Week 1

$u_{\xi\xi}\xi_x^2 + u_\xi \xi_{xx} + 2u_{\xi\tau}\xi_x\tau_x + u_{\tau\tau}\tau_x^2 + u_\tau \tau_{xx} - (u_{\xi\xi}\xi_t^2 + u_\xi\xi_{tt} + 2u_{\xi\tau}\xi_t\tau_t + u_{\tau\tau}\tau_t^2 + u_\tau\tau_{tt}) =$

$a u_{xx} + 2b u_{xt} + c u_{tt} = \alpha u_{\xi\xi} + 2\beta u_{\xi\tau} + \gamma u_{\tau\tau} =$

$\xi_x^2 u_{\xi\xi} + 2\xi_x\tau_x u_{\xi\tau} + \tau_x^2 u_{\tau\tau} - \xi_t^2 u_{\xi\xi} - 2\xi_t\tau_t u_{\xi\tau} - \tau_t^2 u_{\tau\tau} + \xi_{xx}u_\xi + \tau_{xx}u_\tau - \xi_{tt}u_\xi - \tau_{tt}u_\tau$

$= (\xi_x^2 - \xi_t^2)u_{\xi\xi} + 2(\xi_x\tau_x - \xi_t\tau_t)u_{\xi\tau} + (\tau_x^2 - \tau_t^2)u_{\tau\tau} + \xi_{xx}u_\xi + \tau_{xx}u_\tau - \xi_{tt}u_\xi - \tau_{tt}u_\tau$

$= 1\frac{\partial^2 v}{\partial \xi^2} - 1\frac{\partial^2 v}{\partial \tau^2} = 1 v_{\xi\xi} - 1 v_{\tau\tau} = 0 = RHS \therefore$

$\xi_x^2 - \xi_t^2 = 1, \quad \xi_x\tau_x - \xi_t\tau_t = 0, \quad \tau_x^2 - \tau_t^2 = -1 \therefore$

$\xi = Ax + Bt \therefore \xi_x = A, \ \xi_t = B \qquad \tau = Cx + Dt \therefore \tau_x = C, \ \tau_t = D \therefore$

$A^2 - B^2 = 1, \ AC - BD = 0, \ \cancel{C^2 - D^2 = 0} \ C^2 - D^2 = -1 \therefore B > 0 \therefore$

$A^2 = \cosh^2\phi \quad \therefore \cosh^2\phi - \sinh^2\phi = 1 \quad \therefore \cosh^2\phi - 1 = \sinh^2\phi \therefore$

$A^2 - 1 = B^2 = \cosh^2\phi - 1 = \sinh^2\phi \quad \therefore B = \sinh\phi$

$C > 0, \ D > 0 \quad \therefore$ is $D = \cosh\psi \therefore D^2 = \cosh^2\psi \therefore D^2 - 1 = C^2 \therefore$

$D^2 - 1 = \cosh^2\psi - 1 = C^2 = \sinh^2\psi \quad \therefore C = \sinh\psi$ for some $\psi > 0 \therefore$

$AC - BD = 0 \quad \therefore AC = BD \therefore \frac{B}{A} = \frac{C}{D} = \frac{\sinh\phi}{\cosh\phi} = \frac{\sinh\psi}{\cosh\psi} = \cancel{\tanh\phi}$

$\tanh\phi = \tanh\psi \quad \therefore \phi = \psi \therefore$

$\cancel{A = C = \sinh\phi, \ B = D}$ $\quad A = D = \cosh\phi, \ B = C = \sinh\phi$

$\frac{\partial x}{\partial \rho} = \frac{\partial}{\partial \rho}(\rho + t\delta(\rho)) = \frac{\partial}{\partial\rho}(\rho) + \frac{\partial}{\partial\rho}(t\delta(\rho)) = 1 + \frac{\partial}{\partial\rho}(t)\delta(\rho) + t\frac{\partial}{\partial\rho}\delta(\rho) =$

$1 + 0 + t\delta'(\rho) = 1 + t\delta'(\rho) = \frac{\partial x}{\partial \rho} \quad \therefore \frac{\partial \rho}{\partial x} = \frac{1}{1 + t\delta'(\rho)}$

$\frac{\partial x}{\partial t} = \frac{\partial}{\partial t}(\rho + t\delta(\rho)) = 0 = \frac{\partial \rho}{\partial t} + \frac{\partial}{\partial t}(t\delta(\rho)) = \frac{\partial\rho}{\partial t} + \frac{\partial}{\partial t}(t)\delta(\rho) + t\frac{\partial}{\partial t}\delta(\rho) =$

$\frac{\partial\rho}{\partial t} + \delta(\rho) + \frac{\partial}{\partial t}\delta(\rho) = \cancel{\frac{\partial \rho}{\partial t}} \ \frac{\partial\rho}{\partial t} + \delta(\rho) + \frac{\partial}{\partial\rho}\delta(\rho)\frac{\partial\rho}{\partial t} = \frac{\partial\rho}{\partial t} + \delta(\rho) + \frac{\partial\rho}{\partial t}\delta'(\rho) = 0$

$= \frac{\partial\rho}{\partial t}(1 + \delta'(\rho)) + \delta(\rho) \quad \therefore \quad -\delta(\rho) = \frac{\partial}{\partial\rho}(1 + \delta'(\rho)) \quad \therefore \quad -\frac{\delta(\rho)}{1 + \delta'(\rho)} = \frac{\partial\rho}{\partial t}$

$\frac{\partial x}{\partial x} = \frac{\partial}{\partial x}(\rho + t\delta(\rho)) = \frac{\partial\rho}{\partial x} + \frac{\partial}{\partial x}(t\delta(\rho)) = 1 = \frac{\partial\rho}{\partial x} + \frac{\partial t}{\partial x}\delta(\rho) + t\frac{\partial}{\partial x}\delta(\rho) =$

$\frac{\partial\rho}{\partial x} + (0)\delta(\rho) + t\frac{\partial}{\partial\rho}\delta(\rho)\frac{\partial\rho}{\partial x} = \frac{\partial\rho}{\partial x} + t\delta'(\rho)\frac{\partial\rho}{\partial x} = 1 = (1 + t\delta'(\rho))\frac{\partial\rho}{\partial x} \therefore$

$\frac{\partial\rho}{\partial x} = \frac{1}{1 + t\delta'(\rho)}$

$u = \delta(\rho) \quad \therefore \quad u_t = \frac{\partial}{\partial t}\delta(\rho) \qquad u_x = \frac{\partial}{\partial x}\delta(\rho)$

$u_t = \frac{\partial}{\partial t}(\delta(\rho)) = \frac{\partial}{\partial\rho}(\delta(\rho))\frac{\partial\rho}{\partial t} = \delta'(\rho)\frac{-\delta(\rho)}{1 + t\delta'(\rho)}$

$u_x = \frac{\partial}{\partial x}(\delta(\rho)) = \frac{\partial}{\partial\rho}(\delta(\rho))\frac{\partial\rho}{\partial x} = \delta'(\rho)\frac{1}{1 + t\delta'(\rho)} \quad \therefore$

$LHS = u_t + uu_x = f'(p)\frac{-f(p)}{1+tf'(p)} + f(p)f'(p)\frac{1}{1+tf'(p)} =$

$(1-1)\frac{f(p)f'(p)}{1+tf'(p)} = 0 = RHS$

$u = u(x,t) = f(p(x,t)) \therefore$

$u(x,0) = u(x,t=0) = f(p(x,t=0)) = f(x) = f(p(x))$

$\frac{\partial u}{\partial t} = \frac{\partial f(x)}{\partial t} = 0 \quad , \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}f(x) = \frac{\partial}{\partial x}f(p(x)) = \frac{\partial}{\partial p}f(p(x))\Big|\frac{\partial p(x)}{\partial x} =$

$f'(p(x))\frac{\partial p(x)}{\partial x}\Big|_{t=0} = f'(p(x))\frac{\partial p(x,t=0)}{\partial x} = f'(p))\frac{1}{1+(0)f'(p)} = f'(p) = f'(p(x,0))$

$LHS = 0 + f(x)f'(p(x,0)) = f(x)f'(p(x,0))$

$u(x,t) = f(p(x,t)) \qquad \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = 0 \qquad u(x,0) = f(x)$

$u(x,0) = f(x) \quad \therefore \frac{\partial}{\partial t}u(x,0) = \frac{\partial}{\partial t}f(x) = 0$

$\cancel{\frac{\partial u}{\partial x}} \quad \frac{\partial}{\partial x}u(x,0) = \frac{\partial}{\partial x}f(x) = f'(x) \quad \therefore$

$LHS = 0 + f(x)f'(x) = \cancel{0}\, f(x)f'(x)$

$\frac{\partial u}{\partial t} = \frac{\partial}{\partial t}f(x) = \frac{\partial}{\partial t}f(p(x)) = \frac{\partial}{\partial p}f(p(x))\frac{\partial p(x)}{\partial t} = f'(p(x))\frac{-f(p)}{1+(0)f'(p)} = -f'(p)f(p)$

$\therefore \frac{\partial u}{\partial x} = \frac{\partial}{\partial x}f(x)f'(x) \therefore$

$LHS = -f'(p)f(p) + f(x)f'(x) = (1-1)f'(p)f(x) = 0$

Let $f(x) = A\tanh(x) \quad \therefore$

$u(x,0) = A\tanh(x) \therefore \qquad \frac{d}{dx}\tanh(x) = 1-\tanh^2(x)$

$u(x,0) = f(x) = A\tanh(x) \quad \therefore$

$\cancel{u_x(x,0) =} \quad \frac{\partial}{\partial x}[u(x,0)] = \frac{\partial}{\partial x}[A\tanh(x)] = A(1-\tanh^2(x)) \quad \&$

$-1 \leq \tanh(x) < 1 \quad \therefore \tanh^2(x) < 1 \therefore$

~~1-tanh~~ $\quad -1 < -\tanh^2(x) \quad \therefore \quad 0 < 1-\tanh^2(x) \quad \therefore$

$0 < A(1-\tanh^2(x))$ for $A > 0 \quad \therefore$ ~~if A~~ as $A \to \infty$:

~~1 tanh~~ $A(1-\tanh^2(x)) \to \infty \quad \therefore \quad u_x(x,t) \to \infty$ with $A \to \infty$ as

$x \to x_*, \; t \to t_* \qquad t_* \to \infty \qquad f(x_*) = A\tanh(x_*) \therefore$

$A = f(x_*)/\tanh(x_*)$

\Week3/ \1a/ $a(x,y)u_{xx}+2b(x,y)u_{xy}+c(x,y)u_{yy}=f(x,y)$

transforms the eqn using 2 coords $\xi=\xi(x,y)$ & $\eta=\eta(x,y)$ to form

$$u_x=\frac{\partial u}{\partial x}=\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x}+\frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x}=u_\xi \xi_x+u_\eta \eta_x$$

$u_y=u_\xi \xi_y+u_\eta \eta_y$

$u_{xx}=\frac{\partial}{\partial x}(u_x)=\cancel{\frac{\partial}{\partial x}} \frac{\partial}{\partial x}(u_\xi \xi_x+u_\eta \eta_x)=$

$\frac{\partial}{\partial x}(u_\xi \xi_x)+\frac{\partial}{\partial x}(u_\eta \eta_x)=\frac{\partial}{\partial x}(u_\xi)\xi_x+u_\xi\frac{\partial}{\partial x}(\xi_x)+\frac{\partial}{\partial x}(u_\eta)\eta_x+u_\eta\frac{\partial}{\partial x}(\eta_x)$

$=\frac{\partial}{\partial \xi}(u_\xi)\xi_x\xi_x+u_\xi \xi_{xx}+\frac{\partial}{\partial \eta}(u_\eta)\frac{\partial \eta}{\partial x}\eta_x+u_\eta\eta_{xx}=$

$u_{\xi\xi}\xi_x^2+u_\xi\xi_{xx}+u_{\eta\eta}\eta_x^2+u_\eta\eta_{xx}$

or

$u_{xx}=\frac{\partial}{\partial x}(u_x)=\frac{\partial}{\partial x}(u_\xi\xi_x+u_\eta\eta_x)=$

$\frac{\partial}{\partial x}(u_\xi\xi_x)+\frac{\partial}{\partial x}(u_\eta\eta_x)=\frac{\partial}{\partial x}(u_\xi)\xi_x+u_\xi\frac{\partial}{\partial x}(\xi_x)+\frac{\partial}{\partial x}(u_\eta)\eta_x+u_\eta\frac{\partial}{\partial x}(\eta_x)=$

$\left[\frac{\partial}{\partial \xi}(u_\xi)\xi_x+\frac{\partial}{\partial \eta}(u_\xi)\eta_x\right]\xi_x+u_\xi\xi_{xx}+\left[\frac{\partial}{\partial \xi}(u_\eta)\xi_x+\frac{\partial}{\partial \eta}(u_\eta)\eta_x\right]\eta_x+u_\eta\eta_{xx}=$

$\left[u_{\xi\xi}\xi_x+u_{\xi\eta}\eta_x\right]\xi_x+u_\xi\xi_{xx}+\left[u_{\xi\eta}\xi_x+u_{\eta\eta}\eta_x\right]\eta_x+u_\eta\eta_{xx}=$

$u_{\xi\xi}\xi_x^2+u_{\xi\eta}\xi_x\eta_x+u_\xi\xi_{xx}+u_{\xi\eta}\xi_x\eta_x+u_{\eta\eta}\eta_x^2+u_\eta\eta_{xx}=$

~~\1a 2nd/~~ $u_{\xi\xi}\xi_x^2+2u_{\xi\eta}\xi_x\eta_x+u_\xi\xi_{xx}+u_{\eta\eta}\eta_x^2+u_\eta\eta_{xx}$

\1a retry/ $(x,y)\to(\xi,\eta)$ $u_x=u_\xi\xi_x+u_\eta\eta_x$ $u_y=u_\xi\xi_y+u_\eta\eta_y$

$u_{xx}=\partial_x[u_\xi\xi_x+u_\eta\eta_x]=\partial_x[u_\xi]\xi_x+u_\xi\xi_{xx}+\partial_x[u_\eta]\eta_x+u_\eta\eta_{xx}=$

$[u_{\xi\xi}\xi_x+u_{\xi\eta}\eta_x]\xi_x+u_\xi\xi_{xx}+[u_{\eta\xi}\xi_x+u_{\eta\eta}\eta_x]\eta_x+u_\eta\eta_{xx}=$

$\xi_x^2u_{\xi\xi}+2\xi_x\eta_xu_{\xi\eta}+\eta_x^2u_{\eta\eta}+\xi_{xx}u_\xi+\eta_{xx}u_\eta$

$\{u_{yy}=\xi_y^2u_{\xi\xi}+2\xi_y\eta_yu_{\xi\eta}+\eta_y^2u_{\eta\eta}\}$

$u_{yy}=\xi_y^2u_{\xi\xi}+2\xi_y\eta_yu_{\xi\eta}+\eta_y^2u_{\eta\eta}+\xi_{yy}u_\xi+\eta_{yy}u_\eta$

$u_{xy}=\xi_x\xi_yu_{\xi\xi}+(\xi_x\eta_y+\xi_y\eta_x)u_{\xi\eta}+\eta_x$

Sub into ODE: $LHS=au_{xx}+2bu_{xy}+cu_{yy}=$

$a[\xi_x^2u_{\xi\xi}+2\xi_x\eta_xu_{\xi\eta}+\eta_xu_{\eta\eta}]+2b[\xi_x\xi_yu_{\xi\xi}+(\xi_x\eta_y+\xi_y\eta_x)u_{\xi\eta}+\eta_x\eta_yu_{\eta\eta}]+$
$c[\xi_y^2u_{\xi\xi}+2\xi_y\eta_yu_{\xi\eta}+\eta_y^2u_{\eta\eta}]+LOT=RHS=f$

collecting similar terms $\alpha u_{\xi\xi}+2\beta u_{\xi\eta}+\gamma u_{\eta\eta}=g$ where

~~$\alpha=a\xi_x^2$~~ $\alpha=a\xi_x^2+2b\xi_x\xi_y+c\xi_y^2$

$\beta=a\xi_x\eta_x+b(\xi_x\eta_y+\xi_y\eta_x)+c\xi_y\eta_y$

$\gamma = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 \qquad g = f \qquad -\text{L.O.T}$ (from Z RHS)

$a = a(x,y) \quad b = b(x,y) \quad c = c(x,y) \qquad (x,y) \to (\xi,\eta)$

$\therefore \alpha = \alpha(\xi,\eta), \ \beta = \beta(\xi,\eta), \ \gamma = \gamma(\xi,\eta)$

\1a Sol/ $u_x = u_\xi \xi_x + u_\eta \eta_x \qquad u_y = u_\xi \xi_y + u_\eta \eta_y$

$u_{xx} = \frac{\partial}{\partial x}(u_\xi \xi_x + u_\eta \eta_x) =$

$u_{\xi\xi}\xi_x^2 + 2u_{\xi\eta}\xi_x\eta_x + u_{\eta\eta}\eta_x^2 + u_\xi \xi_{xx} + u_\eta \eta_{xx}$

$u_{yy} = \partial_y(u_\xi \xi_y + u_\eta \eta_y) =$

$u_{\xi\xi}\xi_y^2 + 2u_{\xi\eta}\xi_y\eta_y + u_{\eta\eta}\eta_y^2 + u_\xi \xi_{yy} + u_\eta \eta_{yy}$

$u_{xy} = \partial_y(u_\xi \xi_x + u_\eta \eta_x) =$

$u_{\xi\xi}\xi_x\xi_y + u_{\xi\eta}(\xi_x\eta_y + \xi_y\eta_x) + u_{\eta\eta}\eta_x\eta_y + u_\xi \xi_{xy} + u_\eta \eta_{xy}$

inserting into Z PDE leads to:

$\alpha u_{\xi\xi} + 2\beta u_{\xi\eta} + \gamma u_{\eta\eta} + A u_\xi + B u_\eta = F$ where:

$\alpha = a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2, \quad \beta = a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y,$

~~$\gamma = a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy}$~~ $\quad B = a\eta_{xx} + 2b\eta_{xy} + c\eta_{yy}$

$\gamma = a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2 \quad A = a\xi_{xx} + 2b\xi_{xy} + c\xi_{yy}, \quad F = g$

\1b/ $\beta^2 - \alpha\gamma = (b^2 - ac)J^2 \Rightarrow \text{sign}(\beta^2 - \alpha\gamma) = \text{sign}(b^2 - ac)$

$\because J = \frac{\partial(\xi,\eta)}{\partial(x,y)} = \begin{vmatrix} \xi_x & \xi_y \\ \eta_x & \eta_y \end{vmatrix} = \xi_x\eta_y - \xi_y\eta_x$ $\therefore J \neq 0 \quad \text{sign}(\beta^2 - \alpha\gamma) = \text{sign}(b^2 - ac)$

$\because \beta^2 - \alpha\gamma = (b^2 - ac)J^2 \ \therefore$

$J^2 = \xi_x^2\eta_y^2 - 2\xi_x\xi_y\eta_x\eta_y + \xi_y^2\eta_x^2 \quad (>0) \qquad \beta^2 - \alpha\gamma =$

$[a\xi_x\eta_x + b(\xi_x\eta_y + \xi_y\eta_x) + c\xi_y\eta_y]^2 - [a\xi_x^2 + 2b\xi_x\xi_y + c\xi_y^2][a\eta_x^2 + 2b\eta_x\eta_y + c\eta_y^2] =$

$a^2[\xi_x^2\eta_x^2 - \xi_x^2\eta_x^2] + c^2[\xi_y^2\eta_y^2 - \xi_y^2\eta_y^2] + b^2[\xi_x^2\eta_y^2 + 2\xi_x\xi_y\eta_x\eta_y + \xi_y^2\eta_x^2 - 4\xi_x\xi_y\eta_x\eta_y] +$

$ab[2\xi_x^2\eta_x\eta_y + 2\xi_x\xi_y\eta_x^2 - 2\xi_x^2\eta_x\eta_y - 2\xi_x\xi_y\eta_x^2] +$

$bc[2\xi_y^2\eta_y\eta_x + 2\xi_y\xi_x\eta_y^2 - 2\xi_y^2\eta_y\eta_x - 2\xi_y\xi_x\eta_y^2] + ac[2\xi_x\xi_y\eta_x\eta_y - \xi_x^2\eta_y^2 - \xi_y^2\eta_x^2] =$

$b^2(\xi_x^2\eta_y^2 - 2\xi_x\xi_y\eta_x\eta_y + \xi_y^2\eta_x^2 + \xi_y^2\eta_x^2) - ac(\xi_x^2\eta_y^2 - 2\xi_x\xi_y\eta_x\eta_y + \xi_y^2\eta_x^2) = (b^2 - ac)J^2$

\1b Sol/ using Z expressions for α, β, γ find

$\beta^2 - \alpha\gamma = (b^2 - ac)\left(\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x}\right)^2 = (b^2 - ac)(\xi_x\eta_y - \xi_y\eta_x)^2 \ \therefore$

Z sign of $(\beta^2 - \alpha\gamma)$ is Z same as that of $(b^2 - ac)$ as long as $\frac{\partial\xi}{\partial x}\frac{\partial\eta}{\partial y} - \frac{\partial\xi}{\partial y}\frac{\partial\eta}{\partial x} \neq 0$

\Week 3/ \1e/ is $a(x,y)u_{xx}+2b(x,y)u_{xy}+c(x,y)u_{yy}=f(x,y,u,u_x,u_y)$ ∴ $(x,y)\mapsto(\xi,\eta)$ ~~st~~ ∴ is $\xi(x,y)=\text{Const}$

- $\eta(x,y)=\text{Const}$

$\nabla\xi \nparallel \nabla\eta \Leftrightarrow \frac{\xi_x}{\xi_y} \neq \frac{\eta_x}{\eta_y} \Leftrightarrow \det\left[\frac{\partial(\xi,\eta)}{\partial(x,y)}\right] \neq 0$

\1c sol/ is Z relations $\xi=\xi(x,y)=\text{const}$ implicitly defines Z

func $y=y(x)$ ∴ $\xi_x+\xi_y\frac{dy}{dx}=0$ or $\frac{dy}{dx}=-\frac{\partial\xi}{\partial x}\Big/\frac{\partial\xi}{\partial y}$

sub above into $a\left(\frac{dy}{dx}\right)^2-2b\frac{dy}{dx}+c=0$ ∴

$a(-\xi_x/\xi_y)^2-2b(-\xi_x/\xi_y)+c=0$ or

$a(\xi_x)^2+2b\xi_x\xi_y+c\xi_y^2=0$ ie $\alpha=0$ Similarly, $\gamma=0$

\2/ $au_{xx}+2xu_{xy}+yu_{yy}=-yu_x-xu_y-xyu=0$

$a=1$ is $b=2x$ is $c=y$ ∴

$b^2-ac=(2x)^2-1y=4x^2-y$

∴ $4x^2-y>0$ is $4x^2>y$ ∴ $(2x)^2>y$ ∴

$2x<-y^{1/2}$ ~~or~~ or $2x>+y^{1/2}$ ∴ is hyperbolic is $x>\frac{1}{2}y^{1/2}$, $x<-\frac{1}{2}y^{1/2}$

$b^2-ac=4x^2-y=0$ ∴ $4x^2=y$ ∴ $x^2=\frac{1}{4}y$ ∴ $x=\pm\frac{1}{2}y^{1/2}$ is parabolic

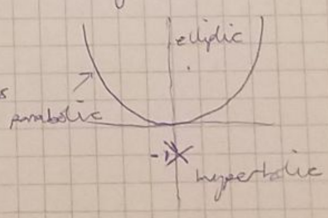
$4x^2-y<0$ ∴ $4x^2<y$ ∴ $x^2<\frac{1}{4}y$ ∴ $-\frac{1}{2}y^{1/2}<x<\frac{1}{2}y^{1/2}$ is elliptic

\2/ Z type of PDE $au_{xx}+2bu_{xy}+cu_{yy}=f(x,y,u,u_x,u_y)$ is defined by Z discriminant $D=b^2-ac$ which for Z given eqn is $D=x^2-y$ ∴ Z PDE is parabolic on Z parabola $y=x^2$, elliptic above this parabola ie $y>x^2$, & hyperbolic below this parabola ie $y<x^2$

Z charac eqn is $a\left(\frac{dy}{dx}\right)^2-2b\left(\frac{dy}{dx}\right)+c=0$ ∴

which for our PDE is $\left(\frac{dy}{dx}\right)^2-2x\left(\frac{dy}{dx}\right)+y=0$ that gives

$\frac{dy}{dx}=x\pm\sqrt{x^2-y}$



- subing $x=0$, $y=-1$ into Z two expressions for dy/dx gives slopes ±1

\3a/ $u_{xx} - 3u_{xt} - 4u_{tt} = 0$ if $u(x,t) = f(t + \alpha x)$ f arb func

$u_x = \alpha f'(t + \alpha x)$ $\therefore u_{xx} = \alpha^2 f''(t + \alpha x)$

$u_t = f'(t + \alpha x)$ $\therefore u_{tt} = f''(t + \alpha x)$, $u_{xt} = \alpha f''(t + \alpha x)$ $\therefore$

$LHS = \alpha^2 f'' - 3\alpha f'' - 4f'' = (\alpha^2 - 3\alpha - 4) f''$ $\therefore (\alpha^2 - 3\alpha - 4) = 0 =$

$(\alpha - 4)(\alpha + 1) = 0$ $\therefore \alpha = 4, \alpha = -1$ $\therefore$

$u(x,t) = f_1(t + 4x) + f_2(t - x)$

\3a sol/ $u(x,t) = f(x + \alpha t)$ $\therefore$

$(1 - 3\alpha - 4\alpha^2) f''(x + \alpha t) = (1 + \alpha)(1 - 4\alpha) f'(x + \alpha t) = 0$ for arbit f if

$\alpha = \alpha_1 = -1$, $\alpha = 1/4$ $\therefore$ GS: $u(x,t) = f_1(x - t) + f_2(4x + t)$

f_1, f_2 arbit funcs (other sols methods are acceptable)

\3b/ $u(x,0) = u(x, t=0) = f_1(0 + 4x) + f_2(0 - x) =$

$f_1(4x) + f_2(-x) = x^2$

$\frac{\partial u}{\partial t}(x,0) = \frac{\partial u}{\partial t}(x, t=0) = f_1'(t + 4x) + f_2'(t - x)\Big|_{t=0} =$

$f_1'(0 + 4x) + f_2'(0 - x) = f_1'(4x) + f_2'(-x) = e^x$

\3b sol/ 2 time derivative of 2 GS is

$u_t(x,y) = -f_1'(x - t) + f_2'(4x + t)$ $\therefore$ applying IC at $t = 0$:

$f_1(x) + f_2(4x) = 4x^2$, $-f_1'(x) + f_2'(4x) = e^x$ or

$f_1'(x) + 4f_2'(4x) = 2x$, $-f_1'(x) + f_2'(4x) = e^x$ $\therefore$ solving 2 above system

gives: $f_1'(x) = \frac{2x}{5} - 4e^x/5$ $f_2'(4x) = \frac{2x}{5} + e^x/5$ $\therefore$

2 2nd eqn can be written as $f_2'(s) = \frac{2}{20}s + \frac{1}{5}e^{s/4}$ $\therefore$

integrating then yields $f_1(x) = \frac{x^2}{5} - \frac{4}{5}e^x$, $f_2(s) = \frac{s^2}{20} + \frac{4}{5}e^{s/4}$

(or by elimination) 2 sol is $\therefore$

$$u(x,t) = \frac{(x-t)^2}{5} - \frac{4}{5}e^{x-t} + \frac{(4x+t)^2}{20} + \frac{4}{5}e^{(4x+t)/4}$$

\3b retry/ if $u(x,t) = f_1(t + 4x) + f_2(t - x)$ $\therefore$

$u(x,0) = f_1(4x) + f_2(-x) = x^2$ $\frac{\partial u}{\partial t}(x,t) = f_1'(t + 4x) + f_2'(t - x)$ $\therefore$

$\frac{\partial u}{\partial t}(x,0) = f_1'(4x) + f_2'(-x) = e^x$

$\frac{\partial}{\partial x}[f_1(4x) + f_2(-x)] = \frac{\partial}{\partial x}(x^2) = 4f_1'(4x) - f_2'(-x) = 2x$ $\therefore$

$4f_1'(4x) - 2x = f_2'(-x)$ $\therefore f_1'(4x) + 4f_1'(4x) - 2x = e^x$ $\therefore 5f_1'(4x) = e^x + 2x$ $\therefore$

$f_1'(4x) = \frac{1}{5}e^x + \frac{2}{5}x$ $\therefore f_2'(-x) = 4\left[\frac{1}{5}e^x + \frac{2}{5}x\right] - 2x = \frac{4}{5}e^x + \frac{8}{5}x - 2x = \frac{4}{5}e^x - \frac{2}{5}x$ $\therefore$

Week 3 Sheet / $g_2'(x) = \frac{4}{5}e^{-x} - \frac{2}{5}(-x) = \frac{4}{5}e^{-x} + \frac{2}{5}x$

$\therefore \; g_1'(x) = \frac{1}{5}e^{x/4} + \frac{2}{5}\left(\frac{x}{4}\right) = \frac{1}{5}e^{x/4} + \frac{1}{10}x \; \therefore$

$g_1(x) = \int \frac{1}{5}e^{x/4} + \frac{1}{10}x \, dx = \frac{4}{5}e^{x/4} + \frac{1}{20}x^2$

$g_2(x) = \int \frac{4}{5}e^{-x} + \frac{2}{5}x \, dx = -\frac{4}{5}e^{-x} + \frac{1}{5}x^2 \; \therefore$

$g_1(t+4x) = \frac{4}{5}e^{\frac{1}{4}(t+4x)} + \frac{1}{20}(t+4x)^2$

$g_2(t-x) = -\frac{4}{5}e^{-(t-x)} + \frac{1}{5}(t-x)^2 \; \therefore$

$u(x,t) = \frac{4}{5}e^{\frac{1}{4}(t+4x)} + \frac{1}{20}(t+4x)^2 - \frac{4}{5}e^{-(t-x)} + \frac{1}{5}(t-x)^2$

4a. $-u_{xx} - 2 \cdot 2u_{xy} - u_{yy} = -2\sqrt{3}\,u_y - u \; \therefore$

$a = -1, \; b = -2, \; c = -1 \; \therefore$

$b^2 - 4ac = (-2)^2 - 4(-1)(-1) = 4 - 4 = 0$ $\therefore$ curve parabolic

$\therefore$ charac eqn: $a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + (c) = 0 \; \therefore$

$-1\left(\frac{dy}{dx}\right)^2 + 4\frac{dy}{dx} - 1 = 0 \; \therefore$

$\frac{dy}{dx} = \frac{-4 \pm \sqrt{16 - 4(-1)(-1)}}{2(-1)} = \frac{-4}{-2} \pm \frac{\sqrt{12}}{-2} = 2 \mp \sqrt{3} = 2 \mp 3^{1/2} \; \therefore$

$y = (2 \mp 3^{1/2})x + C \; \therefore \; y_1 = (2 - 3^{1/2})x + C, \; y_2 = (2 + 3^{1/2})x + C \; \therefore$

$y_1 - (2 - \sqrt{3})x = C, \; y_2 - (2 + \sqrt{3})x = C \; \therefore$

$\xi = y + (\sqrt{3} - 2)x, \; \eta = y - (\sqrt{3} + 2)x$

4a. Sol discriminant is: $b^2 - ac = (-2)^2 - (-1)(-1) = 3 > 0$ so Z eqn is hyperbolic $\therefore$ Z charac eqn is $-\left(\frac{dy}{dx}\right)^2 + 4\frac{dy}{dx} - 1 = 0$ which has sol $\left(\frac{dy}{dx}\right) = 2 - \sqrt{3}$; $\left(\frac{dy}{dx}\right) = 2 + \sqrt{3}$ these yield charac coords $\xi = y + (\sqrt{3} - 2)x$, $\eta = y - (\sqrt{3} + 2)x$

4b. Charac eqn $a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0$

$\therefore \; -1\left(\frac{dy}{dx}\right)^2 + 4\frac{dy}{dx} - 1 = 0 \; \therefore$

$(x,y) \mapsto (\xi, \eta)$ $\quad u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi(1) + u_\eta(1) = u_\xi + u_\eta$

$u_{yy} = \partial_y(u_\xi + u_\eta) = \partial_y u_\xi + \partial_y u_\eta =$

$u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y + u_{\xi\eta}\xi_y + u_{\eta\eta}\eta_y = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$\xi_x = (\sqrt{3}-2) \qquad \eta_x = (-\sqrt{3}-2)$

$u_x = u_\xi \xi_x + u_\eta \eta_x = (\sqrt{3}-2)u_\xi + (-\sqrt{3}-2)u_\eta$

$u_{xx} = \partial_x((\sqrt{3}-2)u_\xi + (-\sqrt{3}-2)u_\eta) = \partial_x((\sqrt{3}-2)u_\xi) + \partial_x((-\sqrt{3}-2)u_\eta) =$

$(\sqrt{3}-2)\partial_x u_\xi + (-\sqrt{3}-2)\partial_x u_\eta =$

$(\sqrt{3}-2)[u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x] + (-\sqrt{3}-2)[u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x] =$

$(\sqrt{3}-2)[(\sqrt{3}-2)u_{\xi\xi} + (-\sqrt{3}-2)u_{\xi\eta}] + (-\sqrt{3}-2)[(\sqrt{3}-2)u_{\xi\eta} + (-\sqrt{3}-2)u_{\eta\eta}] =$

$(\sqrt{3}-2)^2 u_{\xi\xi} + (\sqrt{3}-2)(-\sqrt{3}-2)u_{\xi\eta} + (\sqrt{3}-2)(-\sqrt{3}-2)u_{\xi\eta} + (-\sqrt{3}-2)^2 u_{\eta\eta} =$

$(\sqrt{3}-2)^2 u_{\xi\xi} + 2(\sqrt{3}-2)(-\sqrt{3}-2)u_{\xi\eta} + (-\sqrt{3}-2)^2 u_{\eta\eta}$

$u_{xy} = \partial_x u_y = \partial_x(u_\xi + u_\eta) = \partial_x u_\xi + \partial_x u_\eta =$

$u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x =$

$(\sqrt{3}-2)u_{\xi\xi} + (-\sqrt{3}-2)u_{\xi\eta} + (\sqrt{3}-2)u_{\xi\eta} + (-\sqrt{3}-2)u_{\eta\eta} =$

$(\sqrt{3}-2)u_{\xi\xi} - 4u_{\xi\eta} + (-\sqrt{3}-2)u_{\eta\eta}$ $\therefore$ into PDE: LHS $=$

$-(\sqrt{3}-2)^2 u_{\xi\xi} + 2(\sqrt{3}-2)(-\sqrt{3}-2)u_{\xi\eta} + (-\sqrt{3}-2)^2 u_{\eta\eta} - 4(\sqrt{3}-2)u_{\xi\xi} + 16u_{\xi\eta} - 4(-\sqrt{3}-2)u_{\eta\eta} -$

$u_{\xi\xi} - 2u_{\xi\eta} - u_{\eta\eta} + (2\sqrt{3})u_\xi + 2\sqrt{3}u_\eta - u =$

$[-(\sqrt{3}-2)^2 - 4(\sqrt{3}-2) + 1]u_{\xi\xi} + [2(\sqrt{3}-2)(-\sqrt{3}-2) + 16 - 2]u_{\xi\eta} + [(-\sqrt{3}-2)^2 - 4(-\sqrt{3}-2) - 1]u_{\eta\eta} +$

$(2\sqrt{3})u_\xi + 2\sqrt{3}u_\eta - u = 0 = $ RHS $\therefore$

4b Sol ~~use~~ by 2 chain rule:

$u_x = (\sqrt{3}-2)u_\xi - (\sqrt{3}+2)u_\eta$

$u_{xy} = (\sqrt{3}-2)u_{\xi\xi} - 4u_{\xi\eta} - (2+\sqrt{3})u_{\eta\eta}$

$u_{xx} = (\sqrt{3}-2)^2 u_{\xi\xi} + (2+\sqrt{3})^2 u_{\eta\eta} - 2(\sqrt{3}-2)(2+\sqrt{3})u_{\xi\eta}$

$u_y = u_\xi + u_\eta \qquad u_{yy} = u_{\xi\xi} + u_{\eta\eta} + 2u_{\xi\eta}$ $\therefore$ sub into eqn yields after simplification $-12u_{\xi\eta} = 2\sqrt{3}(u_\xi + u_\eta) + u$

4c $-12u_{\xi\eta} = 2\sqrt{3}(u_\xi + u_\eta) + u$

$u = e^{\lambda\xi+\mu\eta}v \quad \therefore \quad u_\xi = \partial_\xi(e^{\lambda\xi+\mu\eta}v) = v\lambda e^{\lambda\xi+\mu\eta} + e^{\lambda\xi+\mu\eta}v_\xi$

$u_\eta = \partial_\eta(e^{\lambda\xi+\mu\eta}v) = \mu e^{\lambda\xi+\mu\eta}v + e^{\lambda\xi+\mu\eta}v_\eta$

$u_{\xi\eta} = \partial_\xi(\mu e^{\lambda\xi+\mu\eta}v + e^{\lambda\xi+\mu\eta}v_\eta) = \partial_\xi(\mu e^{\lambda\xi+\mu\eta}v) + \partial_\xi(e^{\lambda\xi+\mu\eta}v_\eta) =$

$\mu\lambda e^{\lambda\xi+\mu\eta}v + \mu e^{\lambda\xi+\mu\eta}v_\xi + \lambda e^{\lambda\xi+\mu\eta}v_\eta + e^{\lambda\xi+\mu\eta}v_{\eta\xi} \therefore$

Week 3 Sheet / ~~$\frac{1}{2}[\mu\lambda e^{\lambda\xi+\mu\eta}]$~~ into PDE: LHS =

$-\sqrt{2}\left[\mu\lambda e^{\lambda\xi+\mu\eta} v + \mu e^{\lambda\xi+\mu\eta} v_\xi + \lambda e^{\lambda\xi+\mu\eta} v_\eta + e^{\lambda\xi+\mu\eta} v_{\xi\eta}\right] =$

$2\sqrt{3}\left[\lambda e^{\lambda\xi+\mu\eta} + e^{\lambda\xi+\mu\eta} v_{\xi} + \mu e^{\lambda\xi+\mu\eta} v + e^{\lambda\xi+\mu\eta} v_\eta\right] + e^{\lambda\xi+\mu\eta} v$

4c sol / by using the suggested sub, can eliminate the first order derivatives like v_η; note: $e^{\lambda\xi+\mu\eta} v$ is

$u_\xi = e^{\lambda\xi+\mu\eta}(v_\xi + \lambda v) \qquad u_\eta = e^{\lambda\xi+\mu\eta}(v_\eta + \mu v)$

$u_{\xi\eta} = e^{\lambda\xi+\mu\eta}(v_{\xi\eta} + \mu v_\xi + \lambda v_\eta + \mu\lambda v)$

$\{u_{\xi\eta} = \partial_\eta(e^{\lambda\xi+\mu\eta}(v_\xi + \lambda v)) =$

$(v_\xi + \lambda v)\partial_\eta[e^{\lambda\xi+\mu\eta}] + e^{\lambda\xi+\mu\eta}\partial_\eta(v_\xi + \lambda v) =$

$(v_\xi + \lambda v)\mu e^{\lambda\xi+\mu\eta} + e^{\lambda\xi+\mu\eta} v_{\xi\eta} + \lambda v_\eta =$

$e^{\lambda\xi+\mu\eta}(v_{\xi\eta} + \mu v_\xi + \lambda v_\eta + \mu\lambda v)$

$u_{\xi\eta} = \partial_\xi[\mu e^{\lambda\xi+\mu\eta} v + e^{\lambda\xi+\mu\eta} v_\eta] =$

$\partial_\xi(\mu e^{\lambda\xi+\mu\eta} v) + \partial_\xi(e^{\lambda\xi+\mu\eta} v_\eta) =$

$\mu\lambda e^{\lambda\xi+\mu\eta} v + \mu e^{\lambda\xi+\mu\eta} v_\xi + \lambda e^{\lambda\xi+\mu\eta} v_\eta + e^{\lambda\xi+\mu\eta} v_{\xi\eta} =$

$e^{\lambda\xi+\mu\eta}(v_{\xi\eta} + \mu v_\xi + \lambda v_\eta + \mu\lambda v)\}$ inserting these into the eqn gives:

$-\sqrt{2}(u_{\xi\eta} + \mu v_\xi + \lambda v_\eta + \mu\lambda v) = 2\sqrt{3}[v_\xi + v_\eta + (\lambda+\mu)v] + v$

if we take $\lambda = \mu = -\sqrt{3}/6$ then $v_{\xi\eta} = 0$ which can be integrated

wrt ξ & η giving rise to the GS of the form

$u(x,y) = e^{-\sqrt{3}(y-2x)/3}\{f_1(y+(\sqrt{3}-2)x) + f_2(y-(\sqrt{3}+2)x)\}$ where f_1 & f_2 are two arbitrary funcs of one variable

4c relook / $-\sqrt{2}[\mu\lambda v + \mu v_\xi + \lambda v_\eta + v_{\xi\eta}] =$

$2\sqrt{3}(v\lambda + v_\xi + \mu v + v_\eta) + v$ ~~$\frac{1}{\sqrt{2}} 2\sqrt{3}[\lambda v]$~~ $\therefore$

$\mu\lambda v + \mu v_\xi + \lambda v_\eta + v_{\xi\eta} =$ ~~$\frac{1}{\sqrt{2}}$~~

$-\frac{1}{\sqrt{2}}2\sqrt{3}\lambda v - \frac{1}{\sqrt{2}}2\sqrt{3}v_\xi - \frac{1}{\sqrt{2}}2\sqrt{3}\mu v - \frac{1}{\sqrt{2}}2\sqrt{3}v_\eta - \frac{1}{\sqrt{2}}v \;\therefore$

$v_{\xi\eta} = \left[-\mu\lambda - \frac{1}{\sqrt{2}}2\sqrt{3}\lambda - \frac{1}{\sqrt{2}}2\sqrt{3}\mu - \frac{1}{\sqrt{2}}\right]v + \left[-\mu - \frac{1}{\sqrt{2}}2\sqrt{3}\right]v_\xi + \left[-\lambda - \frac{1}{\sqrt{2}}2\sqrt{3}\right]v_\eta = 0 = \text{RHS}$

$-\mu - \frac{1}{\sqrt{2}}2\sqrt{3} = 0 \;\therefore\; \mu = -\sqrt{3}/6$, $\quad -\lambda - \frac{1}{\sqrt{2}}2\sqrt{3} = 0 \;\therefore\; \lambda = -\sqrt{3}/6$

$\therefore \frac{\partial}{\partial\xi}\left(\frac{\partial}{\partial\eta}v\right) = 0 \;\therefore\; \frac{\partial}{\partial\eta}v = F(\eta) \;\therefore\; v = f_1(\xi) + f_2(\eta) = v(\xi,\eta) \;\therefore$

$u(x,y) = e^{-\frac{1}{6}\sqrt{3}[y+(\sqrt{3}-2)x] - \frac{1}{6}\sqrt{3}[y-(\sqrt{3}+2)x]}\{f_1(y+(\sqrt{3}-2)x) + f_2(y-(\sqrt{3}+2)x)\}$ f_1, f_2 arbit

$= u(x,y) = e^{-\sqrt{3}(y-2x)/3}\{ f_1(y+(\sqrt{3}-2)x) + f_2(y-(\sqrt{3}+2)x)\}$ f_1 & f_2 arbit

\5a/ $x^2 u_{xx} - u_{yy} = 6e^{2y} - 2xu_x + u_y = f \quad (x>0) \quad \therefore$

$a u_{xx} + 2b u_{xy} + c u_{yy} = x^2 u_{xx} - u_{yy} \quad \therefore$

$a = x^2,\ b = 0,\ c = -1 \quad \therefore \quad b^2 - ac = 0^2 - x^2(-1) = x^2 \quad (x>0) \quad \therefore$

$x^2 > 0$ ∴ hyperbolic ∴ Charac eqn:

$a\left(\frac{dy}{dx}\right)^2 - 2b\frac{dy}{dx} + c = 0 = x^2\left(\frac{dy}{dx}\right)^2 - 1 \quad \therefore \quad 1 = x^2\left(\frac{dy}{dx}\right)^2 \quad \therefore$

$\frac{1}{x^2} = \left(\frac{dy}{dx}\right)^2 \quad \therefore \quad \pm\frac{1}{x} = \frac{dy}{dx} \quad \therefore \quad \frac{dy}{dx} = \frac{1}{x},\ \frac{dy}{dx} = -\frac{1}{x} \quad \therefore$

$y = \ln|x| + C,\ -\ln|x| + C = y = \ln|x^{-1}| + C \quad (x>0) \quad \therefore$

$y = \ln(x) + C,\ y = \ln\left(\frac{1}{x}\right) + C \quad \therefore$

$C = y - \ln(x),\ C = y - \ln\left(\frac{1}{x}\right) \quad \therefore$

$\xi = y - \ln(x),\ \eta = y - \ln\left(\frac{1}{x}\right) = y - \ln(x^{-1}) = y + \ln(x)$

are 2 the charac coords

\5b/ $\xi_y = 1,\ \eta_y = 1,\ \xi_x = -\frac{1}{x},\ \eta_x = \frac{1}{x} \quad \therefore$

$u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi + u_\eta \quad \therefore$

$u_{yy} = u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y + u_{\xi\eta}\xi_y + u_{\eta\eta}\eta_y = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$u_{xy} = \partial_x(u_\xi + u_\eta) = \partial_x u_\xi + \partial_x u_\eta$

$u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x =$

$-\frac{1}{x}u_{\xi\xi} + \frac{1}{x}u_{\xi\eta} - \frac{1}{x}u_{\xi\eta} + \frac{1}{x}u_{\eta\eta} = -\frac{1}{x}u_{\xi\xi} + \frac{1}{x}u_{\eta\eta}$

$u_x = u_\xi \xi_x + u_\eta \eta_x = -\frac{1}{x}u_\xi + \frac{1}{x}u_\eta$

$u_{xx} = \partial_x\left(-\frac{1}{x}u_\xi + \frac{1}{x}u_\eta\right) = \partial_x(-x^{-1}u_\xi + x^{-1}u_\eta) =$

$\partial_x(-x^{-1}u_\xi) + \partial_x(x^{-1}u_\eta) =$

$x^{-2}u_\xi - x^{-1}\partial_x u_\xi - x^{-2}u_\eta + x^{-1}\partial_x u_\eta =$

$x^{-2}u_\xi - x^{-1}[u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x] - x^{-2}u_\eta + x^{-1}[u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x] =$

$x^{-2}u_\xi - x^{-1}\left[-\frac{1}{x}u_{\xi\xi} + \frac{1}{x}u_{\xi\eta}\right] - x^{-2}u_\eta + x^{-1}\left[-\frac{1}{x}u_{\xi\eta} + \frac{1}{x}u_{\eta\eta}\right] =$

$\frac{1}{x^2}u_\xi + \frac{1}{x^2}u_{\xi\xi} - \frac{1}{x^2}u_{\xi\eta} - \frac{1}{x^2}u_\eta - \frac{1}{x^2}u_{\xi\eta} + \frac{1}{x^2}u_{\eta\eta} =$

$\frac{1}{x^2}u_\xi + \frac{1}{x^2}u_{\xi\xi} - \frac{2}{x^2}u_{\xi\eta} - \frac{1}{x^2}u_\eta + \frac{1}{x^2}u_{\eta\eta}$ ∴ into PDE: LHS =

$x^2\left[\frac{1}{x^2}u_\xi + \frac{1}{x^2}u_{\xi\xi} - \frac{2}{x^2}u_{\xi\eta} - \frac{1}{x^2}u_\eta + \frac{1}{x^2}u_{\eta\eta}\right] - [u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}] +$

$2x\left[-\frac{1}{x}u_\xi + \frac{1}{x}u_\eta\right] - u_\xi - u_\eta =$

Week 3 Sheet

$u_\xi + u_{\xi\xi} - 2u_{\xi\eta} - u_\eta + u_{\eta\eta} - u_{\xi\xi} - 2u_{\xi\eta} - \cancel{u_{\eta\eta}} - 2u_\xi + 2u_\eta - \cancel{u_\xi} - \cancel{u_\eta} =$

$-4u_{\xi\eta} - 2u_\xi = 6e^{2(\frac{1}{2}(\xi+\eta))} = 6e^{\xi+\eta} = RHS \therefore$

$-2u_\xi - 6e^{\xi+\eta} = 4u_{\xi\eta} \therefore$

$-\frac{1}{2}u_\xi - \frac{3}{2}e^{\xi+\eta} = u_{\xi\eta} \therefore$

$\int u_{\xi\eta}\,\partial\xi = \int -\frac{1}{2}u_\xi - \frac{3}{2}e^{\xi+\eta}\,\partial\xi = -\frac{1}{2}\int u_\xi\,\partial\xi - \frac{3}{2}\int e^{\xi+\eta}\,\partial\xi =$

$u_\eta = -\frac{1}{2}u - \frac{3}{2}e^{\xi+\eta} \therefore$

$\int u_\eta\,\partial\eta = \int -\frac{1}{2}u - \frac{3}{2}e^{\xi+\eta}\,\partial\eta = -\frac{1}{2}\int u\,\partial\eta - \frac{3}{2}\int e^{\xi+\eta}\,\partial\eta =$

$u_{\xi\eta} = -\frac{1}{2}u_\xi - \frac{3}{2}e^{\xi+\eta} = 2\,\frac{1}{2}u_{\xi\eta} = f(\xi, \eta, u_\xi) \therefore$

$a = 0,\ b = \frac{1}{2},\ c = 0 \therefore b^2 - ac = \frac{1}{4} \therefore$

$a\left(\frac{d\xi}{d\eta}\right)^2 - 2b\frac{d\xi}{d\eta} + c = 0 = -2\frac{d\xi}{d\eta} = 0 \therefore$

$\frac{d\xi}{d\eta} = 0 \quad \therefore \xi = f(\xi) \qquad 1 \times \xi = \xi$

5b sol we have $\xi_x = 1/x$, $\xi_y = 1$, $\eta_x = -\frac{1}{x}$, $\eta_y = 1$ & $\therefore$ by 2 chain rule, $u_x = \xi_x u_\xi + \eta_x u_\eta = \frac{1}{x}u_\xi - \frac{1}{x}u_\eta$,

$u_y = \xi_y u_\xi + \eta_y u_\eta = u_\xi + u_\eta$,

$u_{xx} = -\frac{1}{x^2}u_\xi + \frac{1}{x^2}u_\eta + \frac{1}{x}\left(\frac{1}{x}\partial\xi - \frac{1}{x}\partial\eta\right)u_\xi - \frac{1}{x}\left(\frac{1}{x}\partial\xi - \frac{1}{x}\partial\eta\right)u_\eta =$

$\frac{1}{x^2}(u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} - u_\xi + u_\eta)$

$u_{yy} = (\partial\xi + \partial\eta)(u_\xi + u_\eta) = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$ upon subing these into 2 eqn, we get $x^2 u_{xx} - u_{yy} + 2xu_x - u_y = 6e^{2y}$

$x^2 \cdot \frac{1}{x^2}(\cancel{u_{\xi\xi}} - 2u_{\xi\eta} + \cancel{u_{\eta\eta}} + -u_\xi + u_\eta) - (\cancel{u_{\xi\xi}} + 2u_{\xi\eta} + \cancel{u_{\eta\eta}}) +$

$2x \cdot \frac{1}{x}(u_\xi - u_\eta) - (u_\xi + u_\eta) = 6e^{2y}$ which simplifies to

$-4u_{\xi\eta} - 2u_\eta = 4xe^y = 6e^{\xi+\eta}$ or $u_{\xi\eta} + \frac{1}{2}u_\eta = -\frac{3}{2}e^{\xi+\eta}$

5b retry $u_{\xi\eta} = -\frac{1}{2}u_\eta - \frac{3}{2}e^{\xi+\eta}$, $\xi = y + \ln x$, $\eta = y - \ln x \therefore$

$v = u_\eta \therefore u_{\xi\eta} = v_\xi \therefore v_\xi = -\frac{1}{2}v - \frac{3}{2}e^{\xi+\eta} \therefore$

$v_\xi + 0v_\eta + \frac{1}{2}v = -\frac{3}{2}e^{\xi+\eta} = av_\xi + bv_\eta + cv = g(\xi, \eta) \therefore$

$a = 1,\ b = 0,\ c = \frac{1}{2} \therefore$ let ~~$\xi = as + bt$, $\eta = bs - at$~~ $\therefore$

~~$\xi_s = a$, $\xi_t = b$, $\eta_s = b$, $\eta_t = -a$~~ $\therefore$

$\cancel{v_\xi = v_t \xi_t}$ $s = a\xi + b\eta$, $t = b\xi - a\eta$ ∴

$s_\xi = a$, $s_\eta = b$, $t_\xi = b$, $t_\eta = -a$ ∴

$v_\xi = v_s s_\xi + v_t t_\xi$ ∴

$a(v_s s_\xi + v_t t_\xi) + cv = g(\xi,\eta) = \tilde{g}(s(\xi,\eta), t(\xi,\eta)) = \tilde{g}(s,t) =$

$a(a v_s + b v_t) = a^2 v_s + ab v_t + cv$

\ 5b Sol/ take $v = u_\eta$ then $v_\xi + \frac{1}{2}v = -\frac{3}{2}e^{\xi+\eta}$ solving this hyperbolic scalar gives:

$e^{\xi/2} v_\xi + \frac{1}{2}e^{\xi/2} v = \frac{\partial}{\partial \xi}(e^{\xi/2} v) = -\frac{3}{2}e^{3\xi/2+\eta}$ ∴

$\left\{ v_\xi + \frac{1}{2}v = -\frac{3}{2}e^{\xi+\eta} \right.$ ∴ $h(\xi)v_\xi + h(\xi)\frac{1}{2}v = h(\xi)(-\frac{3}{2})e^{\xi+\eta} =$

$\frac{\partial}{\partial \xi}(h(\xi)v) = h(\xi)v_\xi + h'(\xi)v$ ∴

$h(\xi)\frac{1}{2} = h'(\xi)$ ∴ $\frac{h'(\xi)}{h(\xi)} = \frac{1}{2}$ ∴ $\int \frac{h'(\xi)}{h(\xi)} d\xi = \int \frac{1}{2} d\xi$ ∴

$\ln|h(\xi)| = \frac{1}{2}\xi$ ∴ $h(\xi) = e^{\frac{1}{2}\xi}$ ∴

$\frac{\partial}{\partial \xi}(e^{\frac{1}{2}\xi} v) = -\frac{3}{2}e^{\frac{1}{2}\xi}e^{\xi+\eta} = -\frac{3}{2}e^{\frac{3}{2}\xi+\eta}$ ∴

$\int e^{\frac{1}{2}\xi} v \, d\xi = \int -\frac{3}{2}e^{\frac{3}{2}\xi+\eta} d\xi = e^{\frac{1}{2}\xi} v = -e^{\frac{3}{2}\xi+\eta} + F(\eta)$

$v = -e^{-\frac{1}{2}\xi}e^{\frac{3}{2}\xi+\eta} + e^{-\frac{1}{2}\xi}F(\eta) = -e^{\xi+\eta} + e^{-\frac{1}{2}\xi}F(\eta) = u_\eta$ ∴

$u = \int u_\eta \, \partial\eta = \int -e^{\xi+\eta} + e^{-\frac{1}{2}\xi}F(\eta)\partial\eta =$

$-\int e^{\xi+\eta}\partial\eta + e^{-\frac{1}{2}\xi}\int F(\eta)\partial\eta =$

$-e^{\xi+\eta} + e^{-\frac{1}{2}\xi}[\tilde{f}(\eta) + \tilde{g}(\xi)] =$

$-e^{2y} + e^{-\frac{1}{2}(y+\ln x)}[\tilde{f}(y-\ln x) + \tilde{g}(y+\ln x)] =$

$-e^{2y} + e^{-\frac{1}{2}y + -\frac{1}{2}\ln x}[\tilde{f}(\ln(e^y)-\ln x) + \tilde{g}(\ln(e^y)+\ln x)] =$

$-e^{2y} + e^{-\frac{1}{2}y}e^{-\frac{1}{2}\ln x}[\tilde{f}(e^y/x) + \tilde{g}(x e^y)] =$

$-e^{2y} + e^{-\frac{1}{2}y}e^{-\ln x}e^{+\frac{1}{2}\ln x}[\tilde{f}((\cancel{x}/e^y)^{-1}) + \tilde{g}(x e^y)] =$

$-e^{2y} + (e^{\ln x})^{-1}e^{\frac{1}{2}\ln x}e^{-\frac{1}{2}y}[\tilde{f}((x/e^y)^{-1}) + \tilde{g}(x e^y)] =$

$\cancel{e^2 - e^{2y} + x^{-1}e^{\frac{1}{2}x}e^{-\frac{1}{2}y} + x^{-1}e^{\ln x}e^{-\frac{1}{2}y}}$

$-e^{2y} + x^{-1}e^{\frac{1}{2}\ln x}e^{-\frac{1}{2}y}\tilde{f}((x/e^y)^{-1}) + x^{-1}e^{\frac{1}{2}\ln x}e^{-\frac{1}{2}y}\tilde{g}(x e^{+y}) =$

$-e^{2y} + x^{-1}f(x/e^y) + g(x e^y) = -e^{2y} + x^{-1}f(x e^{-y}) + g(x e^y)$

\ 5b Sol/ $\frac{\partial}{\partial \xi}(e^{\xi/2}v) = -\frac{3}{2}e^{3\xi/2+\eta}$ ∴ $e^{\xi/2}v = -e^{3\xi/2+\eta} + A(\eta)$ ∴

Week 3 Sheet $\therefore r = -e^{\xi+\eta} + e^{-\xi/2} A(\eta) = u_\eta \therefore$

$u = \int u_\eta \, d\eta = -e^{\xi+\eta} + e^{-\xi/2} B(\eta) + C(\xi) \quad [B'(\eta) = A(\eta)]$ or in Z

original variables ($\xi = y + \ln x$, $\eta = y - \ln x$):

$u = -e^{-2y} + x^{-1/2} e^{-y/2} B(y - \ln x) + C(y + \ln x)$ Z required sol is obtained here by setting $B(z) = e^{z/2} f(e^{-z})$ & $C(z) = g(e^{z})$

5c $u(x,y) = -e^{2y} + x^{-1} \cdot f(xe^{-y}) + g(xe^{y}) \therefore$

$u(x,0) = u(x, y=0) = -e^{0} + x^{-1} f(xe^{0}) + g(xe^{0}) = -1 + x^{-1} f(x) + g(x) = 0$

$u_y = -2e^{-2y} + x^{-1} f'(xe^{-y})(-xe^{-y}) + g'(xe^{y}) xe^{y} \therefore$

$u_y(x,0) = u_y(x, y=0) = -2e^{0} + x^{-1} f'(xe^{0})(-xe^{0}) + g'(xe^{0}) xe^{0} =$

$-2 + x^{-1} f'(x)(-x) + g'(x)x = 0 \therefore$

$\frac{\partial}{\partial x}(u(x,0)) = \frac{\partial}{\partial x}(0) = 0 = \frac{\partial}{\partial x}(-1 + x^{-1} f(x) + g(x)) = -x^{-2} f(x) + x^{-1} f'(x) + g'(x)$

$\therefore -2 + x^{-1} f'(x)(-x) + g'(x)x = -x^{-2} f(x) + x^{-1} f'(x) + g'(x) \therefore$

$\cancel{-2x^{-1} + x^{-2} f'}$

$-2x^{-1} - x^{-1} f'(x) + g'(x) = 0 \therefore g'(x) = -2x^{-1} + x^{-1} f'(x) \therefore$

$x^{-2} f(x) - x^{-1} f'(x) = g'(x) \therefore$

$x^{-2} f(x) - x^{-1} f'(x) = -2x^{-1} + x^{-1} f'(x) \therefore$

$x^{-2} f(x) = -2x^{-1} + 2x^{-1} f'(x) \therefore$

$f(x) = -2x + 2x f'(x) \therefore f(x) - 2x f'(x) = -2x \therefore$

$IF = e^{\int -2x\, dx} = e^{-x^2} \therefore \frac{d}{dx}(e^{-x^2} f(x)) = -2xe^{-x^2} \therefore$

$e^{-x^2} f(x) = \int -2xe^{-x^2} dx = e^{-x^2} + C \therefore$

$f(x) = 1 + Ce^{x^2} \therefore \cancel{g'(x) = -2x^{-1} + x^{-1}}$

$g(x) = 1 - x^{-1} f(x) = 1 - x^{-1}[e^{+x^2} C + 1] = 1 - \frac{1}{x} - C\frac{1}{x} e^{x^2} \therefore$

with Z $\therefore f(xe^{-y}) = 1 + Ce^{(xe^{-y})^2}$

$g(xe^{y}) = 1 - \frac{1}{xe^{y}} - C\frac{1}{xe^{y}} e^{(xe^{y})^2} \therefore$

$u(x,y) = -e^{2y} + x^{-1}[1 + Ce^{(xe^{-y})^2}] + 1 - \frac{1}{xe^{y}} - C\frac{1}{xe^{y}} e^{(xe^{-y})^2}$

5c sol/ diffing Z GS $u = -e^{2y} + x^{-1} f(xe^{-y}) + g(xe^{y})$ wrt y get:

$u_y = -2e^{2y} - e^{-y} f'(xe^{-y}) + xe^{y} g'(xe^{y}) \therefore$ Z IC state:

$-1 + x^{-1} f(x) + g(x) = 0$, $-2 - f'(x) + xg'(x) = 0$ this system can be solve by elimination, say sind $g(x)$ in terms of $f(y)$ from Z first eqn $g(x) = 1 - x^{-1} f(x)$ ∴ $g'(x) = x^{-2} f(x) - x^{-1} f'(x)$ & subbing that into Z 2nd eqn get $-f'(x) + x^{-1} f(x) - f'(x) = 2$ which is a 1st order linear ODE for $f(x)$: $f'(x) - \frac{1}{2x} f(x) = -1$ solved by integrating factor $x^{-1/2} f'(x) - \frac{1}{2} x^{-3/2} f(x) = \frac{d}{dx}\left(x^{-1/2} f(x)\right) = -x^{-1/2}$ ∴

$x^{-1/2} f(x) = -\int x^{-1/2}\,dx = -2x^{1/2} + A$, $A = \text{const}$ ∴ $f(x) = -2x + Ax^{1/2}$ ∴

$g(x) = 1 - x^{-1} f(x) = 3 - Ax^{-1/2}$ & Z sol is $u(x,y) =$
$-2e^{2y} + x^{-1}\left[2xe^{-y} + Ax^{1/2}e^{-y/2}\right] + 3 - Ax^{-1/2}e^{-y/2} = -e^{2y} - 2e^{-y} + 3$

\6 / $v_{tt} = c^2 v_{xx}$ change of variabs $v(x,t) = -w(\xi,\tau)$
$x = -\xi$, $t = \tau$ ∴ $v = -w$ ∴ $v_t = w_\tau \tau_t = -w_\tau$
$v_{tt} = -w_{\tau\tau}$ ∴
$v_x = -w_\xi \underbrace{\xi_x}_{-1} = w_\xi$ $v_{xx} = w_{\xi\xi} \underbrace{\xi_x}_{-1} = -w_{\xi\xi}$ ∴ into PDE:
$c^2 v_{xx} - v_{tt} = 0$ $c^2(-w_{\xi\xi}) - (-w_{\tau\tau}) = 0$ ∴ $c^2 w_{\xi\xi} - w_{\tau\tau} = 0$ ∴
$w(\xi,\tau)$ satisfies Z same PDE as $v(x,t)$ ∴ PDE is invariant wrt change of variables

IC1: $v(x,0) = \phi(x) = v(x,t)\big|_{t=0} = -w(\xi,\tau)\big|_{\tau=0} = -w(\xi,0) =$
$\phi(-\xi) = -\phi(\xi)$ ∴
$w(\xi,0) = \phi(\xi)$ ∴ $v(\xi,\tau)$ satisfies Z same IC1 as $u(x,t)$

IC2: $w(\xi,\tau)$ satisfies Z same IC2 as $v(x,t)$ ∵
$\frac{\partial v}{\partial t}(x,0) = \psi(x) = \frac{\partial v(x,t)}{\partial t}\Big|_{t=0} = -\frac{\partial w(\xi,\tau)}{\partial \tau}\Big|_{\tau=0} = -\frac{w(\xi,0)}{\partial \tau} = \psi(-\xi) = -\psi(\xi)$
∴ $\frac{\partial}{\partial \tau} w(\xi,0) = \psi(\xi)$ ∴

$w(\xi,\tau)$ satisfies Z same IVP as $u(x,t)$, so to IVP is unique
⇒ $u : (x,t) \mapsto u(x,t)$ $v : (\xi,\tau) \mapsto v(\xi,\tau)$ are one & Z same func

\6 sol / Z dirichlet BC implies Z odd extension of both Z IC to Z whole line: $\phi_{odd}(x) = A\cos kx$, $\psi_{odd}(x) = -k\cos kx$ for $x < 0$;
& $\phi_{odd}(x) = -A\cos kx$, $\psi_{odd}(x) = k\cos kx$ for $x > 0$
indeed let $u(x,t)$ be Z sol of $u_{tt} - c^2 u_{xx} = 0$ $-\infty < x < \infty$
subject to Z IC $u(x,0) = \phi_{odd}(x)$, $u_t(x,0) = \psi_{odd}(x)$ then $u(x,t)$ is an odd func of x, & all continuous odd funcs vanish at zero ∴ Z BC $u(0,t) = 0$ is satisfied automatically

\Week 3 Sheet / Desine $v(x,t) = u(x,t)$ Sor $-\infty < x < 0$ 2 $t > 0$, restricting u to Z negative half-line $(x<0)$. Use Z

d'Alembert's Sormula $v(x,t) = u(x,t) =$

$$\frac{1}{2}\left[\phi_{odd}(x+ct) + \phi_{odd}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(s)\,ds$$

when $ct < |x|$, st $x+ct<0$ 2 $x-ct<0$ have:

$$v(x,t) = \frac{1}{2}\left[A\cos k(x+ct) + A\cos k(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\left[-k\cos(ks)\right]ds$$

$\therefore$ aster carrying out integration obtain: $v(x,t) =$

$$\frac{A}{2}\left[\cos k(x+ct) + \cos k(x-ct)\right] - \frac{1}{2c}\left[\sin k(x+ct) - \sin k(x-ct)\right] =$$

$A\cos(kx)\cos(kct) - \frac{1}{c}\cos(kx)\sin(kct)$ Sor $ct < |x|$

when $ct > |x|$ st $x+ct>0$ 2 $x-ct<0$ have: $v(x,t) =$

$$\frac{1}{2}\left[-A\cos k(x+ct) + A\cos k(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{0}\left[-k\cos(ks)\right]ds + \frac{1}{2c}\int_{0}^{x+ct}\left[k\cos(ks)\right]ds$$

$\therefore$ aster integration, obtain: $v(x,t) =$

$$\frac{A}{2}\left[-\cos k(x+ct) + \cos k(x-ct)\right] + \frac{1}{2c}\left[\sin k(x-ct) + \sin k(x+ct)\right]$$

Sor $ct > |x|$ Z complete sol is given by Z pair of Z Sormulas

\6 Sol / use Z d'Alembert Sormula $v(x,t) = u(x,t) =$

$\frac{1}{2}\left[\phi_{odd}(x+ct) + \phi_{odd}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi_{odd}(s)\,ds$ when $ct < |x|$, st

$x+ct<0$ 2 $x-ct<0$ have $v(x,t) =$

$$\frac{1}{2}\left[A\cos k(x+ct) + A\cos k(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\left[-k\cos(ks)\right]ds \quad \therefore$$

$v(x,t) =$

$$\frac{A}{2}\left[\cos k(x+ct) + \cos k(x-ct)\right] - \frac{1}{2c}\left[\sin k(x+ct) - \sin k(x-ct)\right] =$$

$A\cos(kx)\cos(kct) - \frac{1}{c}\cos(kx)\sin(kct)$

$\cancel{=}\ \frac{A}{2}\cos(kx+kct) + \frac{A}{2}\cos(kx-kct) - \frac{1}{2c}\sin(kx+kct) + \frac{1}{2c}\sin(kx-kct) =$

$\frac{A}{2}\cos(kx)\cos(kct) - \frac{A}{2}\sin(kx)\sin(kct) + \frac{A}{2}\cos(kx)\cos(kct) + \frac{A}{2}\sin(kx)\sin(kct) +$

$\frac{-1}{2c}\sin(kx)\cos(kct) - \frac{1}{2c}\cos(kx)\sin(kct) + \frac{1}{2c}\sin(kx)\cos(kct) - \frac{1}{2c}\cos(kx)\sin(kct) =$

$A\cos(kx)\cos(kct) - \frac{1}{c}\cos(kx)\sin(kct)$ Sor $ct < |x|$

when $ct > |x|$, st $x+ct>0$ 2 $x-ct<0$ have $v(x,t) =$

$$\frac{1}{2}\left[-A\cos k(x+ct) + A\cos k(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{0}\left[-k\cos(ks)\right]ds + \frac{1}{2c}\int_{0}^{x+ct}\left[k\cos(ks)\right]ds \quad \therefore$$

$v(x,t) = \frac{A}{2}\left[-\cos k(x+ct) + \cos k(x-ct)\right] + \frac{1}{2c}\left[\sin k(x-ct) + \sin k(x+ct)\right]$ Sor $ct > |x|$

Z complete sol is given by Z pair of Z Sormulas

7a / $u_{tt} - c^2 u_{xx} = 0 \quad 0 \le x \le L$

put $u = X(x)T(t) = u(x,t)$ $\therefore u_{xx} = X''T$, $u_{tt} = XT''$ $\therefore$

$XT'' - c^2 X''T = 0 \therefore XT'' = c^2 X''T \therefore$

$\frac{T''}{c^2 T} = \frac{X''}{X} = -\lambda$ = Constant $\therefore X'' + \lambda X = 0$, $T'' + \lambda c^2 T = 0$

$X(0) = 0$, $X(L) = 0$ BCs $\therefore$ Cases $\lambda < 0$, $\lambda = 0$, $\lambda > 0$ $\therefore$

• $\lambda < 0$: let $\lambda = -a^2$ $\therefore X'' - a^2 X = 0 \therefore X'' = a^2 X \therefore$

$\therefore$ if $X = e^{px}$ $X'' = p^2 e^{px} \therefore p^2 e^{px} - a^2 e^{px} = 0 = p^2 - a^2 \therefore p = a, p = -a$;

$X = e^{ax}$, $X = e^{-ax}$ $\therefore X = Ae^{ax} + Be^{-ax}$ $\therefore$

BCs: $X(0) = Ae^0 + Be^0 = A + B = 0 \therefore A = -B \therefore X = -Be^{ax} + Be^{-ax}$ $\therefore$

$X(L) = -Be^{aL} + Be^{-aL} = 0 \therefore Be^{-aL} = Be^{aL} \therefore$

$-Be^{2aL} + B = 0 \therefore B(-e^{aL} + e^{-aL}) = 0 = B(-e^{2aL} + 1) = 0 \therefore$ ~~B~~

$0 \ne L \therefore -e^{2aL} + 1 \ne 0 \therefore B = 0 \therefore A = 0 \therefore$

for $\lambda < 0$ no nontrivial Sol

• $\lambda = 0$: $X'' + 0X = 0 = X''$ $\therefore p^2 e^{px} = 0 = p^2 \therefore p_1 = 0, p_2 = 0 \therefore$

$X = Ae^{0x} + Bxe^{0x} = A + Bx = X$ $\therefore$ BCs $X(0) = X(L) = 0$ $\therefore$

$A + B(0) = 0 = A$ $\therefore Bx = X \therefore BL = 0 = B \therefore$

$\lambda = 0$ has no non trivial Sols

• $\lambda > 0$: let $\lambda = a^2$ $a \in \mathbb{R}$ $\therefore$ ~~p^2~~ $X'' + a^2 X = 0 \therefore p^2 e^{px} + a^2 e^{px} = 0 =$

$p^2 + a^2 = 0 \therefore p^2 = -a^2 \therefore p = +ia, p = -ia \therefore$

$X = \cancel{A}De^{iax} + \cancel{B}C e^{-iax} = D\cos(ax) + iD\sin(ax) + C\cos(-ax) + Ci\sin(-ax) =$

$D\cos(ax) + iD\sin(ax) + C\cos(ax) - Ci\sin(ax) =$

$A\cos(ax) + B\sin(ax) = X$ $\therefore$ BCs $\therefore$

$X(0) = A\cos(0) + B\sin(0) = A = 0$ $\therefore X(L) = A\cos(aL) + B\sin(aL) = B\sin(aL) = 0$

$\therefore aL = n\pi \therefore a = \frac{n\pi}{L}$ $n = 1, 2, 3, \ldots$ $\therefore X(x) = \sin\left(\frac{n\pi}{L}x\right) \therefore$

$\lambda = a^2 = \left(\frac{n\pi}{L}\right)^2 \Rightarrow$

$T'' + c^2\lambda T = 0 = T'' + c^2 a^2 T = 0 \therefore T = e^{pt} \therefore p^2 e^{pt} = T'' \therefore$

$p^2 e^{pt} + c^2 a^2 e^{pt} = 0 = p^2 + c^2 a^2 \therefore p^2 = -c^2 a^2 \therefore p = ica, p = -ica$

$\therefore T(t) = C_2 e^{icat} + D_2 e^{-icat} = A\cos(cat) + B\sin(cat) =$

$A\cos\left(c\frac{n\pi}{L}t\right) + B\sin\left(c\frac{n\pi}{L}t\right)$ $\therefore$ our seperated sol is

Zero $\therefore$ Z BC $u(0,t) = 0$ is always one

Week 3 Sheet / $u(x,t) = T(t)X(x) =$
$\left[A\cos\left(c\frac{n\pi}{L}t\right) + B\sin\left(c\frac{n\pi}{L}t\right)\right]\sin\left(\frac{n\pi}{L}x\right)$ $n \in \mathbb{N}$ ∴ G.S =

$u(x,t) = \sum_{n=1}^{\infty}\left[A_n\sin\left(\frac{n\pi}{L}ct\right) + B_n\cos\left(\frac{n\pi}{L}ct\right)\right]\sin\left(\frac{n\pi}{L}x\right)$ ... ICs: $u(x,0)=0$

∴ $\frac{\partial u}{\partial t}(x,0) = \tau\,\delta(x - x_0)$ at $t=0$

$\frac{\partial u}{\partial t} = \sum_{n=1}^{\infty}\left[\frac{n\pi}{L}cA_n\cos\left(\frac{n\pi}{L}ct\right) - \frac{n\pi}{L}cB_n\sin\left(\frac{n\pi}{L}ct\right)\right]\sin\left(\frac{n\pi}{L}x\right)$ ∴

$u_t(x,0) = \sum_{n=1}^{\infty}\left[\frac{n\pi}{L}cA_n\cos(0) - \frac{n\pi}{L}cB_n\sin(0)\right]\sin\left(\frac{n\pi}{L}x\right) =$

$\sum_{n=1}^{\infty}\frac{n\pi}{L}cA_n\sin\left(\frac{n\pi}{L}x\right) = 0 = \sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{L}x\right) = 0$ ∴ $A_n = 0$ ∴

$u(x,t) = \sum_{n=1}^{\infty}B_n\cos\left(\frac{n\pi}{L}ct\right)\sin\left(\frac{n\pi}{L}x\right)$ ∴

$u(x,0) = \sum_{n=1}^{\infty}B_n\cos\left(\frac{n\pi}{L}c(0)\right)\sin\left(\frac{n\pi}{L}x\right) = \sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi}{L}x\right) = \frac{4hx(L-x)}{L^2}$

7a Sol / By Z method of separation of variables $u = X(x)T(t)$ ⇒
$XT'' - c^2TX'' = 0$, $T''/c^2T = X''/X = -\lambda$ where λ is constant while Z
eqn $X'' + \lambda X(x) = 0$ after using Z BC $X(0) = X(L) = 0$ has Z sol
$X_n = \sin\left(\frac{n\pi x}{L}\right)$ $n = 1, 2, \ldots$ Z eqn for T $T'' + c^2\left(\frac{n\pi}{L}\right)^2T = 0$

has sols $T_n(t) = A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi c}{L}t\right)$ ∴ sol such that Z BCs
satisfying Z BC is $u(x,t) =$
$\sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{n\pi ct}{L}\right) + B_n\sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi x}{L}\right)$ making use of Z IC
gives $\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right) = \frac{4hx(L-x)}{L^2}$ & $\sum_{n=1}^{\infty}B_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right) = 0$

{ $u(x,0) = \sum_{n=1}^{\infty}\left[A_n\cos(0) + B_n\sin(0)\right]\sin\left(\frac{n\pi x}{L}\right) = \sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi x}{L}\right) = \frac{4hx(L-x)}{L^2} = g(x)$

$u_t = \sum_{n=1}^{\infty}\left[-\frac{n\pi c}{L}A_n\sin\left(\frac{n\pi c}{L}t\right) + \frac{n\pi c}{L}B_n\cos\left(\frac{n\pi c}{L}t\right)\right]\sin\left(\frac{n\pi x}{L}\right)$ ∴

$u_t(x,0) = \sum_{n=1}^{\infty}\left[-\frac{n\pi c}{L}A_n\sin(0) + \frac{n\pi c}{L}B_n\cos(0)\right]\sin\left(\frac{n\pi}{L}x\right) =$

$\sum_{n=1}^{\infty}\frac{n\pi c}{L}B_n\sin\left(\frac{n\pi}{L}x\right) = 0 = \sum_{n=1}^{\infty}B_n\sin\left(\frac{n\pi}{L}x\right)$ } using Z standard formula for
coeffs of Z sin-series Fourier, get $A_n = \frac{2}{L}\int_0^L\frac{4hx(L-x)}{L^2}\sin\left(\frac{n\pi}{L}x\right)dx$
{ $A_n = \frac{1}{L}\int_{-L}^{L}\frac{4hx(L-x)}{L^2}\cos\left(\frac{n\pi}{L}x\right)$, $B_n = \frac{1}{L}\int_{-L}^{L}\frac{4hx(L-x)}{L^2}\sin\left(\frac{n\pi}{L}x\right)dx$ } where Z
integration can be readily carried out by sub $s = x/L$ & integration by

parts, giving (we skip Z standard steps):

$A_n = \frac{16h}{(n\pi)^3}[1-(-1)^n] \quad n=1,2,3,\dots$

$\therefore A_{2n}=0$, it is convenient to let $n=2k+1$,

$A_{2k+1}=\frac{32h}{[\pi(2k+1)]^3} \quad k=0,1,2,3,\dots$ Z other BC, straightforwardly

gives $B_n=0$ Z Sol satisfying Z IC & BC is

$u=\sum_{k=0}^{\infty}\frac{32h}{[\pi(2k+1)]^3}\cos\left(\frac{(2k+1)\pi ct}{L}\right)\sin\left(\frac{(2k+1)\pi x}{L}\right)$

\7b/ $u(x,t)=\sum_{n=1}^{\infty}\left(A_n\cos\left(\frac{n\pi ct}{L}\right)+B_n\sin\left(\frac{n\pi ct}{L}\right)\right)\sin\left(\frac{n\pi}{L}x\right)$

$u(x,0)=\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right) \quad 0\neq L$

$u_t(x,0)=0 \quad \therefore$

$u_t(x,t)=\sum_{n=1}^{\infty}\left[-\frac{n\pi c}{L}A_n\sin\left(\frac{n\pi ct}{L}\right)+\frac{n\pi c}{L}B_n\cos\left(\frac{n\pi ct}{L}\right)\right]\sin\left(\frac{n\pi}{L}x\right)$

$u(x,0)=u(x,t=0)=\sum_{n=1}^{\infty}[A_n\cos(0)+B_n\sin(0)]\sin\left(\frac{n\pi}{L}x\right)=\sum_{n=1}^{\infty}A_n\sin\left(\frac{n\pi}{L}x\right)=$

$\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)=\phi(x) \quad \therefore$

$A_n=\frac{1}{L}\int_{-L}^{L}\phi(x)\sin\left(\frac{n\pi}{L}x\right)dx=\frac{1}{L}\int_{-L}^{L}\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$

$\sum_{n=1}^{\infty}B_n\left(\frac{n\pi c}{L}\right)\sin\left(\frac{n\pi x}{L}\right)=0$

$A_n=\frac{2}{L}\int_{0}^{L}\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{n\pi x}{L}\right)dx$

\7b Sol/ recycle previous result with exception of Z coeffs of A_n which are Z only data that depends on Z changed IC this time, do not need to ~~integrate~~ calc Z integrals to find Z Fourier coeffs, as Z IC initial datum is expanded in Z sin-series Fourier by simple observation, using elementary trig

$u(x,0)=\sin(5\pi x/L)\cos(\pi x/L)=\frac{1}{2}\left(\sin\left(\frac{4\pi x}{L}\right)+\sin\left(\frac{6\pi x}{L}\right)\right)$

$\{2\sin x\cos y=\sin(x-y)+\sin(x+y) \quad \therefore \sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)=$

$\frac{1}{2}\cdot 2\sin\left(\frac{5\pi x}{L}\right)\cos\left(\frac{\pi x}{L}\right)=\frac{1}{2}\left[\sin\left(\frac{5\pi x}{L}-\frac{\pi x}{L}\right)+\sin\left(\frac{5\pi x}{L}+\frac{\pi x}{L}\right)\right]=$

$\frac{1}{2}\left[\sin\left(\frac{4\pi x}{L}\right)+\sin\left(\frac{6\pi x}{L}\right)\right]\}$ so we immediately have that $A_4=\frac{1}{2}$, $A_6=\frac{1}{2}$

& all other $A_n=0$. Z Sol $\therefore$ consists of just two terms instead of

zero ...

Week 3 Sheet 3 / on infinite series

$\{ B_n = 0, \sum_{n=1}^{\infty} A_n \cos(0) \sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\left(\sin\left(\frac{4\pi x}{L}\right) + \sin\left(\frac{6\pi x}{L}\right)\right) =$

$\sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{L}\right) = \frac{1}{2}\sin\left(\frac{4\pi x}{L}\right) + \frac{1}{2}\sin\left(\frac{6\pi x}{L}\right) = A_4 \sin\left(\frac{4\pi x}{L}\right) + A_6 \sin\left(\frac{6\pi x}{L}\right)$

$\therefore A_4 = \frac{1}{2}$, $A_6 = \frac{1}{2}$ $\therefore$ all other $A_n = 0$ $\therefore$ $A_n = 0 \ \forall n \in \mathbb{N} \setminus \{4, 6\} \}$

$\therefore u(x,t) = \frac{1}{2}\cos\left(\frac{4\pi ct}{L}\right)\sin\left(\frac{4\pi x}{L}\right) + \frac{1}{2}\cos\left(\frac{6\pi ct}{L}\right)\sin\left(\frac{6\pi x}{L}\right)$ $\therefore$

$\{ = \sum_{n=1}^{\infty} A_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi}{L}x\right) = \sum_{n=4,6} A_n \cos\left(\frac{n\pi ct}{L}\right)\sin\left(\frac{n\pi x}{L}\right) =$

$A_4 \cos\left(\frac{4\pi ct}{L}\right)\sin\left(\frac{4\pi x}{L}\right) + A_6 \cos\left(\frac{6\pi ct}{L}\right)\sin\left(\frac{6\pi x}{L}\right) \}$

8 a / $u_{tt} - c^2 \nabla^2 u = 0$ $\therefore u_{tt} = c^2 \nabla^2 u$ $\therefore$

$E(t) = \int_{R^3} \left(\frac{1}{2}u_t^2 + \frac{1}{2}c^2(\nabla u)^2\right) dV$

$E_V = \iiint_V \left(\frac{u_t^2}{2} + \frac{c^2}{2}(\nabla u)^2\right) dV$. $\frac{dE_V}{dt} = \iiint_V (u_t u_{tt} + c^2 \nabla u \cdot \nabla u_t) dV =$

$\frac{dE_V}{dt} = \iiint_V (u_t c^2 \nabla^2 u + c^2 \nabla u \cdot \nabla u_t = c^2 \iiint_V (u_t \nabla^2 u + \nabla u \cdot \nabla u_t) dV =$

$c^2 \iiint_V (u_t \nabla \cdot \nabla u + \nabla u \cdot \nabla u_t) dV = c^2 \iiint_V \nabla \cdot (u_t \nabla u) dV =$ (by divergence thm)

$c^2 \iint_S u_t \nabla u \cdot \vec{n} \, dS = c^2 \iint_C u_t \frac{\partial u}{\partial r} = \frac{dE_V}{dt}$

8 a SS1 / apply Z divergence thm to Z eqn over Z spherical domain $V = \{\vec{x} : |\vec{x}| \le r\}$ $\int_V u_{tt} \, dV = c^2 \int_V \nabla^2 u \, dV = c^2 \int_V \nabla \cdot \nabla u \, dV =$

(by divergence thm) $\{ = c^2 \iint_S \nabla u \cdot \vec{n} \, dS = \}$ $c^2 \int_S \underline{n} \cdot \nabla u \, dS$

where S denotes Z boundary of V ie Z surf of Z sphere with center $\vec{x} = 0$ & radius r desired in Z question, Z $\underline{n}$ is Z normal vector to this surf $\therefore$ denote: $\bar{u}(r,t) = \frac{1}{4r^2\pi} \iint_{|\vec{x}|=r} u(\vec{x},t) \, dS =$

$\frac{1}{4\pi} \int_0^{2\pi} \int_0^{\pi} u(\vec{x},t) \sin\theta \, d\theta \, d\phi$

$\{ dV = r^2 d\Omega \, dr = r^2 \sin\theta \, d\theta \, d\phi$ $dS = R^2 \sin\theta \, d\theta \, d\phi$ $\iiint_V u_{tt} \, dV = c^2 \iiint_V \nabla^2 u \, dV =$

$c^2 \iiint_V \nabla \cdot (\nabla u) \, dV = c^2 \oiint \nabla u \cdot \underline{n} \, dS = \int_0^R \oiint_\Omega u_{tt}(r, \theta, \phi, t) r^2 \, dr \, d\Omega =$

$c^2 \oiint_\Omega u_r(R, \theta, \phi, t) R^2 \, d\Omega$ $\Omega = \sin\theta \, d\theta \, d\phi$ $\therefore$ $dS = R^2 d\Omega$ $\therefore$

$\bar{u}(r,t) = \frac{1}{4\pi r^2} \oiint_{|x|=r} u(x,t) \, dS = \frac{1}{4\pi r^2} \oiint u(r, \phi, \theta, t) r^2 \, d\Omega = \frac{1}{4\pi} \oiint u(r, \phi, \theta, t) \, d\Omega$

$$\frac{\partial^2}{\partial t^2}\int_0^R\left[\oiint u\,d\Omega\right]r^2\,dr = \frac{\partial^2}{\partial t^2}\int_0^R\left[4\pi\bar{u}(r,t)\right]r^2\,dr = 4\pi\frac{\partial^2}{\partial t^2}\int_0^R\bar{u}(r,t)r^2\,dr$$

$$= c^2R^2\frac{\partial}{\partial r}\oiint u\,d\Omega = c^2R^2\frac{\partial}{\partial r}\left[4\pi\bar{u}(r,t)\Big|_{r=R^2}\right] = 4\pi c^2R^2\left(\frac{\partial\bar{u}}{\partial r}\right)\Big|_{r=R} \therefore$$

$$\left.\int_0^R u_{tt}(r,t)\,r^2\,dr = c^2R^2\,\bar{u}_r(R,t)\quad\right\}$$

$$\bar{u}(r,t) = \frac{1}{4r^2\pi}\iint_{|\vec{x}|=r}u(\vec{x},t)\,dS = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}u(\vec{x},t)\sin\theta\,d\theta\,d\phi$$

$$\left\{\frac{1}{4r^2\pi}\iint_{|\vec{x}|=r}u(\vec{x},t)\,dS = \frac{1}{4\pi r^2}\int_0^{2\pi}\int_0^{\pi}u(\vec{x},t)\,r^2\sin\theta\,d\theta\,d\phi = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}u(\vec{x},t)\sin\theta\,d\theta\,d\phi\right\}$$

clearly have: $\bar{u}(0,t) = u(\vec{0},t)\ \therefore$

$$\left\{\bar{u}(0,t) = \bar{u}(\cancel{x}=0,t)\ \cancel{=\phi},\quad u(\vec{0},t) = u(\vec{x}=\vec{0},t)\right.$$

$$\left.\bar{u}(r=0,t) = \frac{1}{4\pi}\iint_{|\vec{x}|=0}u(\vec{x},t)\,dS = u(\vec{x}=\vec{0},t) = u(\vec{0},t)\right\}\ \text{then}$$

$$\int_0^r\frac{\partial^2}{\partial t^2}\left[\bar{u}(r,t)\right]r^2\,dr = \int_0^r\frac{\partial^2\bar{u}(r,t)}{\partial t}r^2\,dr = c^2\frac{\partial\bar{u}(r,t)}{\partial r}r^2$$

$$\left\{\int_V\frac{\partial^2u}{\partial t^2}\,dV = c^2\int_S\vec{n}\cdot\nabla u\,dS\ \therefore\ \int_V\frac{\partial^2\bar{u}}{\partial t^2}\,dV = c^2\int_S\vec{n}\cdot\nabla\bar{u}\,dS\right.$$

$$\bar{u}(r,t) = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}u(\vec{x},t)\sin\theta\,d\theta\,d\phi\ \therefore$$

$$\frac{\partial\bar{u}(r,t)}{\partial t} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{\partial u(\vec{x},t)}{\partial t}\sin\theta\,d\theta\,d\phi\ \therefore$$

$$\frac{\partial^2\bar{u}(r,t)}{\partial t^2} = \frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{\partial^2u(\vec{x},t)}{\partial t^2}\sin\theta\,d\theta\,d\phi\ \therefore$$

$$\int_0^r\frac{\partial^2\bar{u}(r,t)}{\partial t^2}r^2\,dr = \int_0^r\frac{1}{4\pi}\int_0^{2\pi}\int_0^{\pi}\frac{\partial^2u(x,t)}{\partial t^2}\sin\theta\,d\theta\,d\phi\,r^2\,dr =$$

$$\frac{1}{4\pi}\int_0^r\int_0^{2\pi}\int_0^{\pi}\frac{\partial^2u(x,t)}{\partial t^2}r^2\sin\theta\,d\theta\,d\phi\,d\cancel{r} = \frac{1}{4\pi}\int_V\frac{\partial^2u(x,t)}{\partial t^2}\,dV =$$

$$\frac{1}{4\pi}c^2\int_S n\cdot\nabla u\,dS\ \Big\}\ \int_0^r\frac{\partial^2\bar{u}(r,t)}{\partial t^2}r^2\,dr = c^2\frac{\partial\bar{u}(r,t)}{\partial r}r^2$$ $\therefore$ diffing it

wrt r yields: $$r^2\frac{\partial^2\bar{u}(r,t)}{\partial t^2} = c^2\left(r^2\frac{\partial^2\bar{u}(r,t)}{\partial r^2} + 2r\frac{\partial\bar{u}(r,t)}{\partial r}\right)$$

$$\left\{\frac{\partial}{\partial r}\left[\int_0^r\frac{\partial^2\bar{u}(r,t)}{\partial t^2}r^2\,dr\right] = r^2\frac{\partial^2\bar{u}(r,t)}{\partial t^2} = \frac{\partial}{\partial r}\left[c^2\frac{\partial\bar{u}(r,t)}{\partial r}r^2\right]\right\}$$

8b $u(\vec{x},0) = \phi(\vec{x})$, $u_t(\vec{x},0) = 0\ \therefore$

$$u(x,t) = \frac{1}{2}\left[\phi(x+ct) + \phi(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,dS$$

Week 3 Sheet / $\bar{u}(r,t) = \frac{1}{4r^2\pi}\iint_S u(\vec{x},t)\,dS$

8bsol / now sub $v(r,t) = r\bar{u}(r,t)$ into the eqn.

- $\frac{\partial^2 v(r,t)}{\partial t^2} = c^2 \frac{\partial^2 v(r,t)}{\partial r^2}$, $0 \le r \le \infty$

$\{ u_{tt} = c^2 \nabla^2 u$ $\quad r^2 \bar{u}_{tt} = c^2[r^2 \bar{u}_{rr} + 2r\bar{u}_r]$ $\therefore$

$v(r,t) = r\bar{u}(r,t)$ $\therefore$ $v_{tt} = r\bar{u}_{tt}$, $v_r = \partial_r[r\bar{u}] = r\bar{u}_r + \bar{u}_r$ $\therefore$

$v_{rr} = \partial_r[r\bar{u}_r + \bar{u}_r] = \partial_r(r\bar{u}_r) + \partial_r \bar{u}_r = r\bar{u}_{rr} + \bar{u}_r + \bar{u}_{rr} = (1+r)\bar{u}_{rr} + \bar{u}_r$

~~v_{tt}~~ $v_t = \partial_t(r\bar{u}(r,t)) = r\bar{u}_t(r,t)$ $\therefore$ $v_{tt} = r\bar{u}_{tt} = rc^2\nabla^2 u$

$v = ru$ $\therefore$ $\frac{1}{r}v = \bar{u}$ $\therefore$ $u_t = \frac{\partial}{\partial t}(\frac{1}{r}v) = \frac{1}{r}v_t$ $\therefore$ $\bar{u}_{tt} = \frac{1}{r}v_{tt}$

$\bar{u}_r = \partial_r(\frac{1}{r}v) = \partial_r(r^{-1}v) = -r^{-2}v + r^{-1}v_r$ $\therefore$

$\bar{u}_{rr} = \partial_r[-r^{-2}v + r^{-1}v_r] = \partial_r[-r^{-2}v] + \partial_r[r^{-1}v_r] =$

$2r^{-3}v + -r^{-2}v_r - r^{-2}v_r + r^{-1}v_{rr} = 2r^{-3}v - 2r^{-2}v_r + r^{-1}v_{rr}$ $\therefore$

$r^2\bar{u}_{tt} = r^2\frac{1}{r}v_{tt} = rv_{tt} = c^2[\cancel{2r^{-1}v} - \cancel{2v_r} + rv_{rr} - \cancel{2r^{-1}v} + \cancel{2v_r}] =$

$c^2[rv_{rr}] = rv_{tt}$ $\therefore$

~~$\therefore$~~ $v_{tt} = c^2 v_{rr} = \frac{\partial^2 v(r,t)}{\partial t^2} = c^2\frac{\partial^2 v(r,t)}{\partial r^2}$ $\quad 0 \le r < \infty \}$

where clearly $v(0,t) = 0$ $\{ v(0,t) = v(r=0,t) = r\bar{u}(r=0,t) = r\bar{u}(0,t) =$
$ru(\vec{0},t) = ru(\vec{x}=\vec{0},t)$ $\quad 0 \le r \le \infty$ $\therefore$ at $r=0$: $v(0,t)=0 \}$ $\therefore$

$v(0,t) = 0$ with IC $v(r,0) = r\bar{\phi}(r)$ $\{ u(\vec{x},0) = \phi(\vec{x})$ $\therefore$

$\bar{u}(r,0) = \frac{1}{4r^2\pi}\iint_S u(\vec{x},0)\,dS = \frac{1}{4r^2\pi}\iint_S \phi(\vec{x})\,dS = \bar{\phi}(r)$ $\therefore$

$v(r,0) = r\bar{u}(r,0) = r\bar{\phi}(r) \}$ $\therefore$ it is reduced to a half-line prob in

1 dimen. For $0 \le r \le ct$, the d'Alembert formula on the half line

with Dirichlet boundary: $v(r,t) = \frac{1}{2}[\phi(r+ct) - \phi(-r+ct)]$

in the current problem $\{ \phi(-(r-ct)) = \phi(-r+ct) \}$

- taking $x \to r$ & $\phi(s) \to s\bar{\phi}(s)$ gives

$v(r,t) = \frac{1}{2}[(r+ct)\bar{\phi}(r+ct) - (-r+ct)\bar{\phi}(-r+ct)]$ at the origin $r=0$,

the relationship between u & v cannot be used directly. take the limit

$\{ \phi(r+ct) \to (r+ct)\bar{\phi}(r+ct)$ $\quad \phi(-r+ct) \to (-r+ct)\bar{\phi}(-r+ct)$

$$\bar{u}(0,t) = \lim_{r\to 0} \frac{v(r,t)}{r} = \lim_{r\to 0} \frac{(ct+r)\bar{\phi}(ct+r) - (ct-r)\bar{\phi}(ct-r)}{2r}$$

$$\left\{ \bar{u} = \frac{v}{r} \therefore \frac{v(0,t)}{r} = \lim_{r\to 0} \frac{v(0,t)}{r} = \lim_{r\to 0} \frac{\frac{1}{2}\left[(r+ct)\bar{\phi}(r+ct) - (-r+ct)\bar{\phi}(-r+ct)\right]}{r} = \lim_{r\to 0} \bar{u}(r,t) = \bar{u}(0,t) \right\}$$

$$= \lim_{r\to 0} \frac{\delta(\xi+r) - \delta(\xi-r)}{2r}\Bigg|_{\xi=ct,\ \delta(\xi)=\xi\bar{\phi}(\xi)} = \delta'(\xi) = \frac{d}{d\xi}\left(\xi\bar{\phi}(\xi)\right)\Bigg|_{\xi=ct} =$$

$$\frac{1}{c}\frac{d}{dt}\left(ct\,\bar{\phi}(ct)\right) = \frac{d}{dt}\left(t\,\bar{\phi}(ct)\right) \quad \left\{ \phi(s) \to s\,\cancel{\bar{\phi}}(s) \therefore \right.$$

$$\delta(\xi) = \xi\bar{\phi}(\xi) \quad \therefore \lim_{r\to 0} \frac{(ct+r)\bar{\phi}(ct+r) - (ct-r)\bar{\phi}(ct-r)}{2r} =$$

$$\lim_{r\to 0} \frac{\bar{\phi}(ct+r) - \bar{\phi}(ct-r)}{2r} = \lim_{r\to 0} \frac{\delta(ct+r) - \delta(ct-r)}{2r} = \lim_{r\to 0} \frac{\delta(\xi+r) - \delta(\xi-r)}{2r}\Bigg|_{\xi=ct,\ \delta(\xi)=\xi\bar{\phi}(\xi)}$$

$$= \lim_{r\to 0} \frac{\left[\delta(\xi+r) - \delta(\xi-r)\right]'}{[2r]'} = \lim_{r\to 0} \frac{\delta'(\xi+r) - -\delta'(\xi-r)}{2} = \cancel{\lim_{r\to 0} \delta'}$$

$$\lim_{r\to 0} \frac{\delta'(\xi+r) + \delta'(\xi-r)}{2} = \left(\delta'(\xi+0) + \delta'(\xi-0)\right)/2 = \left(\delta'(\xi) + \delta'(\xi)\right)/2 = \delta'(\xi) =$$

$$\frac{d}{d\xi}\left(\delta(\xi)\right) = \frac{d}{d\xi}\left(\xi\bar{\phi}(\xi)\right)\Bigg|_{\xi=ct} = \frac{d}{d(ct)}\left((ct)\bar{\phi}(ct)\right) =$$

$$\xi\bar{\phi}'(\xi) + \bar{\phi}(\xi) = \frac{1}{c}\frac{d}{dt}\left((ct)\bar{\phi}(ct)\right) = \frac{1}{c}c\frac{d}{dt}\left(t\bar{\phi}(ct)\right) = \frac{\partial}{\partial t}\left(t\bar{\phi}(ct)\right) \Big\}$$

where $\delta'(\xi)$ emerges as Z preceding expression is in fact a def of Z derivative, or applying Z L'hopital's rule $\therefore$ Z sol $u(0,t)$ is

$$\bar{u}(0,t) = \frac{\partial v}{\partial r}(0,t) = \frac{\partial}{\partial t}\left[t\,\bar{\phi}(ct)\right] = \frac{\partial}{\partial t}\left[\frac{t}{4\pi(ct)^2}\iint_{|\vec{x}|=ct} \phi(\vec{x})\,dS\right]$$

$$\left\{ \bar{u}(0,t) = \frac{d}{dt}\left(t\bar{\phi}(ct)\right) = \frac{\partial}{\partial t}\left[t\bar{\phi}(ct)\right] = \cancel{\frac{1}{4\pi}\iint_S u(0,}\right.$$

$$\therefore \bar{\phi}(ct) = \frac{1}{4\pi(ct)^2}\iint_{|\vec{x}|=ct} \phi(\vec{x})\,dS = \bar{u}(r=ct,t) = \frac{1}{4(ct)^2\pi}\iint_S u(\vec{x}=ct,t)\,dS =$$

$$\frac{1}{4(ct)^2\pi}\iint_{|\vec{x}|=ct} u(ct,t)\,dS \qquad \phi(\vec{x}) = u(\vec{x},0) \therefore \frac{1}{4\pi(ct)^2}\iint_{|\vec{x}|=ct}\phi(\vec{x})\,dS = \frac{1}{4\pi(ct)^2}\iint_{|\vec{x}|=ct} u(\vec{x},0)\,dS \Big\}$$

where S denotes Z spherical surf with center $\vec{x}=0$ & radius ct. this can be translated to any spatial pt $\vec{x}_0$ $\therefore$

$$u(\vec{x}_0,t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\iint_{|\vec{x}-\vec{x}_0|=ct} \phi(\vec{x})\,dA\right]$$

where A denotes Z spherical surf with center $\vec{x}_0$ & radius ct.

~~8C~~ for (i): $|\vec{x}_0| > \rho + ct$, $|\vec{x} - \vec{x}_0| = ct$ $\therefore$ $ct =$

$$|\vec{x} - \vec{x}_0| \geq \big||\vec{x}| - |\vec{x}_0|\big| = \big||\vec{x}| - \rho - ct\big|$$

$\exists$ a pt st $\vec{x} - x_0 = ct$ $\therefore$ $|\vec{x}_0| > \rho + |\vec{x} - \vec{x}_0|$

Week 3 sheet / $|\vec{x}_0| > \rho + ct$ ∴

for $|\vec{x}| < \rho$: $\phi(\vec{x}) = Q$ ∴

$$u(\vec{x}_0, t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\iint_A Q \, dA\right]$$

for $|\vec{x}| > \rho$: $\phi(\vec{x}) = 0$ ∴

$$u(\vec{x}_0, t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t^2}\iint_A 0 \, dA\right] = \frac{\partial}{\partial t} 0 = 0$$

for case (ii) : $|\vec{x}_0| + ct < \rho$ ∴ $|\vec{x}_0| < \rho - ct$, $|\vec{x}_0| \geq 0$ ∴

$ct \geq 0$, $\rho > 0$ ∴ $\rho > ct$ ∴

for $|\vec{x}| \cancel{>} \rho$: $\phi(\vec{x}) = 0$ ∴ $u(\vec{x}_0, t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t^2}\iint_A \cancel{Q} \, dA\right] = 0$

~~for $|\vec{x}| > \rho$~~

~~for $|\vec{x}| < \rho$~~ : $|\vec{x}_0| + ct < \rho < |\vec{x}|$ ∴ $|\vec{x}_0| < |\vec{x}| - ct$ ∴ $|\vec{x}_0| < |\vec{x}|$

for $|\vec{x}| < \rho$: $\phi(\vec{x}) = Q$ $u(\vec{x}_0, t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t^2}\iint_A Q \, dA\right]$

8c Sol / (i) when $|\vec{x}_0| > \rho + ct$, the intersection of the sphere of radius ct centered at $\vec{x}_0$ & the sphere of radius ρ centered at the origin is empty $u(\vec{x}_0, t) = 0$

{∵ $\rho + ct > ct$ ∴ sphere center $\rho + ct$, radius ct can not touch sphere center 0 , radius ρ}

(ii) when $|\vec{x}_0| + ct \cancel{>} \rho$, A lies within the sphere of radius ρ

{$|\vec{x}_0| + ct < \rho$ ∴ sphere of center $\vec{x}_0$, radius ct is inside sphere center 0 , radius ρ} ∴ $u(\vec{x}_0, t) = \frac{\partial}{\partial t}\left[\frac{1}{4\pi c^2 t}\int_{|\vec{x} - \vec{x}_0| = ct} Q \, dA\right] =$

$$\frac{\partial}{\partial t}\left[\frac{Q}{4\pi c^2 t}\int_{|\vec{x} - \vec{x}_0| = ct} dA\right] = \frac{\partial}{\partial t}\left[\frac{Q}{4\pi c^2 t}(4\pi (ct)^2)\right] = \left(\frac{\partial}{\partial t}\left[\frac{Q}{4\pi c^2 t}\int_r dA\right] = \right.$$

$\frac{\partial}{\partial t}\left[\frac{Q}{4\pi c^2 t} 4\pi r^2\right]$ } ∵ $4\pi r^2$ is area of sphere }

which gives $u(\vec{x}_0, t) = \frac{\partial}{\partial t}(Qt) = Q$ {$u(\vec{x}_0, t) = \frac{\partial}{\partial t}\left[\frac{Q}{4\pi c^2 t} 4\pi c^2 t^2\right] =$

$\frac{\partial}{\partial t}\left[\frac{Q}{c^2 t} c^2 t^2 = \frac{\partial}{\partial t}[Qt]\right] = Q$ }

\Week 5 Sheet/

\1a/ $3u_{xx}+6u_{xy}+3u_{yy}-u_x-4u_y+u=0$ $\therefore$

$3u_{xx}+6u_{xy}+3u_{yy}=4u_x-4u_y-u=f(u_x,u_y,u)$ $\therefore$

discriminant is $D=b^2-ac$ for

$au_{xx}+2bu_{xy}+cu_{yy}=f(u_x,u_y,u)$ $\therefore$

$a=3$, $2b=6$ $\therefore$ $b=3$, $c=3$ $\therefore$

$D=(3)^2-3\times3=9-9=0$ $\therefore$ eqn is elliptic $\therefore$

\1a sol/ $a\left(\frac{dy}{dx}\right)^2-2b\frac{dy}{dx}+c$ $\therefore$ charac eqn is $3\left(\frac{dy}{dx}\right)^2-6\frac{dy}{dx}+3=0$

which is a quadratic eqn for $y'(x)$, has one double root

$\left\{\frac{dy}{dx}=\frac{+6\pm\sqrt{36-4\times3\times3}}{2\times3}=1\pm\sqrt{0}=1 \quad \because \left(3\frac{dy}{dx}-3\right)\left(\frac{dy}{dx}-1\right)=0=3\left[\left(\frac{dy}{dx}\right)^2-2\frac{dy}{dx}+1\right]=0\right\}$

$\therefore$ Z PDE is parabolic (or argue $b^2-ac=(6/2)^2-3\cdot3=0$ $\therefore$ eqn is parabolic

\1a cont/ charac eqn $a\left(\frac{dy}{dx}\right)^2-2b\frac{dy}{dx}+c=0$

\1a sol/ Z charac eqn is $3\left(\frac{dy}{dx}\right)^2-6\frac{dy}{dx}+3=0$ which has a quadratic eqn for $y'(x)$, has one double root $\therefore$ Z PDE parabolic (or equiv: $b^2-ac=(6/2)^2-3\cdot3=0$) Z charac eqn is solved as $3\left(\frac{dy}{dx}-1\right)^2=0$ $\therefore$ $\frac{dy}{dx}=1$ $\therefore$ $y=x+\xi$ $\therefore$ $\xi=y-x$

& Z 2nd characteristic can be chosen arbitly as long as it is independent of ξ; so $\eta=x$ is a valid choice: in that case

$\frac{\partial(\xi,\eta)}{\partial(x,y)}=\begin{vmatrix}-1 & 1\\ 1 & 0\end{vmatrix}=-1\neq0$ for independence

\1b sol/ by chain rule: $u_x=u_\xi\xi_x+u_\eta\eta_x=u_\eta-u_\xi$

$u_y=u_\xi\xi_y+u_\eta\eta_y=u_\xi$

$u_{xx}=\left(\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\xi}\right)^2u=u_{\eta\eta}-2u_{\xi\eta}+u_{\xi\xi}$

$u_{xy}=\left(\frac{\partial}{\partial\xi}\right)\left(\frac{\partial}{\partial\eta}-\frac{\partial}{\partial\xi}\right)u=u_{\xi\eta}-u_{\xi\xi}$ $\quad u_{yy}=u_{\xi\xi}$ $\therefore$ subbing into eqn yields Z canonical form $u_{\eta\eta}=u_\xi+\frac{1}{3}u_\eta-\frac{1}{3}u=0$ QED

\1c sol/ Z sub yields: $u(\xi,\eta)=e^{\lambda\xi+\mu\eta}v(\xi,\eta)$.

$u_\xi=e^{\lambda\xi+\mu\eta}(v_\xi+\lambda v)$ $\quad u_\eta=e^{\lambda\xi+\mu\eta}(v_\eta+\mu v)$

$u_{\eta\eta}=e^{\lambda\xi+\mu\eta}(v_{\eta\eta}+2\mu v_\eta+\mu^2v)$ $\therefore$

~~Week 5 sheet~~

$(v_{\eta\eta}+2\mu v_\eta+\mu^2 v)=(v_\xi+\lambda v)+\frac{1}{3}(v_\eta+\mu v)-\frac{1}{3}v$ or

$v_{\eta\eta}=v_\xi+v_\eta\left(\frac{1}{3}-2\mu\right)+v\left(\lambda+\frac{1}{3}\mu-\frac{1}{3}-\mu^2\right)$ ∴ need to arrange that

$\frac{1}{3}-2\mu=0$, $\lambda+\frac{1}{3}\mu-\frac{1}{3}-\mu^2=0$ ∴

$\mu=\frac{1}{6}$, $\lambda=\mu^2+\frac{1}{3}-\frac{1}{3}\mu=\frac{11}{36}$ ∴ Z eqn for v is

$v_\xi=\nu v_{\eta\eta}$ for $\nu=1$

~~Z Sol~~ let u_1, u_2 be two sols of Z eqn, that is:

$u_{1_t}-Du_{1_{xx}}=f$ $u_1(0,t)=u_1(L,t)$, $u_{1_x}(0,t)=u_{1_x}(L,t)$

$u_{2_t}-Du_{2_{xx}}=f$ $u_2(0,t)=u_2(L,t)$ $u_{2_x}(0,t)=u_{2_x}(L,t)$ ∴ Z

difference $w=u_1-u_2$ satisfies $w_t-Dw_{xx}=0$, $\{w_t=Dw_{xx}\}$

$0\le x\le L$; $w(0,t)=w(L,t)$, $w_x(0,t)=w_x(L,t)$. in terms of

$w(x,t)$, Z LHS in Z required inequality is $E(t)=\int_0^L w^2(x,t)\,dt$

diffing it wrt time, have, with account of Z PDE satisfied by w

$\frac{dE}{dt}=2\int_0^L ww_t\,dx=2D\int_0^L ww_{xx}\,dx$ which can be transformed using

integration by parts formula as $\ldots=2D\left\{\left[ww_x\right]_0^L-\int_0^L w_x^2\,dx\right\}$

according to Z BCs, $w(0,t)=w(L,t)$ & $w_x(0,t)=w_x(L,t)$ ∴

$w(0,t)w_x(0,t)=w(L,t)w_x(L,t)$, that is $\left[ww_x\right]_0^L=0$, so we can

continue that $\frac{dE}{dt}=\ldots=-2D\int_0^L w_x^2\,dx\le 0$ ∵ ~~In~~ of non-nega func

$w_x^2(x,t)$ is non-nega ∴ $E(t)$ is non-increasing, so $E(t)\le E(0)$

$\left\{\frac{dE(t)}{dt}\le 0 \therefore E(t_2)<E(t_1) \text{ for } t_1<t_2\right\}$ ∴ $E(t)\le E(0)$ which renders

Z requested inequility with $C=1$, after subing Z def of E in

terms of W & w in terms of u_1 & u_2. Clearly Z inequility becomes

equality for $t=0$ & $C=1$, & making $C<1$ would make inequility

false already at $t=0$ $\left\{\int_0^L[u_1(x,0)-u_2(x,0)]^2\,dx\le C\int_0^L[\phi_1(x)-\phi_2(x)]^2\,dx\right.$

$\therefore \int_0^L[\phi_1(x)-\phi_2(x)]^2\,dx\le C\int_0^L[\phi_1(x)-\phi_2(x)]^2\,dx$ is equality for $\left.C=1\right\}$,

& making $C<1$ would make inequality false already at $t=0$ & ∴

$C=1$ is Z smallest val.

\3 sol/ it is convenient to reformulate Z required property of Z sol as $u(1+x,t)=-u(1-x,t)$ $x\in[-1,1]$, $t\in[0,\infty)$

$\{u(x,t)=-u(2-(x),t)$ $\therefore u((x+1),t)=-u(2-(x+1),t)=-u(1-x,t)\}$

that is Z sol is expected to be odd about $x=1$ $\{g(-x)=-g(x)$ but about $x=1\}$ Z key observation is that Z IC $f(x)=u(x,0)$ has this property: $f(1-x)=(1-x)((1-x)-1)((1-x)-2)=(1-x)(-x)(1+x)$, $f(1+x)=(1+x)((1+x)-1)((1+x)-2)=(1+x)(x)(1-x)=-f(1-x)$ $\forall x$, so Z required property is true for $t=0$ & we need to show that this remains true for $t>0$. Let $v(\xi,t)=-u(x,t)$ where $\xi=2-x$. Shall show $v(x,t)\equiv u(x,t)$ which is equiv to Z required $u(x,t)\equiv-u(2-x,t)$ first, look at Z BCs:

$v(0,t)=-u(2,t)=0$ & similarly $v(2,t)=-u(0,t)=0$ so $v(\xi,t)$ satisfies Z same Dirichlet BCs as $u(x,t)$. For Z IC:

$v(\xi,0)=-u(2-\xi,0)=-f(2-\xi)=f(\xi)$ according to Z already established property of f. So Z IC for $v(\xi,t)$ is also exactly Z same as that for $u(x,t)$.

now look at Z eqn for $v(\xi,t)$. note $\partial\xi/\partial x=-1$ $\therefore$

$v_{\xi\xi}=\partial_x((-u)_x)=\frac{\partial}{\partial x}(v_\xi)=-v_{\xi\xi}$ & $v_t=-u_t$ $\therefore v_t=v_{\xi\xi}$ $\therefore$

$v(\xi,t)$ satisfies Z same eqn with Z same initial & boundary data as $u(x,t)$, up to Z names of Z variabs. $\therefore$ Z sol of Z given BVP is unique, sure v must be Z same as sure u, as promised □

\4a sol/ this can be asserted by subing Z proposed sol into Z given ODE, BCs & IC, & using Z assumed properties of $u_{1,2}(x,t)$.

For Z ODE, have $LHS=u_{3_t}=\partial_t(u_1+u_2)=u_{1_t}+u_{2_t}=u_{1_{xx}}+u_{2_{xx}}=\partial_{xx}(u_1+u_2)=u_{3_{xx}}=RHS$, by linearity of derivs & using Z assumps that u_1 & u_2 satisfy Z ODE. For Z left BC, have

$u_3(0,t)=u_1(0,t)+u_2(0,t)=\alpha_1+\alpha_2=\alpha_3$. For Z right BC similarly:

$u_3(L,t)=u_1(L,t)+u_2(L,t)=\beta_1+\beta_2=\beta_3$ & finally for Z IC

$u_3(x,0)=u_1(x,0)+u_2(x,0)=f_1(x)+f_2(x)=f_3(x)$ where we have use Z assumps that u_1 & u_2 satisfy their respective BCs & ICs.

\Week 5 Sheet/ Z given fncn satisfies Z required ODE, Z BCs & Z IC, ie is a sol □

\4 b sol/ finding sols of Z form $u(x,t)=P(x)Q(t)$, ie two ODEs linked via a const param λ: $Q'(t)+\lambda Q(t)=0$, $P''(x)+\lambda P(x)=0$ & Z ODE for $P(x)$ has BCs $P'(0)=P'(L)=0$

Z BVP for $P(x)$ has non trivial sols (eigfuncs) $P_n(x)=\cos(n\pi x/L)$ for Z param vals $\lambda_n=n^2\pi^2/L^2$, $n=0,1,2,3,\ldots$ Z corres time part of Z sol then works out as $Q_n(t)=e^{-\lambda_n t}=e^{-(n^2\pi^2/L^2)t}$, up to an arbit const factor. Linear combination of such sols $P_n(x)Q_n(t)$ make a family of sols satisfying Z PDE & Z homog BCs: $u(x,t)=A_0+\sum_{n=1}^{\infty}A_n\exp\left[-\frac{n^2\pi^2}{L^2}t\right]\cos\left(\frac{n\pi x}{L}\right)$ if $f(x)$ is given, then A_n can be found via standard formulas for Z cosine series, emerging from this answer at $t=0$, that is $A_0=\frac{1}{L}\int_0^L f(x)\,dx$, $A_n=\frac{2}{L}\int_0^L f(x)\cos\left(\frac{n\pi x}{L}\right)dx$, $n\geq 1$

\4 c sol/ $\because u_t=0$, since $u(x,t)$ is in fact func of only x, $u(x,t)=U(x)$ subbing that into PDE gives: $U''=0$, $U'(0)=H$, $U'(L)=H$, $U(x)=f(x)$. Z G.S of Z ODE is $U(x)=C_1+C_2x$ $\therefore$ both BCs demand $C_2=H$, & C_1 remains undetermined $\therefore$ Z sol is $u(x,t)=C_1+Hx$ $C_1\in\mathbb{R}$, const C_1 is arbit ie cannot be established based on Z given data. incidentally $f(x)=C_1+Hx$ too

\4 d sol/ using Z proposition proved in part (a), let us consider
- sol to part (b) as $u_1(x,t)$, corres to $\alpha_1=\beta_1=0$ & $f_1(x)$ still to be identified;
- sol of part (c) as $u_2(x,t)$ corres to $\alpha_2=\beta_2=H$ & $f_2(x)=C_1+Hx$
- sol of Z present probl as $u_3(x,t)$, corres to $\alpha_3=\beta_3=H$ & $f_3(x)=0$. clearly have $\alpha_3=\alpha_1+\alpha_2$ & $\beta_3=\beta_1+\beta_2$

$\therefore$ to apply Z proposition from (a), only need to arrange that $f_3(x)=f_1(x)+f_2(x)$. $\because$ $f_3(x)$ is a fixed fn & $f_2(x)=C_1+Hx$ is fixed up to an additive const C_1, just need to set

$f_1(x) = f_3(x) - f_2(x) = -C_3 - Hx$ then according to results to part (b),

$u_1(x,t) = A_0 + \sum_{n=1}^{\infty} A_n \exp\left[-\frac{n^2\pi^2}{L^2}t\right]\cos\left(\frac{n\pi x}{L}\right),$

$A_0 = \frac{1}{L}\int_0^L f_1(x)\,dx$, $A_n = \frac{2}{L}\int_0^L f_1(x)\cos\left(\frac{n\pi x}{L}\right)dx$, $n \geq 1$ computing of Z integrals gives $A_0 = \frac{1}{L}\int_0^L(-C_3 - Hx)\,dx = -C_3 - \frac{1}{2}HL$,

$A_n = \frac{2}{L}\int_0^L(-C_3 - Hx)\cos\left(\frac{n\pi x}{L}\right)dx = \cdots = \begin{cases} 4HL/n^2\pi^2, & n = 2m+1 \\ 0, & n = 2m \end{cases}$

combining ∴ found results together : $u(x,t) = u_3(x,t) =$
$u_1(x,t) + u_2(x,t) =$
$H\left(x - \frac{L}{2}\right) + \frac{4}{\pi^2}HL\sum_{m=0}^{\infty}\frac{1}{(2m+1)^2}\cos\left(\frac{(2m+1)\pi x}{L}\right)\exp\left[-\frac{(2m+1)^2\pi^2}{L^2}t\right]$

note C_3 in u_2 cancelled with $-C_3$ in u_1 ∴ Z sol to this probl is unique despite Z sol in (c) defined only upto a const, Z probl with given BCs conserves Z total heat in Z syst, & this amount is determined by ICs, which have been specified here but not in (c)

\5 sol/ consider Z probl for Z unknown func $v(x,t)$:

$v_t - Dv_{xx} = 0 \quad x \in (-\infty, \infty) \qquad v(x,0) = Q\delta(x - x_0) + Q\delta(x + x_0)$ ⊛

Z eqn is invariant wrt $x \mapsto -x$ & Z IC is even in x ∴ Z sol is even in x. ∴ Z sol of this probl is even & contly diffable at $x=0$, we have $v_x(0,t) = 0 \quad \forall t \geq 0$ ∴ Z restriction of $v(x,t)$ to $x \geq 0$ provides sol $u(x,t)$ to Z original probl.

Z sol to ⊛ is given by $v(x,t) = \frac{1}{\sqrt{4D\pi t}}\int_{-\infty}^{\infty} v(s,0)e^{-(x-s)^2/(4Dt)}\,ds$ Subbing here Z IC $v(s,0)$: $v(x,t) =$

$\frac{1}{\sqrt{4D\pi t}}\left\{Q\int_{-\infty}^{\infty}\delta(s - x_0)e^{-(x-s)^2/(4Dt)}\,ds + Q\int_{-\infty}^{\infty}\delta(s + x_0)e^{-(x-s)^2/(4Dt)}\,ds\right\}$

which using Z properties of δ-func evaluated as: $v(x,t) =$
$\frac{Q}{\sqrt{4D\pi t}}\left\{e^{-(x-x_0)^2/(4Dt)} + e^{-(x+x_0)^2/(4Dt)}\right\} = u(x,t)$ for $x \geq 0$.

viscid burgers eqn / $u_t + uu_x - u_{xx} = 0$ is nonlinear but is is semilinear $\because$ u_{xx} is highest order & has linear coeffs

$\therefore$ is a hyperbolic func eg parabolic

Self-similar sols: $u \mapsto \lambda \tilde{u}$, $x \mapsto \mu \tilde{x}$; $t \mapsto \nu \tilde{t}$

choose λ, μ, ν st

$\tilde{u}_{\tilde{t}} + \tilde{u}\tilde{u}_{\tilde{x}} - \tilde{u}_{\tilde{x}\tilde{x}} = 0$

~~use~~ this sols to find sols of Z form

$u(x,t) = t^{\alpha} f(\xi)$; $\xi = xt^{\beta}$, α, β const to be found ; $f(\cdot)$ is a func to be found

(Say is $t = \nu\tilde{t}$; $x = \nu^{-\beta}\tilde{x}$

$u = \nu^{\alpha}\tilde{u} \Rightarrow \tilde{u} = \tilde{t}^{\alpha} f(\tilde{x}\tilde{t}^{\beta})$)

Sub into PDE ; $u = t^{\alpha} f(\xi)$; $\xi = xt^{\beta}$

$\frac{\partial u}{\partial x} = t^{\alpha} f'(\xi)\frac{\partial \xi}{\partial x} = t^{\alpha+\beta} f'(\xi)$ $\{x = \xi t^{-\beta}\}$

$u_{xx} = \dots = t^{\alpha+2\beta} f''(\xi)$

$u_t = \alpha t^{\alpha-1} f(\xi) + t^{\alpha} f'(\xi)\xi_t = t^{\alpha}\beta t^{\beta-1} \underbrace{x}_{\xi t^{-\beta}} f'(\xi) + \alpha t^{\alpha-1} f(\xi)$

$= \beta\xi t^{\alpha-1} f'(\xi) + \alpha t^{\alpha-1} f(\xi) = t^{\alpha-1}[\beta\xi f'(\xi) + \alpha f(\xi)]$

$u_t + uu_x - u_{xx} = t^{\alpha-1}[\beta\xi f'(\xi) + \alpha f(\xi)] + t^{2\alpha+\beta} f(\xi)f'(\xi) - t^{\alpha+2\beta} f''(\xi) = 0$ $\forall t, \xi$

need $\alpha - 1 = 2\alpha + \beta = \alpha + 2\beta \Longrightarrow \alpha = -\frac{1}{2}$, $\beta = -\frac{1}{2}$ $\therefore$

$f'' - f'f + \frac{1}{2}\xi f' + \frac{1}{2} f = 0$ ODE for $f(\xi)$

propagating waves / $u \mapsto \tilde{u}$, $x \mapsto \tilde{x} - a$, $t \mapsto \tilde{t} - \beta$

look for sols $u(x,t) = U(\xi)$ $\xi = x - ct$ Speed c = const

$c = ?$ $U(\xi) = ?$

$u_t = -cU(\xi)$ $u_x = U'(\xi)$ $u_{xx} = U''(\xi)$

$\therefore u_t + uu_x - u_{xx} = 0$ $\therefore -cU'(\xi) + UU'(\xi) - U''(\xi) = 0$

ODE for c = const ~~any~~

for apps $|U(\xi)| < \infty$ $\forall \xi$ ODE autonomous - (doesn't depend on ξ)

Standard reduc order subs: / $\frac{dU}{d\xi} = p(U)$ $p(\cdot)$ to be found

$\frac{d^2U}{d\xi^2} = \frac{dp}{dU}\underbrace{\frac{dU}{d\xi}}_{p} = p\frac{dp}{dU}$ Sub into ODE: $-cp + Up - p\frac{dp}{dU} = 0$ $\therefore$

1st order ODE

$P\left(-\frac{dP}{dU}+U-C\right)=0$ $\therefore P=0$ or $\frac{dP}{dU}=U-C$

$\Downarrow$

$U=\text{const}$ $\therefore P=\int(U-C)\,dU=$

$\frac{1}{2}U^2-CU+D$ $D\in\mathbb{R}$

$=\frac{dU}{d\xi}$

$|U(\pm\infty)|<\infty$ $\therefore U(\infty)=\text{const}$, $U(-\infty)=\text{const}$

$U'(\pm\infty)=0$ $\therefore$ $\frac{1}{2}U^2-CU+D=0$ should have roots

So $P(U)=\frac{1}{2}(U-A)(U-B)$ $\therefore A+B=2C$, $AB=2D$

$\frac{dU}{d\xi}=\frac{1}{2}(U-A)(U-B)$ $\int\frac{dU}{(U-A)(U-B)}=\int\frac{1}{2}d\xi$ by partial fracs

$\therefore \frac{1}{A-B}\ln\left|\frac{U-A}{U-B}\right|=\frac{\xi-\xi_0}{2}$ $\xi_0\in\mathbb{R}$

resolve wrt $U(\xi)$ $\therefore$ ~~$U=A-B\varepsilon$~~ $U=\frac{A-B\varepsilon}{1-\varepsilon}$

where $\varepsilon=\pm\exp\left[\frac{(A-B)(\xi-\xi_0)}{2}\right]$

$|U(\xi)|<\infty$ $\forall\xi$ $\therefore$ $\varepsilon=-\exp[\quad]$ $\therefore$

$U(\xi)=u(x,t)=\frac{A+B\exp[(A-B)(\xi-\xi_0)/2]}{1+\exp[(A-B)(\xi-\xi_0)/2]}$,

$\xi=x-Ct=x-\frac{A+B}{2}t$ $C=\frac{A+B}{2}$ propagate from

VB: GS/ $u_t+uu_x-u_{xx}=0$ "Cole Hopf" transformation

in two steps: ①: $u\mapsto w$ $u=\frac{\partial w}{\partial x}$ $\therefore$

$u_t=w_{tx}$, $u_x=w_{xx}$, $u_{xx}=w_{xxx}$ $\therefore$ Sub:

$w_{tx}+w_xw_{xx}-w_{xxx}=0=\frac{\partial}{\partial x}\left[w_t+\frac{1}{2}w_x^2-w_{xx}\right]$ $\therefore$ int by x:

$w_t+\frac{1}{2}w_x^2-w_{xx}=Q(t)$ without loss of generality $Q(t)\equiv 0$

otherwise $w(x,t)=\tilde{w}(x,t)+\int Q(t)\,dt$ $\Rightarrow$

$\tilde{w}_t+\frac{1}{2}\tilde{w}_x^2-\tilde{w}_{xx}=0$ $\therefore$

$w_t+\frac{1}{2}w_x^2-w_{xx}=0$ (also called Burgers eqn, alternative form)

②: nonlinear sols: $w\mapsto\phi$; $w=\alpha\ln\phi$ $\{\phi>0\ \forall x,t$, $\alpha=$ const TBD$\}$

$\therefore$ ~~$w_t=\alpha\phi_t$~~ $w_t=\alpha\frac{\phi_t}{\phi}$, $w_x=\alpha\frac{\phi_x}{\phi}$ $\therefore$

$w_{xx} = \alpha \frac{\theta\theta_{xx} - \theta_x\theta_x}{\theta^2} = \alpha\left[\frac{\theta_{xx}}{\theta} - \frac{\theta_x^2}{\theta^2}\right]$ ∴ Sub in:

- $w_t + \frac{1}{2}w_x^2 - w_{xx} = 0$ ∴

$$\alpha\frac{\theta_t}{\theta} + \underbrace{\frac{1}{2}\alpha^2\frac{\theta_x^2}{\theta^2} - \alpha\frac{\theta_{xx}}{\theta} + \alpha\frac{\theta_x^2}{\theta^2}}_{\text{nonlinear}} = 0$$

∴ choose $\frac{1}{2}\alpha^2 + \alpha = 0$ ∴

$\alpha = 0$ or $\alpha = -2$ ($\alpha = 0$ not interesting) ∴ $\alpha = -2$ ∴

then $\theta_t - \theta_{xx} = 0$ is 1D HE know how to solve ∴

For any sol $\theta(x,t)$ $u(x,t) = \frac{\partial}{\partial x}\left[-2\ln\theta\right] = -2\frac{\theta_x}{\theta}$

eg ~~$\theta(x,t) = e^{Ax+A^2t}$~~

$\theta(x,t) = C_1 e^{Ax+A^2t} + C_2 e^{Bx+B^2t}$ ∴ gives earlier sol, where

ξ_0 defined by C_1, C_2 $c = (A+B)/2$ $\xi = x - ct$

$u(x,t) = U(\xi)$ $\xi = x - ct$ ~~$u = \dots$~~

Korteweg–de Vries eqn (KdV) / $u_t + uu_x + u_{xxx} = 0$

eg shallow layer of fluid $u \equiv$ height of this layer

propagating wave sols

$u(x,t) = U(\xi)$; $\xi = x - ct$ $x \in \mathbb{R}$

- BC: $u(\pm\infty, t) = 0$ $u_x(\pm\infty, t) = 0$ $u_{xx}(\pm\infty, t) = 0$

∴ $U(\pm\infty) = 0$; $U'(\pm\infty) = 0$ $U''(\pm\infty) = 0$

$u_t = -cU'(\xi)$ $u_x = U'(\xi)$ $u_{xxx} = U'''(\xi)$ ∴ Sub in:

$U''' + UU' - cU' = 0$

note: $\frac{d}{d\xi}\left[U'' + \frac{1}{2}U^2 - cU\right] = 0$ ∴

$U'' + \frac{1}{2}U^2 - cU = A = \text{const}$

BC: $U(\pm\infty) = 0$; $U''(\pm\infty) = 0$ $\Rightarrow A = 0$ ∴ $\{c > 0\}$

- $U'' + \frac{1}{2}U^2 - cU = 0$ ∴ For convenience $U(\xi) \to V(z)$ ∴

where $z = \xi\sqrt{c}$; $U = -3cV$ ∴ by chain rule:

$\frac{d^2V}{dz^2} = V + \frac{3}{2}V^2$ $\frac{dU}{d\xi} = \frac{dU}{dV}\frac{dV}{dz}\frac{dz}{d\xi} = (-3c)\frac{dV}{dz}\sqrt{c} = -3c^{3/2}\frac{dV}{dz}$ ∴

$\frac{d^2U}{d\xi^2} = 9C^3\frac{d^2v}{dz^2}$ ∴ $\frac{d^2U}{d\xi^2} + \frac{1}{2}U^2 - CU = 0$ ∴

$9C^3\frac{d^2v}{dz^2} + \frac{1}{2}(-3Cv)^2 - C(-3Cv) = 0$ ∴

$9C^3\frac{d^2v}{dz^2} + \frac{9C^2}{2}v^2 + 3C^3v = 0$

∴ $\frac{d^2v}{dz^2} = v + \frac{3}{2}v^2$ ∴ reduction of order :

$\frac{dv}{dz} = p(v)$ ∴ $\frac{d^2v}{dz^2} = \frac{dp}{dv}\frac{dv}{dz} = p\frac{dp}{dv}$ ∴

$p\frac{dp}{dv} = v + \frac{3}{2}v^2$ ∴ $\int p\,dp = \int\left(v + \frac{3}{2}v^2\right)dv$ ∴

$\frac{p^2}{2} = \frac{v^2}{2} + \frac{v^3}{2} + \frac{B}{2}$ ∴ $p = \frac{dv}{dz}$ ∴

BC $v(\pm\infty) = 0$; $v'(\pm\infty) = 0 = p$ ∴ $B = 0$ ∴

$p^2 = v^2 + v^3 = \left(\frac{dv}{dz}\right)^2$ ∴

$\frac{dv}{dz} = \pm\sqrt{v^2+v^3}$ ∴ $\frac{dv}{dz} = \sigma v\sqrt{v+1}$ $\sigma = \pm 1$ ∴

$\int\frac{dv}{v\sqrt{v+1}} = \int\sigma\,dz$ ∴ $\sigma(z - z_0) = \int\frac{dv}{v\sqrt{v+1}}$ $\left\{ w = \sqrt{v+1} \therefore v+1 = w^2 \therefore v = w^2-1 \therefore dv = 2w\,dw \right\}$

$= \int\frac{2\cancel{w}\,dw}{(w^2-1)\cancel{w}} = \ln\left|\frac{w-1}{w-1}\right|$ resolve wrt w:

$w = \frac{1+De^{\sigma z}}{1-De^{\sigma z}}$ where $D = \pm e^{-\sigma z_0}$

For no nonzero element $D = -e^{-\sigma z_0}$

~~u_t~~ kdv $u_t + uu_x + u_{xxx} = 0$

propagating wave sols $u(x,t) = U(\xi)$ $\xi = x - ct$ ...

$\frac{d^2U}{d\xi^2} + \frac{1}{2}U^2 \cancel{\xi} - CU = 0$

$U(\xi) = -3Cv(z)$ $z = \xi\sqrt{C}$

$\frac{dU}{d\xi} = -3C\frac{dv}{dz}\frac{dz}{d\xi} = -3C^{3/2}\frac{dv}{dz}$

$\frac{d^2U}{d\xi^2} = -3C^{3/2}\frac{d^2v}{dz^2}\frac{dz}{d\xi} = -3C^2\frac{d^2v}{dz^2}$ ∴

$U^2 = 9C^2v^2$ sub into ODE :

$$-3\cancel{c^2}\frac{d^2V}{dz^2}+\frac{9}{2}\cancel{c^2}V^2+3c^2V=0$$

$$\frac{d^2V}{dz^2}=\frac{3}{2}V^2+V \quad \text{as promised}$$

$$\int_V \frac{dV}{V\sqrt{V+1}}=\int \sigma\, dz \quad \{\sigma=\pm1\}$$

$\therefore w=\sqrt{V-1}$ $\therefore$ ...

$$w=\frac{1+De^{\sigma z}}{1-De^{\sigma z}} \text{, where } D=\text{const}=\pm e^{-\sigma z_0} \quad \{z_0=\text{const}\}$$

$|w(z)|<\infty \ \forall z \Rightarrow$ take "$-$" take minus

Let $e^{\sigma(z-z_0)}=\varepsilon^2>0$ $\therefore$

$$w=\frac{1-\varepsilon^2}{1+\varepsilon^2} \qquad V=w^2-1=\left(\frac{1-\varepsilon^2}{1+\varepsilon^2}\right)^2-1$$

$$=\dots=-\left(\frac{2}{\varepsilon+\varepsilon^{-1}}\right)^2 \quad \because = \frac{1-2\varepsilon^2+\varepsilon^4-(1-2\varepsilon^2-\varepsilon^4}{(1+\varepsilon^2)^2}=\frac{-4\varepsilon^2}{(1+\varepsilon^2)^2}=$$

$$-\left(\frac{2\varepsilon}{1+\varepsilon^2}\right)^2=-\left(\frac{2}{\varepsilon+\varepsilon^{-1}}\right)^2$$

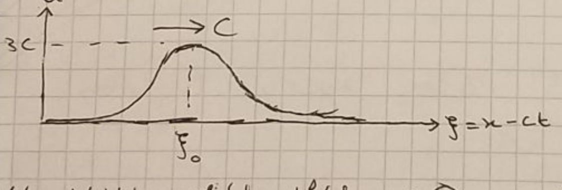
$$\varepsilon=e^{\sigma(z-z_0)/2}$$

reminder: ~~eh~~ $\cosh x=\dfrac{e^x+e^{-x}}{2}$ $\quad \therefore \dfrac{2}{e^x+e^{-x}}=\dfrac{1}{\cosh x}=\operatorname{sech} x$

$\therefore V(z)=-\operatorname{sech}^2(\sigma(z-z_0)/2)$ $\therefore$

~~U~~ $U=-3cV(z)$ $/z=\xi\sqrt{c}$ $=3c\operatorname{sech}^2\left(\frac{\sqrt{c}}{2}(\xi-\xi_0)\right)=u(x,t)$

$\{\xi_0=z_0/\sqrt{c}\}$ $\therefore$ $\xi=x+ct$



"Sol"

$u_t+uu_x-\alpha u_{xx}+\beta u_{xxx}=0 \qquad \alpha=1,\beta=0 \Rightarrow$ V. Burgers

$\alpha=0,\beta=1$: KdV

$\therefore u_t=-uu_x+\alpha u_{xx}-\beta u_{xxx}$

$M=\int_{-\infty}^{\infty}u\,dx \quad \therefore \dfrac{dM}{dt}=\int_{-\infty}^{\infty}u_t\,dx=$ (appropriate BC)

$$-\int uu_x\,dx+\alpha\underbrace{\int u_{xx}\,dx}_{I_1}-\beta\underbrace{\int u_{xxx}\,dx}_{I_2} \quad \therefore$$

$-\int u u_x \, dx = -\frac{1}{2}\left[u^2\right]_{-\infty}^{\infty} = 0$

$I_1 = \int_{-\infty}^{\infty} u_{xx} \, dx = \left[u_x\right]_{-\infty}^{\infty} = 0$

$I_2 = \int_{-\infty}^{\infty} u_{xxx} \, dx = \left[u_{xx}\right]_{-\infty}^{\infty} = 0$ $\therefore$

$\frac{dM}{dt} = 0$ both VB & kdV

$E = \int_{-\infty}^{\infty} \frac{u^2}{2} \, dx \qquad \frac{dE}{dt} = \int_{-\infty}^{\infty} u u_t \, dx =$

$$-\underbrace{\int_{-\infty}^{\infty} u^2 u_x \, dx}_{=\left[\frac{u^3}{3}\right]_{-\infty}^{\infty} = 0} + \alpha \underbrace{\int_{-\infty}^{\infty} u u_{xx} \, dx}_{I_3} - \beta \underbrace{\int_{-\infty}^{\infty} u u_{xxx} \, dx}_{I_4}$$

$$I_3 = \int_{-\infty}^{\infty} u u_{xx} \, dx = \int_{-\infty}^{\infty} u \, d(u_x) = \underbrace{\left[u u_x\right]_{-\infty}^{\infty}}_{=0} - \int_{-\infty}^{\infty} u_x^2 \, dx \leq 0$$

$$I_4 = \int_{-\infty}^{\infty} u u_{xxx} \, dx = \int_{-\infty}^{\infty} u \, d(u_{xx}) = \underbrace{\left[u u_{xx}\right]_{-\infty}^{\infty}}_{=0 \text{ by BC}} - \int_{-\infty}^{\infty} u_{xx} u_x \, dx$$

$$= -\int_{x=-\infty}^{\infty} u_x \, d(u_x) = -\left[\frac{u_x^2}{2}\right]_{-\infty}^{\infty} = 0 \text{ by BC}$$

so: For VB: $\frac{dE}{dt} \leq 0$ (as in 1DHE)

for kdV $\frac{dE}{dt} = 0$

also: If u is thickness of a layer, B: total momentum, & total energy as

$W(t) = \int_{-\infty}^{\infty} \left(\frac{1}{2} u_x^2 - \frac{1}{6} u^3\right) dx$

$\frac{dW}{dt} = \cdots = 0$

\revision: / \Sheet 2/ \Q4c/

$u_{\xi\eta} = -\frac{1}{12}\left[2\sqrt{3}(u_\xi + u_\eta) + u\right]$ $u(\xi,\eta) = e^{\lambda\xi+\mu\eta} v(\xi,\eta)$

$u_\xi = \lambda e^{\lambda\xi+\mu\eta} v + e^{\lambda\xi+\mu\eta} v_\xi$

$u_\eta = \mu e^{\lambda\xi+\mu\eta} v + e^{\lambda\xi+\mu\eta} v_\eta$

$v_{\xi\eta} = \mu\lambda e^{\lambda\xi+\mu\eta} v + \mu e^{\lambda\xi+\mu\eta} v_\xi + \lambda e^{\lambda\xi+\mu\eta} v_\eta + e^{\lambda\xi+\mu\eta} v_{\xi\eta}$

$e^{\lambda\xi+\mu\eta}$: common factor ∴

$v_{\xi\eta} + \lambda v_\eta + \mu v_\xi + \lambda\mu v = -\frac{1}{12}\left[2\sqrt{3}(v_\xi + \lambda v) + 2\sqrt{3}(v_\eta + \mu v) + v\right]$ ∴ simplify:

$v_{\xi\eta} + v_\xi\left[\mu + \frac{\sqrt{3}}{6}\right] + v_\eta\left[\lambda + \frac{\sqrt{3}}{6}\right] + v\left[\lambda\mu + \frac{\sqrt{3}}{6}\lambda + \frac{\sqrt{3}}{6}\mu + \frac{1}{12}\right] = 0$

Let $\mu = -\sqrt{3}/6$; $\lambda = -\sqrt{3}/6$ ∴

$A = \left(\frac{\sqrt{3}}{6}\right)^2 - \left(\frac{\sqrt{3}}{6}\right)^2 - \left(\frac{\sqrt{3}}{6}\right)^2 + \frac{1}{12} = -\left(\frac{\sqrt{3}}{6}\right)^2 + \frac{1}{12} = -\frac{3}{36} + \frac{1}{12} = 0$ ∴

$v_{\xi\eta} = 0$ □ QED

\Kirchhoff's formula for 3DWE/ $c^2\nabla^2 u = u_{tt}$

$u(\bar{x}, 0) = 0$; $u_t(\bar{x}, 0) = \psi(\bar{x})$

$u(\bar{x}, t) = ?$

consider $u(\bar{0}, t) = ?$ (WLOG)

spherical ball: $V = \{\bar{x} \mid |\bar{x}| < R\}$

$\int_V u_{tt}\, dV = c^2 \int_V \nabla^2 u\, dV = c^2 \int_S \nabla u \cdot \vec{n}\, dS = \int_V u_{tt}\, dV = \frac{\partial^2}{\partial t^2}\int_V u\, dV =$

$c^2 \int_{\partial V} \nabla u \cdot \vec{n}\, dS = c^2 \int_{\partial V} \frac{\partial u}{\partial r}\, dS$ ∴ let us use spher coord:

$dV = r^2\, dr\, d\Omega$: $d\Omega = \sin\phi\, d\phi\, d\theta$ "solid angle

$dS = R^2\, d\Omega$ ∴

$\frac{\partial^2}{\partial t^2}\int_0^R \left(\int_\Omega u\, d\Omega\right) r^2\, dr = c^2 \int_\Omega u_r R^2\, d\Omega = c^2 R^2 \frac{\partial}{\partial r}\left(\int_\Omega u\, d\Omega\right)\Big|_{r = \Omega}$

"spherical coord" $\bar{u}(r,t) = \frac{1}{4\pi}\int_\Omega u(r,\theta,\phi)\, d\Omega = \frac{1}{4\pi r^2}\int_{|\vec{r}|} u(\vec{r})\, d^2 S$

$\frac{\partial^2}{\partial t^2}\int_0^R \bar{u}(r,t)\, dr = c^2 R^2 \frac{\partial}{\partial r} u(r,t)\Big|_{r=R}$

$\frac{\partial^2 u}{\partial t^2} = \frac{\partial^2 \bar{u}}{\partial R^2} c^2 R^2 + 2c^2 R \frac{\partial \bar{u}}{\partial R}$ $R \to r$ $\frac{\partial^2 \bar{u}}{\partial t^2} = c^2 r^2 \frac{\partial^2 \bar{u}}{\partial r^2} + 2c^2 r \frac{\partial \bar{u}}{\partial r}$

$v(r,t) = r\bar{u}(r,t)$ ∴

$\frac{\partial^2 v}{\partial t^2} = r\frac{\partial^2 \bar{u}}{\partial t^2}$ $\qquad \frac{\partial v}{\partial r} = r\frac{\partial \bar{u}}{\partial r} + \bar{u}$ $\therefore$

$\frac{\partial^2 v}{\partial r^2} = r\frac{\partial^2 \bar{u}}{\partial r^2} + \underbrace{\frac{\partial \bar{u}}{\partial r} + \frac{\partial \bar{u}}{\partial r}}_{\times 2}$ $\quad \therefore \frac{\partial^2 v}{\partial t^2} = c^2 \frac{\partial^2 v}{\partial r^2}$ is 1DWE! $r \in (0, \infty)$

BC at $r=0$:

$v = r\bar{u}$; $r = 0 \Rightarrow v = 0$ $\qquad \frac{\partial^2 u}{\partial r^2} = c^2 \frac{\partial^2 v}{\partial r^2}$ $\quad v(0, t)$

odd extension to $r \in (-\infty, \infty)$

~~and~~ ~~use~~ D'A' formula

$v(r,t) = \frac{\partial}{\partial t}\left[\frac{1}{2c}\int_{r+ct}^{r+ct}\overset{=0}{\phi(s)}\,ds + \frac{1}{2c}\int_{ct-r}^{ct+r}\psi(s)\,ds\right]$ (need $r=0$)

$v(r,t) = \frac{1}{2c}\int_{ct-r}^{ct+r}\bar{\psi}(s)\,ds$

$\bar{\psi}(s) = v(s,0) = s\,\bar{u}(s,0)$

$v(r,t) = \frac{1}{2c}\int_{ct-r}^{ct+r} s\,\bar{u}(s,0)\,ds$ $\qquad v(0,t) = 0$ (BC)

need $\bar{u}(0,t) = ?$ $\qquad \bar{u}(r,t) = \frac{1}{r}v(r,t)$ $\therefore$

$\bar{u}(r,t) = \lim_{r\to 0}\frac{1}{r}v(r,t)$ $\quad$ {L'hopitals ~~rode~~: }

$= \lim_{r\to 0} v_r(r,t) = \frac{1}{2c}\lim_{r\to 0}\left[(ct+r)\,\bar{u}(ct+r,0) - (ct-r)\,\bar{u}(ct-r,0)\right]$

$= \frac{1}{2c}\left[2ct\,\bar{u}(ct,0)\right] = t\,\bar{u}(ct,0)$

$\therefore u(0,t) = \frac{1}{4\pi(ct)^2}\int_{|\vec{r}|=ct}\psi(\vec{r},0)\,d^2s$ $\quad \square$

\final sheet/ \Q3b/ $u_t + 6uu_x + u_{xxx} = 0$ $\therefore$

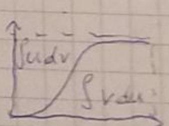
$u_t = -6uu_x - u_{xxx}$ $\quad \therefore W = \int_{-\infty}^{\infty}(2u^3 - u_x^2)\,dx$

$\frac{dW}{dt} = \int_{-\infty}^{\infty}(6u^2u_t - 2u_xu_{xt})\,dx = \int_{-\infty}^{\infty}6u^2u_t\,dx - 2\overbrace{\int_{-\infty}^{\infty}u_x\,d[u_t]}^{A} =$ $\quad$ { $[u_xu_t]_{-\infty}^{\infty} = 0$

$\int_{-\infty}^{\infty}6u^2u_t\,dx + 2\int_{-\infty}^{\infty}u_tu_{xx}\,dx = \int_{-\infty}^{\infty}(6u^2 + 2u_{xx})\,u_t\,dx =$

$-\int_{-\infty}^{\infty}(6u^2 + 2u_{xx})(6uu_x + u_{xxx})\,dx = -I_1 - I_2 - I_3 - I_4$

{ IBP: $\int uv'\,dx = [uv] - \int vu'\,dx = \int u\,dv = uv - \int v\,du$



$\left\{ \therefore A: \int_{-\infty}^{\infty} u_x u_{xt}\,dx \right.$ $\quad \left\{ \int U W'\,dx = UW - \int W U'\,dx \therefore U = u_x \quad W = u_t \right\}$

$= \cancel{[u_x u_t] \int_{-\infty}^{\infty} u_t u_{xx}}$

$= \underbrace{\left[u_x u_t\right]_{-\infty}^{\infty}}_{=0\,(\text{by BC})} - \int_{-\infty}^{\infty} u_t u_{xx}\,dx \Big\}$

$\therefore I_1 = 3 \times \int_{-\infty}^{\infty} u^2 \cdot u u_x\,dx = 36 \int_{-\infty}^{\infty} u^3\,du = 9\left[u^4\right]_{-\infty}^{\infty} = 0$

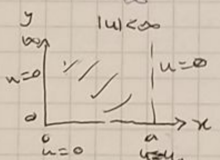
$I_2 = 12 \int_{-\infty}^{\infty} u u_x u_{xx}\,dx = \ldots$

$I_3 = 6 \int_{-\infty}^{\infty} u^2 u_{xxx}\,dx = \ldots$

$I_4 = 2\int_{-\infty}^{\infty} u_{xx} u_{xxx}\,dx = 2\int_{-\infty}^{\infty} u_{xx}\,d\left[u_{xx}\right] = \frac{1}{2}\left[u_{xx}^2\right]_{-\infty}^{\infty} = 0$

$I_3 = 6\int_{-\infty}^{\infty} u^2\,d\left[u_{xx}\right] = -12\int_{-\infty}^{\infty} u_{xx} u u_x\,dx \Rightarrow I_2 + I_3 = 0 \Rightarrow$

$\frac{dW}{dt} = 0 \quad \square$



\Q4/ $u_{xx} + u_{yy} = 0$ $\qquad u(0,y) = 0,\ u(a,y) = 0$

$u(x,0) = -\frac{u_0}{a} x$

$|u(x,\infty)| < \infty.$

\Sol/ $u(x,y) = X(x)Y(y) \qquad X''(x)Y(y) + Y''(y)X(x) = 0 \therefore$

$\frac{X''(x)}{X(x)} = -\frac{Y''(y)}{Y(y)} = -\lambda = \text{Constant}$

$X''(x) + \lambda X(x) = 0 \qquad X(0) = 0 \ ; \ X(a) = 0$

$\lambda = \left(\frac{n\pi}{a}\right)^2 \qquad X_n = \sin\left(\frac{n\pi x}{a}\right) \qquad n \in \mathbb{Z}_+$

$Y''(y) - \lambda Y(y) = 0 \qquad \lambda = (n\pi/a)^2$

$Y(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}$

$|Y(\infty)| < \infty \Rightarrow A_n = 0 \qquad \therefore LC \therefore$

$u(x,y) = \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) e^{-n\pi y/a}$ $\quad$ Satisfies homog BC

$(x=0;\ x=a;\ y=\infty)$

For $y=0$: $\quad \sum_{n=1}^{\infty} B_n \sin\left(\frac{n\pi x}{a}\right) = u(x,0) = -u_0 x/a \therefore$

$$B_n = \frac{2}{a}\int_0^a u(x,0)\sin\left(\frac{n\pi k}{a}\right)dx = -\frac{2u_0}{a^2}\int_0^a x\sin\left(\frac{n\pi x}{a}\right)dx = \dots =$$

$$\frac{2u_0}{n\pi}(-1)^n \quad \therefore$$

$$u = \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi k}{a}\right)e^{-n\pi y/a} \Rightarrow$$

$$u = -\frac{2u_0}{\pi}\arctan\left(\frac{\sin(\pi x/a)}{\cos(\pi x/a)+\exp(\pi y/a)}\right) \quad \therefore$$

$$\Phi = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\sin\left(\frac{n\pi x}{a}\right)e^{-n\pi y/a}$$

$$\psi = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{\frac{in\pi x}{a}}e^{-n\pi y/a}$$ (by eulers eqn $\emptyset = I_m(\psi)$)

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{\frac{in\pi}{a}(x+iy)} \qquad \{z = x+iy;\}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}\left(e^{i\pi z/a}\right)^n \qquad \left\{Z = \frac{i\pi z}{a}\right\}$$

$$= \sum_{n=1}^{\infty}\frac{(-1)^n}{n}Z^n$$

$u_{xx}+u_{yy}=0 \qquad x\in(0,a) \quad y\in(0,+\infty)$

$u(0,y)=u(a,y)=0 \qquad |u(x,\infty)|<\infty;$

$u(x,0) = -\frac{u_0}{a}x$

\Sol/ $u = \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{\frac{-n\pi y}{a}}\sin\left(\frac{n\pi x}{a}\right) = \frac{2u_0}{\pi}\Phi$

$\Phi = I_m\psi \quad ; \psi = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}Z^n$

$Z = e^{i\pi z/a} \quad ; z = x+iy$

$\psi = \sum_{n=1}^{\infty}\frac{1}{n}(-Z)^n = \psi(Z)$

$\frac{d\psi}{dZ} = -\sum_{n=1}^{\infty}(-Z)^n = -\frac{1}{1+Z} \quad \therefore$

$\psi = \int\frac{d\psi}{dZ}dZ = -\ln(1+Z)+A$

AB $Z=0 \Rightarrow \psi=0 \Rightarrow A=0 \quad \therefore$

$\psi = -\ln(1-Z)$

reminder $\Phi = I_m\psi \quad ; Z = \exp\left(\frac{i\pi}{a}(x+iy)\right)$

$\Psi = \Xi + i\,\Phi \qquad \Xi = Re(\Psi)$

$e^{-\Psi} = 1 + Z$

$e^{-\Xi}(\cos\phi + i\sin\phi) = 1 + e^{-y\pi/a}\left[\cos\left(\frac{\pi x}{a}\right) + i\sin\left(\frac{\pi x}{a}\right)\right]$

$1 + e^{-y\pi/a}\cos\left(\frac{\pi x}{a}\right) = e^{-\Xi}\cos(\phi)$

$e^{-y\pi/a}\sin\left(\frac{\pi x}{a}\right) = e^{-\Xi}\sin(\phi)$

divide: $\tan\phi = \dfrac{e^{-y\pi/a}\sin\left(\frac{\pi x}{a}\right)}{1 + e^{-y\pi/a}\cos\left(\frac{\pi x}{a}\right)} =$

$\dfrac{\sin(\pi x/a)}{\cos\left(\frac{\pi x}{a}\right) + e^{y\pi/a}}$

$\phi = \arctan\left[\dfrac{\sin(\pi x/a)}{\cos\left(\frac{\pi x}{a}\right) + e^{\pi y/a}}\right]$ $\qquad u = \frac{2u_0}{\pi}\Phi \quad \square$ QED

\CH1Q8, P06Q1 / $u_t + uu_x = 0$

first $u \to C \qquad u_t + C u_x = 0$

$u = f(x - Ct$

now $C \to u$

$u = u(x,t)$? what is corres $x = (u, t=0)$

$p = p(x,t)$ ∴ coord at x axis, crossing

2 charac Ω (x,t)

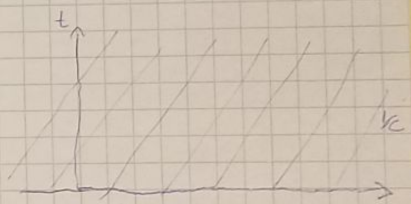
I C: $u(x,0) = f(x)$

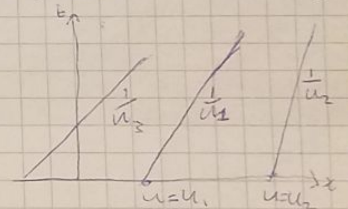
2 slope of charac is $\frac{1}{u(p,0)} = \frac{1}{f(p)}$ ∴

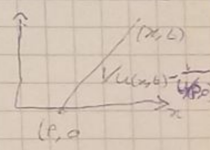
$x = p + t\, f(p)$

if f is given, then ∴ defines $p = p(x,t)$

$\frac{\partial u}{\partial x} = \cdots = \dfrac{f'(p(x,t))}{\underbrace{1 + t f'(p(x,t))}} \to = 0 \Rightarrow \frac{\partial u}{\partial x} = \infty$







\May2021 PP / $yu_x - xu_y = 0 \qquad (y, -x)\cdot\nabla u = 0$

$(dx, dy) \parallel (y, -x) \quad \cancel{\frac{dy}{dx} = -\frac{y}{x}} \quad \frac{dx}{y} = -\frac{dy}{x}$ ∴ $\int y\,dy = -\int x\,dx$

$y^2 = -x^2 + C \qquad x^2 + y^2 = C$ implicit sol ∴

GS: $u = f(x^2+y^2)$

BC (i) $u(x, \sqrt{4+x^2}) = x^2$ $\therefore y = \sqrt{4+x^2}$

$u(x,y) = f(x^2+y^2) = f(\underbrace{x^2+4+x^2}_{s}) = x^2$ $\therefore$

$f(s) = x^2$; $s = 4+2x^2$ $\therefore x^2 = \frac{s-4}{2}$ $\therefore$

$f(s) = \frac{s-4}{2}$ $\therefore$

$u(x,y) = \frac{x^2+y^2-4}{2}$

(ii) $y = \sqrt{4-x^2}$ $u(x,y) = f(x^2+y^2) = f(x^2+4-x^2) =$

$= f(4) = x^2$ - impossible

$\therefore$ no sols

(Boundary is charac line)

\Week 5 sheet/

\6 Sol/ $u(x,t)=v(x,t)h(t)$ gives $u_t = h(t)v_t + h'(t)v$, ~~u_x~~ $u_x = h(t)v_x$,

$u_{xx} = h(t)v_{xx}$ Subing them into eqn:

$h(t)v_t + h'(t)v - Dh(t)v_{xx} + e^{-pt}h(t)v = 0$ Z 2nd & 4th term will ~~be~~ cancel out if $h'(t)/h(t) = -e^{-pt}$ which is a 1st order linear ODE for $h(t)$, which can be solved say by seperation of vars $\therefore$

$h(t) = A\exp(\frac{1}{p}e^{-pt})$ (choice of A is immaterial as long as $A \neq 0$)

$\therefore$ Z common factor $h(t)$ can be cancelled out & have:

$v_t - Dv_{xx} = 0$ according to Z sub made, Z IC for v is

$v(x,0) = \frac{1}{h(0)}u(x,0) = \frac{e^{-1/p}}{A}\phi(x)$ $\therefore$ $v(x,t) =$

$\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty}\frac{e^{-1/p}}{A}\phi(s)e^{-(x-s)^2/(4Dt)}\,ds$ $\therefore$ Sol is $u(x,t) = v(x,t)h(t) =$

$\frac{1}{\sqrt{4\pi Dt}}\exp\left(\frac{e^{-pt}-1}{p}\right)\int_{-\infty}^{\infty}\phi(s)e^{-(x-s)^2/(4Dt)}\,ds$

\7 Sol/ Let $u(r,t) = v(r,t)/r$; then $u_t = \frac{1}{r}v_t$, $u_r = \frac{1}{r}v_r - \frac{1}{r^2}v$,

$u_{rr} = \frac{1}{r}v_{rr} - \frac{2}{r^2}v_r + \frac{2}{r^3}v$ & Z eqn becomes: $v_t = Dv_{rr}$ $0 \le r \le R$

note: Z transformation $v = ru$ was used at some pt in derivation of Kirchhoff formula as Z spatial part, ie Z 3 dimen Laplacian is Z same there & in Z current case $\therefore$ $v(r,t) = ru(r,t)$ & $|u(0,t)| < \infty$, Z BCs are $v(0,t)=0$, $v(R,t)=0$ using seperation of variables sub $v(r,t) = X(r)T(t)$, have an ODE for time part

$\frac{dT}{dt} = -\lambda DT$ & a BVP for space part $X'' + \lambda X = 0$, $X(0)=0$, $X(R)=0$

by Z standard method, obtain sols of this BVP as

$X_n(r) = \sin(\frac{n\pi r}{R})$, $n = 1,2,3,\ldots$ & then Z sol to this part is

$T_n(t) = \exp(-\frac{Dn^2\pi^2 t}{R^2})$ (up to a const factor) this yields Z GS for Z PDE+BC $v(r,t) = \sum_{n=1}^{\infty} C_n \exp(-\frac{Dn^2\pi^2 t}{R^2})\sin(\frac{n\pi r}{R})$ Z IC for

Z transformed eqn is $v(r,0) = ru(r,0) = 4\sin^3(\frac{\pi r}{R})$ which means

$\sum_{n=1}^{\infty} C_n \sin(\frac{n\pi r}{R}) = 4\sin^3(\frac{\pi r}{R})$ using Z trig identity

$\sin^3(x) = \frac{1}{4}(3\sin(x) - \sin(3x))$

for $x=\pi r/R$ immediately obtain by observation that

$C_n = \begin{cases} 3, & n=1 \\ -1, & n=3 \\ 0, & \text{otherwise} \end{cases}$ Z ~~soln~~ sol for Z IVP for $u(r,t)=v(r,t)/r$ is

$$u(r,t) = \frac{1}{r}\sum_{n=1}^{\infty} C_n \exp\left(-\frac{Dn^2\pi^2 t}{R^2}\right)\sin\frac{n\pi r}{R} =$$

$$\frac{3}{r}\exp\left(-\frac{D\pi^2 t}{R^2}\right)\sin\left(\frac{\pi r}{R}\right) - \frac{1}{r}\exp\left(-\frac{9D\pi^2 t}{R^2}\right)\sin\left(\frac{3\pi r}{R}\right)$$

\Week 7 sheet/

\1i sol/ under Z transformation $\xi = x+a$, $\eta = y+b$ ∴

$u_x = u_\xi$, $u_{xx} = u_{\xi\xi}$ $u_y = u_\eta$ $u_{yy} = u_{\eta\eta}$ ie

$u_{xx}+u_{yy} = u_{\xi\xi}+u_{\eta\eta} = 0$ Z eqn is invariant under translation in Z plane.

\1ii sol/ under reflection in Z x-direction $\xi = -x$, $\eta = y$

have: $u_x = -u_\xi$, $u_{xx} = u_{\xi\xi}$ $u_y = u_\eta$ $u_{yy} = u_{\eta\eta}$ ie

$u_{xx}+u_{yy} = u_{\xi\xi}+u_{\eta\eta} = 0$ Z eqn is invariant under reflections in Z x-direction. similarly for reflections in y direction

\1iii/ under rotations in Z plane through an angle θ

$\xi = x\cos\theta + y\sin\theta$, $\eta = -x\sin\theta + y\cos\theta$ ∴

$u_x = u_\xi\cos\theta - u_\eta\sin\theta$, $u_y = u_\xi\sin\theta + u_\eta\cos\theta$

$u_{xx} = \partial_\xi(u_\xi\cos\theta - u_\eta\sin\theta)\cos\theta - \partial_\eta(u_\xi\cos\theta - u_\eta\sin\theta)\sin\theta$

$u_{xx} = (u_{\xi\xi}\cos\theta - u_{\xi\eta}\sin\theta)\cos\theta - (u_{\xi\eta}\cos\theta - u_{\eta\eta}\sin\theta)\sin\theta$

$u_{xx} = u_{\xi\xi}\cos^2\theta - 2u_{\xi\eta}\sin\theta\cos\theta + u_{\eta\eta}\sin^2\theta$ similarly

$u_{yy} = \partial_\xi(u_\xi\sin\theta + u_\eta\cos\theta)\sin\theta + \partial_\eta(u_\xi\sin\theta + u_\eta\cos\theta)\cos\theta$

$u_{yy} = u_{\xi\xi}\sin^2\theta + 2u_{\xi\eta}\sin\theta\cos\theta + u_{\eta\eta}\cos^2\theta$ leads to

$u_{xx}+u_{yy} = (u_{\xi\xi}+u_{\eta\eta})(\sin^2\theta+\cos^2\theta) = u_{\xi\xi}+u_{\eta\eta} = 0$ ie Z eqn is invariant under rotation in Z plane. Z Laplace operator is associated with any isotropic sol that is without preferred direction

Week 7 Sheet

7/ Let $u(r,t) = v(r,t)/r$ $\therefore$

$u_t = \frac{1}{r}v_t$, $u_r = \frac{1}{r}v_r - \frac{1}{r^2}v$, $u_{rr} = \frac{1}{r}v_{rr} - \frac{2}{r^2}v_r + \frac{2}{r^3}V$ $\therefore$

i) $\frac{\partial v}{\partial t} = Dv_{rr}$ $\quad 0 \le r \le R$ $\quad \because v = ru$ $\therefore$

$v(r,t) = ru(r,t)$, $|u(0,t)| < \infty$

BC: $v(0,t) = 0$, $v(R,t) = 0$ $\therefore$

$v(r,t) = X(r)T(t)$ $\therefore$ $\frac{dT}{dt} = -\lambda DT$,

$X'' + \lambda X = 0$, $X(0) = 0$ $X(R) = 0$ $\therefore$

$X_n(r) = \sin\left(\frac{n\pi r}{R}\right)$, $n \in \mathbb{N}$ $\therefore$

$T_n(t) = \exp\left(-\frac{Dn^2\pi^2 t}{R^2}\right)$ upto a constant factor $\therefore$

$$v(r,t) = \sum_{n=1}^{\infty} C_n \exp\left(-\frac{Dn^2\pi^2 t}{R^2}\right)\sin\left(\frac{n\pi r}{R}\right)$$

IC: $v(r,0) = ru(r,0) = 4\sin^3\left(\frac{\pi r}{R}\right)$ $\therefore$

$v(r,0) = \sum_{n=1}^{\infty} C_n(e^0)\sin\left(\frac{n\pi r}{R}\right) = \sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi r}{R}\right) = 4\sin^3\left(\frac{\pi r}{R}\right)$ $\therefore$

$\sin^3(x) = \frac{1}{4}(3\sin x - \sin(3x))$ $\therefore$

$\sum_{n=1}^{\infty} C_n \sin\left(\frac{n\pi r}{R}\right) = 4\left[\frac{1}{4}\left(3\sin\frac{\pi r}{R} - \sin\left(3\frac{\pi r}{R}\right)\right)\right] = 3\sin x - \sin\left(\frac{3\pi r}{R}\right)$ $\therefore$

$\therefore C_n = \begin{cases} 3, & n=1 \\ -1, & n=3 \\ 0, & \text{otherwise} \end{cases}$ $\quad \therefore$ $u(r,t) = v(r,t)/r$ $\therefore$

$$u(r,t) = \frac{1}{r}\sum_{n=1}^{\infty} C_n \exp\left(-\frac{Dn^2\pi^2 t}{R^2}\right)\sin\left(\frac{n\pi r}{R}\right) =$$

$$\frac{3}{r}\exp\left(-\frac{D\pi^2 t}{R^2}\right)\sin\left(\frac{\pi r}{R}\right) - \frac{1}{r}\exp\left(\frac{-9D\pi^2 t}{R^2}\right)\sin\left(\frac{3\pi r}{R}\right)$$

CW2 Q8:

$u_t = D\left(u_{rr} + \frac{2}{r}u_r\right)$ $\quad 0 \le r \le R$ $\quad D > 0$

$|u(0,t)| < \infty$ $\quad u_r(R,t) = 0$ $\quad$ Let $u(r,t) = \frac{w(r,t)}{r}$

$u_t = \frac{1}{r}w_t$, $u_r = \frac{1}{r}w_r - \frac{1}{r^2}w$, $u_{rr} = \frac{1}{r}w_{rr} - \frac{2}{r^2}w_r + \frac{2}{r^3}w$ $\therefore$

$w_t = Dw_{rr}$ $\quad 0 \le r < R$ $\quad w(r,t) = ru(r,t)$ $\quad |u(0,t)| < \infty$,

$u_r(R,t) = 0$ $\therefore$ $w(0,t) = 0u(0,t) = 0$ $\quad w_r = ru_r + u$ $\therefore$

$u_r = \frac{1}{r}w_r - \frac{1}{r^2}w$ $\therefore$ $r^2u_r = rw_r - w$ $\therefore$

at $r = R$: $R^2u_r(R,t) = Rw_r(R,t) - w(R,t) = R^2(0) = 0$ $\therefore$

$w = X(r)T(t)$ $\therefore$ $T'(t)X(r) = DX''(r)T(t) = -\lambda$ $\therefore$

$T' + D\lambda T = 0$, $X'' + \lambda X = 0$ $\therefore$

Case 1: $\lambda < 0$ let $p = \sqrt{-\lambda} = 0$ $\therefore$

$X'' - p^2 X = 0$ $\therefore X(r) = C_1 \cosh(pr) + C_2 \sinh(pr)$ $\therefore$

$r = 0$ $\therefore X(0) - 0$ $\therefore$ $C_1 \cosh(p \times 0) + C_2 \sinh(p \times 0) = 0$ $\therefore C_1 = 0$

$\therefore X'(r) = p[C_1 \sinh(pr) + C_2 \cosh(pr)]$ $\therefore$ $RX'(R) - X(R) = 0$ $\therefore$

$pR C_2 \cosh(pR) - C_2 \sinh(pR) = 0$ $\therefore$ $C_2(pR \cosh(pR) - \sinh(pR)] = 0$ $\therefore$

$C_2 = 0$ $\therefore C_1 = C_2 = 0$ $\therefore$ no eigvals

Case 2: $\lambda = 0$ $\therefore$ $X'' = 0$ $\therefore X(r) = C_1 + C_2 r$ $\therefore$

$X(0) = 0$ $\therefore C_1 = 0$ $\therefore$ $RX'(R) - X(R) = 0$ $\therefore$ $RC_2 - RC_2 = 0$ $\therefore$

C_2 is arbitrary $\therefore$ Let $C_2 = 1$ $\therefore$ $\lambda = 0$ is an eigenvalue $\therefore$

$X(r) = r$ is the corresponding eigfunc $X(r) = 0 + 1r = r$

Case 3: $\lambda > 0$: let $\omega = \sqrt{\lambda}$ $\therefore X'' + \omega^2 \lambda = 0$ $\therefore$

$X(r) = C_1 \cos(\omega r) + C_2 \sin(\omega r)$ $\therefore$ BC1: $X(0) = 0$ $\therefore C_1 = 0$

BC2: $RX'(R) - X(R) = 0$ $\therefore$

$RC_2 \omega \cos(\omega R) - C_2 \sin(\omega R) = 0$ $\therefore$

$R\omega C_2 \cos(\omega R) - C_2 \frac{\cos(\omega R)}{\cos(\omega R)} \sin(\omega R) = R\omega C_2 \cos(\omega R) - C_2 \cos(\omega R) \tan(\omega R) = 0$

$\therefore C_2 \cos(\omega R)(R\omega - \tan(\omega R)) = 0$ $\therefore$ Let ω_n equal the n^{th}

sol to $\tan(\omega R) = \omega R$ $\therefore$ have C_2 is arbitrary. $\therefore$ let $C_2 = 1$

$\therefore$ $X_n(r) = \sin(\omega_n r)$ $\therefore$

For $T' + \omega_n^2 DT = 0$ $T_n(t) = Ce^{-\omega_n^2 Dt}$ $\therefore$ Z GS is:

$$W(r,t) = A_0 r + \sum_{n=0}^{\infty} B_n \sin(\omega_n r) \exp(-\omega_n^2 Dt) \quad \therefore$$

$$u(r,t) = \frac{1}{r} W = A_0 + \frac{1}{r} \sum_{n=0}^{\infty} B_n \sin(\omega_n r) \exp(-\omega_n^2 Dt) \quad \therefore$$

$$u(r,0) = A_0 + \frac{1}{r} \sum_{n=0}^{\infty} B_n \sin(\omega_n r) = 2 + \frac{3}{r} \sin\left(\frac{qr}{R}\right)$$, where $\tan q = q$ $\therefore$

$$A_0 r + \sum_{n=0}^{\infty} B_n \sin(\omega_n r) = 2r + 3\sin\left(\frac{qr}{R}\right) \quad \therefore$$

$$\sum_{n=0}^{\infty} B_n \sin(\omega_n r) = r(2 + A_0) + 3\sin\left(\frac{qr}{R}\right)$$ have eigfuncs

$\phi(n) = \sin(\omega_n r)\,dr$ $\therefore \int_0^R \sin^2(\omega_n r)\,dr = \frac{1}{2}\int_0^R [1 - \cos(2\omega_n r)]\,dr$

$= \frac{R}{2} - \frac{1}{2\omega_n} \tan(\omega_n R) \cos^2(\omega_n R) = \frac{R}{2} - \frac{1}{2\omega_n} R\omega_n \left[\frac{1}{1 + (R\omega_n)^2}\right] = \frac{R}{2} - \frac{R}{2(1 + (R\omega_n)^2)}$

$$\therefore B_n = \left[\frac{R}{2} - \frac{R}{2(1 + (R\omega_n)^2)}\right]^{-1} \cdot \int_0^R r \sin(\omega_n r) + 3\sin(\omega_n r) \sin\left(\frac{qr}{R}\right) dr =$$

$$\left[\frac{R}{2} - \frac{R}{2(1+(R\omega_n)^2)}\right]^{-1} \int_0^R r\sin(\omega_n r) + \frac{3}{2}\cos\left(\omega_n r - \frac{q r}{R}\right) - \frac{3}{2}\cos\left(\omega_n r + \frac{q r}{R}\right) dr =$$

$$\left[\frac{R}{2} - \frac{R}{2(1+(R\omega_n)^2)}\right]^{-1} \left[\sin(\omega_n r) - \frac{r}{\omega_n}\cos(\omega_n r) + \frac{3}{2}R\frac{\sin\left(\omega_n r - \frac{q r}{R}\right)}{R\omega_n - q} - \frac{3}{2}R\frac{\sin\left(\omega_n r + \frac{q r}{R}\right)}{R\omega_n + q}\right]_0^R$$

* $\tan(\omega_n R) = \omega_n R$, set $q = \omega_n R$ ∴

$$\left[\frac{R}{2} - \frac{R}{2(1+q^2)}\right]^{-1}\left[\sin q - \frac{R}{\omega_n}\cos q + \frac{3}{2}\frac{\sin(q-q)}{q-q} - \frac{3}{2}\frac{\sin(q+q)}{2q}\right] =$$

$$\left[\frac{R}{2} - \frac{R}{1+q^2}\right]^{-1}\left[\sin q - \frac{R}{\omega_n}\cos q - \frac{3}{2}\frac{\sin(2q)}{2q}\right] =$$

$$\left[\frac{R}{2} - \frac{R}{2(1+q^2)}\right]^{-1}\left[\sin q - \frac{R}{\omega_n}\cos q - \frac{3}{2q}\sin q\cos q\right] =$$

$$\left[\frac{R}{2} - \frac{R}{2(1+q^2)}\right]^{-1}\left[\sin q - \frac{R}{\omega_n(1+q^2)^{1/2}} - \frac{3\sin q}{2q(1+q^2)^{1/2}}\right] \therefore \text{ sol is:}$$

$$u(r,t) = A_0 + \frac{1}{r}\sum_{n=1}^{\infty}\left[\frac{R}{2} - \frac{R}{2(1+q^2)}\right]^{-1}\left[\sin q - \frac{R}{\omega_n(1+q^2)^{1/2}} - \frac{3\sin q}{2q(1+q^2)^{1/2}}\right]\sin(\omega_n r)\exp(-\omega_n^2 D t)$$

CW2 Q4 ∂V is the boundary of V.

$u^2(\vec{x},t) \geq 0 \ \forall(\vec{x},t)$ ∴ $E(t) \geq 0$ ∴ diffing wrt t:

$\frac{dE}{dt} = \frac{1}{2}\int_V 2uu_t \, dV = \int_V uu_t \, dV$ & $\frac{\partial u}{\partial t} = \nabla^2 u$ ∴

$\frac{dE}{dt} = \int_V u\nabla^2 u \, dV$

using ~~the~~ divergence ~~theorem~~ then: $\int_S u\nabla u \cdot \vec{n} \, dS =$
$\int_V [u\nabla u + \nabla_i \cdot \vec{x}_i] \, dV = \int_V (u\nabla^2 u + |\nabla u|^2) \, dV$ due to this domain.

$S = \partial V$ ∴ $\frac{dE}{dt} = \int_V u\nabla^2 u \, dV = \int_{\partial V} u\nabla u \cdot \vec{n} \, d(\partial V) - \int_V |\nabla u|^2 \, dV$

using BC (a), have $u=0$ when $\vec{x} \in \partial V$ ∴

$\frac{dE}{dt} = -\int_V |\nabla u|^2 \, dV \leq 0$ ∴ $\frac{dE}{dt} \leq 0$ for BC (a)

using BC (b) have $\vec{n}\cdot\nabla u = 0$ when $\vec{x} \in \partial V$ & $\vec{n}\cdot\nabla u = \nabla u \cdot \vec{n}$ ∴

$\frac{dE}{dt} = -\int_V |\nabla u|^2 \, dV \leq 0$ ∴ $\frac{dE}{dt} \leq 0$ for BC (b)

CW2 Q4 ∂V is the boundary of V ∴ diffing wrt t:

$\frac{dE}{dt} = \frac{1}{2}\int_V 2uu_t \, dV = \int_V uu_t \, dV$ & $\frac{\partial u}{\partial t} = \nabla^2 u$ ∴

$\frac{dE}{dt} = \int_V u\nabla^2 u \, dV$ ∴ $\int_S u\nabla u \cdot \vec{n} \, dS = \int_V (u\nabla^2 u + \nabla u \nabla u) \, dV$ ∴

$S = \partial V \quad \therefore \frac{dE}{dt} = \int_V u\nabla^2 u = \int_{\partial V} u\nabla u \cdot \vec{n}\, d(\partial V) - \int_V |\nabla u|^2 dV$

$\therefore$ Dirichlet BC, $u=0$ when $\vec{x} \in \partial V$ $\therefore$

$\frac{dE}{dt} = -\int_V |\nabla u|^2 \quad \therefore \leq 0 \quad \therefore \frac{dE}{dt} \leq 0$ for dirichlet BC

For Neumann BC $\vec{n} \cdot \nabla u = 0$ when $\vec{x} \in \partial V$

$\therefore \frac{dE}{dt} = -\int_V |\nabla u|^2 dV \quad \therefore \leq 0 \quad \therefore$

$\frac{dE}{dt} \leq 0$ for Neumann BC

divergence Theorem: $\int_S F \cdot \hat{n}\, dS = \int_V \nabla \cdot F\, dV$

$\frac{dE}{dt} = \int_V u\nabla^2 u\, dV$

$\therefore \quad \int_S (u\nabla u \cdot \vec{n})\, dS = \int_S (u\nabla u) \cdot \vec{n}\, dS = \int_V (u\nabla^2 u + \nabla u \nabla u) = \int_V (u\nabla^2 u + (\nabla u)^2) dV$

$\int_S (u\nabla u \cdot \vec{n})\, dS = \int_V \nabla \cdot (u\nabla u)\, dV = \left\{ \int_V \nabla u \cdot (\nabla \times u) - u \cdot (\nabla \times \nabla u)\, dV \right\}$

$\nabla^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \quad \therefore$

$\nabla u = \frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} \quad \therefore$

$u\nabla^2 u = u\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) = u\partial_{xx}u + u\partial_{yy}u + u\partial_{zz}u \quad \therefore$

$\nabla u \nabla u = (\partial_x u + \partial_y u + \partial_z u)(\partial_x u + \partial_y u + \partial_z u) =$

$(\partial_x u)^2 + \partial_x u \partial_y u + \partial_x u \partial_z u + \partial_x u \partial_y u + (\partial_y u)^2 + \partial_y u \partial_z u + \partial_x u \partial_z u + \partial_y u \partial_z u + (\partial_z u)^2$

$= 2\partial_x u \partial_y u + 2\partial_x u \partial_z u + 2\partial_y u \partial_z u + (\partial_x u)^2 + (\partial_y u)^2 + (\partial_z u)^2$

$u\nabla u = u(\partial_x u + \partial_y u + \partial_z u) = u\partial_x u + u\partial_y u + u\partial_z u \quad \therefore$

$\nabla \cdot (u\nabla u) = \nabla \cdot (u\partial_x u + u\partial_y u + u\partial_z u) =$

$\partial_x(u\partial_x u) + \partial_y(u\partial_y u) + \partial_z(u\partial_z u) =$

$(\partial_x u)^2 + u\partial_{xx}u + (\partial_y u)^2 + u\partial_{yy}u + (\partial_z u)^2 + u\partial_{zz}u$

$\int_V \nabla \cdot F\, dV = \int_S F \cdot \vec{n}\, dS$

$\int_S (u\nabla u \cdot \vec{n})\, dS = \int_V \nabla \cdot (u\nabla u)\, dV = \int_V [(u\nabla^2 u) + (\partial_x u)^2 + (\partial_y u)^2 + (\partial_z u)^2]\, dV$

$= \int_{\partial V} \vec{n} \cdot u\nabla u)\, d(\partial V) \quad \therefore \int_V u\nabla^2 u\, dV = \int_{\partial V} u\nabla u \cdot \vec{n}\, d(\partial V) - \int (\partial_x u)^2 + (\partial_y u)^2 + (\partial_z u)^2\, dV$

Week 7 Sheet /1i/ $u_{xx}+u_{yy}=0$ $\xi=x+a$, $\eta=y+b$ $\therefore$

$u_x=u_\xi$, $u_{xx}=u_{\xi\xi}$, $u_y=u_\eta$, $u_{yy}=u_{\eta\eta}$ $\therefore$ into PDE:

$u_{xx}+u_{yy}=u_{\xi\xi}+u_{\eta\eta}$ $\therefore$ Z eqn is invariant under translation in Z plane

/1ii/ under reflection in Z x-direction, $\xi=-x$, $\eta=y$ $\therefore$

$u_x=-u_\xi$, $u_{xx}=u_{\xi\xi}$, $u_y=u_\eta$, $u_{yy}=u_{\eta\eta}$ $\therefore$ into PDE:

$u_{xx}+u_{yy}=u_{\xi\xi}+u_{\eta\eta}=0$ Z eqn is invariant under reflections in Z x-direction $\therefore$ by symmetry: also for reflections in Z y direction

/1iii/ under rotations in Z plane through angle θ:

$\xi=x\cos\theta+y\sin\theta$, $\eta=-x\sin\theta+y\cos\theta$ $\therefore$ $u_x=u_\xi\cos\theta-u_\eta\sin\theta$,

$u_y=u_\xi\sin\theta+u_\eta\cos\theta$, $u_{xx}=\frac{\partial}{\partial\xi}(u_\xi\cos\theta-u_\eta\sin\theta)\cos\theta-\frac{\partial}{\partial\eta}(u_\xi\cos\theta-u_\eta\sin\theta)\sin\theta$

$\therefore u_{xx}=(u_{\xi\xi}\cos\theta-u_{\xi\eta}\sin\theta)\cos\theta-(u_{\xi\eta}\cos\theta-u_{\eta\eta}\sin\theta)\sin\theta$ $\therefore$

$u_{xx}=u_{\xi\xi}\cos^2\theta-2u_{\xi\eta}\sin\theta\cos\theta+u_{\eta\eta}\sin^2\theta$

$\{$ is $\cancel{u_x=\frac{\partial}{\partial\xi}(u)\frac{\partial x}{\partial\xi}+\frac{\partial}{\partial\eta}(u)\frac{\partial x}{\partial\eta}}$ X $\therefore$ $\cancel{x_\xi=\cos\theta,\ x_\eta}$ $\}$

$\{$ is $u_x=\frac{\partial}{\partial\xi}(u)\frac{\partial\xi}{\partial x}+\frac{\partial}{\partial\eta}(u)\frac{\partial\eta}{\partial x}$ $\therefore$ $\xi_x=\cos\theta$, $\xi_y=\sin\theta$, $\eta_x=-\sin\theta$, $\eta_y=\cos\theta$ $\therefore$

$u_x=u_\xi\cos\theta-u_\eta\sin\theta$ $\therefore$ $u_y=u_\xi\sin\theta+u_\eta\cos\theta$ $\therefore$

$u_{xx}=\frac{\partial}{\partial x}(u_x)=\frac{\partial}{\partial\xi}(u_x)\frac{\partial\xi}{\partial x}+\frac{\partial}{\partial\eta}(u_x)\frac{\partial\eta}{\partial x}=\frac{\partial}{\partial\xi}(u_x)\cos\theta-\frac{\partial}{\partial\eta}(u_x)\sin\theta=$

$\frac{\partial}{\partial\xi}(u_\xi\cos\theta-u_\eta\sin\theta)\cos\theta-\frac{\partial}{\partial\eta}(u_\xi\cos\theta-u_\eta\sin\theta)\sin\theta=$

$u_{xx}=(u_{\xi\xi}\cos\theta-u_{\xi\eta}\sin\theta)\cos\theta-(u_{\xi\eta}\cos\theta-u_{\eta\eta}\sin\theta)\sin\theta=$

$u_{xx}=u_{\xi\xi}\cos^2\theta-u_{\xi\eta}\sin\theta\cos\theta-u_{\xi\eta}\sin\theta\cos\theta+u_{\eta\eta}\sin\theta\sin\theta=$

$u_{xx}=u_{\xi\xi}\cos^2\theta-2u_{\xi\eta}\sin\theta\cos\theta+u_{\eta\eta}\sin^2\theta$ $\}$ &

$u_{yy}=\frac{\partial}{\partial\xi}(u_\xi\sin\theta+u_\eta\cos\theta)\sin\theta+\frac{\partial}{\partial\eta}(u_\xi\sin\theta+u_\eta\cos\theta)\cos\theta=$

$\frac{\partial}{\partial\xi}(u_y)\frac{\partial\xi}{\partial y}+\frac{\partial}{\partial\eta}(u_y)\eta_y=\frac{\partial}{\partial\xi}(u_y)\sin\theta+\frac{\partial}{\partial\eta}(u_y)\cos\theta=$

$u_{yy}=(u_{\xi\xi}\sin\theta+u_{\xi\eta}\cos\theta)\sin\theta+(u_{\xi\eta}\sin\theta+u_{\eta\eta}\cos\theta)\cos\theta=$

$u_{\xi\xi}\sin^2\theta+u_{\xi\eta}\sin\theta\cos\theta+u_{\xi\eta}\sin\theta\cos\theta+u_{\eta\eta}\cos^2\theta=$

$u_{yy}=u_{\xi\xi}\sin^2\theta+2u_{\xi\eta}\sin\theta\cos\theta+u_{\eta\eta}\cos^2\theta$ $\therefore$ into PDE:

$u_{xx}+u_{yy}=\cancel{u_{\xi\xi}\cos^2\theta-2u_{\xi\eta}\sin\theta\cos\theta+u_{\eta\eta}\sin^2\theta}$

$u_{\xi\xi}\cos^2\theta-2u_{\xi\eta}\sin\theta\cos\theta+u_{\eta\eta}\sin^2\theta+u_{\xi\xi}\sin^2\theta+2u_{\xi\eta}\sin\theta\cos\theta+u_{\eta\eta}\cos^2\theta=$

$u_{\xi\xi}\cos^2\theta+u_{\xi\xi}\sin^2\theta+u_{\eta\eta}\cos^2\theta+u_{\eta\eta}\sin^2\theta=$

$u_{\xi\xi}(\sin^2\theta+\cos^2\theta)+u_{\eta\eta}(\sin^2\theta+\cos^2\theta)=u_{\xi\xi}+u_{\eta\eta}=0$ $\therefore$

Z eqn is invariant under rotation in Z plane. Z Laplace operator is associated with any isotropic situation, that is without preferred direction

$\nabla^2 u = 0 \quad x,y \in (0,1) \quad u(x,0) = f_1(x) \quad u(x,1) = f_2(x),\ u(0,y) = f_3(y),\ u(1,y) = f_4(y)$

first off, we note that due to linearity & homogeneity of Z PDE, Z sol to Z given probl can be obtained as Z sum of sols of four probls, each set in Z unit square, & in each having nontrivial BC only on one side of Z square eg: probl 1: $\nabla^2 u_1 = 0,\ x,y \in (0,1)$,

$u_1(x,0) = f_1(x),\ u_1(x,1) = u_1(0,y) = u_1(1,y) = 0$

probl 2: $\nabla^2 u_2 = 0\ x,y \in (0,1),\ u_2(x,0) = 0,\ u_2(x,1) = f_2(x)$,

$u_2(0,y) = u_2(1,y) = 0$

probl 3: $\nabla^2 u_3 = 0 \quad x,y \in (0,1) \quad u_3(x,0) = u_3(x,1) = 0,\ u_3(0,y) = f_3(y),\ u_3(1,y) = 0$

probl 4: $\nabla^2 u_4 = 0,\ x,y \in (0,1) \quad u_4(x,0) = u_4(x,1) = u_4(0,y) = 0,\ u_4(1,y) = f_4(y)$

& Z sol to Z proble is $u(x,y) = u_1(x,y) + u_2(x,y) + u_3(x,y) + u_4(x,y)$.

Next, we note probl 1 is a special case ~~of~~ for $a=0,\ b=1,\ L=1,\ \&\ f(x) = f_1(x)$,

$\therefore$ Z sol to it is $u_1(x,y) = \sum_{n=1}^{\infty} \frac{F_{1,n}}{\sinh(k_n L)} \sinh(k_n(L-y)) \sin(k_n(x-a)) \Big|_{k_n = n\pi/(b-a)}$

or for Z present vals of Z params

$u_1(x,y) = \sum_{n=1}^{\infty} \frac{F_{1,n}}{\sinh(n\pi)} \sinh(n\pi(1-y)) \sin(n\pi x)$ here & later use notation

$F_{k,n}$ for Z n-th Fourier coeffs of Z k-th boundary data:

$F_{k,n} = \frac{2}{b-a} \int_a^b f_k(s) \sin\left(\frac{n\pi(x-a)}{b-a}\right) ds = 2\int_0^1 f_k(s) \sin(n\pi s)\, ds$

probl 2 can be transformed to probl 1 (up to Z names of Z variabls) by a transformation $(x,y) \mapsto (x, 1-y)$, which is a combination of reflection in Z y direction & a translation by Z vec $(0,1)$, $\therefore$ Z above invariance results apply. that is, Z formula for u_2 is obtained from Z above formula for u_1 by replacing y with $1-y$ & $F_{1,n}$ with $F_{2,n}$:

$u_2(x,y) = \sum_{n=1}^{\infty} \frac{F_{2,n}}{\sinh(n\pi)} \sinh(n\pi y) \sin(n\pi x)$.

probl 3 is transformed to proble 1 by swapping Z coords $(x,y) \mapsto (y,x)$ which can be achived by a rotation by $90°$ & a reflection, $\therefore$ Z above invariance results apply, & we deduce that

$u_3(x,y) = \sum_{n=1}^{\infty} \frac{F_{3,n}}{\sinh(n\pi)} \sinh(n\pi(1-x)) \sin(n\pi y)$.

\Week 7 Sheet/ likewise probl 4 is identical probl 2 up to Z swap of Z coords, ∴

n) $u_4(x,y)=\sum_{n=1}^{\infty}\frac{F_{4,n}}{\sinh(n\pi)}\sinh(n\pi x)\sin(n\pi y)$. finally, Z sol to Z original probl is Z sum of Z sols,

$$u(x,y)=\sum_{n=1}^{\infty}\frac{1}{\sinh(n\pi)}\Big[F_{1,n}\sinh(n\pi(1-y))\sin(n\pi x)+F_{2,n}\sinh(n\pi y)\sin(n\pi x)+$$
$$F_{3,n}\sinh(n\pi(1-x))\sin(n\pi y)+F_{4,n}\sinh(n\pi x)\sin(n\pi y)\Big].$$

\2/ let u_1 & u_2 be two different sols of Z eqn satisfying

$u_{1xx}+u_{1yy}+u_{1zz}=f(x,y,z)$, $u_{2xx}+u_{2yy}+u_{2zz}=f(x,y,z)$

$u_{1\vec{n}}=g(x,y,z)$, $u_{2\vec{n}}=g(x,y,z)$ on ∂D.

Let $w=u_1-u_2$. then w satisfies $w_{xx}+w_{yy}+w_{zz}=0$

{ $w_{xx}+w_{yy}+w_{zz}=(u_1-u_2)_{xx}+(u_1-u_2)_{yy}+(u_1-u_2)_{zz}=$
$u_{1xx}-u_{2xx}+u_{1yy}-u_{2yy}+u_{1zz}-u_{2zz}=(u_{1xx}+u_{1yy}+u_{1zz})-(u_{2xx}+u_{2yy}+u_{2zz})=$
$f(x,y,z)-f(x,y,z)=0$ } with $w_{\vec{n}}=0$ on ∂D.

{ $w_{\vec{n}}=(u_1-u_2)_{\vec{n}}=u_{1\vec{n}}-u_{2\vec{n}}=g(x,y,z)-g(x,y,z)=0$ on ∂D }

∴ use Z divergence thm: $\int_D\nabla\cdot\vec{F}\,dV=\int_{\partial D}\vec{F}\cdot\vec{n}\,dS$ ∴ by taking $\vec{F}=w\nabla w$, we obtain: $\int_D\nabla\cdot(w\nabla w)\,dV=\int_{\partial D}(w\nabla w)\cdot\vec{n}\,dS=\int_{\partial D}w\frac{\partial w}{\partial\vec{n}}\,dS=0=$ $\int_{\partial D}w(0)\,dS=\int_{\partial D}0\,dS=0$. by Z BC of w . on Z other hand.

$\int_D\nabla\cdot(w\nabla w)\,dV=\int_D[\nabla w\cdot\nabla w+w\nabla^2 w]\,dV=\int_D|\nabla w|^2\,dV$ ∵ of $\nabla^2 w=0$ ∴

{ $\int_D[\nabla w\cdot\nabla w+w\nabla^2 w]\,dV=\int_D[\nabla w\cdot\nabla w+w(0)]\,dV=\int_D[\nabla w\cdot\nabla w]\,dV=$ $\int_D|\nabla w|^2\,dV$ } ∴ implies that: $\int_D|\nabla w|^2\,dV=0$ ∴ $\nabla w=0$ everywhere in D. this last property means that w must be const everywhere in D. ∵ $w=u_1-u_2$, this implies that Z sol of Z original PDE, if it exists, is unique up to an additive const. Z fact that Z const maybe arbit, is verified by direct calc: if u_1 is a sol of Z probl, then $u_2=u_1+C$ is also a sol for any const C.

\3a/ $2u_{xx}+2u_{xy}+u_{yy}+6u_x+2u_y+u=0$. hyte: $b^2-ac=-1<0$

{ $2u_{xx}+2u_{xy}+u_{yy}=-6u_x-2u_y-u=au_{xx}+2bu_{xy}+cu_{yy}$ ∴
$a=2,\ b=1,\ c=1$ ∴ $b^2-ac=1^2-2(1)=1-2=-1<0$ } ∴ it is an elliptic

eqn everywhere. Z charac eqn is $a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0 =$
$2\left(\frac{dy}{dx}\right)^2 - 2\frac{dy}{dx} + 1 = 0$ with Z Sol: $\frac{dy}{dx} = \frac{2 \pm \sqrt{4 - 4(2)(1)}}{2(2)} = (1 \pm i)/2$,
$\left\{ \frac{dy}{dx} = \frac{1}{2} + \frac{1}{2}i \right.$, $\frac{dy}{dx} = \frac{1}{2} - \frac{1}{2}i$ $\therefore$ $\frac{dy}{dx} = (1 \pm i)/2$ $\therefore \int \frac{dy}{dx}\,dx = \int 1\,dy =$
$\int (1 \pm i)/2\,dx = y = (1 \pm i)/2\,x + C$ $\therefore$ $y = (\frac{1}{2} + \frac{1}{2}i)x + C$, $y = (\frac{1}{2} - \frac{1}{2}i)x + C$ $\therefore$
$\therefore C = y - (1+i)x/2$, $C = y - (1-i)x/2 \}$ $\therefore$ giving Z complex coords
$\xi = y - (1+i)x/2$, $\eta = y - (1-i)x/2$ which can be written in Z real form
$\mu = (\xi + \eta)/2 = y - \frac{x}{2}$, $\nu = (\xi - \eta)/(2i) = -\frac{x}{2}$
$\{ \therefore \xi + \eta = y - (1+i)x/2 + y - (1-i)x/2 = 2y - \frac{x}{2} - \frac{x}{2}i - \frac{x}{2} + \frac{x}{2}i = 2y - x$ $\therefore$
$\mu = (\xi + \eta)/2 = (2y - x)/2 = y - \frac{1}{2}x$ & $\xi - \eta = y - (1+i)\frac{x}{2} - y + (1-i)x/2 =$
$y - \frac{x}{2} - \frac{x}{2}i - y + \frac{x}{2} - \frac{x}{2}i = -xi$ $\therefore \nu = (\xi - \eta)/2i = -\frac{xi}{2i} = -x/2 \}$

3b/ can transform Z PDE: $u_x = u_\mu(-\frac{1}{2}) - u_\nu(\frac{1}{2})$,
$u_y = u_\mu$, $u_{xy} = u_{\mu\mu}(-\frac{1}{2}) - u_{\mu\nu}(\frac{1}{2})$,
$u_{xx} = u_{\mu\mu}(\frac{1}{4}) + u_{\mu\nu}(\frac{1}{2}) + u_{\nu\nu}(\frac{1}{4})$, $u_{yy} = u_{\mu\mu}$ $\therefore$ subs of them into Z eqn
yields Z promised canonical form. $u_{\mu\mu} + u_{\nu\nu} = 2(u_\mu + 3u_\nu - u)$

3c/ $u(\mu, \nu) = e^{\rho\mu + \sigma\nu} v(\mu, \nu)$, $u_\mu = e^{\rho\mu + \sigma\nu}(v_\mu + \rho v)$;
$u_\nu = e^{\rho\mu + \sigma\nu}(v_\nu + \sigma v)$, $u_{\mu\mu} = e^{\rho\mu + \sigma\nu}(v_{\mu\mu} + 2\rho v_\mu + \rho^2 v)$,
$u_{\nu\nu} = e^{\rho\mu + \sigma\nu}(v_{\nu\nu} + 2\sigma v_\nu + \sigma^2 v)$ $\therefore$ into eqn;
$v_{\mu\mu} + v_{\nu\nu} + 2\rho v_\mu + 2\sigma v_\nu + (\sigma^2 + \rho^2)v = 2[v_\mu - 3v_\nu + \rho v - 3\sigma v - v]$,
leading to $v_{\mu\mu} + v_{\nu\nu} - 8v = 0$ if we take $\sigma = 3$ & $\rho = 1$.

4/ this example is in fact a variation for Z case $L \to \infty$ $\therefore$
seperating variabs: $u = X(x)Y(y)$ $\therefore$ $u_{xx} = X''(x)Y(y)$, $u_{yy} = X(x)Y''(y)$ $\therefore$
$u_{xx} + u_{yy} = X''Y + XY'' = 0$ $\therefore$ $XY'' = -X''Y$ $\therefore$ $Y''/Y = -X''/X = \lambda = \text{const}$
$\therefore Y'' - \lambda Y = 0$, $X'' + \lambda X = 0$, $X(0) = X(a) = 0$ $\therefore$ Sol to eigfunc prob
for X with proofs not needed ever! $\therefore$ For $\lambda \leq 0$, there are no nontrivial
Sol for $X(x)$ that would satisfy Z BCs $X(0) = 0$, $X(a) = 0$.
Let $\lambda = q^2 > 0$, $\therefore X'' + q^2 X = 0$ $\therefore X(x) = C_1 \cos qx + C_2 \sin qx$. Z BCs give
$X(0) = 0$ $\therefore C_1 = 0$, $X(a) = 0$ $\therefore C_2 \sin qa = 0$ $\therefore q = n\pi/a$ $\{\therefore qa = n\pi\}$ ie

Week 7 Sheet / $\lambda = q^2 = (n\pi)^2/a^2$, $n = 1, 2, 3, \ldots$ ∴ Z required eigfuncs are $X(x) = C_2 \sin\frac{n\pi x}{a}$ with Z eigvals $q = \frac{n\pi}{a}$. Eid:

$Y'' - \frac{n^2\pi^2}{a^2} Y = 0$, with sols of Z form

$Y(y) = A_n e^{n\pi y/a} + B_n e^{-n\pi y/a}$ ∴ applying Z bounded condition at y→∞, obtain $A_n = 0$ $\{ \because |Y(y=\infty)| < \infty$ & as $y \to \infty$: $e^{n\pi y/a} = \infty$ ∴ $A_n = 0 \}$.

∴ by Z superposition principle, Z GS to Z homog part of Z prob (PDE, BC at $x=0$, BC at $x=a$, & BC at $y \to \infty$) is then:

$u = \sum_{n=1}^{\infty} B_n e^{-n\pi y/a} \sin\frac{n\pi x}{a}$ ∴ making use of Z BC $u(x,0) = -u_0 x/a$:

$\sum_{n=1}^{\infty} B_n \sin\frac{n\pi x}{a} = -u_0 x/a$ ∴ with proof not needed!:

$B_n = \frac{2}{a}\int_0^a (-u_0 x/a)\sin\left(\frac{n\pi x}{a}\right) = \frac{2u_0}{n\pi}(-1)^n$ ∴ Z required sol is:

$u = \frac{2u_0}{\pi}\sum_{n=1}^{\infty}\frac{(-1)^n}{n} e^{-n\pi y/a}\sin\left(\frac{n\pi x}{a}\right)$ & above sol can be simplified further, using Z same idea use to derive Z poisson's formula for Z disk. ∴ Let $\Phi = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} e^{-n\pi y/a}\sin\left(\frac{n\pi x}{a}\right)$. note this is related to ~~$\Psi = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}(e^{\pi i z/a})^n$~~ $\Psi = \sum_{n=1}^{\infty}\frac{(-1)^n}{n} e^{-n\pi y/a + in\pi x/a} =$

$\sum_{n=1}^{\infty}\frac{(-1)^n}{n} e^{in\pi(x+iy)/a}$ in Z following way $\Phi = \mathrm{Im}(\Psi)$ (or could just use Euler's complex exponentials for Z sin). Further more, introduce then $z = x+iy$, $Z = e^{i\pi z/a}$

$\Psi = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}(e^{\pi i z/a})^n = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}Z^n$ this series is not simply a geometric progression ∵ of Z factor $\frac{1}{n}$ but it is related to one. deriv Z series wrt Z: $\frac{d\Psi}{dZ} = \sum_{n=1}^{\infty}(-1)^n Z^{n-1} = \sum_{n=1}^{\infty}(-1)(-1)^{n-1}Z^{n-1} = -\sum_{n=1}^{\infty}(-1)^{n-1}Z^{n-1} =$

$-\sum_{n=0}^{\infty}(-1)^{n+1-1}Z^{n+1-1} = -\sum_{n=0}^{\infty}(-1)^n Z^n = -\sum_{m=0}^{\infty}(-1)^m Z^m = -\left(\frac{1}{1+Z}\right)$ ∴ integrate taking into account $\Psi(Z=0) = 0$ $\{ \because \Psi(Z=0) = \sum_{n=1}^{\infty}\frac{(-1)^n}{n}(0)^n = 0 \}$

$\Psi = -\ln(1+Z) = -\ln(1+e^{i\pi z/a})$ ∴ $1 + e^{i\pi(x+iy)/a} = e^{-\Psi}$ ∴

$\left(1 + e^{-\pi y/a}\cos\frac{\pi x}{a}\right) + i e^{-\pi y/a}\sin\frac{\pi x}{a} = e^{-(\mathrm{Re}(\Psi) + i\,\mathrm{Im}(\Psi))} = e^{-(\mathrm{Re}(\Psi) + i\Phi)} =$

$e^{-\mathrm{Re}(\Psi)}(\cos\Phi - i\sin\Phi) = e^{-\mathrm{Re}(\Psi)}e^{i(-\Phi)} = e^{-\mathrm{Re}(\Psi)}(\cos(-\Phi) + i\sin(\Phi)) =$

$\left(e^{-\mathrm{Re}(\Psi)}\cos(\Phi)\right) + i\left(e^{-\mathrm{Re}(\Psi)}\sin\Phi\right) = \left(1 + e^{-\pi y/a}\cos\frac{\pi x}{a}\right) + i\left(e^{-\pi y/a}\sin\frac{\pi x}{a}\right)$ ∴

taking Z ratio of Z imaginary part of Z real part for both sides of Z eqn:

$\frac{\mathrm{Im}}{\mathrm{Re}} = \frac{e^{-\pi y/a}\sin\left(\frac{\pi x}{a}\right)}{1 + e^{-\pi y/a}\cos\left(\frac{\pi x}{a}\right)} = -\frac{\sin\Phi}{\cos\Phi}$ ∴ $\frac{\sin\frac{\pi x}{a}}{e^{\pi y/a} + \cos\frac{\pi x}{a}} = -\tan\Phi$ ∴ Z sol can be written as $u(x,y) = \frac{2u_0}{\pi}\Phi = -\frac{2u_0}{\pi}\arctan\left(\frac{\sin(\pi x/a)}{\cos(\pi x/a) + e^{\pi y/a}}\right)$

\5/ by seperation of varials: $u = X(x)Y(y)$ $\therefore Y''/Y = -X''/X = k^2 > 0$

$\therefore$ $Y'' - k^2 Y = 0$, $X'' + k^2 X = 0$ which has sols in Z form $X(x) = Ce^{ikx}$,

$Y(y) = C_1(k)e^{-|k|y} + C_2(k)e^{|k|y}$. use Z condition $u(x,\infty) \neq \infty$, then

$C_2(k) = 0$. A sol of Z eqn is $u_k = C_3(k)e^{-|k|y}e^{ikx}$, $(C_3 = CC_1)$ $\therefore$

$\therefore$ Z GS can be written as Z integral:

$\{ C_2(k) = 0 \therefore Y(y) = C_1(k)e^{-|k|y}$ $\therefore$ $u = X(x)Y(y) = Ce^{ikx}C_1(k)e^{-|k|y} =$

$CC_1e^{-|k|y}e^{ikx} = C_3(k)e^{-|k|y}e^{ikx} = u_k \}$ $\therefore$

$u(x,y) = \int_{-\infty}^{\infty} C_3(k)e^{ikx}e^{-|k|y}dk$, we can use Z BC to determine $C_3(k)$:

$u(x,0) = \int_{-\infty}^{\infty} C_3(k)e^{ikx}e^{ikx}dk = f(x)$ where Z Fourier transform of $f(x)$ is

$C_3(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s)e^{-iks}ds$ $\therefore$ Z Sol is:

$u(x,y) = \int_{-\infty}^{\infty}\left(\frac{1}{2\pi}\int_{\infty}^{\infty} f(s)e^{-iks}ds\right)e^{ikx}e^{-|k|y}dk$ or

$u(x,y) = \int_{-\infty}^{\infty} f(s)\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ik(x-s)}e^{-|k|y}dk\Big]ds$ but

$\int_{-\infty}^{\infty} e^{ik\beta}e^{-|k|y}dk = 2\int_0^{\infty}\cos k\beta\, e^{-ky}dk = 2\frac{y}{\beta}\int_0^{\infty}\sin k\beta\, e^{-ky}dk =$

$-2\frac{y}{\beta^2}\left[-1 + y\int_0^{\infty}\cos k\beta\, e^{-ky}dk\right]$, & $\int_0^{\infty}\cos k\beta\, e^{-ky}dk = \frac{y}{y^2+\beta^2}$ $\therefore$

$u(x,y) = \int_{-\infty}^{\infty} f(s)\left[\frac{1}{2\pi}\frac{2y}{(x-s)^2+y^2}\right]ds = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(s)}{(x-s)^2+y^2}ds$ $\therefore$ now:

$\therefore$ apply Z BC $f(x) = u_j$, $x_j < x < x_{j+1}$, $j = 0, 1, 2, 3, \ldots$ $\therefore$

$u(x,y) = y\sum_{j=0}^{n}\frac{u_j}{\pi}\int_{x_j}^{x_{j+1}}\frac{1}{(s-x)^2+y^2}ds =$

$\sum_{j=0}^{n}\frac{u_j}{\pi}\left[\arctan\left(\frac{x_{j+1}-x}{y}\right) - \arctan\left(\frac{x_j - x}{y}\right)\right] = \sum_{j=0}^{n}\frac{u_j\theta_j}{\pi}$ where θ_j is Z angle betw

Z line connecting Z pt (x,y) & Z pt $(x_j,0)$, & Z line connecting

(x,y) to Z $(x_{j+1},0)$.

\6/ seperate varials in polar coords: $u(r,\theta) = R(r)\Theta(\theta)$ $\therefore$

$u_{rr} = R''(r)\Theta(\theta)$, $u_r = R'(r)\Theta(\theta)$, $u_{\theta\theta} = R(r)\Theta''(\theta)$ $\therefore$

$\Theta\frac{d^2R}{dr^2} + \frac{\Theta}{r}\frac{dR}{dr} + \frac{R}{r^2}\frac{d^2\Theta}{d\theta^2} = 0$ $\therefore$ $\Theta\frac{d^2R}{dr^2} + \frac{\Theta}{r}\frac{dR}{dr} = -\frac{R}{r^2}\frac{d^2\Theta}{d\theta^2}$ $\therefore$

$(r^2/R)\frac{d^2R}{dr^2} + (r/R)\frac{dR}{dr} = -(1/\Theta)\frac{d^2\Theta}{d\theta^2} = \lambda = \text{Const}$ $\therefore$

$r^2\frac{d^2R}{dr^2} + r\frac{dR}{dr} - \lambda R = 0$ & $\frac{d^2\Theta}{d\theta^2} + \lambda\Theta = 0$ ~~const~~ For Θ, have Z periodic

Week 7 Sheet / BC: $\Theta(\theta)=\Theta(\theta+2\pi)$, which require $\sqrt{\lambda}=n$, $n\in\mathbb{Z}_+$ $\{\Theta(\theta)=C_1\cos(\sqrt{\lambda}\theta)+C_2\sin(\sqrt{\lambda}\theta)$ ∴

$C_1\cos(0)+C_2\sin(0)=C_1\cos(\sqrt{\lambda}\,2\pi)+C_2\sin(\sqrt{\lambda}\,2\pi)$ ∴

$\sqrt{\lambda}\,2\pi=N\pi$ ∴ $\cos(\sqrt{\lambda}\,2\pi)=1$ ∴ $\sqrt{\lambda}\,2\pi=N2\pi$ ∴ $\sqrt{\lambda}=n,\ n\in\mathbb{Z}_+\}$,

& $\Theta=A\cos(n\theta)+B\sin(n\theta)$. Z eqn for R is of Z Euler type.

Let $R(r)=r^m$, it reduces to $m(m-1)r^m+mr^m-\lambda r^m=$

$m(m-1)r^m+mr^m-n^2r^m=0$ ∴ $m\pm n$ ∴

$u=A_n\cos(n\theta)r^{-n}+B_n\sin(n\theta)r^{-n}+C_n\cos(n\theta)r^n+D_n\sin(n\theta)r^n$ for $n\geq 1$, &

$u=A_0+C_0\ln(r)$ for $n=0$. Z requirement that Z sol is finite at $r=\infty$

implies $C_n=D_n=0\ \forall n$ ∴ summing Z sols:

$u=A_0+\sum_{n=1}^{\infty}(A_n\cos(n\theta)+B_n\sin(n\theta))r^{-n}$. setting $r=a$:

$f(\theta)=A_0+\sum_{n=1}^{\infty}(A_n\cos(n\theta)+B_n\sin(n\theta))a^{-n}$ this is Z fourier series for a

2π-periodic f, so $A_0=\frac{1}{2\pi}\int_0^{2\pi}f(\theta)d\theta$, $A_n=\frac{a^n}{\pi}\int_0^{2\pi}f(\theta)\cos n\theta\,d\theta$,

$B_n=\frac{a^n}{\pi}\int_0^{2\pi}f(\theta)\sin(n\theta)d\theta$ after inserting them into u, get:

$u=\frac{1}{2\pi}\int_0^{2\pi}f(\xi)d\xi+\sum_{n=1}^{\infty}\frac{a^n}{\pi r^n}\int_0^{2\pi}f(\xi)[\cos n\xi\cos(n\theta)+\sin n\xi\sin(n\theta)]d\xi=$

$\frac{1}{2\pi}\int_0^{2\pi}f(\xi)\left[1+2\sum_{n=1}^{\infty}\frac{a^n}{r^n}\cos n(\theta-\xi)\right]d\xi$. using Euler's complex exponentials:

$\left[1+2\sum_{n=1}^{\infty}\frac{a^n}{r^n}\cos(\theta-\xi)\right]=1+\sum_{n=1}^{\infty}\frac{a^n}{r^n}e^{in(\theta-\xi)}+\sum_{n=1}^{\infty}\frac{a^n}{r^n}e^{-in(\theta-\xi)}$, that is,

two (complex) geometric progressions: $\frac{1}{1-x}=\sum_{n=0}^{\infty}x^n$, $|x|<1$, ∴

$$\frac{(a/r)e^{i(\theta-\xi)}}{1-(a/r)e^{i(\theta-\xi)}}=\left(\frac{a}{r}\right)e^{i(\theta-\xi)}\sum_{n=0}^{\infty}\frac{a^n}{r^n}e^{in(\theta-\xi)}=\sum_{n=1}^{\infty}\frac{a^n}{r^n}e^{in(\theta-\xi)}$$

$$\frac{(a/r)e^{-i(\theta-\xi)}}{1-(a/r)e^{-i(\theta-\xi)}}=\left(\frac{a}{r}\right)e^{-i(\theta-\xi)}\sum_{n=0}^{\infty}\frac{a^n}{r^n}e^{in(\theta-\xi)}=\sum_{n=1}^{\infty}\frac{a^n}{r^n}e^{-in(\theta-\xi)}\ ∴$$

$$1+\sum_{n=1}^{\infty}\frac{a^n}{r^n}e^{in(\theta-\xi)}+\sum_{n=1}^{\infty}\frac{a^n}{r^n}e^{-in(\theta-\xi)}=1+\frac{ae^{i(\theta-\xi)}}{r-ae^{i(\theta-\xi)}}+\frac{ae^{-i(\theta-\xi)}}{r-ae^{-i(\theta-\xi)}}=$$

$$\frac{[r-ae^{i(\theta-\xi)}][r-ae^{-i(\theta-\xi)}]+ae^{i(\theta-\xi)}[r-ae^{-i(\theta-\xi)}]+ae^{-i(\theta-\xi)}[r+ae^{-i(\theta-\xi)}]}{[r-ae^{i(\theta-\xi)}][r-ae^{-i(\theta-\xi)}]}$$

$$=\frac{r^2-a^2}{a^2-2ar\cos(\theta-\xi)+r^2}\ ∴\ u=\frac{r^2-a^2}{2\pi}\int_0^{2\pi}\frac{f(\xi)}{a^2-2ar\cos(\theta-\xi)+r^2}d\xi,\ \text{for } r>a,$$

is poisson's formula for Z exterior of Z circle

$w_t = Dw_{xx}$ ∴ $w(x,t)=\frac{1}{\sqrt{4D\pi t}}\int_{-\infty}^{\infty} w(s,0)e^{-(x-s)^2/(4Dt)}ds$ $w(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} w(s,0)e^{-\frac{(x-s)^2}{4Dt}}ds$

$w_t = Dw_{xx}$ ∴ $w(x,t)=\frac{1}{\sqrt{4D\pi t}}\int_{-\infty}^{\infty} w(s,0)e^{-(x-s)^2/(4Dt)}ds$

$w_t = Dw_{xx}$ $w(x,t)=\frac{1}{\sqrt{4D\pi t}}\int_{-\infty}^{\infty} w(s,0)e^{-(x-s)^2/(4Dt)}ds$ $w(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} w(s,0)e^{-\frac{(x-s)^2}{4Dt}}ds$

$w_t = Dw_{xx}$ $w(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} w(s,0)e^{-(x-s)^2/(4Dt)}ds$ $w(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} w(s,0)e^{-\frac{(x-s)^2}{4Dt}}ds$

$w_t = Dw_{xx}$ $w(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} w(s,0)e^{-(x-s)^2/(4Dt)}ds$ $w(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} w(s,0)e^{-\frac{(x-s)^2}{4Dt}}ds$

$w_t = Dw_{xx}$ $w(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} w(s,0)e^{-\frac{(x-s)^2}{4Dt}}ds$ $w_t = Dw_{xx}$ $w(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} w(s,0)e^{-\frac{(x-s)^2}{4Dt}}ds$

$w_t = Dw_{xx}$ $w(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} w(s,0)e^{-\frac{(x-s)^2}{4Dt}}ds$ $w_t = Dw_{xx}$ $w(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} w(s,0)e^{-\frac{(x-s)^2}{4Dt}}ds$

$w_t = Dw_{xx}$ $w(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} w(s,0)e^{-\frac{(x-s)^2}{4Dt}}ds$ $w_t = Dw_{xx}$ $w(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} w(s,0)e^{-\frac{(x-s)^2}{4Dt}}ds$

$w_t = Dw_{xx}$ $w(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} w(s,0)e^{-\frac{(x-s)^2}{4Dt}}ds$ $w_t = Dw_{xx}$ $w(x,t)=\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} w(s,0)e^{-\frac{(x-s)^2}{4Dt}}ds$

\7/ 2 relevant formula for Z sol of Z Laplace eqn in Z upper half-plane with BC $u(x,0)=f(x)$ is $u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(s)}{(x-s)^2+y^2}ds$

Subbing here $f(s)=\frac{1}{1+s^2}$:

$u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{1}{(1+s^2)[(x-s)^2+y^2]}ds$ ∴ by partial fractions or Cauchy residue

thm: $u(x,y)=\frac{y+1}{x^2+(y+1)^2}$ ∴ verify by sub ∴ deriv:

$u_x=\frac{-2x(y+1)}{[x^2+(y+1)^2]^2}$, $u_{xx}=\frac{2[3x^2-(y+1)^2](y+1)}{[x^2+(y+1)^2]^3}$

$u_y=\frac{x^2-(y+1)^2}{[x^2+(y+1)^2]^2}$, $u_{yy}=\frac{2[(y+1)^2-3x^2](y+1)}{(x^2+(y+1)^2)^3}$ ∴ $u_{xx}+u_{yy}=0$ 2 sub:

$y=0$ into sol: get $u=\frac{1}{x^2+1}$ & $u(x,\infty)=\lim_{y\to\infty}u(x,y)=\lim_{y\to\infty}\frac{1}{(y+1)+\left(\frac{x^2}{y+1}\right)}=0$,

So Z BC is satisfied as well.

\Sheet 1/ \1/ let $x+y=\xi$, $x-y=\eta$ ∴

$\xi_x=1$, $\xi_y=1$, $\eta_x=1$, $\eta_y=-1$ ∴ $u_x=u_\xi\xi_x+u_\eta\eta_x=u_\xi+u_\eta$

$u_y=u_\xi\xi_y+u_\eta\eta_y=u_\xi-u_\eta$ ∴ $u_x+2u_y=$

$u_\xi+u_\eta+2(u_\xi-u_\eta)=3u_\xi-u_\eta$

\Sheet 1/ \1/ let $\xi = x+2y$, $\eta = x-2y$ ∴ $\xi_x = 1$, $\xi_y = 2$, $\eta_x = 1$, $\eta_y = -2$ ∴

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$,

$u_y = u_\xi \xi_y + u_\eta \eta_y = 2u_\xi - 2u_\eta$ ∴

$u_x + 2u_y = u_\xi + u_\eta + 2(2u_\xi - 2u_\eta) = u_\xi + u_\eta + 4u_\xi - 4u_\eta = 5u_\xi - 3u_\eta$

let ~~$\xi = 2x - 2y$, $\eta =$~~ $\xi = x+y$, $\eta = 2x - y$ ∴ $\xi_x = 1$, $\xi_y = 1$, $\eta_x = 2$, $\eta_y = -1$ ∴

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + 2u_\eta$, $u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi - u_\eta$ ∴

$u_x + 2u_y = u_\xi + 2u_\eta + 2(u_\xi - u_\eta) = 3u_\xi + 2u_\eta - 2u_\eta = 3u_\xi = 0$ ∴

$u_\xi = 0$ ∴ $u = f(\eta) = f(2x-y)$ f arbit ∴

C ~~$= 2x$~~ $C = f(2x-y)$

\1b/ $(u_x, u_y) \cdot (1, 2) = 0$ ∴ $\frac{dy}{dx} = \frac{2}{1} = 2$ ∴

$\int \frac{dy}{dx} dx = \int 1 dy = \int 2 dx = y = 2x + C$ ∴

$y - 2x = C$ ∴ $u = f(y-2x)$ ∴

at $y = 0$: $u(x,0) = f(0-2x) = f(-2x) = e^x$

let ~~$x = -\frac{1}{2}s$~~ ∴ $-2x = s$ ∴ $x = -\frac{1}{2}s$ ∴ ~~$f(s) =$~~

$f(-2x) = f(-2(-\frac{1}{2}s)) = f(s) = e^{-\frac{1}{2}s}$ ∴ $f(y-2x) = e^{-\frac{1}{2}(y-2x)} = e^{-\frac{1}{2}y + x}$

$u(x,y) = e^{-\frac{1}{2}y + x} = e^{x - \frac{1}{2}y}$

\1c/ ~~eqn~~ equiv to $(1,2) \cdot \nabla u = 0$ ∴ charac curves satisfy

$(dx, dy) \parallel (1,2)$ or $\frac{dy}{dx} = 2$ ∴ charac curves have form $y = 2x + c$

or $C = y - 2x$ ∴ GS: $u = f(y-2x)$ ∴ $y = 0$: $u = f(-2x) = e^x = e^{-\frac{1}{2}(-2x)}$

$f(s) = e^{-\frac{1}{2}s}$ ∴ $u = f(y-2x) = e^{-s/2}\big|_{s=y-2x} = e^{x-y/2}$

eqn equiv to $(1,2) \cdot \nabla u = 0$ ∴ charac curves satisfy

$\frac{dy}{dx} = \frac{2}{1} = 2$ ∴ charac curves form $y = \int 2 dx = 2x + C$ ∴

$y - 2x = C$ ∴ GS: $u = f(y-2x)$

$u(x,0) = f(0-2x) = f(-2x) = e^x$ ∴ let $-2x = s$ ∴ $x = -\frac{1}{2}s$ ∴

$f(s) = e^{-\frac{1}{2}s}$ ∴ $u(x,y) = f(y-2x) = e^{-\frac{1}{2}(y-2x)} = e^{x - \frac{1}{2}y}$

\2a/ ∴ $(\cos^2 x, y) \cdot \nabla u = 0$ ∴ $\frac{dy}{dx} = \frac{y}{\cos^2 x}$ ∴ $\frac{1}{y}\frac{dy}{dx} = \sec^2 x$

$\frac{d}{dx}\tan x = \sec^2 x$ ∴ $\int \frac{1}{y} dy = \int \sec^2 x \, dx = \ln|y| = \tan x + C$

$C = \ln|y| - \tan x$ is the charact curve ∴

$u(x,y) = f(\ln|y| - \tan x)$ ∴

$u(0,y) = f(\ln|y| - \tan(0)) = f(\ln|y|) = y^2$ ∴ 1)

let $\ln|y| = s$ ∴ $e^{\ln|y|} = |y| = e^s > 0$ ∴ ~~$y = e^s$~~ $y = \pm e^s$ ∴

~~$f(s) =$~~ $f(\ln|y|) =$ ~~e^s~~ $f(s) = y^2 = (\pm e^s)^2 = e^{2s}$ ∴

~~f~~ $u(x,y) = e^{2(\ln|y| - \tan x)}$

$u(x,0) = x^2 = f(\ln|0| - \tan x)$ is divergent ∴ no solution

\2 Sol/ $(\cos^2 x, y) \cdot \nabla u = 0$ ∴ Charac Curves $\frac{dy}{dx} = \frac{y}{\cos^2 x}$ ∴

$\int \frac{1}{y}\,dy = \int \frac{1}{\cos^2 x}\,dx$ ∴ $\ln|y| = \tan x + C_1$ ∴ $|y| = e^{\tan x + C_1} = C_2 e^{\tan x}$

∴ $y = Ce^{\tan x}$ ∴

$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\cos^2 x}$ ∴ $\int \frac{1}{y}\,dy = \int \frac{1}{\cos^2 x}\,dx = \ln|y| = \tan x + C_1$ ∴

$y = 0$ is the Special solution ∴

$|y| = e^{\tan x + c} = C_2 e^{\tan x}$ ∴ $y = Ce^{\tan x}$ ∴

$\frac{y}{e^{\tan x}} = C$ is Z charac curve ∴ $u(x,t) = f\left(\frac{y}{e^{\tan x}}\right)$ ∴

$u(0,y) = y^2 = f\left(\frac{y}{e^{\tan(0)}}\right) = f\left(\frac{y}{e^0}\right) = f(y) = y^2$ ∴ let $s = y$ ∴

$f(s) = s^2$ ∴ $f\left(\frac{y}{e^{\tan x}}\right) = \left(\frac{y}{e^{\tan x}}\right)^2 = \frac{y^2}{e^{2\tan x}} = u(x,y) = y^2 e^{-2\tan x}$

$u(x,0) = y^2 = f\left(\frac{0}{e^{\tan x}}\right) = f(0) = y^2$ which is impossible ∴ no sols

∴ eqn: $(\cos^2 x, y) \cdot \nabla u = 0$ ∴ $\frac{dy}{dx} = \frac{y}{\cos^2 x}$ ∴ $\int \frac{1}{y}\,dy = \int \frac{1}{\cos^2 x}\,dx = \ln|y| = \tan x + C_1$

∴ $y = Ce^{\tan x}$ ∴ Special sol $y(x) = 0$ is covered with $C = 0$ ∴

~~y~~ $C = ye^{-\tan x}$ ∴ GS: $u(x,y) = f(ye^{-\tan x})$ ∴

$u(0,y) = f(ye^{-\tan 0}) = f(y) = y^2$ ∴ $s = y$ ∴ $f(s) = s^2$ ∴ $u(x,y) = y^2 e^{-2\tan x}$

$u(x,0) = f((0)e^{-\tan x}) = f(0) = x^2$ which cannot be satisfied ∵ f cannot take different values for the same argument of 0 ∴ solution does not exist ∵ the boundary cond is set on $y = 0$ which is a charac curve.

\3/ eqn equiv: $(1, xy^3) \cdot \nabla u$ ∴ Charac curves satisfy

$\frac{dy}{dx} = \frac{xy^3}{1} = xy^3$ ∴ $\frac{1}{y^3}\frac{dy}{dx} = x$ ∴ $\int \frac{1}{y^3}\,dy = \int x\,dx = \frac{1}{-2}y^{-2} = \frac{1}{2}x^2 + C$ ∴

Charac curves: $-\frac{1}{2}y^{-2} - \frac{1}{2}x^2 = C$ ∴ $u(x,y) = f\left(-\frac{1}{2}y^{-2} - \frac{1}{2}x^2\right)$

Sheet 1/ ∴ $u_x = -\frac{1}{2}2x f(-\frac{1}{2}y^{-2} - \frac{1}{2}x^2) = -x f(-\frac{1}{2}y^{-2} - \frac{1}{2}x^2)$

$u_y = y^{-3} f(-\frac{1}{2}y^{-2} - \frac{1}{2}x^2)$ ∴

i) $u_x + xy^3 u_y = -x f(-\frac{1}{2}y^{-2} - \frac{1}{2}x^2) + xy^3 y^{-3} f(-\frac{1}{2}y^{-2} - \frac{1}{2}x^2) =$

$-x f(-\frac{1}{2}y^{-2} - \frac{1}{2}x^2) + x f(-\frac{1}{2}y^{-2} - \frac{1}{2}x^2) = 0 = 0$ LHS = RHS

$\int \frac{1}{y^3}\,dy = \int x\,dx$ ∴ $-\frac{1}{2y^2} = \frac{1}{2}x^2 + A = \frac{x^2 - c}{2}$ ∴ $\frac{1}{y^2} = c - x^2$ ∴ $\frac{1}{c - x^2} = y^2$ ∴

$y = \pm\frac{1}{\sqrt{c - x^2}}$ ∴ ~~x~~ $c = x^2 + \frac{1}{y^2}$ ∴ $u = f(x^2 + \frac{1}{y^2})$ f arbit ∴

$u_y = \frac{-2}{y^3} f'$, $u_x = 2x f'$ ∴ $u_x + u_y = (+2x + xy^3(-\frac{2}{y^3}))f' = (2x - 2x)f' = 0$

4/ let $\xi = x + 3y$ ∴ $\eta = 3x - y$ ∴ $\xi_x = 1$, $\xi_y = 3$, $\eta_x = 3$, $\eta_y = -1$ ∴

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + 3u_\eta$, $u_y = u_\xi \xi_y + u_\eta \eta_y = 3u_\xi - u_\eta$ ∴

$u_x + 3u_y + 10u + 20e^{10x} = u_\xi + 3u_\eta + 3(3u_\xi - u_\eta) + 10u = 20e^x =$

$u_\xi + 3u_\eta + 9u_\xi - 3u_\eta + 10u = 10u_\xi + 10u = 20e^x$ ∴ $u_\xi + u = 10e^x$

∴ $\xi - 3y = x$ ∴ $\eta = 3(\xi - 3y) - y = 3\xi - 9y - y = \eta = 3\xi - 10y$ ∴

$10y = 3\xi - \eta$ ∴ $y = \frac{3}{10}\xi - \frac{1}{10}\eta$ ∴ $\xi - 3(\frac{3}{10}\xi - \frac{1}{10}\eta) = x = \xi - \frac{9}{10}\xi + \frac{3}{10}\eta = \frac{1}{10}\xi + \frac{3}{10}\eta = x$

∴ $u_\xi + u = 10e^{\frac{1}{10}\xi + \frac{3}{10}\eta}$ ∴

IF $= e^{\int 1\,d\xi} = e^\xi$ ∴ $\frac{\partial}{\partial\xi}(\xi u) = 10\xi e^{\frac{1}{10}\xi + \frac{3}{10}\eta}$ ∴

$\xi u = \int 10\xi e^{\frac{1}{10}\xi + \frac{3}{10}\eta}\,\partial\xi = c + [10\xi \frac{1}{(\frac{1}{10})} e^{\frac{1}{10}\xi + \frac{3}{10}\eta}] - \int 10 \frac{1}{(\frac{1}{10})} e^{\frac{1}{10}\xi + \frac{3}{10}\eta}\,\partial\xi$

$= 100\xi e^{\frac{1}{10}\xi + \frac{3}{10}\eta} - \int 100 e^{\frac{1}{10}\xi + \frac{3}{10}\eta}\,\partial\xi + c =$

$100\xi e^{\frac{1}{10}\xi + \frac{3}{10}\eta} - 1000 e^{\frac{1}{10}\xi + \frac{3}{10}\eta} + c = \xi u$

4 redo/ $10u_\xi + 10u = 20e^x$ ∴ $u_\xi + u = 2e^x$ ∴

$\xi - 3y = x$ ∴ $\eta = 3(\xi - 3y) - y = 3\xi - 9y - y = 3\xi - 10y$ ∴ $10y = 3\xi - \eta$ ∴ $y = \frac{3}{10}\xi - \frac{1}{10}\eta$

∴ $\xi - 3(\frac{3}{10}\xi - \frac{1}{10}\eta) = x = \frac{1}{10}\xi + \frac{3}{10}\eta$ ∴

$u_\xi + u = 2e^{\frac{1}{10}\xi + \frac{3}{10}\eta}$ ∴ IF $= e^{\int 1\,d\xi} = e^\xi$ ∴

$\frac{\partial}{\partial\xi}(e^\xi u) = 2e^\xi e^{\frac{1}{10}\xi + \frac{3}{10}\eta} = 2e^{\frac{11}{10}\xi + \frac{3}{10}\eta}$ ∴

$e^\xi u = \int 2e^{\frac{11}{10}\xi + \frac{3}{10}\eta}\,\partial\xi = 2(\frac{10}{11})e^{\frac{11}{10}\xi + \frac{3}{10}\eta} + c$ ∴

$u = e^{-\xi} e^{\frac{11}{10}\xi + \frac{3}{10}\eta} + ce^{-\xi} = e^{\frac{1}{10}\xi + \frac{3}{10}\eta} + ce^{-\xi} =$

$e^{\frac{1}{10}(x+3y) - \frac{3}{10}(3x-y)} + ce^{-(x+3y)} = e^{\frac{1}{10}x + \frac{3}{10}y - \frac{9}{10}x + \frac{3}{10}y} + ce^{-x-3y} =$

$e^{\frac{8}{10}x + \frac{6}{10}y} + ce^{-x-3y}$

\4 sol/ $u_\xi + u = 2e^{\xi+3\eta} \therefore IF = e^{\int 1 \, d\xi} = e^{\xi} \therefore$

$\frac{\partial}{\partial\xi}(e^{\xi}u) = 2e^{\xi}e^{\xi+3\eta} \quad \therefore \xi = x+3y, \eta = 3x-y \therefore y = 3x-\eta \therefore$

$\xi = x+3(3x-\eta) = x-9x+3\eta = -8x+3\eta \therefore \cancel{\xi - 3\eta} \; 8x = -\xi+3\eta$ X

$\xi = x+3y, \eta = 3x-y \therefore y = 3x-\eta \therefore$

$\xi = x+3(3x-\eta) = x+9x-3\eta = \xi = 10x-3\eta \therefore \xi+3\eta = 10x \therefore \frac{1}{10}\xi + \frac{3}{10}\eta = x$

$\therefore u_\xi + u = 2e^{10x} = 2e^{10(\frac{1}{10}\xi + \frac{3}{10}\eta)} = 2e^{\xi+3\eta} \therefore$

$\frac{\partial}{\partial\xi}(e^{\xi}u) = 2e^{\xi}e^{\xi+3\eta} = 2e^{2\xi+3\eta} \therefore$

$e^{\xi}u = \int 2e^{2\xi+3\eta}\,\partial\xi = 2\frac{1}{2}e^{2\xi+3\eta} + f(\eta) = e^{2\xi+3\eta} + f(\eta) \therefore$

$u = e^{-\xi}e^{2\xi+3\eta} + e^{-\xi}f(\eta) = e^{\xi+3\eta} + e^{-\xi}f(\eta) =$

$e^{10x} + e^{-(x+3y)}f(3x-y) = e^{10x} + e^{-x-3y}f(3x-y)$ ✓

\5a/ let $a=2$, $b=\frac{5}{2}$ $c=-3 \therefore \left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} + 3 = 0 \therefore$

$\frac{dy}{dx} = \frac{5 \pm \sqrt{25-4(1)(3)}}{2(1)} = \frac{5 \pm \sqrt{13}}{2} = \frac{5}{2} \pm \frac{\sqrt{13}}{2} \therefore$

$y = \int \left(\frac{5}{2} \pm \frac{\sqrt{13}}{3}\right)dx = \left(\frac{5}{2} \pm \frac{\sqrt{13}}{2}\right)x + C \therefore C = y - \left(\frac{5}{2} \pm \frac{\sqrt{13}}{2}\right)x \therefore$

$u = f\left(y - \left(\frac{5}{2} \pm \frac{\sqrt{13}}{2}x\right)\right)$

let $u = \phi(x+\alpha y) \therefore$ $u_{xx} = \phi''$, $u_{xy} = \alpha\phi''$, $u_{yy} = \alpha^2\phi'' \therefore$

\5a/ let $u = \phi(\alpha x+y) \therefore u_{xx} = \alpha^2\phi''$, $u_{xy} = \alpha\phi''$, $u_{yy} = \phi'' \therefore$

$2(\alpha^2\phi'') + 5(\alpha\phi'') - 3\phi'' = 2\alpha^2 + 5\alpha - 3 = 0 \therefore$

$\alpha = \frac{-5 \pm \sqrt{25-4(2)(-3)}}{2(1)} \quad \frac{-5}{2} \pm \frac{\sqrt{14}}{2} = -\frac{5}{2} \pm \frac{\sqrt{14}}{2}$

$2\alpha^2 + 5\alpha - 3 = 0 \therefore \alpha = \frac{-5 \pm \sqrt{(5)^2-4(2)(-3)}}{2(2)} = \frac{-5 \pm \sqrt{25+24}}{4} = \frac{-5 \pm \sqrt{49}}{4} =$

$\frac{-5 \pm \sqrt{49}}{2} = \frac{-5 \pm 7}{4} = \alpha \therefore \alpha = \frac{1}{2}, \alpha = -3 \therefore 0 = 2\alpha^2 - 5\alpha - 3 = (2\alpha+1)(\alpha-3)$

$0 = 2\alpha^2 + 5\alpha - 3 = (2\alpha-1)(\alpha+3) \therefore \alpha = \frac{1}{2}, \alpha = -3 \therefore$

$u = f\left(\frac{1}{2}x+y\right) + g(-3x+y)$ f, g arbit

\5b/ let $\xi = x-4y$, $\eta = -4x-y \therefore \xi_x = 1$, $\xi_y = -4$, $\eta_x = -4$, $\eta_y = -1$

$u_x = u_\xi\xi_x + u_\eta\eta_x = u_\xi - 4u_\eta$, $u_y = u_\xi\xi_y + u_\eta\eta_y = -4u_\xi - u_\eta \therefore$

$u_x - 4u_y = u_\xi - 4u_\eta - 4(-4u_\xi - u_\eta) = u_\xi - 4u_\eta + 16u_\xi + 4u_\eta = 17u_\xi \therefore$

$u_x - 4u_y - 3u = 17u_\xi - 3u = 0 = u_\xi - \frac{3}{17}u \therefore$

$IF = e^{\int -\frac{3}{17}d\xi} = e^{-\frac{3}{17}\xi} \therefore$

$\frac{\partial}{\partial\xi}\left(e^{-\frac{3}{17}\xi}u\right) = 0 \therefore e^{-\frac{3}{17}\xi}u = f(\eta) \therefore u = e^{+\frac{3}{17}\xi}f(\eta) \therefore$

\Sheet 1/ $u(x,y) = e^{\frac{3}{17}(x-4y)} g(-4x-y) = e^{\frac{3}{17}x - \frac{12}{17}y} g(-4x-y)$

$= e^{\frac{3}{17}(x-4y)} g(4x+y)$

\5c/ let $u = \varnothing(\alpha x + y) \therefore$

$u_x = \alpha\varnothing'$, $u_y = \varnothing'$, $u_{xx} = \alpha^2\varnothing''$, $u_{yy} = \varnothing'' \therefore$

$x^2\alpha^2\varnothing'' - 4y^2\varnothing'' + x\alpha\varnothing' - 4y\varnothing' =$

let $\xi = x - 4y \therefore$ $\eta = -4x - y \therefore$

$\xi_x = 1$, $\xi_y = -4$, $\eta_x = -4$, $\eta_y = -1 \therefore$

$u_x = u_\xi\xi_x + u_\eta\eta_x = u_\xi - 4u_\eta$, $u_y = u_\xi\xi_y + u_\eta\eta_y = -4u_\xi - u_\eta \therefore$

$u_{xx} = u_{\xi\xi}\xi_x - 4u_{\xi\eta}\xi_x + u_{\xi\eta}\eta_x - 4u_{\eta\eta}\eta_x = u_{\xi\xi} - 4u_{\xi\eta} - 4u_{\xi\eta} + 16u_{\eta\eta} =$

$u_{\xi\xi} - 8u_{\xi\eta} + 16u_{\eta\eta}$

$u_{yy} = -4u_{\xi\xi}\xi_y - u_{\xi\eta}\xi_y - 4u_{\xi\eta}\eta_y - u_{\eta\eta}\eta_y = 16u_{\xi\xi} + 4u_{\xi\eta} + 4u_{\xi\eta} + u_{\eta\eta} =$

$16u_{\xi\xi} + 8u_{\xi\eta} + u_{\eta\eta} \therefore$

$x^2u_{xx} - 4y^2u_{yy} + xu_x - 4yu_y =$

$x^2[u_{\xi\xi} - 8u_{\xi\eta} + 16u_{\eta\eta}] - 4y^2[16u_{\xi\xi} + 8u_{\xi\eta} - u_{\eta\eta}] + x[u_\xi - 4u_\eta] - 4y[-4u_\xi - u_\eta] =$

$x^2u_{\xi\xi} - 8x^2u_{\xi\eta} + 16x^2u_{\eta\eta} - 64y^2u_{\xi\xi} - 32y^2u_{\xi\eta} - 4y^2u_{\eta\eta} + xu_\xi - 4xu_\eta + 16yu_\xi + 4yu_\eta =$

$[x^2 - 64y^2]u_{\xi\xi} + [-8x^2 - 32y^2]u_{\xi\eta} + [16x^2 - 4y^2]u_{\eta\eta} + [x + 16y]u_\xi + [-4x + 4y]u_\eta = 0$

$\therefore x + 16y = 0$, $-4x + 4y = 0$?

\5c sol/ $x = e^s$ $y = e^t \therefore \ln x = s$, $\ln y = t \therefore s_x = \frac{1}{x}$, $t_y = \frac{1}{y}$, $s_{xx} = \frac{-1}{x^2}$, $s_{yy} = \frac{-1}{y^2}$

$u_x = u_s s_x + u_t t_x = \frac{1}{x}u_s$, $u_y = u_t t_y = \frac{1}{y}u_t$; $\cancel{u_{xx} = \frac{2}{x^2}}$

$u_{xx} = \partial_s u_x s_x = \partial_s\left(\frac{1}{x}u_s\right)\frac{1}{x} = \cancel{\partial_s\left(\frac{1}{x}\right)u_s\frac{1}{x}} + \frac{1}{x}\partial_s u_s\frac{1}{x} = \frac{1}{x^2}u_{ss}$,

$u_{yy} = \partial_t u_y t_y = \partial_t\left(\frac{1}{y}u_t\right)\frac{1}{y} = \frac{1}{y^2}u_{tt} \therefore$

$x^2u_{xx} - 4y^2u_{yy} + xu_x - 4yu_y =$

$x^2\left(\frac{1}{x^2}u_{ss}\right) - 4y^2\left(\frac{1}{y^2}u_{tt}\right) + x\left(\frac{1}{x}u_s\right) - 4y\left(\frac{1}{y}u_t\right) =$

$u_{ss} - 4u_{tt} + u_s - 4u_t = 0$

\CW1/ $u_x - 3u_y = 0 \therefore$ eqn equiv to $(1,-3)\cdot\nabla u = 0 \therefore \frac{dy}{dx} = \left(\frac{-3}{1}\right) = -3$

satisfies charac curves $\therefore \int 1\,dy = \int -3\,dx = y = -3x + C$ is charac curve

$\therefore y + 3x = C \therefore u(x,y) = f(y + 3x)$ for arbitrary func $\therefore$

$u(x,0) = f(0 + 3x) = f(3x) = \cos(2x)$ $\therefore$ let $s = 3x \therefore \frac{s}{3} = x \therefore$

$f(s) = \cos\left(2\frac{s}{3}\right) = \cos\left(\frac{2}{3}s\right) \therefore u(x,y) = f(y + 3x) = \cos\left(\frac{2}{3}(y + 3x)\right) = \cos\left(\frac{2}{3}y + 2x\right)$

\1 Sol/ $u_x - 3u_y = 0$ ∴ $(1,-3)\cdot(u_x, u_y) = 0 = (1,-3)\cdot(\nabla u) = 0$ ∴

(dx, dy) parallel to $(1,-3)$ ∴ u = constant along $(1,-3)$ ∴

$\frac{dy}{dx} = \frac{-3}{1} = -3 = \frac{dy}{dx}$ ∴

$\int 1\,dy = \int -3\,dx = y = -3x + C$ C is arbit const is charac curves ∴

$y + 3x = C$ is charac curves ∴

$u(x,y) = f(y+3x)$ f arbit func is GS ∴

$u(x,0) = f(0+3x) = f(3x) = \cos(2x)$ ∴

let $s = 3x$ ∴ $\frac{s}{3} = x$ ∴ $f(s) = \cos(2(\frac{s}{3})) = \cos(\frac{2}{3}s)$ ∴

$u(x,y) = f(y+3x) = \cos(\frac{2}{3}(y+3x)) = \cos(\frac{2}{3}y + 2x)$

\2a/ ∴ $((1+(x)^2), y)\cdot(u_x, u_y) = ((1+(x)^2), y)\cdot\nabla u$ ∴ $\frac{dy}{dx} = \frac{y}{1+x^2}$ ∴

$\int \frac{1}{y} dy = \int \frac{1}{1+x^2} dx = y = \arctan x + C$ ∴

$y - \arctan x = C$ ∴ $u(x,y) = f(y - \arctan x)$ ∴

$u(0,y) = \ln(y) = f(y - \arctan(0)) = f(y)$ ∴ let $s = y$ ∴

$f(s) = \ln(s)$ ∴ $u(x,y) = f(y - \arctan x) = \ln(y - \arctan x)$

\2b/ $u(x,0) = f(0 - \arctan(x)) = \ln x = f(-\arctan x)$ ∴ ~~let s = x~~ ∴

~~ln(s) = f(−arctan s)~~ ∴ $-\arctan x = s$ ∴ $\arctan x = -s$ ∴

~~tan~~ $x = \tan(-s)$ ∴ $\ln(\tan(-s)) = f(s)$ ∴

$u(x,y) = f(y - \arctan x) = \ln(\tan(-(y - \arctan x))) = \ln(\tan(-y + \arctan x))$

\2 redo/ ∴ $(1+x^2, y)\cdot\nabla u$ ∴ $\frac{dy}{dx} = \frac{y}{1+x^2}$ ∴ $\cancel{\int \frac{1}{y} dy}$ $\frac{1}{y}\frac{dy}{dx} = \frac{1}{1+x^2}$ ∴

$\int \frac{1}{y} dy = \int \frac{1}{1+x^2} dx = \ln|y| = \arctan x + C_1$ ∴

$|y| = e^{\arctan x + C_1} = e^{C_1} e^{\arctan x} = C_2 e^{\arctan x}$ ∴ $y = C e^{\arctan x}$ ∴

$y e^{-\arctan x} = C$ ∴ $u(x,y) = f(y e^{-\arctan x})$ ∴

$u(0,y) = \ln y = f(y e^{-\arctan(0)}) = f(y e^0) = f(y)$ ∴ let $y = s$ ∴

$f(s) = \ln y$ ∴ $f(y e^{-\arctan x}) = \ln(y e^{-\arctan x}) = \ln(y) - \arctan x = u(x,t)$

∴ $u(x,0) = \ln x = f((0) e^{-\arctan x}) = f(0)$ ∴ has no solutions

∵ $y = 0$ is the special sol ∴ the BC is on the charac curve ∴

no sols exist