- differentiation $\mathcal{L}\left(\frac{df(\cdot)}{dt}\right) = s\mathcal{L}(f(\cdot)) - f(0) \leftarrow IC$
- Final val thm $\lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$ these are useful in studing

LTI systems

$\mathcal{L}\left(\frac{d^2 f}{dt^2}\right) = s\mathcal{L}\left(\frac{df}{dt}\right) - \frac{df}{dt}(0) = s\left[s\mathcal{L}(f) - f(0)\right] - \frac{df}{dt}(0) =$

$s^2\mathcal{L}(s) - sf(0) - \dot{f}(0)$ generally following identical route as in

$\frac{d^2 f}{dt^2}$: $\mathcal{L}\left(\frac{d^n f(\cdot)}{dt^n}\right) = s^n\mathcal{L}(f) - s^{n-1}f(0) - s^{n-2}\dot{f}(0) - s^{n-3}\ddot{f}(0) - \dots - \underbrace{f^{n-1}(0)}_{\left(\frac{d^{n-1}f}{dt^{n-1}}\right)\Big|_{(0)}}$

with assumption $f(0) = \dot{f}(0) = \ddot{f}(0) = \dots = 0$

$\mathcal{L}\left(\frac{d^n f(\cdot)}{dt^n}\right) = s^n\mathcal{L}(f)$ L.T. converts derivatives into multiplication by

s & builds in ICs

real space: high ODE for $y(t)$ $\xrightarrow{\mathcal{L}(\cdot)}$ LT space: Algebraic eqn for $Y(s)$ $\Rightarrow$ solve & rearrange $\rightarrow Y(s) = \dots$

Sol $y(t) \xleftarrow{\mathcal{L}^{-1}(\cdot)} Y(s) = \dots$

$\frac{d^2 x(t)}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x(t) = \omega_n^2 f(t)$ ICs $x(0) = \dot{x}(0) = 0$

Taking Laplace transform $\mathcal{L}\left[\frac{d^2x}{dt^2} + 2\zeta\omega_n\frac{dx}{dt} + \omega_n^2 x\right] = \mathcal{L}\left[\omega_n^2 f(t)\right]$

$s^2X(s) + 2\zeta\omega_n sX(s) + \omega_n^2 X(s) = \omega_n^2 F(s)$, $X(s) = \mathcal{L}(x(t))$ & $F(s) = \mathcal{L}(F(t))$

$(s^2 + 2\zeta\omega_n s + \omega_n^2)X(s) = \omega_n^2 F(s)$

$G(s) = \frac{X(s)}{F(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ } is ratio of polys in 's' is called transfer func of Z system

Numerical Ex/ $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = e^{4t}$ $y(0) = 10$ $\frac{dy}{dt}(0) = 8$

for considering generic case

Taking L.T. $\mathcal{L}\left\{\frac{d^2y}{dt^2}\right\} + 5\mathcal{L}\left[\frac{dy}{dt}\right] + 6\mathcal{L}[y] = \mathcal{L}[e^{4t}]$ ∴

$\left(s^2Y(s) - sY(0) - \frac{dY}{dt}(0)\right) + \left(5sY(s) - 5Y(0)\right) + 6Y(s) = \frac{1}{s-4}$

$(s^2 + 5s + 6)Y(s) = sY(0) + 5Y(0) + \frac{dY}{dt}(0) + \frac{1}{s-4} = (s+5)10 + 8 + \frac{1}{s-4}$

$Y(s) = \frac{1}{(s+2)(s+3)(s-4)} + \frac{10s+58}{(s+2)(s+3)} = \frac{10s^2 + 18s - 231}{(s+2)(s-3)(s-4)} =$

$\frac{227}{6(s+2)} - \frac{195}{7(s+3)} + \frac{1}{42(s-4)}$ (doing partial fractions) ∴ taking $\mathcal{L}^{-1}(\cdot)$

Sol to our ODE is $y(t) = \underbrace{\frac{227}{6}e^{-2t} - \frac{195}{7}e^{-3t}}_{\text{from ICs part : C.F.}} + \underbrace{\frac{1}{42}e^{4t}}_{\text{from Z \& Z nonhomogeneous part : P.I.}}$

Consider a single input , single output dynamical system

input $u(t)$ $\rightarrow$ $\boxed{\text{System}}$ $\rightarrow$ output $y(t)$ System is represented by:

$\dot{x}(t) = Ax(t) + bu(t)$
$y(t) = Cx(t)$ $\Big\}$ ①

$x(t) \in \mathbb{R}^n$ (states)
$u(t) \in \mathbb{R}$ (control input) $y(t) \in \mathbb{R}$ (output)

$A \in \mathbb{R}^{n \times m}$ $b \in \mathbb{R}^{n \times 1}$ & $C \in \mathbb{R}^{1 \times n}$ assume IC $x(0) = 0$ (unless specified, may consider ICs 0)

taking Laplace of ① yields $sX(s) = AX(s) + bU(s)$, $Y(s) = CX(s)$

$\therefore$ ~~sX~~ $sX(s) - AX(s) = bU(s)$, $(s\underbrace{I_n}_{\text{identity mat of n dimen}} - A)X(s) = bU(s)$

$\therefore X(s) = (sI_n - A)^{-1} bU(s)$

output $Y(s) = CX(s) = \underbrace{C(sI_n - A)^{-1}}_{\text{system}} \underbrace{bU(s)}_{\text{input}}$ $\quad U(s) \rightarrow \boxed{\text{system}} \rightarrow Y(s)$

$\frac{Y(s)}{U(s)} = C(sI_n - A)^{-1}b$ ← transfer func representation this is transfer func betw $Y(s)$ & $U(s)$ & is a ratio of poly in 's'.

revisit $\frac{d^2x}{dt^2} + 2\zeta\omega_n \frac{dx}{dt} + \omega_n^2 x = \omega_n^2 \delta(t)$ $\quad \delta(t) \xrightarrow{\text{input}} \boxed{\text{system}} \xrightarrow[\text{output}]{y(t) = x(t)}$

state space form $\dot{x} = Ax + Bu$ $y = Cx$ Triplet (A, B, C)

$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}$, $B = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix}$, $C = [1 \;\; 0]$ $\quad x = [x \;\; \dot{x}]^T$

⇒ Converting to TF (transfer func) representation

$\frac{Y(s)}{U(s)} = C(sI_n - A)^{-1}B$ use A, B, C from above
$\vec{F}(s)$

$(sI_n - A)^{-1} = \begin{bmatrix} s & 1 \\ \omega_n^2 & s + 2\zeta\omega_n \end{bmatrix}^{-1} = \frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \begin{bmatrix} s + 2\zeta\omega_n & 1 \\ -\omega_n^2 & s \end{bmatrix}$

$B = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix}$ & $C = [1 \;\; 0]$ $\therefore \frac{Y(s)}{F(s)} = [1 \;\; 0] \left(\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \begin{bmatrix} s + 2\zeta\omega_n & 1 \\ -\omega_n^2 & s \end{bmatrix} \right) \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix} =$

$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$.

$\frac{d^2x}{dt^2} + 2\zeta\omega_n \frac{dx}{dt} + \omega_n^2 x = \omega_n^2 \delta(t)$, $x = 0 = \dot{x}(0) = 0$

state space form $\dot{x} = Ax + Bu$ $Y = Cx$ $x = [x \;\; \dot{x}]^T$

$A = \begin{bmatrix} 0 & 1 \\ -\omega_n^2 & -2\zeta\omega_n \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ \omega_n^2 \end{bmatrix}$ $C = [1 \;\; 0]$

transfer funct
$G(s) = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$ equivalent $\equiv$

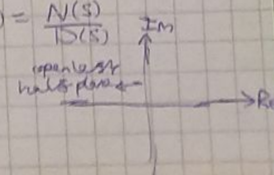
input $\rightarrow \boxed{\text{linear system}} \rightarrow$ output

ordinary differential eqn (ODE) $\xrightarrow{\mathcal{L}(s) \text{ Laplace transform}}$ transfer func form

compact form

state space model $\Longleftrightarrow$ $G(s) = C(sI_n - A)^{-1}B$ ratio of polys 's' $G(s) = \frac{N(s)}{D(s)}$

$\dot{x} = Ax + Bu$ $y = Cx + Du$
represented also by (A, B, C) triplet



stability: eigvals of $[A]$
by charac eqn $\det(sI - A) = 0$ λ is eigvals $Re(\lambda) < 0$

& for transfer func form: Charac eqn $D(s)=0$ (poly) roots ~~of poly~~ of this poly poles of transfer func $Re(\text{poles}) < 0$

\Ex/ state space (A,B,C) $A=\begin{bmatrix}-7 & -12\\ 1 & 0\end{bmatrix}$ $B=\begin{bmatrix}1\\0\end{bmatrix}$ $C=[1 \quad 2]$ $\Longrightarrow G(s)=C(sI-A)^{-1}B$

transfer func $\frac{x(s)}{u(s)}=\frac{s+2}{s^2+7s+12}$

eignals $\det(\lambda I_n - A)=0$ (Charac eqn) $\lambda_{1,2}=\{-3,-4\}$

eignals stable node at equilib

~~$G(s)=\frac{s+2}{s^2}$~~ (Charac eqn $\lambda^2+7\lambda+12=0$)

$G(s)=\frac{s+2}{s^2+7s+12}=\frac{N(s)}{D(s)}$

$D(s)=s^2+7s+12$ ∴ Charac eqn

$D(s)=0$ & solving for s yields poles as -3 & -4

For a triple (A,B,C) $\dot{x}=Ax+Bu$ $x\in\mathbb{R}^n$ $u \rightarrow \boxed{LTI} \rightarrow y$

$y=Cx$ $y\in\mathbb{R}$, $u\in\mathbb{R}$

associated transfer func representation $G(s)=C(sI_n-A)^{-1}B$

this representation is unique

but it is possible to have many different triple representation (A,B,C) give rise to same transfer func

uniqueness of state space repertation: suppose T is a square & invertible Mat let T has dimens of A

(A,B,C) state space triple → transfer func representation →

$T \Rightarrow (TAT^{-1}, TB, CT^{-1})$ transformed state space triple using square invertible T → lead to same transfer func representation

$TT^{-1}=I_n$

$G(s)=CT^{-1}\times(sI_n-TAT^{-1})^{-1}\times TB = CT^{-1}[sTT^{-1}-TAT^{-1}]^{-1}\cdot TB$

$=CT^{-1}[T(sI_n-A)T^{-1}]^{-1}TB$

$G(s)=C(sI-A)^{-1}B$

$(AB)^{-1}=B^{-1}A^{-1}$

$=CT^{-1}[T(sI_n-A)T^{-1}]^{-1}TB=$

$CT^{-1}[(sI_n-A)T^{-1}]^{-1}T^{-1}TB$

$=CT^{-1}T(sI_n-A)^{-1}T^{-1}TB=C(sI_n-A)^{-1}B \Longleftarrow$ not unique transfer func (A,B,C) triple

\Ex/ of non unique state space representation/ plant Mat $A=\begin{bmatrix}0 & 1\\ -1 & -2\end{bmatrix}$ input Mat $B=\begin{bmatrix}0\\1\end{bmatrix}$

output Mat $C=[1 \quad 0]$ triple (A,B,C) $\tilde{A}=\begin{bmatrix}-1 & 0\\ -1 & -1\end{bmatrix}$ $\tilde{B}=\begin{bmatrix}-1\\-1\end{bmatrix}$ $\tilde{C}=[-1 \quad -1]$ triple $(\tilde{A},\tilde{B},\tilde{C})$

Both (A,B,C) & $(\tilde{A},\tilde{B},\tilde{C})$ yield same transfer func use $D=0$

$G(s)=\frac{1}{s^2+2s+1}$ ← this is unique

Controllability / LTI system: $\dot{x}(t) = Ax(t) + Bu(t)$, $x(t) \in \mathbb{R}^n$

$u(t) \in \mathbb{R}^n$ output eqn is not important now

- Sol to state space LTI system is ~~$x(t) = e^{A(t-t_0)}$~~

$$x(t) = \underbrace{e^{A(t-t_0)}x(t_0)}_{\text{depending on IC}} + \underbrace{\int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau}_{\text{due to forcing or control input}}$$

Let us desire a set which represents all those pts that can be reached in time T from origin — reachable set

$$R(T) = \left\{ x_f \in \mathbb{R}^n \mid x_f = \int_0^T e^{A(t-\tau)}Bu(\tau)\,d\tau \right\} \quad (\because e^{A(t-t_0)} \times 0 = 0)$$

for some control $u(t) \in U$ during time interval $[0, T]$. System is controllable if $R(T) \equiv \mathbb{R}^n$ ie any pt in $\mathbb{R}^n$ is reachable in time T

If it is possible to find a control signal $u(t)$ which drive Z state from origin at time $t = t_0$ to anywhere in $\mathbb{R}^n$ during $(0, T]$ is this possible? given (A, B, C) under what conditions Z system is said to be controllable?

For $\dot{x}(t) = Ax(t) + Bu(t)$ know Sol is $x(t) = e^{A(t-t_0)}\overset{IC}{\overbrace{x(t_0)}} + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)\,d\tau$

at time T, $x(T) = x^* = e^{A(T-t_0)}x(t_0) + \int_{t_0}^{T} e^{A(T-\tau)}Bu(\tau)\,d\tau = x^* =$

$$= e^{AT} \times e^{-At_0}x(t_0) + \int_{t_0}^{T} e^{A(T-\tau)}Bu(\tau)\,d\tau$$

$$e^{-AT}x^* = e^{-At_0}x(t_0) + \cancel{e^{-At}} \int_{t_0}^{T} \cancel{e^{AT}} \times e^{-A\tau}Bu(\tau)\,d\tau \quad \therefore$$

cancels

$$e^{-AT}x^* - e^{-At_0}x(t_0) = \int_0^T \overbrace{e^{-A\tau}Bu(\tau)}^{\text{crucial part}}\,d\tau$$

{ recall Cayley-Hamilton thm: $e^{-A\tau} = \sum_{i=0}^{n-1} \alpha_i(\tau)A^i$ where $\alpha_i(\tau)$, $i = 0, 1, \ldots, n-1$ scalar time funcs, depending on coeffs of charac poly of A } { for e^{-At} }

$$e^{-AT}x^* - e^{-At_0}x(t_0) = \int_{t_0}^{T} e^{-A\tau}Bu(\tau)\,d\tau = \int_{t_0}^{T} \sum_{i=0}^{n-1} \alpha_i(\tau)A^iBu(\tau)\,d\tau =$$

$$\sum_{i=1}^{n-1} A^iB \int_{t_0}^{T} \alpha_i(\tau)u(\tau)\,d\tau = \underbrace{\left[A^0B \vdots AB \vdots A^2B \vdots \cdots \vdots A^{n-1}B\right]}_{\text{Mat}} \begin{bmatrix} \int_{t_0}^{T} \alpha_0(\tau)u(\tau)\,d\tau \\ \int_{t_0}^{T} \alpha_1(\tau)u(\tau)\,d\tau \\ \vdots \\ \int_{t_0}^{T} \alpha_{n-1}(\tau)u(\tau)\,d\tau \end{bmatrix}$$

components of funcs of required control inputs (I/P) $\tau \in (t_0, T)$

- $$= \overbrace{e^{-AT}x^* - e^{-At_0}x(t_0)}^{\text{known} \therefore \text{const vec}}$$

$$\text{Const vec}^{n \times 1} = \overset{\text{full rank}}{M^{n \times n \cdot m}} \cdot \begin{bmatrix} g_1(u(\tau)) \\ \vdots \\ g_{n-1}(u(\tau)) \end{bmatrix}^{n \times m \times 1}$$

Sol exists iff rank$(M) = n$

\Def:/ A matrix $(\overset{\text{Plant Mat}}{A}, \overset{\text{input Mat}}{B})$ that forms an n-dimen LTI System, $\dot{x} = Ax + Bu$
is said to be <u>controllable</u> iff controllability Mat
$M = [B \vdots AB \vdots A^2B \vdots \cdots \vdots A^{n-1}B]$ {here n is dimen of state}
is full Rank ie $\text{rank}(M) = n$
also (A, B) <u>is</u> <u>controllable</u> iff Z {t_0, T} eigvals of $(A-BK)$
can be assigned arbitly by choice of K. For single I/P
single O/P, test for controllability is that $\det(M) \neq 0$

\Ex/ consider $\dot{x}_1 = x_2$ $\dot{x}_2 = u$ ie $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
controllability mat is $M = [B \vdots AB]$ {∵ 2 dimen System}
$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ {clearly rank(M)=2} check for controllability $\det(M) = -1 \neq 0$ ∴
controllable
\Ex/ consider $\tilde{A} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $\tilde{B} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ controllability mat $M = [B \ AB]$
$\det(M) = \det\begin{bmatrix} 0 & 0 \\ 1 & 1 \end{bmatrix} = 0$ ∴ not controllable ∴
$(A-BK)$ can't be assigned arbitly

\Week 5/ ① linear control laws state feedback design → pole placement design ② Lipschitz condition
\linear control laws state feedback - pole placements design/
consider LTI system ① $\begin{cases} \dot{x} = Ax + Bu & x \in \mathbb{R}^n \\ y = Cx & (A, B, C) \end{cases}$ ⇒ let $r(t)$ be a <u>reference</u>
<u>profile</u> which Z syst is expected to follow - this is <u>tracking</u> probl
$r(t) = 0$ is a special case, called <u>regulation</u> probl
consider single I/P single O/P (SISO) case: Define ②: $u = -Kx$
is state feedback control with ② in ① yields:
$\dot{x} = Ax - BKx$ $\dot{x} = \underbrace{(A - BK)}_{A_{cl}} x$ Def: this is <u>closed loop syst</u>
Suppose $r = 0$: looking at regulation probl given $\dot{x} = Ax + Bu$
$u = -Kx$ to be determined ⇒ <u>objective</u>: with 'u': • closed loop syst
A_{cl} <u>is</u> <u>stable</u> • syst (closed loop) possess <u>specific time</u>
<u>response</u> <u>behaviour</u>, which is indicated by desired eigvals or
pole locations

$\dot{x} = \overbrace{(A-BK)}^{A_{cl}}x \qquad + A_{cl}$ + ~~hurwitz~~ ie stable $Re(\lambda(A_{cl})) < 0$

charac eqn (poly in 'S') $\det(S\overset{\text{identity Mat of correct dimen}}{I} - (A-BK)) = 0$

how to achive this?

\pole placement prob/ $\dot{x} = Ax + Bu \quad x \in \mathbb{R}^n \quad y = Cx$

find $u = -Kx$ st closed loop poles (eigvals) are at $\mu_1, \mu_2, \dots, \mu_n$

desired locations defines the specific behaviour for syst

Assumptions: • (A, B) controllable $\Rightarrow M = [B \vdots AB \vdots \dots \vdots A^{n-1}B]$

rank(M) full rank

• assume μ_i, $i = 1, \dots, n$ are distinct

• entire state available for feedback control design

• Control I/P is unconstrained. (ie no amplitude or rate limits on signal u)

\pole placement Design steps/ $\dot{x} = Ax + Bu$

step 1: check controllability of (A, B) $\quad M = [B \vdots AB \vdots A^2B \vdots \dots A^{n-1}B]$

rank(M) = n   in single I/P single O/P case $\det(M) \neq 0$ also ok

step 2: define $K = [k_1, k_2, \dots, k_n]$ $\because x \in \mathbb{R}^n$ vec of variabs k_i

step 3: closed loop syst $\dot{x} = (A-BK)x$ charac poly is $\det(SI - (A-BK))$

$= \prod_{i=1}^{n}(S - \mu_i)$ ← known desired poles given   which is the desired charac poly $\because \mu_1, \mu_2, \dots, \mu_n$ are distinct desired poles

step 4: solve for variab k_i's by matching coeffs of polys

$u = -Kx$ state feedback control law

\Ex 1:/ $\dot{x} = Ax + Bu \quad y = Cx \quad A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \quad C = [1 \ \ 0]$

find $u = -Kx$ st closed loop eigvals are -4 & -5 :

step 1: $[B \vdots AB]$: $M = \left[\begin{matrix} 0 \\ 1 \end{matrix} \vdots \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right] = \begin{bmatrix} 0 & 2 \\ 1 & 1 \end{bmatrix}$ rank(M) = full rank (2)

also $\det(M) \neq 0$ (-2) [single I/P sing O/P] $\therefore$ Controllable

step 2: $K = [k_1 \ \ k_2]$

step 3: desired charac poly $(S+4)(S+5) = S^2 + 9S + 20$ ①

charac poly of closed loop syst $A_{cl}^{+} = \begin{bmatrix} 1 & 2 \\ 2-k_1 & 1-k_2 \end{bmatrix}$ is

$A_{cl} = (A - BK) \quad \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} - \begin{bmatrix} 0 \\ 1 \end{bmatrix}[k_1 \ \ k_2]$

$S^2 + (k_2 - 2)S - 3 + 2k_1 - k_2$ ②

Comparing coeffs of ① & ② $\quad -2+k_2=9 \quad \therefore k_2=11 \qquad -3+2k_1-k_2=20$

$\therefore k_1=17 \quad \therefore$ Control law $u=[-17 \;\; -11]x$

\Towards a general case:/ $\dot{x}=Ax+Bu \quad x\in\mathbb{R}^n \quad u=-Kx$

$u=\mathbb{R}^m,\ m=1 \quad y=Cx$ suppose A, B posses a specific structure as follows:

$$A=\begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & \cdots & 0 \\ \vdots & & & & & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_n & -a_{n-1} & -a_{n-2} & \cdots & & -a_1 \end{bmatrix} \qquad B=\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix}$$

Controllable canonical form

$\det(sI-A) = \longrightarrow s^n+a_1s^{n-1}+a_2s^{n-2}+\dots+a_n$

$\Rightarrow$ so what happens? Write: $\dot{x}=(A-BK)x=A_{CL}x \qquad K=[k_1 \; k_2 \; \cdots \; k_n]$

(sing I/P sing O/P considered)

$$\text{where } A_{CL}=\begin{array}{l} \\ \\ \text{one row}\Big\{ \end{array}\begin{bmatrix} 0_{n-1} & \vdots & \mathbb{I}_{n-1} & & & \\ - & \vdots & - & - & - & - \\ -a_n & \vdots & -a_{n-1} & -a_{n-2} & \cdots & -a_1 \end{bmatrix}_{n \text{ columns}} - \begin{bmatrix} 0_{n-1} & \vdots & 0_{n-1} \\ - & \vdots & - \\ k_1 & \vdots & k_2 \; \cdots \; k_n \end{bmatrix}_{n \text{ columns}} \Big\} \text{only one row}$$

$$A_{CL}=\begin{bmatrix} 0_{n-1} & \vdots & & \mathbb{I}_{n-1} & & \\ - & \vdots & - & - & - & - \\ (-a_n-k_1) & \vdots & (-a_{n-1}-k_2) & (-a_{n-3}-k_3) & \cdots & (-a_1-k_n) \end{bmatrix}$$

known $\quad$ variab to be determined

$\because$ know $\mu_1, \dots, \mu_n$ (known) are distinct desired poles

$\prod_{i=1}^{n}(s-\mu_i) = s^n+\alpha_1 s^{n-1}+\alpha_2 s^{n-2}+\dots+\alpha_n$ like earlier

$\because$ know, $\mu_1, \dots, \mu_n$ (known) are distinct desired poles

$\prod_{i=1}^{n}(s-\mu_i) = s^n+\alpha_1 s^{n-1}+\alpha_2 s^{n-2}+\dots+\alpha_n$, like earlier

associated syst mat will be $\begin{bmatrix} 0_{n-1} & \vdots & \mathbb{I}_{n-1} & & & \\ - & \vdots & - & - & - & - \\ -\alpha_n & \vdots & -\alpha_{n-1} & -\alpha_{n-2} & \cdots & -\alpha_1 \end{bmatrix} \equiv A_{CL}$ Comparing two mats yields

$$\begin{aligned} a_n+k_1 &= \alpha_n \\ a_{n-1}+k_2 &= \alpha_{n-1} \\ &\vdots \\ a_1+k_n &= \alpha_1 \end{aligned} \qquad \Longrightarrow \qquad \begin{aligned} k_1 &= \alpha_n-a_n \\ k_2 &= \alpha_{n-1}-a_{n-1} \\ &\vdots \\ k_n &= \alpha_1-a_1 \end{aligned} \qquad u=-Kx$$

known $\because$ associate with $\det(sI-A)$ $\qquad$ known $\qquad$ straightforward calc

how to obtain this struc ie controllable canonical form $\therefore$ define square invertible mat T let $x=T\tilde{x}$ $\ni x\in\mathbb{R}^n$ & $\tilde{x}$ represents states in transformed coords take deriv w.r.t time

$\dot{x}=T\dot{\tilde{x}} \quad \because T$ is invertible $\dot{\tilde{x}}=T^{-1}\dot{x}=T^{-1}(Ax+Bu) \leftarrow \because \dot{x}=Ax+Bu$

$=T^{-1}Ax+T^{-1}Bu=T^{-1}AT\tilde{x}+T^{-1}Bu \quad (\because x=T\tilde{x})$

$\dot{\tilde{x}}=(T^{-1}AT)\tilde{x}+(T^{-1}B)u$ is in transformed coords

how determine T?

$[T]$ is square, invertible mat $\quad \dot{x}=Ax+Bu \quad$ define $T=MW$ $\because$

M is controllability mat $\quad M=[B \,\vdots\, AB \,\vdots\, A^2B \,\vdots\, A^{n-1}B]$, &

$\dot{V}$ is nega definite for $\beta \le 0$ when $\beta > 0$

$\dot{V} \le -2(\alpha - \beta)\|x\|^2, \forall \|x\|_2 \le r$

$\therefore$ $\dot{V}$ is nega definite when $r^2 < \frac{\alpha}{\beta}$

$\because$ $V(x) = \|x\|^2$ can write domain of attracts

$\Omega := \{ x \in \mathbb{R}^2 \mid \|x\|^2 < \frac{\alpha}{\beta} \}$.

\Back stepping control / Consider a nonlinear syst

Consider a nonlinear syst $\dot{\eta} = f(\eta) + g(\eta)\xi$ ①

$\dot{\xi} = u$ ② for $\eta \in \mathbb{R}^n$ $\xi \in \mathbb{R}$ $\therefore$ statespace $[\eta \ \xi]^T \in \mathbb{R}^{n+1}$

$u \in \mathbb{R}$ is control input $u \leftarrow$ single control input

$\dot{\xi} = \frac{d\xi}{dt}$ $\xi = \int \frac{d\xi}{dt}$

① is subsyst ①, ② is subsyst ②

$f(\cdot)$ $D \to \mathbb{R}^n$, $g(\cdot) \to \mathbb{R}^n$ are smooth funcs, & D contains

in domain $D \subseteq \mathbb{R}^n$

origin

$\eta = 0$ $f(\eta) = 0$ design objective

design state feedback control 'u' to stabilize Z

nonlinear syst ① - ② about origin

assumpts are: $f(\cdot)$ is known $g(\cdot)$ is known all states

η^T, ξ are measured

step 1: in ① ie $\dot{\eta} = f(\eta) + g(\eta)\xi$ $\xi \in \mathbb{R}$

treat ξ as a "virtual input"

Let ① be stabilized by a smooth state feed back control $\xi = \phi(\eta)$

& $\phi(0) = 0$ implies $\dot{\eta} = f(\eta) + g(\eta)\phi(\eta)$ is asymp stable $\phi: \mathbb{R}^n \to \mathbb{R}$

Suppose have a smooth posi definite Lyapunov func $V(\eta)$

that satisfy $\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] \le -W(\eta)$, $\forall \eta \in D$, where $W(\eta)$ is a

choice tobe determined

posi definite func

here $W(\cdot)$ introduces certain level of performance $\therefore$ virtual input;

at this step get or determine $\phi(\eta)$ that stabilize subsyst ①

asymp stable

step 2 Let us rewrite $\dot{\eta} = f(\eta) + g(\eta)\xi$, $\dot{\xi} = u$ as
in original syst

$\dot{\eta} = f(\eta) + g(\eta)\xi$

$\therefore \dot{\eta} = f(\eta) + g(\eta)\xi + g(\eta)\phi(\eta) - g(\eta)\phi(\eta)$

$\dot{\xi} = u$ define $z = \xi - \phi(\eta)$ $\quad g(\eta)\xi - g(\eta)\phi(\eta)$ is

$g(\eta)[\xi - \phi(\eta)] = g(\eta)z$ $\therefore$ Z dynamics is rewritten as

$\dot{\eta} = [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z$ ⑥ $\quad \dot{z} = \dot{\xi} - \dot{\phi}(\cdot)$ ⑦ $= u - \dot{\phi}(\eta)$

where $\dot{\phi} = \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\phi]$ this is called back stepping

suppose $v = u - \dot{\phi}$ $\therefore$ ⑥ remains same ⑦ becomes $\dot{z} = v$

in $\dot{\eta} = f(\eta) + g(\eta)\phi(\eta) + g(\eta)z$, $\dot{z} = v$ has a stable origin

step 3: define a composite Lyapunov func which is posi desinite

$V_c(\eta^T, z) = V(\eta) + \frac{1}{2}z^2$ ← original one

taking deriv of $V_c(\cdot)$ along ⑧ - ⑨ yields

$\dot{V}_c(\cdot) = \underbrace{\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)]}_{\le -W(\eta) \text{ negi desinite}} + \underbrace{\frac{\partial V}{\partial \eta}g(\eta)z + zv}_{\text{sign indesinite}}$ $\quad$ η dynamics

~~design~~ v - one can pick st $\dot{V}_c(\cdot)$ is negi desinite

$\dot{V}_c(\cdot) \le -W(\eta) + \frac{\partial V}{\partial \eta}(g(\eta))z + zv$ ← designed

choose $v = -\frac{\partial V}{\partial \eta}g(\eta) - kz$ ← known terms $\quad$ where k is scalar

with this v $\quad \dot{V}_c(\cdot) \le -W(\eta) \underbrace{- kz^2}_{\text{negi desinite}}$

desinite implies Z origin $(\eta^T = 0_n, z = 0)$ which is negi is asymp stable

Sub v, z, $\dot{\phi}(\cdot)$ $\quad u = \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi] - \frac{\partial V}{\partial \eta}g(\eta) - k[\xi - \phi(\eta)]$ is

Z back stepping control law to be used in ① - ②

$W(\eta) + kz^2$ is posi desinite is 0 only when $\eta = 0$ & $z = 0$

\Ex Q 1:/ Consider $\dot{x}_1 = \underbrace{x_1^2 - x_1^3}_{f(x_1)} + x_2$ ← virtual input ξ, $g(\cdot) = 1$ $\qquad \dot{x}_2 = u$

x_1 is like η $\qquad$ x_2 is like ξ in our theorey

$f(\eta) = x_1^2 - x_1^3$; $g(\eta) = 1$

Consider $\dot{x}_1 = x_1^2 + x_1^3 + x_2$ $\quad$ treate x_2 as 'virtual control'

need to design $x_2 = \phi(x_1)$ $\quad$ Let $x_2 = \frac{-x_1^2 - x_1}{\phi(x_1)}$ is choice given or

$V(\cdot)$ & $W(\cdot)$ given

with this virtual control $\phi(\cdot)$ subsyst ① dynamics

$\dot{x}_1 = x_1^2 - x_1^3 - x_1^2 - x_1 = -x_1^3 - x_1$

$\dot{V}(x_1) = x_1\dot{x}_1 = -x_1^2 - x_1^4$ for $\dot{V}(x_1) \leq -W(x_1)$ ∴

∴ $\dot{V}(x) = -x_1^2 - x_1^4 \leq -2V$ ∴ origin is stable exponentially & globally

to do backstepping, define $z_2 = x_2 - \phi(x_1)$ (is like $\dot{\xi} - \phi(\xi)$ terminology)

$z_2 = x_2 + x_1^2 + x_1$ $\quad \dot{z}_2 = \dot{x}_2 + \dot{x}_1 + 2x_1\dot{x}_1$

$\dot{x}_1 = -x_1 - x_1^3 + z_2$ $\quad \dot{z}_2 = \dot{x}_2 + \dot{x}_1 + 2x_1\dot{x}_1 = u + (1+2x_1)(-x_1 - x_1^3 + z_2)$

defining composite Lyapunov func $V_c(x_1, z) = V(x_1) + \frac{1}{2}z_2^2$

(will be given) $\quad \dot{V}_c = \dot{x}_1 x_1 + z_2\dot{z}_2 =$

$x_1(-x_1 - x^3 + z_2) + z_2(\cancel{u+}(1+2x_1)(-x_1 - x_1^3 + z_2)) =$

$\underbrace{-x_1^2 - x_1^4}_{\leq W(\cdot)} + z_2\underbrace{[x_1 + (1+2x_1)(-x_1 - x_1^3 + z_2) + u]}_{-z_2^2}$ ← unknown to be derived

$\dot{V}_c$ as negi definite $u = -x_1 - (1+2x_1)(-x_1 - x_1^3 - z_2) - z_2$

give us $\dot{V}_c = -x_1^2 - x_1^4 - z_2^2$ which is negi definite & ∴ Globally asymp stable

\Ex/ \Q1/ consider 2nd order dynamical syst $\dot{x}_1 = x_1 \underbrace{x_2}_{\phi(x_1) = -x_1^2}$ ①

$\dot{x}_2 = x_1 + u$ ②

\a/ show $x_2 = \phi(x_1) = -x_1^2$ is a virtual control making sub syst ① stable asymptly

\b/ define $z_2 = x_2 - \phi(x_1)$ ~~must~~ & use a composite Lyapunov func $V_c(x_1, z_2) = \frac{1}{2}(x_1^2 + z_2^2)$ & find $u = \psi(x_1, x_2)$ st

$\dot{V}_c(x_1, z_2) = -x_1^4 - z_2^2$

\a/ with $x_2 = -x_1^2$ given

$\dot{x}_1 = x_1 x_2 \Rightarrow \dot{x}_1 = -x_1^3$ choose candidate Lyapunov func $\frac{1}{2}x_1^2$ which is p.d.f. [∵ composite Lyapu func $V_c(x_1, z_2)$ given as $\frac{1}{2}(x_1^2 + \cancel{z_2^2})$] we can show $\dot{V}(x)$ is n.d.f. globally ∴

$\dot{V}(x) = x_1\dot{x}_1 = -x_1^4$ & ∴ $\dot{x}_1 = x_1 x_2$ with 'virtual control' $x_2 = -x_1^2$ is globally asymp stab

\b/ define $z_2 = x_2 - \phi(x_1)$ $\quad \dot{z}_2 = \dot{x}_2 - \dot{\phi}(x_1)$ $\quad \{\dot{x}_1 = x_1 x_2,\ \dot{x}_2 = x_1 + u\}$

$= \dot{x}_2 - 2x_1\dot{x}_1 = x_1 + 2x_1^2 x_2 + u$ ∴

$\dot{x}_1 = x_1 \overbrace{x_2}^{z_2 + \phi(x_1)}$

$\dot{x} = f(x) \rightarrow x_1, z_2$

$x = x_1, x_2$

$\dot{z}_2 = x_1 + 2x_1^2 + x_2 + u$

$V_c(\cdot) = \frac{1}{2}(x_1^2 + z_2^2)$ posi d.f.

$V_c \to \infty$ as $\|x\| \to \infty$ $\quad x = (x_1, z_2)^T$

$\dot{V}_c = x_1 \dot{x} + z_2 \dot{z}_2 = -x_1^4 - z_2^2$ ← derived (or given)

$= x_1(x_1 x_2) + z_2(x_1 + 2x_1^2 x_2 + u)$

$= x_1^2 x_2 + z_2 x_1 + z_2 \cancel{2} x_1^2 x_2 + z_2 u$

$= x_1(x_1 x_2) + z_2(x_1 + 2x_1^2 x_2 + u) = -x_1^4 - z_2^2$ add & subtract

$= -x_1^4 + x_1^2 x_2 + x_1^4 + z_2(\overbrace{x_1 + 2x_1^2 x_2 + u}^{-z_2})$ → freedom to be designed

$= -x_1^4 + x_1^2(\overbrace{x_2 + x_1^2}^{z_2}) + z_2(x_1 + 2x_1^2 x_2 + u) =$

$-x_1^4 + [(x_1^2 + x_1 + 2x_1^2 x_2 + u) z_2]$ to have

$= -x_1^4 - z_2^2 \therefore$

$x_1^2 + x_1 + 2x_1^2 x_2 + u = -z_2$

$u = -z_2 - (x_1^2 + x_1 + 2x_1^2 x_2)$

$\cancel{= -(x_1 + x_2)}$ $\quad u = \psi(x_1, x_2) = -(x_1 + x_2 + 2x_1^2 x_2)$

\May 2020 pp/

\Week 6/ \Q4 from week 6/

$\ddot{\theta}(t) + \dot{\theta}(t) + \frac{g}{l}\sin\theta = 0 \rightarrow$ dynamics

- Candidate Lyapunov func

$V(\theta,\dot{\theta}) = \overbrace{\frac{1}{2}\dot{\theta}^2}^{+ve} + \frac{g}{l}\overbrace{(1-\cos\theta)}^{+ve}$

$V(\theta,\dot{\theta}) = 0$ only at $\theta = \dot{\theta} = 0$ in a domain $D = \{(\theta,\dot{\theta}) \mid |\theta| < \pi\}$

$V(\theta,\dot{\theta})$ is positive in $D \setminus \{0\}$

$\because \dot{\theta}^2$ is +ve and $(1-\cos\theta)$ term +ve ($\cos\theta$ upper bounded by 1)

$\therefore$ Sum of two +ve terms.

$\therefore V(\theta,\dot{\theta})$ is p.d.f in D

Taking time derivative $\dot{V}(\theta,\dot{\theta}) = \dot{\theta}\ddot{\theta} + \frac{g}{l}\dot{\theta}\sin\theta$

subbing from eqn of dynamics (given)

$= \dot{\theta}(-\dot{\theta} - \frac{g}{l}\sin\theta) + \frac{g}{l}\dot{\theta}\sin\theta = -\dot{\theta}^2 \leq 0$

$\dot{V}(\theta,\dot{\theta})$ cant be claimed as negative definite (though $-\dot{\theta}^2$ is negative term) for the system, since

$\dot{V}(\theta,\dot{\theta}) = 0$ for $\dot{\theta} = 0$ and no matter what θ values are $\therefore$

$\therefore \dot{V}(\theta,\dot{\theta})$ is negative semi definite & conclusion is that Z origin is Lyapunov Stable

using $V(\theta,\dot{\theta}) = \frac{1}{2}\dot{\theta}^2 + \frac{g}{l}(1-\cos\theta)$

remark: Lyapunov conditions are only sufficient conditions

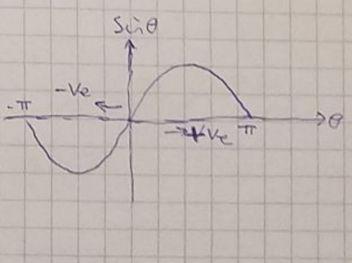
(ii) $V(\theta,\dot{\theta}) = \underbrace{\frac{1}{2}\dot{\theta}^2 + \frac{2g}{l}(1-\cos\theta)}_{\text{was considered in part (i)}} + \frac{1}{2}(\theta+\dot{\theta})^2$

$V(\theta,\dot{\theta})$ is p.d.f in D (argue like in part (i))

Taking derivative $\dot{V}(\theta,\dot{\theta}) = \theta\ddot{\theta} + (\theta+\dot{\theta})(\theta+\ddot{\theta}) + \frac{2g}{l}\dot{\theta}\sin\theta$

Like earlier substitute for $\ddot{\theta}$ from dynamics

and bit of algebra yields $= \underbrace{-\dot{\theta}^2}_{\text{like in part (i)}} - \frac{g}{l}\theta\sin\theta$



$\dot{\theta}^2$ is +Ve $\forall\theta$ $\theta\sin\theta$

θ	+Ve	-Ve
$\sin\theta$	+Ve	-Ve
$\theta\sin\theta$	+Ve	+Ve

$\dot{V}(\theta,\dot{\theta}) < 0 \quad \forall(\theta,\dot{\theta}) \in D \setminus \{0\}_{\{(0,0)\}}$

since $V(\theta,\dot\theta)$ is p.d.s and $\dot V(\theta,\dot\theta)$ is n.d.s. in D the origin of the System in D is Asymptotically Stable in $D=\{(\theta,\dot\theta)^T \mid |\theta|<\pi\}$

\Q6 from week 6/ $\Sigma=\begin{cases}\dot x_1=-4x_1+x_1x_2^2\\ \dot x_2=-x_2+x_1\end{cases}$

Candidate 'L' func $V(x_1,x_2)=x_1^2+x_2^2$ $\qquad V=x_1^2+x_2^2$ Sum of square terms

$x_1=x_2=0 \qquad V=0$

$V(x_1,x_2)>0 \quad \forall (x_1,x_2)\neq\{(0,0)\} \qquad V(0,0)=0 \;\therefore V(\cdot)$ is p.d.s

~~taking~~ $\{V(\cdot)>0 \;\forall x\in\mathbb{R}^2\setminus\{0\} \quad V(0)=0,\; x=0 \quad V(\cdot)$ is p.d.s$\}$

taking deriv of $V(\cdot)$ along Σ $\qquad \dot V(\cdot)=2x_1\dot x_1+2x_2\dot x_2$ Sub $\dot x_1$ & $\dot x_2$ from Σ: $\quad \dot V(\cdot)=2x_1(-4x_1+x_1x_2^2)+2x_2(-x_2+x_1)=$

$-8x_1^2+2x_1^2x_2^2+2x_2x_1-2x_2^2=$

$-7x_1^2-x_1^2+2x_1^2x_2^2+2x_1x_2-x_2^2-x_2^2=$

$-7x_1^2-x_2^2+2x_1^2x_2^2-(x_1^2+x_2^2-2x_1x_2)=$

$-7x_1^2-x_2^2+2x_1^2x_2^2-\underbrace{(x_1^2+x_2^2-2x_1x_2)}_{(x_1-x_2)^2}$ Lowerbounded by 0

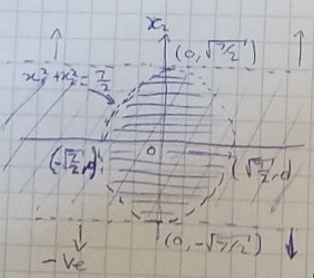
$\leq -7x_1^2-x_2^2+2x_1^2x_2^2$

$\dot V(\cdot)\leq -7x_1^2-x_2^2+2x_1^2x_2^2$

←OR→

$\dot V(\cdot)=-x_1^2(7-2x_2^2)-x_2^2$ ~~$\dot V(\cdot)=2M$~~

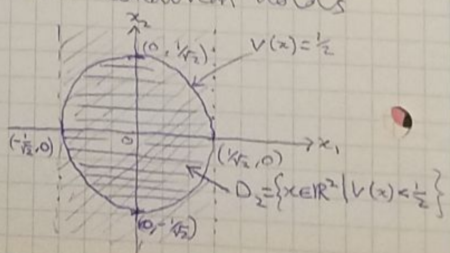
$\dot V(\cdot)<0$ if $(7-2x_2^2)$ is guaranteed to be positive which implies $\dot V(\cdot)<0$ if $x_1^2+x_2^2<\frac{7}{2}$



then we can desire an invariant set $D_1\{x\in\mathbb{R}^2 \mid V(x)<\frac{7}{2}\}$ in which $V(\cdot)$ is p.d.s & $\dot V(\cdot)$ is n.d.s. Conditions holds

$\dot V(\cdot)=-7x_1^2-x_2^2(1-2x_1^2)$

$\dot V(\cdot)<0$ if $(1-2x_1^2)$ is guaranteed to be +ve which implies $\dot V(\cdot)<0$ if $x_1^2+x_2^2<\frac12$ then we can desire an invariant $D_2=\{x\in\mathbb{R}^2 \mid V(x)<\frac12\}$ in which $V(\cdot)$ is p.d.s & $\dot V(\cdot)$ is n.d.s. Condition holds



Week 1 Sheet

i/ $y^{(n)} = \dfrac{d^n y}{dt^n}$

~~$\delta = f\,\dot{y}$~~ $\quad y^{(1)} = g(y, u)$

$y^{(2)} = g(y, \dot{y}, u) \qquad y^{(3)} = g(y, \dot{y}, \ddot{y}, u) \qquad y^{(4)} = g(y, \dot{y}, \ddot{y}, y^{(3)}, u)$

$$\begin{bmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \\ y^{(4)} \\ \vdots \\ y^{(n)} \end{bmatrix} = \begin{bmatrix} g(y, u) \\ g(y, \dot{y}, u) \\ g(y, \dot{y}, \ddot{y}, u) \\ g(y, \dot{y}, \ddot{y}, y^{(3)}, u) \\ \vdots \\ g(y, \dot{y}, \dots, y^{(n-1)}, u) \end{bmatrix}$$

i Sol/ let $x_1 = y \quad x_2 = \dfrac{dy}{dt} \quad \dots \quad x_n = \dfrac{d^{n-1} y}{dt^{n-1}}$

taking derivatives of Z new variables

$\dot{x}_1 = \dot{y}$

$\dot{x}_2 = \ddot{y} \dots \therefore \dot{x}_{n-1} = \dfrac{d^{n-1} y}{dt^{n-1}} \qquad \dot{x}_n = \dfrac{d}{dt}\left(\dfrac{d^{n-1} y}{dt^{n-1}}\right) = y^{(n)} \therefore$

$y^{(n)} = g(y, \dot{y}, \dots, y^{(n-1)}, u)$ given $\therefore$ sub variables:

$$\left.\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= x_3 \\ &\vdots \\ \dot{x}_{n-1} &= x_n \\ \dot{x}_n &= g(x_1, x_2, \dots, x_n, u) \end{aligned}\right]$$

is represented as $\dot{x} = F(x, u)$

$X = [x_1; x_2, \dots, x_n]^T$

output is said to be y, ie x_1

So $y = x_1 \}$ $\quad y = CX$ where $C = [1 \ 0 \ \dots \ 0]_{1\times n}$

1/ let $x_1 = y$, $x_2 = \dfrac{dy}{dt}$, $x_3 = \dfrac{d}{dt}\left(\dfrac{dy}{dt}\right) = \dfrac{d^2 y}{dt^2} \ \dots \ x_n = \dfrac{d^{n-1} y}{dt^{n-1}} \quad \therefore$

$\dot{x}_1 = \dot{y} = \dfrac{dx}{dt} = \dfrac{dy}{dt} \therefore \dot{y} = x_2 \therefore \quad \ddot{y} = \dot{x}_2 \therefore \quad y^{(3)} = \dot{x}_3 \ \dots \ y^{(n)} = \dot{x}_n \therefore$

$\dot{x}_{n-1} = \dfrac{d^{n-1} y}{dt^{n-1}} = y^{(n-1)} \therefore \dot{x}_n = \dfrac{d}{dt}\left(y^{(n-1)}\right) = y^{(n)} = g(y, \dot{y}, \dots, y^{(n-1)}, u) \therefore$

$$\dot{X} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ g(y, \dot{y}, \dots, y^{(n-1)}, u) \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ g(x_1, x_2, \dots, x_{n-1}, u) \end{bmatrix}$$

$\therefore \ x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \therefore \quad y = CX \qquad C = [1 \ 0 \ \dots \ 0]_{1\times n}$

\2/ $I\ddot{q}_1 = -MgL\sin q_1 - k(q_1 - q_2)$ ∴

$\ddot{q}_1 = -\frac{1}{I}MgL\sin q_1 - \frac{1}{I}k(q_1 - q_2)$ $J\ddot{q}_2 \cancel{\cdots} k(q_1 - q_2)$ ∴

$\ddot{q}_2 = J^{-1}u + J^{-1}k(q_1 - q_2)$ ∴

$\dot{q}_1 = \frac{d}{dt}q_1 \qquad \ddot{q}_1 = \frac{d^2}{dt^2}q_1$

$$\begin{bmatrix}\ddot{q}_1\\ \ddot{q}_2\end{bmatrix} = \begin{bmatrix}-\frac{1}{I}MgL\sin q_1 - I^{-1}k(q_1 - q_2)\\ J^{-1}u + J^{-1}k(q_1 - q_2)\end{bmatrix}$$

\2 sol/ Let $x_1 = q_1$, $x_2 = \dot{q}_1$, $x_3 = q_2$, $x_4 = \dot{q}_2$ ∴ taking derivs,

$\dot{x}_1 = x_2 \qquad \dot{x}_2 = -\frac{Mg}{I}\sin x_1 - \frac{k}{I}(x_1 - x_3)$, $\dot{x}_3 = x_4$

$\dot{x}_4 = \frac{k}{J}(x_1 - x_3) + \frac{1}{J}u$

state variables $x = [q_1, \dot{q}_1, q_2, \dot{q}_2] \in \mathbb{R}^4$

~~\2/ Let $x_1 \Sigma q$~~ (Z order of Z states could be different accordingly Z RHS of $\dot{x} = f(x)$ will change)

\2/ Let $x_1 = q_1$, $x_3 = q_2$ ∴ $\dot{q}_1 = \frac{d}{dt}q_1 = \frac{d}{dt}x_1 = \dot{x}_1 = x_2$

$\dot{q}_2 = \frac{d}{dt}q_2 = \frac{d}{dt}x_3 = \dot{x}_3 = x_4$ ∴

$\dot{x}_1 = \frac{d}{dt}x_1 = \frac{d}{dt}q_1 = \dot{q}_1 = x_2$

$\dot{x}_3 = \dot{q}_2 = x_4$

$\ddot{q}_1 = -I^{-1}MgL\sin q_1 - I^{-1}k(q_1 - q_2)$

$\ddot{q}_2 = J^{-1}u + J^{-1}k(q_1 - q_2)$ ∴

$\dot{x}_2 = \frac{d}{dt}x_2 = \frac{d}{dt}\dot{q}_1 = \ddot{q}_1 = -I^{-1}MgL\sin q_1 - I^{-1}k(q_1 - q_2)$

$\dot{x}_4 = \frac{d}{dt}x_4 = \frac{d}{dt}\dot{q}_2 = \ddot{q}_2 = J^{-1}u + J^{-1}k(q_1 - q_2)$ ∴

$$x = \begin{bmatrix}x_1\\x_2\\x_3\\x_4\end{bmatrix} = \begin{bmatrix}q_1\\ \dot{q}_1\\ q_2\\ \dot{q}_2\end{bmatrix}$$ ∴ $x = [q_1, \dot{q}_1, q_2, \dot{q}_2] \in \mathbb{R}^4$

\Week 1 Sheet/

\3/ $\ddot{y} = g - M^{-1}ky - M^{-1}c_1\dot{y} - M^{-1}c_2\dot{y}|\dot{y}|$ ∴

$x_1 = y$ ∴ $\cancel{x_2}$ $\dot{y} = \frac{d}{dt}y = \frac{d}{dt}x_1 = \dot{x} = x_2$

$\ddot{y} = \frac{d}{dt}\dot{y} = \frac{d}{dt}x_2 = \dot{x}_2 = x_3$

$x_1 = y$ ∴ $\dot{x}_1 = \dot{y} = x_2$

$\dot{x}_2 = \ddot{y} = g - M^{-1}ky - M^{-1}c_1\dot{y} - M^{-1}c_2\dot{y}|\dot{y}|$ ∴

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} y \\ \dot{y} \end{bmatrix}$ ∴ $x = [y, \dot{y}] \in \mathbb{R}^2$

\3 sol/ $M\ddot{y} = Mg - ky - c_1\dot{y} - c_2\dot{y}|\dot{y}|$

Let $x_1 = y$ $x_2 = \dot{y}$ then $\dot{x}_1 = x_2$

$\dot{x}_2 = -\frac{k}{M}x_1 - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2| + g$

states $x = [y \;\; \dot{y}] \in \mathbb{R}^2$

$x = [x_1 \;\; x_2]$

\3/ let $x_1 = y$ ∴ $\cancel{\dot{x}_{1}}$ $\dot{y} = \frac{d}{dt}y = \frac{d}{dt}x_1 = \dot{x}_1 = x_2$ ∴

$\ddot{y} = \frac{d}{dt}\dot{y} = \frac{d}{dt}x_2 = \dot{x}_2 = -\frac{k}{M}x_1 - \frac{c_1}{M}x_2 - \frac{c_2}{M}x_2|x_2| + g$

∴ states $[y, \dot{y}] \in \mathbb{R}$ ∴ $x = [x_1 \;\; x_2]$

\4a/ let $x_1 = r$ ∴ $\dot{r} = \frac{d}{dt}r = \frac{d}{dt}x_1 = \dot{x}_1 = x_2$

$\ddot{r} = \frac{d}{dt}\dot{r} = \frac{d}{dt}x_2 = \dot{x}_2 = r(t)\dot{\theta}^2(t) - \gamma r^{-2}(t)\, u_1(t)$

$\ddot{\theta}(t) = -2\frac{1}{r(t)}\dot{r}(t)\dot{\theta}(t) + r(t)^{-1}u_2(t)$ ∴

$\theta = x_3$ ∴ $\dot{\theta} = \dot{x}_3 = x_4$ ∴ $\ddot{\theta} = \dot{x}_4 = -2r^{-1}(t)\dot{r}(t)\dot{\theta}(t) + r^{-1}(t)u_2(t)$

compact state : $x = [r, \dot{r}, \theta \;\; \dot{\theta}]$ ∴ $x = [x_1 \;\; x_2 \;\; x_3 \;\; x_4]$

$$\left.\begin{aligned} \ddot{r}(t) &= r(t)\dot{\theta}^2(t) - \gamma r^{-2}(t) + u_1(t) \\ r(t)\ddot{\theta}(t) &= -2\dot{r}(t)\dot{\theta}(t) + u_2(t) \end{aligned}\right\} ①$$

~~4a sol~~ From ① it can be seen that

\4a sol/ this is a nonlinear system of fourth order $\ddot{r}(t) = \frac{d^2 r(t)}{dt^2}$

$\ddot{\theta}(t) = \frac{d^2\theta(t)}{dt^2}$ introduce state variabs $x_1 = r(t)$ $x_2 = \dot{r}(t)$

$x_3 = \theta(t)$ $\dot{x}_3 = \dot{\theta}(t)$ taking time derivs of these state variabs yields

$\dot{x}_1 = \dot{r}(t) = x_2(t)$ ← from def of state vector

$\dot{x}_2 = \ddot{r}(t) = r(t)\dot{\theta}^2(t) - \gamma r^{-2}(t) + u_1(t)$

$\dot{x}_3 = \dot{\theta}(t) = x_4(t)$ $\quad \dot{x}_4 = \ddot{\theta}(t) = \frac{1}{r(t)}[-2\dot{r}(t)\dot{\theta}(t) + u_2(t)]$ Subbing state variables

for $r(t)$, $\dot{\theta}(t)$, $\dot{r}(t)$, ~~$\theta(t)$~~ etc

$\dot{x}_1 = x_2 \quad \dot{x}_2 = x_1 x_4^2 - \gamma x_1^{-2} + u_1 \quad \dot{x}_3 = x_4$

$\dot{x}_4 = -2x_2 x_4 x_1^{-1} + u_2 x_1^{-1}$ ∴ above is in ∴ compact state space form

\4a redo/ $x_1 = r(t)$, $x_2 = \dot{r}(t)$ $\quad x_3 = \theta(t)$ $\quad \dot{x}_3 = \dot{\theta}(t)$ ∴

$\dot{x}_1 = \dot{r}(t) = x_2$

$\dot{x}_2 = \ddot{r}(t) = r(t)\dot{\theta}^2(t) - \gamma r^{-2}(t) + u_1(t)$

$\dot{x}_3 = \dot{\theta}(t) = x_4(t)$ ∴ $\qquad \dot{x}(t) = F(x, u)$

$\dot{x}_4 = \ddot{\theta}(t) = \frac{1}{r(t)}[-2\dot{r}(t)\dot{\theta}(t) + u_2(t)]$ ∴ $\quad x = [x_1, x_2, x_3, x_4]^T = [r(t)\ \dot{r}(t)\ \theta(t)\ \dot{\theta}(t)]$

$F = [f_1(\cdot), f_2(\cdot), f_3(\cdot), f_4(\cdot)]^T$

for $r(t)$, $\dot{\theta}(t)$, $\dot{r}(t)$, etc $\qquad u = [u_1, u_2]^T$

$\dot{x}_1 = x_2 \quad \dot{x}_2 = x_1 x_4^2 - \gamma x_1^{-2} + u_1 \quad \dot{x}_3 = x_4$

$\dot{x}_4 = -2x_2 x_4 x_1^{-1} + u_2 x_1^{-1}$ is compact state space form

\4b/ $\dot{\tilde{x}}(t) = [0, 0, \sqrt{\gamma}, 0]^T$

\4b sol/ $\dot{x}(t) = F(x, u)$ where $x = [x_1, x_2, x_3, x_4]^T$ is

$[r(t)\ \dot{r}(t)\ \ \theta(t)\ \ \dot{\theta}(t)]^T \qquad F = [f_1(\cdot), f_2(\cdot)\ f_3(\cdot), f_4(\cdot)]^T$

$u = [u_1, u_2]^T \qquad F = [f_1(\cdot), f_2(\cdot), f_3(\cdot), f_4(\cdot)]^T$

is $\tilde{x}(t) = [1\ \ 0\ \ \sqrt{\gamma}(t)\ \ \sqrt{\gamma}]^T \qquad \tilde{u}(0) = (0, 0)^T$ ∴ linearised

eqn about this trajectory is $\dot{x} = Ax + Bu$ where

$$A = \begin{bmatrix} \partial f_1/\partial x_1 & \partial_{x_2} f_1 & \partial f_1/\partial x_3 & \partial f_1/\partial x_4 \\ \partial f_2/\partial x_1 & \partial_{x_2} f_2 & \partial_{x_3}(f_2) & \partial f_2/\partial x_4 \\ \partial f_3/\partial x_1 & \partial_{x_2} f_3 & (f_3)_{x_3} & \partial f_3/\partial x_4 \\ \partial f_4/\partial x_1 & \partial_{x_2} f_4 & \partial f_4/\partial x_3 & \partial f_4/\partial x_4 \end{bmatrix}\Bigg|_{(\tilde{x}(0), \tilde{u}(0))} =$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ x_4^2 + 2\gamma x_1^{-3} & 0 & 0 & 2x_1 x_4 \\ 0 & 0 & 0 & 1 \\ -2x_2 x_4 x_1^{-2} - u_2 x_1^{-2} & -2x_4 x_1^{-1} & 0 & -2x_2 x_1^{-1} \end{bmatrix}\Bigg|_{(\tilde{x}(0), \tilde{u}(0))}$$ and

$$B = \begin{bmatrix} \partial f_1/\partial u_1 & \partial f_1/\partial u_2 \\ \partial f_2/\partial u_1 & \partial f_2/\partial u_2 \\ \partial f_3/\partial u_1 & \partial f_3/\partial u_2 \\ \partial f_4/\partial u_1 & \partial f_4/\partial u_2 \end{bmatrix}(\tilde{x}(0), \tilde{u}(0)) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & x_1^{-1} \end{bmatrix}\Bigg|_{(\tilde{x}(0), \tilde{u}(0))} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix}$$

A2 ∴

\week 1 sheet/ $A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\gamma & 0 & 0 & 2\sqrt{\gamma} \\ 0 & 0 & 0 & 1 \\ 0 & -2\sqrt{\gamma} & 0 & 0 \end{bmatrix}$ states $x = [r, \dot{r}, \theta, \dot{\theta}]$

yields $\dot{x} = Ax + Bu$

\4b/ $\tilde{x}(t) = [1 \quad 0 \quad \sqrt{\gamma}t \quad \sqrt{\gamma}]^T$ $\tilde{u}(0) = (0, 0)^T$ & linearised

~~lin~~ eqn about this trajectory is $\dot{x} = Ax + Bu$ where

$$A = \begin{bmatrix} \partial\dot{x}_1/\partial x_1 & \partial\dot{x}_1/\partial x_2 & \partial\dot{x}_1/\partial x_3 & \partial\dot{x}_1/\partial x_4 \\ \dot{x}_2/\partial x_1 & \partial\dot{x}_2/\partial x_2 & \partial\dot{x}_2/\partial x_3 & \partial\dot{x}_2/\partial x_4 \\ \partial x_3(\dot{x}_3) & \partial\dot{x}_3/\partial x_2 & \partial\dot{x}_3/\partial x_3 & \partial\dot{x}_3/\partial x_4 \\ \partial x_1 \dot{x}_4 & \partial\dot{x}_4/\partial x_2 & \partial\dot{x}_4/\partial x_3 & (\dot{x}_4)x_4 \end{bmatrix}\Bigg|(\tilde{x}(t), \tilde{u}(t)) =$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ x_4^2 + 2\gamma x_1^{-3} & 0 & 0 & 2x_1x_4 \\ 0 & 0 & 0 & 1 \\ -2x_2x_4x_1^{-2} - u_2x_1^{-2} & -2x_4x_1^{-1} & 0 & -2x_2x_1^{-1} \end{bmatrix}\Bigg|(\tilde{x}(t), \tilde{u}(t)) =$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\gamma & 0 & 0 & 2\sqrt{\gamma} \\ 0 & 0 & 0 & 1 \\ 0 & -2\sqrt{\gamma} & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} (\dot{x}_1)u_1 & (\dot{x}_1)u_2 \\ (\dot{x}_2)u_1 & (\dot{x}_2)u_2 \\ (\dot{x}_3)u_1 & (\dot{x}_3)u_2 \\ (\dot{x}_4)u_1 & (\dot{x}_4)u_2 \end{bmatrix}\Bigg|(\tilde{x}(t), \tilde{u}(t)) =$$

$$\begin{bmatrix} (\dot{x}_1)u_1 & (\dot{x}_1)u_2 \\ (\dot{x}_2)u_1 & (\dot{x}_2)u_2 \\ (\dot{x}_3)u_1 & (\dot{x}_3)u_2 \\ (\dot{x}_4)u_1 & (\dot{x}_4)u_2 \end{bmatrix}\Bigg|(\tilde{x}(t), \tilde{u}(t)) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & x_1^{-1} \end{bmatrix}\Bigg|(\tilde{x}(t), \tilde{u}(t)) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \therefore \dot{x} = Ax + Bu$$

\4c sol/ Distance $r(t)$ & angle $\theta(t)$ are measured so

$y(t) = [r(t) \quad \theta(t)]^T$ i.e. need to write $y = Cx$ where

$y \in \mathbb{R}^2$ & $x \in \mathbb{R}^4$ so determine a C matrix $C \in \mathbb{R}^{2\times4}$ so

$$\begin{bmatrix} r(t) \\ \theta(t) \end{bmatrix} = C\begin{bmatrix} r(t) \\ \dot{r}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix} \therefore \text{where } C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

\4c resol/ $\therefore y(t) = [r(t) \quad \theta(t)]^T$ $\therefore y = Cx \therefore$

$y \in \mathbb{R}^2$ & $x \in \mathbb{R}^4$ $\therefore$ C matrix $C = \mathbb{R}^{2\times4}$ $\therefore$

~~If~~ $x = [r(t) \quad \dot{r}(t) \quad \theta(t) \quad \dot{\theta}(t)]^T \therefore$

$$\begin{bmatrix} r(t) \\ \theta(t) \end{bmatrix} = C\begin{bmatrix} r(t) \\ \dot{r}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix} \therefore C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

\5a sol/ $m\ddot{x} = -k_1 x - k_2 x^3$ let $x_1 = x$, $x_2 = \dot{x}$ then

$\dot{x}_1 = x_2$

$\dot{x}_2 = \ddot{x} = -\frac{k_1}{m}x - \frac{k_2}{m}x^3 = -\frac{k_1}{m}x_1 - \frac{k_2}{m}x_1^3 = -\frac{k_1}{m}x_1 - \frac{k_2}{m}x_1^3$

is in compact form with states $X = [x_1, x_2]$

\5a resol/ let $x_1 = x$ ∴ $x_2 = \dot{x}$ then

$\dot{x} = \frac{d}{dt}x = \frac{d}{dt}x_1 = \dot{x}_1 = x_2$ ∴

$\dot{x}_2 = \frac{d}{dt}x_2 = \frac{d}{dt}\dot{x} = \ddot{x} = -\frac{k_1}{m}x - \frac{k_2}{m}x^3 = -\frac{k_1}{m}x_1 - \frac{k_2}{m}x_1^3$

is in compact form with states $X = [x_1, x_2]$

\5b sol/ For equilibrium pts solve

$x_2 = 0$ ∴ $-\frac{k_1}{m}x_1 - \frac{k_2}{m}x_1^3 = 0 \Longrightarrow x_1^* = \pm\sqrt{-k_1/k_2}$, 0

which is real only if k_1 & k_2 are of opposite signs

$k_1 = -1$, $k_2 = 1/2$ so $(0,0)$ $(\sqrt{2}, 0)$ $(-\sqrt{2}, 0)$

\5b resol/ for equilib pts solve $x_2 = 0$ ∴

$\frac{dx_2}{dt} = \frac{d}{dt}(0) = 0 = \dot{x}_2$ ∴

$-\frac{k_1}{m}x_1 - \frac{k_2}{m}x_1^3 = 0 = x_1\left(-\frac{k_1}{m} - \frac{k_2}{m}x_1^2\right)$ ∴ $x_1 = 0$, $-\frac{k_1}{m} - \frac{k_2}{m}x_1^2 = 0$ ∴

$-\frac{k_1}{m} = \frac{k_2}{m}x_1^2$ ∴ $-\frac{k_1}{k_2} = x_1^2$ ∴ $x_1 = +\sqrt{-\frac{k_1}{k_2}}$, $x_2 = -\sqrt{-\frac{k_1}{k_2}}$

∴ $k_1 = -1$, $k_2 = 1/2$ ∴ $x_1 = \sqrt{-\frac{-1}{1/2}} = \sqrt{2}$, $x_1 = -\sqrt{-\frac{-1}{1/2}} = -\sqrt{2}$, $x_1 = 0$ ∴

$(0,0)$, $(\sqrt{2}, 0)$, $(-\sqrt{2}, 0)$

\5c sol/ For linearised plant, determine Jacobian $A = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix}$

$f_1(\cdot) := x_2$ $f_2(\cdot) := -\frac{k_1}{m}x_1 - \frac{k_2}{m}x_1^3$ ∴ $A = \begin{bmatrix} 0 & 1 \\ -k_1 - 3k_2(x_1)^2 & 0 \end{bmatrix}$

evaluating A about equilibrium:

∴ at $(0,0)$: $A_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -k_1 & 0 \end{bmatrix}$ given $k_1 = -1$ so $A_{(0,0)} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\dot{X} = A_{(0,0)}X$

at $(\pm\sqrt{2}, 0)$ $A_{(\pm\sqrt{2},0)} = \begin{bmatrix} 0 & 1 \\ -k_1 - 3k_2 \cdot 2 & 0 \end{bmatrix}$ since $k_1 = -1$, $k_2 = \frac{+1}{2}$ ∴

$\dot{X} = A_{(\pm\sqrt{2},0)}X$ where $A_{(\pm\sqrt{2},0)} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$

\5c resol/ For linearised plant, determine Jacobian $A = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix}$

$f_1 = x_2$, $f_2 = \dot{x}_2 = -\frac{k_1}{m}x_1 - \frac{k_2}{m}x_1^3$ ∴ $A = \begin{bmatrix} 0 & 1 \\ -k_1 - 3k_2(x_1)^2 & 0 \end{bmatrix}$ ∴

evaluating A about equilibriums:

Week 1 Sheet / at $(0,0)$: $A_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -k_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ ∴ $\dot{x} = A_{(0,0)} X$

at $(\pm\sqrt{2}, 0)$: $A_{(\pm\sqrt{2},0)} = \begin{bmatrix} 0 & 1 \\ -k_1 - 3k_2 \cdot 2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ ∴ $\dot{x} = A_{(\pm\sqrt{2},0)} X$

Week 2 Sheet /

1a sol / $\dot{x}_1 = -x_1 + x_2$ $\quad \dot{x}_2 = -0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3$ ∴

$0 = -x_1 + x_2 \Rightarrow x_1 = x_2$ $\quad , 0 = -0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3 \Rightarrow$

$-0.1x_1^3 - x_1^2 - 2.1x_1$ ∴ $x_1(0.1x_1^2 - x_1 - 2.1) = 0$

3 equilibrium pts are $(0,0)^T$, $(-2.76, -2.76)^T$, $(-7.23, -7.23)^T$

$\frac{\partial f}{\partial x} = \begin{bmatrix} -1 & 1 \\ 0.1 - 2x_1 - 0.3x_1^2 & -2 \end{bmatrix}$ $\quad \left.\frac{\partial f}{\partial x}\right|_{(0,0)^T} = \begin{bmatrix} -1 & 1 \\ 0.1 & -2 \end{bmatrix} \Rightarrow$

$\lambda_{1,2} = -2.09$ and -0.9 ∴ $(0,0)$ stable node

$\left.\frac{\partial f}{\partial x}\right|_{(-2.76,-2.76)^T} = \begin{bmatrix} -1 & 1 \\ 0.1 + 2 \cdot 2.76 - 2.76^2 \cdot 0.3 & -2 \end{bmatrix} \Rightarrow$

$\lambda_{1,2} = -3.39$ and 0.393 ∴ $(-2.76, -2.76)^T$ is saddle

$\left.\frac{\partial f}{\partial x}\right|_{(-7.23,-7.23)^T} = \begin{bmatrix} -1 & 1 \\ -1.168 & -2 \end{bmatrix}$ & $\lambda_{1,2} = -1.5 \pm 0.958i$ ∴

$Re(\lambda) < 0$ ∴ $(-7.23, -7.23)$ is stable focus

1b sol / $0 = x_1(1 + x_2)$ ① $\quad 0 = -x_2 + x_2^2 + x_1x_2 - x_1^3$ ② ∴

①: $x_1 = 0$ or $x_2 = -1$ ∴

$x_1 = 0 \Rightarrow 0 = -x_2 + x_2^2 \Rightarrow x_2 = 0$ or $x_2 = 1$,

$x_2 = -1 \Rightarrow 0 = 2 - x_1 - x_1^3 \Rightarrow x_1 = 1$ ∴ have 3 equilibrium pts

$(0,0)^T$ $\quad (0,1)^T$, $(1,-1)^T$ $\quad$ & Jacobian of f $\frac{\partial f}{\partial x} = \begin{bmatrix} 1 + x_2 & x_1 \\ x_2 - 3x_1^2 & -1 + 2x_2 + x_1 \end{bmatrix}$

$\left.\frac{\partial f}{\partial x}\right|_{(0,0)} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \Rightarrow \lambda_{1,2} = 1, -1$ ∴ $(0,0)^T$ is ∴ saddle

(+ve & -ve real eigenvals)

$\left.\frac{\partial f}{\partial x}\right|_{(0,1)^T} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix} \Rightarrow \lambda_{1,2} = 2, 1$ $\quad$ 2 +eigenvals $(0,1)^T$ is unstable node

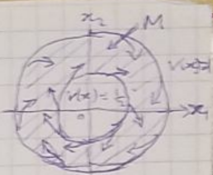
$\left.\frac{\partial f}{\partial x}\right|_{(1,-1)^T} = \begin{bmatrix} 0 & 1 \\ -4 & -2 \end{bmatrix} \Rightarrow \lambda_{1,2} = -1 \pm i\sqrt{3}$ $\quad Re(\lambda) < 0$ ∴ $(1,-1)^T$ is stable focus

1c sol / $0 = (x_1 - x_2)(1 - x_1^2 - x_2^2)$ $\quad 0 = (x_1 + x_2)(1 - x_1^2 - x_2^2)$

$\{x_1^2 + x_2^2 = 1\}$ is an equilibrium set $\quad (0,0)$ is an isolated equilibrium pt

$\left.\frac{\partial f}{\partial x}\right|_{(0,0)^T} = \begin{bmatrix} 1 - 3x_1^2 - x_2^2 + 2x_1x_2 & -2x_1x_2 - 1 + x_1^2 + 3x_2^2 \\ 1 - 3x_1^2 - x_2^2 - 2x_1x_2 & -2x_1x_2 + 1 - x_1^2 - 3x_2^2 \end{bmatrix}_{(0,0)^T} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ ∴

\Week 3 sheet/ if $r = \frac{1}{\sqrt{2}}$ $f(x) \cdot \nabla V(x) = x_1^2 \geq 0$



$V(x) = 1$ $f(x) \cdot \nabla V(x) \leq 0$ inward direction

$V(x) = \frac{1}{2}$ $f(x) \cdot \nabla V(x) \geq 0$ outward direction

Further note that system $\dot{x} = f(x)$ has unique equilibri at origin $(0,0)^T$ & this is excluded in Z annulus $\frac{1}{\sqrt{2}} \leq r < 1$ considered.
∴ no equili in Z considered set ∴ by PB (Poincare Bendixon) criterion, ∃ there must be atleast one periodic orbit, closed path - between Z circles, ie in set M (closed bounded)
$M = \{ x \in \mathbb{R}^2 \mid \frac{1}{2} \leq V(x) = x_1^2 + x_2^2 \leq 1 \}$ here $x_1^2 + x_2^2 = r^2$ ∴ $\frac{1}{\sqrt{2}} < r < 1$
at least one periodic sol exists

\4 or 5=1/ $\dot{x}_1 = x_2 \cos x_1$ $\dot{x}_2 = \sin x_1$ ∴ equili pts are $(\pm n\pi, 0)$ for $n = 0, 1, 2, \ldots$ linearisation provides $\begin{bmatrix} 0 & \alpha \\ \alpha & 0 \end{bmatrix}$ where α is +1, or -1 (flip)
inside any periodic orbit γ, there must be atleast one equili pt. Suppose Z equili pts inside γ are ~~that~~ hyperbolic ie is Jacobian at that pt has no eignvals on Z imaginary axis - then if N is number of nodes, & foci & S is number of saddles, it must be that $N - S = 1$. A periodic orbit must enclose equili pt st $N - S = 1$ ∴ there are no periodic orbit.

\Week 7 Q1/ ∴ diss eqn u is control input

$M(q)$ is symmetric Mat

$\dot{M} - 2C$ is a skew symmetric Mat

$\dot{M}$ is Z total deriv of $M(q)$ wrt time t

D is posi semi definite symmetric Mat

$P(q)$ is a posi definite func

$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + D\dot{q} + g(q) = u$

$M(q) = M(q)^T$ symmetric P.d.f. $\forall q \in \mathbb{R}^n$

$C(q, \dot{q})q$ - force [C] has property $\dot{M} - 2C$ is skew symmetric $\forall q, \dot{q} \in \mathbb{R}^n$

{ Skew symmetric Mat $A \Rightarrow A^T = -A \quad a_{ji} = -a_{ij}$ }

$V(\cdot) = \frac{1}{2}\dot{q}^T M(q)\dot{q} + P(q)$ ← p.d.f.

Symmetric p.d.f $\forall q \in \mathbb{R}^m$

quadratic form p.d.f

$V(\cdot) = 0$ only for $q = \dot{q} = 0$

$\therefore V(\cdot)$ is a p.d.f taking deriv of V along dynamics

$$\dot{V} = \frac{1}{2}\dot{q}^T\left[M(q)\frac{d}{dt}(\dot{q}) + \frac{d}{dt}(M(q))\dot{q}\right] = \frac{1}{2}\ddot{q}^T M(q)\dot{q} + \frac{\partial P(q)}{\partial q}\dot{q} =$$

$$\frac{1}{2}\dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}\dot{M}(q)\dot{q} + \frac{1}{2}\ddot{q}^T M(q)\dot{q} + \frac{\partial P(q)}{\partial q}\dot{q} \qquad \{(AB)^T = B^T A^T$$

$$(\ddot{q}^T(M(q)\dot{q}))^T = (M(q)\dot{q})^T\ddot{q} = \dot{q}^T M^T(q)\ddot{q} = \dot{q}^T M(q)\ddot{q} \qquad \{\because M(q) = M^T(q)\}$$

$$= \dot{q}^T M(q)\ddot{q} + \frac{1}{2}\dot{q}\dot{M}(q)\dot{q} + g^T(\cdot)\dot{q}$$

we will sub from dynamics ($\because \frac{\partial P(q)}{\partial q} = g^T(\cdot)$ by def)

$M(q)\ddot{q} + C(q,\dot{q})\dot{q} + D\dot{q} + g(\cdot) = 0 \quad (u = -k_q\dot{q})$

$$= \dot{q}^T[-C(q,\dot{q})\dot{q} - D\dot{q} - g(\cdot)] + \frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q} + g^T(\cdot)\dot{q} =$$

$$-\dot{q}^T C(\cdot)\dot{q} - \dot{q}^T D\dot{q} - \dot{q}^T g(\cdot) + \frac{1}{2}\dot{q}^T\dot{M}(\cdot)\dot{q} + g^T(\cdot)\dot{q} =$$

$$\underbrace{\frac{1}{2}\dot{q}^T(\dot{M} - 2C)\dot{q}}_{0} - \dot{q}^T D\dot{q} \underbrace{- \dot{q}^T g(\cdot) + g^T(\cdot)\dot{q}}$$

$0 \because \dot{M} - 2C$ is skewsymmetric $\qquad (g^T(\cdot)\dot{q})^T = \dot{q}^T(g^T(\cdot))^T = \dot{q}^T g(\cdot)$

$\dot{V}(\cdot) = -\dot{q}^T D\dot{q}$ ← p.s.d.f $\qquad \dot{V}(\cdot) \leq 0$

$V(\cdot)$ p.d.f & $\dot{V}(\cdot)$ nega semi $\quad \therefore$ origin in this case $u = 0$ is Lyapunov stab if we use $u = -k_q\dot{q}$ a control law given

$\dot{V}(\cdot) = -\dot{q}(k_q - D)\dot{q}$ provided k_q p.d.f

$\dot{V}(\cdot) \leq 0$ ($\because$ indep of q)

$S = \{(q,\dot{q}) \mid \dot{V}(\cdot) \equiv 0\}$

$\dot{V}(\cdot) \equiv 0 \Rightarrow \dot{q} = 0 \rightarrow \ddot{q} = 0$

$M(q)\ddot{q} + C(\cdot)\dot{q} + D\dot{q} + g(\cdot) = -k_q\dot{q} \Rightarrow g(\cdot) = 0$ note $g(q) = 0 \Rightarrow q = 0$ (by def of $g(\cdot)$) $\therefore$ by LaSalle invariance corollary we can argue $\therefore$ origin $(q,\dot{q}) : (0,0)$ is Asymp stab

Week 8 Q1 $\dot{x}_1 = -x_2$, $\dot{x}_2 = x_1 + (x_1^2 - 1)x_2$ • assess asymp stability (method of linearisation)

• Determine region of attraction (esti)

$$\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \underbrace{\begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}}_{\text{linear part}} + \underbrace{\begin{bmatrix}0\\ x_1^2 x_2\end{bmatrix}}_{\text{nonlinearity}}$$

$\dot{x} = Ax + g(x)$, $x = (x_1, x_2)^T$

Z matrix $[A]$ has eignls $\frac{-1 \pm i\sqrt{3}}{2}$ ∴

$Re(\lambda_i(A)) < 0$, $i = 1, 2$. Can say $[A]$ is hurwitz & ∴

Z linear part above $\dot{x} = Ax$ is asymp stab (global)

by converse Lyapunov thm $\exists$ a symmetric posi desinite P

satisfies $A^T P + PA = -Q$, for some $Q = Q^T > 0$

let Q be $I_{2\times2} \Rightarrow A^T P + PA = -I$ choose $P = \begin{bmatrix}P_{11} & P_{12}\\ P_{12} & P_{22}\end{bmatrix}$ ∵ $P = P^T$

we know A & ∴ by solving simultaneous eqns

$\Omega_c \{x \in D \mid V(x) \leq \lambda_{min}(P) r^2\}$

$P = \begin{bmatrix}3/2 & -1/2\\ -1/2 & 1\end{bmatrix}$ $\lambda_{min}(P) = 0.691$ (min eignl)

so Z Lyapunov func $\dot{x} = Ax$ is $V = x^T P x = \begin{bmatrix}x_1 & x_2\end{bmatrix}\begin{bmatrix}1.5 & -0.5\\ -0.5 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$

$V(x) = 1.5x_1^2 + x_2^2 - x_1x_2$

$V(x) = x^T P x$ p.d.s

deriv along Z traject of $\dot{V}(x) = 3x_1\dot{x}_1 + 2x_2\dot{x}_2 - x_1\dot{x}_2 - \dot{x}_1x_2$

$= (3x_1 - x_2)\dot{x}_1 + (-x_1 + 2x_2)\dot{x}_2 =$

$(3x_1 - x_2)(-x_2) + (-x_1 + 2x_2)(x_1 + (x_1^2 - 1)x_2) =$

$-3x_1x_2 + x_2^2 - x_1^2 - x_1^3x_2 + x_1x_2 + 2x_1x_2 + 2x_2^2x_1^2 - 2x_2^2 =$

$-x_1^2 - x_2^2 - x_1^3x_2 + 2x_1^2x_2^2 =$

$-(x_1^2 + x_2^2) - x_1^2x_2(x_1 - 2x_2) \leq -\|x\|^2 + \|x_1\| \|x_1x_2\| \|(x_1 - 2x_2)\|$

$\leq -\|x\|^2 + \|x_1\| \|x_1x_2\| \|x_1 - 2x_2\|$

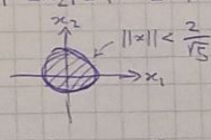
$\{ |x_1| \leq \|x\|$ euclidean norm

$|x_1x_2| \leq \frac{\|x\|^2}{2}$

$|x_1 - 2x_2| \leq \sqrt{1 - 2^2}\|x\| \leq \sqrt{5}\|x\|$

$\leq -\|x\|^2 + \|x\| \frac{\|x\|^2}{2} \sqrt{5}\|x\|$

$\leq -\underbrace{\left(1 - \frac{\sqrt{5}}{2}\|x\|^2\right)}_{\text{+ve provide}}\|x\|^2$



$1 - \frac{\sqrt{5}}{2}\|x\|^2 = 0$ $0 \leq \|x\|^2 < \frac{2}{\sqrt{5}}$

Choose $\frac{2}{\sqrt{5}} = r^2$ & following Z arguments $\forall \gamma > 0$ $\exists r > 0$ st

$\|g(x)\| < \gamma \|x\|, \ \forall \|x\| < r$ $\qquad g(x) = \begin{bmatrix} 0 \\ x_1^2 x_2 \end{bmatrix}$

can write 2 ests as region of attraction

$\Omega_c = \{x \in \mathbb{R}^n \mid V(x) < c\}$ where $c = \lambda_{min}(P) r^2$

$2\sin x \cos x = \sin 2x$ $\qquad \sin(x+y) = \sin x \cos y + \cos x \sin y$

$\sin(x-y) \sin x \cos y - \cos x \sin y$ ∴

$\sin\left(\frac{x+y}{2}\right) = \sin\left(\frac{x}{2} + \frac{y}{2}\right) = \sin\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right) + \cos\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)$

~~sin~~ $\cos(x-y) = \sin x \sin y + \cos x \cos y$ ∴

$\cos\left(\frac{x-y}{2}\right) = \cos\left(\frac{x}{2} - \frac{y}{2}\right) = \sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right) + \cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)$

$2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right) =$

$2\left(\sin\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right) + \cos\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\right)\left(\sin\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right) + \cos\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)\right) =$

$2\left[\sin^2\left(\frac{x}{2}\right)\cos\left(\frac{y}{2}\right)\sin\left(\frac{y}{2}\right) + \sin\left(\frac{x}{2}\right)\cos^2\left(\frac{y}{2}\right)\cos\left(\frac{x}{2}\right) + \sin\left(\frac{x}{2}\right)\cos\left(\frac{x}{2}\right)\sin^2\left(\frac{y}{2}\right) + \cos^2\left(\frac{x}{2}\right)\sin\left(\frac{y}{2}\right)\cos\left(\frac{y}{2}\right)\right]$

$\sin x \cos x = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$

Week 5 Sheet

1a $\dot{x} = Ax + Bu$ $\quad y = Cx$ $\quad A = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$,

$C = [0 \ \ 1]$ ∴ $\dot{x} = \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u$ $\qquad y = [0 \ \ 1] x$

Controllability mat $M = [B \ \vdots \ AB] = \left[\begin{matrix} 1 \\ 0 \end{matrix} \ \ \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

to check controllable or not $\text{rank}(M) = 2 =$ full rank ∴ Controllable

or $\det(M) = 1 \neq 0$ ∴ Controllable

1b desired poles: 4 & -5 ∴ associated characteristic polynomial

$s^2 + 9s + 20 = 0$ ∴ $(s+4)(s+5)$ with $K = [k_1 \ \ k_2]$ (gains). 2 closed loop characteristic polynomial is $\det(sI - (A - BK)) = 0$ ∴

$\det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -(1+k_1) & -(2+k_2) \\ 1 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} s+(1+k_1) & 2+k_2 \\ 1 & s \end{bmatrix}\right) = 0$ ∵ $BK = \begin{bmatrix} 1 \\ 0 \end{bmatrix}[k_1 \ \ k_2] = \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix}$

∴ $s^2 + (1+k_1)s + (2+k_2) = 0$ ∴ $1+k_1 = 9$ & $2 + k_2 = 20$ ∴

$k_1 = 8$ & $k_2 = 18$ ∴ $K = [8 \ , 18]$ ∴ $u = -[8 \ \ 18]x$

1c transfer function associated is T.F $G(s) = C(sI - A)^{-1}B =$

$[1 \ \ 0] \cdot \text{inv}\begin{pmatrix} s+1 & 2 \\ -1 & s \end{pmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix} = [1 \ \ 0] \cdot \frac{1}{s^2+s+2}\begin{bmatrix} s & -2 \\ 1 & s+1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\qquad \det(sI - A) = \frac{1}{s^2+s+2}$

$= \frac{1}{s^2+s+2}[s \ \ -2]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{s}{s^2+s+2}$

\Week 5 Sheet/ $x = T\tilde{x}$ ∴ $T = MW$, M is Controllability Matr

$M = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ full rank ∴ $n=2$ ∴ $W = \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}$

$\det(sI-A) = s^2 + a_1 s + a_2 = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} s+1 & +2 \\ -1 & s \end{bmatrix}\right) =$

$(s+1)s - (-1)2 = s^2 + s + 2$ ∴ $a_1 = 1$ ∴

$W = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ ✓ ∴ $T = MW = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

\1a/ controllability mat : $M = [B \vdots AB] = \left[\begin{bmatrix} 1 \\ 0 \end{bmatrix} \vdots \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right] =$

$\begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ is full rank matr ∴ $\text{rank}(M) = 2$ ∴ Controllable

or $\det(M) = 1 \neq 0$ ∴ Controllable

\1b/ desired poles: 4 & -5 ∴ associated charac poly: $(s-4)(s+5) =$

$s^2 + 9s + 20 = 0$ ∴ with $K = [k_1 \ \ k_2]$; 2 closed loop

charac poly is $\det(sI - (A - BK)) = 0 = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \end{bmatrix}[k_1 \ \ k_2]\right)\right) =$

$\det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \left(\begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} k_1 & k_2 \\ 0 & 0 \end{bmatrix}\right)\right) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -(1+k_1) & -(2+k_2) \\ -1 & 0 \end{bmatrix}\right) =$

$\det\left(\begin{bmatrix} s+(1+k_1) & 2+k_2 \\ 1 & s \end{bmatrix}\right) = 0 = s^2 + (1+k_1)s + (2+k_2) = s^2 + 9s + 20$ ∴

$1 + k_1 = 9$ & $2 + k_2 = 20$ ∴ $k_1 = 8$ & $k_2 = 18$ ∴ $K = [8 \ \ 18]$ ∴

$u = -[8 \ \ 18]x$

\1c/ transfer function is T.F. $G(s) = c(sI-A)^{-1}B =$

$[0 \ \ 1]\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} =$

$[0 \ \ 1]\left(\begin{bmatrix} s+1 & 2 \\ -1 & s \end{bmatrix}\right)^{-1}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = [0 \ \ 1]\frac{1}{s^2+s+2}\begin{bmatrix} s & -2 \\ 1 & s+1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} \cancel{4}$

but is $C = [1 \ \ 0]$; $G(s) = [1 \ \ 0]\frac{1}{s^2+s+2}\begin{bmatrix} s & -2 \\ 1 & s+1 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} =$

$\frac{1}{s^2+s+2}[s \ \ -2]\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \frac{1}{s^2+s+2}[s] = \frac{s}{s^2+s+2}$

\1d/ $x = T\tilde{x}$ ∴ $T = MW$ ∴ M is controllability mat $M = [B \vdots AB] = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$

is full rank ∴ $n=2$ ∴ $W = \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}$ ∴

~~det~~ $\det(sI-A) = s^2 + a_1 s + a_2 = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & -2 \\ 1 & 0 \end{bmatrix}\right) = \det\left(\begin{bmatrix} s+1 & 2 \\ -1 & s \end{bmatrix}\right) =$

$(s+1)s - (-1)2 = s^2 + s + 2 = s^2 + a_1 s + a_2$ ∴ $a_1 = 1$ ∴

$W = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ ∴ $T = MW = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

\2/ charac eqn is $\det\begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix} = s^2+1=0$

$\{\det(sI-A)=0=\det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}-\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\right)=\det\left(\begin{bmatrix} s & -1 \\ 1 & s \end{bmatrix}\right)=0=$

$s^2-1(-1)=s^2+1=0\}$

Controllability Mat: $M=[B \vdots AB]=\left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \vdots \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]=\left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \vdots \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right]=$

$M=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\{\det(M)=0(0)-1(1)=-1\neq 0$ $\therefore$ M is full rank $\therefore$

$\text{rank}(M)=2$ $\therefore$ $n=2$ $\therefore$ $W=\begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}$ $\therefore$

$\det(sI-A)=s^2+a_1s+a_2=s^2+1=s^2+0s+1=0$ $\therefore$ $a_1=0$ $\therefore$ $W=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$

$\therefore$ $T=MW=\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}=\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}=I=$ identity mat $\therefore$

A, B is in Canonical form.

\3/ controllability mat is $M=[B \vdots AB]=\left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \vdots \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right]=$

$\left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \vdots \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right]=M=\begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}$ $\{\det(M)=-1(0)-1(0)=0-0=0$ $\therefore$

not full rank $\therefore$ $\text{rank}(M)\neq 2$ $\therefore$ $\text{rank}(M)=1$ So Z syst is not controllable.

$A-BK=\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}-\begin{bmatrix} 0 \\ 1 \end{bmatrix}\begin{bmatrix} k_1 & k_2 \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}-\begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix}=\begin{bmatrix} -1 & 0 \\ 1-k_1 & -1-k_2 \end{bmatrix}$ where $K=\begin{bmatrix} k_1 & k_2 \end{bmatrix}$

$\therefore$ Z charac eqn is $\det(sI-(A-BK))=0=\det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix}-\begin{bmatrix} -1 & 0 \\ 1-k_1 & -1-k_2 \end{bmatrix}\right)=$

$\det\begin{bmatrix} s+1 & 0 \\ k_1-1 & s+1+k_2 \end{bmatrix}=0=(s+1)(s+1+k_2)=0$ $\therefore$ clearly $\lambda=-1$ is an eignval for all k_1 and k_2 $\therefore$ -1 is a closed loop eignval $\forall\, k_1$ and k_2 $\therefore$ $\forall K$

\4/ f_1+f_2, f_1f_2, $f_1 \circ f_2$: $\forall x_0 \in \mathbb{R}$ $\exists$ +ve r, L_1, L_2; k_1 and k_2 st

$|f_1(x)-f_1(y)|\leq L_1|x-y|$ and $|f_2(x)-f_2(y)|\leq L_2|x-y|$ $\{\because f_1$ and f_2 are locally Lipschitz$\}$ $\therefore$ $|f_1(x)|\leq k_1$, $|f_2(x)|\leq k_2$ $\forall x,y\in\{x\in\mathbb{R} \mid |x_0-y_0|<r\}$

for $f: f_1+f_2$: $|f(x)-f(y)|=|f_1(x)-f_1(y)-f_2(x)-f_2(y)|=$

$|(f_1(x)+f_2(x))-(f_1(y)+f_2(y))|=|(f_1(x)-f_1(y))+(f_2(x)-f_2(y))|\leq$

$|f_1(x)-f_1(y)|+|f_2(x)-f_2(y)|\leq L_1|x-y|+|f_2(x)-f_2(y)|\leq$

$L_1|x-y|+L_2|x-y|=(L_1+L_2)|x-y|$ $\therefore$ $|f(x)-f(y)|\leq(L_1+L_2)|x-y|$ for $f=f_1+f_2$ $\therefore$ f_1+f_2 is locally Lipschitz

for $f=f_1f_2$: $|f(x)-f(y)|=|f_1(x)f_2(x)-f_1(y)f_2(y)|=$

$|f_1(x)f_2(x)+(f_1(x)f_2(y)-f_1(x)f_2(y))-f_1(y)f_2(y)|=$

$|(f_1(x)f_2(x)-f_1(x)f_2(y))+(f_1(x)f_2(y)-f_1(y)f_2(y))|\leq|f_1(x)f_2(x)-f_1(x)f_2(y)|+|f_1(x)f_2(y)-f_1(y)f_2(y)|$

Week 5 Sheet $= |f_1(x)||f_2(x)-f_2(y)| + |f_2(y)||f_1(x)-f_1(y)| \le$
$K_1 |f_2(x) - f_2(y)| + |f_2(y)||f_1(x) - f_1(y)| \le K_1 L_2 |x-y| + |f_2(y)||f_1(x) - f_1(y)| \le$
$K_1 L_2 |x-y| + K_2 |f_1(x) - f_1(y)| \le K_1 L_2 |x-y| + K_2 L_1 |x-y| =$
$(K_1 L_2 + K_2 L_1)|x-y|$ $\because |f_1(x)| \le K_1$, $|f_2(y)| \le K_2$ $\therefore$
$|f(x) - f(y)| \le (K_1 L_2 + K_2 L_1)|x-y|$ for $f = f_1 f_2$ $\therefore f_1 f_2$ is locally Lipschitz

for $f = f_1 \circ f_2$: $|f(x) - f(y)| = |f_1(f_2(x)) - f_1(f_2(y))| \le L_2 |f_1(x) - f_1(y)|$
$\left\{ \because |f_1(x) - f_1(y)| \le L_2 |x-y| \therefore |f_1(f_2(x)) - f_1(f_2(y))| \le L_2 |f_2(x) - f_2(y)| \right\} \therefore$
$|f(x) - f(y)| \le L_2 |f_1(x) - f_1(y)| \le L_2 L_1 |x-y|$ $\therefore f = f_1 \circ f_2$ is locally Lipschitz

5 $\dot{x} = f(x)$ $\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}$ $\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -x_1 + \varepsilon(1 - x_1^2)x_2 \end{aligned}$ $\therefore f(x) = \begin{bmatrix} x_2 \\ -x_1 - \varepsilon(1 - x_1^2)x_2 \end{bmatrix} \therefore$

$\frac{\partial f}{\partial x} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix}$ $\because f(x) = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix}$ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \therefore$

$\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ -1 - 2\varepsilon x_1 x_2 & -\varepsilon(1 - x_1^2) \end{bmatrix}$ $\frac{\partial f}{\partial x}$ is continuous on D_r $\therefore$

$f(\cdot)$ is locally Lipschitz on D_r $\forall r > 0$
$\left[\frac{\partial f}{\partial x}\right]$ is not globally bounded $\therefore f(\cdot)$ is not globally Lipschitz

6 $x^2 + |x|$ $|x|$ is not continuously diffable at $x=0$ but is globally Lipschitz $\because ||x| - |y|| \le |x-y|$ $\forall x, y \in \mathbb{R}$. term x^2 is contly diffable, but its ~~partially~~ partial deriv is not globally bounded.
$\therefore f = x^2 + |x|$ is not contly diffable at $x=0$. it is contly diffable on a domain that does not include $x=0$. it is locally Lipschitz, $\therefore$ continuous. but not globally Lipschitz

$f = \sin(x)\,\mathrm{sgn}(x)$ $\therefore$ if both x, y non negative:
$|f(x) - f(y)| = |\sin(x)\,\mathrm{sgn}(x) - \sin(y)\,\mathrm{sgn}(y)| = |\sin(x) - \sin(y)| \le |x-y|$
$\because \sin(x)$ is bounded $\therefore -1 \le \sin(x) \le 1$
if $x \ge 0$, y ~~<~~ then $|f(x) - f(y)| = |\sin(x) - \sin y| = \left|2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right|$
$\le 2\left|\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)\right| \le |x-y|$ $\because$ obviously bounded $\because -1 \le \cos x \le 1$
$\therefore$ have $|f(x) - f(y)| \le |x-y|$ $\forall x, y$ $\therefore f$ is locally Lipschitz and continuous. it is not contly diffable at $x=0$ $\because$ two limits

exist for $x \to 0^+$ & $x \to 0^-$

$f = -x + a\sin x$: Conty diffable, ~~locally~~ Lipschitz

$\frac{df}{dx} = -1 + a\cos x$ $\left|\frac{\partial f}{dx}\right|$ bounded globally $\therefore$ globally Lipschitz

$\left|\frac{\partial f}{\partial x}\right|$ bounded globally $\therefore$ globally Lipschitz
$\left|\frac{\partial f}{\partial x}\right|$ bounded globally $\therefore$ globally Lipschitz
$\left|\frac{\partial f}{\partial x}\right|$ bounded globally $\therefore$ globally Lipschitz
$\left|\frac{\partial f}{\partial x}\right|$ bounded globally $\therefore$ globally Lipschitz $\left|\frac{\partial f}{\partial x}\right|$ bounded globally $\therefore$ globally Lipschitz $\left|\frac{\partial f}{\partial x}\right|$ bounded globally $\therefore$ globally Lipschitz
$\left|\frac{\partial f}{\partial x}\right|$ bounded globally $\therefore$ globally Lipschitz $\left|\frac{\partial f}{\partial x}\right|$ bounded globally $\therefore$ globally Lipschitz $\left|\frac{\partial f}{\partial x}\right|$ bounded globally $\therefore$ globally Lipschitz

\Week 6 Sheet/ $A = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}$ $\therefore$ $eig(A) = 0.2984, 6.7016$ &
$A^T = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix}^T = \begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} = A^T$ $\therefore$ $\lambda_i(A) > 0\ \forall i$ & $A = A^T$ $\therefore$ A is posi desi

$P = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix}$ $\therefore$ $eig(P) = 0.1049, 3.6027, 5.29$ & $P^T = \begin{bmatrix} 4 & 1 & 2 \\ 1 & 3 & -1 \\ 2 & -1 & 2 \end{bmatrix} = P$
(symmetric $\therefore$ $\lambda_i(P) > 0\ \forall i$ & $P = P^T$ $\therefore$ P is posi desi

\2.1/ $V(x_1, x_2) = (x_1 + x_2)^2$ $\therefore$ $V(0) = V(0,0) = 0$ & $V(x_1, x_2) > 0\ \forall$
$\forall x_1, x_2 \in \mathbb{R} \setminus \{0\}$ $\therefore$ globally posi desi

\2.2/ $V(x_1, x_2) = x_1^2 + 2x_1x_2 + 3x_2^2 = \cancel{(x_1 + x_2^2)}\ x_1^2 + 2x_1x_2 + x_2^2 + 2x_2^2 =$
$(x_1 + x_2)^2 + 2x_2^2 > 0\ \forall x_1, x_2 \in \mathbb{R} \setminus \{0\}$ and $\therefore$ $V(0) = 0$ &
$V(x_1, x_2) > 0\ \forall x_1, x_2 \in \mathbb{R} \setminus \{0\}$ $\therefore$ globally posi desi

\2.3/ $V(x_1, x_2) = (x_1 - x_2)^2$ $\therefore$ $V(0) = 0$ & if $x_1 = x_2$: $V(x) = 0$
$\therefore$ $V(x_1, x_2)$ is not posi desi $\therefore$
$V(x_1, x_2) \geq 0$ $\therefore$ $V(x_1, x_2)$ is $\therefore$ posi semi desi
$\therefore$ $V(x_1, x_2) \geq 0\ \forall (x_1, x_2) \in \mathbb{R}$

\3/ $V(x_1, x_2, x_3) = \frac{1}{2}(\alpha_1 x_1^2 + \alpha_2 x_2^2 + \alpha_3 x_3^2)$ $\therefore$ $V(0) = 0$ & $V(x) > 0$ for $x \neq 0$
$\therefore$ V is posi desi $\because$ V is sum of squared terms & $V(0,0,0) = 0$
provided $\alpha_i > 0$, at $x_1 = x_2 = x_3 = 0$ $\therefore$ taking deriv wrt time, or along trajectory of dynamics : $\dot{V} = \sum_{i=1}^{3} \alpha_i x_i \dot{x}_i$ $\therefore$ subing $\dot{x}_i$ from

Week 6 Sheet / $\dot{x} = f(x),\ x \in \mathbb{R}^3$: $\dot{V} = \sum_{i=1}^{3} \alpha_i x_i \dot{x}_i =$

$\alpha_1 x_1 (I_{23} x_3 x_2) + \alpha_2 x_2 (I_{31} x_3 x_1) + \alpha_3 x_3 (I_{12} x_1 x_2) = \dot{V} =$

$(\alpha_1 I_{23} + \alpha_2 I_{31} + \alpha_3 I_{12})(x_1 x_2 x_3)$ → sign indefinite term in $\mathbb{R}^3$

choice $\alpha_1 I_{23} + \alpha_2 I_{31} + \alpha_3 I_{12} = 0$ ∵ $I_{31} < 0$. it is always possible to pick a set of $\alpha_1, \alpha_2, \alpha_3$ st V is posi desi (required $\alpha_i > 0$) & $\dot{V}$ is nega semi desi, ie

$\dot{V} = 0\ \forall x_1, x_2, x_3$ for Z choice of I_{31} st $\alpha_1 I_{23} + \alpha_2 I_{31} + \alpha_3 I_{12} = 0$

∴ Lyapunov stable: ie $\alpha_2 I_{31}$ dominates & equal to $-(\alpha_1 I_{23} + \alpha_3 I_{12})$

4.1 / $\ddot{\theta} + \dot{\theta} + \frac{g}{l}\sin\theta = 0$ ① , $V(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 + \frac{g}{l}(1 - \cos\theta)$ ∴

$V(\theta, \dot{\theta}) = 0$ only at $\theta = \dot{\theta} = 0$ in a domain $D = \{(\theta, \dot{\theta}) \mid |\theta| < \pi\}$ &

$V(\theta, \dot{\theta}) > 0\ \forall (\theta, \dot{\theta}) \in D \setminus \{(0,0)\}$ ∵ $\dot{\theta}^2$ is the square term &

$-1 \le \cos\theta \le 1$, $\cos\theta$ is bounded term ∴ $V(\theta, \dot{\theta})$ is posi desi in D

taking time deriv ie: $\dot{V}(\theta, \dot{\theta}) = \dot{\theta}\ddot{\theta} + \frac{g}{l}\dot{\theta}\sin\theta$ subbing from ① for $\ddot{\theta}$:

$\ddot{\theta} = -\dot{\theta} - \frac{g}{l}\sin\theta$ ∴ $\dot{V}(\theta, \dot{\theta}) = \dot{\theta}\ddot{\theta} + \frac{g}{l}\dot{\theta}\sin\theta =$

$\dot{\theta}(-\dot{\theta} - \frac{g}{l}\sin\theta) + \frac{g}{l}\dot{\theta}\sin\theta = -\dot{\theta}^2 - \frac{g}{l}\dot{\theta}\sin\theta + \frac{g}{l}\dot{\theta}\sin\theta = -\dot{\theta}^2 \le 0$,

$\dot{V}(\theta, \dot{\theta})$ cant be claimed as nega desi for syst ∵

$\dot{V}(\theta, \dot{\theta}) = 0$ for $\dot{\theta} = 0$ no matter what θ values are.

∴ $\dot{V}(\theta, \dot{\theta})$ is nega semi desi & ∴ only conclusion is that Z origin is Lyapunov stable for $V(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 + \frac{g}{l}(1 - \cos\theta)$

∵ Lyapunov conditions are only sufficiency conditions.

4.2 / $\ddot{\theta} + \dot{\theta} + \frac{g}{l}\sin(\theta) = 0$, $V(\theta, \dot{\theta}) = \frac{1}{2}\dot{\theta}^2 + \frac{1}{2}(\theta + \dot{\theta})^2 + \frac{2g}{l}(1 - \cos\theta)$

∴ $-1 \le \cos\theta \le 1$ ∴ ~~$1 - \cos\theta$~~ $0 \le 1 - \cos\theta \le 2$ ∵ $\cos\theta$ is bounded by 1 ∴ $|\cos\theta| \le 1$

$V(\theta, \dot{\theta})$ is posi desi ∵ each term of sum is posi & $V(0,0) = 0$ only at $\theta = \dot{\theta} = 0$ in D. $V(\theta, \dot{\theta}) > 0\ \forall (\theta, \dot{\theta}) \in D \setminus \{(0,0)\}$ ∴

V is posi desi in domain D ∴ taking deriv of V along Z trajec of ① yields: $\dot{V}(\theta, \dot{\theta}) = \dot{\theta}\ddot{\theta} + (\theta + \dot{\theta})(\dot{\theta} + \ddot{\theta}) + \frac{2g}{l}\dot{\theta}\sin\theta$ ∴ subing $\ddot{\theta}$ from ① :

$= \dot{\theta}(-\dot{\theta} - \frac{g}{l}\sin\theta) + (\theta + \dot{\theta})(\dot{\theta} + -\dot{\theta} - \frac{g}{l}\sin\theta) + \frac{2g}{l}\dot{\theta}\sin\theta =$

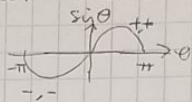
$-\dot{\theta}^2 - \frac{g}{l}\dot{\theta}\sin\theta + (\theta + \dot{\theta})(-\frac{g}{l}\sin\theta) + \frac{2g}{l}\dot{\theta}\sin\theta =$

$-\dot{\theta}^2 - \frac{g}{l}\dot{\theta}\sin\theta - \frac{g}{l}\theta\sin\theta - \frac{g}{l}\dot{\theta}\sin\theta + \frac{2g}{l}\dot{\theta}\sin\theta = -\dot{\theta}^2 - \frac{g}{l}\theta\sin\theta$ ∴ note

$+\dot{\theta}^2$ is +ve (square term) $\therefore$ $-\dot{\theta}^2$ is −ve ,

$\theta \sin\theta$ is +ve term no matter what θ is $\because$ product of two odd funcs $\because$

θ	+ve	−ve
$\sin\theta$	+ve	−ve



$\therefore$

$\dot{V}(\theta,\dot{\theta}) = -\dot{\theta}^2 - \frac{g}{l}\theta\sin\theta < 0 \quad \forall(\theta,\dot{\theta}) \in D \setminus \{(0,0)\}$ $\because$ V is posi desi in D & $\dot{V}$ is nega desi in D $\therefore$ the origin of the syst is an (locally) asymp stable equili in $D = \{(\theta,\dot{\theta})^T \mid |\theta| < \pi\}$

\5/ $\dot{x}_1 = -x_1 + x_2^3$, $\dot{x}_2 = -x_1 - x_2$ $\quad V(x_1,x_2) = \frac{1}{2}x_1^2 + \frac{1}{4}x_2^4$ $\therefore \because$ sum of +ve terms

$V(x)$ is posi desi & $V(x) \to \infty$ as $\|x\| \to \infty$ (ie a radially ~~unbo~~ unbounded func) taking deriv along the trajecs of syst:

$\dot{V}(x) = x_1\dot{x}_1 + \cancel{2}x_2^3\dot{x}_2 = x_1(-x_1 + x_2^3) + \cancel{2}x_2^3(-x_1 - x_2) = \cancel{2x_1^2 \, 2x_2^4}$

$-x_1^2 + x_1x_2^3 - x_1x_2^3 - x_2^4 = \cancel{x_1^2 \, 2x} \; -x_1^2 - x_2^4 < 0 \; \forall (x_1,x_2) \neq (0,0)$ or $\forall (x_1,x_2) \in \mathbb{R}^2 \setminus \{0,0\}$ $\therefore \dot{V}(x_1,x_2)$ nega desi $\therefore$ zero sol to syst is globally <u>asyp</u> stable

\6/ $\dot{x}_1 = -4x_1 + x_1x_2^2$, $\dot{x}_2 = -x_2 + x_1$ $\quad V(x) = x_1^2 + x_2^2$. $\therefore \Sigma \begin{cases} \dot{x}_1 = -4x_1 + x_1x_2^2 \\ \dot{x}_2 = -x_2 + x_1 \end{cases}$

$V = x_1^2 + x_2^2$ $\therefore \because$ sum of squares, $V(x_1,x_2) > 0 \;\; \forall (x_1,x_2) \neq \{0,0\}$, $V(0,0) = 0$

$\therefore V > 0 \;\; \forall x \in \mathbb{R}^2 \setminus \{0\}$, ~~&~~ $V = 0$ for $x = 0$ $\therefore$ V is posi desi $\therefore$

taking along the trajecs of the syst Σ : $\dot{V} = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 =$

$2x_1(-4x_1 + x_1x_2^2) + 2x_2(-x_2 + x_1)$ $\quad$ (by sub for $\dot{x}_1$ & $\dot{x}_2$)

$= -8x_1^2 + 2x_1x_2^2 + 2x_1x_2 - 2x_2^2 =$

$= -7x_1^2 - x_1^2 + 2x_1^2x_2^2 - x_2^2 - x_2^2 + 2x_1x_2$

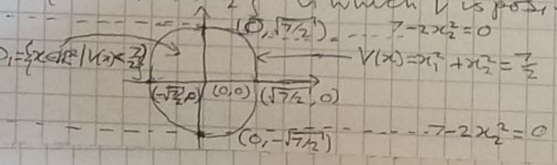
$= -7x_1^2 - x_2^2 + 2x_1^2x_2^2 - (x_1^2 \cancel{+} - 2x_1x_2 + x_2^2) = -7x_1^2 - x_2^2 + 2x_1^2x_2^2 - (x_1 - x_2)^2 \leq$

$-7x_1^2 - x_2^2 + 2x_1^2x_2^2$ $\quad$ ($\because (x_1 - x_2)^2$ is +ve term & it is minus)

$= -x_1^2(7 - 2x_2^2) - x_2^2 < 0$ if $(7 - 2x_2^2)$ is guaranteed to be +ve $\therefore$

$7 - 2x_2^2 > 0 \;\therefore\; 7 > 2x_2^2 \;\therefore\; \cancel{\tfrac{7}{2} x_2^2} \;\; \frac{7}{2} > x_2^2 \;\therefore\;$ if $\frac{7}{2} > x_1^2 + x_2^2$ $\therefore$

$\dot{V} < 0$ if $x_1^2 + x_2^2 < \frac{7}{2}$ $\therefore$ Can define an invariant set $(x_1^2 + x_2^2) = V(x) < \frac{7}{2}$; $D_1 = \{x \in \mathbb{R}^2 \mid V(x) < \frac{7}{2}\}$ in which V is posi desi & $\dot{V}$ is nega desi holds.



Week 7 Sheet / 1 / $M(q)\ddot{q}+C(q,\dot{q})\dot{q}+D\dot{q}+g(q)=u$

$M(q)$ is a symmetric inertia mat $\therefore M(q)=M(q)^T$. $M(q)$ is posi desi $\forall q \in \mathbb{R}^n$

$C(q,\dot{q})\dot{q}$ is force $[C]$ has property $\dot{M}-2C$ is skew symmetric mat $\forall q, \dot{q} \in \mathbb{R}^n$

recall skew symmetric mat $A \Rightarrow A^T=-A \quad a_{ji}=-a_{ij}$

$V(q,\dot{q})=\frac{1}{2}\dot{q}^T M(q)\dot{q}+P(q)$ is posi desi $\because \dot{q}^T M(q)\dot{q}$ is quadratic form

& $M(q)$ posi desi $\Big\{ \because \dot{q}^T\dot{q}$ is posi desi & $M(q)$ is posi desi $\therefore$

$\dot{q}^T M(q)\dot{q}$ is posi desi $\Big\}$ & $P(q)$ is posi desi since $P(q)$ is +ve $\therefore$

V is posi desi $\therefore$ taking ~~the~~ deriv of V along Z dynamics:

$$\dot{V}=\frac{d}{dt}V=\frac{d}{dt}\left[\frac{1}{2}\dot{q}^T M(q)\dot{q}+P(q)\right]=\frac{1}{2}\frac{d}{dt}\left[\dot{q}^T M(q)\dot{q}\right]+\frac{d}{dt}\left[P(q)\right]=$$

$$\frac{1}{2}\dot{q}^T\frac{d}{dt}\left[M(q)\dot{q}\right]+\frac{1}{2}\ddot{q}^T M(q)\dot{q}+\frac{\partial}{\partial q}\left[P(q)\right]\frac{\partial q}{\partial t}=$$

$$\frac{1}{2}\dot{q}^T\left[M(q)\frac{d}{dt}[\dot{q}]+\frac{d}{dt}[M(q)]\dot{q}\right]+\frac{1}{2}\ddot{q}^T M(q)\dot{q}+\frac{\partial}{\partial q}\left[P(q)\right]\dot{q}=$$

$$\frac{1}{2}\dot{q}^T M(q)\ddot{q}+\cancel{\tfrac{1}{2}\dot{M}(q)}\ \frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q}+\frac{1}{2}\ddot{q}^T M(q)\dot{q}+\frac{\partial P(q)}{\partial q}\dot{q}=\dot{V}$$

note: $(AB)^T=B^T A^T \therefore$

$$\Big(\overbrace{(\ddot{q}^T)}^{A}\overbrace{(M(q)\dot{q})}^{B}\Big)^T=(M(q)\dot{q})^T(\ddot{q}^T)^T=(M(q)\dot{q})^T\ddot{q}=(\dot{q})^T(M(q))^T\ddot{q}=\cancel{\dot{q}^T M^T(q)}$$

$$\dot{q}^T M^T(q)\ddot{q}=\dot{q}^T M(q)\ddot{q} \quad (\because M(q)=M^T(q))$$

& $g(q)=\left(\frac{\partial P(q)}{\partial q}\right)^T \therefore \frac{\partial P(q)}{\partial q}=g^T(q) \therefore$

$$\dot{V}=\frac{1}{2}\dot{q}^T M(q)\ddot{q}+\frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q}+\frac{1}{2}\ddot{q}^T M(q)\dot{q}+\frac{\partial P(q)}{\partial q}\dot{q}=$$

$$\frac{1}{2}\dot{q}^T M(q)\ddot{q}+\frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q}+\frac{1}{2}\ddot{q}^T M(q)\dot{q}+g^T(q)\dot{q} \quad \therefore$$

$$(\ddot{q}^T M(q)\dot{q})^T=\dot{q}^T M(q)\ddot{q} \quad \therefore \quad \ddot{q}^T M(q)\dot{q}=(\dot{q}^T M(q)\ddot{q})^T \quad \therefore$$

$$\dot{V}=\frac{1}{2}\dot{q}^T M(q)\ddot{q}+\frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q}+\frac{1}{2}\ddot{q}^T M(q)\dot{q}+\frac{\partial P(q)}{\partial q}\dot{q}=$$

$$\frac{1}{2}\dot{q}^T M(q)\ddot{q}+\frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q}+\frac{1}{2}(\dot{q}^T M(q)\ddot{q})^T+\frac{\partial P(q)}{\partial q}\dot{q}=$$

$$\frac{1}{2}\left[\dot{q}^T M(q)\ddot{q}+(\dot{q}^T M(q)\ddot{q})^T\right]+\frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q}+\cancel{\frac{\partial P(q)}{\partial q}\dot{q}}\ g^T(q)\dot{q}=$$

$$\dot{q}^T M(q)\ddot{q}+\frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q}+g^T(q)\dot{q}=\dot{V} \quad \therefore \because$$

$$M(q)\ddot{q}=-C(q,\dot{q})\dot{q}-D\dot{q}-g(q) \quad \because$$

$M(q)\ddot{q}+C(q,\dot{q})\dot{q}+D\dot{q}+g(q)=0 \quad \because u=0$ when $V(q,\dot{q})=\frac{1}{2}\dot{q}^T M(q)\dot{q}+P(q) \therefore$

$\dot{V} = \dot{q}^T[M(q)\ddot{q}] + \frac{1}{2}\dot{q}^T \dot{M}(q)\dot{q} + g^T(q)\dot{q} =$

$\dot{q}^T[-C(q,\dot{q})\dot{q} - D\dot{q} - g(q)] + \frac{1}{2}\dot{q}^T\dot{M}(q)\dot{q} + g^T(q)\dot{q} =$

$-\dot{q}^T C(q,\dot{q})\dot{q} - \dot{q}^T D\dot{q} - \dot{q}^T g(q) + \frac{1}{2}\dot{q}^T(\dot{M}(q))\dot{q} + g^T(q)\dot{q} =$

$\frac{1}{2}\dot{q}^T[\dot{M}(q)]\dot{q} + \frac{1}{2}\dot{q}^T[2C(q,\dot{q})]\dot{q} - \dot{q}^T D\dot{q} - \dot{q}^T g(q) + g^T(q)\dot{q} =$

$\frac{1}{2}\dot{q}^T[\dot{M}(q) - 2C] - \dot{q}^T D\dot{q} - \dot{q}^T g(q) + g^T(q)\dot{q} = \dot{V}$

$\therefore$ at the origin: $(q=0, \dot{q}=0)$:

$\dot{M}-2C$ is skew symmetric $\therefore$ $\dot{M}(q)-2C=0$ at ~~for~~ skew symmetry at everywhere including origin $\forall q, \dot{q} \in \mathbb{R}^n$ $\therefore$

$\dot{V} = \frac{1}{2}\dot{q}^T[0] - \dot{q}^T D\dot{q} - \dot{q}^T g(q) + g^T(q)\dot{q} = \cancel{V}$

$-\dot{q}^T D\dot{q} - \dot{q}^T g(q) + g^T(q)\dot{q} = \dot{V}$

& $\dot{q}^T g(q)$, $g^T(q)\dot{q}$ are identical $\{[(\dot{q}^T) g(q)]^T = g^T(q)\,\dot{q}^T$ & $g(q=0)=g(0)=0\}$

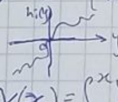
$\therefore$ $-\dot{q}^T g(q) + g^T(q)\dot{q} = 0$ $\therefore$

$\dot{V} = -\dot{q}^T D\dot{q}$ $\therefore$ $\dot{q}^T D\dot{q}$ is in quadratic form & D is posi semidesi $\therefore$

$\therefore$ $+\dot{q}^T D\dot{q}$ is posi semidesi $\therefore$ $-\dot{q}^T D\dot{q}$ is nega semi desi $\therefore$

$\dot{V} \leq 0$ $\therefore$ V is posi desi & $\dot{V}$ is nega semidesi $\therefore$ Z origin is Lyapunov stable

Q2/ $\Sigma: \begin{cases} \dot{x}_1 = x_2 \\ \dot{x}_2 = -h_1(x_1) - h_2(x_2) \end{cases}$ $\because$ given $h_1(0)=0$, $y h_i(y) > 0$ $\forall y \neq 0$ & $y \in (-a, a)$



$\therefore$ $\because$ $y h_i(y) > 0$ for $y<0$ $\therefore$ $h_i(y) < 0$, for $y>0$: $y h_i(y) > 0$ $\therefore$ $h_i(y) > 0$

$\therefore$ $V(x) = \int_0^{x_1} h_1(y)dy + \frac{1}{2}x_2^2$ $\therefore$ V is contly diffsable, & posi desi at least locally here $\because y \in (-a, a)$ $\therefore$ for $0<y<a$: ~~$h_1(y)$~~ $\therefore y>0$ $\therefore h_1(y) > 0$

$\therefore$ $\int_0^{x_1} h_1(y)dy > 0$ & for $x_2 > 0$: $\frac{1}{2}x_2^2 > 0$ $\therefore$ V is posi desi $\therefore$

taking time deriv: $\dot{V}(x) = \frac{d}{dt}\left[\int_0^{x_1} h_1(y)dy\right] + \frac{d}{dt}\left(\frac{1}{2}x_2^2\right) =$

$\frac{\partial}{\partial x_1}\left[\int_0^{x_1} h_1(y)dy\right]\frac{\partial x_1}{\partial t} + \frac{1}{2}\frac{\partial}{\partial x_2}[x_2^2]\frac{\partial x_2}{\partial t} - \frac{\partial x_1}{\partial t}\left[h_1(x_1)\frac{\partial x_1}{\partial x_1} - h_1(0)\frac{\partial(0)}{\partial x_1}\right] + \frac{2}{2}x_2\dot{x}_2 =$

$[h_1(x_1) - 0]\dot{x}_1 + x_2\dot{x}_2 = h_1(x_1)\dot{x}_1 + x_2\dot{x}_2 = \dot{V}(x) =$

$h_1(x_1)x_2 + x_2(-h_1(x_1) - h_2(x_2)) = h_1(x_1)x_2 - x_2 h(x_1) - h_2(x_2)x_2 = h_1(x_1)x_2 - h(x_1)x_2 - h_2(x_2)x_2$

$= -x_2 h_2(x_2) = \dot{V}(x)$ $\therefore$ $\because$ $y h_i(y) > 0$ $\therefore$ $y h_2(y) > 0$ $\therefore$ $x_2 h_2(x_2) > 0$ $\therefore$

$-x_2 h_2(x_2) < 0$ ~~$\forall x_2 \neq 0$~~ $\therefore$ $-x_2 h_2(x_2)$ is nega semi desi $\therefore$

$\dot{V}(x) \leq 0$ nega semi desi ($\because$ indep of x_1) $\therefore$

\Week 7 Sheet/ by Lyapunov argument $\because$ V is posi desi &

$\dot{V}$ is nega semi desi $\therefore$ origin is Lyapunov stable $\therefore$

only Lyapunov stable $\because$ indep of x $\therefore$ only nega semidesi $\dot{V}(x)=0$ everywhere x_1

$\therefore$ as $x_2 \to 0$: $\dot{V}(x)=0$ $\because h_1(0)=0$

Consider a set consists of (x_1, x_2) st $x_2=0$ & x_1 be any value in $\mathbb{R}$ $\therefore$ $\therefore \dot{V}(x)=0$

everywhere $\dot{V}(x)=0$ $\therefore$ Suppose trajectory $x(t) \in \mathbb{R}^2$ starts in set

$S=\{x \in D \mid \dot{V}(x)=0\}$ then $x_2=0$ from Σ.

$\dot{x}_1=0$ $\therefore$ x_1 is a constant but is this const is nonzero, then

from Σ dynamics, $\dot{x}_2=-h_1(x_1)-h_2(x_2)$ $\therefore \dot{x}_2 \neq 0$, in term x_2 will be a non

zero val. $\therefore$ trivial sol has to be origin $\therefore$ by Lasalle's invariance

principle argument, can conclude Σ origin Locally asymp stable.

\3/ $V(x)=\frac{1}{2}x_1^2+\frac{1}{2\gamma}(x_2-b)^2$ $\quad b>a, \gamma>0$ $\quad x_1=y$ & $x_2=k(t)$ $\therefore$

V is contly diffable & posi desi (sum of square terms & given facts)

$\because x_1^2 = y^2 \geq 0$, $(x_2-b)^2 \geq 0$ $\therefore V \geq 0$, $V(0,b)=0$ $\therefore$ V is posi desi $\because$

taking deriv of $V(x)$ along trajectories of $\left.\begin{array}{l}\dot{y}=ay+u\\ u=-ky\\ \dot{k}=\gamma y^2\end{array}\right\}$ $\begin{array}{l}\dot{y}=(a-k(t))y\\ \dot{k}=\gamma y^2\end{array}$ $\therefore$

$\dot{V}(y,k(t))=\frac{d}{dt}\left[\frac{1}{2}y^2\right]+\frac{d}{dt}\left[\frac{1}{2\gamma}(k(t)-b)^2\right]=y\dot{y}+\frac{1}{\gamma}(k(t)-b)\dot{k}(t)=$

$(y)\left[-(k(t)-a)y\right]+\frac{1}{\gamma}(k(t)-b)\gamma y^2=-(y^2)\left[k(t)-a\right]+(k(t)-b)y^2=$

$\left[k(t)-b-k(t)+a\right]y^2=-(b-a)y^2 \leq 0$ $\because b>a$ $\therefore b-a>0$, $y^2 \geq 0$ $\therefore$

$-(b-a)y^2 \leq 0$ from nega semidesi perspective $\therefore$

Consider $S=\{(y,k) \in D \mid \dot{V}(y,k)=0\}$ when $y=0$, no matter what k.

to find Ω compact set, link this with Σ given $V(x)$ $\because$

V is contly diffable & posi desi. $\Omega_c=\{(y,k) \in D \mid V(y,k) \leq c\}$ this

Ω_c is compact the invariant set $\because V(x)$ is posi desi & $\dot{V}(x) \leq 0$ in

Ω_c. $\therefore$ $S=\{(y,k) \in \Omega_c \mid y=0\}$ is invariant set. by Lasalle's thm

every trajec in Ω_c approaches S as $t \to \infty$ as $y(t) \to 0$ as $t \to \infty$ $\therefore$

Locally asymp stable

\4/ $\begin{array}{l}\dot{x}_1=x_1(k^2-x_1^2-x_2^2)+x_2(x_1^2+x_2^2+k^2)\\ \dot{x}_2=-x_1(x_1^2+x_2^2+k^2)+x_2(k^2-x_1^2-x_2^2)\end{array}$ $\quad k=0$ & $k \neq 0$: $\quad V(x)=x_1^2+x_2^2$ $\therefore$

$V(x)$ is posi desi $\because x_1^2 \geq 0$, $x_2^2 \geq 0$ $\therefore V(x) \geq 0$, $V(0)=0$ $\therefore$

now taking the deriv of $V(x)$ along Z trajecs of Z dynamics, ie time deriv of $V(x)$: $\dot V(x) = 2x_1\dot x_1 + 2x_2\dot x_2 =$

$2x_1[x_1(k^2-x_1^2-x_2^2)+x_2(x_1^2+x_2^2+k^2)] + 2x_2[-x_1(x_1^2+x_2^2+k^2)+x_2(k^2-x_1^2-x_2^2)] =$

$2x_1[x_1(k^2-x_1^2-x_2^2)] + 2x_2[x_2(k^2-x_1^2-x_2^2)] =$

$2x_1^2(k^2-x_1^2-x_2^2)+2x_2^2(k^2-x_1^2-x_2^2) = 2(x_1^2+x_2^2)(k^2-x_1^2-x_2^2) =$

$2(x_1^2+x_2^2)(k^2-(x_1^2+x_2^2)) = 2V(k^2-V) = \dot V$ ∴

two cases:

(i) if $k=0$, then $\dot V = 2V(0^2-V) = -2V^2$ ∵ $V^2 \geq 0$ ∴ $-2V^2 \leq 0$ ∴ $\dot V \leq 0$ ∴ $\dot V < 0$ $\forall x \in \mathbb{R}^2\setminus\{0\}$, $V(0)=0$ note: $V(x)\to\infty$ as $\|x\|\to\infty$

(ii) if $k\neq 0$ then $\dot V<0$ for $V>k^2$ or $\|x\|>|k|$ (ie $D=\{x\in\mathbb{R}^2 \mid V(x)>k^2\}$)

notice that for $k\neq 0$, have $\dot V<0$ within neighbourhood of $x=0$ of radius $|k|$ and cannot find an invariant set D st $x=0$ contained in D, & $\dot V<0$ on D ∵ here Z requirement is $\|x\|>|k|$, not $\|x\|<|k|$ which would have given us a region $V(x)\leq k^2$)

if $k=0$, Z origin is Globally asymp stable from argument in (i) ∵ V is posi defi, $\dot V$ is nega defi.

\5a/ $\dot x_1 = -x_1+x_2^2$, $\dot x_2 = -x_2$; $V(x)=x_1^2+x_2^2$ & equilib $x=(0,0)$ ∴ $V(x)>0\ \forall x\in\mathbb{R}^2\setminus\{0\}$ & $V(0)=0$ ∴ V is posi defi

$\dot V(x) = 2x_1\dot x_1 + 2x_2\dot x_2 = 2x_1(-x_1+x_2^2)+2x_2(-x_2) = -2x_1^2+2x_1x_2^2-2x_2^2 =$
$-2x_1^2-2x_2^2(1-x_1)$ ∴ $\dot V(x)<0$ if $x_1\leq 1$, ∴ $x=(0,0)$ is an asymp stable equili in a domain $D=\{x\in\mathbb{R}^2 \mid V(x)<1\}$ ∵ $x_1<1$ ∴ $x_1^2<1$ ∴ $\{x_1^2+x_2^2<1\}\subseteq\{x_1^2<1\}$

\5b/ $\dot x_1 = (x_1-x_2)(x_1^2+x_2^2-1)$, $\dot x_2 = (x_1+x_2)(x_1^2+x_2^2-1)$. $V(x)=x_1^2+x_2^2$ ∴ $V(x)$ is posi defi ∴

$\dot V(x) = 2x_1\dot x_1 + 2x_2\dot x_2 =$

$2x_1[(x_1-x_2)(x_1^2+x_2^2-1)] + 2x_2[(x_1+x_2)(x_1^2+x_2^2-1)] =$

$2x_1[x_1(x_1^2+x_2^2-1)] + 2x_2[x_2(x_1^2+x_2^2-1)] = 2x_1^2(x_1^2+x_2^2-1)+2x_2^2(x_1^2+x_2^2-1) =$

$2(x_1^2+x_2^2)(x_1^2+x_2^2-1) = 2V(V-1) < 0$ if $V<1$ ∴ origin is an asymp stable equili in $D=\{x\in\mathbb{R}^2 \mid V(x)<1\}$

$\dot V(x)\leq 0$ nega semi defi ∵ indep ...

\Week 7 Sheet/ ~~5x~~/ $\dot{x}_1 = -x_1 + x_1^2 x_2$, $\dot{x}_2 = +x_1 - x_2$ $V(x) = x_1^2 + x_2^2$ ∴ $x_1^2 \geq 0$, $x_2^2 \geq 0$ ∴

$V(x) \geq 0$, $V(0) = 0$, $V(x) > 0 \ \forall x \in \mathbb{R}^2 \setminus \{0\}$ ∴ $V(x)$ is posi desi ∴

taking the derivative of V along the trajectories of the system: $\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1(-x_1 + x_2 x_1^2) + 2x_2(x_1 - x_2) =$

$2[-(x_1^2 + x_2^2) + x_1 x_2 (1 + x_1^2)]$ ∴ if $\|x\| < 1$ then: $(x_1^2 + x_2^2) > x_1 x_2 (1 + x_1^2)$

∵ $1 + x_1^2 > 0$ if $x_1 > 0, x_2 < 0$: $x_1 x_2 < 0$ ∴ $x_1 x_2 (1 + x_1^2) < 0$ ∴

$(x_1^2 + x_2^2) > x_1 x_2 (1 + x_1^2) = x_1 x_2 + x_1 x_1^2 x_2 = x_1 x_2 + x_1^3 x_2$ ∴ $x_1 < 1, x_2 < 1$ ∴

$x_1^3 < x_1^2$ ∴ $x_1^3 x_2 < x_1^2 x_2 < x_1^2$ ∴ $x_1^2 + x_2^2 > x_1 x_2 (1 + x_1^2)$ ∴

$\dot{V}(x) < 0$ if $\|x\| < 1$ ∵ $\dot{V} = 2[-(x_1^2 + x_2^2) + x_1 x_2 (1 + x_1^2)]$ ∴

$-(x_1^2 + x_2^2) + x_1 x_2 (1 + x_1^2) < 0$ ∴ $\dot{V} < 0$ ∴ ~~x~~ $x = (0,0)$ is Locally asymp Stable

$D = \{x \in \mathbb{R}^2 \mid V(x) < 1\}$

\5d/ $\dot{x}_1 = -x_1 - x_2$, $\dot{x}_2 = x_1 - x_2^3$ $V(x) = x_1^2 + x_2^2$ ∴ V is posi desi ∴ taking deriv:

$\dot{V}(x) = 2x_1\dot{x}_1 + 2x_2\dot{x}_2 = 2x_1[-x_1 - x_2] + 2x_2[x_1 - x_2^3] =$

$-2x_1^2 - 2x_1x_2 + 2x_1x_2 - 2x_2^4 = -2x_1^2 - 2x_2^4 = -2(x_1^2 + x_2^4) = -2(x_1^2 + (x_2^2)^2) \leq 0$ ∴

$\dot{V}(x) < 0$ if $x \in \mathbb{R} \setminus \{0\}$. $\dot{V}(0) = 0$ ∴ origin is globally asymp stable.

\3/ V is contly dissable & posi desi,

$\dot{V} \leq 0$ from a nega semi desi perspective

{Lasalle's thm: V is contly dissable & posi desi &

$\dot{V} \leq 0$ ∴ let $S = \{x \in D \mid \dot{V} = 0\}$ ∴ suppose no sol sol can identically stay in S other than Z trivial sol $x(t) = 0$ ∴ origin is locally asymp Stable

\2/ to make origin ~~asymp~~ globally asymp stable: release the restriction on y ∴ $y \in \mathbb{R}$ & $V(x) \to \infty$ as $\|x\| \to \infty$ ∴ radially unbounded

\3/ Ω_c is compact posi invariant set ∴ V is posi desi &

$\dot{V} \leq 0$ in Ω_c ∴ S is invariant set ∴ every trajec in Ω_c approaches S as $t \to \infty$ as $y(t) \to 0$ as $t \to \infty$ ∴ Locally asymp stable.

V posi desi $\dot{V}$ nega semi desi Lyapunov stable

V posi desi $\dot{V}$ nega semi desi asymp Stable $V > 0$ $\dot{V} \leq 0$ Lyapunov Stable

V posi desi $\dot{V}$ semi desi ∴ globally asymp Stabe

$V > 0$ $\dot{V} \leq 0$ Lyapunov stable $V > 0$ $\dot{V} < 0$ Lyapunov stable

$V > 0$ $\dot{V} < 0$ globally asymp stable $V > 0$ $\dot{V} \leq 0$ Locally asymp Stable

V posi desi $\dot{V}$ nega desi ∴ globally asymp stable $V > 0$ $\dot{V} \leq 0$ Lyapunov Stable

Sheet week 8 / Q1/ $\dot{x}_1 = -x_2$, $\dot{x}_2 = x_1 + (x_1^2 - 1)x_2$ $\therefore x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \therefore \dot{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

$$\begin{bmatrix} -x_2 \\ x_1 + x_1^2 x_2 - x_2 \end{bmatrix} = \begin{bmatrix} 0x_1 - 1x_2 \\ 1x_1 - 1x_2 + x_1^2 x_2 \end{bmatrix} = \begin{bmatrix} 0x_1 - 1x_2 \\ 1x_1 - 1x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x_1^2 x_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x_1^2 x_2 \end{bmatrix} = Ax + g(x)$$

$\therefore A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ $g(x) = \begin{bmatrix} 0 \\ x_1^2 x_2 \end{bmatrix}$ $\therefore \det(A - \lambda I) = \begin{vmatrix} 0-\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & -1 \\ 1 & -1-\lambda \end{vmatrix} =$

$-\lambda(-1-\lambda) - (1)(-1) = 0 = \lambda + \lambda^2 + 1 = \lambda^2 + \lambda + 1 = 0 \therefore \lambda = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{-3}}{2} \therefore$

Mat A has eigvals $\frac{-1 \pm i\sqrt{3}}{2} = \frac{-1}{2} \pm i\sqrt{3}$ $\therefore$ $Re(\lambda_i(A)) < 0$, $i = 1, 2$ $\therefore$

$[A]$ is hurwitz $\therefore$ & $\therefore$ Z linear part alone : $\dot{x} = Ax$ is asymp stable (origin) $\therefore$

by converse Lyapunov argument, $\exists P = P^T > 0$ $\forall Q = Q^T > 0$ satisfying Lyapunov eqn $A^T P + PA = -Q$ let Q be I $\therefore A^T P + PA = -I \therefore$

Choose $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$ $\because P = P^T$ $\therefore$ we know $[A]$ $\therefore$ subing A & P in Z

Lyapunov eqn: $P = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1 \end{bmatrix}$ $\Big\{$ is $A^T = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \therefore A^T P + PA = -I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} =$

$$\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} + \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} P_{12} & P_{22} \\ -P_{11} - P_{12} & -P_{12} - P_{22} \end{bmatrix} + \begin{bmatrix} -P_{12} & P_{11} - P_{12} \\ -P_{22} & P_{12} - P_{22} \end{bmatrix} = \begin{bmatrix} 0 & P_{11} - P_{12} + P_{22} \\ -P_{11} - P_{12} - P_{12} & -2P_{22} \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \Big\}$$

$A = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$ $A^T P + PA = -I$ $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$ $P = P^T$ $\therefore P = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1 \end{bmatrix} \therefore$

$\lambda_{min}(P) = 0.691$ (min eig val) $\therefore P = P^T > 0 \therefore$

Z Lyapunov for LTI syst $\dot{x} = Ax$ is $V(x) = X^T P X$ where $X = [x_1 \; x_2]^T$ & $P = \begin{bmatrix} 1.5 & -0.5 \\ -0.5 & 1 \end{bmatrix}$ one can also write $V(x) = 1.5x_1^2 + x_2^2 - x_1 x_2$

taking $\dot{V}(x)$ along Z dynamics of system:

$\dot{V}(x) = 3x_1\dot{x}_1 + 2x_2\dot{x}_2 - x_1\dot{x}_2 - \dot{x}_1 x_2 =$

$(3x_1 - x_2)(-x_2) + (-x_1 + 2x_2)(x_1 + (x_1^2 - 1)x_2) =$

$-3x_1x_2 + x_2^2 - x_1^2 - x_1^3 x_2 + x_1 x_2 + 2x_1 x_2 + 2x_2^2 x_1^2 - 2x_2^2 =$

$-x_1^2 - x_2^2 - x_1^3 x_2 + 2x_1^2 x_2^2 = -(x_1^2 + x_2^2) - x_1^2 x_2 (x_1 - 2x_2) \le$

$\|x\|^2 + \|x\| \|x_1 x_2\| \, |(x_1 - 2x_2)|$

here we know $|x_1| \le \|x\|$, $|x_1 x_2| \le \|x\|^2/2$, $|x_1 - 2x_2| \le \sqrt{1^2 + 2^2} \|x\| \le \sqrt{5} \|x\|$

$\therefore$ making use of these :

$\dot{V}(x) \le -\|x\|^2 + \|x\| \frac{1}{2} \|x\|^2 \sqrt{5} \|x\| \le -\|x\|^2 + \frac{\sqrt{5}}{2} \|x\|^4 \le -\|x\|^2 (\overbrace{1 - \frac{\sqrt{5}}{2} \|x\|^2}^{\text{+ve provided}}) \therefore$

$0 \le \|x\|^2 < \frac{2}{\sqrt{5}}$ $\therefore$ choose $\frac{2}{\sqrt{5}} = r^2$ & following Z arguments of $\forall r > 0$, $\exists r > 0$ st $\|g(x)\| < r\|x\|$, $\forall \|x\| < r$: can write Z esti of Z region of attraction stability as $D = \{x \in \mathbb{R}^2 \mid V(x) < c\}$ where $c = \lambda_{min}(P) \cdot r^2 = \frac{(0.691)2}{\sqrt{5}}$

Week 8 Sheet 2 $\quad \dot{x}_1 = x_2$, $\dot{x}_2 = -2x_1 - 3x_2 + x_2|\sin x_2|$ $\therefore x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \therefore$

$$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -2x_1 - 3x_2 + x_2|\sin x_2| \end{bmatrix} = Ax + g(x) = \begin{bmatrix} x_2 \\ -2x_1 - 3x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2|\sin x_2| \end{bmatrix} =$$

$$\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ x_2|\sin x_2| \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2|\sin x_2| \quad \text{ie } \dot{x} = Ax + B\tilde{g}(x)$$

here $[A]$ is stable. by converse Lyapunov argume

$\Big\{ A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \therefore \det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = -\lambda(-3-\lambda) - 1(-2) = 3\lambda + \lambda^2 + 3 =$

$\lambda^2 + 3\lambda + 3 = 0 \quad \therefore \lambda_i = \dfrac{-3 \pm \sqrt{9 - 4(3)(1)}}{2(1)} = -\dfrac{3}{2} \pm \dfrac{\sqrt{-3}}{2} = -\dfrac{3}{2} \pm \dfrac{\sqrt{3}}{2}i \therefore$

$Re(\lambda_i(A)) < 0 \quad i = 1, 2 \therefore A$ is stable $\therefore$ by converse Lyapunov thm:

$\exists$ a mat $[P]$: $P = P^T > 0$ & a choice of $Q = Q^T > 0$ satisfying

$A^T P + PA = -Q$ let $Q = I$ $\therefore$ solving Z Lyapunov eqn : $P = \begin{bmatrix} 5/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$

$\Big\{ P = P^T \therefore P = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \therefore A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \therefore A^T = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix} \therefore$

$$A^T P + PA = -I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 1 & -3 \end{bmatrix}\begin{bmatrix} a & b \\ b & d \end{bmatrix} + \begin{bmatrix} a & b \\ b & d \end{bmatrix}\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} = \begin{bmatrix} -2b & -2d \\ a-3b & b-3d \end{bmatrix} + \begin{bmatrix} -2b & -3b \\ -2d & b-3d \end{bmatrix} =$$

$$\begin{bmatrix} -4b & -3b-2d \\ a-3b-2d & 2b-6d \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \therefore -4b = -1 \therefore b = \tfrac{1}{4} \therefore$$

$-3b - 2d = 0 \therefore -2d = 3(\tfrac{1}{4}) = \tfrac{3}{4} \therefore d = -\tfrac{3}{8}$

$2b - 6d = -1 \therefore 6d = 2(\tfrac{1}{4}) + 1 = \tfrac{1}{2} + 1 = \tfrac{3}{2} \therefore d = \tfrac{3}{6} \cdot \tfrac{1}{2} = \tfrac{1}{2} \times \tfrac{1}{2} = \tfrac{1}{4}$

$a - 3b - 2d = 0 \therefore a = 3b + 2d \therefore a = \tfrac{5}{4} = 3(\tfrac{1}{4}) + 2(\tfrac{1}{4}) \therefore P = \begin{bmatrix} 5/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix} \Big\} \therefore$

write Z Lyapunov func $V(x) = x^T P x$ where $P = \begin{bmatrix} 5/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$ taking Z

deriv of V along Z trajecs of Z syst :

$$\dot{V}(x) = x^T(PA + A^T P)x + 2x^T P g(x) = -x^T x + 2\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} x|\sin x_2|$$

$$= -x^T x + 2\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix} x_2|\sin x_2| = -x^T x + 2x_1 P_{12} x_2|\sin x_2| + 2x_2 P_{22} x_2|\sin x_2| = \dot{V}(x)$$

note $\tfrac{1}{4} = P_{22} = P_{12} > 0$ & $|\sin x_2| < 1 \ \forall x_2 \therefore |\sin x_2| < 1 \ \forall x_2 \therefore$

$\dot{V}(x) = -x^T x + 2x_1 P_{12} x_2|\sin x_2| + 2x_2 P_{22} x_2|\sin x_2| \leq$

$-x_1^2 - x_2^2 + 2x_1 P_{12} x_2|\sin x_2| + 2x_2 P_{22} x_2 \leq$

$\underbrace{-x_1^2 - (1 - 2P_{22})x_2^2}_{\text{can argue neg defi}} + \underbrace{2x_1 P_{12} x_2|\sin x_2|}_{\text{is sign indefi}}$

$\Big\{$ Def / can make use of Young's inequality : $2Z^T Y \leq \frac{1}{\beta}\|Z\|^2 + \beta\|Y\|^2 \ \forall \beta > 0, \forall Z, Y \in \mathbb{R}^m \Big\}$

$\dot{V}(x) = -x_1^2 - (1 - 2P_{22})x_2^2 + 2x_1 P_{12} x_2 |\sin x_2|$

$\therefore$ making use of Young's inequality in 2nd term ie $2\underbrace{x_1 P_{12}}_{Z^T}\underbrace{x_2|\sin x_2|}_{Y} \therefore$

$$2\overbrace{x_1 P_{12}}^{Z \cdot Z^T} x_2|\sin x_2| \leq \frac{1}{\beta}\underbrace{P_{12}^2 x_1^2}_{\|Z\|^2} + \beta \underbrace{x_2^2|\sin x_2|^2}_{\|Y\|^2} \leq \frac{1}{\beta}P_{12}^2 x_1^2 + \beta x_2^2 \quad \therefore \text{using this :}$$

$\dot{V}(x) \leq -x_1^2 - (1-2P_{22})x_2^2 + \frac{1}{\beta}P_{12}^2 x_1^2 + \beta x_2^2 = -(1-\frac{1}{\beta}P_{12}^2)x_1^2 - (1-2P_{22}-\beta)x_2^2$

$\dot{V}(x)$ is nega desi : $1-\frac{1}{\beta}P_{12}^2$ is +ve & $1-2P_{22}-\beta$ is +ve

$\beta > P_{12}^2$ & $\beta < 1-2P_{22}$ ∴ $\frac{1}{16} < \beta < \frac{1}{2}$

\Q1/ $\dot{x}_1 = -x_2$, $\dot{x}_2 = x_1 + (x_1^2 - 1)x_2$ $\quad \dot{x} = \begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} + \begin{bmatrix}0\\ x_1^2 x_2\end{bmatrix} = \overset{\text{linear part}}{Ax} + g(x)$ ← nonlinear part

Z mat $[A]$ has eig vals $\frac{-1\pm\sqrt{3}i}{2}$ ∴ $Re(\lambda_i(A)) < 0$, $i=1,2$ ∴ $[A]$ is hurwitz

& ∴ Z linear part alone $\dot{x} = Ax$ ∴ $\begin{bmatrix}\dot{x}_1\\ \dot{x}_2\end{bmatrix} = \begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix}$ is globally asymp stable (origin)

∴ by Converse Lyapunov argument, $\exists\ P = P^T > 0$ for any $Q = Q^T > 0$ satisfying Lyapunov eqn $A^T P + PA = -Q$

∴ let $Q = I$ ∴ $A^T P + PA = -I$ ∴

choose $P = \begin{bmatrix}P_1 & P_{12}\\ P_{12} & P_{22}\end{bmatrix}$ ∴ $P = P^T$ ∴

$A = \begin{bmatrix}0 & -1\\ 1 & -1\end{bmatrix}$ ∴ know $[A]$ ∴ Sub A & P in Z lyapunov eqn ∴

$P = \begin{bmatrix}3/2 & -1/2\\ -1/2 & 1\end{bmatrix}$ ∴

$\lambda_{min}(P) = 0.691$ (min eigval) ∴ ~~∴ P>~~

∴ $P = P^T > 0$ ∴

Z lyapunov func for LTI syst $\dot{x} = Ax$:

$V(x) = x^T P x$ where $x = [x_1 \quad x_2]^T$ & $P = \begin{bmatrix}1.5 & -0.5\\ -0.5 & 1\end{bmatrix}$ one can write

$V(x) = 1.5x_1^2 + x_2^2 - x_1 x_2$ $\quad \{$ ∴ $V(x) = x^T P x = [x_1 \quad x_2]\begin{bmatrix}1.5 & -0.5\\ -0.5 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\end{bmatrix} =$

$[1.5x_1 - 0.5x_2 \quad -0.5x_1 + x_2]\begin{bmatrix}x_1\\ x_2\end{bmatrix} = 1.5x_1^2 + x_2^2 - x_1 x_2 \}$

∴ taking $\dot{V}(x)$ along Z dynamics of original system:

$\dot{V}(x) = 3x_1\dot{x} + 2x_2\dot{x}_2 - x\dot{x}_2 - \dot{x}_1 x_2 = (3x_1 - x_2)\dot{x}_1 + (-x_1 + 2x_2)\dot{x}_2 =$ (Sub in Syst)

$(3x_1 - x_2)(-x_2) + (-x_1 + 2x_2)(x_1 + (x_1^2 - 1)x_2) =$

$-3x_1x_2 + x_2^2 - x_1^2 - x_1^3x_2 + x_1x_2 + 2x_1x_2 + 2x_2^2x_1^2 - 2x_2^2 = -x_1^2 - x_2^2 - x_1^3x_2 + 2x_1^2x_2^2 =$

$-\underbrace{(x_1^2 + x_2^2)}_{\|x\|^2} - (x_1^2 x_2)(x_1 - 2x_2)$ $\qquad$ (∴ $\|AB\| \leq \|A\|\|B\|$ ∴)

$\leq -\|x\|^2 + (-x_1^2x_2)(x_1 - 2x_2) \leq -\|x\|^2 + \|-x_1^2x_2\|\|x_1 - 2x_2\| = -\|x\|^2 + \|x_1^2x_2\|\|x_1 - 2x_2\|$

$\leq -\|x\|^2 + \|x_1(x_1x_2)\|\|x_1 - 2x_2\| \leq -\|x\|^2 + \|x_1\|\|x_1x_2\|\|x_1 - 2x_2\|$

here we know : $|x_1| \leq \|x\|$ & $|x_1x_2| \leq \|x\|^2/2$ ∵ $x_1^2 + x_2^2 + 2x_1x_2 = \|x\|^2$

∴ $x_1x_2 + \frac{1}{2}(x_1^2 + x_2^2) = \|x\|^2/2$ ∴ $|x_1x_2| \leq \|x\|^2/2$ ∴

$|x_1 - 2x_2| \leq \sqrt{1^2 + (-2)^2}\|x\| = \sqrt{1^2 + 2^2}\|x\| = \sqrt{5}\|x\|$ ∴ making use of these

\week 8 sheet/ $\dot{V}(x) \leq -\|x\|^2 + \|x_1 x_2\| \, |x_1 - 2x_2| \leq$

$-\|x\|^2 + \|x\| \frac{1}{2}\|x\|^2 \sqrt{5}\|x\| \leq -\|x\|^2 + \frac{\sqrt{5}}{2}\|x\|^4 \leq -\|x\|^2\left(1 - \frac{\sqrt{5}}{2}\|x\|^2\right)$

+ve provided $0 \leq \|x\|^2 \leq \frac{2}{\sqrt{5}}$

$\therefore$ have accounted $g(x)$ in Z dynamics $\therefore$ $\dot{V}(x) \leq 0$ is $0 \leq \|x\|^2 \leq \frac{2}{\sqrt{5}}$

$\therefore$ asymp stable $\therefore$

choose $\frac{2}{\sqrt{5}} = r^2$ & following Z arguments of any $\gamma > 0$, $\exists r > 0$ st $\|g(x)\| < \gamma\|x\|$ $\forall \|x\| < r$ $\therefore$

let $\frac{2}{\sqrt{5}} = r^2$ $\therefore$ $0 \leq \|x\|^2 \leq \frac{2}{\sqrt{5}} = r^2$ $\therefore$ for $\gamma > 0$, $\exists r > 0$ st $\|g(x)\| < \gamma\|x\|$, $\forall \|x\| < r$ $\therefore$

can write Z esti of Z region of attraction ~~of~~ stability as

$D = \left\{x \in \mathbb{R}^2 \,\middle|\, V(x) < C\right\}$ where $C = \lambda_{min}(P) \cdot r^2 = \frac{0.691 \cdot 2}{\sqrt{5}} = 0.691 \cdot \frac{2}{\sqrt{5}}$

\2/ $\dot{x}_1 = x_2$
$\dot{x}_2 = -2x_1 - 3x_2 + x_2|\sin x_2|$

$\underbrace{\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}}_{\dot{x}} = \underbrace{\begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{x} + \underbrace{\begin{bmatrix} 0 \\ x_2|\sin x_2| \end{bmatrix}}_{g(x)}$ form.

$\begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \underbrace{x_2|\sin x_2|}_{g(x)}$ ie $\dot{x} = Ax + Bg(x)$ $\therefore$

$A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $\therefore$ $\det(A - \lambda I) = \det\begin{bmatrix} -\lambda & 1 \\ -2 & -3-\lambda \end{bmatrix} = -\lambda(-3-\lambda) - 1(-2) = 3\lambda + \lambda^2 + 3 = \lambda^2 + 3\lambda + 3 = 0$ $\therefore$

$\lambda_i = \frac{-3 \pm \sqrt{9 - 4(3)(1)}}{2(1)} = -\frac{3}{2} \pm \frac{\sqrt{3}}{2} = -\frac{3}{2} \pm \frac{\sqrt{3}}{2}i$ $\therefore$ $Re(\lambda_i(A)) < 0$, $i = 1, 2$ $\therefore$

$[A]$ is stable $\therefore$

by converse Lyapunov argument, $\exists$ a mat $[P]$ st $P = P^T > 0$ & a choice of $Q = Q^T > 0$ satisfying $A^T P + PA = -Q$, $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$, $A = \begin{bmatrix} 0 & 1 \\ -2 & -3 \end{bmatrix}$ $\therefore$

let $Q = I$ $\therefore$ solving Z Lyapunov eqn $A^T P + PA = -I$ $\therefore$

$P = \begin{bmatrix} 5/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$ $\therefore$

write Lyapunov func $V(x) = x^T P x$, where $P = \begin{bmatrix} 5/4 & 1/4 \\ 1/4 & 1/4 \end{bmatrix}$ $\therefore$ taking Z deriv of V along Z trajectories of Z dynamics:

$\dot{V}(x) = x^T P \dot{x} + \dot{x}^T P x = x^T P[Ax + g(x)] + [Ax + g(x)]^T P x$

$= x^T(PA + A^T P) + 2x^T P g(x) = -x^T Q x + 2x^T P g(x)$ & choose $\gamma < \frac{1}{2}\frac{\lambda_{min}(Q)}{\|P\|}$

$\therefore \dot{V}(x) = x^T(PA + A^T P)x + 2x^T P g(x)$ $\therefore$ $\dot{V}(x) \leq -x^T Q x + 2\|x\| \|P\| \|g(x)\|$ $\therefore$

$\dot{V}(x) = x^T(\overbrace{PA + A^T P}^{-I})x + 2x^T P g(x) = -x^T x + 2\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} x_2|\sin x_2| =$

$-x^T x + 2\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} P_{12} \\ P_{22} \end{bmatrix} x_2|\sin x_2| =$

$-x^T x + 2x_1 P_{12} x_2|\sin x_2| + 2x_2 P_{22} x_2|\sin x_2|$

note $\frac{1}{4} = P_{22} = P_{12} > 0$ & $|\sin x_2| < 1$ $\forall x_2$ $\therefore$

$\dot{V}(x) = -\underbrace{x^T x}_{\|x\|^2} + 2x_1 P_{12} x_2 |\sin x_2| + 2x_2 P_{22} x_2 |\sin x_2| = -\begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + 2x_1 P_{12} x_2 |\sin x_2| + 2x_2 P_{22} x_2 |\sin x_2|$

$\leq -x_1^2 - x_2^2 + 2x_1 P_{12} x_2 |\sin x_2| + 2x_2 P_{22} x_2$ $\quad (\because |\sin x_2| < 1 \ \forall x_2)$

$\leq \cancel{2x_1^2 - x_2^2 + 2x_1 P_{12} x_2 + 2x_2 P_{22} x_2}$ $\leq \underbrace{-x_1^2 - (1-2P_{22})x_2^2}_{\text{nega desi}} + \underbrace{2x_1 P_{12} x_2 |\sin x_2|}_{\text{sign indesi}}$

$\therefore$ by youngs inequality $2Z^T Y \leq \frac{1}{\beta}\|Z\|^2 + \beta\|Y\|^2 \quad \forall \beta > 0 \quad \forall Z, Y \in \mathbb{R}^m$

$\therefore 2\underbrace{x_1 P_{12}}_{Z^T} \underbrace{x_2 |\sin x_2|}_{Y} = 2Z^T Y \leq \frac{1}{\beta}\|Z\|^2 + \beta\|Y\|^2 = \cancel{\frac{1}{\beta}\|[x_1 P_{12}]^T\|^2 \|\beta x_2 |\sin x_2|\|}$

$\frac{1}{\beta}\|[x_1 P_{12}]^T\|^2 + \beta\|x_2 |\sin x_2|\|^2 = \frac{1}{\beta}\|P_{12} x_1\|^2 + \beta x_2^2 |\sin x_2|^2 = \frac{1}{\beta} P_{12}^2 x_1^2 + \beta x_2^2 |\sin x_2|^2$

$\leq \frac{1}{\beta} P_{12}^2 x_1^2 + \beta x_2^2$ $\quad , \beta > 0 \quad \therefore$

$\dot{V}(x) \leq -x_1^2 - (1-2P_{22})x_2^2 + 2x_1 P_{12} x_2 |\sin x_2| \leq$

$-x_1^2 - (1-2P_{22})x_2^2 + \frac{1}{\beta} P_{12}^2 x_1^2 + \beta x_2^2 |\sin x_2|^2 \leq$

$-x_1^2 - (1-2P_{22})x_2^2 + \frac{1}{\beta} P_{12}^2 x_1^2 + \beta x_2^2 = -\underbrace{(1 - \frac{1}{\beta} P_{12}^2)}_{+ve} x_1^2 - \underbrace{(1 - 2P_{22} - \beta)}_{+ve} x_2^2$

$\dot{V}(x)$ is nega desi provided: $1 - \frac{1}{\beta} P_{12}^2$ is +ve & $1 - 2P_{22} - \beta$ is +ve

$\therefore 1 - \frac{1}{\beta} P_{12}^2 > 0 \quad \therefore 1 > \frac{1}{\beta} P_{12}^2 \quad \therefore \beta > P_{12}^2 \quad \therefore \beta > (\frac{1}{4})^2 \quad \therefore \beta > \frac{1}{16}$ &

$1 - 2P_{22} - \beta > 0 \quad \therefore 1 - 2P_{22} > \beta \quad \therefore 1 - 2P_{22} = 1 - 2(\frac{1}{4}) = 1 - \frac{1}{2} = \frac{1}{2} > \beta \quad \therefore$

$\frac{1}{16} < \beta < \frac{1}{2}$ will have asymp stability ~~at~~ of origin.

\3/ $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \therefore x^T = \begin{bmatrix} x_1 & x_2 \end{bmatrix} \quad \therefore \det(A - \lambda I) = \det\begin{bmatrix} 0-\lambda & -1 \\ 1 & -1-\lambda \end{bmatrix} = \det\begin{bmatrix} -\lambda & -1 \\ 1 & -1-\lambda \end{bmatrix}$

$= (-\lambda)(-1-\lambda) - 1(-1) = \lambda + \lambda^2 + 1 = \lambda^2 + \lambda + 1 = 0 \quad \therefore \lambda = \frac{-1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{-1 \pm \sqrt{1-4}}{2} = \frac{-1}{2} \pm \frac{\sqrt{-3}}{2} =$

$-\frac{1}{2} \pm \frac{\sqrt{3}}{2} i \quad \therefore \mathrm{Re}(\lambda_i(A)) = -\frac{1}{2} < 0, \ i = 1, 2 \quad \therefore A$ is hurwitz $\therefore$

linear system is globally asymp stable.

by converse Lyapunov argument: $V(x) = x^T P x$ where $P = P^T > 0$

$Q = Q^T > 0$ satisfying $PA + A^T P = -Q \quad \therefore P = P^T \quad \therefore P = \begin{bmatrix} a & b \\ b & d \end{bmatrix}$

let $Q = I \quad \therefore A^T = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix} \quad \therefore PA + A^T P = -I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} a & b \\ b & d \end{bmatrix}\begin{bmatrix} 0 & -1 \\ +1 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ b & d \end{bmatrix} =$

$\begin{bmatrix} b & -a-b \\ d & -b-d \end{bmatrix} + \begin{bmatrix} b & d \\ -a-b & -b-d \end{bmatrix} = \begin{bmatrix} 2b & -a-b+d \\ -a-b+d & -2b-2d \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \quad \therefore$

$2b = -1 \quad \therefore b = -\frac{1}{2} \quad \therefore -2b - 2d = -1 = -2(-\frac{1}{2}) - 2d = 1 - 2d = -1 \quad \therefore 2 = 2d \quad \therefore d = 1 \quad \therefore$

$-a - b + d = 0 = -a + \frac{1}{2} + 1 = -a + \frac{3}{2} = 0 \quad \therefore a = \frac{3}{2} \quad \therefore$

$P = \begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1 \end{bmatrix} \quad \therefore V(x) = x^T P x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 3/2 & -1/2 \\ -1/2 & 1 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} \frac{3}{2}x_1 - \frac{1}{2}x_2 \\ -\frac{1}{2}x_1 + x_2 \end{bmatrix} =$

$\frac{3}{2}x_1^2 - \frac{1}{2}x_1 x_2 - \frac{1}{2}x_1 x_2 + x_2^2 = \frac{3}{2}x_1^2 + x_2^2 - x_1 x_2$ $\quad , \det(P) = \det\begin{bmatrix} \frac{3}{2} - \lambda & -1/2 \\ -1/2 & 1-\lambda \end{bmatrix} =$

$(\frac{3}{2} - \lambda)(1 - \lambda) - (-\frac{1}{2})(-\frac{1}{2}) = \frac{3}{2} - \frac{5}{2}\lambda + \lambda^2 - \frac{1}{4} = \lambda^2 - \frac{5}{2}\lambda + \frac{5}{4} = 0 \quad \therefore \lambda = \left(\frac{5}{2} \pm \sqrt{(\frac{5}{2})^2 - 4(1)(\frac{5}{4})}\right)/2$

$\lambda_{min}(P) = \frac{5 - \sqrt{5}}{4} \approx 0.691 > 0 \quad \therefore P$ is posi desi.

\Week 8 Sheet / \4a /

$\dot{x}_1 = -x_1 + x_2^2$, $\dot{x}_2 = 2x_1 - x_2$ ∴ Z dynamical sys is: $\dot{x} = Ax + g(x)$ where $A = \begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix}$ & $g(x) = \begin{bmatrix} x_2^2 \\ 0 \end{bmatrix}$ linear & nonlinear part respectively ∴ $eig(A) = -1, -1$ ∴ Z origin is asymp stable ∵ A is a hurwitz mat ∵ $Re(\lambda_i(A)) < 0$

∴ Solving lyapunov eqn: $PA + A^T P = -I$ ∵ choice of Q is I ∴

$\left\{ P = \begin{bmatrix} a & b \\ b & d \end{bmatrix} \right.$ ∵ $P = P^T > 0$ ∴ $A^T = \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}$ ∴ $PA + A^T P = -I = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} =$

$\begin{bmatrix} a & b \\ b & d \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 2 & -1 \end{bmatrix} + \begin{bmatrix} -1 & 2 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a & b \\ b & d \end{bmatrix} = \begin{bmatrix} -a+2b & -b \\ -b+2d & -d \end{bmatrix} + \begin{bmatrix} -a+2b & -b+2d \\ -b & -d \end{bmatrix} = \begin{bmatrix} -2a+4b & -2b+2d \\ -2b+2d & -2d \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}$ ∴

$-2d = -1$ ∴ $d = \frac{1}{2}$ ∴ $0 = -2b + 2d = -2b + 1$ ∴ $2b = 1$ ∴ $b = \frac{1}{2}$ ∴

$-2a + 4b = -1 = -2a + 2$ ∴ $2a = 3$ ∴ $a = 3/2$ ∴ $\left.\right\}$ $P = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$ ∴

$\left\{ \det(P - \lambda I) = \det\begin{bmatrix} 3/2 - \lambda & 1/2 \\ 1/2 & \frac{1}{2} - \lambda \end{bmatrix} = (\frac{3}{2} - \lambda)(\frac{1}{2} - \lambda) - \frac{1}{2}(\frac{1}{2}) = \lambda^2 + \frac{3}{4} - 2\lambda - \frac{1}{4} = \lambda^2 - 2\lambda + \frac{1}{2} = 0 \right.$

∴ $\lambda = \frac{2 \pm \sqrt{4 - 4(\frac{1}{2})}}{2} = 1 \pm \frac{\sqrt{2}}{2} \}$ eigval of P are $\frac{2 \pm \sqrt{2}}{2}$ ∴

$1 \pm \frac{\sqrt{2}}{2}$ ∴ $\|P\| = 1 + \frac{\sqrt{2}}{2}$ ∵ $1 + \frac{\sqrt{2}}{2} > 1 - \frac{\sqrt{2}}{2}$

Z lyapunov func associated with $\dot{x} = Ax$ is $V = x^T P x$ where $P = \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}$

∴ $V(x) = x^T P x = \begin{bmatrix} x_1 & x_2 \end{bmatrix}\begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} \frac{3}{2}x_1 + \frac{1}{2}x_2 & \frac{1}{2}x_1 + \frac{1}{2}x_2 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =$

$\frac{3}{2}x_1^2 + \frac{1}{2}x_1x_2 + \frac{1}{2}x_1 \cancel{x_2} + \frac{1}{2}x_2^2 = \frac{3}{2}x_1^2 + x_1x_2 + \frac{1}{2}x_2^2 = \frac{1}{2}(3x_1^2 + 2x_1x_2 + x_2^2)$ ∴ ∵ $\frac{\|g(x)\|}{\|x\|} \to 0$ as $\|x\| \to 0$, can

to find a bound on nonlinearity/perturbation

find $\forall \|x\| < r$ st $\|g(x)\| < \gamma\|x\|$ ∴ here $\gamma < \frac{1}{2\|P\|}$

∵ $x_2^2 \le \|x\|^2$ ∴ $\frac{\|g(x)\|}{\|x\|} = \frac{x_2^2}{\|x\|} \le \frac{\|x\|^2}{\|x\|} = \|x\|$ ∴

$\frac{\|g(x)\|}{\|x\|} < \gamma$ when $\|x\| < \gamma$ ∴ $\gamma = r = \frac{1}{2\|P\|} = \frac{1}{2 + \sqrt{2}}$ ∴

$V(x) = \frac{1}{2}(3x_1^2 + 2x_1x_2 + x_2^2)$ ∴ can now write an invariant set, in general,

$D = \left\{ x \in \mathbb{R}^2 \mid V < \lambda_{min}(P)\, r^2 \right\}$ where $\lambda_{min}(P) = 1 - \frac{\sqrt{2}}{2}$, $r^2 = \left(\frac{1}{2 + \sqrt{2}}\right)^2$ ∴

$D = \left\{ x \in \mathbb{R}^2 \mid (3x_1^2 + 2x_1x_2 + x_2^2) < \frac{2 - \sqrt{2}}{6 + 4\sqrt{2}} \right\}$

\4b /

$\dot{x}_1 = -2x_1 + x_2 + x_1^3 x_2^2$, $\dot{x}_2 = -x_1 - 2x_2 + x_1^2 x_2^3$ $\dot{x} = Ax + G(x)$ where $A = \begin{bmatrix} -2 & 1 \\ -1 & -2 \end{bmatrix}$ ∴ $\lambda_i(A) = -2 \pm i$ ∴

$Re(\lambda_i(A)) < 0$ ∴ $[A]$ is Hurwitz ∴ Solving $A^T P + PA = -I$: $P = \begin{bmatrix} 0.25 & 0 \\ 0 & 0.25 \end{bmatrix}$

has eigvals $0.25 = \|P\|$ ∴ $V = \frac{1}{4}(x_1^2 + x_2^2)$ & ∵ $V(x) = x^T P x$ ∴ & ∵

$\frac{\|g(x)\|}{\|x\|} \to 0$ as $\|x\| \to 0$ ∵ $g(x) = \begin{bmatrix} x_1^3 x_2^2 \\ x_1^2 x_2^3 \end{bmatrix}$ ∴ $\|g(x)\| = x_1^3 x_2^2 + x_1^2 x_2^3 =$

$x_1^2 x_2^2 (x_1 + x_2) = (x_1 x_2)(x_1 x_2)(x_1 + x_2) \le \frac{1}{2}\|x\|(x_1 x_2)(x_1 + x_2) \le \|x\|^2 (x_1 + x_2) \le \|x\|^2 \|x\| = \|x\|^3$ ∴

$\frac{\|g(x)\|}{\|x\|} \le \frac{\|x\|^3}{\|x\|} = \|x\|^2 \to 0$ as $\|x\|$ ∴

can find $\forall \|x\|<r$ st $\|g(x)\| < \gamma\|x\|$ here $\gamma < \frac{1}{2\|x\|}$ ∴

$\frac{\|g(x)\|}{\|x\|} = x_1^2 x_2^2 \le \frac{1}{2}(x_1^4 + x_2^4)$ $\quad \{ g(x) = \begin{bmatrix} x_1^3 x_2^2 \\ x_1^2 x_2^3 \end{bmatrix} \quad \frac{\|g(x)\|}{\|x\|} = x_1^2 x_2^2$

is $\|g(x)\| = x_1^3 x_2^2 + x_1^2 x_2^3 = x_1^2 x_2^2 (x_1 + x_2)$ ∴ $\frac{\|g(x)\|}{(x_1+x_2)} = \frac{x_1^2 x_2^2 (x_1+x_2)}{x_1+x_2} = x_1^2 + x_2^2$

$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ ∴ $\|x\| = x_1 + x_2 \}$

$\frac{\|g(x)\|}{\|x\|} = x_1^2 x_2^2 \le \frac{1}{2}(x_1^4 + x_2^4) \le \frac{1}{2}(x_1^4 + 2x_1^2 x_2^2 + x_2^4) = \frac{1}{2}(x_1^2 + x_2^2)^2 = \frac{1}{2}\|x\|^4$ ∴

$\frac{\|g(x)\|}{\|x\|} \le \frac{1}{2}\|x\|^4 \to 0$ as $\|x\| \to 0$ ∴ find $\forall \|x\|<r$ st $\|g(x)\| < \gamma\|x\|$ ∴

$\frac{\|g(x)\|}{\|x\|} < \gamma$ when $\|x\|<r$ ∴ $\gamma < \frac{1}{2\|P\|}$ ∴

$\frac{\|g(x)\|}{\|x\|} < \gamma < \frac{1}{2\|P\|} = \frac{1}{2(0.25)} = 2$ when ~~...~~ $\|x\| < (2\gamma)^{1/4} = \sqrt{2}$ $= (2\times 2)^{1/4} = 4^{1/4} = \sqrt{2}$

∴ can write an invariant set in general, $D = \{x \in \mathbb{R}^2 \mid V < \lambda_{min}(P) r^2\}$ ∴

~~$D = \{$~~ $\lambda_{min}(P) = 0.5$ ∴ $\frac{1}{4}(x_1^2 + x_2^2) < 0.25(\sqrt{2})^2 = 0.25(2) = \frac{1}{2}$ ∴ $x_1^2 + x_2^2 = 2$ ∴

$D = \{x \in \mathbb{R}^2 \mid x_1^2 + x_2^2 < 2\}$

\5a/ $\dot{x}_1 = -x_1 + x_2$
$\dot{x}_2 = (x_1 + x_2)\sin x_1 - 3x_2$ ∴ $0 = -x_1 + x_2$ ∴ $x_1 = x_2$ ∴ $0 = (x_1 + x_2)\sin x_1 - 3x_2$ ∴

$(x_1 + x_1)\sin x_1 - 3x_1 = 0 = 2x_1 \sin x_1 - 3x_1 = x_1(2\sin x_1 - 3)$ ∴ $x_1 = 0$ ∴ $x_2 = 0$ ∴

origin is unique equili pt ∴ $\frac{\partial f}{\partial x} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ \sin x_1 + (x_1+x_2)\cos x_1 & \sin x_1 - 3 \end{bmatrix}$

∴ $A = \frac{\partial f}{\partial x}(0) = \begin{bmatrix} -1 & 1 \\ 0 & -3 \end{bmatrix}$ ∴ A is hurwitz ∴ origin is locally asymp stable.

to show Global asymp stability: let choose $V(x) = \frac{1}{2}(x_1^2 + x_2^2)$ which is a posi defi func & radially unbounded. ∵ $V > 0 \ \forall x \in \mathbb{R}^2 \setminus \{0\}$, $V(0) = 0$ &

$V \to \infty$ as $\|x\| \to \infty$ ∴ taking deriv along the trajecs:

$\dot{V} = \frac{1}{2}2x_1\dot{x}_1 + \frac{1}{2}2x_2\dot{x}_2 = x_1\dot{x}_1 + x_2\dot{x}_2 = -x_1^2 + 2x_1x_2(1 + \sin x_1) - (3 - \sin x_1)x_2^2 \le$

$-x_1^2 + 2|x_1||x_2| - 2x_2^2 \quad = -\begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} |x_1| \\ |x_2| \end{bmatrix} < 0 \quad \forall x \ne 0$

~~5b/ $\dot{x}_1 = -x_1^3 + x_2$, $\dot{x}_2 = -ax_1 - bx_2$~~ which is in quadratic form & is $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = B$:

~~...~~ $\begin{vmatrix} 1-\lambda & -1 \\ -1 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) + 1(-1) = \lambda^2 - 3\lambda + 2 - 1 = \lambda^2 - 3\lambda + 1 = 0$ ∴

$\lambda = \frac{3 \pm \sqrt{9 - 4(1)(1)}}{2} = \frac{3 \pm \sqrt{5}}{2} > 0$ ∴ $\lambda_{min}(B) > 0$ ∴ $\begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix}$ is posi defi mat ∴

V is posi defi, radially unbounded, $\dot{V}$ is nega defi ∴

the origin is globally asymp stable

\5b/ $\dot{x}_1 = -x_1^3 + x_2$
$\dot{x}_2 = -ax_1 - bx_2$ ∴ $\dot{x}_1 = 0 = -x_1^3 + x_2$, $\dot{x}_2 = 0 = -ax_1 - bx_2$ ∴ $x_2 = -\frac{a}{b}x_1$ ∴

$-x_1^3 - \frac{b}{a}x_1 = x_1(x_1^2 + \frac{a}{b}) = 0$ ∴ $x_1 = 0$ ∴ $x_2 = 0$ ∴ origin is the unique equili pt

∴ $\frac{\partial f}{\partial x} = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix} = \begin{bmatrix} -3x_1^2 & 1 \\ -a & -b \end{bmatrix}$ $\quad a, b > 0$ ∴

Week 8 Sheet

$A = \frac{\partial f}{\partial x}(0) = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}$ ∴ $\det(A) = \begin{vmatrix} -\lambda & 1 \\ -a & -b-\lambda \end{vmatrix} =$

$-\lambda(-b-\lambda) - 1(-a) = \lambda^2 + b\lambda + a = 0$ ∴ $\lambda = \frac{-b \pm \sqrt{b^2-4a}}{2}$ ∴

$\lambda_{min}(A) = \frac{-b}{2} - \frac{\sqrt{b^2-4a}}{2} < 0$ ∵ $Re(\lambda_i(A)) < 0$, $i = 1,2$ ∴ A is Hurwitz

& ∴ origin is asymp stable ∴ to show globally asymp stability:

Let $V(x) = \frac{1}{2}(x_1^2 + \alpha x_2^2)$ $\alpha > 0$ ∴ V is posi defi & radially unbounded

∵ $V > 0$ $\forall x \in \mathbb{R}^2 \setminus \{0\}$, $V \to \infty$ as $\|x\| \to \infty$ ∴

taking deriv of V:

~~$\dot{V} = x_1^4 + x_1 x_2 + 1$~~ $\dot{V} = x_1 \dot{x}_1 + \alpha x_2 \dot{x}_2 = -x_1^4 + x_1 x_2 (1 - a\alpha) - b\alpha x_2^2$ ∴

taking $\alpha = \frac{1}{a}$: $\dot{V}(x) = -x_1^4 - \frac{b}{a} x_2^2 < 0$ $\forall x \neq 0$ ∴ origin is globally asymp stable ∵ $\frac{b}{a} > 0$ ∴ V is posi defi & unbounded, & $\dot{V}$ is nega defi

Week 9 Sheet

1a

$\dot{x}_1 = \underbrace{-\alpha x_1 - \omega x_2}_{\text{linear part}} + \underbrace{(\beta x_1 - \gamma x_2)(x_1^2 + x_2^2)}_{\text{nonlinear part}}$

$\dot{x}_2 = \omega x_1 - \alpha x_2 + (\gamma x_1 + \beta x_2)(x_1^2 + x_2^2)$ ∴

structure is indicated ∴

note: $x_1^2 + x_2^2 = \|x\|_2^2$ ∴ eqn as: $\dot{x} = Ax + \|x\|_2^2 Bx$ where

$x = [x_1 \ \ x_2]^T$, $A = \begin{bmatrix} -\alpha & -\omega \\ -\omega & -\alpha \end{bmatrix}$, $B = \begin{bmatrix} \beta & -\gamma \\ \gamma & \beta \end{bmatrix}$ ∴ $|B| = \beta^2 + \gamma^2$

choose $V(x) = x^T x$ which is posi defi & for ~~nonlinear~~ nominal linear syst $A + A^T = -2\alpha I$ $V(x) = x^T x$ is Lyapunov funct ∴

(converse lyapunov argument) $\dot{V}(x) \leq -2\alpha\|x\|^2 + 2\|B\|\,\|x\|^4 \leq -2\alpha\|x\|^2 + 2\underbrace{\sqrt{\beta^2+\gamma^2}}_{\|B\|}\|x\|^4$

∵ $\dot{V} = \dot{x}^T x + x^T \dot{x}$ ∴

for $\|x\| \leq r$: $\dot{V} \leq -2\alpha\|x\|^2 + 2r^2\sqrt{\beta^2+\gamma^2}\,\|x\|^2 = -2(\alpha - r^2\sqrt{\beta^2+\gamma^2})\|x\|^2 < 0$

for $\alpha - r^2\sqrt{\beta^2+\gamma^2} > 0$ ∴ $\sqrt{\beta^2+\gamma^2} < \frac{\alpha}{r^2}$ ∴ calcing the deriv of V directly

$\dot{V} = -2\alpha\|x\|^2 + 2\beta\|x\|^4$ when $\beta \leq 0$ $\dot{V} \leq -2\alpha\|x\|^2$ & the origin is globally exponentially stable ∵ chosen V's also radially unbounded

∵ $V \to \infty$ as $\|x\| \to \infty$ when $\beta > 0$ ∴

$\dot{V} \leq -2(\alpha - \beta)\|x\|^2$, $\forall \|x\|_2 \leq r$ ∴ $\dot{V}$ is nega defi when $r^2 < \frac{\alpha}{\beta}$ ∴

$V(x) = \|x\|^2$ we could conclude that the set $\Omega = \{x \in \mathbb{R}^2 \mid \|x\|^2 < \frac{\alpha}{\beta}\}$ is included in the region of attraction ∴ result of (b) are less conservative than (a).

2

$\dot{x}_1 = -x_1 + x_2 - x_3$

$\dot{x}_2 = -x_1 x_3 - x_2 + u$

$\dot{x}_3 = -x_1 + u$

$y = x_3$ ∴ taking the deriv of y wrt time:

$\dot{y} = \frac{d}{dt}(y) = \frac{d}{dt}(x_3) = \dot{x}_3 = -x_1 + u$ ∴ explicit presence of u ∴

Z syst has a relative degree of 1 in R^3 $\{ \because \dot{y} = -x_1 + u \}$

\3a/ $\dot{x}_1 = -2x_1 + ax_2 + \sin x_1$
$\dot{x}_2 = -x_2 \cos x_1 + u\cos(2x_1)$ Z nonlinearity cannot be cancelled by control input u. define a new set of variables $z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$ ∴

$z_1 = x_1$
$z_2 = ax_2 + \sin x_1$ ∴

$\dot{z}_1 = \dot{x}_1 = -2\underbrace{x_1}_{z_1} + \underbrace{ax_2 + \sin x_1}_{z_2} = -2z_1 + z_2$,

$\dot{z}_2 = a\dot{x}_2 + \cos(x_1)\dot{x}_1 = a(-x_2\cos x_1 + u\cos(2x_1)) + \cos(x_1)\dot{x}_1 =$

$-ax_2\cos x_1 + au\cos(2x_1) + \cos(x_1)(-2x_1 + ax_2 + \sin x_1) =$

$-ax_2\cos(x_1) + au\cos(2x_1) + ax_2\cos x_1 - 2x_1\cos x_1 + \cos x_1 \sin x_1 =$

$-2x_1\cos x_1 + \cos x_1 \sin x_1 + au\cos(2x_1) =$

$\dot{z}_2 = -2z_1\cos z_1 + \cos z_1 \sin z_1 + ua\cos(2z_1)$ ∴
$\dot{z}_1 = -2z_1 + z_2$
$\dot{z}_2 = -2z_1\cos z_1 + \cos z_1 \sin z_1 + ua\cos(2z_1)$ ∴

choose $u = \frac{1}{a\cos(2z_1)}(\nu - \cos z_1 \sin z_1 + 2z_1\cos z_1)$ ∴

$\dot{z}_2 = -2z_1\cos z_1 + \cos z_1 \sin z_1 + \frac{1}{a\cos(2z_1)}(\nu - \cos z_1 \sin z_1 + 2z_1 \cos z_1)a\cos(2z_1) =$

$-2z_1\cos z_1 + \cos z_1 \sin z_1 + \nu - \cos z_1 \sin z_1 + 2z_1\cos z_1 = \dot{z}_2 = \nu$ ∴

$\dot{z}_1 = -2z_1 + z_2$
$\dot{z}_2 = \nu$ ∴ it is now possible to pick feedback gains k_1 & k_2, in $\nu = -k_1z_1 - k_2z_2$ to place Z poles properly. when states z_1 & z_2 converge to zero original states x_1, x_2 will also converges to zero

\CW my answers/ \1a/ $u=0$ ∴ $\dot{x}_1 = 2x_2 - 2x_1 + x_1 - x_1^3$
$\dot{x}_2 = -2x_2 + 2x_1 + x_2 - x_2^3$ ∴

equilibria when $(\dot{x}_1, \dot{x}_2) = (0,0)$ ∴ let: $\dot{x}_1 = 2x_2 - x_1 - x_1^3 = 0$ ∴ $\frac{1}{2}x_1^3 + \frac{1}{2}x_1 = x_2$ ∴

$\dot{x}_2 = -x_2 + 2x_1 - x_2^3 = 0 = -(\frac{1}{2}x_1^3 + \frac{1}{2}x_1) + 2x_1 - (\frac{1}{2}x_1^3 + \frac{1}{2}x_1)^3$ ∴

if $x_1 = 0$: $-(\frac{1}{2}(0)^3 + \frac{1}{2}(0)) + 2(0) - (\frac{1}{2}(0)^3 + \frac{1}{2}(0))^3 = 0 = \dot{x}_2$ ∴ $\frac{1}{2}(0)^3 + \frac{1}{2}(0) = 0 = x_2$

∴ $\dot{x}_1 = 2(0) - (0) - (0)^3 = 0$ ∴ $(\dot{x}_1, \dot{x}_2) = (0,0)$ when $(x_1 = 0, x_2 = 0) = (0,0)$ ∴ is an equili pt

if $x_1 = 1$: $-(\frac{1}{2}(1)^3 + \frac{1}{2}(1)) + 2(1) - (\frac{1}{2}(1)^3 + \frac{1}{2}(1))^3 = -1 + 2 - 1 = 0 = \dot{x}_2$ ∴ $\frac{1}{2}(1)^3 + \frac{1}{2}(1) = x_2 = 1$

∴ $\dot{x}_1 = 2(1) - 1 - (1)^3 = 2 - 1 - 1 = 0$ ∴ $(\dot{x}_1, \dot{x}_2) = (0,0)$ when $(x_1 = 1, x_2 = 1) = (1,1)$ ∴ is equili pt

if $x_1 = -1$: $\frac{1}{2}(-1)^3 + \frac{1}{2}(-1) = x_2 = -1$ ∴ $2(-1) - (-1) - (-1)^3 = \dot{x}_1 = 0$,

$-(\frac{1}{2}(-1)^3 + \frac{1}{2}(-1)) + 2(-1) - (\frac{1}{2}(-1)^3 + \frac{1}{2}(-1))^3 = 1 - 2 + 1 = \dot{x}_2 = 0$ ∴ $(x_1 - 1, x_2 = -1) = (-1,-1)$ is an equili pt ∴

Z equili are $(0,0)$, $(1,1)$, $(-1,-1)$

\CW my answers/\1b/ $x=\begin{bmatrix}x_1\\x_2\end{bmatrix}$ $f=\begin{bmatrix}f_1\\f_2\end{bmatrix}=\dot{x}=\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}=\begin{bmatrix}2x_2-x_1-x_1^3\\-x_2+2x_1-x_2^3\end{bmatrix}$ ∴

$f_1=2x_2-x_1-x_1^3$, $f_2=-x_2+2x_1-x_2^3$ ∴

$\frac{\partial f_1}{\partial x_1}=-1-3x_1^2$, $\frac{\partial f_2}{\partial x_1}=2$, $\frac{\partial f_1}{\partial x_2}=2$, $\frac{\partial f_2}{\partial x_2}=-1-3x_2^2$ ∴

$\frac{\partial f}{\partial x}=\frac{\partial}{\partial x}\begin{bmatrix}f_1\\f_2\end{bmatrix}=\begin{bmatrix}\partial f_1/\partial x_1 & \partial f_1/\partial x_2\\\partial f_2/\partial x_1 & \partial f_2/\partial x_2\end{bmatrix}=\begin{bmatrix}-1-3x_1^2 & 2\\2 & -1-3x_2^2\end{bmatrix}$ ∴

for equilibria (0,0): $\frac{\partial f}{\partial x}\Big|_{(0,0)}=\begin{bmatrix}-1-3(0)^2 & 2\\2 & -1-3(0)^2\end{bmatrix}=\begin{bmatrix}-1 & 2\\2 & -1\end{bmatrix}$ ∴

$\frac{\partial f}{\partial x}\Big|_{(0,0)}-\lambda I=\begin{bmatrix}-1-\lambda & 2\\2 & -1-\lambda\end{bmatrix}$ ∴ $\det\left(\frac{\partial f}{\partial x}\Big|_{(0,0)}-\lambda I\right)=\begin{vmatrix}-1-\lambda & 2\\2 & -1-\lambda\end{vmatrix}=(-1-\lambda)(-1-\lambda)-2(2)=$

$1+\lambda^2+2\lambda-4=\lambda^2+2\lambda-3=(\lambda+3)(\lambda-1)=0$ ∴ $\lambda_1=-3$, $\lambda_2=1$ ∴ $\lambda_1\in\mathbb{R}_{<0}$, $\lambda_2\in\mathbb{R}_{>0}$ ∴

Z equili (0,0) is a saddle point

for equili (1,1): $\frac{\partial f}{\partial x}\Big|_{(1,1)}=\begin{bmatrix}-1-3x_1^2 & 2\\2 & -1-3x_2^2\end{bmatrix}\Bigg|_{(1,1)}=\begin{bmatrix}-1-3(1)^2 & 2\\2 & -1-3(1)^2\end{bmatrix}=\begin{bmatrix}-4 & 2\\2 & -4\end{bmatrix}$

∴ for its eigenvalue: $\det\left(\frac{\partial f}{\partial x}\Big|_{(1,1)}-\lambda I\right)=\begin{vmatrix}-4-\lambda & 2\\2 & -4-\lambda\end{vmatrix}=(-4-\lambda)(-4-\lambda)-2(2)=$

$16+\lambda^2-8\lambda-4=\lambda^2+8\lambda+12=(\lambda+6)(\lambda+2)=0$ ∴ $\lambda_1=-6$, $\lambda_2=-2$ ∴ $\lambda_1,\lambda_2\in\mathbb{R}_{<0}$ ∴

Z equili (1,1) is a stable node

for equili (-1,-1): $\frac{\partial f}{\partial x}\Big|_{(-1,-1)}=\begin{bmatrix}-1-3x_1^2 & 2\\2 & -1-3x_2^2\end{bmatrix}\Bigg|_{(-1,-1)}=\begin{bmatrix}-1-3(-1)^2 & 2\\2 & -1-3(-1)^2\end{bmatrix}=\begin{bmatrix}-4 & 2\\2 & -4\end{bmatrix}$ ∴

same as before : for its eigenvalue: $\det\left(\frac{\partial f}{\partial x}\Big|_{(-1,-1)}-\lambda I\right)=\begin{vmatrix}-4-\lambda & 2\\2 & -4-\lambda\end{vmatrix}=$

$(\lambda+6)(\lambda+2)=0$ ∴ $\lambda_1=-6$, $\lambda_2=-2$ ∴ $\lambda_1,\lambda_2\in\mathbb{R}_{<0}$ ∴ Z equili (-1,-1) is a stable node

\1c/ Z equilibria (0,0) has eigenvals: $\lambda_1=-3$, $\lambda_2=1$

Z equilibria (1,1) has eigenvals $\lambda_1=-6$, $\lambda_2=-2$

Z equilibria (-1,-1) has eigenval $\lambda_1=-6$, $\lambda_2=-2$ ∴ all Z equili pts have at least one eigenval that is nega ∴ they all have at least one ~~equili~~ eigenval that is not posi ∴ none of Z equili are unstable focus or unstable nodes. ∴ By Z Poincare-Bendixson criterion; Z criterion for Z existence of a limit cycle is not fulfilled ∴

Z system does not have limit cycles.

\1d/ $x=\begin{bmatrix}x_1\\x_2\end{bmatrix}$ ∴ $\dot{x}=\frac{d}{dt}x=\frac{d}{dt}\begin{bmatrix}x_1\\x_2\end{bmatrix}=\begin{bmatrix}\dot{x}_1\\\dot{x}_2\end{bmatrix}=Ax+Bu=\begin{bmatrix}2(x_2-x_1)+x_1(1-x_1^2)\\-2(x_2-x_1)+x_2(1-x_2^2)+u\end{bmatrix}=$

$\begin{bmatrix}2(x_2-x_1)+x_1(1-x_1^2)\\-2(x_2-x_1)+x_2(1-x_2^2)\end{bmatrix}+\begin{bmatrix}0\\u\end{bmatrix}=\begin{bmatrix}2x_2-x_1-x_1^3\\-x_2+2x_1-x_2^3\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}u=\begin{bmatrix}(-1-x_1^2)x_2+2(x_2)\\(2)x_1+(-1-x_2^2)x_2\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}u=$

$\begin{bmatrix}-1-x_1^2 & 2\\2 & -1-x_2^2\end{bmatrix}\begin{bmatrix}x_1\\x_2\end{bmatrix}+\begin{bmatrix}0\\1\end{bmatrix}u=A\begin{bmatrix}x_1\\x_2\end{bmatrix}+Bu=Ax+Bu$ ∴ $A=\begin{bmatrix}-1-x_1^2 & 2\\2 & -1-x_2^2\end{bmatrix}$, $B=\begin{bmatrix}0\\1\end{bmatrix}$

$\therefore$ about Z origin is $(x_1, x_2) = (0,0)$ $\therefore$ $A = \begin{bmatrix} -1-0^2 & 2 \\ 2 & -1-0^2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ $B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

\1e/ controllability matrix is: $M = [B \vdots AB] = \left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \vdots \begin{bmatrix} -1-x_1^2 & 2 \\ 2 & -1-x_1^2 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right] =$

$\left[\begin{bmatrix} 0 \\ 1 \end{bmatrix} \vdots \begin{bmatrix} 2 \\ -1-x_1^2 \end{bmatrix} \right] = \begin{bmatrix} 0 & 2 \\ 1 & -1-x_1^2 \end{bmatrix}$ $\therefore$ Rank$(M) = 2$ is full rank Matrix. $\therefore$

$\det(M) = \begin{vmatrix} 0 & 2 \\ 1 & -1-x_1^2 \end{vmatrix} = 0(-1-x_1^2) - 1(2) = -2 \neq 0$ $\therefore$ controllable. $\therefore$ Z pair (A, B) is controllable. about Z origin is $(x_1, x_2) = (0,0)$ $\therefore$ $M = \begin{bmatrix} 0 & 2 \\ 1 & -1-0^2 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$ &

Rank$(M) = 2$ $\therefore$ Z pair (A, B) is controllable

\1f/ $K = [k_1 \ k_2]$ eignvals $-5, -8$ $\therefore$ desired charac poly: $(s+5)(s+8) =$

$s^2 + 5s + 8s + 40 = s^2 + 13s + 40$ $\therefore$ $A^+_{cl}\big|_{(0,0)} = A - BK\big|_{(0,0)} = \begin{bmatrix} -1-x_1^2 & 2 \\ 2 & -1-x_2^2 \end{bmatrix}\Bigg|_{(0,0)} - \begin{bmatrix} 0 \\ 1 \end{bmatrix} [k_1 \ k_2]$

$= \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} - \begin{bmatrix} 0 & 0 \\ k_1 & k_2 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 2-k_1 & -1-k_2 \end{bmatrix}$ $\therefore$ for charac poly of closed loop:

$\det(sI - A^+_{cl})\big|_{(0,0)} = \det(sI - (A - BK))\big|_{(0,0)} = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 2-k_1 & -1-k_2 \end{bmatrix} \right) = \det\left(\begin{bmatrix} s+1 & -2 \\ -2+k_1 & s+1+k_2 \end{bmatrix} \right)$

$= (s+1)(s+1+k_2) - (-2)(-2+k_1) = s^2 + s + k_2 s + s + 1 + k_2 - 4 + 2k_1 =$

$s^2 + 2s + k_2 s + (-3 + 2k_1 + k_2) = s^2 + (2+k_2)s + (-3 + 2k_1 + k_2) = s^2 + 13s + 40$ $\therefore$

~~13s~~ $13 = 2 + k_2$, $-3 + 2k_1 + k_2 = 40$ $\therefore$ $11 = k_2$ $\therefore$

$2k_1 + 11 = 43$ $\therefore$ $2k_1 = 32$ $\therefore$ $k_1 = 16$ $\therefore$

$U = -Kx = -[k_1 \ k_2]x = -[16 \ 11]x = [-16 \ -11]x$

\1g/ $T = M \times W$ $M = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix}$ at $(0,0)$ $\therefore$ full rank $\therefore$ $n = 2$ $\therefore$

$W = \begin{bmatrix} a_{n-1} & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & 1 \\ 1 & 0 \end{bmatrix}$ where $\det(sI - A) = s^2 + a_1 s + a_2$ $\therefore$ $A = \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix}$ $\therefore$

$\det(sI - A) = \det\left(\begin{bmatrix} s & 0 \\ 0 & s \end{bmatrix} - \begin{bmatrix} -1 & 2 \\ 2 & -1 \end{bmatrix} \right) = \det\left(\begin{bmatrix} s+1 & -2 \\ -2 & s+1 \end{bmatrix} \right) = (s+1)(s+1) - (-2)(-2) =$

$s^2 + s + s + 1 - 4 = s^2 + 2s - 3 = s^2 + a_1 s + a_2$ $\therefore$ $a_1 = 2$ $\therefore$ $W = \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix}$ $\therefore$

Z transformation Matrix is: $T = M \times W = \begin{bmatrix} 0 & 2 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 1 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 1 & 1 \end{bmatrix}$ about $(0,0)$

\2/ $f = \begin{bmatrix} f_1 \\ f_2 \end{bmatrix} = \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + x_2(2 - 3x_1^2 - 2x_2^2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} x_2 \\ -x_1 + 2x_2 - 3x_1^2 x_2 - 2x_2^3 \end{bmatrix}$ $\therefore$

$x_2 = 0$ $\therefore$ $-x_1 + x_2(2 - 3x_1^2 - 2x_2^2) = 0 = -x_1 + 0(2 - 3(x_1^2) - 2(0)^2) = -x_1 + 0 = 0 = -x_1$ $\therefore$

$x_1 = 0$ $\therefore$ equili pt at $(x_1, x_2) = (0,0)$ is a unique equili $\therefore$ Jacobian:

$\frac{\partial f}{\partial x}\Big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1-6x_1x_2 & 2-3x_1^2-6x_2^2 \end{bmatrix}\Bigg|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1-6(0)(0) & 2-3(0)^2-6(0)^2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 2 \end{bmatrix}$ $\therefore$ eignvals by:

$\begin{vmatrix} 0-\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = \begin{vmatrix} -\lambda & 1 \\ -1 & 2-\lambda \end{vmatrix} = -\lambda(2-\lambda) - 1(-1) = \lambda^2 - 2\lambda + 1 = (\lambda-1)(\lambda-1) = 0$ $\therefore$ eignvals: $\lambda_1 = 1, \lambda_2 = 1$

$\therefore$ $\lambda_1, \lambda_2 > 0$ $\therefore$ equilibria $(x_1, x_2) = (0,0)$ is an unstable node $\therefore$

\CW Myanswers/ let $V = x_1^2 + x_2^2$ ∴ $\nabla V(x) = \nabla V = \nabla V(x_1, x_2) = \nabla(x_1^2 + x_2^2) =$
$\left[\frac{\partial}{\partial x_1}(x_1^2 + x_2^2) \quad \frac{\partial}{\partial x_2}(x_1^2 + x_2^2)\right] = [2x_1 \quad 2x_2]$ ∴ $f = f(x)$ ∴
$f(x) \cdot \nabla V(x) = [x \quad -x_1 + 2x_2 - 3x_1^2 x_2 - 2x_2^3] \cdot [2x_1 \quad 2x_2] =$
$2x_1x_2 + 2x_2(-x_1 + 2x_2 - 3x_1^2 x_2 - 2x_2^3) = 2x_1x_2 - 2x_1x_2 + 4x_2^2 - 6x_1^2x_2^2 - 4x_2^4 =$
$4x_2^2 - 6x_1^2x_2^2 - 4x_2^4 = 2x_2^2(2 - 3x_1^2 - 2x_2^2) = 2x_2^2(2 - x_1^2 - 2x_1^2 - 2x_2^2) =$
$2x_2^2(2 - x_1^2 - 2(x_1^2 + x_2^2)) = 2x_2^2(2 - x_1^2 - 2V) = f \cdot \nabla V$ ∴
$2x_2^2 \geq 0$ ∴ $f(x) \cdot \nabla V(x) \leq 0$ if $2 - x_1^2 - 2V \leq 0$ ∴ $2 - x_1^2 \leq 2V$ ∴ $1 - \frac{1}{2}x_1^2 \leq V$
& $\frac{1}{2}x_1^2 \geq 0$ ∴ $1 - \frac{1}{2}x_1^2 \leq 1$ ∴ $1 - \frac{1}{2}x_1^2 \leq V$ if $V \geq 1$ ∴
if $V \geq 1$: $f(x) \cdot \nabla V(x) \leq 0$ ∴ on a closed bounded set
$M = \{x \in \mathbb{R}^2 \mid V(x) = x_1^2 + x_2^2 \leq 1\}$ which contains one equili, which is unstable, Z poincaré–Bendixson Criterion holds ∴ it is possible to ensure all trajectories are trapped inside M. ∴ it can be concluded ∃ a periodic ~~or~~ orbit in $M = \{x \in \mathbb{R}^2 \mid V(x) = x_1^2 + x_2^2 \leq 1\}$ ∴
Z system has a periodic orbit.

\3/ $f(x) = \frac{1}{x^2} = x^{-2}$ ∴ f is Lipschitz cont on an interval if f is Lipschitz cont ∀ pts in that interval. f is Lipschitz cont ∀ pts in an interval if ∃ a Lipschitz const $L > 0$ & a neighood that is a subset of that interval of that pt s.t. $\|f(x) - f(y)\| \leq L\|x - y\|$ for x, y in that neighbood ∴ without loss of generallity let $x, y \in [\frac{1}{2}, 2]$ ∴ $x, y \in I$ & $x \geq y$ ∴ $f(x) = \frac{1}{x^2}$, $f(y) = \frac{1}{y^2}$
∴ $x, y \geq \frac{1}{2}$ ∴ $x \leq 2x^2$, $y \leq 2y^2$ ∴ $x^2 \leq 2x^3$, $y^2 \leq 2y^3$ ∴
$x^2 \leq 4x^2y^2$, $y^2 \leq 4x^2y^2$ ∴ $x^2 \leq 8x^3y^2$, $y^2 \leq 8x^2y^3$ ∴
if $L_f = 64$: $L_f(x - y) = 64(x - y)$. $f(x) - f(y) = \frac{1}{x^2} - \frac{1}{y^2} = \left(\frac{1}{x^2} - \frac{1}{y^2}\right)\frac{x^2y^2}{x^2y^2} =$
$\frac{y^2}{x^2y^2} - \frac{x^2}{x^2y^2} = -(x^2 - y^2)\frac{1}{x^2y^2}$ ∴ $x^2 - y^2 \leq 8x^3y^2 - 8x^2y^3 = 8(x^3y^2 - x^2y^3) \leq 64(x^3y^2 - x^2y^3)$
∴ i' $x^2 > 0$, $y^2 > 0$ ∵ $x, y \geq \frac{1}{2}$ ∴ $x^2y^2 > 0$ ∴
$(x^2 - y^2)\frac{1}{x^2y^2} \leq 64(x^3y^2 - x^2y^3)\frac{1}{x^2y^2}$ ∴ $\left(\frac{x^2}{x^2y^2} - \frac{y^2}{x^2y^2}\right) = \left(\frac{1}{y^2} - \frac{1}{x^2}\right) = -\left(\frac{1}{x^2} - \frac{1}{y^2}\right) \leq$
$64(x - y)$ ∴ $|-(\frac{1}{x^2} - \frac{1}{y^2})| = |\frac{1}{x^2} - \frac{1}{y^2}| = |f(x) - f(y)| \leq |64(x - y)| = 64|x - y|$
∴ $|f(x) - f(y)| \leq 64|x - y| = L_f|x - y|$ ∴ $f(x)$ is Lipschitz cont on Z interval $I = [\frac{1}{2}, 2]$ with Lipschitz const $L_f = 64$ ∴ Z $|f(x) - f(y)| \leq 64|x - y| \leq 128|x - y| = L_f|x - y|$ ∴ $f(x)$ is Lipschitz cont on Z interval $I = [\frac{1}{2}, 2]$ with Lipschitz

Corresponding $L_g = 128$ $\because$ $128 > 64$.

\4/ $V(0,0) = 0$ & $V(x_1, x_2) = x_1^2 + x_2^2 > 0$ $\forall (x_1, x_2) \in \mathbb{R}^2 \setminus \{0,0\}$ $\therefore$

V is posi defi $\therefore$ taking Z deriv of V along Z dynamics of Z

Syst: $\dot{V} = \dot{V}(x_1, x_2) = \frac{d}{dt} V(x_1, x_2) = \frac{d}{dt}[x_1^2 + x_2^2] = 2x_1 \frac{d}{dt}(x_1) + 2x_2 \frac{d}{dt}(x_2) =$

$2x_1 \dot{x}_1 + 2x_2 \dot{x}_2 = 2x_1(-3x_1 + x_1 x_2) + 2x_2(x_1 - 2x_2) = -6x_1^2 + 2x_1^2 x_2 + 2x_1 x_2 - 4x_2^2 =$

$(-5x_1^2 - x_1^2) + 2x_1^2 x_2 + (-3x_2^2 - x_2^2 + 2x_1 x_2 = (-5x_1^2 + 2x_1^2 x_2 - 3x_2^2) + (-x_1^2 + 2x_1 x_2 - x_2^2) =$

$(-5x_1^2 + 2x_1^2 x_2 - 3x_2^2) - (x_1^2 - 2x_1 x_2 + x_2^2) = (-5x_1^2 + 2x_1^2 x_2 - 3x_2^2) - (x_1 - x_2)^2$

& $(x_1 - x_2)^2 \geq 0$ $\therefore (x_1 - x_2)^2$ is lower bounded by zero $\therefore$

$\dot{V} = (-5x_1^2 + 2x_1^2 x_2 - 3x_2^2 - (x_1 - x_2)^2 \leq -5x_1^2 + 2x_1^2 x_2 - 3x_2^2 = -x_1^2(5 - 2x_2) - 3x_2^2$ $\therefore$

$\dot{V} \leq -x_1^2(5 - 2x_2) - 3x_2^2$ $\therefore$ & $x_2^2, x_1^2 \in \mathbb{R}_{\geq 0}$ $\therefore$ $-3x_2^2 \leq 0$, $-x_1^2 \leq 0$ $\therefore$

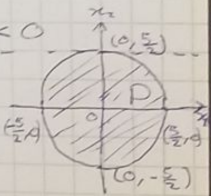
if $5 - 2x_2 > 0$: $\dot{V} < 0$ $\therefore$ $5 > 2x_2$ $\therefore$ $\frac{5}{2} > x_2$ $\therefore$ $(\frac{5}{2})^2 > (x_2)^2$ $\therefore$ $\frac{25}{4} > x_2^2$

$\therefore \frac{25}{4} > x_1^2 + x_2^2$ $\therefore$ for $x_1^2 + x_2^2 < \frac{25}{4}$: $\dot{V} < 0$ $\therefore$ Z origin is locally asymp

stable with invariant set $D = \{x \in \mathbb{R}^2 \mid V(x) < \frac{25}{4}\}$ in which V is posi defi

$\because V(0,0) = 0^2 + 0^2 = 0$ & $x_1^2 \geq 0$, $x_2^2 \geq 0$ $\therefore$ for $x_1, x_2 \neq 0$: $V(x_1, x_2) = x_1^2 + x_2^2 > 0$

& $\dot{V}$ is negat defi $\because$ $\dot{V}(0,0)$ & $\dot{V} \leq 0$ & for $x_1, x_2 \neq 0$: $\dot{V} < 0$



\5a/ $M, g, k, c_1, c_2 \in \mathbb{R}_{>0}$ $M\ddot{y} = Mg - c_1 \dot{y} - c_2 \dot{y}|\dot{y}| - ky$ $\therefore$

$\ddot{y} = M^{-1}Mg - M^{-1}c_1 \dot{y} - M^{-1} c_2 \dot{y}|\dot{y}| - M^{-1}ky$ $\therefore$

$\ddot{y} = g - M^{-1}c_1 \dot{y} - M^{-1}c_2 \dot{y}|\dot{y}| - M^{-1}ky$ $\therefore$ $x_1 = y - Mgk^{-1}$, $x_2 = \dot{y}$ $\therefore$

$\dot{x}_1 = \frac{d}{dt} x_1 = \frac{d}{dt}(y - Mgk^{-1}) = \frac{d}{dt} y = \dot{y} = x_2$

$\dot{x}_2 = \frac{d}{dt}(x_2) = \frac{d}{dt}(\dot{y}) = \ddot{y} = g - M^{-1}c_1 \dot{y} - M^{-1}c_2 \dot{y}|\dot{y}| - M^{-1}ky$ & $(x_1 + Mgk^{-1}) = y$ $\therefore$

$\dot{x}_2 = g - M^{-1}c_1 x_2 - M^{-1}c_2 x_2|x_2| - M^{-1}k(x_1 + Mgk^{-1}) =$

$g - M^{-1}c_1 x_2 - M^{-1}c_2 x_2|x_2| - M^{-1}kx_1 - M^{-1}kMgk^{-1} = \underset{\sim}{g} - M^{-1}c_1 x_2 - M^{-1}c_2 x_2|x_2| - M^{-1}kx_1 - \underset{\sim}{g}$

$= -M^{-1}c_1 x_2 - M^{-1}c_2 x_2|x_2| - M^{-1}kx_1 = \dot{x}_2$ $\therefore$

$\dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} = \begin{bmatrix} x_2 \\ -M^{-1}c_1 x_2 - M^{-1}c_2 x_2|x_2| - M^{-1}kx_1 \end{bmatrix}$ $\therefore$ Compact state space: $x = [y \quad \dot{y}] \in \mathbb{R}^2$, $\downarrow$ $x = [x_1 \quad x_2]$

\5b/ Z origin of $\dot{x} = f(x)$ is globally asymp stable if it is Lyapunov stable & $\forall x(0) \in \mathbb{R}^n$, $\lim_{t \to \infty} x(t) = 0$. Z origin of $\dot{x} = f(x)$ is Lyapunov stable, if

$\forall \varepsilon > 0$, $\exists \delta = \delta(\varepsilon) > 0$ st if $\|x(0)\| < \delta$ $\therefore$ $\|x(t)\| < \varepsilon$, $\forall t \geq 0$ $\therefore$ origin is

$(x_1, x_2) = (0,0)$. $\dot{x}|_{(0,0)} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix}\Big|_{(0,0)} = \begin{bmatrix} x \\ -M^{-1}c_1 x_2 - M^{-1}c_2 x_2|x_2| - M^{-1}kx_1 \end{bmatrix}\Big|_{(0,0)} =$

$\begin{bmatrix} 0 \\ -M^{-1}c_1(0) - M^{-1}c_2(0)|0| - M^{-1}k(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\therefore$

CW My answers/ $\dot{x}_1 = 0, \dot{x}_2 = 0$ at Z origin $\therefore (x_1, x_2)$ is an equili pt

for $(x_1, x_2) = (0,0)$. $x_1^2 \geq 0$ & $x_2^2 \geq 0 \therefore V(x_1, x_2) = a x_1^2 + b x_2^2 \geq 0$ for $a, b > 0$

$V(x_1 = 0, x_2 = 0) = a(0)^2 + b(0)^2 = 0 \therefore V(x_1, x_2) > 0, \forall (x_1, x_2) \in \mathbb{R}^2 \setminus \{(0,0)\}$

$\dot{V}(x_1, x_2) = \frac{d}{dt} V(x_1, x_2) = \frac{d}{dt}(a x_1^2 + b x_2^2) = a \frac{d}{dt}(x_1^2) + b \frac{d}{dt}(x_2^2) =$

$2 a x_1 \frac{d}{dt}(x_1) + 2 b x_2 \frac{d}{dt}(x_2) = 2 a x_1 \dot{x}_1 + 2 b x_2 \dot{x}_2 =$

$2 a x_1 x_2 + 2 b x_2 (-M^{-1} c_1 x_2 - M^{-1} c_2 x_2 |x_2| - M^{-1} k x_1) =$

$2 a x_1 x_2 - 2M^{-1} c_1 b x_2^2 - 2M^{-1} c_2 b x_2^2 |x_2| - 2M^{-1} k b x_1 x_2 =$

$(a - M^{-1} k b) 2 x_1 x_2 - 2M^{-1}(c_1 b x_2^2 + c_2 b x_2^2 |x_2|) = (a - M^{-1} k b) 2 x_1 x_2 - 2M^{-1} x_2^2 (c_1 b + c_2 b |x_2|)$

$= (a - M^{-1} k b) 2 x_1 x_2 - 2M^{-1} b x_2^2 (c_1 + c_2 |x_2|) = \dot{V}(x_1, x_2)$ $\therefore$ let $a = M^{-1} k b \therefore \dot{V}(x_1, x_2) =$

$-2M^{-1} b x_2^2 (c_1 + c_2 |x_2|) \because x_2^2 \geq 0, |x_2| \geq 0, M > 0 \therefore M^{-1} > 0, b > 0 \therefore$

$-2M^{-1} b x_2^2 (c_1 + c_2 |x_2|) \leq 0 \therefore \dot{V}(x_1, x_2) \leq 0 \therefore$ for $a, b > 0$ & $a = M^{-1} k b$;

Z origin $(x_1, x_2) = (0,0)$ is a globally asymp stable equili pt.

$\because \dot{V}(x_1, 0) = 0$ & $\dot{V} < 0$ for $x_2 \neq 0 \therefore$ ~~globally~~ negen desi but only can be said to be globally asymp stable & not globally stable $\because$ $\dot{V}$ is indep of x_1.

Week 1 Sheet/ $y^{(n)} = \frac{d^n y}{dt^n} \therefore y^{(1)} = g(y, u)$, $y^{(2)} = g(y, \dot{y}, u)$, $y^{(3)} = g(y, \dot{y}, \ddot{y}, u)$

$y^{(4)} = g(y, \dot{y}, \ddot{y}, y^{(3)}, u) \therefore \begin{bmatrix} y^{(1)} \\ y^{(2)} \\ y^{(3)} \\ \vdots \\ y^{(n)} \end{bmatrix} = \begin{bmatrix} g(y,u) \\ g(y, \dot{y}, u) \\ g(y, \dot{y}, \ddot{y}, u) \\ g(y, \dot{y}, \dots, y^{(n-1)}, u) \end{bmatrix}$ let $x_1 = y$, ~~x_2~~ $x_2 = \frac{dy}{dt} \cdots x_n = \frac{d^{n-1} y}{dt^{n-1}}$

$\therefore$ taking derivs of Z new variables: $\dot{x}_1 = \dot{y}$, $\dot{x}_2 = \ddot{y} \cdots \dot{x}_{n-1} = \frac{d^{n-1} y}{dt^{n-1}}$,

$\dot{x}_n = \frac{d}{dt}\left(\frac{d^{n-1} y}{dt^{n-1}}\right) = y^{(n)} = g(y, \dot{y}, \dots, y^{(n-1)}, u) \therefore \dot{x} = \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_{n-1} \\ \dot{x}_n \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ g(y, \dot{y}, \dots, y^{(n-1)}, u) \end{bmatrix} = \begin{bmatrix} x_2 \\ x_3 \\ \vdots \\ x_n \\ g(x_1, x_2, \dots, x_{n-1}, u) \end{bmatrix}$

$\therefore x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \therefore y = Cx \quad C = [1 \; 0 \; \cdots \; 0]_{1 \times n} \quad \therefore y = x_1$

2/ let $x_1 = q_1$, $x_3 = q_2 \therefore \dot{q}_1 = \frac{d}{dt} q_1 = \frac{d}{dt} x_1 = \dot{x}_1 = x_2$, $\dot{q}_2 = \frac{d}{dt} q_2 = \frac{d}{dt} x_3 = \dot{x}_3 = x_4 \therefore$

$\dot{x}_1 = \frac{d}{dt} x_1 = \frac{d}{dt} q_1 = \dot{q}_1 = x_2 \therefore \dot{x}_3 = \dot{q}_2 = x_4 \therefore \ddot{q}_2 = J^{-1} u + J^{-1} k (q_1 - q_2)$,

$\ddot{q}_1 = -I^{-1} MgL \sin q_1 - I^{-1} k (q_1 - q_2) \therefore \dot{x}_2 = \frac{d}{dt} x_2 = \frac{d}{dt} \dot{q}_1 = \ddot{q}_1 = -I^{-1} MgL \sin q_1 - I^{-1} k (q_1 - q_2)$

$\therefore \dot{x}_4 = \frac{d}{dt} x_4 = \frac{d}{dt} \dot{q}_2 = \ddot{q}_2 = J^{-1} u + J^{-1} k (q_1 - q_2) \therefore x = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} q_1 \\ \dot{q}_1 \\ q_2 \\ \dot{q}_2 \end{bmatrix} \therefore x = [q_1, \dot{q}_1, q_2, \dot{q}_2] \in \mathbb{R}^4$

3/ let $x_1 = y \therefore \dot{y} = \frac{d}{dt} y = \frac{d}{dt} x_1 = \dot{x}_1 = x_2 \therefore \ddot{y} = \frac{d}{dt} \dot{y} = \frac{d}{dt} x_2 = \dot{x}_2 = -\frac{k}{m} x_1 - \frac{c_1}{m} x_2 - \frac{c_2}{m} x_2 |x_2| + g$

$\therefore$ states $[y \;\; \dot{y}] \in \mathbb{R} \therefore x = [x_1 \;\; x_2]$

$x = [y \;\; \dot{y}] \in \mathbb{R}^2$

$x = [x_1 \;\; x_2]$

4a/ $x_1 = r(t)$, $x_2 = \dot{r}(t)$, $x_3 = \theta(t)$, $x_4 = \dot{\theta}(t)$ $\therefore$ $\dot{x}_1 = \dot{r}(t) = x_2$,

$\dot{x}_2 = \ddot{r}(t) = r(t)\dot{\theta}^2(t) - \gamma r^{-2}(t) + u_1(t)$ $\therefore \dot{x}(t) = F(x,u)$

$\dot{x}_3 = \dot{\theta}(t) = x_4$ $\dot{x}_4 = \ddot{\theta}(t) = \frac{1}{r(t)}[-2\dot{r}(t)\dot{\theta}(t) + u_2(t)]$ $\therefore x = [x_1, x_2, x_3, x_4]^T =$

$[r(t)\ \dot{r}(t)\ \theta(t)\ \dot{\theta}(t)]$ for $r(t)$ $\dot{\theta}(t)$, $\dot{r}(t)$ etc $F = [f_1(\cdot)\ f_2(\cdot), f_3, f_4]^T$

$\dot{x}_1 = x_2$ $\dot{x}_2 = x_1 x_4^2 - \gamma x_1^{-2} + u_1$ $\dot{x}_3 = x_4$ $\therefore \dot{x}_4 = -2x_2 x_4 x_1^{-1} + u_2 x_1^{-1}$ is compact

State space form $\dot{x}(t) = F(x,u)$ where $x = [x_1\ x_2\ x_3\ x_4]^T = [r(t)\ \dot{r}(t)\ \theta(t)\ \dot{\theta}(t)]^T$

for $F = [f_1\ f_2\ f_3\ f_4]^T$ $\therefore u = [u_1\ u_2]^T$

4b/ $\tilde{x}(t) = [1\ 0\ \sqrt{\gamma}t\ \sqrt{\gamma}]^T$ $\tilde{u}(t) = (0,0)^T$ 2 linearised eqn about this trajectory is $\dot{x} = Ax + Bu$ where

$$A = \begin{bmatrix} \partial\dot{x}_1/\partial x_1 & \partial\dot{x}_1/\partial x_2 & \partial\dot{x}_1/\partial x_3 & \partial\dot{x}_1/\partial x_4 \\ \partial\dot{x}_2/\partial x_1 & \partial\dot{x}_2/\partial x_2 & \partial\dot{x}_2/\partial x_3 & \partial\dot{x}_2/\partial x_4 \\ \partial\dot{x}_3/\partial x_1 & \partial\dot{x}_3/\partial x_2 & \partial\dot{x}_3/\partial x_3 & \partial\dot{x}_3/\partial x_4 \\ \partial\dot{x}_4/\partial x_1 & \partial\dot{x}_4/\partial x_2 & \partial\dot{x}_4/\partial x_3 & \partial\dot{x}_4/\partial x_4 \end{bmatrix}\Bigg|_{(\tilde{x}(t),\tilde{u}(t))} =$$

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ x_4^2 + 2\gamma x_1^{-3} & 0 & 0 & 2x_1x_4 \\ 0 & 0 & 0 & 1 \\ -2x_2x_4x_1^{-2} - u_2x_1^{-2} & -2x_4x_1^{-1} & 0 & -2x_2x_1^{-1} \end{bmatrix}\Bigg|_{(\tilde{x}(t),\tilde{u}(t))} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 3\gamma & 0 & 0 & 2\sqrt{\gamma} \\ 0 & 0 & 0 & 1 \\ 0 & -2\sqrt{\gamma} & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} (f_1)_{u_1} & (f_1)_{u_2} \\ (f_2)_{u_1} & (f_2)_{u_2} \\ (f_3)_{u_1} & (f_3)_{u_2} \\ (f_4)_{u_1} & (f_4)_{u_2} \end{bmatrix}\Bigg|_{(\tilde{x}(t),\tilde{u}(t))} =$$

$$\begin{bmatrix} (\dot{x}_1)_{u_1} & (\dot{x}_1)_{u_2} \\ (\dot{x}_2)_{u_1} & (\dot{x}_2)_{u_2} \\ (\dot{x}_3)_{u_1} & (\dot{x}_3)_{u_2} \\ (\dot{x}_4)_{u_1} & (\dot{x}_4)_{u_2} \end{bmatrix}\Bigg|_{(\tilde{x}(t),\tilde{u}(t))} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & x_1^{-1} \end{bmatrix}\Bigg|_{(\tilde{x}(t),\tilde{u}(t))} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \therefore \dot{x} = Ax + Bu$$

4c/ $y(t) = [r(t)\ \theta(t)]^T$ $\therefore y = Cx$ $\therefore$ $y \in \mathbb{R}^2$ & $x \in \mathbb{R}^4$ $\therefore$ C mat $C = \mathbb{R}^{2\times 4}$ $\therefore$

$x = [r(t)\ \dot{r}(t)\ \theta(t)\ \dot{\theta}(t)]^T$ $\therefore$ $\begin{bmatrix} r(t) \\ \theta(t) \end{bmatrix} = C\begin{bmatrix} r(t) \\ \dot{r}(t) \\ \theta(t) \\ \dot{\theta}(t) \end{bmatrix}$ $\therefore$ $C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$

5a/ let $x_1 = x$ $\therefore$ $x_2 = \dot{x}$ $\therefore$ $\dot{x} = \frac{d}{dt}x = \frac{d}{dt}x_1 = \dot{x}_1 = x_2$ $\therefore$ $\dot{x}_2 = \frac{d}{dt}x_2 = \frac{d}{dt}\dot{x} = \ddot{x} = -\frac{k_1}{m}x - \frac{k_2}{m}x^3$ $= -\frac{k_1}{m}x_1 - \frac{k_2}{m}x_1^3$ is in compact form with states $x = [x_1, x_2]$

5b/ for equilib pts solve $x_2 = 0$ $\therefore$ $\frac{dx_2}{dt} = \frac{d}{dt}(0) = 0 = \dot{x}_2$ $\therefore$

$-\frac{k_1}{m}x_1 - \frac{k_2}{m}x_1^3 = 0 = x_1\left(-\frac{k_1}{m} - \frac{k_2}{m}x_1^2\right)$ $\therefore$ $x_1 = 0$, $-\frac{k_1}{m} - \frac{k_2}{m}x_1^2 = 0$ $\therefore$ $-\frac{k_1}{m} = \frac{k_2}{m}x_1^2$ $\therefore$ $-\frac{k_1}{k_2} = x_1^2$ $\therefore$

$x_1 = +\sqrt{-\frac{k_1}{k_2}}$, $x_2 = -\sqrt{-\frac{k_1}{k_2}}$ $\therefore$ $k_1 = -1$, $k_2 = \frac{1}{2}$ $\therefore$ $k_1 = \sqrt{-\frac{-1}{1/2}} = \sqrt{2}$ $\therefore$ $x_1 = \sqrt{-\frac{1}{1/2}} = -\sqrt{2}$, $x_2 = 0$ $\therefore$

$(0,0)$, $(\sqrt{2},0)$, $(-\sqrt{2},0)$

5c/ for linearised plant, determine Jacobian $A = \begin{bmatrix} \partial f_1/\partial x_1 & \partial f_1/\partial x_2 \\ \partial f_2/\partial x_1 & \partial f_2/\partial x_2 \end{bmatrix}$

$f_1 = x_2$, $f_2 = \dot{x}_2 = -\frac{k_1}{m}x_1 - \frac{k_2}{m}x_1^3$ $\therefore$ $A = \begin{bmatrix} 0 & 1 \\ -k_1 - 3k_2x_1^2 & 0 \end{bmatrix}$ $\therefore$ evaluating A about equilibs:

at $(0,0)$: $A_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -k_1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ $\dot{X} = A_{(0,0)}X$, at $(\pm\sqrt{2},0)$: $A_{(\pm\sqrt{2},0)} = \begin{bmatrix} 0 & 1 \\ -k_1 - 3k_2(2) & 0 \end{bmatrix} =$ $\begin{bmatrix} 0 & 1 \\ -2 & 0 \end{bmatrix}$ $\therefore \dot{x} = A_{(\pm\sqrt{2},0)}X$

Week 2/ $\dot{x}_1 = -x_1 + x_2$, $\dot{x}_2 = -0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3$ $\therefore$ $0 = -x_1 + x_2$ $\therefore$ $x_1 = x_2$ $\therefore$ $0 = -0.1x_1 - 2x_2 - x_1^2 - 0.1x_1^3$

$= -0.1x_1^3 - x_1^2 - 2.1x_1$ $\therefore$ $x(0.1x_1^2 - x_1 - 2.1) = 0$ 3 equili pts are $(0,0)^T$, $(-2.76, -2.76)^T$, $(-7.23, -7.23)^T$

\ Week2 Sheet / \ 3a sol / $\dot{x}_1 = x_2 \quad \dot{x}_2 = -x_1 - 2\tan^{-1}(x_1 + x_2)$

To determine equili (∵ $\dot{x} = f(x)$ form) set $\dot{x} = 0$ ie

$x_2 = 0 \quad -x_1 - 2\tan^{-1}(x_1 + x_2) = 0 \Rightarrow x_1 = 2\tan^{-1}(x_1) \quad \therefore \tan\left(\frac{x_1}{2}\right) = x_1 \Longrightarrow$

$x_1 \approx \pm 2.33$ So System has 3 equili pts

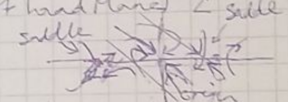
they are at $(x_1^*, x_2^*)^T$ respectively $(0,0)^T \ (2.33, 0)^T, (-2.33, 0)^T$

Jacobian Matrix $\frac{\partial f}{\partial x} = \begin{bmatrix} 0 & 1 \\ 1 - 2/(1+(x_1+x_2)^2) & -2/(1+(x_1+x_2)^2) \end{bmatrix}$ $\quad \frac{\partial f}{\partial x}\Big|_{(0,0)^T} = \begin{bmatrix} 0 & +1 \\ -1 & -2 \end{bmatrix} \Rightarrow$

$\lambda_{1,2} = -1, -1$ multiple eigvals at -1 leading to Stable Star

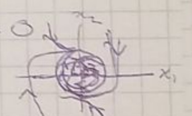
$\frac{\partial f}{\partial x}\Big|_{(2.33, 0)^T} = \begin{bmatrix} 0 & 1 \\ 0.6892 & -0.3108 \end{bmatrix} \Rightarrow \lambda_{1,2} = 0.6892, -1$ this means

$(2.33, 0)^T$ is a Saddle as one eigval on LHP (Left hand Plane) & other RHP (right hand plane)



similarly $(-2.33, 0)^T$ is also a Saddle ∴ in Z given diagram

\ 3b sol / $0 = x_2 \quad 0 = -x_1 + x_2(1 - 3x_1^2 - 2x_2^2) \Rightarrow x_2 = 0$ & $x_1 = 0$



there is an unique equili pt at Z origin

$\frac{\partial f}{\partial x}\Big|_{(0,0)} = \begin{bmatrix} 0 & 1 \\ -1 & 1 \end{bmatrix} \Rightarrow \lambda_{1,2} = \frac{1}{2} \pm i\sqrt{3}/2 \Rightarrow (0,0)^T$ is an unstable focus

there is a periodic orbit around Z origin from Z diagram

(periodic orbit & existance

\ 4 sol / $\dot{x}_1 = -x_1^2 + x_1 + \sin x_2 \leftarrow f_1 \quad \dot{x}_2 = \cos x_2 - x_1^3 - 5x_2 \leftarrow f_2 \ \therefore$

at equili $x_0 = (1, 0)^T$ Jacobian. $\frac{\partial f}{\partial x} = \begin{bmatrix} -2x_1 + 1 & \cos x_2 \\ -3x_1^2 & -\sin x_2 - 5 \end{bmatrix}\Bigg|_{(1,0)} = \begin{bmatrix} -1 & 1 \\ -3 & -5 \end{bmatrix}$

eigvals are -2 & -4 ∵ both eigval are real negu

$(1, 0)^T$ is a stable node

\ 5a sol / False ∵ from given diagram it can be seen Q_2 is Saddle

\ 5b sol / True

\ 5c sol / False ∵ $\dot{x}_1 = x_2 \quad \dot{x}_2 = -h(x_1, x_2)\sin x_2^t - 5x_1 x_2$ has explicit appearence of 't' on RHS of dynamics. this is not autonomous System. non-autonomous - time varying system

\ 5d sol / equili subspace corres to when one or both eigvals are zero. $\pm i\beta$ corres to center

Week 3 sheet

1 Sol $\ddot{y}+y=\varepsilon\dot{y}(1-y^2-\dot{y}^2)$ let $x_1=y$ $x_2=\dot{y}$ $\therefore \left.\begin{array}{l}\dot{x}_1=x_2\\ \dot{x}_2=-x_1+\varepsilon x_2(1-x_1^2-x_2^2)\end{array}\right\}$

of $\dot{x}=f(x)$ $x\in\mathbb{R}^2$ form

choose $V(x)=x_1^2+x_2^2=C$ diffable let us evaluate now on $V(x)=C$

$$f(x)\cdot\nabla V(x)=\overbrace{2x_1}^{\partial V/\partial x_1}\overbrace{x_2}^{f_1(\cdot)}+\overbrace{2x_2}^{\partial V/\partial x_2}\underbrace{(-x_1+\varepsilon x_2(1-x_1^2-x_2^2))}_{f_2(\cdot)}$$

$=2\varepsilon x_2^2(1-x_1^2-x_2^2)=2\varepsilon x_2^2(1-V)$ $\therefore$

$f(x)\cdot\nabla V(x)\le 0$ for $V(x)\ge 1$ in particular, all trajectories starting in $M=\{x\in\mathbb{R}^2 \mid V(x)\le 1\}$ (a closed bounded set) stay in M $\forall$ future times

Further, M contains only one equili at origin. linearisation about origin yields. Matrix $\begin{bmatrix}0 & 1\\ -1 & \varepsilon\end{bmatrix}$ & origin is unstable node or focus for $\varepsilon>0$. By PB (poincare Bendixson) criterion, there orbit in M

2 Sol $\left.\begin{array}{l}\dot{x}_1=x_2\\ \dot{x}_2=-(1+x_1^2+x_1^4)x_2-x_1\end{array}\right\}$ ① System given in ① is an autonomous dynamical system of dimens 2. <u>Bendixson criterion</u> for 2nd order autonomous is as follows:

if on a simply connected region D of 2 x_1-x_2 plane, $\frac{\partial f_1}{\partial x_1}+\frac{\partial f_2}{\partial x_2}$ is <u>not</u> <u>identically zero</u>, & <u>doesnot change</u> sign, then 2 system has no <u>close Trajectory</u> lying entirely in D

$\frac{\partial f_1}{\partial x_1}=0$ $\frac{\partial f_2}{\partial x_2}=-(1+x_1^2+x_1^4)$ $\frac{\partial f_1}{\partial x_1}+\frac{\partial f_2}{\partial x_2}\le -1<0$ $\forall x_1, x_2$

(remark: $x_1^2+x_1^4$ posi terms. smallest possible term for $(1+x_1^2+x_1^4)$ is 1)

$\frac{\partial f_1}{\partial x_1}+\frac{\partial f_2}{\partial x_2}$ does not change sign in $\mathbb{R}^2$ & $\therefore$ system has no close trajectory

3 Sol $\left.\begin{array}{l}\dot{x}_1=x_1-x_2-x_1(x_1^2+2x_2^2)\\ \dot{x}_2=x_1+x_2-x_2(x_1^2+2x_2^2)\end{array}\right\}$ ① Consider $V(x)=x_1^2+x_2^2=r^2$ where $\frac{1}{\sqrt{2}}<r<1$ above system is in $\dot{x}=f(x)$ form. V is contly diffible vector field $f(x)$ at x on curve $V(x)$

$$f(x)\cdot\nabla V(x)=\overbrace{2x_1}^{\partial V/\partial x_1}\overbrace{(x_1-x_2-x_1(x_1^2+2x_2^2))}^{f_1(\cdot)}+\overbrace{2x_2}^{\partial V/\partial x_2}\underbrace{(x_1+x_2-x_2(x_1^2+2x_2^2))}_{f_2(\cdot)}$$

$=2x_1^2-\cancel{2x_1x_2}-2x_1^2\underbrace{(x_1^2+x_2^2)}+2x_2^2-2x_2^2\underbrace{(x_1^2+2x_2^2)}_{\text{common}}+\underbrace{\cancel{2x_2x_1}}_{\text{cancels}}$

$=2(x_1^2+x_2^2)[1-(x_1^2+2x_2^2)]=2r^2[1-r^2-x_2^2]$ $\{\because x_1^2+x_2^2=r^2$ from det of $V(x)\}$

when $r=1$: $f(x)\cdot\nabla V(x)=-2x_2^2\le 0$ likewise, subing for x_2^2 with $r^2-x_1^2$

$f(x)\cdot\nabla V(x)=2r^2(1-2r^2+x_1^2)$