Week 4 Sheet

1a vs $\underline{\omega} = \nabla \times \underline{u}$ $\quad \underline{u} = V(R,t)\hat{\underline{\phi}}$ in cylindrical polars (R, ϕ, z)

$$\omega = \nabla \times \underline{u} = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial/\partial R & \partial/\partial\phi & \partial/\partial z \\ u_R & Ru_\phi & F_z \end{vmatrix} = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & R\cdot V & 0 \end{vmatrix} =$$

$$\frac{1}{R}\left\{\hat{\underline{R}}\left[\partial_\phi 0 - \partial_z(R\cdot V)\right] - R\hat{\underline{\phi}}\left[\partial_R 0 - \partial_z 0\right] + \hat{\underline{z}}\left[\partial_R(R\cdot V) - \partial_\phi 0\right]\right\} =$$

$$\frac{1}{R}\left\{\hat{\underline{R}}\left[0-0\right] - R\hat{\underline{\phi}}\left[0\right] + \hat{\underline{z}}\left[V\frac{\partial R}{\partial R} + R\frac{\partial V(R,t)}{\partial R}\right]\right\} =$$

$$\frac{1}{R}\left\{\hat{\underline{z}}V + R\frac{\partial V(R,t)}{\partial t}\hat{\underline{z}}\right\} = \hat{\underline{z}}\left(\frac{1}{R}V(R,t) + \frac{\partial V(R,t)}{\partial t}\right) = \hat{\underline{z}}\,\omega = \underline{\omega} = \hat{\underline{z}}\,\omega(R,t)$$

where $\omega = \frac{1}{R}V(R,t) + \frac{\partial V(R,t)}{\partial t} = \frac{1}{R}\frac{\partial}{\partial R}\left(R\cdot V(R,t)\right)$ $\therefore$ $\omega = \omega(R,t)$ only

1b consider each of the terms in the N-S eqn:

$$\partial_t \underline{u} = \partial_t(V\hat{\underline{\phi}}) = \hat{\underline{\phi}}\,\partial_t V = \hat{\underline{\phi}}\,\partial_t V(R,t)$$

$$\underline{u}\times\underline{\omega} = \underline{u}\times(\omega(R,t)\hat{\underline{z}}) = (V(R,t)\hat{\underline{\phi}})\times(\omega(R,t)\hat{\underline{z}}) = \begin{vmatrix} \hat{\underline{R}} & \hat{\underline{\phi}} & \hat{\underline{z}} \\ u_R & u_\phi & u_z \\ \omega_R & \omega_\phi & \omega_z \end{vmatrix} = \begin{vmatrix} \hat{\underline{R}} & \hat{\underline{\phi}} & \hat{\underline{z}} \\ 0 & V & 0 \\ 0 & 0 & \omega \end{vmatrix}$$

$$= \hat{\underline{R}}\,V\omega = \hat{\underline{R}}\,V(R,t)\,\omega(R,t) \qquad u^2 = |\underline{u}|^2 = \left(\sqrt{V(R,t)^2}\right)^2 = (V(R,t))^2 = V^2$$

$$\nabla\left(\frac{p}{\rho} + \frac{V^2}{2}\right) = \frac{\partial}{\partial R}\left(\frac{p}{\rho} + \frac{V^2}{2}\right)\hat{\underline{R}} + \frac{1}{R}\frac{\partial}{\partial\phi}\left(\frac{p}{\rho} + \frac{V^2}{2}\right)\hat{\underline{\phi}} + \frac{\partial}{\partial z}\left(\frac{p}{\rho} + \frac{V^2}{2}\right)\hat{\underline{z}}$$

$$= \frac{\partial}{\partial R}\left(\frac{p}{\rho} + \frac{V^2}{2}\right)\hat{\underline{R}} + \frac{1}{R}\frac{\partial}{\partial\phi}\left(\frac{p}{\rho}\right)\hat{\underline{\phi}} + \frac{\partial}{\partial z}\left(\frac{p}{\rho}\right)\hat{\underline{z}}$$

$$\nabla\times\underline{\omega} = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & (R)(0) & \omega \end{vmatrix} = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & 0 & \omega \end{vmatrix} = \frac{1}{R}\hat{\underline{R}}\,\partial_\phi\omega - \frac{1}{R}R\,\hat{\underline{\phi}}\,\partial_R\omega =$$

$$0 - \hat{\underline{\phi}}\,\partial_R\,\omega(R,t) = -\hat{\underline{\phi}}\frac{\partial\omega(R,t)}{\partial R}$$ $\therefore$ taking all contributions in the ϕ-direction:

$$\cancel{\hat{\underline{\phi}}\,\partial_t V(R,t)}\ \ \partial_t V(R,t) - 0 = -\frac{1}{R}\frac{\partial}{\partial\phi}\left(\frac{p}{\rho}\right) + \nu\frac{\partial\omega(R,t)}{\partial R} = \frac{\partial V(R,t)}{\partial t} = \nu\frac{\partial\omega(R,t)}{\partial R}$$

$\therefore \frac{\partial V(R,t)}{\partial t} = \nu\frac{\partial\omega(R,t)}{\partial R}$ assuming there's no ϕ-component of the pressure gradient $=$ i.e. $\frac{\partial p}{\partial\phi} = 0$

1c $\therefore V(R,t) = \frac{\Gamma_0 F(\xi)}{2\pi R}$ $\quad \Gamma_0$ is a const $\quad \xi = R/\sqrt{4\nu t}$ $\therefore$

$$\omega = \frac{1}{R}\frac{\partial}{\partial R}(RV) = \frac{1}{R}\frac{\partial}{\partial R}\left(\frac{\Gamma_0 F(\xi)}{2\pi}\right) = \frac{1}{R}\frac{\Gamma_0}{2\pi}\frac{\partial}{\partial R}\left(F(\xi)\right) = \frac{\Gamma_0}{2\pi R}\frac{dF}{d\xi}\frac{\partial\xi}{\partial R} =$$

$$\frac{\Gamma_0}{2\pi R}\frac{dF}{d\xi}\frac{\partial\xi}{\partial R} = \frac{\Gamma_0}{2\pi R}F'\frac{\partial(R/\sqrt{4\nu t})}{\partial R} = \frac{\Gamma_0}{2\pi R}F'\frac{1}{\sqrt{4\nu t}} \qquad \left\{R = \xi\sqrt{4\nu t}\right\}$$

$$\frac{\Gamma_0}{2\pi\,\xi\sqrt{4\nu t}}F'\frac{1}{\sqrt{4\nu t}} = \frac{\Gamma_0}{2\pi\,\xi\,4\nu t}F' = \frac{\Gamma_0 F'(\xi)}{8\pi\,\xi\,\nu t} = \omega$$

\ week 4 Sheet / \ 1d / $\dfrac{\Gamma_0 F'(\xi)}{8\pi \xi \nu t} = \omega \qquad \dfrac{\partial v}{\partial t} = \nu \dfrac{\partial \omega}{\partial R}$

$v(R,t) = \dfrac{\Gamma_0 F(\xi)}{2\pi R} \quad \therefore \quad \dfrac{\partial v(R,t)}{\partial t} = \dfrac{\Gamma_0}{2\pi R} \dfrac{\partial F(\xi)}{\partial t} = \dfrac{\Gamma_0}{2\pi R} \dfrac{\partial F(\xi)}{\partial \xi} \dfrac{\partial \xi}{\partial t} =$

$\dfrac{\Gamma_0}{2\pi R} \dfrac{dF(\xi)}{d\xi} \dfrac{\partial \left(\frac{R}{\sqrt{4\nu}} t^{-1/2} \right)}{\partial t} = \dfrac{\Gamma_0}{2\pi R} (F')(-\tfrac{1}{2}) t^{-3/2} \dfrac{R}{\sqrt{4\nu}} = \dfrac{\Gamma_0}{2\pi R} (F')(-\tfrac{1}{2}) \xi t^{-1} \therefore$

$\dfrac{\partial v}{\partial t} = -\dfrac{\Gamma_0}{4\pi R t} \xi F' \quad \therefore$

$\cancel{\dfrac{\partial v}{\partial t}} \Rightarrow \dfrac{1}{\nu} \dfrac{\partial v}{\partial t} = \dfrac{\partial \omega}{\partial R} = -\dfrac{\Gamma_0}{4\pi R \nu t} \xi F'$

$\dfrac{\partial \omega}{\partial R} = \dfrac{\partial}{\partial R} \left(\dfrac{\Gamma_0 F'(\xi)}{8\pi \xi \nu t} \right) = \dfrac{\Gamma_0}{8\pi \nu t} \dfrac{\partial}{\partial R} \left(\dfrac{F'}{\xi} \right) = \dfrac{\Gamma_0}{8\pi \nu t} \dfrac{\partial}{\partial \xi} \left(\dfrac{F'}{\xi} \right) \dfrac{\partial \xi}{\partial R} =$

$\dfrac{\Gamma_0}{8\pi \nu t} \dfrac{\partial}{\partial \xi} \left(\dfrac{F'}{\xi} \right) \dfrac{\partial}{\partial R} \left(\dfrac{R}{\sqrt{4\nu t}} \right) = \dfrac{\Gamma_0}{8\pi \nu t} \dfrac{\partial}{\partial \xi} \left(\dfrac{F'}{\xi} \right) \dfrac{1}{\sqrt{4\nu t}} \qquad \therefore \dfrac{\partial v}{\partial t} = \nu \dfrac{\partial \omega}{\partial R}$

$-\dfrac{\Gamma_0}{4\pi R t} \xi F' = \nu \dfrac{\Gamma_0}{8\pi \nu t} \dfrac{\partial}{\partial \xi} \left(\dfrac{F'}{\xi} \right) \dfrac{1}{\sqrt{4\nu t}} = \dfrac{\Gamma_0}{8\pi t} \dfrac{\partial}{\partial \xi} \left(\dfrac{F'}{\xi} \right) \dfrac{1}{\sqrt{4\nu t}}$

$\therefore -\dfrac{1}{R} \xi F' = \dfrac{1}{2} \dfrac{\partial}{\partial \xi} \left(\dfrac{F'}{\xi} \right) \dfrac{1}{\sqrt{4\nu t}} \quad \therefore$

$-\dfrac{\sqrt{4\nu t}}{R} \xi 2F' = \dfrac{\partial}{\partial \xi} \left(\dfrac{F'}{\xi} \right) \cancel{\dfrac{\sqrt{4\nu t}}{\sqrt{4\nu t}}} = -\dfrac{1}{\xi} \xi 2F' = -2F' \quad \therefore$

$2F' + \dfrac{\partial}{\partial \xi} \left(\dfrac{F'}{\xi} \right) = 0 \quad \therefore$

$\Gamma = 2\pi R v = \Gamma_0 F(\xi) \to \Gamma_0$ as $R \to \infty$

$\Longrightarrow F(\xi) \to 1$ as $\xi \to \infty$

& $\Gamma_0 F(\xi) \to 0$ as $R \to 0 \Longrightarrow F(0) \to 0$ as $\xi \to 0$

\ 2 / $\left(\int_{-\infty}^{\infty} e^{-x^2} dx \right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2} dx \right) \left(\int_{-\infty}^{\infty} e^{-y^2} dy \right) =$

$\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-(x^2+y^2)} dx\, dy \qquad \{ r^2 = x^2 + y^2 \quad r\sin\phi = y \quad r\cos\phi = x \}$

$= 2\pi \int_{\phi=0}^{2\pi} \int_{r=0}^{\infty} e^{-r^2} r\, dr\, d\phi$ {using polar coords (r, ϕ), $J = \begin{vmatrix} \partial_x r & \partial_y r \\ \partial_x \phi & \partial_y \phi \end{vmatrix}$

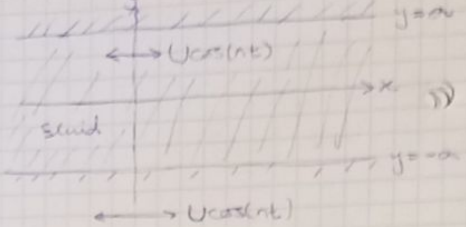
$0 \le \phi < 2\pi \quad 0 \le x \le \infty \quad 0 \le y < \infty \therefore 0 \le r < \infty$ }

$= 2\pi \int_{r=0}^{\infty} r e^{-r^2} dr = -\dfrac{2\pi}{2} \left[e^{-r^2} \right]_0^{\infty} = -\pi(0-1) = \pi$

$\therefore \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

\Week 5 sheet/

\1a/



\1b/ NS-eqn: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho \underline{f} + \mu\nabla^2\underline{u}$

but no boundary forces ∴ no gravity ∴

$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \mu\nabla^2\underline{u}$, $\underline{u} = u\,\underline{i}$ ∴

$\underline{u} = u(y,t)\,\underline{i}$ gives $\frac{\partial \underline{u}}{\partial t} = \frac{\partial u}{\partial t}\underline{i}$, $\nabla^2\underline{u} = \nabla^2 u\,\underline{i}$

$\underline{u}\cdot\nabla\underline{u} = u\frac{\partial}{\partial x}(u\underline{i}) = u\frac{\partial u}{\partial x}\underline{i} = 0$ since u is indep of x

∴ $\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = \frac{\partial u}{\partial t}\underline{i} + 0 = \frac{\partial u}{\partial t}\underline{i}$ ∴ only the x-component of the

NS-eqn is non trivial ∴ $\frac{\partial p}{\partial x} = 0$ since pressure is indep of x

∴ $\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu\nabla^2 u$ ∴ $\rho\frac{\partial u}{\partial t} = \mu\nabla^2 u$ ∴

$\frac{\partial u}{\partial t} = \frac{\mu}{\rho}\nabla^2 u = \frac{\mu}{\rho}\nabla^2 u(y,t) = \nu\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$

\1c/ U, n const ∴ $u(y,t) = \Re\left[UF(y)e^{int}\right]$ ∴ $\frac{\partial u}{\partial t} = \partial_t u = \Re\left[UF(y)in e^{int}\right]$

$\frac{\partial^2 u}{\partial y^2} = \partial_{yy}u = \Re\left[UF''(y)e^{int}\right]$ ∴

$\Re\left[UF(y)in e^{int}\right] = \Re\left[UF''(y)e^{int}\right]\nu$ ∴

$UF(y)in e^{int} = UF''(y)e^{int}\nu$ ∴ $inF(y) = \nu F''(y)$ ∴

$F''(y) - \frac{n}{\nu}iF(y) = 0$ is the required ODE

Boundary conditions: $u(y=-a,t) = U\cos(nt)$ ∴

~~[illegible]~~ $\Re\left[UF(-a)e^{int}\right] = U\cos(nt)$ ∴

$\Re\left[UF(-a)(\cos nt + i\sin nt)\right] = UF(-a)\cos(nt) = U\cos(nt)$ ∴

$F(-a) = 1$

for $y=a$: $U(y=a,t) = U\cos(nt) = \Re\left[UF(a)e^{int}\right] = \Re\left[UF(a)(\cos nt + i\sin nt)\right]$

$= UF(a)\cos nt$ ∴ $F(a) = 1$

\1d/ $F''(y) - \frac{n}{\nu}iF(y) = 0$ is an ODE of 2nd order with const coeffs.

∴ seek sols of form: $F = e^{my}$ ∴ $F' = me^{my}$ ∴ $F'' = m^2e^{my}$ ∴

$m^2e^{my} - \frac{n}{\nu}ie^{my} = 0$ ∴ $m^2 - \frac{n}{\nu}i$ ∴ $m^2 = \frac{n}{\nu}i$ ∴ $m = \pm\sqrt{\frac{n}{\nu}i}$ ∴

Week 5 Sheet / $\frac{1}{\delta} = \sqrt{\frac{n}{2\nu}}$ $\therefore$ $\frac{i}{\delta} = \sqrt{\frac{n}{2\nu}}\, i$ $\therefore$

$\frac{(1+i)}{\delta} = \frac{1}{\delta} + \frac{i}{\delta} = \sqrt{\frac{n}{2\nu}} + \sqrt{\frac{n}{2\nu}}\, i = (1+i)\sqrt{\frac{n}{2\nu}} = \sqrt{\frac{n}{2\nu}(1+i)^2} = \sqrt{\frac{n}{\nu} i}$

$m = \pm\sqrt{\frac{n}{\nu} i} = \pm\sqrt{\frac{n}{2\nu} 2i} = \pm \frac{1}{\delta}\sqrt{2i}$

$\arg(2i) = \frac{\pi}{2}$ $\quad |2i| = 2$ $\quad \therefore \sqrt{2i} = \therefore \arg(\sqrt{2i}) = \frac{\pi}{4}$ $\quad |\sqrt{2i}| = \sqrt{2}$

$\therefore \arg(2i) = \arg(\sqrt{2i}\cdot\sqrt{2i}) = 2\arg(\sqrt{2i}) = \frac{\pi}{2}$ $\therefore \sqrt{2i} = \sqrt{2}\left(\cos\left(\frac{\pi}{4}\right) + i\sin\frac{\pi}{4}\right) = 1+i$

$\therefore \pm\frac{1}{\delta}(1+i) = m$ $\therefore$

$F = Ae^{(1+i)y/\delta} + Be^{-(1+i)y/\delta}$ $\therefore$ apply boundary conditions $\therefore$

~~A-2B=~~ $F(a) = 1 = Ae^{(1+i)a/\delta} + Be^{-(1+i)a/\delta} = Ae^{\frac{a}{\delta}}e^{ia/\delta} + Be^{-\frac{a}{\delta}}e^{-i\frac{a}{\delta}} =$

~~$Ae^{a/\delta}$ A+B~~ $(Ae^{a/\delta} + Be^{-a/\delta})\cos\frac{a}{\delta} + i(Ae^{a/\delta} - Be^{-a/\delta})\sin\frac{a}{\delta} = 1$

$F(-a) = 1 = Ae^{(1+i)\frac{-a}{\delta}} + Be^{-(1+i)\frac{-a}{\delta}} = (Ae^{-a/\delta} + Be^{+a/\delta})\cos\frac{a}{\delta} + i(Ae^{-a/\delta} + Be^{a/\delta})\sin\frac{a}{\delta} = 1$

$\therefore A = B = \left(e^{(1+i)\frac{a}{\delta}} + e^{-(1+i)\frac{a}{\delta}}\right)^{-1} = \left(2\cdot\frac{1}{2}\left(e^{(1+i)\frac{a}{\delta}} + e^{-(1+i)\frac{a}{\delta}}\right)\right)^{-1} = \left(2\cosh\left[(1+i)\frac{a}{\delta}\right]\right)^{-1}$

$A = B = \frac{2}{\cosh\left[(1+i)\frac{a}{\delta}\right]}$ $\therefore$

$F = \frac{2}{\cosh((1+i)\frac{a}{\delta})}e^{(1+i)y/\delta} + \frac{2}{\cosh((1+i)\frac{a}{\delta})}e^{-(1+i)y/\delta} = \frac{2}{\cosh((1+i)\frac{a}{\delta})}\left(e^{(1+i)\frac{a}{\delta}} + e^{-(1+i)\frac{a}{\delta}}\right)$

$F = \frac{\cosh[(1+i)y/\delta]}{\cosh[(1+i)a/\delta]}$ $\qquad \delta = \sqrt{\frac{2\nu}{n}}$ $\therefore$ 2 required complex number $q = 1+i$

1e/ given $F(y) \approx \exp\left(\frac{-(1+i)(y+a)}{\delta}\right)$ for $y \approx -a$ & large $\frac{a}{\delta}$ $\therefore$

$u(y\approx -a) \approx \mathbb{R}\left[UF(y\approx -a)e^{int}\right] = \mathbb{R}\left[U\exp\left(\frac{-(1+i)(y+a)}{\delta}\right)\exp(int)\right] =$

$U\exp\left(\frac{-(y+a)}{\delta}\right)\mathbb{R}\left[\exp\left(\frac{-i(y+a)}{\delta} + int\right)\right] = U\exp\left(\frac{-(y+a)}{\delta}\right)\cos\left(\frac{-(y+a)}{\delta} + nt\right)$

1δ/ $u(y=0, t) = \mathbb{R}\left[UF(0)e^{int}\right]$

$F(0) = \frac{\cosh[(1+i)0/\delta]}{\cosh[(1+i)\frac{a}{\delta}]} = \frac{\cosh 0}{\cosh[(1+i)\frac{a}{\delta}]} = \frac{1}{\frac{1}{2}\left(e^{(1+i)\frac{a}{\delta}} + e^{-(1+i)\frac{a}{\delta}}\right)} =$

$\frac{2}{e^{(1+i)\frac{a}{\delta}} + e^{-(1+i)\frac{a}{\delta}}} \cdot \frac{\left(e^{(1-i)\frac{a}{\delta}} + e^{-(1-i)\frac{a}{\delta}}\right)}{\left(e^{(1-i)\frac{a}{\delta}} + e^{-(1-i)\frac{a}{\delta}}\right)} = \frac{e^{(1-i)\frac{a}{\delta}} + e^{-(1-i)\frac{a}{\delta}}}{\cosh\left(\frac{2a}{\delta}\right) + \cos\left(\frac{2a}{\delta}\right)}$ $\therefore$

$u(0,t) = \frac{U}{\cosh\left(\frac{2a}{\delta}\right) + \cos\left(\frac{2a}{\delta}\right)}\mathbb{R}\left[e^{(1-i)\frac{a}{\delta} + int} + e^{-(1-i)\frac{a}{\delta} + int}\right] =$

$\frac{U}{\cosh\left(\frac{2a}{\delta}\right) + \cos\left(\frac{2a}{\delta}\right)}\left[e^{\frac{a}{\delta}}\cos\left(nt - \frac{a}{\delta}\right) + e^{-\frac{a}{\delta}}\cos\left(nt + \frac{a}{\delta}\right)\right]$

\Week 7 sheet/

\1a/ $\nabla \cdot \underline{U} = \nabla \cdot (U\hat{\underline{R}} + 0\hat{\underline{\phi}}) = \frac{\partial}{\partial R}(U\hat{\underline{R}}) = \hat{\underline{R}}\cancel{\frac{\partial}{\partial R}(U)} = 0 \therefore$

$U\hat{\underline{R}}$ is constant with respect to R $\therefore U = \frac{c}{R}$ but

flow is directed in the negative directn $\therefore U = -\frac{Q}{R}$, $Q > 0$

~~where~~ $\cancel{c = -Q}$

\1a sol/ in polar coords $\underline{U} = U\hat{\underline{R}}$ with $\nabla \cdot \underline{U} = \left\{ \frac{1}{R}\frac{\partial}{\partial R}(RU_R) + \frac{1}{R}\frac{\partial U_\phi}{\partial \phi} \right\} =$

$\frac{1}{R}\frac{\partial}{\partial R}(RU) = 0$ $\{$since incompressibility conditn $\therefore \nabla \cdot \underline{U} = 0\}$ $\therefore$

$RU = \text{Const}$ $\therefore U = \frac{c}{R}$ $\therefore$ for a radially inwards flow $\therefore c < 0$ $\therefore$

$U = -\frac{Q}{R}$ with $Q > 0$

\1b/ $\underline{U} = U\hat{\underline{R}} = -\frac{Q}{R}\hat{\underline{R}} = \frac{-Q}{\sqrt{x^2+y^2}}(\cos\phi\,\underline{i} + \sin\phi\,\underline{j})$ $\{$since $R = \sqrt{x^2+y^2}$ &

$\hat{\underline{R}} = \cos\phi\,\underline{i} + \sin\phi\,\underline{j}$ in cartesian since even though 2 main

flow is in polar coords, 2 boundary layer flow is in cartesian$\}$

$\approx -\frac{Q}{x}\,\underline{i}$ near 2 boundary layer where $y \approx 0$ $\phi \approx 0$

$\{\cos\phi = 1$ $\underline{U} = \frac{-Q}{\sqrt{x^2+0^2}}(\cos 0\,\underline{i} + \sin 0\,\underline{j}) = \frac{-Q}{x}(\underline{i} + 0\underline{j}) = -\frac{Q}{x}\,\underline{i}\}$

$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial^2 y}$ $\therefore$ since analysing boundary layer form on $y=0$

then from $u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{dp}{dx}$ $\therefore -\frac{Q}{x}\frac{\partial}{\partial x}\left(-\frac{Q}{x}\right) = -\frac{1}{\rho}\frac{dp}{dx} \Rightarrow$

$\frac{Q^2}{x}\frac{\partial}{\partial x}\left(\frac{1}{x}\right) = -\frac{1}{\rho}\frac{dp}{dx} \Rightarrow \frac{Q^2}{x}\left(-\frac{1}{x^2}\right) = -\frac{1}{\rho}\frac{dp}{dx}$ $\therefore \frac{1}{\rho}\frac{dp}{dx} = \frac{Q^2}{x^3}$

$\{\underline{U} = -\frac{Q}{x}\,\underline{i}$ $\therefore \frac{\partial U}{\partial y} = 0$ $\therefore \frac{\partial^2 U}{\partial y^2} = 0$ $\therefore$ remains: $u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{dp}{dx}\}$

~~VEL~~ [mainstream flow is eventually inviscid, so 2 bernoulli func is a const

$\frac{p}{\rho} + \frac{1}{2}u^2 = \text{Const} \Rightarrow \cancel{\frac{1}{\rho}\frac{dp}{dx}}$ $\frac{1}{\rho}\frac{dp}{dx} = -\frac{1}{2}\cdot 2u\frac{du}{dx}$]

\1c/ $\underline{u} = \nabla \times \Psi\underline{k} = (\partial_y\Psi, -\partial_x\Psi, 0)$ $\{$def of a streamfunc$\}$

$\Psi = -\sqrt{\nu Q}\,F(\eta)$ $\eta = \frac{y}{\delta(x)}$ $\delta(x) = x\sqrt{\frac{\nu}{Q}}$ $\therefore$

$\frac{\partial\Psi}{\partial y} = \frac{\partial\Psi}{\partial\eta}\frac{\partial\eta}{\partial y} = -\sqrt{\nu Q}\frac{dF}{d\eta}\frac{\partial\eta}{\partial y}$ $\left\{\frac{\partial\Psi}{\partial\eta} = \frac{\partial}{\partial\eta}\left(-\sqrt{\nu Q}\,F(\eta)\right) = -\sqrt{\nu Q}\frac{\partial F(\eta)}{\partial\eta}\right\}$

where $\frac{\partial\eta}{\partial y} = \frac{1}{\delta(x)} = \frac{1}{\delta}$ $\therefore$ $\underline{u} = (u, v, 0)$ $\therefore u = -\sqrt{\nu Q}\frac{dF}{d\eta}\frac{1}{\delta} = -\frac{\sqrt{\nu Q}}{\delta}F'$

$\frac{\partial\Psi}{\partial x} = \frac{\partial\Psi}{\partial\eta}\frac{\partial\eta}{\partial x} = \frac{\partial}{\partial\eta}\left(-\sqrt{\nu Q}\,F(\eta)\right)\frac{\partial\eta}{\partial x} = -\sqrt{\nu Q}\frac{\partial F(\eta)}{\partial\eta}\frac{\partial\eta}{\partial x} = -\sqrt{\nu Q}\,F'\frac{\partial\eta}{\partial x}$

$\frac{\partial\eta}{\partial x} = \frac{\partial}{\partial x}\left(\frac{y}{\delta(x)}\right) = y\frac{\partial}{\partial x}\left(\frac{1}{\delta(x)}\right) = y\frac{\partial[(\delta(x))^{-1}]}{\partial x} = y\left(-(\delta(x))^{-2}\right)\frac{d\delta(x)}{dx} = -\frac{y}{\delta^2}\left(\frac{d\delta(x)}{dx}\right) =$

week 7 sheet / $-\frac{y}{\delta^2}\frac{d}{dx}\left(x\sqrt{\frac{\nu}{Q}}\right) = -\frac{y}{\delta^2}\sqrt{\frac{\nu}{Q}} = -\frac{\eta}{\delta}\sqrt{\frac{\nu}{Q}}$

$\therefore \frac{\partial \psi}{\partial x} = -\sqrt{\nu Q}\,F'\left(-\frac{\eta}{\delta}\sqrt{\frac{\nu}{Q}}\right) = \nu\frac{\eta}{\delta}F'$ $\therefore$

$v = -\frac{\partial \psi}{\partial x} = -\nu\frac{\eta}{\delta}F'$

d/ subbing into Z boundary layer eqn requires

$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{Q^2}{x^3} + \nu\frac{\partial^2 u}{\partial y^2}$ $\left\{ u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2} \quad \left[-\frac{1}{\rho}\frac{dp}{dx} = \frac{Q^2}{x^3}\right.\right\}$

$\therefore u = -\frac{\sqrt{\nu Q}}{\delta}F'$ $\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(-\frac{\sqrt{\nu Q}}{\delta}F'\right) = -\sqrt{\nu Q}\frac{\partial}{\partial x}\left(\frac{F'}{\delta}\right) =$

$-\sqrt{\nu Q}\frac{\partial}{\partial x}\left(\frac{1}{\delta}F'\right) = \cancel{-\sqrt{\nu Q}\left[\frac{\partial}{\partial x}\right.}$ $-\sqrt{\nu Q}\left[\frac{1}{\delta}\frac{\partial}{\partial x}F'(\eta) + F'\frac{\partial}{\partial x}\left(\frac{1}{\delta(x)}\right)\right] =$

$-\sqrt{\nu Q}\left[\frac{1}{\delta}\frac{\partial F'}{\partial \eta}\frac{\partial \eta}{\partial x} + F'\left(-\frac{1}{\delta^2(x)}\right)\frac{\partial \delta(x)}{\partial x}\right] = -\sqrt{\nu Q}\left[\frac{1}{\delta}F''\left(-\frac{\eta}{\delta}\sqrt{\frac{\nu}{Q}}\right) + F'\left(-\frac{1}{\delta^2(x)}\right)\sqrt{\frac{\nu}{Q}}\right]$

$= -\nu\left[-\frac{\eta}{\delta^2}F'' + -\frac{1}{\delta^2}F'\right] = \nu\left[\frac{1}{\delta^2}F' + \frac{\eta}{\delta^2}F''\right]$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left(-\frac{\sqrt{\nu Q}}{\delta}F'\right) = -\frac{\sqrt{\nu Q}}{\delta}\frac{\partial}{\partial y}F' = -\frac{\sqrt{\nu Q}}{\delta}\frac{\partial F'(\eta)}{\partial \eta}\frac{\partial \eta}{\partial y} = -\frac{\sqrt{\nu Q}}{\delta}F''\frac{1}{\delta} = -\frac{\sqrt{\nu Q}}{\delta^2}F''$

$\therefore \frac{\partial^2 u}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) = \frac{\partial}{\partial y}\left(-\frac{\sqrt{\nu Q}}{\delta^2}F''\right) = -\frac{\sqrt{\nu Q}}{\delta^2}\frac{\partial}{\partial y}(F'') = -\frac{\sqrt{\nu Q}}{\delta^2}\frac{\partial F''(\eta)}{\partial \eta}\frac{\partial \eta}{\partial y} =$

$-\frac{\sqrt{\nu Q}}{\delta^2}F'''\frac{1}{\delta} = -\frac{\sqrt{\nu Q}}{\delta^3}F'''$ $\therefore$ sub into boundary layer eqn:

$-\frac{\sqrt{\nu Q}}{\delta}F'\nu\left[\frac{F'}{\delta^2} + \frac{\eta F''}{\delta^2}\right] + \nu\frac{\eta}{\delta}F'\frac{\sqrt{\nu Q}}{\delta^2}F'' = -\frac{Q^2}{x^3} - \nu\frac{\sqrt{\nu Q}}{\delta^3}F'''$ $\therefore \cdot\delta^3$:

$-\nu\sqrt{\nu Q}(F')^2 - \sqrt{\nu Q}\,\nu F'\eta F'' + \eta\nu\sqrt{\nu Q}F'F'' = -Q^2\frac{\nu^{3/2}}{Q^{3/2}} - \nu\sqrt{\nu Q}F'''$ $\therefore \div\nu^{3/2}Q^{1/2}$:

$-(F')^2 - \eta F'F'' + \eta F'F'' = -1 - F''' \Rightarrow -(F')^2 = -1 - F''' \Rightarrow$

$F''' - (F')^2 + 1 = 0$

~~e/~~ • on $y=0$ $u=0 \Rightarrow F'(0)=0$ $v=0$ automatically satisfied

since $\eta = 0$ $\Big\{$ on $y=0$ $\eta = \frac{y}{\delta(x)}$ $\therefore$ $\eta = 0$ $u = -\frac{\sqrt{\nu Q}}{\delta(x)}F'$

$v = -\nu\frac{\eta}{\delta(x)}F'$ $u = \frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}\left(-\sqrt{\nu Q}F(\eta)\right) = -\frac{\sqrt{\nu Q}}{\delta(x)}F'$ at $y=0$ is the

boundary layer $\therefore$ is Z stream func doesn't apply since its not

inviscid mainstream flow $\therefore$ at $y=0$: $u=0$ $\therefore$ $u = -\frac{\sqrt{\nu Q}}{\delta(x)}F'(\eta) = 0$ $\therefore$

~~$F'(\eta)=0$~~ $\eta = \frac{y}{\delta(x)} = \frac{0}{\delta(x)} = 0 = \eta$ $\therefore$ $u = -\frac{\sqrt{\nu Q}}{\delta(x)}F'(0) = 0$ $\therefore F'(0) = 0$

$\eta = 0$; $v = -\nu\eta\frac{1}{\delta}F' = -\nu 0\frac{1}{\delta}F' = 0 = v \Big\}$

• Matching onto mainstream: $u = -\frac{\sqrt{\nu Q}}{\delta}F'(\eta\to\infty) \to -\frac{Q}{x}$

$\Big\{ u = -\frac{Q}{x}F'(\eta)$ $\therefore$ ~~$F'(\eta)$~~ $\eta = \frac{y}{\delta(x)}$ $\therefore$ $\delta(x) = x\sqrt{\frac{\nu}{Q}}$ $\therefore$ $\eta = y\frac{1}{x}\sqrt{\frac{Q}{\nu}}$ $\therefore$

$\eta\to\infty$ is $\lim_{x\to 0^+} x$ or $y\to\infty$ $\therefore$ $y = x\tan(\alpha)$ $\therefore$ $x\to\infty$ but $F'(\eta\to\infty) = 1$ $\therefore$

$F'(\eta = 0) = 0$ at y $\eta \Rightarrow \eta = \infty$ $\therefore$ $\eta = \frac{y}{\delta(x)}$ $\therefore$ $y = \infty$ $\therefore$

~~u = f~~ $\therefore$ ~~u =~~ $u = -\frac{Q}{x} = u(\eta \to \infty)$ $u = -\frac{Q}{x} F'(\eta)$ $\therefore$

$-\frac{Q}{x} F'(\eta \to \infty) = -\frac{Q}{x}$ $\therefore$ $F'(\eta \to \infty) = 1$

Since $u \approx U = -\frac{Q}{x}$ on the edge of the layer $\frac{du}{dy} = 0 \Rightarrow F''(\eta \to \infty) = 0$

$U = -\frac{Q}{R}$ & $R = \sqrt{x^2 + y^2}$ $\therefore$ as $y = 0$ $\sqrt{x^2 + y^2} = x$ $\therefore$ $U = -\frac{Q}{x}$ on the

boundary layer $\therefore$ ~~u =~~ $u(\eta \to \infty) = -\frac{Q}{x} F'(\eta \to \infty) = -\frac{Q}{x}$ $\therefore$

$u \approx U = -\frac{Q}{x}$ $\therefore$ $\frac{du}{dy} = \frac{d(-\frac{Q}{x})}{dy} = 0$ $\therefore$ $F''(\eta) = \frac{dF'(\eta)}{d\eta}$ $\therefore$ $F''(\eta \to \infty) = 0$

\1e/ $G = F'$ $\therefore$ $G' = F''$ $F''' - (F')^2 + 1 = 0$ $F''' = (F'')'$ $\therefore$

$G'' = F'''$ $\therefore$ $G'' - (G)^2 + 1 = 0$ $\therefore$ $G'G'' - G'(G)^2 + G' = 0$ $\therefore$

$\left\{ \frac{\partial}{\partial \eta}\left[\frac{(G')^2}{2}\right] = \frac{2G'}{2} \frac{\partial}{\partial \eta}(G') = G'G'' \right.$, $\left. \frac{\partial}{\partial \eta}\left(\frac{G^3}{3}\right) = \frac{3G^2}{3} G' = G^2 G' \right\}$

$\cancel{\frac{(G')^2}{2} - \frac{G^3}{3}} + \frac{\partial}{\partial \eta}\left[\frac{(G')^2}{2}\right] - \frac{\partial}{\partial \eta}\left(\frac{G^3}{3}\right) + G' = 0$ $\therefore$

$\frac{(G')^2}{2} - \frac{G^3}{3} + G = \text{Const}$ $\therefore$ $F'(\infty) = 1$, $F''(\infty) = 0$

$G(\infty) = 1$, $G'(\infty) = 0$; $G(\infty)$

$\frac{0^2}{2} - \frac{1^3}{3} + 1 = \text{Const} = \frac{2}{3}$ $\therefore$ $\frac{(G')^2}{2} - \frac{G^3}{3} + G = \frac{2}{3}$ $\times 3$

$(G')^2 - \frac{2}{3}G^3 + 2G = \frac{4}{3}$ $\therefore$ $(G')^2 = \frac{4}{3} - 2G + \frac{2}{3}G^3$ $\therefore$

$(G')^2 = \frac{2}{3}(2 - 3G + G^3) = \frac{2}{3}(2 + G)(1 + G^2 - 2G) = \frac{2}{3}(1 - G)^2(2 + G) + (G')^2$

$\{ (1 - G)^2 = 1 + G^2 - 2G$ $(G - 1)^2 = G^2 + 1 - 2G \}$

\1f/ $H^2 = G + 2$ $\{$ $\therefore$ $H = \sqrt{G + 2} = (G + 2)^{1/2}$ $\therefore$ $H' = \frac{1}{2}(G + 2)^{-1/2}$

$HH' = (G + 2)^{1/2} \frac{1}{2}(G + 2)^{-1/2} = \frac{1}{2}$ $\therefore$ $2HH' = 2 \cdot \frac{1}{2} = 1$ X $\}$

$\frac{d}{d\eta}(H^2) = 2HH' = \frac{d}{d\eta}(G + 2) = G'$ $\{$ vs $y(t), x(t)$ $y^2 = x + 2$ $\therefore$ $y = (x + 2)^{1/2}$

$y(t) = (x(t) + 2)^{1/2}$ $\therefore$ $\dot{y} = \frac{1}{2}\dot{x}$ $\frac{dy}{dt} = \frac{d}{dt}\left[(x + 2)^{1/2}\right] = \frac{1}{2}(x + 2)^{-1/2}\dot{x}$

$\therefore$ ~~2t~~ ~~2H~~ $2y\dot{y} = 2(x + 2)^{1/2}\frac{1}{2}(x + 2)^{-1/2}\dot{x} = \dot{x} = \frac{dx}{dt}$ $\}$ $\therefore$

$2HH' = G'$ $\therefore$ $(G')^2 = (2HH')^2 = 4H^2(H')^2$ $\therefore$ $\{$ assuming $H^2 \neq 0$ $\therefore$ $G \neq -2$ $\}$

$\frac{2}{3}(1 - G)^2(2 + G) = (G')^2 = 4H^2(H')^2 = \frac{2}{3}(3 - H^2)H^2$ $\therefore$ $4(H')^2 = \frac{2}{3}(3 - H^2)$ $\therefore$ $2 - \frac{2}{3}H^2$

$\therefore$ $(H')^2 = \frac{1}{2} - \frac{1}{6}H^2 = \frac{1}{6}(-H^2 + 3)^2$ $\therefore$ $H' = \pm\frac{1}{\sqrt{6}}(3 - H^2)$ $\therefore$

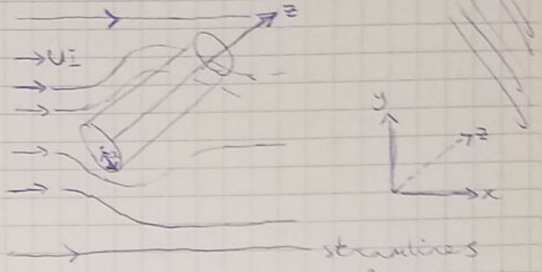
$H = \sqrt{3}\tanh(I)$ $\therefore$ $H' = \sqrt{3}(1 - \tanh^2(I))I'$ $\left\{ \frac{d}{dx}\tanh x = 1 - \tanh^2 x \right\}$ $\therefore$

$\sqrt{3}(1 - \tanh^2 I)\, I' = \pm\frac{1}{\sqrt{6}} \left| 3 - 3\tanh^2 I \right|$ $\therefore$ $\sqrt{3} I' = \pm\frac{3}{\sqrt{6}}$ $\therefore$ $I' = \pm\frac{1}{\sqrt{2}}$

\Week 7 Sheet/ $I = \pm\frac{1}{\sqrt{2}}\eta + c \therefore H = \sqrt{3}\tanh\left[\pm\frac{1}{\sqrt{2}} + c\right]$

$\therefore F' = G = H^2 - 2 = 3\tanh^2\left[c \pm \frac{\eta}{\sqrt{2}}\right] - 2$

\Week 8 Sheet/\1a/



\1b/ $\underline{u} = 0$ on $R = a$ $\Rightarrow$ $\left\{ \underline{u} = \nabla\times(\Psi\hat{\underline{z}}) = \nabla\times(0\hat{\underline{R}} + 0\hat{\underline{\phi}} + \Psi\hat{\underline{z}}) = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & 0 & \Psi \end{vmatrix} \right.$

$= \frac{1}{R}\left[\hat{\underline{R}}\left(\frac{\partial\Psi}{\partial\phi} - \frac{\partial 0}{\partial z}\right) - R\hat{\underline{\phi}}\left(\frac{\partial\Psi}{\partial R} - \frac{\partial 0}{\partial z}\right) + \hat{\underline{z}}\left(\frac{\partial 0}{\partial R} - \frac{\partial 0}{\partial\phi}\right)\right] =$

$\left.\frac{1}{R}\left[\hat{\underline{R}}\frac{\partial\Psi}{\partial\phi} - R\hat{\underline{\phi}}\frac{\partial\Psi}{\partial R}\right] = \frac{1}{R}\hat{\underline{R}}\frac{\partial\Psi}{\partial\phi} - \hat{\underline{\phi}}\frac{\partial\Psi}{\partial R} = \underline{u} = (u, v)\right\} \therefore$

$u = \frac{1}{R}\frac{\partial\Psi}{\partial\phi} = 0$ on $R = a$ & $v = -\frac{\partial\Psi}{\partial R} = 0$ on $R = a$

since $\underline{u} = \nabla\times(\Psi\hat{\underline{z}})$ & $\Psi = \Psi(R, \phi)$

also, as $R\to\infty$ $\underline{u} \to U\underline{i} = U(\cos\phi\hat{\underline{R}} - \sin\phi\hat{\underline{\phi}})$

$\left\{ \underline{u} = \nabla\times(\Psi\hat{\underline{z}}) = \frac{1}{R}\hat{\underline{R}}\frac{\partial\Psi}{\partial\phi} - \hat{\underline{\phi}}\frac{\partial\Psi}{\partial R} \right.$ $\therefore$ as $R\to\infty$: $\underline{u} \to 0\hat{\underline{R}}\frac{\partial\Psi}{\partial\phi} - \hat{\underline{\phi}}\frac{\partial\Psi}{\partial R} = -\hat{\underline{\phi}}\frac{\partial\Psi}{\partial R}$

$\therefore \left\{ \underline{i} = \cos\phi\hat{\underline{R}} - \sin\phi\hat{\underline{\phi}} = \cos\phi(\cos\phi\underline{i} + \sin\phi\underline{j}) - \sin\phi(-\sin\phi\underline{i} + \cos\phi\underline{j}) \right.$

$= \cos^2\phi\underline{i} + \sin\phi\cos\phi\underline{j} + \sin^2\phi\underline{i} - \sin\phi\cos\phi\underline{j} = (\cos^2\phi + \sin^2\phi)\underline{i} + 0\underline{j} = 1\underline{i}$

$\left.\therefore U\underline{i} = U(\cos\phi\hat{\underline{R}} - \sin\phi\hat{\underline{\phi}})\right\}$ at large radii the flow velocity is in the x-direction only & has magnitude U $\therefore$ as $R\to\infty$: $\underline{u}\to U\underline{i} = U(\cos\phi\hat{\underline{R}} - \sin\phi\hat{\underline{\phi}})$ } $\therefore$

$\nabla\times(\Psi\hat{\underline{z}}) = \underline{u} = U\underline{i} = U(\cos\phi\hat{\underline{R}} - \sin\phi\hat{\underline{\phi}}) = \frac{1}{R}\hat{\underline{R}}\frac{\partial\Psi}{\partial\phi} - \hat{\underline{\phi}}\frac{\partial\Psi}{\partial R}$ $\therefore$

$\frac{1}{R}\frac{\partial\Psi}{\partial\phi} \approx U\cos\phi$ & $\frac{\partial\Psi}{\partial R} \approx U\sin\phi$ as $R\to\infty$ $\therefore$

integrating the last eqn wrt R gives: $\left\{\int\frac{\partial\Psi}{\partial R}dR = \int U\sin\phi\, dR\right\}$

$\Psi = UR\sin\phi + g(\phi)$ $\therefore$ $\frac{\partial\Psi}{\partial\phi} = \frac{\partial}{\partial\phi}(UR\sin\phi + g(\phi)) = UR\cos\phi + \frac{dg(\phi)}{d\phi}$

$\Rightarrow \frac{\partial\Psi}{\partial\phi} \approx UR\cos\phi$ $\therefore UR\cos\phi = UR\cos\phi + \frac{dg(\phi)}{d\phi} = \frac{\partial\Psi}{\partial\phi}$ $\therefore \frac{dg(\phi)}{d\phi} = 0$ using first condition at large R $\therefore$ $\left\{? \frac{dg(\phi)}{d\phi} = 0 \Rightarrow g(\phi) = 0\right\}$

$\Psi \approx UR\sin\phi$ as $R\to\infty$

1c/ sols with form: $\psi = f(R)\sin\phi$ with trial form. $f(R) = R^m$ $\therefore$

$\psi = R^m \sin\phi$ trial form sols $\therefore$ you should find $\nabla^2\psi = (m^2-1)R^{m-2}\sin\phi$

$\left\{ \nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = \nabla\cdot\nabla \quad \therefore \nabla^2 f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2} \right\} \therefore$

$\{$in plane polar coords (R, ϕ): $\nabla\Omega = \frac{\partial\Omega}{\partial R}\hat{\underline{R}} + \frac{1}{R}\frac{\partial\Omega}{\partial\phi}\hat{\underline{\phi}}$ &

$\nabla^2\Omega = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\Omega}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2\Omega}{\partial\phi^2} \quad \therefore \quad \nabla^2\psi = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial\psi}{\partial R}\right) + \frac{1}{R^2}\left(\frac{\partial^2\psi}{\partial\phi^2}\right) \therefore$

$\nabla^2\psi = \nabla^2(R^m\sin\phi) = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}(R^m\sin\phi)\right) + \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2}(R^m\sin\phi) =$

$\frac{1}{R}\frac{\partial}{\partial R}(R\sin\phi\; mR^{m-1}) + \frac{1}{R^2}R^m\frac{\partial}{\partial\phi}(\cos\phi) =$

$\frac{1}{R}m\sin\phi\frac{\partial}{\partial R}(R^m) + \frac{1}{R^2}R^m(-\sin\phi) = \frac{1}{R}m\sin\phi\,(mR^{m-1}) - \sin\phi(R^{m-2}) =$

$m^2R^{m-2}\sin\phi - R^{m-2}\sin\phi = (m^2-1)R^{m-2}\sin\phi$ ✓ $\therefore$

$\nabla^2\nabla^2\psi = \nabla^4\psi = \nabla^2(\nabla^2\psi) = \nabla^2\left[(m^2-1)R^{m-2}\sin\phi\right] =$

$\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial}{\partial R}\left[(m^2-1)R^{m-2}\sin\phi\right]\right) + \frac{1}{R^2}\frac{\partial}{\partial\phi^2}\left[(m^2-1)R^{m-2}\sin\phi\right] =$

$\cancel{\frac{1}{R}\frac{\partial}{\partial R}\left(R^{m-1}\sin\phi\frac{\partial}{\partial R}\right.}\; \frac{1}{R}\frac{\partial}{\partial R}\left(R(m^2-1)\sin\phi\frac{\partial R^{m-2}}{\partial R}\right) + R^{m-4}(m^2-1)\frac{\partial}{\partial\phi^2}\sin\phi =$

$\frac{1}{R}(m^2-1)\sin\phi\frac{\partial}{\partial R}\left(R(m-2)R^{m-3}\right) + R^{m-4}(m^2-1)(-\sin\phi) =$

$\frac{1}{R}(m^2-1)(m-2)\sin\phi\frac{\partial}{\partial R}(R^{m-2}) - R^{m-4}(m^2-1)\sin\phi =$

$\frac{1}{R}(m^2-1)(m-2)\sin\phi\,(m-2)R^{m-3} - R^{m-4}(m^2-1)\sin\phi =$

$R^{m-4}(m^2-1)(m^2-4m+4)\sin\phi - (1)R^{m-4}(m^2-1)\sin\phi =$

$R^{m-4}(m^2-1)\sin\phi\left[m^2-4m+4-1\right] = R^{m-4}(m^2-1)\sin\phi\,(m^2-4m+3) =$

$R^{m-4}(m^2-1)(m-1)(m-3)\sin\phi = \nabla^4\psi \quad \therefore \quad$ & so $\nabla^4\psi = 0 \quad \therefore$

$m^2-1=0 \quad \therefore m = \pm 1$ or $m=1, m=3$ & we have 2 general sol:

$f(R) = \frac{A}{R} + BR + CR\ln R + DR^3 \qquad \{ f(R) = R^m \therefore$

$f(R) = AR^{-1} + BR^1 + CR^1(\ln R) + DR^3$ since multiple $m=1 \therefore$ do $BR^1 + CR^1\ln R \}$

1d/ for $f(R) \simeq UR$ as $R\to\infty$ must take $B=U$ & $C=D=0$

$\{ B=U$ & $C=D=0 \therefore f(R) = AR^{-1} + UR \}$

also $\frac{\partial\psi}{\partial\phi} = 0$ on $R=0 \Rightarrow$ ~~A~~ $f(a) = \frac{A}{a} + Ua = 0$

$\{$require $\psi \simeq UR\sin\phi$ as $R\to\infty$ & $\psi = f(R)\sin\phi \therefore f(R) = UR =$

$AR^{-1} + BR^1 + CR^1\ln R + DR^3 = AR^{-1} + BR^1 + 0R^1\ln R + 0R^3 = BR^1 \therefore B = U$

& $u = \frac{1}{R}\frac{\partial\psi}{\partial\phi} = 0$ on $R=a \quad \therefore \frac{\partial\psi}{\partial\phi} = 0 \quad \therefore f(R=a) = Aa^{-1} + Ba^1 = \frac{A}{a} + Ua = 0 \}$

$\{ \psi = f(R)\sin\phi \therefore \frac{\partial\psi}{\partial\phi} = \frac{\partial}{\partial\phi}(f(R)\sin\phi) = f(R)\cos\phi = \left(\frac{A}{a} + Ua\right)\cos\phi = 0 \therefore$

\Week 8 Sheet/ $\frac{A}{a} + Ua = 0 \quad \therefore \quad -\frac{\partial \psi}{\partial R} = 0 = \frac{\partial \psi}{\partial R} \quad \therefore$

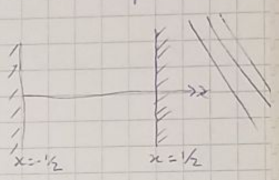
$\frac{\partial \psi}{\partial R} = \frac{\partial}{\partial R}(f(R)\sin\phi) = \sin\phi \frac{\partial f(R)}{\partial R} = \sin\phi\left(-\frac{A}{R^2} + B + C\ln R + C + 3DR^2\right) =$

$\sin\phi\left(-\frac{A}{R^2} + U\right) \quad \therefore \quad \frac{\partial \psi}{\partial R} = 0$ on $R = a \quad \therefore \sin(\phi)\left(-\frac{A}{a^2} + U\right) = 0 = -\frac{A}{a^2} + U$

$= f'(a) \quad \therefore$ have $\frac{A}{a} + Ua = 0 \quad \& \quad -\frac{A}{a^2} + U = 0 \quad \therefore$

$A = -Ua^2 \quad \& \quad A = Ua^2$ } these are satisfied for incompatible $\hat{\phi}$

vals of A so a sol doesn't exist



\Week 9/ \1a/ $\nabla^4 \psi = \nabla^2(\nabla^2 \psi) = 0$,

$\underline{u} = \nabla \times (\psi \underline{k}) = (\partial_y \psi, -\partial_x \psi, 0) \quad \therefore$ take $\psi(x,y) = f(x)e^{\lambda y}$, $\lambda = $ const

$\therefore \nabla^4 \psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)^2 [f(x)e^{\lambda y}] = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)[(f'' + \lambda^2 f)e^{\lambda y}] =$

$\left\{\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right)\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right)[f(x)e^{\lambda y}] = \left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right)\left[e^{\lambda y}\frac{\partial f(x)}{\partial x^2} + f(x)\frac{\partial}{\partial y^2}(e^{\lambda y})\right] =\right.$

$\left(\frac{\partial}{\partial x^2} + \frac{\partial}{\partial y^2}\right)[e^{\lambda y} f''(x) + \lambda^2 f(x) e^{\lambda y}] = \Big\} \; e^{\lambda y}\frac{\partial}{\partial x^2}(f'' + \lambda^2 f) + (f'' + \lambda^2 f)\frac{\partial}{\partial y^2}(e^{\lambda y}) =$

$(f'''' + \lambda^2 f'' + \lambda^2 f'' + \lambda^4 f)e^{\lambda y} \quad \therefore \nabla^4 \psi = 0$ requires $\cancel{f'''' + 2\lambda^2 f'' + \lambda^4 f = 0}$.

$(f^{(4)} + 2\lambda^2 f'' + \lambda^4 f)e^{\lambda y} = 0 \quad \therefore \quad f'''' + 2\lambda^2 f'' + \lambda^4 f = 0 \quad \therefore$

seek sols of form $f(x) = e^{px}$, $p = $ const $\therefore \quad f''(x) = p^2 e^{px}$, $f''''(x) = p^4 e^{px} \therefore$

$p^4 e^{px} + 2\lambda p^2 e^{px} + \lambda^4 e^{px} = 0 = p^4 + 2p^2\lambda + \lambda^4 = 0 \quad \therefore (p^2 + \lambda^2)^2 = 0 = (p^2 + \lambda^2)$

$\therefore p^2 = -\lambda^2 \quad \therefore \quad p = \pm i\lambda$ twice; each root is repeated $\therefore$

2 general sol is: $\cancel{f(x) = e^{i\lambda x}}$ $f(x) = A_2 e^{i\lambda x} + B_2 x e^{i\lambda x} + C_2 e^{-i\lambda x} + D_2 x e^{-i\lambda x} =$

$A_2\cos\lambda x + iA_2\sin\lambda x + B_2 x\cos\lambda x + iB_2 x\sin\lambda x + C_2\cos(-\lambda x) + iC_2\sin(-\lambda x) + D_2 x\cos(-\lambda x) + iD_2 x\sin(-\lambda x)$

$= A_2\cos\lambda x + iA_2\sin\lambda x + B_2 x\cos\lambda x + iB_2 x\sin\lambda x + C_2\cos\lambda x - iC_2\sin\lambda x + D_2 x\cos\lambda x - iD_2 x\sin\lambda x =$

$(A_2 + C_2)\cos\lambda x + (iA_2 - iC_2)\sin\lambda x + x(B_2 + D_2)\cos\lambda x + x(iB_2 - iD_2)\sin\lambda x =$

$\cancel{A\cos\lambda x}$ $A\sin\lambda x + B\cos\lambda x + Cx\sin\lambda x + Dx\cos\lambda x$

\1b/ no slip conditions on $x = -\frac{1}{2}$ & $x = \frac{1}{2}$ $\therefore$ for even sols $f(-x) = f(x)$ $\therefore$

take $A = D = 0 \quad \therefore$ no-slip at $x = \pm\frac{1}{2}$:

$u = 0 \Rightarrow \underline{u} = (\partial_y \psi, -\partial_x \psi, 0) = (u, v, 0) \quad \therefore \quad u = \partial_y \psi \quad \therefore$

$\psi = f(x)e^{\lambda y} = (B\cos\lambda x + Cx\sin\lambda x)e^{\lambda y}$ {since $(B\cos\lambda x + Cx\sin\lambda x)$ is an even func}

$\therefore \quad u = \partial_y \psi = \lambda f(x) e^{\lambda y} = 0 \quad \therefore \quad \lambda f(\pm\frac{1}{2}) = 0 \quad \therefore \quad B\cos(\pm\frac{\lambda}{2}) + C(\pm\frac{1}{2})\sin(\pm\frac{\lambda}{2}) =$

$B\cos\frac{\lambda}{2} + \frac{C}{2}\sin\frac{\lambda}{2} = 0$ ⊕

$v = 0 \quad \therefore \quad v = -\partial_x \psi = -f'(x)e^{\lambda y} = 0 \quad \therefore \quad -f'(\pm\frac{1}{2}) = 0 \quad \therefore \cancel{= f(x) \, Bx}$

$= f'(x) = -\left[-B\lambda \sin\lambda x + C\sin\lambda x + C\lambda x\cos\lambda x\right] = B\lambda\sin\lambda x - C\lambda x\cos\lambda x - C\sin\lambda x \therefore$

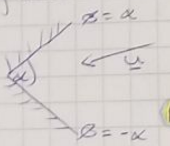
$= f'(x) = f(\pm\frac{1}{2}) = 0 = f'(\pm\frac{1}{2}) = -B\lambda\sin(\pm\frac{\lambda}{2}) + C\lambda(\pm\frac{1}{2})\cos(\pm\frac{\lambda}{2}) + C\sin(\pm\frac{\lambda}{2}) =$

$= B\lambda\sin(\frac{\lambda}{2}) + \frac{C}{2}\lambda\cos\frac{\lambda}{2} + C\sin\frac{\lambda}{2} = 0$ ④ $\therefore$ using ① & ④ :

$\cos\frac{\lambda}{2}\left[\frac{\lambda}{2}\cos\frac{\lambda}{2} + \sin\frac{\lambda}{2}\right] + \frac{\lambda}{2}\sin\frac{\lambda}{2}\sin\frac{\lambda}{2} = 0 \therefore$

$\frac{\lambda}{2} + \cos\frac{\lambda}{2}\sin\frac{\lambda}{2} = 0 \therefore \lambda + \sin\lambda = 0$

1e for odd sols, take $B = C = 0$ $\therefore$ similar calculation to above gives: $\sin\lambda - \lambda = 0$ $\therefore$ to show graphically that Z only real root is $\lambda = 0$, plot $\lambda, -\lambda$ & $\sin\lambda$ & show that they only intersect at $\lambda = 0$



2a as $\psi = f(\phi)$ $\therefore$ f is indep of R $\therefore$

$\underline{u} = \nabla \times (\psi\hat{\underline{z}}) = \left(\frac{1}{R}\frac{\partial\psi}{\partial\phi}, -\frac{\partial\psi}{\partial R}\right) = (u_R, u_\phi) \therefore$

$u_R = \frac{1}{R}\frac{\partial\psi}{\partial\phi}$ & $u_\phi = -\frac{\partial\psi}{\partial R}$, $\psi = f(\phi) \therefore \frac{\partial\psi}{\partial R} = \frac{\partial f(\phi)}{\partial R} = 0 \therefore u_\phi = 0$

& $u_R = \frac{1}{R}\frac{\partial f(\phi)}{\partial\phi} = \frac{1}{R}\frac{\partial f}{\partial\phi} \therefore \underline{u} = u_R(R,\phi)\hat{\underline{R}}$

2b $\nabla^2\psi = \left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2}\right)f(\phi) = \frac{1}{R^2}\frac{d^2 f}{d\phi^2}$ $\because$ f is indep of R $\therefore$

$\nabla^4\psi = \left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2}\right)\left(\frac{1}{R^2}\frac{d^2 f}{d\phi^2}\right) = \left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial\phi^2}\right)\left(\frac{1}{R^2}f''\right) =$

$f''\frac{\partial^2(\frac{1}{R^2})}{\partial R^2} + f''\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{1}{R^2}\right) + \frac{1}{R^2}\frac{1}{R^2}\frac{\partial^2}{\partial\phi^2}(f'') =$

~~$f''(-2)\frac{\partial}{\partial R}(R^{-3})$~~ $f''\frac{\partial}{\partial R}(-2R^{-3}) + f''\frac{1}{R}(-2R^{-3}) + \frac{1}{R^4}f'''' =$

$f''6R^{-4} + -2f''R^{-4} + \frac{1}{R^4}f'''' = \frac{1}{R^4}(4f'' + f'''')$ & $\nabla^4\psi = 0 \therefore$

$\frac{1}{R^4}(4f'' + f'''') = 0 \therefore 4f'' + f'''' = 0$

2c no-slip condition $\therefore$ require $u_R(\phi = \pm\alpha) = 0$ $\because$ $\underline{u}(\phi = \pm\alpha) = 0 \therefore$

$\underline{u}(\phi = \pm\alpha) = (u_R, 0) = u_R(R, \phi = \pm\alpha)\hat{\underline{R}} = 0 \therefore u_R(\phi = \pm\alpha) = 0 \therefore$

$u_R = \frac{1}{R}\frac{\partial\psi}{\partial\phi}$, $\psi = f(\phi) \therefore u_R = \frac{1}{R}\frac{\partial f(\phi)}{\partial\phi} \therefore u_R(\phi = \pm\alpha) = 0 = \frac{1}{R}\frac{\partial f(\phi = \pm\alpha)}{\partial\phi} \therefore$

$\frac{\partial}{\partial\phi}f(\phi = \pm\alpha) = 0 \therefore \frac{df}{d\phi} = 0$ for $\phi = \pm\alpha$ $\therefore$ searching for sols of $f'''' + 4f'' = 0$

of Z form $f = e^{M\phi} \therefore f'''' = M^4 e^{M\phi}$, $f'' = M^2 e^{M\phi} \therefore$

$M^4 e^{M\phi} + 4M^2 e^{M\phi} = 0 = M^4 + 4M^2 = M^2(M^2 + 4) \therefore M^2 = 0$, $M^2 = -4 \therefore$

$M = 0, 0, \pm 2i \therefore f = f(\phi) = Ee^{2i\phi} + Fe^{-2i\phi} + Ce^{0\phi} + D\phi e^{0\phi} =$

$E\cos 2\phi + iE\sin 2\phi + F\cos(-2\phi) + iF\sin(-2\phi) + C + D\phi =$

$E\cos 2\phi + iE\sin 2\phi + F\cos(2\phi) - iF\sin 2\phi + C + D\phi =$

$(E + F)\cos 2\phi + (iE - iF)\sin 2\phi + C + D\phi = A\cos 2\phi + B\sin 2\phi + C + D\phi \therefore$

\Week 9/ now set $\frac{\partial f}{\partial \theta} = 0$ on $\theta = \pm\alpha$ to find:

$$\frac{\partial f}{\partial \theta} = \frac{\partial}{\partial \theta}\left(A\cos 2\theta + B\sin 2\theta + C + D\theta\right) =$$

$$-2A\sin 2\theta + 2B\cos 2\theta + D \quad \therefore$$

$$\frac{\partial f(\theta = +\alpha)}{\partial \theta} = -2A\sin 2\alpha + 2B\cos 2\alpha + D = 0 \quad \&$$

$$\frac{\partial f(\theta = -\alpha)}{\partial \theta} = -2A\sin(-2\alpha) + 2B\cos(-2\alpha) + D = 2A\sin 2\alpha + 2B\cos 2\alpha + D = 0 \quad \therefore$$

$$D = -2B\cos 2\alpha \quad \& \quad A = 0 \quad \& \therefore$$

$$u_r = \frac{1}{r}\frac{\partial f}{\partial \theta} = -\frac{C}{R}\left(\cos 2\theta - \cos 2\alpha\right), \quad C = -2B$$

\Week 10 sheet/ $\underline{u} = (u, v, 0) = \hat{i}\,u(x,y,t) + \hat{j}\,v(x,y,t) + \hat{k}\,0$

$$\underline{\omega} = \nabla \times \underline{u} = \omega \underline{k}$$

\1a/ $\underline{\omega} = \nabla \times \underline{u} = \nabla \times \left[\alpha y \hat{\underline{i}} + \frac{U\alpha t}{1+\alpha^2 t^2}\sin[m(x-\alpha y t)]\,\hat{\underline{i}} + \frac{U}{1+\alpha^2 t^2}\sin[m(x-\alpha y t)]\,\hat{\underline{j}}\right]$

$$= \nabla \times \left\{\hat{\underline{i}}\left[\alpha y + \frac{U\alpha t}{1+\alpha^2 t^2}\sin[m(x-\alpha y t)]\right] + \frac{U}{1+\alpha^2 t^2}\sin[m(x-\alpha y t)]\,\hat{\underline{j}}\right\} =$$

$$\begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \partial_x & \partial_y & \partial_z \\ \alpha y + \frac{U\alpha t}{1+\alpha^2 t^2}\sin[m(x-\alpha y t)] & \frac{U}{1+\alpha^2 t^2}\sin[m(x-\alpha y t)] & 0 \end{vmatrix} = \quad \therefore$$

$\hat{\underline{z}}$ comp: $\frac{\partial}{\partial x}\left\{\frac{U}{1+\alpha^2 t^2}\sin[m(x-\alpha y t)]\right\} - \frac{\partial}{\partial y}\left\{\alpha y + \frac{U\alpha t}{1+\alpha^2 t^2}\sin[m(x-\alpha y t)]\right\} =$

$$\frac{U}{1+\alpha^2 t^2}\frac{\partial}{\partial x}\sin[mx - m\alpha y t] - \alpha\frac{\partial}{\partial y}y + \frac{U\alpha t}{1+\alpha^2 t^2}\frac{\partial}{\partial y}\sin[mx - m\alpha y t] =$$

$$\frac{U}{1+\alpha^2 t^2}m\cos[mx - m\alpha y t] - \alpha + \frac{U\alpha t}{1+\alpha^2 t^2}(-m\alpha t)\cos[mx - m\alpha y t] =$$

$$-\alpha + \left\{\frac{Um}{1+\alpha^2 t^2} + \frac{-U\alpha^2 t^2 m}{1+\alpha^2 t^2}\right\}\cos[mx - m\alpha y t] = \quad \therefore \hat{z}\text{ comp: } \omega = -\alpha + Um\cos[m(x-\alpha y t)]$$

\1b/ inviscid $\therefore \nu = 0 \quad \therefore \frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\nabla\underline{u} = \frac{\partial \underline{\omega}}{\partial t} + \underline{u}\cdot\nabla\underline{\omega} \quad \therefore$

$\therefore$ if $\underline{u}$ & ω satisfy it: $\omega \cdot \nabla \underline{u} = \frac{\partial \omega}{\partial t} + \underline{u}\cdot\nabla\omega \quad \therefore$

but $\frac{\partial \omega}{\partial t} + \underline{u}\cdot\nabla\omega = \nu\nabla^2\omega = 0\,\nabla^2\omega = 0 = \frac{\partial \omega}{\partial t} + \underline{u}\cdot\nabla\omega$

$$\therefore \frac{\partial \omega}{\partial t} = (-\alpha y m)Um\sin[m(x-\alpha y t)] = \alpha y m^2 U\sin[m(x-\alpha y t)]$$

\1c/ $\frac{\partial \omega}{\partial t} + \underline{u}\cdot\nabla\omega = \nu\nabla^2\omega \quad \therefore \frac{\partial \omega}{\partial t} = \frac{\partial}{\partial t}\left[-\alpha + Um\cos[m(x-\alpha y t)]\,F(t)\right]$

$$= \frac{\partial}{\partial t}\left[-\alpha + Um\cos[mx - \alpha m y t]\,F(t)\right] =$$

$$Um\cos[mx - \alpha m y t]\,F'(t) + (-1)(-\alpha m y)Um\sin[mx - \alpha m y t]\,F(t) =$$

$Um\cos[mx-\alpha myt]F'(t) + \alpha m^2 \cancel{y} y \sin[mx-\alpha myt]F(t)\alpha$

$\underline{u}\cdot\nabla = \left(\alpha y + \frac{U\alpha t}{1+\alpha^2t^2}\sin[mx-\alpha myt]F(t), \frac{U}{1+\alpha^2t^2}\sin[m\cancel{x}-\alpha ytm]F(t), 0\right)(\partial x, \partial y, \partial z) =$

$\left(\left[\alpha y + \frac{U\alpha t}{1+\alpha^2t^2}\sin[mx-\alpha myt\right]\partial x, \frac{U}{1+\alpha^2t^2}\sin[mx-\alpha ytm]F(t)\partial y, 0\right)$

$\therefore \underline{u}\cdot\nabla\omega = (\underline{u}\cdot\nabla)(-\alpha + Um\cos[mx-\alpha yt]F(t)) =$

$\left[\alpha y + \frac{U\alpha t}{1+\alpha^2t^2}\sin[mx-\alpha myt]\right](-Um^2)\sin[mx-\alpha ymt]F(t) +$

$\frac{U}{1+\alpha^2t^2}\sin[mx-\alpha ytm]F(t)(-1)(-\alpha Um^2)\sin[mx-\alpha myt]F(t) \cancel{+} 0 =$

$\frac{U}{1+\alpha^2t^2}\sin[mx-\alpha ytm]F(t) + \alpha Um^2\sin[mx-\alpha myt]F(t)$

\1d/ $F(t) = \exp\left[-\nu m^2\left(t+\frac{1}{3}\alpha^2t^3\right)\right] = e^{-\nu m^2t+\frac{1}{3}\alpha^2t^3} \therefore$

$\cancel{\int F(t)\,dt =}$ $\frac{d}{dt}F(t) = \frac{d}{dt}\left(e^{-\nu m^2t+\frac{1}{3}\alpha^2t^3}\right) = \cancel{\frac{d}{dt}(e}$

$\frac{d}{dt}\left(-\nu m^2t+\frac{1}{3}\alpha^2t^3\right)e^{-\nu m^2t+\frac{1}{3}\alpha^2t^3} = (-\nu m^2+\alpha^2t^2)e^{-\nu m^2t+\frac{1}{3}\alpha^2t^3}$

\2a/ for $R^2+z^2 \le a^2$: $\Psi = \frac{3}{4}UR^2a^{-2}(a^2-R^2-z^2) \therefore$

$\underline{u} = \nabla(R^{-1}\Psi\hat{\underline{\phi}}) = \nabla\left[R^{-1}\frac{3}{4}UR^2a^{-2}(a^2-R^2-z^2)\hat{\underline{\phi}}\right] = \nabla\left[\left(0, R^{+1}\frac{3}{4}Ua^{-2}(a^2-R^2-z^2), 0\right)\right]$

$= \nabla\left(0\hat{\underline{R}} + \left[\frac{3}{4}UR - \frac{3}{4}Ua^{-2}R^3 - \frac{3}{4}Ua^{-2}Rz^2\right]\hat{\underline{\phi}} + 0\hat{\underline{z}}\right) =$ X

$\underline{u} = \nabla\times(R^{-1}\Psi\hat{\underline{\phi}}) = \nabla\times\left(0\hat{\underline{R}} + \left[\frac{3}{4}UR - \frac{3}{4}Ua^{-2}R^3 - \frac{3}{4}Ua^{-2}Rz^2\right]\hat{\underline{\phi}} + 0\hat{\underline{z}}\right) =$

$$\frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & \frac{3}{4}UR^2 - \frac{3}{4}Ua^{-2}R^4 - \frac{3}{4}Ua^{-2}R^2z^2 & 0 \end{vmatrix} =$$

$-\frac{1}{R}\hat{\underline{R}}\frac{\partial}{\partial z}\left(\frac{3}{4}UR^2 - \frac{3}{4}Ua^{-2}R^4 - \frac{3}{4}Ua^{-2}R^2z^2\right) + \frac{1}{R}\hat{\underline{z}}\frac{\partial}{\partial R}\left(\frac{3}{4}UR^2 - \frac{3}{4}Ua^{-2}R^4 - \frac{3}{4}Ua^{-2}R^2z^2\right)$

$= -\frac{1}{R}\hat{\underline{R}}\left(-\frac{3}{4}\right)Ua^{-2}R^2 2z + \frac{1}{R}\hat{\underline{z}}\left(\frac{3}{2}UR - 3Ua^{-2}R^3 - \frac{3}{2}Ua^{-2}Rz^2\right) =$

$\frac{3}{2}Ua^{-2}Rz\hat{\underline{R}} + \left(\frac{3}{2}U\cancel{R} - 3Ua^{-2}R^2 - \frac{3}{2}Ua^{-2}z^2\right)\hat{\underline{z}}$ when $R^2+z^2 \le a^2$

& for $R^2+z^2 > a^2$: $\Psi = -\frac{1}{2}UR^2\left[1-a^3(R^2+z^2)^{-3/2}\right] \therefore$

$\underline{u} = \nabla\times\left(R^{-1}\left(-\frac{1}{2}\right)UR^2\left[1-a^3(R^2+z^2)^{-3/2}\right]\hat{\underline{\phi}}\right) = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & -\frac{1}{2}UR^2\left[1-a^3(R^2+z^2)^{-3/2}\right] & 0 \end{vmatrix} =$

$-\frac{\hat{\underline{R}}}{R}\frac{\partial}{\partial z}\left(-\frac{1}{2}UR^2\left[1-a^3(R^2+z^2)^{-3/2}\right]\right) + \frac{1}{R}\hat{\underline{z}}\frac{\partial}{\partial R}\left(-\frac{1}{2}UR^2\left[1-a^3(R^2+z^2)^{-3/2}\right]\right) =$

$-\frac{1}{R}\hat{\underline{R}}\left(-\frac{1}{2}\right)UR^2\left(-\frac{3}{2}\right)\left[a^3(R^2+z^2)^{-5/2}\right]2z + \frac{1}{R}\hat{\underline{z}}\left(-\frac{1}{2}U\right)\left(2R\left[1-a^3(R^2+z^2)^{-3/2}\right] + R^2\left(-a^3(R^2+z^2)^{-5/2}2R\right)\right)$

$= -\frac{3}{2}RUa^3(R^2+z^2)^{-5/2}z\hat{\underline{R}} + \left(\cancel{2}U\left[1-a^3(R^2+z^2)^{-3/2}\right] + \cancel{2}Ua^3R^2(R^2+z^2)^{-5/2}\right)\hat{\underline{z}}$

\Week 10 Sheet/ ∴

$$\underline{u} = \begin{cases} \frac{3}{2}Ua^{-2}Rz\hat{\underline{R}} + \left(\frac{3}{2}U - 3Ua^{-2}R^2 - \frac{3}{2}Ua^{-2}z^2\right)\hat{\underline{z}}, & R^2+z^2 \le a^2 \\ -\frac{3}{2}RUa^3(R^2+z^2)^{-5/2}z\hat{\underline{R}} + \left(-U\left[1-a^3(R^2+z^2)^{-3/2}\right] + Ua^3R^2(R^2+z^2)^{-5/2}\right)\hat{\underline{z}}, & R^2+z^2 > a^2 \end{cases}$$

\1b/ ∴ at $R^2+z^2=a^2$: $\lim_{R^2+z^2\to(a^2)^-} = \lim_{R^2+z^2\to a^2}$

$$\lim_{a^2\to(R^2+z^2)^+} \underline{u} = \lim_{a^2\to R^2+z^2}\left[\frac{3}{2}Ua^{-2}Rz\hat{\underline{R}} + \left(\frac{3}{2}U - 3Ua^{-2}R^2 - \frac{3}{2}Ua^{-2}z^2\right)\hat{\underline{z}}\right] =$$

$$\left[\frac{3}{2}U\frac{1}{R^2+z^2}Rz\hat{\underline{R}} + \left(\frac{3}{2}U - 3U\frac{1}{R^2+z^2}R^2 - \frac{3}{2}U\frac{1}{R^2+z^2}z^2\right)\hat{\underline{z}}\right] = \lim_{a^2\to(R^2+z^2)^+}\underline{u}$$

$$\lim_{a^2\to(R^2+z^2)^-}\underline{u} = \lim_{a^2\to(R^2+z^2)} -\frac{3}{2}RUa^3(R^2+z^2)^{-5/2}z\hat{\underline{R}} + \left(-U\left[1-a^3(R^2+z^2)^{-3/2}\right] + Ua^3R^2(R^2+z^2)^{-5/2}\right)\hat{\underline{z}} =$$

$$-\frac{3}{2}RU(R^2+z^2)^3(R^2+z^2)^{-5/2}z\hat{\underline{R}} + \left(-U\left[1-(R^2+z^2)^3(R^2+z^2)^{-3/2}\right] + U(R^2+z^2)^3R^2(R^2+z^2)^{-5/2}\right)\hat{\underline{z}} =$$

$$-\frac{3}{2}RU(R^2+z^2)^{+1/2}z\hat{\underline{R}} + \left(-U\left[1-(R^2+z^2)^{+3/2}\right] + UR^2(R^2+z^2)^{1/2}\right)\hat{\underline{z}} = \lim_{a^2\to(R^2+z^2)^-}\underline{u}$$

∴ as $\lim_{a^2\to(R^2+z^2)^-}\underline{u} = \lim_{a^2\to(R^2+z^2)^+}\underline{u}$: $\hat{\underline{R}}$ comp:

$$-\frac{3}{2}RU(R^2+z^2)^{1/2}z = \frac{3}{2}U(R^2+z^2)^{-1}Rz$$

$\hat{\underline{z}}$ comp:

$$\left(-U\left[1-(R^2+z^2)^{3/2}\right] + UR^2(R^2+z^2)^{1/2}\right) = \frac{3}{2}U - 3U(R^2+z^2)^{-1}R^2 - \frac{3}{2}U(R^2+z^2)^{-1}z^2$$

\2c/ inside Z vortex when $R^2+z^2 \le a^2$, outside when $R^2+z^2 > a^2$

$\underline{\omega} = \omega\hat{\underline{\phi}} = \nabla\times\underline{u}$ ∴

for $R^2+z^2 \le a^2$: $\underline{\omega} = \nabla\times\underline{u} = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial/\partial z \\ u & 0 & w \end{vmatrix} = \omega\hat{\underline{\phi}}$ at $R^2+z^2 \le a^2$: $\frac{\omega}{R} = \text{constant}$

for $R^2+z^2 \ge a^2$: $\underline{\omega} = \nabla\times\underline{u} = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ U & 0 & w \end{vmatrix} = \omega\hat{\underline{\phi}}$ ∴ at $R^2+z^2 > a^2$: $\frac{\omega}{R} = 0$

\Week 11 Sheet/ \1a/ $\xi = x - ct$ ∴ $\frac{\partial\xi}{\partial t} = -c$ $\frac{\partial\xi}{\partial x} = 1$ $\eta = x + ct$ ∴ $\frac{\partial\eta}{\partial t} = c$

$$\frac{\partial\eta}{\partial x} = 1 \therefore \frac{\partial\beta}{\partial t} = \frac{\partial\beta}{\partial\xi}\frac{\partial\xi}{\partial t} = \frac{\partial\beta}{\partial\xi}(-c) = -c\frac{\partial\beta}{\partial\xi} \therefore \frac{\partial^2\beta}{\partial t^2} = \frac{\partial}{\partial t}\left(-c\frac{\partial\beta}{\partial\xi}\right) = -c\frac{\partial}{\partial t}\left(\frac{\partial\beta}{\partial\xi}\right) =$$

$$-c\frac{\partial}{\partial\eta}\left(\frac{\partial\beta}{\partial\xi}\right)\frac{\partial\eta}{\partial t} = -c\frac{\partial^2\beta}{\partial\xi\partial\eta}c = -c^2\frac{\partial^2\beta}{\partial\xi\partial\eta} = \frac{\partial^2\beta}{\partial t^2} = c^2\frac{\partial^2\beta}{\partial x^2} \therefore$$

$$+\frac{\partial^2\beta}{\partial\xi\partial\eta} = -\frac{\partial^2\beta}{\partial x^2}, \quad \frac{\partial\beta}{\partial x} = \frac{\partial\beta}{\partial\eta}\frac{\partial\eta}{\partial x} = \frac{\partial\beta}{\partial\eta}\cdot 1 = \frac{\partial\beta}{\partial\eta} \therefore \frac{\partial^2\beta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\beta}{\partial\eta}\right) =$$

$$\frac{\partial}{\partial\xi}\left(\frac{\partial\beta}{\partial\eta}\right)\frac{\partial\xi}{\partial x} = \frac{\partial^2\beta}{\partial\xi\partial\eta}\cdot 1 = \frac{\partial^2\beta}{\partial\xi\partial\eta} = \frac{\partial^2\beta}{\partial x^2} \therefore -\frac{\partial^2\beta}{\partial x^2} = -\frac{\partial^2\beta}{\partial\xi\partial\eta} \therefore$$

$$\frac{\partial^2\beta}{\partial\xi\partial\eta} = -\frac{\partial^2\beta}{\partial\xi\partial\eta} \therefore \frac{\partial^2\beta}{\partial\xi\partial\eta} = 0$$

1b/ $\frac{\partial^2 \beta}{\partial \xi \partial \eta} = 0 \therefore \int \frac{\partial^2 \beta}{\partial \xi \partial \eta} \partial \xi = \int 0 \, \partial \xi = g(\eta) = \frac{\partial \beta}{\partial \eta} \therefore$

$\int \frac{\partial \beta}{\partial \eta} \partial \eta = \int g(\eta) \partial \eta = \beta = g(\eta) + f(\xi)$, f, g arbit $\therefore$

$\beta = f(\xi) + g(\eta) = f(x - ct) + g(x + ct)$

1c/ 1 dimens wave eqn: $\frac{\partial^2 \beta}{\partial t^2} = c^2 \frac{\partial^2 \beta}{\partial x^2} \therefore$

$\frac{\partial \beta}{\partial t} = \frac{\partial}{\partial t} f(x-ct) + \frac{\partial}{\partial t} g(x+ct) \therefore$

$\frac{\partial^2 \beta}{\partial t^2} = \frac{\partial}{\partial t^2} f(x-ct) + \frac{\partial}{\partial t^2} g(x+ct)$

$\frac{\partial^2 \beta}{\partial x^2} = \frac{\partial}{\partial x^2} f(x-ct) + \frac{\partial}{\partial x^2} g(x+ct) \therefore$

$\left\{ \frac{\partial}{\partial t^2} f(x-ct) = c^2 f_2(x-ct) \right\}$ is $\frac{\partial}{\partial t} f(t) = f'(t) = f'(x) = \frac{\partial}{\partial x} f(x) \therefore$

$\frac{\partial^2}{\partial t^2} f(x-ct) = c^2 f''(x-ct)$, $\frac{\partial^2}{\partial t^2} g(x+ct) = c^2 g''(x+ct)$

$\frac{\partial^2}{\partial x^2} f(x-ct) = f''(x-ct)$, $\frac{\partial}{\partial t^2} g(x+ct) = g''(x+ct) \therefore$

$\frac{\partial^2 \beta}{\partial t^2} = c^2 f''(x-ct) + c^2 g''(x+ct) = c^2 \left[f''(x-ct) + g''(x+ct) \right] = c^2 \left[\frac{\partial^2}{\partial x^2} f(x-ct) + \frac{\partial^2}{\partial x^2} g(x+ct) \right] =$

$c^2 \frac{\partial^2}{\partial x^2} \left[f(x-ct) + g(x+ct) \right] = c^2 \frac{\partial^2 \beta}{\partial x^2} \therefore \beta$ satisfies 1 dimens wave eqn

2/ $\frac{\partial^2 \beta}{\partial t^2} = c^2 \nabla^2 \beta \therefore$ ~~β~~ 3 dimens $\therefore \beta = a e^{i(\underline{k} \cdot \underline{r} - \omega t)} = a e^{i(k_1 x + k_2 y + k_3 z - \omega t)} \therefore$

$\frac{\partial \beta}{\partial t} = -a\omega e^{i(\underline{k} \cdot \underline{r} - \omega t)}$ $\quad \{ \underline{r} = x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}} \} \therefore$

$\frac{\partial \beta^2}{\partial t^2} = a\omega^2 e^{i(\underline{k} \cdot \underline{r} - \omega t)} = \omega^2 \beta$

$\nabla^2 \beta \therefore \quad \nabla \beta = \frac{\partial^2 \beta}{\partial x^2} + \frac{\partial^2 \beta}{\partial y^2} + \frac{\partial^2 \beta}{\partial z^2} =$

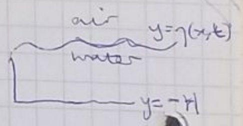
$a k_1^2 e^{i(\underline{k} \cdot \underline{r} - \omega t)} + a k_2^2 e^{i(\underline{k} \cdot \underline{r} - \omega t)} + a k_3^2 e^{i(\underline{k} \cdot \underline{r} - \omega t)} = (k_1^2 + k_2^2 + k_3^2) \beta \therefore$

$c^2 \nabla^2 \beta = c^2 (k_1^2 + k_2^2 + k_3^2) \beta = c^2 |k|^2 \beta \therefore$ for $\omega^2 = c^2 |k|^2 : \frac{\partial^2 \beta}{\partial t^2} = c^2 \nabla^2 \beta \therefore$

$\omega = \pm c|k| = \omega(k) = \pm ck \quad \therefore \frac{\omega}{k} = \pm c$, $\frac{d\omega}{dk} = \pm c \therefore$

$c_g = \frac{d\omega}{dk} = \pm c$, $c_p = \frac{\omega}{k} = \pm c \therefore c_p = c_g \therefore$ non dispersive

3/ vorticity $\underline{\zeta} = \nabla \times \underline{u} \quad \underline{u} = (u(x,y,t), v(x,y,t), 0) \therefore$



$\underline{\zeta} = \nabla \times \underline{u} = + \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right) \hat{\underline{k}} = 0$ is ~~$\phi = 0$~~ $\underline{\zeta} = 0 \therefore \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y}$

$\nabla \phi = \underline{u} \therefore \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0$ velocity potential ϕ st $\underline{u} = \nabla \phi = \left(\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, 0 \right) =$

$(u, v, 0) \Rightarrow \frac{\partial u}{\partial y} = \frac{\partial}{\partial y} \frac{\partial \phi}{\partial x} = \frac{\partial^2 \phi}{\partial y \partial x}$, $\frac{\partial v}{\partial x} = \frac{\partial^2 \phi}{\partial y \partial x} \therefore \nabla^2 \phi = \nabla \cdot (\nabla \phi) = \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} =$

$\nabla \cdot \underline{u} = 0 \therefore$

Week 11 Sheet/ $F = F(x,y,t) = y - \eta(x,t)$ $\quad \frac{DF}{Dt} = 0 \therefore$

$\frac{\partial F}{\partial t} + \underline{u}\cdot\nabla F = 0 \quad \therefore \frac{\partial F}{\partial t} = \frac{\partial}{\partial t}(y - \eta(x,t)) = -\frac{\partial \eta}{\partial t}$

$\underline{u}\cdot\nabla F = (u\frac{\partial}{\partial x}, v\frac{\partial}{\partial y}, 0)F = u\frac{\partial F}{\partial x} + v\frac{\partial F}{\partial y} = -u\frac{\partial \eta}{\partial x} + v \therefore$

$-\frac{\partial \eta}{\partial t} - u\frac{\partial \eta}{\partial x} + v = 0 \quad \therefore \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} = v$ on $y = \eta(x,t)$

$\therefore$ wave small amplitude $\therefore u, v, \eta$ small $\therefore$

neglect $u\frac{\partial \eta}{\partial x} \therefore \frac{\partial \eta}{\partial t} = v(x,y,t) = v$ on $y = \eta(x,t) \therefore$

$\cancel{v(x,y,t)} = v(x, y=\eta, t) = v(x, y=0, t) + \eta\frac{\partial v(x, y=0, t)}{\partial y} + \dots$

$\therefore$ neglecting $\eta\frac{\partial v}{\partial y}(y=0)$: $v(x, y=\eta, t) = v(x, y=0, t) = \frac{\partial \eta}{\partial t} \therefore \frac{\partial \eta}{\partial t} = v$ on $y = 0$

$\underline{u} = \nabla\phi = (\frac{\partial \phi}{\partial x}, \frac{\partial \phi}{\partial y}, 0) = (u, v, 0) \therefore \frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial y}$ on $y = 0$

but because depth of fluid is finite: $\frac{\partial \phi}{\partial y} = 0$ $\{v = 0\}$ on $y = -H$ {noslip condition} $\therefore$

$\eta = A\cos(kx - \omega t)$ for $\phi = \phi(x,y,t)$

inviscid N-S eqn: $\cancel{\frac{\partial u}{\partial t} + \nabla H}$ $\rho(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \rho g \therefore$

$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\nabla\frac{p}{\rho} + g = -\nabla\frac{p}{\rho} - \nabla\Pi = -\nabla(\frac{p}{\rho} + \Pi) = \frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} \therefore$

$\underline{u}\cdot\nabla\underline{u} = \nabla(\frac{|\underline{u}|^2}{2}) - \underline{u}\times(\nabla\times\underline{u}) \quad \nabla\times\underline{u} = \xi \therefore$

$\underline{u}\cdot\nabla\underline{u} = \nabla(\frac{\cancel{\underline{u}\cdot\underline{u}}}{2}) - \underline{u}\times(\nabla\times\underline{u}) = \nabla(\frac{|\underline{u}|^2}{2}) - \underline{u}\times\xi$ irrotational $\therefore \xi = 0 \therefore$

$\therefore \underline{u}\cdot\nabla\underline{u} = \nabla(\frac{|\underline{u}|^2}{2}) \therefore \frac{\partial \underline{u}}{\partial t} + \nabla(\frac{|\underline{u}|^2}{2}) = -\nabla(\frac{p}{\rho} + \Pi) \therefore$

$\frac{\partial \underline{u}}{\partial t} + \nabla(\frac{1}{2}|\underline{u}|^2 + \frac{p}{\rho} + \Pi) = 0 \therefore \cancel{\frac{\partial}{\partial t}\nabla\phi} + \cancel{\nabla}[\nabla(\frac{1}{2}|\underline{u}|^2 + \frac{p}{\rho} + \Pi)] = 0 =$

$\nabla[\frac{\partial \phi}{\partial t} + \frac{1}{2}|\underline{u}|^2 + \frac{p}{\rho} + \Pi] = 0 \quad \therefore f(t) = \frac{\partial \phi}{\partial t} + \frac{1}{2}|\underline{u}|^2 + \frac{p}{\rho} + \Pi \therefore$

at $y = \eta(x,t)$: $p = p_0 \therefore$ is $\frac{p_0}{\rho} = f(t)$: $\frac{\partial \phi}{\partial t} + \frac{1}{2}|\underline{u}^2| + \Pi = 0 = \frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + v^2) + \Pi$

$= 0 = \frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2 + v^2) + g\eta$ on $y = \eta\cancel{(x,t)}$ $\therefore$ small waves amplitude: neglect:

u^2 & v^2 $\therefore \frac{\partial \phi}{\partial t} + g\eta = 0$ on $y = \eta\cancel{(x,t)}$

& $\frac{\partial \phi}{\partial y} = 0$ on $y = -H$, $\eta = A\cos(kx - \omega t)$ for $\phi = \phi(x,y,t) \therefore$

$\phi = f(y)\eta = f(y)\cancel{A}\sin(kx - \omega t)$ $\quad \frac{\partial \eta}{\partial t} = \frac{\partial \phi}{\partial y}$ on $y = 0$ $\therefore \frac{\partial \eta}{\partial t} = A\omega\sin(kx - \omega t) \therefore$

$\frac{\partial \phi}{\partial y} = A\omega\sin(kx - \omega t) \therefore \phi = f(y)\sin(kx - \omega t)$

$\therefore \frac{\partial^2}{\partial x^2}\phi = -f(y)k^2\sin(kx - \omega t)$, $\frac{\partial^2}{\partial y^2}\phi = f''\sin(kx - \omega t) \therefore$

$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 = -f(y)k^2\sin(kx - \omega t) + f''(y)\sin(kx - \omega t) = 0 = f''(y) - k^2 f(y) \therefore$

$f(y) = e^{qy} \quad \therefore f''(y) = q^2 e^{qy} \quad \therefore q^2 e^{qy} - k^2 e^{qy} = 0 = q^2 - k^2 = (q - k)(q + k) \therefore q = k, -k$

$\therefore f(y) = Ae^{ky} + Be^{-ky} \quad \therefore \phi = [Ae^{ky} + Be^{-ky}]\sin(kx - \omega t) \quad \therefore$ take $k > 0$

$\frac{\partial \phi}{\partial y} = 0$ on $y = -H$, $\frac{\partial \phi}{\partial t} = -g\eta$ on $y = 0 \quad \therefore$

at $y = -H$: $\quad \frac{\partial \phi}{\partial y} = [Ake^{ky} - Bke^{-ky}]\sin(kx - \omega t) \quad \therefore$

$\cancel{[Ak(1) - Bk]\sin(kx - \omega t)} \quad [Ake^{-kH} - Bke^{kH}]\sin(kx - \omega t) = 0 \quad \forall (kx - \omega t) \therefore$

$Ake^{-kH} - Bke^{kH} = 0 \quad \therefore Ak\frac{1}{e^{kH}} = Bke^{kH} \quad \therefore \quad \frac{Ak}{Bk} = (e^{kH})^2 = \frac{A}{B} \quad \therefore$

$\pm\sqrt{\frac{A}{B}} = e^{kH} = \left(\frac{A}{B}\right)^{1/2} \quad \therefore kH = \ln\left[\left(\frac{A}{B}\right)^{1/2}\right] = \frac{1}{2}\ln\left(\frac{A}{B}\right) = \frac{1}{2}\ln A - \frac{1}{2}\ln B = kH$

$A = Be^{2kH} \quad \therefore$

$\cancel{\phi = Ae^{ky} +} \quad \phi = [Be^{2kH}e^{ky} + Be^{-ky}]\sin(kx - \omega t)$

$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}$ on $y = 0 \quad \therefore \frac{\partial \phi}{\partial y} = A\omega\sin(kx - \omega t) \quad \therefore$

$\frac{\partial \phi}{\partial y} = [Bke^{2kH}e^{ky} - Bke^{-ky}]\sin(kx - \omega t) \quad \therefore$

$[Bke^{2kH} - Bk]\sin(kx - \omega t) = A\omega\sin(kx - \omega t) \quad \therefore \quad Bk[e^{2kH} - 1] = A\omega \quad \therefore$

$B\frac{1}{\omega}k[e^{2kH} - 1] = A = Be^{2kH} \quad \therefore$

$\frac{1}{\omega}k[e^{2kH} - 1] = e^{2kH} = \frac{k}{\omega}e^{2kH} - \frac{k}{\omega} \quad \therefore \frac{k}{\omega} = \left[\frac{k}{\omega} - 1\right]e^{2kH} \quad \therefore$

$\omega = \frac{k(e^{2kH} - 1)}{e^{2kH}}$ X

$f(y) = Ce^{ky} + De^{-ky} \quad \therefore \frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} = A\omega\sin(kx - \omega t)$ on $y = 0 \quad \therefore$

$[Cke^{k0} - Dke^{-k0}]\sin(kx - \omega t) = A\omega\sin(kx - \omega t) \quad \therefore$

$Cke^{k0} - Dke^{-k0} = A\omega = Ck - Dk \quad \therefore A\omega + Dk = Ck \quad \therefore C = \frac{A\omega}{k} + D$

$\frac{\partial \phi}{\partial t} + g\eta = 0$ on $y = 0 \quad \therefore \frac{\partial \phi}{\partial t} = -g\eta \quad \therefore$

$\phi = \left[\left(\frac{A\omega}{k} + D\right)e^{ky} + De^{-ky}\right]\sin(kx - \omega t) \quad \therefore$

$\frac{\partial \phi}{\partial t} = -\left[\left(\frac{A\omega}{k} + D\right)e^{ky} + De^{-ky}\right]\omega\cos(kx - \omega t) \quad \therefore$

$-\omega\left[\left(\frac{A\omega}{k} + D\right)e^{k0} + De^{-k0}\right]\cos(kx - \omega t) = gA\cos(kx - \omega t) \quad \therefore$

$-\omega\left(\frac{A\omega}{k} + D\right) - \omega D = gA = -\frac{A}{k}\omega^2 - 2D\omega \quad \therefore$

$-\frac{A}{k}\omega^2 - 2D\omega - gA = 0 \quad \therefore \omega = \left(2D \pm \sqrt{4D^2 - 4\left(-\frac{A}{k}\right)(-gA)}\right) / 2\left(-\frac{A}{k}\right) \quad \therefore$

$2D\omega = -\frac{A}{k}\omega^2 - gA = -A\left(\frac{\omega^2}{k} + g\right) \quad \therefore D = -\frac{A}{2}\left(\frac{\omega}{k} + \frac{g}{\omega}\right)$

$C = -\frac{A}{2}\left(\frac{\omega}{k} + \frac{g}{\omega}\right) + \frac{A\omega}{k} = -\frac{A}{2}\left(\frac{\omega}{k} + \frac{g}{\omega} - \frac{2\omega}{k}\right) = -\frac{A}{2}\left(\frac{g}{\omega} - \frac{\omega}{k}\right)$

$\tanh kH = \frac{1 - e^{-2kH}}{1 + e^{-2kH}} \qquad \omega^2 = gk\frac{1 - e^{-2kH}}{1 + e^{-2kH}} = gk\tanh kH \quad \therefore \omega = \sqrt{gk\tanh kH}$

\Week 11 sheet/ \1a/ $\partial_{tt}\beta = c^2\partial_{xx}\beta$ for $\beta(x,t)$ let

$\xi = x-ct$ & $\eta = x+ct$ then $\frac{\partial\beta}{\partial x} = \frac{\partial\beta}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\beta}{\partial\eta}\frac{\partial\eta}{\partial x} = \frac{\partial\beta}{\partial\xi}\cdot 1 + \frac{\partial\beta}{\partial\eta}\cdot 1$

) $\{\beta = \beta(x,t)$, $\xi = \xi(x,t)$ $\eta = \eta(x,t)$ $\therefore \frac{\partial\beta}{\partial x} = \frac{\partial\beta}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\beta}{\partial\eta}\frac{\partial\eta}{\partial x}\}$ $\therefore$

$\cancel{\frac{\partial^2\beta}{\partial x^2} = \frac{\partial}{\partial\xi}\left(\frac{\partial\beta}{\partial}\right.}$ $\{\therefore \frac{\partial^2\beta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\beta}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\beta}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\beta}{\partial\eta}\frac{\partial\eta}{\partial x}\right) =$

$\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)\left(\frac{\partial\beta}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\beta}{\partial\eta}\frac{\partial\eta}{\partial x}\right)\left(\frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial x}\right)$ X $\therefore \frac{\partial\beta}{\partial x} = \left(\left(\frac{\partial}{\partial\xi} + \frac{\partial}{\partial\eta}\right)(\beta)\right)\left(\frac{\partial\xi}{\partial x} + \frac{\partial\eta}{\partial x}\right)$ X $\}$

$\therefore \frac{\partial^2\beta}{\partial x^2} = \{\frac{\partial\beta}{\partial x} = \frac{\partial\beta}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\beta}{\partial\eta}\frac{\partial\eta}{\partial x}$ $\therefore \frac{\partial^2\beta}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial\beta}{\partial x}\right) = \frac{\partial}{\partial x}\left(\frac{\partial\beta}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\beta}{\partial\eta}\frac{\partial\eta}{\partial x}\right) =$

$\frac{\partial}{\partial\xi}\left(\frac{\partial\beta}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\beta}{\partial\eta}\frac{\partial\eta}{\partial x}\right)\frac{\partial\xi}{\partial x} + \frac{\partial}{\partial\eta}\left(\frac{\partial\beta}{\partial\xi}\frac{\partial\xi}{\partial x} + \frac{\partial\beta}{\partial\eta}\frac{\partial\eta}{\partial x}\right)\frac{\partial\eta}{\partial x}\}$

$\frac{\partial\beta}{\partial x} = \frac{\partial\beta}{\partial\xi} + \frac{\partial\beta}{\partial\eta}$ $\therefore \frac{\partial^2\beta}{\partial x^2} = \frac{\partial}{\partial\xi}\left(\frac{\partial\beta}{\partial x}\right)\frac{\partial\xi}{\partial x} + \frac{\partial}{\partial\eta}\left(\frac{\partial\beta}{\partial x}\right)\frac{\partial\eta}{\partial x} = \frac{\partial}{\partial\xi}\left(\frac{\partial\beta}{\partial\xi} + \frac{\partial\beta}{\partial\eta}\right)\cdot 1 + \frac{\partial}{\partial\eta}\left(\frac{\partial\beta}{\partial\xi} + \frac{\partial\beta}{\partial\eta}\right)\cdot 1$

$= \frac{\partial^2\beta}{\partial\xi^2} + 2\frac{\partial^2\beta}{\partial\xi\partial\eta} + \frac{\partial^2\beta}{\partial\eta^2}$

similarly: $\frac{\partial^2\beta}{\partial t^2} = c^2\left(\frac{\partial^2\beta}{\partial\eta^2} + \frac{\partial^2\beta}{\partial\xi^2} - 2\frac{\partial^2\beta}{\partial\xi\partial\eta}\right)$ $\{\frac{\partial\beta}{\partial t} = \frac{\partial\beta}{\partial\xi}\frac{\partial\xi}{\partial t} + \frac{\partial\beta}{\partial\eta}\frac{\partial\eta}{\partial t} =$

$-c\frac{\partial\beta}{\partial\xi} + c\frac{\partial\beta}{\partial\eta} = c\left(\frac{\partial\beta}{\partial\eta} - \frac{\partial\beta}{\partial\xi}\right)$ $\therefore \frac{\partial^2\beta}{\partial t^2} = \frac{\partial}{\partial\xi}\left[c\left(\frac{\partial\beta}{\partial\eta} - \frac{\partial\beta}{\partial\xi}\right)\right]\frac{\partial\xi}{\partial t} + \frac{\partial}{\partial\eta}\left[c\left(\frac{\partial\beta}{\partial\eta} - \frac{\partial\beta}{\partial\xi}\right)\right]\frac{\partial\eta}{\partial t} =$

$-c^2\left(\frac{\partial^2\beta}{\partial\xi\partial\eta} - \frac{\partial^2\beta}{\partial\xi^2}\right) + c^2\left(\frac{\partial^2\beta}{\partial\eta^2} - \frac{\partial^2\beta}{\partial\xi\partial\eta}\right) = c^2\left(\frac{\partial^2\beta}{\partial\xi^2} - 2\frac{\partial^2\beta}{\partial\xi\partial\eta} + \frac{\partial^2\beta}{\partial\eta^2}\right)$ $\therefore$ from $\frac{\partial^2\beta}{\partial t^2} = c^2\frac{\partial^2\beta}{\partial x^2}$:

$c^2\left(\frac{\partial^2\beta}{\partial\eta^2} + \frac{\partial^2\beta}{\partial\xi^2} - 2\frac{\partial^2\beta}{\partial\xi\partial\eta}\right) = c^2\left(\frac{\partial^2\beta}{\partial\xi^2} + 2\frac{\partial^2\beta}{\partial\xi\partial\eta} + \frac{\partial^2\beta}{\partial\eta^2}\right)$ $\therefore -2\frac{\partial^2\beta}{\partial\xi\partial\eta} = 2\frac{\partial^2\beta}{\partial\xi\partial\eta}$ $\therefore$

$\frac{\partial^2\beta}{\partial\xi\partial\eta} = 0$ ⊛

\1b/ integrate ⊛: $\cancel{\frac{\partial\beta}{\partial\xi}}$ wrt η: $\frac{\partial\beta}{\partial\xi} = F(\xi)$ F is arbit $\therefore$ integrate wrt ξ:

$\beta = \int F(\xi)\,d\xi + g(\eta)$ g is arbit $\therefore$ $\beta = f(\xi) + g(\eta) = f(x-ct) + g(x+ct)$

$f(\xi) = \int F(\xi)\,d\xi$

\1c/ $\beta = f(x-ct) + g(x+ct)$ $\therefore \frac{\partial\beta}{\partial x} = f'\frac{\partial}{\partial x}(x-ct) + g'(x+ct) = f' + g'$

f', g' are derivative wrt 2 argument of 2 func, $\frac{\partial\beta}{\partial t} = -cf' + cg'$

$\therefore \frac{\partial^2\beta}{\partial x^2} = f'' + g''$, $\frac{\partial^2\beta}{\partial t^2} = +c^2f'' + c^2g''$ $\therefore$ for $\partial_{tt}\beta + -c^2\partial_{xx} = 0$

$\therefore -c^2(f'' + g'') + +c^2f'' + c^2g'' = -c^2f'' + 0 - c^2g'' + c^2f'' + c^2g'' = 0$ $\therefore$ given $\beta(x,t)$)

is a sol

\2/ $\beta = ae^{i(\underline{k}\cdot\underline{r} - \omega t)}$ $\therefore \frac{\partial\beta}{\partial t} = ae^{i(\underline{k}\cdot\underline{r} - \omega t)}(-i\omega) = -i\omega\beta$,

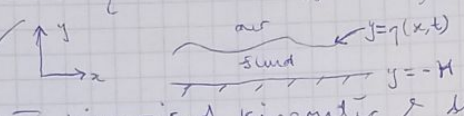
$\frac{\partial\beta}{\partial x} = ae^{i(\underline{k}\cdot\underline{r} - \omega t)}(ik_1)$ $\because \underline{k} = (k_1, k_2, k_3)$ $\therefore \frac{\partial^2\beta}{\partial x^2} = (ik_1)^2\beta = -k_1^2\beta$ &

$\nabla^2\beta = -k_1^2\beta - k_2^2\beta - k_3^2\beta = -k^2\beta$ $k = \sqrt{k_1^2 + k_2^2 + k_3^2} = |\underline{k}|$ $\therefore$ sub into 3d wave eqn:

$\frac{\partial^2 \beta}{\partial t^2} = c^2 \nabla^2 \beta \therefore \quad (-i\omega)^2 \beta = c^2(-k^2)\beta \quad \therefore (-i\omega)^2 = c^2(-k^2) \Rightarrow -\omega^2 = -c^2k^2 \Rightarrow$

$\omega = \pm ck \quad \{\nabla^2 \beta = \partial_{xx}\beta + \partial_{yy}\beta + \partial_{zz}\beta\}$

\3/



$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2}$ from $\underline{u} = \nabla \phi$

with Z linearised kinematic & dynamic conditions

$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}$ on $y=0$ $\quad \frac{\partial \phi}{\partial t} + g\eta = 0$ on $y = 0$. but now $\because$ Z depth of fluid is finite, have $\frac{\partial \phi}{\partial y} = 0$ on $y = -H$ from no normal flow on Z sea bed note eqn is already linear

as in Z no sea bed: seek a sinusoidal travelling wave sol Z free surf $\eta = A\cos(kx - \omega t)$ corres: $\phi = F(y)\sin(kx - \omega t)$

$\left\{ \frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t} \Rightarrow \frac{\partial \eta}{\partial t} = \frac{\partial}{\partial t}(A\cos(kx - \omega t)) = A\omega \sin(kx - \omega t) = \frac{\partial \phi}{\partial y} \therefore \phi = f(y) A\omega \sin(kx - \omega t) \right.$

$\left[\therefore \phi = F(y)\sin(kx - \omega t) \right\} \quad \therefore$

require $F'' - k^2 F = 0 \quad \therefore F = Ce^{ky} + De^{-ky}$ consts C & D $\therefore$ take $k > 0$ without loss of generality $\therefore$

$\frac{\partial \phi}{\partial y} = 0$ on $y = -H \Rightarrow Cke^{-kH} - Dke^{kH} = 0$

$\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}$ on $y = 0 \Rightarrow Cke^0 - Dke^0 = A\omega \quad \therefore$ solve C & D

$Ck - Dk = A\omega \therefore Ck = A\omega + Dk \therefore C = A\frac{\omega}{k} + D \therefore (A\frac{\omega}{k} + D)ke^{-kH} - Dke^{kH} = 0 \therefore$

$(A\omega + Dk)e^{-kH} - Dke^{kH} = 0 = A\omega + Dk - Dke^{2kH} = A\omega + D(k - ke^{2kH}) = 0 \therefore$

$D = \frac{-A\omega}{k - ke^{2kH}} = -\frac{A\omega}{k}\frac{1}{1 - e^{2kH}} = D \therefore C = \frac{A\omega}{k} - \frac{A\omega}{k}\frac{1}{1 - e^{2kH}} = \frac{A\omega}{k}\left(1 + \frac{-1}{1 - e^{2kH}}\right)$

$\frac{A\omega}{k}\frac{-e^{2kH}}{1 - e^{2kH}} = C \quad \therefore \cancel{A\omega = A\omega \frac{-e^{2kH}}{1 - e^{2kH}}}$

$F(y)\sin(kx - \omega t) = \phi = \left(\frac{A\omega}{k}\frac{-e^{2kH}}{1 - e^{2kH}}e^{ky} + -\frac{A\omega}{k}\frac{1}{1 - e^{2kH}}e^{-ky}\right)\sin(kx - \omega t) \therefore$

$\frac{\partial \phi}{\partial t} = \left(\frac{A\omega}{k}\frac{-e^{2kH}}{1 - e^{2kH}}e^{ky} - \frac{A\omega}{k}\frac{1}{1 - e^{2kH}}e^{-ky}\right)(-\omega)\cos(kx - \omega t) \quad \therefore$ on $y = 0$:

$\left(\frac{A\omega}{k}\frac{-e^{2kH}}{1 - e^{2kH}} - \frac{A\omega}{k}\frac{1}{1 - e^{2kH}}\right)(-\omega)\cos(kx - \omega t) = -gA\cos(kx - \omega t) \therefore$

$+\omega^2\left(\frac{-e^{2kH}}{1 - e^{2kH}} - \frac{1}{1 - e^{2kH}}\right) = +gk = \omega^2(-1)\left(\frac{1 + e^{2kH}}{1 - e^{2kH}}\right) \therefore$

$\cancel{-gk(} \quad -gk\left(\frac{1 - e^{2kH}}{1 + e^{2kH}}\right) = \omega^2 = -gk\left(\frac{1 - e^{kH}e^{kH}}{1 + e^{kH}e^{kH}}\right) = -gk\left(\frac{1 - e^p e^p}{1 + e^p e^p}\right) =$

$-gk\left(\frac{(\frac{1}{e^p} - e^p)e^p}{(\frac{1}{e^p} + e^p)e^p}\right) = gk\left(\frac{e^{2kH} - 1}{e^{2kH} + 1}\right) = gk\left(\frac{e^{kH}(e^{kH} - \frac{1}{e^{kH}})}{e^{kH}(e^{kH} + \frac{1}{e^{kH}})}\right) = gk\left(\frac{e^{kH} - e^{-kH}}{e^{kH} + e^{-kH}}\right) =$

$gk\frac{\sinh(kH)}{\cosh(kH)} = gk\tanh(kH) = \omega^2 \quad \therefore \omega = \sqrt{gk\tanh(kH)}$ Z neglected Z neg square root also possible & corres to a wave propagating in Z opposite direction

\Week 10 Sheet/ \1a/ $\underline{\omega} = \nabla \times \underline{u} = \nabla \times \left(\alpha y \,\underline{\hat{i}} + \frac{U}{1+\alpha^2 t^2} \sin[m(x-\alpha y t)](\alpha t\,\underline{\hat{i}} + \underline{\hat{j}})\right) =$

$\nabla \times \left(\left(\alpha y + \frac{U\alpha t}{1+\alpha^2 t^2} \sin[m(x-\alpha y t)]\right)\underline{\hat{i}} + \frac{U}{1+\alpha^2 t^2} \sin[m(x-\alpha y t)]\,\underline{\hat{j}}\right) = \underline{u} = (u, v, 0)$

is $\underline{\omega} = \omega \underline{\hat{k}}$ $\therefore$ ω is Z component of $\nabla \times \underline{u}$ $\therefore$ $\omega = \frac{\partial v}{\partial x} - \frac{\partial}{\partial y} u$ $\quad \underline{u} = (u, v, 0)$

$\therefore \frac{\partial v}{\partial x} = \frac{\partial}{\partial x}\left(\frac{U}{1+\alpha^2 t^2} \sin[m(x-\alpha y t)]\right) = \frac{Um}{1+\alpha^2 t^2} \cos[m(x-\alpha y t)]$

$\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}\left\{\alpha y + \frac{U\alpha t}{1+\alpha^2 t^2} \sin[m(x-\alpha y t)]\right\} = \alpha + \frac{-U\alpha^2 t^2 m}{1+\alpha^2 t^2} \cos[m(x-\alpha y t)] =$

$\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = \frac{Um}{1+\alpha^2 t^2} \cos[m(x-\alpha y t)] - \alpha + \frac{U}{1+\alpha^2 t^2}(\alpha^2 t^2 m) \cos[m(x-\alpha y t)] =$

$-\alpha + \frac{U}{1+\alpha^2 t^2}(m + \alpha^2 t^2 m) \cos[m(x-\alpha y t)] = -\alpha + \frac{U}{1+\alpha^2 t^2} m(1+\alpha^2 t^2) \cos[m(x-\alpha y t)] =$

$-\alpha + Um \cos[m(x-\alpha y t)] = \omega$

\1b/ inviscid $\therefore$ $\nu = 0$, $\frac{D\underline{\omega}}{Dt} = \frac{\partial \underline{\omega}}{\partial t} + \underline{u}\cdot\nabla\underline{\omega} = \cancel{\underline{\omega}\cdot\nabla\underline{u}}$ $\nu \nabla^2 \omega = 0$

if $\underline{u}$ & ω satisfy it: $\frac{\partial \omega}{\partial t} + \underline{u}\cdot\nabla\omega = \omega\cdot\nabla\underline{u}$

$= \frac{\partial \omega}{\partial t} + (u, v, 0)\cdot\nabla\omega = \frac{\partial \omega}{\partial t} + \left[(u, v, 0)\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\right]\omega = \frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = \omega\cdot\nabla\underline{u}$

$= (\omega\cdot\nabla)\underline{u} = \left(\omega\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)\right)\underline{u} = \left(\omega\frac{\partial}{\partial x}, \omega\frac{\partial}{\partial y}, \omega\frac{\partial}{\partial z}\right)\underline{u} = \left(\omega\frac{\partial}{\partial x}, \omega\frac{\partial}{\partial y}, \omega\frac{\partial}{\partial z}\right)(u, v, 0)$

$\therefore \frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} = 0$ $\therefore$

$\frac{\partial \omega}{\partial t} = \frac{\partial}{\partial t}\left\{-\alpha + Um\cos[m(x-\alpha y t)]\right\} = -Um\sin[m(x-\alpha y t)](-m\alpha y) =$

$Um^2 \alpha y \sin[m(x-\alpha y t)]$

$\frac{\partial \omega}{\partial x} = \frac{\partial}{\partial x}\left\{-\alpha + Um\cos[m(x-\alpha y t)]\right\} = -Um\sin[m(x-\alpha y t)]m = -Um^2\sin[m(x-\alpha y t)]$

$\therefore$ by symmetry: $\frac{\partial \omega}{\partial y} = Um^2 \alpha t \sin[m(x-\alpha y t)]$

$\therefore u\frac{\partial \omega}{\partial x} = \left\{\alpha y + \frac{U\alpha t}{1+\alpha^2 t^2}\sin[m(x-\alpha y t)]\right\}\left\{-Um^2\sin[m(x-\alpha y t)]\right\} =$

$-Um^2 \alpha y \sin[m(x-\alpha y t)] - \frac{U^2 m^2 \alpha t}{1+\alpha^2 t^2} \sin^2[m(x-\alpha y t)]$

$v\frac{\partial \omega}{\partial y} = \left(\frac{U}{1+\alpha^2 t^2}\sin[m(x-\alpha y t)]\right)\left(Um^2 \alpha t \sin[m(x-\alpha y t)]\right) = \frac{U^2 m^2 \alpha t}{1+\alpha^2 t^2}\sin^2[m(x-\alpha y t)]$

$\therefore \frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y} =$

$Um^2 \alpha y \sin[m(x-\alpha y t)] + -Um^2 \alpha y \sin[m(x-\alpha y t)] - \frac{U^2 m^2 \alpha t}{1+\alpha^2 t^2}\sin^2[m(x-\alpha y t)] + \frac{U^2 m^2 \alpha t}{1+\alpha^2 t^2}\sin^2[m(x-\alpha y t)]$

$= 0 + 0 = 0 = \frac{\partial \omega}{\partial t} + u\frac{\partial \omega}{\partial x} + v\frac{\partial \omega}{\partial y}$ $\therefore$ $=0$ as is Z case for inviscid flow ($\nu = 0$)

\c/ $\frac{\partial \omega}{\partial t} + \underline{u} \cdot \nabla \omega = \nu \nabla^2 \omega$ $\quad \underline{u} = \left\{ \alpha y + \frac{U \alpha t}{1+\alpha^2 t^2} \sin[m(x-\alpha y t)] F(t) \right\} \hat{\underline{i}} + \frac{U}{1+\alpha^2 t^2} \sin[m(x-\alpha y t)] F(t) \hat{\underline{j}}$

$\therefore \cancel{\frac{\partial \omega}{\partial t}} + \underline{u} \cdot \nabla \omega = (u, v, 0) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \omega = u \frac{\partial \omega}{\partial x} + v \frac{\partial}{\partial y}$ $\therefore$

$\frac{\partial \omega}{\partial t} = U m \cos[m(x-\alpha y t)] F'(t) + U m^2 \alpha y \sin[m(x-\alpha y t)] F(t)$

$\frac{\partial \omega}{\partial t} + \underline{u} \cdot \nabla \omega = \nu \nabla^2 \omega = \frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y}$

$\frac{\partial \omega}{\partial x} = -U m^2 \sin[m(x-\alpha y t)] F(t)$

$\frac{\partial \omega}{\partial y} = U m^2 \sin[m(x-\alpha y t)] F(t)$ $\therefore$

$u \frac{\partial \omega}{\partial x} = \left\{ \alpha y + \frac{U \alpha t}{1+\alpha^2 t^2} \sin[m(x-\alpha y t)] F(t) \right\} \left\{ -U m^2 \sin[m(x-\alpha y t)] F(t) \right\} =$

$-U m^2 \alpha y \sin[m(x-\alpha y t)] F(t) - \frac{U^2 m^2 \alpha t}{1+\alpha^2 t^2} \sin^2[m(x-\alpha y t)] F^2(t)$

$v \frac{\partial \omega}{\partial y} = \left\{ \frac{U}{1+\alpha^2 t^2} \sin[m(x-\alpha y t)] F(t) \right\} \left\{ U m^2 \sin[m(x-\alpha y t)] F(t) \right\} = \frac{U^2 m^2}{1+\alpha^2 t^2} \sin^2[m(x-\alpha y t)] F^2(t)$

$\nabla^2 \omega = \nabla \cdot \nabla \omega = \frac{\partial^2}{\partial x^2} \omega + \frac{\partial^2}{\partial y^2} \omega + \frac{\partial^2}{\partial z^2} \omega = \frac{\partial^2}{\partial x^2} \omega + \frac{\partial^2}{\partial y^2} \omega$ $\therefore$

$\frac{\partial^2}{\partial x^2} \omega = \frac{\partial}{\partial x}\left(-U m^2 \sin[m(x-\alpha y t)] F(t)\right) = -U m^3 \cos[m(x-\alpha y t)] F(t)$

$\frac{\partial^2}{\partial y^2} \omega = \frac{\partial}{\partial y}\left(U m^2 \sin[m(x-\alpha y t)] F(t)\right) = -U m^3 \alpha t \cos[m(x-\alpha y t)] F(t)$ $\therefore$

$\frac{\partial \omega}{\partial t} + \underline{u} \cdot \nabla \omega = \frac{\partial \omega}{\partial t} + u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = U m \cos[m(x-\alpha y t)] F'(t) + U m^2 \alpha y \sin[m(x-\alpha y t)] F(t) -$
$U m^2 \alpha y \sin[m(x-\alpha y t)] F(t) - \frac{U^2 m^2 \alpha t}{1+\alpha^2 t^2} \sin^2[m(x-\alpha y t)] F^2(t) + \frac{U^2 m^2}{1+\alpha^2 t^2} \sin^2[m(x-\alpha y t)] F^2(t)$

$= \nu \nabla^2 \omega = \nu \left(\frac{\partial^2 \omega}{\partial x^2} + \frac{\partial^2 \omega}{\partial y^2} \right) = -\nu U m^3 \cos[m(x-\alpha y t)] F(t) - \nu U m^3 \alpha t \cos[m(x-\alpha y t)] F(t)$

$= U m \cos[m(x-\alpha y t)] F'(t) + \frac{U^2 m^2}{1+\alpha^2 t^2} (1-\alpha t) \sin^2[m(x-\alpha y t)] F^2(t) =$

$-\nu U m^3 (1+\alpha t) \cos[m(x-\alpha y t)] F(t)$ $\therefore$

$\frac{dF}{dt} = -\nu m^2 (1+\alpha^2 t^2) F = F'(t) = \frac{dF(t)}{dt} = \frac{\partial F(t)}{\partial t} = \frac{\partial F}{\partial t}$

\!d/ $\therefore \frac{d}{dx}[\ln(f(x))] = \frac{f'(x)}{f(x)}$ $\therefore \frac{F'(t)}{F(t)} = -\nu m^2 (1+\alpha^2 t^2)$ $\therefore$

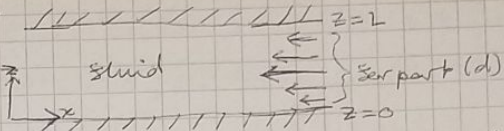
$\int \frac{F'(t)}{F(t)} dt = \int -\nu m^2 (1+\alpha^2 t^2) dt = -\nu m^2 (t + \frac{1}{3}\alpha^2 t^3) + C_1 = \ln|F(t)|$ $\therefore$

$|F(t)| = e^{-\nu m^2 (t + \frac{1}{3}\alpha^2 t^3) + C_1} = F(t) = C_2 e^{-\nu m^2 (t + \frac{1}{3}\alpha^2 t^3)}$ $\quad$, $F(0) = 1$ $\therefore$

$F(0) = C_2 e^{-\nu m^2 (0 + \frac{1}{3}\alpha^2 0^3)} = F(t=0) = C_2 e^{-\nu m^2 (0)} = 1 = C_2 e^0 = C_2 = 1$ $\therefore$

$F(t) = 1 \cdot e^{-\nu m^2 (t + \frac{1}{3}\alpha^2 t^3)} = e^{-\nu m^2 (t + \frac{1}{3}\alpha^2 t^3)} = F(t)$ $\quad$ {t^3 term represents viscous dissusion enhanced by a shear α}

\CW 1 Sols/ \1a/



\1b/ $\nabla p = \frac{dp}{dx}\underline{i}$ with $\frac{dp}{dx} > 0 \therefore -\nabla p$ is in the ~~regan~~ $-x$-direction & provides $\therefore$ the force driving the ~~flow~~ (gravity neglected & the boundaries are stationary) so $\underline{u} = u\,\underline{i}$ expected with $u<0$
u expected to be indep of time (steady given) & x (no special x locations) but dependent on z due to boundaries at $z = 0, L$ $\therefore$
$u = u(z)\underline{i}$, $u < 0$ where $u = 0$ (no slip)

\1c/ Navier-Stokes: $\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\underline{u}$ with gravity neglected $\therefore$ ~~$\partial_t \underline{u}$~~ $\partial_t\underline{u} = 0$, $\underline{u}\cdot\nabla\underline{u} = 0 \therefore 0 + 0 = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\underline{u} = 0$

only the x-component is non-trivial: $-\frac{1}{\rho}\frac{dp}{dx} + \frac{\mu}{\rho}\frac{d^2u}{dz^2} = 0 \therefore$
$-\frac{dp}{dx} + \mu\frac{d^2u}{dz^2} = 0 \therefore \mu\frac{d^2u}{dz^2} = \frac{dp}{dx} \therefore \frac{d^2u}{dz^2} = \frac{1}{\mu}\frac{dp}{dx} \therefore$
$\frac{du}{dz} = \frac{1}{\mu}\frac{dp}{dx}z + c$ c = constant $\therefore u(z) = \frac{1}{\mu}\frac{dp}{dx}\frac{z^2}{2} + cz + d$, d = const
BCs: no slip $u(0) = 0 \therefore d = 0$, $u(L) = 0 \therefore c = -\frac{1}{\mu}\frac{dp}{dx}\frac{L}{2} \therefore$
$u(z) = \frac{1}{\mu}\frac{dp}{dx}\left(\frac{z^2}{2} - \frac{Lz}{2}\right)$

\1c my attempt/ gravity is to be neglected $\therefore \rho\left(\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \mu\nabla^2\underline{u} \therefore$
steady fluid flow $\therefore \underline{u}$ is indep of time $\therefore \frac{\partial\underline{u}}{\partial t} = 0 \therefore \underline{u}\cdot\nabla\underline{u} = (\underline{u}\cdot\nabla)\underline{u} \therefore$
let $\underline{u} = u(z)\hat{\underline{i}} \therefore \underline{u}\cdot\nabla = (u(z)\hat{\underline{i}})\cdot\nabla = u(z)\frac{\partial}{\partial x} \therefore \underline{u}\cdot\nabla\underline{u} = \left(u(z)\frac{\partial}{\partial x}\right)\underline{u} =$
$\left(u(z)\frac{\partial}{\partial x}\right)(u(z)\hat{\underline{i}}) = u(z)\frac{\partial}{\partial x}(u(z)\hat{\underline{i}}) = u(z)\hat{\underline{i}}\frac{\partial}{\partial x}(u(z)) = u(z)\hat{\underline{i}}(0) = 0 = \underline{u}\cdot\nabla\underline{u} \therefore$
$\rho(0) = -\nabla p + \mu\nabla^2\underline{u} = 0 \therefore \mu\nabla^2\underline{u} = \nabla p = \frac{dp}{dx}\hat{\underline{i}} = \mu\nabla^2(u(z)\hat{\underline{i}}) \therefore$
$\nabla^2\underline{u} = \nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{u}) \therefore \nabla\cdot\underline{u} = \nabla\cdot(u(z)\hat{\underline{i}}) = \frac{\partial}{\partial x}u(z) = 0 \therefore \nabla(\nabla\cdot\underline{u}) = \nabla(0) = 0 \therefore$
$\nabla\times\underline{u} = \nabla\times(u(z)\hat{\underline{i}}) = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial_x & \partial_y & \partial_z\\ u(z) & 0 & 0\end{vmatrix} = i(\partial_y(0) - \partial_z(0)) - j(\partial_x(0) - \partial_z(u(z))) + \hat{k}(\partial_x(0) - \partial_y(u(z))) =$
$\hat{\underline{i}}(0) - \hat{\underline{j}}\left(0 - \frac{\partial}{\partial z}(u(z))\right) + \hat{\underline{k}}(0 - 0) = \partial_z(u(z))\hat{\underline{j}} = \nabla\times\underline{u} \therefore \nabla\times(\nabla\times u) = \nabla\times\left[\partial_z(u(z))\hat{\underline{j}}\right] = \begin{vmatrix}\hat{i} & \hat{j} & \hat{k}\\ \partial_x & \partial_y & \partial_z\\ 0 & \partial_z u(z) & 0\end{vmatrix} =$
$\hat{\underline{i}}\left[\partial_y(0) - \partial_z(\partial_z(u(z)))\right] - \hat{\underline{j}}\left[\partial_x(0) - \partial_z(0)\right] + \hat{\underline{k}}\left[\partial_x(\partial_z(u(z))) - \partial_y(0)\right] =$
$\hat{\underline{i}}\left[0 - \partial_{zz}(u(z))\right] - \hat{\underline{j}}[0 - 0] + \hat{\underline{k}}\left[\partial_z(\partial_x(u(z))) - 0\right] = -\partial_{zz}(u(z))\hat{\underline{i}} + \underline{k}\left[\partial_z(0)\right] = -\hat{\underline{i}}\frac{\partial^2}{\partial z^2}u(z) = \nabla\times(\nabla\times\underline{u})$
$\therefore \nabla^2\underline{u} = 0 - -\hat{\underline{i}}\frac{\partial^2}{\partial z^2}u(z) = \hat{\underline{i}}\frac{\partial^2}{\partial z^2}u(z) = \nabla^2\left[u(z)\hat{\underline{i}}\right] \therefore \frac{dp}{dx}\hat{\underline{i}} = \mu\hat{\underline{i}}\frac{\partial^2}{\partial z^2}u(z) \therefore \frac{dp}{dx} = \mu\frac{\partial^2}{\partial z^2}u(z)$
$\therefore \frac{dp}{dx}$ is indep of $x \therefore \frac{dp}{dx}$ is either a func of time or const $\therefore \frac{dp}{dx} = F(t) \therefore$
$F(t) = \mu\frac{\partial^2}{\partial z^2}u(z) \therefore \int F(t)\,dz = \int\mu\frac{\partial^2}{\partial z^2}u(z)\,dz = \mu\int\frac{\partial^2}{\partial z^2}u(z)\,dz = \mu\frac{\partial}{\partial z}u(z) = F(t)z + c_1 = \mu\frac{\partial}{\partial z}u(z) \therefore$
$\int zF(t) + c_1\,dz = \int\mu\frac{\partial}{\partial z}(u) = \mu u(z) = \frac{1}{2}z^2F(t) + c_1z + c_2 \therefore$ stationary boundaries $\therefore$

by Z no slip BCs: $u(z=0)=0$ & $u(z=L)=0$ $\therefore$ at $z=0$: $\mu u(z=0)=\frac{1}{2}(0)^2F(t)+C_1(0)+C_2=$

$\mu(0)=0=0+0+C_2=C_2=0$ $\therefore$ $\mu u(z)=\frac{1}{2}z^2F(t)+C_1z$ $\therefore$ at $z=L$: $\mu u(z=L)=\mu(0)=0=$

$\frac{1}{2}L^2F(t)+C_1L$ $\therefore$ $-\frac{1}{2}L^2F(t)=C_1L$ $\therefore$ $C_1=-\frac{1}{2}LF(t)$ $\therefore$ $\mu u(z)=\frac{1}{2}z^2F(t)-\frac{1}{2}LF(t)z$ $\therefore$

$u(z)=\frac{1}{2\mu}z^2F(t)-\frac{1}{2\mu}LF(t)z$

\1c sol/ take $\underline{u}=u(z)\underline{i}$ $\therefore$ $\underline{u}\cdot\nabla=u\partial_x$ $\therefore$ $\underline{u}\cdot\nabla\underline{u}=u\partial_x(u(z)\underline{i})=\underline{0}$ $\therefore$

$\partial_t\underline{u}=\partial_t(u(z)\underline{i})=0$ & $\nabla^2\underline{u}=\frac{d^2u}{dz^2}\underline{i}$

$\{\nabla\cdot(u(z)\underline{i})=\nabla\cdot(u(z)\underline{i}+0\underline{j}+0\underline{k})=(\partial_x\ \partial_y\ \partial_z)(u(z)\ 0\ 0)=\partial_x u(z)$

$\nabla^2\underline{u}=\nabla^2(u(z)\underline{i})=\nabla^2(u(z)\underline{i}+0\underline{j}+0\underline{k})=$

$(\partial_{xx}u(z)+\partial_{yy}u(z)+\partial_{zz}(u))\underline{i}+(\partial_{xx}0+\partial_{yy}0+\partial_{zz}0)\underline{j}+(\partial_{xx}0+\partial_{yy}0+\partial_{zz}0)\underline{k}=$

$(0+0+\partial_{zz}(u(z))\underline{i}+0\underline{j}+0\underline{k}=\partial_{zz}(u(z))\underline{i}\}$

Navier-Stokes: $\partial_t\underline{u}+\underline{u}\cdot\nabla\underline{u}=-\frac{1}{\rho}\nabla p+\frac{\mu}{\rho}\nabla^2\underline{u}$ with gravity neglected $\therefore$

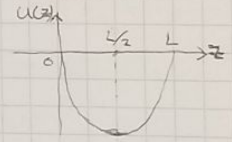
$0=-\frac{1}{\rho}\nabla p+\frac{\mu}{\rho}\nabla^2\underline{u}=-\frac{1}{\rho}\nabla p+\frac{\mu}{\rho}\frac{d^2u}{dz^2}\underline{i}=0$ only x-component is nontrivial $\therefore$

$-\frac{1}{\rho}\frac{dp}{dx}\underline{i}+\frac{\mu}{\rho}\frac{d^2u(z)}{dz^2}\underline{i}=0$ $\therefore$ $-\frac{1}{\rho}\frac{dp}{dx}+\frac{\mu}{\rho}\frac{d^2u}{dz^2}=0$ $\therefore$ $-\frac{dp}{dx}+\mu\frac{d^2u}{dz^2}=0$ $\therefore$

$\frac{du}{dz}=\frac{1}{\mu}\frac{dp}{dx}z+c$ $\therefore$ $\frac{d^2u}{dz^2}=\frac{1}{\mu}\frac{dp}{dx}$ $\therefore$ $\frac{du}{dz}=\frac{1}{\mu}\frac{dp}{dx}z+c$ $c=$ const $\therefore$

$u(z)=\frac{1}{\mu}\frac{dp}{dx}\frac{z^2}{2}+cz+d$, $d=$ const $\therefore$ BCs: no slip $u(0)=0$ $\therefore$ $d=0$ $\therefore$

$u(L)=0$ $\therefore$ $c=-\frac{1}{\mu}\frac{dp}{dx}\frac{L}{2}$ $\therefore$ $u(z)=\frac{1}{\mu}\frac{dp}{dx}\left(\frac{z^2}{2}-\frac{Lz}{2}\right)$



\1d/ $\frac{1}{\mu}\frac{dp}{dx}>0$, $z-L\leq 0$ for $z\in[0,L]$ $\therefore$ $u(z)\leq 0$

$\frac{du}{dz}=\frac{1}{\mu}\frac{dp}{dx}(z-\frac{L}{2})=0$ at $z=\frac{L}{2}$ $\quad\frac{d^2u}{dz^2}\Big|_{L/2}=\frac{1}{\mu}\frac{dp}{dx}>0$ $\therefore$ minimum

$|\underline{u}|$ largest at $z=L/2$

\2a/

$\underline{u}=r^2\cos\theta\,\hat{\underline{r}}+\frac{1}{r}\hat{\underline{\theta}}+\frac{1}{\sin\theta}\hat{\underline{\phi}}$ spherical polar coords (r,θ,ϕ)

$\underline{\omega}=\nabla\times\underline{u}=\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\,\hat{\underline{\phi}}\\ \partial_r & \partial_\theta & \partial_\phi\\ r^2\cos\theta & r\frac{1}{r} & r\sin\theta\frac{1}{\sin\theta}\end{vmatrix}=\frac{1}{r^2\sin\theta}\{\hat{\underline{r}}\cdot 0-r\hat{\underline{\theta}}\cdot 1+r\sin\theta\,\hat{\underline{\phi}}(0+r^2\sin\theta)\}=$

$-\frac{1}{r\sin\theta}\hat{\underline{\theta}}+r\sin\theta\,\hat{\underline{\phi}}$

\2b/ $\underline{u}=R\cos\phi\,\hat{\underline{R}}+\sin\phi\,\hat{\underline{\phi}}$ in cylindrical polars (R,ϕ,z) $\therefore$

$\underline{u}\cdot\nabla=R\cos\phi\,\partial_R+\frac{\sin\phi}{R}\partial_\phi$ $\because$ $\nabla=\hat{\underline{R}}\partial_R+\frac{1}{R}\hat{\underline{\phi}}\partial_\phi+\hat{\underline{z}}\partial_z$ $\because$ $\nabla\Omega=\partial_R\Omega\hat{\underline{R}}+\frac{1}{R}\partial_\phi\Omega\hat{\underline{\phi}}+\partial_z\Omega\hat{\underline{z}}$

$\underline{u}\cdot\nabla\underline{u}=(\underline{u}\cdot\nabla)\underline{u}=(R\cos\phi\,\partial_R+\frac{\sin\theta}{R}\partial_\phi)(R\cos\phi\,\hat{\underline{R}}+\sin\phi\,\hat{\underline{\phi}})=$

$(R\cos\phi\,\partial_R+\frac{\sin\phi}{R}\partial_\phi)(R\cos\phi\,\hat{\underline{R}})+(R\cos\phi\,\partial_R+\frac{\sin\phi}{R}\partial_\phi)(\sin\phi\,\hat{\underline{\phi}})=$

$R\cos\phi\,\partial_R(R\cos\phi\,\hat{\underline{R}})+\frac{\sin\phi}{R}\partial_\phi(R\cos\phi\,\hat{\underline{R}})+R\cos\phi\,\partial_R(\sin\phi\,\hat{\underline{\phi}})+\frac{\sin\phi}{R}\partial_\phi(\sin\phi\,\hat{\underline{\phi}})=$

$R\cos\phi\,\hat{\underline{R}}\cos\phi\,\partial_R R+\frac{\sin\phi}{R}R\hat{\underline{R}}\partial_\phi\cos\phi+\frac{\sin\phi}{R}R\cos\phi\,\partial_\phi\hat{\underline{R}}+0+\frac{\sin\phi}{R}\hat{\underline{\phi}}\partial_\phi\sin\phi+\frac{\sin\phi}{R}\sin\phi\,\partial_\phi\hat{\underline{\phi}}=$

\CW1 SOLS/ $R\cos\phi\,\hat{R} - \sin\phi\sin\phi\,\hat{R} + \sin\phi\cos\phi\,\hat{\phi} + \frac{\sin\phi\cos\phi}{R}\hat{\phi} + -\hat{R}\frac{\sin^2\phi}{R} =$

$\hat{R}\left(R\cos^2\phi - \frac{\sin^2\phi}{R} - \sin^2\phi\right) + \hat{\phi}\left(\sin\phi\cos\phi + \frac{\sin\phi\cos\phi}{R}\right) =$

$\hat{R}R\cos^2\phi + \frac{\sin\phi}{R}R\left(-\sin\phi\hat{R} + \cos\phi\frac{\partial\hat{R}}{\partial\phi}\right) + \frac{\sin\phi}{R}\left(\cos\phi\,\hat{\phi} + \sin\phi\frac{\partial\hat{\phi}}{\partial\phi}\right) =$

$\hat{R}R\cos^2\phi - \sin^2\phi\hat{R} + \sin\phi\cos\phi\hat{\phi} + \frac{\sin\phi\cos\phi}{R}\hat{\phi} - \frac{\sin^2\phi}{R}\hat{R} =$

$\left(R\cos^2\phi - \left(1+\frac{1}{R}\right)\sin^2\phi\right)\hat{R} + \left(1+\frac{1}{R}\right)\sin\phi\cos\phi\,\hat{\phi}$

$\nabla = \hat{R}\partial_R + \frac{1}{R}\hat{\phi}\partial_\phi + \hat{z}\partial_z \quad \because \nabla\Omega = \frac{\partial\Omega}{\partial R}\hat{R} + \frac{1}{R}\frac{\partial\Omega}{\partial\phi}\hat{\phi} + \frac{\partial\Omega}{\partial z}\hat{z} \quad \therefore$

$\nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z \quad \therefore \nabla\Omega = \frac{\partial\Omega}{\partial r}\hat{r} + \frac{1}{r}\frac{\partial\Omega}{\partial\theta}\hat{\theta} + \frac{1}{r\sin\theta}\frac{\partial\Omega}{\partial\phi}\hat{\phi} =$

$\hat{r}\partial_r\Omega + \hat{\theta}\frac{1}{r}\partial_\theta\Omega + \hat{\phi}\frac{1}{r\sin\theta}\partial_\phi\Omega \therefore \nabla = \hat{r}\partial_r + \hat{\theta}\frac{1}{r}\partial_\theta + \hat{\phi}\frac{1}{r\sin\theta}\partial_\phi$

$\nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z \quad \nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z \quad \nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z$

$\nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z \quad \nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z \quad \nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z \quad \nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z$

$\nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z \quad \nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z \quad \nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z \quad \nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z$

$\nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z \quad \nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z \quad \nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z \quad \nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z$

$\nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z \quad \nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z \quad \nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z \quad \nabla = \hat{R}\partial_R + \hat{\phi}\frac{1}{R}\partial_\phi + \hat{z}\partial_z$

$\nabla = \hat{r}\partial_r + \hat{\theta}\frac{1}{r}\partial_\theta + \hat{\phi}\frac{1}{r\sin\theta}\partial_\phi \quad \nabla = \hat{r}\partial_r + \hat{\theta}\frac{1}{r}\partial_\theta + \hat{\phi}\frac{1}{r\sin\theta}\partial_\phi \quad \nabla = \hat{r}\partial_r + \hat{\theta}\frac{1}{r}\partial_\theta + \hat{\phi}\frac{1}{r\sin\theta}\partial_\phi$

$\nabla = \hat{r}\partial_r + \hat{\theta}\frac{1}{r}\partial_\theta + \hat{\phi}\frac{1}{r\sin\theta}\partial_\phi \quad \nabla = \hat{r}\partial_r + \hat{\theta}\frac{1}{r}\partial_\theta + \hat{\phi}\frac{1}{r\sin\theta}\partial_\phi \quad \nabla = \hat{r}\partial_r + \hat{\theta}\frac{1}{r}\partial_\theta + \hat{\phi}\frac{1}{r\sin\theta}\partial_\phi$

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\3a/ $\underline{u} = \nabla\times\left(\frac{\psi}{R}\hat{\phi}\right)$ cylindrical polars (R,ϕ,z) $\therefore \underline{u} = \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\phi} & \hat{z} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & R\frac{\psi}{R} & 0 \end{vmatrix} =$

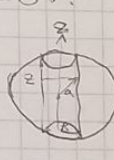
$\frac{1}{R}\left(-\hat{R}\frac{\partial\psi}{\partial z} - R\hat{\phi}\cdot 0 + \hat{z}\frac{\partial\psi}{\partial R}\right) \therefore \underline{u} = u\hat{R} + w\hat{z}$ where $\psi = -\frac{R^2}{10}(a^2 - R^2 - z^2) \therefore$

$u = -\frac{1}{R}\frac{\partial\psi}{\partial z} = -\frac{1}{R}\left(-\frac{R^2}{10}(-2z)\right) = -\frac{Rz}{5}, \quad w = \frac{1}{R}\frac{\partial\psi}{\partial R} =$

$\frac{1}{R}\left\{\left(-\frac{R^2}{10}\right)(-2R) + (a^2 - R^2 - z^2)\left(\frac{-2R}{10}\right)\right\} = \frac{1}{R}\left\{+\frac{R^3}{5} - \frac{R}{5}(a^2 - R^2 - z^2)\right\} = \frac{1}{5}(2R^2 - a^2 + z^2)$

Stagnation pts have $\underline{u} = 0 \therefore u = w = 0 \therefore u = 0 \therefore R = 0$ or $z = 0 \therefore$

for $R = 0$, $w = 0 \therefore z^2 = a^2 \therefore$ Stagnation pts $R = 0, z = \pm a$



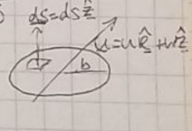
for $z = 0$, $w = 0 \therefore 2R^2 = a^2 \therefore$ Stagnation pts $R = \pm a/\sqrt{2}, z = 0 \therefore$

$(0, \phi, a), (0, \phi, -a), (a/\sqrt{2}, \phi, 0) \because R > 0$

\3b/ $|\underline{u}|^2 = \frac{R^2z^2}{25} + \frac{1}{25}(2R^2 - a^2 + z^2)^2$ on a spherical surface radius a $\therefore$

$R^2 + z^2 = a^2 \therefore |\underline{u}|^2 = \frac{1}{25}(R^2z^2 + (R^2)^2) = \frac{R^2}{25}(z^2 + R^2) = \frac{R^2a^2}{25}$ & $|\underline{u}| = \frac{Ra}{5}$

$\therefore$ max value is at $R = a$ ie at z equator $\therefore$



\3c/ mass flux $\rho\int_S \underline{u}\cdot d\underline{S} = \rho\int \underline{u}\cdot dS\,\hat{z} = \rho\int \underline{u}\cdot\hat{z}\,dS =$

$\rho\int \underline{u}\cdot(0\hat{R} + 0\hat{\phi} + 1\hat{z})dS = \rho\int w\,dS = \rho\int_{\phi=0}^{2\pi}\int_{R=0}^{b} wR\,dR\,d\phi = \rho\int_0^{2\pi}\int_0^b \frac{1}{R}\frac{\partial\psi}{\partial R}R\,dR\,d\phi =$

$\rho\int_0^{2\pi}\int_0^b \frac{\partial\Psi}{\partial R}\, dR\, d\phi = \rho 2\pi(\Psi(R=b)-\Psi(R=0)) = \rho 2\pi \Psi(R=b)$ $(\because \Psi(0)=0 \therefore)$

$= \rho 2\pi\left(\frac{-b^2}{10}\right)(a^2-b^2\cancel{4}-z^2) = \frac{\pi\rho b^2}{5}(b^2+z^2-a^2)$

Check: by Stokes thm: $\rho\int_S \underline{u}\cdot d\underline{S} = \rho\int_S \nabla\times\left(\frac{\Psi}{R}\hat{\underline{\phi}}\right)\cdot d\underline{S} = \rho\int_C \frac{\Psi}{R}\hat{\underline{\phi}}\cdot d\underline{l}$

where z direction around z closed curve C is related to $d\underline{S}$ in a righthanded sense $= \rho\int_{\phi=0}^{2\pi}\frac{\Psi}{R}\hat{\underline{\phi}}\cdot(R\,d\phi\,\hat{\underline{\phi}})$ at $R=b$ $d\underline{l} = R\,d\phi\,\hat{\underline{\phi}}$

$= \rho\int_{\phi=0}^{2\pi}\Psi\,\hat{\underline{\phi}}\cdot(d\phi\,\hat{\underline{\phi}}) = \rho\int_{\phi=0}^{2\pi}\Psi\, d\phi$ at $R=b$

$= \rho 2\pi\Psi(R=b)$ as above.

$d\underline{S} = \hat{\underline{n}}\,dS$ $\int_S \nabla\times\left(\frac{\Psi}{R}\hat{\underline{\phi}}\right)\cdot d\underline{S} = \int_C \frac{\Psi}{R}\hat{\underline{\phi}}\cdot d\underline{l}$ $d\underline{l} = R\,d\phi\,\hat{\underline{\phi}}$

$dS = R\,dR\,d\phi$ $d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{n}\,dS$ $\oiint_S \underline{F}\cdot d\underline{S}$ $d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$

$d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$

Mass flux $= \oiint_S \underline{F}\cdot d\underline{S}$ $\oiint_S \underline{F}\cdot d\underline{S}$ $d\underline{S}=\hat{\underline{n}}\,dS$ $\iint_S \underline{F}\cdot d\underline{S}$ $\hat{\underline{n}}\,dS = d\underline{S}$ $\iint_S \underline{F}\cdot d\underline{S}$ $d\underline{S}=\hat{\underline{n}}\,dS$

$dS = R\,dR\,d\phi$ $dS = R\,dR\,d\phi$ $dS = R\,dR\,d\phi$ $dS = R\,dR\,d\phi$ $dS = R\,dR\,d\phi$ $dS = R\,dR\,d\phi$

$\int_S \nabla\times\left(\frac{\Psi}{R}\hat{\underline{\phi}}\right)\cdot d\underline{S} = \int_C \frac{\Psi}{R}\hat{\underline{\phi}}\cdot d\underline{l} = \int_0^{2\pi}\frac{\Psi}{R}\hat{\underline{\phi}}\cdot(R\,d\phi\,\hat{\underline{\phi}})$ $\int_S \nabla\times\left(\frac{\Psi}{R}\hat{\underline{\phi}}\right)\cdot d\underline{S} = \int_C \frac{\Psi}{R}\hat{\underline{\phi}}\cdot d\underline{l} = \int_0^{2\pi}\frac{\Psi}{R}\hat{\underline{\phi}}\cdot(R\,d\phi\,\hat{\underline{\phi}})$

$d\underline{l} = R\,d\phi\,\hat{\underline{\phi}}$ $d\underline{l} = R\,d\phi\,\hat{\underline{\phi}}$ $d\underline{l} = R\,d\phi\,\hat{\underline{\phi}}$ $d\underline{l} = R\,d\phi\,\hat{\underline{\phi}}$ $d\underline{l} = R\,d\phi\,\hat{\underline{\phi}}$ $d\underline{l} = R\,d\phi\,\hat{\underline{\phi}}$

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$d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$ $d\underline{S}=\hat{\underline{n}}\,dS$

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\4a/ $\underline{u} = E(x,-y)$ E is const $\nabla\cdot\underline{u} = \partial_x(Ex)+\partial_y(-Ey)+\partial_z(0) =$

$E-E=0$ $\therefore$ incompressible

\4b/ inviscid Navier-Stokes without gravity $\rho(\partial_t\underline{u}+\underline{u}\cdot\nabla\underline{u}) = -\nabla p + \mu\nabla^2\underline{u}$

$\therefore \nabla^2\underline{u}=0$ $\because$ inviscid $\therefore \mu=0$, $\partial_t\underline{u}=0$ $\because$ $\underline{u}$ is indep of time $\therefore$

$\rho(\underline{u}\cdot\nabla\underline{u}) = -\nabla p$ $\therefore$ $\underline{u}\cdot\nabla = E(x\partial_x - y\partial_y)$ $\therefore \underline{u}\cdot\nabla\underline{u} = \cancel{E(x\partial_x - y\partial_y)}$

$E(x\partial_x-y\partial_y)(Ex\,\underline{i})+E(x\partial_x-y\partial_y)(-Ey\,\underline{j}) = ExE\,\underline{i} + E(-y)(-E)\,\underline{j} = E^2(x\,\underline{i}+y\,\underline{j})$

$\therefore \rho(E^2(x\,\underline{i}+y\,\underline{j})) = -\nabla p$ $\therefore$ x comp of N-S: $\rho E^2 x = -\frac{\partial p}{\partial x}$ $\therefore$

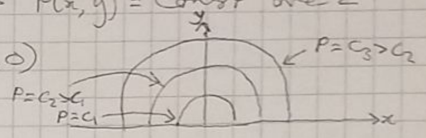
$\frac{\partial p}{\partial x} = -\rho E^2 x$ $\therefore$ $p = -\rho E^2\frac{x^2}{2}+\delta(y)$ $\therefore$ $\partial_y p = \frac{d\delta}{dy}$

y comp of N-S: $\rho E^2 y = -\frac{\partial p}{\partial y} = -\frac{d\delta}{dy}$ $\therefore \frac{d\delta}{dy} = -\rho E^2 y$ $\therefore$ $\delta = -\rho E^2\frac{y^2}{2}+c$

c is const $\therefore$ $p = -\rho\frac{E^2}{2}\frac{x^2}{2}-\rho E^2\frac{y^2}{2}+c = \frac{-\rho E^2}{2}(x^2+y^2)+c = p = -\frac{\rho E^2}{2}(x^2+y^2)+p_0$

where $p_0 = c = p(x=0, y=0)$

CW1 Sols / 4c / pressure contours $p(x,y) = \text{const}$ are the curves $x^2+y^2=\text{const}$ ie circles centre $(0,0)$



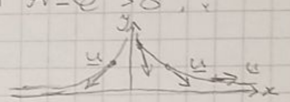
4d / streamlines: $\frac{dx}{u} = \frac{dy}{v}$ $\therefore$ $\frac{dx}{Ex} = \frac{dy}{-Ey}$ $\therefore$ $\frac{dx}{x} = \frac{dy}{-y}$ $\therefore$

$\int \frac{1}{x}\,dx = \int -y\,dy$ $\therefore$ $\left\{ \int \frac{1}{x}\,dx\,\frac{dx}{dx} = \int \frac{1}{y}\,dy\,dx \right.$ ✗ $\frac{dy}{dx} = \frac{-y}{x}$ $\therefore -\frac{1}{y}\frac{dy}{dx} = \frac{1}{x}$ $\therefore$

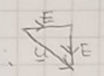
$\left. \int -\frac{1}{y}\frac{dy}{dx}\,dx = \int \frac{1}{x}\,dx = \int -\frac{1}{y}\,dy \right\}$ $\therefore \ln|x| = -\ln|y| + c$ $\therefore$

$e^{\ln|x|} = e^{-\ln|y|+c} = e^c e^{\ln|y|^{-1}} = e^c|y|^{-1} = e^c/|y| = A/|y| = |x|$ $A = e^c > 0$ $\therefore$

$|y| = A/|x|$ ie hyperbolic



4e / $\underline{u} = E(x,-y)$ at $(x,y)=(1,1)$ $\underline{u} = E(1,-1)$ $\therefore$

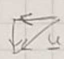


for $E>0$

as $x \to 0^+$ $\underline{u} \to E(0,-y)$ $\therefore \downarrow \underline{u}$ for $E, y>0$

as $x \to \infty$ $\underline{u} \to (\infty, 0)$ $\therefore \downarrow \underline{u}$ for $E, y>0$

$y \to 0^+$ at $(x,y) = (-1,1)$ $\underline{u} = E(-1,1)$ $\therefore$



for $E>0$

4f / $y=0$ is a solid boundary $\therefore$ no flow through $y=0$ $\therefore v=0$ at $y=0$

$\therefore -Ey=0$ at $y=0$

$y=0$ is a stationary boundary $\therefore$ no slip $\therefore u=0$ at $y=0$ $\therefore Ex=0$ at $y=0$ $\therefore$ ✗ when $E=0$ but then $\underline{u}=0$ everywhere. The given velocity $\underline{u} = E(x,-y)$ satisfies no normal flow but doesn't satisfy the no slip condition (inviscid)

$\nabla \cdot \underline{u} = 0$ $\therefore$ incompressible $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \mu\nabla^2\underline{u}$

$\partial_t \underline{u} = 0$ $\therefore$ steady $\mu = 0$ $\therefore$ inviscid $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \mu\nabla^2\underline{u}$

$\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \mu\nabla^2\underline{u}$ $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = \rho g - \nabla p + \mu\nabla^2\underline{u}$ $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = \rho g - \nabla p + \mu\nabla^2\underline{u}$

$\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \rho g + \mu\nabla^2\underline{u}$ $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = \rho g - \nabla p + \mu\nabla^2\underline{u}$ $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = \rho g - \nabla p + \mu\nabla^2\underline{u}$

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$\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = \rho g - \nabla p + \mu\nabla^2\underline{u}$ $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = \rho g - \nabla p + \mu\nabla^2\underline{u}$ $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = \rho g - \nabla p + \mu\nabla^2\underline{u}$

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\CW2 sols/ \1a/ $e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) = D^s_{ij}$ $\quad \xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = D^a_{ij}$

$e = \frac{1}{2}\begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix}$ with $e_{ij} = e_{ji}$ (ie $e_{12}=e_{21}$, $e_{13}=e_{31}$ $e_{23}=e_{32}$) $\therefore$ 6 indep components

\1b/ $\xi = \frac{1}{2}\begin{bmatrix} 0 & \xi_{12} & \xi_{13} \\ -\xi_{12} & 0 & \xi_{23} \\ -\xi_{13} & -\xi_{23} & 0 \end{bmatrix}$ $\because \xi_{ji} = -\xi_{ij}$ $\therefore$ 3 indep components

\1c/ irrotational flows have $\underline{\omega} = \nabla \times \underline{u} = 0$ writing $\underline{u} = \nabla \phi$ $\therefore$

$\nabla \times \underline{u} = \nabla \times \nabla \phi = 0 \because$ curlgrad $= 0$ $\quad$ cdg $\quad \therefore u_i = \frac{\partial \phi}{\partial x_i} \therefore$

$\xi_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_i}\right) - \frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial x_j}\right)\right) = \frac{1}{2}\left(\frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_i}\right) - \frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_j}\right)\right) = \frac{1}{2}(0) = 0$ on

interchanging 2 order of 2 derivs

\1d/ incompressible flows have $\nabla \cdot \underline{u} = 0$, $\therefore \frac{\partial u_i}{\partial x_i} = 0 \therefore$ then

$e_{ii} = e_{11} + e_{22} + e_{33} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0$ $\therefore e_{11} = -(e_{22}+e_{33})$ say $\therefore$ there are 1

fewer indep components than in general $\therefore$ 5 indep components using part (a)

\1e/ $\delta u_i = \delta x_j D_{ij} = \delta x_j (D^s_{ij} + D^a_{ij}) = \delta u^s_i + \delta u^a_i$ where $\delta u^s_i = \delta x_j D^s_{ij}$ &

$\delta u^a_i = \delta x_j D^a_{ij}$ $\quad \delta u^a_i = \delta x_j D^a_{ij} = \delta x_j \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\delta x_j \frac{\partial u_i}{\partial x_j} - \frac{1}{2}\delta x_j \frac{\partial u_j}{\partial x_i}$

now consider $(\underline{\omega} \times \delta \underline{x})_i = \varepsilon_{ijk}\omega_j \delta x_k = \varepsilon_{ijk}\left(\varepsilon_{jmn}\frac{\partial u_n}{\partial x_m}\right)\delta x_k =$

$\varepsilon_{jki}\varepsilon_{jmn}\frac{\partial u_n}{\partial x_m}\delta x_k = (\delta_{km}\delta_{in} - \delta_{im}\delta_{kn})\frac{\partial u_n}{\partial x_m}\delta x_k =$

$\delta_{km}\delta_{in}\frac{\partial u_n}{\partial x_m}\delta x_k - \delta_{im}\delta_{kn}\frac{\partial u_n}{\partial x_m}\delta x_k = \frac{\partial u_i}{\partial x_k}\delta x_k - \frac{\partial u_k}{\partial x_i}\delta x_k = 2\delta u^a_i$ $\because$ 2 dummy

variab k can be called j instead $\therefore \delta u^a_i = \frac{1}{2}(\underline{\omega} \times \delta \underline{x})_i$

\1f/ $\underline{u} = (ax_1 + \gamma x_2, bx_2 - \gamma x_1, cx_3)$ $\quad e_{ij} = \frac{1}{2}\begin{bmatrix} 2a & \gamma+(-\gamma) & 0 \\ -\gamma+\gamma & 2b & 0 \\ 0 & 0 & 2c \end{bmatrix} =$

$\begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$ $\quad \xi_{ij} = \frac{1}{2}\begin{bmatrix} 0 & \gamma-(-\gamma) & 0 \\ -\gamma-\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \gamma & 0 \\ -\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ $\quad \xi_{ij} = \frac{1}{2}\begin{bmatrix} 0 & \gamma-(-\gamma) & 0 \\ -\gamma-\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & \gamma & 0 \\ -\gamma & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\delta u^s_i = \delta x_j D^s_{ij} = \delta x_j e_{ij}$ $\therefore i=1$ $\delta u^s_1 = \delta x_j e_{1j} = \delta x_1 e_{11}$ $(\because e_{12} = e_{13} = 0)$

$= a\delta x_1$ $\quad i=2: \delta u^s_2 = \delta x_j e_{2j} = \delta x_2 e_{22}$ $(\because e_{21} = e_{23} = 0)$ $= b\delta x_2$

$i=3$ $\quad \delta u^s_3 = \delta x_j e_{3j} = \delta x_3 e_{33} = c\delta x_3$ $\therefore \delta \underline{u}^s = (a\delta x_1, b\delta x_2, c\delta x_3)$ similarly

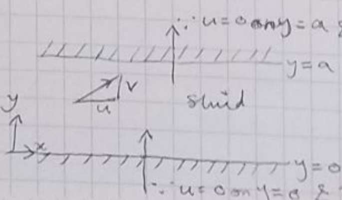
$\delta u^a_i = \delta x_j D^a_{ij} = \delta x_j \xi_{ij}$ $\therefore i=1$ $\delta u^a_1 = \delta x_j \xi_{1j} = \delta x_2 \xi_{12}$ $(\because \xi_{11} = \xi_{13} = 0)$

$= -\gamma\delta x_1$ $\quad i=3: \delta u^a_3 = \delta x_j \xi_{3j} = 0$ $(\because \xi_{3j} = 0)$ $\therefore \delta u^a = (\gamma\delta x_2, -\gamma\delta x_1, 0)$ $\therefore$

$\delta \underline{u}^s + \delta \underline{u}^a = (a\delta x_1, b\delta x_2, c\delta x_3) + (\gamma\delta x_2, -\gamma\delta x_1, 0) =$

$(a\delta x_1 + \gamma\delta x_2, b\delta x_2 - \gamma\delta x_1, c\delta x_3)$

\2a/



$u = u(y)\underline{i} + v\underline{j}$
with $v>0$
$u(0) = u(a) = 0$

always move on

\revision Session 1/ more marks for method most of all

say why crossed out terms eg ∴ flow is steady or $\nabla u = 0$ & show it
more questions in reading materials
) Search ECM3707 fluid dynamics to 2015 onwards

\introduction:/ Fundamentals (VCA) (vector calculus:)

Navier-stokes eqn $\underbrace{\rho}_{\text{density}}\left(\frac{\partial \underline{u}}{\partial t} + \underbrace{\underline{u}}_{\text{velocity}} \cdot \nabla \underline{u}\right) = -\nabla p + \rho \underline{g} + \mu \nabla^2 \underline{u}$ (remember) ∴

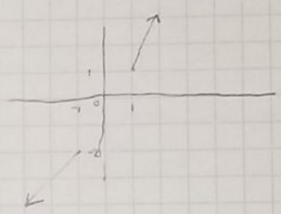
continuity (conservation of mass:) $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0$ ∴ (remember) ∴

$\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \underline{u} + \underline{u} \cdot \nabla \rho = 0$ (identity iii)

$\therefore \frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u} = 0$ $\quad \because \frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u} \cdot \nabla$ "material derivative

incompressible flows $\frac{D\rho}{Dt} = 0 \quad \therefore \nabla \cdot \underline{u} = 0$

\visualising flows / steamlines / pathlines /



- velocity: $\underline{u} = (x, 2y)$ $\quad \therefore$ at $(x,y) = (1,1)$: $\underline{u} = (1,2)$

at $(-2,-1)$: $u = (-2,-2)$

\streamlines/: curves drawn in the fluid at a fixed time that have $\underline{u}$ as tangents: $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{dw}$ $\quad \underline{u} = (u, v, w)$

$u = (x, y(t+1)) = (x, yc)$ ∵ $t+1 = c$ constant at streamlines

$\int \frac{dx}{x} = \int \frac{dy}{yc} \therefore \ln x = \frac{1}{c}\ln y + d = \ln y^{1/c} + d \therefore e^{\ln x} = e^{\ln y^{1/c} + d} = A e^{\ln y^{1/c}} = A y^{1/c} = x$

to draw need to be told the time to draw it.

\pathlines/ show the path of a fluid parcel as a function of time

$\frac{dx}{dt} = u \quad \frac{dy}{dt} = v \quad \frac{dz}{dt} = w \quad \therefore \frac{dx}{dt} = x \therefore \int \frac{dx}{x} = \int dt \therefore \ln x = t + c \therefore$

$x = Ae^t$,

$\frac{dy}{dt} = y(t+1) \therefore \int \frac{dy}{y} = \int (t+1)\,dt \therefore \ln y = \frac{t^2}{2} + t + C \therefore y = e^{\frac{t^2}{2} + t} \tilde{A}$

\Suffix Notation/ $\underline{x} = (x, y, z) \rightarrow (x_1, x_2, x_3)$ ∴ ith component of $\underline{x}$ is x_i $(i = 1,2,3)$ ∴ $\underline{u} = (u, v, w) \rightarrow (u_1, u_2, u_3)$ ∴ ith component of $\underline{u}$ is u_i $(i=1,2,3)$ ∴

$\nabla = \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial x}, \frac{\partial}{\partial z}\right) \rightarrow \left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}\right)$ ∴ of ∇ is: $\nabla_i = \frac{\partial}{\partial x_i} = \partial_i$ $(i = 1,2,3)$

$\therefore \nabla \cdot \underline{u} = 0 \therefore \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0 \rightarrow \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0 \therefore$

- $\sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} = 0 \therefore \frac{\partial u_i}{\partial x_i} = 0$ eg write down Z vorticity eqn in suffix notation is

$\therefore \frac{\partial \underline{\omega}}{\partial t} = \nabla \times (u \times \underline{\omega}) + \nu \nabla^2 \underline{\omega}$, $\underline{\omega} = \nabla \times \underline{u}$ ∴ ith component ∴ $\nu = \mu/\rho$

$\left[\frac{\partial \underline{\omega}}{\partial t}\right]_i = [\nabla \times (\underline{u} \times \underline{\omega})]_i + [\nu \nabla^2 \underline{\omega}_i] \rightarrow \frac{\partial \omega_i}{\partial t} = [\nabla \times (\underline{u} \times \underline{\omega})]_i + \nu \nabla^2 \omega_i$

$$\nabla^2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

recall $(\underline{a} \times \underline{b})_i = \varepsilon_{ijk} a_j b_k$ $\therefore$ $[\nabla \times (\underline{u} \times \underline{\omega})]_i = \varepsilon_{ijk} \nabla_j (\underline{u} \times \underline{\omega})_k =$

$\varepsilon_{ijk} \nabla_j (\varepsilon_{klm} u_l \omega_m) = \varepsilon_{kij} \varepsilon_{klm} \nabla_j (u_l \omega_m) = \varepsilon_{kij} \varepsilon_{klm} \frac{\partial}{\partial x_j} (u_l \omega_m) =$

$\varepsilon_{kij} \varepsilon_{klm} \frac{\partial}{\partial x_j} (u_l \omega_m) = (\delta_{il} \delta_{jm} - \delta_{im} \delta_{jl}) \frac{\partial}{\partial x_j} (u_l \omega_m) =$ (eqn 2.20 remember)

$\delta_{il} \delta_{jm} \frac{\partial}{\partial x_j} (u_l \omega_m) - \delta_{im} \delta_{jl} \frac{\partial}{\partial x_j} (u_l \omega_m) =$

requires $i = l$, $j = m$ $\therefore$ $l \to i$, $m \to j$

$\frac{\partial}{\partial x_j} (u_i \omega_j) - \frac{\partial}{\partial x_j} (u_j \omega_i)$ $\therefore$

$\frac{\partial \omega_i}{\partial t} = \frac{\partial}{\partial x_j} (u_i \omega_j) - \frac{\partial}{\partial x_j} (u_j \omega_i) + \nu \nabla^2 \omega_i$

$= u_i \frac{\partial \omega_j}{\partial x_j} + \omega_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial \omega_i}{\partial x_j} - \omega_i \frac{\partial u_j}{\partial x_j} + \nu \nabla^2 \omega_i$

$\nabla \cdot \underline{\omega} = \nabla \cdot (\nabla \times \underline{u}) = 0$ $\qquad$ $\nabla \cdot \underline{u} = 0$ $\therefore$

for incompressible flow:

$\frac{\partial \omega_i}{\partial t} = \omega_j \frac{\partial u_i}{\partial x_j} - u_j \frac{\partial \omega_i}{\partial x_j} + \nu \nabla^2 \omega_i$

$u = (x, y)$ $\quad u_i = x_i$

\exact sols/ (i) given $\underline{u}(x, t) = \cdots$ show it solves Navier Stokes

sub in & show LHS = RHS

$\underline{u} \cdot \nabla \underline{u} \neq (\nabla \cdot \underline{u}) \underline{u}$ but $\underline{u} \cdot \nabla \underline{u} = (\underline{u} \cdot \nabla) \underline{u} = D \underline{u} = D(u \underline{i} + v \underline{j} + w \underline{k}) =$

$D(u \underline{i}) + D(v \underline{j}) + D(w \underline{k}) =$

$u D \underline{i} + \underline{i} D u + \underline{j} D v + v D \underline{j} + w D \underline{k} + \underline{k} D w$ $\qquad$ $\underline{i} = (1, 0, 0)$

in cartesian: $D \underline{i} = 0$, $D \underline{j} = 0$, $D \underline{k} = 0$ $\therefore$

$= \underline{i} D u + \underline{j} D v + \underline{k} D w$

$u = (x, y^2, 0)$ $\therefore$

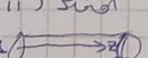
$D = \underline{u} \cdot \nabla = \begin{pmatrix} x \\ y^2 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} \partial/\partial x \\ \partial/\partial y \\ \partial/\partial z \end{pmatrix} = x \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} + 0 \frac{\partial}{\partial z}$ $\therefore$

$D u = \left(x \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} \right) u = \left(x \frac{\partial}{\partial x} + y^2 \frac{\partial}{\partial y} \right) x = x + 0 = x$

in cylindrical polars: $\underline{u} \cdot \nabla = u \frac{\partial}{\partial R} + v \frac{\partial}{\partial \theta} + w \frac{\partial}{\partial z}$ $\therefore$

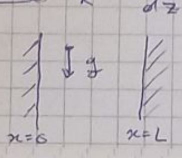
$(\underline{u} \cdot \nabla) \hat{\underline{R}} = \underbrace{u \frac{\partial}{\partial R} \hat{\underline{R}}}_{=0} + v \frac{1}{R} \frac{\partial \hat{\underline{R}}}{\partial \theta} + \underbrace{w \frac{\partial}{\partial z} \hat{\underline{R}}}_{=0} = \frac{v}{R} \underline{\theta}$

(ii) find a sol of Navier-Stokes. flow moves only as forces act

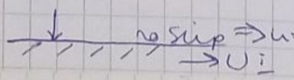


P_R $\quad \frac{dP}{dz} \neq 0$ $\quad \nabla P = \frac{dP}{dz} \hat{\underline{z}}$ expect $\underline{u} = W \hat{\underline{z}}$ $\quad W(R)$

(ii)



$\underline{u} = W(R) \hat{\underline{z}}$ Sub into N-S to find $W(R)$



Dimensional analysis & similarity sols / Dimensions (units) of each of ...

~~Operations~~ Questions in N-S $(u, p, \rho, \nu, x, t, \dots)$

$\{LT^{-2} = [\nu] L^{-1} T^{-1} \therefore$
$[\nu] = L^2 T^{-1}$

1) $[x] = L \quad [t] = T \quad [\underline{u}] = [u] = \frac{\text{distance}}{\text{time}} = LT^{-1}$

$\left[\frac{\partial u}{\partial t}\right] = \frac{1}{T} = \frac{LT^{-1}}{T} \qquad [\nu \nabla^2 u] = [\nu] L^{-2} L T^{-1} \iff \frac{\partial u}{\partial t} \therefore [\nu] = \dots$

a) non dimensionalise an eqn: change of variables $x' = \frac{x}{L}$, $t' = \frac{t}{T} \therefore \dots$

$dx' = \frac{1}{L} dx \therefore \frac{d}{dx} = \frac{1}{L}\frac{d}{dx'} \to \nabla \to \frac{1}{L}\nabla'$, $\frac{\partial}{\partial t} \to \frac{1}{T}\frac{\partial}{\partial t'}$

$\to$ N-S except $Re = \frac{UL}{\nu}$ $(Re \gg 1, Re \ll 1)$

(b) Buckingham-Pi theorem $\underline{u} = \underline{u}(x, y, z, t, \nu, \dots) \to \Pi = \Phi(\Pi_1, \Pi_2, \Pi_3, \dots \Pi_4)$

PDE $\to$ ODE $\quad \underline{u} \to \Pi = \frac{u}{[u]}$

n params in total; divided into indep params k (x, t) & dependent params m (ν)

$\therefore n = k + m \quad \therefore \quad \Pi = \Phi(\Pi_1, \Pi_2, \dots \Pi_m)$

Solve ODE - Constant coeffs $f'' - a^2 f = 0 \therefore$ sols $f(x) = e^{px} \therefore p^2 - a^2 = 0 \therefore$

$p = \pm a \therefore f(x) = Ae^{ax} + Be^{-ax}$

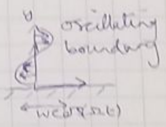
- Seperation of variabls: $f'' + 2xf' = 0 \therefore g = f' \therefore g' + 2xg = 0 \therefore$

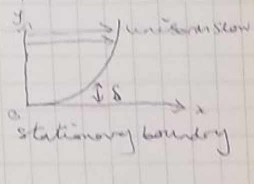
$\frac{dg}{dx} = -2xg \quad \therefore \quad \frac{dg}{g} = -2x\,dx \therefore \int \frac{dg}{g} = \int -2x\,dx \therefore \ln g = \frac{-2x^2}{2} + c \therefore$

$g = e^{-x^2} e^c$

- variable coeffs $f'' + 2x^{-2} f = 0$ seek sols $f(x) = x^p \quad f' = px^{p-1} \therefore$

$f'' = p(p-1)x^{p-2} \to f(x) = Ax^{p_1} + Bx^{p_2}$





Boundary layers: / ex flow near to object

- Derive a simplified version of N-S for flow within the boundary layer $\therefore$ use $\delta \ll L \therefore$

$\rho(\cancel{\partial_t u}_{\text{steady}} + u\partial_x u + v\partial_y u) = -\partial_x P + \nu(\partial_{xx} u + \underbrace{\partial_{yy} u}_{\text{neglect}})$

x comp $u \gg v \to$ Boundary layer eqn

Stokes flow: / $0 = -\nabla p + \mu \nabla^2 \underline{u}$ $\quad \therefore$ inertia terms $(\partial_t u + u \cdot \nabla u = 0)$

$\therefore Re \ll 1$: $Re = \frac{UL}{\nu}$

(i) $\underline{u} = \nabla \times (\psi \hat{\underline{z}}) \quad \psi = \psi(x, y) \quad = \begin{vmatrix} \partial_x & \partial_y & 0 \\ 0 & 0 & \psi \end{vmatrix} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0\right)$

$\to \nabla^4 \psi = 0$

(ii) $\underline{u} = u\hat{\underline{r}} + v\hat{\underline{\theta}} = \nabla \times \left(\frac{\psi}{r \sin\theta}\hat{\underline{\phi}}\right) \to E^4 \psi = 0 \Rightarrow E^2(E^2\psi) = 0$ (formula)

$\underline{dS} = \hat{n}\, dS$

Vorticity dynamics: / $\underline{\omega} = \nabla \times \underline{u}$ deriv vorticity eqn from N-S

- vorticity ~~flux~~ = circulation flux: $\int_S \underline{\omega} \cdot \underline{dS} = \int_S (\nabla \times \underline{u}) \cdot dS =$

$\int_S (\nabla \times \underline{u}) \cdot \hat{\underline{n}} \, dS = \oint_S \underline{u} \cdot d\underline{l}$ by stokes thm (formula)

waves eqn $\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2}$ $\xi(x,t)$ D'Alemberts sol $\xi(x,t) = g(x-ct) + h(x+ct)$

$\xi = a\cos[kx - \omega t]$ k wavenumber, ω frequency are constants

$= \mathbb{R}[A e^{i(kx-\omega t)}]$ $e^{ikx} = \cos kx + i \sin kx$ $A = A_R + i A_I$

$= (A_R + i A_I)(\cos(kx-\omega t) + i \sin(kx - \omega t))$

$= A_R \cos(kx - \omega t) - A_I \sin(kx - \omega t) \therefore$

$\frac{\partial \xi}{\partial t} = \mathbb{R}\left[A \frac{\partial}{\partial t} e^{i(kx-\omega t)}\right] = \mathbb{R}\left[A e^{i(kx-\omega t)}(-i\omega)\right] = -i\omega \xi$

$\rightarrow$ $\frac{\partial \xi}{\partial x} \rightarrow ik\xi$

· surface gravity

$\nabla^2 \underline{u} = \nabla^2(u\underline{i} + v\underline{j}) = \nabla^2(u\underline{i}) + \nabla^2(v\underline{j}) = \underline{i}\nabla^2 u + \underline{j}\nabla^2 v$

$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}$

$\nabla^2 \underline{u} = \nabla(u\hat{\underline{R}} + v\hat{\underline{\theta}}) \neq \hat{\underline{R}}\nabla^2 u + \hat{\underline{\theta}}\nabla^2 v$

identity (viii) : $\nabla^2 \underline{u} = \nabla(\cancel{\nabla \cdot \underline{u}}) - \nabla \times (\nabla \times \underline{u})$
incompressible

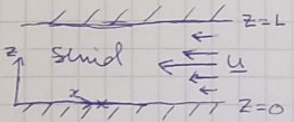
variables : velocity vec: $\underline{u}(\underline{x}, t) = (u, v, w) =$
$u(x,y,z,t)\hat{\underline{i}} + v(x,y,z,t)\hat{\underline{j}} + w(x,y,z,t)\hat{\underline{k}}$

pressure scalar $p = p(x,y,z,t) \therefore$ force $-\nabla p$ directed from regions of high to low pressure

density scalar $\rho = \rho(x,y,z,t)$

vec ∇p is perpendicular to level surfaces $p = \text{const}$

Mass continuity



\CW1/\1a/

\1b/ $\nabla p = \frac{dp}{dx}\underline{i}$ with $\frac{dp}{dx} > 0 \therefore -\nabla p$ is in the negative x-direction and provides the force driving the flow (gravity is neglected and the boundaries are stationary) $\therefore \underline{u} = u\,\underline{i}$ expected with $u < 0$, u expected to be independent of time (steady given) and x ($\because$ no special x locations) but dependent of z due to boundaries at $z = 0, L$ $\therefore \underline{u} = u(z)\,\underline{i}$, $u < 0$, where $u(0) = u(L) = 0$ (no slip)

\CW1/ \1e/ let $\underline{u} = u(z)\underline{i}$ $\therefore$ $\underline{u}\cdot\nabla = (u(z)\underline{i})\cdot\nabla = u(z)\partial_x$ $\therefore$

$\underline{u}\cdot\nabla\underline{u} = (\underline{u}\cdot\nabla)\underline{u} = u(z)\partial_x(u(z)\underline{i}) = \underline{0}$ $\therefore$

) Navier Stokes: $\partial_t\underline{u} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla P + \frac{\mu}{\rho}\nabla^2\underline{u}$, gravity neglected

$\therefore$ $\underline{u}\cdot\nabla\underline{u} = 0$, $\partial_t\underline{u} = 0$ $\therefore$

$0 = -\frac{1}{\rho}\nabla P + \frac{\mu}{\rho}\nabla^2\underline{u}$ $\therefore$ only x-component is non trivial $\therefore$

$-\nabla P + \mu\nabla^2\underline{u} = 0$ $\therefore$ $-\nabla P + \mu\nabla^2(u(z)\hat{\underline{i}}) = -\nabla P + \mu(\partial_{xx} + \partial_{yy} + \partial_{zz})u(z)\hat{\underline{i}} =$

$-\nabla P + \mu\partial_{zz}u(z)\hat{\underline{i}} = 0$ $\therefore$

$\mu\partial_{zz}u(z)\hat{\underline{i}} = \nabla P = \frac{\partial P}{\partial x}\hat{\underline{i}} + \frac{\partial P}{\partial y}\hat{\underline{j}} + \partial_z P\hat{\underline{k}}$ $\therefore$ $\frac{\partial P}{\partial y} = 0$, $\partial_z P = 0$ $\therefore$

$\mu\partial_{zz}(u(z))\hat{\underline{i}} = \frac{\partial P}{\partial x}\hat{\underline{i}}$ $\therefore$ $\mu\partial_{zz}u(z) = \partial_x P$ $\therefore$

$\partial_z\partial_z u(z) = \frac{1}{\mu}\partial_x P$ $\therefore$ $\partial_z u(z) = \frac{1}{\mu}\partial_x(P)z + C$ $\therefore$

$\partial_{zz}u(z) = \frac{1}{\mu}\partial_x P = \frac{d}{dz}\left(\frac{d}{dz}u(z)\right)$ $\therefore$

$\frac{d}{dz}(u(z)) = \frac{1}{\mu}\partial_x P z + C$ $\therefore$ $u(z) = \frac{1}{\mu}\frac{dP}{dx}\frac{z^2}{2} + cz + d$, c, d constants

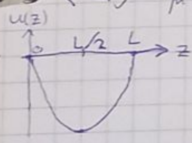
$\therefore$ BCs: no slip: $u(0) = 0$ $\therefore$ $u(0) = 0 = \frac{1}{\mu}\frac{dP}{dx}\frac{0^2}{2} + c(0) + d = d = 0$ $\therefore$

$u(z) = \frac{1}{\mu}\frac{dP}{dx}\frac{z^2}{2} + cz$ $\therefore$ $u(L) = 0 = \frac{1}{\mu}\frac{dP}{dx}\frac{L^2}{2} + cL$ $\therefore$

$\frac{1}{\mu}\frac{dP}{dx}\frac{L}{2} + c = 0$ $\therefore$ $c = -\frac{L}{2\mu}\frac{dP}{dz} = -\frac{1}{\mu}\frac{dP}{dx}\frac{L}{2}$ $\therefore$

$u(z) = \frac{1}{\mu}\frac{dP}{dx}\frac{z^2}{2} - \frac{1}{\mu}\frac{dP}{dx}\frac{L}{2}z = \frac{1}{\mu}\frac{dP}{dx}\left(\frac{z^2}{2} - \frac{Lz}{2}\right) = u(z)$

\1d/ $\frac{1}{\mu}\frac{dP}{dx} > 0$ $\therefore$ $z - L \leq 0$ $\because$ $z \in [0, L]$ $\therefore$ $u(z) \leq 0$ (as expected)

$\therefore$ 

$\frac{du}{dz} = \frac{1}{\mu}\frac{dP}{dx}\left(z - \frac{L}{2}\right) = 0$ at $z = \frac{L}{2}$ $\therefore$

$\left.\frac{d^2u}{dz^2}\right|_{L/2} = \left.\frac{1}{\mu}\frac{dP}{dx}(1)\right|_{L/2} = \frac{1}{\mu}\frac{dP}{dx} > 0$ $\therefore$ minimum $\therefore$

$|\underline{u}|$ largest at $z = \frac{L}{2}$

\2a/ $\underline{\omega} = \nabla\times\underline{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\hat{\underline{\phi}} \\ \partial_r & \partial_\theta & \partial_\phi \\ r^2\cos\theta & \cancel{r}\frac{1}{r} & r\cancel{\sin\theta}\frac{1}{\cancel{\sin\theta}} \end{vmatrix} =$

$\frac{1}{r^2\sin\theta}\left[\hat{\underline{r}}(0) - r\hat{\underline{\theta}}(1) + r\sin\theta\hat{\underline{\phi}}(0 + r^2\sin\theta)\right] = -\frac{1}{r\sin\theta}\hat{\underline{\theta}} + r\sin\theta\hat{\underline{\phi}}$

\2b/ $\underline{u}\cdot\nabla\underline{u} = (\underline{u}\cdot\nabla)\underline{u}$ $\therefore$ $(\underline{u}\cdot\nabla) = R\cos\phi\,\partial_R + \frac{\sin\phi}{R}\partial_\phi$ $\because$

$\nabla = \hat{\underline{R}}\partial_R + \frac{1}{R}\hat{\underline{\phi}}\partial_\phi + \hat{\underline{z}}\partial_z$ $\therefore$

$\underline{u}\cdot\nabla\underline{u} = \left(R\cos\phi\,\partial_R + \frac{\sin\phi}{R}\partial_\phi\right)(R\cos\phi\hat{\underline{R}}) + \left(R\cos\phi\,\partial_R + \frac{\sin\phi}{R}\partial_\phi\right)(\sin\phi\hat{\underline{\phi}}) =$

$\hat{\underline{R}} R\cos^2\phi + \frac{\sin\phi}{R}\cdot R\left(-\sin\phi\,\hat{\underline{R}} + \cos\phi\frac{\partial\hat{\underline{R}}}{\partial\phi}\right) + \frac{\sin\phi}{R}\left(\cos\phi\,\hat{\underline{\phi}} + \sin\phi\frac{\partial\hat{\underline{\phi}}}{\partial\phi}\right) =$

$\hat{\underline{R}} R\cos^2\phi - \sin^2\phi\,\hat{\underline{R}} + \sin\phi\cos\phi\,\hat{\underline{\phi}} + \frac{\sin\phi}{R}\cos\phi\,\hat{\underline{\phi}} - \frac{\sin^2\phi}{R} =$

$\left(R\cos^2\phi - \left(1+\frac{1}{R}\sin^2\phi\right)\right)\hat{\underline{R}} + \left(1+\frac{1}{R}\right)\sin\phi\cos\phi\,\hat{\underline{\phi}}$

\3a/ $\therefore \underline{u} = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & R\frac{\psi}{R} & 0 \end{vmatrix} = \frac{1}{R}\left(-\hat{\underline{R}}\frac{\partial\psi}{\partial z} - R\hat{\underline{\phi}}\cdot 0 + \hat{\underline{z}}\frac{\partial\psi}{\partial R}\right) = \underline{u} = u\hat{\underline{R}} + w\hat{\underline{z}} \therefore$

$\psi = -\frac{R^2}{10}(a^2 - R^2 - z^2) \quad \therefore u = -\frac{1}{R}\frac{\partial\psi}{\partial z} = -\frac{1}{R}\left(-\frac{R^2}{10}\right)(-2z) = -\frac{Rz}{5},$

$w = \frac{1}{R}\frac{\partial\psi}{\partial R} = \frac{1}{R}\left[\left(-\frac{R^2}{10}\right)(-2R) + (a^2 - R^2 - z^2)\left(-\frac{2R}{10}\right)\right] = \frac{1}{R}\left[\frac{R^3}{5} - \frac{R}{5}(a^2 - R^2 - z^2)\right] =$

$\frac{1}{5}(2R^2 - a^2 + z^2) \therefore$

Stagnation points have $\underline{u} = 0 \therefore u = w = 0 \therefore u = 0 = -\frac{1}{5}Rz \therefore$

$u = 0 \therefore R = 0$ or $z = 0 \therefore$

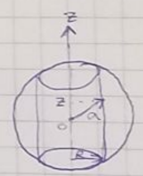
for $R = 0$: $w = 0 = \frac{1}{5}(2(0)^2 - a^2 + z^2) = 0 = -a^2 + z^2 \therefore z = \pm a \therefore$

stagnation points $R = 0, z = \pm a$

for $z = 0$: $w = 0 = \frac{1}{5}(2R^2 - a^2) = 0 = 2R^2 - a^2 \quad \therefore R^2 = \frac{a^2}{2} \therefore R = \pm\frac{a}{\sqrt{2}} \therefore$

stagnation points $R = \pm\frac{a}{\sqrt{2}}, z = 0 \therefore$

$(0, \phi, a), (0, \phi, -a), \left(\frac{a}{\sqrt{2}}, \phi, 0\right) \because R > 0$



\3b/ $|\underline{u}|^2 = \frac{R^2z^2}{25} + \frac{1}{25}(2R^2 - a^2 + z^2)^2$

on a spherical surface radius a: $R^2 + z^2 = a^2 \therefore$

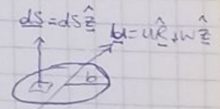
$|\underline{u}|^2 = \frac{R^2z^2}{25} + \frac{1}{25}(R^2 - a^2 + a^2)^2 = \frac{R^2z^2}{25} + \frac{1}{25}(R^2)^2 = \frac{1}{25}(R^2z^2 + (R^2)^2)$

$= \frac{R^2}{25}(z^2 + R^2) = \frac{R^2}{25}a^2 = \frac{R^2a^2}{25} = |\underline{u}|^2 \because |\underline{u}| > 0 \therefore |\underline{u}| = \frac{Ra}{5} \therefore$

$\frac{d|\underline{u}|}{dR} = \frac{a}{5} > 0 \quad \therefore \quad |\underline{u}(R=a)| > |\underline{u}(R=b)|$ for $a > b \therefore$

$\max|\underline{u}| = |\underline{u}(R=a)| = \frac{aa}{5} = \frac{a^2}{5} \therefore$

Max value at $R = a$ eg the equator



\3c/ Mass flux $\rho\int\underline{u}\cdot d\underline{S}$ $\therefore d\underline{S} = \hat{\underline{n}}\,dS \therefore \hat{\underline{n}} = \hat{\underline{z}} \therefore$

$\rho\int\underline{u}\cdot d\underline{S} = \rho\int\underline{u}\cdot\hat{\underline{z}}\,dS = \rho\int(u\hat{\underline{R}} + w\hat{\underline{z}})\cdot\hat{\underline{z}}\,dS = \rho\int w\,dS =$

$\rho\int_{\phi=0}^{2\pi}\int_{R=0}^{b} wR\,dR\,d\phi = \rho\int_{\phi=0}^{2\pi}\int_{R=0}^{b}\left(\frac{1}{R}\frac{\partial\psi}{\partial R}\right)R\,dR\,d\phi = \rho\int_{\phi=0}^{2\pi}\int_{R=0}^{b}\frac{\partial\psi}{\partial R}\,dR\,d\phi =$

$\rho 2\pi\int_{R=0}^{b}\frac{\partial\psi}{\partial R}\,dR = 2\pi\rho\left[\psi\right]_{R=0}^{b} = 2\pi\rho\left[\psi(R=b) - \psi(R=0)\right] \therefore$

\CW1/ $\psi = -\frac{R^2}{10}(a^2 - R^2 - z^2)$ ∴

$\psi(R=0) = -\frac{0^2}{10}(a^2 - (0)^2 - z^2) = 0$, $\psi(R=b) = -\frac{b^2}{10}(a^2 - b^2 - z^2)$ ∴

$\rho\int \underline{u}\cdot d\underline{S} = 2\pi\rho[\psi(R=b) - \psi(R=0)] =$

$2\pi\rho[\psi(R=b) - 0] = 2\pi\rho[\psi(R=b)] = 2\pi\rho\psi(R=b) =$

$2\pi\rho\left[-\frac{b^2}{10}(a^2 - b^2 - z^2)\right] = 2\pi\rho\left(-\frac{b^2}{10}\right)(a^2 - b^2 - z^2) = \frac{\pi\rho b^2}{5}(b^2 + z^2 - a^2)$

∴ check ∴ Stoke's theorem: $\int_C \underline{F}\cdot d\underline{L} = \int_S (\nabla\times\underline{F})\cdot\hat{\underline{n}}\, dS = \int_S (\nabla\times\underline{F})\cdot d\underline{S}$ ∴

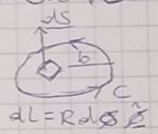
$\rho\int \underline{u}\cdot d\underline{S} = \rho\int_S (\nabla\times(\frac{\psi}{R}\hat{\underline{\phi}})) = \rho\int_C (\frac{\psi}{R}\hat{\underline{\phi}})\cdot d\underline{L} = \int_0^{2\pi}(\frac{\psi}{R}\hat{\underline{\phi}})\cdot b\hat{\underline{\phi}}\, d\phi =$

$b\rho\int_0^{2\pi}\frac{\psi}{R}\, d\phi$ at $R=b$:

$= b\rho\int_0^{2\pi}\frac{\psi}{b}\, d\phi = \rho\int_0^{2\pi}\psi\, d\phi$

check: by Stokes theorem: $\rho\int_S \underline{u}\cdot d\underline{S} = \rho\int_S \nabla\times(\frac{\psi}{R}\hat{\underline{\phi}})\cdot d\underline{S} =$

$\rho\int_C \frac{\psi}{R}\hat{\underline{\phi}}\cdot d\underline{L}$ where the direction around the closed curves C is related to $d\underline{S}$ in a right-hand sense ∴



$= \rho\int_{\phi=0}^{2\pi}\frac{\psi}{R}\hat{\underline{\phi}}\cdot R\hat{\underline{\phi}}\, d\phi$ at $R=b$ ∴

$= \rho\int_{\phi=0}^{2\pi}\frac{\psi}{R}R\, d\phi = \rho\int_{\phi=0}^{2\pi}\psi\, d\phi =$

$\rho\psi\int_{\phi=0}^{2\pi} 1\, d\phi = 2\pi\rho\psi$ at $R=b$ =

$2\pi\rho\psi(R=b)$ as above

\4a/ $\nabla\cdot\underline{u} = \partial_x(Ex) + \partial_y(-E) + \partial_z(0) = E - E = 0$ ∴ incompressible

\4b/ inviscid N-S without gravity : $\rho(\partial_t\underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \mu\nabla^2\underline{u}$

∴ inviscid ∴ $\mu = 0$ ∴ $\mu\nabla^2\underline{u} = 0$, $\frac{\partial\underline{u}}{\partial t} = \frac{\partial}{\partial t}(Ex, -Ey) = 0$ ∴

$\rho(\underline{u}\cdot\nabla\underline{u}) = -\nabla p$ ∴

$(\underline{u}\cdot\nabla)\underline{u} = \underline{u}\cdot\nabla\underline{u}$ ∴ $(\underline{u}\cdot\nabla) = (Ex\hat{\underline{i}} - Ey\hat{\underline{j}})\cdot(\hat{\underline{i}}\frac{\partial}{\partial x} + \hat{\underline{j}}\frac{\partial}{\partial y}) = Ex\frac{\partial}{\partial x} - Ey\frac{\partial}{\partial y} =$

$E(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y})$ ∴

$\underline{u}\cdot\nabla\underline{u} = E(x\frac{\partial}{\partial x} - y\frac{\partial}{\partial y})(Ex\hat{\underline{i}}) + E(x\partial_x - y\partial_y)(-Ey\hat{\underline{j}}) =$

$E^2x\hat{\underline{i}} + E^2y\hat{\underline{j}} = ExE\hat{\underline{i}} + E(-y)(-E)\hat{\underline{j}} = E^2(x\hat{\underline{i}} + y\hat{\underline{j}})$ ∴

$\rho(\underline{u}\cdot\nabla\underline{u}) = -\nabla p$ ∴ $\rho E^2(x\hat{\underline{i}} + y\hat{\underline{j}}) = -\frac{\partial p}{\partial x}\hat{\underline{i}} - \frac{\partial p}{\partial y}\hat{\underline{j}}$ ∴

x comp: $\rho E^2 x = -\frac{\partial p}{\partial x}$ ∴ $-\rho E^2 x = \frac{\partial p}{\partial x}$ ∴ $p = -\rho\frac{E^2x^2}{2} + f(y)$ ∴

$\frac{\partial p}{\partial y} = \frac{\partial f(y)}{\partial y} = \frac{df}{dy}$ ∴

y comp: $\rho E^2 y = -\frac{\partial p}{\partial y}$ ∴ $-\rho E^2 y = \frac{\partial p}{\partial y}$ ∴

$p = -\rho \frac{E^2 y^2}{2}$ ∴ $-\rho E^2 y = \frac{\partial p}{\partial y} = \frac{df}{dy}$ ∴

$f = -\rho \frac{E^2 y^2}{2} + C$ C is constant ∴

$p = -\rho \frac{E^2 x^2}{2} + f(y) = -\rho \frac{E^2 x^2}{2} - \rho \frac{E^2 y^2}{2} + C = -\frac{\rho E^2}{2}(x^2+y^2) + C$

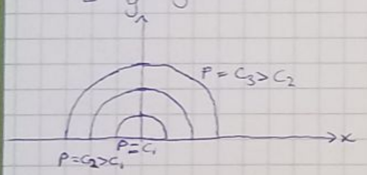
∴ $p(0,0) = P_0 = -\frac{\rho E^2}{2}(0^2+y^2) + C = C = P_0$ ∴

$p = -\frac{\rho E^2}{2}(x^2+y^2) + P_0$, $P_0 = C = p(x=0, y=0)$

\4c/ pressure contours $p(x,y) = \text{constant}$ are the curves

$x^2+y^2 = \text{Const}$ ∵ $p(x,y) = -\frac{\rho E^2}{2}(x^2+y^2) + P_0 = C_1$ ∴

$-\frac{\rho E^2}{2}(x^2+y^2) = C_2$ ∴ $x^2+y^2 = C_3 = \text{Const}$ ie circles centre (0,0)

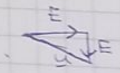


\4d/ streamlines : $\frac{dx}{u} = \frac{dy}{v}$ ∴ $\frac{dx}{Ex} = \frac{dy}{-Ey}$ ∴ $\cancel{\frac{1}{x}dx}$ $\frac{1}{x}dx = -\frac{1}{y}dy$ ∴

$\ln|x| = -\ln|y| + c$ ∴ $e^{\ln|x|} = |x| = e^{-\ln|y|+c} = \cancel{e^c e}$

$e^c e^{-\ln|y|} = \cancel{A}$ $A e^{\ln(|y|^{-1})} = A|y|^{-1} = \frac{A}{|y|}$, $A = e^c > 0$ ∴

$|y| = \frac{A}{|x|}$ ∴ hyperbolic ∴

\4e/ $\underline{u} = E(x, -y)$ at $(x,y) = (1,1)$: $\underline{u} = E(+1, -1)$ ∴  for $E > 0$

as $x \to 0^+$, $\underline{u} \to E(0, -y)$ ∴ $\downarrow \underline{u}$ for $E, y > 0$

as $x \to +\infty$ $\underline{u} \to E(\infty, 0)$ ∴

$y \to 0^+$ $\underline{u} \to E(\infty, 0)$ ∴ $\longrightarrow \underline{u}$

at $(x,y) = (-1,1)$ $\underline{u} = E(-1,-1)$ ∴ $\swarrow \underline{u}$ for $E > 0$

\4f/ $y=0$ is a solid boundary ∴ no flow through $y=0$ ∴

$v = 0$ at $y = 0$ ∴ $-Ey = 0$ at $y = 0$ ✓

$y=0$ is a stationary boundary ∴ no slip ∴ $\underline{u} = 0$ at $y=0$ ∴ $Ex = 0$ at $y = 0$

✗ when $E = 0$ but then $\underline{u} = 0$ everywhere. The given velocity $\underline{u} = E(x, -y)$ satisfies no normal flow but doesn't satisfy the no slip condition (inviscid)

\CW2/ \1a/ $e = \frac{1}{2}\begin{bmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{bmatrix}$ with $e_{ij} = e_{ji}$

(ie $e_{12} = e_{21}$, $e_{13} = e_{31}$, $e_{23} = e_{32}$) $\therefore$ 6 independent components

\1b/ $\xi = \frac{1}{2}\begin{bmatrix} 0 & \xi_{12} & \xi_{13} \\ -\xi_{12} & 0 & \xi_{23} \\ -\xi_{13} & -\xi_{23} & 0 \end{bmatrix}$ $\because \xi_{ji} = -\xi_{ij}$ $\therefore$ 3 independent components

\1c/ irrotational flows have $\underline{\omega} = \nabla \times \underline{u} = 0$

writing $\underline{u} = \nabla \phi$ $\therefore \nabla \times \underline{u} = \nabla \times \nabla \phi \equiv 0$ $\because$ curl grad $\equiv 0$ $\therefore$

$u_i = \frac{\partial \phi}{\partial x_i}$ $\therefore \xi_{ij} = \frac{1}{2}\left(\frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_i}\right) - \frac{\partial}{\partial x_i}\left(\frac{\partial \phi}{\partial x_j}\right)\right) \equiv 0$ as interchanging the of the derivatives

\1d/ incompressible flows have $\nabla \cdot \underline{u} = 0$ $\therefore \frac{\partial u_i}{\partial x_i} = 0$ $\therefore$

$e_{ii} = e_{11} + e_{22} + e_{33} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3} = 0$ $\therefore e_{11} = -(e_{22} + e_{33})$ $\therefore$

1 fewer independent components in general $\therefore$ 5 independent components using part (a)

\1e/ $\delta u_i = \delta x_j D_{ij} = \delta x_j (D^s_{ij} + D^a_{ij}) = \delta x_j D^s_{ij} + \delta x_j D^a_{ij} =$

$\delta u_i^s + \delta u_i^a$ $\therefore$

$\delta u_i^a = \delta x_j D^a_{ij} = \delta x_j \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \frac{1}{2}\delta x_j \frac{\partial u_i}{\partial x_j} - \frac{1}{2}\delta x_j \frac{\partial u_j}{\partial x_i}$

consider $(\underline{\omega} \times \delta \underline{x})_i = \varepsilon_{ijk}\omega_j \delta x_k = \varepsilon_{ijk}(\nabla \times \underline{u})_j \delta x_k =$

$\varepsilon_{ijk}(\varepsilon_{jmn}\nabla_m u_n)\delta x_k = \varepsilon_{ijk}(\varepsilon_{jmn}\partial_m u_n)\delta x_k =$

$\varepsilon_{ijk}\left(\varepsilon_{jmn}\frac{\partial u_n}{\partial x_m}\right)\delta x_k = \varepsilon_{jki}\varepsilon_{jmn}\frac{\partial u_n}{\partial x_m}\delta x_k =$

$(\delta_{km}\delta_{in} - \delta_{km}\delta_{im})\frac{\partial u_n}{\partial x_m}\delta x_k =$

$\delta_{km}\delta_{in}\frac{\partial u_n}{\partial x_m}\delta x_k - \delta_{im}\delta_{kn}\frac{\partial u_n}{\partial x_m}\delta x_k =$

$\frac{\partial u_i}{\partial x_k}\delta x_k - \frac{\partial u_k}{\partial x_i}\delta x_k = \frac{\partial u_i}{\partial x_j}\delta x_j - \frac{\partial u_j}{\partial x_i}\delta x_j$ $\because$ dummy variable $k \to j$ instead

$= \delta x_j\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) = \cancel{\delta x_j \frac{2}{2}}\ \cancel{\delta x_j \frac{2}{2}}\ 2\delta x_j \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right) =$

$2\delta x_j D^a_{ij} = 2\delta u_i^a$ $\therefore$

$\delta u_i^a = \frac{1}{2}(\underline{\omega} \times \delta \underline{x})_i$

\1f/ $e_{ij} = \frac{1}{2}\begin{pmatrix} 2a & \delta + (-\delta) & 0 \\ -\delta + \delta & 2b & 0 \\ 0 & 0 & 2c \end{pmatrix} = \begin{pmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{pmatrix}$

$\xi_{ij} = \frac{1}{2}\begin{pmatrix} 0 & \delta - (-\delta) & 0 \\ -\delta - \delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \delta & 0 \\ -\delta & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$

$\delta u_i^s = \delta x_j D_{ij}^s = \delta x_j e_{ij}$ $\therefore i=1:$ $\delta u_1^s = \delta x_j e_{1j} =$

$\delta x_1 e_{11} + \delta x_2 e_{12} + \delta x_3 e_{13} = \delta x_1 e_{11} + \delta x_2 (0) + \delta x_3 (0) = \delta x_1 e_{11}$ $\because e_{12} = e_{13} = 0$

$\therefore \delta u_1^s = \delta x_1 e_{11} = \delta x_1 (a) = a\,\delta x_1$

for $i=2:$ $\delta u_2^s = \delta x_j e_{2j} = \delta x_1 e_{21} + \delta x_2 e_{22} + \delta x_3 e_{23} =$

$\delta x_1 (0) + \delta x_2 e_{22} + \delta x_3 (0)$ $\because e_{21} = e_{23} = 0$

$= \delta x_2 e_{22} = \delta x_2 (b) = b\,\delta x_2$

for $i=3:$ $\delta u_3^s = \delta x_j e_{3j} = \delta x_1 e_{31} + \delta x_2 e_{32} + \delta x_3 e_{33} =$

$\delta x_1 (0) + \delta x_2 (0) + \delta x_3 e_{33}$ $\because e_{31} = e_{32} = 0$

$= \delta x_3 e_{33} = c\,\delta x_3$ $\therefore$

$\delta \underline{u}^s = (a\,\delta x_1, b\,\delta x_2, c\,\delta x_3)$

$\delta u_i^a = \delta x_j D_{ij}^a = \delta x_j \xi_{ij}$

$\therefore$ for $i=1:$ $\delta u_1^a = \delta x_j \xi_{1j} = \delta x_1 \xi_{11} + \delta x_2 \xi_{12} + \delta x_3 \xi_{13}$ $\because \xi_{11} = \xi_{13} = 0$

$= \delta x_1 (0) + \delta x_2 \xi_{12} + \delta x_3 (0) = \delta x_2 \xi_{12} = \delta x_2 (\gamma) = \gamma\,\delta x_2$

for $i=2:$ $\delta u_2^a = \delta x_j \xi_{2j} = \delta x_1 \xi_{21} + \delta x_2 \xi_{22} + \delta x_3 \xi_{23} =$

$\delta x_1 (-\gamma) + \delta x_2 (0) + \delta x_3 (0)$ $\because \xi_{22} = \xi_{23} = 0$

$= \delta x_1 (-\gamma) = -\gamma\,\delta x_1$

for $i=3:$ $\delta u_3^a = \delta x_j \xi_{3j} = \delta x_1 \xi_{31} + \delta x_2 \xi_{32} + \delta x_3 \xi_{33} =$

$\delta x_1 (0) + \delta x_2 (0) + \delta x_3 (0)$ $\because \xi_{3j} = 0$

$= 0 + 0 + 0 = 0$ $\therefore$

$\delta \underline{u}^a = (\gamma\,\delta x_2, -\gamma\,\delta x_1, 0)$ $\therefore$

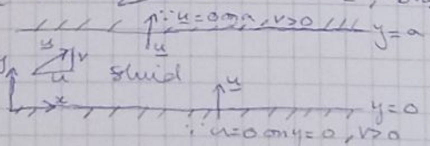
$\delta \underline{u}^s + \delta \underline{u}^a = (a\,\delta x_1, b\,\delta x_2, c\,\delta x_3) + (\gamma\,\delta x_2, -\gamma\,\delta x_1, 0) =$

$(a\,\delta x_1 + \gamma\,\delta x_2, b\,\delta x_2 - \gamma\,\delta x_1, c\,\delta x_3)$ $\therefore$

for $\delta x_i = x_i$:

$\delta \underline{u}^s + \delta \underline{u}^a = \cancel{(a\,\delta x_1 + \gamma}\,(a x_1 + \gamma x_2, b x_2 - \gamma x_1, c x_3) = \underline{u}$

\2a/



$\underline{u} = u(y)\hat{\underline{i}} + v\hat{\underline{j}}$, with $v>0$

$u(0) = u(a) = 0$

\2b/ N-S: $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \rho\underline{g} + \mu\nabla^2\underline{u}$ $\therefore$

neglect gravity $\therefore \underline{g} = 0 \therefore \rho\underline{g} = 0$, steady flow $\therefore \partial_t \underline{u} = 0$

\CW2/ $\therefore \rho(\underline{u}\cdot\nabla\underline{u}) = -\nabla p + \mu\nabla^2\underline{u} \because$

$\underline{u}\cdot\nabla\underline{u} = (\underline{u}\cdot\nabla)\underline{u} \quad \therefore \quad \underline{u}\cdot\nabla = (u\hat{\underline{i}} + v\hat{\underline{j}})\cdot(\partial_x\hat{\underline{i}} + \partial_y\hat{\underline{j}}) = u\partial_x + v\partial_y \therefore$

$\underline{u}\cdot\nabla\underline{u} = (u\partial_x + v\partial_y)(u(y)\hat{\underline{i}} + v\hat{\underline{j}}) = (u\partial_x + v\partial_y)(u(y)\hat{\underline{i}}) \quad \because v = \text{const}$

$= v\partial_y u(y)\hat{\underline{i}} = v\frac{du}{dy}\hat{\underline{i}} \therefore$

$\rho(v\frac{du}{dy}\hat{\underline{i}}) = -\nabla p + \mu\nabla^2\underline{u} \therefore$

x comp: $0 + \rho v\frac{du}{dy} = -\frac{\partial p}{\partial x} + \mu(\frac{d^2}{dx^2} + \frac{d^2}{dy^2} + \frac{d^2}{dz^2})u(y) =$

$-\frac{\partial p}{\partial x} + \mu\frac{d^2}{dy^2}u(y) = -\frac{\partial p}{\partial x} + \mu\frac{d^2u}{dy^2} = \rho v\frac{\partial u(y)}{\partial y}$

y comp $0 = -\frac{\partial p}{\partial y} + \mu\nabla^2 v = -\frac{\partial p}{\partial y} = 0$ ~~$\therefore p$ is~~ $\therefore \frac{\partial p}{\partial y} = 0 \therefore$

p is independent of y $\therefore$

Seperate variables of x component $\therefore$

$\frac{\partial p}{\partial x} = \mu\frac{d^2u(y)}{dy^2} - \rho v\frac{\partial u(y)}{\partial y}$ is function of y only $\therefore$

$\frac{\partial p}{\partial x}$ is not a function of x and p is independent of y $\therefore$

$\therefore \frac{\partial p}{\partial x}$ independent of y $\therefore \frac{\partial p}{\partial x} = \text{constant} = -G$, say.

$\therefore p = p(x) \therefore \frac{\partial p(x)}{\partial x} = -G \therefore \frac{\partial(-Gx)}{\partial x} = -G \therefore$ PZ

x component of N-S: $-G = \mu\frac{d^2u(y)}{dy^2} - \rho v\frac{du(y)}{dy} \therefore$

$\mu u'' - \rho v u' = -G$

\4c/ $[a] = L$, $[v] = LT^{-1}$, $[\mu] = [\rho]T^{-1}L^2 \because [\rho] = ML^{-3} \therefore$

$[\rho\partial_t\underline{u}] \sim [\mu\nabla^2\underline{u}] \quad \therefore [\rho\partial_t\underline{u}] \sim [\mu]L^{-2}LT^{-1} = L^{-1}T^{-1}[\mu] \therefore$

$[\rho\partial_t\underline{u}] = [\rho]T^{-1}LT^{-1} \therefore [\rho]L^2T^{-1} = [\mu] = ML^{-3}L^2T^{-1} = ML^{-1}T^{-1}$

$\because$ density = mass/volume $\therefore$

$[G] = [\frac{dp}{dx}] \therefore \quad [\nabla p] \sim [\rho\underline{u}\cdot\nabla\underline{u}] = ML^{-3}LT^{-1}L^{-1}LT^{-1} = ML^{-2}T^{-2}$

or $[\nabla p] \sim [\rho\partial_t\underline{u}] = ML^{-3}T^{-1}LT^{-1} = ML^{-2}T^{-2} \therefore$

$[G] = [\rho][\partial_t\underline{u}] = [\rho]T^{-1}LT^{-1} = LT^{-2} = ML^{-3}L^{+1}T^{-2} =$

\4d/

independent $a, v, \rho \quad \therefore$ 3 have independent dimension

$\therefore$ Dependent dimension: $b_1 = G \therefore [G] = [\rho][v]^2[a]^{-1} \therefore$

$\Pi_1 = \frac{G}{\rho v^2 a^{-1}} = \frac{Ga}{\rho v^2}$

$b_2 = \mu \quad [\mu] = [\rho][V][a] \quad \therefore \Pi_2 = \frac{\mu}{\rho V a}$

\4e/ $u'' - \frac{\rho V}{\mu} u' = -\frac{G}{\mu} \quad \therefore$

homogeneous part: RHS = 0 $\therefore$ let $u = e^{sy} \quad \therefore s^2 - \frac{\rho V}{\mu} s = 0 \therefore$

$s = 0, \quad s = \frac{\rho V}{\mu} \quad \therefore \quad u = A + Be^{y\rho V/\mu} \quad \therefore$

PI: $u = Cy \quad \therefore u' = C \quad \therefore \quad u'' = 0 \quad \therefore$ into ODE:

$0 - \frac{\rho V}{\mu} C = -\frac{G}{\mu} \quad \therefore \quad C = \frac{G}{\rho V} \quad \therefore$

$u(y) = A + Be^{y\rho V/\mu} + \frac{G}{\rho V} y \quad \therefore$

BC: $\quad u(0) = 0 = A + B \quad \therefore \quad A = -B \quad \therefore$

$u(y) = B(-1 + e^{y\rho V/\mu}) + \frac{G}{\rho V} y \quad \therefore$

$u(a) = 0 = B(\cancel{1}e^{a\rho V/\mu}) + \frac{G}{\rho V} a \quad \therefore$

$\cancel{Be^{a\rho V/\mu}} = -\frac{Ga}{\rho V} = B(-1 + e^{a\rho V/\mu}) \quad \therefore \quad B = -\frac{Ga}{\rho V} \frac{1}{(-1 + e^{a\rho V/\mu})} \quad \therefore$

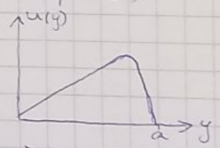
$u(y) = -\frac{Ga}{\rho V} \frac{-1 + e^{(y\rho V/\mu)}}{-1 + e^{a\rho V/\mu}} + \frac{G}{\rho V} y =$

$\frac{G}{\rho V}\left(y - \frac{(-a + ae^{y\rho V/\mu})}{-1 + e^{a\rho V/\mu}}\right) = \frac{G}{\rho V}\left(y - \frac{(-a + ae^{y\rho V/\mu})}{-1 + \exp(R_e)}\right) = \frac{G}{\rho V}\left(y - \frac{(a - ae^{y\rho V/\mu})}{1 - \exp(R_e)}\right),$

$R_e = a\rho V/\mu = \frac{aV}{\nu} \qquad \therefore u(y) = \frac{Ga}{\rho V}\left(\frac{y}{a} - \frac{1 - e^{y\rho V/\mu}}{1 - \exp(R_e)}\right)$

\4S/ $\therefore V = 10^{-4}, \quad \mu = 10^{-3}, \quad \rho = 10^3, \therefore a = 1 \quad \therefore$

$R_e = \frac{aV\rho}{\mu} = \frac{1 \times 10^{-4} 10^3}{10^{-3}} = 10^2 \quad \therefore$



R_e large $\therefore$ $1 - e^{R_e}$ is large $\therefore$ $u \approx \frac{Gy}{\rho V}$ (linear in y) apart from close to $y = a$, where $u = 0 \quad \therefore \quad u(y=0) = 0 = u(y=a)$,

$u(y) \approx \frac{Gy}{\rho V} \quad \therefore$

\Week 1/ \1ai/ $\underline{\omega} = \nabla \times \underline{u}$,

$\underline{u} \times (\nabla \times \underline{u}) + \underline{u} \times (\nabla \times \underline{u}) = \nabla(\underline{u} \cdot \underline{u}) - (\underline{u} \cdot \nabla)\underline{u} - (\underline{u} \cdot \nabla)\underline{u}$

$2\underline{u} \times (\nabla \times \underline{u}) = \nabla(\underline{u} \cdot \underline{u}) - 2(\underline{u} \cdot \nabla)\underline{u} \quad \therefore$

$(\underline{u} \cdot \nabla)\underline{u} = \underline{u} \cdot \nabla \underline{u} = \frac{1}{2}\nabla(\underline{u} \cdot \underline{u}) - \underline{u} \times (\nabla \times \underline{u}) = \frac{1}{2}\nabla u^2 - \underline{u} \times \underline{\omega} \quad \because$ ~~u·u =~~

$\underline{u} \cdot \underline{u} = |\underline{u}|^2 = u^2$

\1aii/ $\nabla \times (\underline{u} \cdot \nabla \underline{u}) = \nabla \times (\nabla \frac{u^2}{2}) - \nabla \times (\underline{u} \times \underline{\omega}) \quad \therefore$

$\nabla \times (\underline{u} \cdot \nabla \underline{u}) = -\nabla \times (\underline{u} \times \underline{\omega}) \quad \because \quad \nabla \times \nabla \phi = \underline{0} \quad \forall \phi$

\week 1/ \1a iii/ $\nabla^2 \underline{u} = \nabla(\nabla\cdot\underline{u}) - \nabla\times\underline{\omega}$ $\therefore$

$\nabla\times\nabla^2\underline{u} = \nabla\times\nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{\omega}) = -\nabla\times(\nabla\times\underline{\omega})$ $\quad\because \nabla\times\nabla\phi \equiv \underline{0}$

$\Rightarrow = -\nabla(\nabla\cdot\underline{\omega}) + \nabla^2\underline{\omega} = \nabla^2\underline{\omega}$ $\because \nabla\cdot\underline{\omega} = \nabla\cdot(\nabla\times\underline{u}) = 0$ $\quad\because \nabla\cdot\nabla\times\underline{u} = 0$

\1b/ N-S: $\partial_t\underline{u} + \underline{u}\cdot\nabla\underline{u} = -\nabla\left(\frac{p+\rho g z}{\rho}\right) + \nu\nabla^2\underline{u}$, $\nu = \mu/\rho$ $\therefore$

$\partial_t\underline{\omega} + \nabla\times(\underline{u}\cdot\nabla\underline{u}) = 0 + \nu\nabla\times\nabla^2\underline{u}$ $\quad\because \nabla\times\nabla\phi = 0$ $\therefore$

$\therefore \partial_t\underline{\omega} = -\nabla\times(\underline{u}\cdot\nabla\underline{u}) + \nu\nabla\times\nabla^2\underline{u}$ $\therefore$

$\nabla\times(\underline{u}\cdot\nabla\underline{u}) = -\nabla\times(\underline{u}\times\underline{\omega})$ $\therefore$

$\partial_t\underline{\omega} = \nabla\times(\underline{u}\times\underline{\omega}) + \nu\nabla\times\nabla^2\underline{u}$ $\therefore$ $\qquad \frac{D\rho}{Dt} = \frac{\partial\rho}{\partial t} + \underline{u}\cdot\nabla\rho = 0$

$\nabla\times\nabla^2\underline{u} = \nabla^2\underline{\omega}$ $\therefore$

$\partial_t\underline{\omega} = \nabla\times(\underline{u}\times\underline{\omega}) + \nu\nabla^2\underline{\omega}$ $\qquad \frac{\partial\rho}{\partial t} + \nabla\cdot(\rho\underline{u}) = 0$

\1c/ $\nabla\times(\underline{u}\times\underline{\omega}) = \underbrace{\underline{u}(\nabla\cdot\underline{\omega})}_{=0 \because \nabla\cdot\nabla\times\underline{u}} - \underbrace{\underline{\omega}(\nabla\cdot\underline{u})}_{\because \nabla\cdot\underline{u}=0} + \underline{\omega}\cdot\nabla\underline{u} - \underline{u}\cdot\nabla\underline{\omega} = \underline{\omega}\cdot\nabla\underline{u} - \underline{u}\cdot\nabla\underline{\omega}$

$\therefore \partial_t\underline{\omega} + (\underline{u}\cdot\nabla)\underline{\omega} = (\underline{\omega}\cdot\nabla)\underline{u} + \nu\nabla^2\underline{\omega}$ $\qquad \partial_t\underline{\omega} = \nabla\times(\underline{u}\times\underline{\omega}) + \nu\nabla^2\underline{\omega}$

\2a/ $\nabla\times(\phi\underline{v}) = \phi\nabla\times\underline{v} + \nabla\phi\times\underline{v}$ $\therefore$

$[\nabla\times(\phi\underline{v})]_i = \varepsilon_{ijk}\nabla_j(\phi\underline{v})_k = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\phi v_k) =$

$\varepsilon_{ijk}\left\{\phi\frac{\partial v_k}{\partial x_j} + v_k\frac{\partial\phi}{\partial x_j}\right\} = \varepsilon_{ijk}\phi\frac{\partial v_k}{\partial x_j} + \varepsilon_{ijk}v_k\frac{\partial\phi}{\partial x_j} =$

$\phi\varepsilon_{ijk}\frac{\partial}{\partial x_j}v_k + \varepsilon_{ijk}v_k\frac{\partial}{\partial x_j}(\phi) =$

$\phi\varepsilon_{ijk}\nabla_j v_k + \varepsilon_{ijk}v_k\nabla_j(\phi) =$

$\cancel{\phi\varepsilon_{ijk}\nabla_j v_k + \varepsilon_{ijk}v}$ $\phi\varepsilon_{ijk}\nabla_j v_k - \varepsilon_{ikj}v_k\nabla_j(\phi) =$

$\phi[\nabla\times\underline{v}]_i - [\underline{v}\times\nabla\phi]_i = [\phi\nabla\times\underline{v} - \underline{v}\times\nabla\phi]_i$ $\therefore$

$\nabla\times(\phi\underline{v}) = \phi\nabla\times\underline{v} - \underline{v}\times\nabla\phi$ X

$[\nabla\times(\phi\underline{v})]_i = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\phi\underline{v})_k = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\phi v_k) = \varepsilon_{ijk}\left\{\phi\frac{\partial v_k}{\partial x_j} + v_k\frac{\partial\phi}{\partial x_j}\right\} =$

$\varepsilon_{ijk}\phi\frac{\partial v_k}{\partial x_j} + \varepsilon_{ijk}v_k\frac{\partial\phi}{\partial x_j} = \phi\varepsilon_{ijk}\frac{\partial v_k}{\partial x_j} + \varepsilon_{ijk}v_k\frac{\partial\phi}{\partial x_j} =$

$[\phi\nabla\times\underline{v}]_i + \varepsilon_{ijk}v_k\frac{\partial\phi}{\partial x_j} = [\phi\nabla\times\underline{v}]_i + \varepsilon_{ijk}\frac{\partial\phi}{\partial x_j}v_k = [\phi\nabla\times\underline{v}]_i + [\nabla\phi\times\underline{v}]_i$

\2b/ $\nabla\cdot\nabla\times\underline{v} = 0$ $\therefore$ $[\nabla\cdot(\nabla\times\underline{v})]_i = \frac{\partial}{\partial x_i}\left(\varepsilon_{ijk}\frac{\partial v_k}{\partial x_j}\right) =$

$\varepsilon_{ijk}\frac{\partial^2 v_k}{\partial x_i\partial x_j} = \varepsilon_{ij1}\frac{\partial^2 v_1}{\partial x_i\partial x_j} + \varepsilon_{ij2}\frac{\partial^2 v_2}{\partial x_i\partial x_j} + \varepsilon_{ij3}\frac{\partial^2 v_3}{\partial x_i\partial x_j}$ $\therefore$

$\varepsilon_{ij1}\frac{\partial^2 v_1}{\partial x_i\partial x_j} = \varepsilon_{231}\frac{\partial^2 v_1}{\partial x_2\partial x_3} + \varepsilon_{321}\frac{\partial^2 v_1}{\partial x_3\partial x_2} = \frac{\partial^2 v_1}{\partial x_2\partial x_3} - \frac{\partial^2 v_1}{\partial x_3\partial x_2} = 0$ $\therefore$

$\therefore \varepsilon_{ij2} \frac{\partial^2 V_2}{\partial x_i \partial x_j} = 0$, $\varepsilon_{ij3} \frac{\partial^2 V_3}{\partial x_i \partial x_j} = 0$ $\therefore$ $\nabla \cdot (\nabla \times \underline{V}) = 0$

\3/ The divergence theorem is $\int_V \nabla \cdot \underline{F}\, dV = \int_S \underline{F} \cdot d\underline{S}$

\2a/ $[\nabla \times (\phi \underline{V})]_i = \varepsilon_{ijk} \nabla_j (\phi \underline{V})_k = \varepsilon_{ijk} \frac{\partial}{\partial x_j} (\phi V_k) =$

$\varepsilon_{ijk}\left(\left(\frac{\partial}{\partial x_j}\right)(\phi) V_k + \phi \frac{\partial V_k}{\partial x_j}\right) = \varepsilon_{ijk} \frac{\partial \phi}{\partial x_j} V_k + \varepsilon_{ijk} \phi \frac{\partial V_k}{\partial x_j} =$

$\cancel{\varepsilon_{ijk}}$ $\phi\, \varepsilon_{ijk} \frac{\partial}{\partial x_j}(V_k) + \varepsilon_{ijk} \frac{\partial}{\partial x_j}(\phi) V_k = [\phi \nabla \times \underline{V}]_i + [\nabla \phi \times \underline{V}]_i = [\phi \nabla \times \underline{V} + \nabla \phi \times \underline{V}]_i$

\3/ divergence thm: $\int_V \nabla \cdot \underline{F}\, dV = \int_S \underline{F} \cdot d\underline{S}$, where surface S encloses the volume V and $d\underline{S} = \hat{\underline{n}}\, dS$, where $\hat{\underline{n}}$ is the unit outward normal to S. Take $\underline{F} = \underline{a}\, \phi(\underline{x})$ where $\underline{a}$ is constant

$\int_V \nabla \cdot (\underline{a} \phi) dV = \int_S (\underline{a} \phi) \cdot \hat{\underline{n}}\, dS$ $\therefore$

$\int_V \underbrace{\phi \nabla \cdot \underline{a}}_{=0 \because \underline{a} = \text{const}} + \underline{a} \cdot \nabla \phi\, dV = \int_S \phi\, \underline{a} \cdot \hat{\underline{n}}\, dS$ $\therefore \int_V \underline{a} \cdot \nabla \phi\, dV = \int_S \phi\, \underline{a} \cdot \hat{\underline{n}}\, dS$

now take $\underline{a} = \underline{i} = (1,0,0)$ to obtain $\int_V \frac{\partial \phi}{\partial x} dV = \int \phi\, \hat{n}_x\, dS$,

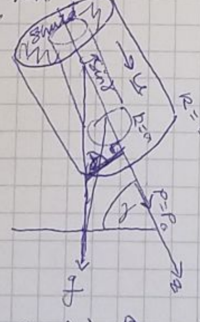
$\hat{n}_x$ is x component of $\hat{\underline{n}}$ $\therefore$

taking $\underline{a} = \hat{j} = (0,1,0)$ gives $\int_V \frac{\partial \phi}{\partial y} dV = \int \phi\, \hat{n}_y\, dS$

and $\underline{a} = \underline{k} = (0,0,1)$ gives $\int_V \frac{\partial \phi}{\partial z} dV = \int \phi\, \hat{n}_z\, dS$ $\therefore$

$\int_V \frac{\partial \phi}{\partial x_i} dV = \int \phi\, \hat{n}_i\, dS$ for $i = 1,2,3$ $\therefore$ as vec: $\int_V \nabla \phi\, dV = \int_S \phi\, \hat{\underline{n}}\, dS$

\Week 2/



N-S: $\rho(\underline{u}_t + \underline{u} \cdot \nabla \underline{u}) = -\nabla p + \rho \underline{g} + \mu \nabla^2 \underline{u}$

$\therefore$ z comp:

$\rho(\underbrace{\partial_t W}_{=0 \because \text{steady}} + [\underline{u} \cdot \nabla \underline{u}]_z) = -\partial_z p + \rho g \sin\gamma + \mu[\nabla^2 \underline{u}]_z$

$\therefore$ $\rho([\underline{u} \cdot \nabla \underline{u}]_z) = -\partial_z p + \rho g \sin\gamma + \mu[\nabla^2 \underline{u}]_z$

$\therefore$ for $\underline{u} = W(R)\hat{\underline{z}}$:

$[\underline{u} \cdot \nabla]_z = [(W(R)\hat{\underline{z}}) \cdot (\frac{\partial}{\partial z}\hat{\underline{z}})]_z = [W(R)\frac{\partial}{\partial z}]_z$ $\therefore$

$[\underline{u} \cdot \nabla \underline{u}]_z = [W(R)\frac{\partial}{\partial z}(W(R)\hat{\underline{z}})]_z = 0$, $\therefore$ $[\underline{u} \cdot \nabla \underline{u}]_z = 0$ $\therefore$

$(\nabla^2 \underline{u})_z =$ $\therefore$ $(\nabla^2 \underline{u}) = \nabla(\nabla \cdot \underline{u}) - \nabla \times (\nabla \times \underline{u})$ $\therefore$

$\nabla \cdot \underline{u} = 0 \therefore \nabla(\nabla \cdot \underline{u}) = 0 \therefore$ $\nabla^2 \underline{u} = -\nabla \times (\nabla \times \underline{u})$ $\therefore$

for $\underline{u} = W(R)\hat{\underline{z}}$ $\therefore$ $\nabla \times \underline{u} = \nabla \times (W(R)\hat{\underline{z}}) = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & 0 & W(R) \end{vmatrix} =$

$\frac{1}{R}[\hat{\underline{R}}(0) - R\hat{\underline{\phi}}(\partial_R W(R)) + \hat{\underline{z}}(0)] = -\partial_R(W(R))\hat{\underline{\phi}}$ $\therefore$

\Week 2/ $\therefore \nabla \times (\nabla \times \underline{u}) = \nabla \times (-\partial_R (W(R)) \hat{\underline{\phi}}) = \frac{1}{R} \begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & -R\partial_R(W(R)) & 0 \end{vmatrix} =$

$\frac{1}{R}\left[\hat{\underline{R}}(0) - R\hat{\underline{\phi}}(0) + \hat{\underline{z}}(-\partial_R(W(R)) - R\partial_{RR}(W(R))\right]$

$\Rightarrow = \left(-\frac{1}{R}\partial_R(W(R)) - \partial_{RR}(W(R))\right)\hat{\underline{z}} \quad \therefore$

$[\nabla^2 \underline{u}]_z = \frac{1}{R}\partial_R(W(R)) + \partial_{RR}(W(R))$ X

$(\nabla^2 \underline{u})_z = \frac{1}{R}\frac{d}{dR}\left(R\frac{dW}{dR}\right) \quad \therefore$

$0 = -\frac{\partial p}{\partial z} + \rho g \sin\gamma + \frac{\mu}{R}\frac{d}{dR}\left(R\frac{dW}{dR}\right)$

\1b/ seperation of ~~pariables~~ variables gives

$\frac{dp}{dz} = \rho g \sin\gamma + \frac{\mu}{R}\frac{d}{dR}\left(R\frac{dW(R)}{dR}\right) \quad \therefore \frac{dp}{dz}$ independent of $z \quad \therefore$

$\cancel{\frac{dp}{dz} = \partial z} \quad \therefore \quad \cancel{p = p(R, \phi)} \quad \therefore$ set $\frac{\partial p}{\partial z} = f(R) \quad \therefore$

$p(z=0) = p(z=L) = p_0 \quad \therefore \quad \therefore \; p = f(R)z + C \quad \therefore$

$p(z=0) = \cancel{f(R)z + C} \; p_0 = f(R)(0) + C = C = p_0 \quad \therefore$

$p = f(R)z + p_0 \quad \therefore p(z=L) = p_0 = f(R)L + p_0 = p_0 \quad \therefore f(R)L = 0 \quad \therefore f(R) = 0$

$\therefore p = p_0 \quad \therefore \; p = p_0 = \text{constant} \quad \therefore \nabla p = 0$

\1c/ $\frac{dp}{dz} = \rho g \sin\gamma + \frac{\mu}{R}\frac{d}{dR}\left(R\frac{dW}{dR}\right) \quad \therefore \quad \frac{dp}{dz} = 0 \quad \therefore$

$0 = \rho g \sin\gamma + \frac{\mu}{R}\frac{d}{dR}\left(R\frac{dW}{dR}\right) \quad \therefore$

$-\rho g \sin\gamma = \frac{\mu}{R}\frac{d}{dR}\left(R\frac{dW}{dR}\right) \quad \therefore -\rho g \sin\gamma \, R = \mu\frac{d}{dR}\left(R\frac{dW}{dR}\right) \quad \therefore$

$-\frac{\rho g \sin\gamma}{\mu}R = \frac{d}{dR}\left(R\frac{dW(R)}{dR}\right) \quad \therefore A - \frac{\rho g \sin}{2\mu}R^2 = R\frac{dW(R)}{dR} \quad \therefore$

$-\frac{\rho g \sin\gamma}{2\mu}R + \frac{A}{R} = \frac{dW(R)}{dR} \quad \therefore -\frac{\rho g \sin\gamma}{4\mu}R^2 + A\ln R + B = W(R) \quad \therefore$

\1d/ no slip BCs $\therefore$ at $R=a$; $\underline{u}=0$, at $R=b$; $\underline{u}=0 \quad \therefore$

$W(R=a)=0$, $W(R=b)=0 \quad \therefore \quad W(R=a) = 0 = -\frac{\rho g \sin\gamma}{4\mu}a^2 + A\ln(a) + B$,

$W(R=b) = 0 = -\frac{\rho g \sin\gamma}{4\mu}b^2 + A\ln b + B \quad \therefore$

$\frac{\rho g \sin\gamma}{4\mu}a^2 - A\ln a = B \quad \therefore \quad 0 = -\frac{\rho g \sin\gamma}{4\mu}b^2 + A\ln b + \frac{\rho g \sin\gamma}{4\mu}a^2 - A\ln a \quad \therefore$

$\frac{\rho g \sin\gamma}{4\mu}a^2 - A\ln a = B \quad \therefore 0 = -\frac{\rho g \sin\gamma}{4\mu}b^2 + A\ln b + \frac{\rho g \sin\gamma}{4\mu}a^2 - A\ln a \quad \therefore$

$\frac{\rho g \sin\gamma}{4\mu}(b^2 - a^2) = A(\ln b - \ln a) = A\ln\left(\frac{b}{a}\right) \quad \therefore$

$\frac{\rho g \sin\gamma}{4\mu}(b^2 - a^2)\frac{1}{\ln\left(\frac{b}{a}\right)} = A \quad \therefore$

$W(R) = -\frac{\rho g \sin\gamma}{4\mu}R^2 + \frac{\rho g \sin\gamma}{4\mu\ln\left(\frac{b}{a}\right)}(b^2 - a^2)\ln R + \frac{\rho g \sin\gamma}{4\mu}a^2 - \frac{\rho g \sin\gamma(b^2 - a^2)}{4\mu\ln\left(\frac{b}{a}\right)}\ln(a)$

(W2Q2)

$$= -\frac{\rho g \sin\gamma}{4\mu}\left[R^2 - \frac{(b^2-a^2)\ln R}{\ln(\frac{b}{a})} - a^2 + \frac{(b^2-a^2)}{\ln(\frac{b}{a})}\ln a\right] =$$

$$w(R) = \frac{\rho g \sin\gamma}{4\mu}\left[R^2 - \frac{b^2 \ln R}{\ln(\frac{b}{a})} - a^2 + \frac{(b^2-a^2)}{\ln(\frac{b}{a})}\ln a\right] =$$

$$w(R) = -\frac{\rho g \sin\gamma}{4\mu}\left[R^2 - \frac{b^2\ln R}{\ln(\frac{b}{a})} + \frac{a^2 \ln R}{\ln(\frac{b}{a})} - a^2 + \frac{b^2 \ln a}{\ln(\frac{b}{a})} - \frac{a^2}{\ln(\frac{b}{a})}\ln a\right] =$$

$$-\frac{\rho g \sin\gamma}{4\mu}\left[R^2 + (a^2\ln R - b^2 \ln R + b^2 \ln a - a^2 \ln a - a^2 \ln(\frac{b}{a}))/\ln(b/a)\right] =$$

$$-\frac{\rho g \sin\gamma}{4\mu}\left[R^2 - (b^2\ln(\frac{R}{a}) - a^2 \ln(\frac{R}{b}) + a^2 \ln a - a^2 \ln a)/\ln(b/a)\right] =$$

$$-\frac{\rho g \sin\gamma}{4\mu}\left[R^2 - \left[(b^2 \ln(R/a) - a^2\ln(R/b)\right]\right]/\ln(b/a)$$

\2/ $v = f(R)g(t) \therefore \frac{dv}{dt} = f\frac{dg}{dt}$, $\frac{dv}{dR} = gf' \therefore \frac{d^2v}{dR^2} = gf''$ is

$g(t) = e^{-int} \therefore \frac{dg}{dt} = -ine^{-int} = -ing \therefore -ingf = \nu(gf'' + \frac{1}{R}gf' - \frac{1}{R^2}fg)$

$-inf = \nu f'' + \frac{\nu}{R}f' - \frac{1}{R^2}\nu f \therefore 0 = \nu f'' + \nu\frac{1}{R}f' + (in - \nu\frac{1}{R^2})f \therefore$

$0 = \nu R^2 f'' + \nu R f' + (inR^2 - \nu)f \therefore 0 = R^2 f'' + Rf' + (\frac{in}{\nu}R^2 - 1)f \therefore$

$\alpha^2 = \frac{in}{\nu} \therefore 0 = R^2 f'' + Rf' + (\alpha^2 R^2 - 1)f \therefore$

is $S = \alpha R \therefore \frac{dS}{dR} = \alpha \therefore$

$\frac{d}{dR} = \alpha\frac{d}{dS} \therefore \frac{d^2}{dR^2}\ \alpha^2\frac{d^2}{dS^2} \therefore 0 = R^2\alpha^2\frac{d^2}{dS^2}f + R\alpha\frac{d}{dS}f + (\alpha^2R^2 - 1)f \therefore$

$S^2\frac{d^2f}{dS^2} + S\frac{df}{dS} + (S^2 - 1)f = 0 \therefore$

Sols: $f = J_1$, $f = Y_1 \therefore$

GS: $f = AJ_1 + BY_1$

\week 4/

\1a/ $\underline{\omega} = \nabla\times\underline{u} \therefore \underline{u} = v(R,t)\hat{\underline{\phi}}$ in cylindrical polars $(R, \phi, z) \therefore$

$$\underline{\omega} = \nabla\times\underline{u} = \frac{1}{R}\begin{vmatrix}\hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ u_R & Ru_\phi & F_z\end{vmatrix} = \frac{1}{R}\begin{vmatrix}\hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & Rv & 0\end{vmatrix} =$$

$$\frac{1}{R}\left\{\hat{\underline{R}}\left[\partial_\phi(0) - \partial_z(Rv)\right] - R\hat{\underline{\phi}}\left[\partial_R(0) - \partial_z(0)\right] + \hat{\underline{z}}\left[\partial_R(Rv) - \partial_\phi(0)\right]\right\} =$$

$$\frac{1}{R}\left\{\hat{\underline{R}}[0-0] - R\hat{\underline{\phi}}[0] + \hat{\underline{z}}\left[v\frac{\partial R}{\partial R} + R\frac{\partial v(R,t)}{\partial R}\right]\right\} = \frac{1}{R}\left\{\hat{\underline{z}}v + R\frac{\partial v(R,t)}{\partial t}\hat{\underline{z}}\right\} =$$

$\hat{\underline{z}}\left(\frac{1}{R}v(R,t) + \frac{\partial v(R,t)}{\partial t}\right) = \hat{\underline{z}}\omega = \underline{\omega} = \hat{\underline{z}}\omega(R,t)$,

$\omega = \frac{1}{R}v(R,t) + \frac{\partial v(R,t)}{\partial t} = \frac{1}{R}\frac{\partial}{\partial R}(Rv(R,t)) \therefore \omega = \omega(R,t)$ only

$$\underline{\omega} = \nabla\times\underline{u} = \frac{1}{R}\begin{vmatrix}\hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & Rv & 0\end{vmatrix} = \frac{1}{R}\left\{\hat{\underline{R}}\cdot 0 - R\hat{\underline{\phi}}\cdot 0 + \hat{\underline{z}}\partial_R(Rv)\right\} = \frac{1}{R}\partial_R(Rv)\hat{\underline{z}} = \omega\hat{\underline{z}} = \underline{\omega}$$

$\therefore \omega = \omega(R,t)$ any

\1b/ Consider each of the terms in the N-S equation:

\Week 24/ $\partial_t \underline{u} = \partial_t (v\hat{\underline{\phi}}) = \hat{\underline{\phi}}\, \partial_t v = \hat{\underline{\phi}}\, \partial_t v(R,t)$ ∴ $\begin{vmatrix} \hat{R} & \hat{\phi} & \hat{z} \\ u_R & u_\phi & u_z \\ \omega_R & \omega_\phi & \omega_z \end{vmatrix} = \begin{vmatrix} \hat{R} & \hat{\phi} & \hat{z} \\ 0 & v & 0 \\ 0 & 0 & \omega \end{vmatrix}$

$\underline{u} \times \underline{\omega} = \underline{u} \times (\underline{\omega}) = \underline{u} \times (\omega(R,t)\hat{\underline{z}}) = (v(R,t)\,\hat{\underline{\phi}}) \times (\omega(R,t)\hat{\underline{z}}) =$

$= \hat{\underline{R}}\, v\omega = \hat{\underline{R}}\, v(R,t)\,\omega(R,t)$ ∴

$u^2 = |\underline{u}|^2 = (\sqrt{v(R,t)^2})^2 = (v(R,t))^2 = v^2$ ∴

$\nabla\left(\frac{p}{\rho} + \frac{v^2}{2}\right) = \frac{\partial}{\partial R}\left(\frac{p}{\rho} + \frac{v^2}{2}\right)\hat{\underline{R}} + \frac{1}{R}\frac{\partial}{\partial \phi}\left(\frac{p}{\rho} + \frac{v^2}{2}\right)\hat{\underline{\phi}} + \frac{\partial}{\partial z}\left(\frac{p}{\rho} + \frac{v^2}{2}\right)\hat{\underline{z}}$

$= \frac{\partial}{\partial R}\left(\frac{p}{\rho} + \frac{v^2}{2}\right)\hat{\underline{R}} + \frac{1}{R}\frac{\partial}{\partial \phi}\left(\frac{p}{\rho}\right)\hat{\underline{\phi}} + \frac{\partial}{\partial z}\left(\frac{p}{\rho}\right)\hat{\underline{z}}$ ∴

$\nabla \times \underline{\omega} = \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\phi} & \hat{z} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & R(0) & \omega \end{vmatrix} = \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\phi} & \hat{z} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & 0 & \omega \end{vmatrix} = \frac{1}{R}\hat{\underline{R}}\,\partial_\phi \omega - \frac{1}{R}R\hat{\underline{\phi}}\,\partial_R \omega =$

$0 - \hat{\underline{\phi}}\,\partial_R \omega(R,t) = -\hat{\underline{\phi}}\frac{\partial \omega(R,t)}{\partial R}$ ∴ taking all contributions in the ϕ direction:

$\partial_t v(R,t) - 0 = -\frac{1}{R}\frac{\partial}{\partial \phi}\left(\frac{p}{\rho}\right) + \nu\frac{\partial \omega(R,t)}{\partial R} = \frac{\partial v(R,t)}{\partial t} = \nu\frac{\partial \omega(R,t)}{\partial R}$

∴ $\frac{\partial p}{\partial \phi} = 0$

\1b/ N-S: $\partial_t = \partial_t(v\hat{\underline{\phi}}) = \hat{\underline{\phi}}\,\partial_t v$ ∴ $\underline{u} \times \underline{\omega} = \begin{vmatrix} \hat{R} & \hat{\phi} & \hat{z} \\ 0 & v & 0 \\ 0 & 0 & \omega \end{vmatrix} = v\omega\hat{\underline{R}}$

$\nabla\left(\frac{p}{\rho} + \frac{v^2}{2}\right) = \frac{\partial}{\partial R}\left(\frac{p}{\rho} + \frac{v^2}{2}\right)\hat{\underline{R}} + \frac{1}{R}\frac{\partial}{\partial \phi}\left(\frac{p}{\rho} + \underbrace{\cancel{\frac{v^2}{2}}}_{\text{indep of } \phi}\right)\hat{\underline{\phi}} + \frac{\partial}{\partial z}\left(\frac{p}{\rho} + \underbrace{\cancel{\frac{v^2}{2}}}_{\text{indep of } z}\right)\hat{\underline{z}}$

$\nabla \times \underline{\omega} = \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\phi} & \hat{z} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & 0 & \omega \end{vmatrix} = -\frac{1}{R}R\hat{\underline{\phi}}\frac{\partial \omega}{\partial R}$ ∵ $\omega = \omega(R,t)$ only

taking all contributions in the ϕ direction:

$\frac{\partial v}{\partial t} - 0 = -\frac{1}{R}\frac{\partial}{\partial \phi}\left(\frac{p}{\rho}\right) + \nu\frac{\partial \omega}{\partial R}$ ∴ $\frac{\partial v}{\partial t} = \nu\frac{\partial \omega}{\partial R}$ ∴ $\frac{dp}{d\phi} = 0$

\1c/ $v(R,t) = \frac{\Gamma_0 F(\xi)}{2\pi R}$, $\Gamma_0 = \text{const}$, $\xi = R/\sqrt{4\nu t}$ ∴

$\omega = \frac{1}{R}\frac{\partial}{\partial R}(Rv) = \frac{1}{R}\frac{\partial}{\partial R}\left(\frac{\Gamma_0 F(\xi)}{2\pi}\right) = \frac{1}{R}\frac{\Gamma_0}{2\pi}\frac{\partial}{\partial R}(F(\xi)) = \frac{\Gamma_0}{2\pi R}\frac{\partial F}{\partial \xi}\frac{\partial \xi}{\partial R} =$

$\frac{\Gamma_0}{2\pi R}\frac{dF}{d\xi}\frac{d\xi}{dR} = \frac{\Gamma_0}{2\pi R}F'\frac{\partial(R/\sqrt{4\nu t})}{\partial R} = \frac{\Gamma_0}{2\pi R}F'\frac{1}{\sqrt{4\nu t}}$ $\{R = \xi\sqrt{4\nu t}\}$ ∴

$\frac{\Gamma_0}{2\pi\xi\sqrt{4\nu t}}F'\frac{1}{\sqrt{4\nu t}} = \frac{\Gamma_0}{2\pi\xi 4\nu t}F' = \frac{\Gamma_0 F'(\xi)}{8\pi\xi\nu t} = \omega$

\1d/ $\frac{\Gamma_0 F'(\xi)}{8\pi\xi\nu t} = \omega$ ∴ $\frac{\partial v}{\partial t} = \nu\frac{\partial \omega}{\partial R}$

$v(R,t) = \frac{\Gamma_0 F(\xi)}{2\pi R}$ ∴

$\frac{\partial v(R,t)}{\partial t} = \frac{\Gamma_0}{2\pi R}\frac{\partial F(\xi)}{\partial t} = \frac{\Gamma_0}{2\pi R}\frac{\partial F(\xi)}{\partial \xi}\frac{\partial \xi}{\partial t} = \frac{\Gamma_0}{2\pi R}\frac{dF(\xi)}{d\xi}\,d\left(\frac{R}{\sqrt{4\nu}}t^{-1/2}\right) =$

$\frac{\Gamma_0}{2\pi R}(F')\left(-\frac{1}{2}\right)t^{-3/2}\frac{R}{\sqrt{4\nu}} = \frac{\Gamma_0}{2\pi R}(F')\left(-\frac{1}{2}\right)\xi t^{-1}$ ∴

$\frac{\partial v}{\partial t} = -\frac{\Gamma_0}{4\pi R t}\xi F'$ ∴

$\frac{1}{\nu}\frac{\partial v}{\partial t} = \frac{\partial \omega}{\partial R} = -\frac{\Gamma_0}{4\pi R\nu t}\xi F'$

\Q1/

$$\frac{\partial \omega}{\partial R} = \frac{\partial}{\partial R}\left(\frac{\Gamma_0 F'(\xi)}{8\pi \xi \nu t}\right) = \frac{\Gamma_0}{8\pi \nu t}\frac{\partial}{\partial R}\left(\frac{F'}{\xi}\right) = \frac{\Gamma_0}{8\pi \nu t}\frac{\partial}{\partial \xi}\left(\frac{F'}{\xi}\right)\frac{\partial \xi}{\partial R} =$$

$$\frac{\Gamma_0}{8\pi \nu t}\frac{\partial}{\partial \xi}\left(\frac{F'}{\xi}\right)\frac{\partial}{\partial R}\left(\frac{R}{\sqrt{4\nu t}}\right) = \frac{\Gamma_0}{8\pi \nu t}\frac{\partial}{\partial \xi}\left(\frac{F'}{\xi}\right)\frac{\partial \xi}{\partial R} = \frac{\Gamma_0}{8\pi \nu t}\frac{\partial}{\partial \xi}\left(\frac{F'}{\xi}\right)\frac{\partial \xi}{\partial R} =$$

$$\frac{\Gamma_0}{8\pi \nu t}\frac{\partial}{\partial \xi}\left(\frac{F'}{\xi}\right)\frac{1}{\sqrt{4\nu t}} \quad \therefore \frac{\partial v}{\partial t} = \nu\frac{\partial \omega}{\partial R} \therefore$$

$$-\frac{\Gamma_0}{4\pi R t}\xi F' = \nu\frac{\Gamma_0}{8\pi \nu t}\frac{\partial}{\partial \xi}\left(\frac{F'}{\xi}\right)\frac{1}{\sqrt{4\nu t}} = \frac{\Gamma_0}{8\pi t}\frac{\partial}{\partial \xi}\left(\frac{F'}{\xi}\right)\frac{1}{\sqrt{4\nu t}} \therefore$$

$$-\frac{1}{R}\cancel{\xi F'} \cancel{=} \nu\frac{\Gamma_0}{8\pi \nu t}\frac{\partial}{\partial \xi}\left(\frac{F'}{\xi}\right)\frac{1}{\sqrt{4\nu t}} = \frac{\Gamma_0}{8\pi t}\frac{\partial}{\partial \xi}\left(\frac{F'}{\xi}\right)\frac{1}{\sqrt{4\nu t}}$$

$$-\frac{1}{R}\xi F' = \frac{1}{2}\frac{\partial}{\partial \xi}\left(\frac{F'}{\xi}\right)\frac{1}{\sqrt{4\nu t}}$$

$$-\frac{\sqrt{4\nu t}}{R}\xi 2F' = \frac{\partial}{\partial \xi}\left(\frac{F'}{\xi}\right) = -\frac{1}{\xi}\xi 2F' = -2F' \therefore$$

$$2F' + \frac{\partial}{\partial \xi}\left(\frac{F'}{\xi}\right) = 0 \therefore$$

$\Gamma = 2\pi R v = \Gamma_0 F(\xi) \to \Gamma_0$ as $R \to \infty$ $\therefore$

$F(\xi) \to 1$ as $\xi \to \infty$

and $\Gamma_0 F(\xi) \to 0$ as $R \to 0$ $\therefore$ $F(0) \to 0$ as $\xi \to 0$

\1c/ $v(R,t) = \frac{\Gamma_0 F(\xi)}{2\pi R}$, where $\Gamma_0 = \text{const}$, $\xi = \frac{R}{\sqrt{4\nu t}}$.

$$\omega = \frac{1}{R}\frac{\partial}{\partial R}(Rv) = \frac{1}{R}\frac{\partial}{\partial R}\left(\frac{\Gamma_0 F(\xi)}{2\pi}\right) = \frac{\Gamma_0}{2\pi R}\frac{dF}{d\xi}\frac{d\xi}{dR}$$ by the chain rule

$$= \frac{\Gamma_0}{2\pi R}F'\frac{1}{\sqrt{4\nu t}} = \frac{\Gamma_0 F'}{2\pi \xi 4\nu t} = \frac{\Gamma_0 F'}{8\pi \nu t \xi}$$

\1d/ $\frac{\partial v}{\partial t} = \nu\frac{\partial \omega}{\partial R}$ $\quad \frac{\partial v}{\partial t} = \frac{\Gamma_0}{2\pi R}\frac{\partial F}{\partial t} = \frac{\Gamma_0}{2\pi R}\frac{\partial F}{\partial \xi}\frac{\partial \xi}{\partial t}$.

$$\frac{\partial \xi}{\partial t} = \frac{\partial}{\partial t}\left(\frac{R t^{-1/2}}{\sqrt{4\nu}}\right) = \frac{R}{\sqrt{4\nu}}\left(-\frac{1}{2}t^{-3/2}\right) = -\frac{1}{2}\frac{\xi}{t} \therefore$$

$$\frac{\partial v}{\partial t} = -\frac{\Gamma_0}{4\pi R t}F'\xi,$$

$$\frac{\partial \omega}{\partial R} = \frac{\Gamma_0}{8\pi \nu t}\frac{\partial}{\partial R}\left(\frac{F'}{\xi}\right) = \frac{\Gamma_0}{8\pi \nu t}\frac{\partial}{\partial \xi}\left(\frac{F'}{\xi}\right)\frac{\partial \xi}{\partial R} = \frac{\Gamma_0}{8\pi \nu t}\frac{\partial}{\partial \xi}\left(\frac{F'}{\xi}\right)\frac{1}{\sqrt{4\nu t}}$$

$$-\frac{\Gamma_0}{4\pi R \cancel{t}}F'\xi = \frac{\nu\Gamma_0}{8\pi \nu \cancel{t}}\frac{1}{\sqrt{4\nu t}}\frac{\partial}{\partial \xi}\left(\frac{F'}{\xi}\right) \therefore$$

$$2F' + \frac{d}{d\xi}\left(\frac{F'}{\xi}\right) = 0$$

$\Gamma = 2\pi R v = \Gamma_0 F(\xi) \to \Gamma_0$ as $R \to \infty$ $\therefore$

$F(\xi) \to 1$ as $\xi \to \infty$,

$\Gamma_0 F(\xi) \to 0$ as $R \to 0$ $\therefore$

$F(0) \to 0$ as $\xi \to 0$

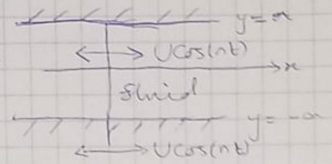
\2/ $$\left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)^2 = \left(\int_{-\infty}^{\infty} e^{-x^2}dx\right)\left(\int_{-\infty}^{\infty} e^{-y^2}dy\right) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-(x^2+y^2)}dx\,dy$$

$\{r^2 = x^2+y^2,\ r\sin\theta = y,\ r\cos\theta = x\}$ $$= 2\pi\int_{\theta=0}^{2\pi}\int_{r=0}^{\infty} e^{-r^2} r\,dr\,d\theta$$

\week 4\ $J = \begin{vmatrix} \partial_x r & \partial_y r \\ \partial_x \theta & \partial_y \theta \end{vmatrix} \ldots$ $\therefore 2\pi \int_{r=0}^{\infty} r e^{-r^2} dr = -\frac{2\pi}{2}\left[e^{-r^2}\right]_0^{\infty} = -\pi(0-1) = \pi$

$\therefore \int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$

\week 5\ \10\



\1b\ NS: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho \underline{g} + \mu\nabla^2\underline{u}$

but no boundary forces $\therefore$ no gravity $\therefore$

$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \mu\nabla^2\underline{u}$, $\underline{u} = u\hat{\underline{i}}$ $\therefore$

$u = u(y,t)\hat{\underline{i}}$ gives $\frac{\partial \underline{u}}{\partial t} = \frac{\partial u}{\partial t}\hat{\underline{i}}$,

$\nabla^2\underline{u} = \nabla^2 u\hat{\underline{i}}$,

$\underline{u}\cdot\nabla\underline{u} = u\frac{\partial}{\partial x}(u\hat{\underline{i}}) = u\frac{\partial u}{\partial x}\hat{\underline{i}} = 0$ $\because$ u is independent of x

$\therefore \frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = \frac{\partial u}{\partial t}\hat{\underline{i}} + 0 = \frac{\partial u}{\partial t}\hat{\underline{i}}$ $\therefore$ only the x-component of the NS is nontrivial $\therefore$ $\frac{\partial p}{\partial x} = 0$ $\therefore$ pressure is independent of x

$\therefore \rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu\nabla^2 u$ $\therefore \rho\frac{\partial u}{\partial t} = \mu\nabla^2 u$ $\therefore$

$\frac{\partial u}{\partial t} = \frac{\mu}{\rho}\nabla^2 u = \frac{\mu}{\rho}\nabla^2 u(y,t) \Rightarrow \nu\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$

\1b\ NS: $\rho\left\{\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right\} = -\nabla p + \underbrace{\cancel{\rho \underline{g}}}_{\text{no boundary forces}} + \mu\nabla^2\underline{u}$

$\underline{u} = u(y,t)\hat{\underline{i}}$ $\therefore$ $\frac{\partial \underline{u}}{\partial t} = \frac{\partial u}{\partial t}\hat{\underline{i}}$, $\nabla^2\underline{u} = \nabla^2 u\hat{\underline{i}}$

$\underline{u}\cdot\nabla\underline{u} = u\frac{\partial}{\partial x}(u\hat{\underline{i}}) = u\frac{\partial u}{\partial x}\hat{\underline{i}} = \underline{0}$ $\because$ u is independent of x

$\therefore$ only the x-component of the N-S is nontrivial

$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu\nabla^2 u$ but pressure in dependence of x given $\therefore$

$\frac{\partial u}{\partial t} = \nu\frac{\partial^2 u}{\partial y^2}$.

\1b\ ~~take 2u=R~~ NS: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho \underline{g} + \mu\nabla^2\underline{u}$ $\therefore$

no boundary forces $\therefore$ no gravity $\therefore$

$\rho(\underline{u}_t + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \mu\nabla^2\underline{u}$, $\underline{u} = u\hat{\underline{i}}$ $\therefore$

$\underline{u} = u(y,t)\underline{i}$ gives $\frac{\partial \underline{u}}{\partial t} = \frac{\partial u}{\partial t}\underline{i}$, $\nabla^2\underline{u} = \nabla u\,\underline{i}$

$\underline{u}\cdot\nabla\underline{u} = u\frac{\partial}{\partial x}(u\underline{i}) = u\frac{\partial u}{\partial x}\underline{i} = 0$ $\because$ u is indep of x $\therefore$

$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = \frac{\partial u}{\partial t}\underline{i} + 0 = \frac{\partial u}{\partial t}\hat{\underline{i}}$ $\therefore$ only the x component of the N-S is non trivial $\therefore$ $\frac{\partial p}{\partial x} = 0$ $\therefore$ pressure is indep of x $\therefore$

$\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu\nabla^2 u$ $\therefore$ $\rho\frac{\partial u}{\partial t} = \mu\nabla^2 u$ $\therefore$

$\frac{\partial u}{\partial t} = \frac{\mu}{\rho}\nabla^2 u = \frac{\mu}{\rho}\nabla^2 u(y,t) = \nu\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$

\16/ NS: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla P + \rho \underline{g} + \mu\nabla^2\underline{u}$ ∴

no boundary forces ∴ no gravity ∴

$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla P + \mu\nabla^2\underline{u}$, $\underline{u} = u\hat{i}$.

$\underline{u} = u(y,t)\underline{i}$ ∴ $\frac{\partial \underline{u}}{\partial t} = \frac{\partial u}{\partial t}\underline{i}$, $\nabla^2\underline{u} = \nabla^2 u\underline{i} = \frac{\partial}{\partial y^2}u\,\underline{i}$

$\underline{u}\cdot\nabla = u\frac{\partial}{\partial x}$ ∴ $\underline{u}\cdot\nabla\underline{u} = u\frac{\partial}{\partial x}(u\hat{i}) = u\frac{\partial u}{\partial x}\hat{i} = 0$. u is indep of x ∴

$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = \frac{\partial u}{\partial t}\underline{i} + 0 = \frac{\partial u}{\partial t}\hat{i}$ ∴ only the x comp is nontrivial ∴

$\frac{\partial P}{\partial x} = 0$ ∵ pressure is indep of x ∴

$\rho\frac{\partial u}{\partial t} = -\frac{\partial P}{\partial x} + \mu\nabla^2 u$ ∴ $\rho\frac{\partial u}{\partial t} = \mu\nabla^2 u$ ∴

$\frac{\partial u}{\partial t} = \frac{\mu}{\rho}\nabla^2 u = \frac{\mu}{\rho}\nabla^2 u(y,t) = \nu\frac{\partial^2 u}{\partial y^2} = \frac{\partial u}{\partial t}$

\1C/ U, n const ∴ $u(y,t) = \mathbb{R}[UF(y)e^{int}]$ ∴

$\frac{\partial u}{\partial t} = \partial_t u = \mathbb{R}[UF(y)ine^{int}]$ ∴

$\frac{\partial^2 u}{\partial y^2} = \partial_{yy}u = \mathbb{R}[UF''(y)e^{int}]$ ∴

$\mathbb{R}[UF(y)ine^{int}] = \mathbb{R}[UF''(y)e^{int}]\nu$ ∴

$UF(y)ine^{int} = UF''(y)e^{int}\nu$ ∴ $inF(y) = \nu F''(y)$ ∴

$F''(y) - \frac{n}{\nu}iF(y) = 0$ is ODE

BCs: $u(y=-a,t) = U\cos(nt)$ ∴

$\mathbb{R}[UF(-a)e^{int}] = U\cos(nt) = \mathbb{R}[UF(-a)(\cos(nt) + i\sin(nt))] =$

$UF(-a)\cos(nt) = U\cos(nt)$ ∴

$F(-a) = 1$

\1C/ take $u = \mathbb{R}[UF(y)e^{int}]$ (U, n const) ∴

$u_t = \partial_t u = \mathbb{R}[UFine^{int}]$, $\partial_{yy}u = \mathbb{R}[UF''e^{int}]$ ∴ $u_t = \nu u_{yy}$ ∴

$inF = \nu F''$ ∴

$F'' - \frac{in}{\nu}F = 0$ is ODE

BCs: $u(y=-a,t) = U\cos(nt)$ ∴ $\mathbb{R}[UF(-a)e^{int}] = U\cos(nt)$ ∴

$\cancel{U}F(-a)\cos(nt) = \cancel{U}\cos(nt)$ ∴ $F(-a) = 1$ ∴

for $y = a$: $u(y=a,t) = U\cos(nt)$ ∴

$\mathbb{R}[UF(a)e^{int}] = U\cos(nt)$ ∴

$\cancel{U}F(a)\cos(nt) = \cancel{U}\cancel{\cos(nt)}$ ∴ $F(a) = 1$

\week 5/ \1d/ $F''(y) - \frac{n}{\nu} i F(y) = 0 \therefore$ let $F = e^{my} \therefore$

$F' = m e^{my} \quad \therefore \quad F'' = m^2 e^{my} \quad \therefore \quad m^2 e^{my} - \frac{n}{\nu} i e^{my} = 0 \quad \therefore \quad m^2 - \frac{n}{\nu} i \therefore$

$\Rightarrow m^2 = \frac{n}{\nu} i \quad \therefore \quad m = \pm\sqrt{\frac{n}{\nu} i} \quad \therefore$

$\frac{1}{\delta} = \sqrt{\frac{n}{2\nu}} \quad \therefore \quad \frac{i}{\delta} = \sqrt{\frac{n}{2\nu}}\, i \quad \therefore$

$\frac{(1+i)}{\delta} = \frac{1}{\delta} + \frac{i}{\delta} = \sqrt{\frac{n}{2\nu}} + \sqrt{\frac{n}{2\nu}}\, i = (1+i)\sqrt{\frac{n}{2\nu}} = \sqrt{\frac{n}{2\nu}(1+i)^2} = \sqrt{\frac{n}{\nu} i} \quad \therefore$

$m \pm\sqrt{\frac{n}{\nu} i} = \pm\sqrt{\frac{n}{2\nu} 2i} = \pm \frac{1}{\delta}\sqrt{2i} \quad \therefore$

$\arg(2i) = \frac{\pi}{2} \quad \therefore \quad |2i| = 2 \quad \therefore \quad \sqrt{2i} = \arg(\sqrt{2i}) = \frac{\pi}{4} \quad \therefore \quad |\sqrt{2i}| = \sqrt{2} \quad \therefore$

$\arg(2i) = \arg(\sqrt{2i} \cdot \sqrt{2i}) = 2\arg(\sqrt{2i}) = \frac{\pi}{2} \therefore \sqrt{2i} = \sqrt{2}\left(\cos(\frac{\pi}{4}) + i \sin\frac{\pi}{4}\right) = 1+i$

$\therefore \pm \frac{1}{\delta}(1+i) = m \therefore$

$F = Ae^{(1+i)y/\delta} + Be^{-(1+i)y/\delta} \quad \therefore$ apply BCs $\therefore$

$F(a) = 1 = Ae^{(1+i)a/\delta} + Be^{-(1+i)a/\delta} = Ae^{a/\delta}e^{ia/\delta} + Be^{-a/\delta}e^{-ia/\delta} =$

$(Ae^{a/\delta} + Be^{-a/\delta})\cos\frac{a}{\delta} + i(Ae^{a/\delta} - Be^{-a/\delta})\sin(\frac{a}{\delta}) = 1$

$F(-a) = 1 = Ae^{(1+i)-\frac{a}{\delta}} + Be^{-(1+i)-\frac{a}{\delta}} = (Ae^{-a/\delta} + Be^{a/\delta})\cos\frac{a}{\delta} + i(-Ae^{-a/\delta} + Be^{a/\delta})\sin\frac{a}{\delta} = 1$

$\therefore A = B = (e^{(1+i)\frac{a}{\delta}} + e^{-(1+i)\frac{a}{\delta}})^{-1} = (2 \cdot \frac{1}{2}(e^{(1+i)\frac{a}{\delta}} + e^{-(1+i)\frac{a}{\delta}}))^{-1} = (2\cosh((1+i)\frac{a}{\delta}))^{-1} \therefore$

$A = B = \frac{2}{\cosh[(1+i)\frac{a}{\delta}]}$

$F = \frac{2}{\cosh((1+i)\frac{a}{\delta})} e^{(1+i)y/\delta} + \frac{2}{\cosh((1+i)\frac{a}{\delta})} e^{-(1+i)y/\delta} =$

$\frac{2}{\cosh((1+i)\frac{a}{\delta})}\left(e^{(1+i)\frac{a}{\delta}} + e^{-(1+i)\frac{a}{\delta}}\right)$

$F = \frac{\cosh[(1+i)y/\delta]}{\cosh[(1+i)a/\delta]} \quad \therefore \quad \delta = \sqrt{\frac{2\nu}{n}} \quad \therefore \quad q = 1+i \therefore$

\1d/ let $F = e^{my} \quad \therefore m^2 = \frac{in}{\nu} \quad \therefore m = \pm\sqrt{\frac{in}{\nu}} = \pm\frac{(1+i)}{\delta}$; $\delta = \sqrt{\frac{2\nu}{n}} \quad \therefore$

$F = Ae^{(1+i)y/\delta} + Be^{-(1+i)y/\delta} \therefore$

$A = B = \frac{1}{e^{(1+i)a/\delta} + e^{-(1+i)a/\delta}} = \frac{2}{\cosh[(1+i)\frac{a}{\delta}]} \therefore$

$F = \frac{\cosh[(1+i)y/\delta]}{\cosh[(1+i)a/\delta]}$, ~~$\delta$~~ $\delta = \sqrt{\frac{2\nu}{n}}$, $q = 1+i$

\1e/ $F(y) \approx \exp\left(-\frac{(1+i)(y+a)}{\delta}\right)$ for $y \approx -a$, large $a/\delta \therefore$

$u(y \approx -a) \approx \mathbb{R}\left[U\exp\left(\frac{-(1+i)(y+a)}{\delta}\right)\exp(int)\right] =$

$U\exp\left(-\frac{(y+a)}{\delta}\right)\mathbb{R}\left[\exp\left(-i\frac{(y+a)}{\delta} + int\right)\right] = U\exp\left(-\frac{(y+a)}{\delta}\right)\cos\left(-\frac{(y+a)}{\delta} + nt\right) =$

$U\exp\left(-\frac{(y+a)}{\delta}\right)\cos\left(-\frac{(y+a)}{\delta} + nt\right)$

\1e/ $F(y) \approx \exp\left(-\frac{(1+i)(y+a)}{\delta}\right)$ for ~~dy~~ $y \approx -a$, large $\frac{a}{\delta} \therefore$

$u(y \approx -a) \approx \mathbb{R}\left[UF(y \approx -a)e^{int}\right] = \mathbb{R}\left[U\exp\left(-\frac{(1+i)(y+a)}{\delta}\right)\exp(int)\right] =$

$U\exp\left(-\frac{(y+a)}{\delta}\right)\mathbb{R}\left[\exp\left(-\frac{i(y+a)}{\delta} + int\right)\right] = U\exp\left(-\frac{(y+a)}{\delta}\right)\cos\left(-\frac{(y+a)}{\delta} + nt\right)$

$\lambda \mid \delta \quad u(y=0, t) = \mathbb{R}\left[UF(0)e^{int}\right] \quad , F(0) = \frac{\cosh(0)}{\cosh\left(\frac{(1+i)a}{\delta}\right)} = \frac{1}{\frac{1}{2}\left[e^{(1+i)a/\delta} + e^{-(1+i)\frac{a}{\delta}}\right]}$

$\therefore F(0) = \frac{2}{e^{(1+i)a/\delta} + e^{-(1+i)a/\delta}} \cdot \frac{(e^{(1-i)a/\delta} + e^{-(1-i)a/\delta})}{(e^{(1-i)a/\delta} + e^{-(1-i)a/\delta})} \therefore$

$F(0) = \frac{e^{(1-i)a/\delta} + e^{-(1-i)a/\delta}}{\cosh\left(2\frac{a}{\delta}\right) + \cos\left(\frac{2a}{\delta}\right)} \quad \therefore u(0,t) = \frac{U\mathbb{R}\left[e^{(1-i)a/\delta + int} + e^{-(1-i)\frac{a}{\delta} + int}\right]}{\cosh\left(\frac{2a}{\delta}\right) + \cos\left(\frac{2a}{\delta}\right)}$

$= \frac{U}{\cos\left(\frac{2a}{\delta}\right) + \cos\left(\frac{2a}{\delta}\right)}\left[e^{\frac{a}{\delta}}\cos\left(nt - \frac{a}{\delta}\right) + e^{-\frac{a}{\delta}}\cos\left(nt + \frac{a}{\delta}\right)\right]$

velocity $\underline{u}(\underline{x}, t) = (u, v, w) = u(x,y,z,t)\hat{i} + v(x,y,z,t)\hat{j} + w(x,y,z,t)\hat{k} \therefore$

pressure $p = p(x,y,z,t)$: force $-\nabla p$

density $\rho = \rho(x,y,z,t)$

∇p is perpendicular to level surfaces $p =$ constant

Flux rate of flow of the quantity through the surface

rate that volume crosses dS $\therefore$ volume flux:

$(\hat{n} \cdot \underline{u})\,dS = \underline{u} \cdot d\underline{S}$

Mass flux: $\rho\underline{u} \cdot d\underline{S} \therefore$

total mass flux: $\rho\int_S \underline{u} \cdot d\underline{S}$

$\frac{D\rho}{Dt} + \rho\nabla \cdot \underline{u} = 0 \quad , \frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u} \cdot \nabla \quad \therefore \partial_t\rho + \nabla \cdot (\rho\underline{u}) = 0,$

$\frac{D\rho}{Dt} = \partial_t\rho + u \cdot \nabla\rho = 0$

streamlines $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w} \quad , u, v, w \neq 0$

if $u = (\underline{x}, t)$ set $t =$ const

only stagnation points ($\underline{u} = \underline{0}$) can streamlines cross

expect steady motion velocity $= (\underline{x}, t) \therefore$

$\underline{r}(\underline{x}_0, t) \therefore \frac{d\vec{r}}{ds}$ must be parallel to velocity $\therefore \frac{d\vec{r}}{ds} = \lambda\underline{u}$

$\underline{F} = m\underline{a} \quad \therefore \underline{x} \to \underline{x} + \delta\underline{x} \quad , t + \delta t \therefore$

$\underline{u}(\underline{x} + \delta\underline{x}, t + \delta t) - \underline{u}(\underline{x}, t) \approx \cancel{\underline{u}(\underline{x},t) + \delta x_1\frac{\partial u}{\partial x_1}} + \partial$

$u(\underline{x}, t) + \delta x_1\frac{\partial u}{\partial x_1} + \delta x_2\frac{\partial u}{\partial x_2} + \delta x_3\frac{\partial u}{\partial x_3} + \delta t\frac{\partial u}{\partial t} - u(x, t)$

$\frac{Du}{Dt} = \frac{\partial u}{\partial t} + \underline{u} \cdot \nabla\underline{u}$ is acceleration

Pressure force: $\delta V(-\nabla p)$

gravitational force $\delta M g = \rho \delta V g$

→ viscous force $\delta V \mu \nabla^2 \underline{u}$

$\frac{\partial \rho}{\partial t} = 0$, $\nabla \rho = 0$

$\rho\left(\frac{\partial u}{\partial t} + \underline{u} \cdot \nabla \underline{u}\right) = -\nabla p + \rho g + \mu \nabla^2 \underline{u}$, $\nabla \cdot \underline{u} = 0$

$\delta_{ij} = \begin{cases} 1 & \text{if } i=j \\ 0 & \text{if } i \neq j \end{cases}$ δ_{ij} is symmetric ($\delta_{ij} = \delta_{ij}$)

$(\underline{a} \times \underline{b})_i = \varepsilon_{ijk} a_j b_k$ cross product

$\underline{a} \cdot (\underline{b} \times \underline{c}) = a_i (\underline{b} \times \underline{c})_i = a_i \varepsilon_{ijk} b_j c_k = \varepsilon_{ijk} a_i b_j c_k =$

$\varepsilon_{jki} a_i b_j c_k = b_j \varepsilon_{jki} c_k a_i = b_j \cdot (\underline{c} \times \underline{a})_j = b_i \cdot (\underline{c} \times \underline{a})_i = [\underline{b} \cdot (\underline{c} \times \underline{a})]_i$

$\nabla \cdot \underline{u} = \frac{\partial u}{\partial x_i}$ $[\nabla \times \underline{u}]_i = \varepsilon_{ijk} \nabla_j u_k = \varepsilon_{ijk} \frac{\partial}{\partial x_j}(u_k) = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}$

$\nabla^2 = \frac{\partial^2}{\partial x_i \partial x_i}$ $\therefore \nabla \phi = \frac{\partial^2 \phi}{\partial x_i \partial x_i}$, $[\nabla^2 F]_j = \frac{\partial^2 F_j}{\partial x_i \partial x_i}$

$\underline{u} \cdot \nabla \underline{u}$ vanishes,

let $\underline{u} = u(y,t)\hat{\underline{i}} \neq 0$ $\therefore$ y comp : $p = -\rho g y + f(x,t)$

x comp : $\rho \frac{\partial u}{\partial t} - \mu \frac{\partial^2 u}{\partial y^2} = -\frac{\partial p}{\partial x}$, $-\frac{\partial p}{\partial x} = G(t)$

$\therefore \frac{\partial p}{\partial x}$ must be indep of x

No slip , regularity , Free surface : no stress $\mu \frac{\partial u}{\partial y} = 0$

if $\Omega_1 = \Omega_2 = \Omega$, $B = 0$, $A = \Omega$ $\therefore V = R\Omega$

if $A = 0$ let $B = \frac{\Gamma}{2\pi}$ $\therefore v = \frac{\Gamma}{2\pi R}$ $\therefore$

flow of a line vortex at the origin with circulation

$\oint_c \underline{u} \cdot d\underline{r} = \int_{\phi=0}^{2\pi} \underline{u} \cdot (R\, d\phi\, \hat{\underline{\phi}}) = \int_{\phi=0}^{2\pi} \frac{\Gamma}{2\pi R} R\, d\phi = \Gamma$

$\underline{u}(\underline{x} + \delta \underline{x}, t) = \underline{u}(x,t) + \delta \underline{u}$, $\delta \underline{u} \approx \delta x_j \frac{\partial u}{\partial x_j}$

$D_{ij} = \frac{\partial u_i}{\partial x_j}$ $\therefore D_{ij} = e_{ij} + \xi_{ij}$ $\therefore$

$e_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)$, $\xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$

total momentum : $\underline{P} = \int_{V(t)} \rho \underline{u}\, dV$ $\therefore \frac{dP}{dt} = \int_V \rho \frac{Du}{Dt} dV$

force on dS from fluid outside V: $\Sigma(\underline{x}, t, \hat{n})\, dS$,

$F_S = \int_S \Sigma(\underline{x}, t, \hat{\underline{n}})\, dS$

$\mathring{g} = \nabla(\underline{g} \cdot \underline{x}) \quad \therefore -\nabla p + \rho \underline{g} = -\nabla P \quad , P = p - \rho \underline{g} \cdot \underline{x}$

$y = f(a_1, a_2 \ldots a_k, b_1, b_2, \ldots, b_m)$

$[y] = [a_1]^p \cdots [a_k]^r \quad \therefore \qquad \Pi = \frac{y}{a_1^p \cdots a_k^r}$

$\Pi_1 = \frac{b_1}{a_1^{p_1} - \cdots a_k^{r_1}} \quad \ldots \quad \Pi_m = \frac{b_m}{a_1^{p_m} \cdots a_k^{r_m}}$

$\frac{\partial \underline{u}'}{\partial t} + (\underline{u}' \cdot \nabla') \underline{u}' = -\nabla' P' + Re^{-1} \nabla'^2 \underline{u}' \quad , Re^{-1} = \frac{\mu}{UL\rho} \quad , \therefore$

$R_e = \frac{\rho U L}{\mu}$

$[\varnothing] = L^\alpha M^\beta T^\gamma$

$\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-s^2)\, ds \, , \quad \operatorname{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-s^2)\, ds \, ,$

$\operatorname{erf}(0) = 0 \, , \quad \operatorname{erfc}(x \to \infty) = 0$

$\partial_t u = \nu \partial_{yy} u \quad , u = \mathcal{R}[\hat{u}(y) \exp(i \Omega t)]$

$\mathrm{is} \quad u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} = \nu \nabla^2 w \quad , \psi(x, y) \quad , \underline{u} = \nabla \times (\psi \, \hat{\underline{k}})$

$\frac{\delta}{L} << 1 \quad \therefore \quad u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = \frac{G}{\rho} + \nu \partial_{yy} u \quad , G = -\partial_x P$

$\mathrm{if} \quad \underline{u} = (U, 0) \qquad \underline{u} = U \hat{\underline{i}} \quad \therefore u = \partial_y \psi \, , \quad v = -\partial_x \psi$

$\partial_t \underline{u} + \underline{u} \cdot \nabla \underline{u} = -\frac{1}{\rho} \nabla + \nu \nabla^2 \underline{u} \quad , \qquad \nabla \cdot \underline{u} = 0 \quad , \nu = \mu / \rho$

$\underline{u} = \nabla \times \psi(\underline{k}) = (\partial_y \psi, -\partial_x \psi, 0) \quad \therefore$

$\nabla \cdot \underline{u} = 0 \quad \therefore \quad -\nabla p + \mu \nabla^2 (\nabla \times (\psi \underline{k})) = 0$

$\mathrm{if} \quad 0 = -\nabla p + \mu \nabla^2 \underline{u} \quad , \nabla \cdot \underline{u} \, 0 \, , \quad \underline{u} = G(x)$ $\therefore$ unique $\therefore$ $\underline{W} = u_1 - u_2$

$\therefore$ divergence thm

$d F_i = \sigma_{ij} \hat{n}_j \, dS = \left[-p \delta_{ij} + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] \hat{n}_j \, dS$

$\rho(\partial_t \underline{u} + \underline{u} \cdot \nabla \underline{u}) = -\nabla p + \rho \underline{g} + \mu \nabla^2 \underline{u} \quad , \underline{g} = -\nabla \Pi \quad \therefore$

$\partial_t \underline{u} + \underline{u} \cdot \nabla \underline{u} = -\nabla P + \nu \nabla^2 \underline{u} \quad , \quad P = \frac{p}{\rho} + \Pi \, , \quad \nu = \frac{\mu}{\rho}$

$D_{ij} = \frac{\partial u_i}{\partial x_j} = e_{ij} + \xi_{ij}$

$(\underline{\omega} \cdot \nabla \underline{u})_i = \underline{\omega} \cdot \nabla u_i = \omega_j \frac{\partial u_i}{\partial x_j} = \omega_j D_{ij}$

$\partial_t \underline{\omega} = \nabla \times (\underline{u} \times \underline{\omega}) + \nu \nabla^2 \underline{\omega} \quad , \quad \frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = 0 \, , \quad \partial_t \rho + \underline{u} \cdot \nabla \rho = 0$

$\partial_t \underline{u} + \underline{u} \cdot \nabla \underline{u} = -\nabla p + \rho \underline{g} + \mu \nabla^2 \underline{u}$

$\mathrm{is} \quad \frac{D \omega}{D t} = \frac{\partial \omega}{\partial t} + \underline{u} \cdot \nabla \underline{\omega} = \underline{\omega} \cdot \nabla \underline{u}$ for large $R_e = \frac{UL}{\nu}$

$\Gamma = \oint_C \underline{u} \cdot \underline{dr}$

$\int_S \underline{\omega} \cdot d\underline{S} = \int_S \underline{\omega} \cdot \hat{\underline{n}} \, dS = \int_S \nabla \times \underline{u} \cdot \hat{\underline{n}} \, dS = \oint_C \underline{u} \cdot d\underline{r} = \Gamma = \mathrm{Const}$

$T \propto \frac{1}{\omega}$, $\partial_{tt}\phi = c^2 \partial_{xx}\phi$, $\phi = f(x-ct) + g(x+ct)$

amplit a, wavenumber k , wavelength $= \lambda = \frac{2\pi}{k}$

frequency ω , argument $k(x-ct)$

$\partial_{tt}\phi = c^2 \nabla^2 \phi$, $\phi = a e^{i(\underline{k}\cdot\underline{r} - \omega t)}$, $\underline{k} = k\hat{\underline{n}} = \frac{\omega}{c}\hat{\underline{n}}$

$H = \frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2 + \Pi$

$\frac{\partial\phi}{\partial t} + \frac{1}{2}(u^2+v^2) + g\eta = 0$, $y = \eta(x,t)$

neglect higher order terms

$\eta = A\cos(kx - \omega t)$ speed $c = \frac{\omega}{k}$

dispersion relation $\omega = \omega(k)$ groupspeed $c_g = \frac{d\omega}{dk}$

$c_p = \frac{\omega}{k}$ phase speed

Week 7/ 1a/ $\underline{u} = U\hat{\underline{R}}$, $\nabla\cdot\underline{u} = \frac{1}{R}\frac{\partial}{\partial R}(RU) = 0 \therefore RU = \text{const} \therefore$

$U = \frac{C}{R}$ $\therefore C<0 \therefore U = -\frac{Q}{R}$, $Q>0$

1b/ $\underline{U} = U\hat{\underline{R}} = -\frac{Q}{\sqrt{x^2+y^2}}(\cos\theta\,\underline{i} + \sin\theta\,\underline{j}) \approx -\frac{Q}{x}\underline{i}$; $y \approx 0$, $\theta \approx 0$

$\therefore U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{dp}{dx}$ $\therefore -\frac{Q}{x}\frac{\partial}{\partial x}\left(-\frac{Q}{x}\right) = -\frac{1}{\rho}\frac{dp}{dx}$ $\therefore \frac{Q^2}{x}\left(-\frac{1}{x^2}\right) = -\frac{1}{\rho}\frac{dp}{dx}$ $\therefore$

$\frac{1}{\rho}\frac{dp}{dx} = \frac{Q^2}{x^3}$ $\therefore \frac{p}{\rho} + \frac{1}{2}u^2 = \text{const}$ $\therefore \frac{1}{\rho}\frac{dp}{dx} = -\frac{1}{2}\cdot 2u\frac{du}{dx}$

1c/ $\underline{u} = \nabla\times\psi\underline{k} = (\partial_y\psi, -\partial_x\psi, 0)$ $\therefore$

$\psi = -\sqrt{\nu Q}\,F(\eta)$ $\eta = y/\delta(x)$, $\delta(x) = x\sqrt{\frac{\nu}{Q}}$

$\frac{\partial\psi}{\partial y} = -\sqrt{\nu Q}\frac{dF}{d\eta}\frac{d\eta}{dy}$, $\frac{\partial\eta}{\partial y} = \frac{1}{\delta}$ $\therefore u = -\frac{\sqrt{\nu Q}\,F'}{\delta}$

$\frac{\partial\psi}{\partial x} = -\sqrt{\nu Q}\,F'\frac{\partial\eta}{\partial x}$, $\frac{\partial\eta}{\partial x} = y\left(-\frac{1}{\delta^2}\right)\frac{d\delta}{dx} = -\frac{y}{\delta^2}\sqrt{\frac{\nu}{Q}} = -\frac{\eta}{\delta}\sqrt{\frac{\nu}{Q}}$ $\therefore$

$v = -\frac{\partial\psi}{\partial x} = \sqrt{\nu Q}\,F'\left(-\frac{\eta}{\delta}\right)\sqrt{\frac{\nu}{Q}} = -\frac{\eta\nu F'}{\delta}$

1d/ $u\partial_x u + v\partial_y u = -\frac{Q^2}{x^3} + \nu\partial_{yy}u$ $\therefore$

$\partial_x u = -\sqrt{\nu Q}\,\partial_x\left(\frac{F'}{\delta}\right) = -\sqrt{\nu Q}\left[\frac{1}{\delta}F''\frac{\partial\eta}{\partial x} + F'\frac{\partial\delta}{\partial x}\left(-\frac{1}{\delta^2}\right)\right] =$

$-\sqrt{\nu \cancel{Q}}\left[-\frac{F''}{\delta}\frac{\eta}{\delta}\sqrt{\frac{\nu}{\cancel{Q}}} - \frac{F'}{\delta^2}\sqrt{\frac{\nu}{\cancel{Q}}}\right] = \nu\left[\frac{F'}{\delta^2} + \frac{F''\eta}{\delta^2}\right]$ $\therefore$

$\partial_y u = -\frac{\sqrt{\nu Q}}{\delta}\partial_y F' = -\frac{\sqrt{\nu Q}}{\delta}F''\frac{1}{\delta}$ $\therefore$

$\partial_{yy}u = -\frac{\sqrt{\nu Q}}{\delta^2}\frac{F'''}{\delta}$ $\therefore$

$-\frac{\sqrt{\nu Q}}{\delta}F'\nu\left[\frac{F'}{\delta^2} + \frac{F''\eta}{\delta^2}\right] + \left(-\frac{\eta\nu F'}{\delta}\right)\left(-\frac{\sqrt{\nu Q}\,F''}{\delta^2}\right) = -\frac{Q^2}{x^3} + \nu\left(-\frac{\sqrt{\nu Q}}{\delta^3}F'''\right)$ ..