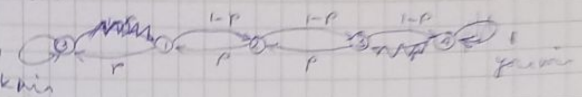
$\tilde{p} = (p_1 \;\; p_2 \;\; p_3)$ ∴ $\tilde{p}T = \tilde{p}$ $\quad (p_1 \;\; p_2 \;\; p_3)\begin{pmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 0 & 1/2 \\ 0 & 0 & 1 \end{pmatrix} = (p_1 \;\; p_2 \;\; p_3)$

∴ $\tilde{p} = (0, 0, 1)$

\1di/ 

state = #H − #T

\1dii/ P(bank wins in 1 round) = 0, P(bank wins in 2 rounds) = p^2 $\quad p^2 = p^2$

\1diii/ $2 - 0 = 2$ is even ∴ P(start at 2 and bank wins in 3 rounds) = 0

\1div/ $G_Y(\theta) = E(\theta^Y)$

$\frac{dp_0}{dt} = p\,p_1 = 0 \therefore p_1 = 0$, $\frac{dp_2}{dt} = (1-p)p_1 + pp_2 - (1-p)p_2 + p_3 = p_3 - p_2 = 0 \therefore p_3 = p_2$

$\frac{dp_3}{dt} = (1-p)p_2 - p_3 - (1-p)p_4 = (1-p)p_3 - p_3 - (1-p)p_3 = -p_3 = 0 \therefore p_3 = p_2 = 0$ ∴

$\sum_{n=1}^{4} p_n = 1 = p_0 + p_1 + p_2 + p_3 + p_4$ $\quad \frac{dp_1}{dt} = -pp_1 - (1-p)p_1 + pp_2 = 0 - p_1 + pp_2$

$\frac{dp_4}{dt} = 0 = p_3(1-p) \quad p_3 = 0$

$p_1 = p_2 = p_3 = 0$ ∴ $p_0 + p_4 = 1$ ∴ $\beta = p_0$ ∴ $\alpha = p_4$ ∴ $\alpha + \beta = 1$ ∴

$G_X(\theta) = E(\theta^Y) = p_0\theta^0 + p_4\theta^4 = p_0 + p_4\theta^4 = \beta + \alpha\theta^4$

\2ai/ $G_Y(\theta) = E(\theta^Y) = E[E[\theta^Y | N=n]] = \sum_{n=0}^{\infty} E(\theta^{x_1+\dots+x_n})P(N=n) =$

$\sum_{n=0}^{\infty} E[(\theta^{x_1})\theta^{x_2}\dots\theta^{x_n}]P(N=n) = \sum_{n=0}^{\infty} E[\theta^{x_1}]E(\theta^{x_2})\dots E(\theta^{x_n})P(N=n) =$

$\sum_{n=0}^{\infty} \underbrace{E(\theta^X)\dots E(\theta^X)}_{n\text{ times}} P(N=n) = \sum_{n=0}^{\infty} \underbrace{G_X(\theta)\dots G_X(\theta)}_{n\text{ times}} P(N=n) =$

$\sum_{n=0}^{\infty} (G_X(\theta))^n P(N=n) = G_N(G_X(\theta))$

\2aii/ $Z \sim Poi(\lambda)$ ∴ $f_Z(z) = \frac{e^{-\lambda}(\lambda)^z}{z!}$ ∴

∴ $G_Z(\theta) = e^{-\lambda}\sum_{n=0}^{\infty}\frac{1}{n!}(\lambda\theta)^n = e^{-\lambda}e^{\lambda\theta} = e^{\lambda\theta - \lambda} = e^{\lambda(\theta-1)}$

$G_Z(\theta) = E(\theta^Z) = \sum_{k=0}^{\infty}P(Z=k)\theta^k = \sum_{n=0}^{\infty}e^{-\lambda}\frac{\lambda^k}{k!}\theta^k = e^{-\lambda}\sum_{n=0}^{\infty}\frac{1}{k!}(\lambda\theta)^k =$

$e^{-\lambda}e^{\lambda\theta} = e^{\lambda(\theta-1)}$

\2aiii/ ∴ $\lambda = 2$ ∴ $G_N(G_X(\theta)) = 0.5 + 0.3(e^{\lambda(\theta-1)}) + 0.2(e^{\lambda(\theta-1)})^2 =$

$0.5 + 0.3e^{-\lambda}e^{\lambda\theta} + 0.2e^{-2\lambda}e^{2\lambda\theta}$ ∴

$= 0.5 + 0.3e^{-\lambda}\sum_{n=0}^{\infty}\frac{1}{n!}(\lambda\theta)^n + 0.2e^{-2\lambda}\sum_{n=0}^{\infty}\frac{1}{n!}(2\lambda\theta)^n$ ∴

∴ $P(Y=2) = 0.3e^{-\lambda}\frac{1}{2!}\lambda^2 + 0.2e^{-2\lambda}\frac{1}{2!}(2\lambda)^2 = 0.3e^{-\lambda}\frac{\lambda^2}{2} + 0.2e^{-2\lambda}\frac{(2\lambda)^2}{2}$

\2bi/ $P(X>1) = 1 - (P(X \le 1)) = 1 - (P(X=0) + P(X=1))$ ∴

$P(X=0) + P(X=1) = e^{-2}2^0\frac{1}{0!} + e^{-2}2^1\frac{1}{1!} = 0.406$ ∴ $P(X>1) = 1 - 0.406 = 0.594$

\PP2019/ \1ai/ $E(Y) = 2E(X) = 2[0.2(1)+(0.4)(5)] = 4.4$

$P(Y=2) = P(2X=2) = P(X=1) = 0.2$ $\quad E(X^3) = P(X=x)x^3$ $\quad \{0, 1, 5\}$ $\{0.4, 0.2, 0.4\}$

\1aii/ $E(Y) = E(X^3)+2$ ∴ $\quad x^3 \in \{0, 1, 125\}$

$E(X^3) = 0.2 \times 1^3 + 0.4 \times 5^3 = 50.2$ ∴ $E(Y) = 52.2$

$P(Y=2) = P(X^3+2=2) = P(X^3=0) = P(X=0) = 0.4$

\1aiii/ $G_{T_Y}(\theta) = G_{T_{X_1+X_2+X_3}}(\theta) = G_{T_{X_1}}(\theta)G_{T_{X_2}}(\theta)G_{T_{X_3}}(\theta) = (G_{T_X}(\theta))^3$ ∴

∴ $E(Y) = G_{T_Y}'(1) = 3(G_{T_X}(1))^2 G_{T_X}'(1) = 3(1)^2 E(X) = 3(0.2 \times 1 + 0.4 \times 5) = 3 \times 2.2 = 6.6$

$P(Y=2)$ ∴ $G_{T_Y}(\theta) = (0.4 + 0.2\theta + 0.4\theta^5)^3 =$

$(0.2\theta)^2(0.4)^1 \frac{3!}{2!1!} + \dots = 0.048\theta^2 + \dots$ ∴ $P(Y=2) = 0.048$

\1bi/ $\lambda_P = 0.05 \times 100 = 5$ ∴ $\frac{5}{100} = 0.05 = P(\text{next is purple})$

observations independent ∴ $P(\text{purple}) = 0.05$

\1bii/ $\frac{1}{100}$ hours time T between bee sightings exponential distribution

∴ $E(T) = \frac{1}{\lambda} = \frac{1}{100}$ hr

\1biii/ $\lambda_R = 30$ ∴ $P(N_R(\frac{1}{4})=2) = e^{-(30 \times \frac{1}{4})}(30 \times \frac{1}{4})^2 \frac{1}{2!} = 0.0156$ (3 s.f.)

\1biv/ $\lambda_P = 5$ ∴ $P((N_P(\frac{1}{4})=1) \mid (N_P(1)=2)) =$

$P((N_P(\frac{1}{4})=1) \cap (N_P(1)=2)) / P(N_P(1)=2) = \frac{P(N_P(\frac{1}{4})=1)\,P(N_P(\frac{3}{4})=1)}{P(N_P(1)=2)}$

$= e^{-30 \times \frac{1}{4}}(30 \times \frac{1}{4})^1 \frac{1}{1!} \times e^{-30 \times \frac{3}{4}}(30 \times \frac{3}{4})^1 \frac{1}{1!} \times \left(\frac{1}{e^{-30 \times 1}(30 \times 1)^2 \frac{1}{2!}}\right) = \frac{3}{8} = 0.375$

purple: $\lambda_P = 5/\text{hr}$ ∴ binomial distribution

$P(X=1) = \binom{2}{1}p^1(1-p)^1$, $p = 0.25$

∴ $P(X=1) = 2 \times 0.25 \times 0.75 = 0.375$

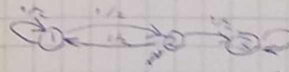
\1ci a/ state is absorbing if $T_{ij} = 0 \; \forall i \neq j$, $T_{ii} = 1$

Absorbing if $T_{ii} = 1$

\1ci b/ State is transient if probability of first return f_i is $f_i^{(n)}$ after n steps and $f_i = \sum_{n=1}^{\infty} f_i^{(n)} < 1$ ∴ transient

let $f_i^{(n)}$ = prob of first return to i at time n, $f_i = \sum_{n=1}^{\infty} f_i^{(n)}$ eventual return, transient is $f_i < 1$

\1ci c/ State is recurrent if $f_i = \sum_{n=1}^{\infty} f_i^{(n)} = 1$ recurrent is $f_i = 1$

\1cii/  state 2: $f_2^{(1)} = 0$, $f_2^{(2)} = \frac{1}{2} \cdot \frac{1}{2}$

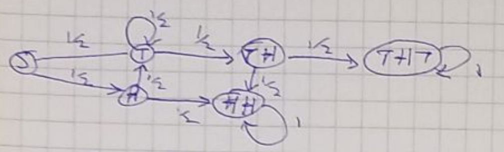
$f_2^{(3)} = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}$, $f_2^{(4)} = \frac{1}{2} \cdot (\frac{1}{2})^2 \cdot \frac{1}{2}$ ∴ $f_2^{(n)} = (\frac{1}{2})^n$ ∴ $f_2 = \sum_{n=2}^{\infty} f_2^{(n)} = \sum_{n=2}^{\infty} (\frac{1}{2})^n = \frac{1/4}{1-\frac{1}{2}} = \frac{1}{2} = \frac{1}{2} < 1$

∴ transient

\PP2015/ $D_S = 1 + \frac{1}{2}D_T + \frac{1}{2}D_H = 1 + \frac{1}{2}(4) + \frac{1}{2}(6) = 1 + 2 + 3 = 6$ tosses

E(number of tosses to see two tails)

\4 c/



∴ let THT be winning ∴

let θ_i be probability to win from state i ∴

$\theta_{HH} = 0$, $\theta_{THT} = 1$ ∴

$\theta_S = \frac{1}{2}\theta_T + \frac{1}{2}\theta_H$, $\theta_H = \frac{1}{2}\theta_T + \frac{1}{2}\theta_{HH} = \frac{1}{2}\theta_T$

$\theta_T = \frac{1}{2}\theta_T + \frac{1}{2}\theta_{TH}$ $\quad \theta_{TH} = \frac{1}{2}\theta_{THT} + \frac{1}{2}\theta_{HH} = \frac{1}{2}\theta_{THT} = \frac{1}{2}$ ∴

$\frac{1}{2}\theta_T = \frac{1}{2}\theta_{TH} = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$ ∴ $\theta_T = \frac{1}{2}$ ∴

$\theta_H = \frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$ ∴

$\theta_S = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{4}\right) = \frac{1}{4} + \frac{1}{8} = \frac{3}{8}$ = P(THT appears before HH

\pp2019/ for $\mu=1$: $1=\sum_{n=0}^{N} p_n = \sum_{n=\emptyset}^{N} 2\left(\frac{1}{\mu}\right)^n p_0 + p_0 = 2p_0\left(\sum_{n=0}^{N}\left(\frac{1}{\mu}\right)^n - 1\right) + p_0 =$

$2p_0\left(\frac{1-\left(\frac{1}{\mu}\right)^{N+1}}{1-\frac{1}{\mu}} - 1\right) + p_0 = p_0\left[2\left(\frac{\mu-\left(\frac{1}{\mu}\right)^N}{\mu-1} - \frac{\mu-1}{\mu-1}\right)+1\right] =$

$p_0\left[\frac{\mu-2\left(\frac{1}{\mu}\right)^N+1}{\mu-1} + \frac{\mu-1}{\mu-1}\right] = p_0\left[\frac{2\mu-2\left(\frac{1}{\mu}\right)^N}{\mu-1}\right] = 1 \therefore$

$\frac{\mu-1}{2\mu-2\left(\frac{1}{\mu}\right)^N} = p_0 \therefore \quad p_n = 2\left(\frac{1}{\mu}\right)^n \frac{\mu-1}{2\mu-2\left(\frac{1}{\mu}\right)^N} \; ; n \geq 1$

~~lim~~ for $\mu=1$: $\lim_{\mu\to 1} \frac{\mu-1}{2\mu-2\left(\frac{1}{\mu}\right)^N} = \lim_{\mu\to 1}\frac{\mu-1}{2\mu-2\mu^{-N}} = \lim_{\mu\to 1}\frac{1}{2-2(-N)\mu^{-N-1}} =$

$\frac{1}{2+2N(1)^{-N-1}} = \frac{1}{2+2N} = \frac{1}{2N+2} \therefore$

$p_n = 2\left(\frac{1}{1}\right)^n\left(\frac{1}{2N+2}\right) = \frac{1}{N+1} \; ; n \geq 1$

\3aii/ ~~E(x)=~~ $G_X(\theta) = \left(\frac{\mu-1}{\mu+1}\right)\left(1+2\sum_{n=1}^{\infty}\frac{\theta^n}{\mu^n}\right)$

$E(X) = G_X'(1) = \frac{2}{\mu}\frac{\mu-1}{\mu+1}\frac{1}{\left(1-\frac{1}{\mu}\right)^2} \therefore$

\3bii/ $G_{X_N}(\theta) = \sum_{n=0}^{N}\theta^n P(X_N=n) = \sum_{n=\emptyset}^{N}\theta^n 2\left(\frac{1}{\mu}\right)^n\frac{\mu-1}{2\mu-2\left(\frac{1}{\mu}\right)^N} + \frac{\mu-1}{2\mu-2\left(\frac{1}{\mu}\right)^N}$

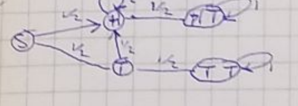
$2\frac{\mu-1}{2\mu-2\left(\frac{1}{\mu}\right)^N}\sum_{n=1}^{N}\left(\frac{\theta}{\mu}\right)^n + \frac{\mu-1}{2\mu-2\left(\frac{1}{\mu}\right)^N} = \frac{\mu-1}{\mu-\left(\frac{1}{\mu}\right)^N}\left(\frac{1-\left(\frac{\theta}{\mu}\right)^{N+1}}{1-\frac{\theta}{\mu}} - 1\right) + \frac{\mu}{2\mu-2\left(\frac{1}{\mu}\right)^N}$

~~for~~ $N=\infty$: $p_n = \frac{2}{\mu}\left(\frac{\mu-1}{\mu+1}\right)$, $N<\infty$: $p_n^{(N)} = 2\left(\frac{1}{\mu}\right)^n \frac{1-\frac{1}{\mu}}{1-\frac{1}{\mu}-2\left(\frac{1}{\mu}\right)^{N+1}}$

$\therefore \lim_{N\to\infty} p_n^{(N)} = \lim_{N\to\infty} 2\left(\frac{1}{\mu}\right)^n \frac{1-\frac{1}{\mu}}{1+\frac{1}{\mu}-2\left(\frac{1}{\mu}\right)^{N+1}} = \frac{1-\frac{1}{\mu}}{1+\frac{1}{\mu}}2\left(\frac{1}{\mu}\right)^n = 2\frac{\mu-1}{\mu+1}\left(\frac{1}{\mu}\right)^n = p_n$

$\therefore p_n^{(N)} \to p_n$ as $n\to\infty$

\4ai/



$\therefore$ let θ_i be probability of winning from i

$\therefore \theta_i = \sum_j \theta_j T_{ij}$

$\therefore \theta_{HT}=0$, $\theta_{TT}=1 \therefore \quad \theta_H = \frac{1}{2}\theta_H + \frac{1}{2}\theta_{HT} \therefore \frac{1}{2}\theta_H = \frac{1}{2}\theta_{HT} = \frac{1}{2}(0) = 0 = \theta_H$

$\therefore \theta_T = \frac{1}{2}\theta_H + \frac{1}{2}\theta_{TT} = \frac{1}{2}(0)+\frac{1}{2}(1) = \frac{1}{2} \therefore$

$\theta_S = \frac{1}{2}\theta_H + \frac{1}{2}\theta_T = \frac{1}{2}(0)+\frac{1}{2}\left(\frac{1}{2}\right) = \frac{1}{4}$ is probability of TT before HT

\4aii/ let $D_i = 1+\sum_j \theta_j T_{ij}$ is expected tosses to end $\therefore$

$D_{HT}=0$, $D_{TT}=0$, $D_H = 1+\frac{1}{2}D_{HT}+\frac{1}{2}D_H \therefore \frac{1}{2}D_H = 1+\frac{1}{2}(0) = 1 \therefore D_H = 2 \therefore$

$D_T = 1+\frac{1}{2}D_H+\frac{1}{2}D_{TT} = 1+\frac{1}{2}(2)+\frac{1}{2}(0) = 2 \quad \therefore D_S = 1+\frac{1}{2}D_H+\frac{1}{2}D_T = 1+\frac{1}{2}(2)+\frac{1}{2}(2) = 3$

\3aii/ $G_{X}(\theta) = \sum_{n=0}^{\infty} \theta^n P_n = \sum_{n=\phi}^{\infty} \theta^n P_n + P_0 = \sum_{n=1}^{\infty} \left(\frac{1}{\mu}\right)^n \frac{2\mu-2}{3\mu-1}\theta^n + \frac{\mu-1}{3\mu-1} =$

$\frac{2\mu-2}{3\mu-1}\left[\frac{1}{1-\frac{1}{\mu}} - 1\right] + \frac{\mu-1}{3\mu-1} = \frac{2\mu-2}{3\mu-1}\left[\frac{\mu}{\mu-1} - 1\right] + \frac{\mu-1}{3\mu-1} =$

$\cancel{\frac{3}{\mu}} \quad \frac{1}{3\mu-1}\left(\left[2\mu-2\right]\left[\frac{\mu-(\mu-1)}{\mu-1}\right] + \mu - 1\right) =$

$\frac{1}{3\mu-1}\left(2\mu-2\left[\frac{1}{\mu-1}\right] + \mu - 1\right) = \frac{1}{3\mu-1}\left(\frac{2\mu-2+(\mu-1)^2}{\mu-1}\right) =$

$\frac{2\mu-2+\mu^2+1-2\mu}{\mu-1} = \frac{\mu^2-1}{\mu-1} = \frac{(\mu+1)(\mu-1)}{\mu-1} = \frac{\mu+1}{1}$

\3aii/ $G_{X}(\theta) = \sum_{n=0}^{\infty} \theta^n P_n = \sum_{n=1}^{\infty} \theta^n P_n + P_0 = \sum_{n=1}^{\infty} \left(\frac{1}{\mu}\right)^n \frac{2\mu-2}{3\mu-1}\theta^n + \frac{\mu-1}{3\mu-1}$

$\frac{2\mu-2}{3\mu-1} \sum_{n=1}^{\infty} \left(\frac{1}{\mu}\theta\right)^n + \frac{\mu-1}{3\mu-1} = \frac{2\mu-2}{3\mu-1} \frac{1}{1-\left(\frac{\theta}{\mu}\right)} - 1 + \frac{\mu-1}{3\mu-1} =$

$\cancel{3\mu-2} \quad \frac{1}{3\mu-1}\left[(2\mu-2)\frac{\mu}{\mu-\theta} + \mu - 1\right] - \frac{(3\mu-1)}{3\mu-1} = \frac{1}{3\mu-1}\left[(2\mu-2)\frac{\mu}{\mu-\theta} + \mu - 1 - 3\mu + 1\right] =$

$\frac{1}{3\mu-1}\left[\frac{(2\mu^2-2\mu)}{\mu-\theta} - 2\mu\right] = \frac{1}{3\mu-1}\left[\frac{(2\mu^2-2\mu) - 2\mu(\mu-\theta)}{\mu-\theta}\right] = \frac{2\mu^2-2\mu-2\mu^2+2\mu\theta}{\mu-\theta} =$

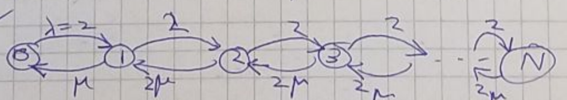
$\frac{-2\mu+2\mu\theta}{\mu-\theta} = \frac{2\mu(\theta-1)}{\mu-\theta} \therefore$

Mean of System Size $= E$ (System Size$] = L_s = E(X) = G_X'(\theta)\big|_{\theta=1} =$

$\frac{d}{d\theta}\left[(-2\mu+2\mu\theta)(\mu-\theta)^{-1}\right]\Big|_{\theta=1} = (+2\mu)(\mu-\theta)^{-1} + (-2\mu+2\mu\theta)(-1)(\mu-\theta)^{-2}(-1)\Big|_{\theta=1} =$

$\cancel{(2\mu(\mu-2)} \quad \frac{2\mu}{\mu-1} + \frac{-2\mu+2\mu}{(\mu-1)^2} = \frac{2\mu}{\mu-1} + \frac{0}{(\mu-1)^2} = \frac{2\mu}{\mu-1}$ X

$\sum_{n\geq 1}^{\infty}\left(\frac{1}{\mu}\right)^n = \sum_{n=0}^{\infty}\left(\frac{1}{\mu}\right)^n - \left(\frac{1}{\mu}\right)^0 = \frac{1}{1-\frac{1}{\mu}} - 1 = \frac{1}{1-\frac{1}{\mu}} - \frac{1-\frac{1}{\mu}}{1-\frac{1}{\mu}} = \frac{1-1+\frac{1}{\mu}}{1-\frac{1}{\mu}} = \frac{\frac{1}{\mu}}{1-\frac{1}{\mu}} =$

\3bi/ 

steady state: $\frac{dP_0}{dt} = \mu P_1 - 2P_0 = 0 \therefore \quad P_1 = \frac{2}{\mu}P_0 \therefore$

$\frac{dP_1}{dt} = \cancel{2}\mu P_2 - 2P_1 - (\mu P_1 - 2P_0) = 0 = 2\mu P_2 - 2P_1 \quad \therefore \quad P_2 = \frac{1}{\mu}P_1 = 2\left(\frac{1}{\mu}\right)^2 P_0$

$\frac{dP_N}{dt} = 2P_{N-1} - 2\mu P_N = 0 \therefore P_N = \frac{1}{\mu}P_{N-1} = \frac{1}{\mu}2\left(\frac{1}{\mu}\right)^{N-1}P_0 = 2\left(\frac{1}{\mu}\right)^N P_0 \therefore$

$P_n = 2\left(\frac{1}{\mu}\right)^n P_0 \quad ; n \geq 1 \therefore \quad \sum_{n=0}^{\cancel{N}} P_n = 1$

if $\mu = 1 \quad P_n = 2(1)^n P_0$

\PP2019/ $0.6e^{-2}\theta^2 + 1.6e^{-4}\theta^2 + \ldots = (0.6e^{-2} + 1.6e^{-4})\theta^2 + \ldots \therefore$

$P(Y=2) = 0.6e^{-2} + 1.6e^{-4} = 0.111\ (3\text{S.F.})$ ~~0.0368~~

$E(x_i) = 2$

\2bi/ $P(S_1 > 1) = 1 - P(S_1 \leq 1) = 1 - P(S_1 = 1, 0) = 1 - P(S_1 = 1 \cup S_1 = 0) = 1 - (P(S_1 = 1) + P(S_1 = 0)) \therefore$

$G_X(\theta) = e^{\lambda(\theta - 1)} \quad \therefore \lambda = \frac{1}{2} \therefore$

let Y be offspring $\therefore G_Y(\theta) = (G_X(\theta))^3 = (e^{\lambda\theta - \lambda})^3 = (e^{-\lambda}e^{\lambda\theta})^3 =$

$e^{-3\lambda}e^{3\lambda\theta} = e^{-3\lambda}\sum_{n=0}^{\infty}\frac{(3\lambda\theta)^n}{n!} = e^{-3\lambda}\frac{(3\lambda\theta)^0}{0!} + e^{-3\lambda}\frac{3\lambda\theta}{1!} + \ldots =$

$e^{-3\lambda} + 3\lambda e^{-3\lambda}\theta + \ldots \therefore$

$P(S_1 > 1) = 1 - (e^{-3\lambda} + 3\lambda e^{-3\lambda}) = 1 - (e^{-3(\frac{1}{2})} + \frac{3}{2}e^{-3(\frac{1}{2})}) = \cancel{0.893\ (3\text{S.F.})}$

$\cancel{= 0.107\ (3\text{S.F.})} = 0.442\ (3\text{S.F.})$

\2bii/ $E(S_4) = 3 \cdot (E(X))^4 = \cancel{3\lambda^4}$

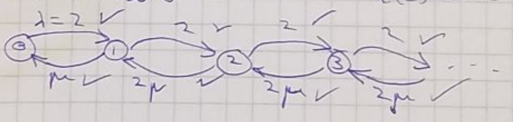
$\therefore E(X) = G_X'(\theta)\big|_{\theta=1} = \lambda \therefore$

$E(S_4) = 3\lambda^4 = 3(\frac{1}{2})^4 = \frac{3}{16} = 0.188\ (3\text{S.F.})$

\2biii/ $E(X) = \lambda = 2 > 1 \therefore \quad G_X'(1) = \mu_X > 1 \therefore$

P(ultimate extinction) $= e < 1 \therefore G_X(e) - e = 0$

\3ai/



$\therefore \frac{dP_0}{dt} = \mu P_1 - 2P_0 = 0 \therefore P_1 = \frac{2}{\mu}P_0$

$\frac{dP_1}{dt} = 2\mu P_2 - 2P_1 - (\mu P_1 - 2P_0) = 0 = 2\mu P_2 - 2P_1 \therefore \quad 2P_1 = 2\mu P_2 \therefore$

$P_2 = \frac{1}{\mu}P_1 = \frac{1}{\mu}\frac{2}{\mu}P_0 = (\frac{1}{\mu})^2 2P_0 = 2(\frac{1}{\mu})^2 P_0$

$\frac{dP_2}{dt} = 2\mu P_3 - 2P_2 - (2\mu P_2 - 2P_1) = 0 = 2\mu P_3 - 2P_2 \quad \therefore 2P_2 = 2\mu P_3 \therefore$

$P_3 = \frac{1}{\mu}P_2 = \frac{1}{\mu}(\frac{1}{\mu})^2 2P_0 = 2(\frac{1}{\mu})^3 P_0 \therefore$

$1 = P_0\left[1 + 2\sum_{n=1}^{\infty}(\frac{1}{\mu})^n\right] = P_0\left[1 + 2\left(\frac{1}{1-\frac{1}{\mu}} - 1\right)\right] =$

$P_1 = 2(\frac{1}{\mu})^1 P_0 \therefore \quad P_n = 2(\frac{1}{\mu})^n P_0, \ n \geq 1 \therefore \quad \sum_{n=0}^{\infty}P_n = 1 = P_0\left[1 + 2\left(\frac{\mu - (\mu - 1)}{\mu - 1}\right)\right] =$

$P_0\left[1 + 2\left(\frac{1}{\mu-1}\right.\right.$

$P_0 + \sum_{n=1}^{\infty}P_n = P_0 + \sum_{n=1}^{\infty}2(\frac{1}{\mu})^n P_0 = P_0 + 2P_0\sum_{n=1}^{\infty}(\frac{1}{\mu})^n = P_0\left[1 + 2\frac{1}{1-(\frac{1}{\mu})}\right] =$

$P_0\left[1 + 2\frac{\mu}{\mu - 1}\right] = P_0\left[\frac{\mu - 1 + 2\mu}{\mu - 1}\right] = P_0\left[\frac{3\mu - 1}{\mu - 1}\right] = 1$

$\therefore P_0 = \frac{\mu - 1}{3\mu - 1} \quad \therefore P_n = 2(\frac{1}{\mu})^n\frac{\mu - 1}{3\mu - 1} = (\frac{1}{\mu})^n\frac{2\mu - 2}{3\mu - 1}, \ n \geq 1$

\2ai/ $G_Y(\theta) = E(\theta^Y) = E[E(\theta^Y | N)] = E[E(\theta^{X_1+X_2+\cdots+X_N} | N)] =$

$E[E(\theta^{X_1+X_2+\cdots+X_k} | N=k)] = E[E(\underbrace{\theta^{X_1}\theta^{X_2}\theta^{X_3}\cdots\theta^{X_k}}_{k\text{ times}} | N=k)] =$

$E[E(\theta^{X_1})E(\theta^{X_2})\cdots E(\theta^{X_k}) | N=k] =$

$E[\underbrace{E(\theta^X)E(\theta^X)\cdots E(\theta^X)}_{k\text{ times}} | N=k] = E[(E(\theta^X))^k | N=k] =$

$E[(G_X(\theta))^k | N=k] = E[(G_X(\theta))^N] = G_N(G_X(\theta))$

\2ai/ $G_Y(\theta) = E(\theta^Y) = E[E(\theta^Y|N)] = \sum_{k=0}^{\infty} E(\theta^Y | N=k)P(N=k) =$

$\sum_{k=0}^{\infty} E(\theta^{X_1+X_2+\cdots+X_N} | N=k)P(N=k) = \sum_{k=0}^{\infty} E(\theta^{X_1+X_2+\cdots+X_k})P(N=k) =$

$\sum_{k=0}^{\infty} E(\theta^{X_1}\theta^{X_2}\cdots\theta^{X_k})P(N=k) = \cancel{\sum_{k=0}^{\infty} E(\underbrace{\theta^X\theta^X\cdots\theta^X}_{k\text{ times}}}$

$\sum_{k=0}^{\infty} E(\theta^{X_1})E(\theta^{X_2})\cdots E(\theta^{X_k})P(N=k) = \sum_{k=0}^{\infty} \underbrace{E(\theta^X)E(\theta^X)\cdots E(\theta^X)}_{k\text{ times}} P(N=k) =$

$\sum_{k=0}^{\infty} (E(\theta^X))^k P(N=k) = \sum_{k=0}^{\infty} (G_X(\theta))^k P(N=k) = E((G_X(\theta))^N) = G_N(G_X(\theta))$

\2ai/ $G_Y(\theta) = E(\theta^Y) = E[E(\theta^Y | N)] = \sum_{n=0}^{\infty} E(\theta^Y | N=n)P(N=n) =$

$\sum_{n=0}^{\infty} E(\theta^{X_1+X_2+\cdots+X_N} | N=n)P(N=n) = \sum_{n=0}^{\infty} E(\theta^{X_1+\cdots+X_n} \cancel{|N=n})P(N=n) =$

$\sum_{n=0}^{\infty} E(\theta^{X_1}\theta^{X_2}\cdots\theta^{X_n})P(N=n) = \sum_{n=0}^{\infty} E(\theta^{X_1})E(\theta^{X_2})\cdots E(\theta^{X_n})P(N=n) =$

$\sum_{n=0}^{\infty} E(\theta^X)E(\theta^X)E(\theta^X)\cdots E(\theta^X)P(N=n) = \sum_{n=0}^{\infty} (E(\theta^X))^n P(N=n) =$

$\sum_{n=0}^{\infty} (G_X(\theta))^n P(N=n) = \cancel{G_N}\ E((G_X(\theta))^N) = G_N(G_X(\theta))$

$\sum_{n=0}^{\infty} \theta^n P(Z=n)$

\2aii/ $f_Z(\theta) = \frac{e^{-\lambda}\lambda^n}{n!}$ $\therefore$ $G_Z(\theta) = E(\theta^Z) = \sum_{n=0}^{\infty} \theta^n p_n = \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n}{n!}\theta^n =$

$e^{-\lambda}\sum_{n=0}^{\infty} \frac{1}{n!}(\lambda\theta)^n = e^{-\lambda}e^{\lambda\theta} = e^{\lambda\theta-\lambda} = e^{\lambda(\theta-1)}$

\2aiii/ $E(X_i) = \lambda = 2$ $\quad G_Y(\theta) = G_N(G_X(\theta)) =$

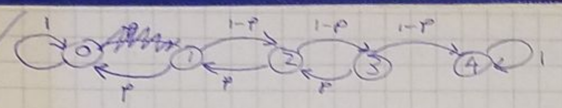
$0.5 + 0.3(G_X(\theta)) + 0.2(G_X(\theta))^2 =$

$0.5 + 0.3e^{2(\theta-1)} + 0.2(e^{2(\theta-1)})^2 = 0.5 + 0.3e^{2\theta-2} + 0.2e^{4(\theta-1)}$

$\therefore G_Y(\theta) = 0.3e^{-2}e^{2\theta} + 0.2e^{-4}e^{4\theta} + \dots =$

$0.3e^{-2}\sum_{n=0}^{\infty} \frac{(2\theta)^n}{n!} + 0.2e^{-4}\sum_{n=0}^{\infty} \frac{(4\theta)^n}{n!} + \dots =$

$0.3e^{-2}\frac{(2\theta)^2}{2!} + 0.2e^{-4}\frac{(4\theta)^2}{2!} + \dots = 0.3e^{-2}\frac{4\theta^2}{2} + 0.2e^{-4}\frac{16}{2}\theta^2 + \dots =$

\1di/  State = #H − #T

starting at 2 is a random walk

\1dii/ starting at 2 to get to 0:

$P(2\to 0 \text{ in } 2 \text{ steps} \mid 2) = P(2\to 1\to 0 \mid 2) = p\times p = p^2$

\1diii/ starting at 2 for bank to win means ending at 0 ∴ 2, 0 are even but 3 is odd ∴

$P(2\to 0 \text{ in } 3 \text{ steps}) = 0$

\1div/ $\sum_{n=0}^{4} P_n = 1 = P_0 + P_1 + P_2 + P_3 + P_4$ ∴

$G_Y = E(\theta^Y) = \cancel{P_0\theta^{Y}+p} \sum_{n=0}^{4} P_n\theta^n$ ∴

steady state: $\cancel{\frac{dP_0}{dt} = Pd}$

$\frac{dP_1}{dt} = pP_2 - pP_1 - (1-p)P_1 = pP_2 - P_1 = 0$ ∴ $\frac{1}{p}P_1 = P_2$

$\frac{dP_2}{dt} = \cancel{pP_3 - pP_2} - pP_3 - (1-p)P_2 - (pP_2 - (1-p)P_1) = 0 = \cancel{pP_3 - P_2}$

$-pP_1 = -(1-p)P_1 + pP_2$ ∴ $-pP_1 + P_1 - pP_1 = pP_2 = (1-2p)P_1$

$\frac{dP_0}{dt} = +pP_1 = 0$ ∴ $P_1 = 0$ ∴

$\frac{dP_1}{dt} = -pP_1 - (1-p)P_1 + pP_2 = 0 = pP_2$ ∴ $P_2 = 0$ ∴

$\cancel{\frac{dP_2}{dt}}$ $\frac{dP_4}{dt} = (1-p)P_3 = 0 = P_3$ ∴

$P_1 = P_2 = P_3 = 0$ ∴ $\cancel{P_0 +}$ $\sum_{n=0}^{4} P_n = 1 = P_0 + P_4 + \sum_{n=1}^{3} P_n = P_0 + P_4 = 1$ ∴

$P_4 = 1 - P_0$ ∴ let $P_0 = \alpha$, $P_4 = \beta$ ∴

$G_Y(\theta) = \alpha\theta^0 + \beta\theta^4 = \alpha + \beta\theta^4$

\2a/ $G_Y(\theta) = E(\theta^Y) = E(\theta^{X_1+X_2+X_3+\dots+X_N}) \cancel{= E(}$

$E(N) = \sum_{n=0}^{\infty} P_n n$ ∴ $G_Y(\theta) = E(\theta^{X_1}\theta^{X_2}\dots\theta^{X_N}) = E(\theta^{X_1}\theta^{X_2}\dots\theta^{X_k} \mid N=k) =$

$E(\underbrace{\theta^{X}\theta^{X}\dots\theta^{X}}_{k\text{ times}} \mid N=k) = E((\theta^X)^k \mid N=k)$

$G_N(\theta) = E(\theta^N)$ ∴ $G_N(G_X(\theta)) = E((G_X(\theta))^N) =$

$E((E(\theta^X))^N) = E(\underbrace{E(\theta^X)E(\theta^X)\dots E(\theta^X)}_{N\text{ times}})$

$G_Y(\theta) = E(\theta^Y) = E(\theta^{X_1+X_2+\dots+X_N}) \cancel{= E[E(\theta}$

$\cancel{G_Y}$

\1biii/ $\lambda_r = 0.3\times 100 = 30/hr$ ∴

$P(N_r(t=\frac{15}{60})=2) = P_r(N_r(t=\frac{1}{4})=2) = e^{-30\times\frac{1}{4}}\times(30\times\frac{1}{4})^2\frac{1}{2!} = 0.0156\ (3s.f.)$

\1biv/ $P((N_p(t=\frac{1}{4})=1) \mid N_p(t=1)=2) \sim Bin(2,\frac{15}{60}) = Bin(2,\frac{1}{4})$ ∴ follows Binomial ∴

$P(N_p(t=\frac{1}{4})=1 \mid N_p(t=1)=2) = \binom{2}{1}(\frac{1}{4})^1(1-\frac{1}{4})^{2-1} = \frac{2!}{1!(2-1)!}(\frac{1}{4})(\frac{3}{4})^1 = \frac{3}{8} = 0.375$

$P(N_p(t=\frac{1}{4})=1 \mid N_p(t=1)=2) = \frac{P(N_p(t=\frac{1}{4})=1,\ N_p(t=1)=2)}{P(N_p(t=1)=2)} =$

$\frac{P(N_p(t=\frac{1}{4})=1,\ N_p(t=\frac{3}{4})=1)}{P(N_p(t=1)=2)} = \frac{P(N_p(t=\frac{1}{4})=1)P(N_p(t=\frac{3}{4})=1)}{P(N_p(t=1)=2)} =$

$e^{-30\times\frac{1}{4}}(30\times\frac{1}{4})^1\frac{1}{1!}e^{-30\times\frac{3}{4}}(30\times\frac{3}{4})^1\frac{1}{1!}\frac{1}{e^{-30\times1}(30\times1)^2\frac{1}{2!}} = \frac{3}{8} = 0.375$

\1ci/ A: absorbing state is: $T_{ii}=1$, $T_{ij}=0$ for $i\neq j$

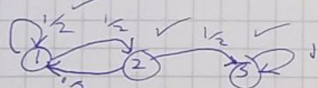
B: let $f_i^{(n)} = P$(first return to state i starting at it in n steps)

∴ $f_i = P$(~~first~~ return to state i starting at it) ∴

$f_i = \sum_{n=1}^{\infty} f_i^{(n)}$ ∴

transient state is $f_i < 1$

C: recurrent state is $f_i = \sum_{n=1}^{\infty} f_i^{(n)} = 1$

\1cii/ 

∴ ③ is an irreducible subchain

for ②: $f_2^{(1)}=0$, $f_2^{(2)}=\frac{1}{2}\times\frac{1}{2}$, $f_2^{(3)}=\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}$,

$f_2^{(4)}=\frac{1}{2}\times(\frac{1}{2})^2\times\frac{1}{2}$ ∴ $f_2^{(n)}=\frac{1}{2}\times(\frac{1}{2})^{n-2}\times\frac{1}{2}$ for $n\geq 2$ ∴

$f_2 = \sum_{n=1}^{\infty} f_2^{(n)} = \sum_{n=2}^{\infty}\frac{1}{2}(\frac{1}{2})^{n-2}\times\frac{1}{2} = \frac{1}{4}\sum_{n=2}^{\infty}(\frac{1}{2})^{n-2} = \frac{1}{4}\sum_{n=0}^{\infty}(\frac{1}{2})^n =$

$\frac{1}{4}\frac{1}{1-\frac{1}{2}} = \frac{1}{2} < 1$ ∴ ② is ~~transient~~ not recurrent ∴

② is transient

\1ciii/ let steady state $\tilde{P}=(P_1\ \ P_2\ \ P_3)$ ∴ $\tilde{P}T=\tilde{P}=(P_1\ \ P_2\ \ P_3)\begin{pmatrix}\frac{1}{2}&\frac{1}{2}&0\\ \frac{1}{2}&0&\frac{1}{2}\\ 0&0&1\end{pmatrix} =$

$(\frac{1}{2}P_1+\frac{1}{2}P_2\quad \frac{1}{2}P_1\quad \frac{1}{2}P_2+P_3) = (P_1\ \ P_2\ \ P_3)$ ∴ $P_1+P_2+P_3=1$ ∴

$\frac{1}{2}P_1+\frac{1}{2}P_2=P_1$ ∴ $\frac{1}{2}P_1=P_2$ ∴ $\frac{1}{2}P_2=P_1$ ∴ $\frac{1}{2}(\frac{1}{2}P_2)=\frac{1}{4}P_2=P_2$ ∴ $\frac{3}{4}P_2=0$ ∴ $P_2=0$

∴ $P_1=0$ ∴ $P_3=1$ ∴ $\tilde{P}=(0\ \ 0\ \ 1)$

2019/

$$\begin{tikzcd} S \arrow[r, "p"] \arrow[d, "q"'] & H \arrow[loop above, "p"] \arrow[r, "q"] & HT \\ & T \arrow[r, "q"] & TT \end{tikzcd}$$

$q = 1-p \therefore \theta_S = p\theta_H + q\theta_T$

$\theta_H = p\theta_H + q\theta_{HT} \therefore \theta_H = 0 \quad \theta_T = q$

$\theta_T = p\theta_H + q\theta_{TT} \quad \theta_{HT} = 0, \theta_{TT} = 1 \therefore \theta_H = 0, \theta_S = q^2 \therefore$

$\therefore$ equal chance to win is $q^2 = \frac{1}{2} \therefore q = \frac{1}{\sqrt{2}} \therefore p = 1 - \frac{1}{\sqrt{2}}$

$D_S = 1 + pD_H + qD_T$

$D_H = 1 + pD_H + qD_{HT}$

$D_T = 1 + pD_H + qD_{TT}, \quad D_{HT} = D_{TT} = 0 \therefore$

$D_T = 1 + pD_H, \quad D_H = 1 + pD_H \therefore (1-p)D_H = 1 \therefore D_H = \frac{1}{1-p}$

$D_T = 1 + p\left(\frac{1}{1-p}\right) \therefore D_S = 1 + \frac{1}{1-p}p + q\left(1 + p\left(\frac{1}{1-p}\right)\right)$

1ai/ $E(Y) = E(2X) = 2E(X) \therefore$

$E(X) = 0.4(0) + 0.2(1) + 0.4(5) = 2.2 \therefore$

$E(Y) = 2 \times 2.2 = 4.4$

$P(Y=2) = P(2X=2) = P(X=1) = 0.2$

1aii/ $E(Y) = E(X^3+2) = E(X^3) + 2 \therefore$

$E(X^3) = 0.4(0^3) + 0.2(1^3) + 0.4(5^3) = 50.2 \therefore$

$E(Y) = 50.2 + 2 = 52.2$

$P(Y=2) = P(X^3+2=2) = P(X^3=0) = P(X=0) = 0.4$

1aiii/ $E(X_1+X_2+X_3) = E(X_1) + E(X_2) + E(X_3) = E(X) + E(X) + E(X) =$

$3E(X) = 3 \times 2.2 = 6.6$

$G_Y(\theta) = G_{X_1+X_2+X_3}(\theta) = G_{X_1}(\theta)G_{X_2}(\theta)G_{X_3}(\theta) = G_X(\theta)G_X(\theta)G_X(\theta) = (G_X(\theta))^3$

$= (0.4 + 0.2\theta + 0.4\theta^5)^3 = (0.2\theta)^2(0.4)^1\frac{3!}{2!1!} + \dots = 0.04\theta^2(1.2) + \dots =$

$0.048\theta^2 + \dots \therefore$

$P(Y=2) = 0.048$

1bi/ $\therefore \lambda = 100/hr \therefore \lambda_{Pur} = \lambda_P = 0.05 \times 100 = 5/hr \therefore$

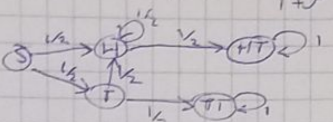
$\frac{0.05}{1} = \frac{\lambda_P}{\lambda} = \frac{5}{100} = 0.05 = P(\text{next bee is purple})$

1bii/ $\lambda = 100/hr \therefore \frac{1}{\lambda} = \frac{1}{100}$ hrs $= \frac{60}{100}$ minutes $= 0.6$ minutes

$= 0.6 \times 60$ secs $= 36$ seconds

for $N \to \infty$ in (a) : $P_n = 2\rho^n \frac{1-\rho}{1+\rho}$ $\therefore$ $P_n^{(N)} \to P_n$ as $n \to \infty$

\4ai/

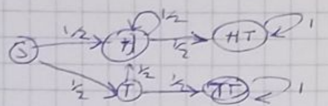


$\therefore$ θ_i is ~~pro~~ P(win from i) $\therefore$ $\theta_{TT} = 1$, $\theta_{HT} = 0$ $\therefore$

$\theta_S = \frac{1}{2}H + \frac{1}{2}\theta_T$, $\theta_H = \frac{1}{2}\theta_H + \frac{1}{2}\theta_{TH} = \frac{1}{2}\theta_H$ $\therefore$ $\theta_H = 0$

$\theta_T = \frac{1}{2}\theta_H + \frac{1}{2}\theta_{TT} = \frac{1}{2}\theta_{TT} = \frac{1}{2}$ $\therefore$ $\theta_S = \frac{1}{2}(0) + \frac{1}{2}(\frac{1}{2}) = \frac{1}{4}$ ✓

\4aii/

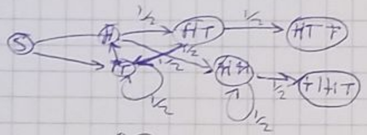


$D_S = \frac{1}{2}D_H + \frac{1}{2}D_T + 1$, $D_H = \frac{1}{2}D_H + \frac{1}{2}D_{HT} + 1$ $\therefore$ $D_{HT} = 0$, $D_{TT} = 0$ $\therefore$

$\frac{1}{2}D_H = \frac{1}{2}(0) + 1 = 0 + 1$, $D_H = 0 + 1 = 1$,

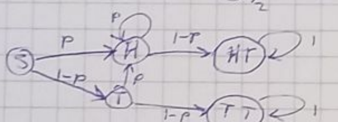
$D_T = \frac{1}{2}D_{TT} + \frac{1}{2}D_H + 1 = \frac{1}{2}(1) + 1 = \frac{3}{2}$

\4aiii/



no change $\therefore$ game is equivalent starting from H (both need heads)

\4bi/



$\therefore$ $\theta_i = P(\text{win from } i)$ $\therefore$

let $\theta_{HT} = 1$, $\theta_{TT} = 0$, $\theta_S = \frac{1}{2}$ $\therefore$

$\theta_S = \frac{1}{2} = p\theta_H + (1-p)\theta_T$ $\therefore$ $\theta_H = p\theta_H + (1-p)\theta_{HT} = p\,\theta_H + (1-p)1 = p\theta_H + 1 - p$

$(1-p)\theta_H = 1-p$ $\therefore$ $\theta_H = 1$ $\therefore$

$\theta_T = p\theta_H + (1-p)\theta_{TT} = p\theta_H = p$ $\therefore$

$\frac{1}{2} = p + (1-p)p = p + p - p^2 = 2p - p^2$ $\therefore$ $p^2 - 2p + \frac{1}{2} = 0$ $\therefore$ $p = \frac{2 \pm \sqrt{4 - 4(1)(\frac{1}{2})}}{2(1)}$

$= \frac{2 \pm \sqrt{2}}{2}$ $\therefore$ $p = \frac{2-\sqrt{2}}{2}$, $p = \frac{2+\sqrt{2}}{2} > 1$ $\therefore$

$p = \frac{2-\sqrt{2}}{2} \approx 0.293$ (3 s.f.)

\4bii/ $D_{HT} = 0$, $D_{TT} = 0$ $p = 0.293$ $\therefore$ $1 - p = \frac{\sqrt{2}}{2}$ $\therefore$

$D_H = pD_H + \frac{\sqrt{2}}{2}D_{HT} + 1 = pD_H + \frac{\sqrt{2}}{2}(0) + 1 = pD_H + 1$ $\therefore$ $(1-p)D_H = \frac{\sqrt{2}}{2}D_H = 1$ $\therefore$ $D_H = \sqrt{2}$

$D_T = 1 + D_H + \frac{\sqrt{2}}{2}D_{TT} = 1 + \sqrt{2}$ $\therefore$

$D_S = p + \frac{\sqrt{2}}{2}D_T + 1 = p + \frac{\sqrt{2}}{2}(1+\sqrt{2}) + 1 = \frac{2-\sqrt{2}}{2} + \frac{\sqrt{2}}{2}(1+\sqrt{2}) + 1 = 3$

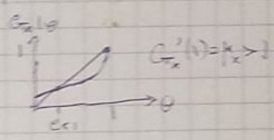
\PP2019/ \2bi/ $G_{S_1}(\theta) = G_N(G_X(\theta)) = (e^{\lambda(\theta-1)})^3 = e^{3\lambda(\theta-1)} =$

$e^{-3\lambda} e^{3\lambda\theta} = e^{-3\lambda} \sum_{n=0}^{\infty} \frac{(3\lambda\theta)^n}{n!}$, $\lambda = 2$ ∴

$P(S_1 > 1) = 1 - P(S_1 \leq 1) = 1 - (e^{-3\lambda}(1 + 3\lambda))$

\2bii/ $\mu_s = \mu_N (\mu_X)^2$ ∴ $\mu_N = G_N'(\theta)|_{\theta=1} = 3\theta^2|_{\theta=1} = 3$

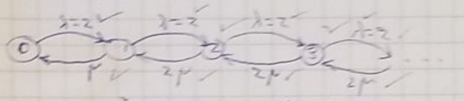
$\mu_X = G_X'(\theta)|_{\theta=1} = 2$ ∴ $\mu_s = 3 \cdot (2)^2 = 48$



\2biii/ $\mu_X = E(X) = 2 > 1$ ∴ $E(X) > 1$ ∴ $e < 1$ ∴

∴ equilibrium with $e < 1$

\3ai/



$\frac{dP_0}{dt} = -2P_0 + \mu P_1 = 0$ ∴ $\mu P_1 = 2P_0$ $P_1 = \frac{2}{\mu} P_0$,

$\frac{dP_1}{dt} = 2P_0 - \mu P_1 - 2P_1 + 2\mu P_2 = 0 = -2P_1 + 2\mu P_2 = 0$ $2P_1 = 2\mu P_2$ ∴ $\frac{1}{\mu} P_1 = P_2$

$\frac{dP_2}{dt} = 2P_1 - 2\mu P_2 - 2P_2 + 2\mu P_3 = 0 = -2P_2 + 2\mu P_3$ ∴ $2P_2 = 2\mu P_3$ ∴ $\frac{1}{\mu} P_2 = P_3$ ∴

$\frac{1}{\mu} P_{n-1} = P_n$ for $n \geq 2$ ∴ $P_2 = \frac{1}{\mu} \frac{2}{\mu} P_0 = \frac{2}{\mu^2} P_0$ ∴

$\frac{2}{\mu^n} P_0 = P_n$ for $n \geq 1$ ∴

$\sum_{i=0}^{\infty} P_n = P_0 + P_1 + \sum_{n=2}^{\infty} \frac{2}{\mu^n} P_0 = P_0 + \cancel{\frac{2}{\mu} P_0}$ $\sum_{n=1}^{\infty} \frac{2}{\mu^n} P_0 = P_0 + 2 \sum_{n=1}^{\infty} \frac{1}{\mu^n} P_0 = P_0 + 2 \frac{\left(\frac{1}{\mu}\right)}{1 - \left(\frac{1}{\mu}\right)} P_0 \neq 1$

$= P_0 + \frac{1}{\mu - 1} P_0 = \frac{\mu - 1 + 1}{\mu - 1} P_0 = \frac{\mu}{\mu - 1} = 1$ ∴ $P_0 = \frac{\mu - 1}{\mu}$

∴ $2P_0 = \mu P_1$, $2P_1 = 2\mu P_2$, ... $2P_{n-1} = 2\mu P_n$ ∴ $P_n = 2\left(\frac{1}{\mu}\right)^n P_0$ ∴

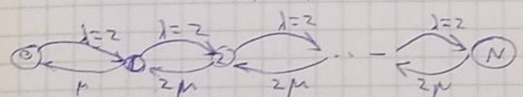
$P_0\left(1 + 2\sum_{n \geq 1}^{\infty} \left(\frac{1}{\mu}\right)^n\right) = P_0\left(1 + \frac{2\frac{1}{\mu}}{1 - \frac{1}{\mu}}\right) = P_0\left(\frac{\mu+1}{\mu-1}\right) = 1$ ∴ $P_0 = \frac{\mu - 1}{\mu + 1}$ ∴ $P_n = \frac{2}{\mu^n}\left(\frac{\mu-1}{\mu+1}\right)$

$P_n = \frac{2}{\mu^n}\left(\frac{\mu-1}{\mu+1}\right)$

\3aii/ ~~Et~~ $G_X(\theta) = \sum_{n=0}^{\infty} P(X=n)\theta^n = \sum_{n=0}^{\infty} P_n \theta^n = \left(\frac{\mu-1}{\mu+1}\right)\left(1 + 2\sum_{n=1}^{\infty} \frac{\theta^n}{\mu^n}\right) =$

$E(X) = G_X^{(1)}(1) = \frac{2}{\mu}\left(\frac{\mu-1}{\mu+1}\right) \sum_{n=1}^{\infty} k\left(\frac{\theta}{\mu}\right)^{k-1}\Big|_{\theta=1} = \frac{2}{\mu}\left(\frac{\mu-1}{\mu+1}\right) \frac{1}{\left(1 - \frac{1}{\mu}\right)^2}$

\3bi/



∴

$2P_0 = \mu P_1$, $2P_{n-1} = 2\mu P_n$ ∴ $P_n = 2\left(\frac{1}{\mu}\right)^n P_0$ for $n \geq 1$ ∴

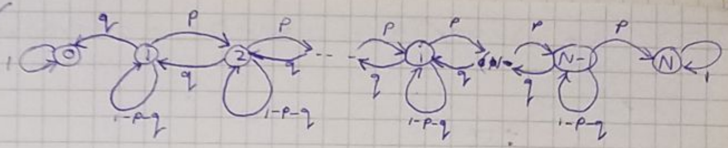
for $\mu \neq 1$: $P_0^{(N)} = \left(1 + 2\sum_{n=1}^{N} \left(\frac{1}{\mu}\right)^n\right) = P_0^{(N)}\left(1 + \frac{2}{\mu} \sum_{n=0}^{N} \left(\frac{1}{\mu}\right)^n\right) = P_0^{(N)}\left(1 + \frac{2}{\mu}\left(\frac{1 - \left(\frac{1}{\mu}\right)^N}{1 - \left(\frac{1}{\mu}\right)}\right)\right)^{-1}$

for $\mu = 1$: $2P_0^{(N)} = P_1^{(N)}$, $P_{n-1}^{(N)} = P_n^{(N)}$ ∴ $P_n^{(N)} = \frac{1}{N+1}$

\3bii/ $\rho = \frac{1}{\mu}$ ∴ $P_0^{(N)} = \frac{1}{1 + 2\rho\left(\frac{1 - \rho^N}{1 - \rho}\right)} = \frac{1 - \rho}{1 + \rho - 2\rho^{N+1}}$

$P_n^{(N)} = 2\rho^n \frac{1 - \rho}{1 + \rho - 2\rho^{N+1}}$, $G_X(\theta) = \sum_{n \geq 0} P_n^{(N)} \theta^n$

\ip2022/ \1di/



\1dii/ let $\theta_0 = 0$, $\theta_N = 1$ $\therefore$

$\theta_i = p\theta_{i+1} + q\theta_{i-1} + (1-p-q)\theta_i$ $\therefore$

let $\theta_i = A\lambda^i$ $\therefore$ $\theta_{i-1} = A\lambda^{i-1}$ $\therefore$ $\theta_i = A\lambda^{i-1}\lambda$ $\therefore$ $\theta_{i+1} = A\lambda^{i-1}\lambda^2$ $\therefore$

$0 = p\theta_{i+1} + (1-p-q)\theta_i - \theta_i + q\theta_{i-1} = p\theta_{i+1} + (-p-q)\theta_i + q\theta_{i-1} =$

$pA\lambda^{i-1}\lambda^2 + (-p-q)A\lambda^{i-1}\lambda + qA\lambda^{i-1} =$

$A\lambda^{i-1}\left[p\lambda^2 + (-p-q)\lambda + q\right] = 0 \Rightarrow p\lambda^2 + (-p-q)\lambda + q$ $\therefore$

$\lambda = \left[-(-p-q) \pm ((-p-q)^2 - 4pq)^{1/2}\right]/(2p) =$

$\left[p+q \pm (p^2+q^2+2pq-4pq)^{1/2}\right]/(2p) =$

$\left[p+q \pm ((p^2+q^2-2pq))^{1/2}\right]/(2p) =$

$\left[p+q \pm ((p-q)^2)^{1/2}\right]/(2p) =$

$\left[p+q \pm (p-q)\right]/(2p) = \dfrac{p+q\pm(p-q)}{2p}$

$\therefore$ $\lambda_+ = \dfrac{p+q+(p-q)}{2p} = \dfrac{2p}{2p} = 1$, $\lambda_- = \dfrac{p+q-p+q}{2p} = \dfrac{2q}{2p} = \dfrac{q}{p} = \rho$ $\therefore$

$\theta_i = A(1)^i + B(\rho)^i = A + B(\rho)^i$ $\therefore$

$\theta_0 = 0 = A + B(\rho)^0 = A + B(1) = A + B = 0$ $\therefore$ $A = -B$ $\therefore$

$\theta_i = B(-1+\rho^i)$ $\therefore$ $\theta_N = 1 = B(-1+\rho^N)$ $\therefore$ $B = \dfrac{1}{\rho^N - 1}$ $\therefore$

$\theta_i = \dfrac{\rho^i - 1}{\rho^N - 1} = \dfrac{1-\rho^i}{1-\rho^N}$

\1diii/ $\therefore$ $N = 100$, $i = 10$ $\therefore$

$P(\text{winning} \mid i=10, N=100) = \theta_{10}\big|_{N=100} = \dfrac{1-\rho^{10}}{1-\rho^{100}}$

$\therefore$ win, lose, draw equal $\therefore$ $p = q$, $p+q+1-p-q = 1$ $\therefore$

$p = q = 1-p-q$ $\therefore$ $p = 1-2q$ $\therefore$ $q = 1-2q$ $\therefore$ $3q = 1$ $\therefore$ $q = \frac{1}{3}$ $\therefore$

$p = \frac{1}{3}$ $\therefore$ $\rho = \dfrac{q}{p} = \dfrac{(1/3)}{(1/3)} = 1$ $\therefore$

$P(\text{winning} \mid i=10, N=100) = \lim_{\rho\to 1} \dfrac{1-\rho^{10}}{1-\rho^{100}} = \lim_{\rho \to 1} \dfrac{-10\rho^9}{-100\rho^{99}} = \dfrac{-10(1)^9}{-100(1)^{99}} = 0.1$

$\therefore$ $P(\text{losing} \mid i=10, N=100) = 1 - P(\text{winning} \mid i=10, N=100) = 1 - 0.1 = 0.9$

$$e = \frac{-(2qp-1) \pm \sqrt{(2qp-1)^2 - 4(1)(q^2)}}{2(1)} = \frac{-2qp+1}{2} \pm \frac{\sqrt{4q^2p^2+1-4qp-4q^2}}{2}$$

$e^2 + (2qp-1)e + q^2 = 0 \therefore \quad 1-q=p \therefore$

~~ez~~ for $e=1$: $1^2 + (2qp-1)1 + q^2 = 1 + 2qp - 1 + q^2 = 2qp + q^2 =$

$2q(1-q) + q^2 = 2q - 2q^2 + q^2 = 2q - q^2 = 2(1-p) - (1-p)^2 =$

$2 - 2p - 1 - p^2 + 2p = 1 - p^2$

for $e = \frac{q}{p}$: $\left(\frac{q}{p}\right)^2 + (2qp-1)\frac{q}{p} + q^2 = \frac{q^2}{p^2} + 2q^2 - \frac{q}{p} + q^2 = \frac{q^2}{p^2} + 3q^2 - \frac{q}{p}$

$4q^2p^2 + 1 - 4qp - 4q^2 = 4(1-p)^2p^2 + 1 - 4(1-p)p - 4(1-p)^2 =$

$4(1+p^2-2p)p^2 + 1 - 4(p-p^2) - 4(1+p^2-2p) =$

$(4+4p^2-8p)p^2 + 1 - 4p + 4p^2 - 4 - 4p^2 + 8p =$

$4p^2 + 4p^4 - 8p^3 + 1 - 4p + 4p^2 - 4 - 4p^2 + 8p$ ~~ez~~

\1biii/ $k=2 \therefore G_X(\theta) = (1-p+p\theta)^2 \therefore$ ~~ez~~ $p > \frac{1}{k} = \frac{1}{2} \therefore e < 1 \therefore$

$G_X(e) - e = 0 = (1-p+pe)^2 - e = (1-p)^2 + p^2e^2 + 2(1-p)pe - e =$

$p^2e^2 + (2p-2p^2)e - e + (1+p^2-2p) =$

$p^2e^2 + (-2p^2+2p-1)e + (1+p^2-2p) = 0 \therefore$

$e = \left[-(-2p^2+2p-1) \pm \left((-2p^2+2p-1)^2 - 4p^2(1+p^2-2p)\right)^{1/2}\right] / (2p^2) \therefore$

$2p^2e = (2p^2-2p+1) \pm (4p^4 + 1 + 4p^2 - 8p^3 + 4p^2 - 4p - 4p^2 - 4p^4 + 8p^3)^{1/2} =$

$(2p^2-2p+1) \pm (4p^2-4p+1)^{1/2} = (2p^2-2p+1) \pm ((2p-1)^2)^{1/2} =$

$(2p^2-2p+1) \pm (2p-1) \therefore$

$2p^2e = 2p^2 - 2p + 1 + 2p - 1 = 2p^2$, $2p^2e = 2p^2 - 2p + 1 - 2p + 1 = 2p^2 - 4p + 2 = 2(p^2-2p+1)$

$\therefore e = \frac{2p^2}{2p^2} = 1$, $e = \frac{2(p^2-2p+1)}{2p^2} = \frac{p^2-2p+1}{p^2} = \frac{(p-1)^2}{p^2} < 1 \therefore$

take ~~negative~~ $e_- \therefore \quad e = \frac{(p-1)^2}{p^2}$ is P(ultimate extinction

\1biv/ $\therefore$ let N be initial population size $\therefore G_N(\theta) = e^{4(\theta-1)}$

$\therefore E(N) = G_N'(1) = 4e^{4(\theta-1)}\big|_{\theta=1} = 4e^0 = 4 \therefore$

for $k=5, p=0.1$ $\therefore E(X) = 0.5 \therefore E(\text{old } S_n) = (E(X))^n = 0.5^n \therefore$

$E(\text{new } S_n) = E(N)E(\text{old } S_n) = 4 \cdot 0.5^n \therefore$

$E(\text{new } S_5) = 4 \times 0.5^5 = 0.125$

\PP2022/ \1ai/ $E(Y)=E(5X)=5E(X)$

$E(X)=G_X'(\theta)|_{\theta=1}=\cancel{0.2}\,2\times0.4\theta+4\times0.2\theta^3+0.2\times6\theta^5|_{\theta=1}=$

$\cancel{0.2}\,0.8+0.8+1.2=2.8$ ∴

$E(Y)=5\times2.8=14$

$P(Y\le5)=1-P(Y\ge6)=\cancel{1-P}\ P(5X\le5)=P(X\le1)=P(X=0)=0.2$

\1aii/ $E(X^2+1)=E(Y)=E(X^2)+1$

$E(X^2)=0.2\times0^2+0.4\times2^2+0.2\times4^2+0.2\times6^2=12$ ∴

$E(Y)=12+1=13$

$P(Y\le5)=P(X^2+1\le5)=P(X^2\le2)$

∴ $x\in\{0,2,4,6\}$ ∴ $x^2\in\{0,4,16,36\}$ ∴

$P(Y\le5)=P(X^2\le2)=P(X\le\sqrt{2})=P(X\le1)=P(X=0)=0.2$

\1aiii/ $E(Y)=E(X_1+X_2+X_3)=E(X_1)+E(X_2)+E(X_3)=E(X)+E(X)+E(X)$ by independence

$=3E(X)=3\times2.8=8.4$

$G_Y(\theta)=G_{X_1+X_2+X_3}(\theta)=G_{X_1}(\theta)G_{X_2}(\theta)G_{X_3}(\theta)=G_X(\theta)G_X(\theta)G_X(\theta)=(G_X(\theta))^3$ ∴

$P((Y)\le5)=P(X_1+X_2+X_3\le5)$ ∴

$G_Y(\theta)=(G_X(\theta))^3=(0.2+0.4\theta^2+0.2\theta^4+0.2\theta^6)^3=$

$0.2^3\frac{3!}{3!}+0.2^1\times(0.4\theta^2)^2\frac{3!}{1!2!}+0.2^2\times(0.4\theta^2)^1\frac{3!}{2!1!}+0.2^2\times(0.2\theta^4)^1\frac{3!}{2!1!}+\dots=$

$0.008+0.096\theta^4+0.048\theta^2+0.024\theta^4+\dots=$

$0.008+0.048\theta^2+0.12\theta^4+\dots$ ∴

$P(Y\le5)=0.008+0.048+0.12=0.176$

\1bi/ ∴ $G_X(\theta)=(1-p+p\theta)^k$

∴ for $k=5$ $p=0.1$. $E(X)=G_X'(\theta)|_{\theta=1}=\frac{d}{d\theta}((1-0.1+0.1\theta)^5)|_{\theta=1}=$

$\frac{d}{d\theta}((0.9+0.1\theta)^5|_{\theta=1}=5(0.9+0.1\theta)^4(0.1)|_{\theta=1}=5(0.9+0.1)^4(0.1)=0.5$

\1bii/ extinction guaranteed ∴ $e=1$ for $E(X)<1$ ∴

$G_X(\theta)=(1-p+p\theta)^k$ ∴ $X\sim$ Binomial (k,p) ∴

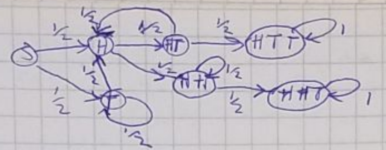
$E(X)=kp<1$ ∴ $0\le p\le1$ ∴ $p<\frac{1}{k}$ ∴ $0\le p<\frac{1}{k}$ for guaranteed

\1biii/ $k=2$ ∴ $G_X(\theta)=(1-p+p\theta)^2$ ∴ let $1-p=q$ ∴ $G_X(\theta)=(q+p\theta)^2$

∴ for not guaranteed ∴ $p>\frac{1}{k}=\frac{1}{2}$ ∴ $e<1$ ∴

$G_X(e)-e=0=q^2+p^2e^2+2qpe-e=0=p^2e^2+(2qp-1)e+q^2=0$ ∴

\4 a iii/



$\therefore \theta_{HTT} = 1, \quad \theta_{HHT} = 0 \therefore$

$\theta_{HH} = \frac{1}{2}\theta_{HH} + \frac{1}{2}\theta_{HHT} = \frac{1}{2}\theta_{HH} + 0 = \frac{1}{2}\theta_{HH} \quad \therefore \frac{1}{2}\theta_{HH} = 0 \quad \therefore \theta_{HH} = 0 \therefore$

$\theta_{HT} = \frac{1}{2}\theta_{H} + \frac{1}{2}\theta_{HTT} = \frac{1}{2}\theta_{H} + \frac{1}{2} \quad \therefore$

$\theta_{H} = \frac{1}{2}\theta_{HT} + \frac{1}{2}\theta_{HH} = \frac{1}{2}\theta_{HT} = \frac{1}{2}\left(\frac{1}{2}\theta_{H} + \frac{1}{2}\right) = \frac{1}{4}\theta_{H} + \frac{1}{4} \therefore$

$\frac{3}{4}\theta_{H} = \frac{1}{4} \quad \therefore \theta_{H} = \frac{1}{3} \quad \therefore$

$\theta_{T} = \frac{1}{2}\theta_{T} + \frac{1}{2}\theta_{H} \quad \therefore \frac{1}{2}\theta_{T} = \frac{1}{2}\theta_{H} = \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{6} \quad \therefore \theta_{T} = \frac{1}{3} \therefore$

$\theta_{S} = \frac{1}{2}\theta_{T} + \frac{1}{2}\theta_{H} = \frac{1}{2}\left(\frac{1}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{3} = P(\text{HTT appears before HHT})$

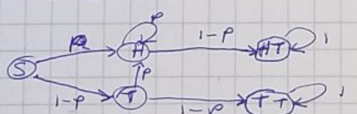
$\therefore$ ~~Pt~~ from (a i) $P(\text{TT appears before HT}) = \frac{1}{4} \therefore$

$\frac{1}{4} < \frac{1}{3} \therefore$

$P(\text{HTT appears before HHT}) > P(\text{TT appears before HT})$ X

no change $\because$ the games are equivalent $\because$ starting from H both players need TT before HT

\4 b i/



$\therefore$ ~~Chance~~ $\theta_S = \frac{1}{2} \therefore$ let $\theta_{HT} = 0, \quad \theta_{TT} = 1 \therefore$

$\theta_H + p\theta_H + (1-p)\theta_{HT} = p\theta_H \quad \therefore (1-p)\theta_H = 0 \quad \therefore \theta_H = 0 \therefore$

$\theta_T = p\theta_H + (1-p)\theta_{TT} = (1-p) \therefore$

$\theta_S = p\theta_H + (1-p)\theta_T = p(0) + (1-p)(1-p) = 1 - 2p + p^2 = \frac{1}{2} \therefore$

$p^2 - 2p + \frac{1}{2} = 0 \therefore$

$p = \frac{2 \pm \sqrt{4 - 4(1)(\frac{1}{2})}}{2(1)} = \frac{2 \pm \sqrt{2}}{2} \quad \therefore p = \frac{2-\sqrt{2}}{2} = 0.293, \quad p = \frac{2+\sqrt{2}}{2} = 1.7 + > 1$

$p < 1 \therefore \quad p = \frac{2-\sqrt{2}}{2} = 0.293$ (3 s.f.) $= 1 - \frac{1}{\sqrt{2}}$ ✓

\4 b ii/ $\therefore D_{HT} = 0, \quad D_{TT} = 0 \therefore \quad D_H = 1 + pD_H + (1-p)D_{HT} = 1 + pD_H$ ~~+(1-p)~~ $\therefore$

$(1-p)D_H = 1 \quad \therefore D_H = \frac{1}{1-p}, \quad D_T = pD_H + (1-p)D_{TT} + 1 = pD_H + 1 = \frac{p}{1-p} + 1 = \frac{p + 1 - p}{1-p} = \frac{1}{1-p}$

$\therefore D_S = pD_H + (1-p)D_T + 1 = p\frac{1}{1-p} + \frac{1-p}{1-p} + 1 = \frac{p}{1-p} + 1 + 1 = \frac{p}{1-p} + 2 = \frac{p + 2 - 2p}{1-p} = \frac{2-p}{1-p} \therefore$

$p = \frac{2-\sqrt{2}}{2} \quad \therefore D_S = \frac{2 - \left(\frac{2-\sqrt{2}}{2}\right)}{1 - \left(\frac{2-\sqrt{2}}{2}\right)} = 1 + \sqrt{2} = 2.41$ (3 s.f.) is E(steps to end) ✓

\PP 2022/

~~$\lambda_{B+C} = 4+2 = 6$~~

~~$P(N_A(1)=5)$~~ $P(N_{B+C})$

$\cap$ $P(N_A(1)=5)\left[P(N_B(1)=1)+P(N_C(1)=1)\right] =$

$e^{-5\times1}(5\times1)^5 \frac{1}{5!}\left[e^{-4\times1}(4\times1)^1\frac{1}{1!}+e^{-2\times1}(2\times1)^1\frac{1}{1!}\right] = 0.0603$ $\therefore$

$P(6$ mins over 1 hour with 5 from A$) = \frac{0.0603}{0.0411} = 1.47$ X

\2bii/ P(6 mins over 1 hour with 5 from A) =

P(5 mins from A in 1 hour and 1 min from B or C) =

P(5 mins from A in 1 hour)[P(1 mins from B in 1 hr) + P(1 min from C in 1 hr)] =

$P(N_A(1)=5)\left[P(N_B(1)=1)+P(N_C(1)=1)\right] =$

$e^{-5\times1}(5\times1)^5\frac{1}{5!}\left[e^{-4\times1}(4\times1)^1\frac{1}{1!}+e^{-2\times1}(2\times1)^1\frac{1}{1!}\right] = 0.0603$

\2bii/ P(5 mins on A in 1 hour | 6 total mins in 1 hour) =

P(5 mins on A in 1 hour $\cap$ 6 total mins in 1 hour)/P(6 total mins in 1 hour)

= ~~P(5 mins~~ P(5 mins on A in 1 hour $\cap$ 1 min on B or C in 1 hour)/$P(N_\lambda(1)=6)$

= P(5 mins on A in 1 hour)[P(1 min on B in 1 hour) + P(1 min on C in 1 hour)]/$P(N_\lambda(1)=6)$

$= P(N_A(1)=5)\left[P(N_B(1)=1)+P(N_C(1)=1)\right]/P(N_\lambda(1)=6)$

$e^{-(5\times1)}(5\times1)^5\frac{1}{5!}\left[e^{-(4\times1)}(4\times1)^1\frac{1}{1!}+e^{-(2\times1)}(2\times1)^1\frac{1}{1!}\right]\frac{1}{e^{-(11\times1)}(11\times1)^6\frac{1}{6!}} = \frac{0.0603}{0.0411} = 1.47$ X

\2biii/ P(5 mins on A in t=1 | 6 mins on all in t=1) =

P(5 mins on A in t=1 $\cap$ 6 mins on all in t=1)/P(6 mins on all in t=1) =

P(5 mins on A in t=1 $\cap$ 1 min on B or C in t=1)/$P(N_\lambda(1)=6)$ =

P(5 mins on A in t=1)P(1 min on B or C in t=1)/$P(N_\lambda(1)=6)$ =

$P(N_A(1)=5)P(N_{B+C}(1)=1)/P(N_\lambda(1)=6)$

$\therefore \lambda_{B+C} = \lambda_B+\lambda_C = 4+2 = 6 \therefore P(N_{B+C}(1)=1) = e^{-(6\times1)}(6\times1)^1\times\frac{1}{1!} = 0.01487$

$P(N_A(1)=5) = e^{-(5\times1)}(5\times1)^5\frac{1}{5!} = 0.1755$

~~$P(N_\lambda$~~ $\lambda = \lambda_A+\lambda_B+\lambda_C = 5+4+2 = 11 \therefore P(N_\lambda(1)=6) = e^{-(11\times1)}(11\times1)^6\frac{1}{6!} = 0.04109$

$\therefore$ P(5 mins on A in 1 hour | 6 total mins in 1 hour) =

$0.1755\times0.01487\times\frac{1}{0.04109} = 0.0635$

\2bi/ $\lambda = \lambda_A + \lambda_B + \lambda_C = 5+4+2 = 11$ ∴

$E(\text{time between wins}) = \frac{1}{\lambda} = \frac{1}{11}$ hours $= 0.0910$ hours $= 5.45$ minutes

\2bii/ ~~P(N~~ P(≥2 wins on B and 2 wins on C in $\frac{1}{2}$ hour) =

$P(N_B(\frac{1}{2}) \geq 2 \cap N_C(\frac{1}{2}) \geq 2) = P(N_B(\frac{1}{2}) \geq 2) P(N_C(\frac{1}{2}) \geq 2) =$

$P(N_B(\frac{1}{2}) \geq 2) = 1 - P(N_B(\frac{1}{2}) \leq 1) =$

$1 - [e^{-\lambda_B \frac{1}{2}}(1 + \frac{(\lambda_B \frac{1}{2})^1}{1!})] = 1 - [e^{-4 \times \frac{1}{2}}(1 + \frac{(4 \times \frac{1}{2})^1}{1})] = 0.594$

$P(N_C(\frac{1}{2}) \geq 2) = 1 - P(N_C(\frac{1}{2}) \leq 1) = 1 - e^{-\lambda_C \frac{1}{2}}(1 + \frac{(\lambda_C \frac{1}{2})^1}{1!}) =$

$1 - e^{-2 \times \frac{1}{2}}(1 + \frac{(2 \times \frac{1}{2})^1}{1}) = 0.264$ ∴

P(2 or more wins on B, and on C in $\frac{1}{2}$ hour) $= 0.594 \times 0.264 = 0.157$

\1biii/ $P(N_A(1) = 5 \mid N(1) = 6) = \frac{P(N_A(1)=5, N(1)=6)}{P(N(1)=6)} =$

$\frac{P(N_A(1)=5, N_{B+C}(1)=1)}{P(N(1)=6)} = \frac{e^{-5}(5)^5 \frac{1}{5!} \times \frac{e^{-(4+2)}(4+2)^1}{1!}}{e^{-11}(11)^6 \frac{1}{6!}} = \frac{0.00261}{0.0411} = 0.0635$

\2bi/ $\lambda = \lambda_A + \lambda_B + \lambda_C = 5+4+2 = 11$ per hour ∴

Mean $= \frac{1}{\text{rate}} = \frac{1}{11}$ hours $= 5.45$ minutes

\2bii/ ~~P(N~~ P(at least two wins on B in 30 mins) =

$P(N_B(\frac{1}{2}) \geq 2) = 1 - P(N_B(\frac{1}{2}) \leq 1) = 1 - e^{-(4 \times \frac{1}{2})}[(4 \times \frac{1}{2})^0 \frac{1}{0!} + (4 \times \frac{1}{2})^1 \times \frac{1}{1!}] = 0.594$

P(at least two wins on C in 30 minutes) =

$P(N_C(\frac{1}{2}) \geq 2) = 1 - P(N_C(\frac{1}{2}) \leq 1) = 1 - e^{-(2 \times \frac{1}{2})}[(2 \times \frac{1}{2})^0 \frac{1}{0!} + (2 \times \frac{1}{2})^1 \frac{1}{1!}] = 0.264$ ∴

P(at least two wins on B and on C in 30 minutes) =

$P(N_B(\frac{1}{2}) \geq 2) P(N_C(\frac{1}{2}) \geq 2) = 0.594 \times 0.264 = 0.157$

\1biii/ ~~this follows a binomial distribution with n=6, 1/6~~

P(5 wins on A in past hour | 6 wins total in past hour) =

P(5 wins on A in past hour ∩ 6 wins total in past h)/P(6 wins total in past hour)

∴ P(6 wins in past hour) $= P(N_\lambda(1) = 6) = e^{-(11 \times 1)}(11 \times 1)^6 \frac{1}{6!} = 0.0411$

P(5 wins on A in past hour ∩ 6 wins total in past hour) =

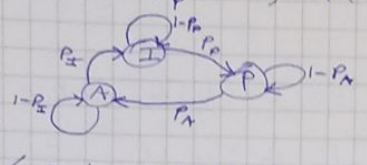
P(5 wins on A in past hour ∩ 1 win on B or C in past hour) =

P(5 wins on A in 1 hour) P(1 win on B or C in 1 hour) =

\1ci/ $\mathrm{Exp}(\lambda) \sim \lambda e^{-\lambda x}$ $\quad P_I + P_P + P_A = 1$ ∴

~~P(I~~ let Idle, Primed, Active = I, P, A ∴

$P(I \to P \mid I) = P_P$, $\quad P(P \to A \mid P) = P_A$, $\quad P(A \to I \mid A) = P_I$ ∴



\1cii/ $P_I(t+\delta t) = P($machine is idle at ~~state~~ time $t+\delta t)$

∴ $f_I(t) = \lambda_I e^{-\lambda_I t} = 3e^{-3t}$ ∴

$P_I(t+\delta t) = f_I(t+\delta t) = 3e^{-3(t+\delta t)} = 3e^{-3t-3\delta t} = 3e^{-3t}e^{-3\delta t} =$

$3e^{-3t}\sum_{n=0}^{\infty}\frac{(-3\delta t)^n}{n!} = 3e^{-3t}\left[\frac{(-3\delta t)^0}{0!} + \frac{(-3\delta t)^1}{1!} + \dots\right] =$

$3e^{-3t}\left[1 + -3\delta t + \dots\right] \approx 3e^{-3t}\left[1 - 3\delta t\right] = 3e^{-3t} - 3^2 e^{-3t}\delta t =$

$3e^{-3t} - 9e^{-3t}\delta t$ ∴

$f_P(t) = 1e^{-1t} = e^{-t}$ ∴

by symmetry: $P_P(t+\delta t) \approx 1e^{-t} - 1^2 e^{-1t}\delta t = e^{-t} - e^{-t}\delta t$

$f_A(t) = 2e^{-2t}$ ∴

by symmetry: $P_A(t+\delta t) \approx 2e^{-2t} - 2^2 e^{-2t}\delta t = 2e^{-2t} - 4e^{-2t}\delta t$

\1ciii/ $P_I(t+\delta t) =$ ~~$P_I(t) \times P($no state change in time $\delta t)$~~ by t.o.t.p.

~~by t.o.t.p.~~ =

$P_I(t) \times P($no state change in time $\delta t) +$

$P_A(t) \times P($state change ~~of~~ A to I in time $\delta t) +$

$P_P(t) \times P($state changes ~~of~~ P to I in time $\delta t)$ ~~=~~

~~$P_I(t)$~~ ∴ $P(N(\delta t) = 0) = e^{-\lambda \delta t}(\lambda \delta t)^0 \frac{1}{0!} = e^{-\lambda \delta t} = \sum_{n=0}^{\infty}\frac{(-\lambda \delta t)^n}{n!} \approx 1 - \lambda \delta t$

$P(N(\delta t) = 1) = e^{-\lambda \delta t}(\lambda \delta t)^1 \frac{1}{1!} = \lambda \delta t e^{-\lambda \delta t} = \lambda \delta t \sum_{n=0}^{\infty}\frac{(-\lambda \delta t)^n}{n!} \approx \lambda \delta t$

$P(N(\delta t) = 2) = e^{-\lambda \delta t}(\lambda \delta t)^2 \frac{1}{2!} \approx 0$ ∴

$P_I(t+\delta t) = P_I(t)(1 - \lambda_I \delta t) + P_A(t)(\lambda_I \delta t) = (1 - \lambda_I \delta t)P_I + \lambda_I \delta t P_A =$

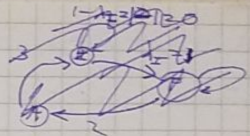
$P_I(1 - 3\delta t) + 3P_A \delta t$ ∴ $\quad$ by symmetry: $P_I(t+\delta t) = P_I + (-3P_I + 3P_A)\delta t$

$P_A(t+\delta t) = P_A(1 - 2\delta t) + 2P_P \delta t = P_A + (-2P_A + 3P_P)\delta t$

$P_P(t+\delta t) = P_P(1 - \delta t) + P_I \delta t = P_P + (-P_P + P_I)\delta t$

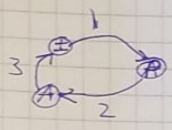
\PP2022/ ~~$\frac{dP_I}{dt}$~~

\1civ/ in steady state : ~~$\frac{dP_I}{dt} = -P_I$~~



~~$\frac{dP_I}{dt} = \lambda_I P_I$~~ $\frac{dP_I(t)}{dt} =$

\1ci/ is $X \sim Exp(\lambda)$ then $f_x(x;\lambda) = \lambda e^{-\lambda x}$ $\therefore E(X) = \frac{1}{\lambda}$ $\therefore$



\1cii/ $P_I(t+\delta t) = \cancel{e^{-1\delta t}} = \cancel{e^{-\delta t}}$ $1(e^{-1(t+\delta t)}) = e^{-t - \delta t} = e^{-t}e^{-\delta t}$

$= e^{-t}\sum_{i=1}^{\infty}\frac{(-\delta t)^n}{n!} = e^{-t}[1 + -\delta t + \dots] \approx e^{-t}[1 - \delta t] = e^{-t} - e^{-t}\delta t$

$P_P(t+\delta t) = \cancel{2e^{-2\delta t}}$ $2(e^{-2(t+\delta t)}) = 2e^{-2t-2\delta t} = 2e^{-2t}e^{-2\delta t} =$

$2e^{-2t}\sum_{n=0}^{\infty}\frac{(-2\delta t)^n}{n!} = 2e^{-2t}[1 - 2\delta t + \dots] \approx 2e^{-2t}[1 - 2\delta t] =$

$2e^{-2t} - 4e^{-2t}\delta t$

$P_A(t+\delta t) = 3(e^{-3(t+\delta t)}) = 3e^{-3t}e^{-3\delta t} \approx 3e^{-3t}[1 - 3\delta t] =$

$3e^{-3t} - 9e^{-3t}\delta t$

\1ciii/ $\therefore$ $\frac{dP_I}{dt} = -1P_P$ and $\frac{dP_P}{dt} = -2P_A$, $\frac{dP_A}{dt} = -3P_I$

$\frac{dP_I}{dt} + \frac{dP_P}{dt} + \frac{dP_A}{dt} = 0 = -P_P - 2P_A - 3P_I$

$P_I + P_P + P_A = 1$

\1civ/ ~~$P_I(t+\delta t) = P_I$~~ - $T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 3 & 0 & 0 \end{bmatrix}$

$\tilde{P} = (P_I \;\; P_P \;\; P_A) = \tilde{P}T = (P_I \;\; P_P \;\; P_A)\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 3 & 0 & 0 \end{bmatrix} = (3P_A \;\; P_I \;\; 2P_P)$

$P_I + P_P + P_A = 1$ $\therefore$ $P_I = 3P_A$, $P_P = P_I$ $2P_P = P_A$ $\therefore$ $\frac{1}{3}P_I = P_A$

~~$P_I + P_I + P_I = 1 \quad 3P_I \quad \therefore P_I = \frac{1}{3}$~~ $\therefore$

$P_I + P_I + \frac{1}{3}P_I = 1 = \frac{7}{3}P_I$ $\therefore$ $\frac{3}{7} = P_I$ $\therefore$

$P_P = \frac{3}{7}$ $\therefore$ $\frac{1}{3} \times \frac{3}{7} = P_A = \frac{1}{7}$ $\therefore$

$\tilde{P} = (\frac{3}{7}, \frac{3}{7}, \frac{1}{7})$ for steady state

\2ai/ memoryless : $P(T > t+s \mid T > s) = P(T > t)$

\2aii/ $P(T > t+s \mid T > s) = \frac{P(T > t+s, T > s)}{P(T > S)} = \frac{P(T > t+S)}{P(T > S)}$ $P(T \le s) = F_T(s) = \int_0^s \lambda e^{-\lambda t} dt =$

$[-e^{-\lambda t}]_0^s = 1 - e^{-\lambda s}$ $\therefore$ $P(T > S) = e^{-\lambda s}$ $\therefore$ $\frac{P(T > t+S)}{P(T > S)} = \frac{e^{-\lambda(t+S)}}{e^{-\lambda s}} = \frac{e^{-\lambda t}e^{-\lambda s}}{e^{-\lambda s}} = e^{-\lambda t} = P(T > t)$

\PP2022/ \3g/ for $P_n = P > 0$, $n > 2$: ∴

in steady state : $\frac{dP_0}{dt} = -\lambda P_0 + \mu P_1 = 0$ ∴ $\mu P_1 = \lambda P_0$ ∴ $P_1 = \frac{\lambda}{\mu} P_0$ ∴

$\frac{dP_1}{dt} = -\lambda P_1 + 2\mu P_2 - \mu P_1 = 0$ ∴ $2\mu P_2 = \lambda P_1 + \mu P_1 = (\lambda + \mu) P_1$

∴ $P_1 = 2\frac{\lambda}{2\mu} P_0 = 2\rho P_0$, $P_2 = \left(\frac{\lambda}{2\mu} + \frac{\mu}{2\mu}\right) P_1 = \left(\rho + \frac{1}{2}\right) 2\rho P_0 = (2\rho^2 + \rho) P_0$

∴ $\sum_{n=0}^{\infty} P_n = P_0 + P_1 + P_2 + \sum_{n=3}^{\infty} P_n = P_0 + P_1 + P_2 + \sum_{n=3}^{\infty} P = P_0 + P_1 + P_2 + P \sum_{n=3}^{\infty} 1 =$

$P_0 + P_1 + P_2 + P(\infty) = P_0 + P_1 + P_2 + \infty = \infty \neq 1$ but

$\sum_{n=0}^{\infty} P_n = 1$ need to be true to be admissible ∴ $P_n = P > 0$ for $n > 2$

is not admissible in steady state because it means all the states for $n > 2$ will share a total probability equally divided by those infinite states for $P > 0$ which is impossible

\3h/ ~~P0~~ $P_1 = 2\rho P_0$, $P_2 = (2\rho^2 + \rho) P_0$ ∴ $P_n = A P_{n-1}$; $n > 2$ ∴

let $1 = \sum_{n=0}^{\infty} P_n = P_0 + P_1 + P_2 + \sum_{n=3}^{\infty} P_n = P_0 + P_1 + P_2 + \sum_{n=3}^{\infty} A P_{n-1} = P_0 + P_1 + P_2 + A \sum_{n=3}^{\infty} P_{n-1} =$

$P_0 + P_1 + P_2 + A \sum_{n=2}^{\infty} P_n = P_0 + P_1 + P_2 + A\left[P_2 + P_3 + P_4 + \dots\right] =$

$P_0 + P_1 + P_2 + A\left[P_2 + AP_2 + AP_3 + AP_4 + \dots\right] = P_0 + P_1 + P_2 + A\left[P_2 + AP_2 + AAP_2 + AAAP_2 + \dots\right] =$

$P_0 + P_1 + P_2 + A\left[P_2 + AP_2 + A^2 P_2 + A^3 P_2 + \dots\right] =$

$P_0 + P_1 + P_2 + A \sum_{n=0}^{\infty} P_2 A^n = P_0 + P_1 + P_2 + A P_2 \sum_{n=0}^{\infty} A^n = P_0 + P_1 + P_2 + P_2 A\left[\frac{1}{1-A}\right]$ for $|A| < 1$

$= P_0 + 2\rho P_0 + P_2 + \frac{A}{1-A} P_2 = P_0 + 2\rho P_0 + \left[1 + \frac{A}{1-A}\right] P_2 = P_0 + 2\rho P_0 + \left[\frac{1-A}{1-A} + \frac{A}{1-A}\right] P_2$

$= P_0 + 2\rho P_0 + \left[\frac{1}{1-A}\right] P_2 = P_0 + 2\rho P_0 + \left[\frac{1}{1-A}\right](2\rho^2 + \rho) P_0 =$

$P_0\left(1 + 2\rho + \left[\frac{1}{1-A}\right](2\rho^2 + \rho)\right) = 1$ ∴

~~$P_0 = P$~~ $P_0 = \left(1 + 2\rho + \left(\frac{1}{1-A}\right)(2\rho^2 + \rho)\right)^{-1}$ ∴

$P_2 = (2\rho^2 + \rho) P_0 = \frac{2\rho^2 + \rho}{1 + 2\rho + \left(\frac{1}{1-A}\right)(2\rho^2 + \rho)}$ ∴

$P_n = A P_{n-1}$, $n > 2$ ∴ $P_3 = A P_2$, $P_4 = A^2 P_2$ ∴ $P_n = A^{n-2} P_2$ ∴

$P_n = \frac{2\rho^2 + \rho}{1 + 2\rho + \left(\frac{1}{1-A}\right)(2\rho^2 + \rho)} A^{n-2}$ for $n > 2$

3d/ $\rho = \frac{\lambda}{2\mu}$, $P_3/P_0 = \left[\frac{1}{2\mu}\left(\frac{(\lambda+2\mu)(\lambda^2+\lambda\mu)}{2\mu^2} - \lambda P_0\right)\right] / P_0 =$

$$\frac{1}{2\mu}\left(\frac{(\lambda+2\mu)(\lambda^2+\lambda\mu)}{2\mu^2} - \lambda\right) = \frac{1}{2\mu}\left(\frac{(\lambda+2\mu)(\lambda^2+\lambda\mu)}{2\mu^2}\right) - \frac{\lambda}{2\mu} =$$

$$\frac{1}{2\mu}\left(\frac{\lambda^3+\lambda^2\mu+2\lambda^2\mu+2\lambda\mu^2}{2\mu^2}\right) - \rho =$$

$$\frac{\lambda^3+\lambda\mu+2\lambda^2\mu+2\lambda\mu^2}{(2\mu)^2\mu} - \rho =$$

$$\frac{\lambda^2\lambda}{(2\mu)^2\mu} + \frac{\lambda\mu}{(2\mu)(2\mu)\mu} + 2\frac{\lambda^2\mu}{(2\mu)^2\mu} + \frac{2\lambda\mu^2}{(2\mu)(2\mu)\mu} - \rho =$$

$$\rho^2(2)\frac{\lambda}{2\mu} + \frac{\rho}{2\mu} + 2\rho^2 + \frac{2\rho\mu}{2\mu} - \rho =$$

$$2\rho^2\rho + \frac{\rho}{2\mu} + 2\rho^2 + \cancel{\rho} - \rho = 2\rho^3 + \frac{\rho}{2\mu} + 2\rho^2$$

3e/ for $n>2$: $\frac{dP_n}{dt} = -\lambda P_n + 2\mu P_{n+1} - 2\mu P_n + \lambda P_{n-2} = 0$ in steady state $\therefore$ $2\mu P_{n+1} = \lambda P_n + 2\mu P_n - \lambda P_{n-2} = (\lambda+2\mu)P_n - \lambda P_{n-2}$ $\therefore$

$2\mu P_n = (\lambda+2\mu)P_{n-1} - \lambda P_{n-3}$ $\therefore$

$P_n = \frac{1}{2\mu}(\lambda+2\mu)P_{n-1} - \frac{\lambda}{2\mu}P_{n-3} = \left(\frac{\lambda}{2\mu} + \frac{2\mu}{2\mu}\right)P_{n-1} - \rho P_{n-3} = (\rho+1)P_{n-1} - \rho P_{n-3}$

for $n>2$

3f/ for $n>2$ $P_n = AP_{n-1}$ $A = \text{constant}$ $P_n \neq 0$,

$\sum_{n=0}^{\infty} P_n = 1$ $\quad P_1 = \frac{\lambda}{\mu}P_0 = \frac{2\lambda}{2\mu}P_0 = 2\rho P_0$ $\therefore$

$$\sum_{n=0}^{\infty} P_n = P_0 + P_1 + \sum_{n=0}^{\infty} P_n = P_0 + P_1 + P_2 + \sum_{n=3}^{\infty} P_n$$

$$P_2 = \frac{\lambda^2+\lambda\mu}{2\mu^2}P_0 = \left(\frac{2\lambda^2}{(2\mu)^2} + \frac{\lambda\mu}{\mu^2}\right)P_0 = \left(2\left(\frac{\lambda}{2\mu}\right)^2 + \frac{2\lambda}{2\mu}\right)P_0 = (2\rho^2+2\rho)P_0 \therefore$$

$$1 = \sum_{n=0}^{\infty} P_n = P_0 + 2\rho P_0 + (2\rho^2+2\rho)P_0 + \sum_{n=3}^{\infty} P_n = P_0 + P_1 + P_2 + \sum_{n=3}^{\infty} AP_{n-1} =$$

$$P_0 + P_1 + P_2 + A\sum_{n=3}^{\infty} P_{n-1} = P_0 + P_1 + P_2 + A\sum_{n=2}^{\infty} P_n = P_0 + P_1 + P_2 + A\left[P_2 + AP_2 + A^2P_2 + A^3P_2 + \dots\right] =$$

$$P_0 + P_1 + P_2 + A\left[\sum_{n=0}^{\infty} A^n P_2\right] = P_0 + P_1 + P_2 + AP_2\sum_{n=0}^{\infty} A^n = P_0 + P_1 + P_2 + AP_2\frac{1}{1-A} \quad \text{for } |A|<1$$

$$= P_0 + P_1 + \left(1 + \frac{A}{1-A}\right)P_2 = P_0 + 2\rho P_0 + (2\rho^2+2\rho)\left(1+\frac{A}{1-A}\right)P_0 = 1 =$$

$$P_0\left[1 + 2\rho + (2\rho^2+2\rho)\left(1+\frac{A}{1-A}\right)\right] \therefore \quad P_0 = \left[1+2\rho+(2\rho^2+2\rho)\left(1+\frac{A}{1-A}\right)\right]^{-1}$$

$\therefore$ polynomial ~~…~~ of: $1 + A + A^2 + A^3 + \dots = \frac{1}{1-A}$ needs to be satisfied for $P_n \neq 0$ for $n>2$

\PP2022/ $P_2 = \alpha P_1 + (1-\beta)P_2 = \alpha P_1 + P_2 - \beta P_2$ ∴

$\beta P_2 = \alpha P_1$ ∴ $\frac{\beta}{\alpha}P_2 = P_1$ ∴ $\frac{\alpha}{\beta}P_1 = P_2$ ∴

$P_3 = \beta P_2 + (1-\gamma)P_3 = \beta P_2 + P_3 - \gamma P_3$ ∴ $\gamma P_3 = \beta P_2$ ∴

$1 = P_1 + P_2 + P_3$ ∴

$\cancel{1 = P_1 + \frac{\alpha}{\gamma}P}$ $1 = P_1 + \frac{\alpha}{\beta}P_1 + \frac{\alpha}{\gamma}P_1 = P_1\left(1+\frac{\alpha}{\beta}+\frac{\alpha}{\gamma}\right)$ ∴

$P_1 = \frac{1}{1+\frac{\alpha}{\beta}+\frac{\alpha}{\gamma}}$ ∴

$\frac{\alpha}{\beta}P_1 = \frac{\alpha}{\beta\left(1+\frac{\alpha}{\beta}+\frac{\alpha}{\gamma}\right)} = \frac{\alpha}{\beta+\alpha+\frac{\alpha\beta}{\gamma}} = P_2$ ∴

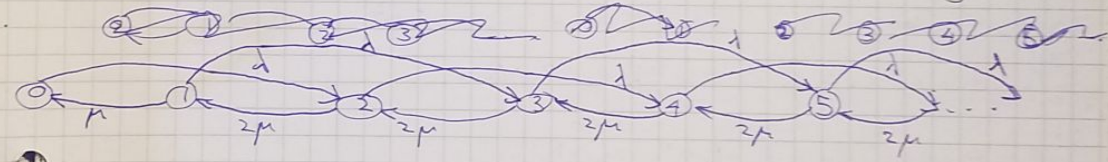
$\frac{\alpha}{\gamma}P_1 = \frac{\alpha}{\gamma\left(1+\frac{\alpha}{\beta}+\frac{\alpha}{\gamma}\right)} = \frac{\alpha}{\gamma+\frac{\alpha\gamma}{\beta}+\alpha} = P_3$ ∴

$\tilde{P} = (P_1 \quad P_2 \quad P_3) = \left(\frac{1}{1+\frac{\alpha}{\beta}+\frac{\alpha}{\gamma}} \quad \frac{\alpha}{\beta+\alpha+\frac{\alpha\beta}{\gamma}} \quad \frac{\alpha}{\gamma+\frac{\alpha\gamma}{\beta}+\alpha}\right)$

\3a/ a ~~$G_1/G_2/r$ queue is one that~~ $M(\lambda)/M(\mu)/r$ queue is one whose arrival process is governed by a probability distribution $M(\lambda)$, service process is governed by a probability distribution $M(\mu)$. r is number of servers

\3b/ This is a M/M/2 queue with infinite capacity ∴

∴



in steady state: $\frac{dP_0}{dt} = -\lambda P_0 + \mu P_1 = 0$ ∴ $\lambda P_0 = \mu P_1$ ∴ $\frac{\lambda}{\mu}P_0 = P_1$

$\frac{dP_1}{dt} = -\lambda P_1 + 2\mu P_2 - \mu P_1 = 0$ ∴ $2\mu P_2 = \lambda P_1 + \mu P_1 = (\lambda+\mu)P_1$ ∴

$P_2 = \frac{\lambda+\mu}{2\mu}P_1 = \frac{\lambda+\mu}{2\mu}\frac{\lambda}{\mu}P_0 = \frac{\lambda^2+\lambda\mu}{2\mu^2}P_0$

$\frac{dP_2}{dt} = -\lambda P_2 + 2\mu P_3 + \lambda P_0 - 2\mu P_2 = 0$ ∴ $2\mu P_3 = \lambda P_2 + 2\mu P_2 - \lambda P_0 = (\lambda+2\mu)P_2 - \lambda P_0$

$= (\lambda+2\mu)\frac{\lambda^2+\lambda\mu}{2\mu^2}P_0 - \lambda P_0 = \left(\frac{(\lambda+2\mu)(\lambda^2+\lambda\mu)}{2\mu^2} - \lambda\right)P_0 = \cancel{2}$ ∴

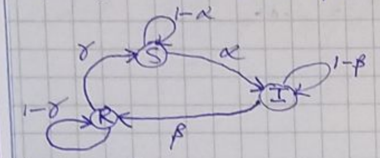
$P_3 = \frac{1}{2\mu}\left(\frac{(\lambda+2\mu)(\lambda^2+\lambda\mu)}{2\mu^2} - \lambda\right)P_0$

$\frac{dP_3}{dt} = -\lambda P_3 + 2\mu P_4 - 2\mu P_3 + \lambda P_1 = 0$ ∴ $2\mu P_4 = \lambda P_3 + 2\mu P_3 - \lambda P_1 = (\lambda+2\mu)P_3 - \lambda P_1$

$= (\lambda+2\mu)\frac{1}{2\mu}\left(\frac{(\lambda+2\mu)(\lambda^2+\lambda\mu)}{2\mu^2} - \lambda\right)P_0 - \lambda\frac{\lambda}{\mu}P_0$ ∴

$\cancel{P_4 = (\lambda+2\mu)}$ $P_4 = \frac{1}{2\mu}(\lambda+2\mu)\frac{1}{2\mu}\left(\frac{(\lambda+2\mu)(\lambda^2+\lambda\mu)}{2\mu^2} - \lambda\right)P_0 - \lambda\frac{\lambda}{\mu}P_0$

\4a/ let susceptible, infected, recovered be S, I, R ∴



\4b/ none of the probabilities depend on time ∴

α, β, γ are fixed ~~in~~ time ∴ constant

∴ the transition matrix has $T_{ii} \neq 1$ and $T_{ij} > 0$

for $i = S, I, R$ ∴ none of the states are transient

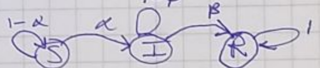
∴ none of the states are absorbing ∴

S, I, R form a irreducible subchain ∴

let $f_i^{(n)} = P($of return to i starting at i be first return in n steps$)$ ∴

$f_i = P($eventual return to i$)$ ∴

$f_i = \sum_{n=1}^{\infty} f_i^{(n)} = 1$ ∴ all the states are recurrent.

if $P(R \to S \mid R) = 0$ then  ∴

Ⓡ forms a irreducible subchain ∴

$T_{Rj} = 0 \ \forall j \neq R$ ∴ $T_{RR} = 1$ ∴ R is absorbing ∴

S and I are transient ∴ ~~they~~

S and I are not recurrent

\4c/ ~~$f_S^{(2)} =$~~ $P($first return to ~~S~~ S for $n > 2) =$

$P($first return to S for $n \geq 3) = 1 - P($first return to S for $n \leq 2) =$

$1 - \left(f_S^{(1)} + f_S^{(2)}\right) = 1 - ((1-\alpha) + 0) = 1 - 1 + \alpha = \alpha$

$P($first return to S for $n > 3) = P($first return to S for $n \geq 4) =$

$1 - P($first return to S for $n \leq 3) = 1 - \left((1-\alpha) + 0 + \alpha\beta\gamma\right) =$

$1 - 1 + \alpha + \alpha\beta\gamma = \alpha + \alpha\beta\gamma = \alpha(1 + \beta\gamma)$

\4d/ for states $\{S, I, R\}$: $T = \begin{bmatrix} 1-\alpha & \alpha & 0 \\ 0 & 1-\beta & \beta \\ \gamma & 0 & 1-\gamma \end{bmatrix}$

\4e/ in steady state $\tilde{p} = \tilde{p}T$, $\tilde{p}$ is steady state vector ∴

$\tilde{p} = (p_1 \ \ p_2 \ \ p_3) = (p_1 \ \ p_2 \ \ p_3)\begin{bmatrix} 1-\alpha & \alpha & 0 \\ 0 & 1-\beta & \beta \\ \gamma & 0 & 1-\gamma \end{bmatrix} = \left((1-\alpha)p_1 + \gamma p_3 \quad \alpha p_1 + (1-\beta)p_2 \quad \beta p_2 + (1-\gamma)p_3\right) = \tilde{p}$

∴ $p_1 = (1-\alpha)p_1 + \gamma p_3 = p_1 - \alpha p_1 + \gamma p_3$ ∴ $\alpha p_1 = \gamma p_3$ ∴ ~~$p_1 =$~~ $\frac{\alpha}{\gamma} p_1 = p_3$

$2P_0 = P_1$, $2P_1 = 2P_2$, $\frac{4}{3}P_2 = P_3$

$P_1 = 2P_0$ $P_2 = 2P_0$, $P_3 = \frac{8}{3}P_0$ ∴ $\sum P_i = 1$ ∴ $P_0(1+2+2+\frac{8}{3}) = 1$

∴ $P_0 = \frac{3}{23}$, $P_1 = \frac{6}{23} = P_2$, $P_3 = \frac{8}{23}$

\3aii/ $L_s = \sum_n nP_n = \frac{6}{23} + \frac{12}{23} + \frac{24}{23} = \frac{42}{23} = 1.826$,

$L_q = \sum_{n=1} (n-1)P_n = P_2 + 2P_3 = \frac{6}{23} + \frac{16}{23} = \frac{22}{23} = 0.956$

\1ciii/

Littles thm: $L_s = \lambda_{eff} W_s$ $L_q = \lambda_{eff} W_q$

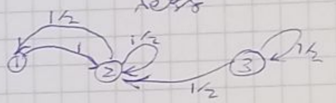
∴ $W_{s,q}$ = waiting time in system / queue,

λ_{eff} = effective arrival rate ∴

$\lambda_{eff} = \sum$ (arrival rate to state n)(prob customer stays given n) $\times P_n$ =

$2P_0 + 2P_1 + \frac{4}{3}P_2 = 1.130$, $W_s = \frac{L_s}{\lambda_{eff}} = 1.615$, $W_q = \frac{L_q}{\lambda_{eff}} = 0.846$

\1di/ $T = \begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix}$



\1dii/ $P(X_3 = 1 | X_0 = 1) = 1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$ ✓ $P(X_3 = 3 | X_0 = 1) = 0$ ✓

$\tilde{p} = (P_0, P_1, P_2)$ ∴ $\tilde{p} T = \tilde{p} = (P_1, P_2, P_3)\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{pmatrix} = (P_1, P_2, P_3)$

$P_3 = 0$ (∵ ③ is transient) ∴

solve $(P_1\ P_2)\begin{pmatrix} 0 & 1 \\ 1/2 & 1/2 \end{pmatrix} = (P_1, P_2)$ ∴ $\frac{1}{2}P_2 = P_1$, $P_1 + \frac{1}{2}P_2 = P_2$ ∴

$P_1 + P_2 = 1$ ∴ $P_1 + 2P_1 = 1$, $P_1 = \frac{1}{3}$, $P_2 = \frac{2}{3}$ ∴

$\tilde{p} = (\frac{1}{3}, \frac{2}{3}, 0)$ is st st probs

\2ai/ $G_X(\theta) = \frac{1}{4} + \frac{1}{4}\theta + \frac{1}{4}\theta^2 + \frac{1}{4}\theta^3 + \frac{1}{4}\theta^4$

$Y = X_1 + X_2$, $E(Y|X_1)(x) = \sum_{y=0}^{2} y P(Y = y | X_1 = x) = \sum_{y=0}^{2} y \frac{P(Y = y \text{ and } X = x)}{P(X = x)}$

for $X_1 = 0$ ∴, $E(Y|X_1) = \sum_{y=0}^{2} y \frac{P(Y=y, X_1 = 0)}{P(X_1 = 0)} = \frac{P(Y = 1, X = 0)}{P(X_1 = 0)}$ all other terms 0

$= \frac{P(X_2 = 1, X_1 = 0)}{P(X_1 = 0)} = \frac{1/4}{1/2} = 1/2$

for $X_1 = 1$, $E(Y|X_1) = \sum_{y=0}^{2} y \frac{P(Y = y, X_1 = 1)}{P(X_1 = 0)}$ ∴

$E(Y|X_1 = 1) = \frac{P(Y = 1, X_1 = 1)}{P(X_1 = 1)} + 2\frac{P(Y = 2, X_1 = 1)}{P(X_1 = 1)} = \frac{P(X_2 = 0, X_1 = 1)}{P(X_1 = 1)} + \frac{2P(X_2 = 1, X_1 = 1)}{P(X_1 = 1)}$

$= \frac{1/4}{1/2} + \frac{2/4}{1/2} = 3/2$

26

\PP2012 / \1ai/ $E(X) = 0.3(1) + 0.3(2) + 0.4(3) = 2.1$ ∴

$E(S_3) = 2.1^3 = 9.26$ (3 s.f.)

\1aii/

$E(X) > 1$ ∴ $e < 1$ ∴ $G_x(e) = e = 0$ ∴ $0.3e + 0.3e^2 + 0.4e^3 - e = 0$

$= 0.4e^3 + 0.3e^2 - 0.7e = 0$ ∴ $4e^3 + 3e^2 - 7e = 0$

∴ $(e-1)(4e^2 + 7e) \cancel{+4e^2} = 0$ ∴ $4e^2 + 7e = 0 = e(4e + 7) = 0$

∴ $4e + 7 = 0$ ∴ $e = -\frac{7}{4}$ but $e \geq 0$ ∴ $e = 0$ is smallest root

∴ $P(X=0) = 0$ ∴ $e = 0$

\1aiii/ $G_{S_2} = G_x(G_x(s)) = (0.3)(0.3s + 0.3s^2 + 0.4s^3) +$

$0.3(0.3s + 0.3s^2 + 0.4s^3)^2 + 0.4(0.3s + 0.3s^2 + 0.4s^3)^3 =$

$0.3(0.3s + 0.3s^2) + 0.3((0.3s)^2) \cancel{+0.3s + 0.3(0.3)} + \ldots =$

$0.09s + 0.117s^2 + \ldots$ ∴

$P(S_2 \leq 2) = 0.09 + 0.117 = 0.207$

\1bi/ $R_t \sim Poi(3t)$, $G \sim Po(2t)$, t = time (hours)

$P(R_t = 3)$; $t = 0.5$

$= \frac{(3t)^3 e^{-3t}}{3!} = 0.126$

\1bii/ $P(R_t + G_t = 4)$, $t = 0.75$, $R_t + G_t \sim Poi(5t)$ ∴

$P(R_t + G_t = 4) = (5t)^4 e^{-5t} \frac{1}{4!} = 0.194$

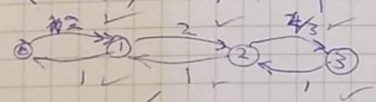
\biii/ $N = 5$, $T = 2$, $t = \frac{1}{2}$, $r = 2$ ∴ $P(X=r) = \binom{N}{r} p_t^r (1-p_t)^{N-r}$

$p_t = \frac{t}{T} = \frac{1}{4}$ ∴ $P(X=2) = \binom{5}{2}\left(\frac{1}{4}\right)^2\left(\frac{3}{4}\right)^3 = 0.264$

\1biv/ $P(t_G < t_R) = P(\text{Green and no red} \mid 1 \text{ in time } t) =$

$\frac{\lambda_G e^{-\lambda_G t} e^{-\lambda_R t}}{(\lambda_G + \lambda_R) e^{-(\lambda_G + \lambda_R)t}} = \frac{\lambda_G}{\lambda_G + \lambda_R} = \frac{2}{2+3} = 0.4$

\1ci/ M/M/1 queue finite capacity



\1cii/ $2P_0 = P_1$, ∴ $2P_1 = P_2$ ∴ $2(2P_0) = 4P_0 = P_2$,

$\frac{4}{3}P_2 = P_3$ ∴ $\frac{4}{3}(4P_0) = P_3 = \frac{16}{3}P_0$ ∴

$\sum_{n=0}^{3} P_n = 1 = P_0(\cancel{1+2}\ldots + 4 + \frac{16}{3}) = 1 = \frac{37}{3}P_0$ ∴ $\frac{3}{37} = P_0$ ∴

$\cancel{P_1 = \frac{4}{3}}$, $\cancel{P_2 = \frac{48}{3}}$, $\frac{6}{37} = P_1$, $\frac{12}{37} = P_2$, $\frac{16}{37} = P_3$

\2ci/ ~~P(3 Spins in 2 hours) = P(~~ P(More then 3 Spins in 2 hours) =

P(~~4~~ or more Spins in two hours) =

1 − P(3 or less spins in two hours)

P(More then 3 bells in two hours) =

~~1~~ P(3 or less bells in two hours) = 1 − P($N_B(t=2) \leq 3$) =

$1 - e^{-(4\times2)}\left[(4\times2)^3 \frac{1}{3!} + (4\times2)^2 \frac{1}{2!} + (4\times2)^1 \frac{1}{1!} + (4\times2)^0 \frac{1}{0!}\right] = 0.958$

∴ P(3 or less bells in two hours) = 0.0424

The dice has PGF $G_N = \frac{1}{6}\theta + \frac{1}{6}\theta^2 + \frac{1}{6}\theta^3 + \frac{1}{6}\theta^4 + \frac{1}{6}\theta^5 + \frac{1}{6}\theta^6$

\3ci/

∴ ~~dicerest~~ dice outcome has PGF: $G_D(\theta) = \frac{5}{6}\theta^0 + \frac{1}{6}\theta^1 = \frac{5}{6} + \frac{1}{6}\theta$

let Y be the number of Spins obtained

let D be is dice is 6 or not ∴ $D_i \in \{0, 1\}$

P(More then 3 spins in two hours) =

1 − P(3 or less Spins in two hours)

P(3 or less spins in two hours)

P(in $\frac{1}{2}$hr at least 2 wins on B and C)

$= P(N_B(\tfrac{1}{2}) \geq 2 \cap N_C(\tfrac{1}{2}) \geq 2) =$

$P(N_B(\tfrac{1}{2}) \geq 2)P(N_C(\tfrac{1}{2}) \geq 2) = [1 - P(N_B \leq 1)][1 - P(N_C(\tfrac{1}{2}) \leq 1)]$

$= \left[1 - e^{-4\times\frac{1}{2}}(1 + (4\times\tfrac{1}{2})^1 \tfrac{1}{1!})\right]\left[1 - e^{-2\times\frac{1}{2}}(1 + (2\times\tfrac{1}{2}))\right] = 0.157$

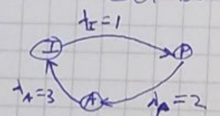
P(3 or less bells) + P(4 or More bells but 3 or less 6 on all dices)

P($N_B(2) \leq 3$) + P(3 or less dice 6 | 4 or More bells) =

P($N_B(2) \leq 3$) + P(3 or less dice 6 | 4 or More dice rolls)

\2cii/ P(bell but no Spin) = $\frac{5}{6}$

\1ei/



steady state: $\frac{dP_I}{dt} = 3P_A - P_I = 0$ ∴

$P_I = 3P_A$, $\frac{dP_P}{dt} = P_I - 2P_P = 0$ ∴ $2P_P = P_I$

∴ $\frac{1}{3}P_I = P_A$, $\frac{1}{2}P_I = P_P$ ∴ $P_I + P_A + P_P = 1 = P_I + \frac{1}{3}P_I + \frac{1}{2}P_I = P_I(\frac{11}{6}) = 1$

∴ $P_I = \frac{6}{11}$ ∴ $P_P = \frac{1}{2}\frac{6}{11} = \frac{3}{11}$, $P_A = \frac{2}{11}$

\2bi/ $\lambda = \lambda_A + \lambda_B + \lambda_C = 5 + 4 + 2 = 11$/hr ∴ E(time between wins) = $\frac{1}{\lambda} = \frac{1}{11}$ hours

\2bii/ P(in $\frac{1}{2}$ hr 2 wins on B and 2 wins on C) = $P((N_B(\tfrac{1}{2}) = 2) \cap (N_C(\tfrac{1}{2}) = 2)) =$

$P(N_B(\tfrac{1}{2}) = 2)P(N_C(\tfrac{1}{2}) = 2) = e^{-4\times\frac{1}{2}}(4\times\tfrac{1}{2})^2 \frac{1}{2!} e^{-2\times\frac{1}{2}}(2\times\tfrac{1}{2})^2 \frac{1}{2!} = 0.0498$

\2biii/

for $n = 6$, $p = \frac{\lambda_A}{\lambda} = \frac{5}{11}$ follows a Binomial: $r \sim Bin(n=6, p=\frac{5}{11})$ ∴

∴ P(5 wins on A | 6 wins) = $P(r=5) = \binom{6}{5}(\frac{5}{11})^5(1-\frac{5}{11})^1 = 0.0635$

\PP 2012/ \3ai/ $G_{T_N}(s) = \frac{1}{4}(s+s^2+s^3+s^4)$

$\therefore G_{T}(s) = G_{T_N}(G_{T_T}(s)) = \frac{1}{4}(G_X(s)+G_X(s)^2+(G_X(s))^3+(G_X(s))^4)$

~~\2011/~~

\1ai/ $E(X) = 0.3(1)+0.3(2)+0.4(3) = 2.1$ $\therefore$

$E(S_3) = (E(X))^3 = 9.26$ (3 s.f.)

\1aii/ $E(X) = 2.1 > 1$ $\therefore$ p(ultimate extinction) $= e < 1$ $\therefore$ ~~e>0~~

$G_X(e) - e = 0.3e + 0.3e^2 + 0.4e^3 - e = -0.7e + 0.3e^2 + 0.4e^3 =$

$e(0.4e^2 + 0.3e - 0.7) = 0$ $\therefore$

~~P(X)~~ $P(X=0) = 0$ $\therefore$

$e = 0$

\1aiii/ $P(S_2 \leq 2) = P(S_2 = 2) + P(S_2 = 1) + P(S_2 = 0)$

$P(X=0) = 0$ $\therefore$ $P(S_2 = 0) = 0$,

~~P(X≥1)~~ $G_{S_2}(s) = G_X(G_X(s)) = G_T(0.3s + 0.3s^2 + 0.4s^3) =$

~~for 0.3(0.3+~~ $G_T(0.3s + 0.3s^2) + \ldots =$

$0.3(0.3s + 0.3s^2) + 0.3(0.3s)^2 + \ldots =$

$0.09s + 0.117s^2 + \ldots$ $\therefore$

$P(S_2 \leq 2) = 0.09 + 0.117 = 0.207$

\1bi/ P(3 red in 30 minutes) $= P(N_R(\frac{1}{2}) = 3)$ ~~⇒~~ $\lambda_r = 3$

$= e^{-(3\times\frac{1}{2})}(3\times\frac{1}{2})^3\frac{1}{3!} = 0.126$

\1bii/ $\lambda = \lambda_r + \lambda_g = 3 + 2 = 5$ $\because$ $\lambda_g = 2$ $\therefore$

P(4 buses in 45 minutes) $= P(N_\lambda(\frac{45}{60}) = 4) = P(N_\lambda(\frac{3}{4}) = 4) =$

$e^{-(5\times\frac{3}{4})}(5\times\frac{3}{4})^4\frac{1}{4!} = 0.194$

\1biii/ P(2 red in first 30 minutes | 5 red in 2 hours) $= P_1$

$\therefore$ ~~Sub~~ number of arrivals in time T then for $t \leq T$.

Number of arrivals follows a Binomial distribution $\therefore$ ~~At~~ $n=5$, $p = \frac{30}{120} = \frac{1}{4}$

$\therefore$ follows $Bin(5, \frac{1}{4})$ $\therefore$

~~$P_1 \geq \frac{1}{2}$~~ $P_1 = \binom{5}{2}(\frac{1}{4})^2(1-\frac{1}{4})^{5-2} = \frac{5!}{2!(5-2)!} = 10\times(\frac{1}{4})^2\times(\frac{3}{4})^3 = 0.264$

~~= P(X~~ $\because$ $N=5$, $T=2$, $t=\frac{1}{2}$, $r=2$ $\therefore$ ~~P(X≥~~ $X \sim Bin(5, \frac{1}{4})$ $\therefore$

$P(X=2) = 0.264$

PP2020 1a i $E(Y) = E(3X) = 3E(X)$ $P(Y>2) = P(3X>2) = 0.2+0.3+0.4=0.9$

$E(X) = 0.2(1) + 0.3(2) + 0.4(3) = 2 \therefore E(Y) = 3(2) = 6$,

1a ii $E(Y) = E(X^2+1) = E(X^2)+1$, $X \in \{0,1,2,3\}$ $3X \in \{0,3,6,9\}$

$E(X^2) = 0.2(1^2) + 0.3(2^2) + 0.4(3^2) = 5 \therefore E(Y) = 5+1 = 6$

$P(Y>2) = P(X^2+1>2) = P(X^2>1) = 1-P(X^2 \le 1) = 1-(P(X^2=0)+P(X^2=1)) =$

$1-(0.1+0.2) = 1-0.3 = 0.7$

1a iii $G_Y(\theta) = G_{X_1+X_2}(\theta) = G_{X_1}(\theta)G_{X_2}(\theta) = (G_X(\theta))G_X(\theta) = (G_X(\theta))^2$

$\therefore G_Y'(1) = E(Y) = 2G_X(\theta)G_X'(\theta)|_{\theta=1} = 2G_X(1)G_X'(1)$

$\therefore G_X(1) = 1$, $G'(1) = E(X) = 2 \therefore$

$E(Y) = 2 \times 2 = 4$

$P(Y>2) = P(X_1+X_2>2)$ $\therefore$ ~~θ^2 coefficient~~

$= 1-P(Y \le 2)$ $\therefore G_Y(\theta) = (0.1+0.2\theta+0.3\theta^2+0.4\theta^3)^2 =$

$0.1^2 + (0.2)^1(0.1)^1 2! + \ldots = (0.1+0.2\theta+0.3\theta^2)^2 + \ldots =$

$0.1^2 + 0.1(0.2)\theta + 0.1(0.3)\theta^2 + 0.1(0.2\theta) + (0.2\theta)^2 + 0.1(0.3\theta^2) =$

$0.01 + 0.04\theta + 0.1\theta^2 \therefore$

$P(Y>2) = 1-(0.01+0.04+0.1) = 0.85$

1b i $P(T \le t) = \int_0^t \lambda e^{-\lambda x}dx = [-e^{-\lambda x}]_{x=0}^t = [-e^{-\lambda t} + e^{-\lambda 0}] =$

$-e^{-\lambda t} + 1 \therefore$ $P(T>t) = 1-P(T \le t) = 1+e^{-\lambda t}-1 = e^{-\lambda t}$

$$P(T>s+t \mid T>s) = \frac{P((T>s+t)\cap(T>s))}{P(T>s)} = \frac{P(T>s+t)}{P(T>s)}$$

$\therefore P(T>s) = e^{-\lambda s}$, $P(T>s+t) = e^{-\lambda(s+t)} = e^{-\lambda s-\lambda t}$.

$P(T>s+t \mid T>s) = e^{-\lambda s-\lambda t}\frac{1}{e^{-\lambda s}} = e^{-\lambda t} = P(T>t)$

$\int_t^\infty \lambda e^{-\lambda x}dx = 0-(-e^{-\lambda t}) = e^{-\lambda t}$

1b ii $M_T(t) = E(e^{tX}) = M_T(t) = E(e^{tT}) = \int_0^\infty \lambda e^{-\lambda t}e^{tT}dt$

$M_T(t) = E(e^{tT}) = \int_0^\infty e^{tx}\lambda e^{-\lambda x}dx = \lambda\int_0^\infty e^{(t-\lambda)x}dx = \frac{\lambda}{\lambda-t}$ for $t<\lambda$

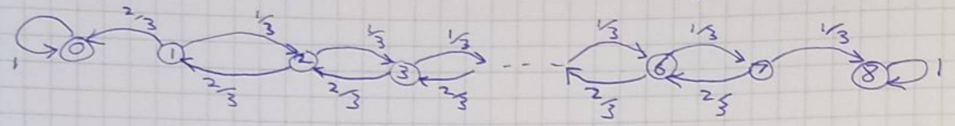
$\therefore E(T) = M_T'(t)|_{t=0} = \frac{\lambda}{(\lambda-t)^2}|_{t=0} = \frac{1}{\lambda}$

$Var(T) = M_T''(t)-(M_T'(t))^2|_{t=0} = \frac{2}{\lambda^2} - \frac{1}{\lambda^2} = \frac{1}{\lambda^2}$,

\4c/ let $U = X_n - Y_n$ ∴ $U \in \{-5, -4, -3, -2, -1, 0, 1, 2, 3\}$

∴ let ~~U~~ $Z = U + 5$ ∴ $Z \in \{0, 1, 2, 3, \dots, 7, 8\}$ ∴

Z forms a random walk: $P(\text{tail}) = \frac{2}{3}$, $P(\text{head}) = \frac{1}{3}$ ∴



∴ let ~~θ~~ ⑧ be winning is $X = Y_n + 3$ ∴

let $\theta_i = P(\text{winning from } i)$ ∴

$\theta_i = \frac{1}{3}\theta_{i+1} + \frac{2}{3}\theta_{i-1}$ ∴ $\theta_0 = 0$, $\theta_N = 1$, $N = 8$ ∴

let $\theta_i = A\lambda^i = A\lambda\lambda^{i-1}$, $\theta_{i-1} = A\lambda^{i-1}$, $\theta_{i+1} = A\lambda^{i+1} = A\lambda^2\lambda^{i-1}$ ∴

$\frac{1}{3}\theta_{i+1} + \frac{2}{3}\theta_{i-1} - \theta_i = 0$ ∴

$\frac{1}{3}A\lambda^2\lambda^{i-1} + \frac{2}{3}A\lambda^{i-1} - A\lambda\lambda^{i-1} = A\lambda^{i-1}\left(\frac{1}{3}\lambda^2 - \lambda + \frac{2}{3}\right) = 0$ ∴

$\frac{1}{3}\lambda^2 - \lambda + \frac{2}{3} = 0 \Rightarrow \lambda^2 - 3\lambda + 2 = (\lambda - 1)(\lambda - 2) = 0$

$\cancel{1 \pm \sqrt{1 - 4(\frac{1}{3})(\frac{2}{3})}}$ ∴ $\lambda = 1$, $\lambda = 2$

$\theta_i = A(1)^i + B(2)^i = A + B(2)^i$ ∴

$\theta_0 = 0 = A(1)^0 + B(2)^0 = A + B$ ∴ $A = -B$ ∴

$\theta_i = -B + B(2)^i = B(-1 + 2^i)$ $\theta_N = 1 = -B + B(2)^N = B(-1 + 2^N)$ ∴

$B = \frac{1}{-1 + 2^N}$ ∴

$\theta_i = \frac{-1 + 2^i}{-1 + 2^N} = \frac{1 - 2^i}{1 - 2^N}$ ∴

$N = 8$, start at $U = 0$ ∴ $Z = 0 + 5 = 5$ ∴

$\theta_5 = \frac{1 - 2^5}{1 - 2^8} = \frac{31}{255} = P(X_n = Y_n + 5 \text{ starting at } 0)$ ∴

$P(Y_n = X_n + 5) = 1 - P(X_n = Y_n + 5) = 1 - \frac{31}{255} = \frac{224}{255} = 0.878$ ✓

\PP2012/ \4aii/ let $f_i^{(n)} = P$(of first return in n steps to state i)

$\therefore$ ~~fi~~ $f_i = P$(of eventual return to state i) $\therefore$

$$f_i = \sum_{n=1}^{\infty} f_i^{(n)} \therefore$$

if $f_i < 1$ then state i is transient

\4aiii/ state i is periodic if

$f_i^{(n)} \geq 0$ for $n = km$, k is: $k \in \mathbb{N}$, $k > 1$, $m \in \mathbb{N}$, k is constant

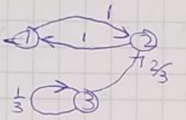
and $f_i^{(n)} = 0 \ \forall n \neq km$ period k

\4aiii/ A state i is periodic with period $k > 1$ if $(T^n)_{ii} = 0$

$\forall n \neq k, 2k, 3k, \ldots$ but $(T^n)_{ii} > 0 \ \forall n = k, 2k, 3k, \ldots$

\4aiv/ a state is recurrent if $f_i = 1 = P$(of eventual return to i)

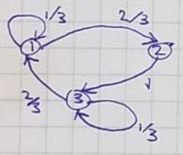
\4b) for $T^{(1)}$:



$\therefore \{①, ②\}$ is an irreducible subchain.

③ is transient ① and ② are periodic and recurrent and null recurrent

for $T^{(2)}$:



$\therefore \{1, 2, 3\}$ are a irreducible subchain aperiodic and recurrent and positively recurrent

$\therefore$ let $f_1^{(n)} = P$(first return to 1 in n steps) $\therefore$

$f_1 = P$(eventual return to 1) $\therefore$

$$f_1 = \sum_{n=1}^{\infty} f_1^{(n)} = \frac{1}{3} + 0 + \frac{2}{3}\times1\times\frac{2}{3} + \frac{2}{3}\times1\times\frac{1}{3}\times\frac{2}{3} + \frac{2}{3}\times1\times\left(\frac{1}{3}\right)^2 \times \frac{2}{3} + \frac{2}{3}\times1\times\left(\frac{1}{3}\right)^3\times\frac{2}{3} + \cdots$$

$$= \frac{1}{3} + \sum_{n=0}^{\infty} \frac{2}{3}\times\left(\frac{1}{3}\right)^n \times\frac{2}{3} = \frac{1}{3} + \frac{4}{9}\sum_{n=0}^{\infty}\left(\frac{1}{3}\right)^n = \frac{1}{3} + \frac{4}{9}\,\frac{1}{1-\left(\frac{1}{3}\right)} = 1 \therefore$$

$f_1 = 1 = P$(eventual return to 1) $\therefore$ ① is recurrent

let $\mu_1 = \sum_{n=1}^{\infty} f_1^{(n)} n = E$(first return to 1 steps) $\therefore$

① is part of a irreducible suchain with fixed probabilities

$\therefore \mu_1 = E$(first return to 1) $< \infty$ $\therefore$

① is not null recurrent $\therefore$ ① is positively recurrent

\2b/ $\therefore G_X(\theta) = \frac{1}{2} + \frac{1}{2}\theta$

$G_Y(\theta) = \cancel{G_{X}(\theta)} \; G_{X_1+X_2}(\theta) = G_{X_1}(\theta)\, G_{X_2}(\theta)$ by independence

$= G_X(\theta)\, G_X(\theta) = (G_X(\theta))^2 \;\therefore$

$E(Y) = G_Y'(\theta)|_{\theta=1} = 2G_X(\theta)G_X'(\theta)|_{\theta=1} = 2(G_X(1)\, G_X'(\theta)|_{\theta=1} = 2(1)\, G_X'(\theta)|_{\theta=1}$

$= 2\left[\frac{1}{2}\right]|_{\theta=1} = 1$

$E(Y|X_1) = E(Y|X_1 = x)$ $\quad \because X_1 \in \{0,1\}, X_2 \in \{0,1\} \therefore Y \in \{0,1,2\} \therefore$

$E(Y|X_1) = E(Y|X_1 = x) = \sum_{y=0}^{2} y P(Y=y|X_1=x) = \sum_{y=0}^{2} y \frac{P(Y=y \cap X_1 = x)}{P(X_1=x)} =$

$\sum_{y=0}^{2} y \frac{P(Y=y \text{ and } X=x)}{P(X=x)}$

$\therefore$ for $X_1 = 0$: $E(Y|X_1) = E(Y|X_1=0) = \cancel{\sum_{y=0}^{3} y \frac{P(Y=y, X_1=0)}{P(X_1=0)}} = \sum_{y=0}^{2} y \frac{P(Y=y, X_1=0)}{P(X_1=0)}$

$(0) \times \frac{P(Y=0, X_1=0)}{P(X_1=0)} + 1 \times \frac{P(Y=1, X_1=0)}{P(X_1=0)} + 2 \times \frac{P(Y=2, X_1=0)}{P(X_1=0)} =$

$0 + \frac{P(Y=1, X_1=0)}{P(X_1=0)} + 2 \times \frac{0}{P(X_1=0)} = \frac{P(Y=1, X_1=0)}{P(X_1=0)} = \cancel{\frac{P(X_2=1)}{P(X_1=0)} = \frac{1/2}{1/2} = 1}$

$\frac{P(X_2=1, X_1=0)}{P(X_1=0)} = \frac{P(X_2=1)P(X_1=0)}{P(X_1=0)} = P(X_2=1) = \frac{1}{2}$

for $X_1 = 1$: $E(Y|X_1) = \cancel{E(X_1} \; E(Y|X_1=1) = \sum_{y=0}^{2} y \frac{P(Y=y, X_1=1)}{P(X_1=1)} =$

$0 \times \frac{P(Y=0, X_1=1)}{P(X_1=1)} + 1 \times \frac{P(Y=1, X_1=1)}{P(X_1=1)} + 2\frac{P(Y=2, X_1=1)}{P(X_1=1)} =$

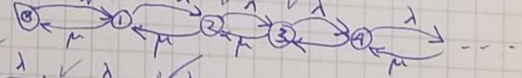
$0 + \frac{P(Y=1, X_1=1)}{P(X_1=1)} + 2\frac{P(Y=2, X_1=1)}{P(X_1=1)} = \frac{P(X_2=0, X_1=1)}{P(X_1=1)} + \frac{2P(X_2=1, X_1=1)}{P(X_1=1)} =$

$\frac{P(X_2=0)P(X_1=1)}{P(X_1=1)} + 2P(X_2=1)\frac{P(X_1=1)}{P(X_1=1)} = P(X_2=0) + 2P(X_2=1) = \frac{1}{2} + 2 \times \frac{1}{2} = \frac{3}{2}$

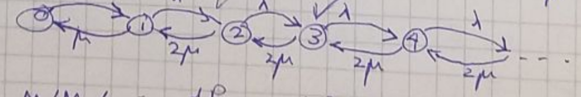
$\therefore E(E(Y|X_1)) = \sum_{x=0}^{1} E(Y|X_1=x)P(X_1=x) =$

$E(Y|X_1=0)P(X_1=0) + E(Y|X_1=1)P(X_1=1) = \frac{1}{2}\times\frac{1}{2} + \frac{3}{2}\times\frac{1}{2} = 1 = E(Y)$

\3a/ for M/M/1:



for M/M/2:



\3b/ for M/M/1: $\frac{dP_0}{dt} = -\lambda P_0 + \mu P_1 = 0 \;\therefore \mu P_1 = \lambda P_0 \;\therefore$ let $\rho = \frac{\lambda}{\mu} \;\therefore$

$P_1 = \frac{\lambda}{\mu}P_0 = \rho P_0 \quad \frac{dP_1}{dt} = -\lambda P_1 + \mu P_2 + \lambda P_0 - \mu P_1 = 0 = -\lambda P_1 + \mu P_2 \;\therefore \mu P_2 = \lambda P_1 \;\therefore$

\3e/ M/M/2 queue has $W_s = L_s \frac{1}{\lambda_{eff}} = \frac{\rho}{2-\rho}\frac{1}{\lambda_{eff}}$

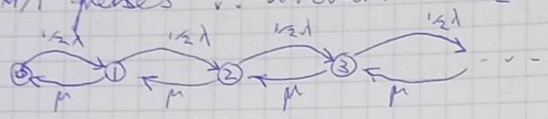
$L_s^{(2)} = \frac{\rho}{2-\rho}$ for $\rho = \frac{\lambda}{\mu}$

$\lambda_{eff} = \sum_{n=0}^{\infty} \lambda_n P_n = \sum_{n=0}^{\infty} \lambda P_n = \lambda \sum_{n=0}^{\infty} P_n = \lambda(1) = \lambda$ ∴

$W_s^{(2)} = \frac{\rho}{2-\rho}\frac{1}{\lambda}$

for ~~two~~ two M/M/1 queues ∴ arrival rate $= \frac{1}{2}\lambda$ ∴

each is:



∴ $\sum_{n=1}^{\infty} P_n = 1$, steady state: $\frac{dP_0}{dt} = -\frac{1}{2}\lambda P_0 + \mu P_1$ ∴ $\mu P_1 = \frac{1}{2}\lambda P_0$

∴ let $\rho = \frac{\lambda}{\mu}$ ∴ $P_1 = \frac{1}{2}\frac{\lambda}{\mu}P_0 = \frac{1}{2}\rho P_0$ ∴

$\frac{dP_1}{dt} = -\frac{1}{2}\lambda P_1 + \mu P_2 + \frac{1}{2}\lambda P_0 - \mu P_1 = 0 = -\frac{1}{2}\lambda P_1 + \mu P_2$ ∴ $\mu P_2 = \frac{1}{2}\lambda P_1$ ∴

$P_2 = \frac{1}{2}\frac{\lambda}{\mu}P_1 = \frac{1}{2}\rho P_1 = \frac{1}{2}\rho\frac{1}{2}\rho P_0 = (\frac{1}{2}\rho)^2 P_0$ ∴

$P_n = (\frac{1}{2}\rho)^n P_0$ ∴

$\sum_{n=0}^{\infty} P_n = 1 = \sum_{n=0}^{\infty}(\frac{1}{2}\rho)^n P_0 = P_0 \sum_{n=0}^{\infty}(\frac{1}{2}\rho)^n = P_0\frac{1}{1-\frac{1}{2}\rho} = P_0\frac{2}{2-\rho}$ ∴

$\frac{2-\rho}{2} = P_0$ ∴ $P_n = \frac{2-\rho}{2}(\frac{1}{2}\rho)^n$ ∴

$L_s^{(1)} = E(\text{length of system}) = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n\frac{2-\rho}{2}(\frac{1}{2}\rho)^n = \frac{2-\rho}{2}\sum_{n=0}^{\infty} n(\frac{1}{2}\rho)^n =$

$\frac{2-\rho}{2}\frac{(\frac{1}{2}\rho)}{(1-(\frac{1}{2}\rho))^2} = \frac{(2-\rho)2(\frac{1}{2}\rho)}{4(1-\rho+\frac{1}{4}\rho^2)} = \frac{(2-\rho)\rho}{4-4\rho+\rho^2} = \frac{(2-\rho)\rho}{(2-\rho)^2} = \frac{\rho}{2-\rho}$ ∴

$\lambda_{eff} = \sum_{n=0}^{\infty}\lambda_n P_n = \sum_{n=0}^{\infty}\frac{1}{2}\lambda P_n = \frac{1}{2}\lambda\sum_{n=0}^{\infty}P_n = \frac{1}{2}\lambda(1) = \frac{1}{2}\lambda$ ∴

~~Ls~~ $W_s^{(1)} = L_s^{(1)}\frac{1}{\lambda_{eff}} = L_s^{(1)}\frac{1}{(\frac{1}{2}\lambda)} = \frac{2}{\lambda}L_s^{(1)} = \frac{2}{\lambda}\frac{\rho}{2-\rho} = 2\frac{\rho}{2-\rho}\frac{1}{\lambda}$

∴ $\frac{W_s^{(1)}}{W_s^{(2)}} = \frac{2\frac{\rho}{2-\rho}\frac{1}{\lambda}}{\frac{\rho}{2-\rho}\frac{1}{\lambda}} = 2$ ∴ $W_s^{(1)} = 2W_s^{(2)}$ ∴ $W_s^{(1)} > W_s^{(2)}$ ∴

The waiting time of the two M/M/1 queues is double that of the M/M/2 queue ∴ the M/M/2 queue is more efficient

\4a i/ An absorbing state is: $T_{ii} = 1$ and $T_{ij} = 0 \ \forall i \neq j$ for state i

\PP2012/ $P_2 = \frac{\lambda}{\mu} P_1 = \rho P_1 = \rho \rho P_0 = \rho^2 P_0$ ∴

$P_n = \rho^n P_0 \quad n \geq 0$ ∴ $\sum_{n=0}^{\infty} P_n = 1 = \sum_{n=0}^{\infty} \rho^n P_0 = P_0 \sum_{n=0}^{\infty} \rho^n = P_0 \frac{1}{1-\rho} = 1$ ∴

$P_0 = 1-\rho$ ∴ $P_n = \rho^n (1-\rho)$ ✓

for M/M/2: $\frac{dP_0}{dt} = -\lambda P_0 + \mu P_1 = 0$ ∴ $\mu P_1 = \lambda P_0$ ∴ $P_1 = \frac{\lambda}{\mu} P_0 = \rho P_0$

for $\rho = \frac{\lambda}{\mu}$ ∴

$\frac{dP_1}{dt} = -\lambda P_1 + 2\mu P_2 + \lambda P_0 - \mu P_1 = 0 = -\lambda P_1 + 2\mu P_2$ ∴ $2\mu P_2 = \lambda P_1$ ∴

$P_2 = \frac{1}{2}\frac{\lambda}{\mu} P_1 = \frac{1}{2}\rho P_1 = \frac{1}{2}\rho\rho P_0 = \frac{1}{2}\rho^2 P_0 = 2\frac{1}{4}\rho^2 P_0 = 2(\frac{1}{2}\rho)^2 P_0 = 2(\frac{\rho}{2})^2 P_0$

$\frac{dP_2}{dt} = -\lambda P_2 + 2\mu P_3 + \lambda P_1 - 2\mu P_2 = 0 = -\lambda P_2 + 2\mu P_3$ ∴ $2\mu P_3 = \lambda P_2$ ∴

$P_3 = \frac{\lambda}{2\mu} P_2 = \cancel{\frac{\lambda}{2\mu}} \frac{1}{2}\rho P_2 = \frac{1}{2}\rho\, 2(\frac{\rho}{2})^2 P_0 = 2(\frac{\rho}{2})^3 P_0$ ∴

$P_n = 2(\frac{\rho}{2})^n P_0$ ∴ $\sum_{n=0}^{\infty} P_n = 1 = \sum_{n=0}^{\infty} 2(\frac{\rho}{2})^n P_0 = 2P_0 \sum_{n=0}^{\infty} (\frac{\rho}{2})^2 =$

$2P_0 \frac{1}{1-(\frac{\rho}{2})} = \cancel{4} \ \frac{4}{2} P_0 \frac{1}{1-\frac{\rho}{2}} = \frac{4P_0}{2-\rho} = 1$ ∴ $P_0 = \frac{2-\rho}{4}$ ∴

$P_n = 2(\frac{\rho}{2})^n \frac{2-\rho}{4} = (\frac{\rho}{2})^n \frac{2-\rho}{2}$

\3d/ Littles theorem is: E(waiting time in system) =

$W_s = L_s \frac{1}{\lambda_{eff}}$ = E(length of system)/λ_{eff} ,

E(waiting time in queue) = $W_q = L_q/\lambda_{eff}$ =

E(length of queue)/λ_{eff}

where $\lambda_{eff} = \sum_{n=0}^{\infty} P_n \lambda_n$ = effective arrival rate

\3c/ ~~B~~ for M/M/1: E(system size) = $\sum_{n=0}^{\infty} P_n n =$ ✓✓

$\sum_{n=0}^{\infty} n\rho^n(1-\rho) = (1-\rho)\sum_{n=0}^{\infty} n\rho^n = (1-\rho)\frac{\rho}{(1-\rho)^2} = \frac{\rho}{1-\rho} = L_s$

for M/M/2: E(system size) = $\sum_{n=0}^{\infty} P_n n =$

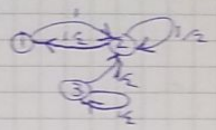
$\sum_{n=0}^{\infty} n(\frac{\rho}{2})^n \frac{2-\rho}{2} = \frac{2-\rho}{2}\sum_{n=0}^{\infty} n(\frac{\rho}{2})^n = \frac{2-\rho}{2} \frac{(\frac{\rho}{2})}{(1-(\frac{\rho}{2}))^2} =$

$\frac{(2-\rho)\rho}{4(1-\rho+\frac{\rho^2}{4})} = \frac{(2-\rho)\rho}{4-4\rho+\rho^2} = \frac{(2-\rho)\rho}{(2-\rho)^2} = \frac{\rho}{2-\rho} = L_s$

\3d/ littles theorem: $L_s = \lambda_{eff} W_s$, $L_q = \lambda_{eff} W_q$, λ_{eff} is effective arrival rate

L_s/L_q are length of system/queue. W_s/W_q is waiting time of system/queue

\PP2012/\1di/



\1dii/ $P(X_3=1|X_0=1) = \cancel{P(1\to}\ P(1\to2\to2\to1) = 1\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{4}$

$P(X_3=3|X_0=1) = P(1\to2\to2\to3) = 1\times\frac{1}{2}\times 0 = 0$

\1diii/ ~~state~~ steady state: $\tilde{p} = \cancel{p} (p_1 \quad p_2 \quad p_3) =$

$$(p_1 \quad p_2 \quad p_3)\begin{pmatrix} 0 & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix} = \left(\frac{p_2}{2} \quad p_1+\frac{1}{2}p_2+\frac{1}{2}p_3 \quad \frac{1}{2}p_3\right) = (p_1 \quad p_2 \quad p_3)$$

$\therefore p_1 = \frac{1}{2}p_2, \quad p_1+\frac{1}{2}p_2+\frac{1}{2}p_3 = p_2, \quad \frac{1}{2}p_3 = p_3 \therefore 0 = \frac{1}{2}p_3 \therefore p_3 = 0$

$\therefore \sum_{n=1}^{3} p_n = p_1+p_2+p_3 = 1 = p_1+p_2 \therefore 1-p_2 = p_1 \therefore$

$1-p_2 = \frac{1}{2}p_2 \therefore 1 = \frac{3}{2}p_2 \therefore \frac{2}{3} = p_2 \therefore 1-\frac{2}{3} = \frac{1}{3} = p_1 \therefore \tilde{p} = \left(\frac{1}{3} \quad \frac{2}{3} \quad 0\right)$

\2ai/ $G_N(\theta) = P(N=1)\theta^1 + P(N=2)\theta^2 + P(N=3)\theta^3 + P(N=4)\theta^4 =$

$\frac{1}{4}\theta + \frac{1}{4}\theta^2 + \frac{1}{4}\theta^3 + \frac{1}{4}\theta^4 \therefore$

$\cancel{G_Y(\theta}\ G_Y(S) = G_N(G_X(S)) = G_N(0.1+0.4S+0.5S^2) =$

$\frac{1}{4}(0.1+0.4S+0.5S^2)^1 + \frac{1}{4}(0.1+0.4S+0.5S^2)^2 + \frac{1}{4}(0.1+0.4S+0.5S^2)^3 + \frac{1}{4}(0.1+0.4S+0.5S^2)^4$

\2aii/ $P(Y\le1) = P(Y=0)+P(Y=1) \therefore$

$\cancel{P(}\ \cancel{G_Y(\theta}\ G_Y(S) = \frac{1}{4}(0.1+0.4S) + \frac{1}{4}\left(0.1^2+0.1\times0.4S\times\frac{2!}{1!1!}\right) + \frac{1}{4}\left(0.1^3+0.1^2\times0.4S\times\frac{3!}{2!1!}\right)$

$+ \frac{1}{4}\left(0.1^4+0.1^3\times0.4S\times\frac{4!}{3!\times1!}\right) + \dots =$

$\frac{1}{40}+\frac{1}{10}S+\frac{1}{400}+\frac{1}{50}S+\frac{1}{4000}+\frac{3}{1000}S + \dots = \frac{111}{4000}+\frac{123}{1000}S + \dots$

$\therefore P(Y\le1) = \frac{111}{4000}+\frac{123}{1000} = 0.151$

\2aiii/

$E(Y) = G_Y'(S)|_{S=1} = \frac{\partial}{\partial S}(G_N(G_X(S)))|_{S=1} = G_N'(G_X(S))G_X'(S)|_{S=1}$

$= G_N'(1)G_X'(S)|_{S=1} = E(N)E(X) = 2.5\times(0.4+0.5\times2) = 2.5\times1.4 = 3.5$

$G_Y''(S)|_{S=1} = E(X^2-X) = E(Y^2)-E(Y) \therefore$

$G_Y''(S)|_{S=1} = G_N''(G_X(S))(G_X'(S))^2|_{S=1} = \left(\frac{2}{4}+\frac{3}{4}\times2\theta+\frac{4}{4}\times3\theta^2\right)(1.4)^2|_{S=1} =$

$\left(\frac{2}{4}+\frac{3}{4}\times2+\frac{4}{4}\times3\right)(1.4)^2 = 9.8 = E(Y^2)-E(Y) \therefore$

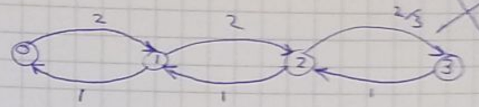
$E(Y^2) = 9.8+E(Y) = 9.8+3.5 = 13.3 \therefore$

~~E~~ $\mathrm{var}(Y) = E(Y^2)-E(Y)^2 = 13.3-3.5^2 = 1.05$

\1b.v/ P(first bursts arrive is green) = $\cancel{\frac{\lambda_g}{\lambda_r+1}}$ $\frac{\lambda_g}{\lambda} = \frac{\lambda_g}{\lambda_r + \lambda_g} =$

$\frac{2}{3+2} = 0.4$ ✓

\1ci/



is a M/M/1 queue finite capacity

$\cancel{\frac{dP_0}{dt}}$ steady state : $\frac{dP_0}{dt} = 1P_1 - 2P_0 = 0 \therefore P_1 = 2P_0$,

$\frac{dP_1}{dt} = 1P_2 - 2P_1 - (P_1 - 2P_0) = 0 = P_2 - 2P_1 \quad \therefore 2P_1 = P_2 = 2(2P_0) = 4P_0$

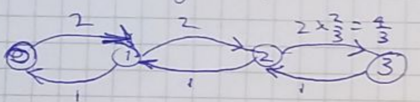
$\frac{dP_3}{dt} = -(P_3 - \frac{2}{3}P_2) = 0 = -P_3 + \frac{2}{3}P_2 \quad \therefore P_3 = \frac{2}{3}P_2 = \frac{2}{3}(4P_0) = \frac{8}{3}P_0$

$\therefore \sum_{n=0}^{3} P_n = 1 = P_0(1 + 2 + 4 + \frac{8}{3}) = \frac{29}{3}P_0 \quad \therefore P_0 = \frac{3}{29}$

~~\1c~~ $\therefore P_1 = \frac{6}{29}, \quad P_2 = \frac{12}{29}, \quad P_3 = \frac{8}{29}$

\1ci/ if n=2 : ~~P(~~ P(decide not to join) = $\frac{1}{3}$ $\therefore$

P(decide to join) = 1 − P(decide not to join) = $1 - \frac{1}{3} = \frac{2}{3}$ $\therefore$



is M/M/1 queue finite capacity $\therefore$

Steady state : $\frac{dP_0}{dt} = -2P_0 + P_1 = 0 \therefore 2P_0 = P_1$,

$\frac{dP_1}{dt} = -(-2P_0 + P_1) - 2P_1 + P_2 = 0 = -2P_1 + P_2 \quad \therefore P_2 = 2P_1 = 2(2P_0) = 4P_0$

$\frac{dP_3}{dt} = \frac{4}{3}P_2 - P_3 = 0 \quad \therefore P_3 = \frac{4}{3}P_2 = (\frac{4}{3})4P_0 = \frac{16}{3}P_0 \therefore$

$\sum_{n=0}^{3} P_n = 1 = P_0\left[1 + 2 + 4 + \frac{16}{3}\right] = 1 = P_0(\frac{37}{3}) \quad \therefore \frac{3}{37} = P_0 \therefore$

$P_1 = \frac{6}{37}, \quad P_2 = \frac{12}{37}, \quad P_3 = \frac{16}{37}$

\1cii/ E(system size) = $E(n) = \sum_{n=0}^{3} nP_n = 0P_0 + 1P_1 + 2P_2 + 3P_3 =$

$\frac{6}{37} + 2 \times \frac{12}{37} + 3 \times \frac{16}{37} = \frac{78}{37} = L_s$

E(queue size) = $\sum_{n=1}^{3}(n-1)P_n = 0P_1 + 1P_2 + 2P_3 = P_2 + 2P_3 = \frac{12}{37} + 2 \cdot \frac{16}{37} = \frac{44}{37} = L_q$

\1ciii/ $\lambda_{eff} = \sum_{n=0}^{3} P_n\lambda_n = P_0\lambda_0 + P_1\lambda_1 + P_2\lambda_2 =$

$\frac{3}{37} \times 2 + \frac{6}{37} \times 2 + \frac{16}{37} \times \frac{4}{3} = \frac{118}{111} \therefore$

E(time in garage) = $L_s \frac{1}{\lambda_{eff}} = \frac{78}{37} \times \frac{1}{(118/111)} = 117/59 = W_s$

E(time in queue) = $L_q \frac{1}{\lambda_{eff}} = \frac{44}{37} \times \frac{1}{(118/111)} = \frac{66}{59}$

\PP2020/ \1ai/ $E(Y) = E(3X) = 3E(X)$

$\therefore E(X) = 0.1(0) + 0.2(1) + 0.3(2) + 0.4(3) = 2 \therefore$

$E(Y) = 3 \times 2 = 6$ ✓ $P(Y > 2) = P(3X > 2) = P(X > \frac{2}{3}) = P(X \geq 1)$ ✓

$\cancel{P(Y>2) = 1 - P(Y \leq 2) = 1 - (P(Y=0 \cup 1 \cup 2))} = 0.2 + 0.3 + 0.4 = 0.9$

$\cancel{1 - (P(Y=0) + P(Y=1) + P(Y=2)) = 1 - (0.1 + 0.2 + 0.3) = 0.4} \therefore$

$\cancel{P(Y \geq 2) = P(Y \geq 3) = P(Y = 3) = 0.4}$

\1aii/ $E(X^2) = 0.1(0^2) + 0.2(1^2) + 0.3(2^2) + 0.4(3^2) = 5 \therefore$

$E(Y) = E(X^2+1) = E(X^2) + 1 = 5 + 1 = 6$

$P(Y > 2) = P(X^2 + 1 > 2) = P(X^2 > 1) = P(X > 1) = P(X \geq 2) = 0.3 + 0.4 = 0.7$ ✓

\1aiii/ $E(Y) = E(X_1 + X_2) = E(X_1) + E(X_2) = E(X) + E(X) = 2E(X) = 2 \times 2 = 4$

$G_Y(\theta) = G_{X_1+X_2}(\theta) = G_{X_1}(\theta) G_{X_2}(\theta) = G_X(\theta) G_X(\theta) = (G(\theta))^2$

$\therefore P(Y > 2) = P(Y \geq 3) = 1 - P(Y \leq 2) \therefore$

$G_Y(\theta) = \cancel{(0.1 + 0.2\theta + 0.3\theta^2 + 0.4\theta^3)}$

$0.1 + 0.2\theta + 0.3\theta^2 + (0.1 + 0.2\theta + 0.3\theta^2)0.2 + 0.3(0.1 + 0.2\theta + 0.3\theta^2)^2 + 0.4(0.1 + 0.2\theta + 0.3\theta^2)^3 + \dots$

$= 0.1 + 0.2\theta + 0.3\theta^2 + 0.02 + \frac{1}{25}\theta + \frac{3}{50}\theta^2 + 0.3 \times 0.1^2 + 0.3 \times 0.2^2\theta^2 + 0.3 \times 0.1 \times 0.2\theta \times \frac{2!}{1!1!} +$

$0.4 \times 0.1^3 + 0.3 \times 0.3\theta^2 \times 0.1 \times \frac{2!}{1!1!} + 0.4 \times 0.2\theta \times 0.1^2 \times \frac{3!}{1!2!} + 0.4 \times 0.3\theta^2 \times 0.1^2 \times \frac{3!}{1!2!} + \dots =$

$0.1234 + 0.2\theta + \frac{1}{25}\theta + \frac{3}{250}\theta + \frac{3}{1250}\theta + 0.3\theta^2 + \frac{3}{50}\theta^2 + \frac{3}{250}\theta^2 + \frac{9}{500}\theta^2 + \frac{9}{2500}\theta^2 + \dots$

$0.123 + \frac{159}{625}\theta + \frac{246}{625}\theta^2 + \dots$

$P(Y > 2) = 1 - \left(0.123 + \frac{159}{625} + \frac{246}{625}\right) = 0.229$

\1bi/ $P(T > s + t \mid T > s) = \frac{P(T > s+t \cap T > s)}{P(T > s)} = \frac{P(T > s+t)}{P(T > s)} = \frac{\lambda e^{-\lambda(s+t)}}{\lambda e^{-\lambda s}} =$

$\lambda e^{-\lambda s} e^{-\lambda t} / \lambda e^{-\lambda s} = e^{-\lambda t} = P(T > t) \therefore$ T is memoryless

$P(T > t) = 1 - P(T \leq t) = 1 - \int_0^t \lambda e^{-\lambda x} dx = 1 - \left[e^{-\lambda x}\right]_0^t = 1 + \left[e^{-\lambda t} - e^0\right] =$

$1 + e^{-\lambda t} - 1 = e^{-\lambda t}$

\1bii/

$M_T(t) = E[e^{tT}] = \int_0^\infty e^{tT} \lambda e^{-\lambda T} dT = \int_0^\infty \lambda e^{tT - \lambda T} dT = \int_0^\infty \lambda e^{(t-\lambda)T} dT$

$= \left[\frac{\lambda}{t-\lambda} e^{(t-\lambda)T}\right]_0^\infty = \frac{\lambda}{t-\lambda}(0) - \frac{\lambda}{t-\lambda} e^{(t-\lambda)0} = -\frac{\lambda}{t-\lambda} = \frac{\lambda}{\lambda - t}$ for $t < \lambda$

\1biii/ $f_{T_1}(t) = 2e^{-2t}$, $f_{T_2}(x) = 3e^{-3t} \therefore$

$Y = \max(T_1, T_2) \therefore f_Y(t) = \int_0^{T_1} \int_0^\infty f_{T_1}(t) f_{T_2}(t) dt$

$\sum_{n=0}^{3} P_n = 1 = P_0 + P_1 + P_2 + P_3 = P_0\left[1 + \frac{P_1}{P_0} + \frac{P_2}{P_0} + \frac{P_3}{P_0}\right] = \left[1 + \rho + \rho^2 + \frac{2\rho}{3} + \frac{\rho^3}{3} + \frac{8}{9}\rho^2\right]$

$\therefore P_0 = \left[1 + \frac{5\rho}{3} + \frac{17}{9}\rho^2 + \frac{\rho^3}{3}\right]^{-1} \therefore$

$P_1 = \rho P_0$, $P_2 = (\rho^2 + \frac{2\rho}{3})P_0$, $P_3 = (\frac{\rho^3}{3} + \frac{8}{9}\rho^2)P_0 \therefore$

\3bii/ $L_s = E(x) = \sum_{n=0}^{3} n P_n = P_1 + 2P_2 + 3P_3 =$

$\rho P_0 + 2(\rho^2 + \frac{2\rho}{3})P_0 + 3(\frac{\rho^3}{3} + \frac{8}{9}\rho^2)P_0 =$

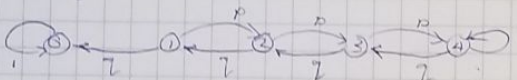
$P_0(\rho + 2\rho^2 + \frac{4\rho}{3} + \rho^3 + \frac{8}{3}\rho^2) = P_0(\frac{7}{3}\rho + \frac{14}{3}\rho^2 + \rho^3) =$

$\left[1 + \frac{5\rho}{3} + \frac{17}{9}\rho^2 + \frac{\rho^3}{3}\right]^{-1}(\frac{7}{3}\rho + \frac{14}{3}\rho^2 + \rho^3)$

$\lambda_{eff} = \sum_{n=0}^{3} \lambda_n P_n = \lambda P_0 + \lambda P_1 + \frac{1}{3}\lambda P_2 = \lambda(P_0 + \rho P_0 + \rho^2 P_0 = \frac{\lambda(1 - \rho + \rho^2)}{1 + \rho + \rho^2 + \frac{1}{3}\rho^3}$

$\therefore W_s = L_s \lambda_{eff}^{-1}$

$L_s = P_1 + 2P_2 + 3P_3$

\4ai/  transition probability only depends on previous state $\therefore$ markov

for $p \neq q$: let $\theta_i = A\lambda^i \therefore \theta_i = q\theta_{i-1} + p\theta_{i+1} \therefore$

$0 = q\theta_{i-1} - \theta_i + p\theta_{i+1} = 0 = Aq\lambda^{i-1} - A\lambda^i + pA\lambda^{i+1} = A\lambda^{i-1}(q - \lambda + p\lambda^2) = 0 \therefore$

$p\lambda^2 - \lambda + q = 0 \therefore \lambda = \frac{1 \pm \sqrt{1 - 4pq}}{2p}$ $\quad 1 = p + q \therefore 1 - q = p \therefore 1 - p = q \therefore$

~~$\lambda = \frac{1 \pm \sqrt{p - 4(1-q)q}}{2}$~~ $\lambda = \frac{1 \pm \sqrt{1 - 4p(1-p)}}{2p} = \frac{1 \pm \sqrt{1 - 4p + 4p^2}}{2p}$

$4p^2 - 4p + 1 = (2p - 1)(2p - 1) = (2p - 1)^2 \therefore \lambda \pm \frac{1 \pm (2p - 1)}{2p} \therefore$

$\lambda_+ = \frac{1 + (2p - 1)}{2p} = \frac{2p}{2p} = 1$, $\lambda_- = \frac{1 - (2p - 1)}{2p} = \frac{1 - 2p + 1}{2p} = \frac{2(1 - p)}{2p} = \frac{q}{p} = \rho \therefore$

$\theta_i = A + B\rho^i \therefore$ $\theta_0 = 0$, $\theta_4 = 1 \therefore$

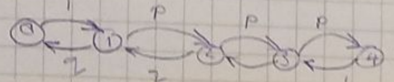
$\theta_0 = 0 = A + B \therefore A = -B \therefore -B + B\rho^i = B(-1 + \rho^i) = \theta_i \therefore$

$\theta_4 = B(-1 + \rho^4) = 1 \therefore B = \frac{1}{-1 + \rho^4} \therefore$

$\theta_i = \frac{(-1 + \rho^i)}{-1 + \rho^4} \therefore \theta_1 = \frac{-1 + \rho^1}{-1 + \rho^4} = \frac{\rho - 1}{\rho^4 - 1} = \frac{\rho - 1}{(\rho - 1)(\rho^3 + 1 + \rho^2 + \rho)} = \frac{1}{\rho^3 + \rho^2 + \rho + 1}$

$\therefore$ for $p = q$: $\theta_i = \frac{i}{N} \therefore \theta_1 = \frac{1}{4}$ $\quad 1 = p + q = p + p = 2p \therefore \frac{1}{2} = p = q$

$\theta_i = \frac{1}{2}\theta_{i-1} + \frac{1}{2}\theta_{i+1} \therefore \theta_1 = \frac{1}{4}$

\4bii/  $\therefore$

$$T = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ q & 0 & p & 0 & 0 \\ 0 & q & 0 & p & 0 \\ 0 & 0 & q & 0 & p \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \therefore \tilde{p}T = \tilde{p} \therefore \text{for } p = \frac{2}{3}:$$

$\tilde{p} = (\frac{1}{30}, \frac{1}{10}, \frac{2}{10}, \frac{4}{10}, \frac{4}{15})$

\PP2020/ $\because S_0 = 1 \therefore S_1 = X \therefore G_{S_1}(\theta) = G_X(\theta)$

$\therefore$ if $G_{S_n}(\theta) = G_X^n(\theta) \therefore S_{n+1} = X_1 + \dots + X_{S_n} \therefore$

$G_{S_{n+1}}(\theta) = G_{S_n}(G_X(\theta)) = G_X^n(G_X(\theta)) = G_X^n(G_X(\theta)) = G_X^{n+1}(\theta)$ $\therefore$ induction

\2bi/ $G_{S_0}(\theta) = \theta^3$, $G_X(\theta) = \theta^2$ $\therefore (\theta^2)^3 = \theta^6 \therefore$

$\therefore G_{S_2}(\theta) = \theta^{12} \therefore P(S_2 > 1) = 1$

$G_{S_2} = G_X^2(\theta^3) = G_X(G_X(\theta^3)) = G_X((\theta^3)^2) = G_X(\theta^6) = (\theta^6)^2 = \theta^{12} \therefore$

$P(S_2 > 1) = 1 \therefore G_{S_2} = G_{S_0}(G_X^2(\theta)) = G_{S_0}(G_X(G_X(\theta))) =$

$G_{S_0}(G_X(\theta^2)) = G_{S_0}((\theta^2)^2) = G_{S_0}(\theta^4) = (\theta^4)^3 = \theta^{12}$

\2bii/ Mean population size : (everything is with prob 1) :

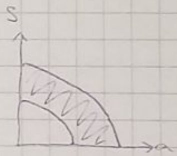
$\mu_4 = S_4 = 3 \cdot 2^4 = 48$

\2biii/ $\because P(X=0) = 0 \therefore e = 0$

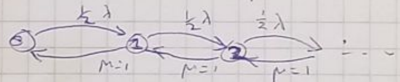
\2c/ $s > 0,\ a > 0 \therefore s^2 + a \leq 1 \therefore s^2 - 1 \leq a$

now $e < 1$ if $\mu_X = E(X) < 1 \therefore E(X) = 2(1 - s^2 - a) < 1 \therefore$

$\frac{1}{2} - s^2 < a \therefore$



\3ai/



$\frac{dP_0}{dt} = -\frac{1}{2}P_0 + P_1 = 0 \therefore P_1 = \frac{1}{2}P_0$

$\frac{dP_1}{dt} = \frac{1}{2}\lambda P_0 - P_1 - \frac{1}{2}\lambda P_1 + P_2 = 0 = -\frac{1}{2}\lambda P_1 + P_2 \therefore \frac{1}{2}\lambda P_{n-1} = \mu P_n \therefore$

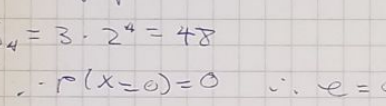
let $\rho = \frac{1}{2}\frac{\lambda}{\mu} \therefore P_n = \rho P_{n-1} = \rho^n P_0 \therefore 1 = \sum_{n=0}^{\infty} P_n = \sum_{n=0}^{\infty} \rho^n P_0 = P_0 \sum_{n=0}^{\infty} \rho^n$

$= \frac{P_0}{1-\rho}$ for $|\rho| < 1 \therefore P_0 = 1 - \rho \therefore P_n = (1-\rho)\rho^n$

\3aii/ $G_X(\theta) = \sum_{k=0}^{\infty} \theta^k P_k = \sum_{k=0}^{\infty} (1-\rho)\rho^k \theta^k = (1-\rho) \sum_{k=0}^{\infty} (\rho\theta)^k = (1-\rho)\frac{1}{1-\rho\theta}$

$|\rho\theta| < 1 \therefore E(X) = G_X'(1) = (1-\rho)\frac{\rho}{(1-\rho)^2} = \frac{\rho}{1-\rho}$ for $\rho = \frac{1}{2}\frac{\lambda}{\mu}$

\3bi/



$\frac{dP_0}{dt} = \mu P_1 - \lambda P_0 = 0 \therefore \mu P_1 = \lambda P_0 \therefore \rho = \frac{\lambda}{\mu} \therefore \rho P_0 = P_1$

$\frac{dP_1}{dt} = \mu P_2 + \frac{\lambda}{3}P_0 - (\mu + \frac{\lambda}{3} + \frac{2\lambda}{3})P_1 = 0$

$P_2 = (1+\rho)P_1 - \frac{\rho}{3}P_0 = (\rho + \rho^2 - \frac{\rho}{3})P_0 = (\rho^2 + \frac{2\rho}{3})P_0$

$\frac{dP_3}{dt} = \frac{2\lambda}{3}P_1 + \frac{\lambda}{3}P_2 - \mu P_3 = 0 \therefore P_3 = \frac{\rho}{3}P_2 + \frac{2\rho}{3}P_1 = \left[\frac{\rho}{3}(\rho^2 + \frac{2\rho}{3}) + \frac{2\rho^2}{3}\right]P_0$

$= \left[\frac{\rho^3}{3} + \frac{8}{9}\rho^2\right]P_0 \therefore$

\1biii/ $T_1 \sim Exp(\lambda_1)$, $T_2 \sim Exp(\lambda_2)$ $\therefore$

$f_{T_1,T_2}(t_1,t_2) = \lambda_1 \lambda_2 e^{-\lambda_1 t_1} e^{-\lambda_2 t_2}$ for $t_1, t_2 \geq 0$, 0 otherwise

$\therefore$ let $Y = \max\{T_1, T_2\}$ $\therefore$ $P(Y<5) = P((T_1<5) \cap (T_2<5)) =$

$\int_0^5\int_0^5 \lambda_1 \lambda_2 e^{-\lambda_1 t_1} e^{-\lambda_2 t_2} dt_1 dt_2 = \lambda_1 \int_0^5 e^{-\lambda_1 t_1} \left[e^{-\lambda_2 t_2}\right]_0^5 dt_1 =$

$\lambda \int_0^5 e^{-\lambda_1 t}(1-e^{-5\lambda_2})dt = (1-e^{-5\lambda_2})\left[-e^{-\lambda_1 t_1}\right]_0^5 = (1-e^{-5\lambda_1})(1-e^{-5\lambda_2})$,

$\lambda_1 = 2$, $\lambda_2 = 3$ $\therefore$ $P(Y<5) = (1-e^{-10})(1-e^{-15})$

\1ci/ $\lambda = 50$ /hour $\therefore$ $\lambda_1 = 0.8 \times 50 = 40$/hour

$\lambda_2 = 10$/hour $\therefore$ poisson processes are independent $\therefore$ 0.2

$= P$(next exam is from M2)

\1cii/ $\lambda_1 = 40$/hour $\therefore$ $\frac{1}{40}$ hours $= 1.5$ minutes

\1ciii/ $\lambda_1 = 40$ $\therefore$ $P(N_1(\frac{1}{2})=25)$ $\therefore$ Poi$(\lambda, t) \sim \frac{e^{-\lambda_1 t}(\lambda t)^x}{x!}$

$\therefore$ $P(N_1(\frac{1}{2})=25) = e^{-40(\frac{1}{2})}(40\times\frac{1}{2})^{25}\frac{1}{25!} = 0.0446$ (3S.F.)

\1civ/ $P((N_2(\frac{1}{4})=5) \mid N_2(1)=20) =$

$P((N_2(\frac{1}{4})=5) \cap (N_2(1)=20)) / P(N_2(1)=20)$ is

binomial with $n=20$, $p = \frac{1/4}{1} = \frac{1}{4}$ $\therefore$

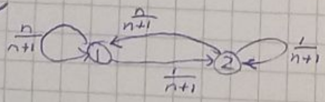
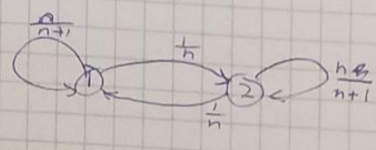
let $Y \sim Bin(20, \frac{1}{4})$ $\therefore$

probability $= P(Y=5) = \binom{20}{5}(1-\frac{1}{4})^{20-5}(\frac{1}{4})^5 = 0.202$ (3S.F.)

\1di/ A: let Transition Matrix T_{ij} be probability of transition from states i to j $\therefore$ states communicate is either $T_{ij}>0$ or $T_{ji}>0$ communicate is $T_{ij}>0$ and $T_{ji}>0$

B: a state is absorbing is $T_{ij}=0$ $\forall i \neq j$ and $T_{ii}=1$

C: a state is recurrent is probability of first return to i in n steps $f_i^{(n)}$ $\therefore$ probability of eventually return $f_i = \sum_{n=1}^{\infty} f_i^{(n)} = 1$ then i is recurrent

\1dii A/  $\therefore$ yes 

\PP2020/ $\sum_{n=0}^{\infty} P_n = 1 = P_0 + \sum_{n=1}^{\infty} P_n = P_0 + \sum_{n=1}^{\infty} 2\left(\frac{\lambda}{2}\right)^n P_0 = P_0 + 2P_0 \sum_{n=1}^{\infty}\left(\frac{\lambda}{2}\right)^n =$

$\cancel{P_0 + 2P_0 + \sum_{n=0}^{\infty}}$ $P_0 + 2P_0 \sum_{n=0}^{\infty}\left(\frac{\lambda}{2}\right)^n - 2P_0 = P_0 - 2P_0 + 2P_0 \frac{1}{1-\left(\frac{\lambda}{2}\right)} =$

$$P_0\left(1 - 2 + \frac{2}{1-\frac{\lambda}{2}}\right) = 1 = P_0\left(-1 + \frac{2}{1-\frac{\lambda}{2}}\right) \therefore$$

$$\frac{2}{1-\frac{\lambda}{2}} = \frac{4}{2-\lambda} \therefore \quad -1 + \frac{2}{1-\frac{\lambda}{2}} = \frac{\lambda-2}{2-\lambda} + \frac{4}{2-\lambda} = \frac{\lambda+2}{2-\lambda} \therefore$$

$$P_0\left(\frac{\lambda+2}{2-\lambda}\right) = 1 \therefore \quad P_0 = \frac{2-\lambda}{\lambda+2} \therefore \quad P_n = 2\left(\frac{\lambda}{2}\right)^n \frac{2-\lambda}{\lambda+2} \; ; n \geq 1$$

\3aii/ $G_X(\theta) = E(\theta^X) = \sum_{n=0}^{\infty} P_n \theta^n = \sum_{n=0}^{\infty} 2\left(\frac{\lambda}{2}\right)^n \frac{2-\lambda}{2+\lambda}\theta^n = 2\frac{2-\lambda}{2+\lambda}\sum_{n=0}^{\infty}\left(\frac{\lambda}{2}\theta\right)^n =$

$$2\frac{2-\lambda}{2+\lambda}\frac{1}{1-\frac{\lambda\theta}{2}} = \frac{4}{2-\lambda\theta}\frac{2-\lambda}{2+\lambda} = \frac{2-\lambda}{2+\lambda}4(2-\lambda\theta)^{-1} \therefore$$

$$L_s = E(X) = G_X'(\theta)\big|_{\theta=1} \therefore$$

$$G_X'(\theta) = \frac{d}{d\theta}\frac{2-\lambda}{2+\lambda}4(2-\lambda\theta)^{-1} = -4\frac{2-\lambda}{2+\lambda}(2-\lambda\theta)^{-2}(-\lambda) \therefore$$

$$L_s = G_X'(\theta)\big|_{\theta=1} = E(X) = \cancel{4}4\lambda\frac{2-\lambda}{2+\lambda}(2-\lambda)^{-2} = \frac{4\lambda}{(2+\lambda)(2-\lambda)}$$

\3bi/
$$\begin{tikzcd}
0 \arrow[r, "\frac{1}{3}\lambda"] \arrow[rr, bend left, "\frac{2}{3}\lambda"] & 1 \arrow[r] & 2
\end{tikzcd}$$

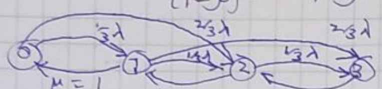
$G_X(\theta) = E(\theta^X) = P_n \theta^n = \sum_{n=0}^{\infty} P_n \theta^n$

$\rho = \frac{1}{2}\frac{\lambda}{\mu} = \frac{\lambda}{2} \therefore P_0 = 1-\rho, \; P_n = (1-\rho)\rho^n \therefore$

$G_X(\theta) = \sum_{k=0}^{\infty} \theta^k P_n = \sum_{n=0}^{\infty} (1-\rho)\rho^k \theta^k = (1-\rho)\frac{1}{1-\rho\theta} \therefore$

$E(X) = G_X'(1) = (1-\rho)\frac{\rho}{(1-\rho)^2} = \frac{\rho}{1-\rho}$

\3bi/



St St: $\frac{dP_0}{dt} = -\frac{2}{3}\lambda P_0 - \frac{1}{3}\lambda P_0 + P_1 = -P_0 + P_1 = 0 \therefore \; P_1 = \frac{P_0}{\lambda} \therefore$

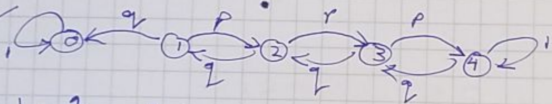
$\frac{dP_1}{dt} = -\frac{2}{3}\lambda P_1 - \frac{1}{3}\lambda P_1 + 1P_2 - 1P_1 + \frac{1}{3}\lambda P_0 = -\lambda P_1 + P_2 - P_1 + \frac{1}{3}P_0 = 0 \therefore$

$P_2 = (1+\lambda)P_1 - \frac{1}{3}P_0 = (1+\lambda)\frac{P_0}{\lambda} - \frac{1}{3}P_0 = \cancel{(\frac{2}{3}+\lambda)P_0} \; (\frac{1}{\lambda} + \frac{2}{3})P_0$

$\therefore \quad \frac{dP_3}{dt} = -P_3 + \frac{2}{3}\lambda P_1 + \frac{1}{3}\lambda P_2 = 0 \therefore$

$P_3 = \frac{2}{3}\lambda P_1 + \frac{1}{3}\lambda P_2 = \frac{2}{3}\lambda P_1$

\4ai/



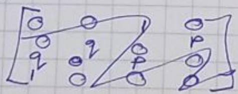
$\lambda = 1, \quad \lambda = \frac{q}{p} = \rho \therefore$

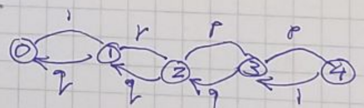
$\theta_i = \frac{1-\rho^i}{1-\rho^4} \therefore \quad \therefore \theta_1 =$

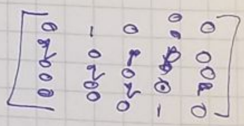
for $p \neq q$: $\theta_1 = \frac{1-\rho^1}{-1+\rho^4}$

for $p = q$: $\theta_i = \frac{i}{N} \therefore \quad \theta_1 = \frac{1}{4} \therefore$

\4b/



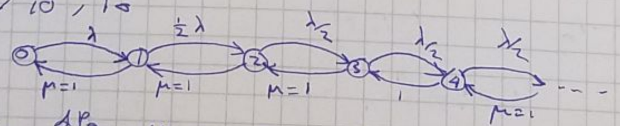




$\therefore \tilde{P} = (P_0 \; P_1 \; P_2 \; P_3 \; P_4) = \tilde{P}\,T \therefore$

$\hat{P} = (\frac{1}{30}, \frac{1}{10}, \frac{3}{10}$

\3ai/

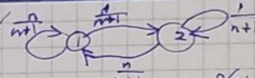


Steady State: $\frac{dP_0}{dt} = -\lambda P_0 + P_1 = 0 \therefore P_1 = \lambda P_0$

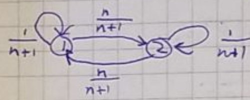
$\frac{dP_1}{dt} = -\frac{\lambda}{2}P_1 + 1P_2 + \lambda P_0 - P_1 = 0 = -\frac{\lambda}{2}P_1 + P_2 \therefore \quad P_2 = \frac{\lambda}{2}P_1 = \frac{\lambda}{2}\lambda P_0 = \frac{\lambda^2}{2}P_0$

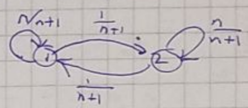
$\frac{dP_2}{dt} = -\frac{\lambda}{2}P_2 + P_3 - P_2 + \frac{\lambda}{2}P_1 = 0 = -\frac{\lambda}{2}P_2 + P_3 \quad \therefore \quad P_3 = \frac{\lambda}{2}P_2 = \frac{\lambda}{2}\frac{\lambda^2}{2}P_0 = \frac{\lambda^3}{2^2}P_0 = 2\left(\frac{\lambda}{2}\right)^3 P_0$

$\therefore \; P_1 = \lambda P_0, \quad P_n = 2\left(\frac{\lambda}{2}\right)^n P_0 \; ; n \geq 1 \therefore$

\PP2020/ \1dii A/ 

$\therefore$ yes

\1dii B/ 



$\therefore$ yes

\2a/ $S_0 = 1 \therefore S_1 = X \therefore G_{S_1}(\theta) = G_X(\theta) \therefore$

assume $G_{S_n}(\theta) = G_X^n(\theta)$, $S_{n+1} = X_1 + \dots + X_{S_n} \therefore$

$G_{S_{n+1}}(\theta) = G_X^{n+1}(\theta) = G_X(G_X^n(\theta)) = G_{S_n}(G_X(\theta)) = G_X^n(G_X(\theta)) = G_X^{n+1}(\theta) \therefore$

$G_{S_{n+1}}(\theta) = G_{S_n}(G_X(\theta)) = G_X^n(G_X(\theta)) = G_X^{n+1}(\theta)$

\2bi/ $\therefore$ $G_{S_0}(\theta) = \theta^3$, $G_X(\theta) = \theta^2 \therefore$

$G_{S_1}(\theta) = G_X(G_{S_0}(\theta)) = G_X(\theta^3) = (\theta^3)^2 = \theta^6 \therefore$

$G_{S_2}(\theta) = G_X(G_{S_1}(\theta)) = (\theta^6)^2 = \theta^{12} \therefore$

$P(S_2 > 1) = P(S_2 = 12) = 1$

$E(X) = 1 \times 2 = 2 \therefore$

\2bii/

$E(S_4) = 3 \times 2^4 = 48$

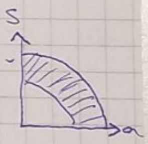
\2biii/ $P(X=0) = 0 \therefore e = 0$

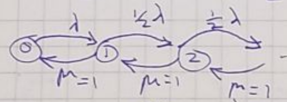
\2c/ $0 \leq S^2 + a \leq 1$, $0 \leq 1 - S^2 - a \leq 1 \therefore$

$S^2 \geq 0 \quad \therefore 1 - S^2 \leq a$,

romis $e < 1 \therefore \mu_x = E(X) < 1 \therefore E(X) = (1 - S^2 - a)2 < 1 \therefore$

$1 - S^2 - a < \frac{1}{2} \therefore$ $\frac{1}{2} < S^2 + a \therefore \frac{1}{2} - S^2 < a \therefore$



\3ai/ 

$\dots \therefore$

steady state $\therefore \frac{dP_0}{dt} = -\lambda P_0 + P_1 = 0 \therefore P_1 = \lambda P_0 \therefore$

$\frac{dP_1}{dt} = -\frac{1}{2}\lambda P_1 + P_2 + \lambda P_0 - P_1 = 0 = -\frac{1}{2}\lambda P_1 + P_2 \therefore \quad P_2 = \frac{1}{2}\lambda P_1 = 2\left(\frac{\lambda}{2}\right)^2 P_0 \therefore$

$\frac{dP_2}{dt} = -\frac{\lambda}{2}P_2 + P_3 + \frac{\lambda}{2}P_1 - P_2 = 0 = -\frac{\lambda}{2}P_2 + P_3 \therefore$

$P_3 = \frac{\lambda}{2}P_2 = 2\left(\frac{\lambda}{2}\right)^3 P_0 \quad \therefore \quad P_n = 2\left(\frac{\lambda}{2}\right)^n P_0 \therefore$

$\sum_{n=0}^{\infty} P_n = 1 = \sum_{n=0}^{\infty} 2\left(\frac{\lambda}{2}\right)^n P_0 = 2P_0 \frac{1}{1 - \frac{\lambda}{2}} = \frac{4}{2-\lambda} P_0 = 1 \therefore P_0 = \frac{2-\lambda}{4}$

$\therefore P_n = 2\left(\frac{\lambda}{2}\right)^n P_0 = 2\left(\frac{\lambda}{2}\right)^n \frac{2-\lambda}{4}$

Joint density $f_{XY}(x,y) = \lambda_1 \lambda_2 e^{-\lambda_1 x} e^{-\lambda_2 y}$ $\therefore$

$f_Y = f_{T_1 T_2}(x,t) = \lambda_{T_1} \lambda_{T_2} e^{-\lambda_{T_1} x} e^{-\lambda_{T_2} t} = 2 \times 3 e^{-2x} e^{-3t} = 6 e^{-2x} e^{-3t}$

$\therefore \quad P(Y<5) = \int_0^5 \int_0^5 f_Y \, dx\,dt = \int_0^5 \int_0^5 6 e^{-2x} e^{-3t} \, dx\,dt =$

$\int_0^5 \left[-3 e^{-2x} e^{-3t}\right]_{x=0}^5 dt = \int_0^5 e^{-3t}(-3)\left[e^{-10} - e^0\right] dt =$

$\int_0^5 -3\left[e^{-10}-1\right] e^{-3t} dt = -3\left[e^{-10}-1\right]\left[-\frac{1}{3} e^{-3t}\right]_0^5 = \left[e^{-10}-1\right]\left[e^{-15} - e^0\right] =$

$\left[e^{-10}-1\right]\left(e^{-15}-1\right)$

\1ei/ $\therefore \lambda = 50/hr$ $\lambda_1 = 0.8 \times \cancel{50}$ $\lambda = 0.8 \times 50 = 40/\cancel{hr}$,

$\lambda_2 = 0.2 \times \lambda = 0.2 \times 50 = 10/hr$

$P(\text{next exam is from } M_2) = 0.2$

\1eii/ ~~E~~ Mean time of $M_1 = E(\text{time for } M_1) = \frac{1}{\lambda_1} = \frac{1}{40}$ hours $=$

1.5 Minutes

\1eiii/ $P(N_1(\frac{1}{2}) = 25) = P(25\ M_1 \text{ in } \frac{1}{2} \text{ hour}) =$

$e^{-\lambda_1 t}(\lambda t)^{25} \frac{1}{25!} = e^{-40 \times \frac{1}{2}} (40 \times \frac{1}{2})^{25} \times \frac{1}{25!} = 0.0446$

\1eiv/ $P(5 M_2 \text{ in } \frac{1}{4} hr \mid 20 M_2 \text{ in } 1 hr)$

$\therefore n = 20$, $p = \frac{1/4}{1} = \frac{1}{4}$ $\therefore$ follows Binomial$(n=20, p=\frac{1}{4})$ $\therefore$

$P(5 M_2 \text{ in } \frac{1}{4} hr \mid 20 M_2 \text{ in } 1 hr) = P(r=5 \mid n=20, p=\frac{1}{4}) =$

$\binom{20}{5} (\frac{1}{4})^5 (1-\frac{1}{4})^{20-5} = 0.202$

\1di/ ~~A)~~ State i communicates with state j is

$T_{ij} \neq 0$ and $T_{ji} \neq 0$ with T_{ij} is probability transition matrix

B) absorbing state is $T_{ii} = 1$, $T_{ij} = 0$ $\forall i \neq j$

C) let $P(\text{first return to } i \text{ in } n \text{ steps}) = f_i^{(n)}$ $\therefore$

$P(\text{eventual return to } i) = f_i$ $\therefore$ is

$f_i = \sum_{n=1}^{\infty} f_i^{(n)} = 1$ then state is recurrent

\2aiii/ $E(Y) = E(X_1+X_2) = 2E(X) = 2 \times 2 = 4$, $G_Y(\theta) = G_{X_1}(\theta) G_{X_2}(\theta) =$

$= (G_X(\theta))^2 = (0.1 + 0.2\theta + 0.3\theta^2)^2 + \ldots =$

$0.1^2 + 0.2\theta^2 + 0.1 \times 0.2\theta \times 2 + 0.3 \times 0.1 \times 2 \times \theta^2 + \ldots = 0.01 + 0.04\theta + 0.1\theta^2 + \ldots$

$P(Y>2) = 1 - P(Y \leq 2) = 1 - (0.01 + 0.04 + 0.1) = 0.85$