sheet 1

\formative 3d/ $Pr(q_{0.05} \le T_2 \le q_{0.95}) = 0.9 = Pr(q_{0.05} \le \cancel{X_0}/\bar{X} \le q_{0.95})$

$\therefore Pr(q_{0.05}\bar{X} \le X_0 \le q_{0.95}\bar{X}) = 0.9$ $\therefore$ 2 90% CI for X_0: $(q_{0.05}\bar{X}, q_{0.95}\bar{X})$

\4a/ from Delta Method: $\text{bias}(\hat{\phi}) \approx \frac{\phi}{2n}$ $\text{var}(\hat{\phi}) \approx \frac{\phi^2}{2n}$ $\therefore sd(\hat{\phi}) = \frac{\phi}{\sqrt{2n}}$

bias corrected estimator $\tilde{\phi} = (1 - \frac{1}{2n})\phi$

\4bi/ $\therefore$ mean(times) = 1.066 sd(times) = 0.612122

gamma(α, β) $\alpha = \frac{\text{mean}^2}{\text{var}} = \frac{\mu^2}{\sigma^2}$ $\beta = \frac{\mu}{\sigma^2}$ $\therefore$ $\frac{\mu}{\beta} = \sigma^2$, $\sigma^2 = \frac{\mu^2}{\alpha}$ $\therefore$

$\frac{\mu^2}{\alpha} = \frac{\mu}{\beta}$ $\therefore$ $\mu = \frac{\alpha}{\beta}$

$\therefore \alpha = 2$ $\therefore$ $\mu = 1.066$ $\therefore$ $\frac{1.066}{2} = 0.533 = \beta = \phi$ $\therefore$ gamma(2, 0.533)

$\text{var}(\hat{\phi}) \approx \frac{\phi^2}{2n}$ $\therefore sd(\hat{\phi}) = \frac{\phi}{\sqrt{2n}}$ $\therefore$ is $\bar{x} = \frac{2}{\phi}$ $\therefore$ $\hat{\phi} = \frac{2}{\bar{x}} = \frac{2}{1.066} = 1.8762$

$\therefore \frac{1.8762}{\sqrt{2 \cdot 10}} = 0.4195$

\4bii/ for Ga(a, b) >> qgamma(p, a, b) $\therefore$ Ga$(2, \phi)$

$\therefore$ qgamma$(p, 2, \phi)$ $\therefore$ qgamma(0.9, 2, 1.8762) = 2.073221

\4biii/ \4biv/

\Sheet 2/ $f(x; \phi) = \phi^2 x e^{-\phi x}$, $x > 0$ given $x_1, \ldots, x_m$

\1a/ $L(\phi; \underline{x}) = \prod_{i=1}^{m} f(x_i; \phi) = \cancel{\phi^2 x_i e} \prod_{i=1}^{m} \phi^2 x_i e^{-\phi x_i}$ ① $=$

$\phi^{2n}\left(\prod_{i=1}^{m} x_i\right) e^{\sum_{i=1}^{m}(-\phi x_i)} = \phi^{2m}\left(\prod_{i=1}^{m} x_i\right) e^{-\phi \sum_{i=1}^{m} x_i} = \phi^{2m}\left(\prod_{i=1}^{m} x_i\right) e^{-\phi m \bar{x}}$

$= \phi^{2m}\left(\prod_{i=1}^{m} x_i\right) e^{-\phi \sum_{i=1}^{m} x_i}$ ② $\therefore$

$l(\phi; \underline{x}) = 2m \ln\phi + \sum_{i=1}^{m} \ln x_i - \phi \sum_{i=1}^{m} x_i$ ① $\therefore$

$l'(\phi; \underline{x}) = \frac{2m}{\phi} - \sum_{i=1}^{m} x_i$ ①, $l''(\phi; \underline{x}) = -\frac{2m}{\phi^2}$ ① < 0

Solve $l'(\hat{\phi}; \underline{x}) = 0$ ① $\therefore \frac{2m}{\hat{\phi}} - \sum_{i=1}^{m} x_i = 0$ $\therefore \hat{\phi} = \frac{2m}{\sum_{i=1}^{m} x_i}$ ① $= \frac{2}{\bar{x}}$

$\therefore l''(\hat{\phi}; \underline{x}) < 0$ ① $\therefore \hat{\phi}$ is the mle

now $I(\phi) = -E[l''(\phi; \underline{x})]$ ① $= -E\left[-\frac{2m}{\phi^2}\right] = \frac{2m}{\phi^2}$ ① {more info as m increases} $\therefore$ 2 asymptotic distri of $\hat{\phi}$ is $N(\phi, I(\phi)^{-1})$ ③

$\therefore N\left(\phi, \left(\frac{2m}{\phi^2}\right)^{-1}\right)$ $\therefore$ $N\left(\phi, \frac{\phi^2}{2m}\right)$

\1b/ $W = (\hat{\theta} - \theta_0)^2 / I(\hat{\theta})^{-1}$ $\quad \{I(\phi) = \frac{2m}{\phi^2}\}$ $\quad H_0: \phi = \phi_0 \quad H_1: \phi \neq \phi_0$ ②

the Wald test stat is $W = (\hat{\phi} - \phi_0)^2 / I(\hat{\phi})^{-1}$

$= (\hat{\phi} - \phi_0)^2 \frac{2m}{\hat{\phi}^2} = 2m\left(1 - \frac{\phi_0}{\hat{\phi}}\right)^2$ ①

the score test stat is: $S = U(\phi_0)^2 / I(\phi_0)$ ② $\quad \{U = l'\}$

$\Rightarrow \left\{ l'(\phi; x) = \frac{2m}{\phi} - \sum_{i=1}^{m} x_i \right\}$ $\quad S = \left(\frac{2m}{\phi_0} - m\bar{x}\right)^2 / \frac{2m}{\phi_0^2}$

$= \left(\frac{2m}{\phi_0} - \frac{2m}{\hat{\phi}}\right)^2 \frac{\phi_0^2}{2m} = 2m\left(1 - \frac{\phi_0}{\hat{\phi}}\right)^2 = W$ ①

the LR test is: $2[l(\hat{\phi}) - l(\phi_0)]$ ② $\quad \left\{ l(\phi) = 2m\ln\phi + \cancel{\left(\sum_{i=1}^{m} \ln x_i\right)} - \phi \sum_{i=1}^{m} x_i \right\}$

$= 2\left[2m\ln\hat{\phi} - m\bar{x}\hat{\phi} - 2m\ln\phi_0 + m\bar{x}\phi_0\right] =$

$2m\left[2\ln\left(\frac{\hat{\phi}}{\phi_0}\right) + \bar{x}(\phi_0 - \hat{\phi})\right]$ ①

these all have χ^2_1 {since one degree of freedom since one param}

null distri (approx, when m is large) ②

\Sheet 1/ \4b.i & iii & iv/ use (3d)

\4 b iii/ >> x = c(

>> n=length(x) $\qquad$ >> phi=2/mean(x) $\qquad V(\hat{\phi}) \approx \phi^2/(2n)$

>> phi/ sqrt(2*n)

\4 b ii/ >> qgamma(c(0.5, 0.95), 2*n, 1)/sum(x)

\Sheet 2 ~~A/~~ \1c/ asympdistri of MLE: $N\left(\phi, \frac{\phi^2}{2m}\right)$

$\{\mu : n(\bar{x} - \mu)^2/\sigma^2 < c\} = (\bar{x} - \sigma\sqrt{c/n}, \bar{x} + \sigma\sqrt{c/n})$ where c is the α-quantile

of the χ^2_1 distri $\therefore c = z^2_{\alpha/2}$ ~~for~~ $\therefore$

$\bar{x} = \phi$, $\sigma = \sqrt{\frac{\phi^2}{2m}} = \frac{\phi}{\sqrt{2m}}$ for α-quantile $\therefore n = m$

$\therefore \left(\phi - \sqrt{\frac{\phi^2}{2m}}\sqrt{z^2_{\alpha/2}/m}, \phi + \sqrt{\frac{\phi^2}{2m}}\sqrt{z^2_{\alpha/2}/m}\right)$

\3~~a~~/ >> x = c(. . .) $\qquad$ >> sd(x) $\Rrightarrow$ 0.612122 is a esti for

standard error of the data

>> mean(x) $\Rrightarrow$ 1.066 is a pt esti of the data

\3 ~~A/~~ \1c/ $f(x;\phi) = \phi^2 x e^{-\phi x}$, $x > 0$ $\quad$ MLE $\hat{\phi} = 2/\bar{x}$ & $I = 2m/\phi^2$

$T(\underline{X}, \theta) = (\hat{\theta} - \theta)^T I(\theta)(\hat{\theta} - \theta)$ $\qquad$ degrees of freedom 1

$S(\underline{X}) = \{\theta ; (\hat{\theta} - \theta)^T I(\theta)(\hat{\theta} - \theta) < c\}$ $\qquad \chi^2_1$

\Sheet 2/ let c be the α-quantile (1) of χ^2_1 (1)

an approx α-CI for ϕ is $\{\phi ; (\hat{\phi}-\phi)^2 I(\phi) < c\}$ (2)

now $(\hat{\phi}-\phi)^2 I(\phi) < c$ $\therefore (\hat{\phi}-\phi)^2 \frac{2m}{\phi^2} < c$ $\quad \left(\frac{\hat{\phi}}{\phi}-1\right)^2 < \frac{c}{2m}$

$\therefore -\sqrt{\frac{c}{2m}} < \frac{\hat{\phi}}{\phi} - 1 < \sqrt{\frac{c}{2m}}$ $\quad \therefore \frac{\hat{\phi}}{1+\sqrt{\frac{c}{2m}}} < \phi < \frac{\hat{\phi}}{1-\sqrt{\frac{c}{2m}}}$

$\therefore$ the interval is $\left(\frac{\hat{\phi}}{1+\sqrt{\frac{c}{2m}}}, \frac{\hat{\phi}}{1-\sqrt{\frac{c}{2m}}}\right)$ (3)

the Wald test stat is $(\hat{\phi}-\phi_0)^2 I(\hat{\phi}) = 2m(1-\phi_0/\hat{\phi})^2$ (2)

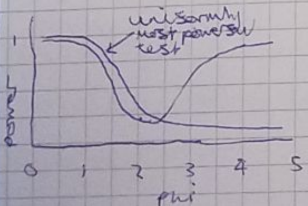
$\therefore$ an approx α-CI for ϕ is $\{\phi : 2m(1-\phi/\hat{\phi})^2 < c\}$ $\therefore$

$2m(1-\phi/\hat{\phi})^2 < c$ $\quad \therefore -\sqrt{\frac{c}{2m}} < 1 - \frac{\phi}{\hat{\phi}} < \sqrt{\frac{c}{2m}}$ $\quad \therefore -1-\sqrt{\frac{c}{2m}} < -\frac{\phi}{\hat{\phi}} < -1+\sqrt{\frac{c}{2m}}$ $\therefore$

$\hat{\phi}$ is pos $\therefore -\hat{\phi}$ is neg $\quad \therefore \left(1-\sqrt{\frac{c}{2m}}\right)\hat{\phi} < \phi < \left(1+\sqrt{\frac{c}{2m}}\right)\hat{\phi}$ (3) {sanity check it does contain $\hat{\phi}$} $\quad \left(\left(1-\sqrt{\frac{c}{2m}}\right)\hat{\phi}, \left(1+\sqrt{\frac{c}{2m}}\right)\hat{\phi}\right)$

3b/

```
>>m=10    >>nsim=10000    >>phi = 1
>> crit = qchisq(0.95,1)
>> x = rgamma(m,2,phi)                >>phi0 = 2    >>phi.hat = 2/mean(x)
>> sc = 2*m*(1- phi0/phi.hat)^2     # score test stats #
>> ~~for (i in 1:nsim~~    >> sc = numeric(nsim)
>> for (i in 1:nsim) { }
>> for (i in 1:nsim) {    >> x=rgamma(m, 2, phi[i])
>> mean(sc>crit) => 0.9088   # is power of test ∴ 90% correctly rejects null
```



$\therefore$ power at $\phi = 2$ is 0.05 but further away from $\phi = 2$ want to reject null more often

can do also uniformly most powerful test >>ump = 1-pgamma(crit,2,

ump always above power for hypoth for $\phi < 2$ but can't tell you about $\phi > 2$

\Sheet 2/ \2a/ let $\theta = (\gamma, \phi)$ $\therefore$ $f(y_i ; (\gamma, \phi)) = f(y_i ; \theta)$ =

$\therefore L(\gamma, \phi ; \underline{x}, \underline{y}) = L(\theta ; \underline{x}, \underline{y}) = \prod_{i=1}^{n} f(y_i ; (\gamma, \phi)) =$

$\prod_{i=1}^{n} \gamma^2 \phi^2 y_i e^{-\gamma\phi y_i} = \gamma^{2n} \phi^{2n} e^{-\sum_{i=1}^{n} \gamma\phi y_i} \prod_{i=1}^{n} y_i = \gamma^{2n} \phi^{2n} e^{-\gamma\phi \sum_{i=1}^{n} y_i} \prod_{i=1}^{n} y_i$ $\therefore$

$l(\gamma, \phi) = \ln[L(\gamma, \phi ; \underline{x}, y)] = \ln\left(\gamma^{2n} \phi^{2n} e^{-\gamma\phi \sum_{i=1}^{n} y_i} \prod_{i=1}^{n} y_i\right) =$

$2n \ln\gamma + 2n \ln\phi - \gamma\phi \sum_{i=1}^{n} y_i + \sum_{i=1}^{n} \ln y_i = l(\gamma, \phi)$

$\frac{\partial l(\gamma, \phi)}{\partial \gamma} = \frac{2n}{\gamma} - \phi \sum_{i=1}^{n} y_i$, $\frac{\partial l(\gamma, \phi)}{\partial \phi} = \frac{2n}{\phi} - \gamma \sum_{i=1}^{n} y_i$

\2nd Sol/ $L(\gamma, \phi ; x, y) = \prod_{i=1}^{m} f(x_i ; \phi) \prod_{i=1}^{n} f(y_i ; \gamma, \phi)$

$= \prod_{i=1}^{m} \phi^2 x_i e^{-\phi x_i} \prod_{i=1}^{n} \gamma^2 \phi^2 y_i e^{-\gamma\phi y_i} =$

$\phi^{2(m+n)} \gamma^{2n} \left(\prod_{i=1}^{m} x_i \prod_{i=1}^{n} y_i\right) e^{-\phi \sum_{i=1}^{m} x_i - \gamma\phi \sum_{i=1}^{n} y_i}$ now:

$l(\gamma, \phi) = 2(m+n)\log\phi + 2n\log\gamma - \phi \sum x_i - \gamma\phi \sum y_i + \text{Const}$ $\therefore$

$\frac{\partial l}{\partial \gamma} = \frac{2n}{\gamma} + \phi \sum_{i=1}^{n} y_i$ $\frac{\partial l}{\partial \phi} = \frac{2(m+n)}{\phi} - \sum_{i=1}^{m} x_i - \gamma \sum_{i=1}^{n} y_i$

Solve $\begin{cases} \frac{\partial l}{\partial \gamma} = 0 \\ \frac{\partial l}{\partial \phi} = 0 \end{cases}$ for (γ, ϕ)

$\therefore$ have: $\hat{\phi} = \frac{2n}{\hat{\gamma} \sum_{i=1}^{n} y_i}$ & $\frac{2(m+n)\hat{\gamma} \sum_{i=1}^{n} y_i}{2n} - \sum_{i=1}^{m} x_i - \hat{\gamma} \sum_{i=1}^{n} y_i = 0$ $\therefore$

$\hat{\gamma} = \sum_{i=1}^{m} x_i \Big/ \left(\frac{m+n}{n} \sum_{i=1}^{n} y_i - \sum_{i=1}^{n} y_i\right) = \sum_{i=1}^{m} x_i \Big/ \left(\frac{m}{n} \sum_{i=1}^{n} y_i\right) = \frac{n \sum_{i=1}^{m} x_i}{m \sum_{i=1}^{n} y_i} = \bar{x}/\bar{y}$.

& $\hat{\gamma} = \frac{2}{\bar{x}}$

asymp distri is $N\left(\binom{\gamma}{\phi}, I(\gamma, \phi)^{-1}\right)$ where

$$I(\gamma, \phi) = \begin{pmatrix} -E\left(\frac{\partial^2 l}{\partial \gamma^2}\right) & -E\left(\frac{\partial^2 l}{\partial \phi \partial \gamma}\right) \\ -E\left(\frac{\partial^2 l}{\partial \phi \partial \gamma}\right) & -E\left(\frac{\partial^2 l}{\partial \phi^2}\right) \end{pmatrix} = \begin{pmatrix} 2n/\gamma^2 & E\left(\sum_{i=1}^{n} Y_i\right) \\ E\left(\sum_{i=1}^{n} Y_i\right) & 2(m+n)/\phi^2 \end{pmatrix} = \begin{pmatrix} 2n/\gamma^2 & 2n/(\gamma\phi) \\ 2n(\gamma\phi) & 2(m+n)/\phi^2 \end{pmatrix}$$

since $E(Y_i) = \frac{2}{\gamma\phi}$

\2b/ have $\frac{\partial l}{\partial \phi} = \frac{2(m+n)}{\phi} - \sum x_i - \gamma \sum y_i$ $\therefore$

$\hat{\phi}(\gamma) = \frac{2(m+n)}{\sum_{i=1}^{m} x_i + \gamma \sum_{i=1}^{n} y_i}$ $\therefore$ $l_p(\gamma) = l(\gamma, \hat{\phi}(\gamma)) =$

$2(m+n) \log\left[\frac{2(m+n)}{\sum_{i=1}^{m} x_i + \gamma \sum_{i=1}^{n} y_i}\right] + 2n\log\gamma - \frac{2(m+n)}{\sum x_i + \gamma \sum y_i}\left[\sum x_i + \gamma \sum y_i\right] + \text{const} =$

$2(m+n)\log 2(m+n) - 2(m+n)\log\left(\sum x_i + \gamma \sum y_i\right) + 2n\log\gamma - 2(m+n) + \text{const} =$

\Sheet 2/ $2n\log\gamma - 2(m+n)\log\left(\sum x_i + \gamma \sum y_i\right) + \text{const}$

\2c/ $H_0 : \gamma = 1 \quad H_1 : \gamma \neq 1$ 2 LR test stat:

$$2\left[l_p(\hat\gamma) - l_p(1)\right] =$$

$$2\left[2n\log\hat\gamma - 2(m+n)\log\left(\sum x_i + \hat\gamma\sum y_i\right) + 2(m+n)\log\left(\sum x_i + \sum y_i\right)\right] =$$

$$4\left[n\log\hat\gamma + (m+n)\log\left(\frac{\sum x_i + \sum y_i}{\sum x_i + \hat\gamma \sum y_i}\right)\right]$$

2 null distri is χ^2_1

\3a/ 2 quasi-score func is $G = \mu_\theta^T \Sigma^{-1}(X-\mu)$

have $X = \begin{pmatrix} X_1 \\ \vdots \\ X_m \\ Y_1 \\ \vdots \\ Y_n \end{pmatrix}$, $\mu = \begin{pmatrix} 2/\phi \\ \vdots \\ 2/\phi \\ 2/(\gamma\phi) \\ \vdots \\ 2/(\gamma\phi) \end{pmatrix}$ $\Sigma = \begin{pmatrix} 2/\phi^2 & & & & 0 \\ & \ddots & & & \\ & & 2/\phi^2 & & \\ & & & 2/(\gamma\phi)^2 & \\ 0 & & & & \ddots \\ & & & & & 2/(\gamma\phi)^2 \end{pmatrix}$

$\Sigma^{-1} = \begin{pmatrix} \phi^2/2 & & & & 0 \\ & \ddots & & & \\ & & \phi^2/2 & & \\ 0 & & & (\gamma\phi)^2/2 & \\ & & & & \ddots \\ & & & & & (\gamma\phi)^2/2 \end{pmatrix}$ $\therefore$

~~$\mu_\theta = \begin{pmatrix} 0 & -2/\phi^2 \\ \vdots & \vdots \\ 0 & -2/\phi^2 \\ -2/\gamma^2\phi & -2 \\ \vdots & \\ -2/\gamma^2\phi & \end{pmatrix}$~~

$\mu_\theta = \begin{pmatrix} 0 & -2/\phi^2 \\ \vdots & \vdots \\ 0 & -2/\phi^2 \\ -2/\gamma^2\phi & -2/\gamma\phi^2 \\ \vdots & \vdots \\ -2/\gamma^2\phi & -2/\gamma\phi^2 \end{pmatrix}$

\Sheet 3/ \1b/ 10000 resamples suggested

$1.88 - 0.1 = 1.78$ {minus 2 bias?}

$Ga(2, \hat\phi)$

```
>> x=c(...)    >> B=10000    >> M=length(x)
```

Bias is $E(\hat\theta) - \theta$ ie $E[\hat\theta(X)] - \theta(F)$ where $X \sim F$

Bootstrap estimate of bias is $E[\hat\theta(X^*)] - \theta(\hat F)$

where $X^* \sim \hat F$

Parametric: $F(x;\theta)$ & $\hat F(x) = F(x;\hat\theta)$ so $\theta(\hat F) = \hat\theta$

non-parametric: $\hat F$ is 2 e.d.f. of $(x_1, \dots, x_n)$

eg $\theta(F)$ is 2 expectation of F then $\theta(\hat F)$ is 2 expectation of $\hat F$

ie $\sum_{i=1}^n x_i P(X^* = x_i) = \frac{1}{n}\sum_{i=1}^n x_i = \bar x$

∴ Coursework use R for Jackknife

Can be asked to do Jackknife in exam

Sheet 3 / 1c / $f(x;\phi)=\phi^2 x e^{-\phi x}$ $\quad \hat{\phi}=2/\bar{X}=\frac{2m}{\sum_{i=1}^{m} X_i}$ $\quad \hat{\phi}_{-i}=\frac{2(m-1)}{\sum_{j\neq i} X_j}=$

$\frac{2(m-1)}{(\sum_{j=1}^{m} X_j)-X_i}$ $\quad \hat{\phi}_i=m\hat{\phi}-(m-1)\hat{\phi}_{-i}$ $\quad \hat{\phi}_J=\frac{1}{m}\sum_{i=1}^{m}\hat{\phi}_i=m\hat{\phi}-\frac{m-1}{m}\sum_{i=1}^{m}\hat{\phi}_{-i}$

i	1	2	3	4	5	6	7	8	9	10
x_i	1.79	0.39 ...								
$\sum_{j=1}^{m} x_j - x_i$	8.77	10.32 ...								
$\hat{\phi}_{-i}$	2.03	1.74 ...								
$\hat{\phi}_i$	0.50	3.06 ...								

$\{\hat{\phi}_J=\frac{1}{m}\sum_{i=1}^{m}\hat{\phi}_i=1.81\}$

$(\hat{\phi}_i-\hat{\phi}_J)^2$ 1.73 1.56 ...

∴ $\hat{\phi}_J=1.81$, standard error is 0.35 ∴ 90% CI is: $\hat{\phi}_J \pm 0.35\, t_{0.95}(m-1)$

$\{t_{0.95}(m-1)=t_{0.95}(10-1)=t_{0.95}(9)=1.833$ ∴ CI: $1.81 \pm 0.35(1.833)=(1.18, 2.45)$

```
>>x <- c(...)        {>>jackknife}
>>m <- length(x)   >>phi.hat <- 2/mean(x)     >>phi.loo <- 2*(m-1)/(sum(x)-x)
>>pseudo <- m*phi.hat-(m-1)*phi.loo
>>phi.jac <- mean(pseudo)   >>se.jac <- sd(pseudo)/sqrt(m)

>>Jackknise <- func(x) {   ...
>>c(estimate=phi.jac, se=se.jac)    >>}
>>jac <- jackknise(x)
>>jac[1]+j[2]*qt(c(0.05,0.95),m-1)
```

2a / $(\hat{\theta}_J-\theta_0)/\sqrt{\hat{V}_J}$ ∴

```
>>t <- (jac[1]-2)/jac[2]
>>2*(1-pt(abs(t),m-1))
```

2b /

```
>>B <- 10000   >>phi0 <- 2   >>sc <- 2*m*(1-phi0/phi.hat)^2
>>lr <- 4*m*(log(phi.hat/phi0)+phi0/phi.hat-1)
>>sc.star <- lr.star <- numeric(B)
```

Sheet 3/ >>for(b in 1:B) { >> x.star <- rgamma(m, 2, phi0)

>> phi.star <- 2/mean(x.star) >> sc.star[b] <- 2*m*(1-phi0/phi.star)^2

>> lr.star[b] <- 4*m*(log(phi.star/phi0)+phi0/phi.star-1) >> }

>> mean(sc.star >= sc) % is p val % ⇒ 0.7684

>> mean(lr.star >= lr) % likelihood ratio test stat % ⇒ 0.7732

>> hist(sc.star) >> sc ⇒ 0.08712

% ∴ dont reject null hypoth %

>> 1-pchisq(sc, 1) ⇒ 0.7678709

>> 1-pchisq(lr, 1) ⇒ 0.7726529 % eg very good approx %

2c/ y <- c(...) n <- length(y) >> xbar <- mean(x) >> ybar <- mean(y)

>> lr <- 4*n*log(xbar/ybar + 4*(m+n)*log((m*xbar+n*y)bar

>> phi0.hat <- 2/mean(c(x,y)) >> lr.star <- numeric(B)

>> for (b in 1:B) { >> x.star <- rgamma(n, 2, phi0.hat)

>> y.star <- rgamma(m, 2, phi0.hat

>> xbar <- mean(x.star >> ybar <- mean

>> lr.star[b] <- 4*n*log(xbar/ybar)+4*(m+n)*(log((m*xbar - >> }

>> mean(lr.star >= lr)

Sheet 3/ 3bi/ $X \sim Ga(2, \phi)$ with pdf $\phi^2 x e^{-\phi x}$, $x>0$

{Scale model is $X = \sigma Z$, where Z has a distri that doesnt depend on (any unknown) params} let $Z = \phi x$ we have

$P(Z \le z) = P(\phi X \le z) = P(X \le \frac{z}{\phi}) = \int_0^{z/\phi} \phi^2 x e^{-\phi x} dx$ $\{y = \phi x\}$

$= \int_0^z \phi \cdot y e^{-y} \frac{dy}{\phi} = \int_0^z y e^{-y} dy$ doesnt depend on ϕ ∴ is a scale model

the pdf of Z is $z e^{-z}$ so $Z \sim Ga(2,1)$

3bii/ an ancillary stat is T $(X_0, X_1, \dots, X_m)$ ∴ is $T = \frac{X_0}{\bar{X}}$

$T = \frac{Z_0/\phi}{\frac{1}{m}\sum_{i=1} \frac{Z_i}{\phi}} = \frac{Z_0}{\bar{Z}}$ ∴ T is ancillary

~~Pt~~

$P(t_{0.05} < T < t_{0.95}) = 0.9$ ie $P(t_{0.05}\bar{X} < X_0 < t_{0.95}\bar{X}) = 0.9$

\Sheet 2 \1a\ $f(x;\phi)=\phi^2 x e^{-\phi x}$ for $x>0$ indep $\therefore$

$L(\phi;\underline{x})=\prod_{i=1}^{m} f(x_i;\phi) = \cancel{\phi^2 x_i e^{-\phi x_i}} = \prod_{i=1}^{n}\phi^2 x_i e^{-\phi x_i} = \phi^{2n}\prod_{i=1}^{n}(x_i)\, e^{\sum_{i=1}^{n}-\phi x_i} =$

$\phi^{2m} e^{-\phi\sum_{i=1}^{n} x_i}\prod_{i=1}^{m}(x_i)$ $\therefore$ $\cancel{\log(\phi;\underline{x})}$ $\log L(\phi;\underline{x}) = l(\phi;\underline{x}) =$

$\ln\left(\phi^{2n} e^{-\phi\sum_{i=1}^{n}x_i}\prod_{i=1}^{m}(x_i)\right) = \ln(\phi^{2m}) + \ln\left(e^{-\phi\sum_{i=1}^{m}x_i}\right) + \ln\left(\prod_{i=1}^{m}(x_i)\right) =$

$2m\ln\phi - \phi\sum_{i=1}^{2}x_i + (\ln(x_1)+\ln(x_2)+\ldots+\ln x_m) = 2m\ln\phi - \phi\sum_{i=1}^{m}x_i + \sum_{i=1}^{m}\ln x_i$ $\therefore$

$\frac{d}{d\phi}l(\phi;\underline{x}) = 2m\frac{1}{\phi} - \sum_{i=1}^{m}x_i$ $\therefore 2m\frac{1}{\hat\phi} - \sum_{i=1}^{m}x_i = 0$ $\therefore$ $2m\frac{1}{\hat\phi} = \sum_{i=1}^{m}x_i$ $\therefore$

$\frac{2m}{\sum_{i=1}^{n}x_i} = \hat\phi = \frac{2m}{m\frac{1}{m}\sum_{i=1}^{2}x_i} = \frac{2m}{n\bar{x}} = \frac{2}{\bar{x}}$ $\therefore \bar{x} = \frac{2}{\hat\phi}$

$\therefore \frac{d^2}{d\phi^2}l(\phi;\underline{x}) = -2m\phi^{-2}$ $\therefore l''(\hat\phi;\underline{x}) = -2m\left(\frac{2}{\bar{x}}\right)^{-2} = -2n\left(\frac{\bar{x}^2}{4}\right) = -\left(\frac{n\bar{x}^2}{2}\right)$ $\therefore$

$\frac{n\bar{x}^2}{2} > 0 \therefore -\frac{n\bar{x}^2}{2} < 0 \therefore l''(\hat\phi;\underline{x}) < 0 \therefore \hat\phi = \frac{2}{\bar{x}}$ is MLE of $L(\phi;\underline{x})$

$L(\phi) = \prod_{i=1}^{n} f(x_i;\phi) \propto \phi^{2m}e^{-\phi m\bar{x}}$ $\therefore$ $l(\phi) = 2m\log\phi - m\bar{x}\phi$ $\therefore l'(\phi) = 2m\phi^{-1} - m\bar{x}$

$\therefore l''(\phi) = -2m\phi^{-2}$ $\because l''(\phi) < 0$ Solving $l'(\hat\phi) = 0$ yields Z MLE,

$\hat\phi = \frac{2}{\bar{x}}$. Z expected information is $I(\theta) = E[J(\phi)]$

$J(\phi) = -\frac{\partial^2 l}{\partial\phi^2}$ $\therefore$ $I = E\left[-\frac{\partial^2 l}{\partial\phi^2}\right] = -E\left[\frac{\partial^2 l}{\partial\phi^2}\right]$ $\therefore$

$\frac{\partial^2 l(\hat\phi;\underline{x})}{\partial\phi^2} = l''(\hat\phi;\underline{x}) = -\left(\frac{n\bar{x}^2}{2}\right)$ $\therefore$ $J(\hat\phi) = \frac{n\bar{x}^2}{2}$ $\therefore E(J(\hat\phi)] = I(\hat\phi) = E\left(\frac{n\bar{x}^2}{2}\right) =$

$\frac{n}{2}E(\bar{x}^2) = \frac{n}{2}\bar{x}^2 = I(\hat\phi)$ is Z expected information X

expected information is $I(\phi) = -E[l''(\phi)] = -E[-2m\phi^{-2}] = 2m\phi^{-2}E[1] =$

$2m\phi^{-2}$ $\therefore$

Z asymtatic distri of Z MLE is $\left\{\lim_{m\to\infty}\frac{2}{\bar{x}} = \lim_{m\to\infty}\hat\phi = \frac{2}{\bar{x}}\right\}$ X

$\left\{N(\hat\phi, I(\phi)^{-1}) = N\left(\frac{2}{\bar{x}}, \frac{\phi^2}{2m}\right)\right\}$ X Z asymptotic distri of Z MLE is $N(\phi, \phi^2/(2m))$

\1b\ ~~wald~~ test stat $W = (\hat\phi - \phi_0)^2 I(\hat\phi) = \left(\frac{2}{\bar{x}} - \phi_0\right)^2 2m\phi_0^{-2} =$

$\left(\frac{4}{\bar{x}^2} + \phi_0^2 - \frac{4\phi_0}{\bar{x}}\right)\cancel{2m\frac{1}{\phi_0^2}} = \cancel{2m}\ \frac{8m}{\bar{x}^2\phi_0^2} + 2m - \frac{8m}{\bar{x}\phi_0}$ X

$W = (\hat\phi - \phi_0)^2 I(\hat\phi) = \{(\hat\phi - \phi_0)^2 2m\hat\phi^{-2} = 2m(\hat\phi - \phi_0)^2\left(\frac{1}{\hat\phi}\right)^2 = 2m\left((\hat\phi - \phi_0)\frac{1}{\hat\phi}\right)^2 =$

$= 2m(1 - \phi_0/\hat\phi)^2$

Score test stat : $S = U(\phi_0)^2 / I(\phi_0)$ $U(\phi_0) = l'(\phi_0)$ $\therefore U(\phi_0) = \frac{2m}{\phi_0} - m\bar{x}$

$\therefore S = \left(\frac{2m}{\phi_0} - m\bar{x}\right)^2\left(\frac{\phi_0^2}{2m}\right) = \frac{1}{2m}\cancel{(2m\phi_0 - 2m\bar{x}\phi_0^2)}\ \frac{1}{2m}\left(\left(\frac{2m}{\phi_0} - m\bar{x}\right)\phi_0\right)^2 = \frac{1}{2m}(2m - m\bar{x}\phi_0)^2$

$S = U(\phi_0)^2 / I(\phi_0) = (2m\phi_0^{-1} - m\bar{x})^2\phi_0^2/2m = W = 2m(1 - \phi_0/\hat\phi)^2$

\Sheet 2/

likelihood ratio test stat: $-2\log\Lambda = 2[l(\hat\phi) - l(\phi_0)] =$

$2[2m\ln\hat\phi - m\bar{x}\hat\phi - 2m\ln\phi_0 + m\bar{x}\phi_0] = 2[2m(\ln\hat\phi - \ln\phi_0) - m\bar{x}(\hat\phi - \phi_0)] =$

$= 2\left[2m\ln\frac{\hat\phi}{\phi_0} - m\bar{x}(\hat\phi - \phi_0)\right]$ X

$-2\log\Lambda = 2[l(\hat\phi) - l(\phi_0)] = 2\left[2m\ln\hat\phi - \hat\phi m\bar{x} + \sum_{i=1}^{m}\ln x_i - 2m\ln\phi_0 + \phi_0 m\bar{x} - \sum_{i=1}^{m}\ln x_i\right] =$

$2[2m(\ln\hat\phi - \ln\phi_0) + m\bar{x}(-\hat\phi + \phi_0)] = 2\left[2m\ln\frac{\hat\phi}{\phi_0} + m\frac{2}{\hat\phi}(-\hat\phi + \phi_0)\right] =$

$2m\left[2\ln\frac{\hat\phi}{\phi_0} + 2\frac{\phi_0}{\hat\phi} - 2\right] = 4m\left[\ln\frac{\hat\phi}{\phi_0} + \frac{\phi_0}{\hat\phi} - 1\right]$

$W = 2m(1 - \phi_0/\hat\phi)^2$ ∴ null distri $\phi = \phi_0$: $W = 2m(1 - \phi/\hat\phi)^2$

Z null distri of all three test stats are approx χ^2_1 distris when m is large

\1c/ asymp distri $N\left(\hat\phi, \frac{\phi^2}{2m}\right) = N\left(\frac{2}{\bar{x}}, \frac{\phi^2}{2m}\right)$

$S(\phi) = \left(\frac{2}{\bar{x}} - z_{1-\alpha}\frac{\sigma}{\sqrt{m}}, \frac{2}{\bar{x}} - z_\alpha\frac{\sigma}{\sqrt{m}}\right) = \left(\frac{2}{\bar{x}} - z_{1-\alpha}\frac{\phi}{\sqrt{2m}\sqrt{m}}, \frac{2}{\bar{x}} - z_\alpha\frac{\phi}{\sqrt{2m}\sqrt{m}}\right) =$

$\left(\frac{2}{\bar{x}} + z_\alpha\frac{\phi}{m\sqrt{2}}, \frac{2}{\bar{x}} - z_\alpha\frac{\phi}{m\sqrt{2}}\right)$

let c be Z α-quantile of Z χ^2_1 distri

an approx α-CI for ϕ is: $\{W = (\hat\phi - \phi_0)^2 I(\hat\phi)$ ∴ $\{\phi : (\hat\phi - \phi)^2 I(\hat\phi) < c\}\}$

$\{\phi : (\hat\phi - \phi)^2 I(\phi) < c\} = \{\phi : 2m(1 - \phi_0/\hat\phi)^2 < c\}$ Z inequality is satisfied if

$\{\ \{\phi : (1 - \phi_0/\hat\phi)^2 < \frac{c}{2m}\}$ ∴ $\left(-\sqrt{\frac{c}{2m}} < (1 - \phi_0/\hat\phi) < \sqrt{\frac{c}{2m}}\right) =$

$\left(-\sqrt{\frac{c}{2m}} - 1 < -\frac{\phi_0}{\hat\phi} < \sqrt{\frac{c}{2m}} - 1\right) = \left(\left(-\sqrt{\frac{c}{2m}} - 1\right)\hat\phi < -\phi < \left(\sqrt{\frac{c}{2m}} - 1\right)\hat\phi\right) =$

$\left(-\left(\sqrt{\frac{c}{2m}} - 1\right)\hat\phi < \phi < -\left(\sqrt{\frac{c}{2m}} - 1\right)\hat\phi\right) = \left(\left(-\sqrt{\frac{c}{2m}} + 1\right)\hat\phi < \phi < \left(+\sqrt{\frac{c}{2m}} + 1\right)\hat\phi\right)\}$

$\frac{\hat\phi}{1 + \sqrt{c/(2m)}} < \phi < \frac{\hat\phi}{1 - \sqrt{c/(2m)}}$ ∴ Z interval is $\left(\frac{\hat\phi}{1 + \sqrt{c/(2m)}}, \frac{\hat\phi}{1 - \sqrt{c/(2m)}}\right)$ ∵

α CI for ϕ is $\{\phi : (\hat\phi - \phi)^2 I(\phi) < c\} = \{\phi : 2m(\hat\phi/\phi - 1)^2 < c\}$ ∵ $\phi = \phi_0$

another approx α-CI for ϕ is $\{\phi : 2m(1 - \phi/\hat\phi)^2 < c\}$ $\{\{\phi : W < c\}\}$

∴ $\hat\phi[1 - \sqrt{c/2m}] < \phi < \hat\phi[1 + \sqrt{c/2m}]$ ∴ $(\hat\phi[1 - \sqrt{c/(2m)}], \hat\phi[1 + \sqrt{c/2m}])$

\2a/ $f(y_i; \gamma, \phi) = \gamma^2\phi^2 y_i e^{-\gamma\phi y_i}$ ∴ $L(\gamma, \phi) = \prod_{i=1}^{m} f(x_i; \phi)\prod_{i=1}^{n} f(y_i; \phi, \gamma) =$

$\phi^{2m} e^{-\phi\sum_{i=1}^{m} x_i}\prod_{i=1}^{m}(x_i)\prod_{i=1}^{n}\gamma^2\phi^2 y_i e^{-\gamma\phi y_i} = \phi^{2m} e^{-\phi\sum_{i=1}^{m} x_i}\prod_{i=1}^{m}(x_i)\gamma^{2n}\phi^{2n} e^{\sum_{i=1}^{n} -\gamma\phi y_i}\prod_{i=1}^{n}(y_i) =$

$\phi^{2m} e^{-\phi m\bar{x}}\prod_{i=1}^{m}(x_i)\gamma^{2n}\phi^{2n} e^{-\gamma\phi\sum_{i=1}^{n} y_i}\prod_{i=1}^{n}(y_i) = \phi^{2(m+n)}\gamma^{2n} e^{-\phi m\bar{x}}\prod_{i=1}^{m}(x_i) e^{-\gamma\phi n\bar{y}}\prod_{i=1}^{n} y_i$

$\gamma^{2n}\phi^{2(m+n)} e^{-\phi(m\bar{x} + \gamma n\bar{y})}\prod_{i=1}^{m}(x_i)\prod_{i=1}^{n} y_i \propto \gamma^{2n}\phi^{2(m+n)} e^{-\phi(m\bar{x} + \gamma n\bar{y})}$ ∴

$l(\gamma, \phi; \underline{x}, \underline{y}) = l(\gamma, \phi) = \log L(\gamma, \phi) = \ln(\gamma^{2n}) + \ln(\phi^{2(m+n)}) - \phi(m\bar{x} + \gamma n\bar{y}) =$

$2n\ln\gamma + 2(m+n)\ln\phi - \phi(m\bar{x} + \gamma n\bar{y}) = 2n\ln\gamma + 2(m+n)\ln\phi - m\bar{x}\phi - n\bar{y}\gamma\phi$ ∴

first derivatives $\frac{d}{d\gamma} l(\gamma,\phi) = \frac{2n}{\gamma} - n\bar{y}\phi$ & $\cancel{z}$ $\cancel{z} = l_\gamma(\gamma,\phi)$

$\frac{d}{d\phi} l(\gamma,\phi) = \frac{2(m+n)}{\phi} - m\bar{x} - n\bar{y}\gamma \doteq l_\phi(\gamma,\phi)$ $\therefore$

{solving $l_\gamma(\hat{\gamma},\phi)=0$: $\frac{2n}{\hat{\gamma}} - n\bar{y}\phi = 0$ $\therefore$ $\frac{2n}{\hat{\gamma}} = n\bar{y}\phi$ $\therefore$ $\frac{2n}{n\bar{y}\phi} = \hat{\gamma} = \frac{2}{\bar{y}\phi}$

& $l_\phi(\gamma,\hat{\phi})=0$: $\frac{2(m+n)}{\hat{\phi}} - m\bar{x} - n\bar{y}\gamma = 0$ $\therefore$ $\frac{2(m+n)}{\hat{\phi}} = m\bar{x} + n\bar{y}\gamma$ $\therefore$

$\frac{2(m+n)}{m\bar{x}+n\bar{y}\gamma} = \hat{\phi}$}

solving $l_\gamma = 0$ & $l_\phi = 0$: $\frac{2n}{\gamma} - n\bar{y}\phi = 0$, $\frac{2(m+n)}{\phi} - m\bar{x} - n\bar{y}\gamma = 0$ $\therefore$

$\frac{2n}{\gamma} = n\bar{y}\phi$ $\therefore$ $\frac{2n}{\gamma n\bar{y}} = \phi = \frac{2}{\bar{y}\gamma}$ $\therefore$ $\frac{2(m+n)\bar{y}\gamma}{2} - m\bar{x} - n\bar{y}\gamma = 0 =$

$-m\bar{x} + \bar{y}\gamma(m+n-n) = -m\bar{x} + \bar{y}\gamma(m) = m(-\bar{x}+\bar{y}\gamma) = 0$ $\therefore m\bar{y}\gamma = m\bar{x}$ $\therefore$

$\hat{\gamma} = \frac{\bar{x}}{\bar{y}}$ $\therefore$ $\hat{\phi} = \frac{2\bar{y}}{\bar{y}\bar{x}} = \frac{2}{\bar{x}}$ $\therefore$

$l_{\gamma\gamma} = \frac{d^2}{d\gamma^2} l(\gamma,\phi) = -2n\gamma^{-2}$ $\quad l_{\gamma\phi} = -n\bar{y}$ $\quad l_{\phi\phi} = -2(m+n)\phi^{-2}$ $\therefore$

Z expected information is $I(\gamma,\phi) = \begin{bmatrix} -E(l_{\gamma\gamma}) & -E(l_{\gamma\phi}) \\ -E(l_{\gamma\phi}) & -E(l_{\phi\phi}) \end{bmatrix} =$

$\begin{bmatrix} -E(-2n\gamma^{-2}) & -E(-n\bar{y}) \\ -E(-n\bar{y}) & -E(-2(m+n)\phi^{-2}) \end{bmatrix} = \begin{bmatrix} 2n\gamma^{-2} & nE(\bar{y}) \\ nE(\bar{y}) & 2(m+n)\phi^{-2} \end{bmatrix}$ $\quad$ {$\because E(\bar{Y}) = E(Y_1) = \frac{2}{\gamma\phi}$

$\because Ga(2,\gamma\phi) = Ga(\alpha,\beta)$ & $\therefore E(Y_1) = \frac{\alpha}{\beta} = E(\bar{Y})$} $\quad \therefore I(\gamma,\phi) = \begin{bmatrix} 2n\gamma^{-2} & 2n(\gamma\phi)^{-1} \\ 2n(\gamma\phi)^{-1} & 2(m+n)\phi^{-2} \end{bmatrix}$

{$\because$ found $E[Ga(2,\phi)] = \frac{2}{\phi}$ $\because Ga(2,\phi)$ was MLE $\hat{\phi} = \frac{2}{\bar{x}}$ $\therefore \bar{x} = \frac{2}{\phi}$ $\therefore$

~~Ga(2,~~ $Ga(2,\gamma\phi)$ $\therefore \bar{y} = \frac{2}{\gamma\phi}$}

$\therefore$ Z asymptotic distri is: {$MVN\left((\gamma,\phi) = \left(\frac{\bar{x}}{\bar{y}}, \frac{2}{\bar{x}}\right), I(\gamma,\phi)^{-1}\right)$}

~~MVN~~ $I(\gamma,\phi)^{-1} = \frac{1}{2mn}\begin{bmatrix} (m+n)\gamma^2 & -n\gamma\phi \\ -n\gamma\phi & n\phi^2 \end{bmatrix}$

asymp distri of $(\hat{\gamma},\hat{\phi})$ is $N((\gamma,\phi), I(\gamma,\phi)^{-1}) = N\left((\gamma,\phi), \frac{1}{2mn}\begin{bmatrix} (m+n)\gamma^2 & -n\gamma\phi \\ -n\gamma\phi & n\phi^2 \end{bmatrix}\right)$

\2b/ {$l(\gamma,\hat{\phi})$, $\hat{\phi} = \frac{2}{\bar{x}}$ $\therefore l(\gamma,\hat{\phi};\underline{x},\underline{y}) = 2n\ln\gamma + 2(m+n)\ln\left(\frac{2}{\bar{x}}\right) - m\bar{x}\frac{2}{\bar{x}} - n\bar{y}\gamma\frac{2}{\bar{x}}$

$= 2n\ln\gamma + 2(m+n)\ln\left(\frac{2}{\bar{x}}\right) - 2m - \frac{2n\bar{y}\gamma}{\bar{x}} = 0$}

solving $l_\phi = 0$ for ϕ yields $\hat{\phi}(\gamma) = \frac{2(m+n)}{m\bar{x}+\gamma n\bar{y}}$ $\therefore$ Z profile log-likhood

for γ is $l_p(\gamma) = l(\gamma,\hat{\phi}(\gamma)) = 2n\ln\gamma - 2(m+n)\ln(m\bar{x}+\gamma n\bar{y}) +$ constant

\2c/ $2\log\Lambda = 2\left[l_p(\hat{\gamma}) - l_p(\gamma_0)\right] =$

$2\left[2n\ln\hat{\gamma} - 2(m+n)\ln(m\bar{x}+\hat{\gamma}n\bar{y}) + C_1 - 2n\ln\gamma_0 + 2(m+n)\ln(m\bar{x}+\gamma_0 n\bar{y}) - C_1\right] =$

$2\left[2n\ln\hat{\gamma} - 2(m+n)\ln(m\bar{x}+\hat{\gamma}n\bar{y}) - 2n\ln\gamma_0 + 2(m+n)\ln(m\bar{x}+\gamma_0 n\bar{y})\right] =$

$4n\ln\left(\frac{\bar{x}}{\bar{y}}\right) + 4(m+n)\ln\left[\frac{m\bar{x}+n\bar{y}}{(m+n)\bar{x}}\right]$ & its null distri is approx χ^2_1 when m & n are large

\Sheet 2/ \3a/ $G(\theta;\underline{X}) = \mu_\theta^T \Sigma^{-1}(\underline{X}-\mu)$ $\quad G(\hat\theta;\underline{X})=0$ is quasi likhood esti

, write $\underline{Z}=(x_1,\dots,x_m,y_1,\dots,y_n)$ & $\theta=(\gamma,\phi)$ $\therefore$ $\{E(X)=\frac{2}{\phi}, E(Y)=\frac{2}{\gamma\phi}$ $\therefore$

$\therefore E(\theta)=\left(\frac{2}{\phi},\frac{2}{\gamma\phi}\right)\}$ $\therefore \mu=\left(\frac{2}{\phi},\dots,\frac{2}{\phi},\frac{2}{\gamma\phi},\dots,\frac{2}{\gamma\phi}\right)$

$\{\therefore \mathrm{var}(X)=\frac{2}{\phi^2}, \mathrm{var}(Y)=\frac{2}{(\gamma\phi)^2}$ $\therefore \Sigma = \mathrm{var}(\theta) = \left\{\frac{2}{\phi^2},\dots,\frac{2}{\phi^2},\frac{2}{(\gamma\phi)^2},\dots,\frac{2}{(\gamma\phi)^2}\right\}\}$

$\Sigma = \mathrm{diag}\left(\frac{2}{\phi^2},\dots,\frac{2}{\phi^2},\frac{2}{(\gamma\phi)^2},\dots,\frac{2}{(\gamma\phi)^2}\right)$ $\therefore$

$$\mu_\theta^T = \begin{bmatrix}\partial\mu_1/\partial\gamma & \dots & \partial\mu_{m+n}/\partial\gamma \\ \partial\mu_1/\partial\phi & \dots & \partial\mu_{m+n}/\partial\phi\end{bmatrix} = \begin{bmatrix} 0 & \dots & 0 & -\frac{2}{\gamma^2\phi} & \dots & -\frac{2}{\gamma^2\phi} \\ -\frac{2}{\phi^2} & \dots & -\frac{2}{\phi^2} & -\frac{2}{\gamma\phi^2} & \dots & -\frac{2}{\gamma\phi^2}\end{bmatrix},$$

$\Sigma^{-1} = \mathrm{diag}\left(\frac{\phi^2}{2},\dots,\frac{\phi^2}{2},\frac{(\gamma\phi)^2}{2},\dots,\frac{(\gamma\phi)^2}{2}\right)$ & $G(\theta;\underline{Z}) = \mu_\theta^T\Sigma^{-1}(\underline{Z}-\mu) =$

$$\left\{\begin{bmatrix} 0 & \dots & 0 & -\frac{2}{\gamma^2\phi} & \dots & -\frac{2}{\gamma^2\phi} \\ -\frac{2}{\phi^2} & \dots & -\frac{2}{\phi^2} & -\frac{2}{\gamma\phi^2} & \dots & -\frac{2}{\gamma\phi^2}\end{bmatrix}\left(\frac{\phi^2}{2},\dots,\frac{\phi^2}{2},\frac{(\gamma\phi)^2}{2},\dots,\frac{(\gamma\phi)^2}{2}\right)\left(x_1-\frac{2}{\phi},\dots,x_m-\frac{2}{\phi},y_1-\frac{2}{\gamma\phi},\dots,y_n-\frac{2}{\gamma\phi}\right)\right\}$$

$= \begin{bmatrix} 2n/\gamma - n\bar y\phi \\ -m\bar x - n\bar y\gamma + 2(m+n)/\phi \end{bmatrix}$ $(=\{0\})$ solving $G(\theta;\underline{Z})=0$ shows

$\left\{\begin{bmatrix} 2n/\gamma - n\bar y\phi \\ -m\bar x - n\bar y\gamma + 2(m+n)/\phi \end{bmatrix} = \begin{bmatrix}0\\0\end{bmatrix} = 0\right.$ $\therefore \frac{2n}{\gamma} - n\bar y\phi = 0$, $-m\bar x - n\bar y\gamma + \frac{2(m+n)}{\phi} = 0$ $\therefore$

$\frac{2n}{\gamma} = n\bar y\phi$ $\therefore \frac{2n}{n\bar y\gamma} = \phi = \frac{2}{\bar y\gamma}$ $\therefore +m\bar x + n\bar y\gamma = \frac{2(m+n)\bar y\gamma}{2} = (m+n)\bar y\gamma$ $\therefore$

$m\bar x = (m+n)\bar y\gamma - n\bar y\gamma = \gamma[(m+n)\bar y - n\bar y] = m\bar x = m\bar y\gamma$ $\therefore \hat\gamma = \frac{\bar x}{\bar y}$ $\therefore$

$\hat\phi = \frac{2}{\bar y}\frac{\bar y}{\bar x} = \frac{2}{\bar x} = \hat\phi\}$ $\therefore$ shows $G(\theta;\underline{Z})=0$ shows Z quasi-likhood esti is Z same as Z max likhood esti as (part 2a)

\3b/ $m=10$; $\phi_0=2$ $\therefore$ >> rgamma(m, 2, phi)

$\therefore$ >> crit = qgamma(0.95, 2*m, 2)

>> power.ump = 1-pgamma(crit, 2*m, phi)

likelihood ratio test, score test

uniformly most powerful test

Z power funcs: Z Wald test is Z same as Z score test for this ex & $\therefore$ comment only on Z score & likhood ratio tests. Z sizes of Z two tests differ slightly from 5% $\because$ only an approx to Z null distri is used: Z monte Carlo approx's to Z sizes are 4.7% for Z score test & 5.1% for Z LR test. Z score test is slightly more powerful than Z LR test at detecting when $\phi<2$. But is less powerful at detecting when $\phi>2$. Both tests are slightly less power but than Z UMP test at detecting when $\phi<2$ as expected. (Z UMP test has low power for $\phi>2$ ∵ it was designed for Z one-sided alternative $H_1: \phi<2$.) Might prefer Z LR test to Z score test in this ex ∵ its size is closer to 5%, it has better power when $\phi>2$, Z has similar power when $\phi<2$

~~Monte Carl~~ {Marks for Caption: Monte Carlo approxs to Z power funcs of Z score & LR tests from part 1b. Z power of Z UMP test for a one sided alternative is also shown}

∴ power at $\phi=2$ is 0.05 but further away from $\phi=2$ want to reject null more often

\3ci/ {>>phi_hat = 2/mean(x) ⇒ 1.87} MLE is $\hat{\phi}=1.88$ & Z asymptotic estied standard error is $\left\{ I(\theta)^{-1} = \frac{\phi^2}{2m} \therefore \right.$ ~~estd SE~~ estied SE: $\left.\frac{\hat{\phi}}{\sqrt{2m}} = 0.419 \right\}$

estied SE: $\hat{\phi}/\sqrt{2m} = 0.42$

\3cii/ Z Wald & Score test stat stats are 0.087 with p-val 0.77. Alternatively, Z lihood ratio test stat is 0.083 also with p-val 0.77. neither test provides any evidence to reject Z null hypoth at any reasonable significance level

\3ciii/ $\left\{ \frac{\hat{\phi}}{1+\sqrt{c/2m}} < \phi < \frac{\hat{\phi}}{1-\sqrt{c/2m}} \right.$ or $\left(\hat{\phi}\left[1-\sqrt{c/(2m)}\right], \hat{\phi}\left[1+\sqrt{c/(2m)}\right]\right)$

c is α-quantile of χ^2_1 distri

$(1-2\alpha)$ CI ∴ 90% ⇒ $\alpha = 5\%$. ∴ }

MLE based 90% CI is (1.37, 2.97) Alternatively, Z Wald-based 90% CI is (1.19, 2.57)

\3civ/ $\left\{ Ga(2, \gamma\phi) \qquad \hat{\gamma} = \frac{\bar{x}}{\bar{y}}, \hat{\phi} = 2/\bar{x} \qquad \therefore \hat{\gamma}\hat{\phi} = \frac{\bar{x}}{\bar{y}}\frac{2}{\bar{x}} = \frac{2}{\bar{y}} \right\}$

test $H_0: \gamma=1$ against $H_1: \gamma\neq1$ using Z LR test from part 2c (H_0 corres with p-value 0.030. if we use Z χ^2_1 null distri. (Z samples may be too small for this to be a good approx to Z null distri.) Conclude there is evidence at 5% level that Z times take different time to complete

{0.0304 ~~∉~~ is not in (1.186, 2.13) 0.0304 ∉ (1.186, 2.13)

large

\Sheet 3/ \1a/ {bootstrapping is to use the current data to produce new test data then do analysis on comparing Z two data sets then use Z new data to calc distri stats}

Bootstrap resampling means to generate new samples of Z same size as Z original sample either by sampling from Z empirical distri of Z original sample (nonparam resampling) or by sampling from Z distri of a fitted probab model (param resampling) if Z stat of interest is evaluated for each of many such samplings then Z empirical distri of these bootstrap stats is an esti of Z sampling distri of Z original stat

\1b/ { • use $G_a(2, \hat{\theta})$ $\quad \hat{\theta} = \frac{2}{\bar{x}}$ to make 1000 sets of samples then find Z average of all these samples { $\text{bias}(\hat{\theta}) = E(\hat{\theta}) - \theta$ }

bias-corrected estimator $\tilde{\theta} = \hat{\theta} - \frac{1}{2}\widehat{\text{var}}(-\hat{\theta})h''(\hat{\theta})$

$t_i^* = \frac{2}{\bar{x}_j^*} - \hat{\theta}$ $\quad \tilde{\theta} = \hat{\theta} - \bar{t}^* = \hat{\theta} - \sum_{i=1}^{b} t_i^* = \hat{\theta} - \frac{1}{b}\sum_{i=1}^{b}\left(\frac{2}{\frac{1}{m}\sum_{j=1}^{m} x_{ij}^*} - \hat{\theta}\right) =$

$\cancel{\hat{\theta} - \frac{2}{mb}\sum_{i=1}^{b}\left(\frac{1}{\sum_{j=1}^{m}}\right.}$ $\hat{\theta} - \frac{2m}{b}\sum_{i=1}^{b}\left(\frac{1}{\sum_{j=1}^{m} x_{ij}^*}\right) + \frac{1}{b}\sum_{i=1}^{b}(\hat{\theta}) = \hat{\theta} - \frac{2m}{b}\sum_{i=1}^{b}\left(\frac{1}{\sum_{j=1}^{m} x_{ij}^*}\right) + \frac{1}{b}b\hat{\theta} =$

$2\hat{\theta} - \frac{2m}{b}\sum_{i=1}^{b}\left(\frac{1}{\sum_{j=1}^{m} x_{ij}^*}\right)$ }

\1c/ { $\sum_{j=1}^{m}(x_j) = 10.66$ $\quad \hat{\theta} = \frac{2}{\bar{x}} = \frac{2m}{\sum_{i=1}^{m} x_i} = 1.876$

i	1	2	3	4	5	6	7	8	9	10
x_i	1.77	0.34	0.43	1.03	0.57	1.27	0.42	1.94	1.73	1.14
$\sum_{j=1}^{m}(x_j) - x_i = \sum_{j \neq i} x_j$	8.87	10.32	10.23	9.63	10.09	9.39	10.24	8.72	8.93	9.52
$\hat{\theta}_{-i} = \frac{2}{\frac{1}{(m-1)}\sum_{j \neq i} x_j}$	2.03	1.74	1.76	1.87	1.78	1.92	1.76	2.06	2.02	1.89
$\hat{\theta}_i = m\hat{\theta} - (m-1)\hat{\theta}_{-i}$	0.50	3.06	2.92	1.93	2.74	1.48	2.92	0.22	0.58	1.75
$(\hat{\theta}_i - \hat{\theta}_J)^2$	1.72	1.56	1.23	0.014	0.86	0.109	1.23	2.53	1.51	0.0036

$\hat{\theta}_J = \frac{1}{m}\sum_{i=1}^{m}\hat{\theta}_i = 1.81$

$\therefore \sum_{i=1}^{m}\left[(\hat{\theta}_i - \hat{\theta}_J)^2\right] = 10.81$ $\quad \therefore$ { SE $\hat{\theta}_J = 1.81$ $\quad$ SE: 0.35 }

jackknife version of $\hat{\theta}$ is $\hat{\theta}_J = \frac{1}{m}\sum_{i=1}^{m}\hat{\theta}_i = 1.81$, $m = 10$, $\hat{\theta}_i = m\hat{\theta} - (m-1)\hat{\theta}_{-i}$

& $\hat{\theta}_{-i} = \frac{2(m-1)}{\sum_{j=1}^{m} x_j - x_i}$ $\quad$ Z jackknife esti of Z SE is $\sqrt{v_J^2} = 0.35$ where

$\hat{V}_J = \frac{1}{m(m-1)} \sum_{i=1}^{m} (\hat{\theta}_i - \hat{\theta}_J)^2$

Z jackknife 90% CI for θ is $\hat{\theta}_J \pm 1.833\sqrt{\hat{V}_J} = (1.18, 2.45)$, 1.833 is Z 95% quantile of Z $Stu(m-1)$ distri {df = m-1 : >>qt(0.95, df=9) ⇒ 1.833113} {stu(9) for 95% quantile on formula sheet : df = m-1 = 9 ∴ $\nu = 9$ 90% CI ∴ 95% quantile ∴ ~~0.05~~ $\alpha = 0.05$ ∴ $t_{\alpha;\nu} = t_{0.05;9} = 1.833$ for Z ~~10%~~ 5% ~~sig~~ value

\2a/ {jackknife test of $H_0: \theta = 2$, $H_1: \theta \neq 2$ ∴ $\hat{\theta}_J = 1.81$ SE = 0.35 ∴ as 90% CI ∴ $\alpha = 0.05$ ∴ ~~$t_{m-1,\alpha} = 2t_{9,2\alpha}$~~ $= 1.833 = t_{\alpha, m-1} = t_{\alpha, 9} = t_{0.05, 9}$ ∴ $(2 - 1.833 \times 0.35, 2 + 1.833 \times 0.35) = (1.358, 2.641)$ ∴ $1.81 = \hat{\theta}_J \in (1.358, 2.641)$ ∴ do not reject H_0, no evidence that $\theta \neq 2$}

Z test stat is $(\hat{\theta}_J - 2)/\sqrt{\hat{V}_J} = -0.54$ {kind like $z = \frac{x-\mu}{\sigma}$} with p-val 2*(1-pt(abs(-0.54),9)) = 0.60 {>>pt(q, df)   q is quantile & pt gives Z distri func pt(0.54,9) = 0.699} ∴ for Z two-sided test ∴ there's insufficient evidence to reject Z null hypoth that $\theta = 2$ {test stat $\frac{\hat{\theta}_J - \theta_0}{\sqrt{\hat{V}_J}}$ is approx $Stu(n-1)$ ∴ compare -0.54 to a $Stu(m-1)$ distri to obtain a p-val of 2*(1-pt(|-0.54|, 9)) = 0.602 ∴ not significant}

\2b/ Z score test stat is $2m(1 - \theta_0/\hat{\theta})^2 = 0.087$ {$2 \times 10(1 - 2/\hat{\theta}) = 2 \times 10(1 - 2/1.88)^2 = 0.0816$ ~~$(2)(10)(1 - \frac{2}{...}$~~ ~~$2(10)(1 - \frac{2}{1.81})^2 =$~~ $2(10)(1 - \frac{2}{1.81})^2 = 0.226$} with p-val 0.77 {2*(1-pt((0.087),9)) = 0.932}

& Z likhood ratio test stat {$4m\left[\ln\frac{\hat{\theta}}{\theta_0} + \frac{\theta_0}{\hat{\theta}} - 1\right] = 4(10)\left[\ln\left(\frac{1.81}{2}\right) - \frac{2}{1.81} - 1\right] =$ ~~0.21~~ 0.21} is $4m\left[\log\left(\frac{\hat{\theta}}{\theta_0}\right) + \frac{\theta_0}{\hat{\theta}} - 1\right] = 0.083$ with p-val 0.77, for either test, there is insufficient evidence to reject Z null hypoth that $\theta = 2$ using R code: as expected these are Z same results as those obtained for Wald & score & likhood ratio test stats & p-vals

\2e/ {$\hat{\theta} = \frac{2}{\bar{y}}$} Z test stat is $4n\log(\bar{x}/\bar{y}) + 4(m+n)\log\left[\frac{m\bar{x} + n\bar{y}}{(m+n)\bar{x}}\right] = 4.69$ with p-val 0.031 {0.031 < 0.05} there is evidence to reject Z null hypoth at 5% level & conclude that Z there are different time for tasks \\\

\Sheet 3/ \3a/ {basic bootstrap interval is $(\hat{\theta}-\hat{q}_{1-\alpha}, \hat{\theta}-\hat{q}_{\alpha})$

studentised bootstrap interval $(\hat{\theta}-\hat{q}_{1-\alpha}s, \hat{\theta}-\hat{q}_{\alpha}s)$

percentile bootstrap interval $(\hat{\theta}^*_{(\alpha B)}, \hat{\theta}^*_{((1-\alpha)B)})$ }

for param resampling, Z 90% CI are (0.92, 2.41) (basic)

(1.24, 2.62) (studentised) (1.37, 2.84) (percentile). for non param

resampling, Z intervals are (1.15, 2.29) (basic) (1.35, 2.41) (studentised)

, (1.46, 2.60) (percentile)

\3bi/ $X \sim Ga(2, \theta)$ $\therefore f(x, \theta) = \theta^2 x e^{-\theta x}$, $x>0$ scale model is $X = \sigma Z$,

, Z has a distri doesnt depend on (any unknown) params $\therefore$

let $Z = \theta x$ $\therefore P(Z \leq z) = P(\theta x \leq z) = P(x \leq \frac{z}{\theta}) = \int_0^{z/\theta} \theta^2 x e^{-\theta x} dx =$ $\{y = \theta x\}$

$\{0$ to z, $\frac{1}{\theta} dy = dx\}$ $= \int_0^z \theta y e^{-y} \frac{1}{\theta} dy = \int_0^z y e^{-y} dy$ doesnt depend

on θ $\therefore$ is a scale model $\therefore$ Z pdf of Z is ze^{-z} $\therefore Z \sim Ga(2,1)$

show $Ga(2, \theta)$ distri is Z distri of Z/θ, Z has $Ga(2,1)$ distri

$\therefore X_i = Z_i/\theta_i$ $\therefore Pr(Z_i \leq z) = Pr(X_i \leq z/\theta) = \int_0^{z/\theta} f(x;\theta) dx = \int_0^{z/\theta} \theta^2 x e^{-\theta x} dx = \int_0^z t e^{-t} dt$

$\therefore$ Z final integrand is Z density of a $Ga(2,1)$ distri

$\{Ga(2, \theta): f(x, \theta) = \theta^2 x e^{-\theta x}$ $\therefore Ga(2,1) = 1^2 x e^{-1x} = xe^{-x}\}$ $\therefore Z_i$ has Z $Ga(2,1)$

distri

\3bii/ Ancillary stat T $\therefore (X_0, X_1, \dots, X_m)$ $\therefore T = \frac{X_0}{\bar{x}}$ $\therefore$

$$T = \frac{Z_0 \theta}{\frac{1}{m}\sum_{i=1}^{m} \frac{Z_i}{\theta}} = \frac{Z_0}{\bar{Z}}$$ $\therefore$ T is ancillary

$P(t_{0.05} < T < t_{0.95}) = 0.9$ ie $P(t_{0.05}\bar{X} < X_0 < t_{0.95}\bar{X}) = 0.9$ }

Z stat $T = X_0/\bar{X}$ is ancillary $\therefore X_0/\bar{X} = Z_0/\bar{Z}$, Z_i have $Ga(2,1)$

distris, if q_p denotes Z p-quantile of T then $0.90 = Pr$

$Pr(q_{0.05} < X_0/\bar{X} < q_{0.95}) = Pr(q_{0.05}\bar{X} < X_0 < q_{0.95}\bar{X})$ & $\therefore$ 90% prediction interval

is $(\hat{q}_{0.05}\bar{X}, \hat{q}_{0.95}\bar{X})$, $\hat{q}_p$ is Z p-quantile of Z bootstrap version of

T, that is of $T^* = X_0^*/\bar{X}^*$, where Z $\bar{X}^*_i$ are indep with distri

$Ga(2, \hat{\theta})$ $\therefore$ interval (0.19, 2.78) seconds

\3c/ coverages of 88.5% (basic), 87.8% (studentised), 88.2%

(percentile). Z coverages of all three intervals are slightly low, but Z

Coverage of Z Studentised interval is very close to Z nominal 90%.

Sheet 1: preparatory: Q1 - 3.29a / Likelihood:

$L(\theta) = \prod_{i=1}^n f(x_i;\theta) = \prod_{i=1}^n \theta x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^n x_i\right)^{\theta-1}$ ∴

Loglikelihood $l(\theta) = n\ln\theta + (\theta-1)\sum_{i=1}^n \ln x_i$ ∴

& Score: $l'(\theta) = \frac{n}{\theta} + \sum_{i=1}^n \ln x_i$, 2nd deriv: $l''(\theta) = -\frac{n}{\theta^2} < 0$ ∴

$l'(\hat\theta) = \frac{n}{\hat\theta} + \sum_{i=1}^n \ln x_i$ is a maximum for $\hat\theta$ ∴

$l'(\hat\theta)=0$ yields Z mle: $\hat\theta = -n/\sum_{i=1}^n \ln x_i$

Q1. 3.29b / expectation is: $E(X) = \int_0^1 x f(x;\theta)\,dx = \int_0^1 \theta x^\theta\,dx = \int_0^1 \theta e^{\ln(x^\theta)}\,dx =$
$\int_0^1 \theta e^{\theta\ln x}\,dx = \int_0^1 \theta x^\theta\,dx = \theta\int_0^1 x^\theta\,dx = \theta\left[\frac{1}{\theta+1}x^{\theta+1}\right]_0^1 = \frac{\theta}{\theta+1}$ equating to $\bar X$ & solving for θ yields Z method of moments estimator $\{E(X) = \frac{\theta}{\theta+1} = \bar X = \frac{\hat\theta}{\hat\theta+1}$ ∴
$(\hat\theta+1)\bar X = \hat\theta = \hat\theta\bar X + \bar X$ ∴ $\hat\theta - \hat\theta\bar X = \bar X = \hat\theta(1-\bar X)$ ∴ $\hat\theta = \bar X/(1-\bar X)\}$ $\hat\theta = \bar X/(1-\bar X)$

2. 2.3 / $E(X_1) = 4\theta$, $var(X_1) = 4\theta^2$, $n=30$: $\bar x = 11.59$ ∴
$E(\hat\theta) = E(\bar X/4) = E(\frac{1}{4}\bar X) = E(\frac{1}{4}\frac{1}{n}\sum_{i=1}^n X_i) = E(\frac{1}{4n}\sum_{i=1}^n X_i) = \frac{1}{4n}\sum_{i=1}^n E(X_i) =$
$\frac{1}{4n}\sum_{i=1}^n (4\theta) = \frac{1}{4n}n4\theta = \frac{4n}{4n}\theta = \theta$ ∴ st $\hat\theta$ is unbiased $\{Bias(\hat\theta) = E(\hat\theta) - \theta = \theta - \theta = 0$ ∴ unbiased$\}$ & : $var(\hat\theta) = var(\bar X/4) = var(\frac{1}{4n}\sum_{i=1}^n X_i) = (\frac{1}{4n})^2 var(\sum_{i=1}^n X_i) =$
$\frac{1}{16n^2} var(\sum_{i=1}^n X_i) = \frac{1}{16n^2}\sum_{i=1}^n var(X_i) = \frac{1}{16n^2}\sum_{i=1}^n 4\theta^2 = \frac{1}{16n^2}n4\theta^2 =$
$\frac{1}{4n}\theta^2 = \frac{\theta^2}{4n}$

$\frac{\theta^2}{4n} \to 0$ as $n\to\infty$ ∴ $var(\hat\theta)\to 0$ as $n\to\infty$ st $\hat\theta$ is consistent

For Z given data, $\hat\theta = \bar x/4 = 2.9$ with estimated standard error
$\sqrt{\hat\theta^2/(4n)} = \sqrt{2.9^2/(4\times 30)} = 0.26$

3 / $h(\theta) = \theta(1-\theta)$ ∴ $h'(\theta) = -\theta + 1 - \theta = 1 - 2\theta$ ∴ $h''(\theta) = -2$ &
$var(\hat\theta) = var(X_1)/n = \theta(1-\theta)/n$ $\{\hat\theta = \bar X$ ∴ $var(\hat\theta) = var(\bar X) = var(\frac{1}{n}\sum_{i=1}^n X_i) =$
$\frac{1}{n^2}var(\sum_{i=1}^n X_i) = \frac{1}{n^2}\sum_{i=1}^n var(X_i) = \frac{1}{n^2}\sum_{i=1}^n var(X_1) = \frac{1}{n^2}n\,var(X_1) = \frac{1}{n}var(X_1) = \frac{1}{n}\theta(1-\theta)\}$
∴ $var(\hat\phi) \approx [h'(\theta)]^2 var(\hat\theta) = (1-2\theta)^2\theta(1-\theta)/n$

$h(\hat\theta) \approx h(\theta) + (\hat\theta - \theta)h'(\theta)$ $var(h(\hat\theta)) \approx var(h(\theta) + (\hat\theta-\theta)h'(\theta)) = var(\hat\theta h'(\theta)) =$
$[h'(\theta)]^2 var(\hat\theta)$ ∴ $\hat\phi = h(\hat\theta)$ $h(\hat\theta) \approx h(\theta) + (\hat\theta-\theta)h'(\theta) + \frac{1}{2}(\hat\theta-\theta)^2 h''(\theta)$
$E(\hat\phi) \approx h(\theta) + (E(\hat\theta)-\theta)h'(\theta) + \frac{1}{2}E((\hat\theta-\theta)^2)h''(\theta) = \phi + \frac{1}{2}var(\hat\theta)h''(\theta)$ ∴
$Bias(\hat\phi) = E(\hat\phi) - \phi \approx \frac{1}{2}var(\hat\theta)h''(\theta)$ ∴ $\tilde\phi = \hat\phi - Bias(\hat\phi) = \hat\phi - (E(\hat\phi) - \phi) = \hat\phi - \frac{1}{2}var(\hat\theta)h''(\hat\theta)$

Sheet 1 / bias corrected estimator: $\tilde{\theta} = \hat{\theta} - \frac{1}{2} \text{var}(\hat{\theta}) h''(\hat{\theta}) =$
$\hat{\theta} - \frac{1}{2}\theta(1-\theta)\frac{1}{n}(-2) = \hat{\theta} + \theta(1-\theta)\frac{1}{n} = \hat{\theta} + \hat{\theta}\frac{1}{n} = (1+\frac{1}{n})\hat{\theta}$

4.2.3 / likelihood $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} e^{-(x_i-\theta)} e^{-e^{-(x_i-\theta)}} =$
$e^{\sum_{i=1}^{n}(-(x_i-\theta))} e^{\sum_{i=1}^{n}(-e^{-(x_i-\theta)})} = e^{\sum_{i=1}^{n}(\theta-x_i)} e^{\sum_{i=1}^{n}(-e^{-(x_i-\theta)})} = e^{n\theta - \sum_{i=1}^{n} x_i} e^{-\sum_{i=1}^{n} e^{\theta-x_i}} =$
$e^{n\theta - \sum_{i=1}^{n} x_i} \exp(-\sum_{i=1}^{n} e^{\theta - x_i})$ $\therefore$ loglikelihood: $l(\theta) = n\theta - \sum_{i=1}^{n} x_i - \sum_{i=1}^{n} e^{\theta-x_i} =$
$n\theta - \sum_{i=1}^{n} x_i - e^{\theta}\sum_{i=1}^{n} e^{-x_i}$ $\therefore l'(\theta) = n - e^{\theta}\sum_{i=1}^{n} e^{-x_i}$ $\therefore$

$l''(\theta) = -e^{\theta}\sum_{i=1}^{n} e^{-x_i}$ $\therefore$ Z expected information:
$I(\theta) = -E(l''(\theta)) = n e^{\theta} E(e^{-X_1}) = n$ $\therefore$
$E(e^{-X_1}) = \int_{-\infty}^{\infty} e^{-2x+\theta} \exp(-e^{-x+\theta})\, dx = \int_0^{\infty} e^{-\theta} y e^{-y}\, dy = e^{-\theta}$
where $y = \exp(\theta - x)$ $\therefore$ Z CRLB is $I(\theta)^{-1} = 1/n$

$\{ l''(\theta) = -e^{\theta}\sum_{i=1}^{n} e^{-x_i}$ $\therefore E(l''(\theta)) = E(-e^{\theta}\sum_{i=1}^{n} e^{-x_i}) = -e^{\theta} E(\sum_{i=1}^{n} e^{-x_i}) =$
$-e^{\theta}\sum_{i=1}^{n} E(e^{-X_i}) = -e^{\theta}\sum_{i=1}^{n} E(e^{-X_1}) = \cancel{-n e^{\theta} E(} -e^{\theta} n E(e^{-X_1}) \neq -E(l''(\theta))$ $\therefore$

$E(e^{-X_1}) = \int_{-\infty}^{\infty} e^{-x_1} f(x;\theta)\, dx = \int_{-\infty}^{\infty} e^{-x_1} e^{-(x-\theta)} e^{-e^{-(x-\theta)}}\, dx =$
$\int_{-\infty}^{\infty} e^{-x_1} e^{-(x-\theta)} \exp(-e^{(-x+\theta)})\, dx = \int_{-\infty}^{\infty} e^{-x} e^{-x+\theta} \exp(-e^{-x+\theta})\, dx = E(e^{-X_1})$ $\therefore$
let $y = e^{\theta - x} = \exp(\theta - x) = e^{-x+\theta}$ $\therefore dy = -e^{-x+\theta} dx$ $\therefore e^{-x+\theta} dx = -dy$ $\therefore$

at $x = -\infty$: $e^{-x+\theta} = e^{--\infty+\theta} = e^{\infty} = \infty$
at $x = \infty$: $e^{-x+\theta} = e^{-\infty+\theta} = e^{-\infty} = 0$ $\therefore$
$E(e^{-X_1}) = \int_{-\infty}^{\infty} e^{-x} e^{-x+\theta} \exp(-e^{-x+\theta})\, dx = \int_{-\infty}^{\infty} e^{-2x+\theta} \exp(-e^{-x+\theta})\, dx =$
$\int_{+\infty}^{0} e^{-x} \exp(-y)(-1)\, dy = \int_0^{\infty} e^{x} e^{-y}\, dy$ X

$E(e^{-X_1}) = \int_{-\infty}^{\infty} e^{-2x+\theta} \exp(-e^{-x+\theta})\, dx = \int_{-\infty}^{\infty} e^{-2x+\theta} e^{-e^{-x+\theta}}\, dx = \int_{-\infty}^{\infty} e^{-2x+\theta} e^{-e^{\theta-x}}\, dx$
$y = \exp(\theta - x) = e^{\theta - x}$ $\therefore x = -\infty \to y = \infty$, $x = \infty \to y = 0$
$\frac{dy}{dx} = -e^{\theta-x} = -y$ $\therefore -\frac{1}{y} dy = dx$ $\ln y = \theta - x$ $\therefore x = \theta - \ln y$ $\therefore$
$2x = 2\theta - 2\ln y$ $\therefore -2x = -2\theta + 2\ln y$ $\therefore -2x+\theta = -2\theta + 2\ln y + \theta = -\theta + 2\ln y =$
$-\theta + \ln y^2 = \cancel{e^{\ln(-\theta)}}\ \ln(e^{-\theta}) + \ln y^2 = \ln(e^{-\theta} y^2)$ $\therefore e^{-2x+\theta} = e^{\ln(e^{-\theta} y^2)} = e^{-\theta} y^2$ $\therefore$

$E(e^{-X_1}) = \int_{-\infty}^{\infty} e^{-2x+\theta} e^{-e^{\theta-x}} dx = \int_{\infty}^{0} e^{-\theta} y^2 e^{-y} (-\frac{1}{y}) dy = e^{-\theta} \int_{\infty}^{0} -y e^{-y} dy =$

$e^{-\theta} \int_0^{\infty} y e^{-y} dy = e^{-\theta} \left[-y e^{-y}\right]_0^{\infty} - e^{-\theta} \int_0^{\infty} -e^{-y} dy =$

$-e^{-\theta}[-0+0] - e^{-\theta}\left[e^{-y}\right]_0^{\infty} = -e^{-\theta}[0 - e^0] = -e^{-\theta}(-1) = e^{-\theta} = E(e^{-X_1})$ ∴

$E(l''(\theta)) = -e^{\theta} n E(e^{-X_1}) = -e^{\theta} n e^{-\theta} = -n$ ∴

$I(\theta) = -E(l''(\theta)) = --n = n$ ∴

CRLB is: $I(\theta)^{-1} = \frac{1}{I(\theta)} = \frac{1}{n}$.

$l'(\theta) = n - e^{\theta} \sum_{i=1}^{n} e^{-x_i} = n\left(1 - \frac{1}{n} e^{\theta} \sum_{i=1}^{n} e^{-x_i}\right) \neq n(\hat{\theta} - \theta) = I(\theta)(\hat{\theta} - \theta)$ ∴

Z score $l'(\theta)$, cannot be written in Z form $I(\theta)(\hat{\theta} - \theta)$ & ∴ by thm 1.2: there is no estimator for θ that is both unbiased & efficient

\5.4.5a/ $f(x;\theta) = \frac{1}{2\theta^3} x^2 e^{-x/\theta}$ ∴ likelihood: $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) =$

$\prod_{i=1}^{n} \frac{1}{2\theta^3} x_i^2 e^{-x_i/\theta} = \prod_{i=1}^{n} \theta^{-3} e^{-x_i/\theta} \frac{x_i^2}{2} = \theta^{-3n} e^{\sum_{i=1}^{n} -x_i/\theta} \prod_{i=1}^{n} \frac{x_i^2}{2} =$

$\theta^{-3n} e^{-n\bar{x}/\theta} \prod_{i=1}^{n} \frac{x_i^2}{2} = L(\theta)$ ∴ & Z likelihood ratio is:

$\Lambda(\underline{x}) = L(\theta_1)/L(\theta_0)$ ∴

$L(\theta_1) = \theta_1^{-3n} e^{-n\bar{x}/\theta_1} \prod_{i=1}^{n} \frac{x_i^2}{2}$, $L(\theta_0) = \theta_0^{-3n} e^{-n\bar{x}/\theta_0} \prod_{i=1}^{n} \frac{x_i^2}{2}$ ∴

$\Lambda(\underline{x}) = L(\theta_1)/L(\theta_0) = \left(\theta_1^{-3n} e^{-n\bar{x}/\theta_1} \prod_{i=1}^{n} \left(\frac{x_i^2}{2}\right)\right) / \left(\theta_0^{-3n} e^{-n\bar{x}/\theta_0} \prod_{i=1}^{n} \frac{x_i^2}{2}\right) =$

$\frac{\theta_0^{3n}}{\theta_1^{3n}} e^{-n\bar{x}\frac{1}{\theta_1} + n\bar{x}\frac{1}{\theta_0}} = \left(\frac{\theta_0}{\theta_1}\right)^{3n} e^{\left(\frac{1}{\theta_0} - \frac{1}{\theta_1}\right) n\bar{x}}$ since $\theta_1 > \theta_0$: $\Lambda(\underline{x}) \geq c$ iff $\bar{x} \geq d$ & ∴ Z most powerful critical region has Z form $\{\underline{x} : \bar{x} \geq d\}$

\5.4.5b/ For a test of size α, requires d to satisfy

$\alpha = Pr(\bar{X} \geq d; \theta_0) = Pr(2\sum_{i=1}^{n} X_i/\theta_0 \geq 2nd/\theta_0; \theta_0)$ ∴ $2nd/\theta_0$ must be Z $(1-\alpha)$ quantile $c_{1-\alpha}$ of Z χ^2_{6n} distri ie $d = c_{1-\alpha}\theta_0/(2n)$

when $n=4$, $\theta_0 = 1$, $\alpha = 0.05$: $d = 36.415/8 = 4.55$ ∴ Z power is -

$Pr(\bar{X} \geq d; \theta_1) = Pr(2\sum_{i=1}^{n} X_i/\theta_1 \geq 2nd/\theta_1; \theta_1)$ & $2nd/\theta_1 = c_{1-\alpha}\theta_0/\theta_1 =$

$36.415/3 = 12.14$ with $2\sum_{i=1}^{n} X_i/\theta_1 \sim \chi^2_{6n}$ ∴ Z power is 0.978

\5.4.5c/ ∵ d is indep of θ_1, this test is Z uniformly most powerful for $H_1': \theta > \theta_0$ for Z given data, $\bar{x} \geq 4.55$ & ∴ we reject H_0 in favour of H_1' at Z 5% level

\Sheet 1\ \6.5.1/ $f(x;\theta) = (\frac{1}{\theta})e^{-x/\theta}$ Let c_α be Z α-quantile of χ^2_{2n} ∴ $Pr(c_\alpha < 2n\bar{x}/\theta < c_{1-\alpha}) = 1-2\alpha$ ∴

$$Pr\left(\frac{c_\alpha}{2n\bar{x}} < \frac{1}{\theta} < \frac{c_{1-\alpha}}{2n\bar{x}}\right) = 1-2\alpha$$ ∴

~~Pr(~~ $Pr\left(\frac{2n\bar{x}}{c_{1-\alpha}} < \theta < \frac{2n\bar{x}}{c_\alpha}\right) = 1-2\alpha$ ∴ $Pr(2n\bar{x}/c_{1-\alpha} < \theta < 2n\bar{x}/c_\alpha) = 1-2\alpha$ & ∴ a $(1-2\alpha)$ CI for θ is $(2n\bar{x}/c_{1-\alpha}, 2n\bar{x}/c_\alpha)$.

\5.4.5/ Z critical region for Z test of size α was $C(\theta_0) = \{\underline{x} : \bar{x} \geq c_{1-\alpha}\theta_0/(2n)\}$ & ∴ a $(1-\alpha)$ CI for θ is $S(\underline{x}) = \{\theta : \bar{x} < c_{1-\alpha}\theta/(2n)\} = (2n\bar{x}/c_{1-\alpha}, \infty)$ for Z given data, this interval is $(8(7.39)/36.415, \infty) = (1.62, \infty)$

\7/ $f(x;\theta) = \frac{1}{\theta}$ $\hat{\theta} = \max\{x_1, \dots, x_n\}$ ∴ Z distri func of $\hat{\theta}$ is $Pr(\hat{\theta} \leq t) = Pr(\max\{x_1, \dots, x_n\} \leq t) = \prod_{i=1}^{n} Pr(X_i \leq t) = Pr(X_1 \leq t)^n = (t/\theta)^n$ { ∴ $Pr(x_1 \leq t) = \int_0^t f(x;\theta)\,dx = \int_0^t \frac{1}{\theta}\,dx = \frac{1}{\theta}[x]_0^t = \frac{t}{\theta}$ } ∴

for $0 < t < \theta$, show Z distri func of T is indep of θ. note that T can be any posi number, but X_0 & $\hat{\theta}$ lie on $(0, \theta)$. (explains Z upper limit x must lie betw 0 & θ but if $x > \theta t$, then $x/t > \theta$ & $Pr(\hat{\theta} \geq x/t) = 0$)

∴ $Pr(T \leq t) = Pr(\hat{\theta} \geq X_0/t) = \int_0^{\min\{\theta, \theta t\}} Pr(\hat{\theta} \geq x/t) f(x;\theta)\,dx$ (by indep of X_0 & $\hat{\theta}$)

$$= \theta^{-1}\int_0^{\min\{\theta,\theta t\}} [1 - x^n/(\theta t)^n]\,dx = \min\{1, t\} - \theta^{-n-1}t^{-n}\left[x^{n+1}/(n+1)\right]_0^{\min\{\theta,\theta t\}} =$$

$$\min\{1, t\} - \frac{\min\{1,t\}^{n+1}}{(n+1)t^n} = \begin{cases} \frac{nt}{n+1} & \text{if } 0 \leq t \leq 1 \\ 1 - \frac{1}{(n+1)t^n} & \text{if } t > 1 \end{cases}$$

this distri doesn't depend on θ & ∴ T is ancillary.

Z α-quantile of T satisfies $\alpha = Pr(T \leq t_\alpha) = nt_\alpha/(n+1)$ if $0 \leq t_\alpha \leq 1$ ie if $0 \leq \alpha \leq n/(n+1)$. this is Z case if $\alpha \leq \frac{1}{2}$ & ∴ $t_\alpha = \alpha(n+1)/n$.

similarly, Z $(1-\alpha)$ quantile of T satisfies $1-\alpha = Pr(T \leq t_{1-\alpha}) = nt_{1-\alpha}/(n+1)$ if $0 \leq t_{1-\alpha} \leq 1$ ie if $0 \leq 1-\alpha \leq n/(n+1)$. this is Z case if $\alpha \geq 1/(n+1)$ & ∴ $t_{1-\alpha} = (1-\alpha)(n+1)/n$ ∴ $1-2\alpha = Pr(t_\alpha < T < t_{1-\alpha}) =$

$Pr(\alpha(n+1)/n < X_0/\hat{\theta} < (1-\alpha)(n+1)/n) = Pr(\alpha(1+1/n)\hat{\theta} < X_0 < (1-\alpha)(1+\frac{1}{n})\hat{\theta})$ as required. for Z data, $n=7$, $\hat{\theta} = 4.72$, $\alpha = 0.1$ does lie betw $1/(n+1)$ & $\frac{1}{2}$ ∴

Z 80% pred interv is $(0.1(1+\frac{1}{7})4.72, 0.9(1+\frac{1}{7})4.72) = (0.57, 4.72)$

Sheet 1 Extra exercises: / 1.3.1, a, b, c / a /

$f(x;\theta) = \theta(1-\theta)^{x-1}$ ∴ Likelihood is $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} \theta(1-\theta)^{x_i-1} =$
$\theta^n (1-\theta)^{\sum_{i=1}^{n} x_i - 1} = \theta^n (1-\theta)^{-n+n\bar{x}} = \theta^n (1-\theta)^{n(\bar{x}-1)}$ ∴ loglikelihood:
$l(\theta) = n\ln\theta + n(\bar{x}-1)\ln(1-\theta)$ ∴ & score $l'(\theta) = \frac{n}{\theta} - n(\bar{x}-1)/(1-\theta)$
Z 2nd deriv: $l''(\theta) = -n/\theta^2 - n(\bar{x}-1)/(1-\theta)^2 < 0$ st solving $l'(\hat{\theta}) = 0$ ∴
yields Z max likelihood esti mle: $l'(\hat{\theta}) = 0$ ∴ $n(1-\hat{\theta}) - n(\bar{x}-1)\hat{\theta} = 0$
∴ $\hat{\theta} = 1/\bar{x}$

b / $t \frac{1}{\frac{\theta}{2} - -\frac{\theta}{2}} = 1 = t\frac{1}{\frac{\theta}{2}+\frac{\theta}{2}} = \frac{1}{\theta}t$ ∴ $t = \theta$ ∴ $f(x;\theta) = \theta$, $-\frac{\theta}{2} \le x \le \frac{\theta}{2}$

Z density is $f(x;\theta) = \frac{1}{\theta}$ for $-\frac{\theta}{2} < x < \frac{\theta}{2}$ & ∴ Z lihood is:
$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} \frac{1}{\theta} = \left(\frac{1}{\theta}\right)^n = \theta^{-n}$ ∴ $L(\theta) = \begin{cases} \theta^{-n} & \text{if } -\frac{\theta}{2} < x_{(1)} \text{ \& } x_{(n)} < \frac{\theta}{2} \\ 0 & \text{, otherwise} \end{cases}$

$= \begin{cases} \theta^{-n} & \text{if } \theta > \max\{-2x_{(1)}, 2x_{(n)}\} \\ 0 & \text{otherwise} \end{cases}$

where $x_{(1)} = \min\{x_1, \dots, x_n\}$ & $x_{(n)} = \max\{x_1, \dots, x_n\}$ this lihood is
∴ maxed at $\hat{\theta} = \max\{-2x_{(1)}, 2x_{(n)}\}$

3 / $f(x|\theta) = \frac{1}{\theta^2} x e^{-x/\theta}$ ∴ $L(\theta) = \prod_{i=1}^{n} f(x_i|\theta) = \prod_{i=1}^{n} \frac{1}{\theta^2} x_i e^{-x_i/\theta} =$
$\theta^{-2n} e^{\sum_{i=1}^{n} x_i/\theta} \prod_{i=1}^{n} x_i = \theta^{-2n} e^{n\frac{1}{\theta}\bar{x}} \prod_{i=1}^{n} x_i$ ∴ $L(\theta) \propto \theta^{-2n} e^{-n\bar{x}/\theta}$ with

log-likelihood $l(\theta) = \text{Constant} - 2n\ln\theta - \frac{n\bar{x}}{\theta}$ & score:
$l'(\theta) = -\frac{2n}{\theta} + n\bar{x}\frac{1}{\theta^2}$ ∴ solving $l'(\hat{\theta}) = 0$ yields a stationary pt at
$\hat{\theta} = \frac{\bar{x}}{2}$, Z 2nd deriv: $l''(\theta) = 2n\frac{1}{\theta^2} - 2n\bar{x}\frac{1}{\theta^3}$ &
$l''(\hat{\theta}) = \frac{8n}{\bar{x}^2} - \frac{16n}{\bar{x}^2} = -\frac{8n}{\bar{x}^2} < 0$ st $\hat{\theta}$ is Z maximum lihood esti

d / $f(x;\theta) = \theta^x \exp(-\theta) x!$ $x \ge 0$ ∴ lihood: $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) =$
$\prod_{i=1}^{n} \theta^{x_i} e^{-\theta} \frac{1}{x_i!} = e^{-\theta n} \theta^{\sum_{i=1}^{n} x_i} \prod_{i=1}^{n} \frac{1}{x_i!} = e^{-\theta n} \theta^{n\bar{x}} \prod_{i=1}^{n} \frac{1}{x_i!}$ ∴
$L(\theta) \propto \theta^{n\bar{x}} e^{-n\theta}$ ∴ loglihood: $l(\theta) = \text{Constant} + n\bar{x}\log\theta - n\theta$ & score:
$l'(\theta) = \frac{n\bar{x}}{\theta} - n$ ∴ 2nd deriv: $l''(\theta) = -n\bar{x}/\theta^2 < 0$. st solving $l'(\hat{\theta}) = 0$
yields Z mle $\hat{\theta} = \bar{x}$

1.3.8 / $f(x;\theta) = \frac{1}{\theta_2} e^{-(x-\theta_1)/\theta_2}$ ∴ $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) =$
$\prod_{i=1}^{n} \frac{1}{\theta_2} e^{-(x_i-\theta_1)/\theta_2} = \left(\frac{1}{\theta_2}\right)^n e^{\sum_{i=1}^{n} -(x_i-\theta_1)/\theta_2} = \theta_2^{-n} e^{-\frac{1}{\theta_2} n(\bar{x}-\theta_1)}$ ∴
$L(\theta) = \begin{cases} \theta_2^{-n} e^{-n(\bar{x}-\theta_1)/\theta_2} & \text{if } \theta_1 \le x_{(1)} \\ 0 & \text{otherwise} \end{cases}$ ∴ loglihood:

\sheet 1 extra exercises/ $l(\theta)=\begin{cases} -n\ln\theta_2-\theta_2^{-1}n(\bar{x}-\theta_1) & \text{if } \theta_1\le x_{(1)} \\ -\infty & \text{otherwise} \end{cases}$

whatever θ_2, Z loglihood is maximised when θ_1 is as large as possible, st $\hat{\theta}_1=x_{(1)}$ $\therefore$ $\frac{\partial l}{\partial\theta_2}=-\frac{n}{\theta_2}+\frac{n(\bar{x}-\theta_1)}{\theta_2^2}$ $\therefore$ setting to equal zero: yields $\hat{\theta}_2=\bar{x}-\hat{\theta}_1$

\1.3.21/ Let N be Z number of children diseased in a family of k children. independ prob p $\therefore$ $N\sim Bin(k,p)$ & $Pr(N\ge 1)=1-Pr(N=0)=1-(1-p)^k$

Z conditional prob of observing r abnormal children in a family of size k given that atleast one child is: $Pr(N=r|N\ge 1)=\frac{Pr(N=r \,\&\, N\ge 1)}{Pr(N\ge 1)}=\frac{Pr(N=r)}{Pr(N\ge 1)}=\frac{\binom{k}{r}p^r(1-p)^{k-r}}{1-(1-p)^k}$ for $r=1,\dots,k$ $\therefore$ Z likelihood is:

$L(r;p)=L(p)=\prod_{i=1}^{n} f(r_i;p)=\prod_{i=1}^{n}\frac{\binom{k}{r_i}p^{r_i}(1-p)^{k-r_i}}{1-(1-p)^k}$ $\therefore$ loglihood;

$l(p)=\sum_{i=1}^{n}\ln\binom{k}{r_i}+\ln(p)\sum_{i=1}^{n}r_i+\ln(1-p)\sum_{i=1}^{n}(k-r_i)-n\ln(1-(1-p)^k)$ & score:

$l'(p)=p^{-1}\sum_{i=1}^{n}r_i-(1-p)^{-1}\sum_{i=1}^{n}(k-r_i)-\frac{nk(1-p)^{k-1}}{1-(1-p)^k}$ Z eqn $l'(\hat{p})=0$ holds iff

$nk\hat{p}=(1-(1-\hat{p})^k)\sum_{i=1}^{n}r_i$

\2.3.26, 3.27/a/ Z expectation is $E(X)=\sum_{x=1}^{\infty}x\theta(1-\theta)^{x-1}=\sum_{x=0}^{\infty}(x+1)\theta(1-\theta)^{x}=(1-\theta)\sum_{x=1}^{\infty}x\theta(1-\theta)^{x-1}+\theta\sum_{x=0}^{\infty}(1-\theta)^x=(1-\theta)E(X)+1$ $\therefore$

$E(X)=\frac{1}{\theta}$ equating to $\bar{X}$ $\therefore$ mme: $\hat{\theta}=\frac{1}{\bar{X}}$

\b/ this distri is bounded & symmetric about zero so $E(X)=0$, as this doesnt involve θ, it doesnt help with finding an estimator. So need to use Z 2nd moment instead; Z density is $f(x;\theta)=\frac{1}{\theta}$ $\therefore$

$E(X^2)=\int_{-\theta/2}^{\theta/2}x^2\frac{1}{\theta}\,dx=\frac{\theta^2}{12}$ equating to $\sum_{i=1}^{n}X_i^2/n=\frac{1}{n}\sum_{i=1}^{n}X_i^2$ & solving for θ

yields Z method of moments estimator $\hat{\theta}=\sqrt{12\sum_{i=1}^{n}X_i^2/n}$

\c/ Z expectation is $E(X)=\int_0^{\infty}x\frac{1}{\theta^2}xe^{-x/\theta}\,dx=\int_0^{\infty}\frac{x^2}{\theta^2}e^{-x/\theta}\,dx=\theta\int_0^{\infty}y^2e^{-y}\,dy$

$=\theta\Gamma(3)=2!\theta=2\theta$, $y=\frac{x}{\theta}$ $\therefore$ equating to $\bar{X}$ $\therefore$

Method of moments esti: $\hat{\theta}=\bar{X}/2$

\d/ told $E(X)=\theta$ ∴ $\hat{\theta}=\overline{X}$

\2.3.27/ $(\theta+\frac{1}{2}-\theta+\frac{1}{2})t=1=1t=t=1$ ∴ $f(x;\theta)=1$ for $\theta-\frac{1}{2}<x<\theta+\frac{1}{2}$ ∴

lihood $L(\theta)=\prod_{i=1}^{n}1=1$ ∴ $L(\theta)=\begin{cases}1 & \text{if } \theta-\frac{1}{2}<x_{(1)}<x_{(n)}<\theta+\frac{1}{2}\\ 0 & \text{otherwise.}\end{cases}$

this lihood is flat for $x_{(n)}-\frac{1}{2}<\theta<x_{(1)}+\frac{1}{2}$ & ∴ Z mle is any value in this interval. Z expectation of this unif distri is $E(X)=\theta$ & ∴ Z mme is Z sample mean.

\3/ an estior is consistent if Z prob of Z estior being inside any small interval containing Z true param val converges to 1 as Z sample size increases. this is desirable ∵ it means that large enough samples are likely to yield estis that are close to Z true param val being estied

\4.2.1/ Z distri func: $Pr(X_1\le x)=\int_0^x \frac{1}{\theta}dx=\frac{x}{\theta}$ Z expectation: $E(X_1)=\int_0^\theta \frac{x}{\theta}dx=\frac{\theta}{2}$, $E(X_1^2)=\int_0^\theta \frac{x^2}{\theta}dx=\frac{\theta^2}{3}$ st Z $var(X_1)=\frac{\theta^2}{3}-\left(\frac{\theta}{2}\right)^2=\frac{\theta^2}{12}$. Z distri func of $X_{(n)}$ is $Pr(X_{(n)}\le x)=$ $Pr(X_1\le x,\dots,X_n\le x)=\prod_{i=1}^{n}Pr(X_i\le x)=\frac{x^n}{\theta^n}$ for $0<x<\theta$ & ∴ its density: $f(x;\theta)=\frac{d}{dx}P(X_{(n)}\le x)=nx^{n-1}/\theta^n$ for $0<x<\theta$

if $2\overline{X}$: $E(2\overline{X})=2E(\overline{X})=2E(X)=\theta$ & $var(2\overline{X})=4var(\overline{X})=4var(X_1)/n=\theta^2/(3n)$ ∵ $E(2\overline{X})\to\theta$ & $var(2\overline{X})\to 0$ as $n\to\infty$, know $2\overline{X}$ is consistent for θ

for $(n+1)X_{(n)}/n$: $E\left(\frac{n+1}{n}X_{(n)}\right)=\frac{n+1}{n}\int_0^\theta x\frac{nx^{n-1}}{\theta^n}dx=\left[\frac{x^{n+1}}{\theta^n}\right]_0^\theta=\theta$ & $var\left(\frac{n+1}{n}X_{(n)}\right)=\left(\frac{n+1}{n}\right)^2\int_0^\theta x^2\frac{nx^{n-1}}{\theta^n}dx-\theta^2=\dots=\frac{\theta^2}{n(n+2)}$ ∵ $E((n+1)X_{(n)}/n)\to\theta$ & $var((n+1)X_{(n)}/n)\to 0$ as $n\to\infty$ ∴ know $(n+1)X_{(n)}/n$ is consistent for θ. Z relative efficiency of $2\overline{X}$ to $(n+1)X_{(n)}/n$ is $\frac{\theta^2}{n(n+2)}\Big/\frac{\theta^2}{3n}=\frac{3}{n+2}$, ∴ $\frac{3}{n+2}<1$ when $n>1$ & $\frac{3}{n+2}\to 0$ as $n\to\infty$ ∴ $(n+1)X_{(n)}/n$ is a much better estior than $2\overline{X}$

\4.2.2/ $E(\tilde{\theta})=k_1E(\hat{\theta}_1)+k_2E(\hat{\theta}_2)=k_1\theta+k_2\theta$ ∴ required $k_1+k_2=1$ for $\tilde{\theta}$ to be unbiased ie $\tilde{\theta}=k_1\hat{\theta}_1+(1-k_1)\hat{\theta}_2$ ∴ $var(\tilde{\theta})=k_1^2 var(\hat{\theta}_1)+k_2^2 var(\hat{\theta}_2)=k_1^2\sigma_1^2+(1-k_1)^2\sigma_2^2$ & $\frac{d}{dk_1}var(\tilde{\theta})=0$ iff $k_1=\sigma_2^2/(\sigma_1^2+\sigma_2^2)$ & Z 2nd deriv

\sheet1 extra excersises/ as Z variance is positive
$\therefore$ Z variance is minimised when $\tilde{\theta} = \frac{\sigma_2^2}{\sigma_1^2+\sigma_2^2}\hat{\theta}_1 + \frac{\sigma_1^2}{\sigma_1^2+\sigma_2^2}\hat{\theta}_2$

\4.2.4/ $E(\hat{\theta})$ Z expectation is $E(\hat{\theta}) = \frac{n-1}{n}\theta + \frac{1}{n}(\theta+n) = \theta + 1 \neq \theta$
& $\therefore$ Z meansquared error doesn't converge to zero. Let $n > \varepsilon$ $\therefore$
$\Pr(|\hat{\theta}-\theta| \le \varepsilon) = \Pr(-\varepsilon \le \hat{\theta}-\theta \le \varepsilon) = \Pr(\theta-\varepsilon \le \hat{\theta} \le \theta+\varepsilon) = \frac{n-1}{n} \to 1$
& $\therefore$ $\hat{\theta}$ is consistent

\5/ $h(\theta) = \theta^2$, $h'(\theta) = 2\theta$, $h''(\theta) = 2$ & $\text{var}(\hat{\theta}) = \text{var}(X_1)/n = \phi/n$ $\therefore$
$\text{var}(\hat{\phi}) \approx (h'(\theta))^2\,\text{var}(\hat{\theta}) = 4\phi^2/n$ & Z bias corrected estimator is
$\tilde{\phi} = \hat{\phi} - \frac{1}{2}\widehat{\text{var}}(\hat{\theta})h''(\hat{\theta}) = \hat{\phi} - \hat{\phi}/n = (1-1/n)\hat{\phi}$
$\tilde{\phi} = \hat{\phi} - \frac{1}{2}\widehat{\text{var}}(\hat{\theta})h''(\hat{\theta})$ $\tilde{\phi} = \hat{\phi} - \frac{1}{2}\text{var}(\hat{\theta})h''(\hat{\theta})$
$\text{var}(\hat{\phi}) = (h'(\theta))^2\,\text{var}(\hat{\theta})$ $\text{var}(\hat{\phi}) \approx (h'(\theta))^2\,\text{var}(\hat{\theta})$
$\tilde{\phi} = \hat{\phi} - \frac{1}{2}\text{var}(\hat{\theta})h''(\hat{\theta})$ $\text{var}(\hat{\phi}) \approx (h'(\theta))^2\,\text{var}(\hat{\theta})$ $\tilde{\phi} = \hat{\phi} - \frac{1}{2}\text{var}(\hat{\theta})h''(\hat{\theta})$
$\text{var}(\hat{\phi}) = (h'(\theta))^2\,\text{var}(\hat{\theta})$ $\tilde{\phi} = \hat{\phi} - \frac{1}{2}\text{var}(\hat{\theta})h''(\hat{\theta})$ $\tilde{\phi} = \hat{\phi} - \frac{1}{2}\text{var}(\hat{\theta})h''(\hat{\theta})$
$\text{var}(\hat{\phi}) = (h'(\theta))^2\,\text{var}(\hat{\theta})$ $\tilde{\phi} = \hat{\phi} - \frac{1}{2}\text{var}(\hat{\theta})h''(\hat{\theta})$ $\tilde{\phi} = \hat{\phi} - \frac{1}{2}\text{var}(\hat{\theta})h''(\hat{\theta})$
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$h(\hat{\theta}) \approx h(\theta) + (\hat{\theta}-\theta)h'(\theta)$ $h(\hat{\theta}) \approx h(\theta) + (\hat{\theta}-\theta)h'(\theta)$ $h(\hat{\theta}) \approx h(\theta) + (\hat{\theta}-\theta)h'(\theta)$
$h(\hat{\theta}) \approx h(\theta) + (\hat{\theta}-\theta)h'(\theta)$ $h(\hat{\theta}) \approx h(\theta) + (\hat{\theta}-\theta)h'(\theta)$ $h(\hat{\theta}) \approx h(\theta) + (\hat{\theta}-\theta)h'(\theta)$
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$\tilde{\phi} = \hat{\phi} - \frac{1}{2}\text{var}(\hat{\theta})h''(\hat{\theta})$ $\text{var}(\hat{\phi}) = (h'(\theta))^2\,\text{var}(\hat{\theta})$ $\text{var}(\hat{\phi}) = (h'(\theta))^2\,\text{var}(\hat{\theta})$
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\6.2.6a/ Z loglikelihood is $l(\theta) = n\bar{x}\ln\theta + n(1-\bar{x})\ln(1-\theta)$ $\therefore$
$L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n}\theta^{x_i}(1-\theta)^{1-x_i} = \theta^{\sum_{i=1}^{n}x_i}(1-\theta)^{\sum_{i=1}^{n}(1-x_i)} = \theta^{n\bar{x}}(1-\theta)^{n-n\bar{x}}$ $\therefore$
Score: $l'(\theta) = \frac{n\bar{x}}{\theta} - \frac{n(1-\bar{x})}{1-\theta} = \frac{n}{\theta(1-\theta)}(\bar{x}-\theta) = b(\hat{\theta}-\theta)$ this has Z form
$b(\hat{\theta}-\theta)$ required $\therefore$ $b = \frac{n}{\theta(1-\theta)}$, $\hat{\theta} = \bar{x}$ st Z estimator $\hat{\theta} = \bar{x}$ attains Z
CRLB of $\frac{\theta(1-\theta)}{n}$ $\because$ $b^{-1} = \frac{1}{b} = \left(\frac{n}{\theta(1-\theta)}\right)^{-1} = \frac{\theta(1-\theta)}{n}$

\6.2.6b/ if we desire $\phi = \theta^2$ then Z loglikelihood is:
$l(\phi) = \frac{n\bar{x}}{2}\ln\phi + n(1-\bar{x})\ln(1-\phi^{1/2})$ $\therefore$

$L(\theta) = \theta^{n\bar{x}}(1-\theta)^{n(1-\bar{x})} = (\phi^{1/2})^{n\bar{x}}(1-\phi^{1/2})^{n(1-\bar{x})} = L(\phi)$ $\therefore$ ~~ϕ~~ $\theta = \phi^{1/2}$ $\therefore$

~~$L(\theta)$ = n~~ $L(\phi) = \phi^{\frac{1}{2}n\bar{x}}(1-\phi^{1/2})^{n(1-\bar{x})}$ $\therefore$

$l(\phi) = \frac{n\bar{x}}{2}\ln\phi + n(1-\bar{x})\ln(1-\phi^{1/2})$ $\therefore$

$l'(\phi) = \frac{\frac{1}{2}n\bar{x}}{\phi} + n(1-\bar{x})\frac{1}{1-\phi^{1/2}}(-\frac{1}{2}\phi^{-1/2}) \neq b(\hat{\phi}-\phi)$ $\therefore$ Z score cannot be written in Z form $b(\hat{\phi}-\phi)$ & $\therefore$ there is no estimator for θ^2 that attains Z CRLB

\6.2.9/ a bound for Z variance of $\hat{\theta}$ is $\mathrm{cov}(\hat{\theta},U)^2/\mathrm{var}(U)$, where $\mathrm{var}(U) = I(\theta)$, $\therefore$ $\hat{\theta}$ is unbiased $\therefore$ $\mathrm{cov}(\hat{\theta},U) = 1$ $\therefore$ usual CRLB.

but $\hat{\theta}$ is biased $\therefore$ this changes Z covariance. $\therefore$ $E(U) = 0$ as usual st

$\mathrm{cov}(\hat{\theta},U) = E(\hat{\theta}U) = \int\hat{\theta}\frac{\partial l}{\partial\theta}L\,d\underline{x} = \int\hat{\theta}\frac{\partial L}{\partial\theta}\,d\underline{x} = \frac{\partial}{\partial\theta}\int\hat{\theta}L\,d\underline{x} = \frac{\partial}{\partial\theta}E(\hat{\theta})$

under regularity conditions. but now $E(\hat{\theta}) = \int\hat{\theta}L(\theta)d\underline{x} = \theta + b(\theta)$ st

$\mathrm{cov}(\hat{\theta},U) = \frac{\partial}{\partial\theta}[\theta + b(\theta)] = 1 + b'(\theta)$ & Z CRLB for biased estimators is

$[1+b'(\theta)]^2/I(\theta)$ $\therefore$ $\mathrm{var}(\hat{\theta}) \geq (1+\frac{\partial b(\theta)}{\partial\theta})^2 I^{-1}(\theta)$

\6.3.28a/ $f(x) = e^{-(x-\theta)}$, $x>\theta$ $\therefore$ Z likelihood: $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} e^{-(x_i-\theta)} =$

$e^{\sum_{i=1}^{n}-(x_i-\theta)} = e^{-n(\bar{x}-\theta)}$ for $\theta < x_{(1)}$ $\therefore$ $L(\theta) = \begin{cases} e^{-n(\bar{x}-\theta)}, & \text{for } \theta < x_{(1)} \\ 0, & \text{otherwise} \end{cases}$

Which is maximised at $\hat{\theta} = x_{(1)}$

\6.3.28b/ Z distri func of $\hat{\theta}$ satisfies $P_r(\hat{\theta}>x) = P_r(X_1>x,\ldots,X_n>x) =$

$\prod_{i=1}^{n}P_r(X_i>x) = P_r(X_1>x)^n$ &

$P_r(X_1>x) = \int_x^\infty e^{-(x-\theta)}dx = [\frac{1}{-1}e^{-(x-\theta)}]_x^\infty = -1(0) + e^{-(x-\theta)} = e^{-(x-\theta)}$ $\therefore$

$Pr(\hat{\theta}>x) = (e^{-(x-\theta)})^n = e^{-n(x-\theta)}$ $\therefore$

$P_r(\hat{\theta}\leq x) = 1 - P(\hat{\theta}>x) = 1 - e^{-n(x-\theta)}$ $\therefore$ Z density of $\hat{\theta}$ is

$\frac{\partial}{\partial x}P_r(\hat{\theta}\leq x) = \frac{\partial}{\partial x}[1 - e^{-n(x-\theta)}] = ne^{-n(x-\theta)}$ for $x>\theta$. $\therefore$

$E(\hat{\theta}) = \int_\theta^\infty xne^{-n(x-\theta)}dx = [xn\frac{1}{-n}e^{-n(x-\theta)}]_\theta^\infty - \int_\theta^\infty n\frac{1}{-n}e^{-n(x-\theta)}dx =$

$0 + \theta + \int_\theta^\infty e^{-n(x-\theta)}dx = \theta + [-\frac{1}{n}e^{-n(x-\theta)}]_\theta^\infty = \theta + \frac{1}{n}e^0 = \theta + \frac{1}{n}$ $\therefore$ $\hat{\theta}$ is biased

$\therefore$ $E(\hat{\theta}) - \theta = \theta + \frac{1}{n} - \theta = \frac{1}{n} \neq 0$ but is asymp unbiased, &

$E(\hat{\theta}^2) = \int_\theta^\infty x^2ne^{-n(x-\theta)}dx = \ldots = \theta^2 + 2\theta/n + 2/n^2$ $\therefore$ $\mathrm{var}(\hat{\theta}) = E(\hat{\theta}^2) - E(\hat{\theta})^2$

$= \theta^2 + \frac{2\theta}{n} + \frac{2}{n^2} - (\theta+\frac{1}{n})^2 = \ldots = \frac{1}{n^2} \to 0$ as $n\to\infty$ $\therefore$ $\hat{\theta}$ is consistent. Z estimator $\tilde{\theta} = \hat{\theta} - \frac{1}{n}$ would be unbiased & consistent with Z same variance as $\hat{\theta}$.

\Sheet1 extra Ex/ \6.3.28c/ $E(X_1) = \int_\theta^\infty x e^{-(x-\theta)}\,dx = \dots = \theta + 1$ &

$E(X_1^2) = \int_\theta^\infty x^2 e^{-(x-\theta)}\,dx = \dots = \theta^2 + 2\theta + 2$ ∴ $var(X_1) = E(X_1^2) - E(X_1)^2 = 1$ ∴

Z MoM estimator is: $\tilde{\theta} = \bar{X} - 1$, which is unbiased (unlike $\hat{\theta}$) but its variance, $var(\bar{X}) = \frac{1}{n}$ is an order of magnitude larger than Z variance of $\hat{\theta}$. it is inappropriate to compare Z variances of Z CRLB ∵ Z range of possible x vals depends on θ for this distri, which means that Z regularity conditions required for Z CRLB don't hold.

\7.4.1a/ CR is of form $C = \{\underline{x} : \Lambda(\underline{x}) \geq c\}$, where $\Lambda(\underline{x}) = L(\theta_1)/L(\theta_0)$

$f(x;\theta) = \frac{\theta^x e^{-\theta}}{x!}$ ∴ Z likelihood: $L(\theta) = \prod_{i=1}^n f(x_i;\theta) = \prod_{i=1}^n \frac{\theta^{x_i} e^{-\theta}}{x_i!} =$ $\theta^{\sum_{i=1}^n x_i} e^{-n\theta} \prod_{i=1}^n \frac{1}{x_i!}$ ∴ Z likelihood ratio is: $\Lambda(\underline{x}) \propto (\theta_1/\theta_0)^{\sum_{i=1}^n x_i}$ ∵ $\theta_1 > \theta_0$

∵ $L(\theta_1) = \theta_1^{\sum_{i=1}^n} e^{-n\theta_1} \prod_{i=1}^n \frac{1}{x_i!}$ & $L(\theta_0) = \theta_0^{\sum_{i=1}^n x_i} e^{-n\theta_0} \prod_{i=1}^n \frac{1}{x_i!}$ ∴

$$\Lambda(\underline{x}) = \frac{L(\theta_1)}{L(\theta_0)} = \frac{\theta_1^{\sum_{i=1}^n x_i} e^{-n\theta_1} \prod_{i=1}^n \frac{1}{x_i!}}{\theta_0^{\sum_{i=1}^n x_i} e^{-n\theta_0} \prod_{i=1}^n \frac{1}{x_i!}} = \left(\frac{\theta_1}{\theta_0}\right)^{\sum_{i=1}^n x_i} e^{-n(\theta_1 - \theta_0)} \propto \left(\frac{\theta_1}{\theta_0}\right)^{\sum_{i=1}^n x_i}$$ ∴ $\Lambda(\underline{x}) \geq c$

iff $\sum_{i=1}^n x_i \geq d$ & ∴ CR is $\{\underline{x} : \sum_{i=1}^n x_i \geq d\}$ ∵ $\sum_{i=1}^n X_i \sim Poi(n\theta)$, a test of size α requires d to be Z upper α-quantile of Z $Poi(n\theta_0)$ distri

\7.4.1b/ $f(x;\theta) = \lambda e^{-\lambda x} = \left(\frac{1}{\theta}\right) e^{-\left(\frac{1}{\theta}\right)x} = \frac{1}{\theta} e^{-x/\theta}$ for $\lambda = \frac{1}{\theta}$, $\theta_1 > \theta_0$ ∴

likelihood: $L(\theta) = \prod_{i=1}^n f(x_i;\theta) = \prod_{i=1}^n \frac{1}{\theta} e^{-x_i/\theta} = \theta^{-n} e^{\sum_{i=1}^n -x_i/\theta} = \theta^{-n} e^{-\frac{1}{\theta}\sum_{i=1}^n x_i}$ ∴

Z likelihood ratio: $\Lambda(\underline{x}) = \frac{L(\theta_1)}{L(\theta_0)} = \frac{\theta_1^{-n} e^{-\frac{1}{\theta_1}\sum_{i=1}^n x_i}}{\theta_0^{-n} e^{-\frac{1}{\theta_0}\sum_{i=1}^n x_i}} = \left(\frac{\theta_1}{\theta_0}\right)^{-n} e^{-\sum_{i=1}^n (x_i)\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)} \propto$ $e^{-\left(\frac{1}{\theta_1} - \frac{1}{\theta_0}\right)\sum_{i=1}^n x_i}$ ∵ $\theta_1 > \theta_0$ ∴ $\Lambda(\underline{x}) \geq c$ iff $\sum_{i=1}^n x_i \geq d$ & ∴ Z CR is $\{\underline{x} : \sum_{i=1}^n x_i \geq d\}$ ∵ $\sum_{i=1}^n X_i \sim Ga(n,\theta)$, a test of size α requires d to be Z upper α-quantile of Z $Ga(n,\theta_0)$ distri.

\7.4.1c/ $f(x;\theta) = \theta e^{-\theta x}$ $\theta_1 > \theta_0$ ∴ $L(\theta) = \prod_{i=1}^n f(x_i;\theta) = \prod_{i=1}^n \theta e^{-\theta x_i} =$ $\theta^n e^{-\theta\sum_{i=1}^n x_i}$ ∴ Z likelihood ratio is $\Lambda(\underline{x}) = \frac{L(\theta_1)}{L(\theta_0)} = \frac{\theta_1^n e^{-\theta_1\sum_{i=1}^n x_i}}{\theta_0^n e^{-\theta_0\sum_{i=1}^n x_i}} =$ $\left(\frac{\theta_1}{\theta_0}\right)^n e^{-(\theta_1-\theta_0)\sum_{i=1}^n x_i} \propto e^{-(\theta_1-\theta_0)\sum_{i=1}^n x_i}$ ∵ $\theta_1 > \theta_0$ ∴

$\Lambda(\underline{x}) \geq c$ iff $\sum_{i=1}^n x_i \leq d$ & ∴ Z CR is $\{\underline{x} : \sum_{i=1}^n x_i \leq d\}$ ∴ $\sum_{i=1}^n X_i \sim Ga(n,\theta)$, a test of size α requires d to be Z α-quantile of Z $Ga(n,\theta_0)$ distri.

\7. 4.3/ $\theta_1 > \theta_0$ Z likelihood is : $L(\theta) = \prod_{i=1}^{n} \frac{\theta^\lambda}{\Gamma(\lambda)} x_i^{\lambda-1} e^{-\theta x_i} =$

$\frac{\theta^{n\lambda}}{\Gamma(\lambda)^n} e^{-\theta \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i^{\lambda-1} = \theta^{n\lambda} \left(\prod_{i=1}^{n} x_i \right)^{\lambda-1} e^{-\theta \sum_{i=1}^{n} x_i} \Gamma(\lambda)^{-n}$ ∴

Z likelihood ratio is : $\Lambda(\underline{x}) = \frac{L(\theta_1)}{L(\theta_0)} \propto e^{-(\theta_1-\theta_0)\sum_{i=1}^{n} x_i}$ ∵ $\theta_1 > \theta_0$, ∴ Z most powerful CR is of Z form $\{x : \sum_{i=1}^{n} x_i \le C\}$ ∵ $\Lambda(\underline{x}) \ge c$ iff $\sum_{i=1}^{n} x_i \le C$.

when $\lambda = 1$: $f(x;\theta) = \theta e^{-\theta x}$ ∴ C is Z α-quantile of an $Exp(\theta_0)$ distri. Z distri has distri func $Pr(X \le x) = 1 - e^{-\theta_0 x}$ & ∴

$\alpha = 1 - e^{-\theta_0 C}$ ∴ $C = -\ln(1-\alpha)/\theta_0$ ∵ $f(x;\theta_0) = \theta_0 e^{-\theta_0 x}$ ∴

$P(X \le x) = \int_0^x \theta_0 e^{-\theta_0 x} dx = \left[-e^{-\theta_0 x} \right]_0^x = -e^{-\theta_0 x} - -1 = 1 - e^{-\theta_0 x}$ ∴ α quantile:

$P(X \le C) = \alpha = 1 - e^{-\theta_0 C}$ ∴ $e^{-\theta_0 C} = 1 - \alpha$ ∴ $-\theta_0 C = \ln(1-\alpha)$ ∴ $C = -\ln(1-\alpha)/\theta_0$

∴ Z power is $Pr[X \le -\ln(1-\alpha)/\theta_0 ; \theta_1] = 1 - e^{(\frac{\theta_1}{\theta_0})\ln(1-\alpha)} = 1 - (1-\alpha)^{\theta_1/\theta_0}$

∵ when $\theta = \theta_1$, X has an $Exp(\theta_1)$ distri ∴

$X \sim Exp(\theta_1)$ ∴ $Pr(X \le x) = 1 - e^{-\theta_1 x}$ ∴ $Pr(X \le -\ln(1-\alpha)/\theta_0) = 1 - e^{-\theta_1(-\ln(1-\alpha)/\theta_0)} =$

$1 - e^{+(\frac{\theta_1}{\theta_0})\ln(1-\alpha)} = 1 - e^{\ln[(1-\alpha)^{(\theta_1/\theta_0)}]} = 1 - (1-\alpha)^{\theta_1/\theta_0}$

\8/ Z power of a test is Z prob of correctly rejecting Z null hypoth when Z alternative hypoth is true. if Z alternative hypoth specifies more than one val for Z param, then Z power may depend on which of those param vals is true. A test is uniformly most powerful if for all param vals, specified by Z alternative hypoth, its power is at least as great as that of all other tests of Z same size

\9.4.8/ we are testing Z null hypoth $H_0 : f(x) = 1$ against Z alternative $H_1 : f(x;\theta) = \frac{\theta e^{\theta x}}{e^\theta - 1}$ for $0 < x < 1$, where $\theta > 0$. note $f(x) = 1$ is Z limit of $f(x;\theta)$ as $\theta \to 0$, ∴ is like testing $H_0 : \theta = 0$ against $H_1 : \theta > 0$.

Z likelihood is $L(\theta) = 1$ when $\theta = 0$ ∵ $\prod_{i=1}^{n} 1 = L(\theta) = 1^n = 1$ &

$L(\theta) = \prod_{i=1}^{n} \frac{1}{e^\theta - 1} \theta e^{\theta x_i} = \frac{\theta^n}{(e^\theta - 1)^n} e^{\theta \sum_{i=1}^{n} x_i} = \frac{\theta^n e^{n\bar{x}\theta}}{(e^\theta - 1)^n}$ when $\theta > 0$. ∴

Consider $H_0 : \theta = 0$ & $H_1 : \theta = \theta_1$ where $\theta_1 > 0$ ∴ $\Lambda(\underline{x}) = \frac{L(\theta_1)}{L(\theta_0)} =$

$\frac{\theta_1^n e^{n\bar{x}\theta_1}}{(e^{\theta_1} - 1)^n} / 1 = \frac{\theta_1^n e^{n\bar{x}\theta_1}}{(e^{\theta_1} - 1)^n}$ & Z most powerful test has CR of Z form $\{\underline{x} : \bar{x} \ge d\}$ ∵ d is indep of θ_1, this test is uniformly most powerful for $H_1 : \theta > 0$. When n is large, Z central limit thm

\Sheet 1 extra Ex/ tells us that Z null distri of $\bar{X}$ is approx $N(\frac{1}{2}, \frac{1}{12n})$ ∴ $E(X)=\frac{1}{2}$ & $var(X)=\frac{1}{12}$ when H_0 is true ∴ d would approx equal Z upper $\frac{1}{2}\alpha$-quantile of Z $N(\frac{1}{2}, \frac{1}{12n})$ distri

\9; 4.9/ $f(x;\theta)=\frac{\theta^x e^{-\theta}}{x!}$ ∴ if $\theta_1>\theta_0$ then Z most powerful test has CR of Z form $\{\underline{x}: \bar{x}\geq d\}$ but if $\theta_1<\theta_0$ then Z most powerful test has CR of Z form $\{\underline{x}: \bar{x}\leq d\}$ ∴ Z forms of these two tests differ, there is no uniformly most powerful test for Z two-sided hypoth $H_1: \theta\neq\theta_0$. Z power of Z proposed test is

$1-Pr(C_1<Y<C_2;\theta)=1-\sum_{y=C_1}^{C_2-1} Pr(Y=y)=1-\sum_{j=C_1}^{C_2-1}\frac{(n\theta)^j e^{-n\theta}}{j!}$ as required

\10; 5.2/ $f(x;\theta)=\theta e^{-\theta x}$ ∴ for $f(x;\theta)=(\frac{1}{\theta})e^{-x/\theta}$ ∴ Z variance of this exponential distri is θ^2 & ∴ a $(1-2\alpha)$ CI for Z variance is (L,U) if $Pr(L<\theta^2<U)=1-2\alpha$. we can find L & U just by squaring terms in Z corres eqn ie

$Pr[(2n\bar{X}/c_{1-\alpha})^2<\theta^2<(2n\bar{X}/c_\alpha)^2]=1-2\alpha$ st $L=(2n\bar{X}/c_{1-\alpha})^2$ &

$U=(2n\bar{X}/c_\alpha)^2$

\10; 5.11/ $f(x;\theta)=\frac{\theta^x e^{-\theta}}{x!}$ ∴ consider a test of size α for Z hypoth $H_0: \theta=\theta_0$ against $H_1: \theta\neq\theta_0$ at CR is

$C_\alpha(\theta_0)=\{x: Pr(X\leq x;\theta_0)<\frac{\alpha}{2}$ or $Pr(X\geq x;\theta_0)<\frac{\alpha}{2}\}$ ie we would reject H_0 if x were too small or too large. A $(1-\alpha)$ confidence set is ∴

$\{\theta: x\in C_\alpha(\theta)\}=\{\theta; Pr(X\leq x;\theta)\geq\frac{\alpha}{2}$ & $Pr(X\geq x;\theta)\geq\frac{\alpha}{2}\}$. now let θ_U satisfy $Pr(X\leq x;\theta_U)=\frac{\alpha}{2}$. then ∵ $Pr(X\leq x;\theta)$ decreases as θ increases, $Pr(X\leq x;\theta)\leq\frac{\alpha}{2}$ for $\theta>\theta_U$. & let θ_L satisfy $Pr(X\geq x;\theta_L)=\frac{\alpha}{2}$. then ∵ $Pr(X\geq x;\theta)$ decreases as decreases, $Pr(X\geq x;\theta)<\frac{\alpha}{2}$ for $\theta<\theta_L$ ∴, Z confidence set is Z interval (θ_L,θ_U) where θ_L & θ_U satisfy Z eqn

\10; 5.12/ iterating Z recurrence relation: $Pr(Y_{2k}\geq c)=\sum_{j=1}^{k-1}\frac{(c/2)^j e^{-c/2}}{j!}+Pr(Y_2\geq c)$ & $Pr(Y_2\geq c)=\frac{1}{2}\int_c^\infty e^{-\frac{x}{2}}dx=e^{-\frac{c}{2}}$ st $Pr(Y_{2k}\geq c)=\sum_{j=1}^{k-1}\frac{(c/2)^j e^{-c/2}}{j!}$ now

$(c/2)^j e^{-c/2}\frac{1}{j!}$ is Z prob mass func for a $Poi(\frac{c}{2})$ r.v. evaluated at j & ∴

$Pr(Y_{2k} \geq c) = Pr(Poi(\frac{c}{2}) \leq k-1]$. ∴ need θ_U st $\frac{\alpha}{2} = \sum_{i=0}^{x} \frac{\theta_U^i e^{-\theta_U}}{i!} =$

$Pr[Poi(\theta_U) \leq x]$. ∴ is equiv to $\frac{\alpha}{2} = Pr(Y_{2x+2} \geq 2\theta_U)$. ∴

$2\theta_U = \chi^2_{2x+2;\alpha/2}$ is Z upper $\frac{\alpha}{2}$ quantile of a χ^2_{2x+2} distri

& similarly $2\theta_L = \chi^2_{2x;(1-\alpha/2)}$.

\10; S.14/ have $X \sim N(\theta, \theta)$ approx, if is write $z = z_{1-\alpha} = -z_\alpha$ for Z $(1-\alpha)$ quantile of a $N(0,1)$ distri then $1-2\alpha \approx Pr[-z < (X-\theta)/\sqrt{\theta} < z] =$ $Pr[(X-\theta)^2/\theta < z^2] = Pr[\theta^2 - (2X+z^2)\theta + X^2 < 0]$. Z roots of Z quadratic ∴ defines a $(1-2\alpha)$ CI for θ. These roots are $\frac{(2X+z^2) \pm \sqrt{(2X+z)^2 - 4X^2}}{2}$ if Z number of breakdowns for each vehicle is indep $Poi(\theta)$ then Z total number of breakdowns is $X = 1(190) + 2(53) + \ldots + 5(2) = 341$ & for a 95% CI we set $\alpha = 0.025$ st $z = 1.960$. ∴ solving for roots ∴ interval $(306.7, 379.2)$ for Z val of 550θ. ∴ a 95% CI for Z expected number of breakdowns per vehicle, θ, is $(\frac{306.7}{550}, \frac{379.2}{550}) = (0.558, 0.689)$.

\11/ ~~prediction interval~~ Setting $a_0 = 1$ & $a_i = -\frac{1}{n}$ for $i = 1, \ldots, n$ we find that T has a $Cau(0, 2\sigma)$ distri, which is indep of μ & ∴ T is ancillary with distri func $Pr(T \leq t) = \frac{1}{2} + \frac{1}{\pi}\tan^{-1}(\frac{t}{2\sigma})$ ∴ Solving $Pr(T \leq t_p) = p$ shows Z p-quantile of T is $t_p = 2\sigma\tan[\pi(p - \frac{1}{2})]$ st $1 - 2\alpha = Pr(t_\alpha < T < t_{1-\alpha}) =$ $Pr(2\sigma\tan[\pi(\alpha - \frac{1}{2})] < X_0 - \bar{X} < 2\sigma\tan[\pi(\frac{1}{2} - \alpha)]) =$ $Pr(\bar{X} + 2\sigma\tan[\pi(\alpha - \frac{1}{2})] < X_0 < \bar{X} + 2\sigma\tan[\pi(\frac{1}{2} - \alpha)])$. & Z prediction interval is $(\bar{X} + 2\sigma\tan[\pi(\alpha - \frac{1}{2})], \bar{X} + 2\sigma\tan[\pi(\frac{1}{2} - \alpha)])$.

\12/ prediction interval is: $(\bar{Y} + t_{\alpha;n-1} S\sqrt{1+\frac{1}{n}}, \bar{Y} + t_{1-\alpha;n-1} S\sqrt{1+\frac{1}{n}})$ where S is Z sample standard deviation of $Y_1, \ldots, Y_n$. This means

$Pr(\bar{Y} + t_{\alpha;n-1} S\sqrt{1+\frac{1}{n}} < Y_0 < \bar{Y} + t_{1-\alpha;n-1} S\sqrt{1+\frac{1}{n}}) = 1 - 2\alpha$ ∴

$1 - 2\alpha = Pr(\exp\{\bar{Y} + t_{\alpha;n-1} S\sqrt{1+\frac{1}{n}}\} < \exp\{Y_0\} < \exp\{\bar{Y} + t_{1-\alpha;n-1} S\sqrt{1+\frac{1}{n}}\}) =$

$Pr(\tilde{X}\exp\{t_{\alpha;n-1} S\sqrt{1+\frac{1}{n}}\} < X_0 < \tilde{X}\exp\{t_{1-\alpha;n-1} S\sqrt{1+\frac{1}{n}}\})$ where

$\tilde{X} = \prod_{i=1}^{n} X_i^{1/n}$ is Z geometric mean of $X_1, \ldots, X_n$.

$b(\hat{\theta}-\theta)$ $b(\hat{\theta}-\theta)$ $b(\hat{\theta}-\theta)$ $b(\hat{\theta}-\theta)$ $b(\hat{\theta}-\theta)$ $b(\hat{\theta}-\theta)$ $b(\hat{\theta}-\theta)$ $b(\hat{\theta}-\theta)$

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$var(\hat{\theta}) \to 0; n \to \infty$ ∴ $\hat{\theta}$ consistent $var(\hat{\theta}) \to 0$ consistent $var(\hat{\theta}) \to 0$ ∴ consistent

$E(\hat{\theta}) - \theta = 0$ unbiased asymp unbiased $E(\hat{\theta}) - \theta = 0$ unbiased asymp unbiased

$b(\hat\theta-\theta)$ $b(\hat\theta-\theta)$ $b(\hat\theta-\theta)$ $b(\hat\theta-\theta)$ $b(\hat\theta-\theta)$ $b(\hat\theta-\theta)$ $b(\hat\theta-\theta)$ $b(\hat\theta-\theta)$ $b(\hat\theta-\theta)$ $b(\hat\theta-\theta)$ $b(\hat\theta-\theta)$ $b(\hat\theta-\theta)$ $b(\hat\theta-\theta)$

$(E(\hat\theta)-\theta$ asymp unbiased $E(\hat\theta)-\theta$ asymp unbiased $E(\hat\theta)-\theta$ asymp unbiased $E(\hat\theta)-\theta$ asymp unbiased
$var(\hat\theta)\to 0$ consistent $var(\hat\theta)\to 0$ consistent $var(\hat\theta)\to 0$ consistent $var(\hat\theta)\to 0$ consistent

Sheet 2 preparatory / 1 ; 3.18 / $f(x;\mu,\sigma^2)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma}(x-\mu)^2}$ ∴

$L(\mu)=\prod_{i=1}^{n} f(x_i;\mu,\sigma^2)=\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma}(x_i-\mu)^2}=\left(\frac{1}{\sqrt{2\pi}\sigma}\right)^n e^{\frac{-1}{2\sigma}\sum_{i=1}^{n}(x_i-\mu)^2}$ ∴

is $f(x;\mu,\mu^2)=\frac{1}{\sqrt{2\pi}\mu}e^{-\frac{1}{2\mu}(x-\mu)^2}$ ∴

$L(\mu)=\left(\frac{1}{\sqrt{2\pi}\mu}\right)^n e^{\frac{-1}{2\mu}\sum_{i=1}^{n}(x_i-\mu)^2}=(2\pi\mu^2)^{-n/2}e^{-\frac{1}{2\mu}\sum_{i=1}^{n}(x_i-\mu)^2}$ with log-likelihood:

$l(\mu)=\ln(L(\mu))=-\frac{n}{2}\ln(2\pi\mu^2)-\frac{1}{2\mu}\sum_{i=1}^{n}(x_i-\mu)^2=-n\ln\mu-\frac{1}{2\mu^2}\sum_{i=1}^{n}(x_i-\mu)^2+\text{constant}$

& score $l'(\mu)=-\frac{n}{\mu}+\frac{1}{\mu^3}\sum_{i=1}^{n}(x_i-\mu)^2+\frac{1}{\mu^2}\sum_{i=1}^{n}(x_i-\mu)$ ∴ $l'(\hat\mu)=0$ ∴

$-n\hat\mu^2+\sum_{i=1}^{n}(x_i-\mu^2)+\hat\mu\sum_{i=1}^{n}(x_i-\hat\mu)=0$ ∴ $-n\hat\mu^2-\hat\mu\sum_{i=1}^{n}x_i+\sum_{i=1}^{n}x_i^2=0$ ∴

$$\hat\mu=\frac{1}{2n}\left[-\sum_{i=1}^{n}x_i\pm\sqrt{\left(\sum_{i=1}^{n}x_i\right)^2+4n\sum_{i=1}^{n}x_i^2}\right]$$

take the positive root ∵ μ is the standard deviation. The 2nd deriv is $l''(\mu)=$
$-\frac{3}{\mu^4}\sum_{i=1}^{n}(x_i-\mu)^2-\frac{2}{\mu^3}\sum_{i=1}^{n}(x_i-\mu)-\frac{2}{\mu^3}\sum_{i=1}^{n}(x_i-\mu)$ & ∵ $E(X_i)=\mu$,

$-E[l''(\mu)]=\frac{3n}{\mu^4}E[(X_i-\mu)^2]=\frac{3n}{\mu^4}var(X_i)=\frac{3n}{\mu^2}$ ∴ the asymp variance is
$I(\mu)^{-1}=\mu^2/(3n)$. usually $\hat\mu=\bar X$ with variance $\sigma^2/n=\mu^2/n$ ∵ $\sigma=\mu$. this is 3 times greater than $I(\mu)^{-1}$ above.

2 ; 4.19a / $f(x;\theta)=\frac{\theta^x e^{-\theta}}{x!}$ ∴ the likelihood is $L(\theta)=\prod_{i=1}^{n}\frac{\theta^{x_i}e^{-\theta}}{x_i!}=$
$e^{-n\theta}\theta^{\sum_{i=1}^{n}x_i}\prod_{i=1}^{n}\frac{1}{x_i!}$ ∴ loglikelihood: $l(\theta)=-n\theta+\left(\sum_{i=1}^{n}[x_i]\right)\ln(\theta)+\text{constant}$ ∴
score: $U(\theta)=l'(\theta)=-n+\frac{1}{\theta}\sum_{i=1}^{n}x_i$ ∴ $l''(\theta)=-\frac{1}{\theta^2}\sum_{i=1}^{n}x_i$ ∴
expected info: $I(\theta)=-E\left(-\theta^{-2}\sum_{i=1}^{n}x_i\right)=+\frac{1}{\theta^2}E\left(\sum_{i=1}^{n}x_i\right)=\frac{1}{\theta^2}\sum_{i=1}^{n}E(x_i)=$

$\frac{1}{\theta^2}nE(x_1)=\frac{1}{\theta^2}n\theta=n/\theta$ ∴ the score test stat is: $S=U(\theta_0)^2/I(\theta_0)$
$=\left(-n+\frac{1}{\theta_0}\sum_{i=1}^{n}x_i\right)^2/(n/\theta_0)=\frac{\theta_0}{n}\left(\frac{1}{\theta_0}n\bar x-n\right)^2=\frac{\theta_0}{n}\left(\frac{n\bar x}{\theta_0}-n\right)^2=\frac{n}{\theta_0}(\bar x-\theta_0)^2$ whose null distri is approx χ_1^2 when n is large. The critical val for a test of size 5% is C=3.84. ∴ for the given data, S=7.5 which exceeds C & ∴ would reject H_0 at the 5% level. ∵ $\frac{30}{1}(1.5-1)^2=7.5$

2 ; 4.19b / $l'(\hat\theta)=0=-n+\frac{1}{\hat\theta}\sum_{i=1}^{n}x_i=-n+\frac{1}{\hat\theta}n\bar x=0$ ∴ $1=\frac{1}{\hat\theta}\bar x$ ∴ $\hat\theta=\bar x$ is the mle
∴ the Wald test stat is $W=(\hat\theta-\theta_0)^2 I(\hat\theta)=(\bar x-\theta_0)^2\frac{n}{\hat\theta}=(\bar x-\theta_0)^2\frac{n}{\bar x}=$

$n(\bar{x}-\theta_0)^2/\bar{x}$. for Z given data, $W=5.0$, which also exceeds $c=3.84$. & ∴ we would reject H_0 at Z 5% level ∵ $30(1.5-1)^2/1.5=5$

\2 ; 4.19c/ Z likood ratio is : $\Lambda=\frac{L(\theta_0)}{L(\hat{\theta})}=\frac{\theta_0^{\sum_{i=1}^{n}x_i}e^{-n\theta_0}\prod_{i=1}^{n}\left(\frac{1}{x_i!}\right)}{\hat{\theta}^{\sum_{i=1}^{n}x_i}e^{-n\hat{\theta}}\prod_{i=1}^{n}\left(\frac{1}{x_i!}\right)}=$ $\left(\frac{\theta_0}{\hat{\theta}}\right)^{\sum_{i=1}^{n}x_i}e^{-n(\theta_0-\hat{\theta})}=\left(\frac{\theta_0}{\bar{x}}\right)^{n\bar{x}}e^{-n(\theta_0-\bar{x})}$ & ∴ Z likood ratio test stat is $-2\ln\Lambda=2n(\theta_0-\bar{x})-2n\bar{x}\ln(\theta_0/\bar{x})=-2\left[-n(\theta_0-\bar{x})+n\bar{x}\ln\left(\frac{\theta_0}{\bar{x}}\right)\right]$ ∴ for Z given data : Z LR test stat is $-2\ln\Lambda=2(30)(1-1.5)-2(30)1.5\ln\left(\frac{1}{1.5}\right)\approx 6.492\approx 6.5$, which also exceeds $c=3.84$, & ∴ would again reject H_0 at Z 5% level

\3 ; 5.4~~x~~, 4.19/ $f(x;\theta)=\theta x^{\theta-1}$ ∴ mle is $\hat{\theta}=-\frac{n}{\sum_{i=1}^{n}\ln x_i}$ & $I(\theta)=-E(l''(\theta))=\frac{n}{\theta^2}$ st Z asymp distri of $\hat{\theta}$ is $N(\theta,\theta^2/n)$. an approximate $(1-\alpha)$ CI for θ is given by $\{\theta:(\hat{\theta}-\theta)^2 I(\theta)\le c\}=\{\theta:n(\hat{\theta}-\theta)^2/\theta^2\le c\}$ where c is Z $(1-\alpha)$ quantile of Z χ^2_1 distri. this inequality is satisfied iff $(1-c/n)\theta^2-2\hat{\theta}\theta+\hat{\theta}^2\le 0$ ∴ iff θ lies between Z roots ∴ $\frac{\hat{\theta}\pm\hat{\theta}\sqrt{c/n}}{1-c/n}=\frac{\hat{\theta}}{1\mp\sqrt{c/n}}$ ∴ Z interval is $\left(\frac{\hat{\theta}}{1+\sqrt{c/n}},\frac{\hat{\theta}}{1-\sqrt{c/n}}\right)$

\3 ; 4.19/ for wald test of size α, Z CR is $C(\theta_0)=\{\underline{x}:n(\bar{x}-\theta_0)^2/\bar{x}\ge c\}$, where c is Z $(1-\alpha)$ quantile of Z χ^2_1 distri, & ∴ $(1-\alpha)$ confidence set is $S(\underline{x})=\{\theta:n(\bar{x}-\theta)^2/\bar{x}<c\}$ this latter inequality is satisfied iff $-\sqrt{c\bar{x}/n}<\bar{x}-\theta<\sqrt{c\bar{x}/n}$ ∴ st Z interval is $\bar{x}-\sqrt{c\bar{x}/n}<\theta<\bar{x}+\sqrt{c\bar{x}/n}$ for a 90% CI $c=2.71$, when $n=30$ & $\bar{x}=1.5$, Z interval is : $(1.13, 1.87)$. for Z score test, Z CR is $C(\theta_0)=\{\underline{x}:n(\bar{x}-\theta_0)^2/\theta_0\ge c\}$ & so Z $(1-\alpha)$ confidence set is $S(\underline{x})=\{\theta:n(\bar{x}-\theta)^2/\theta<c\}$. this inequality is satisfied iff $\theta^2-(2\bar{x}+c/n)\theta+\bar{x}^2<0$ which happens when θ lies betwe Z roots of Z quadratic ie $\theta_L<\theta<\theta_U$, where $\theta_L=\bar{x}+\frac{c}{2n}-\sqrt{\frac{c}{n}\left(\bar{x}+\frac{c}{4n}\right)}$ $\theta_U=\bar{x}+\frac{c}{2n}+\sqrt{\frac{c}{n}\left(\bar{x}+\frac{c}{4n}\right)}$, when $n=30$ & $\bar{x}=1.5$, this interval is $(1.17, 1.92)$

\4 ; 3.3a/ Z likood for p is Z product of Z three mass funcs $L(p)=\prod_{i=1}^{n}f(x_i;p)=\prod_{i=1}^{3}\binom{n_i}{x_i}p^{x_i}(1-p)^{n_i-x_i}$ ∴ Z loglikood is : $l(p)=\ln(p)\sum_{i=1}^{3}x_i+\ln(1-p)\sum_{i=1}^{3}(n_i-x_i)+\text{constant}$ with derivs :

\Sheet 2 prep 3a/ $L'(p) = p^{-1}\sum_{i=1}^{3} x_i - (1-p)^{-1}\sum_{i=1}^{3}(n_i - x_i)$ & ∴

$L''(p) = -p^{-2}\sum_{i=1}^{3} x_i - (1-p)^{-2}\sum_{i=1}^{3}(n_i - x_i) \le 0$ ∴ Solving $L'(p) = 0$: mle:

$$\hat{p} = \sum_{i=1}^{3} x_i \Big/ \sum_{i=1}^{3} n_i$$

\4; 3.3b/ now ∃ two params & Z lihood:

$L(\alpha,\beta) \propto \alpha^{x_1+x_3}(1-\alpha)^{n_1+n_3-x_1-x_3}(\alpha+\beta)^{x_2}(1-\alpha-\beta)^{n_2-x_2}$ with loglihood:

$l(\alpha,\beta) = \text{Const} + (\log\alpha)(x_1+x_3) + \log(1-\alpha)(n_1+n_3-x_1-x_3) + x_2\ln(\alpha+\beta) + (n_2-x_2)\ln(1-\alpha-\beta)$

∴ $\frac{\partial l}{\partial \beta} = \frac{x_2}{\alpha+\beta} - \frac{n_2-x_2}{1-\alpha-\beta}$ ∴ setting equals zero: ∴ $\hat{\alpha}+\hat{\beta} = \frac{x_2}{n_2}$ ∴

$\frac{\partial l}{\partial \alpha}(\hat{\alpha},\hat{\beta}) = \frac{x_1+x_3}{\hat{\alpha}} - \frac{n_1+n_3-x_1-x_3}{1-\hat{\alpha}}$ & setting equals zero: ∴ $\hat{\alpha} = \frac{x_1+x_3}{n_1+n_3}$ ∴

$\hat{\beta} = \frac{x_2}{n_2} - \frac{x_1+x_3}{n_1+n_3}$

\5; 3.3b/ Z log-lihood: $l(\alpha+\beta) =$

$\text{Const} + (x_1+x_3)\ln(\alpha) + (n_1+n_3-x_1-x_3)\ln(1-\alpha) + x_2\ln(\alpha-\beta) + (n_2-x_2)\ln(1-\alpha-\beta)$ &

$\frac{\partial l}{\partial \beta} = \frac{x_2}{\alpha+\beta} - \frac{n_2-x_2}{1-\alpha-\beta} = \frac{n_2(x_2/n_2 - \alpha - \beta)}{(\alpha+\beta)(1-\alpha-\beta)}$ ∴ setting equals zero: ∴ $\hat{\beta}(\alpha) = \frac{x_2}{n_2} - \alpha$

Z profile log-lihood for α: $l_p(\alpha) = l(\alpha, \hat{\beta}(\alpha)) =$

$\text{Const} + (x_1+x_3)\ln(\alpha) + (n_1+n_3-x_1-x_3)\ln(1-\alpha)$ where all terms that don't involve α have been put into Z Const.

\6; 4.16/ both Z null & alternat hypothes are composite hypothes, so we can use Z lihood ratio. for Z denominator, need to max Z lihood wrt both μ & λ. Z lihood: $L(\mu,\lambda) = \prod_{i=1}^{m} f(x_i;\lambda)\prod_{i=1}^{n} g(y_i;\mu) = \lambda^m e^{-\lambda m\bar{x}}\mu^n e^{-\mu n\bar{y}}$ with loglihood: $l(\mu,\lambda) = m\ln(\lambda) - \lambda m\bar{x} + n\ln(\mu) - \mu n\bar{y}$

& derivs: $\frac{\partial l}{\partial \mu} = n\mu^{-1} - n\bar{y}$ & $\frac{\partial l}{\partial \lambda} = m\lambda^{-1} - m\bar{x}$ ∴ Z max lihood estis: $\hat{\lambda} = \frac{1}{\bar{x}}$ & $\hat{\mu} = \frac{1}{\bar{y}}$. Z denominator of Z lihood ratio is:

$L(\hat{\mu},\hat{\lambda}) = \hat{\lambda}^m e^{-\hat{\lambda}m\bar{x}}\hat{\mu}^n e^{-\hat{\mu}n\bar{y}} = \bar{x}^{-m}e^{-m}\bar{y}^{-n}e^{-n} \propto \bar{x}^{-m}\bar{y}^{-n}$

for Z numerator, need to max $L(\mu,\lambda)$ under Z restriction imposed by $H_0: \lambda = \mu$ ∴ max: $l(\mu,\mu) = (m+n)\ln(\mu) - \mu(m\bar{x}+n\bar{y})$ ∴ Z deriv:

$\frac{\partial l}{\partial \mu} = (m+n)\mu^{-1} - (m\bar{x}+n\bar{y})$ s.t. $\hat{\mu}_0 = (m+n)/(m\bar{x}+n\bar{y})$ &

$L(\hat{\mu}_0,\hat{\mu}_0) = \hat{\mu}_0^m e^{-\hat{\mu}_0 m\bar{x}}\hat{\mu}_0^n e^{-\hat{\mu}_0 n\bar{y}} = \hat{\mu}_0^{m+n}e^{-\hat{\mu}_0(m\bar{x}+n\bar{y})} \propto (m\bar{x}+n\bar{y})^{-(m+n)}$ Z

lihood ratio: ∴ $\Lambda = \frac{L(\hat{\mu}_0,\hat{\mu}_0)}{L(\hat{\mu},\hat{\lambda})} \propto \frac{\bar{x}^m\bar{y}^n}{(m\bar{x}+n\bar{y})^{m+n}} = \frac{1}{(m+n\bar{y}/\bar{x})^m}\frac{1}{(m\bar{x}/\bar{y}+n)^n}$ &

Z data occur only in Z ratio $\bar{y}/\bar{x}$ as required.

\7/ have $\mu = E(\underline{X}) = (\theta, \dots, \theta)$ & $\Sigma = \mathrm{var}(\underline{X}) = \theta^2 I$ ∴

$\mu_\theta = (1, \dots, 1)$, $\Sigma^{-1} = \theta^{-2} I$

& Z quasi-score func:

$$G(\theta; \underline{x}) = \mu_\theta^T \Sigma^{-1}(\underline{x} - \mu) = \frac{1}{\theta^2}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}^T \begin{bmatrix} x_1 - \theta \\ \vdots \\ x_n - \theta\end{bmatrix} = \frac{n(\bar{x} - \theta)}{\theta^2}$$

st Z quasi-lihood estimator: $\hat{\theta} = \bar{x}$, &

$$K(\theta) = \mu_\theta^T \Sigma^{-1} \mu_\theta = \frac{1}{\theta^2}\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix}^T\begin{pmatrix}1\\ \vdots\\ 1\end{pmatrix} = \frac{n}{\theta^2}$$ st Z asymp distri of $\hat{\theta}$ is $N(\theta, \frac{\theta^2}{n})$.

this is Z usual Max lihood estimator for Z expectation of a normal distri when Z model assumes no link betw Z expectation & variance, but differs from Z mle when Z variance is assumed to equal Z square of Z expectation.

\Sheet 2 extra Ex/ \1; 3.16/ $P(X=x) = \frac{-\theta^x}{x \ln(1-\theta)}$ $x \in \mathbb{N}$ ∴

Z lihood: $L(\theta) = \prod_{i=1}^n P(X = x_i) = \prod_{i=1}^n \frac{-\theta^{x_i}}{x_i \ln(1-\theta)} = \frac{1}{(\ln(1-\theta))^n}(-1)^n \theta^{\sum_{i=1}^n x_i} \prod_{i=1}^n \frac{1}{x_i} =$

$(\ln(1-\theta))^{-n}\left(\frac{1}{-1}\right)^n \theta^{n\bar{x}} \prod_{i=1}^n \frac{1}{x_i} = (\ln(1-\theta))^{-n}(-1)^{-n}\theta^{n\bar{x}} \prod_{i=1}^n \frac{1}{x_i} = (-\ln(1-\theta))^{-n}\theta^{n\bar{x}} \prod_{i=1}^n \frac{1}{x_i} \propto$

$(-\ln(1-\theta))^{-n}\theta^{n\bar{x}}$ ∴ Log-lihood:

$l(\theta) = \text{const} - n\ln[-\ln(1-\theta)] + n\bar{x}\ln(\theta)$ & score:

$l'(\theta) = \frac{n\bar{x}}{\theta} + \frac{n}{(1-\theta)\ln(1-\theta)}$ st $l'(\hat{\theta}) = 0$ iff $(1-\hat{\theta})\ln(1-\hat{\theta})\bar{x} + \hat{\theta} = 0$ as required.

now ~~$l''(\theta) = -\frac{n\bar{x}}{\theta^2} + \frac{n}{(1-\theta)^2}\ln(1-\theta)$~~ $l''(\theta) = -\frac{n\bar{x}}{\theta^2} + \frac{n}{(1-\theta)^2\ln(1-\theta)} + \frac{n}{(1-\theta)^2(\ln(1-\theta))^2}$

& $E(X) = \sum_{x=1}^\infty \frac{\theta^x}{-\ln(1-\theta)} = \frac{\theta}{-(1-\theta)\ln(1-\theta)} = \frac{1}{-\ln(1-\theta)}\sum_{x=1}^\infty \theta^x = \frac{1}{-\ln(1-\theta)}\frac{\theta}{1-\theta} =$

$\frac{\theta}{-(1-\theta)\ln(1-\theta)}$ st

$I(\theta) = -E(l''(\theta)) = -E\left(-\frac{n\bar{x}}{\theta^2} + \frac{n}{(1-\theta)^2\ln(1-\theta)} + \frac{n}{(1-\theta)^2(\ln(1-\theta))^2}\right) =$

$E\left(\frac{n\bar{x}}{\theta^2}\right) - E\left(\frac{n}{(1-\theta)^2\ln(1-\theta)}\right) - E\left(\frac{n}{(1-\theta)^2(\ln(1-\theta))^2}\right) =$

$\frac{n}{\theta^2}E(\bar{x}) - \frac{n}{(1-\theta)^2\ln(1-\theta)}E(1) - \frac{n}{(1-\theta)^2(\ln(1-\theta))^2} =$

$\frac{n}{\theta^2}E(x) - \frac{n}{(1-\theta)^2\ln(1-\theta)} - \frac{n}{(1-\theta)^2(\ln(1-\theta))^2} =$

$\frac{n}{\theta^2}\frac{\theta}{-(1-\theta)\ln(1-\theta)} - \frac{n}{(1-\theta)^2\ln(1-\theta)} - \frac{n}{(1-\theta)^2(\ln(1-\theta))^2} = \frac{-n(\theta + \ln(1-\theta))}{\theta(1-\theta)^2(\ln(1-\theta))^2}$

& Z asymp distri of $\hat{\theta}$ is $N(\theta, I(\theta)^{-1}) =$

$N\left(\theta, -\frac{\theta(1-\theta)^2(\ln(1-\theta))^2}{n(\theta + \ln(1-\theta))}\right)$

\Sheet 2 extra Ex/ \1; 3.17a/ when α is known, Z l'hood is

$L(\beta) = \cancel{const + n\alpha \ln(\beta)}$ $L(\beta) = \prod_{i=1}^{n} f(x_i;\beta) = \prod_{i=1}^{n} \frac{\beta^{\alpha} x_i^{\alpha-1} e^{-\beta x_i}}{\Gamma(\alpha)} =$

$\left(\frac{1}{\Gamma(\alpha)}\right)^n (\beta^{\alpha})^n e^{-\beta \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i^{\alpha-1} =$

$\cancel{(\Gamma(\alpha))\sum \beta}$ $(\Gamma(\alpha))^{-n} \beta^{n\alpha} e^{-n\beta\bar{x}} \prod_{i=1}^{n} x_i^{\alpha-1} = \beta^{n\alpha} e^{-\beta n \bar{x}}$ with loglihood:

$l(\beta) = const + n\alpha \ln(\beta) - \beta n \bar{x}$

& score: $l'(\beta) = \frac{n\alpha}{\beta} - n\bar{x}$

& 2nd deriv: $l''(\beta) = -\frac{n\alpha}{\beta^2}$ Z 2nd deriv is nega $\because \alpha > 0$ in gamma distri $\therefore$ solving $l'(\hat{\beta}) = 0$:

mle: $\hat{\beta} = \frac{\alpha}{\bar{x}}$.

Z expected info is $I(\beta) = -E(l''(\beta)) = \frac{n\alpha}{\beta^2}$ & $\therefore$ Z asymp distri of $\hat{\beta}$:

$N(\beta, \beta^2/(n\alpha))$

now let $\gamma = \frac{1}{\beta}$. Z density $\therefore$: $f(x;\gamma) = \gamma^{-\alpha} x^{\alpha-1} e^{-x/\gamma}/\Gamma(\alpha)$ $\therefore$

l'hood: $L(\gamma) \propto \gamma^{-n\alpha} e^{-n\bar{x}/\gamma}$

with loglihood: $l(\gamma) = const - n\alpha \ln(\gamma) - n\bar{x}/\gamma$

Score: $l'(\gamma) = -n\alpha(\gamma) + n\bar{x}(\gamma^2)$

& 2nd deriv: $l''(\gamma) = \frac{n\alpha}{\gamma^2} - \frac{2n\bar{x}}{\gamma^3}$

solving $l'(\hat{\gamma}) = 0$: $\hat{\gamma} = \frac{\bar{x}}{\alpha} = \frac{1}{\hat{\beta}}$ & this is Z mle $\because$

$l''(\hat{\gamma}) = n\alpha^3/\bar{x}^2 - 2n\alpha^3/\bar{x}^2 = -n\alpha^3/\bar{x}^2 < 0$. Z expectation of $\hat{\gamma}$ is $E(\hat{\gamma}) =$

$\frac{E(X)}{\alpha} = \frac{1}{\beta} = \gamma$ & Z variance of $\hat{\gamma}$ is $var(\hat{\gamma}) = \sum_{i=1}^{n} var(X_i)/(n^2\alpha^2) =$

$var(X)/(n\alpha^2) = 1/(n\alpha\beta^2) = \gamma^2/(n\alpha)$ Z expected info is: $I(\gamma) = -E(l''(\gamma)) =$

$-n\alpha/\gamma^2 + 2nE(X)/\gamma^3 = -n\alpha/\gamma^2 + 2n\alpha/\gamma^2 = n\alpha/\gamma^2$ & $\therefore$ Z CRLB is

$\frac{1}{I(\gamma)} = \frac{\gamma^2}{n\alpha}$, which equals Z variance of $\hat{\gamma}$, & $\therefore$ $\hat{\gamma}$ does Z CRLB.

or show Z score can be written as $l'(\gamma) = \frac{n\alpha}{\gamma^2}\left(\frac{\bar{x}}{\alpha} - \gamma\right) = \frac{n\alpha}{\gamma^2}(\hat{\gamma} - \gamma)$

\2/ Z null distri of l'hood-based tests stats (like wald, score, l'hood ratio test stats) can be approxed by their asymp distris, eg their limiting distris as Z sample size increases to infinity. these asymp distris, that is their limiting distris as Z sample size increases to infinity. these asymp distris are often largely indep of Z particular prob model (ex, Z asymp null distri of Z wald, score, likhoodratio test stats are all chi squared distri's) & $\therefore$ we don't have to work out new distri's for each new probab Model. but, Z resulting will give accurate results only when Z sample size is large.

\3 ; 4.15 / Z likhood : $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} c(\theta) d(x_i) = c(\theta)^n \prod_{i=1}^{n} d(x_i)$ $\therefore$

$L(\theta) = \begin{cases} c(\theta)^n \prod_{i=1}^{n} d(x_i), & \text{if } a \le x_{(1)} \le x_{(n)} \le b(\theta) \\ 0, & \text{otherwise} \end{cases}$ $\therefore$ require $\int_a^{b(\theta)} c(\theta) d(x)\,dx = 1$ $\therefore$

$c(\theta) = \frac{1}{\int_a^{b(\theta)} d(x)\,dx}$ now $d(x) > 0$ & $b(\theta)$ is monotonically increasing $\therefore$

$c(\theta)$ decreases as θ increases. $\therefore$ Z likhood is max when θ is ~~...~~ as small possible. this means that Z MLE is $\hat{\theta} = b^{-1}(x_{(n)})$ &

Z likhood ratio test stat is $W = -2\ln\left[c(\theta_0)^n \prod_{i=1}^{n} d(x_i) / c(\hat{\theta})^n \prod_{i=1}^{n} d(x_i)\right]$

$= -2n\ln(c(\theta_0)) - 2n\ln\left[\int_a^{x(n)} d(x)\,dx\right] = -2n\ln\left[\int_a^{x(n)} c(\theta_0) d(x)\,dx\right]$

Let $c(\theta) = \frac{2}{\theta^2}$ & $d(x) = x$, $b(\theta) = \theta$, $a = 0$. then $W = -2n\ln\left[\int_0^{x_{(n)}} 2\theta_0^{-2} x\,dx\right] =$

$-2n\ln(2\theta_0^{-2} x_{(n)}^2/2) = -4n\ln(x_{(n)}/\theta_0)$. now;

$P_r(X_{(n)} \le n) = P_r(X \le x)^n = \left[\int_0^x (2x/\theta^2)\,dx\right]^n = (x/\theta)^{2n}$ for $0 < x < \theta$ $\therefore$

$P_r(W \le w) = P_r[-4n\ln(X_{(n)}/\theta) \le w] = P_r[X_{(n)} \ge \theta\exp(-w/(4n))] = 1 - \exp(-w/2)$

which is distribution func for a χ_2^2 distri

\3; 4.20 a / Z likhood : $L(\theta) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n} \theta e^{-\theta x_i} = \theta^n e^{-\theta \sum_{i=1}^{n} x_i} = \theta^n e^{-n\theta\bar{x}}$

with loglikhood $l(\theta) = n\ln(\theta) - n\theta\bar{x}$ $\therefore$ score : $l'(\theta) = U(\theta) = \frac{n}{\theta} - n\bar{x}$ $\therefore$

expected info : $I(\theta) = -E(l''(\theta)) = -E(-\frac{n}{\theta^2}) = \frac{n}{\theta^2}E(1) = \frac{n}{\theta^2}$ $\therefore$

Z score test stat : $S = U(\theta_0)^2 / I(\theta_0) = (\frac{n}{\theta_0} - n\bar{x})^2(\frac{\theta_0^2}{n}) = n\theta_0^2(\bar{x} - \frac{1}{\theta_0})^2$

\3; 4.20b / $\therefore$ Z mle is $l'(\hat{\theta}) = 0 = \frac{n}{\hat{\theta}} - n\bar{x}$ $\therefore$ $\hat{\theta} = \frac{1}{\bar{x}}$ & Z wald test stat:

$W = (\hat{\theta} - \theta_0)^2 I(\hat{\theta}) = n\bar{x}^2(\frac{1}{\bar{x}} - \theta_0)^2 = n\theta_0^2(\bar{x} - \frac{1}{\theta_0})^2 = S$

\4; 5.6 / Z mle is $\hat{\theta} = \bar{X}$ with asymp distri $N(\theta, \theta^2/n)$ st an approx pivot is $\sqrt{n}(\bar{X} - \theta)/\theta \sim N(0,1)$ when n is large, & $\therefore$, writing z for Z $(1-\frac{\alpha}{2})$ quantile of Z $N(0,1)$ distri, have: $1 - \alpha = P_r[-z < \sqrt{n}(\bar{X}/\theta - 1) < z] =$

$P_r(1 - z/\sqrt{n} < \frac{\bar{X}}{\theta} < 1 + z/\sqrt{n}) = P_r[\bar{X}/(1 + z/\sqrt{n}) < \theta < \bar{X}/(1 - z/\sqrt{n})]$ defines Z CI.

\Sheet 2 Extra Ex/ \4; 5.7/ Z mle, $\hat{\mu}$, for μ is

$$\hat{\mu} = \frac{1}{2n}\left[-\sum_{i=1}^{n} x_i \pm \sqrt{\left(\sum_{i=1}^{n} x_i\right)^2 + 4n\sum_{i=1}^{n} x_i^2}\right]$$ & $I(\mu) = \frac{3n}{\mu^2}$ ∴ an approx $(1-\alpha)$ CI

for μ: $\{\mu : (\hat{\mu}-\mu)^2 I(\mu) \leq c\} = \{\mu : 3n(\hat{\mu}-\mu)^2/\mu^2 \leq c\}$ where c is Z $(1-\alpha)$ quantile of Z χ^2_1 distri. this inequality is satisfied iff $(1-\frac{c}{3n})\mu^2 - 2\hat{\mu}\mu + \hat{\mu}^2 \leq 0$ which occurs iff μ lies betw Z roots

$$\frac{\hat{\mu} \pm \hat{\mu}\sqrt{\frac{c}{3n}}}{1-\frac{c}{3n}} = \frac{\hat{\mu}}{1 \mp \sqrt{\frac{c}{3n}}}$$ & ∴ Z interval: $\left(\frac{\hat{\mu}}{1+\sqrt{\frac{c}{3n}}}, \frac{\hat{\mu}}{1-\sqrt{\frac{c}{3n}}}\right)$

\4; 5.9/ Z asymp distri of Z score is $I(\theta)^{-1/2}U(\theta) \sim N(0,1)$ or, equiv: $I(\theta)^{-1}U(\theta)^2 \sim \chi^2_1$. both are pivot quantities. For a $N(0,\sigma^2)$ distri, Z lihood: $L(\sigma) \propto \sigma^{-n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2}$ ∴ loglihood:

$$l(\sigma) = \text{const} - n\ln(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n} x_i^2$$ ∴ Score:

$$l'(\sigma) = U(\sigma) = -n\sigma^{-1} + \sigma^{-3}\sum_{i=1}^{n} x_i^2$$ ∴ 2nd deriv:

$$l''(\sigma) = n\sigma^{-2} - 3\sigma^{-4}\sum_{i=1}^{n} x_i^2$$ & expected info: $I(\sigma) = -E(l''(\sigma)) =$

$$-E\left(n\sigma^{-2} - 3\sigma^{-4}\sum_{i=1}^{n} x_i^2\right) = -n\sigma^{-2} + 3\sigma^{-4}E\left(\sum_{i=1}^{n} x_i^2\right) = -n\sigma^{-2} + 3\sigma^{-4}\sum_{i=1}^{n}(E(x_i^2)) =$$

$$-n\sigma^{-2} + 3\sigma^{-4}\sum_{i=1}^{n}(E(X_i^2)) = -n\sigma^{-2} + 3\sigma^{-4}\sum_{i=1}^{n}(\operatorname{var}(X_i) + E(X_i)^2) =$$

$$-n\sigma^{-2} + 3\sigma^{-4}\sum_{i=1}^{n}(\sigma^2 + 0^2) = -n\sigma^{-2} + 3\sigma^{-4}\sum_{i=1}^{n}(\sigma^2) = -n\sigma^{-2} + 3\sigma^{-4}n\sigma^2 = 2n\sigma^{-2},$$

an approx $(1-\alpha)$ CI for σ is $\{\sigma : U(\sigma)^2 I(\sigma)^{-1} \leq c\}$ where c is Z $(1-\alpha)$ quantile of Z χ^2_1 distri. So this inequality is satisfied iff

$$\left(1-\frac{2c}{n}\right)\sigma^4 - 2\overline{x^2}\sigma^2 + \overline{x^2}^2 \leq 0$$ where $\overline{x^2} = \sum_{i=1}^{n} x_i^2/n$ this occurs iff σ^2 lies betw Z roots of Z quadratic in σ^2 ∴ Z roots: $\frac{\overline{x^2}}{1 \pm \sqrt{\frac{2c}{n}}}$ st an approx $(1-\alpha)$ CI for σ^2 is Z interval defined by these roots, & Z corres interval for σ is defined by Z square roots of these roots ie Z interval $\left(\sqrt{\frac{\overline{x}}{1+\sqrt{\frac{2c}{n}}}}, \sqrt{\frac{\overline{x^2}}{1-\sqrt{\frac{2c}{n}}}}\right)$.

\4.20 revisited/ for both tests, reject $H_0 : \theta = \theta_0$ in favour of $H_1 : \theta \neq \theta_0$ at level α is $n\theta_0^2\left(\overline{x} - \frac{1}{\theta_0}\right)^2 \geq c$. So a $(1-\alpha)$ confidence set is Z set of θ for which $n\theta^2\left(\overline{x} - \frac{1}{\theta}\right)^2 < c$ ie for which $n\overline{x}^2\theta^2 - 2n\overline{x}\theta + (n-c) < 0$, ie Z interval $\theta_L < \theta < \theta_U$ where $\theta_L = \frac{1}{\overline{x}} - \left(\frac{1}{\overline{x}}\right)\sqrt{\frac{c}{n}}$ $\theta_U = \frac{1}{\overline{x}} + \left(\frac{1}{\overline{x}}\right)\sqrt{\frac{c}{n}}$ are Z roots of Z quadratic.

\5; 3.9/ each X_{ij} is $N(\mu_i, \theta)$ & ∴ has density $f(x_{ij}; \mu_i, \theta) = (2\pi\theta)^{-1/2} e^{-\frac{(x_{ij}-\mu_i)^2}{2\theta}}$ Z lihood is ∴ $L(\mu_1, \dots, \mu_n, \theta) = \prod_{i=1}^{n}\prod_{j=1}^{2} f(x_{ij}; \mu_i, \theta) = (2\pi\theta)^{-n} e^{\left[-\frac{1}{2\theta}\sum_{i,j}(x_{ij}-\mu_i)^2\right]}$ & Z loglihood: $l(\mu_1, \dots, \mu_2, \theta) =$ constant $- n\ln(\theta) - \frac{1}{2\theta}\sum_{i,j}(x_{ij}-\mu_i)^2$ ∴ deriving wrt to μ_1 yields:

$\frac{\partial l}{\partial \mu_1} = \frac{1}{\theta}\sum_{j=1}^{2}(x_{1j}-\mu_1) = \frac{1}{\theta}(x_{11}+x_{12}-2\mu_1)$ & setting equals zero for μ_1: $\hat{\mu}_1 = \frac{x_{11}+x_{12}}{2}$ similarly: $\hat{\mu}_i = (x_{i1}+x_{i2})/2$ for $i = 1, \dots, n$ ∴

$\frac{\partial l}{\partial \theta} = -\frac{n}{\theta} + \frac{1}{2\theta^2} + \frac{1}{2\theta^2}\sum_{i,j}(x_{ij}-\mu_i)^2$ & setting equals zero for θ:

$\hat{\theta} = \frac{1}{2n}\sum_{i,j}(x_{ij}-\hat{\mu}_i)^2 = \frac{1}{2n}\sum_{i=1}^{n}\left[\left(x_{i1} - \frac{x_{i1}+x_{i2}}{2}\right)^2 + \left(x_{i2} - \frac{x_{i1}+x_{i2}}{2}\right)^2\right] =$

$\frac{1}{2n}\sum_{i=1}^{n}\left[\left(\frac{x_{i1}-x_{i2}}{2}\right)^2 + \left(\frac{x_{i2}-x_{i1}}{2}\right)^2\right] = \frac{1}{2n}\sum_{i=1}^{n}\frac{1}{4}\left[\left(\frac{x_{i1}-x_{i2}}{7}\right)^2 + \left(x_{i2}-x_{i1}\right)^2\right] =$

$\frac{1}{4n}\sum_{i=1}^{n}\frac{1}{2}\left[(x_{i1}-x_{i2})^2 + (x_{i1}-x_{i2})^2\right] = \frac{1}{4n}\sum_{i=1}^{n}(x_{i1}-x_{i2})^2 = \frac{1}{4n}\sum_{i=1}^{n} Z_i^2$, $Z_i = X_{i1} - X_{i2}$

\5; 4.1d/ Z only unknown param is μ_2. Z CR is of Z form $C = \{\underline{x} : \Lambda(\underline{x}) \geq c\}$ where $\Lambda(\underline{x}) = f_n(\underline{x}; \mu_2 = \mu_1 + \delta)/f_n(\underline{x}; \mu_2 = \mu_1)$. Z joint density func is $f_n(\underline{x}; \mu_2) = (2\pi\sigma_1^2)^{-n_1/2} e^{-\frac{1}{2\sigma_1^2}\sum_{i=1}^{n}(x_{1i}-\mu_1)^2} \frac{e^{-\frac{1}{2\sigma_2^2}\sum_{i=1}^{n_2}(x_{2i}-\mu_2)^2}}{(2\pi\sigma_2^2)^{n_2/2}}$

& ∴ Z lihood ratio: $\Lambda(\underline{x}) = \frac{f_n(\underline{x}; \mu_2 = \mu_1 + \delta)}{f_n(\underline{x}; \mu_2 = \mu_1)} =$

$e^{-\frac{1}{2\sigma_2^2}\left(-\sum_{i=1}^{n_2}(x_{2i}-\mu_1)^2 + \sum_{i=1}^{n_2}(x_{2i}-\mu_1-\delta)^2\right)} = e^{-\frac{1}{2\sigma_2^2}\left(-2\delta\sum_{i=1}^{n_2}(x_{2i}-\mu) + n_2\delta^2\right)} \propto e^{\frac{\delta}{\sigma_2^2}\sum_{i=1}^{n_2}x_{2i}}$

define $\bar{X}_2 = \sum_{i=1}^{n_2}\frac{X_i}{2}$, then ∵ $\bar{X}_2 \sim N\left(\mu_2, \frac{\sigma_2^2}{n_2}\right)$ & $\delta > 0$, $C = \{\underline{x} : \bar{x}_2 \geq d\}$ where d is Z upper α-quantile of $N(\mu_1, \sigma_2^2/n_2)$ ∴ $d = \mu_1 + \sigma_2 z_{1-\alpha}/\sqrt{n_2}$ where $z_{1-\alpha}$ is Z upper α-quantile of $N(0,1)$.

\5; 4.2a/ $\sigma_1^2 = \sigma_2^2 = \delta = 1$, $n_1 = n_2 = n$, $\alpha = 0.01$; $n = 10$ ∴ Z power is Z prob of rejecting Z null hypoth when Z alternative hypoth is true ie $\Pr(\bar{X}_2 \geq d)$ when $\bar{X}_2 \sim N(\mu_1 + \delta, \sigma_2^2/n_2)$. if we define $Z = (\bar{X}_2 - \mu_1 - \delta)/(\sigma_2/\sqrt{n_2})$ ∴ $Z \sim N(0,1)$ ∴ power: $\Pr(\bar{X}_2 \geq d) = \Pr\left(Z \geq \frac{d-\mu_1-\delta}{\sigma_2/\sqrt{n_2}}\right) = 1 - \Phi\left(z_{1-\alpha} - \frac{\delta\sqrt{n_2}}{\sigma_2}\right)$ ∵ $d = \mu_1 + \sigma_2 z_{1-\alpha}/\sqrt{n_2}$, where Φ is Z cdf of $N(0,1)$ distri ∴ Z power:

$1 - \Phi(z_{0.99} - \sqrt{10}) = 1 - \Phi(2.326 - \sqrt{10}) = 1 - \Phi(-0.836) = 0.798$.

\5; 4.2b/ require $1 - \Phi(z_{0.99} - \sqrt{n}) \geq 0.95$, which holds iff $z_{0.99} - \sqrt{n} \leq \Phi(0.05)$ iff $\sqrt{n} \geq z_{0.99} - z_{0.05}$ iff $n \geq (z_{0.99} - z_{0.05})^2 = 15.77$ ∴ require $n = 16$.

\Sheet 2 extra Ex/ \6/ Z pmf ~~for~~ X_i is $f(x_i;\theta)=\frac{(\theta z_i)^{x} e^{-\theta z_i}}{x_i}$ &

$\therefore$ Z likhood is $L(\theta)=\prod_{i=1}^{n}\frac{1}{x_i!}(\theta z_i)^{x_i}e^{-\theta z_i} \propto \theta^{\sum_{i=1}^{n}x_i}e^{-\theta\sum_{i=1}^{n}z_i}=\theta^{n\bar{x}}e^{-\theta n\bar{z}}$

$\therefore$ Z loglikhood $l(\theta)=\text{const}+n\bar{x}\ln(\theta)-n\bar{z}\theta$ with deriv:

$l'(\theta)=\frac{n\bar{x}}{\theta}-n\bar{z}$ & 2nd deriv $l''(\theta)=-\frac{n\bar{x}}{\theta^2}<0$ st solving $l'(\hat{\theta})=0$:

$\therefore$ mle: $\hat{\theta}=\frac{\bar{x}}{\bar{z}}$ $\therefore$ Z expected info: $I(\theta)=-E(l''(\theta))=$

$-E\left(-\frac{n\bar{x}}{\theta^2}\right)=\frac{1}{\theta^2}E(n\bar{x})=\frac{1}{\theta^2}E\left(\sum_{i=1}^{n}x_i\right)=\frac{1}{\theta^2}\sum_{i=1}^{n}E(x_i)=\frac{1}{\theta^2}\sum_{i=1}^{n}E(X_i)=\frac{1}{\theta^2}\sum_{i=1}^{n}\theta z_i=$

$\frac{1}{\theta^2}n\theta\bar{z}=\frac{n\bar{z}}{\theta}$ st Z asymp distri of $\hat{\theta}$ is $N\left(\theta,\frac{\theta}{n\bar{z}}\right)$.

\7/ Z likhood is: $L(\alpha,\beta)=\prod_{i=1}^{n}f(x_i;\alpha,\beta)=\prod_{i=1}^{n}\frac{1}{\Gamma(\alpha)}\beta^{\alpha}x_i^{\alpha-1}e^{-\beta x_i}=$

$\beta^{n\alpha}\Gamma(\alpha)^{-n}\left(\prod_{i=1}^{n}x_i\right)^{\alpha-1}e^{-\beta\sum_{i=1}^{n}x_i}$ $\therefore$ loglikhood:

$$l(\alpha,\beta)=n\alpha\ln(\beta)-n\ln\Gamma(\alpha)+(\alpha-1)\sum_{i=1}^{n}\ln x_i-\beta\sum_{i=1}^{n}x_i.$$

to find Z profile loglikhood for α need to find an expression for Z val, $\hat{\beta}(\alpha)$ of β that maxs $l(\alpha,\beta)$ when α is fixed $\therefore$

$\frac{\partial l}{\partial\beta}=\frac{n\alpha}{\beta}-\sum_{i=1}^{n}x_i=\frac{n(\alpha-\beta\bar{x})}{\beta}$ & setting equal to zero & solving for β:

$\hat{\beta}(\alpha)=\frac{\alpha}{\bar{x}}$ $\therefore$ Z profile loglikhood for α is: $l_p(\alpha)=l(\alpha,\hat{\beta}(\alpha))=$

$n\alpha\ln\left(\frac{\alpha}{\bar{x}}\right)-n\ln(\Gamma(\alpha))+(\alpha-1)\sum_{i=1}^{n}\ln(x_i)-\frac{\alpha}{\bar{x}}\sum_{i=1}^{n}x_i=$

$n\alpha\ln\left(\frac{\alpha}{\bar{x}}\right)-n\ln(\Gamma(\alpha))+(\alpha-1)\sum_{i=1}^{n}\ln(x_i)-n\alpha$

\8/ Quasi-likhood estimation obtains param estis by solving an estimating eqn, where Z esting func is Z quasi-score func. Z quasi-score func depends on only Z expectations, variances, covariances of Z r.v.'s in Z model, & $\therefore$ ~~quasi~~ quasi-likhood estimation can be used when a full probab model is unspecified. May occure eg if Z process being modelled is particularly complex

\9/ have $\mu=(h^{-1}(\theta z_1),\dots,h^{-1}(\theta z_n))^T$ & $\Sigma=\phi\Sigma_0=$ ~~ϕ diag~~ $\phi\,\text{diag}(v(\mu_1),\dots,v(\mu_n))$ st $\mu_\theta=(z_1/h'(\mu_1),\dots,z_n/h'(\mu_n))^T$, where $h'(\mu)=dh(\mu)/d\mu$ & $\Sigma^{-1}=\phi^{-1}\Sigma_0^{-1}=\phi^{-1}\text{diag}(1/v(\mu_1),\dots,1/v(\mu_n))$.

$\because \frac{dh(\mu)}{d\theta}=\frac{dh(\mu)}{d\mu}\frac{d\mu}{d\theta}$, & $\because \frac{dh(\mu)}{d\theta}=z$ $\therefore$ $\frac{d\mu}{d\theta}=\frac{z}{h'(\mu)}$ $\therefore$, Z quasi-score func is $G(\theta;\underline{X})=\mu_\theta^T\Sigma^{-1}(\underline{X}-\mu)=\dots=\phi^{-1}\sum_{i=1}^{n}\frac{z_i(X_i-\mu_i)}{v(\mu_i)h'(\mu_i)}$, Z asymp variance is Z reciprocal of $K(\theta)=\mu_\theta^T\Sigma^{-1}\mu_\theta=\dots=$

$\phi^{-1} \sum_{i=1}^{n} \frac{z_i^2}{V(\mu_i)h'(\mu_i)^2}$ & an esti for ϕ is $\hat{\phi} = \frac{1}{n-1}(\underline{X}-\hat{\mu})^T \hat{\Sigma}_0^{-1}(\underline{X}-\hat{\mu}) = \dots =$

$\frac{1}{n-1}\sum_{i=1}^{n}\frac{(X_i-\hat{\mu}_i)^2}{V(\hat{\mu}_i)}$ where $\hat{\mu}_i = h^{-1}(\hat{\theta} z_i)$

\Sheet 3 prepara/ have: $\bar{x} = \frac{5}{10} = 0.5$ & $\hat{\phi} = \bar{x}^2 = 0.25$. for Z param bootstrap, esti Z Ber(θ) distri by Z Ber($\hat{\theta}$) distri, where $\hat{\theta} = \bar{x} = 0.5$. Z bootstrap version of Z test stat $T = \hat{\phi} - \phi = \bar{X}^2 - \theta^2$ is $T^* = \bar{X}^{*2} - \hat{\theta}^2$, where $X_1^*, \dots, X_n^*$ are indep Ber($\hat{\theta}$) r.v.s

Z bias-corrected esti is $\hat{\phi} - E(T^*)$ & Z estied standard error is Z square root of $var(T^*)$ ∴ approx Z quantities by resampling in Z following to obtain Z bias-corrected esti 0.22 & estied SE 0.16.

\2a/ have $T^* = \sum_{i=1}^{n} X_i^{*2}/n$, where $X_1^*, \dots, X_n^*$ are indep with common mass func $Pr(X_i^* = x_i) = \frac{1}{n}$ for $i = 1, \dots, n$ ∴ nonparam bootstrap

∴ $E(T^*) = E\left(\sum_{i=1}^{n} \frac{X_i^{*2}}{n}\right) = \frac{1}{n}E\left(\sum_{i=1} X_i^{*2}\right) = \frac{1}{n}\sum_{i=1} E(X_i^{*2}) = E(X_i^{*2}) = \sum_{i=1} x_i^2 Pr(X_i^* = x_i) = \frac{1}{n}\sum_{i=1}^{n} x_i^2$, are told that T is an estimator for Z variance of Z X_i & ∴ Z bias of T is $E(T) - var(X_1)$. Z bootstrap esti of this bias is $E(T^*) - var(X_1^*)$ ∴ $var(X_1^*) = E(X_1^{*2}) - E(X_1^*)^2 = E(T^*) - \left(\sum_{i=1}^{n} x_i Pr(X_1^* = x_i)\right)^2 = E(T^*) - \bar{x}^2$ & ∴ Z estied bias is $E(T^*) - (E(T^*) - \bar{x}^2) = \bar{x}^2$ as required

\2b/ Z nonparam bootstrap esti of Z variance of T is $var(T^*) = \frac{1}{n^2}\sum_{i=1}^{n} var(X_i^{*2}) = \frac{1}{n} var(X_1^{*2}) = \frac{1}{n}\left(E(X_1^{*4}) - E(X_1^{*2})^2\right) = \frac{1}{n}\left(\frac{1}{n}\sum_{i=1}^{n} x_i^4 - \left(\frac{1}{n}\sum_{i=1}^{n} x_i^2\right)^2\right) = \frac{1}{n^2}\left[\sum_{i=1}^{n} x_i^4 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i^2\right)^2\right]$ & Z estied SE is Z square root of this expression.

\~~Z~~/ $\int_0^x e^{x-t} y(t)\,dt = f(x)$ ∴ deriv:

$e^{x-x} y(x) + \int_0^x \frac{\partial(e^{x-t})}{\partial x} y(t)\,dt = f'(x) = \cancel{e^{x-x}}$ $e^0 y(x) + \int_0^x e^{x-t} y(t)\,dt =$

$y(x) + f(x) = f'(x)$

\3/ note: $\hat{\theta} = \sum_{i=1} y_i / \sum_{i=1} x_i = 117/24.4 = 4.795$. Z jackknife version of $\hat{\theta}$ is $\hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n} \hat{\theta}_i = 4.80$, where $n = 6$, $\hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$ &

$\hat{\theta}_{-i} = \left(\left(\sum_{j=1}^{n} y_j\right) - y_i\right) / \left(\left(\sum_{j=1}^{n} x_j\right) - x_i\right)$. Z jackknife esti of Z SE is $\sqrt{\hat{v}_J} = 0.33$, where $\hat{v}_J = \frac{1}{n(n-1)}\sum_{i=1}^{n}(\hat{\theta}_i - \hat{\theta}_J)^2$ Z jackknife 95% CI for θ is $\hat{\theta}_J \pm 2.571\sqrt{\hat{v}_J} = (3.96, 5.64)$ where 2.571 is Z 97.5% quantile of Z Stu(n-1) distri $t_{0.025,5}$

Sheet 3 prepara/

x_i	3.4	7.2	4.6	1.3	5.5	2.4
y_i	14	32	23	3	31	14
$\sum x_j - x_i$	21	17.2	19.8	23.1	18.9	22
$\sum y_j - x_i$	103	85	95	114	86	103
$\frac{\sum x_j - x_i}{\sum y_j - x_i} = \hat{\theta}_{-i}$	4.905	4.942	4.747	4.935	4.55	4.682
$n\hat{\theta} - (n-1)\hat{\theta}_{-i} = \hat{\theta}_i$	4.247	4.061	5.033	4.095	6.019	5.361

$\therefore \hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i = \frac{1}{6}(4.247 + 4.061 + 5.033 + 4.095 + 6.019 + 5.361) = 4.8$

$\therefore n = 6$ $\therefore$ jackknife est. for $SE \approx \sqrt{\hat{v}_J}$ $\therefore$

$\hat{v}_J = \frac{1}{n(n-1)}\sum_{i=1}^{n}(\hat{\theta}_i - \hat{\theta}_J)^2$ $\therefore$

$(\hat{\theta}_i - \hat{\theta}_J)^2$: 0.306 0.546 0.0543 0.497 1.486 0.315 $\therefore$

$\sqrt{\hat{v}_J} = \left(\frac{1}{6(6-1)}(0.302 + 0.546 + 0.00543 + 0.497 + 1.486 + 0.315)\right)^{1/2} = 0.33$ $\therefore$

$n = 6$, ~~ν~~ 95% CI, $\nu = 1 - 5 = 2.5\%$ $\therefore$ $t_{0.025,5} = 2.571$ $\therefore$

95% CI : $\hat{\theta}_J \pm t_{0.025,5}\sqrt{\hat{v}_J}$ $\hat{\theta}_J \pm 2.571\sqrt{\hat{v}_J} = 4.8 \pm 2.571(0.33) = 4.8 \pm 0.848 =$
$(3.96, 5.64)$

$\hat{\theta} = \sum_{i=1}^{n} y_i$ $\therefore$ $\hat{\theta}_{-i} = \sum_{j=1}^{n} y_j - y_i$ $\quad \hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$ $\quad \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i = \hat{\theta}_J$ $\quad \hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$

$\frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i = \hat{\theta}_J$ $\quad \hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$ $\quad \hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$ $\quad \hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$ $\quad \hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$

$\hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$ $\quad \hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$ $\quad \hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$ $\quad \hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$ $\quad \hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$

$\hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$ $\quad \hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$ $\quad \hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$ $\quad \hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i$ $\quad \hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i$ $\quad \hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i$ $\quad \hat{\theta}_J = \sum\hat{\theta}_i$

4/ $\hat{\phi} = \left(\sum_{i=1}^{n} X_i/n\right)^2 = (S/n)^2$ & $\therefore$ $\hat{\phi}_{-i} = \left(\frac{1}{n-1}\sum_{j\neq i}^{n} X_j\right)^2 = \left(\frac{1}{n-1}\left[\left(\sum_{j=1}^{n} X_j\right) - X_i\right]\right)^2 =$
$\left(\frac{1}{n-1}\left[\left(\sum_{i=1}^{n} X_i\right) - X_i\right]\right)^2 = \left(\frac{1}{n-1}[S - X_i]\right)^2 = \left(\frac{S - X_i}{n-1}\right)^2$ $\therefore$

$\hat{\phi}_i = n\hat{\phi} - (n-1)\hat{\phi}_{-i}$ $\therefore$ ~~$\hat{\phi}_J$~~ = the jackknife estimator is : $\hat{\phi}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\phi}_i =$
$\frac{1}{n}\sum_{i=1}^{n}\left[n\hat{\phi} - (n-1)\hat{\phi}_{-i}\right] = \frac{1}{n}\sum_{i=1}^{n}\left[n\left(\frac{S}{n}\right)^2 - (n-1)\left(\frac{S - X_i}{n-1}\right)^2\right] = \frac{n}{n}\sum_{i=1}^{n}\left(\frac{S}{n}\right)^2 - \frac{n-1}{n}\sum_{i=1}^{n}\left(\frac{S - X_i}{n-1}\right)^2 =$
$\sum_{i=1}^{n}\frac{S^2}{n^2} - \frac{n-1}{n}\sum_{i=1}^{n}\left(\frac{S - X_i}{n-1}\right)^2 = \frac{S^2}{n} - \frac{1}{n(n-1)}\sum_{i=1}^{n}(S - X_i)^2 = \frac{1}{n}S^2 - \frac{1}{n(n-1)}\sum_{i=1}^{n}(S^2 - 2SX_i + X_i^2) =$
$\frac{1}{n}S^2 - \frac{1}{n-1}S^2 + \frac{2S}{(n-1)}\frac{1}{n}\sum_{i=1}^{n}X_i - \frac{1}{n(n-1)}\sum_{i=1}^{n}X_i^2 = \frac{1}{n}S^2 - \frac{1}{n-1}S^2 + \frac{2}{n(n-1)}S^2 - \frac{1}{n(n-1)}\sum_{i=1}^{n}X_i^2 =$
$\frac{n-1-n}{n(n-1)}S^2 + \frac{2S^2}{n(n-1)} - \frac{1}{n(n-1)}\sum_{i=1}^{n}X_i^2 = \frac{1}{n(n-1)}S^2 - \frac{1}{n(n-1)}\sum_{i=1}^{n}X_i^2 = \frac{1}{n(n-1)}S^2 - \frac{1}{n(n-1)}\sum_{i=1}^{n}X_i = \hat{\phi}_J$
$\because X_i^2 = X_i$ for Bernoulli r.v.'s $\therefore$ $\hat{\phi}_J = \frac{1}{n(n-1)}S^2 - \frac{1}{n(n-1)}\sum_{i=1}^{n}X_i = \frac{1}{n(n-1)}\left(S^2 - \sum_{i=1}^{n}X_i\right) =$
$\frac{1}{n(n-1)}(S^2 - S) = \frac{S(S-1)}{n(n-1)}$ $\therefore$ for data : $\hat{\phi} = \left(\frac{5}{10}\right)^2 = \frac{1}{4}$ $\therefore$ $\hat{\phi}_J = \frac{(5\times4)}{10\times9} = \frac{2}{9} \approx 0.222$

5a/ a monte carlo test would simulate a large number of samples of size n from the poisson distri with param θ_0 & calc the wald test stat for each sample. The p-value would be the proportion of these resampled test stats that exceed the observed val of the test stat

\6/ resample both $x_1^*, \ldots, x_m^*$ & $y_1^*, \ldots, y_n^*$ from Exp($\hat{\mu}_0$), calc Z test stat for each resample, & compute Z p-value as Z proportion of resampled test stats that exceed Z observed test stat

\7/ 90% CI for ϕ (−0.14, 0.48) (basic), (−0.055, 0.91) (studentised), (0.04, 0.64) (percentile). only Z percentile bootstrap guarantees an interval that is a subset of (0,1), Z set of possible vals for ϕ

\8/ simulate many samples of size 10 from a Ber(0.5) distri & compute Z percentile bootstrap 90% CI for ϕ for each sample ∴ calc Z proportion of times that $\phi = \theta^2 = 0.25$ lies outside Z interval. find that ϕ lies below Z lower limit about 5.3% of Z time & above Z ~~...~~ upper limit about 5.1% of Z time, ∴ Z coverage is about 100−5.3−5.1 = 89.6%, close to Z nominal 90%.

\9a/ let $X_i = \sigma Z_i$, where Z distri of Z_i contains no unknown params ∴ $\bar{X} = \sigma \bar{Z}$ & $T = (X_0/\bar{X}) = (X_0)/(\bar{X}) = (\sigma Z_0)/(\sigma \bar{Z}) = \frac{Z_0}{\bar{Z}}$ which depends only on Z Z_i & ∴ is indep of σ ∴ $X_0/\bar{X} = T$ is ancillary for scale models

\9bi/ let $\sigma = 1/\theta$ & $X = \sigma Z$. Show that Z has a distri that is indep of θ ∴ $P(Z \le z) = P\left(\frac{X}{\sigma} \le z\right) = P\left(\frac{X}{(1/\theta)} \le z\right) = P(\theta X \le z) = P(X \le z/\theta) = \int_0^{z/\theta} f(x;\theta)\,dx = \int_0^{z/\theta} \theta e^{-\theta x}\,dx = 1 - e^{-z}$ which is indep of any unknown params

\9bii/ if q_p denotes Z p-quantile of Z $T = X_0/\bar{X}$ then $0.8 = \Pr(q_{0.1} < X_0/\bar{X} < q_{0.9}) = \Pr(q_{0.1}\bar{X} < X_0 < q_{0.9}\bar{X})$ & ∴ a 80% prediction interval is $(\hat{q}_{0.1}\bar{X}, \hat{q}_{0.9}\bar{X})$, where $\hat{q}_p$ is Z p-quantile of Z bootstrap version of T, that is of $T^* = X_0^*/\bar{X}^*$, where Z X_i^* are indep with distri Exp($\hat{\theta}$) ∴ (0.25, 6.75)

\Sheet3 Extra Ex\1/ $\bar{x} = 19/5 = 3.8$ ∴ $\hat{\phi} = \exp(-3.8) = e^{-3.8} = 0.022$, Z bias corrected esti is 0.013 (param resampling) or 0.016 (non param resampl) & Z estied SE is 0.03 (param), 0.02 (non param)

\2/ ∴ $\hat{\delta} = 1.64$ ∴ 0.098 (param), 0.068 (nonparam) & SE: 0.52 (param) 0.43 (non param)

\3/ $\hat{\phi} = \left(\sum_{i=1}^{n} X_i/n\right)^2$ & Z. Z nonparam bootstrap version of $\hat{\phi}$ is $\hat{\phi}^* = \left(\sum_{i=1}^{n} X_i^*/n\right)^2$, where $X_1^*, \ldots, X_n^*$ are indep with common mass func $\Pr(X_1^* = x_i) = \frac{1}{n}$, $i = 1, \ldots, n$

\Sheet 3 extra Ex/ Z bootstrap esti of Z bias is $E(T^*)$, where $T^* = \hat{\phi}^* - \phi$, $\therefore E(\hat{\phi}^*) = \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} E(X_i^* X_j^*) = \frac{1}{n^2}\left\{\sum_{i=1}^{n} E(X_i^{*2}) + \sum_{i \neq j} E(X_i^* X_j^*)\right\} =$

$\frac{1}{n}\left\{nE(X_1^{*2}) + n(n-1)E(X_1^*)E(X_j^*)\right\} = \frac{1}{n^2}\left\{n\frac{1}{n}\sum_{i=1}^{n} x_i^2 + n(n-1)\bar{x}^2\right\} = \bar{x}^2 + \frac{\bar{x}(1-\bar{x})}{n}$ &

$\hat{\phi} = \bar{x}^2$ $\therefore$ Z esti of Z bias is $E(T^*) = \bar{x}(1-\bar{x})/n$ $\therefore$ Z bias-corrected esti is $\hat{\phi} - E(T^*) = \bar{x}^2 - \frac{\bar{x}(1-\bar{x})}{n} = \frac{n\bar{x}^2}{n} - \frac{\bar{x} - \cancel{n}\bar{x}^2}{n} = (\bar{x})^2 - \frac{\bar{x}(1-\bar{x})}{n} =$

$\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right)^2 - \frac{1}{n}\left(\sum_{i=1}^{n} x_i\right)\left(1 - \frac{1}{n}\sum_{i=1}^{n} x_i\right)/n = \left(\frac{1}{n}S\right)^2 - \frac{1}{n}S\left(1 - \frac{1}{n}S\right)\frac{1}{n} = \frac{1}{n^2}S^2 - \frac{1}{n^2}S\left(1 - \frac{1}{n}S\right) =$

$\frac{1}{n^2}S^2 - \frac{1}{n^2}S + \frac{1}{n^3}S^2 = \frac{S^2 - S}{n^2} + \frac{S^2}{n^3} = \frac{S(S-1)}{n^2} + \frac{S^2}{n^3}$ $\therefore$ $n = 10$, $S = 5$ $\therefore$ Z bias corrected esti is : $5 \times 4/10^2 + 5^2/10^3 = \frac{9}{40} = 0.225$

\4/ Z pseudo valsare $\hat{\phi}_i = n\hat{\phi} - (n-1)\hat{\phi}_{-i}$, where $\hat{\phi}_{-i} = \exp\{-(n\bar{x} - x_i)/(n-1)\}$ $\therefore$

$\hat{\phi}_J = 0.012$, $\sqrt{\hat{V}_J} = 0.026$ $\quad \hat{V}_J = \frac{1}{n(n-1)}\sum_{i=1}^{n}(\hat{\theta}_i - \hat{\theta}_J)^2 \quad \hat{V}_J = \frac{1}{n(n-1)}\sum_{i=1}^{n}(\hat{\phi}_i - \phi_J)^2$

$\hat{V}_J = \frac{1}{n(n-1)}\sum_{i=1}^{n}(\hat{\phi}_i - \phi_J)^2 \quad \hat{V}_J = \frac{1}{n(n-1)}\sum_{i=1}^{n}(\hat{\theta}_i - \hat{\theta}_J)^2 \quad \hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i} \quad \hat{\phi}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$

\5/ Z esti is Z minimum observation $\therefore$ when $x_{(1)}$ is omitted Z esti $\therefore \hat{\theta}_{-1} = x_{(2)}$ but when $x_{(i)}$ is omitted for any $i = 2, \ldots, n$ Z esti remains $\hat{\theta}_{-i} = x_{(1)}$. Z pseudovals are $\hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i} = \begin{cases} nx_{(1)} - (n-1)x_{(2)} & \text{if } i = 1 \\ x_{(1)} & \text{if } i > 1 \end{cases}$

$\therefore$ Z Jackknife esti is $\hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i = \frac{1}{n}\left(\hat{\theta}_1 + \sum_{i=2}^{n}\hat{\theta}_i\right) = \frac{1}{n}\left\{nx_{(1)} - (n-1)x_{(2)} + (n-1)x_{(1)}\right\} =$ $x_{(1)} - \frac{n-1}{n}(x_{(2)} - x_{(1)})$ $\quad$ Z Jackknife variance is $\hat{V}_J = \frac{1}{n(n-1)}\sum_{i=1}^{n}(\hat{\theta}_i - \hat{\theta}_J)^2 =$

$\frac{1}{n(n-1)}\left\{(\hat{\theta}_1 - \hat{\theta}_J)^2 + \sum_{i=2}^{n}(\hat{\theta}_i - \hat{\theta}_J)^2\right\} = \frac{1}{n(n-1)}\left[\left\{(\frac{1}{n}-1)(n-1)(x_{(2)} - x_{(1)})\right\}^2 + (n-1)\left\{\frac{1}{n}(n-1)(x_{(2)} - x_{(1)})\right\}^2\right] = \ldots =$

$\frac{(n-1)^2(x_{(2)} - x_{(1)})^2}{n^2}$ $\therefore$ Z SE is $\sqrt{\hat{V}_J} = (n-1)(x_{(2)} - x_{(1)})/n$

\6/ Z Monte Carlo simulates new data x^* from Z $Bin(n, \theta_0)$ distri and calcs Z test Stat for each new sample. Z p-val is Z proportion of simulate test Stats that exceed Z observed test stat. $\therefore$ Monte Carlo p-val 0.64 (wald), 0.71 (score, Z χ_1^2 distri p-vals of 0.61 (wald), 0.63 (score)

\7/ reject H_0 when Z test stat is large. when H_0 is true, $\underline{x}$ & $\underline{y}$ both have $Poi(\gamma)$ distris, $\therefore$ resample $\underline{x}^*$ & $\underline{y}^*$ indep from $Poi(\hat{\gamma}_0)$, where $\hat{\gamma}_0 = (m\bar{x} + n\bar{y})/(m+n)$ $\therefore$ p-value 0.031 $\therefore$ sufficient evidence to reject H_0 at Z 5% level

\8/ test stat $t = \hat{\beta}$ & reject H_0 when t is large. when H_0 is true, each x_i has a $N(\alpha, \sigma^2)$ distri & $\therefore$ resample x_i^* from $N(\hat{\alpha}_0, \hat{\sigma}_0^2)$, where $\hat{\alpha}_0$ is Z Sample Mean

& $\hat{\sigma}_0$ is Z sample standard deviation of $x_1, \dots, x_n$. Z p-value is Z proportion of resampled test stats that exceed Z observed test stat. ∴ pval 0.002 ∴ strong evidence to reject H_0 & conclude that ∃ an increasing trend in Z sea-level.

\9; 9.24a/ Z basic bootstrap 90% CI is $(\hat{\theta} - \hat{q}_{0.95}, \hat{\theta} - \hat{q}_{0.05})$ where $\hat{q}_p$ is an esti of Z p-quantile of $\hat{\theta}^* - \hat{\theta}$. Z empirical 5% quantile of $\hat{\theta}^*$ is $\hat{\theta}^*_{0.05(50)} = \hat{\theta}^*_{(2.5)} = (42.8 + 50.2)/2 = 46.5$ & Z empirical 95% quantile of $\hat{\theta}^*$ is $\hat{\theta}^*_{(0.95(50))} = \hat{\theta}_{(47.5)} = (93.4 + 93.9)/2 = 93.65$ st Z basic bootstrap interval is $(77.6 - (93.65 - 77.6), 77.6 - (46.5 - 77.6))$ or $(61.55, 108.7)$

\9; 9.24b/ Z percentile bootstrap 90% CI is $(\hat{\theta}^*_{(2.5)}, \hat{\theta}^*_{(47.5)}) = (46.5, 93.65)$

\10/ ∃ no param model & ∴ use non param resampling to obtain Z intervals (0.93, 7.06) (basic) (4.15, 5.76) (Studentised), (2.53, 8.66) (percentile)

\11/ resample x_i^* from $N(\hat{\alpha} + \hat{\beta} z_i, \hat{\sigma}^2)$ for $i = 1, \dots, n$ ∴ intervals (0.12, 1.02) cm/year (basic), (0.10, 1.05) cm/year (studentised), (0.11, 1.01) cm/year (percentile)

\12/ Z param esti is $\hat{\gamma} = 1.64$. ∴ Z plug in (equal-tailed) 80 prediction interval is defined by Z 0.1 & 0.9 quantiles of Z Beta($\hat{\gamma}$) distri ∴ (0.17, 0.83). for Z bootstrap intervals, resample $x_0^*, x_1^*, \dots, x_n^*$ from Z Beta($\hat{\gamma}$) distri ∴ (0.13, 0.92) (Studentised), (0.13, 0.87) (PIT). these intervals are wider than Z plug in interval as expected

\13/ resample x_i^* from $N(\hat{\alpha} + \hat{\beta} z_i, \hat{\sigma}^2)$ for $i = 0, 1, \dots, n$ where $z_0 = 1982 - 1956 = 26$ is Z covariate for Z year we want to predict. ∴ PIT bootstrap 90% PI (102, 167) cm

\Sheet 1/ \1a/ $Ga(\alpha, \beta) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha - 1} e^{-\beta x}$ ∴ $Ga(2, \phi): f(x; \phi) = \frac{\phi^2}{\Gamma(2)} x^{2-1} e^{-\phi x} = \frac{\phi^2}{(2-1)!} x^1 e^{-\phi x} = \phi^2 x e^{-\phi x}$ ∴ $Ga(2n, \phi)$ ∴ $f(y; \phi) = \frac{\phi^{2n}}{\Gamma(2n)} y^{2n-1} e^{-\phi x}$

$E(x_1) = E(x) = \int_0^\infty x f(x; \phi) dx = \int_0^\infty x \phi^2 x e^{-\phi x} dx = \int_0^\infty \phi^2 x^2 e^{-\phi x} dx$ ∴ let $y = \phi x$ ∴

~~$E(x) = \frac{2}{\phi}$~~ $\frac{1}{\phi} dy = dx$ ∴ $x = 0 \rightarrow y = 0$, $x = \infty \Rightarrow y = \infty$ ∴ $E(x) = \int_0^\infty (\phi x)^2 e^{-(\phi x)} dx = \int_0^\infty y^2 e^{-y} \frac{1}{\phi} dy = \frac{1}{\phi} \int_0^\infty y^2 e^{-y} dy = \frac{1}{\phi}\left([-y^2 e^{-y}]_0^\infty - \int_0^\infty -2y e^{-y} dy \right) = \frac{1}{\phi}\left(\int_0^\infty 2y e^{-y} dy \right) = \frac{1}{\phi}\left([-2y e^{-y}]_0^\infty - \int_0^\infty -2 e^{-y} dy \right) = \frac{1}{\phi}\left([-2e^{-y}]_0^\infty \right) = \frac{1}{\phi}(0 + 2) = \frac{2}{\phi}$ ∴ $\frac{1}{\phi} \int_0^\infty y^2 e^{-y} dy = \frac{1}{\phi} \Gamma(3) = \frac{2!}{\phi} = \frac{2}{\phi}$

∴ ϕ satisfies $\bar{x} = \frac{2}{\phi}$, ie $\hat{\phi} = 2/\bar{x}$

\Sheet 1/ \1b/

let $\theta = \frac{2}{\phi}$ check $\hat\theta = 2/\hat\phi = \bar{x}$ $\therefore$

$E(\hat\theta) = E(\bar{x}) = E\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \frac{1}{n}\sum_{i=1}^{n} E(x_i) = \frac{1}{n}\sum_{i=1}^{n}\frac{2}{\phi} = \frac{1}{n} n \frac{2}{\phi} = \frac{2}{\phi} = \theta$ $\therefore$

bias is : $\text{Bias}(\hat\theta) = E(\hat\theta) - \theta = \theta - \theta = 0$

$E(x^2) = \int_0^\infty x^2 f(x;\phi)\,dx = \int_0^\infty x^2 \phi^2 x e^{-\phi x}\,dx = \frac{1}{\phi}\int_0^\infty \phi^3 x^3 e^{-\phi x}\,dx = \frac{1}{\phi}\int_0^\infty (\phi x)^3 e^{-(\phi x)}\,dx$

$\therefore y = \phi x \therefore \frac{1}{\phi}dy = dx \therefore x = 0 \to y \to 0$, $x = \infty \to y = \infty \therefore E(x^2) = \frac{1}{\phi}\int_0^\infty (\phi x)^3 e^{-(\phi x)}\,dx =$

$\frac{1}{\phi}\int_0^\infty y^3 e^{-y}\,\frac{1}{\phi}\,dy = \cancel{\frac{1}{\phi^2}\int_0^\infty y^{4-1}}\ \frac{1}{\phi^2}\int_0^\infty y^{4-1}e^{-y}\,dy = \frac{1}{\phi^2}\Gamma(4) = \frac{(4-1)!}{\phi^2} = \frac{3!}{\phi^2} = \frac{6}{\phi^2}$ $\therefore$

$\text{var}(x) = E(x^2) - E(x)^2 = \frac{6}{\phi^2} - \left(\frac{2}{\phi}\right)^2 = \frac{6}{\phi^2} - \frac{4}{\phi^2} = \frac{2}{\phi^2}$ &

$\text{var}(\hat\theta) = \text{var}(\bar{x}) = \text{var}\left(\frac{1}{n}\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2}\text{var}\left(\sum_{i=1}^{n} x_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\text{var}(x_i) = \frac{1}{n^2}\sum_{i=1}^{n}\text{var}(x) =$ (independence)

$\frac{1}{n^2}\sum_{i=1}^{n}\frac{2}{\phi^2} = \frac{1}{n^2} n \frac{2}{\phi^2} = \frac{1}{n}\frac{1}{2}\frac{(2)(2)}{\phi^2} = \frac{1}{2n}\left(\frac{2}{\phi}\right)^2 = \frac{1}{2n}\theta^2$ $\therefore$

$\text{var}(\hat\theta) \to 0$ as $n \to \infty$ & $\text{bias}(\hat\theta) \to 0$ as $n \to \infty$ $\therefore \hat\theta$ is consistent

\1c/

$\text{var}[h(\hat\theta)] \approx [h'(\theta)]^2\,\text{var}(\hat\theta)$ $\quad$ $\text{bias}[h(\hat\theta)] \approx \frac{1}{2}h''(\theta)\,\text{var}(\hat\theta)$

$\hat\phi = \frac{2}{\hat\theta}$, $\text{var}(\hat\theta) = \frac{\theta^2}{2n} = \frac{2}{n\phi^2}$ is ~~At~~ $h = \hat\phi \therefore h(\hat\theta) = \hat\phi = \frac{2}{\hat\theta} \therefore h'(\hat\theta) = -\frac{2}{\hat\theta^2}$ $\therefore$

$h(\hat\phi) = -\frac{2}{\hat\phi^2}$ $\therefore$ $\text{var}(\hat\phi) = \text{var}[h(\hat\theta)] \approx [h'(\theta)]^2\,\text{var}(\hat\theta)$ $\therefore$

$\hat\phi = h(\hat\theta) \therefore \phi = h(\theta) = \frac{2}{\theta} \therefore h'(\theta) = \frac{-2}{\theta^2}$ $\therefore$

$\text{var}[h(\hat\theta)] \approx [h'(\theta)]^2\,\text{var}(\hat\theta) = \left(\frac{-2}{\theta^2}\right)^2\frac{\theta^2}{2n} = \frac{4}{\theta^4}\frac{\theta^2}{2n} = \frac{2}{n\theta^2}$ ~~th~~ $\therefore$

$\text{var}(\hat\phi) \approx \frac{2}{n\theta^2} = \frac{1}{2n}\left(\frac{2}{\theta}\right)^2 = \frac{\phi^2}{2n}$

\1d/

$\text{bias}[h(\hat\theta)] \approx \frac{1}{2}h''(\theta)\,\text{var}(\hat\theta)$ $\quad$ $\hat\phi = h(\hat\theta) = \frac{2}{\hat\theta} \therefore \phi = h(\theta) = \frac{2}{\theta}$ $\therefore$

$h''(\theta) = \frac{4}{\theta^3}$ $\therefore$ $\text{bias}(\hat\phi) = \text{bias}[h(\hat\theta)] \approx \frac{1}{2}\frac{4}{\theta^3}\frac{\theta^2}{2n} = \frac{1}{n\theta} = \frac{2}{\theta}\left(\frac{1}{2n}\right) = \frac{\hat\phi}{2n}$

a bias corrected estimator is $\tilde\phi = \hat\phi - \text{bias}(\hat\phi) \approx \hat\phi - \frac{\hat\phi}{2n} = \frac{2n\hat\phi - \hat\phi}{2n} = \left(1 - \frac{1}{2n}\right)\hat\phi$

& $\text{var}(\tilde\phi) = \left[1 - \frac{1}{2n}\right]^2\text{var}(\hat\phi)$

\2a/

$f(x;\phi) = \phi^2 x e^{-\phi x}$, $x > 0$ CRLB for estimator of ϕ $\therefore$

$L(\phi) = \prod_{i=1}^{n} f(x;\phi) = \prod_{i=1}^{n}\phi^2 x_i e^{-\phi x_i} = \phi^{2n}\left(\prod_{i=1}^{n} x_i\right)e^{-\phi\sum_{i=1}^{n} x_i} = \phi^{2n}\left(\prod_{i=1}^{n} x_i\right)e^{-\phi n\bar{x}}$ $\therefore$

$l(\phi) = \ln(L(\phi)) = 2n\ln\phi + \left(\sum_{i=1}^{n}\ln x_i\right) - \phi n\bar{x}$ $\therefore$

$l'(\phi) = \frac{2n}{\phi} - \sum_{i=1}^{n} x_i = \frac{2n}{\phi} - n\bar{x}$ $\therefore$ $l''(\phi) = \frac{-2n}{\phi^2}$ $\therefore$

$I(\phi) = -E(l''(\phi)) = -E\left(-\frac{2n}{\phi^2}\right) = \frac{2n}{\phi^2}E(1) = \frac{2n}{\phi^2}$ $\therefore$ CRLB is $\frac{1}{I(\phi)} = \frac{\phi^2}{2n}$

\2b/

score func is $l'(\phi) = U(\phi) = \frac{2n}{\phi} - n\bar{x} = -n\left(\bar{x} - \frac{2}{\phi}\right) - \frac{n}{2}\left(\frac{2}{\bar{x}} - \frac{4}{\phi}\right) = \frac{-n}{2}\left(\hat\phi - \frac{4}{\phi}\right)$

$\neq b(\hat\phi - \phi)$ $\therefore$ no unbiased & efficient estimator exists for ϕ $\therefore$

score func is $U(\mu) = \frac{n}{\sigma^2}(\bar{x} - \mu)$ & iff $U = b(\hat\phi - \phi)$ $\therefore$ $l'(\phi) \neq I(\phi)(\hat\phi - \phi)$ $\therefore$ no unbiased & efficient estimator exists

\2c/ $H_0: \phi=\phi_0$, $H_1: \phi<\phi_0$ $X_1,\dots,X_n$ iid $f(x;\phi)=\phi^2 x e^{-\phi x}$, $x>0$

consider $H_1': \phi=\phi_1$ where $\phi_1<\phi_0$ $\therefore$ have:

$L(\phi)=\phi^{2n}\left(\prod_{i=1}^{n} x_i\right)e^{-\phi\sum_{i=1}^{n} x_i}$ $\therefore$

$\frac{L(\phi_1)}{L(\phi_0)}=\frac{\phi_1^{2n}\left(\prod_{i=1}^{n}x_i\right)e^{-\phi_1\sum_{i=1}^{n}x_i}}{\phi_0^{2n}\left(\prod_{i=1}^{n}x_i\right)e^{-\phi_0\sum_{i=1}^{n}x_i}}=\left(\frac{\phi_1}{\phi_0}\right)^{2n}e^{(\phi_0-\phi_1)\sum_{i=1}^{n}x_i}$ this ratio is large when $\sum_{i=1}^{n}x_i$ is

large, so Z most powerful test has CR of Z form $\{\underline{x}: \sum_{i=1}^{n}x_i>c\}$, this

region doesnt depend on ϕ_1, so its UMP for H_1

~~[illegible]~~ $\therefore \Lambda(\underline{x})=\frac{L(\phi_1)}{L(\phi_0)}=\left(\frac{\phi_1}{\phi_0}\right)^{2n}e^{(\phi_0-\phi_1)n\bar{x}}$ $\therefore \Lambda(\underline{x})\geq c$ iff $n\bar{x}\geq d$ $\therefore \{\underline{x}: n\bar{x}\geq d\}$

is UMP $\because$ d doesnt depend on ϕ_1

\2d/ for a test of size α, need $P_r\left(\sum_{i=1}^{n}X_i>c;\phi=\phi_0\right)=\alpha$

$\because$ UMP for H_1 & CR $\{\underline{x}: \sum_{i=1}^{n}x_i>c\}$ $\therefore$ test size α: $P_r\left(\sum_{i=1}^{n}X_i>c;\phi=\phi_0\right)=\alpha$ $\therefore$

Z critical val, d, must satisfy $P_r(Y\geq d;\phi_0)=\alpha$ $\therefore Y=\sum_{i=1}^{n}X_i$ $\therefore$

$P_r\left(\sum_{i=1}^{n}X_i>C;\phi=\phi_0\right)=\alpha$ $\therefore$ $P_r(Y>C;\phi=\phi_0)=\alpha$ $\therefore$

$P_r(Y>c;\phi=\phi_0)=\int_d^{\infty}f(y;\phi_0)\,dy=\alpha$ $\therefore$ d is Z upper α-quantile of Z

$Ga(2n,\phi_0)$ distri

\3a/ $T_1=\phi Y$ where $Y\sim Ga(2n,\phi)$ with pdf $f(y;\phi)=\frac{\phi^{2n}}{\Gamma(2n)}y^{2n-1}e^{-\phi y}$, $y>0$

$\therefore$ have $P_r(T_1\leq t)=P_r(\phi Y\leq t)=P_r\left(Y\leq\frac{t}{\phi}\right)=\int_0^{t/\phi}f(y;\phi)\,dy=\int_0^{t/\phi}\frac{\phi^{2n}}{\Gamma(2n)}y^{2n-1}e^{-\phi y}\,dy$ $\therefore$

let $s=\phi y$ $\therefore$ $\frac{1}{\phi}ds=dy$ $\therefore$ $y=0\rightarrow s=0$ $y=\frac{t}{\phi}\rightarrow s=t$ $\therefore$

$P_r(T_1\leq t)=\int_0^{t/\phi}\frac{\phi^{2n}}{\Gamma(2n)}y^{2n-1}e^{-\phi y}\,dy=\int_0^{t/\phi}\phi\frac{1}{\Gamma(2n)}(\phi y)^{2n-1}e^{-(\phi y)}\,dy=\phi\int_0^{t}\frac{1}{\Gamma(2n)}s^{2n-1}e^{-s}\frac{1}{\phi}\,ds=$

$\int_0^{t}\frac{s^{2n-1}}{\Gamma(2n)}e^{-s}\,ds$ $\therefore$ this Cdf is indep of ϕ $\therefore$ $T_1=\phi Y$ is a pivot

$\therefore$ Z density of T_1 is: $\frac{d}{dt}P_r(T_1\leq t)=\frac{1}{\Gamma(2n)}t^{2n-1}e^{-t}$ which descries a $Ga(2n,1)$ distri

\3b/ $T=\phi Y\sim Ga(2n,1)$ $\therefore$ ϕY is $Ga(2n,1)$ is: $\frac{1^{2n}}{\Gamma(2n)}y^{2n-1}e^{-1y}=\frac{1}{\Gamma(2n)}y^{2n-1}e^{-y}$ $\therefore$

$P_r(T_1\leq t)=\int_0^{t}\frac{s^{2n-1}}{\Gamma(2n)}e^{-s}\,ds$ $\therefore$ $P_r(Y\leq y)=\int_0^{y}f(y;\phi)\,dy=\int_0^{y}\frac{\phi^{2n}}{\Gamma(2n)}y^{2n-1}e^{-\phi y}\,dy$

$P_r(X\leq x)=\int_0^{x}f(x;\phi)\,dx=\int_0^{x}\frac{\phi^2}{\Gamma(x)}xe^{-\phi x}\,dx=\int_0^{x}\frac{\phi^2}{\Gamma(2)}x^{2-1}e^{-\phi x}=\int_0^{x}\frac{(\phi x)^2}{\Gamma(2)}x^{-1}e^{-\phi x}\,dx$ is $Ga(2,\phi)$

$\therefore$ $P_r(T_1\leq t)=\int_0^{t}\frac{(1\cdot s)^{2n}}{\Gamma(2n)}s^{-1}e^{-1s}\,ds$ is $Ga(2n,1)$ $\therefore$

$P_r(q_5<T_1<q_{95})=0.9$ $\therefore$ $P_r(q_5<\phi Y<q_{95})=0.9$ $\therefore$ $P_r\left(\frac{q_5}{\phi}<Y<\frac{q_{95}}{\phi}\right)=0.9$ $\therefore$

$P_r\left(\frac{q_5}{Y}<\phi<\frac{q_{95}}{Y}\right)=0.9$ $\therefore$ 90% CI: $(q_5/Y, q_{95}/Y)$ for ϕ

\3c/ $X_0\sim Ga(2,\phi)$ $\therefore$ density $f(x;\phi)=\phi^2 xe^{-\phi x}$ $\therefore$

$P_r(X_0\leq x)=\int_0^{x}f(a;\phi)\,da=\int_0^{x}\phi^2 ae^{-\phi a}\,da$ $\therefore$ let $z=\phi a$ $\therefore$ $\frac{1}{\phi}dz=da$ $\therefore$ $a=0: z=0$, $a=x, z=\phi x$

\Sheet 1/ $P(X_0 \le x) = \int_0^x \theta^2 a e^{-\theta a}\,da = \int_0^{\theta x} \theta(z) e^{-z} \frac{1}{\theta}\,dz = \int_0^{\theta x} z e^{-z}\,dz =$

$\left[-ze^{-z}\right]_0^{\theta x} - \int_0^{\theta x} -e^{-z}\,dz = -\theta x e^{-\theta x} + \int_0^{\theta x} e^{-z}\,dz = -\theta x e^{-\theta x} - \left[e^{-z}\right]_0^{\theta x} =$

$-\theta x e^{-\theta x} - e^{-\theta x} + 1 = 1-(1+\theta x)e^{-\theta x}$

ancillary func $T(X_0, \underline{X})$ whose distri is indep of θ ∴

$T_2 = X_0/\bar{X}$ ∴ $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ ∴

$Pr(T_2 \le t) = Pr(X_0/\bar{X} \le t) = Pr(X_0 \le t\bar{X}) = P(X_0 \le tY\frac{1}{n})$ where $Y = \sum_{i=1}^{n} X_i$ &

$Pr(X \le Y) = \int Pr(X \le y \mid Y=y) Pr(Y=y)\,dy$ ∴

$Pr(X_0 \le tY\frac{1}{n}) = \int_0^\infty Pr(X_0 \le ty\frac{1}{n}) f_Y(y;\theta)\,dy$ ∴

$Pr(X_0 \le ty\frac{1}{n}) = 1-(1+\theta\frac{ty}{n})e^{-\theta\frac{ty}{n}}$ ∴

$Pr(X_0 \le tY\frac{1}{n}) = \int_0^\infty Pr(X_0 \le ty\frac{1}{n}) f_Y(y;\theta)\,dy = \int_0^\infty \left(1-(1+\theta\frac{ty}{n})e^{-\theta\frac{ty}{n}}\right) f_Y(y;\theta)\,dy =$

$\int_0^\infty \left(1-(1+\theta\frac{ty}{n})e^{-\theta\frac{ty}{n}}\right)\frac{\theta^{2n}}{\Gamma(2n)} y^{2n-1} e^{-\theta y}\,dy$

∴ let $z = \theta y$ ∴ $\frac{1}{\theta}dz = dy$ $y=0 \to z=0$, $y=\infty \to z=\infty$ ∴

$Pr(X_0 \le tY\frac{1}{n}) = \int_0^\infty \left(1-(1+t\frac{1}{n}z)e^{-\frac{tz}{n}}\right)\frac{z^{2n}}{\Gamma(2n)} y^{-1}\frac{1}{\theta}e^{-z}\,dz = \int_0^\infty \left(1-(1+\frac{tz}{n})e^{-\frac{tz}{n}}\right)\frac{z^{2n-1}}{\Gamma(2n)}e^{-z}\,dz$

is indep of θ ∴ T_2 is ancillary

\3d/ $Pr(q_{0.05} \le T_2 \le q_{0.95}) = 0.9 = Pr(q_{0.05} \le X_0/\bar{X} \le q_{0.95}) = Pr(q_{0.05}\bar{X} \le X_0 \le q_{0.95}\bar{X}) =$

0.9 ∴ Z 90% CI for X_0: $(q_{0.05}\bar{X}, q_{0.95}\bar{X})$

\4a/ from Delta method: $bias(\hat{\theta}) \approx \frac{\theta}{2n}$, $var(\hat{\theta}) \approx \frac{\theta^2}{2n}$ ∴ $Sd(\hat{\theta}) = \frac{\theta}{\sqrt{2n}}$

biased corrected estimator $\tilde{\theta} = (1-\frac{1}{2n})\hat{\theta}$

Z biases & SE, Z original estimator, $\hat{\theta}$ is positively biased its bias decreases as sample increases. bias-corrected estior $\tilde{\theta}$ appears unbiased for all samples sizes. Z SE of both estiors decreases as sample increases. Z biased corrected estior has smaller SE than original estior but still doesnt attain CRLB

\4bi/ for $Ga(a,b)$ ∴ $Ga(2,\theta)$ ∴ $\hat{\theta} = 1.88$ SE $= 0.42$

or $\tilde{\theta} = 1.78$ SE $= 0.38$

\4bii/ 90 CI θ: $(1.24, 2.63)$

\4biii/ critical value $c = 13.94$ 95% quantile of a $Ga(2n, 2)$ distri

test stat is $\sum_{i=1}^{n} x_i = 10.66$ doesnt exceed cz ∴ conclude insufficient evidence at 5% level to reject null hypoth that $\theta = 2$ in favour of Z alternative hypoth that $\theta < 2$. Power func

\4biv/ 90% prediction interval is (0.19, 2.78) seconds

\Sheet 2/ \1a/ $f(x;\phi)=\phi^2 x e^{-\phi x}$, $x>0$ iid ∴

$L(\phi;x)=\prod_{i=1}^{m} f(x_i;\phi)=\prod_{i=1}^{m}\phi^2 x_i e^{-\phi x_i}=\phi^{2m}\prod_{i=1}^{m}(x_i)e^{-\sum_{i=1}^{m}\phi x_i}=$

$\phi^{2m}e^{-\phi\sum_{i=1}^{m}(x_i)}\prod_{i=1}^{m}(x_i)$ ∴

$l(\phi;\underline{x})=\ln(L(\phi;x))=\ln(\phi^{2m}e^{-\phi\sum_{i=1}^{m}x_i}\prod_{i=1}^{m}(x_i))=\ln(\phi^{2m})+\ln(e^{-\phi\sum_{i=1}^{m}x_i})+\ln(\prod_{i=1}^{m}(x_i))=$

$2m\ln\phi-\phi\sum_{i=1}^{m}x_i+(\ln(x_1)+\ln(x_2)+\dots+\ln(x_m))=$

$2m\ln(\phi)-\phi(\sum_{i=1}^{m}x_i)+\sum_{i=1}^{m}\ln x_i$ ∴

$l'(\phi)=\frac{d}{d\phi}l(\phi;\underline{x})=2m\frac{1}{\phi}-\sum_{i=1}^{m}x_i$ ∴

$2m\frac{1}{\phi}-\sum_{i=1}^{m}x_i=0$ ∴ $2m\frac{1}{\hat\phi}=\sum_{i=1}^{m}x_i$ ∴

$\frac{2m}{\sum_{i=1}^{n}x_i}=\hat\phi=\frac{2m}{m\frac{1}{m}\sum_{i=1}^{m}x_i}=\frac{2m}{m\bar{x}}=\frac{2}{\bar{x}}$ ∴ $\bar{x}=\frac{2}{\hat\phi}$ ∴

$\frac{d^2}{d\phi^2}l(\phi;\underline{x})=-2m\phi^{-2}=l''(\phi)$ ∴

$l''(\hat\phi;\underline{x})=-2m(\frac{2}{\bar{x}})^{-2}=-2m\hat\phi^{-2}=-2m(\frac{\bar{x}^2}{4})=-(\frac{m\bar{x}^2}{2})$ ∴ $\frac{n\bar{x}^2}{2}>0$ ∴ $-\frac{n\bar{x}^2}{2}<0$ ∴

$l''(\hat\phi;\underline{x})<0$ ∴ $\hat\phi=\frac{2}{\bar{x}}$ is mle of $L(\phi;x)$

$I(\phi)=E[J(\phi)]=-E[l''(\phi)]=E[-l''(\phi)]=-E(-2m\phi^{-2}]=2m\phi^{-2}E(1)=2m\phi^{-2}$ ∴

Z asymp distri of Z mle is $N(\phi,\frac{1}{I(\phi)})=N(\phi,1/(2m\phi^{-2}))=N(\phi,\frac{\phi^2}{2m})$

\1b/ Z wald test stat is $W=(\hat\phi-\phi_0)^2 I(\hat\phi)=(\hat\phi-\phi_0)^2 2m\hat\phi^{-2}=$

$2m(\hat\phi-\phi_0)^2(\frac{1}{\hat\phi})^2=2m((\hat\phi-\phi_0)\frac{1}{\hat\phi})^2=2m(1-\frac{\phi_0}{\hat\phi})^2$

score func is $U(\phi)=l'(\phi)$ & ∴ Z score test is:

$S=U(\phi_0)^2/I(\phi_0)=(\frac{2m}{\phi_0}-m\bar{x})^2/I(\phi_0)=(\frac{2m}{\phi_0}-m\bar{x})^2(\frac{\phi_0^2}{2m})=$

$\frac{1}{2m}((\frac{2m}{\phi_0}-m\bar{x})\phi_0)^2=\frac{1}{2m}(2m-m\bar{x}\phi_0)^2=(2m\phi_0^{-1}-m\bar{x})^2\phi_0^2/2m=$

$\frac{\phi_0^2}{2m}(2m(\frac{1}{\phi_0}-\frac{\bar{x}}{2}))^2=\frac{\phi_0^2(2m)^2}{2m}(\frac{1}{\phi_0}-\frac{1}{\hat\phi})^2=2m(\phi_0(\frac{1}{\phi_0}-\frac{1}{\hat\phi}))^2=2m(1-\frac{\phi_0}{\hat\phi})^2=W$

Z likelihood ratio test test is $-2\ln\Lambda(\underline{x})=-2\ln(\frac{L(\phi_0)}{L(\hat\phi)})=-2[\ln(L(\phi_0))-\ln(L(\hat\phi))]=$

$-2[l(\phi_0)-l(\hat\phi)]=2[l(\hat\phi)-l(\phi_0)]=$ &

$2[2m\ln(\hat\phi)-\hat\phi(\sum_{i=1}^{m}x_i)+(\sum_{i=1}^{m}\ln x_i)-2m\ln(\phi_0)+\phi_0(\sum_{i=1}^{m}\ln x_i)-\sum_{i=1}^{m}(\ln(x_i))]=$

$2[2m(\ln(\hat\phi)-\ln(\phi_0))-(\hat\phi-\phi_0)(\sum_{i=1}^{m}\ln x_i)]=4m[\ln(\frac{\hat\phi}{\phi_0})+\frac{\phi_0}{\hat\phi}-1]$

Z null distris of all three test stats are approx χ_1^2 distris when m is large

\1c/ asymp $N(\hat\phi,\frac{\phi^2}{2m})=N(\frac{2}{\bar{x}},\frac{\phi^2}{2m})$ ∴ $(\frac{2}{\bar{x}}-z_{1-\alpha}\frac{\sigma}{\sqrt{m}},\frac{2}{\bar{x}}-z_{\alpha}\frac{\sigma}{\sqrt{m}})=$

\Sheet 2/ $\left(\frac{2}{\bar{x}} - z_{1-\alpha}\frac{\phi}{\sqrt{2m}\sqrt{m}}, \frac{2}{\bar{x}} - z_{\alpha}\frac{\phi}{\sqrt{2m}\sqrt{m}}\right) = \left(\frac{2}{\bar{x}} + z_{\alpha}\frac{\phi}{m\sqrt{2}}, \frac{2}{\bar{x}} - z_{\alpha}\frac{\phi}{m\sqrt{2}}\right)$

c let c be Z α-quantile of Z χ^2_1 distri ∴ an approx α-CI for ϕ is:

$W = (\hat{\phi} - \phi_0)^2 I(\hat{\phi})$ ∴ $\{\phi : (\hat{\phi} - \phi)^2 I(\hat{\phi}) < c\}$ ∴ $\{\phi : (\hat{\phi} - \phi)^2 I(\phi) < c\} =$

$\{\phi : 2m(1 - \phi_0/\hat{\phi})^2 < c\}$ ∴ Z inequality is satisfied iff

$\{\phi : (1 - \frac{\phi_0}{\hat{\phi}})^2 < \frac{c}{2m}\}$ ∴ $\{-\sqrt{\frac{c}{2m}} < (1 - \frac{\phi_0}{\hat{\phi}}) < \sqrt{\frac{c}{2m}}\} = \{\phi : -\sqrt{\frac{c}{2m}} - 1 < \frac{-\phi_0}{\hat{\phi}} < \sqrt{\frac{c}{2m}} - 1\} =$

$\{\phi : (-\sqrt{\frac{c}{2m}} - 1)\hat{\phi} < -\phi < (\sqrt{\frac{c}{2m}} - 1)\hat{\phi}\} = \{\phi : -(\sqrt{\frac{c}{2m}} - 1)\hat{\phi} < \phi < -(-\sqrt{\frac{c}{2m}} - 1)\hat{\phi}\} =$

$\{\phi : (1 - \sqrt{\frac{c}{2m}})\hat{\phi} < \phi < (\sqrt{\frac{c}{2m}} + 1)\hat{\phi}\}$ ∴ $\hat{\phi}(1 - \sqrt{\frac{c}{2m}}) < \phi < \hat{\phi}(1 + \sqrt{\frac{c}{2m}})$ ∴

Z interval is $(\hat{\phi}[1 - \sqrt{\frac{c}{2m}}], \hat{\phi}[1 + \sqrt{c/2m}])$

\1ci/ let c be Z α-quantile of Z χ^2_1 distri ∴

an approx α CI for ϕ is: $\{\phi : (\hat{\phi} - \phi)^2 I(\phi) < c\} = \{\phi : 2m(\frac{\hat{\phi}}{\phi} - 1)^2 < c\}$

∴ inequality holds iff $\frac{\hat{\phi}}{1 - \sqrt{\frac{c}{2m}}} < \phi < \frac{\hat{\phi}}{1 - \sqrt{\frac{c}{2m}}}$ & ∴ Z interval is:

$\left(\frac{\hat{\phi}}{1 + \sqrt{\frac{c}{2m}}}, \frac{\hat{\phi}}{1 - \sqrt{\frac{c}{2m}}}\right)$

\1cii/ another approx α-confidence interval for ϕ is: $\{\phi : 2m(1 - \frac{\phi}{\hat{\phi}})^2 < c\}$

Z inequality is satisfied iff $\phi[1 - \sqrt{\frac{c}{2m}}] < \phi < \hat{\phi}[1 + \sqrt{\frac{c}{2m}}]$ ∴

Z interval is: $(\hat{\phi}[1 - \sqrt{\frac{c}{2m}}], \hat{\phi}[1 + \sqrt{\frac{c}{2m}}])$

\2a/ $f(y_i; \gamma, \phi) = \gamma^2 \phi y e^{-\gamma\phi y}$ for $y > 0$ ∴ $L(\gamma, \phi) = \prod_{i=1}^{m} f(x_i; \phi) \prod_{i=1}^{n} f(y_i; \phi, \gamma)$

$= \phi^{2m} e^{-\phi\sum_{i=1}^{m} x_i} \prod_{i=1}^{m}(x_i) \prod_{i=1}^{n}(\gamma^2 \phi y_i e^{-\gamma\phi y_i}) = \phi^{2m} e^{-\phi\sum_{i=1}^{m} x_i} \prod_{i=1}^{m}(x_i)\gamma^{2n}\phi^{2n} e^{\sum_{i=1}^{n} -\gamma\phi y_i} \prod_{i=1}^{n}(y_i) =$

$\phi^{2m} e^{-\phi m\bar{x}} \prod_{i=1}^{m}(x_i)\gamma^{2m}\phi^{2m} e^{-\gamma\phi\sum_{i=1}^{n} y_i} \prod_{i=1}^{n}(y_i) = \phi^{2(m+n)}\gamma^{2n} e^{-\phi m\bar{x}} \prod_{i=1}^{m}(x_i) e^{-\gamma\phi n\bar{y}} \prod_{i=1}^{n} y_i =$

$\gamma^{2n}\phi^{2(m+n)} e^{-\phi(m\bar{x} + \gamma n\bar{y})} \prod_{i=1}^{m}(x_i) \prod_{i=1}^{n} y_i \propto \gamma^{2n}\phi^{2(m+n)} e^{-\phi(m\bar{x} + \gamma n\bar{y})}$ ∴

$l(\gamma, \phi; \underline{x}, \underline{y}) = l(\gamma, \phi) = \ln(L(\gamma, \phi)) = \ln(\gamma^{2n}) + \ln(\phi^{2(m+n)}) - \phi(m\bar{x} + \gamma n\bar{y}) =$

$2n\ln\gamma + 2(m+n)\ln\phi - \phi(m\bar{x} + \gamma n\bar{y}) = 2n\ln\gamma + 2(m+n)\ln\phi - m\bar{x}\phi - n\bar{y}\gamma\phi$ ∴

first derivs $\frac{\partial}{\partial\gamma} l(\gamma, \phi) = \frac{2n}{\gamma} - n\bar{y}\phi = l_\gamma(\gamma, \phi)$

$\frac{\partial}{\partial\phi} l(\gamma, \phi) = \frac{2(m+n)}{\phi} - m\bar{x} - n\bar{y}\gamma = l_\phi(\gamma, \phi)$ ∴

Solving $l_\gamma(\hat{\gamma}, \phi) = 0$: $\frac{2n}{\hat{\gamma}} - n\bar{y}\phi = 0$ ∴ $\frac{2n}{\hat{\gamma}} = n\bar{y}\phi$ ∴ $\frac{2n}{n\bar{y}\phi} = \hat{\gamma} = \frac{2}{\bar{y}\phi}$ &

$l_\phi(\gamma, \hat{\phi}) = 0$: $\frac{2(m+n)}{\hat{\phi}} - m\bar{x} - n\bar{y}\gamma = 0$ ∴ $\frac{2(m+n)}{\hat{\phi}} = m\bar{x} + n\bar{y}\gamma$ ∴ $\frac{2(m+n)}{m\bar{x} + n\bar{y}\gamma} = \hat{\phi}$ ∴

solving $l_\gamma = 0$ & $l_\phi = 0$: $\frac{2n}{\gamma} - n\bar{y}\phi = 0$ ∴ $\frac{2(m+n)}{\phi} - m\bar{x} - n\bar{y}\gamma = 0$ ∴

$\frac{2n}{\gamma} = n\bar{y}\phi$ ∴ $\frac{2n}{\gamma n\bar{y}} = \phi = \frac{2}{\bar{y}\gamma}$ ∴ $\frac{2(m+n)\bar{y}\gamma}{2} - m\bar{x} - n\bar{y}\gamma = 0 = -m\bar{x} + \bar{y}\gamma(m+n-n) =$

$-m\bar{x} + \bar{y}\gamma(m) = m(-\bar{x} + \bar{y}\gamma) = 0$ ∴ $m\bar{y}\gamma = m\bar{x}$ ∴ $\hat{\gamma} = \frac{\bar{x}}{\bar{y}}$ ∴ $\hat{\phi} = \frac{2\bar{y}}{\bar{y}\bar{x}} = \frac{2}{\bar{x}}$ ∴

$l_{\gamma\gamma} = \frac{\partial^2}{\partial\gamma^2} l(\gamma, \phi) = -2n\gamma^{-2}$, $l_{\gamma\phi} = -n\bar{y}$, $l_{\phi\phi} = -2(m+n)\phi^{-2}$ ∴

Z expected info is: $I(\gamma,\phi)=\begin{bmatrix}-E(l_{\gamma\gamma}) & -E(l_{\gamma\phi})\\ -E(l_{\gamma\phi}) & -E(l_{\phi\phi})\end{bmatrix}=\begin{bmatrix}-E(2n\gamma^{-2}) & E(-n\bar{y})\\ -E(-n\bar{y}) & -E(-2(m+n)\phi^{-2})\end{bmatrix}$
$=\begin{bmatrix}2n\gamma^{-2} & nE(\bar{y})\\ nE(\bar{y}) & 2(m+n)\phi^{-2}\end{bmatrix}$ $\therefore E(\bar{Y})=E(Y_1)=\frac{2}{\gamma\phi}$ $\because Ga(2,\gamma\phi)=Ga(\alpha,\beta)\therefore$
$E(Y_1)=\frac{\alpha}{\beta}=E(\bar{Y})\therefore$
$I(\gamma,\phi)=\begin{bmatrix}2n\gamma^{-2} & 2n(\gamma\phi)^{-1}\\ 2n(\gamma\phi)^{-1} & 2(m+n)\phi^{-2}\end{bmatrix}$ $\therefore$ found $E(Ga(2,\phi))=\frac{2}{\phi}$ $\because Ga(2,\phi)$ has
mle $\hat{\phi}=\frac{2}{\bar{x}}$ $\therefore \bar{x}=\frac{2}{\phi}$ $\therefore$ $G(2,\gamma\phi)$ $\therefore$ $\bar{y}=\frac{2}{\gamma\phi}$ $\therefore$
Z asymp distri is ~~MVN~~ $MVN((\gamma,\phi)=(\frac{\bar{x}}{\bar{y}},\frac{2}{\bar{x}}),I(\gamma,\phi)^{-1})$,
$I(\gamma,\phi)^{-1}=\frac{1}{2mn}\begin{bmatrix}2(m+n)\gamma^2 & -n\gamma\phi\\ -n\gamma\phi & n\phi^2\end{bmatrix}$ asymp distri of $(\hat{\gamma},\hat{\phi})$ is
$N((\gamma,\phi),I(\gamma,\phi)^{-1})=N((\gamma,\phi),\frac{1}{2mn}\begin{bmatrix}(m+n)\gamma^2 & -n\gamma\phi\\ -n\gamma\phi & n\phi^2\end{bmatrix}$

\2b/ $l(\gamma,\hat{\phi})\therefore\hat{\phi}=\frac{2}{\bar{x}}\therefore l(\gamma,\hat{\phi};\underline{x},\underline{y})=2n\ln\gamma+2(m+n)\ln(\frac{2}{\bar{x}})-m\bar{x}\frac{2}{\bar{x}}-n\bar{y}\gamma\frac{2}{\bar{x}}=$
$2n\ln\gamma+2(m+n)\ln(\frac{2}{\bar{x}})-2m-\frac{2n\bar{y}}{\bar{x}}\gamma=0$
$\therefore$ Solving l_ϕ for ϕ yields $\hat{\phi}(\gamma)=\frac{2(m+n)}{m\bar{x}+\gamma n\bar{y}}$ $\therefore$ Z profile loglikhood for γ is
$l_p(\gamma)=l(\gamma,\hat{\phi}(\gamma))=2n\ln\gamma-2(m+n)\ln(m\bar{x}+\gamma n\bar{y})+\text{constant}$
~~2b~~/ Solving $l_\phi=0$ for ϕ yields $\hat{\phi}(\gamma)=2(m+n)/(m\bar{x}+\gamma n\bar{y})$ & $\therefore$ Z profile
loglikhood for γ is: $l_p(\gamma)=l(\gamma,\hat{\phi}(\gamma))=2n\ln\gamma-2(m+n)\ln(m\bar{x}+\gamma n\bar{y})+\text{constant}$

\2c/ $-2\log\Lambda=-2\ln\left(\frac{L_p(\gamma_0)}{L_p(\hat{\gamma})}\right)=-2[l_p(\gamma_0)-l_p(\hat{\gamma})]=2[l_p(\hat{\gamma})-l(\gamma_0)]=$
$2[2n\ln\hat{\gamma}-2(m+n)\ln(m\bar{x}+\hat{\gamma}n\bar{y})+c_1-2n\ln\gamma_0+2(m+n)\ln(m\bar{x}+\gamma_0 n\bar{y})-c_1]=$
$2[2n\ln\hat{\gamma}-2(m+n)\ln(m\bar{x}+\hat{\gamma}n\bar{y})-2n\ln\gamma_0+2(m+n)\ln(m\bar{x}+\gamma_0 n\bar{y})]=$
$4n\ln(\frac{\bar{x}}{\bar{y}})+4(m+n)\ln\left[\frac{m\bar{x}+n\bar{y}}{(m+n)\bar{x}}\right]$ & its null distri is approx χ^2_1 when
m & n are large

\2c/ let $\gamma_0=1$ $\therefore$ Z likhood ratio test stat: $2[l_p(\hat{\gamma})-l_p(\gamma_0)]=$
$4n\ln(\bar{x}/\bar{y})+4(m+n)\ln\left(\frac{m\bar{x}+n\bar{y}}{(m+n)\bar{x}}\right]$ & its null distri is approx χ^2_1 when m & n are large

\3a/ ~~Gt~~ $G(\theta;\underline{x})=\mu_\theta^T\Sigma^{-1}(\underline{x}-\mu)$ $G(\hat{\theta};\underline{x})=0$ is quasi likhood esti $\therefore$
$\underline{z}=(x_1,\ldots,x_m,y_1,\ldots,y_n)$ & $\theta=(\gamma,\phi)$ $\because E(X)=\frac{2}{\phi}$, $E(Y)=\frac{2}{\gamma\phi}$ $\therefore E(\theta)=(\frac{2}{\phi},\frac{2}{\gamma\phi})\therefore$
$\mu=(\frac{2}{\phi},\ldots,\frac{2}{\phi},\frac{2}{\gamma\phi},\ldots,\frac{2}{\gamma\phi})\therefore var(X)=\frac{2}{\phi^2}$, $var(Y)=\frac{2}{(\gamma\phi)^2}\therefore$
$\Sigma=diag(\frac{2}{\phi^2},\ldots,\frac{2}{\phi^2},\frac{2}{(\gamma\phi)^2},\ldots,\frac{2}{(\gamma\phi)^2})\therefore \mu_\theta^T=\begin{bmatrix}\partial\mu_1/\partial\gamma & \ldots & \partial\mu_{m+n}/\partial\gamma\\ \partial\mu_1/\partial\phi & \ldots & \partial\mu_{m+n}/\partial\phi\end{bmatrix}=$
$\begin{bmatrix}0 & \cdots & 0 & -\frac{2}{\gamma^2\phi} & \cdots & -\frac{2}{\gamma^2\phi}\\ -\frac{2}{\phi^2} & \cdots & -\frac{2}{\phi^2} & -\frac{2}{\gamma\phi^2} & \cdots & -\frac{2}{\gamma\phi^2}\end{bmatrix}\therefore\Sigma^{-1}=diag(\frac{\phi^2}{2},\ldots,\frac{\phi^2}{2},\frac{(\gamma\phi)^2}{2},\ldots,\frac{(\gamma\phi)^2}{2})$ &
$G(\theta;z)=\mu_\theta^T\Sigma^{-1}(z-\mu)=\begin{bmatrix}0 & \cdots & 0 & -\frac{2}{\gamma^2\phi^2} & \cdots & -\frac{2}{\gamma^2\phi^2}\\ -\frac{2}{\phi^2} & \cdots & -\frac{2}{\phi^2} & -\frac{2}{\gamma^2\phi^2} & \cdots & -\frac{2}{\gamma^2\phi^2}\end{bmatrix}(\frac{\phi^2}{2},\ldots,\frac{\phi^2}{2},\frac{(\gamma\phi)^2}{2},\ldots,\frac{(\gamma\phi)^2}{2})(x_1-\frac{2}{\phi},\ldots,x_m-\frac{2}{\phi},y_1-\frac{2}{\gamma\phi},\ldots$

\Sheet2/ $= \begin{bmatrix} 2n/\gamma - n\bar{y}\phi \\ -m\bar{x} - n\bar{y}\gamma + 2(m+n)/\phi \end{bmatrix} = 0$ ∴ solving $G(\theta; \underline{z}) = 0$ shows:

$\begin{bmatrix} 2n/\gamma - n\bar{y}\phi \\ -m\bar{x} - n\bar{y}\gamma + 2(m+n)/\phi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = 0$ ∴ $\frac{2n}{\gamma} - n\bar{y}\phi = 0$, $-m\bar{x} - n\bar{y}\gamma + \frac{2(m+n)}{\phi} = 0$ ∴

$\frac{2n}{\gamma} = n\bar{y}\phi$ ∴ $\frac{2n}{n\bar{y}\gamma} = \phi = \frac{2}{\bar{y}\gamma}$ ∴ $+m\bar{x} + n\bar{y}\gamma = \frac{2(m+n)}{2}\bar{y}\gamma = (m+n)\bar{y}\gamma$ ∴

$m\bar{x} = (m+n)\bar{y}\gamma - n\bar{y}\gamma = \gamma[(m+n)\bar{y} - n\bar{y}] = m\bar{x} = m\bar{y}\gamma$ ∴ $\hat{\gamma} = \frac{\bar{x}}{\bar{y}}$ ∴

$\hat{\phi} = \frac{2\bar{y}}{\bar{y}\bar{x}} = \frac{2}{\bar{x}} = \hat{\phi}$ ∴ shows $G(\theta; \underline{z}) = 0$ shows Z quasi-likelihood esti is Z same as Z max likelihood esti

\3b/ $m = 10$: $\phi_0 = 2$ likelihood ratio test, score test uniformly most powerful test Z power funcs : Z wald test is Z same as Z score test for this & ∴ comment only on Z score & likelihood ratio tests. Z sizes of Z two tests differ slightly from 5% ∵ only an approx to Z null distri is used. Z monte carlo approxs to Z sizes are 4.7% for Z score test & 5.1% for Z LR test. Z score test is slightly more powerful than Z LR test at detecting when $\phi < 2$. but less powerful at detecting when $\phi > 2$. Both tests are slightly less powerful than Z UMP test at detecting when $\phi < 2$ (Z UMP test has lower power for $\phi > 2$ ∵ it designed for Z one-sided alternative $H_1: \phi < 2$ ∴ might prefer Z LR test to Z score test ∵ its size is closer to 5%, it has better power when $\phi > 2$, & has similar power when $\phi < 2$ ∴ power at $\phi = 2$ is 0.05 but further away from $\phi = 2$ want to reject null more often.

\3ci/ mle is $\hat{\phi} = 1.888$ Z asym estied SE is $I(\theta)^{-1} = \frac{\phi^2}{2m}$ ∴ estied SE : $\frac{\hat{\phi}}{\sqrt{2m}} = 0.49$

estied SE : $\hat{\phi}/\sqrt{2m} = 0.42$

\3cii/ Z wald & score test stats are 0.087 with p-val 0.77 alternatively, Z likelihood ratio test stat is 0.083 & with p-val 0.77. neither test provides any evidence to reject Z null hypoth at any reasonable significance level

\3ciii/ $\frac{\hat{\phi}}{1+\sqrt{c/2m}} < \phi < \frac{\hat{\phi}}{1-\sqrt{c/2m}}$ or $(\hat{\phi}[1-\sqrt{c/2m}], \hat{\phi}[1-\sqrt{c/2m}])$

c is α quantile of χ_1^2 distri $(1-2\alpha)$ CI ∵ .90 ∴ $\alpha = 5\%$ ∴

mle based 90% is (1.37, 2.97), alternate : Z wald based 90% CI is ~~to~~ (1.19, 2.57)

\3civ/ $Ga(2, \gamma\phi)$ ∴ $\hat{\gamma} = \frac{\bar{x}}{\bar{y}}$, $\hat{\phi} = \frac{2}{\bar{x}}$ ∴ $\hat{\gamma}\hat{\phi} = \frac{\bar{x}}{\bar{y}}\frac{2}{\bar{x}} = \frac{2}{\bar{y}}$ test $H_0: \gamma = 1$ against

H_1: $\gamma \neq$ using Z LR test H_0 correct with p-value 0.030, is use Z χ_1^2 null distri ∴ ~~conclude ∄ that~~ is small ~~evidence~~ even the Z samples maybe too small for this to be a good approx to Z null distri ∴ conclude ∃ significant evidence at 5% level that Z times take different time to complete 0.0304 is not in (1.186, 2.13)
$0.0304 \notin (1.186, 2.13)$

\Sheet 3/ bootstrapping is used to use Z current data to produce new ~~test~~ data then do analysis on comparing Z two data sets then use Z new data to calc distri stats. bootstrap resampling means to generate new samples of Z same size as Z original sample either by sampling from Z empirical distri of Z original sample (non param resampling) or by sampling from Z distri of a fitted prob model (param resampling) ∴ Z stat of interest is evaluated for each of many such samples then Z empirical distri of these bootstrap stats is an esti of Z sampling distri of Z original stat

\1b/ use $Ga(2, \hat{\phi})$ $\hat{\phi} = \frac{2}{\bar{x}}$ to make 1000 sets of samples then find Z average of all these samples ∴ $bias(\hat{\theta}) = E(\hat{\theta}) - \theta$ ∴ bias corrected estior $\tilde{\phi} = \hat{\phi} - \frac{1}{2}\widehat{var}(\hat{\theta})h''(\hat{\theta})$ ∵ $t_i^* = \frac{2}{\bar{x}^*} - \hat{\phi}$ ∴ $\tilde{\phi} = \hat{\phi} - \bar{t}^* = \hat{\phi} - \sum_{i=1}^{b} t_i^* =$

$$\hat{\phi} - \frac{1}{b}\sum_{i=1}^{b}\left(\frac{2}{\frac{1}{m}\sum_{j=1}^{m} x_{ij}^*} - \hat{\phi}\right) = \hat{\phi} - \frac{2m}{b}\sum_{i=1}^{b}\left(\frac{1}{\sum_{j=1}^{m} x_{ij}^*}\right) + \frac{1}{b}\sum_{i=1}^{b}\hat{\phi} =$$

$$\hat{\phi} - \frac{2m}{b}\sum_{i=1}^{b}\left(\frac{1}{\sum_{i=1}^{m} x_{ij}^*}\right) + \frac{1}{b}b\hat{\phi} = 2\hat{\phi} - \frac{2m}{b}\sum_{i=1}^{b}\left(\frac{1}{\sum_{j=1}^{m} x_{ij}^*}\right)$$

\1c/ $\sum_{j=1}^{m}(x_j) = 10.66$ $\hat{\phi} = \frac{2}{\bar{x}} = \frac{2m}{\sum_{i=1}^{m} x_i} = 1.876$

i	1	2	3	4	5	6	7	8	9	10
x_i	1.79	0.34	0.43	1.03	0.57	1.27	0.42	1.94	1.73	1.14
$\sum_{j=1}^{m}(x_j) - x_i = \sum_{j \neq i} x_j$	8.87	10.32	10.23	9.63	10.09	9.39	10.24	8.72	8.93	9.52
$\frac{2}{\frac{1}{(m-1)}\sum_{j \neq i} x_j} = \hat{\phi}_{-i}$	2.03	1.74	1.76	1.87	1.78	1.92	1.76	2.06	2.02	1.89
$m\hat{\phi} - (m-1)\hat{\phi}_{-i} = \hat{\phi}_i$	0.30	3.06	2.92	1.93	2.74	1.48	2.92	0.22	0.58	1.75

$\frac{1}{m}\sum_{i=1}^{m}\hat{\phi}_i = \hat{\phi}_J = 1.81$

	1	2	3	4	5	6	7	8	9	10
$(\hat{\phi}_i - \hat{\phi}_J)^2$	1.72	1.56	1.23	0.014	0.86	0.109	1.23	2.53	1.51	0.0036

\Sheet 3/ $\sum_{i=1}^{m}[(\hat{\phi}_i-\hat{\phi}_J)^2]=10.81$ $\therefore \hat{\phi}_J=1.81$, SE $=0.35$ $\therefore$

Jackknife version of $\hat{\phi}$ is $\hat{\phi}_J=\frac{1}{m}\sum_{i=1}^{m}\hat{\phi}_i=1.81$, $m=10$, $\hat{\phi}_i=m\hat{\phi}-(m-1)\hat{\phi}_{-i}$ &

$\hat{\phi}_{-i}=\frac{2(m-1)}{\sum_{j=1}^{m}x_j-x_i}$ $\because$ Z jackknife esti of Z SE is $\sqrt{\hat{v}_J^2}=0.35$ where

$\hat{v}_J=\frac{1}{m(m-1)}\sum_{i=1}^{m}(\hat{\phi}_i-\hat{\phi}_J)^2$ $\therefore$ Z jackknife 90% CI for ϕ is $\hat{\phi}_J\pm1.833\sqrt{\hat{v}_J}=(1.182,2.45)$,

$\therefore$ 1.833 is Z 95% quantile of Z t_{m-1} distri $\because$ df $=m-1$ $\therefore$ 1.833113 $\therefore$ St(9) for

95% quantile on df $=m-1=90$ $\therefore \nu=9$ $\therefore$ 90% CI $\therefore$ 95 quantile $\therefore \alpha=0.05$ $\therefore$

$t_{\alpha;\nu}=t_{0.05;9}=1.833$ for Z 5% val

\2a/ jackknife test of $H_0:\phi=2$, $H_1:\phi\neq2$ $\therefore \hat{\phi}_J=1.81$ SE $=0.35$ $\therefore$ if

90% CI $\therefore \alpha=0.05$ $\therefore$ $1.833=t_{\alpha,m-1}=t_{\alpha,9}=t_{0.05,9}$ $\therefore$ $(2-1.833\times0.35, 2+1.833\times0.35)=$

$(1.358,2.641)$ $\therefore 1.81=\hat{\phi}_J\in(1.358,2.641)$ $\therefore$ dont reject H_0, no evidence that $\phi\neq2$

Z test stat is $(\hat{\phi}_J-2)/\sqrt{\hat{v}_J}=-0.54$ with pval 0.60 $\therefore$ γ is quantiles

$\therefore$ distri func $\therefore$ 0.699 $\therefore$ for Z two sided test $\therefore$ $\exists$ insufficient

evidence to reject Z null hypoth that $\phi=2$ expect same results

as those obtained by wald & score & LR test stats & p-vals

\2c/ $\hat{\gamma}\hat{\phi}=\frac{2}{\bar{y}}$ $\therefore$ Z test stat is $4n\ln(\bar{x}/\bar{y})+4(m+n)\ln\left[\frac{m\bar{x}+n\bar{y}}{(m+n)\bar{x}}\right]=4.69$

with p-val 0.031 $\therefore 0.031<0.05$ $\therefore$ $\exists$ evidence to reject Z null hypoth

at Z 5% level & conclude that $\exists$ different lines for Z tasks

\3a/ basic bootstrap interval is $(\hat{\phi}-\hat{q}_{1-\alpha}, \hat{\phi}-\hat{q}_{\alpha})$ studentised interval

$(\hat{\phi}-\hat{q}_{1-\alpha}s, \hat{\phi}-\hat{q}_{\alpha}s)$ percentile bootstrap interval $(\hat{\phi}^*_{(\alpha B)}, \hat{\phi}^*_{((1-\alpha)B)})$

for param resampling, Z 90% (0.92, 2.41) basic (1.24, 2.62) (studentised)

(1.34, 2.84) percentile for non param resampling, Z intervals are (1.15, 2.29)

(basic) (1.35, 2.41) (studentised) (1.46, 2.60) (percentile)

\3bi/ $X\sim Ga(2,\phi)$ $\therefore f(x,\phi)=\phi^2xe^{-\phi x}$, $x>0$ scale model is $X=\sigma Z$,

Z has a distri doesnt depend (any unknown) params. let $Z=\phi x$ $\therefore$

$P(Z\leq z)=P(\phi X\leq z)=P(X\leq\frac{z}{\phi})=\int_0^{z/\phi}\phi^2xe^{-\phi x}dx=$ $\because y=\phi x$ 0 to z, $\frac{1}{\phi}dy=dx$

$=\int_0^z\phi ye^{-y}\frac{1}{\phi}dy=\int_0^z ye^{-y}dy$ doesnt depend on ϕ $\therefore$ is a scale model $\therefore$

pdf of Z is ze^{-z} $\therefore Z\sim Ga(2,1)$ $\therefore$ show $Ga(2,\phi)$ distri is Z distri of $\frac{Z}{\phi}$

$\therefore$ Z has $Ga(2,1)$ distri $\therefore X_i=Z_i/\phi_i$ $\therefore Pr(Z_i\leq z)=Pr(X_i\leq z/\phi)=\int_0^{z/\phi}f(x;\phi)dx=$

$\int_0^{z/\phi}\phi^2xe^{-\phi x}dx=\int_0^z te^{-t}dt$ $\therefore$ Z final integrand is Z density of a $Ga(2,1)$ distri

$\therefore Ga(2,\theta): f(x,\theta)=\theta^2 x e^{-\theta x}$ $\therefore Ga(2,1)=1^2 x e^{-1x}=xe^{-x}$ $\therefore z_i$ has Z $Ga(2,1)$ distri

\3bii/ ancillary stat T $\therefore (X_0, X_1, \dots, X_m)$ $\therefore T=\frac{X_0}{\bar{X}}$ $\therefore T=\frac{Z_0/\theta}{\frac{1}{m}\sum_{i=1} \frac{Z_i}{\theta}}=\frac{Z_0}{\bar{Z}}$ $\therefore$

T is ancillary $\therefore P(t_{0.05}<T<t_{0.95})=0.9$ ie $P(t_{0.05}\bar{X}<X_0<t_{0.95}\bar{X})=0.9$

$\therefore$ Z stat $T=\frac{X_0}{\bar{X}}$ is ancillary $\because X_0/\bar{X}=\frac{Z_0}{\bar{Z}}$, Z_i have $Ga(2,1)$ distri,

$\therefore$ if q_p denotes Z p-quantile of T then 0.90 $\therefore$ ~~dat~~

$Pr(q_{0.05}<\frac{X_0}{\bar{X}}<q_{0.95})=Pr(q_{0.05}\bar{X}<X_0<q_{0.95}\bar{X})$ & $\therefore$ 90% Prediction interval

is $(\hat{q}_{0.05}\bar{X}, \hat{q}_{0.95}\bar{X})$, $\hat{q}_p$ is Z p-quantile of Z bootstrap version of T,

eg $T^*=X_0^*/\bar{X}^*$, $\bar{X}_i^*$ are indep with distri $Ga(2,\hat{\theta})$ $\therefore$ interval (0.19, 2.78) seconds

\3c/ Coverages of 88.5% (basic) 89.8% (studentised) 88.2% (percentile)

Z coverages of all three intervals are slightly low, but Z coverage

of Z studentised interval is very close to Z required 90%.

$= Ga(2,1)$ distri

$= \frac{z_0}{z}$ ∴

$) = 0.9$

$(2,1)$ distri,

interval

ison of T,

$.19, 2.78)$ seconds

(percentile):

Coverage

\revision lecture/ $\hat{\theta}$ unbiased & efficient iff $U = I(\theta)(\hat{\theta} - \theta)$

eg $bias(\hat{\theta}) = 0$ & $\frac{1}{I(\theta)\, var(\hat{\theta})} = 1$

\1a/ $l(\theta) = -n \ln \theta - \theta^{-1} \sum_{i=1}^{n} |x_i|$ ∴

$u = l'(\theta) = -\frac{n}{\theta} + \frac{1}{\theta^2} \sum_{i=1}^{n} |x_i|$ ∴

$u(\hat{\theta}) = 0$ iff $-n\hat{\theta} + \sum_{i=1}^{n} |x_i| = 0$ ∴ $\hat{\theta} = \frac{1}{n} \sum_{i=1}^{n} |x_i|$

$u = b(\hat{\theta} - \theta)$ ∴ $\underbrace{u = \frac{n}{\theta^2}}_{b}\underbrace{\left(\frac{1}{n} \sum_{i=1}^{n} |x_i| - \theta\right)}_{\hat{\theta}}$ ∴ $\hat{\theta}$ is unbiased & efficient

$var[h(\hat{\theta})] \approx [h'(\theta)]^2 var(\hat{\theta})$, $E(\hat{\theta}) - \theta \approx \frac{1}{2} \widehat{var}(\hat{\theta}) h''(\hat{\theta})$

∴ $\hat{\theta}$ ∴ $Bias(\hat{\theta}) = E(\hat{\theta}) - \theta$ ∴

Bias corrected estimator is $\hat{\theta} - \widehat{Bias}(\hat{\theta}) = \hat{\theta} - \widehat{Bias}(\theta)$ ∴ eg

$-\frac{\theta}{n}$ ∴ $\hat{\theta} - (-\frac{\theta}{n}) = \hat{\theta} + \frac{\theta}{n}$ ∴ $\hat{\theta} + \frac{\hat{\theta}}{n}$

\PP2020/ $var[h(\hat{\theta})] \approx [h'(\theta)]^2 var(\hat{\theta})$, $E(\hat{\theta}) - \theta \approx \frac{1}{2} \widehat{var}(\hat{\theta}) h''(\hat{\theta})$

$mse(\hat{\theta}) = var(\hat{\theta}) + bias^2(\hat{\theta})$

$\Lambda(\underline{x}) = \frac{L(\theta_1; \underline{x})}{L(\theta_0; \underline{x})}$ $\quad c = \{\underline{x} : \Lambda(\underline{x}) \geq c\}$

$\Lambda = \frac{L(\theta_0)}{L(\hat{\theta})}$ ∴ $-2 \ln \Lambda = 2[l(\hat{\theta}) - l(\theta_0)]$

$\Lambda(\underline{x}) = \frac{L(\theta_1; \underline{x})}{L(\theta_0; \underline{x})}$ ∴ $\{\underline{x} \mid \sum_{i=1}^{n} x_i \leq d\}$

$H_0: \theta = \theta_0 \quad H_1: \theta = \theta_1 \quad (\theta_1 < \theta_0)$ ∴ reject H_0 when $\sum_{i=1}^{n} x_i \leq d$,

where $P(\sum_{i=1}^{n} x_i \leq d; \theta_0) = \alpha$ ← test

now consider $H_1: \theta < \theta_0$ ∴ most powerful test is indep of θ_1 ∴

for $\theta < \theta_0$ ∴ $\sum_{i=1}^{n} x_i \leq d$ is the most powerful test ∴ it is the ~~unifo~~

uniformly most powerful test

location & scale or only scale for pivots

$q = (t_{0.05} < \frac{n}{n + \frac{1}{t}} < t_{0.95})$ ∴ $P(T \leq t) = \frac{n}{n + \frac{1}{t}}$ $\quad T = \frac{X_0}{X_{(n)}}$ ∴

90% PI for X_0 ∴ $0.9 = P(T \leq t)$ ∴ $P(\frac{X_0}{X_{(n)}} \leq t) = \frac{n}{n + \frac{1}{t}}$ ∴

5% quantile satisfies $P(\frac{X_0}{X_{(n)}} \leq t_5) = 0.05$ ∴

$\frac{n}{n + \frac{1}{t_5}} = 0.05$ ∴ $n + \frac{1}{t_5} = \frac{n}{0.05}$ ∴ $t_5 = \left(\frac{n}{0.05} - n\right)^{-1} = \frac{1}{n(\frac{1}{20} - 1)} = -\frac{20}{19n}$

∴ the 95% quantile is $t_{95} = \left(\frac{n}{0.95} - n\right)^{-1} = \frac{1}{n(\frac{20}{19} - 1)} = \frac{19}{n}$ ∴

$P\left(t_5 \le \frac{X_0}{X_{(n)}} \le t_{95}\right) = 0.9 = Pr\left(t_5 X_{(n)} \le X_0 \le t_{95} X_{(n)}\right) = 0.9 \therefore$

a 90% PI is $(t_5 X_{(n)}, t_{95} X_{(n)})$

have $P(X_i \le x) = e^{-\theta/x}$ for $x > 0 \therefore P(X_{(n)} \le x) = P(X_1 \le x)^n = e^{-n\theta/x}$

Now $P(T \le t) = P\left(\frac{X_0}{X_{(n)}} \le t\right) = \int_0^\infty Pr(X_0 \le t X_{(n)} \mid X_{(n)} = x) P(X_{(n)} = x)\, dx$

$= \int_0^\infty P(X_0 \le tx) P(X_{(n)} = x)\, dx$

& $P(X_n = x) = \frac{d}{dx} P(X_{(n)} \le x) = \frac{n\theta}{x^2} e^{-n\theta/x} \therefore$

$P(T \le t) = \int_0^\infty e^{-\theta/tx} \frac{n\theta}{x^2} e^{-n\theta/x}\, dx = n\theta \int_0^\infty \frac{1}{x^2} e^{-\theta(n+\frac{1}{t})/x}\, dx$

$\therefore$ let $y = \theta(n+\frac{1}{t})/x \therefore dy = -\frac{\theta(n+\frac{1}{t})}{x^2} dx \therefore \frac{1}{x^2} dx = -\frac{1}{\theta(n+\frac{1}{t})} dy \therefore$

$P(T \le t) = n\theta \int_\infty^0 \left(-\frac{1}{\theta(n+\frac{1}{t})}\right) e^{-y}\, dy = \frac{n}{n+\frac{1}{t}} \int_0^\infty e^{-y}\, dy = \frac{n}{n+\frac{1}{t}}$

\Ex/ $E(X_i) = \theta$, $var(X_i) = \phi\theta(1-\theta)$, $\mu = (\theta, \dots, \theta)^T$

$\Sigma = \begin{bmatrix} \phi\theta(1-\theta) & & \\ & \ddots & \\ 0 & & \phi\theta(1-\theta) \end{bmatrix} = \phi \begin{bmatrix} \theta(1-\theta) & & 0 \\ & \ddots & \\ 0 & & \theta(1-\theta) \end{bmatrix} = \phi\, \Sigma_0(\theta)$

$\hat{\phi} = \frac{1}{n-p} (\underline{x} - \hat{\mu})^T \hat{\Sigma}_0^{-1} (\underline{x} - \hat{\mu})$ $\therefore \hat{\mu} = (\hat{\theta}, \dots, \hat{\theta})^T$

$\therefore$ for 1 dimension $\hat{\phi} = \frac{1}{n-1} (x - \hat{\mu})^T \hat{\Sigma}_0^{-1} (x - \hat{\mu}) =$

$\frac{1}{n-1} (x_1 - \hat{\theta}, \dots, x_n - \hat{\theta}) \frac{1}{\hat{\theta}(1-\hat{\theta})} \begin{bmatrix} 1 & & \\ & \ddots & \\ 0 & & 1 \end{bmatrix} \begin{bmatrix} x_1 - \hat{\theta} \\ \vdots \\ x_n - \hat{\theta} \end{bmatrix} =$

$\frac{1}{(n-1)\hat{\theta}(1-\hat{\theta})} \sum_{i=1}^n (x_i - \hat{\theta})^2$

\Ex/ $(X_1^* + \dots + X_n^*)^2 = X_1^{*2} + X_1^* X_2^* + \dots + X_1^* X_n^*$
$+ X_2^* X_1^* + X_2^{*2} + \dots + X_2^* X_n^*$
$\vdots$
$X_n^* X_1^* + X_n^* X_2^* + \dots + X_n^{*2}$

$= \left(\sum_{i=1}^n X_i^*\right)^2 =$
$\sum_{i=1}^n X_i^{*2} + \sum_{i \ne j} X_i^* X_j^*$

$\therefore \sum_{i=1}^n X_i^{*2} + \sum_{i \ne j} X_i^* X_j^*$

$\widehat{Bias}(\hat{\theta}) = E[\hat{\theta}^*] - \hat{\theta} \therefore$

Bias corrected estimate is $\hat{\theta} - Bias(\hat{\theta}) = 2\hat{\theta} - E(\hat{\theta}^*)$

B028 Inference PP2021

1a $x>0$ ∴ $E(X_1) = E(X) = \int_0^\infty x f(x;\sigma) = \int_0^\infty x\, 2^{1/2}\pi^{-1/2}\sigma^{-1} e^{-\frac{x^2}{2\sigma^2}}\, dx$

∴ $\frac{d}{dx}\left(-\frac{x^2}{2\sigma^2}\right) = -\frac{1}{2\sigma^2}(2)x = -\frac{1}{\sigma^2}x$ ∴

$E(X_1) = 2^{1/2}\pi^{-1/2}\int_0^\infty \frac{1}{\sigma} x e^{-\frac{x^2}{2\sigma^2}}\, dx = -\sigma\sqrt{2}\frac{1}{\sqrt{\pi}}\int_0^\infty -\frac{1}{\sigma}\frac{1}{\sigma} x e^{-\frac{x^2}{2\sigma^2}}\,dx$

$= -\sigma\sqrt{\frac{2}{\pi}}\int_0^\infty -\frac{1}{\sigma^2} x e^{-\frac{x^2}{2\sigma^2}}\,dx = -\sigma\sqrt{\frac{2}{\pi}}\left[e^{-\frac{x^2}{2\sigma^2}}\right]_{x=0}^\infty =$

$-\sigma\sqrt{\frac{2}{\pi}}\lim_{a\to\infty}\left[e^{-\frac{x^2}{2\sigma^2}}\right]_{x=0}^a = -\sigma\sqrt{\frac{2}{\pi}}\lim_{a\to\infty}\left[e^{-\frac{a^2}{2\sigma^2}} - e^{-\frac{1}{2\sigma^2}(0)^2}\right] =$

$-\sigma\sqrt{\frac{2}{\pi}}\lim_{a\to\infty}\left[e^{-a} - e^0\right] = -\sigma\sqrt{\frac{2}{\pi}}[0-1] = \sigma\sqrt{\frac{2}{\pi}}$ ∴

From Method of Moments: $E(X_1) = \bar{x}$ ∴

$\bar{x} = \hat{\sigma}\sqrt{\frac{2}{\pi}}$ ∴ $\frac{\bar{x}}{\sqrt{\frac{2}{\pi}}} = \hat{\sigma} = \sqrt{\frac{\pi}{2}}\,\bar{x}$

1b bias$(\hat{\sigma}) = E(\hat{\sigma}) - \sigma$ ∴

$E(\hat{\sigma}) = E\left(\sqrt{\frac{\pi}{2}}\,\bar{x}\right) = \sqrt{\frac{\pi}{2}}E(\bar{x}) = \sqrt{\frac{\pi}{2}}E\left(\sum_{i=1}^n x_i\right) = \sqrt{\frac{\pi}{2}}\sum_{i=1}^n E(x_i) =$

$\sqrt{\frac{\pi}{2}}\sum_{i=1}^n E(X_1) = \sqrt{\frac{\pi}{2}}\, n E(X_1) = \sqrt{\frac{\pi}{2}}\, n\sigma\sqrt{\frac{2}{\pi}} = n\sigma$ ∴

bias$(\hat{\sigma}) = E(\hat{\sigma}) - \sigma = n\sigma - \sigma = (n-1)\sigma \neq 0$ ∴ the estimator is not unbiased.

$var(\hat{\sigma}) = var\left(\sqrt{\frac{\pi}{2}}\,\bar{x}\right) = \left(\sqrt{\frac{\pi}{2}}\right)^2 var(\bar{x}) = \frac{\pi}{2}var(\bar{x}) =$

$\frac{\pi}{2}var\left(\sum_{i=1}^n x_i\right) = \frac{\pi}{2}\sum_{i=1}^n var(x_i) = \frac{\pi}{2}\sum_{i=1}^n var(X) = \frac{\pi}{2}n\, var(X)$ ∴

$var(X) = E(X^2) - E(X)^2 = E(X_1^2) - E(X_1)^2 = \sigma^2 - \left(\sigma\sqrt{\frac{2}{\pi}}\right)^2 =$

$\sigma^2 - \sigma^2\frac{2}{\pi} = \left(1-\frac{2}{\pi}\right)\sigma^2$ ∴

$var(\hat{\sigma}) = \frac{\pi}{2}n\left(1-\frac{2}{\pi}\right)\sigma^2$ ∴

$\lim_{n\to\infty} var(\hat{\sigma}) = \lim_{n\to\infty}\frac{\pi}{2}n\left(1-\frac{2}{\pi}\right)\sigma^2 \neq 0$ ∴ the Method of Moments estimator is not consistent

1c $L(\sigma;\underline{x}) = \prod_{i=1}^n f(x_i;\sigma) = \prod_{i=1}^n 2^{1/2}\pi^{-1/2}\sigma^{-1}e^{-\frac{1}{2\sigma^2}x_i^2} =$

$2^{\frac{1}{2}n}\pi^{-\frac{1}{2}n}\sigma^{-n}\prod_{i=1}^n e^{-\frac{1}{2\sigma^2}x_i^2} = 2^{\frac{1}{2}n}\pi^{-\frac{1}{2}n}\sigma^{-n}e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n x_i^2} =$

$2^{\frac{1}{2}n}\pi^{-\frac{1}{2}n}\sigma^{-n}e^{-\frac{1}{2\sigma^2}n\overline{x^2}}$ ∴

the log-likelihood: $l(\sigma;\underline{x}) = \ln(L(\sigma;\underline{x})) =$

$\ln\left[2^{\frac{1}{2}n}\pi^{-\frac{1}{2}n}\sigma^{-n}e^{-\frac{1}{2\sigma^2}n\overline{x^2}}\right] \propto$

$\ln(\sigma^{-n}) + \ln(e^{-\frac{1}{2\sigma^2}n\overline{x^2}}) = -n\ln(\sigma) - \frac{1}{2\sigma^2}n\overline{x^2} \therefore$

$l(\sigma; \underline{x}) = -n\ln(\sigma) - \frac{1}{2\sigma^2}n\overline{x^2} + \text{Constant} \therefore$

$l'(\sigma; \underline{x}) = \frac{\partial l(\sigma)}{\partial\sigma} = -n\frac{1}{\sigma} - \frac{1}{2}n\overline{x^2}(-2)\sigma^{-3} = -n\frac{1}{\sigma} + n\overline{x^2}\sigma^{-3} \therefore$

$l''(\sigma; \underline{x}) = -n(-1)\sigma^{-2} + n\overline{x^2}(-3)\sigma^{-4} = n\sigma^{-2} - 3n\overline{x^2}\sigma^{-4} \therefore$

~~tz~~ $l'(\hat\sigma; \underline{x}) = 0 = -\frac{1}{\hat\sigma} + n\overline{x^2}\frac{1}{\hat\sigma^3} = 0 \therefore \hat\sigma^3 \neq 0 \therefore$

$n\overline{x^2}\frac{1}{\hat\sigma^3} = \frac{1}{\hat\sigma} \therefore n\overline{x^2} = \hat\sigma^2 \therefore \sigma > 0 \therefore$

$\hat\sigma = +\sqrt{n\overline{x^2}} \therefore$

$l''(\hat\sigma; \underline{x}) = n(n\overline{x^2})^{-1} - 3n\overline{x^2}(n\overline{x^2})^{-2} < 0 \therefore$

$\hat\sigma = +\sqrt{n\overline{x^2}}$ is the MLE ~~is~~ of σ.

\1d/ $I(\sigma) = -E(l''(\sigma)) = -E(n\sigma^{-2} - 3n\overline{x^2}\sigma^{-4}) =$

$-n\sigma^{-2}E(1) - 3n\sigma^{-4}E(\overline{x^2}) = -n\sigma^{-2} - 3n\sigma^{-4}E\left(\sum_{i=1}^{n}x_i^2\right) =$

$-n\sigma^{-2} - 3n\sigma^{-4}\sum_{i=1}^{n}E(x_i^2) = -\frac{n}{\sigma^2} - \frac{3n}{\sigma^4}\sum_{i=1}^{n}E(x_1^2) =$

$-\frac{n}{\sigma^2} - \frac{3n}{\sigma^4}nE(x_1^2) = -\frac{n}{\sigma^2} - \frac{3n}{\sigma^4}n\sigma^2 = -\frac{n}{\sigma^2} - \frac{3n^2}{\sigma^2} = -n\frac{1}{\sigma^2}(1-3n) \therefore$

$\frac{1}{I(\sigma)} = -\frac{\sigma^2}{n(1-3n)} =$ ~~the~~ $\frac{\sigma^2}{n(3n-1)} \therefore$

the asymptotic distribution of $\hat\sigma$ is $N\left(\sigma, \frac{1}{I(\sigma)}\right) =$

$N\left(\sigma, -\frac{\sigma^2}{n(1-3n)}\right) = N\left(\sigma, \frac{\sigma^2}{n(3n-1)}\right)$

\1e/ $\lim_{n\to\infty}\frac{1}{I(\sigma)} = \lim_{n\to\infty}\left[+\frac{\sigma^2}{n(3n-1)}\right] = 0 \therefore$

~~the~~ the method of moments estimator is efficient

\1f/ ~~$A(\underline{x}) = 7$~~ ~~At~~ $\Lambda = \frac{L(\sigma_1)}{L(\sigma_0)} = \frac{2^{\frac{1}{2}n}\pi^{-\frac{1}{2}n}\sigma_1^{-n}e^{-\frac{1}{2\sigma_1^2}n\overline{x^2}}}{2^{\frac{1}{2}n}\pi^{-\frac{1}{2}n}\sigma_0^{-n}e^{-\frac{1}{2\sigma_0^2}n\overline{x^2}}}$

$= \left(\frac{\sigma_1}{\sigma_0}\right)^{-n}e^{-\frac{1}{2}n\overline{x^2}\left(\frac{1}{\sigma_1^2} - \frac{1}{\sigma_0^2}\right)} \therefore$ critical region depends on $\overline{x^2} < C \therefore$

critical region: ~~[illegible]~~ $\{\underline{x} : \overline{x^2} < C\}$

\1g/ $\frac{U(\sigma_1)}{U(\sigma_0)}(\sigma_1 - \sigma_0)^2 = \frac{l'(\sigma_0)}{l'(\sigma_1)}(\sigma_1 / \sigma_0)^{2} = \frac{-\frac{n}{\sigma_1} + n\overline{x^2}\sigma_1^{-3}}{-\frac{n}{\sigma_0} + n\overline{x^2}\sigma_0^{-3}}\left(\frac{\sigma_1}{\sigma_0}\right) =$

\3028PP2021/ $\dfrac{-A+n\overline{x^2}\sigma_1^{-2}}{-n+n\overline{x^2}\sigma_0^{-2}}$ is the score test statistic $\therefore$ when n is large, both the wald and score test statistics follow a χ^2_1 chi-squared distribution

\2ai/ $Y = \frac{1}{\sigma^2}\sum_{i=1}^{n} X_i^2$ $\therefore$ let $\frac{X}{\sigma} = Z$ $\therefore$ $\frac{X_i}{\sigma} = Z_i$, $\frac{X_0}{\sigma} = Z_0$

$\therefore Y = \sum_{i=1}^{n}\left(\frac{X_i}{\sigma}\right)^2 = \sum_{i=1}^{n} Z_i \sim \chi^2_n$ $\therefore$ $Y \sim \chi^2_n$ distribution which is independent of any unknown parameters $\therefore$ Y is a pivot

~~2aii~~ \2aii/ $\therefore \frac{1-0.9}{2} = \frac{0.1}{2} = 0.05$ $\therefore$ let $q_{0.05}$ ~~be the~~ and $q_{0.95}$ be the 5% and 95% quantiles $\therefore$ 90% confidence interval is $\left(q_5 \le \sum_{i=1}^{n}\frac{X_i^2}{\sigma^2} \le q_{95}\right) =$ $(q_5 \le Y \le q_{95})$ $\therefore$ CI is: $(q_5/Y,\ q_{95}/Y)$

\2bi/ $\frac{X_0}{\bar{X}} = \frac{X_0}{\sum_{i=1}^{n} X_i} = \frac{Z_0/\sigma}{\sum_{i=1}^{n} Z_i/\sigma} = \frac{Z_0}{\sum_{i=1}^{n} Z_i} = \frac{Z_0}{\bar{Z}}$ is χ^2_{n+1} distribution is ancillary

\2bii/ $\therefore$ 90% confidence interval is $\left(q_5 \le \frac{X_0}{\bar{X}} < q_{95}\right) = \left(q_5 \le \frac{Z_0}{\bar{Z}} \le q_{95}\right) = \therefore$ $q_5\frac{\bar{Z}}{Z_0} < q_{95}\frac{\bar{Z}}{Z_0}$) is the confidence interval.

\3a/ $\therefore$ log-likelihood: $l(\sigma, \phi; \underline{x}, \underline{y}) = l(\sigma, \phi) =$
$\ln(L(\sigma, \phi; \underline{x}, \underline{y})) = \ln\left[(2\pi)^{-n}\sigma^{-2n}\phi^n e^{-(S_x+\phi^2 S_y)/(2\sigma^2)}\right] =$
$-n\ln[2\pi] - 2n\ln[\sigma] + n\ln[\phi] - \frac{S_x+\phi^2 S_y}{2\sigma^2}$ $\therefore$

$\frac{\partial l(\sigma,\phi)}{\partial\sigma} = -2n\frac{1}{\sigma} - \frac{S_x+\phi^2 S_y}{2}(-2)\sigma^{-3} = -2n\frac{1}{\sigma} + (S_x+\phi^2 S_y)\sigma^{-2}$ $\therefore$

$\frac{\partial l(\hat{\sigma},\phi)}{\partial\sigma} = 0 = -2n\frac{1}{\hat{\sigma}} + (S_x+\phi^2 S_y)\hat{\sigma}^{-2}$ $\therefore$

$2n\frac{1}{\hat{\sigma}} = (S_x+\phi^2 S_y)\frac{1}{\hat{\sigma}^2}$ $\therefore 2n\hat{\sigma} = S_x+\phi^2 S_y$ $\therefore \hat{\sigma} = \frac{S_x+\phi^2 S_y}{2n}$

$\therefore l_p(\phi; \underline{x}, \underline{y}) =$ ~~$l(\phi, \hat{\sigma}$~~ $l(\hat{\sigma}, \phi; \underline{x}, \underline{y}) =$
$-n\ln[2\pi] - 2n\ln\left[\frac{S_x+\phi^2 S_y}{2n}\right] + n\ln[\phi] - \frac{S_x+\phi^2 S_y}{2}\left(\frac{S_x+\phi^2 S_y}{2n}\right)^{-2} =$
$-n\ln[2\pi] - 2n\ln\left[\frac{S_x+\phi^2 S_y}{2n}\right] + n\ln[\phi] - 2n^2\frac{1}{S_x+\phi^2 S_y}$

\3b/ $\cancel{A}\ \Lambda = \frac{L(\phi_1)}{L(\phi_0)}$ $\therefore$ $2\ln[\Lambda] = 2\ln\left[\frac{L(\phi_1)}{L(\phi_0)}\right] =$

$2[l(\phi_1) - l(\phi_0)]$ $\therefore$

the test statistic is: $2[l_p(\phi_1) - l_p(\phi_0)] =$

$2[l_p(\phi_1; \underline{x}, \underline{y}) - l_p(\phi_0; \underline{x}, \underline{y})] =$

$2\Big[-n\ln[2\pi] - 2n\ln\left[\frac{S_x + \phi_1^2 S_y}{2n}\right] + n\ln[\phi_1] - \frac{2n^2}{S_x + \phi_1^2 S_y} +$

$n\ln[2\pi] + 2n\ln\left[\frac{S_x + \phi_0^2 S_y}{2n}\right] - n\ln[\phi_0] + \frac{2n^2}{S_x + \phi_0^2 S_y}\Big] =$

$2\left[2n\left(\ln\left[\frac{S_x + \phi_0^2 S_y}{2n}\right] - \ln\left[\frac{S_x + \phi_1^2 S_y}{2n}\right]\right) + n\ln\left[\frac{\phi_1}{\phi_0}\right] + 2n^2\left(\frac{1}{S_x + \phi_0^2 S_y} - \frac{1}{S_x + \phi_1^2 S_y}\right)\right] =$

$2\left[2n\ln\left[\frac{S_x + \phi_0^2 S_y}{S_x + \phi_1^2 S_y}\right] + n\ln\left[\frac{\phi_1}{\phi_0}\right] + 2n^2\left(\frac{1}{S_x + \phi_0^2 S_y} - \frac{1}{S_x + \phi_1^2 S_y}\right)\right] =$

$4n\ln\left[\frac{S_x + \phi_0^2 S_y}{S_x + \phi_1^2 S_y}\right] + 2n\ln\left[\frac{\phi_1}{\phi_0}\right] + 4n^2\left(\frac{1}{S_x + \phi_0^2 S_y} - \frac{1}{S_x + \phi_1^2 S_y}\right)$

\3c/ let Z critical region be c $\therefore$

$Pr(2[l(\phi_1) - l(\phi_0)] < c) = \alpha$ $\therefore$

the ratio test statistic follows a χ^2 distribution $\therefore$

$I^{-1}(\alpha) = 2[l(\phi_1) - l(\phi_0)] < c$ $\therefore$

$I^{-1}(\alpha) < c$ is the confidence interval.

\4a/ for $i=1$: $\hat{\beta}_{-1} = x_{(n)} - x_{(2)}$

for $i = 2, \ldots, n-1$: $\hat{\beta}_{-i} = x_{(n)} - x_{(1)} = \hat{\beta}$, for $i=n$: $\hat{\beta}_{-n} = x_{(n-1)} - x_{(1)}$ $\therefore$

$n\hat{\beta} - (n-1)\hat{\beta}_{-i} = \hat{\beta}_i$ $\therefore$

for $i=1$: $\hat{\beta}_1 = n\hat{\beta} - (n-1)(x_{(n)} - x_{(2)})$,

for $i = 2, \ldots, n-1$: $\hat{\beta}_i = n\hat{\beta} - (n-1)\hat{\beta} = \hat{\beta}$

for $i=n$: $\hat{\beta}_n = n\hat{\beta} - (n-1)(x_{(n-1)} - x_{(1)})$ $\therefore$

$\hat{\beta}_J = \frac{1}{n}\sum_{i=1}^{n}\beta_i = \left(n\hat{\beta} - (n-1)(x_{(n)} - x_{(2)}) + \left(\sum_{i=2}^{n-1}\hat{\beta}\right) + n\hat{\beta} - (n-1)(x_{(n-1)} - x_{(1)})\right)\frac{1}{n}$

$(2n\hat{\beta} + (n-2)\hat{\beta} - (n-1)x_{(n)} - (n-1)x_{(1)} - (n-1)x_{(n-1)} - (n-1)x_{(2)})\frac{1}{n} =$

$(3n\hat{\beta} - 2\hat{\beta} - nx_{(n)} + x_{(n)} - nx_{(1)} + x_{(1)} - nx_{(n-1)} + x_{(n-1)} - nx_{(2)} + x_{(2)})\frac{1}{n} =$

$3\hat{\beta} - x_{(n)} - x_{(1)} - x_{(n-1)} - x_{(2)} + (-2\hat{\beta} + x_{(n)} + x_{(1)} + x_{(n-1)} + x_{(2)})\frac{1}{n} =$

\3028PP2021/ $3(x_{(n)}-x_{(1)})-x_{(n)}-x_{(1)}-x_{(n-1)}-x_{(2)}+(-2(x_{(n)}-x_{(1)})+x_{(n)}+x_{(1)}+x_{(n-1)}+x_{(2)})\frac{1}{n}$

$=3x_{(n)}-3x_{(1)}-x_{(n)}-x_{(1)}-x_{(n-1)}-x_{(2)}+(-2x_{(n)}+2x_{(1)}+x_{(n)}+x_{(1)}+x_{(n-1)}+x_{(2)})\frac{1}{n}=$

$2x_{(n)}-4x_{(1)}-x_{(n-1)}-x_{(2)}+(-x_{(n)}+3x_{(1)}+x_{(n-1)}+x_{(2)})\frac{1}{n}$ $\therefore$

$\hat{\beta}_J > x_{(n)}-x_{(1)}$ $\therefore$ $\hat{\beta}_J \geq \hat{\beta}$ $\forall n$

\4bi/ $n=200$ $\therefore$ $\frac{n}{n+1}E(\hat{\beta}^*)=\frac{200}{201}(5.1)=5.075$, (4.S.F

\4bii/ $SE(\hat{\beta}^*)=\sqrt{var(\hat{\beta}^*)}$ $\therefore$ $\sqrt{\frac{1.1}{200}}=0.07416$ (4.S.F)

\4biii/ $z_{0.95}=1.65$ $\therefore$ $\left(5.1-1.65\sqrt{\frac{1.1}{200}},\ 5.1+1.65\sqrt{\frac{1.1}{200}}\right)=$

$(4.98, 5.22)$ (3S.F.) is the 90% $\therefore$

$z_{0.025}=1.96$ $\therefore$ $\left(5.1-1.96\sqrt{\frac{1.1}{200}},\ 5.1+1.96\sqrt{\frac{1.1}{200}}\right)=(4.95, 5.25)$

(3S.F.) is the 95% confidence interval

\4biv/ $\therefore$ $\left(\frac{n}{n-1}(4.95), \frac{n}{n-1}(5.25)\right)=\left(\frac{200}{199}\times 4.95,\ \frac{200}{199}\times 5.25\right)=$

$(4.98, 5.27)$ is the 95% confidence interval

\4c/ ~~$q_{0.9}$~~ $\frac{1-0.9}{2}=0.05$ $\therefore$ for ~~n=10~~ $n=99$: $q_{0.05}=x_{(5)}$,

$1-0.05=0.95$ $\therefore$ $q_{0.95}=x_{(99)}$ $\therefore$ ~~the new data point has a 5% chance of being to~~ the ~~confidence~~ prediction interval has a 5% chance of being above the ~~5% quantile and 95% chance of being below the~~ data point, and a 95% chance of being below it, $\therefore$ the ~~confidence~~ prediction interval is $(q_{0.05}, q_{0.95})$ where $q_{0.05}, q_{0.95}$ are the 5%, 95% quantiles $\therefore$ $\because$ $q_{0.05}=x_{(5)}$ for $n=99$, and $q_{0.95}=x_{(95)}$ $\therefore$ the prediction interval is $(x_{(5)}, x_{(95)})$ for a further data point.

\1aSol/ Sub $y=x^2/(2\sigma^2)$ $\therefore$ $E(X_1)=\int_0^\infty x f(x;\sigma)dx=$

$\sigma^{-1}\sqrt{\frac{2}{\pi}}\int_0^\infty xe^{-x^2/(2\sigma^2)}dx=\sigma^2\sigma^{-1}\sqrt{\frac{2}{\pi}}\int_0^\infty \frac{1}{\sigma^2}xe^{-x^2/(2\sigma^2)}dx=$

$\sigma\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-x^2/(2\sigma^2)}\frac{1}{\sigma^2}x\,dx=\sigma\sqrt{\frac{2}{\pi}}\int_0^\infty e^{-y}dy=\sigma\sqrt{\frac{2}{\pi}}\left[-e^{-y}\right]_0^\infty=\sigma\sqrt{\frac{2}{\pi}}$

MOM: $\bar{X}=\hat{\sigma}\sqrt{\frac{2}{\pi}}$ $\therefore$

$\therefore$ solving $\bar{X}=\hat{\sigma}\sqrt{\frac{2}{\pi}}$ $\therefore$ the Method of Moments estimator:

$\hat{\sigma}=\bar{X}\sqrt{\frac{\pi}{2}}$

\1b/ $E(\hat{\sigma})=E\left(\bar{X}\sqrt{\frac{\pi}{2}}\right)=E(\bar{X})\sqrt{\frac{\pi}{2}}=E\left(\frac{1}{n}\sum_{i=1}^n X_i\right)\sqrt{\frac{\pi}{2}}=\sqrt{\frac{\pi}{2}}\frac{1}{n}\sum_{i=1}^n E(X_i)=$

$\sqrt{\frac{\pi}{2}}\frac{1}{n}\sum_{i=1}^{n}E(X_1)=\sqrt{\frac{\pi}{2}}\frac{1}{n}nE(X_1)=\sqrt{\frac{\pi}{2}}E(X_1)=\sqrt{\frac{\pi}{2}}\sigma\sqrt{\frac{2}{\pi}}=\sigma$ & ∴

$bias(\hat{\sigma})=E(\hat{\sigma})-\sigma=\sigma-\sigma=0$ ∴ $\hat{\sigma}$ is unbiased.

$var(X_1)=E(X_1^2)-E(X_1)^2=\sigma^2-(\sqrt{\frac{2}{\pi}}\sigma)^2=\sigma^2-\frac{2}{\pi}\sigma^2=\sigma^2(1-\frac{2}{\pi})$ ∴

$var(\hat{\sigma})=var(\sqrt{\frac{\pi}{2}}\bar{X})=(\sqrt{\frac{\pi}{2}})^2var(\bar{X})=\frac{\pi}{2}var(\bar{X})=\frac{\pi}{2}var(\frac{1}{n}\sum_{i=1}^{n}X_i)=$

$\frac{\pi}{2n^2}var(\sum_{i=1}^{n}X_i)=\cancel{\frac{\pi}{2n^2}\ldots\sum_{i=}}\ \frac{\pi}{2n^2}\sum_{i=1}^{n}var(X_i)=\frac{\pi}{2n^2}\sum_{i=1}^{n}var(X_1)=$

$\frac{\pi}{2n^2}n\,var(X_1)=\frac{\pi}{2n}var(X_1)=\frac{\pi}{2n}\sigma^2(1-\frac{2}{\pi})=(\frac{\pi}{2n}-\frac{1}{n})\sigma^2=(\frac{\pi}{2}-1)\frac{\sigma^2}{n}$

$\{var(\hat{\sigma})=\frac{\pi}{2n^2}var(\sum_{i=1}^{n}X_i)\neq\}$ ∴

$bias(\hat{\sigma})=0\to 0$ as $n\to\infty$, $var(\hat{\sigma})=(\frac{\pi}{2}-1)\frac{\sigma^2}{n}\to 0$ as $n\to\infty$ ∴

Z bias & variance converge to zero as $n\to\infty$ & ∴ $\hat{\sigma}$ is consistent

\1c/ Z likehood is $L(\sigma;x)=\prod_{i=1}^{n}f(x_i;\sigma)\propto\sigma^{-n}e^{-\sum_{i=1}^{n}x_i^2/(2\sigma^2)}$

∵ $L(\sigma;x)=\prod_{i=1}^{n}f(x_i;\sigma)=\prod_{i=1}^{n}2^{\frac{1}{2}}\pi^{-1/2}\sigma^{-1}e^{-x_i^2/(2\sigma^2)}\propto\prod_{i=1}^{n}\sigma^{-1}e^{-x_i^2/(2\sigma^2)}=$

$\sigma^{-n}e^{-\sum_{i=1}^{n}x_i^2/(2\sigma^2)}$ ∴ the log likelihood:

$l(\sigma;x)=\ln L(\sigma;x)\propto\ln[\sigma^{-n}e^{-\sum_{i=1}^{n}x_i^2/(2\sigma^2)}]=-n\ln(\sigma)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}x_i^2$ ∴

$l(\sigma;x)=\text{constant}-n\ln(\sigma)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}x_i^2$ ∴

$l'(\sigma;x)=-\frac{n}{\sigma}+\frac{1}{\sigma^3}\sum_{i=1}^{n}x_i^2$ ∴ solve $l'(\hat{\sigma};x)=0$:

$-\frac{n}{\hat{\sigma}}+\frac{1}{\hat{\sigma}^3}\sum_{i=1}^{n}x_i^2=0$ ∴ $\frac{1}{n}\sum_{i=1}^{n}x_i^2=\hat{\sigma}^2$ ∴ $\hat{\sigma}=+(\frac{1}{n}\sum_{i=1}^{n}x_i^2)^{1/2}$ ∵ $\sigma>0$

& $l''(\sigma;x)=\frac{n}{\sigma^2}-\frac{3}{\sigma^4}\sum_{i=1}^{n}x_i^2$ ∴

$l''(\hat{\sigma};x)=\frac{n}{\hat{\sigma}^2}-\frac{3}{\hat{\sigma}^4}\sum_{i=1}^{n}x_i^2=\frac{n}{\frac{1}{n}\sum_{i=1}^{n}x_i^2}-\frac{3}{(\frac{1}{n}\sum_{i=1}^{n}x_i^2)^2}\sum_{i=1}^{n}x_i^2=\frac{n}{\hat{\sigma}^2}-\frac{3}{\hat{\sigma}^4}\sum_{i=1}^{n}x_i^2=$

$\frac{n}{\hat{\sigma}^2}-\frac{3}{\hat{\sigma}^4}(n\hat{\sigma}^2)=\frac{n}{\hat{\sigma}^2}-\frac{3n}{\hat{\sigma}^2}=-\frac{2n}{\hat{\sigma}^2}<0$ ∵ $\frac{2n}{\hat{\sigma}^2}>0$ ∴ $l''(\hat{\sigma};x)<0$ ∴

$\hat{\sigma}$ is the MLE

\1d/ the expected information is $I(\sigma)=-E(l''(\sigma))=$

$-E(\frac{n}{\sigma^2}-\frac{3}{\sigma^4}\sum_{i=1}^{n}x_i^2)=-E(\frac{n}{\sigma^2})-E(-\frac{3}{\sigma^4}\sum_{i=1}^{n}x_i^2)=-\frac{n}{\sigma^2}E(1)+\frac{3}{\sigma^4}E(\sum_{i=1}^{n}x_i^2)=$

$-\frac{n}{\sigma^2}+\frac{3}{\sigma^4}\sum_{i=1}^{n}E(x_i^2)=\cancel{-\frac{n}{\sigma^2}+\frac{3n}{\sigma^4}}\ -\frac{n}{\sigma^2}+\frac{3}{\sigma^4}\sum_{i=1}^{n}E(X_1^2)=-\frac{n}{\sigma^2}+\frac{3n}{\sigma^4}E(X_1^2)=$

$-\frac{n}{\sigma^2}+\frac{3n}{\sigma^4}\sigma^2=-\frac{n}{\sigma^2}+\frac{3n}{\sigma^2}=\frac{2n}{\sigma^2}$ and ∴ the asymptotic distribution of the MLE is $N(\sigma,\frac{1}{I(\sigma)})=N(\sigma,(\frac{2n}{\sigma^2})^{-1})=N(\sigma,\frac{\sigma^2}{2n})$

\1e/ The efficiency of the method of moments estimator is

\3028 PP2021/ $I(\sigma)^{-1}/var(\hat{\sigma}) = \frac{1}{I(\sigma)var(\hat{\sigma})} = \frac{1}{I(\sigma)}\frac{1}{var(\hat{\sigma})} =$

$\frac{\sigma^2}{2n}\frac{1}{var(\hat{\sigma})} = \frac{\sigma^2}{2n}(var(\hat{\sigma}))^{-1} = \frac{\sigma^2}{2n}\left(\left(\frac{\pi}{2}-1\right)\frac{\sigma^2}{n}\right)^{-1} =$

$\frac{\sigma^2}{2n}\left(\frac{\pi}{2}-1\right)^{-1}\frac{n}{\sigma^2} = \frac{1}{2}\frac{1}{\frac{\pi}{2}-1} = \frac{1}{2(\frac{\pi}{2}-1)} = \frac{1}{\pi-2}$

$\because$ the efficiency of an unbiased estimator $\hat{\sigma}$ is $\frac{I(\sigma)^{-1}}{var(\hat{\sigma})}$

\1f/ the likelihood ratio is $\Lambda = \frac{L(\sigma_1)}{L(\sigma_0)} = \left(\frac{\sigma_0}{\sigma_1}\right)^n e^{-\frac{1}{2}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\sum_{i=1}^n x_i^2}$

$= \cancel{b}\ \frac{\sigma_0^n}{\sigma_1^n} e^{\frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\sum_{i=1}^n x_i^2} \therefore \because \sigma_1 > \sigma_0 \therefore \frac{1}{\sigma_1} < \frac{1}{\sigma_0} \therefore \frac{1}{\sigma_1^2} < \frac{1}{\sigma_0^2} \therefore$

$\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0 \therefore \frac{L(\sigma_1)}{L(\sigma_0)}$ increases ~~when~~ as $\sum_{i=1}^n x_i^2$ increases $\therefore$

$\frac{L(\sigma_1)}{L(\sigma_0)}$ is large when $\sum_{i=1}^n x_i^2$ is large $\therefore$ the critical region has the form $\{\underline{x} : \sum_{i=1}^n x_i^2 > c\}$ for a critical value c

\1g/ The wald test statistic is $(\hat{\sigma}-\sigma_0)^2 I(\hat{\sigma}) =$

$(\hat{\sigma}-\sigma_0)^2\frac{2n}{\hat{\sigma}^2} = \left((\hat{\sigma}-\sigma_0)\frac{1}{\hat{\sigma}}\right)^2 2n = \left(1-\frac{\sigma_0}{\hat{\sigma}}\right)^2 2n = 2n\left(1-\frac{\sigma_0}{\hat{\sigma}}\right)^2$

and the score test statistic is $U(\sigma_0)^2/I(\sigma_0) = \frac{(l'(\sigma_0))^2}{I(\sigma_0)} =$

$(l'(\sigma_0))^2\frac{1}{I(\sigma_0)} = (l'(\sigma_0))^2\frac{\sigma_0^2}{2n} = \left(-\frac{n}{\sigma_0}+\frac{1}{\sigma_0^3}\sum_{i=1}^n x_i^2\right)^2\frac{\sigma_0^2}{2n} =$

$\frac{\sigma_0^2}{2n}\left(-\frac{n}{\sigma_0}+\frac{1}{\sigma_0^3}n\hat{\sigma}^2\right)^2 = \frac{\sigma_0^2}{2n}n^2\left(-\frac{1}{\sigma_0}+\frac{\hat{\sigma}^2}{\sigma_0^3}\right)^2 = \frac{n\sigma_0^2}{2}\left(\frac{\hat{\sigma}^2}{\sigma_0^3}-\frac{1}{\sigma_0}\right)^2 =$

$\frac{n}{2}\left(\sigma_0\left(\frac{\hat{\sigma}^2}{\sigma_0^3}-\frac{1}{\sigma_0}\right)\right)^2 = \frac{n}{2}\left(\frac{\hat{\sigma}^2}{\sigma_0^2}-1\right)^2$

their approximate null distributions is χ_1^2 when n is large

\2ai/ we know $\frac{X_i}{\sigma}$ are independent $N(0,1)$ random variables

$\because X_i \sim N(0,\sigma^2) \therefore \frac{X_i-\mu}{\sigma} \sim N(0,1) \therefore \frac{X_i-0}{\sigma} = \frac{X_i}{\sigma} \sim N(0,1)$ and $\therefore$

$Y \sim \chi_n^2 \because \frac{X_i}{\sigma} \sim N(0,1) \therefore \frac{X_i}{\sigma} = Z_i \sim N(0,1) \therefore$

$Y = \sum_{i=1}^n \frac{X_i^2}{\sigma^2} = \sum_{i=1}^n \left(\frac{X_i}{\sigma}\right) = \sum_{i=1}^n Z_i^2 \sim \chi_n^2 \therefore Y \sim \chi_n^2$. the χ_n^2 distribution has no unknown parameters $\therefore$ Y is a pivot.

\2aii/ $0.9 = Pr(q_{0.05} < Y < q_{0.95}) = Pr\left(q_{0.05} < \sum_{i=1}^n \frac{X_i^2}{\sigma^2} < q_{0.95}\right) =$

$Pr\left(q_{0.05} < \frac{1}{\sigma^2}\sum_{i=1}^n X_i^2 < q_{0.95}\right) = Pr\left(\frac{1}{q_{0.95}} < \sigma^2\frac{1}{\sum_{i=1}^n X_i^2} < \frac{1}{q_{0.05}}\right) =$

$Pr\left(\frac{1}{q_{0.95}}\sum_{i=1}^n X_i^2 < \sigma^2 < \frac{1}{q_{0.05}}\sum_{i=1}^n X_i^2\right) = Pr\left(\sum_{i=1}^n X_i^2/q_{0.95} < \sigma^2 < \sum_{i=1}^n X_i^2/q_{0.05}\right) \therefore$

$\sqrt{\frac{\pi}{2}}\frac{1}{n}\sum_{i=1}^{n}E(X_1) = \sqrt{\frac{\pi}{2}}\frac{1}{n}nE(X_1) = \sqrt{\frac{\pi}{2}}E(X_1) = \sqrt{\frac{\pi}{2}}\sigma\sqrt{\frac{2}{\pi}} = \sigma$ & $\therefore$

$\text{bias}(\hat{\sigma}) = E(\hat{\sigma}) - \sigma = \sigma - \sigma = 0$ $\therefore$ $\hat{\sigma}$ is unbiased.,

$\text{var}(X_1) = E(X_1^2) - E(X_1)^2 = \sigma^2 - (\sqrt{\frac{2}{\pi}}\sigma)^2 = \sigma^2 - \frac{2}{\pi}\sigma^2 = \sigma^2(1-\frac{2}{\pi})$ $\therefore$

$\text{var}(\hat{\sigma}) = \text{var}(\sqrt{\frac{\pi}{2}}\bar{X}) = (\sqrt{\frac{\pi}{2}})^2\text{var}(\bar{X}) = \frac{\pi}{2}\text{var}(\bar{X}) = \frac{\pi}{2}\text{var}(\frac{1}{n}\sum_{i=1}^{n}X_i) =$

$\frac{\pi}{2n^2}\text{var}(\sum_{i=1}^{n}X_i) = \cancel{\frac{\pi}{2n^2}\ldots\sum_{i=1}^{n}}\ \frac{\pi}{2n^2}\sum_{i=1}^{n}\text{var}(X_i) = \frac{\pi}{2n^2}\sum_{i=1}^{n}\text{var}(X_1) =$

$\frac{\pi}{2n^2}n\,\text{var}(X_1) = \frac{\pi}{2n}\text{var}(X_1) = \frac{\pi}{2n}\sigma^2(1-\frac{2}{\pi}) = (\frac{\pi}{2n} - \frac{1}{n})\sigma^2 = (\frac{\pi}{2}-1)\frac{\sigma^2}{n}$

$\{\text{var}(\hat{\sigma}) = \frac{\pi}{2n^2}\text{var}(\sum_{i=1}^{n}X_i) \neq\}$ $\therefore$

$\text{bias}(\hat{\sigma}) = 0 \to 0$ as $n \to \infty$, $\text{var}(\hat{\sigma}) = (\frac{\pi}{2}-1)\frac{\sigma^2}{n} \to 0$ as $n \to \infty$ $\therefore$

2 bias & variance converge to zero as $n \to \infty$ & $\therefore$ $\hat{\sigma}$ is consistent

\1c/ 2 lihood is $L(\sigma;\underline{x}) = \prod_{i=1}^{n} f(x_i;\sigma) \propto \sigma^{-n}e^{-\sum_{i=1}^{n}x_i^2/(2\sigma^2)}$

$\therefore$ $L(\sigma;\underline{x}) = \prod_{i=1}^{n} f(x_i;\sigma) = \prod_{i=1}^{n} 2^{\frac{1}{2}}\pi^{-\frac{1}{2}}\sigma^{-1}e^{-x_i^2/(2\sigma^2)} \propto \prod_{i=1}^{n}\sigma^{-1}e^{-x_i^2/(2\sigma^2)} =$

$\sigma^{-n}e^{-\sum_{i=1}^{n}x_i^2/(2\sigma^2)}$ $\therefore$ the log likelihood:

$l(\sigma;\underline{x}) = \ln L(\sigma;\underline{x}) \propto \ln[\sigma^{-n}e^{-\sum_{i=1}^{n}x_i^2/(2\sigma^2)}] = -n\ln(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}x_i^2$ $\therefore$

$l(\sigma;\underline{x}) = \text{constant} - n\ln(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}x_i^2$ $\therefore$

$l'(\sigma;\underline{x}) = -\frac{n}{\sigma} + \frac{1}{\sigma^3}\sum_{i=1}^{n}x_i^2$ $\therefore$ solve $l'(\hat{\sigma};\underline{x}) = 0$:

$-\frac{n}{\hat{\sigma}} + \frac{1}{\hat{\sigma}^3}\sum_{i=1}^{n}x_i^2 = 0$ $\therefore$ $\frac{1}{n}\sum_{i=1}^{n}x_i^2 = \hat{\sigma}^2$ $\therefore$ $\hat{\sigma} = +(\frac{1}{n}\sum_{i=1}^{n}x_i^2)^{1/2}$ $\because \sigma > 0$

& $l''(\sigma;\underline{x}) = \frac{n}{\sigma^2} - \frac{3}{\sigma^4}\sum_{i=1}^{n}x_i^2$ $\therefore$

$l''(\hat{\sigma};\underline{x}) = \frac{n}{\hat{\sigma}^2} - \frac{3}{\hat{\sigma}^4}\sum_{i=1}^{n}x_i^2 = \frac{n}{\frac{1}{n}\sum_{i=1}^{n}x_i^2} - \frac{3}{(\frac{1}{n}\sum_{i=1}^{n}x_i^2)^2}\sum_{i=1}^{n}x_i^2 = \frac{n}{\hat{\sigma}^2} - \frac{3}{\hat{\sigma}^4}\sum_{i=1}^{n}x_i^2 =$

$\frac{n}{\hat{\sigma}^2} - \frac{3}{\hat{\sigma}^4}(n\hat{\sigma}^2) = \frac{n}{\hat{\sigma}^2} - \frac{3n}{\hat{\sigma}^2} = -\frac{2n}{\hat{\sigma}^2} < 0$ $\because$ $\frac{2n}{\hat{\sigma}^2} > 0$ $\therefore$ $l''(\hat{\sigma};\underline{x}) < 0$ $\therefore$

$\hat{\sigma}$ is the mle

\1d/ the expected information is $I(\sigma) = -E(l''(\sigma)) =$

$-E(\frac{n}{\sigma^2} - \frac{3}{\sigma^4}\sum_{i=1}^{n}x_i^2) = -E(\frac{n}{\sigma^2}) - E(-\frac{3}{\sigma^4}\sum_{i=1}^{n}x_i^2) = -\frac{n}{\sigma^2}E(1) + \frac{3}{\sigma^4}E(\sum_{i=1}^{n}x_i^2) =$

$-\frac{n}{\sigma^2} + \frac{3}{\sigma^4}\sum_{i=1}^{n}E(x_i^2) = \cancel{-\frac{n}{\sigma^2} + \frac{3n}{\sigma^4}}\ -\frac{n}{\sigma^2} + \frac{3}{\sigma^4}\sum_{i=1}^{n}E(X_1^2) = -\frac{n}{\sigma^2} + \frac{3n}{\sigma^4}E(X_1^2) =$

$-\frac{n}{\sigma^2} + \frac{3n}{\sigma^4}\sigma^2 = -\frac{n}{\sigma^2} + \frac{3n}{\sigma^2} = \frac{2n}{\sigma^2}$ and $\therefore$ the asymptotic distribution

of the mle is $N(\sigma, \frac{1}{I(\sigma)}) = N(\sigma, (\frac{2n}{\sigma^2})^{-1}) = N(\sigma, \frac{\sigma^2}{2n})$

\1e/ The efficiency of the method of moments estimator is

\3028 PP2021/ $I(\sigma)^{-1}/var(\hat{\sigma}) = \frac{1}{I(\sigma)var(\hat{\sigma})} = \frac{1}{I(\sigma)}\frac{1}{var(\hat{\sigma})} =$

$\frac{\sigma^2}{2n}\frac{1}{var(\hat{\sigma})} = \frac{\sigma^2}{2n}(var(\hat{\sigma}))^{-1} = \frac{\sigma^2}{2n}\left((\frac{\pi}{2}-1)\frac{\sigma^2}{n}\right)^{-1} =$

$\frac{\sigma^2}{2n}(\frac{\pi}{2}-1)^{-1}\frac{n}{\sigma^2} = \frac{1}{2}\frac{1}{\frac{\pi}{2}-1} = \frac{1}{2(\frac{\pi}{2}-1)} = \frac{1}{\pi-2}$

$\because$ the efficiency of an unbiased estimator $\hat{\sigma}$ is $\frac{I(\sigma)^{-1}}{var(\hat{\sigma})}$

\1g/ the likelihood ratio is $\Lambda = \frac{L(\sigma_1)}{L(\sigma_0)} = \left(\frac{\sigma_0}{\sigma_1}\right)^n e^{-\frac{1}{2}\left(\frac{1}{\sigma_1^2}-\frac{1}{\sigma_0^2}\right)\sum_{i=1}^{n} x_i^2}$

$= \frac{\sigma_0^n}{\sigma_1^n} e^{\frac{1}{2}\left(\frac{1}{\sigma_0^2}-\frac{1}{\sigma_1^2}\right)\sum_{i=1}^{n} x_i^2} \therefore \because \sigma_1 > \sigma_0 \therefore \frac{1}{\sigma_1} < \frac{1}{\sigma_0} \therefore \frac{1}{\sigma_1^2} < \frac{1}{\sigma_0^2} \therefore$

$\frac{1}{\sigma_0^2} - \frac{1}{\sigma_1^2} > 0 \therefore \frac{L(\sigma_1)}{L(\sigma_0)}$ increases ~~when~~ as $\sum_{i=1}^{n} x_i^2$ increases $\therefore$

$\frac{L(\sigma_1)}{L(\sigma_0)}$ is large when $\sum_{i=1}^{n} x_i^2$ is large $\therefore$ the critical region has the form $\{\underline{x} : \sum_{i=1}^{n} x_i^2 > c\}$ for a critical value c

\1g/ The wald test statistic is $(\hat{\sigma}-\sigma_0)^2 I(\hat{\sigma}) =$

$(\hat{\sigma}-\sigma_0)^2\frac{2n}{\hat{\sigma}^2} = \left((\hat{\sigma}-\sigma_0)\frac{1}{\hat{\sigma}}\right)^2 2n = \left(1-\frac{\sigma_0}{\hat{\sigma}}\right)^2 2n = 2n\left(1-\frac{\sigma_0}{\hat{\sigma}}\right)^2$

and the score test statistic is $U(\sigma_0)^2/I(\sigma_0) = \frac{(l'(\sigma_0))^2}{I(\sigma_0)} =$

$(l'(\sigma_0))^2\frac{1}{I(\sigma_0)} = (l'(\sigma_0))^2\frac{\sigma_0^2}{2n} = \left(-\frac{n}{\sigma_0} + \frac{1}{\sigma_0^3}\sum_{i=1}^{n} x_i^2\right)^2\frac{\sigma_0^2}{2n} =$

$\frac{\sigma_0^2}{2n}\left(-\frac{n}{\sigma_0} + \frac{1}{\sigma_0^3}n\hat{\sigma}^2\right)^2 = \frac{\sigma_0^2}{2n}n^2\left(-\frac{1}{\sigma_0} + \frac{\hat{\sigma}^2}{\sigma_0^3}\right)^2 = \frac{n\sigma_0^2}{2}\left(\frac{\hat{\sigma}^2}{\sigma_0^3} - \frac{1}{\sigma_0}\right)^2 =$

$\frac{n}{2}\left(\sigma_0\left(\frac{\hat{\sigma}^2}{\sigma_0^3} - \frac{1}{\sigma_0}\right)\right)^2 = \frac{n}{2}\left(\frac{\hat{\sigma}^2}{\sigma_0^2} - 1\right)^2$

their approximate null distributions is χ_1^2 when n is large

\2ai/ we know $\frac{X_i}{\sigma}$ are independent $N(0,1)$ random variables

$\because X_i \sim N(0,\sigma^2) \therefore \frac{X_i-\mu}{\sigma} \sim N(0,1) \therefore \frac{X_i-0}{\sigma} = \frac{X_i}{\sigma} \sim N(0,1)$ and $\therefore$

$Y \sim \chi_n^2 \because \frac{X_i}{\sigma} \sim N(0,1) \therefore \frac{X_i}{\sigma} = Z_i \sim N(0,1) \therefore$

$Y = \sum_{i=1}^{n}\frac{X_i^2}{\sigma^2} = \sum_{i=1}^{n}\left(\frac{X_i}{\sigma}\right) = \sum_{i=1}^{n} Z_i^2 \sim \chi_n^2 \therefore Y \sim \chi_n^2$. the χ_n^2 distribution has no unknown parameters $\therefore$ Y is a pivot.

\2aii/ $0.9 = Pr(q_{0.05} < Y < q_{0.95}) = Pr\left(q_{0.05} < \sum_{i=1}^{n}\frac{X_i^2}{\sigma^2} < q_{0.95}\right) =$

$Pr\left(q_{0.05} < \frac{1}{\sigma^2}\sum_{i=1}^{n} X_i^2 < q_{0.95}\right) = Pr\left(\frac{1}{q_{0.95}} < \sigma^2\frac{1}{\sum_{i=1}^{n} X_i^2} < \frac{1}{q_{0.05}}\right) =$

$Pr\left(\frac{1}{q_{0.95}}\sum_{i=1}^{n} X_i^2 < \sigma^2 < \frac{1}{q_{0.05}}\sum_{i=1}^{n} X_i^2\right) = Pr\left(\sum_{i=1}^{n} X_i^2/q_{0.95} < \sigma^2 < \sum_{i=1}^{n} X_i^2/q_{0.05}\right) \therefore$

a 90% confidence interval for σ^2 is $\left(\sum_{i=1}^{n} x_i^2 / q_{0.95}, \sum_{i=1}^{n} x_i^2 / q_{0.05}\right)$

\2bi/ the ancillary statistic $T = \frac{X_0^2}{W} = \frac{X_0^2}{\sum_{i=1}^{n} X_i^2 / n} = \frac{X_0^2}{\frac{1}{n}\sum_{i=1}^{n} X_i^2}$

where $W = \sum_{i=1}^{n} X_i^2 / n$ $\because \frac{X_i}{\sigma} = Z_i$ $\therefore \frac{X_0}{\sigma} = Z_0$ $\therefore$

$$\frac{Z_0^2}{\bar{Z}^2} = \frac{Z_0}{\frac{1}{n}\sum_{i=1}^{n} Z_i^2} = \frac{\left(\frac{X_0}{\sigma}\right)^2}{\frac{1}{n}\sum_{i=1}^{n}\left(\frac{X_i}{\sigma}\right)^2} = \frac{\left(\frac{X_0^2}{\sigma^2}\right)}{\frac{1}{n}\sum_{i=1}^{n}\frac{X_i^2}{\sigma^2}} = \frac{\left(\frac{X_0^2}{\sigma^2}\right)}{\frac{1}{n}\frac{1}{\sigma^2}\sum_{i=1}^{n} X_i^2} = \frac{X_0^2}{\frac{1}{n}\sum_{i=1}^{n} X_i^2} = T$$

$\therefore$ T is ancillary and has a $F_{1,n}$ distribution $\because$

$\frac{Y_1/1}{Y_2/n}$ has a $F_{1,n}$ distribution and $\therefore$ Z $Y_1 \sim \chi_1^2$, $Y_2 \sim \chi_n^2$ $\therefore$

$\sum_{i=0}^{0} Z_i^2 = Z_0^2 \sim \chi_1^2$, $\sum_{i=1}^{n} Z_i^2 \sim \chi_n^2$ $\therefore \frac{Z_0^2/1}{\frac{1}{n}\sum_{i=1}^{n} Z_i^2} \sim F_{1,n}$

\2bii/ let c^2 denote the 0.9-quantile of $F_{1,n}$ $\therefore$

$0.9 = Pr(T < c^2) = Pr\left(\frac{X_0^2}{W} < c^2\right) = Pr\left(-c < \sqrt{\frac{X_0^2}{W}} < c\right) =$

$Pr\left(-c < \frac{X_0}{\sqrt{W}} < c\right) = Pr(-c\sqrt{W} < X_0 < c\sqrt{W})$ and $\therefore$ a 90% prediction interval for X_0 is $(-c\sqrt{W}, c\sqrt{W})$.

an alternative sol uses $Pr(a^2 < T < b^2) = 0.9$ where a^2 & b^2 are the 0.05 & 0.95 quantiles of $F_{1,n}$ & leads to the prediction set $(-b\sqrt{W}, -a\sqrt{W}) \cup (a\sqrt{W}, b\sqrt{W})$ this is an undesirable set as it fails to contain zero but would still be satisfactory.

\3a/ The log-likelihood is $l(\sigma, \phi; \underline{x}, \underline{y}) =$

$\text{constant} - 2n\ln(\sigma) + n\ln(\phi) - \frac{S_x + \phi^2 S_y}{2\sigma^2}$ $\therefore$

$\frac{\partial l}{\partial \sigma} = l_\sigma = -\frac{2n}{\sigma} + \frac{S_x + \phi^2 S_y}{\sigma^3}$ $\therefore$ solve $l_\sigma = 0$ for σ:

$-\frac{2n}{\hat{\sigma}} + \frac{S_x + \phi^2 S_y}{\hat{\sigma}^3} = 0$ $\therefore$ $S_x + \phi^2 S_y = 2n\hat{\sigma}^2$ $\therefore$ $\frac{S_x + \phi^2 S_y}{2n} = \hat{\sigma}(\phi)^2$ $\therefore$

$l_{\sigma\sigma} = \frac{2n}{\sigma^2} - \frac{3(S_x + \phi^2 S_y)}{\sigma^4}$ $\therefore$ $l_{\sigma\sigma}(\hat{\sigma}(\phi)) = \frac{2n}{\hat{\sigma}(\phi)^2} - \frac{3(S_x + \phi^2 S_y)}{(\hat{\sigma}(\phi)^2)^2} =$

$\frac{2n}{\hat{\sigma}(\phi)^2} - \frac{3(S_x + \phi^2 S_y)}{\left(\frac{S_x + \phi^2 S_y}{2n}\right)^2} = \frac{2n}{\hat{\sigma}(\phi)^2} - 3 \cdot 2n \cdot \frac{2n}{S_x + \phi^2 S_y} = \frac{2n}{\hat{\sigma}(\phi)^2} - 6n\frac{1}{\hat{\sigma}(\phi)^2} =$

$-4n\frac{1}{\hat{\sigma}(\phi)^2} < 0$ $\therefore$ the profile log-likelihood is $\therefore$

$l_p(\phi; \underline{x}, \underline{y}) = l(\hat{\sigma}(\phi), \phi, \underline{x}, \underline{y}) =$ ~~constant - 2nln~~

$\text{constant} - 2n\ln(\hat{\sigma}(\phi)) + n\ln(\phi) - \frac{S_x + \phi^2 S_y}{2\hat{\sigma}(\phi)^2} =$