x_1^2\end{bmatrix}\begin{bmatrix}0\\1\end{bmatrix}\right] = \left[\begin{bmatrix}0\\1\end{bmatrix} \vdots \begin{bmatrix}2\\ -1-x_1^2\end{bmatrix}\right] =$$

$$\begin{bmatrix}0 & 2\\ 1 & -1-x_1^2\end{bmatrix} \therefore$$

Rank$(M) = 2$ is full rank matrix. $\therefore$

$$\det(M) = \begin{vmatrix}0 & 2\\ 1 & -1-x_1^2\end{vmatrix} = 0(-1-x_1^2) - 1\times 2 = -2 \neq 0 \ \therefore \text{Controllable.} \therefore$$

The pair (A, B) is Controllable.

about the origin is $(x_1, x_2) = (0, 0)$ $\therefore$

$$M = \begin{bmatrix}0 & 2\\ 1 & -1-0^2\end{bmatrix} = \begin{bmatrix}0 & 2\\ 1 & -1\end{bmatrix}$$

and Rank$(M) = 2$ $\therefore$

The pair (A, B) is Controllable

Question 18:

$K = [K_1 \quad K_2]$

eigenvalues $-5, -8$ $\therefore$ desired characteristic polynomial:

$(S-5)(S+8) = S^2 + 5S + 8S + 40 = S^2 + 13S + 40$

$$A_{cl}^{+} = A - BK = \begin{bmatrix} -1-x_1^2 & 2 \\ 2 & -1-x_2^2 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} K_1 & K_2 \end{bmatrix} = \begin{bmatrix} -1-x_1^2 & 2 \\ 2 & -1-x_2^2 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ K_1 & K_2 \end{bmatrix} =$$

$$\begin{bmatrix} -1-x_1^2 & 2 \\ 2-K_1 & -1-x_2^2-K_2 \end{bmatrix}$$ $\therefore$ for characteristic polynomial of closed loop:

$$\det(SI - A_{cl}^{+}) = \det(SI - (A-BK)) = \det\left(\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} -1-x_1^2 & 2 \\ 2-K_1 & -1-x_2^2-K_2 \end{bmatrix} \right)$$

$$= \det\left(\begin{bmatrix} S+1+x_1^2 & -2 \\ -2+K_1 & S+1+x_2^2+K_2 \end{bmatrix} \right) = \begin{vmatrix} S+1+x_1^2 & -2 \\ -2+K_1 & S+1+x_2^2+K_2 \end{vmatrix} =$$

$(S+1+x_1^2)(S+1+x_2^2+K_2) - [(-2)(-2+K_1)] =$

$S^2 + S + x_2^2 S + K_2 S + S + 1 + x_2^2 + K_2 + x_1^2 S + x_1^2 + x_1^2 x_2^2 + K_2 x_1^2 - [4 - 2K_1] =$

$S^2 + (1 + x_2^2 + K_2 + 1 + x_1^2) S + (1 + x_2^2 + K_2 + x_1^2 + x_1^2 x_2^2 + K_2 x_1^2 - 4 + 2K_1) =$

$S^2 + (2 + x_1^2 + x_2^2 + K_2) S + (-3 + 2K_1 + K_2 + x_1^2 + K_2 x_1^2 + x_2^2 + x_1^2 x_2^2) =$

$S^2 + 13S + 40$ $\therefore$

$13 = 2 + x_1^2 + x_2^2 + K_2$, $\quad 40 = -3 + 2K_1 + K_2 + x_1^2 + K_2 x_1^2 + x_2^2 + x_1^2 x_2^2$ $\therefore$

$11 - x_1^2 - x_2^2 = K_2$ $\therefore$

$40 = -3 + 2K_1 + (11 - x_1^2 - x_2^2) + x_1^2 + (11 - x_1^2 - x_2^2) x_1^2 + x_2^2 + x_1^2 x_2^2$ $\therefore$

$43 = 2K_1 + 11 - x_1^2 - x_2^2 + 11x_1^2 - x_1^4 - x_1^2 x_2^2 + x_2^2 + x_1^2 x_2^2 + x_1^2 =$

$2K_1 + 11 + 11x_1^2 - x_1^4 - x_1^2 x_2^2 + x_1^2 x_2^2 =$

$2K_1 + 11 + 11x_1^2 - x_1^4$ $\therefore$

$2K_1 = +32 - 11x_1^2 + x_1^4$ $\therefore$

$K_1 = 16 - \frac{11}{2} x_1^2 + \frac{1}{2} x_1^4$ $\therefore$

$u = -Kx = -[K_1 \quad K_2] x =$

$-\left[16 - \frac{11}{2} x_1^2 + \frac{1}{2} x_1^4 \quad 11 - x_1^2 - x_2^2\right] x =$

$\left[-16 + \frac{11}{2} x_1^2 - \frac{1}{2} x_1^4 \quad -11 + x_1^2 + x_2^2\right] x = u$ is control law.

about the origin is $(x_1, x_2) = (0, 0)$ $\therefore$ $u = -Kx =$

$\left[-16 + \frac{11}{2}(0)^2 - \frac{1}{2}(0)^4 \quad -11 + (0)^2 + 0^2\right] x = u = [-16 \quad -11] x$

Question 1g:

$T = M \times W \quad , \quad M = \begin{bmatrix} 0 & 2 \\ 1 & -1-x_1^2 \end{bmatrix}$

full rank $\therefore n=2 \quad \therefore$

$W = \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}$

where $\det(SI-A) = S^2 + a_1 S + a_2 \quad , \quad A = \begin{bmatrix} -1-x_1^2 & 2 \\ 2 & -1-x_2^2 \end{bmatrix} \therefore$

$\det(SI-A) = \det\left(\begin{bmatrix} S & 0 \\ 0 & S \end{bmatrix} - \begin{bmatrix} -1-x_1^2 & 2 \\ 2 & -1-x_2^2 \end{bmatrix}\right) = \det\left(\begin{bmatrix} S+1+x_1^2 & -2 \\ -2 & S+1+x_2^2 \end{bmatrix}\right) =$

$\begin{vmatrix} S+1+x_1^2 & -2 \\ -2 & S+1+x_2^2 \end{vmatrix} = (S+1+x_1^2)(S+1+x_2^2) - (-2)(-2) =$

$S^2 + S + x_2^2 S + S + 1 + x_2^2 + x_1^2 S + x_1^2 + x_1^2 x_2^2 - 4 =$

$S^2 + (1 + x_2^2 + 1 + x_1^2) S + (1 + x_2^2 + x_1^2 + x_1^2 x_2^2 - 4) = S^2 + a_1 S + a_2 \quad \therefore$

$a_1 = 1 + x_2^2 + 1 + x_1^2 = 2 + x_1^2 + x_2^2 \quad \therefore$

$W = \begin{bmatrix} 2 + x_1^2 + x_2^2 & 1 \\ 1 & 0 \end{bmatrix} \quad \therefore$

The transformation matrix is: $T = M \times W =$

$\begin{bmatrix} 0 & 2 \\ 1 & -1-x_1^2 \end{bmatrix} \begin{bmatrix} 2+x_1^2+x_2^2 & 1 \\ 1 & 0 \end{bmatrix} =$

$\begin{bmatrix} 0+2(1) & 0+0 \\ 2+x_1^2+x_2^2-1-x_1^2 & 1+0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1+x_2^2 & 1 \end{bmatrix}$

$\therefore$ about the origin: $(x_1, x_2) = (0,0) \quad \therefore$

$T = \begin{bmatrix} 2 & 0 \\ 1+0^2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$

Question 2:

$$f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} \doteq \dot{x} = \begin{bmatrix}\dot x_1\\ \dot x_2\end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + x_2(2-3x_1^2-2x_2^2)\end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1+2x_2-3x_1^2x_2-2x_2^3\end{bmatrix} \therefore$$

$\therefore x_2 = 0 \quad \therefore -x_1 + x_2(2-3x_1^2-2x_2^2) = 0 =$

$-x_1 + 0(2-3\times x_1^2 - 2 - 2\times 0^2) = -x_1 + 0 = 0 = -x_1 \quad \therefore$

$x_1 = 0 \quad \therefore$

equilibrium point at $(x_1,x_2)=(0,0)$ is a unique equilibrium

$\therefore$ Jacobian $\frac{\partial f}{\partial x}\Big|_{(0,0)} = \begin{bmatrix}0 & 1\\ -1-6x_1x_2 & 2-3x_1^2-6x_2^2\end{bmatrix}\Bigg|_{(0,0)} =$

$$\begin{bmatrix}0&1\\-1-6(0)(0) & 2-3(0)^2-6(0)^2\end{bmatrix} = \begin{bmatrix}0&1\\-1&2\end{bmatrix} \quad \therefore$$

Eigenvalues by: $\begin{vmatrix}0-\lambda & 1\\ -1 & 2-\lambda\end{vmatrix} = \begin{vmatrix}-\lambda & 1 \\ -1 & 2-\lambda\end{vmatrix} = -\lambda(2-\lambda)-1(-1) =$

$\lambda^2-2\lambda+1 = (\lambda-1)(\lambda-1)=0 \quad \therefore$ eigenvalues:

$\lambda_1 = 1, \lambda_2 = 1 \quad \therefore \quad \lambda_1,\lambda_2 \geq 0 \quad \therefore$

equilibria $(x_1,x_2)=(0,0)$ is an unstable node

Let $V = x_1^2+x_2^2 \quad \therefore$

$\nabla V(x) = \nabla V = \nabla V(x_1, x_2) = \nabla(x_1^2+x_2^2) = \left[\frac{\partial}{\partial x_1}(x_1^2+x_2^2) \quad \frac{\partial}{\partial x_2}(x_1^2+x_2^2)\right] = [2x_1 \quad 2x_2]$

$\therefore f = f(x) \quad \therefore \quad f(x)\cdot\nabla V(x) = [x \quad -x_1+2x_2-3x_1^2x_2-2x_2^3]\cdot[2x_1 \quad 2x_2] =$

$2x_1x_2 + 2x_2(-x_1+2x_2-3x_1^2x_2-2x_2^3) = 2x_1x_2 - 2x_1x_2 + 4x_2^2 - 6x_1^2x_2^2 - 4x_2^4 =$

$4x_2^2 - 6x_1^2x_2^2 - 4x_2^4 = 2x_2^2(2-3x_1^2-2x_2^2) = 2x_2^2(2-x_1^2-2x_1^2-2x_2^2) =$

$2x_2^2(2-x_1^2-2(x_1^2+x_2^2)) = 2x_2^2(2-x_1^2-2V) = f\cdot\nabla V \quad \therefore$

$2x_2^2 \geq 0 \quad \therefore \quad f(x)\cdot\nabla V(x) \leq 0$ if $2-x_1^2-2V\leq 0 \quad \therefore 2-x_1^2\leq 2V \quad \therefore 1-\frac{1}{2} x_1^2 \leq V$

and $\frac{1}{2} x_1^2 \geq 0 \quad \therefore \quad 1-\frac{1}{2}x_1^2 \leq 1 \quad \therefore \quad 1-\frac{1}{2} x_1^2 \leq V$ if $V\geq 1 \quad \therefore$

if $V\geq 1 : f(x)\cdot\nabla V(x)\leq 0 \quad \therefore$

on a closed bounded set $M=\{x\in\mathbb{R}^2 \mid V(x)=x_1^2+x_2^2\leq 1\}$, which contains one equilibrium, which is unstable, the poincaré-Bendixson criterion holds. $\therefore$ it is possible to ensure all trajectories are trapped inside M. $\therefore$ It can be concluded that there is a periodic orbit in $M=\{x\in\mathbb{R}^2 \mid V(x)=x_1^2+x_2^2\leq 1\}$

$\therefore$ The system has a periodic orbit.

Question 3:

$f(x) = \frac{1}{x^2} = x^{-2}$ ∴ f is Lipschitz Continuous on an interval if f is Lipschitz continuous at every point in that interval. f is Lipschitz Continuous at a point in an interval if there exits a Lipschitz Constant $L > 0$ and a ~~ozz~~ neighbourhood that is a subset of that interval ~~such~~ of that point such that $\|f(x) - f(y)\| \leq L\|x-y\|$, for x, y in that neighbourhood.

∴ without loss of generallity let $x, y \in [\frac{1}{2}, 2]$ ∴ $x, y \in I$ and $x \geq y$ ∴

$f(x) = \frac{1}{x^2}$, $f(y) = \frac{1}{y^2}$ ∵ $x, y \geq \frac{1}{2}$ ∴

~~$x^2 y^3$~~ $x \leq 2x^2$, $y \leq 2y^2$ ∴ $x^2 \leq 2x^3$, $y^2 \leq 2y^3$ ∴

$x^2 \leq 4x^2y^2$, $y^2 \leq 4x^2y^2$ ∴ $x^2 \leq 8x^3y^2$, $x^2 \leq 8x^2y^3$ ∴

if $L_f = 64$: $L_f(x-y) = 64(x-y)$ ∴

$f(x) - f(y) = \frac{1}{x^2} - \frac{1}{y^2} = (\frac{1}{x^2} - \frac{1}{y^2})\frac{x^2y^2}{x^2y^2} = \frac{y^2}{x^2y^2} - \frac{x^2}{x^2y^2} = -(x^2-y^2)\frac{1}{x^2y^2}$ ∴

$x^2 - y^2 \leq 8x^3y^2 - 8x^2y^3 = 8(x^3y^2 - x^2y^3) \leq 64(x^3y^2 - x^2y^3)$ ∴

∵ $x^2 > 0$, $y^2 > 0$ ∵ $x, y \geq \frac{1}{2}$ ∴ $x^2y^2 > 0$ ∴

$(x^2 - y^2)\frac{1}{x^2y^2} \leq 64(x^3y^2 - x^2y^3)\frac{1}{x^2y^2}$ ∴

$(\frac{x^2}{x^2y^2} - \frac{y^2}{x^2y^2}) = (\frac{1}{y^2} - \frac{1}{x^2}) = -(\frac{1}{x^2} - \frac{1}{y^2}) \leq 64(x-y)$ ∴

$|-(\frac{1}{x^2} - \frac{1}{y^2})| = |\frac{1}{x^2} - \frac{1}{y^2}| = |f(x) - f(y)| \leq |64(x-y)| = 64|x-y|$ ∴

$|f(x) - f(y)| \leq 64|x-y| = L_f|x-y|$ ∴ $f(x)$ is Lipschitz continuous on the interval $I = [\frac{1}{2}; 2]$ with Lipschitz Constant $L_f = 64$. ∴

and $|f(x) - f(y)| \leq 64|x-y| \leq 128|x-y| = L_f|x-y|$ ∴ $f(x)$ is Lipschitz Continuous on the interval $I = [\frac{1}{2}; 2]$ with Lipschitz constant $L_f = 128$.

because $128 > 64$

\Question 4:/

$\dot{V} = \dot{V}(x_1, x_2) = \frac{d}{dt} V(x_1, x_2) = \frac{d}{dt}\left[x_1^2 + x_2^2\right] =$

$2x_1 \frac{d}{dt}(x_1) + 2x_2\left(\frac{d}{dt}(x_2)\right) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 =$

$2x_1(-3x_1 + x_1x_2) + 2x_2(x_1 - 2x_2) =$

$-6x_1^2 + 2x_1^2x_2 + 2x_1x_2 - 4x_2^2 = (-5x_1^2 - x_1^2) + 2x_1^2x_2 + (-3x_2^2 - x_2^2) + 2x_1x_2 =$

$(-5x_1^2 + 2x_1^2x_2 - 3x_2^2) + (-x_1^2 + 2x_1x_2 - x_2^2) =$

$(-5x_1^2 + 2x_1^2x_2 - 3x_2^2) - (x_1^2 - 2x_1x_2 + x_2^2) = (-5x_1^2 + 2x_1^2x_2 - 3x_2^2) - (x_1 - x_2)^2$

and $(x_1 - x_2)^2 \geq 0$ ∴ $(x_1 - x_2)^2$ is lower bounded by zero ∴

$\dot{V} = (-5x_1^2 + 2x_1^2x_2 - 3x_2^2) - (x_1 - x_2)^2 \leq -5x_1^2 + 2x_1^2x_2 - 3x_2^2 =$

$-x_1^2(5 - 2x_2) - 3x_2^2$ ∴

∴ $\dot{V} \leq -x_1^2(5 - 2x_2) - 3x_2^2$

and $x_2^2, x_1^2 \in \mathbb{R}_{\geq 0}$ ∴

$-3x_2^2 \leq 0$, $-x_1^2 \leq 0$ ∴

if $5 - 2x_2 > 0$; $\dot{V} < 0$ ∴

$5 > 2x_2$ ∴ $\frac{5}{2} > x_2$ ∴ $\left(\frac{5}{2}\right)^2 > (x_2)^2$ ∴ $\frac{25}{4} > x_2^2$ ∴

$\frac{25}{4} > x_1^2 + x_2^2$

∴ for $x_1^2 + x_2^2 < \frac{25}{4}$; $\dot{V} < 0$ ∴

The origin is locally asymptotically stable.

With invariant set $D = \left\{x \in \mathbb{R}^2 \middle| V(x) < \frac{25}{4}\right\}$

in which V is positive definite ∴

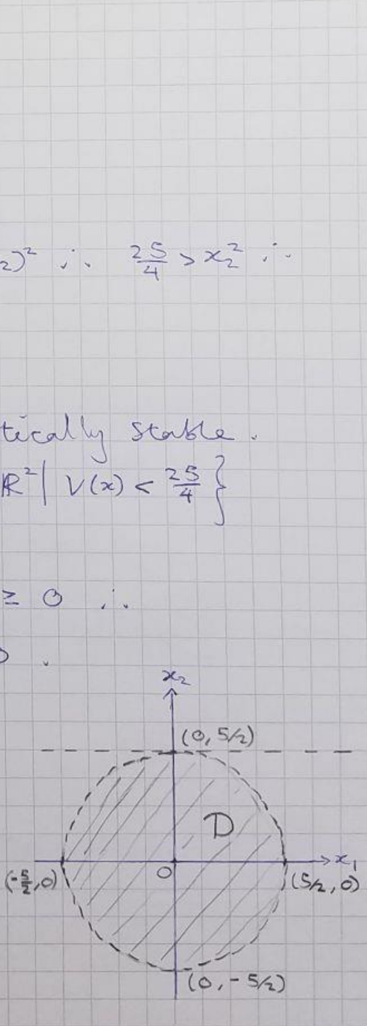
$V(0,0) = 0^2 + 0^2 = 0$ and $x_1^2 \geq 0$, $x_2^2 \geq 0$ ∴

for $x_1, x_2 \neq 0$: $V(x_1, x_2) = x_1^2 + x_2^2 > 0$.

and $\dot{V}$ is negative definite ∴

$\dot{V}(0,0) = 0$ and $\dot{V} \leq 0$ and

for $x_1, x_2 \neq 0$: $\dot{V} < 0$



Q5a:

$$M, g, k, c_1, c_2 \in \mathbb{R}_{>0}$$

$$M\ddot{y} = Mg - c_1\dot{y} - c_2\dot{y}|\dot{y}| - ky \quad \therefore$$

$$\ddot{y} = M^{-1}Mg - M^{-1}c_1\dot{y} - M^{-1}c_2\dot{y}|\dot{y}| - M^{-1}ky \quad \therefore$$

$$\ddot{y} = g - M^{-1}c_1\dot{y} - M^{-1}c_2\dot{y}|\dot{y}| - M^{-1}ky$$

$$x_1 = y - Mgk^{-1} \quad , x_2 = \dot{y} \quad \therefore$$

$$\dot{x}_1 = \frac{d}{dt}x_1 = \frac{d}{dt}(y - Mgk^{-1}) = \frac{d}{dt}y = \dot{y} = x_2$$

$$\dot{x}_2 = \frac{d}{dt}x_2 = \frac{d}{dt}\dot{y} = \ddot{y} = g - M^{-1}c_1\dot{y} - M^{-1}c_2\dot{y}|\dot{y}| - M^{-1}ky$$

and $(x_1 + Mgk^{-1}) = y$ $\therefore$

$$\dot{x}_2 = g - M^{-1}c_1x_2 - M^{-1}c_2x_2|x_2| - M^{-1}k(x_1 + Mgk^{-1}) =$$

$$g - M^{-1}c_1x_2 - M^{-1}c_2x_2|x_2| - M^{-1}kx_1 - M^{-1}kMgk^{-1} =$$

$$g - M^{-1}c_1x_2 - M^{-1}c_2x_2|x_2| - M^{-1}kx_1 - g =$$

$$-M^{-1}c_1x_2 - M^{-1}c_2x_2|x_2| - M^{-1}kx_1 = \dot{x}_2 \quad \therefore$$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -M^{-1}c_1x_2 - M^{-1}c_2x_2|x_2| - M^{-1}kx_1 \end{bmatrix} \quad \therefore$$

Compact state space: $x = \begin{bmatrix} y & \dot{y} \end{bmatrix} \in \mathbb{R}^2$

$$\downarrow \quad \downarrow$$

$$x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}$$

\Question 5b:/ The origin of $\dot{x}=f(x)$ is globally asymptotically stable if it is Lyapunov stable and for all $x(0)\in\mathbb{R}^n$, $\lim_{t\to\infty}x(t)=0$

The origin of $\dot{x}=f(x)$ is Lyapunov stable, if for all $\varepsilon>0$, $\exists\,\delta=\delta(\varepsilon)>0$ such that if $\|x(0)\|<\delta$ then $\|x(t)\|<\varepsilon$, $\forall t\geq 0$.

$\therefore$ origin is $(x_1,x_2)=(0,0)$ $\therefore$

$$f=\begin{bmatrix} f_1 \\ f_2 \end{bmatrix}=\dot{x}=\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} \quad \therefore$$

at origin: $\dot{x}\Big|_{(0,0)}=\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}\Bigg|_{(0,0)}=\begin{bmatrix} x_2 \\ -M^{-1}C_1x_2-M^{-1}C_2x_2|x_2|-M^{-1}kx_1 \end{bmatrix}\Bigg|_{(0,0)}=$

$\begin{bmatrix} 0 \\ -M^{-1}C_1(0)-M^{-1}C_2(0)|0|-M^{-1}k(0) \end{bmatrix}=\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\therefore$ $\dot{x}_1=0$, $\dot{x}_2=0$ at the origin. $\therefore$ (x_1,x_2) is an equilibrium point for $(x_1,x_2)=(0,0)$.

$x_1^2\geq 0$ and $x_2^2\geq 0$ $\therefore$ $V(x_1,x_2)=ax_1^2+bx_2^2\geq 0$ for $a,b>0$.

$V(x_1=0,x_2=0)=a(0)^2+b(0)^2=0$ $\therefore$

$V(x_1,x_2)>0$, $\forall(x,x_2)\in\mathbb{R}^2\setminus\{(0,0)\}$

$\dot{V}(x_1,x_2)=\frac{d}{dt}V(x_1,x_2)=\frac{d}{dt}(ax_1^2+bx_2^2)=a\frac{d}{dt}(x_1^2)+b\frac{d}{dt}(x_2^2)=$

$2ax_1\frac{d}{dt}(x_1)+2bx_2\frac{d}{dt}(x_2)=2ax_1\dot{x}_1+2bx_2\dot{x}_2=$

$2ax_1x_2+2bx_2(-M^{-1}C_1x_2-M^{-1}C_2x_2|x_2|-M^{-1}kx_1)=$

$2ax_1x_2-2M^{-1}C_1bx_2^2-2M^{-1}C_2bx_2^2|x_2|-2M^{-1}kbx_1x_2=$

$(a-M^{-1}kb)2x_1x_2-2M^{-1}(C_1bx_2^2+C_2bx_2^2|x_2|)=$

$(a-M^{-1}kb)2x_1x_2-2M^{-1}x_2^2(C_1b+C_2b|x_2|)=$

$(a-M^{-1}kb)2x_1x_2-2M^{-1}bx_2^2(C_1+C_2|x_2|)=\dot{V}(x_1,x_2)$ $\therefore$

Let $a=M^{-1}kb$ $\therefore$ $\dot{V}(x_1,x_2)=-2M^{-1}bx_2^2(C_1+C_2|x_2|)$ $\therefore$

$x_2^2\geq 0$, $|x_2|\geq 0$, $M>0$ $\therefore M^{-1}>0$, $b>0$ $\therefore$ $-2M^{-1}bx_2^2(C_1+C_2|x_2|)\leq 0$

$\therefore$ $\dot{V}(x_1,x_2)\leq 0$ $\therefore$

for $a,b>0$ and $a=M^{-1}kb$: The origin $(x_1,x_2)=(0,0)$ is a globally asymptotically stable equilibrium point.