Ex 3.9/ if $\hat{\theta} = n^{-1}\sum_{i=1}^{n} x_i$ is Z sample mean then
$\left\{\theta_{-i} = \frac{1}{n-1}\sum_{j\neq i} x_j = \frac{1}{n-1}\left[\sum_{j=1}^{n}(x_j) - x_i\right]\right\}$ $\hat{\theta}_{-i} = \frac{1}{n-1}\sum_{j\neq i} x_j = \frac{1}{n-1}\left(\sum_{j=1}^{n} x_j - x_i\right)$

Can such extra estimators $\hat{\theta}_{-i}$ tell us about Z sampling distri of $\hat{\theta}$? might expect Z difference betw $\hat{\theta}_{-i}$ & $\hat{\theta}$ to tell us about Z effect on Z bias of increasing Z sample size from $n-1$ to n & ∴ may be able to use Z extra estimators to reduce any bias in $\hat{\theta}$.

let $E_n = E(\hat{\theta})$ & suppose $\hat{\theta}$ is asympt unbiased st $E_n \to \theta$ as $n \to \infty$, so $\hat{\theta}$ is unbiased for E_n but possibly biased for θ & we seek a less biased estimator for θ. Show how E_n might vary with sample size & show if we knew E_n & E_{n-1} then could form a linear approx to Z curve & approx θ by Z intercept of our linear approx with Z vertical axis. this intercept is $E_n - \frac{E_{n-1} - E_n}{1/(n-1) - 1/n} \cdot \frac{1}{n} = E_n - (n-1)(E_{n-1} - E_n)$

we don't know E_n Z E_{n-1} but $\hat{\theta}$ is unbiased for E_n Z $\hat{\theta}_{-i}$ is unbiased for E_{n-1} for $i = 1, \dots, n$ ∴ can replace E_n with $\hat{\theta}$ & replace E_{n-1} with Z mean of Z $\hat{\theta}_{-i}$ to obtain Z <u>jackknife estimator</u> for θ,
$\hat{\theta}_J = \hat{\theta} - (n-1)\left(\frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_{-i} - \hat{\theta}\right) = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i$, $\theta_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$ are known as <u>pseudo</u>-vals for $i = 1, \dots, n$

to see Z effect of Z jackknife esti. Suppose Z bias of $\hat{\theta}$ can be expanded in powers of n st $E(\hat{\theta}) = \theta + \frac{a}{n} + \frac{b}{n^2} + O(n^{-3})$ for consts a Z b & $O(n^{-3})$ is terms of order n^{-3} & smaller. ∴ ∃ a const M st $|O(n^{-3})| \leq M n^{-3}$ ∀n ∴ as n gets large, Z first order term is Z bias is a/n, Z second order term is b/n^2 is smaller, & Z $O(n^{-3})$ is smaller still. note if multiply a $O(n^{-k})$ term by n then Z term becomes $O(n^{-k+1})$. Such expansion of bias can be valid ∴ $E(\hat{\theta}_J) = \frac{1}{n}\sum_{i=1}^{n} E(\hat{\theta}_i) = E\left[n\hat{\theta} - (n-1)\hat{\theta}_{-i}\right] = n\left[\theta + \frac{a}{n} + \frac{b}{n^2} + O(n^{-3})\right] - (n-1)\left[\theta + \frac{a}{n-1} + \frac{b}{(n-1)^2} + O(n^{-3})\right] = n\theta + a + \frac{b}{n} - (n-1)\theta - a - \frac{b}{n-1} + O(n^{-2}) = \theta - \frac{b}{n(n-1)} + O(n^{-2}) = \theta + O(n^{-2})$ & Z jackknife esti has removed Z 1st-order bias, a/n, from $\hat{\theta}$.

note: use $(n-1)^{-k} = n^{-k}(1-n^{-1})^{-k} = n^{-k}\left[1 + O(n^{-1})\right]^k = O(n^{-k})$

\Ex 3.10/ $x_1, \dots, x_n$ indep $N(\mu, \sigma^2)$ r.v. Z max likehood esti for Z var is $\hat{\sigma}^2 = \frac{1}{n}\sum_{j=1}^{n}(x_j - \bar{x})^2 = \frac{1}{n}\left[\sum_{j=1}^{n} x_j^2 - \frac{1}{n}\left(\sum_{j=1}^{n} x_j\right)^2\right]$, now $E(\hat{\sigma}^2) = \sigma^2 - \sigma^2/n$ (ex)

$\therefore$ $\hat{\sigma}^2$ is biased but Z jackknife estimator will remove this bias completely. find jackknife version of $\hat{\sigma}^2$:

$\sigma_{-i}^2 = \langle \frac{1}{n-1}\left(\sum_{j=1}^{n}(x_j - \bar{x})^2 - (x_i - \bar{x})^2\right)\rangle = \frac{1}{n-1}\left[\sum_{j \neq i} x_j^2 - \frac{1}{n-1}\left(\sum_{j=1} x_j\right)^2\right]$ & pseudo-vals

$\hat{\sigma}_i^2 = \langle \frac{1}{n} \sum_{i=1}^{n} \sigma_{-i}^2 \rangle = n\hat{\sigma}^2 - (n-1)\hat{\sigma}_{-i}^2 =$

$\sum_{j=1}^{n} x_j^2 - \frac{1}{n}\left(\sum_{j=1}^{n} x_j\right)^2 - \left(\sum_{j=1}^{n} x_j^2 - x_i^2\right) + \frac{1}{n-1}\left(\sum_{j=1}^{n} x_j - x_i\right)^2 =$

$\frac{n}{n-1}x_i^2 + \frac{1}{n(n-1)}\left(\sum_{j=1}^{n} x_j\right)^2 - \frac{2}{n-1}x_i \sum_{j=1}^{n} x_j = \frac{n}{n-1}(x_i^2 - 2x_i\bar{x} + \bar{x}^2) = \frac{n}{n-1}(x_i - \bar{x})^2$ $\therefore$ Z

jackknife estimator is $\hat{\sigma}_J^2 = \langle \frac{1}{n}\sum_{i=1}^{n} \hat{\sigma}_i^2 \rangle = \frac{1}{n}\sum_{i=1}^{n} \hat{\sigma}_i^2 = \frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2$

Z usual unbiased estima for Z var as expected

note $\hat{\theta}_J$ is desired to estima $\hat{\theta}$ also reducing bias, jackknife supports other inferences about θ. like: it provides a way to esti Z var of $\hat{\theta}$. Z jackknife esti for Z var of $\hat{\theta}$ is: $\hat{v}_J = \frac{1}{n(n-1)}\sum_{i=1}^{n}(\hat{\theta}_i - \hat{\theta}_J)^2$.

rough sketch of estima: recall $\hat{\theta}_J$ is Z mean of Z pseudo-vals. Z pseudo-vals are ident distried $\therefore$ is indep (often a good approx) $\therefore$ Z var of $\hat{\theta}_J$ estied by s^2/n, s^2 is Z sample var of Z pseudovals. is just $\hat{v}_J$

Z jackknife esti for Z var be used to obtain a CI for θ, eg: $\because$ $\hat{\theta}_J$ is a mean, might expect for approx large n; $\frac{\hat{\theta}_J - \theta}{\sqrt{\hat{v}_J}}$ has a $Stu(n-1)$ distri, yielding Z $(1-2\alpha)$- CI $\hat{\theta}_J \pm t_\alpha \sqrt{\hat{v}_J}$, t_α is Z α-quantile of Z $Stu(n-1)$ distri. for testing Z null hypoth $H_0: \theta = \theta_0$ might expect (recal Z one-sample t-test) that Z null distri of Z test stat $\frac{\hat{\theta}_J - \theta_0}{\sqrt{v_J}}$ is approx a $Stu(n-1)$ distri. have described Z jackknife for indep iid r.v. for covariate models in which Z distri of X_i depends on a covariate Z_i, Z jackknife proceeds in a similar way by calc $\hat{\theta}_{-i}$ after leaving out ith pair (Z_i, X_i)

\Ex 3.11/ $x_{(1)} \leq \dots \leq x_{(n)}$ be Z order stats for a sample size $n = 2m$ & let $\hat{\theta} = (x_{(m)} + x_{(m+1)})/2$ be an esti of Z median. Show jackknife version of $\hat{\theta}$ is still equal to $\hat{\theta}$. when observation $x_{(i)}$ is omitted Z median of Z remaining $2m-1$ vals is Z mth largest val. is $x_{(m+1)}$ when $1 \leq i \leq m$ & is $x_{(m)}$

when $m+1 \le i \le 2m$ $\therefore$ $\hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n}\left[n\hat{\theta} - (n-1)\hat{\theta}_{-i}\right] = n\hat{\theta} - \frac{n-1}{n}\sum_{i=1}^{n}\hat{\theta}_{-i} =$
$\frac{n}{2}(x_{(m)} + x_{(m+1)}) - \frac{n-1}{n}(mx_{(m)} + mx_{(m+1)}) = m(x_{(m)} + x_{(m+1)}) - (1 - \frac{1}{2m})(mx_{(m)} + mx_{(m+1)}) =$
$\frac{1}{2}(x_{(m)} + x_{(m+1)}) = \hat{\theta}$. Z jackknife esti of Z var is:
$\hat{V}_J = \frac{1}{n(n-1)}\sum_{i=1}^{n}\left[n\hat{\theta} - (n-1)\hat{\theta}_{-i} - \hat{\theta}\right]^2 = \frac{n-1}{n}\sum_{i=1}^{n}(\hat{\theta} - \hat{\theta}_{-i})^2 =$
$\frac{n-1}{n}\left[m\left(\frac{x_{(m)} + x_{(m+1)}}{2} - x_{(m)}\right)^2 + m\left(\frac{x_{(m)} + x_{(m+1)}}{2} - x_{(m+1)}\right)^2\right] =$
$\frac{m(n-1)}{4n}\left[(x_{(m+1)} - x_{(m)})^2 + (x_{(m)} - x_{(m+1)})^2\right] = \frac{m(n-1)}{2n}(x_{(m+1)} - x_{(m)})^2$, & $\because n = 2m$ have:
$\hat{V}_J = (n-1)(x_{(m+1)} - x_{(m)})^2/4$ eg data: ... $\therefore \hat{\theta}_J = (0.57 + 0.63)/2 = 0.60$ &
$\hat{V}_J = 9(0.63 - 0.57)^2/4 = 0.09^2$ st an approx 90% CI for θ is $\hat{\theta}_J \pm t_{0.95}\sqrt{\hat{V}_J} =$
$0.60 \pm 1.833(0.09) = (0.435, 0.765)$ $\{n = 10 \therefore DF = 9\} \therefore t_{\alpha=0.05, \nu=9} = 1.833\}$
a test of $H_0: \theta = 0.5$ against $H_1: \theta \ne 0.5$ compares Z test stat of
$(0.60 - 0.5)/0.09 = 1.111$ to a Stu(9) distri $\{Stu(n-1)\}$ to obtain a
p-val of $2*(1-pt(1.111, 9)) = 0.295$ >> 2*(1-pt(1.111,9)) $\Rightarrow$ 0.295

3.3 Monte Carlo & bootstrap tests / Monte Carlo & bootstrap resampling

use to construct hypoth tests. recall Model data $\underline{x}$ as indep r.v.s $\underline{X}$ with common distri func $F(x;\theta)$ simple case: $H_0: \theta = \theta_0$ suppose chosen a test stat $T(\underline{X})$ st CR take Z form $\{\underline{x} : T(\underline{x}) \ge c\}$ for a test of size α require Z critical val c to satisfy $Pr[T(\underline{X}) \ge c; \theta_0] = \alpha$ find Z p-val $Pr[T(\underline{X}) \ge t; \theta_0]$, $t = T(\underline{x})$ is Z observed val of Z test stat. either case, need to know Z null distri of $T(\underline{X})$, that is Z distri of $T(\underline{X})$ when H_0 is true & $\underline{X}$ has distri given by $F(x;\theta_0)$. what is can't find Z null distri exactly or approx? $\because$ H_0 is simple, it specifies completely a distri for Z data, namely $F(x;\theta_0)$ can simulate samples of size n from this distri: $\underline{x}_b^* = (x_{b1}^*, \ldots, x_{bn}^*)$ for $b = 1, \ldots, B$. Z test stats ~~...~~ $t_b^* = T(\underline{x}_b^*)$ computed from these resamples are then a sample from Z null distri of T & $\therefore$ can use them to form an approx to Z null distri eg Z p-val would be approxed by Z proportion $\frac{1}{B}\sum_{b=1}^{B}\mathbb{1}(t_b^* \ge t)$, where $\mathbb{1}(A) = 1$ is A is true & $\mathbb{1}(A) = 0$ is A is false, & Z CV would be approxed by Z empirical $(1-\alpha)$-quantile $t^*_{((1-\alpha)B)}$, $t^*_{(1)} \le \ldots \le t^*_{(B)}$ are Z order stats. larger vals for B lead to closer approxs to Z p-val & CV. a test based on simulating data from $F(x;\theta_0)$ is known as a <u>Monte Carlo test</u>. Z beauty of Monte Carlo test is that any test stat can be used!

\Ex3.12/ $X_1, \dots, X_n$ be indep $Exp(\theta)$ r.v. & consider $H_0: \theta = \theta_0$ & $H_1: \theta = \theta_1$, $\theta_1 < \theta_0$ in (ex1.28) showed Z most powerful CR is $\{\underline{x} : T(\underline{x}) \le c\}$, $T(\underline{x}) = \sum_{i=1}^{n} x_i$ & noted Z null distri of T is $Ga(n, \theta_0)$. if we didnt know Z null distri then we could use a MonteCarlo test to find Z CV or p-val by resampling from Z $Exp(\theta_0)$ distri. Z following code does this for a data set when $\theta_0 = 1$ & gives a p-val of 0.19. Z approx to Z null distri can be graphed. >>x=c(...) >>B=1000 >>theta0=1

>>n=length(x) %samplesize% >>t=sum(x) %observed test stat%

>>t-star=numeric(B) %vec to store test stat%

>> for (b in 1:B){ % for each b=1,...,B %

>>x-star=rexp(n,1/theta0) % simulate a new sample

>>t-star[b]=sum(x-star) % calc & store test stat%. >>}

>>hist(t-star) %approx null distri% >>mean(t-star<=t) %approx p-val%

>>quantile(t-star, 0.05) %approx 5% CV%

\Ex3.13/ Suppose random sample $(x_1, y_1), \dots, (x_n, y_n)$ from a bivariate normal distri with zero expectations, unit variances & correlation ρ & that we want to test Z null hypoth that Z x&y variables are indep ($\rho = 0$) against a two-sided alternative ($\rho \neq 0$). one possible test uses Z squared sample correlation as a test stat & rejects Z null when it is large. Under Z null, Z x & y variables have indep $N(0,1)$ distris & ∴ can resample x & y data from these distris & calc Z squared correlation for each sample to approx Z null distri. eg sample below yields p-val of 0.44: >>x=c(...) >>y=c(...) >>B=1000

>>n=length(x) %sample size% >>t=cor(x,y)^2 %observed test stat%

>>t-star=numeric(B) %vec to store test stats% >> for (b in 1:B){ %for each b%

>>x-star=rnorm(n) % simulate new x-sample%. y-star=rnorm(n) % simulate new y-sample%.

Ex >>t_Star[b]=Cor(x_Star, y-Star)^2 % calc test Stat% >>}
>>Mean(t-Star>=t) % p-val%

if $H_0: \theta \in \Theta_0$ is composite then H_0 doesnt specify Z distri of Z data completely & ∴ a Monte Carlo test cannot be used; dont know which distri should be used to simulate Z extra samples. one approach in this ~~distri~~ situation is to resample from a esti of Z data distri, where Z esti satisfies H_0. So we simulate from $F(x; \hat{\theta}_0)$, where $\hat{\theta}_0$ is an esti of θ that statisfies $\theta_0 \in \Theta_0$. its known as parametric bootstrap test. eg if $\theta = (\psi, \eta)$ & $H_0: \psi = \psi_0$ st η is unspecified by H_0 then we could simulate from $F(x; \psi_0, \hat{\eta}_0)$, $\hat{\eta}_0$ is Z val of η that maximises $L(\psi_0, \eta)$

Ex3.14/ suppose Z expectations of Z bivariate Normal distri is in (Ex3.13) were ~~ek~~ known. then could resample x from Z $N(\bar{x}, 1)$ distri & resample y indep from Z $N(\bar{y}, 1)$ distri as Z code:

```
>>x-bar=Mean(x) % Sample Mean of x data%   y-bar=Mean(y) % Sample Mean of y%
>> for (b in 1:B) { % for each b%      >>x_Star=rnorm(n, Mean=x-bar)
% simulate new x-Sample%  >>y-Star=rnorm(n, Mean=y-bar)  % Simulate
new y-Sample%  >>t Star[b]=Cor(x-Star, y-Star)^2  % Calc test Stat%  >>}
>>Mean(t-Star>=t)    >>p-val
```

Ex3.15/ $X_1, ..., X_n$ indep $N(\mu, \sigma^2)$ r.v. & consider $H_0: \mu = \mu_0$ against $H_1: \mu \neq \mu_0$. Z null hypoth is composite ∵ Z val of σ is unspecified. in (Ex2.6) we considered a generalised likhood ratio test for this situation, but Z test relied on having a large sample st Z distri of Z test stat would be well approxed by its asympt χ^2_1 distri. an alternative is to use Z same test stat but to use resampling to approx its null distri. Z param bootstrap test could simulate samples from $N(\mu_0, \hat{\sigma}_0^2)$, $\hat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i - \mu_0)^2$ was found in (Ex2.6) to max $L(\mu_0, \sigma^2)$. then compute Z test stat for each resample & find p-vals or CV's as before. Code for: $\mu_0 = 0$ data in (Ex2.6) for

$n=30$, $t=5.61$ & $\hat{\sigma}_0^2=8.99$ Z p-val is 0.020

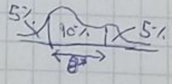
```
>>B=1000  >>Mu0=0  >>n=30  >>sigma0hat = sqrt(8.99)
T = function(x) {  >> sigmahat2=mean((x-mean(x))^2) %define
func to% %calc Z likhood%  >>sigma0hat2 = mean((x-Mu0)^2
%ratio test stat%  >>n*(log(sigma0hat2)-log(sigmahat2))
% -2log(lambda) %  >>}  >>t_star=numeric(B) %vec to store test
stats%  >>for (b in 1:B) { % for each b=1,...,B%
>>x_star=rnorm(n,Mu0,sigma0hat) %simulate a new sample under H0%
>>t_star[b]=T(x_star) %store test stat%  >>}
>>mean(t_star >=5.61) %p-val%
```

in Z nonparam case, dont assume a param model for F, can still formulate hypoth about F, eg that it has a certain expectation - can also desire approp test stats, such as Z sample mean $T(\underline{x})=\bar{x}$ but how can we find Z null distri of T? as for Z param bootstrap test, can try to resample from an estied distri that satisfies H_0. finding such an esti of F in Z nonparam can be tricky: cant just plug in some param estis. suitable estis of F often exist but in which case Z test is known as a nonparametric bootstrap test.

we have described Monte Carlo & bootstrap tests for indep iid r.vs for covariate models in which X_i has distri $F_i(x;\theta)$, Z tests proceeds in a similar manner with Monte Carlo tests simulating x^*_{bi} from $F_i(x;\theta_0)$ & bootstrap tests simulating x^*_{bi} from $F_i(x;\psi_0,\hat{\gamma}_0)$

\Week 11 lec/ \menti/ \1/ $\hat{\theta}=1.5$ let Z 5, 10, 90 & 95% quantiles of $\hat{\theta}^*$ be -1, 0, 2, 3 whats Z percentile bootstrap 90% CI for θ?

$(\hat{\theta}^*_{(\alpha B)}, \hat{\theta}^*_{((1-\alpha)B)})$ (-1, 3)



\2/ $\hat{\theta}=1.5$ quantiles of $\hat{\theta}^*$ -1, 0, 2, 3 ... bootstrap 80% CI? (0, 2)

\3/

3/ $\hat{\theta} = 1.5$ 5, 10, 90, 95% quantiles of $\hat{\theta}^*$ be $-1, 0, 2, 3$ whats basic bootstrap 90% CI? $(2\hat{\theta} - \theta^*_{(\,)}, \,)$

- $(2\hat{\theta} - \underbrace{\hat{\theta}^*_{(\,)}}_{95\%}, 2\hat{\theta} - \underbrace{\hat{\theta}^*_{(\,)}}_{5\%})$ $\therefore (2(1.5)-3, 2(1.5)-(-1)) = (0, 4)$

4/ $1.5 = \hat{\theta}$ 80% CI basic bootstrap? $(3-2, 3-0) = (1, 3)$

5/ let $\bar{x} = 1$, $s = 0.5$ $t = (x_0 - \bar{x})/s$ let the 5 & 95% quantiles of t^* be $-2, 4$ whats the studentised bootstrap 90% PI for X_0?

$X \sim F(\mu, \sigma^2)$ $\frac{X-\mu}{\sigma} \sim F(0,1)$ $(\bar{X} + \hat{q}_{\alpha} s, \bar{X} + q_{1-\alpha} s)$

$(1 + 0.5 \cdot 4, 1 + 0.5 \cdot -2) = (3, 0)$ ✗ $(1 + 0.5(-2), 1 + 0.5(4)) = (0, 3)$ ✓

$(\bar{X} + \hat{q}_{0.05} s, \bar{X} + q_{0.95} s)$ $\therefore (\bar{x} + s q_{0.05}, \bar{x} + s q_{0.95})$

6/ Scale model: $X_i = \sigma Z_i$ for $i = 0, 1, \dots, n$ which are ancillary stats?

Location-Scale Model $X = \mu + \sigma Z$ $\therefore$ distri of Z doesnt depend on the params

$$T = \frac{X_0 - \bar{X}}{s} = \frac{\mu - \sigma Z_0 - \frac{1}{n}\sum_{i=1}^{n}(\mu + \sigma Z_i)}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}\left[\mu + \sigma Z_i - \frac{1}{n}\sum_{j=1}^{n}(\mu + \sigma Z_j)\right]^2}}$$ {replaced all X with $\mu + \sigma Z$}

$$= \frac{Z_0 - \frac{1}{n}\sum_{i=1}^{n} Z_i}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(Z_i - \bar{Z})^2}}$$ doesnt depend on the params

$\therefore$ ~~X_n~~ $X_i = \sigma Z_i$ $\therefore \frac{X_0}{\bar{X}} = \frac{X_0}{\frac{1}{n}\sum_{i=1}^{n} X_i} = \frac{X_0}{\frac{1}{n}\sum_{i=1}^{n}\sigma Z_i} = \frac{X_0}{\frac{1}{n}\sigma\sum_{i=1}^{n} Z_i} = \frac{\sigma Z_0}{\frac{1}{n}\sigma\sum_{i=1}^{n} Z_i} = \frac{Z_0}{\frac{1}{n}\sum_{i=1}^{n} Z_i}$

$\frac{X_0}{s}$ ✓ $\frac{X_0 X_{\text{var}}}{s^2}$ ✓ $\frac{X_0}{\bar{X}}$ ✓ $\frac{X_0}{IQR}$ ✓

$T = \frac{X_0}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}}$ for $\frac{X_0}{s}$ $\therefore = \frac{\sigma Z_0}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(\sigma Z_i - \sigma \bar{Z})^2}}$

$\frac{X_0}{IQR}$ $IQR = X_{(0.75n)} - X_{(0.25n)} = \sigma Z_{(0.75n)} - \sigma Z_{(0.25n)}$

3.4 Bootstrap Confidence intervals

Show how bootstrapping can be used to construct CIs as before. Let $\hat{\theta} = \hat{\theta}(\underline{X})$ be an esti for $\theta = \theta(F)$. If we knew the distri of a suitable func $T = T(\underline{X}, F)$ $\therefore$ could construct CIs for

- θ. eg if q_p denotes the p-quantile of the distri of $T = \hat{\theta} - \theta$ $\therefore$

$\Pr(q_{\alpha} < \hat{\theta} - \theta < q_{1-\alpha}) = 1 - 2\alpha$ $\therefore \Pr(\hat{\theta} - q_{1-\alpha} < \theta < \hat{\theta} - q_{\alpha}) = 1 - 2\alpha$ $\therefore$ the $(1-2\alpha)$ CI is: $(\hat{\theta} - q_{1-\alpha}, \hat{\theta} - q_{\alpha})$. If knew the distri of T then could ~~calc~~ any quantile q_p Z ∴ determine Z CI . ∵ F is unknown, can use bootstrap to esti Z distri of T & ∴ Z quantiles q_p.

as in Section 3.1 let $\hat{F}$ be a param or nonparam esti of F, let $T^* = T(\underline{X}^*, \hat{F}) = \hat{\theta}(\underline{X}^*) - \theta(\hat{F})$ be Z bootstrap version of T, let $\underline{x}_1^*, \ldots, \underline{x}_B^*$ be B bootstrap Samples Simulated from $\hat{F}$, Z let $t_b^* = T(\underline{x}_b^*, \hat{F}) = \hat{\theta}_b^* - \theta(\hat{F})$, $\hat{\theta}_b^* = \hat{\theta}(\underline{x}_b^*)$, ∴ $t_1^*, \ldots, t_B^*$ is a sample from Z distri of T^* & Z empirical p-quantile of this sample is an esti of q_p. if $t_{(1)}^* \leq \ldots \leq t_{(B)}^*$ are Z order Stats then this esti $\hat{q}_p = t^*_{(pB)}$ & an approx $(1-2\alpha)$ - CI is $(\hat{\theta} - \hat{q}_{1-\alpha}, \hat{\theta} - \hat{q}_{\alpha})$. this CI is called a Basic bootstrap interval. if $\theta(\hat{F}) = \hat{\theta}$ then $\hat{q}_p = \hat{\theta}^*_{(pB)} - \hat{\theta}$ Z Z interval becomes $(2\hat{\theta} - \hat{\theta}^*_{((1-\alpha)B)}, 2\hat{\theta} - \hat{\theta}^*_{(\alpha B)})$.

\Ex 3.16/ consider Z exponential model with density $\theta \exp(-\theta x)$. Z MLE is $\hat{\theta} = 1/\bar{x}$ with asympt distri $N(\theta, \theta^2/n)$ ∵ Z expected info is n/θ^2. Z following code compares a 90% CI for θ based on this Normal approx with Z basic (nonparam) bootstrap interval. note $\theta(F) = 1/E(X)$ ∴ $\theta(\hat{F}) = 1/E(X^*) = \hat{\theta}$.

```
>>x=c(...)   >>B=1000   >>alpha=0.05  %set CI%   >>n=length(x)
>>theta_hat=1/mean(x)   >>t_star=numeric(B)   >>for (b in 1:B) {
>>x_star =sample(x, replace =TRUE)  %resample X%
>>t_star[B]=1/mean(x_star)-theta_hat  %calc T*%   >>}
>>bas =theta_hat - quantile(t_star, c(1-alpha, alpha)) %basic interval%
>>mle = theta_hat - qnorm(c(1-alpha, alpha)) * theta_hat/sqrt(n)  %normal%
>>rbind(mle=mle, basic=bas)
```

noted in Section 3.1 that accuracy can be improved if bootstrap pivots Z difference $\hat{\theta} - \theta$ is rarely a pivot so often prefer to use func $T = \frac{\hat{\theta} - \theta}{S}$, S is an estimator for Z standard error of $\hat{\theta}$, if q_p denotes Z p-quantile of this T then $Pr(q_\alpha < \frac{\hat{\theta}-\theta}{S} < q_{1-\alpha}) = 1-2\alpha$ ∴ $Pr(\hat{\theta} - q_{1-\alpha} S < \theta < \hat{\theta} - q_\alpha S) = 1-2\alpha$ st a $(1-2\alpha)$ - CI is $(\hat{\theta} - q_{1-\alpha} S, \hat{\theta} - q_\alpha S)$. ∴ can again esti q_p with Z bootstrap. as before $\hat{q}_p = t^*_{(pB)}$ but now

$t_b^* = \frac{\hat{\theta}_b^* - \theta(\hat{F})}{s_b^*}$, s_b^* is standard error computed from from Z b^{th} bootstrap sample, $\underline{x}_n^*$. Z CI $(\hat{\theta} - \hat{q}_{1-\alpha}s, \hat{\theta} - \hat{q}_{\alpha}s)$ is called a studentised bootstrap interval. studentisedtised bootstrap interval

\Ex 3.17/ Z following coded uses $s = \hat{\theta}/\sqrt{n}$ to obtain Z studentised (nonparam) bootstrap interval for Z data in Ex 3.16

```
>> s = theta_hat/sqrt(n) %.esti standard error%  >> for (b in 1:B) {
>> x_star = sample(x, replace = TRUE) % resample%
theta_star = 1/mean(x_star) %.esti theta%
>> s_star = theta_star/sqrt(n) %.esti SE%
>> t_star[b] = (theta_star - theta_hat)/s_star % calc T* %  >> }
>> stu = theta_hat - quantile(t_star, c(1-alpha, alpha))*s %.interval%
>> rbind(mle = mle, basic = bas, studentised = stu)
```

a third type of bootstrap CI can be constructed is $\theta(\hat{F}) - \hat{\theta}$ & $\exists$ a monotonically increasing func h, s.t. Z distri of $h(\hat{\theta})$ is symmetric about $h(\theta)$. let $T = h(\hat{\theta}) - h(\theta)$ & denote Z p-quantile of T by q_p. Z bootstrap esti of q_p is again $\hat{q}_p = \hat{t}_{(pB)}$ but now $t_b^* = h(\hat{\theta}_b^*) - h(\hat{\theta})$ ∴ by symmetry q_p is also Z p-quantile of $h(\theta) - h(\hat{\theta}) = -T$ s.t. $Pr(q_\alpha < h(\theta) - h(\hat{\theta}) < q_{1-\alpha}) = 1 - 2\alpha$ & a $(1-2\alpha)$ CI for $h(\theta)$ is $(h(\hat{\theta}) + q_\alpha, h(\hat{\theta}) + q_{1-\alpha})$ ∴ subbing bootstrap estis of q_α & $q_{1-\alpha}$ ∴ $h(\hat{\theta}) + \hat{q}_\alpha = h(\hat{\theta}^*_{(\alpha B)})$ & $h(\hat{\theta}) + \hat{q}_{1-\alpha} = h(\hat{\theta}^*_{((1-\alpha)B)})$ ∴ Z interval is $(h(\hat{\theta}^*_{(\alpha B)}), h(\hat{\theta}^*_{((1-\alpha)B)}))$. finally ∵ h is monotonically increasing, a CI for θ is found by inverting Z transformation h to obtain $(\hat{\theta}^*_{(\alpha B)}, \hat{\theta}^*_{((1-\alpha)B)})$. this CI is called a percentile bootstrap interval. note don't actually use h when calcing Z final interval

\Ex 3.18/ Z following code computes Z percentile (nonparam) bootstrap interval for (Ex 3.16)

```
>> theta_star = numeric(B) %.vec to store esti%
>> for (b in 1:B) {  >> x_star = sample(x, replace = TRUE % resample%
>> theta_star[b] = 1/mean(x_star) %.esti theta%. }
```

```
>>per = quantile(theta-star, c(alpha, 1-alpha)) % percentile interval
>>rbind(mle=mle, basic=bas, studentised=stu, percentile=per)
```

have described three types of bootstrap CI for iid r.v. these same intervals can be used for covariate models as long as either Z param case or error sampling described at Z end of Sec 3.1 is used

3.5 bootstrap prediction intervals

/ show ~~bootstrapping~~ used to construct prediction intervals. recall wish to predict X_0 using $\underline{X} = (X_1, \dots, X_n)$, all indep with common distri func $F(x;\theta)$. if knew θ then an equal-tailed $(1-2\alpha)$ prediction interval for X_0 would be $(q_\alpha, q_{1-\alpha})$, q_p denotes Z q-quantile of F st $F(q_p;\theta)=p$. Z difficulty is dont know θ. might try esting Z distri of X_0 by Z distri of X_0^*, Z bootstrap version of X_0. with param bootstrap resampling, X_0^* has distri $F(x;\hat\theta)$ & $\therefore$ our prediction interval would be $(\hat q_\alpha, \hat q_{1-\alpha})$, $\hat q_p$ satisfies $F(\hat q_p;\hat\theta)=p$. is precisely Z plug in prediction interval discussed in Sec 1.7 & found to have poor coverage

can do better if bootstrap other stats of Z form $T(\underline{X}, X_0)$ in a manner similar to studentised bootstrap CI, eg if $T = \frac{X_0 - \bar X}{S}$, $\bar X = \frac{1}{n}\sum_{i=1}^{n} X_i$ & $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar X)^2$ if q_p now denotes Z p-quantile of Z distri of T

$\therefore 1-2\alpha = Pr\left(q_\alpha < \frac{X_0 - \bar X}{S} < q_{1-\alpha}\right) = Pr\left(\bar X + q_\alpha S < X_0 < \bar X + q_{1-\alpha} S\right)$ & a $(1-2\alpha)$ prediction interval for X_0 be $(\bar X + q_\alpha S, \bar X + q_{1-\alpha} S)$ if dont know Z distri of T then can esti its quantiles with Z bootstrap let $T^* = T(\underline{X}^*, X_0^*)$ be Z bootstrap version of T, $\underline{X}^* = (X_1^*, \dots, X_n^*)$ & Z r.v. $X_0^*, X_1^*, \dots, X_n^*$ are indep with distri $F(x;\hat\theta)$ resampling proceeds as: for $b = 1, \dots, B$

① simulate x_{b0}^* & $\underline{x}_b^* = (x_{b1}^*, \dots, x_{bn}^*)$ from $F(x;\hat\theta)$;

② calc $t_b^* = T(\underline{x}_b^*, x_{b0}^*)$ Z bootstrap esti of q_p is then $\hat q_p = t^*_{(pB)}$ & Z studentised bootstrap prediction interval is $(\bar X + \hat q_\alpha S, \bar X + \hat q_{1-\alpha} S)$

\Ex3.19\ let $X_0, X_1, \ldots, X_n$ be indep $N(\mu, \sigma^2)$ r.v. Z following code calcs Z studentised bootstrap 90% prediction interval for X_0 based on a sample of size $n=10$. also calcs Z exact prediction interval for this model that derived in (Ex1.40)

```
>>x=c(...)   >>B=1000   >>n=length(x)   >>alpha=0.05 % define coverage%   >>mu=mean(x)   >>esti mu%   >>sigma=sd(x) %esti sigma%
>>t_star=numeric(B)   >>for (b in 1:B) {   >>x_star=rnorm(n,mu,sigma) % resample X1,...Xn%   >>x0_star=rnorm(1,mu,sigma) % resample X0%
>>t_star[b]=(x0_star-mean(x_star))/sd(x_star) % calc T* %   >>}
>>stu=mu+quantile(t_star,c(alpha,1-alpha))*sigma % studentised%
>>exa=mu+qt(c(alpha,1-alpha),n-1)*sigma*sqrt(1+1/n) %exact%
>>rbind(exact=exa,studentised=stu)
```

noted bootstrap more accurate if T is a pivot. Same idea here: if Z distri of T is indep of θ (T is an ancillary stat) then T & T^* have same distri & Z only bootstrap error arises from B having to be finite. Z stat $T=(X_0-\bar{X})/s$ is exact ancillary in Z foregoing ex $\because$ $T/\sqrt{1+1/n}$ has $\sim$ Stu$(n-1)$ distri as noted in (Ex1.40) $\therefore$ if able to make B infinite in ex above, Z bootstrap interval be Z same as Z exact interval. Z stat $T=(X_0-\bar{X})/s$ is ancillary if Z distri of $X_0, X_1, \ldots, X_n$ is a location-scale model.

\Def3.1\ Z distri of a r.v. X is a <u>location-scale model</u> with location param μ & scale param σ if X has Z same distri as $\mu+\sigma Z$ & Z is a r.v. whose distri contains no unknown params

Z $N(\mu,\sigma^2)$ distri is an example of a location-scale model $\because$ it is Z distri of $\mu+\sigma Z$ when Z has a $N(0,1)$ distri. to see T is ancillary for location-scale models, define $Z_i=(X_i-\mu)/\sigma$ st Z distri of Z_i contains no unknown param $\therefore$ $\bar{X}=\frac{1}{n}\sum_{i=1}^{n}X_i=\frac{1}{n}\sum_{i=1}^{n}(\mu+\sigma Z_i)=\mu+\sigma\bar{Z}$

$S^2=\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2=\frac{1}{n-1}\sum_{i=1}^{n}(\mu+\sigma Z_i-\mu-\sigma\bar{Z})^2=\frac{\sigma^2}{n-1}\sum_{i=1}^{n}(Z_i-\bar{Z})^2$ & $T=\frac{X_0-\bar{X}}{s}=$

$\dfrac{\mu+\sigma Z_0-\mu-\sigma\bar{Z}}{\sigma\sqrt{\sum_{i=1}^{n}(Z_i-\bar{Z})^2/(n-1)}} = \dfrac{Z_0-\bar{Z}}{\sqrt{\sum_{i=1}^{n}(Z_i-\bar{Z})^2/(n-1)}}$ Z distri of T ∴ depend only on the Z_i & ∴ indep of Z params μ & σ

in Z previous ex, knew Z distri of Z ancillary stat was (after scaling by $\sqrt{1+1/n}$) a Stu(n-1) distri. knowing Z distri of Z ancilly stat allowed us to derive Z exact prediction interval in (Ex1.40). for other location scale models, Z distri of Z ancillary stat may be intractable ∴ bootstrapping provides a way of forming Z prediction interval. if an ancillary stat is unavailable then choosing T to be approx ancillary (choosing T have distri approx indep to model params) yield good results. one generally applicable choice is Z <u>probability integral transform</u> (PIT): $T=F(X_0;\hat{\theta})$, $\hat{\theta}=\hat{\theta}(\underline{X})$ is an esti of θ based on $\underline{X}$ to see its approx ancillary note: $\Pr(T\leq t)=\Pr[F(X_0;\hat{\theta})\leq t]$ $=\Pr[X_0\leq F^{-1}(t;\hat{\theta})]\approx\Pr[X_0\leq F^{-1}(t;\theta)]=F[F^{-1}(t;\theta);\theta]=t$ for $0<t<1$ so T is approx uniformly distried on (0,1) if q_p denotes Z p-quantile of Z distri of this alternative T then $1-2\alpha=\Pr[q_\alpha<F(X_0;\hat{\theta})<q_{1-\alpha}]=$ $\Pr[F^{-1}(q_\alpha;\hat{\theta})<X_0<F^{-1}(q_{1-\alpha};\hat{\theta})]$ ∴ a $(1-2\alpha)$ prediction interval for X_0 be $(F^{-1}(q_\alpha;\hat{\theta}), F^{-1}(q_{1-\alpha};\hat{\theta}))$ don't know Z distri of T then can esti its quantiles using Z bootstrap with $T^*=F(X_0^*;\hat{\theta}(\underline{X}^*))$ ∴ Z <u>PIT bootstrap prediction interval</u> is $(F^{-1}(\hat{q}_\alpha;\hat{\theta}), F^{-1}(\hat{q}_{1-\alpha};\hat{\theta}))$, $q_p=t^*_{(pB)}$

Ex3.20/ $X_0, X_1, \dots, X_n$ indep $Ga(\gamma,\delta)$ r.v. Z following code calcs a bootstrap 90% prediction interval for X_0 based on a sample of n=10 Z method of moments estimas $\hat{\gamma}=\bar{X}^2/S^2$ & $\hat{\delta}=\bar{X}/S^2$ for esti Z params

```
>>x=c(...) >>B=1000 >>n=length(x) >>gamma=mean(x)^2/var(x) % esti gamma%
>>delta=mean(x)/var(x) % esti delta% >>t_star=numeric(B) >>for(b in 1:B){
>>x_star=rgamma(n, gamma, delta) % resample X1,...,Xn %
>>x0_star=rgamma(1, gamma, delta) % resample X0 %
>>gamma_star=mean(x_star)^2/var(x_star) % esti gamma%
>>delta_star=mean(x_star)/var(x_star) % esti delta%
```

t_star[b] = pgamma(x0star, gamma_star, delta_star) % calc T^* %
>> gamma quantile(t_star, c(0.05, 0.95)), gamma, delta) % interval %

described bootstrap prediction intervals for iid r.v, x_{bi}^* is simulated from $F(x;\hat{\theta})$ for $i = 0, 1, \ldots, n$ for covariate models in which X_i has distri $F_i(x;\theta)$ similar intervals can be used by simulating x_{bi}^* from $F_i(x;\hat{\theta})$

\Week 12 lecture/ $T = T(X_0, \underline{X})$ if know Z distri of T then we can find t_5 & t_{95} st $Pr(t_5 < T < t_{95}) = 0.90$ T is ancillary

eg: $T = X_0/\bar{X}$ then $Pr(t_5 < \frac{X_0}{\bar{X}} < t_{95}) = 0.90 \iff$

$Pr(\bar{X} t_5 < X_0 < \bar{X} t_{95}) = 0.90$ {desires a 95% prediction interval for X_0}

So $(\bar{X} t_5, \bar{X} t_{95})$ is a 90% P.I. for X_0

$X_0 - \bar{X} = X_0 - \frac{1}{n}(X_1, \ldots, X_n) = \sum_{i=0}^{n} a_i X_i$, $a_0 = 1$, $a_i = -\frac{1}{n}$ for $i = 1, \ldots, n$

$= T = X_0 - \bar{X}$

$= \sum_{i=0}^{n} a_i X_i \sim Cau(0, 2\sigma)$ {from Sheet 1 question}

\Sheet 1 preparatory Q7/ $Uni(0,\theta)$ $f(x;\theta) = 1/\theta$ $0 < x < \theta$ $\hat{\theta} = \max\{X_1, \ldots, X_n\}$

derive Z distri func of $\hat{\theta}$ st $T = X_0/\hat{\theta}$ is ancillary

now $P(\hat{\theta} \le t) = P(X_1 \le t, \ldots, X_n \le t) = \prod_{i=1}^{n} P(X_i \le t)$ by independence

& $P(X_i \le t) = \int_0^t f(x;\theta)\,dx = \int_0^t \frac{1}{\theta}\,dx = \frac{t}{\theta}$ for $0 < t < \theta$

So $P(\hat{\theta} \le t) = \prod_{i=1}^{n} \frac{t}{\theta} = \frac{t^n}{\theta^n}$ for $0 < t < \theta$

let $T = X_0/\hat{\theta}$ {for Scale Model use: $\frac{X_0}{\bar{X}}$ $\left\{= \frac{\sigma Z_0}{\sigma \bar{Z}}\right\}$}

Location Model: $X_0 - \bar{X}$ $\{= (\mu + Z_0) - (\mu + \bar{Z})\}$

Location-Scale Models: $\frac{X_0 - \bar{X}}{S}$

have: $P(T \le t) = P(X_0/\hat{\theta} \le t) = P(\hat{\theta} \ge X_0/t)$ $\left\{= \int_0^{\min\{\theta, \theta t\}} P(\hat{\theta} \ge x/t, X_0 = x)\,dx\right\}$

$\{0 < x < \theta$ & $0 < \hat{\theta} < \theta$ $\therefore 0 < \frac{x}{t} < \theta$ $\therefore 0 < x < \theta t$ $= \int_0^{\min\{\theta, \theta t\}} P(\hat{\theta} \ge x/t, X_0 = x)\,dx$

$= \int_0^{\min\{\theta, \theta t\}} P(\hat{\theta} \ge \frac{x}{t}) f(x;\theta)\,dx = \int_0^{\min\{\theta, \theta t\}} \left[1 - \frac{x^n}{t^n \theta^n}\right] \frac{1}{\theta}\,dx = \frac{1}{\theta} \int_0^{\min\{\theta, \theta t\}} 1\,dx - \frac{1}{\theta^{n+1} t^n} \int_0^{\min\{\theta, \theta t\}} x^n\,dx$

$= \frac{1}{\theta} \min\{\theta, \theta t\} - \frac{1}{\theta^{n+1} t^n} \frac{(\min\{\theta, \theta t\})^{n+1}}{n+1} =$

$\min\{1, t\} - \frac{\min\{1, t\}^{n+1}}{(n+1)t^n} = \begin{cases} \frac{nt}{n+1} & \text{if } 0 < t \le 1 \\ 1 - \frac{1}{(n+1)t^n} & \text{if } t > 1 \end{cases}$ let t_p be Z p-quantile of T

$\therefore P(t_\alpha < T < t_{1-\alpha}) = 1 - 2\alpha$, ie $P(t_\alpha < \frac{X_0}{\hat{\theta}} < t_{1-\alpha}) = 1 - 2\alpha$ ie $P(\hat{\theta} t_\alpha < X_0 < \hat{\theta} t_{1-\alpha}) = 1 - 2\alpha$

\Ex1.4\ $X_0, X_1, \ldots, X_n$ be indep Exponential r.v. distri func $1-\exp(-x/\theta)$ for $x>0$ recall Z sum of n indep Exponential r.v. has a gamma distr eg $\sum_{i=1}^{n} X_i$ has density $f(s)=\frac{s^{n-1}}{\theta^n \Gamma(n)} e^{-s/\theta}$ for $s>0$. show $T=X_0/\bar{X}$ is ancillary & ∴ find a prediction interval for X_0.

writing $\phi=\theta(1+t/n)$ ∴ $Pr(T \le t)=Pr(X_0 \le t\bar{X})=Pr(X_0 \le \frac{t}{n}\sum_{i=1}^{n} X_i)=$
$\int_0^\infty Pr(X_0 \le tS/n \mid \sum_{i=1}^{n} X_i = s) f(s)\, ds$

{ let X & Y be two indep r.v. with pdfs $f_X(x)$ & $f_Y(y)$ ∴ X & Y are indep, their joint pdf, $f_{X,Y}(x,y)$, factorises: $f_{X,Y}(x,y)=f_X(x) f_Y(y)$. now $X \le Y$ is true on Z set $A=\{(x,y): x \le y\}$ & ∴ $Pr(X \le Y)=\int_A f_{X,Y}(x,y)\,dx\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{y} f_{X,Y}(x,y)\,dx\,dy=\int_{-\infty}^{\infty}\int_{-\infty}^{y} f_X(x) f_Y(y)\,dx\,dy=$
$\int_{-\infty}^{\infty} f_Y(y)\int_{-\infty}^{y} f_X(x)\,dx\,dy=\int_{-\infty}^{\infty} f_Y(y) Pr(X \le y)\,dy$ one way to think about this final integral is that we fix $Y=y$. find Z probab ($Pr(X \le y)$) that Z event of interest ($X \le Y$) is true for this val of Y, multiply by Z probab (density) that $Y=y$ & then integrate over y - assumed that Z densities of X & Y are posi on Z whole real line. if Z densities are posi on only subsets of Z real line then Z limits of integration will differ. eg in Ex1.4 want $Pr(X_0 \le tS/n)$, $S=\sum_{i=1}^{n} X_i$ with pdf $f(s)$. Fix $S=s$, find Z probab ($Pr(X_0 \le ts/n)$) that Z event of interest ($X_0 \le tS/n$) is true for this val of S, multiply by Z probab (density) that $S=s$ & then integrate over s to obtain $Pr(X_0 \le tS/n)=\int_0^\infty f(s) Pr(X_0 \le tS/n)\, ds$, here Z possible vals for S are Z posi reals}

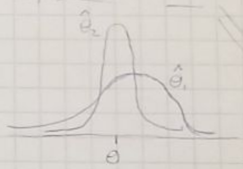
$P(T \le t)=\int_0^\infty Pr(X_0 \le tS/n \mid \sum_{i=1}^{n} X_i = s) f(s)\, ds = \int_0^\infty Pr(X_0 \le ts/n) f(s)\, ds$ (by indep)
$=\int_0^\infty \{1-e^{-ts/n\theta}\} \frac{s^{n-1}}{\theta^n \Gamma(n)} e^{-s/\theta}\, ds = 1-\int_0^\infty \frac{s^{n-1}}{\theta^n \Gamma(n)} e^{-(1+t/n)s/\theta}\, d\theta =$
$1-(1+t/n)^{-n}\int_0^\infty \frac{s^{n-1}}{\phi^n \Gamma(n)} e^{-s/\phi}\, ds = 1-(1+t/n)^{-n}$ ∵ Z last integrand is Z density of a gamma distr ∴ Z distri func of T is indep of θ & ∴ T is ancillary. solving $Pr(T \le t)=p$ for t shows Z p quantile of T is $t_p = n[(1-p)^{-1/n}-1]$ ∴ $Pr(n[(1-\alpha)^{-1/n}-1] < X_0/\bar{X} < n[\alpha^{-1/n}-1])=1-2\alpha$ & multiplying through by $\bar{X}$ yields Z $(1-2\alpha)$-prediction interval $(n[(1-\alpha)^{-1/n}-1]\bar{X},\ n[\alpha^{-1/n}-1]\bar{X})$

Week 1 videos

Def 1.2 / Z **score func** is Z first derivative of Z log-likelihood

$U(\theta; \underline{X}) = \frac{\partial L}{\partial \theta}$ if θ is a vec with ith elem θ_i then $U(\theta; \underline{X})$ is a vec with ith elem $\partial L / \partial \theta_i$. use U & $U(\theta)$ for $U(\theta; \underline{X})$, a rando quantity

Def 1.3 / Z **observed info** is $J(\theta; \underline{X}) = -\frac{\partial^2 L}{\partial \theta^2}$ if θ is a vec then $J(\theta; \underline{X})$ is a matrix with (i,j)th elem $-\partial^2 L / \partial \theta_i \partial \theta_j$. use $J(\theta)$ for $J(\theta; \underline{X})$, a rand quantity. Z **expected info** is $I(\theta) = E[J(\theta)]$



Ex 1.8 / $x_1, \dots, x_n \sim \text{Uni}(0, \theta)$ $\therefore f(x; \theta) = \frac{1}{\theta}$ $\therefore$

$F(x; \theta) = \int_0^x f(t; \theta)\,dt = \int_0^x \frac{1}{\theta}\,dt = \frac{x}{\theta}$ $\therefore$

$P(\hat{\theta} \le x) = P(x_1 \le x, x_2 \le x, \dots, x_n \le x) =$

$P(x_1 \le x) P(x_2 \le x) \dots P(x_n \le x) = \frac{x}{\theta} \cdot \frac{x}{\theta} \cdots \frac{x}{\theta} = \frac{x^n}{\theta^n}$

Ex 1.9 / $E[\hat{\mu}] = E(\bar{X}) = E\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \frac{1}{n} \sum_{i=1}^{n} E(X_i) = \frac{1}{n} \sum_{i=1}^{n} \mu = \mu$

$\text{var}(\hat{\mu}) = \text{var}(\bar{X}) = \text{var}\left(\frac{1}{n} \sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} \text{var}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2} \sum_{i=1}^{n} \text{var}(X_i) =$

$\frac{1}{n^2} n \sigma^2 = \frac{\sigma^2}{n}$

$\text{MSE}(\hat{\theta}) = \text{var}(\hat{\theta}) + \text{bias}(\hat{\theta})^2$ $\quad E(\hat{\theta} - \theta)^2 = \cancel{E(\hat{\theta})^2} \; \cancel{E(\hat{\theta}^2) - 2E(\hat{\theta})\theta + \theta^2}$

$E(\hat{\theta}^2) - 2E(\hat{\theta})\theta + \theta^2 + E(\hat{\theta})^2 - E(\hat{\theta})^2$

Ex 1.16 / $\cancel{h(\hat{\theta})}$ $\hat{\phi} = h(\hat{\theta}) \approx h(\theta) + (\hat{\theta} - \theta) h'(\theta)$

$\text{var}(\hat{\phi}) = \text{var}(h(\hat{\theta})) \approx \text{var}(h(\theta) + \hat{\theta} h'(\theta) - \theta h'(\theta)) =$

$\text{var}(\hat{\theta} h'(\theta)) = (h'(\theta))^2 \text{var}(\hat{\theta})$

$E(\hat{\phi}) = E(h(\hat{\theta})) \approx E\left(h(\theta) + (\hat{\theta} - \theta) h'(\theta) + \frac{1}{2}(\hat{\theta} - \theta)^2 h''(\theta)\right) =$

$h(\theta) + \underbrace{E(\hat{\theta} - \theta)}_{=0} h'(\theta) + \frac{1}{2} E\left[(\hat{\theta} - \theta)^2\right] h''(\theta) =$

$h(\theta) + \frac{1}{2} \text{var}(\hat{\theta}) h''(\theta) = \phi + \frac{1}{2} \text{var}(\hat{\theta}) h''(\theta)$ $\therefore$

$E(\hat{\phi}) - \phi \approx \frac{1}{2} \text{var}(\hat{\theta}) h''(\theta)$

Week 2 vids / $\text{Cov}(\hat{\theta}, U) = 1 \longleftrightarrow \text{var}(\hat{\theta}) \geq \frac{1}{\text{var}(U)}$

$$E(\hat{\theta}) = \int \hat{\theta}(\vec{x})\, f_n(\vec{x};\theta)\, d\vec{x} = \theta$$

$$\frac{\partial}{\partial\theta}\theta = 1 = \frac{\partial}{\partial\theta}\left[\int \hat{\theta}(\vec{x})\, f_n(\vec{x};\theta)\, d\vec{x}\right] = \int \hat{\theta}(\vec{x})\frac{\partial}{\partial\theta} f_n(\vec{x};\theta)\, d\vec{x}$$

$$\left\{ \frac{d}{d\theta}(\log f(\theta)) = \frac{\frac{d}{d\theta} f(\theta)}{f(\theta)} \quad \therefore (\log f)' = \frac{f'}{f} \right\}$$

$$= \int \hat{\theta}(\vec{x})\left(\frac{\partial}{\partial\theta}\log f_n(\vec{x};\theta)\right) f_n(\vec{x};\theta)\, d\vec{x}$$

$$= \int\left[\hat{\theta}(\vec{x})\, U(\theta;\vec{x})\right] f_n(\vec{x};\theta)\, d\vec{x} = E\left[\hat{\theta}(\vec{x})\, U(\theta;\vec{x})\right] = 1$$

$$\text{Cov}(\hat{\theta}, U) = \underbrace{E(\hat{\theta} U)}_{=1} - E(\hat{\theta})E(U)$$

$$E(U) = \int\left[\frac{\partial}{\partial\theta}\log f_n(\vec{x};\theta)\right] f_n(\vec{x};\theta)\, d\vec{x} =$$

$$\int\left[\frac{\frac{\partial}{\partial\theta} f_n(\vec{x};\theta)}{f_n(\vec{x};\theta)}\right] f_n(\vec{x};\theta)\, d\vec{x} = \int \frac{\partial}{\partial\theta} f_n(\vec{x};\theta)\, d\vec{x} =$$

$$\frac{\partial}{\partial\theta}\int f_n(\vec{x};\theta)\, dx = \frac{\partial}{\partial\theta} 1 = 0 \implies \text{Cov}(\hat{\theta}, U) = 1 \implies$$

$$\text{var}(\hat{\theta}) \geq \frac{1}{\text{var}(U)}$$

$$\text{var}(U) = E(U^2) - E(U)^2 = E(U^2) \qquad \left\{ (\log f)' = \frac{f'}{f} \qquad \left(\frac{1}{f}\right)' = -\frac{f'}{f^2} \right\}$$

$$\therefore U^2 = \frac{\frac{\partial^2}{\partial\theta^2} f_n(\vec{x};\theta)}{f_n(\vec{x};\theta)} - \frac{\partial^2}{\partial\theta^2}\log f_n(\vec{x};\theta)$$

$$\left\{ (\log f)'' = ((\log f)')' = \left(\frac{f'}{f}\right)' = \frac{f''}{f} + f'\left(\frac{1}{f}\right)' = \frac{f''}{f} + f'\left(-\frac{f'}{f^2}\right) = \frac{f''}{f} - \left(\frac{f'}{f}\right)^2 = \frac{f''}{f} - \left[(\log f)'\right]^2 \right\}$$

$$= \frac{\frac{\partial^2}{\partial\theta^2} f_n(\vec{x};\theta)}{f_n(\vec{x};\theta)} + J(\theta;\vec{x})$$

$$\therefore E(\ldots) = \int \frac{\partial^2}{\partial\theta^2} f_n(\vec{x};\theta)\, d\vec{x} = \frac{\partial^2}{\partial\theta^2}\int f_n(\vec{x};\theta)\, d\vec{x} = \frac{\partial^2}{\partial\theta^2} 1 = 0 \;\therefore$$

$$\text{var}(U) = E(J(\theta;\vec{x})) = I(\theta) \;\therefore$$

$\text{var}(\hat{\theta}) \geq I(\theta)^{-1}$ Cramér-Rao Lower bound

Ex 1.2 θ / $L(\theta) = L(\theta;\vec{x}) = \prod_{i=1}^{n} f(x_i;\theta) = \prod_{i=1}^{n}\frac{\theta^{x_i} e^{-\theta}}{x_i!} = \frac{\theta^{\sum_{i=1}^{n} x_i} e^{-n\theta}}{\prod_{i=1}^{n} x_i!} \;\therefore$

$$l(\theta;\vec{x}) = \left(\sum_{i=1}^{n} x_i\right)\log\theta - n\theta - \sum_{i=1}^{n}\log(x_i!)$$

$$U(\theta;\vec{x}) = \frac{\sum_{i=1}^{n} x_i}{\theta} - n \quad \therefore \quad J(\theta;\vec{x}) = \frac{\sum_{i=1}^{n} x_i}{\theta^2} \;\therefore$$

$I(\theta) = E(J(\theta; \vec{x})) = \frac{1}{\theta^2}\sum_{i=1}^{n} E(x_i) = \frac{n}{\theta}$

$var(\hat\theta) \geq \frac{\theta}{n}$

$\hat\theta$ is unbiased, efficient $\rightleftharpoons U = I(\theta)(\hat\theta - \theta)$

$E(\hat\theta) = \theta \qquad var(\hat\theta) = \frac{1}{I(\theta)} = \frac{1}{var(U)}$

$\Rightarrow Cor(\hat\theta, U) = 1 \Rightarrow \cancel{U = b\hat\theta} \quad U = a + b\hat\theta$

$E(U) = 0 = a + b\theta \qquad a = -b\theta \qquad U = b(\hat\theta - \theta)$

$var(U) = I(\theta) = b^2 var(\hat\theta - \theta) = b^2 var(\hat\theta) = b^2 \frac{1}{var(U)}$ ∴

$b^2 = var(U)^2 = I(\theta)^2$

$U = I(\theta)(\hat\theta - \theta)$

$U = b(\hat\theta - \theta) \qquad E(U) = 0 \longleftrightarrow E(\hat\theta) = \theta$ ∴ $\hat\theta$ is unbiased

$Cov(\hat\theta, U) = 1 = E(\hat\theta U) \qquad var(U) = I(\theta) = E(U^2) =$

$E(b(\hat\theta - \theta)U) = b\,E(\hat\theta U) - b\theta \underbrace{E(U)}_{=0} = b\underbrace{E(\hat\theta U)}_{=1} = b$

$U = I(\theta)(\hat\theta - \theta)$

$var(U) = I(\theta) = I(\theta)^2 var(\hat\theta) \longleftrightarrow var(\hat\theta) = \frac{1}{I(\theta)}$ ∴ $\hat\theta$ is efficient

Ex 1.22 $\quad U'(\theta) = U(\theta) = \frac{1}{\theta}\sum x_i - n$

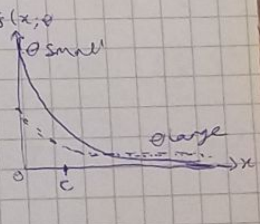
$\{U = b(\hat\theta - \theta)\} \qquad = \frac{1}{\theta}(\sum x_i - n\theta) = \underbrace{\frac{n}{\theta}}_{\text{expected info}}\left(\underbrace{\frac{1}{n}\sum x_i}_{\text{is unbiased efficient estimator for } \theta} - \theta\right)$

test \ H_0	True	False
reject	type 1	✓
dont reject	✓	type 2

$H_0: \theta = \theta_0 \qquad P(\vec{x} \in C; \theta_0)$

$H_0: \theta \in \Theta_0 \qquad \Theta = \{\theta_a, \theta_0\}$

$P(\vec{x} \in C; \theta_a) = \alpha_a \quad P(\vec{x} \in C; \theta_0) = \alpha_0$



Ex 1.27 $\quad X: P(X \leq x; \theta) = 1 - e^{-x/\theta} \qquad x > 0, \theta > 0$

$f(x;\theta) = \frac{1}{\theta}e^{-\frac{x}{\theta}} \qquad E(x) = \theta$

$H_0: \theta > \theta_0 \qquad H_1: \theta \leq \theta_0 \qquad x \qquad C: x \in (0, c)$

$P(X \in C; \theta) = P(X \leq c; \theta) = 1 - e^{-c/\theta}$

$\alpha = \sup_{\theta \in \Theta_0} \qquad \alpha = \sup_{\theta > \theta_0}(1 - e^{-c/\theta}) = 1 - e^{-c/\theta_0}$ ∴ $c = -\theta_0 \log(1 - \alpha)$

$W(\theta)=P(x\in C;\theta)=P(x\le -\theta_0\log(1-\alpha);\theta)=1-e^{+\theta_0\log(1-\alpha)/\theta}$

$=1-(1-\alpha)^{\theta_0/\theta}$

week 3 vids $f_n(\vec{x};\theta_0)$ $f_n(\vec{x};\theta_1)$

$C=\{\vec{x}: \Lambda(\vec{x})\ge c\}$ $\Lambda(\vec{x})=\frac{L(\theta_1;\vec{x})}{L(\theta_0;\vec{x})}=\frac{f_n(\vec{x};\theta_1)}{f_n(\vec{x};\theta_0)}$

$P(\vec{x}\in C;\theta_0)=\alpha$

$P(\vec{x}\in C';\theta_0)=\alpha$ $P(\vec{x}\in C;\theta_1)\ge P(\vec{x}\in C';\theta_1)$

$\Delta=P(\vec{x}\in C;\theta_1)-P(\vec{x}\in C';\theta_1)\ge 0$

$\Delta=P(\vec{x}\in C\cap C';\theta_1)+P(\vec{x}\in C\cap\bar{C}';\theta_1)-P(\vec{x}\in C\cap C';\theta_1)-P(\vec{x}\in C'\cap\bar{C};\theta_1)$

$=P(\vec{x}\in C\cap\bar{C}';\theta_1)-P(\vec{x}\in C'\cap\bar{C};\theta_1)$

everywhere in $C\cap\bar{C}'$ $f_n(\vec{x};\theta_1)\ge c f_n(\vec{x};\theta_0)$

$P(\vec{x}\in C\cap\bar{C}';\theta_1)=\int_{C\cap\bar{C}'} f_n(\vec{x};\theta_1)\,d\vec{x}\ge c\int_{C\cap\bar{C}'} f_n(\vec{x};\theta_0)\,d\vec{x}=cP(\vec{x}\in C\cap\bar{C}';\theta_0)$

P(x

everywhere in $\bar{C}\cap C'$ $f_n(\vec{x};\theta_1)\le c f_n(\vec{x};\theta_0)$

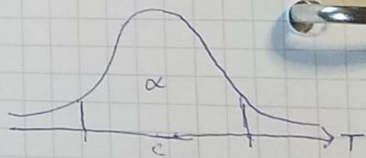
$P(\vec{x}\in C'\cap\bar{C};\theta)\le cP(\vec{x}\in C'\cap\bar{C};\theta_0)$

$\Delta\ge c[P(\vec{x}\in C\cap\bar{C}';\theta_0)-P(\vec{x}\in C'\cap\bar{C};\theta_0)]=$

$c[P(\vec{x}\in C\cap\bar{C}';\theta_0)+P(\vec{x}\in C\cap C';\theta_0)-P(\vec{x}\in C'\cap\bar{C};\theta_0)-P(\vec{x}\in C\cap C';\theta_0)]=$

$c[P(\vec{x}\in C;\theta_0)-P(\vec{x}\in C';\theta_0)]=c[\alpha-\alpha]=0$

$P(S(\vec{x})\ni\theta;\theta)=\int_{\{\vec{x}:\theta\in S(\vec{x})\}} f_n(\vec{x};\theta)\,d\vec{x}$



$T(\vec{x};\theta)$ $P(T(\vec{x};\theta)\in C(\theta);\theta)=\alpha\ \forall\theta$

$S(\vec{x})=\{\theta: T(\vec{x};\theta)\in C(\theta)\}$ $P(S(\vec{x})\ni\theta;\theta)=P(T(\vec{x};\theta)\in C(\theta);\theta)$

$=\alpha\ \forall\theta$ $C(\theta)=C$

$S(\vec{x})=\{\theta: T(\vec{x};\theta)\in C\}$

Ex 1.32 $T=\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}$ $T\sim N(0,1)$ $P(z_\alpha\le T\le z_{1-\alpha})=1-2\alpha=$

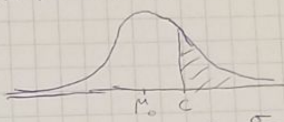
$P(z_\alpha\le\frac{\bar{X}-\mu}{\sigma/\sqrt{n}}\le z_{1-\alpha})$ $P(\bar{X}-z_{1-\alpha}\frac{\sigma}{\sqrt{n}}\le\mu\le\bar{X}-z_\alpha\frac{\sigma}{\sqrt{n}})=1-2\alpha$

$S(\vec{x})=(\bar{X}-z_{1-\alpha}\frac{\sigma}{\sqrt{n}},\ \bar{X}-z_\alpha\frac{\sigma}{\sqrt{n}})$

Week 4 vids / $S(\vec{X}) = \{\theta : \vec{X} \notin C(\theta)\}$

$P(\vec{X} \in C(\theta); \theta) = \alpha$ $\quad P(S(\vec{X}) \ni \theta; \theta) = P(\vec{X} \notin C(\theta); \theta) =$

$1 - P(\vec{X} \in C(\theta); \theta) = 1 - \alpha$



$c = \mu_0 + z_{1-\alpha}\frac{\sigma}{\sqrt{n}}$

Ex 1.35 / $\mu_0 \to \bar{X} \sim N(\mu_0, \frac{\sigma^2}{n})$

$S(\vec{X}) = \{\mu : \bar{X} < \mu + z_{1-\alpha}\frac{\sigma}{\sqrt{n}}\} = (\bar{X} - z_{1-\alpha}\frac{\sigma}{\sqrt{n}}, \infty)$

$H_0 : \mu = \mu_0 \qquad H_1 : \mu > \mu_0$

$H_1 : \mu < \mu_0 \quad S(\vec{X}) = \{\mu : \bar{X} \geq \mu + z_\alpha \frac{\sigma}{\sqrt{n}}\} = (-\infty, \bar{X} - z_\alpha \frac{\sigma}{\sqrt{n}})$

$H_0 : \mu = \mu_0 \qquad H_1 : \mu \neq \mu_0 \qquad N(\mu_0, \frac{\sigma^2}{n})$

$C(\mu_0) = \{\vec{x} : \bar{x} \leq \mu_0 + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} \text{ or } \bar{x} \geq \mu_0 + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\}$

$\S(\vec{X}) = \{\mu : \mu + z_{\alpha/2}\frac{\sigma}{\sqrt{n}} < \bar{X} < \mu + z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}\} = (\bar{X} - z_{1-\frac{\alpha}{2}}\frac{\sigma}{\sqrt{n}}, \bar{X} - z_{\alpha/2}\frac{\sigma}{\sqrt{n}})$

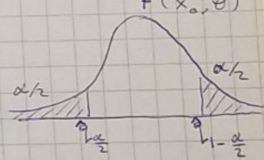
Ex 1.37 / $C(\theta_0) = \{x : P(X \leq x; \theta_0) \leq \frac{\alpha}{2} \text{ or } P(X \geq x; \theta_0) < \frac{\alpha}{2}\}$

$S(X) = (\theta_L, \theta_U) \quad S(X) = \{\theta : P(X > x; \theta) \geq \frac{\alpha}{2} \text{ and } P(X < x; \theta) \geq \frac{\alpha}{2}\}$

$\theta_L : P(X \geq x; \theta_L) = \frac{\alpha}{2} \qquad \theta_U : P(X \leq x; \theta_U) = \frac{\alpha}{2}$

$\sum_{r=x}^{n} \binom{n}{r}\theta_L^r(1-\theta_L)^{n-r} = \frac{\alpha}{2} \qquad \sum_{r=0}^{x}\binom{n}{r}\theta_U^r(1-\theta_U)^{n-r} = \frac{\alpha}{2}$

$R(\vec{X}),\ X_0 \qquad P(X_0 \in R(\vec{X}); \theta) = \iint_{\{\vec{x} : x_0 \in R(\vec{x})\}} f_n(\vec{x}; \theta) f(x_0; \theta)\, d\vec{x}\, dx_0$



Ex 1.38 / $\hat{\mu} = \frac{1}{n}\sum X_i = \bar{X}$

$X_0 \sim N(\bar{X}, \sigma^2) \qquad (\bar{X} + z_\alpha \sigma, \bar{X} + z_{1-\alpha}\sigma)$

$X_0 - \bar{X} \qquad E(X_0 - \bar{X}) = E(X_0) - E(\bar{X}) = \mu - \mu = 0$

$Var(X_0 - \bar{X}) = Var(X_0) + Var(\bar{X}) = \sigma^2 + \frac{\sigma^2}{n} = \sigma^2(1 + \frac{1}{n})$ $\qquad Z =$

$\frac{X - X_0}{\sigma\sqrt{1+\frac{1}{n}}} \sim N(0,1) \qquad P(\bar{X} + \sigma z_\alpha < X_0 < \bar{X} + z_{1-\alpha}\sigma)$

$= P(z_\alpha < \frac{X_0 - \bar{X}}{\sigma} < z_{1-\alpha}) = P(z_\alpha < \sqrt{1+\frac{1}{n}}\, Z < z_{1-\alpha}) = P(\frac{z_\alpha}{\sqrt{1+\frac{1}{n}}} < Z < \frac{z_{1-\alpha}}{\sqrt{1+\frac{1}{n}}}) <$

$P(z_\alpha < Z < z_{1-\alpha}) = 1 - 2\alpha$

Ex 1.41 / $X_0 : P(X_0 < x) = 1 - e^{-\frac{x}{\theta}} \qquad S = \sum_{i=1}^{n} X_i \qquad S \sim Ga(n, \theta)$

$f(s) = \frac{s^{n-1}}{\theta^n \Gamma(n)} e^{-\frac{s}{\theta}} \qquad T = \frac{X_0}{\bar{X}} = \frac{nX_0}{S} \qquad F(t) = P(T < t) = P(X_0 < \frac{S}{n}t) =$

$$\int_{S=0}^{\infty}\int_{x=0}^{\frac{S}{n}t} f(x_0, S;\theta)\,dx_0\,dS = \int_{S=0}^{\infty}\int_{x=0}^{\frac{S}{n}t} f(x_0;\theta)f(S;\theta)\,dx_0\,dS =$$

$$\int_{S=0}^{\infty} P(X_0 \le \tfrac{S}{n}t) f(S;\theta)\,dS = \int_{S=0}^{\infty}(1-e^{-\frac{St}{n\theta}}) f(S;\theta)\,dS = 1-\int_0^{\infty}\frac{S^{n-1}}{\theta^n\Gamma(n)}e^{-\frac{S}{\theta}(1+\frac{t}{n})}\,dS$$

$\therefore Z = S\,\frac{1+\frac{t}{n}}{\theta} \qquad S = Z\,\frac{\theta}{1+\frac{t}{n}} \qquad dS = \frac{\theta}{1+\frac{t}{n}}\,dZ$

$$= 1-\int_0^{\infty} Z^{n-1}\left(\frac{\theta}{1+\frac{t}{n}}\right)^{n-1}\frac{1}{\theta^n\Gamma(n)}e^{-Z}\frac{\theta}{1+\frac{t}{n}}\,dZ = 1-\int_0^{\infty} Z^{n-1}\left(\frac{1}{1+\frac{t}{n}}\right)^{n-1}\frac{1}{\Gamma(n)}e^{-Z}\frac{1}{1+\frac{t}{n}}\,dZ$$

$$= 1-\left(1+\tfrac{t}{n}\right)^{-n}\int_0^{\infty} Z^{n-1}\frac{1}{\Gamma(n)}e^{-Z}\,dZ = 1-\left(1+\tfrac{t}{n}\right)^{-n}$$

$t_p : P(T<t_p) = p = 1-(1+\frac{t_p}{n})^{-n} \qquad t_p = n\left[(1-p)^{-\frac{1}{n}}-1\right]$

$P(t_\alpha < T < t_{1-\alpha}) = 1-2\alpha = P(\bar{X}t_\alpha < X_0 < \bar{X}t_{1-\alpha})$

$R(\vec{x}) = (\bar{x}t_\alpha, \bar{x}t_{1-\alpha})$

$x_1 \cdots x_5$ $\quad \frac{1}{6}\ \underset{x_{(1)}}{\times}\ \frac{1}{6}\ \underset{x_{(2)}}{\times}\ \frac{1}{6}\ \underset{x_{(3)}}{\times}\ \frac{1}{6}\ \underset{x_{(4)}}{\times}\ \frac{1}{6}\ \underset{x_{(5)}}{\times}\ \frac{1}{6}$ $\quad \therefore P(X_{(1)} < X_0 < X_{(5)}) = \frac{2}{3} = \{\frac{1}{6}\times 4\}$

Week 5 vids $\dot{\sim}$ means asymptotic distri $\therefore$

$\bar{X} \dot{\sim} N(\mu, \frac{1}{n}\Sigma) \qquad n\bar{X} \dot{\sim} N(n\mu, n\Sigma) \qquad n\bar{X}-n\mu \dot{\sim} N(0, n\Sigma)$

$\frac{1}{\sqrt{n}}(n\bar{X}-n\mu) \dot{\sim} N(0,\Sigma)$

$X : E(X)=\mu \quad var(X)=\sigma^2 \qquad \therefore Z = \frac{X-\mu}{\sigma} \quad \therefore E(Z)=0, \ var(Z)=1$

$\vec{X} : E(\vec{X})=\vec{\mu} \quad var(\vec{X})=\Sigma \qquad \Sigma^{1/2} : (\Sigma^{1/2})(\Sigma^{1/2}) = \Sigma$

$\vec{Z} = \Sigma^{-1/2}(\vec{X}-\vec{\mu}) \qquad E(\vec{Z}) = \vec{0} \quad var(\vec{Z}) = \mathbb{1} = \mathbb{I}$ (identity matrix)

$\vec{Z} = \begin{pmatrix} Z_1 \\ \vdots \\ Z_p \end{pmatrix} \qquad \vec{Z}^T\vec{Z} = (Z_1^2 + Z_2^2 + \dots + Z_p^2)$

$\vec{Z} = I(\hat\theta)^{1/2}(\hat\theta-\theta_0) \qquad \vec{Z}^T\vec{Z} = (\hat\theta-\theta_0)^T\underbrace{(I(\hat\theta)^{1/2})^T(I(\hat\theta)^{1/2})}_{I(\hat\theta)}(\hat\theta-\theta_0) = W$

$l(\theta_0) \approx l(\hat\theta) + (\theta_0-\hat\theta)^T U(\hat\theta) - \frac{1}{2}(\theta_0-\hat\theta)^T J(\hat\theta)(\theta_0-\hat\theta) \quad \therefore$

$2(l(\hat\theta)-l(\theta_0)) \approx (\theta_0-\hat\theta)^T J(\hat\theta)(\theta_0-\hat\theta) \approx (\theta_0-\hat\theta)^T I(\hat\theta)(\theta_0-\hat\theta) = W$

Ex 2.4 $X_1, \dots, X_n \sim N(\mu, \sigma^2)$ (σ known) $\quad L(\mu) \propto \exp(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2) \quad \therefore$

$l(\mu) = \text{Const} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 = \text{Const} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(X_i-\mu)^2 \quad \therefore$

$U(\mu) = +\frac{1}{\sigma^2}\sum_i (x_i - \mu) = +\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \mu) = +\frac{n}{\sigma^2}(\bar{x} - \mu)$ ∴

$J(\mu) = +\frac{n}{\sigma^2}$ ∴ $I(\mu) = \frac{n}{\sigma^2}$

∴ $W = (\hat{\mu} - \mu_0)^2 \frac{n}{\sigma^2} = \frac{n}{\sigma^2}(\bar{x} - \mu_0)^2$

$S = \frac{U(\mu_0)^2}{I(\mu_0)} = \frac{n^2}{\sigma^4}(\bar{x} - \mu_0)^2 \frac{\sigma^2}{n} = \frac{n}{\sigma^2}(\bar{x} - \mu_0)^2$

$-2\log\Lambda = 2[l(\hat{\mu}) - l(\mu_0)] = \frac{1}{\sigma^2}\sum_i[(x_i - \mu_0)^2 - (x_i - \hat{\mu})^2] =$

$\frac{1}{\sigma^2}\sum_i[-2x_i(\mu_0 - \hat{\mu}) + \mu_0^2 - \hat{\mu}^2] = \frac{\mu_0 - \hat{\mu}}{\sigma^2}\sum_i[-2x_i + \mu_0 + \hat{\mu}] = \frac{\mu_0 - \hat{\mu}}{\sigma^2}\sum_{i=1}^{n}[-2x_i + \mu_0 + \hat{\mu}] =$

$\frac{\mu_0 - \hat{\mu}}{\sigma^2}[-2n\bar{x} + n\mu_0 + n\hat{\mu}] = \frac{n}{\sigma^2}(\mu_0 - \hat{\mu})^2$

Week 7 vids

$\hat{\theta} \sim N(\theta, I(\theta)^{-1})$ $\quad \underbrace{I(\theta)^{1/2}(\hat{\theta} - \theta)}_{\vec{Z}} \sim N(\vec{0}, \mathbb{1})$ $\quad \{Z_i \sim N(0,1)$

$\vec{Z}^T\vec{Z} = \sum_{i=1}^{p} Z_i^2 \sim \chi_p^2$

∴ $(\hat{\theta} - \theta)^T I(\theta)(\hat{\theta} - \theta) \sim \chi_p^2$ $\quad T(\vec{x}, \theta)$ approx pivot

C: $(1-\alpha)$ quantile of χ_p^2

$S(\vec{x}) = \{\theta : T(\vec{x}, \theta) < C\}$ $\quad P(S(\vec{x}) \ni \theta; \theta) = P(T(\vec{x}, \theta) < C; \theta) = 1 - \alpha$

Ex 2.7 $W = S = -2\log\Lambda = n(\bar{x} - \mu_0)^2/\sigma^2$

C: $(1-\alpha)$ quantile of χ_1^2 $\quad \{\mu : n(\bar{x} - \mu)^2/\sigma^2 < C\}$

$n(\bar{x} - \mu^2)/\sigma^2 - C < 0$ $\quad \frac{n}{\sigma^2}\mu^2 - 2\frac{n}{\sigma^2}\bar{x}\mu + \frac{n}{\sigma^2}\bar{x}^2 - C = 0$ ∴

$\mu^2 - 2\bar{x}\mu + \bar{x}^2 - C\frac{\sigma^2}{n} = 0$ ∴ $\mu_{1,2} = \bar{x} \pm \sqrt{\bar{x}^2 - \bar{x}^2 + C\frac{\sigma^2}{n}} = \bar{x} \pm \sqrt{C\frac{\sigma^2}{n}} = \bar{x} \pm \sigma\sqrt{\frac{C}{n}}$

$\left(\bar{x} - \sigma\sqrt{\frac{C}{n}}, \bar{x} + \sigma\sqrt{\frac{C}{n}}\right)$

Ex 2.8 $S(x) = \left\{\theta : \frac{n(\hat{\theta} - \theta)^2}{\theta(1-\theta)} < C\right\}$ $\quad (n+C)\theta^2 - (2n\hat{\theta} + C)\theta + n\hat{\theta}^2 = 0$

$\theta^2\left(\frac{n}{\hat{\theta}(1-\hat{\theta})}\right) + \theta\left(-\frac{2n\hat{\theta}}{\hat{\theta}(1-\hat{\theta})}\right) + \frac{n\hat{\theta}^2}{\hat{\theta}(1-\hat{\theta})} - C = 0$ $\quad \theta_{1,2} = \cdots$

$X_i \sim N(\mu_i, \sigma^2)$ $\quad \mu_i = \alpha + \beta z_i$ $\quad \vec{\theta} = \begin{pmatrix}\alpha \\ \beta \\ \sigma\end{pmatrix}$

Ex 2.11 $\frac{\partial}{\partial\beta} l = \frac{\partial}{\partial\beta}\left(-\frac{n}{2}\log 2\pi - n\log\sigma - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i - \alpha - \beta z_i)^2\right)$

$= +\frac{1}{2\sigma^2}\sum_{i=1}^{n} 2(x_i - \alpha - \beta z_i)z_i = \frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i - \alpha - \beta z_i)z_i$ $\quad \{\overline{xz} = \frac{1}{n}\sum_{i=1}^{n} x_i z_i$

$\overline{z^2} = \frac{1}{n}\sum_{i=1}^{n} z_i^2 \}$ $\{\bar{z} = \frac{1}{n}\sum_{i=1}^{n} z_i\}$

$$= \frac{1}{\sigma^2}(n\overline{xz} - n\alpha\bar{z} - n\beta\overline{z^2})$$

$$J = -\begin{bmatrix} \frac{\partial^2 l}{\partial\alpha^2} & \frac{\partial^2 l}{\partial\alpha\partial\beta} & \frac{\partial^2 l}{\partial\alpha\partial\sigma} \\ & \frac{\partial^2 l}{\partial\beta^2} & \frac{\partial^2 l}{\partial\beta\partial\sigma} \\ \text{Symmetric.} & & \frac{\partial^2 l}{\partial(\sigma)^2} \end{bmatrix}$$

$$-E\left[\frac{\partial^2 l}{\partial\alpha\partial\sigma}\right] = -\frac{2n}{\sigma^3}E(\bar{x} - \alpha - \beta\bar{z})$$

$$\left\{\bar{x} = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n}(\alpha + \beta z_i) = \alpha + \beta\bar{z} \qquad X_i \sim N(\alpha + \beta z_i, \sigma^2)\right\}$$

$$= -\frac{2n}{\sigma^3}(0) = 0$$

\Ex2.12/ $L(\theta; \vec{x}) = \prod_{i=1}^{m} f_1(x_i;\theta)\prod_{i=m+1}^{n} f_2(x_i;\theta) =$

$$\prod_{i=1}^{m}\frac{1}{\theta}e^{-x_i/\theta}\prod_{i=m+1}^{n}\frac{3}{\theta}e^{-3x_i/\theta} \propto \theta^{-m}e^{-\frac{1}{\theta}\sum_{i=1}^{m}x_i}\theta^{-(n-m)}e^{-\frac{3}{\theta}\sum_{i=1}^{n}x_i} =$$

$$\theta^{-n}\exp\left\{-\frac{1}{\theta}\left(\sum_{i=1}^{m}x_i + 3\sum_{i=m+1}^{n}x_i\right)\right\}$$

$$l(\theta;\vec{x}) = \text{Const} - n\log\theta - \left\{\frac{1}{\theta}\left(\sum_{i=1}^{m}x_i + 3\sum_{i=m+1}^{n}x_i\right)\right\}$$

$$\frac{\partial l}{\partial\theta} = -\frac{n}{\theta} + \frac{1}{\theta^2}\left(\sum_{i=1}^{m}x_i + 3\sum_{i=m+1}^{n}x_i\right)$$

$$\hat{\theta} = \frac{1}{n}\left(\sum_{i=1}^{m}x_i + 3\sum_{i=m+1}^{n}x_i\right) \qquad \frac{\partial^2 l}{\partial\theta^2} = \frac{n}{\theta^2} - \frac{2}{\theta^3}(\dots) = \frac{n}{\theta^2} - \frac{2}{\theta^3}n\hat{\theta} \quad \therefore$$

$$\left.\frac{\partial^2 l}{\partial\theta^2}\right|_{\hat{\theta}} = \frac{n}{\hat{\theta}^2} - \frac{2}{\hat{\theta}^3}n\hat{\theta} = -\frac{n}{\hat{\theta}^2} < 0 \qquad \frac{\partial^2 l}{\partial\theta^2} = \frac{n}{\theta^2} - \frac{2n\hat{\theta}}{\theta^3}$$

$$I(\theta) = -E\left(\frac{\partial^2 l}{\partial\theta^2}\right) \qquad \left\{E(\hat{\theta}) = \frac{1}{n}\left(\sum_{i=1}^{m}\underbrace{E(x_i)}_{\theta} + 3\sum_{i=1}^{n}\underbrace{E(x_i)}_{\theta/3}\right) = \frac{1}{n}\sum_{i=1}^{n}\theta = \theta\right\}$$

$$I(\theta) = -\frac{n}{\theta^2} + \frac{2n}{\theta^3}E(\hat{\theta}) = -\frac{n}{\theta^2} + \frac{2n}{\theta^2} = \frac{n}{\theta^2} \quad \therefore$$

$$\hat{\theta} \sim N\left(\theta, \frac{\theta^2}{n}\right)$$

\Week 8 vids/

\Ex2.13/ $L(\mu,\sigma^2) = (2\pi\sigma^2)^{-\frac{n}{2}}\exp\left\{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2\right\}$

$$l(\mu,\sigma^2) = -\frac{n}{2}\log(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2$$

$$\frac{\partial l}{\partial\sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu)^2 = 0 \text{ at } \hat{\sigma}^2 \quad \therefore$$

$$\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2 = \hat{\sigma}^2(\mu) \quad \therefore$$

$$L_p(\mu) = L(\mu, \hat{\sigma}^2(\mu)) \quad \therefore \quad l_p(\mu) = l(\mu, \hat{\sigma}^2(\mu)) =$$

$$-\frac{n}{2}\log\left(2\pi\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2\right)-\frac{1}{2\left(\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2\right)}\sum_{i=1}^{n}(x_i-\mu)^2=$$

$$-\frac{n}{2}\log\left(2\pi\frac{1}{n}\sum_{i=1}^{n}(x_i-\mu)^2\right)-\frac{n}{2}$$

$$\frac{\partial l}{\partial \mu}=\frac{1}{\sigma^2}\sum_{i=1}^{n}(x_i-\mu)=0 \text{ at } \hat{\mu} \quad \therefore \sum_{i=1}^{n}x_i=n\hat{\mu} \Rightarrow \hat{\mu}=\bar{x} \quad \therefore$$

$$L_p(\sigma^2)=l(\hat{\mu},\sigma^2)=-\frac{n}{2}\log(2\pi\sigma^2)-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\bar{x})^2$$

\Ex 2.16/ $n\log\left(\sum_{i=1}^{n}(x_i-\mu_0)^2\right]-n\log(nS^2)>c \quad \therefore$

$$S(\vec{x})=\left\{\mu: n\log\left(\sum_{i=1}^{n}(x_i-\mu)^2\right)-n\log(nS^2)\le c\right\} \qquad \left\{S^2=\frac{1}{n}\sum_{i=1}^{n}(x_i-\bar{x})^2\right\}$$

$$\equiv n\log\frac{\sum_{i=1}^{n}(x_i-\mu)^2}{nS^2}\le c \quad \therefore \quad \sum_{i=1}^{n}(x_i-\mu)^2\le nS^2e^{c/n}$$

$$\mu^2 n+\mu(-2n\bar{x})+\left(\sum_{i=1}^{n}x_i^2\right)-nS^2e^{c/n}=0$$

$$\left(\bar{x}-S\sqrt{e^{c/n}-1},\ \bar{x}+S\sqrt{e^{c/n}-1}\right)$$

\Ex 2.17/ $g(\hat{\theta};\vec{x})=0 \qquad g(\theta;\vec{x})=\begin{pmatrix}\mu-\frac{1}{n}\sum_{i=1}^{n}x_i\\ \sigma^2+\mu^2-\frac{1}{n}\sum_{i=1}^{n}x_i^2\end{pmatrix}$

$$\hat{\mu}=\frac{1}{n}\sum_{i=1}^{n}x_i \qquad \hat{\sigma}^2=\frac{1}{n}\sum_{i=1}^{n}x_i^2-\left(\frac{1}{n}\sum_{i=1}^{n}x_i\right)^2$$

\Ex 2.18/ $g_k(\theta;\vec{x})=E(X^k)-\frac{1}{n}\sum_{i=1}^{n}X_i^k$

$$E(g_k(\theta;\vec{X});\theta)=E(X^k)-\frac{1}{n}\sum_{i=1}^{n}E(X^k)=0$$

$$E(\vec{X})=\mu(\theta) \qquad \mu=\begin{pmatrix}\theta\\ \vdots\\ \theta\end{pmatrix} \qquad \mu=\begin{pmatrix}\theta_1\\ \vdots\\ \theta_1\\ \theta_2\\ \vdots\\ \theta_2\end{pmatrix} \qquad \mu=\begin{pmatrix}1\theta\\ 2\theta\\ 3\theta\\ \vdots\\ n\theta\end{pmatrix}$$

$$\mathrm{var}(\vec{X})=\Sigma(\theta) \qquad \Sigma=\begin{pmatrix}\theta & & 0\\ & \ddots & \\ 0 & & \theta\end{pmatrix} \qquad \Sigma\begin{pmatrix}\theta_1 & & & & 0\\ & \ddots & & & \\ & & \theta_1 & & \\ & & & \theta_2 & \\ 0 & & & & \ddots\ \theta_2\end{pmatrix}$$

$$g(\hat{\theta};\vec{X})=0 \qquad C^T(\vec{X}-\mu)=0\to\hat{\theta}$$

choose C st $\hat{\theta}$ has the smallest possible variance

$$g(\theta;\vec{x})=C^T(\vec{x}-\mu) \qquad C^T=\mu_\theta^T\Sigma^{-1} \qquad [\mu_\theta]_{i,j}=\left(\frac{\partial\mu_i}{\partial\theta_j}\right)$$

$$G=\mu_\theta^T\Sigma^{-1}(\vec{X}-\mu) \qquad E(G)=0 \qquad \mathrm{var}(G)=E(GG^T)-E(G)E(G^T)$$

$$=E(GG^T)=E\left(\mu_\theta^T\Sigma^{-1}(\vec{X}-\mu)(\vec{X}-\mu)^T\Sigma^{-1}\mu_\theta\right)=$$

$$\mu_\theta^T\Sigma^{-1}E\left((\vec{X}-\mu)(\vec{X}-\mu)^T\right)\Sigma^{-1}\mu_\theta=\mu_\theta^T\Sigma^{-1}\mu_\theta$$

$$\frac{\partial G}{\partial \theta} = \left[\frac{\partial}{\partial \theta}\left(\mu_\theta^T \Sigma^{-1}\right)\right](\vec{X}-\mu) \underbrace{- \mu_\theta^T \Sigma^{-1} \mu_\theta}_{-var(G)}$$

$$E\left(\frac{\partial G}{\partial \theta}\right) = -var(G)$$

Ex 2.19 / $E(X_i) = \theta z_i$ $\quad var(X_i) = \theta z_i$ $\quad \mu = \begin{pmatrix} E(X_1) \\ E(X_2) \\ \vdots \\ E(X_n) \end{pmatrix} = \begin{pmatrix} \theta z_1 \\ \theta z_2 \\ \vdots \\ \theta z_n \end{pmatrix}$

$$var(\vec{X}) = \Sigma = \begin{pmatrix} \theta z_1 & & 0 \\ & \theta z_2 & \\ 0 & & \theta z_n \end{pmatrix} \qquad (\mu_\theta)_{i,j} = \left(\frac{\partial \mu_i}{\partial \theta_j}\right) \qquad \mu_\theta = \begin{pmatrix} \partial\mu_1/\partial\theta \\ \vdots \\ \partial\mu_n/\partial\theta \end{pmatrix} = \begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

$$\Sigma^{-1} = (\theta\, diag(z_1 \cdots z_n))^{-1} = \frac{1}{\theta}\, diag\left(\frac{1}{z_1} \cdots \frac{1}{z_n}\right)$$

$$G(\theta; \vec{X}) = \mu_\theta^T \Sigma^{-1}(\vec{X}-\mu) = (z_1 \cdots z_n)\frac{1}{\theta}\begin{pmatrix} 1/z_1 & & \\ & \ddots & \\ & & 1/z_n \end{pmatrix}\left(\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} - \theta\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}\right) =$$

$$(1\ 1 \cdots 1)\frac{1}{\theta}\left(\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} - \theta\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}\right) = \frac{1}{\theta}\left[(1 \cdots 1)\begin{pmatrix} X_1 \\ \vdots \\ X_n \end{pmatrix} - \theta(1 \cdots 1)\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}\right] =$$

$$\frac{1}{\theta}\sum_{i=1}^n X_i - \sum_{i=1}^n z_i = G(\theta; X)$$

$$G = 0 \Rightarrow \hat{\theta} = \frac{\sum_{i=1}^n X_i}{\sum_{i=1}^n z_i} = \frac{\bar{X}}{\bar{z}} \qquad K(\theta) = \mu_\theta^T \Sigma^{-1} \mu_\theta = (z_1 \cdots z_n)\frac{1}{\theta}\begin{pmatrix} 1/z_1 & & \\ & \ddots & \\ & & 1/z_n \end{pmatrix}\begin{pmatrix} z_1 \\ \vdots \\ z_n \end{pmatrix}$$

$$= (z_1 \cdots z_n)\frac{1}{\theta}\begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} = \frac{n\bar{z}}{\theta} \qquad \hat{\theta} \sim N\left(\theta, \frac{\theta}{n\bar{z}}\right) \qquad stderr(\hat{\theta}) = \sqrt{\frac{\hat{\theta}}{n\bar{z}}}$$

Ex 2.20 / $\mu = \begin{pmatrix} \alpha+\beta z_1 \\ \vdots \\ \alpha+\beta z_n \end{pmatrix}$ $\quad \Sigma = \begin{pmatrix} \sigma^2 & & \\ & \ddots & \\ & & \sigma^2 \end{pmatrix} = \sigma^2 \mathbb{1}$ $\quad \theta = \begin{pmatrix} \alpha \\ \beta \\ \sigma^2 \end{pmatrix}$ $\quad \mu_\theta = \begin{pmatrix} \partial\mu_1/\partial\theta_1 & \frac{\partial\mu_1}{\partial\theta_2} & 0 \\ \vdots & \vdots & \vdots \\ \partial\mu_n/\partial\theta_1 & \frac{\partial\mu_n}{\partial\theta_2} & 0 \end{pmatrix} =$

$$\begin{pmatrix} 1 & z_1 & 0 \\ 1 & z_2 & 0 \\ \vdots & \vdots & \vdots \\ 1 & z_n & 0 \end{pmatrix} \qquad \Sigma^{-1} = \frac{1}{\sigma^2}\mathbb{1} \qquad G(\theta; \vec{X}) = \begin{bmatrix} 1 & \cdots & 1 \\ z_1 & & z_n \\ 0 & & 0 \end{bmatrix}\frac{1}{\sigma^2}\mathbb{1}\underbrace{\left[\begin{bmatrix} X_1 \\ \vdots \\ X_n \end{bmatrix} - \begin{bmatrix} \mu_1 \\ \vdots \\ \mu_n \end{bmatrix}\right]}_{\begin{bmatrix} X_1-\alpha-\beta z_1 \\ \vdots \\ X_n-\alpha-\beta z_n \end{bmatrix}}$$

$$= \frac{1}{\sigma^2}\begin{bmatrix} \sum_{i=1}^n (X_i-\alpha-\beta z_i) \\ \sum_{i=1}^n z_i(X_i-\alpha-\beta z_i) \\ 0 \end{bmatrix} = 0 \text{ at } \hat{\theta} \quad \therefore$$

$$n\bar{X} - n\hat{\alpha} - n\hat{\beta}\bar{z} = 0 \quad , \quad n\overline{Xz} - n\hat{\alpha}\bar{z} - n\hat{\beta}\overline{z^2} = 0$$

$$\bar{X}\bar{z} - \hat{\beta}\bar{z}^2 = \overline{Xz} - \hat{\beta}\overline{z^2} \qquad \hat{\beta} = \frac{\overline{Xz} - \bar{X}\bar{z}}{\overline{z^2} - \bar{z}^2} \quad \text{Similarly for } \hat{\alpha}$$

$$K(\theta) = \mu_\theta^T \Sigma^{-1} \mu_\theta = \frac{1}{\sigma^2}\begin{bmatrix} 1 & \cdots & 1 \\ z_1 & \cdots & z_n \end{bmatrix}\begin{bmatrix} 1 & z_1 \\ \vdots & \vdots \\ 1 & z_n \end{bmatrix} = \frac{1}{\sigma^2}\begin{bmatrix} n & n\bar{z} \\ n\bar{z} & n\overline{z^2} \end{bmatrix} = \frac{n}{\sigma^2}\begin{bmatrix} 1 & \bar{z} \\ \bar{z} & \overline{z^2} \end{bmatrix} \quad \therefore \text{ is } \bar{z} = 0$$

$$K^{-1}(\theta) = \frac{\sigma^2}{n}\begin{bmatrix} 1 & 0 \\ 0 & \frac{1}{\overline{z^2}} \end{bmatrix}$$

Week 8 vids / $\sum_{i=1}^n \frac{1}{n}\mathbb{1}(X_i \le x)$

Ex 3.5 / $T^* = \frac{1}{n}\sum_{i=1}^n X_i^*$ $\quad P(X_i^* = X_i) = \frac{1}{n}$ $\quad E(X_i^*) = \sum_{i=1}^n X_i P(X_i = x_i) = \frac{1}{n}\sum_{i=1}^n X_i$

$$var(X_i^*) = \sum_{i=1}^n (X_i - \bar{X})^2 P(X_i^* = X_i) = \frac{1}{n}\sum_{i=1}^n (X_i - \bar{X})^2$$

$E(T^*) = \frac{1}{n}\sum_{i=1}^{n} E(X_i^*) = \bar{X}$ $\quad var(T^*) = \frac{1}{n^2} Var\left(\sum_{i=1}^{n} X_i^*\right) = \frac{1}{n^2}\sum_{i=1}^{n} var(X_i^*) =$

$\frac{1}{n^2}\sum_{i=1}^{n}(X_i - \bar{X})^2$

\Ex 3.6/ $T(\vec{X}, \theta)$ distri of T is indep of θ

$X_1 \cdots X_n \sim N(\mu, \sigma^2)$ $\quad \frac{\bar{X}-\mu}{\sqrt{V/n}} \sim t_{n-1}$

$T(\vec{X}, \theta) \rightarrow$ distri of T is indep of θ

$T^*(\vec{X}^*, \hat{\theta}) \rightarrow$ distri indep of $\hat{\theta}$

\Ex 3.8/ $X_1 \cdots X_n$ $\quad z_1 \cdots z_n$ $\quad X_i^* \sim N(\alpha + \beta z_i, \sigma^2)$

$X_i^* \sim N(\hat{\alpha} + \hat{\beta} z_i, \hat{\sigma}^2)$

\Week 9 vids/

$\hat{\theta} = \frac{1}{n}\sum_{i=1}^{n} X_i$ $\quad \hat{\theta}_{-i} = \frac{1}{n-1}\sum_{j \neq i} X_j = \frac{1}{n-1}\left(\sum_{j=1}^{n} X_j - X_i\right)$

\Ex 3.9/ $E_n = E(\hat{\theta})$ $\quad E_{n-1}$ $\quad \frac{\Delta y}{\Delta x} = \text{const}$ $\quad \frac{E_{n-1} - E_n}{\frac{1}{n-1} - \frac{1}{n}} = \frac{E_n - E_\infty}{\frac{1}{n} - 0}$

$E_\infty = E_n - \frac{1}{n}\frac{E_{n-1} - E_n}{\frac{1}{n-1} - \frac{1}{n}} = E_n - (n-1)(E_{n-1} - E_n)$ $\quad \left\{\frac{n}{n-1} - 1 = \frac{1}{n-1}\right\}$

$= nE_n - (n-1)E_{n-1}$

$E_n \approx \hat{\theta}$ $\quad E_{n-1} \approx \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_{-i}$ $\quad \hat{\theta}_J = \hat{\theta} - (n-1)\left(\frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_{-i} - \hat{\theta}\right) = \frac{1}{n}\sum_{i=1}^{n}\hat{\theta}_i$

$\hat{\theta}_i = n\hat{\theta} - (n-1)\hat{\theta}_{-i}$

\Ex 3.11/ $\hat{\theta}_J = \frac{1}{n}\sum_{i=1}^{n}\left[n\hat{\theta} - (n-1)\hat{\theta}_{-i}\right] = n\hat{\theta} - \frac{n-1}{n}\sum_{i=1}^{n}\hat{\theta}_{-i} =$

$\frac{n}{2}(x_{(m)} + x_{(m+1)}) - \frac{n-1}{n}(m x_{(m)} + m x_{(m+1)}) =$

$m(x_{(m)} + x_{(m+1)}) - \left(1 - \frac{1}{2m}\right)(m x_{(m)} + m x_{(m+1)}) = \frac{1}{2}(x_{(m)} + x_{(m+1)}) = \hat{\theta}$

$\hat{V}_J = \frac{1}{n(n-1)}\sum_{i=1}^{n}\left(n\hat{\theta} - (n-1)\hat{\theta}_{-i} - \hat{\theta}\right)^2 = \frac{n-1}{n}\sum_{i=1}^{n}(\hat{\theta} - \hat{\theta}_{-i})^2 =$

$\frac{n-1}{n}\left[m\left(\frac{x_{(m)} + x_{(m+1)}}{2} - x_{(m)}\right)^2 + m\left(\frac{x_{(m)} + x_{(m+1)}}{2} - x_{(m+1)}\right)^2\right] =$

$\frac{m(n-1)}{4n}\left[(x_{(m+1)} - x_{(m)})^2 + (x_{(m)} - x_{(m+1)})^2\right] = \frac{m(n-1)}{2n}(x_{(m+1)} - x_{(m)})^2$

\Week 10 vids/

$P(q_\alpha < \hat{\theta} - \theta < q_{1-\alpha}) = 1 - 2\alpha$

$P(\hat{\theta} - q_{1-\alpha} < \theta < \hat{\theta} - q_\alpha) = 1 - 2\alpha$ $\quad T(\vec{X}, F) \rightarrow T^*(\vec{X}^*, \hat{F})$

$t_b^* = T(\vec{X}_b^*, \hat{F}) = \hat{\theta}_b^* - \hat{\theta}$ $\quad t_1^* \cdots t_B^* \rightarrow t_{(1)}^* \leq t_{(2)}^* \leq \cdots \leq t_{(B)}^*$

$\hat{q}_p = t_{(pB)}^*$ $\quad P(\hat{q}_\alpha < \hat{\theta} - \theta < \hat{q}_{1-\alpha}) = 1 - 2\alpha$

$\therefore q_p = t_{(pB)}^* = \hat{\theta}_{(pB)}^* - \hat{\theta}$ $\quad P(2\hat{\theta} - \hat{\theta}_{((1-\alpha)B)}^* < \theta < 2\hat{\theta} - \hat{\theta}_{(pB)}^*) = 1 - 2\alpha$

θ $\quad \theta(\vec{x})$ $\quad \hat{\theta}(\vec{x}^*)$ $\quad P(\hat{\theta}^*_{(\alpha B)} < \theta < \hat{\theta}^*_{((1-\alpha)B)})$

~~T the~~ $T = h(\hat{\theta}) - h(\theta)$ $\quad$ T has the same distri as $-T$

$\therefore \; P(h(\hat{\theta}^*_{(\alpha B)}) - h(\hat{\theta}) < h(\theta) - h(\hat{\theta}) < h(\hat{\theta}^*_{((1-\alpha)B)}) - h(\hat{\theta})) =$

$P(h(\hat{\theta}^*_{(\alpha B)}) - h(\hat{\theta}) < h(\hat{\theta}) - h(\theta) < h(\hat{\theta}^*_{((1-\alpha)B)}) - h(\hat{\theta})) \approx 1 - 2\alpha$ $\quad \{$CI eg $\alpha = 0.05\}$

$X_0 \sim F(x;\theta)$ $\quad T = F(X_0;\hat{\theta})$ $\quad P(T \le t) = P(F(X_0;\hat{\theta}) \le t)$ $\quad \{F^{-1}$ exists

$F^{-1}(F(x)) = x\}$

$= P(X_0 \le F^{-1}(t;\hat{\theta})) \approx P(X_0 \le F^{-1}(t;\theta)) = F(F^{-1}(t;\theta)) = t$

$T \sim \mathrm{Unif}(0,1)$ $\quad T \sim \mathrm{Uni}(0,1)$

\ MTH3028 Stats inference CW1 /

\ 1.a.i / $\hat{\mu} = \bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$ $\quad \hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$

$\hat{\phi} = \hat{\mu} / \hat{\sigma}^2$

\ 1.a.ii / ϕ is $\hat{\phi} = \hat{\mu}/\hat{\sigma}^2$ $\therefore \hat{\phi} \approx 0.1268394$, $\hat{\mu} \approx 4.65$

$\sum_{j=1}^{n}(X_j) - X_i = \sum_{j\neq i} X_j$ $\quad \frac{\hat{\mu}}{\frac{1}{(n-1)}\sum_{j\neq i} X_j} \approx \frac{4.65}{\frac{1}{(n-1)}\sum_{j\neq i} X_j} = \hat{\phi}_{-i}$

$$\hat{\phi} = \hat{\mu}/\hat{\sigma}^2 = \frac{\frac{1}{n}\sum_{i=1}^{n} X_i}{\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2}$$

Jackknife is about using all the data points apart from the data point corresponding to that i'th data point $\therefore$

$$\hat{\phi}_{-i} = \frac{\frac{1}{(n-1)}\sum_{j\neq i} X_j}{\frac{1}{(n-1)-1}\,\frac{1}{n-1}\sum_{j\neq i}(X_i - \bar{X}_i)^2}$$

is $\bar{X}_i = \frac{1}{(n-1)}\sum_{j\neq i} X_j$ $\quad \therefore \hat{\phi}_{-i} = \frac{\frac{1}{(n-1)}\sum_{j\neq i} X_j}{\frac{1}{(n-1)-1}\sum_{j\neq i}(X_j - \bar{X}_j)^2}$

$$\sum_{j\neq i} X_j = \sum_{j=1}^{n}[X_j] - X_i$$

$$\sum_{j\neq i}(X_j - \bar{X}_j)^2 = \sum_{j=1}^{n}[(X_j - \bar{X}_j)^2] - (X_j - \bar{X}_j)^2$$

$$\hat{\phi}_{-i} = \frac{\bar{X}_i}{\frac{1}{(n-1)-1}\sum_{j\neq i}(X_j - \bar{X}_j)^2}$$ $\quad \hat{\phi}_i = n\hat{\phi} - (n-1)\hat{\phi}_{-i}$ $\quad \hat{\phi}_J = \frac{1}{n}\sum_{i=1}^{n}\hat{\phi}_i$

\ 1.b.i / $\hat{\phi}_J \pm q_{0.95}\sqrt{\hat{V}_J} = (\hat{\phi}_J - q_{0.95}\sqrt{\hat{V}_J}\ ,\ \hat{\phi}_J + q_{0.95}\sqrt{\hat{V}_J})$ is Jackknife 90% CI $\quad$ 90% CI $\therefore \alpha = 5\%$. $\quad \hat{\phi}_J \pm t_\alpha\sqrt{\hat{V}_J}$

$$\sqrt{\hat{V}_J} = \sqrt{\frac{1}{n(n-1)}\sum_{i=1}^{n}(\hat{\phi}_i - \hat{\phi}_J)^2}$$

95% quantile of the Studedent distribution for degrees of freedom $= df = n-1 = 20-1 = 19$ $\quad \therefore Stu(n-1) \approx 1.729133$

\ 1.b.ii / $(\hat{\phi} - \hat{q}_{1-\alpha}\ ,\ \hat{\phi} - \hat{q}_\alpha)$ $\quad \hat{q}_p = t^*_{(pB)}$

\ 1.C.i / Conduct test $H_0: \phi = 1$ against $H_1: \phi < 1$ at 1% level

, null distri of Z test stat $\frac{\hat{\phi}_J - \phi_0}{\sqrt{\hat{V}_J}} \approx Stu(n-1)$ distri $\therefore$

$(\hat{\phi}_J - 1)/\sqrt{\hat{V}_J}$ $\therefore$ pvalue: ~~>> (1-2pt(abs(-6.84), df~~

>> $1 - pt(abs(\frac{\hat{\phi}_J - 1}{\sqrt{\hat{V}_J}}), df)$ < 0.01 you reject null hyphesis that $\phi = 1$

\1.c.ii/ test statistic is $\hat{\phi}$, and $\hat{\phi} \approx 0.1268374$

$H_0: \phi = 1$, $H_1: \phi < 1$ at 1% significance level

when $\phi = 1$, $X \sim poi(\mu)$ $\hat{\mu}$ is known

$\therefore$ test ~~was~~ $H_0: \phi = 1$, $H_1: \phi < 1$ using parametric bootstrap test with test statistic $\hat{\phi}$

Question:

To determine whether super-spreaders might exist, it is proposed to test the hypothesis $\phi = 1$ against the hypothesis $\phi < 1$ at the 1% significance level.

When $\phi = 1$, a common model assumes that $X_1, \ldots, X_n$ are independent $poi(\mu)$ random variables. Test the hypothesis using a parametric bootstrap test with test statistic $\hat{\phi}$. State clearly your critical value and conclusion. A sample of n may be simulated from the poi(Mu) distribution with the R command rpois(n, Mu).

$\therefore$ simulate samples from poi(μ_0)

$P(X=x) = P(X=x;\mu_0) = \dfrac{\mu_0^x e^{-\mu_0}}{x!}$

if $N(\mu, \sigma^2)$ $L(\mu, \sigma^2) = \prod_{i=1}^{n} \dfrac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2} =$

$\left[(2\pi\sigma^2)^{-1/2}\right]^n e^{\sum_{i=1}^{n} -\frac{1}{2}\left(\frac{x_i-\mu}{\sigma}\right)^2} = (2\pi\sigma^2)^{-n/2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2}$ $\therefore$

$l(\mu_0, \sigma^2) = \ln(L(\mu, \sigma^2)) = \ln\left[(2\pi\sigma^2)^{-n/2}\right] - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 =$

$-\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu)^2 = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(x_i-\mu_0)^2$

$\therefore \dfrac{\partial l(\mu_0, \sigma^2)}{\partial \sigma^2} = -\dfrac{n}{2\sigma^2} + \dfrac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu_0)^2 \doteq \dfrac{\partial l}{\partial \sigma^2}$

$\therefore -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(x_i-\mu_0)^2 = 0$ $\therefore \hat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^{n}(x_i-\mu_0)^2$ &

$l(\mu_0, \hat{\sigma}_0^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\hat{\sigma}_0^2) - \frac{1}{2\hat{\sigma}_0^2}\sum_{i=1}^{n}(x_i-\mu_0)^2 =$

$-\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\hat{\sigma}_0^2) - \frac{1}{2}$ ... $\therefore \hat{\mu} = \bar{x}$

$l(\hat{\mu}, \hat{\sigma}^2) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln(\hat{\sigma}^2) - \frac{n}{2}$ &

$-2\log\Lambda = -2\left[l(\mu_0, \hat{\sigma}_0^2) - l(\hat{\mu}, \hat{\sigma}^2)\right] =$

$-2\left[-\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\hat{\sigma}_0^2 - \frac{n}{2} + \frac{n}{2}\ln(2\pi) + \frac{n}{2}\ln\hat{\sigma}^2 + \frac{n}{2}\right] = -2\left[-\frac{n}{2}\ln\hat{\sigma}_0^2 + \frac{n}{2}\ln\hat{\sigma}^2\right] = 2n\ln(\hat{\sigma}_0/\hat{\sigma})$

3a/ Taylor expansion in terms of $\hat{\mu}, \hat{\sigma}^2$:

Taylor expansion of μ, σ^2 around a point $(\hat{\mu}, \hat{\sigma}^2)$ of $h(\mu, \sigma^2)$:

- $h(\mu, \sigma^2) \approx h(\hat{\mu}, \hat{\sigma}^2) + (\mu - \hat{\mu}) \frac{\partial h}{\partial \mu}(\hat{\mu}, \hat{\sigma}^2) + (\sigma^2 - \hat{\sigma}^2) \frac{\partial h(\hat{\mu}, \hat{\sigma}^2)}{\partial \sigma^2} +$

$\frac{(\mu - \hat{\mu})^2}{2} \frac{\partial^2 h}{\partial \mu^2}(\hat{\mu}, \hat{\sigma}^2) + \frac{(\sigma^2 - \hat{\sigma}^2)^2}{2} \frac{\partial^2 h}{\partial \sigma^4}(\hat{\mu}, \hat{\sigma}^2) + (\mu - \hat{\mu})(\sigma^2 - \hat{\sigma}^2) \frac{\partial^2 h(\hat{\mu}, \hat{\sigma}^2)}{\partial \mu \partial \sigma^2}$

Similar question to 3a: formative sheet 1 Q1c, 1d):

Delta Method: $h(\hat{\theta}) \approx h(\theta) + (\hat{\theta} - \theta) h'(\theta)$ ∴

$\mathrm{Var}[h(\hat{\theta})] \approx \mathrm{var}[h(\theta) + (\hat{\theta} - \theta) h'(\theta)] = \mathrm{var}[\hat{\theta} h'(\theta)] = [h'(\theta)]^2 \mathrm{var}(\hat{\theta})$

$\{$ is $\mathrm{var}[h(\theta) + (\hat{\theta} - \theta) h'(\theta)] = \mathrm{var}[h(\theta)] + \mathrm{var}[(\hat{\theta} - \theta) h'(\theta)] =$

$\mathrm{var}[h(\theta)] + \mathrm{var}[\hat{\theta} h'(\theta)] + \mathrm{var}[-\theta h'(\theta)] = [h(\theta)]^2 \mathrm{var}(1) + [h'(\theta)]^2 \mathrm{var}(\hat{\theta}) + [\theta h'(\theta)]^2 \mathrm{var}(1)$

- $= 0 + [h'(\theta)]^2 \mathrm{var}(\hat{\theta}) + 0 = [h'(\theta)]^2 \mathrm{var}(\hat{\theta}) \}$

$h(\hat{\theta}) \approx h(\theta) + (\hat{\theta} - \theta) h'(\theta) + \frac{1}{2}(\hat{\theta} - \theta)^2 h''(\theta)$ ∴

$E(\hat{\phi}) \approx h(\theta) + [E(\hat{\theta}) - \theta] h'(\theta) + \frac{1}{2} E[(\hat{\theta} - \theta)^2] h''(\theta) = \phi + \frac{1}{2} \mathrm{var}(\hat{\theta}) h''(\theta)$

$\{$ is $E(\hat{\phi}) \approx E[h(\hat{\theta})] \approx E[h(\theta) + (\hat{\theta} - \theta) h'(\theta) + \frac{1}{2}(\hat{\theta} - \theta)^2 h''(\theta)] =$

$E[h(\theta)] + E[(\hat{\theta} - \theta) h'(\theta)] + E[\frac{1}{2}(\hat{\theta} - \theta)^2 h''(\theta)] =$

$h(\theta) + [E(\hat{\theta}) - E(\theta)] h'(\theta) + \frac{1}{2} E[(\hat{\theta} - \theta)^2] h''(\theta) = h(\theta) + [E(\hat{\theta}) - \theta] h'(\theta) + \frac{1}{2} E[(\hat{\theta} - \theta)^2] h''(\theta) \}$

∴ $E(\hat{\phi}) - \phi \approx \frac{1}{2} \widehat{\mathrm{var}}(\hat{\theta}) h''(\hat{\theta}) = \{\phi + \frac{1}{2} \mathrm{var}(\hat{\theta}) h''(\theta) - \phi\}$ ∴

$\tilde{\phi} = \hat{\phi} - \frac{1}{2} \widehat{\mathrm{var}}(\hat{\theta}) h''(\hat{\theta})$

- is $\mathrm{var}(\hat{\phi}) = \mathrm{var}[h(\hat{\theta})] = \mathrm{var}[\frac{2}{\hat{\theta}}] = [h'(\theta)]^2 \mathrm{var}(\hat{\theta}) = [h'(\theta)]^2 \frac{\theta^2}{2n}$

∵ $\phi = \frac{2}{\theta}$ ∴ $\hat{\phi} = \frac{2}{\hat{\theta}}$ ∴ $\mathrm{var}(\hat{\theta}) = \frac{\theta^2}{2n} = \frac{2}{n\phi^2}$

let $\theta = \frac{2}{\phi}$ check $\hat{\theta} = \frac{2}{\hat{\phi}} = \bar{X}$ ∴ have $E(\hat{\theta}) = E(\bar{X}) = E(\frac{1}{n} \sum_{i=1}^{n} X_i) = \frac{1}{n} E(\sum_{i=1}^{n} X_i) =$

$\frac{1}{n} \sum_{i=1}^{n} E(X_i) = \frac{1}{n} \sum_{i=1}^{n} (\frac{2}{\phi}) = n \frac{1}{n} \frac{2}{\phi} = \frac{2}{\phi} = \theta$

∴ is $E(X^2) = \frac{6}{\phi^2}$ ∴ $\mathrm{var}(X) = E(X^2) - E(X)^2 = \frac{6}{\phi^2} - (\frac{2}{\phi})^2 = \frac{6}{\phi^2} - \frac{4}{\phi^2} = \frac{2}{\phi^2}$ ∴

$\mathrm{var}(\hat{\theta}) = \mathrm{var}(\bar{X}) = \mathrm{var}(\frac{1}{n} \sum_{i=1}^{n} X_i) = \frac{1}{n^2} \mathrm{var}(\sum_{i=1}^{n} X_i) = \frac{1}{n^2} \sum_{i=1}^{n} \mathrm{var}(X_i) = \frac{1}{n^2} \sum_{i=1}^{n} (\frac{2}{\phi^2}) =$

$\frac{1}{n^2} n \frac{2}{\phi^2} = \frac{2}{n\phi^2} = \frac{4}{2n\phi^2} = \frac{1}{2n} \frac{4}{\phi^2} = \frac{1}{2n} (\frac{2}{\phi})^2 = \frac{\theta^2}{2n}$ ∴

$\mathrm{var}(\hat{\phi}) = \mathrm{var}[h(\hat{\theta})] = \mathrm{var}[\frac{2}{\hat{\theta}}] = [h'(\theta)]^2 \mathrm{var}(\hat{\theta}) = [h'(\theta)]^2 \frac{\theta^2}{2n}$ ∴

- $h(\hat{\theta}) = \frac{2}{\hat{\theta}}$ ∴ $h(\theta) = \frac{2}{\theta}$ ∴ $h'(\theta) = -\frac{2}{\theta^2}$ ∴

$\mathrm{var}(\hat{\theta}) = (-\frac{2}{\theta^2})^2 \frac{\theta^2}{2n} = \frac{4}{2n\theta^2} = \frac{\phi^2}{2n}$ ∴ $\tilde{\phi} = \hat{\phi} - \frac{1}{2} \widehat{\mathrm{var}}(\hat{\theta}) h''(\hat{\theta}) = \phi - \frac{\phi}{2n} = (1 - \frac{1}{2n})\phi$

3a attempt / $\hat{\theta} = \hat{\mu}/\hat{\sigma}^2$ ∴ $\theta = \mu/\sigma^2$ $\hat{\mu} = \bar{X} = \sum_{i=1}^{n} X_i$

$\theta = \frac{\mu}{\sigma^2}$ ∴ $\cancel{h(\theta) = \mu}$ $h(\mu, \sigma^2) = \frac{\mu}{\sigma^2}$ ∴ $\theta = h(\mu, \sigma^2)$

Bias$(\hat{\theta}) = E(\hat{\theta}) - \theta$

$h(\mu, \sigma^2) \approx h(\hat{\mu}, \hat{\sigma}^2) + (\mu - \hat{\mu})\frac{\partial h(\hat{\mu}, \hat{\sigma}^2)}{\partial \mu} + (\sigma^2 - \hat{\sigma}^2)\frac{\partial h(\hat{\mu}, \hat{\sigma}^2)}{\partial \sigma^2}$

$h(\hat{\mu}, \hat{\sigma}^2) = \frac{\hat{\mu}}{\hat{\sigma}^2}$ $h = \frac{\mu}{\sigma^2}$ ∴ $\frac{\partial h}{\partial \mu} = \frac{1}{\sigma^2}$ $\frac{\partial h}{\partial \sigma^2} = \frac{\partial}{\partial \sigma^2}[\mu(\sigma^2)^{-1}] =$

$\mu \frac{\partial}{\partial \sigma^2}[(\sigma^2)^{-1}] = -\mu(\sigma^2)^{-2} = \frac{-\mu}{\sigma^4}$

is E[$\hat{\theta} = h(\hat{\mu}, \hat{\sigma}^2) = \hat{\mu}/\hat{\sigma}^2$

$h(\hat{\mu}, \hat{\sigma}^2) \approx h(\mu, \sigma^2) + (\hat{\mu} - \mu)\frac{\partial h}{\partial \hat{\mu}}(\mu, \sigma^2) + (\hat{\sigma}^2 - \sigma^2)\frac{\partial h}{\partial \hat{\sigma}^2}(\mu, \sigma^2)$ ∴

$\frac{\partial h}{\partial \hat{\mu}} = \frac{1}{\hat{\sigma}^2}$ $\frac{\partial h}{\partial \hat{\sigma}^2} = \frac{\partial}{\partial \hat{\sigma}^2}[\hat{\mu}(\hat{\sigma}^2)^{-1}] = \hat{\mu}\frac{\partial}{\partial \hat{\sigma}^2}[(\hat{\sigma}^2)^{-1}] = -\hat{\mu}(\hat{\sigma}^2)^{-2} = \frac{-\hat{\mu}}{\hat{\sigma}^4}$ ∴

$h(\hat{\mu}, \hat{\sigma}^2) \approx h(\mu, \sigma^2) + (\hat{\mu} - \mu)\frac{1}{\hat{\sigma}^2}(\mu, \sigma^2) + (\hat{\sigma}^2 - \sigma^2) - \frac{\hat{\mu}}{\hat{\sigma}^4}(\mu, \sigma^2)$

is $h(\hat{\mu}, \hat{\sigma}^2) = h(\mu, \sigma^2) + (\hat{\mu} - \mu)\frac{\partial h(\mu, \sigma^2)}{\partial \hat{\mu}} + (\hat{\sigma}^2 - \sigma^2)\frac{\partial h(\mu, \sigma^2)}{\partial \hat{\sigma}^2}$ ∴

$E(\hat{\theta}) \approx E[h(\hat{\mu}, \hat{\sigma}^2)] \approx E\left[h(\mu, \sigma^2) + (\hat{\mu} - \mu)\frac{\partial h(\mu, \sigma^2)}{\partial \hat{\mu}} + (\hat{\sigma}^2 - \sigma^2)\frac{\partial h(\mu, \sigma^2)}{\partial \hat{\sigma}^2}\right] =$

$E[h(\mu, \sigma^2)] + E\left[(\hat{\mu} - \mu)\frac{\partial h(\mu, \sigma^2)}{\partial \hat{\mu}}\right] + E\left[(\hat{\sigma}^2 - \sigma^2)\frac{\partial h(\mu, \sigma^2)}{\partial \hat{\sigma}^2}\right] =$

$h(\mu, \sigma^2) + E[\hat{\mu} - \mu]\frac{\partial h(\mu, \sigma^2)}{\partial \hat{\mu}} + E[\hat{\sigma}^2 - \sigma^2]\frac{\partial h(\mu, \sigma^2)}{\partial \hat{\sigma}^2} =$

$h(\mu, \sigma^2) + (E[\hat{\mu}] - E[\mu])\frac{\partial h(\mu, \sigma^2)}{\partial \hat{\mu}} + (E[\hat{\sigma}^2] - E[\sigma^2])\frac{\partial h(\mu, \sigma^2)}{\partial \hat{\sigma}^2} =$

$h(\mu, \sigma^2) + (E[\hat{\mu}] - \mu)\frac{\partial h(\mu, \sigma^2)}{\partial \hat{\mu}} + (E[\hat{\sigma}^2] - \sigma^2)\frac{\partial h(\mu, \sigma^2)}{\partial \hat{\sigma}^2} =$

Bias$(\hat{\theta}) = E[\hat{\theta}] - \theta = E\left[\frac{\hat{\mu}}{\hat{\sigma}^2}\right] - \frac{\mu}{\sigma^2}$

$E[\hat{\mu}] = E[\bar{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n}E\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n}\sum_{i=1}^{n} E[X_i] = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{1}{n}n\mu = \mu$

$var[\hat{\mu}] = var[\bar{X}] = var\left[\frac{1}{n}\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}var\left[\sum_{i=1}^{n} X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n} var[X_i] = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 =$

$\frac{1}{n^2}n\sigma^2 = \frac{1}{n}\sigma^2 = \frac{\sigma^2}{n}$

$var[\hat{\sigma}^2] = var\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = \frac{1}{(n-1)^2}var\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = \frac{1}{(n-1)^2}\sum_{i=1}^{n} var[(X_i - \bar{X})^2]$

$= \frac{1}{(n-1)^2}\sum_{i=1}^{n} var[X_i^2 - 2\bar{X}X_i + \bar{X}^2] = \left\{\frac{\mu_4 - \sigma^4}{n} + \frac{2\sigma^4}{n(n-1)}\right\}$

$E[\hat{\sigma}^2] = E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = \left\{\frac{1}{n-1}E\left[\sum_{i=1}^{n}(X_i - \bar{X})^2\right] = \frac{n}{n-1}E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i - \bar{X})^2\right]\right\} =$

$\frac{n}{n-1}\left(1 - \frac{1}{n}\right)\sigma^2 = \sigma^2$

$Cov(\hat{\mu}, \hat{\sigma}^2) = Cov\left(\bar{X}, \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2\right) = Cov(\bar{X}, S^2) = \frac{\mu_3}{n}$

3a $\hat{\phi} = \frac{\hat{\mu}}{\hat{\sigma}^2}$ $\quad \phi = \frac{\mu}{\sigma^2}$ $\quad h(\mu, \sigma^2) = \frac{\mu}{\sigma^2} = \phi$ $\quad h(\mu_0, \sigma_0^2) = \frac{\mu_0}{\sigma_0^2} = \mu_0 \sigma_0^{-2}$

I.c.ii $\phi = 1$ $\quad X_1, \dots, X_n \overset{iid}{\sim} \text{Poi}(\mu)$ $\quad E[X] = \text{var}(X)$

$H_0 : \phi = 1$ $\quad H_1 : \phi = 1$ Test the hypothesis using a parametric bootstrap with test statistic $\hat{\phi}$.

$n = 20$, $\quad t = \phi = 0.1268314$, 0.127 $\quad t = 0.127$

$\mu_0 = 4.65$ Sample mean $\quad \text{Poi}(\text{Mu\_0}) = \text{poi}(\mu_0)$

MLE for μ: $P(x_i | \mu) = \frac{e^{-\mu} \mu^{x_i}}{x_i!}$ $\quad x_i \in \{0, 1, 2, \dots\}$

The likelihood function is given by,

$$L(\mu) = \prod_{i=1}^{n} P(x_i | \mu) = \prod_{i=1}^{n} \frac{e^{-\mu} \mu^{x_i}}{x_i!} = \frac{\prod_{i=1}^{n} e^{-\mu} \prod_{i=1}^{n} \mu^{x_i}}{\prod_{i=1}^{n} x_i!} = \frac{e^{-n\mu} \mu^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}$$

The log-likelihood function is; $l(\mu) = \log L(\mu) = \log\left[\frac{e^{-n\mu} \mu^{\sum_{i=1}^{n} x_i}}{\prod_{i=1}^{n} x_i!}\right] =$

$$\log(e^{-n\mu}) + \log\left(\mu^{\sum_{i=1}^{n} x_i}\right) - \log\left(\prod_{i=1}^{n} x_i!\right) = -n\mu + \left(\sum_{i=1}^{n} x_i\right)\log(\mu) - \log\left(\prod_{i=1}^{n} x_i!\right)$$

$$l'(\mu) = \frac{\partial l(\mu)}{\partial \mu} = \frac{\partial}{\partial \mu}\left[-n\mu + \left(\sum_{i=1}^{n} x_i\right)\log(\mu) - \log\left(\prod_{i=1}^{n} x_i!\right)\right] = -n + \frac{1}{\mu}\sum_{i=1}^{n} x_i$$

$$l'' = \frac{\partial^2 l(\mu)}{\partial \mu^2} = \frac{\partial}{\partial \mu}\left[-n + \frac{1}{\mu}\sum_{i=1}^{n} x_i\right] = -\frac{1}{\mu^2}\sum_{i=1}^{n} x_i < 0$$

$$\left.\frac{\partial l(\mu)}{\partial \mu}\right|_{\mu = \hat{\mu}} = 0 \quad \text{(mle)}$$

$$l(\mu_0) = -n + \frac{1}{\mu_0}\sum_{i=1}^{n} x_i \implies -n + \frac{1}{\mu_0}\sum_{i=1}^{n} x_i = 0 \implies \hat{\mu}_0 = \frac{1}{n}\sum_{i=1}^{n} x_i = \text{Mean}(x)$$

$l(\hat{\mu}) = -n + \frac{1}{\hat{\mu}}\sum_{i=1}^{n} x_i$ and $\phi = \frac{\mu}{\sigma^2}$ $\hat{\phi} = \frac{\hat{\mu}}{\hat{\sigma}^2}$ when $\phi = 1$: $\mu = \sigma^2$ and

$$-2\log\Lambda = -2\left[l(\hat{\mu}_0) - l(\hat{\mu})\right] = -2\left[-n + \frac{1}{\hat{\mu}_0}\sum_{i=1}^{n} x_i - \left(-n + \frac{1}{\hat{\mu}}\sum_{i=1}^{n} x_i\right)\right]$$

$$t = -2\left[\sum_{i=1}^{n} x_i\left(\frac{1}{\hat{\mu}_0} - \frac{1}{\hat{\mu}}\right)\right]$$

$\left\{ 0.478 \quad \phi = 1 \implies \mu = \sigma^2 \quad \hat{\phi} = \frac{\hat{\mu}}{\hat{\sigma}^2} \right\}$

$t = 0.127$ (test statistic) μ_0 assumes $\phi = 1 \implies \mu = \sigma^2$, variance when $\phi = 1 \implies 1 = \frac{\mu}{\sigma^2} \implies \mu = \sigma^2$ $\quad$ p-value ≈ 0.5 $\quad$ 3hits p-value not enough evidence to reject H_0 (should be less than 0.01)

1-C-ii/ ~~[illegible]~~ Test hypothesis $H_0: \phi = 1$ against $H_1: \phi < 1$

at 1% significance level

when $\phi = 1$: $X_1, \dots, X_n \sim Poi(\mu)$ ∴ $E[X] = Var[X]$

$n = 20$ $t = \hat{\phi} = 0.12683394 \simeq 0.127$

$\hat{\mu} = 4.65$ ∴ $\mu_0 = 4.65$ sample mean ∴ $Poi(\mu_{ML\,0}) = Poi(\mu_0)$

The Maximum likelihood estimate for μ: $P(X_i|\mu) = \frac{\mu^{x_i}}{x_i!}e^{-\mu}$ $X_i \geq 0$

∴ ~~Likelihood~~ The likelihood function is:

$L(\mu) = \cancel{\sum_{i=1}^{n}} P(\underline{X}|\mu) = \prod_{i=1}^{n} P(X_i|\mu) = \prod_{i=1}^{n} \frac{\mu^{x_i}}{x_i!}e^{-\mu} = \left(\prod_{i=1}^{n}\left[\frac{1}{x_i!}\right]\right)\prod_{i=1}^{n}\mu^{x_i}\prod_{i=1}^{n}e^{-\mu}$

$= \left(\prod_{i=1}^{n}\left[\frac{1}{x_i!}\right]\right)\mu^{\sum x_i}(e^{-\mu})^n = \left(\prod_{i=1}^{n}\left[\frac{1}{x_i!}\right]\right)\mu^{\sum_{i=1}^{n} x_i}e^{-n\mu}$ ∴

The log-likelihood function is: $l(\mu) = \log(L(\mu)) =$

$\ln\left[\left(\prod_{i=1}^{n}\left[\frac{1}{x_i!}\right]\right)\mu^{\sum_{i=1}^{n} x_i}e^{-n\mu}\right] = \ln\left[\prod_{i=1}^{n}\left[\frac{1}{x_i!}\right]\right] + \ln\left[\mu^{\sum_{i=1}^{n} x_i}\right] + \ln\left[e^{-n\mu}\right] =$

$\ln\left[\prod_{i=1}^{n}\left[\frac{1}{x_i!}\right]\right] + \ln[\mu]\sum_{i=1}^{n}[x_i] - n\mu$ ∴

$l'(\mu) = \frac{\partial l(\mu)}{\partial \mu} = \frac{\partial}{\partial \mu}\left(\ln\left[\prod_{i=1}^{n}\left[\frac{1}{x_i}\right]\right] + \ln[\mu]\sum_{i=1}^{n}[x_i] - n\mu\right) =$

$0 + \frac{1}{\mu}\sum_{i=1}^{n}[x_i] - n = \frac{1}{\mu}\sum_{i=1}^{n}[x_i] - n$ ∴

for maximum likelihood estimate for μ: $\cancel{\mu}$ $l'(\hat{\mu}) = 0$ ∴

$l'(\hat{\mu}) = \frac{1}{\hat{\mu}}\sum_{i=1}^{n}[x_i] - n = 0$ ∴ $\frac{1}{\hat{\mu}}\sum_{i=1}^{n}[x_i] = n$ ∴

$\frac{1}{n}\sum_{i=1}^{n}[x_i] = \hat{\mu}$ is MLE ∴

$l''(\mu) = \frac{\partial}{\partial \mu}(l'(\mu)) = \frac{\partial^2}{\partial \mu^2}l(\mu) = \frac{\partial}{\partial \mu}\left(\frac{1}{\mu}\sum_{i=1}^{n}[x_i] - n\right) = -\frac{1}{\mu^2}\sum_{i=1}^{n}[x_i]$ ∴

$\mu^2 > 0$, $x_i \geq 0$ ∴ $\sum_{i=1}^{n} > 0$ ∴ $-\frac{1}{\mu^2}\sum_{i=1}^{n}[x_i] < 0$ ∴

$-\frac{1}{\hat{\mu}^2}\sum_{i=1}^{n}[x_i] < 0$ ∴ $l''(\hat{\mu}) < 0$ ∴ $\hat{\mu}$ is MLE ∴

The unrestricted maximum likelihood is $L(\hat{\mu})$ where $\hat{\mu} = \bar{x}$ ∴

$\cancel{l(\mu_0) = \ln\left[\prod_{i=1}^{n}\left(\frac{1}{x_i!}\right)\right] + \ln[\mu_0]\sum_{i=1}^{n}[x_i] - n\mu_0}$

$l'(\mu_0) = \frac{1}{\mu_0}\sum_{i=1}^{n}[x_i] - n$ ∴ $l'(\hat{\mu}_0) = \frac{1}{\hat{\mu}_0}\sum_{i=1}^{n}[x_i] - n = 0$ ∴ $\frac{1}{\hat{\mu}_0}\sum_{i=1}^{n}[x_i] = n$

$\frac{1}{n}\sum_{i=1}^{n}[x_i] = \hat{\mu}_0$ ∴ $-2\log\Lambda = -2[l(\mu_0) - l(\hat{\mu})] =$

$-2\left[\ln\left[\prod_{i=1}^{n}\left(\frac{1}{x_i!}\right)\right] - \ln[\mu_0]\sum_{i=1}^{n}[x_i] - n\mu_0 - \ln\left[\prod_{i=1}^{n}\left(\frac{1}{x_i!}\right)\right] + \ln[\hat{\mu}]\sum_{i=1}^{n}[x_i] + n\hat{\mu}\right] =$

$-2\left[-\ln(\mu_0)\sum_{i=1}^{n}[x_i] - n\mu_0 + \ln(\hat{\mu})\sum_{i=1}^{n}[x_i] - n\hat{\mu}\right] = \cancel{-2}[$

1c.ii $l(\mu) = -n\mu + (\sum_{i=1}^{n} x_i)\log(\mu) - \log(\prod_{i=1}^{n} x_i)$

$= l(\mu) = \ln(\prod_{i=1}^{n}[\frac{1}{x_i!}]) + \ln(\mu)\sum_{i=1}^{n}[x_i] - n\mu$ ∴

$\hat{\mu} = \frac{1}{n}\sum_{i=1}^{n}[x_i]$

$\hat{\mu}_0 = \bar{x}$ ∴ likelihood ratio test statistic is

$-2\log\Lambda = 2[l(\hat{\mu}) - l(\mu_0)]$

$l(\hat{\mu}) = \ln(\prod_{i=1}^{n}[\frac{1}{x_i!}]) + \ln(\hat{\mu})\sum_{i=1}^{n}[x_i] - n\hat{\mu}$

$l(\hat{\mu}_0) = \ln(\prod_{i=1}^{n}[\frac{1}{x_i!}]) + \ln(\hat{\mu}_0)\sum_{i=1}^{n}[x_i] - n\hat{\mu}_0$ ∴

$-2\log\Lambda = -2[l(\hat{\mu}_0) - l(\hat{\mu})] =$

$-2[\ln(\prod_{i=1}^{n}[\frac{1}{x_i!}]) + \ln(\hat{\mu}_0)\sum_{i=1}^{n}[x_i] - n\hat{\mu}_0 - \ln(\prod_{i=1}^{n}[\frac{1}{x_i!}]) - \ln(\hat{\mu})\sum_{i=1}^{n}[x_i] + n\hat{\mu}] =$

$-2[\ln(\hat{\mu}_0)\sum_{i=1}^{n}[x_i] - \ln(\hat{\mu})\sum_{i=1}^{n}[x_i] - n\hat{\mu}_0 + n\hat{\mu}] =$

$-2[(\ln(\hat{\mu}_0) - \ln(\hat{\mu}))\sum_{i=1}^{n}[x_i] - n(\hat{\mu}_0 - \hat{\mu})] =$

$-2[\ln(\frac{\hat{\mu}_0}{\hat{\mu}})\sum_{i=1}^{n}[x_i] - n(\hat{\mu}_0 - \hat{\mu})]$

we would reject H_0 in favour of H_1 at level α is the test statistic exceeded the $(1-\alpha)$-quantile of the χ_1^2 distribution

test statistic $\hat{\phi} = \dfrac{\frac{1}{n}\sum_{i=1}^{n} x_i}{\frac{1}{n-1}\sum_{i=1}^{n}(x_i - \bar{x})^2}$

1d Let X_0 denote the number of people infected by the next infectious person. When $\phi < 1$, a common model assumes that $X_1, \ldots, X_n$ are independent with a negative binomial distribution. Use this model to complete a plug-in 90% prediction interval for X_0, where the interval is a closed interval of the form $[0, x]$, and explain why the converage of the interval might not be exactly 90%. For the negative binomial model with $\mu = mu$ and $\phi = phi$, $Pr(X_0 \le x)$ may be computed with the command pnbinom(x, mu*phi/(1-phi), phi).

~~98%~~ plug-in 90% prediction interval for X_0, where the interval is a closed interval of the form $[0, x]$

$\therefore$ if $T = T(X_0, \underline{X})$ is know the distribution of T then we can find t_{90} such that $Pr(0 < T < t_{90}) = 0.9$, T is ancillary

and when $\phi < 1$: $X \sim$ negative binomial $(\hat{\mu}, \hat{\phi})$ $\therefore$ $X \sim NB(\hat{\mu}, \hat{\phi})$

$\therefore NB(X; \hat{\mu}, \hat{\phi}) = \binom{X + \hat{\mu} - 1}{\hat{\mu} - 1} \hat{\phi}^{\hat{\mu}} (1 - \hat{\phi})^{X}$ $\quad X = 0, 1, 2, \ldots$

~~X~~ $X \geq 0$ $\therefore$ $T \geq 0$ $\therefore$ $P(T < 0) = 0$ $\therefore$

90% prediction interval is $Pr(0 < T < t_{90}) = 0.9$ $\therefore$

T is ancillary ~~if T =~~

if: $T = X_0 / \bar{X}$ then $Pr(0 < \frac{X_0}{\bar{X}} < t_{90}) = 0.90$ $\therefore$

$Pr(0 < X_0 < \bar{X} t_{90})$ is a 90% prediction interval for X_0

~~if~~

if: ~~NB~~ $NB(\mu, \phi)$ is a location-scale model $\therefore$ with $Z \sim N(0,1)$:

$Z_i = (X_i - \mu)/\sigma$ so the distribution of Z_i contains no unknown params $\therefore$ $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i = \frac{1}{n}\sum_{i=1}^{n}(\mu + \sigma Z_i) = \mu + \sigma \bar{Z}$

$\hat{\sigma}^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\mu + \sigma Z_i - \mu - \sigma \bar{Z})^2 = \frac{1}{n-1}\sum_{i=1}^{n}(\sigma(Z_i - \bar{Z}))^2 =$

$\frac{\sigma^2}{n-1}\sum_{i=1}^{n}(Z_i - \bar{Z})^2$ and $T = \frac{X_0 - \bar{X}}{\hat{\sigma}} = \frac{\mu + \sigma Z_0 - \mu - \sigma \bar{Z}}{\sigma\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(Z_i - \bar{Z})^2}} = \frac{\sigma(Z_0 - \bar{Z})}{\sigma\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(Z_i - \bar{Z})^2}} =$

$\frac{Z_0 - \bar{Z}}{\sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(Z_i - \bar{Z})^2}}$ The distribution of T depends only on Z_i

$\therefore$ is independent of the parameters μ and ϕ

$\therefore$ T is ancillary $\therefore$ $T = \frac{X_0 - \bar{X}}{\hat{\sigma}}$ $\therefore$

$Pr(0 < T < t_{90}) = 0.9 = Pr(0 < \frac{X_0 - \bar{X}}{\hat{\sigma}} < t_{90}) = Pr(0 < X_0 - \bar{X} < t_{90}\hat{\sigma}) = 0.9 =$

~~Pr(0<~~ $Pr(\bar{X} < X_0 < \bar{X} + t_{90}\hat{\sigma}) = 0.9$ is a 90% prediction interval for X_0

if $T = X_0 / \bar{X}$ $\therefore$ $\cancel{Z}\sigma \sim N(0, \sigma^2)$ $\therefore$ $Z_i \sigma = X_i - \mu$ $\therefore$

$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i =$

\1d/ is $T = \frac{X_0}{\bar{X}}$ is ancillary

$X_{\neq} \geq 0$ is $T = T(X_0, \bar{X})$ then $T \geq 0$ $Pr(t_5 < T < t_{90}) = 0.9$

\2a/ $Bias(\hat{\phi}) = E(\hat{\phi}) - \phi$ $\quad Bias(\hat{\phi}_J) = E(\hat{\phi}_J) - \phi$

\1aii/ $\hat{\phi} = \hat{\mu} / \hat{\sigma}^2$

$$\hat{\sigma}^2_{-i} = \frac{1}{n-1}\left[\sum_{j \neq i} X_j^2 - \frac{1}{n-1}\left(\sum_{j \neq i} X_j\right)^2\right]$$

$$\therefore \hat{\phi}_{-i} = \frac{\frac{1}{n-1}\sum_{j \neq i} X_j}{\hat{\sigma}^2_{-i}} = \frac{\frac{1}{n-1}\sum_{j \neq i} X_j}{\frac{1}{n-1}\left[\sum_{j \neq i} X_j^2 - \frac{1}{n-1}\left(\sum_{j \neq i} X_j\right)^2\right]}$$

$$\sum_{j \neq i} X_j = \sum_{j=1}^{n}[X_j] - X_i \qquad \sum_{j \neq i} X_j^2 = \sum_{j=1}^{n}[X_j^2] - X_i^2$$

$$\therefore \hat{\phi}_{-i} = \frac{\frac{1}{n-1}\left\{\sum_{j=1}^{n}[X_j] - X_i\right\}}{\frac{1}{n-1}\left[\left\{\sum_{j=1}^{n}[X_j^2] - X_i^2\right\} - \frac{1}{n-1}\left(\left\{\sum_{j=1}^{n}[X_j] - X_i\right\}^2\right)\right]}$$

$$\hat{\phi}_i = n\hat{\phi} - (n-1)\hat{\phi}_{-i} \qquad \cancel{\hat{\phi}_J = \frac{1}{n}\sum_{i=1}^{n}[\hat{\phi}_i]}$$

$$\hat{\phi}_J = \frac{1}{n}\sum_{i=1}^{n}[\hat{\phi}_i]$$

$$\hat{\phi} = \hat{\mu} / \hat{\sigma}^2 = Mean(X) / var(X)$$

$$\frac{1}{n-1}\sum_{j \neq i} X_j = \frac{1}{n-1}\left(\sum_{j=1}^{n}[X_j] - X_i\right)$$

$$\frac{1}{n-1}\sum_{j \neq i} X_j^2 = \frac{1}{n-1}\left(\sum_{j=1}^{n}[X_j^2] - X_i^2\right)$$

$$\hat{\phi}_{-i} = Mean(X_{-i}) / var(X_{-i}) = E[X_{-i}] / var[X_{-i}]$$

$$\hat{\phi}_i = n\hat{\phi} - (n-1)\hat{\phi}_{-i} \qquad \hat{\phi}_J = \frac{1}{n}\sum_{i=1}^{n}[\hat{\phi}_i]$$

\2b/ $n=10, 20, \dots, 100$

compute monte carlo approximations to the coverages of the confidence interval

useful material:

notes 1 p36 below Ex 1.37

sheet 3 formative Q3c

sheet 3 preparatory Q8 (rating 4/10)

\3a/ useful material:

sheet 1 Q1 formative 1b, c, d

sheet 1 extra Q5

sheet 1 preparatory Q3

$\hat{\phi} = \hat{\mu}/\hat{\sigma}^2 \quad \therefore \phi = \mu/\sigma^2 \qquad \text{Bias}(\hat{\phi}) = E(\hat{\phi}) - \phi \qquad \phi = h(\mu, \sigma^2)$.

$\hat{\phi} = h(\hat{\mu}, \hat{\sigma}^2) \quad \therefore h(\hat{\mu}, \hat{\sigma}^2) = \hat{\mu}/\hat{\sigma}^2 = \hat{\mu}(\hat{\sigma}^2)^{-1} \quad \therefore$

$$\frac{\partial h}{\partial \mu} = \frac{\partial h(\hat{\mu}, \hat{\sigma}^2)}{\partial \mu} = \frac{\partial}{\partial \mu}\left(\frac{\hat{\mu}}{\hat{\sigma}^2}\right) = \frac{1}{\hat{\sigma}^2} = \frac{\partial h}{\partial \mu}(\hat{\mu}, \hat{\sigma}^2)$$

$$\frac{\partial h}{\partial \sigma^2} = \frac{\partial h(\hat{\mu}, \hat{\sigma}^2)}{\partial \sigma^2} = \frac{\partial}{\partial \sigma^2}\left(\frac{\mu}{\sigma^2}\right)(\hat{\mu}, \hat{\sigma}^2) = -\frac{\hat{\mu}}{\hat{\sigma}^4} = -\hat{\mu}(\hat{\sigma}^2)^{-2} = \frac{\partial h}{\partial \sigma^2}(\hat{\mu}, \hat{\sigma}^2)$$

$$\frac{\partial^2 h}{\partial \mu^2}(\hat{\mu}, \hat{\sigma}^2) = \frac{\partial}{\partial \mu}\left(\frac{\partial h}{\partial \mu}\right)(\hat{\mu}, \hat{\sigma}^2) = \frac{\partial}{\partial \mu}\left(\frac{1}{\sigma^2}\right)(\hat{\mu}, \hat{\sigma}^2) = 0$$

$$\frac{\partial^2 h}{\partial \sigma^4}(\hat{\mu}, \hat{\sigma}^2) = \frac{\partial}{\partial \sigma^2}\left(\frac{\partial h}{\partial \sigma^2}\right)(\hat{\mu}, \hat{\sigma}^2) = \frac{\partial}{\partial \sigma^2}\left(-\mu(\sigma^2)^{-2}\right)(\hat{\mu}, \hat{\sigma}^2) =$$

$$-\mu \frac{\partial}{\partial \sigma^2}((\sigma^2)^{-2})(\hat{\mu}, \hat{\sigma}^2) = 2\mu(\sigma^2)^{-3}(\hat{\mu}, \hat{\sigma}^2) = 2\hat{\mu}(\hat{\sigma}^2)^{-3} = \frac{2\hat{\mu}}{\hat{\sigma}^6}$$

$$\frac{\partial^2 h}{\partial \mu \partial \sigma^2}(\hat{\mu}, \hat{\sigma}^2) = \frac{\partial}{\partial \mu}\left(\frac{\partial h}{\partial \sigma^2}\right)(\hat{\mu}, \hat{\sigma}^2) = \frac{\partial}{\partial \mu}\left(-\mu(\sigma^2)^{-2}\right)(\hat{\mu}, \hat{\sigma}^2) =$$

$$-(\hat{\sigma}^2)^{-2} = -\frac{1}{\hat{\sigma}^4} \quad \therefore$$

$$h(\mu, \sigma^2) \approx \frac{\hat{\mu}}{\hat{\sigma}^2} + (\mu - \hat{\mu})\frac{1}{\hat{\sigma}^2} + (\sigma^2 - \hat{\sigma}^2)\left(-\frac{\hat{\mu}}{\hat{\sigma}^4}\right) +$$

$$\frac{(\mu - \hat{\mu})^2}{2}(0) + \frac{(\sigma^2 - \hat{\sigma}^2)^2}{2}\frac{2\hat{\mu}}{\hat{\sigma}^6} + (\mu - \hat{\mu})(\sigma^2 - \hat{\sigma}^2)\left(-\frac{1}{\hat{\sigma}^4}\right) =$$

\3a conti/ $\frac{\hat{\mu}}{\hat{\sigma}^2} + \mu \frac{1}{\hat{\sigma}^2} - \hat{\mu}\frac{1}{\hat{\sigma}^2} - \sigma^2 \hat{\mu} \frac{1}{\hat{\sigma}^4} + \hat{\mu}\frac{1}{\hat{\sigma}^2} +$

$\cancel{\tfrac{1}{2}}(\sigma^4 - 2\sigma^2\hat{\sigma}^2 + \hat{\sigma}^4)\frac{\hat{\mu}}{\hat{\sigma}^6} + (\mu\sigma^2 - \mu\hat{\sigma}^2 - \sigma^2\hat{\mu} + \mu\hat{\sigma}^2)(-\frac{1}{\hat{\sigma}^4})$

$= (\hat{\mu}\frac{1}{\hat{\sigma}^2} - \hat{\mu}\frac{1}{\hat{\sigma}^2}) + \mu\frac{1}{\hat{\sigma}^2} - \sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4} + \hat{\mu}\frac{1}{\hat{\sigma}^2} +$

$\sigma^4\hat{\mu}\frac{1}{\hat{\sigma}^6} - 2\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4} + \hat{\mu}\frac{1}{\hat{\sigma}^2} - \mu\sigma^2\frac{1}{\hat{\sigma}^4} + \mu\frac{1}{\hat{\sigma}^2} + \sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4} - \mu\frac{1}{\hat{\sigma}^2} =$

$0 + (\mu\frac{1}{\hat{\sigma}^2} + \mu\frac{1}{\hat{\sigma}^2} - \mu\frac{1}{\hat{\sigma}^2}) + (-\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4} - 2\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4} + \sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4}) +$

$(\hat{\mu}\frac{1}{\hat{\sigma}^2} + \hat{\mu}\frac{1}{\hat{\sigma}^2}) + \sigma^4\hat{\mu}\frac{1}{\hat{\sigma}^6} + (-\mu\sigma^2\frac{1}{\hat{\sigma}^4}) =$

$\mu\frac{1}{\hat{\sigma}^2} - 2\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4} + 2\frac{1}{\hat{\sigma}^2} - \sigma^4\hat{\mu}\frac{1}{\hat{\sigma}^6} - \mu\sigma^2\frac{1}{\hat{\sigma}^4} =$

\3a redo/

$\hat{\phi} = \frac{\hat{\mu}}{\hat{\sigma}^2} \quad \therefore \phi = \frac{\mu}{\sigma^2} \quad \therefore \cancel{h(\phi) = \frac{2\mu}{\sigma^2}} \quad \phi = h(\mu, \sigma^2) = \frac{\mu}{\sigma^2} = \mu(\sigma^2)^{-1}$

$h(\hat{\mu}, \hat{\sigma}^2) = \frac{\hat{\mu}}{\hat{\sigma}^2} \qquad \text{Bias}(\hat{\phi}) = E(\hat{\phi}) - \phi$

$\frac{\partial h}{\partial \mu}(\hat{\mu}, \hat{\sigma}^2) = \frac{\partial}{\partial \mu}\left(\frac{\mu}{\sigma^2}\right)(\hat{\mu}, \hat{\sigma}^2) = \frac{1}{\hat{\sigma}^2}$

$\frac{\partial h}{\partial \sigma^2}(\hat{\mu}, \hat{\sigma}^2) = \frac{\partial}{\partial \sigma^2}(\mu(\sigma^2)^{-1})(\hat{\mu}, \hat{\sigma}^2) = \mu\frac{\partial}{\partial \sigma^2}((\sigma^2)^{-1})(\hat{\mu}, \hat{\sigma}^2) = -\hat{\mu}(\hat{\sigma}^2)^{-2}$

$= -\hat{\mu}\frac{1}{\hat{\sigma}^4}$

$\frac{\partial^2 h}{\partial \mu^2}(\hat{\mu}, \hat{\sigma}^2) = \frac{\partial}{\partial \mu}\left(\frac{\partial h}{\partial \mu}\right)(\hat{\mu}, \hat{\sigma}^2) = \frac{\partial}{\partial \mu}\left(\frac{1}{\sigma^2}\right)(\hat{\mu}, \hat{\sigma}^2) = 0$

$\frac{\partial^2 h}{\partial \sigma^4}(\hat{\mu}, \hat{\sigma}^2) = \frac{\partial}{\partial \sigma^2}\left(\frac{\partial h}{\partial \sigma^2}\right)(\hat{\mu}, \hat{\sigma}^2) = \frac{\partial}{\partial \sigma^2}(-\mu(\sigma^2)^{-2}) = -\mu\frac{\partial}{\partial \sigma^2}((\sigma^2)^{-2})(\hat{\mu}, \hat{\sigma}^2)$

$= 2\hat{\mu}(\hat{\sigma}^2)^{-3} = 2\hat{\mu}\frac{1}{\hat{\sigma}^6}$

$\frac{\partial^2 h}{\partial \mu \partial \sigma^2}(\hat{\mu}, \hat{\sigma}^2) = \frac{\partial}{\partial \mu}\left(\frac{\partial h}{\partial \sigma^2}\right)(\hat{\mu}, \hat{\sigma}^2) = \frac{\partial}{\partial \mu}\left(-\mu\frac{1}{\sigma^4}\right) = -\frac{1}{\hat{\sigma}^4} \quad \therefore$

$h(\mu, \sigma^2) \approx \hat{\mu}\frac{1}{\hat{\sigma}^2} + (\mu - \hat{\mu})\frac{1}{\hat{\sigma}^2} + (\sigma^2 - \hat{\sigma}^2)(-\hat{\mu}\frac{1}{\hat{\sigma}^4}) +$

$\frac{(\mu - \hat{\mu})^2}{2}(0) + \frac{(\sigma^2 - \hat{\sigma}^2)^2}{2} 2\hat{\mu}\frac{1}{\hat{\sigma}^6} + (\mu - \hat{\mu})(\sigma^2 - \hat{\sigma}^2)(-\frac{1}{\hat{\sigma}^4}) =$

3a Continued:

$$\hat{\mu}\frac{1}{\hat{\sigma}^2}+\mu\frac{1}{\hat{\sigma}^2}-\hat{\mu}\frac{1}{\hat{\sigma}^2}-\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4}+\hat{\mu}\frac{1}{\hat{\sigma}^2}+$$

$$0+(\sigma^4-2\sigma^2\hat{\sigma}^2+\hat{\sigma}^4)\hat{\mu}\frac{1}{\hat{\sigma}^6}+(\mu\sigma^2-\mu\hat{\sigma}^2-\sigma^2\hat{\mu}+\hat{\mu}\hat{\sigma}^2)\left(-\frac{1}{\hat{\sigma}^4}\right)=$$

$$\mu\frac{1}{\hat{\sigma}^2}-\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4}+\hat{\mu}\frac{1}{\hat{\sigma}^2}+\sigma^4\hat{\mu}\frac{1}{\hat{\sigma}^6}-2\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4}+\hat{\mu}\frac{1}{\hat{\sigma}^2}+$$

$$-\mu\sigma^2\frac{1}{\hat{\sigma}^4}+\mu\frac{1}{\hat{\sigma}^2}+\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4}-\hat{\mu}\frac{1}{\hat{\sigma}^2}=$$

$$2\mu\frac{1}{\hat{\sigma}^2}-\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4}+\hat{\mu}\frac{1}{\hat{\sigma}^2}+\sigma^4\hat{\mu}\frac{1}{\hat{\sigma}^6}-2\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4}+\hat{\mu}\frac{1}{\hat{\sigma}^2}+$$

$$-\mu\sigma^2\frac{1}{\hat{\sigma}^4}+\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4}-\hat{\mu}\frac{1}{\hat{\sigma}^2}=$$

$$2\mu\frac{1}{\hat{\sigma}^2}-\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4}+\sigma^4\hat{\mu}\frac{1}{\hat{\sigma}^6}-2\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4}+\hat{\mu}\frac{1}{\hat{\sigma}^2}+$$

$$-\mu\sigma^2\frac{1}{\hat{\sigma}^4}+\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4}=$$

$$2\mu\frac{1}{\hat{\sigma}^2}+\sigma^4\hat{\mu}\frac{1}{\hat{\sigma}^6}-2\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4}+\hat{\mu}\frac{1}{\hat{\sigma}^2}+-\mu\sigma^2\frac{1}{\hat{\sigma}^4}$$

3a redo again

$$\frac{\partial h}{\partial\mu}=\frac{1}{\hat{\sigma}^2}\qquad\frac{\partial h}{\partial\sigma^2}=-\hat{\mu}\frac{1}{\hat{\sigma}^4}$$

$$\frac{\partial^2 h}{\partial\mu^2}=0\qquad\frac{\partial^2 h}{\partial\sigma^4}=2\hat{\mu}\frac{1}{\hat{\sigma}^6}\qquad\frac{\partial^2 h}{\partial\mu\,\partial\sigma^2}=-\frac{1}{\hat{\sigma}^4}$$

$$h(\mu,\sigma^2)\approx\hat{\mu}\frac{1}{\hat{\sigma}^2}+(\mu-\hat{\mu})\frac{1}{\hat{\sigma}^2}+(\sigma^2-\hat{\sigma}^2)\left(-\hat{\mu}\frac{1}{\hat{\sigma}^4}\right)+$$

$$\frac{(\mu-\hat{\mu})^2}{2}(0)+\frac{(\sigma^2-\hat{\sigma}^2)^2}{2}2\hat{\mu}\frac{1}{\hat{\sigma}^6}+(\mu-\hat{\mu})(\sigma^2-\hat{\sigma}^2)\left(-\frac{1}{\hat{\sigma}^4}\right)=$$

$$\hat{\mu}\frac{1}{\hat{\sigma}^2}+\mu\frac{1}{\hat{\sigma}^2}-\hat{\mu}\frac{1}{\hat{\sigma}^2}-\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4}+\hat{\mu}\frac{1}{\hat{\sigma}^2}+$$

$$(\sigma^4-2\sigma^2\hat{\sigma}^2+\hat{\sigma}^4)\hat{\mu}\frac{1}{\hat{\sigma}^6}+(\mu\sigma^2-\mu\hat{\sigma}^2-\sigma^2\hat{\mu}+\hat{\mu}\hat{\sigma}^2)\left(-\frac{1}{\hat{\sigma}^4}\right)=$$

$$\mu\frac{1}{\hat{\sigma}^2}-\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4}+\hat{\mu}\frac{1}{\hat{\sigma}^2}+\sigma^4\hat{\mu}\frac{1}{\hat{\sigma}^6}-2\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4}+\hat{\mu}\frac{1}{\hat{\sigma}^2}+$$

$$-\mu\sigma^2\frac{1}{\hat{\sigma}^4}+\mu\frac{1}{\hat{\sigma}^2}+\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4}-\hat{\mu}\frac{1}{\hat{\sigma}^2}=$$

\3a continue ⊗/ $2\mu\frac{1}{\hat{\sigma}^2} - \sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4} + \hat{\mu}\frac{1}{\hat{\sigma}^2} + \sigma^4\hat{\mu}\frac{1}{\hat{\sigma}^6} - 2\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4} +$

$\hat{\mu}\frac{1}{\hat{\sigma}^2} + -\mu\sigma^2\frac{1}{\hat{\sigma}^4} + \sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4} - \hat{\mu}\frac{1}{\hat{\sigma}^2} =$

$2\mu\frac{1}{\hat{\sigma}^2} - \sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4} + \sigma^4\hat{\mu}\frac{1}{\hat{\sigma}^6} - 2\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4} +$

$\hat{\mu}\frac{1}{\hat{\sigma}^2} - \mu\sigma^2\frac{1}{\hat{\sigma}^4} + \sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4} =$

$2\mu\frac{1}{\hat{\sigma}^2} + \sigma^4\hat{\mu}\frac{1}{\hat{\sigma}^6} - 2\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4} + \hat{\mu}\frac{1}{\hat{\sigma}^2} - \mu\sigma^2\frac{1}{\hat{\sigma}^4}$ ∴

$h(\mu,\sigma^2) \approx 2\mu\frac{1}{\hat{\sigma}^2} + \sigma^4\hat{\mu}\frac{1}{\hat{\sigma}^6} - 2\sigma^2\hat{\mu}\frac{1}{\hat{\sigma}^4} + \hat{\mu}\frac{1}{\hat{\sigma}^2} - \mu\sigma^2\frac{1}{\hat{\sigma}^2}$ ∴

$\therefore\ h(\hat{\mu},\hat{\sigma}^2) \approx 2\hat{\mu}\frac{1}{\sigma^2} + \hat{\sigma}^4\mu\frac{1}{\sigma^6} - 2\hat{\sigma}^2\mu\frac{1}{\sigma^4} + \mu\frac{1}{\sigma^2} - \hat{\mu}\hat{\sigma}^2\frac{1}{\sigma^2}$

$\cancel{E[\hat{\theta}]}$ $\hat{\theta} = \frac{\hat{\mu}}{\hat{\sigma}^2}$ $\hat{\theta} = h(\hat{\mu},\hat{\sigma}^2) = \frac{\hat{\mu}}{\hat{\sigma}^2}$ $Bias(\hat{\theta}) = E(\hat{\theta}) - \frac{\mu}{\sigma^2}$

$E[\hat{\theta}] = E[h(\hat{\mu},\hat{\sigma}^2)] \approx E\left[2\hat{\mu}\frac{1}{\sigma^2} + \hat{\sigma}^4\mu\frac{1}{\sigma^6} - 2\hat{\sigma}^2\mu\frac{1}{\sigma^4} + \mu\frac{1}{\sigma^2} - \hat{\mu}\hat{\sigma}^2\frac{1}{\sigma^2}\right] =$

$E\left[2\hat{\mu}\frac{1}{\sigma^2}\right] + E\left[\hat{\sigma}^4\mu\frac{1}{\sigma^6}\right] + E\left[-2\hat{\sigma}^2\mu\frac{1}{\sigma^4}\right] + E\left[\mu\frac{1}{\sigma^2}\right] + E\left[-\hat{\mu}\hat{\sigma}^2\frac{1}{\sigma^2}\right] =$

$2\frac{1}{\sigma^2}E[\hat{\mu}] + \mu\frac{1}{\sigma^6}E[\hat{\sigma}^4] - 2\mu\frac{1}{\sigma^4}E[\hat{\sigma}^2] + \mu\frac{1}{\sigma^2}E[1] - \frac{1}{\sigma^2}E[\hat{\mu}\hat{\sigma}^2]$

$E[\hat{\mu}] = E[\bar{X}] = E\left[\frac{1}{n}\sum_{i=1}^{n}X_i\right] = \frac{1}{n}E\left[\sum_{i=1}^{n}X_i\right] = \frac{1}{n}\sum_{i=1}^{n}E[X_i] = \frac{1}{n}\sum_{i=1}^{n}\mu = \frac{1}{n}n\mu = \mu$

$Var[\hat{\mu}] = Var[\bar{X}] = Var\left[\frac{1}{n}\sum_{i=1}^{n}X_i\right] = \frac{1}{n^2}Var\left[\sum_{i=1}^{n}X_i\right] = \frac{1}{n^2}\sum_{i=1}^{n}Var[X_i] = \frac{1}{n^2}\sum_{i=1}^{n}\sigma^2 =$

$\frac{1}{n^2}n\sigma^2 = \frac{1}{n}\sigma^2 = \frac{\sigma^2}{n}$

$Var[\hat{\sigma}^2] = Var\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2\right] = \frac{1}{(n-1)^2}Var\left[\sum_{i=1}^{n}(X_i-\bar{X})^2\right] = \frac{\mu_4-\sigma^4}{n} + \frac{2\sigma^4}{n(n-1)}$

$E[\hat{\sigma}^2] = E\left[\frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2\right] = \frac{1}{n-1}E\left[\sum_{i=1}^{n}(X_i-\bar{X})^2\right] = \frac{n}{n-1}E\left[\frac{1}{n}\sum_{i=1}^{n}(X_i-\bar{X})^2\right] =$

$\frac{n}{n-1}\left(1-\frac{1}{n}\right)\sigma^2 = \frac{n}{n-1}\left(\frac{n-1}{n}\right)\sigma^2 = \sigma^2$

$Cov[\hat{\mu},\hat{\sigma}^2] = Cov\left[\bar{X}, \frac{1}{n-1}\sum_{i=1}^{n}(X_i-\bar{X})^2\right] = Cov[\bar{X}, S^2] = \frac{\mu_3}{n}$

∴

3a ② continued / $E[\hat{\phi}] = E[h(\hat{\mu}, \hat{\sigma}^2)] \approx$

$$2\frac{1}{\sigma^2}E[\hat{\mu}] + \mu\frac{1}{\sigma^6}E[\hat{\sigma}^4] - 2\mu\frac{1}{\sigma^4}E[\hat{\sigma}^2] + \mu\frac{1}{\sigma^2}E[1] - \frac{1}{\sigma^2}E[\hat{\mu}\hat{\sigma}^2]$$

$$= 2\frac{1}{\sigma^2}E[\hat{\mu}] + \mu\frac{1}{\sigma^6}E[\hat{\sigma}^4] - 2\mu\frac{1}{\sigma^4}E[\hat{\sigma}^2] + \mu\frac{1}{\sigma^2} - \frac{1}{\sigma^2}E[\hat{\mu}\hat{\sigma}^2] =$$

$$2\frac{1}{\sigma^2}\mu + \mu\frac{1}{\sigma^6}E[\hat{\sigma}^2\hat{\sigma}^2] - 2\mu\frac{1}{\sigma^4}\sigma^2 + \mu\frac{1}{\sigma^2} - \frac{1}{\sigma^2}E[\hat{\mu}\hat{\sigma}^2] =$$

$$2\mu\frac{1}{\sigma^2} + \mu\frac{1}{\sigma^6}E[\hat{\sigma}^2\hat{\sigma}^2] - 2\mu\frac{1}{\sigma^2} + \mu\frac{1}{\sigma^2} - \frac{1}{\sigma^2}E[\hat{\mu}\hat{\sigma}^2] =$$

$$\mu\frac{1}{\sigma^2} + \mu\frac{1}{\sigma^6}E[\hat{\sigma}^2\hat{\sigma}^2] - \frac{1}{\sigma^2}E[\hat{\mu}\hat{\sigma}^2] =$$

$$\mu\frac{1}{\sigma^2} + \mu\frac{1}{\sigma^6}E[\hat{\sigma}^2]E[\hat{\sigma}^2] - \frac{1}{\sigma^2}E[\hat{\mu}]E[\hat{\sigma}^2] =$$

$$\mu\frac{1}{\sigma^2} + \mu\frac{1}{\sigma^6}\sigma^2\sigma^2 - \frac{1}{\sigma^2}\mu\sigma^2 =$$

$$\mu\frac{1}{\sigma^2} + \mu\frac{1}{\sigma^2} - \mu =$$

$$2\mu\frac{1}{\sigma^2} - \mu \quad \therefore$$

$$E[\hat{\phi}] \approx 2\mu\frac{1}{\sigma^2} - \mu \quad \therefore$$

$$\text{Bias}(\hat{\phi}) = E[\hat{\phi}] - \phi = E[\hat{\phi}] - \frac{\mu}{\sigma^2} =$$

$$2\mu\frac{1}{\sigma^2} - \mu - \mu\frac{1}{\sigma^2} =$$

$$\mu\frac{1}{\sigma^2} - \mu = \mu\left(\frac{1}{\sigma^2} - 1\right) = \mu\left(\frac{1}{\sigma^2} - \frac{\sigma^2}{\sigma^2}\right) = \mu\left(\frac{1-\sigma^2}{\sigma^2}\right)$$

$$= \mu\frac{1}{\sigma^2} - \mu\frac{\sigma^2}{\sigma^2} = \frac{1}{\sigma^2}(\mu - \mu\sigma^2)$$

3a checking: $\frac{2\mu\sigma_0^4 - \mu\sigma^2\sigma_0^2 + (\sigma^2-\sigma_0^2)^2\mu_0}{\sigma_0^6} =$

$2\mu\frac{\sigma_0^4}{\sigma_0^6} - \mu\sigma^2\frac{\sigma_0^2}{\sigma_0^6} + (\sigma^2-\sigma_0^2)^2\mu_0\frac{1}{\sigma_0^6} =$

$2\mu\frac{1}{\sigma_0^2} - \mu\sigma^2\frac{1}{\sigma_0^4} + (\sigma^4 - 2\sigma^2\sigma_0^2 + \sigma_0^4)\mu_0\frac{1}{\sigma_0^6} =$

$2\mu\frac{1}{\sigma_0^2} - \mu\sigma^2\frac{1}{\sigma_0^4} + \sigma^4\mu_0\frac{1}{\sigma_0^6} - 2\sigma^2\sigma_0^2\mu_0\frac{1}{\sigma_0^6} + \sigma_0^4\mu_0\frac{1}{\sigma_0^6} =$

$2\mu\frac{1}{\sigma_0^2} - \mu\sigma^2\frac{1}{\sigma_0^4} + \sigma^4\mu_0\frac{1}{\sigma_0^6} - 2\sigma^2\mu_0\frac{1}{\sigma_0^4} + \mu_0\frac{1}{\sigma_0^2} =$

$\cancel{2\mu\frac{1}{\sigma_0}}$ $\cancel{2\mu}$

$2\mu\frac{1}{\sigma_0^2} + \sigma^4\mu_0\frac{1}{\sigma_0^6} - 2\sigma^2\mu_0\frac{1}{\sigma_0^4} + \mu_0\frac{1}{\sigma_0^2} - \mu\sigma^2\frac{1}{\sigma_0^4}$

3a redo

$h(\mu,\sigma^2) \approx 2\mu\frac{1}{\hat\sigma^2} - \sigma^2\hat\mu\frac{1}{\hat\sigma^4} + \hat\mu\frac{1}{\hat\sigma^2} + \sigma^4\hat\mu\frac{1}{\hat\sigma^6} - 2\sigma^2\hat\mu\frac{1}{\hat\sigma^4} +$
$\hat\mu\frac{1}{\hat\sigma^2} - \mu\sigma^2\frac{1}{\hat\sigma^4} + \sigma^2\hat\mu\frac{1}{\hat\sigma^4} - \hat\mu\frac{1}{\hat\sigma^2} =$

$2\mu\frac{1}{\hat\sigma^2} - \sigma^2\hat\mu\frac{1}{\hat\sigma^4} + \sigma^4\hat\mu\frac{1}{\hat\sigma^6} - 2\sigma^2\hat\mu\frac{1}{\hat\sigma^4} +$
$\hat\mu\frac{1}{\hat\sigma^2} - \mu\sigma^2\frac{1}{\hat\sigma^4} + \sigma^2\hat\mu\frac{1}{\hat\sigma^4} =$

$2\mu\frac{1}{\hat\sigma^2} + \sigma^4\hat\mu\frac{1}{\hat\sigma^6} - 2\sigma^2\hat\mu\frac{1}{\hat\sigma^4} + \hat\mu\frac{1}{\hat\sigma^2} - \mu\sigma^2\frac{1}{\hat\sigma^4} \approx h(\mu,\sigma^2)$ $\therefore$

$h(\hat\mu,\hat\sigma) \approx 2\hat\mu\frac{1}{\sigma^2} + \hat\sigma^4\mu\frac{1}{\sigma^6} - 2\hat\sigma^2\mu\frac{1}{\sigma^4} + \mu\frac{1}{\sigma^2} - \hat\mu\hat\sigma^2\frac{1}{\sigma^4}$ $\therefore$

$\hat\theta = \frac{\hat\mu}{\hat\sigma^2}$ $\quad \hat\theta = h(\hat\mu,\hat\sigma^2) = \frac{\hat\mu}{\hat\sigma^2}$ $\quad \text{Bias}(\hat\theta) = E[\hat\theta] - \theta = E[\hat\theta] - \frac{\mu}{\sigma^2}$ $\therefore$

$E[\hat\theta] = E[h(\hat\mu,\hat\sigma^2)] \approx E\left[2\hat\mu\frac{1}{\sigma^2} + \hat\sigma^4\mu\frac{1}{\sigma^6} - 2\hat\sigma^2\mu\frac{1}{\sigma^4} + \mu\frac{1}{\sigma^2} - \hat\mu\hat\sigma^2\frac{1}{\sigma^4}\right] =$

$E\left[2\hat\mu\frac{1}{\sigma^2}\right] + E\left[\hat\sigma^4\mu\frac{1}{\sigma^6}\right] + E\left[-2\hat\sigma^2\mu\frac{1}{\sigma^4}\right] + E\left[\mu\frac{1}{\sigma^2}\right] + E\left[-\hat\mu\hat\sigma^2\frac{1}{\sigma^4}\right] =$

$2\frac{1}{\sigma^2}E[\hat\mu] + \mu\frac{1}{\sigma^6}E[\hat\sigma^4] - 2\mu\frac{1}{\sigma^4}E[\hat\sigma^2] + \mu\frac{1}{\sigma^2}E[1] - \frac{1}{\sigma^4}E[+\hat\mu\hat\sigma^2] =$

$\cancel{2\frac{1}{\sigma^2}\mu + \mu\frac{1}{\sigma^6}E[\sigma^4] - 2\mu\frac{1}{\sigma^4}\sigma^2}$

$$2\frac{1}{\sigma^2}\mu + \mu\frac{1}{\sigma^6}E[\hat{\sigma}^4] - 2\mu\frac{1}{\sigma^4}E[\hat{\sigma}^2] + \mu\frac{1}{\sigma^2} - \frac{1}{\sigma^4}E[\hat{\mu}\hat{\sigma}^2] =$$

$$2\mu\frac{1}{\sigma^2} + \mu\frac{1}{\sigma^6}E[\hat{\sigma}^4] - 2\mu\frac{1}{\sigma^4}\sigma^2 + \mu\frac{1}{\sigma^2} - \frac{1}{\sigma^4}E[\hat{\mu}\hat{\sigma}^2] =$$

$$3\mu\frac{1}{\sigma^2} + \mu\frac{1}{\sigma^6}E[\hat{\sigma}^4] - 2\mu\frac{1}{\sigma^2} - \frac{1}{\sigma^4}E[\hat{\mu}\hat{\sigma}^2] =$$

$$\mu\frac{1}{\sigma^2} + \mu\frac{1}{\sigma^6}E[\hat{\sigma}^4] - \frac{1}{\sigma^4}E[\hat{\mu}\hat{\sigma}^2] =$$

$$\mu\frac{1}{\sigma^2} + \mu\frac{1}{\sigma^6}E[\hat{\sigma}^2\hat{\sigma}^2] - \frac{1}{\sigma^4}E[\hat{\mu}\hat{\sigma}^2] =$$

$$\mu\frac{1}{\sigma^2} + \mu\frac{1}{\sigma^6}E[\hat{\sigma}^2]E[\hat{\sigma}^2] - \frac{1}{\sigma^4}E[\hat{\mu}]E[\hat{\sigma}^2] =$$

$$\mu\frac{1}{\sigma^2} + \mu\frac{1}{\sigma^6}\sigma^2\sigma^2 - \frac{1}{\sigma^4}\mu\sigma^2 =$$

$$\mu\frac{1}{\sigma^2} + \mu\frac{1}{\sigma^2} - \frac{1}{\sigma^2}\mu =$$

$$\mu\frac{1}{\sigma^2} + \mu\frac{1}{\sigma^2} - \mu\frac{1}{\sigma^2} =$$

$$\mu\frac{1}{\sigma^2} \quad \therefore$$

$$E[\hat{\phi}] \approx \mu\frac{1}{\sigma^2} \quad \therefore$$

$$E[\hat{\phi}] = \mu\frac{1}{\sigma^2} \quad \therefore$$

$$Bias(\hat{\phi}) = E[\hat{\phi}] - \phi = E[\hat{\phi}] - \frac{\mu}{\sigma^2} =$$

$$\mu\frac{1}{\sigma^2} - \mu\frac{1}{\sigma^2} = \underline{\underline{0}}$$

$$E[\hat{\mu}\hat{\sigma}^2] = ?$$

$$E[\hat{\sigma}^4] = ?$$

$\mu_3 = E[(X-\mu)^3]$

$E[XY] = E[X]E[Y]$

$\mu_3 = \frac{1}{n}\sum_{i=1}^{n} X_i^3$

$\mu_2 = E[X^2]$

$var = E[X^2] - E[X]^2$
$= \mu_2 - \mu_1^2$

$E[\hat{\mu}\hat{\sigma}^2]$

$E[\hat{\sigma}^4]$

3a attempting new/ $E[\hat{\mu}] = \mu$ $\quad E[\hat{\sigma}^2] = \sigma^2$ $\therefore$

$\hat{\mu}$ & σ^2 are unbiased estimators $E[\hat{\mu} - \mu] = E[\hat{\mu}] - E[\mu] =$

$E[\hat{\mu}] - \mu = \mu - \mu = 0$

$\hat{\varnothing} = \frac{\hat{\mu}}{\hat{\sigma}^2}$ ~~tw~~

$\cancel{h(\hat{\mu}, \hat{\sigma}^2) \approx}$ u, v unbiased $\therefore E[u - u_0] = E[v - v_0] = 0$

$h(u, v) \approx \ldots$

$$E[h(u,v)] \approx E[h(u_0,v_0)] + E\left[(u-u_0)\frac{\partial h}{\partial u}(u_0,v_0)\right] +$$

$$E\left[(v-v_0)\frac{\partial h}{\partial v}(u_0,v_0)\right] + E\left[\frac{(u-u_0)^2}{2}\frac{\partial^2 h}{\partial u^2}(u_0,v_0)\right] +$$

$$E\left[\frac{(v-v_0)^2}{2}\frac{\partial^2 h}{\partial v^2}(u_0,v_0)\right] + E\left[(u-u_0)(v-v_0)\frac{\partial^2 h}{\partial u \partial v}(u_0,v_0)\right] =$$

$$h(u_0,v_0) + E[(u-u_0)]\frac{\partial h}{\partial u}(u_0,v_0) +$$

$$E[(v-v_0)]\frac{\partial h}{\partial v}(u_0,v_0) + \frac{1}{2}E[(u-u_0)^2]\frac{\partial^2 h}{\partial u^2}(u_0,v_0) +$$

$$\frac{1}{2}E[(v-v_0)^2]\frac{\partial^2 h}{\partial v^2}(u_0,v_0) + E[(u-u_0)(v-v_0)]\frac{\partial^2 h}{\partial u \partial v}(u_0,v_0) =$$

$\cancel{h(u_0,v_0) + \frac{1}{2}E[(}$

$\cancel{h(u_0,v_0) + \frac{1}{2}Var}$

$$E[\hat{\varnothing}] = E[h(\hat{u},\hat{v})] \approx$$

$$E[h(u,v)] + \frac{1}{2}E[(v_0-v)^2]\frac{\partial^2 h}{\partial v^2}(u_0,v_0) + E[(u_0-u)(v_0-v)]\frac{\partial^2 h}{\partial u \partial v}(u_0,v_0)$$

$$= \varnothing + \frac{1}{2}var(\hat{\sigma}^2)\frac{\partial^2 h}{\partial v^2}(u_0,v_0) + Cov(\hat{\mu},\hat{\sigma}^2)\frac{\partial^2 h}{\partial u \partial v}(u_0,v_0)$$

$$= \varnothing + \frac{1}{2}var(\hat{\sigma}^2)\,2\hat{\mu}\frac{1}{\hat{\sigma}^6} + \left[Cov(\hat{\mu},\hat{\sigma}^2)\left(-\frac{1}{\hat{\sigma}^4}\right)\right]$$

$$= \varnothing + \hat{\mu}\frac{1}{\hat{\sigma}^6}var(\hat{\sigma}^2) - \frac{1}{\hat{\sigma}^4}Cov(\hat{\mu},\hat{\sigma}^2)$$

$\hat{\mu} = u \qquad \hat{\sigma} = v \qquad u_0 = \mu \qquad v_0 = \sigma^2$

$\hat{\mu} = u \qquad \hat{\sigma}^2 = v \qquad \mu = u_0 \qquad \sigma^2 = v_0$

$\frac{\partial^2 h}{\partial v^2}(u_0, v_0) = \frac{\partial^2 h}{\partial \hat{\sigma}^4}(\mu, \sigma^2)$

\3b/ $Bias(\hat{\phi}) \approx \hat{\mu} \frac{1}{\hat{\sigma}^6} Var[\hat{\sigma}^2] - \frac{1}{\hat{\sigma}^4} Cov(\hat{\mu}, \hat{\sigma}^2) \quad \therefore$

bias-corrected estimator for ϕ is:

$\tilde{\phi} = \hat{\phi} - Bias(\hat{\phi}) = \hat{\phi} - \frac{\hat{\mu}}{\hat{\sigma}^6} Var[\hat{\sigma}^2] + \frac{1}{\hat{\sigma}^4} Cov(\hat{\mu}, \hat{\sigma}^2) =$

$\frac{\hat{\mu}}{\hat{\sigma}^2} - \frac{1}{\hat{\sigma}^6} Var[\hat{\sigma}^2] + \frac{1}{\hat{\sigma}^4} Cov(\hat{\mu}, \hat{\sigma}^2)$

$m_3 = \frac{1}{n} \sum_{i=1}^{n} [(X_i - \bar{X})^3] \qquad m_4 = \frac{1}{n} \sum_{i=1}^{n} (X_i - \bar{X})^4$

$Cov(\hat{\mu}, \hat{\sigma}^2) = \frac{m_3}{n}$

$Var[\hat{\sigma}^2] = [m_4 - (n-3)\hat{\sigma}^4/(n-1)]/n = [m_4 - (n-3)(\hat{\sigma}^2)^2/(n-1)]/n$

$\therefore \tilde{\phi} = \frac{\hat{\mu}}{\hat{\sigma}^2} - \frac{\hat{\mu}}{(\hat{\sigma}^2)^3} Var[\hat{\sigma}^2] - \frac{1}{(\hat{\sigma}^2)^2} Cov(\hat{\mu}, \hat{\sigma}^2)$

\\ week 1 /

\\ formative / gamma distri $Ga(2, \phi)$

• $f(x;\phi) = \phi^2 x e^{-\phi x}$ for $x>0$ where $\phi>0$ is a param

Let $Y = \sum_{i=1}^{n} X_i$ then Y is a $Ga(2n, \phi)$ r.v. with density

$f(y;\phi) = \dfrac{\phi^{2n}}{\Gamma(2n)} y^{2n-1} e^{-\phi y}$ for $y>0$

\\ 1a / $E(X) = \int_0^\infty x f(x;\phi)\,dx \overset{①}{=} \int_0^\infty \phi^2 x^2 e^{-\phi x}\,dx$ gamma func

$\therefore$ let $y = \phi x$ $\therefore E(X) = \dfrac{1}{\phi}\int_0^\infty y^2 e^{-y}\,dy \overset{②}{=} \dfrac{1}{\phi}\Gamma(3) = \dfrac{1}{\phi} 2! = \dfrac{2}{\phi}$

$\therefore \hat{\phi}$ satisfies $\bar{X} = \dfrac{2}{\hat{\phi}}$ ①, ie $\hat{\phi} = \dfrac{2}{\bar{X}}$ ①

\\ 1b / let $\theta = 2/\phi$ check $\hat{\theta} = 2/\hat{\phi} = \bar{X}$ ①

have $E(\hat{\theta}) = E(\bar{X}) = E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) \overset{①}{=} \frac{1}{n}\sum_{i=1}^{n}\left(\frac{2}{\phi}\right) = \frac{2}{\phi} = \theta$

$\therefore$ bias is $E(\hat{\theta}) - \theta = \theta - \theta = 0$ ①

Now $E(X^2) = \int_0^\infty x^2 f(x;\phi)\,dx = \cdots = \dfrac{6}{\phi^2}$ ① $\therefore$

$var(X) = E(X^2) - E(X)^2 \overset{①}{=} \dfrac{2}{\phi^2}$ ① & $var(\hat{\theta}) = var\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n} var(X_i)$ ①

(by independence) $= \dfrac{2}{n\phi^2} = \dfrac{\theta^2}{2n}$ ① $\therefore var(\hat{\theta}) \to 0$ ① & bias$(\hat{\theta}) \to 0$

as $n \to \infty$ ① $\therefore \hat{\theta}$ is consistent ①

$var[h(\hat{\theta})] \approx [h'(\theta)]^2 var(\hat{\theta})$

$bias[h(\hat{\theta})] \approx \frac{1}{2} h''(\theta) var(\hat{\theta})$

\\ 1c / $var(\hat{\phi})$? ① $\hat{\phi} \overset{②}{=} \dfrac{2}{\hat{\theta}}$, $var(\hat{\theta}) = \dfrac{\theta^2}{2n} = \dfrac{2}{n\phi^2}$

$var(\hat{\phi}) \approx \left(-\dfrac{2}{\theta^2}\right)^2 \dfrac{\theta^2}{2n} = \dfrac{2}{n\theta^2} = \dfrac{\phi^2}{2n}$ ①

\\ 1d / $bias(\hat{\phi}) \approx \dfrac{1}{2}\,\overset{①}{\dfrac{4}{\theta^3}}\,\overset{①}{\dfrac{\theta^2}{2n}} = \dfrac{1}{n\theta} = \dfrac{\phi}{2n}$ ①

a bias-corrected estimator is $\tilde{\phi} = \hat{\phi} - \dfrac{\hat{\phi}}{2n} \overset{①}{=} \left(1 - \dfrac{1}{2n}\right)\hat{\phi}$

2a/ $f(x;\phi) = \phi^2 x e^{-\phi x}$, $x > 0$ CRLB for estimator of ϕ?

CRLB ~~I(theta)~~ $\therefore L(\phi) = \prod_{i=1}^{n} f(x_i;\phi) \overset{①}{=} \prod_{i=1}^{n} \phi^2 x_i e^{-\phi x_i}$ ②

$\phi^{2n}\left(\prod_{i=1}^{n} x_i\right) e^{-\phi \sum_{i=1}^{n} x_i} = \phi^{2n}\left(\prod_{i=1}^{n} x_i\right) e^{-\phi n \bar{y}}$ $\therefore$

$l(\phi) = 2n \log \phi + \sum_{i=1}^{n} \ln x_i - \phi \sum_{i=1}^{n} x_i$ ① $\therefore$

$l'(\phi) = \frac{2n}{\phi} + - \sum_{i=1}^{n} x_i$ ① $\therefore$ $l''(\phi) = -\frac{2n}{\phi^2}$ ① $\therefore$

$I(\phi) = -E\left[l''(\phi)\right] \overset{①}{=} \frac{2n}{\phi^2}$ ① $\therefore$ CRLB is $\frac{1}{I(\phi)} = \frac{\phi^2}{2n}$ ①

2b/ Score func is $U(\mu) = \frac{n}{\sigma^2}(\bar{x} - \mu)$.

an unbiased & efficient estimator exists for ϕ iff

$u = b(\hat{\phi} - \phi)$ ①

now $u = \frac{2n}{\phi} - \sum_{i=1}^{n} x_i$

$= \frac{?}{\phi} - \left(\sum_{i=1}^{n} x_i - \frac{2n}{\phi}\right)$ $\therefore$ no unbiased & efficient estimator exists for ϕ ①

4a/ $\phi = 1$ $\therefore$ >> rgamma (100, 2, 1) >> rgamma (n, 2, phi)

```
>> n=seq(10,100,10)      >>phi=1
>>n=1      >> x=rgamma(n,2,phi)
>> ml = 2/mean(x)
>> bc =(1-1/(2*n))*ml      #bias corrected#      ml=bc=numeric(nsim)
>>nsim=10000      >> for(i in 1:nsim){
>>x=rgamma(n,2,phi)      >> ml[i]=2/mean(x)
>> bc[i]=(1-1/(2*n))*ml[i]      >>}
>>mean(ml)-phi
>>mean(bc)-phi
>> sd(ml)      >> sd(bc)
#now do: #  >> n=seq(10,100,10)
>> for (j in 1:length n(n)) {
```

\2c/ $H_0: \phi = \phi_0$ $H_1: \phi < \phi_0$ $X_1, \dots, X_n$ iid $f(x;\phi) = \phi^2 x e^{-\phi x}$, $x > 0$

Consider $H_1': \phi < \phi_1$ where $\phi_1 < \phi_0$

I have $L(\phi) = \phi^{2n} \left(\prod_{i=1}^{n} x_i \right) e^{-\phi \sum_{i=1}^{n} x_i}$ ∴ $\dfrac{L(\phi_1)}{L(\phi_0)} = \left(\dfrac{\phi_1}{\phi_0} \right)^{2n} e^{(\phi_0 - \phi_1) \sum_{i=1}^{n} x_i}$ ① ①

this ratio is large ① when $\sum_{i=1}^{n} x_i$ is large ① so a most powerful test has critical region of a form $\{ x : \sum_{i=1}^{n} x_i > c \}$ this region doesn't depend on ϕ_1, so it's UMP for H_1 ①

\2d/ for a test of size α, need

$\Pr\left(\sum_{i=1}^{n} X_i > c ; \phi = \phi_0 \right) = \alpha$ ①

\3a/ $T_1 = \phi Y$ where $Y \sim Ga(2n, \phi)$ with pdf

$$f(y;\phi) = \frac{\phi^{2n}}{\Gamma(2n)} y^{2n-1} e^{-\phi y}, \quad y > 0$$

have $\Pr(T_1 \le t) = \Pr(\phi Y \le t) = \Pr(Y \le t/\phi) = \int_0^{t/\phi} f(y;\phi)\, dy$ ①

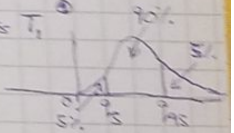
$= \int_0^{t/\phi} \frac{\phi^{2n}}{\Gamma(2n)} y^{2n-1} e^{-\phi y}\, dy$ let $s = \phi y$

$= \int_0^{t} \frac{\phi \cdot s^{2n-1}}{\Gamma(2n)} e^{-s} \frac{ds}{\phi} = \int_0^{t} \frac{s^{2n-1}}{\Gamma(2n)} e^{-s}\, ds$ ① this cdf is indep of ϕ ∴ T_1 is a pivot ①

\3b/ $T_1 = \phi Y \sim Ga(2n, 1)$ ① ∴ $\Pr(q_{0.05} < T_1 < q_{0.95}) = 0.9$ ∴

$\Pr(q_{0.05} < \phi Y < q_{0.95}) = 0.9$ ∴ $\Pr(q_{0.05}/Y < \phi < q_{0.95}/Y) = 0.9$ ∴

~~CI: of a 90% CI: $(q_{0.05}/Y,\ q_{0.95})$~~ let q_p be a p-quantile of T_1 ①



where $\Pr(q_{0.05} < \phi Y) = 0.05$ & $\Pr(\phi Y < q_{0.95}) = 0.05$

$\Pr(q_5 < \phi Y < q_{95}) = 0.9$ ① ∴ $\Pr\left(\frac{q_5}{Y} < \phi < \frac{q_{95}}{Y} \right) = 0.9$ ①

So a 90% confidence interval for ϕ is $(q_5/Y,\ q_{95}/Y)$

week 5 / ~~3∅~~ $Ga(2n, \phi)$ density: $f(y;\phi) = \frac{\phi^{2n}}{\Gamma(2n)} y^{2n-1} e^{-\phi y}$ for $y>0$

$\therefore Ga(2,\phi)$ density $\frac{\phi^2}{\Gamma(2)} y^{2-1} e^{-\phi y}$ for $y>0$ is

$f(y;\phi) = \frac{\phi^2}{(2-1)!} y^1 e^{-\phi y} = \phi^2 y e^{-\phi y}$ for $y>0$

$\therefore Pr(X_0 \le x) = \int_0^x \phi^2 y e^{-\phi y} dy = \phi^2 \int_0^x y e^{-\phi y} dy$

$\int uv\,dx = u(\int v\,dx) - \int u'(\int v\,dx)\,dx \qquad \int v \frac{du}{dx} dx = uv - \int u \frac{dv}{dx} dx$

$f(x;\phi) = \phi^2 x e^{-\phi x} \quad x>0$

$Pr(X_0 \le x) = \int_0^x f(y;\phi) dy = \int_0^x \phi^2 y e^{-\phi y} dy$ let $z = \phi y \therefore$

$= \int_0^{\phi x} \phi z e^{-z} \frac{dz}{\phi} = \int_0^{\phi z} z e^{-z} dz = \left[-ze^{-z}\right]_0^{\phi x} + \int_0^{\phi z} e^{-z} dz =$

$= -\phi x e^{-\phi x} - \left[e^{-z}\right]_0^{\phi z} = 1 - (1+\phi x) e^{-\phi x}$

let $T_2 = X_0 / \bar{X} \therefore Pr(T_2 \le t) = Pr(X_0/\bar{X} \le t) =$

$Pr(X_0 \le t \frac{Y}{n})$ where $Y = \sum_{i=1}^n X_i$

$= \int_0^\infty Pr(X_0 \le ty/n) f_Y(y;\phi) dy = \int_0^\infty \left[1 - (1+\phi \frac{ty}{n}) e^{-\phi t y/n}\right] \frac{\phi^{2n}}{\Gamma(2n)} e^{2n-1} e^{-\phi y} dy$

(let $z = \phi y$) $= \int_0^\infty \left[1 - (1+\frac{tz}{n}) e^{-zt/n}\right] \frac{\phi z^{2n-1}}{\Gamma(2n)} e^{z} \frac{dz}{\phi} =$

$\int_0^\infty [1 - (1 -$

this is independent of ϕ and so T_2 is ancillary

$\bar{X} = \frac{1}{n} \sum_{i=1}^n X_i$ is a random quantity

$E(X) = \frac{2}{\phi}$ is not random quantity

Law of total probability $P(A) = \sum_{i=1}^k Pr(A|B_i) Pr(B_i) = \sum_{i=1}^k Pr(A \cap B_i) \therefore$

$Pr(X \le Y) = \int Pr(X \le y | Y = y) \cdot Pr(Y = y)\, dy$

$Pr(X \le Y) = \int \underbrace{Pr(X \le y | Y = y)}_{= Pr(X \le y) \text{ if } X \,\&\, Y \text{ are indep}} \overbrace{Pr(Y = y)}^{\text{pdf of } Y} dy$

\week 5/ \3d/ $T_2 = X_0 / \bar{X}$ 90% PI for X_0?

let q_5 & q_{95} be the ~~95%~~ 5% & 95% quantiles of T_2

then T_2 $Pr(q_5 \le T_2 \le q_{95}) = 0.9$ iff

$Pr(q_5 \le \frac{X_0}{\bar{X}} \le q_{95}) = 0.9$ iff

$Pr(q_5 \bar{X} \le X_0 \le q_{95} \bar{X}) = 0.9$

so a 90% PI for X_0 is $(q_5 \bar{X}, q_{95} \bar{X})$

\week 1/
\Preparatory ex/
\1a/ $f(x;\theta) = \begin{cases} \theta x^{\theta-1} & 0<x<1, \theta>0 \\ 0 & \text{elsewhere} \end{cases}$

$\therefore L(\theta) = \prod_{i=1}^{n} \theta x_i^{\theta-1} = \theta^n \prod_{i=1}^{n} x_i^{\theta-1} = \theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1} = \prod_{i=1}^{n} f(x_i;\theta)$

$\therefore l(\theta) = \log L(\theta) = \log\left[\theta^n \left(\prod_{i=1}^{n} x_i\right)^{\theta-1}\right] = \ln \theta^n + \ln\left[\left(\prod_{i=1}^{n} x_i\right)^{\theta-1}\right] =$
$n \ln \theta + (\theta-1) \ln\left[\prod_{i=1}^{n} x_i\right] = n \ln \theta + (\theta-1) \sum_{i=1}^{n} \ln x_i$

$\therefore l'(\theta) = n \frac{1}{\theta} + \sum_{i=1}^{n} \ln x_i$ ✓

$\therefore l''(\theta) = -n \frac{1}{\theta^2} < 0 \quad \therefore \quad l'(\hat{\theta}) = 0 \quad \therefore \quad n \frac{1}{\hat{\theta}} + \sum_{i=1}^{n} \ln x_i = 0 \quad \therefore$

$n \frac{1}{\hat{\theta}} = -\sum_{i=1}^{n} \ln x_i \quad \therefore \quad \frac{-n}{\sum_{i=1}^{n} \ln x_i} = \hat{\theta}$ ✓

$E(X) = \int_0^1 x f(x;\theta) dx = \int_0^1 x \theta x^{\theta-1} dx = \int_0^1 \theta x^{\theta} dx = \left[\frac{\theta}{\theta+1} x^{\theta+1}\right]_0^1 =$

$\frac{\theta}{\theta+1}\left[1^{\theta+1} - 0^{\theta+1}\right] = \frac{\theta}{\theta+1}[1-0] = \frac{\theta}{\theta+1}$ $\therefore$ equating to $\bar{X}$ $\therefore$

$\bar{X} = E(X) \therefore \bar{X} = \frac{\hat{\theta}}{\hat{\theta}+1} \therefore \hat{\theta}\bar{X} + \bar{X} = \hat{\theta} \quad \therefore \bar{X} = \hat{\theta} - \hat{\theta}\bar{X} = \hat{\theta}(1-\bar{X}) \therefore$

$\frac{\bar{X}}{1-\bar{X}} = \hat{\theta}$ ✓

\formative ex's/ \1a/ $x > 0 \quad \therefore 0 < x < \infty \quad \therefore E(X) = \int_0^\infty x f(x;\phi) dx$

$= \int_0^\infty x \phi^2 x e^{-\phi x} dx = \phi^2 \int_0^\infty x^2 e^{-\phi x} = \int_0^\infty (\phi x)^2 e^{-\phi x} dx$

$\frac{1}{\phi} dy = dx \therefore$

let $y = \phi x \quad \therefore \quad x=0 \to y=0, \; x=\infty \to y=\infty \quad \therefore \frac{1}{\phi}\int_0^\infty y^2 e^{-y} dy =$

$\frac{1}{\phi}\int_0^\infty y^{3-1} e^{-y} dy = \frac{1}{\phi}\Gamma(3) = \frac{1}{\phi} 2! = \frac{1}{\phi} 2 = \frac{2}{\phi}$ $\therefore$ equating to $\bar{X}$ $\therefore$

$\bar{X} = E(X) \therefore \bar{X} = \frac{2}{\hat{\phi}} \therefore \hat{\phi} = \frac{2}{\bar{X}}$

\1b/ let $\theta = 2/\phi$ check $\hat{\theta} = 2/\hat{\phi} = \bar{X}$ $\therefore$ have $E(\hat{\theta}) = E(\bar{X}) =$

$E\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n} E\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n}\sum_{i=1}^{n} E(X_i) = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{2}{\phi}\right) = \frac{1}{n} n \frac{2}{\phi} = \frac{2}{\phi} = \theta$

$\therefore \text{Bias}(\theta) = E(\hat{\theta}) - \theta = \theta - \theta = 0$

$\therefore$ ~~E(ø²~~ $E(X^2) = \int_0^\infty x^2 f(x;\phi)\,dx = \int_0^\infty x^2 \phi^2 x e^{-\phi x}\,dx = \int_0^\infty (\phi^2 x^3) e^{-\phi x}\,dx =$

$\left[(\phi^2 x^3)\left(-\frac{1}{\phi} e^{-\phi x}\right)\right]_0^\infty - \int_0^\infty (\phi^2 3x^2)\left(-\frac{1}{\phi} e^{-\phi x}\right) dx =$

$\left[(\phi^2 x^3)\left(\frac{1}{-\phi} e^{-\phi x}\right)\right]_0^\infty + 3\frac{1}{\phi}\int_0^\infty \phi^2 x^2 e^{-\phi x}\,dx = \dots = \frac{6}{\phi^2}$

$\therefore \text{var}(X) = E(X^2) - E(X)^2 = \frac{6}{\phi^2} - \left(\frac{2}{\phi}\right)^2 = \frac{2}{\phi^2}$ $\therefore$

$\text{var}(\hat{\theta}) = \text{var}(\bar{X}) = \text{var}\left(\frac{1}{n}\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\text{var}\left(\sum_{i=1}^{n} X_i\right) = \frac{1}{n^2}\sum_{i=1}^{n}\text{var}(X_i) =$

$\frac{1}{n^2}\sum_{i=1}^{n}\frac{2}{\phi^2} = \frac{1}{n^2} n \frac{2}{\phi^2} = \frac{2}{n\phi^2} = \frac{4}{2n\phi^2} = \frac{\theta^2}{2n}$ $\therefore$

$\text{var}(\hat{\theta}) \to 0$ as $n \to \infty$ $\therefore \hat{\theta}$ is consistent ✓

\1c/ $\text{var}(\hat{\phi})$? $\phi = \frac{2}{\theta}$ $\therefore$ $\hat{\phi} = \frac{2}{\hat{\theta}}$ $\therefore \text{var}(\hat{\theta}) = \frac{\theta^2}{2n} = \frac{2}{n\phi^2}$ $\therefore$

$\text{var}(\hat{\phi}) \approx \left(-\frac{2}{\theta^2}\right)^2 \frac{\theta^2}{2n} = \left\{\frac{4}{\theta^4}\frac{\theta^2}{2n} = \frac{2\theta^2}{n\theta^4}\right\} = \frac{2}{n\theta^2} = \frac{4}{2n\theta^2} = \frac{\phi^2}{2n}$

<learn delta method>

\1d/ $\text{Bias}(\hat{\phi}) = E(\hat{\phi}) - \phi = E(\hat{\phi}) - \frac{2}{\theta}$

$\text{bias}(\hat{\phi}) \approx \frac{1}{2}\frac{4}{\theta^3}\frac{\theta^2}{2n} = \frac{1}{n\theta} = \frac{\phi}{2n}$

a bias corrected estimator is $\tilde{\phi} = \hat{\phi} - \frac{\hat{\phi}}{2n} = \left(1 - \frac{1}{2n}\right)\hat{\phi}$

Delta Method: $h(\hat{\theta}) \approx h(\theta) + (\hat{\theta} - \theta)h'(\theta)$

$\therefore \text{var}[h(\hat{\theta})] \approx \text{var}[h(\theta) + (\hat{\theta} - \theta)h'(\theta)] = \text{var}[\hat{\theta}h'(\theta)] = [h'(\theta)]^2\text{var}(\hat{\theta})$

$h(\hat{\theta}) \approx h(\theta) + (\hat{\theta} - \theta)h'(\theta) + \frac{1}{2}(\hat{\theta} - \theta)^2 h''(\theta)$ $\therefore$

$E(\hat{\phi}) \approx h(\theta) + [E(\hat{\theta}) - \theta]h'(\theta) + \frac{1}{2}E[(\hat{\theta} - \theta)^2]h''(\theta) = \phi + \frac{1}{2}\text{var}(\hat{\theta})h''(\theta)$

$E(\hat{\phi}) - \phi \approx \frac{1}{2}\widehat{\text{var}}(\hat{\theta})h''(\hat{\theta})$

$\tilde{\phi} = \hat{\phi} - \frac{1}{2}\widehat{\text{var}}(\hat{\theta})h''(\hat{\theta})$

~~var[~~ is $\text{var}(\hat{\phi}) = \text{var}[h(\hat{\theta})] = \text{var}\left[\frac{2}{\hat{\theta}}\right] = [h'(\theta)]^2\text{var}(\hat{\theta}) = [h'(\theta)]^2\frac{\theta^2}{2n}$

$h(\hat{\theta}) = \frac{2}{\hat{\theta}}$ $\therefore h(\theta) = \frac{2}{\theta}$ $\therefore h'(\theta) = -\frac{2}{\theta^2}$ $\therefore \text{var}(\hat{\theta}) = \left(-\frac{2}{\theta^2}\right)^2\frac{\theta^2}{2n} = \frac{4}{2n\theta^2} = \frac{\phi^2}{2n}$

~~$\tilde{\phi} = \hat{\phi} - \frac{1}{2}\text{var}$~~ $E(\hat{\phi}) - \phi = \text{bias}(\hat{\phi}) = \phi + \frac{1}{2}\text{var}(\hat{\theta})h'' = \frac{1}{2}\frac{\phi^2}{2n}\left(\frac{4}{\theta^3}\right) = \frac{\phi}{2n}$

$\tilde{\phi} = \hat{\phi} - \frac{1}{2}\widehat{\text{var}}(\hat{\theta})h''(\hat{\theta}) = \phi - \frac{\phi}{2n} = \left(1 - \frac{1}{2n}\right)\phi$

Formative 2a/ $I(\phi) = -E\left(\frac{\partial^2 L}{\partial \phi^2}\right)$ $\quad var(\hat{\phi}) \geq I(\phi)^{-1}$

$f(x;\phi) = \phi^2 x e^{-\phi x}$ $\quad L(\phi) = \prod_{i=1}^{n} f(x_i;\phi) = \prod_{i=1}^{n} \phi^2 x_i e^{-\phi x_i} =$

$\phi^{2n} e^{\sum_{i=1}^{n}(-\phi x_i)} \prod_{i=1}^{n} x_i = \phi^{2n} e^{-\phi \sum_{i=1}^{n} x_i} \prod_{i=1}^{n} x_i = \phi^{2n} e^{-\phi n \bar{y}} \prod_{i=1}^{n} x_i$ ∴

$l(\phi) = \operatorname{Log} L(\phi) = \ln(\phi^{2n}) + \ln(e^{-\phi n \bar{y}}) + \ln\left(\prod_{i=1}^{n} x_i\right) =$

$2n \ln\phi + -\phi n \bar{y} + \sum_{i=1}^{n} \ln x_i$ ∴

$l'(\phi) = \frac{2n}{\phi} - n\bar{y}$ ∴ $l''(\phi) = \frac{-2n}{\phi^2}$ ∴ $I(\phi) = -E\left(\frac{-2n}{\phi^2}\right) = \frac{2n}{\phi^2}E(1) = \frac{2n}{\phi^2}$

∴ $CRLB = \frac{1}{I(\phi)} = \frac{\phi^2}{2n}$

2b/ Score func $U(\phi;X) = \frac{\partial l}{\partial \phi} = l'(\phi) = \frac{2n}{\phi} - n\bar{y} = U = \frac{2n}{\phi} - \sum_{i=1}^{n} x_i$

$= -\sum_{i=1}^{n} x_i + \frac{2n}{\phi} = \frac{n}{\phi^2}\left(-\frac{\phi^2}{n}\sum_{i=1}^{n} x_i + 2\phi\right) = -\frac{n}{\phi^2}(\phi^2 \bar{x} - 2\phi) \neq \frac{n}{\sigma^2}(\bar{x} - \mu)$ ∴

no unbiased & efficient estimator exists for ϕ

2c/ $H_0: \phi = \phi_0$ $\quad H_1: \phi < \phi_0$ $\quad X_1, \dots, X_n$ iid $f(x;\phi) = \phi^2 x e^{-\phi x}$, $x > 0$

consider $H_1': \phi < \phi_1$ where $\phi_1 < \phi_0$ have $L(\phi) = \phi^{2n}\left(\prod_{i=1}^{n} x_i\right) e^{-\phi \sum_{i=1}^{n} x_i}$

∴ $L(\phi_1) = \phi_1^{2n}\left(\prod_{i=1}^{n} x_i\right) e^{-\phi_1 \sum_{i=1}^{n} x_i}$ $\quad L(\phi_0) = \phi_0^{2n}\left(\prod_{i=1}^{n} x_i\right) e^{-\phi_0 \sum_{i=1}^{n} x_i}$ ∴

$\frac{L(\phi_1)}{L(\phi_0)} = \frac{\phi_1^{2n}\left(\prod_{i=1}^{n} x_i\right) e^{-\phi_1 \sum_{i=1}^{n} x_i}}{\phi_0^{2n}\left(\prod_{i=1}^{n} x_i\right) e^{-\phi_0 \sum_{i=1}^{n} x_i}} = \left(\frac{\phi_1}{\phi_0}\right)^{2n} e^{(\phi_0 - \phi_1)\sum_{i=1}^{n} x_i}$ this ratio is large

when $\sum_{i=1}^{n} x_i$ is large ∴ Z most powerful test has critical region of Z

form $\{x : \sum_{i=1}^{n} x_i > c\}$ this region doesn't depend on ϕ_1 ∴ Z

uniformly most powerful for H_1

2d/ $Ga(2n, \phi)$ is $f(y;\phi) = \frac{\phi^{2n}}{\Gamma(2n)} y^{2n-1} e^{-\phi y}$ ∴ for a test of size α

need $Pr\left(\sum_{i=1}^{n} x_i > c ; \phi = \phi_0\right) = \alpha$

3a/ $T_1 = \phi Y$ where $Y \sim Ga(2n, \phi)$ with pdf

$f(y;\phi) = \frac{\phi^{2n}}{\Gamma(2n)} y^{2n-1} e^{-\phi y}$, $y > 0$ have $Pr(T_1 \leq t) = Pr(\phi Y \leq t) = Pr\left(Y \leq \frac{t}{\phi}\right) =$

$\int_0^{t/\phi} f(y;\phi)\,dy = \int_0^{t/\phi} \frac{\phi^{2n}}{\Gamma(2n)} y^{2n-1} e^{-\phi y}\,dy$ let $s = \phi y$ ∴ $y = 0 \rightarrow s = 0$, $y = \frac{t}{\phi} \rightarrow s = t$

∴ $\frac{1}{\phi}\,ds = dy$ ∴ $\int_0^{t/\phi} \frac{(\phi y)^{2n}}{\Gamma(2n)} y^{-1} e^{-\phi y}\,dy = \int_0^{t} \frac{s^{2n}}{\Gamma(2n)} y^{-1} e^{-s} \frac{1}{\phi}\,ds =$

$\int_0^{t} \frac{s^{2n}}{\Gamma(2n)} s^{-1} e^{-s}\,ds = \int_0^{t} \frac{s^{2n-1}}{\Gamma(2n)} e^{-s}\,ds$ this cdf is indep of ϕ

∴ T_1 is a pivot

3b/ $T_1 = \phi Y \sim Ga(2n, 1)$ $\left\{ Ga(2n, \phi) \text{ is } f(y; \phi) = \frac{\phi^{2n}}{\Gamma(2n)} y^{2n-1} e^{-\phi y} \right.$

$\therefore \phi Y$ is $\cancel{f(\phi y; \phi)} = Ga(2n, 1)$ ie $\left. \frac{1^{2n}}{\Gamma(2n)} y^{2n-1} e^{-1y} = \frac{1}{\Gamma(2n)} y^{2n-1} e^{-y} \right\}$

$Pr(T_1 \le t) = \int_0^t \frac{s^{2n-1}}{\Gamma(2n)} e^{-s} ds$

$Pr(Y \le y) = \int_0^y f(y; \phi) dy = \int_0^y \frac{\phi^{2n}}{\Gamma(2n)} y^{2n-1} e^{-\phi y} dy$

$Pr(X_i \le x) = \int_0^x f(x; \phi) dx = \int_0^x \frac{\phi^2 x e^{-\phi x}}{\Gamma(2)} dx = \int_0^x \frac{\phi^2}{\Gamma(2)} x^{2-1} e^{-\phi x} dx = \int_0^x \frac{(\phi x)^2}{\Gamma(2)} x^{-1} e^{-\phi x} dx$

is $Ga(2, \phi)$ $\therefore$

$Pr(T_1 \le t) = \int_0^t \cancel{\phi} \frac{(1s)^{2n}}{\Gamma(2n)} s^{-1} e^{-1s} ds$ is $Ga(2n, 1)$ $\therefore$

~~Pr(q~~ $Pr(q_5 < T_1 < q_{95}) = 0.9$ $\therefore$ $Pr(q_5 < \phi Y < q_{95}) = 0.9$ $\therefore$

$Pr\left(\frac{q_5}{\phi} < Y < \frac{q_{95}}{\phi}\right) = 0.9$ $\therefore$ $Pr\left(\frac{q_5}{Y} < \phi < \frac{q_{95}}{Y}\right) = 0.9$ $\therefore$

90% CI : $(q_5/Y, q_{95}/Y)$ for ϕ

3c/ $X_0 \sim Ga(2, \phi)$ $\therefore$ density $f(x; \phi) = \phi^2 x e^{-\phi x}$ $\therefore$

$Pr(X_0 \le x) = \int_0^x f(a; \phi) da = \int_0^x \phi^2 a e^{-\phi a} dy$ let $z = \phi a$ $\therefore$ $\frac{1}{\phi} dz = da$ $\therefore$

$a = 0 \to z = 0$ $a = x \to z = \phi x$ $\therefore$ $\int_0^{\phi x} \phi z e^{-z} \frac{1}{\phi} dz = \int_0^{\phi x} z e^{-z} dz$

$= \left[-z e^{-z}\right]_0^{\phi x} - \int_0^{\phi x} -e^{-z} dz = -\phi x e^{-\phi x} + \int_0^{\phi x} e^{-z} dz = -\phi x e^{-\phi x} - \left[e^{-z}\right]_0^{\phi x} =$

$\cancel{-\phi x e^{-\phi x} - e^{-\phi x} = (\phi x + 1) e^{-\phi x}}$ $-\phi x e^{-\phi x} - e^{-\phi x} + 1 =$

$1 - (1 + \phi x) e^{-\phi x}$ ✓

ancillary is func $T(X_0, \underline{X})$ whose distri is indep of θ

3d/

$\therefore$ $T_2 = X_0 / \bar{X}$ $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ $\therefore$ $Pr(T_2 \le t) = Pr(X_0 / \bar{X} \le t) = Pr(X_0 \le t \bar{X}) =$

$Pr(X_0 \le t Y \frac{1}{n})$ where $Y = \sum_{i=1}^{n} X_i$ & $Pr(X \le Y) = \int Pr(X \le y \mid Y = y) \cdot Pr(Y = y) dy$ $\therefore$

$Pr(X_0 \le t Y \frac{1}{n}) = \int_0^\infty Pr(X_0 \le t y \frac{1}{n}) f_Y(y; \phi) dy$ $\therefore$ $\left\{ Pr(X_0 \le t y \frac{1}{n}) = 1 - (1 + \phi \frac{ty}{n}) e^{-\phi \frac{ty}{n}} \right\}$

$= \int_0^\infty (1 - (1 + \phi \frac{ty}{n}) e^{-\phi \frac{ty}{n}}) f_Y(y; \phi) dy = \int_0^\infty (1 - (1 + \phi \frac{ty}{n}) e^{-\phi \frac{ty}{n}}) \frac{\phi^{2n}}{\Gamma(2n)} y^{2n-1} e^{-\phi y} dy$

(let $z = \phi y$) $\therefore$ $\frac{1}{\phi} dz = dy$ $y = 0 \to z = 0$, $y = \infty \to z = \infty$ $\therefore$

$\int_0^\infty (1 - (1 + t \frac{1}{n} z) e^{-\frac{tz}{n}}) \frac{z^{2n}}{\Gamma(2n)} y^{-1} \frac{1}{\phi} e^{-z} dz = \int_0^\infty (1 - (1 + \frac{tz}{n}) e^{-\frac{tz}{n}}) \frac{z^{2n-1}}{\Gamma(2n)} e^{-z} dz$

is indep of ϕ $\therefore$ T_2 is ancillary