1a/ $\underline{u} = \underline{u}(\underline{x}) = \underline{u}(x_1, x_2, x_3)$, $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ $\therefore$

~~$e_{11} = \frac{1}{2}\frac{\partial u_1}{}$~~ $e_{ii} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i}\right) = \frac{1}{2}\,2\left(\frac{\partial u_i}{\partial x_i}\right) = \frac{\partial u_i}{\partial x_i}$

$e_{ji} = \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right)$

$e_{ii} = \frac{\partial u_i}{\partial x_i} = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = \sum_{i=1}^{3} e_{ii}$ $\therefore$

$e_{11} = \frac{\partial u_1}{\partial x_1}$, $e_{22} = \frac{\partial u_2}{\partial x_2}$, $e_{33} = \frac{\partial u_3}{\partial x_3}$

$e_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \frac{\partial u_1}{\partial x_1}$, $e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right)$ $e_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$

$e_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right)$ $e_{22} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right) = \frac{\partial u_2}{\partial x_2}$ $e_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$

$e_{31} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right)$ $e_{32} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right)$ $e_{33} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_3} + \frac{\partial u_3}{\partial x_3}\right) = \frac{\partial u_3}{\partial x_3}$

$\therefore e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = \cancel{\frac{1}{2}\left(\frac{\partial u_j}{\partial x_j}\right)}\ \frac{1}{2}\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) = e_{ji}$ $\therefore$

$e_{ij} = e_{ji}$ $\therefore$

$e_{12} = e_{21}$, $e_{13} = e_{31}$, $e_{23} = e_{32}$ $\therefore$

$e_{11}, e_{12}, e_{13}, e_{22}, e_{23}, e_{33}$ means e_{ij} only has 6 independent components,

~~with 3 of its components all being independent dependent on at least one of those 6~~

1b/ $\xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ $i = 1,2,3$, $j = 1,2,3$ $\therefore$

$\xi_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} - \frac{\partial u_1}{\partial x_1}\right) = \frac{1}{2}(0) = 0$ $\xi_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right)$ $\xi_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right)$

$\xi_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right)$ $\xi_{22} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_2} - \frac{\partial u_2}{\partial x_2}\right) = 0$ $\xi_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right)$

$\xi_{31} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right)$ $\xi_{32} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)$ $\xi_{33} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_3} - \frac{\partial u_3}{\partial x_3}\right) = 0$ $\therefore$

$\xi_{11} = 0$, $\xi_{22} = 0$, $\xi_{33} = 0$ and $-\xi_{12} = -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2}\right) = \xi_{21}$

and $-\xi_{13} = -\frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} - \frac{\partial u_1}{\partial x_3}\right) = \xi_{31}$ and

$-\xi_{23} = -\frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) = \xi_{32}$ $\therefore -\xi_{12} = \xi_{21}$, $-\xi_{13} = \xi_{31}$,

$-\xi_{23} = \xi_{32}$ and $\xi_{11} = 0$, $\xi_{22} = 0$, $\xi_{33} = 0$ $\therefore -\xi_{ij} = \xi_{ji}$ $\therefore$

ξ_{ij} has 3 non-zero independent components

1c/ using c d g curl of the gradient is always zero

$\therefore$ if $\underline{u} = \nabla \phi$ and vorticity $\underline{\omega} = \nabla \times \underline{u}$ $\therefore$

$\underline{\omega} = \nabla \times (\nabla \phi) = \nabla \times \nabla \phi = 0 = \underline{\omega}$ which shows the flow is irrotational, which is why it is useful to write $\underline{u} = \nabla \phi$ for an irrotational flow.

$\underline{u} = \nabla \phi \quad \therefore \quad \nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, \frac{\partial \phi}{\partial z} \right) = \underline{\hat{i}} \frac{\partial \phi}{\partial x} + \cancel{k}\, \hat{j} \frac{\partial \phi}{\partial y} + \underline{\hat{k}} \frac{\partial \phi}{\partial z} = \underline{u} \quad \therefore$

$\underline{u}_i = \frac{\partial \phi}{\partial x_i} = [\nabla \phi]_i = [\underline{u}]_i \quad \therefore \quad u_i = \frac{\partial \phi}{\partial x_i} = \frac{\partial}{\partial x_i}(\phi)$ and

$\xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} \right) = \frac{1}{2}\left(\frac{\partial}{\partial x_j}[u_i] - \frac{\partial}{\partial x_i}[u_j] \right)$ and

$u_j = \frac{\partial}{\partial x_j}(\phi) \quad \therefore \quad \xi_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial x_j}\left[\frac{\partial}{\partial x_i}(\phi)\right] - \frac{\partial}{\partial x_i}\left[\frac{\partial}{\partial x_j}(\phi)\right] \right) =$

$\frac{1}{2}\left(\frac{\partial^2 \phi}{\partial x_j \partial x_i} - \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) = \frac{1}{2}\left(\frac{\partial^2 \phi}{\partial x_i \partial x_j} - \frac{\partial^2 \phi}{\partial x_i \partial x_j} \right) = \frac{1}{2}(0) = 0 \quad \therefore$

$\xi_{ij} = 0$ for $\underline{u} = \nabla \phi$

\1d/ $\text{flow is incompressible} \therefore \nabla \cdot \underline{u} = 0$

$\underline{u} = (u_1, u_2, u_3) \quad \therefore \quad \nabla \cdot \underline{u} = \nabla \cdot (u_1, u_2, u_3) =$

$\left(\frac{\partial}{\partial x} u_1 + \frac{\partial}{\partial y} u_2 + \frac{\partial}{\partial z} u_3\right) = \frac{\partial}{\partial x} u_1 + \frac{\partial}{\partial y} u_2 + \frac{\partial}{\partial z} u_3 = 0 \therefore$

$[\nabla \cdot \underline{u}]_i = \sum_{i=1}^{3} \frac{\partial}{\partial x_i} u_i = \sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = \frac{\partial u_i}{\partial x_i} = 0 \quad \therefore$

$\frac{\partial u_1}{\partial x_1} = 0, \quad \frac{\partial u_2}{\partial x_2} = 0, \quad \frac{\partial u_3}{\partial x_3} = 0 \quad \therefore$

$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) \quad \therefore \quad e_{ii} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_i} + \frac{\partial u_i}{\partial x_i}\right) = \frac{1}{2}(0+0) = 0 \quad \therefore$

$e_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = 0 \qquad e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) \qquad e_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right)$

$e_{21} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) \qquad e_{22} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right) = 0 \qquad e_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right)$

$e_{31} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) \qquad e_{32} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) \qquad e_{33} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_3} + \frac{\partial u_3}{\partial x_3}\right) = 0 \quad \therefore$

$e_{11} = 0, \quad e_{22} = 0, \quad e_{33} = 0$

and same as (Q1a) :

$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_1} + \frac{\partial u_1}{\partial x_2}\right) = e_{21}$

$e_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3}\right) = e_{31}$

$e_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} + \frac{\partial u_2}{\partial x_3}\right) = e_{32} \quad \therefore$

$+e_{12} = e_{21}, \quad +e_{13} = e_{31}, \quad +e_{23} = e_{32} \quad \therefore$

e_{ij} has 3 independent components for incompressible flow

1e $\delta u_i = \delta x_j D_{ij}$ and $D_{ij} = D^s_{ij} + D^a_{ij}$ $\therefore$

$\delta u_i = \delta x_j (D_{ij}) = \delta x_j (D^s_{ij} + D^a_{ij}) = \delta u_i = \delta x_j D^s_{ij} + \delta x_j D^a_{ij}$

and $\delta x_j D^s_{ij} = \delta u^s_i$, $\delta x_j D^a_{ij} = \delta u^a_i$ $\therefore$

$\delta x_j D^s_{ij} + \delta x_j D^a_{ij} = \delta u^s_i + \delta u^a_i = \delta u_i$

taking the ith component of $\underline{\delta u}^a$: $[\underline{\delta u}^a]_i = \delta u^a_i = \delta x_j D^a_{ij} =$

$\delta x_j (D^a_{ij})$ and $D^a_{ij} = \xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ $\therefore$

$[\underline{\delta u}^a]_i = \delta x_j (D^a_{ij}) = \delta x_j (\xi_{ij}) = \delta x_j \left(\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)\right) = \delta x_j \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) =$

$\frac{1}{2} \delta x_j \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ and

$\underline{u} = (u_1, u_2, u_3) = u_1 \hat{\underline{i}} + u_2 \hat{\underline{j}} + u_3 \hat{\underline{k}}$, $\underline{\omega} = \nabla \times \underline{u} = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ u_1 & u_2 & u_3 \end{vmatrix} =$

$\hat{\underline{i}}\left(\frac{\partial u_3}{\partial y} - \frac{\partial u_2}{\partial z}\right) - \hat{\underline{j}}\left(\frac{\partial u_3}{\partial x} - \frac{\partial u_1}{\partial z}\right) + \hat{\underline{k}}\left(\frac{\partial u_2}{\partial x} - \frac{\partial u_1}{\partial y}\right) = \nabla \times \underline{u} = \underline{\omega}$ and

$[\underline{\omega}]_k = \omega_k = [\nabla \times \underline{u}]_k = (\nabla \times \underline{u})_k = \varepsilon_{klm} \nabla_l u_m = \varepsilon_{klm} \frac{\partial}{\partial x_l} u_m$ and

$[\omega \times \underline{\delta x}]_i = \varepsilon_{ikj} \omega_k \delta x_j = \delta x_j (\varepsilon_{kji} \omega_k)$ $\therefore$

$\varepsilon_{kji} \omega_k = \varepsilon_{kji} (\omega_k) = \varepsilon_{kji} \varepsilon_{klm} \frac{\partial}{\partial x_l} u_m =$

$\varepsilon_{kji} \omega_k (\delta_{jl} \delta_{im} - \delta_{jm} \delta_{il}) \frac{\partial}{\partial x_l} u_m = \delta_{jl} \delta_{im} \frac{\partial}{\partial x_l} u_m - \delta_{jm} \delta_{il} \frac{\partial}{\partial x_l} u_m =$

$\delta_{jl} \delta_{im} \frac{\partial}{\partial x_j} u_i - \delta_{jm} \delta_{il} \frac{\partial}{\partial x_i} u_j = \frac{\partial}{\partial x_j} u_i - \frac{\partial}{\partial x_i} u_j = \frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i} = \varepsilon_{kji} \omega_k$ $\therefore$

$\frac{1}{2} \delta x_j \left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2} \delta x_j (\varepsilon_{kji} \omega_k) = [\underline{\delta u}^a]_i = \frac{1}{2} \varepsilon_{kji} \omega_k \delta x_j =$

$\frac{1}{2} \varepsilon_{ikj} \omega_k \delta x_j = \frac{1}{2} [\underline{\omega} \times \underline{\delta x}]_i = [\frac{1}{2} \underline{\omega} \times \underline{\delta x}]_i$ $\therefore$

$[\underline{\delta u}^a]_i = [\frac{1}{2} \underline{\omega} \times \underline{\delta x}]_i$ $\therefore$

$\underline{\delta u}^a = \frac{1}{2} \underline{\omega} \times \underline{\delta x}$

\1f/ $\underline{u} = (ax_1 + fx_2 \;,\; bx_2 - fx_1 \;,\; cx_3) = (ax_1 + fx_2)\hat{\underline{i}} + (bx_2 - fx_1)\hat{\underline{j}} + (cx_3)\hat{\underline{k}}$ $\therefore$

$u_1 = (ax_1 + fx_2) \;,\; u_2 = (bx_2 - fx_1) \;,\; u_3 = (cx_3)$ and $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$ $\therefore$

$$e_{11} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1} + \frac{\partial u_1}{\partial x_1}\right) = \frac{\partial u_1}{\partial x_1} = \frac{\partial}{\partial x_1}(ax_1 + fx_2) = a$$

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_2}(ax_1 + fx_2) + \frac{\partial}{\partial x_1}(bx_2 - fx_1)\right) = \frac{1}{2}(f - f) = 0$$

$$e_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_3}(ax_1 + fx_2) + \frac{\partial}{\partial x_1}(cx_3)\right) = \frac{1}{2}(0 + 0) = 0$$

$$e_{22} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_2} + \frac{\partial u_2}{\partial x_2}\right) = \frac{\partial u_2}{\partial x_2} = \frac{\partial}{\partial x_2}(bx_2 - fx_1) = b$$

$$e_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_3}(bx_2 - fx_1) + \frac{\partial}{\partial x_2}(cx_3)\right) = \frac{1}{2}(0 + 0) = 0$$

$$e_{33} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_3} + \frac{\partial u_3}{\partial x_3}\right) = \frac{\partial u_3}{\partial x_3} = \frac{\partial}{\partial x_3}(cx_3) = c$$

and from question (1a): $e_{ij} = e_{ji}$ $\therefore$

$e_{12} = e_{21} = 0 \;,\; e_{13} = e_{31} = 0 \;,\; e_{23} = e_{32} = 0$ $\therefore$ $e_{ij} = \begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix} =$

$$e_{ij} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = e_{ij} = D^s_{ij}$$

$\xi_{ij} = D^a_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ and from question (1b): $-\xi_{ij} = \xi_{ji}$ and

$\xi_{11} = 0 \;,\; \xi_{22} = 0 \;,\; \xi_{33} = 0$ $\therefore$

$$\xi_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} - \frac{\partial u_2}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_2}(ax_1 + fx_2) - \frac{\partial}{\partial x_1}(bx_2 - fx_1)\right) = \frac{1}{2}(f - -f) = \frac{1}{2}(2f) = f$$

$$\xi_{13} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} - \frac{\partial u_3}{\partial x_1}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_3}(ax_1 + fx_2) - \frac{\partial}{\partial x_1}(cx_3)\right) = \frac{1}{2}(0 - 0) = 0$$

$$\xi_{23} = \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} - \frac{\partial u_3}{\partial x_2}\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_3}(bx_2 - fx_1) - \frac{\partial}{\partial x_2}(cx_3)\right) = \frac{1}{2}(0 - 0) = 0 \quad \therefore$$

$-\xi_{12} = \xi_{21} = -f \;,\; -\xi_{13} = \xi_{31} = -0 = 0 \;,\; -\xi_{23} = \xi_{32} = -0 = 0$ $\therefore$

$$\xi_{ij} = \begin{bmatrix} 0 & f & 0 \\ -f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \xi_{11} & \xi_{12} & \xi_{13} \\ \xi_{21} & \xi_{22} & \xi_{23} \\ \xi_{31} & \xi_{32} & \xi_{33} \end{bmatrix} = \xi_{ij} = D^a_{ij}$$

$\delta u^s_i = \delta x_j D^s_{ij} = \delta x_j e_{ij}$ and $\underline{x} = (x_1, x_2, x_3)$ $\therefore$ ~~$\delta x_j = [x_1$~~

$\delta x_j = [x_1 \quad x_2 \quad x_3]$ $\therefore$ $\delta u^s_i = \delta x_j e_{ij} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} =$

$[ax_1 \quad bx_2 \quad cx_3] = \delta u^s_i$ $\therefore$

$\delta u^a_i = \delta x_j D^a_{ij} = \delta x_j \xi_{ij} = [x_1 \quad x_2 \quad x_3]\begin{bmatrix} 0 & f & 0 \\ -f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} =$

$[-fx_2 \quad fx_1 \quad 0] = \delta u^a_i$ and

$$D_{ij} = D^s_{ij} + D^a_{ij} = \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} + \begin{bmatrix} 0 & f & 0 \\ -f & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} a & f & 0 \\ -f & b & 0 \\ 0 & 0 & c \end{bmatrix} = D_{ij} \quad \therefore$$

\ 1 5 continued / $\delta x_j D_{ij} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & \delta & 0 \\ -\delta & b & 0 \\ 0 & 0 & c \end{bmatrix} =$

$\begin{bmatrix} ax_1 - \delta x_2 & \delta x_1 + bx_2 & cx_3 \end{bmatrix} = \delta x_j D_{ij} = \delta u_i = \begin{bmatrix} ax_1 - \delta x_2 & \delta x_1 + bx_2 & cx_3 \end{bmatrix}$

and $\delta u_i^s + \delta u_i^a = \begin{bmatrix} ax_1 & bx_2 & cx_3 \end{bmatrix} + \begin{bmatrix} -\delta x_2 & \delta x_1 & 0 \end{bmatrix} =$

$\begin{bmatrix} ax_1 - \delta x_2 & \delta x_1 + bx_2 & cx_3 \end{bmatrix} = (\delta u^s)_i + (\delta u^a)_i = (\delta u^s + \delta u^a)_i$ $\therefore$

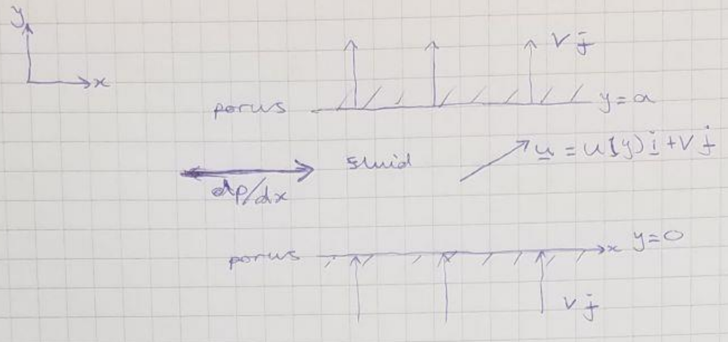
$\delta u_i = (\delta u^s + \delta u^a)_i = (\delta u)_i$ and

$(\delta u^s + \delta u^a)_i$ is the i^{th} component of $\underline{\delta u}^s + \underline{\delta u}^a$ and

$(\delta u)_i$ is the i^{th} component of $\underline{\delta u}$ $\therefore$

$\underline{\delta u} = \underline{\delta u}^s + \underline{\delta u}^a$

2a



2b Navier-Stokes equation: $\rho(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}) = \rho g\underline{j} - \nabla p + \mu\nabla^2\underline{u}$

incompressible flow ∴ $\nabla\cdot\underline{u}=0$

steady flow ∴ $\frac{\partial \underline{u}}{\partial t}=0$

neglect gravity ∴ $\rho \underline{j} = 0$ ∴

equation: $\rho(\underline{u}\cdot\nabla\underline{u}) = - \nabla p + \mu\nabla^2\underline{u}$ ∴ $\underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\underline{u}$ ∴

$\underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{u}$, $\nu = \frac{\mu}{\rho}$ ∴

x component: $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial x^2} + \nu\frac{\partial^2 u}{\partial y^2}$

y component: $u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 v}{\partial x^2} + \nu\frac{\partial^2 v}{\partial y^2}$

but pressure gradient only in x-direction ∴ $\frac{dp}{dy}=0$ & $\frac{dp}{dx}=-G$

∴ x component: $u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y} = +\frac{1}{\rho}G + \nu\frac{\partial^2u}{\partial x^2}+\nu\frac{\partial^2 u}{\partial y^2}$

y component: $u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y} = \cancel{-\frac{1}{\rho}\frac{dp}{dx}} + \nu\frac{\partial^2 v}{\partial x^2} + \nu\frac{\partial^2 v}{\partial y^2}$

Since maintained pressure ∴ $\frac{dp}{dx}=-G$ since pressure pushes fluid in positive x-direction so pressure is stronger as x decreases

and $u=u(y)$ ∴ $\frac{\partial u}{\partial x}=\frac{\partial u(y)}{\partial x}=0$, $\frac{\partial^2 u}{\partial x^2}=\frac{\partial^2 u(y)}{\partial x^2}=0$ and $\frac{\partial u(y)}{\partial y}=\frac{\partial u}{\partial y}=\frac{du}{dy}$,

$\frac{\partial^2 u}{\partial y^2}=\frac{\partial^2 u(y)}{\partial y^2}=\frac{d^2u}{dy^2}$ and $\frac{dp}{dx}=-G$ ∴

x component: $v\frac{du}{dy}=\frac{G}{\rho}+\frac{\mu}{\rho}\frac{d^2u}{dy^2}$ ∴ $\mu\frac{d^2u}{dy^2}-\rho v\frac{du}{dy}+G=0$

~~y component~~ v is a constant ∴ $\frac{\partial v}{\partial x}=0$, $\frac{\partial v}{\partial y}=0$, $\frac{\partial^2 v}{\partial x^2}=0$, $\frac{\partial^2 v}{\partial y^2}=0$ ∴

y component: $u(0)+v(0) = \nu(0)+\nu(0) = 0 = 0$ ∴ for constant G:

x component: $\frac{d^2u}{dy^2}-\frac{\rho v}{\mu}\frac{du}{dy}+\frac{G}{\mu}=0$ is ODE of $u(y)$

\2c/ Give parameters a, v, μ, ρ and G.

v is a constant speed $\therefore [v] = LT^{-1}$

ρ is a density $\therefore [\rho] = ML^{-3}$

$G = -\frac{dp}{dx}$ where p is a pressure and pressure $= \frac{\text{force}}{\text{area}}$ $\therefore$

$[p] = MLT^{-2}L^{-2} = ML^{-1}T^{-2}$ $\therefore$

$[G] = \left[-\frac{dp}{dx}\right] = \left[\frac{dp}{dx}\right] = ML^{-1}T^{-2}L^{-1} = ML^{-2}T^{-2} = [G]$

Kinematic viscosity is ν and $\nu = \frac{\mu}{\rho}$ and $[\nu] = L^2T^{-1}$ $\therefore$

$\nu\rho = \mu$ $\therefore$

μ is a viscosity $\therefore [\mu] = [\nu\rho] = [\nu][\rho] = \cancel{(L^2T^{-1})(ML^{-2}T^{-2})} =$

$(L^2T^{-1})(ML^{-3}) = ML^{-1}T^{-1} = [\mu]$

a is a length $\therefore [a] = L$

2d/ By Buckingham-Π theorem: $q = f(a_1, a_2, \ldots, a_k, b_1, b_2, \ldots, b_m)$ then $\Pi = \Phi(\Pi_1, \Pi_2, \ldots, \Pi_m)$ $\therefore$

Seive parameters: $[a] = L$, $[v] = LT^{-1}$, $[\mu] = ML^{-1}T^{-1}$, $[\rho] = ML^{-3}$, $[G] = ML^{-2}T^{-2}$ $\therefore$

$L^{\alpha} \neq LT^{-1}$ $\therefore [a]^{\alpha} \neq [v]$ $\therefore$ a and v are dimensionally independent

$(L)^{\alpha}(LT^{-1})^{\beta} = L^{\alpha}L^{\beta}T^{-\beta} = L^{\alpha+\beta}T^{-\beta} \neq ML^{-1}T^{-1}$ $\therefore [a]^{\alpha}[v]^{\beta} \neq [\mu]$ $\therefore$

a and v and μ are all dimensionally independent of eachother

$[\rho] = ML^{-3} = M^{\varepsilon}L^{\delta}T^{\Omega} = (L)^{\alpha_2}(LT^{-1})^{\beta_2}(ML^{-1}T^{-1})^{\gamma_2} = [a]^{\alpha_2}[v]^{\beta_2}[\mu]^{\gamma_2}$ $\therefore$

ρ is not dimensionally independent of a, v, μ

$[G] = ML^{-2}T^{-2} = M^{\delta}L^{\varepsilon}T^{\Omega} = (L)^{\alpha_1}(LT^{-1})^{\beta_1}(ML^{-1}T^{-1})^{\gamma_1} = [a]^{\alpha_1}[v]^{\beta_1}[\mu]^{\gamma_1}$ $\therefore$

G is not dimensionally independent of a, v, μ $\therefore$

$n = 5$, $k = 3$, $m = 2$

$a_1 = a$, $a_2 = v$, $a_3 = \mu$, $b_1 = \rho$, $b_2 = G$ $\therefore$

$\Pi_1 = \frac{\rho}{a^{\alpha}v^{\beta}\mu^{\gamma}}$ $\therefore [\rho] = [a]^{\alpha}[v]^{\beta}[\mu]^{\gamma} = ML^{-3} = (L^{\alpha})(LT^{-1})^{\beta}(ML^{-1}T^{-1})^{\gamma} =$
$L^{\alpha}L^{\beta}T^{-\beta}M^{\gamma}L^{-\gamma}T^{-\gamma} = M^{\gamma}L^{\alpha+\beta-\gamma}T^{-\beta-\gamma} = M^{1}L^{-3}T^{0}$ $\therefore$

$\gamma = 1$ $\therefore -\beta-\gamma = 0 = -\beta-1$ $\therefore \beta = -1$ $\therefore \alpha+\beta-\gamma = -3 = \alpha-1-1 = \alpha-2 = -3$ $\therefore$

$\alpha = -1$ $\therefore$ $\Pi_1 = \frac{\rho}{a^{-1}v^{-1}\mu^{1}} = \frac{\rho}{a^{-1}v^{-1}\mu} = \Pi_1$

$\Pi_2 = \frac{G}{a^{\alpha}v^{\beta}\mu^{\gamma}}$ $\therefore [G] = ML^{-2}T^{-2} = M^{\gamma}L^{\alpha+\beta-\gamma}T^{-\beta-\gamma} = M^{1}L^{-2}T^{-2}$ $\therefore$

$\gamma = 1$ $\therefore -\beta-\gamma = -2 = -\beta-1$ $\therefore$ $\cancel{\beta}$ $0 = -\beta+1$ $\therefore$ $\beta = 1$ $\therefore$ $\alpha+\beta-\gamma = -2 =$
$\alpha+1-1 = \alpha = -2$ $\therefore$ $\Pi_2 = \frac{G}{a^{-2}v^{1}\mu^{1}} = \frac{G}{a^{-2}v\mu} = \Pi_2$ $\therefore$

$\Pi = \Phi(\Pi_1, \Pi_2) = \cancel{\left(\frac{\rho}{a^{-1}v^{-1}\mu}\right)}$ $\Phi\left(\frac{\rho}{a^{-1}v^{-1}\mu}, \frac{G}{a^{-2}v\mu}\right) =$
$\Phi\left(\frac{av\rho}{\mu}, \frac{a^2 G}{v\mu}\right) = \Pi$

\2e/ $\frac{d^2u}{dy^2} - \frac{\rho v}{\mu}\frac{du}{dy} + \frac{G_T}{\mu} = 0$ G_T is a constant

$\therefore$ let $q = \frac{\rho v}{\mu}$ $\therefore$ $\frac{d^2u}{dy^2} - q\frac{du}{dy} = -\frac{G_T}{\mu}$ $\therefore$

for complementary function: $u_{CF} = u_{CF}(y)$:

$\frac{d^2u}{dy^2} - q\frac{du}{dy} = 0$ $\therefore$ let $u(y) = e^{my}$ $\therefore$ $\frac{du}{dy} = me^{my}$, $\frac{d^2u}{dy} = m^2e^{my}$ $\therefore$

$m^2e^{my} - qme^{my} = 0 = m^2 - qm = m(m-q) = 0$ $\therefore$ $m=0$, $m-q=0$ $\therefore$

$m=0$, $m=q$ $\therefore$ $u(y) = De^{(0)y} + Ee^{qy} = D + Ee^{qy} = u_{CF}(y) = u_{CF}$

for particular integral: use trial function $u_{TF} = u_{TF}(y) = b + fy$

$\frac{du}{dy} = f$, $\frac{d^2u}{dy^2} = 0$ $\therefore$ $\frac{d^2u}{dy^2} - q\frac{du}{dy} = -\frac{G_T}{\mu}$ $\therefore$

$0 - qf = -\frac{G_T}{\mu}$ $\therefore$ $qf = \frac{G_T}{\mu}$ $\therefore$ $f = \frac{G_T}{\mu q}$ $\therefore$ f is constant $\therefore$

$u_{TF} = b + \frac{G_T}{\mu q}y = u_{PI}$ and $u(y) = u_{CF} + u_{PI}$ $\therefore$

$u(y) = D + Ee^{qy} + b + \frac{G_T}{\mu q}y = F + Ee^{qy} + \frac{G_T}{\mu q}y$ $\therefore$

using boundary conditions: $u(y=0) = 0 = u(0) = F + Ee^{q(0)} + \frac{G_T}{\mu q}(0) =$

$0 = F + E$ $\therefore$ $E = -F$ $\therefore$ $u(y) = F - Fe^{qy} + \frac{G_T}{\mu q}y = F(1-e^{qy}) + \frac{G_T}{\mu q}y = u(y)$

and $u(y=a) = 0 = u(a) = F(1-e^{qa}) + \frac{G_T}{\mu q}a = 0 = u(a)$ $\therefore$

$F(1-e^{qa}) = -\frac{G_T}{\mu q}a$ $\therefore$ $F = -\frac{G_T a}{(1-e^{qa})\mu q}$ $\therefore$

$$u(y) = -\frac{G_T a}{\mu q(1-e^{qa})}(1-e^{qy}) + \frac{G_T}{\mu q}y = \frac{G_T}{\mu q}\left(y - \frac{a(1-e^{qy})}{1-\exp(qa)}\right) =$$

$$u(y) = \frac{G_T}{\mu q}\left([y] - \frac{[a - ae^{qy}]}{1-\exp(qa)}\right) = \frac{G_T\mu}{\mu\rho v}\left([y] - \frac{\left[a - a\exp\left(\frac{\rho v y}{\mu}\right)\right]}{1-\exp\left(\frac{\rho v a}{\mu}\right)}\right) =$$

$$u(y) = \frac{G_T}{\rho v}\left([y] - \frac{\left[a - a\exp\left(\frac{\rho v y}{\mu}\right)\right]}{1-\exp\left(\frac{\rho v a}{\mu}\right)}\right) = u(y) = \frac{G_T}{\rho v}\left([y] - \frac{\left[a - a\exp\left(\frac{\rho v y}{\mu}\right)\right]}{1-\exp(R_e)}\right)$$

where $R_e = \frac{\rho v a}{\mu}$, $A(y) = y$, $B(y) = a - a\exp\left(\frac{\rho v y}{\mu}\right)$, $C = \frac{G_T}{\rho v}$

2b/ Channel has width of unity $\therefore a=1$

The boundaries are porus with fluid passing through them at constant speed $v>0$ and a percolation flow speed of 10^{-4} $\therefore$ $v=10^{-4}$

viscosity is μ $\therefore$ $\mu=10^{-3}$

density is ρ $\therefore$ $\rho=10^{3}$ $\therefore$

$Re=\frac{\rho v a}{\mu}=\frac{10^3(10^{-4})1}{10^{-3}}=10^2=100$ and $\frac{G}{\rho v}=\frac{G}{(10^3)10^{-4}}=\frac{G}{10^{-1}}=10G$ $\therefore$

$\frac{\rho v}{\mu}=\frac{10^3(10^{-4})}{10^{-3}}=10^2=100$ $\therefore$

$\frac{G}{\rho v}\left([y]-\frac{[a-a\exp(\frac{\rho v y}{\mu})]}{1-\exp(Re)}\right)=u(y)=10G\left([y]-\frac{[1-1\exp(100y)]}{1-\exp(100)}\right)=$

$u(y)=10G\left(y-\frac{1-\exp(100y)}{1-\exp(100)}\right)=G(10)\left(y-\frac{1}{1-e^{100}}+\frac{e^{100y}}{1-e^{100}}\right)=$

$G\left(\frac{-10}{1-e^{100}}+\frac{1}{1-e^{100}}e^{100y}+y\right)=u(y)$ $\therefore$

if $e^{100}\approx\infty$ $\therefore$ $1-e^{100}\approx-\infty$ $\therefore$ $\frac{1}{1-e^{100}}\approx-0$ $\therefore$ $\frac{-10}{1-e^{100}}\approx+0$ $\therefore$

$\cancel{\frac{1}{1-e^{100}}e^{100y}}$ note: $\frac{1}{1-e^{100}}e^{100(1)}=\frac{1}{1-e^{100}}e^{100}\approx\frac{e^{100}}{-e^{100}}=-1$

and $e^{100y}=(e^{100})^y=e^{100}(e^{100})^{y-1}$ $\therefore$ $\frac{1}{1-e^{100}}e^{100y}\approx-\infty$ for $y>1$ and

$\frac{1}{1-e^{100}}e^{100y}\approx 0$ for $y\leq 1$ $\therefore$

for $y\leq1$: $u(y)\approx G(y)=Gy$ $\therefore$ $u(y)=Gy$ for $y\leq 1$

and for $y>1$: $u(y)\approx G(-\infty)=-\infty$ $\therefore$ $u(y)=-\infty$ for $y>1$ $\therefore$

for G is a constant and $G>0$ $\therefore$

$u(y=1)=u(1)=G(1)=G$: $(1,G)$

$u(0)=0$: $(0,0)$

$u(a)=0$: $(a,0)$