Sheet 3 (Q3) $\lambda = 10$/hour ~~$\mu = 4$~~

service time of 4 mins ~~μ~~ $\therefore$ 1 per 4 mins $\therefore$ 15 per hour $\therefore$

- Length of Syst is $L_s = E(N) = \frac{\rho}{1-\rho}$ $\qquad \frac{\lambda}{\mu} = \rho = \frac{10}{15} = \frac{2}{3} < 1 \therefore$

$L_s = \frac{2/3}{1-\frac{2}{3}} = 2$ $\qquad \therefore N$ is number in Syst

L_q is length of queue $L_q = \frac{\rho^2}{1-\rho} = \frac{(2/3)^2}{1-(2/3)} = \frac{4}{3}$

W_s is mean waiting time for Syst

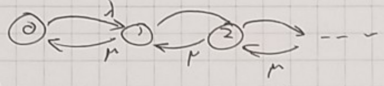
W_q is mean waiting time for queue $\therefore W_q = E(T_q)$

$T_q = T_1 + T_2 + \dots + T_N$ $\quad T_q$ is mean waiting times

$\therefore W_q = L_s/\mu = \frac{2}{15}$ //

~~$\therefore W_s =$~~ $\therefore W_q = \frac{\rho^2}{1-\rho}\frac{1}{\lambda} = \frac{(2/3)^2}{1-\frac{2}{3}}\frac{1}{\lambda} = \frac{2}{15} \therefore$

$W_s = W_q + \frac{1}{\mu} = \frac{\rho}{1-\rho}\frac{1}{\lambda} = \frac{(2/3)}{1-\frac{2}{3}}\frac{1}{10} = 2 \times \frac{1}{10} = \frac{1}{5}$

(Q3 sol)  $\mu = 15$/hour $\therefore \rho = \frac{\lambda}{\mu} = \frac{2}{3} \therefore$

~~P_0 (Q3) $P_0 =$~~ $P_0 = 1 - \rho = \frac{1}{3}$

~~P_1~~ $P_n = \rho^n(1-\rho) = \frac{1}{3}\left(\frac{2}{3}\right)^n$

{ is $\frac{dP_0}{dt} = -\lambda P_0 + \mu P_1 = 0$ $\therefore \lambda P_0 = \mu P_1$ $\quad \therefore \lambda P_1 = \mu P_2 \therefore$

$\rho P_0 = P_1$ $\quad \therefore \rho P_1 = P_2 \therefore \rho^2 P_0 = P_2 \therefore \rho^n P_0 = P_n \therefore \sum_{n=0}^{\infty} P_n = 1 \therefore$

$\sum_{n=0}^{\infty} P_n = P_0 + P_1 + P_2 + \dots = P_0 + \rho P_0 + \rho^2 P_0 = P_0 \sum_{n=1}^{\infty} \rho^n = P_0 \frac{1}{1-\rho} = 1 =$

$P_0 \frac{1}{1-\frac{2}{3}} = 3P_0 = 1$ $\quad \therefore P_0 = \frac{1}{3}$ $\quad \therefore P_n = \rho^n \frac{1}{3} = \left(\frac{2}{3}\right)^n \frac{1}{3} \therefore$

$\therefore P_0 \frac{1}{1-\rho} = 1$ $\quad \therefore P_0 = 1 - \rho$ $\quad \therefore P_n = \rho^n(1-\rho) \therefore$

$L_s = \frac{\rho}{1-\rho} = 2$, $\quad L_q = L_s - \rho = \frac{4}{3}$

for effective arrival rate: $\lambda_{eff} = \sum_{n \geq 0} \lambda_n P_n$ & $\therefore \lambda_n = \lambda \ \forall n$:

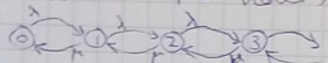
$\lambda_{eff} = \lambda$ by littles thm $\therefore$

- $W_s =$ ~~$\frac{L_q}{\lambda}$~~ $\frac{L_s}{\lambda_{eff}} = 12$ mins $\qquad W_q = \frac{L_q}{\lambda_{eff}} = 8$ mins $\left\{\frac{2}{15} \times 60 = 8\right\}$

$\{0.2 \times 60 = 12\}$

\3 sol/ this is an M/M/1 system with infinite capacity

& $\lambda=10$ (per hour) & $\mu=15$ (per hour) ∴ St St:



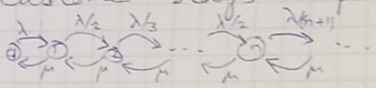
... & as in LN: $\rho = \lambda/\mu = 2/3$ ∴

$P_0 = 1-\rho = \frac{1}{3}$ $\quad P_n = \rho^n(1-\rho) = \frac{1}{3}\left(\frac{2}{3}\right)^n$ $\quad L_s = \frac{\rho}{1-\rho} = 2$ $\quad L_q = L_s - \rho = \frac{4}{3}$

for Z effective arrival rate have: $\lambda_{eff} = \sum_{n\geq 0} \lambda_n P_n$ & ∵ $\lambda_n = \lambda \; \forall n$ ∴

$\lambda_{eff} = \lambda$. by little's thm: $W_s = \frac{L_s}{\lambda_{eff}} = 12$ mins $\quad W_q = \frac{L_q}{\lambda_{eff}} = 8$ mins

now suppose Z modified model where Z customer stays with probab $\frac{1}{n+1}$ if Z syst is in state n: St St diag



in Z st st $\frac{\lambda}{n+1} P_n = \mu P_{n+1}$ & ∴ $P_n = \frac{\rho^n}{n!} P_0 \;\; \forall n \geq 1$ ∴

$1 = P_0\left(\sum_{n\geq 1} \frac{\rho^n}{n!}\right) \Rightarrow P_0 = e^{-\rho}$ $\quad$ ∴ $P_n = \frac{\rho^n}{n!} e^{-\rho}$ ∴

$L_s = \sum_{n=0}^{\infty} n P_n = P_0 \rho e^{\rho} = \rho = \frac{2}{3}$ $\quad L_q = L_s -$ (mean no being served) $=$

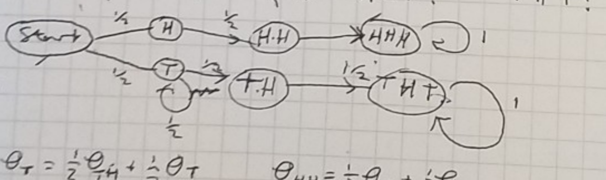
$L_s - (1-P_0) = L_s - 1 + e^{-\rho} \approx 0.18$ ∴

use little's thm: $\lambda_{eff} = \sum_{n=0}^{\infty} \left(\frac{\rho^n e^{-\rho}}{n!}\right) \frac{\lambda}{n+1} = \frac{\lambda e^{-\rho}}{\rho}(e^{\rho}-1) = \frac{\lambda(1-e^{-\rho})}{\rho}$

∴ by little's thm: ~~$L_s = W_s \lambda_s$~~ $L_s = W_s \lambda_{eff}$ & $L_q = W_q \lambda_{eff}$ & ∴

$W_s = \frac{\rho^2}{\lambda(1-e^{-\rho})} \approx 5.48$ mins, $W_q = 1.48$ mins

\Sheet 5/ \4/ HHH? THT?



$\theta_S = \frac{1}{2}\theta_H + \frac{1}{2}\theta_T$

$\theta_H = \frac{1}{2}\theta_{HH} + \frac{1}{2}\theta_T$

$\theta_T = \frac{1}{2}\theta_{TH} + \frac{1}{2}\theta_T$ $\quad \theta_{HH} = \frac{1}{2}\theta_T + \frac{1}{2}\underbrace{\theta_{HHH}}_{=1}$ $\quad \theta_{TH} = \frac{1}{2}\theta_{HH} + \frac{1}{2}\theta_{THT}$

for probab to win set $\theta_{HHH} = 1$ & $\theta_{THT} = 0$ ∴

$\theta_T = \frac{1}{3}$ $\left\{ \theta_T = \frac{1}{2}\theta_{TH} + \frac{1}{2}\theta_T = \frac{1}{2}\left(\frac{1}{2}\theta_{HH} + \frac{1}{2}\theta_{THT}\right) + \frac{1}{2}\left(\frac{1}{2}\theta_{TH} + \frac{1}{2}\theta_T\right) \right\}$

$\theta_{HH} = \frac{1}{2}\theta_T + \frac{1}{2}$ $\quad , \quad \theta_{TH} = \frac{1}{2}\theta_{HH} \Rightarrow \theta_{HH} = 2\theta_T$

$\theta_H = \frac{\theta_{HH}}{2} + \frac{1}{2}\theta_T$ $\quad \theta_T = \frac{1}{2}\theta_{TH} + \frac{1}{2}\theta_T \Rightarrow \theta_T = \theta_{TH}$

∴ $\theta_H = \frac{1}{2}$ $\quad$ ∴ $\theta_S = \frac{1}{2} \times \frac{1}{3} + \frac{1}{2} \times \frac{1}{2} = \frac{5}{12} < \frac{1}{2}$ ∴ More likely

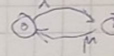
Sheet 3 $P_0 + \frac{\lambda_0}{\mu_1}P_0 + \frac{\lambda_1\lambda_0}{\mu_2\mu_1}P_0 + \frac{\lambda_2\lambda_1\lambda_0}{\mu_3\mu_2\mu_1}P_0 = 1 = \left(1 + \frac{\lambda_0}{\mu_1} + \frac{\lambda_1\lambda_0}{\mu_2\mu_1} + \frac{\lambda_2\lambda_1\lambda_0}{\mu_3\mu_2\mu_1}\right)P_0$

$1 = \left(1 + \frac{3}{2} + \frac{2\cdot3}{2\cdot2} + \frac{1\cdot2\cdot3}{2\cdot2\cdot2}\right)P_0 = \frac{19}{4}P_0 \quad \therefore \quad \cancel{\tfrac{4}{19}P_0} \quad \frac{4}{19} = P_0 \quad \therefore$

$P_1 = \frac{\lambda_0}{\mu_1}\frac{4}{19} = \frac{3}{2}\cdot\frac{4}{19} = \frac{6}{19}$, $P_2 = \frac{\lambda_1}{\mu_2}P_1 = \frac{2}{2}\cdot\frac{6}{19} = \frac{6}{19}$, $P_3 = \frac{\lambda_2}{\mu_3}\cdot\frac{6}{19} = \frac{1}{2}\cdot\frac{6}{19} = \frac{3}{19}$

$\therefore X \equiv$ ~~be~~ no. of machines out of action $\therefore$

$E[X] = (0)\frac{4}{19} + (1)\frac{6}{19} + (2)\frac{6}{19} + (3)\frac{3}{19} = \frac{27}{19} \approx 1.42$ (3 S.F.)

2 Sol For two states write $\lambda = \lambda_0$ & $\mu = \mu_1$ $\therefore$  $\therefore$

$\frac{dP_0}{dt} = \mu P_1 - \lambda P_0 \qquad \frac{dP_1}{dt} = \lambda P_0 - \mu P_1$, $P_0 + P_1 = 1 \quad \therefore P_1 = 1 - P_0$

$\frac{dP_0}{dt} = \mu(1-P_0) - \lambda P_0 = \mu - (\mu+\lambda)P_0 \quad \therefore \quad \frac{dP_0}{dt} = -(\mu+\lambda)P_0$ has sol $P_0(t) = Ae^{-(\mu+\lambda)t}$

& guess $P_0(t) = a + bt \quad \therefore b = 0$, $a = \frac{\mu}{\lambda+\mu} \quad \therefore$

G.S: $P_0(t) = \frac{\mu}{\lambda+\mu} + Ae^{-(\mu+\lambda)t}$,

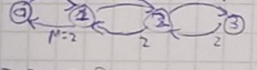
$P_1(t) = 1 - \frac{\mu}{\lambda+\mu} - Ae^{-(\mu+\lambda)t} = \frac{\lambda}{\lambda+\mu} - Ae^{-(\mu+\lambda)t}$

$P_0(0) = 0$, $P_1(0) = 1 \quad \therefore Ae^{-(\mu+t)} \to 0$ as $t \to \infty \quad \therefore$

$P_0(t) \Rightarrow \frac{\mu}{\lambda+\mu}$, $P_1(t) \to \frac{\lambda}{\lambda+\mu} \quad \therefore$

as $t \to \infty$ these are the same steady state probabs that one gets from solving the steady state eqns $\lambda P_0 = \mu P_1$ with $P_0 + P_1 = 1$

1 Sol Let the system state be the number n of machines broken down. The breakdown rate is 1 per hour (per working machine) the repair rate $\mu = (\text{mean repair time})^{-1} = 2 \quad \therefore$

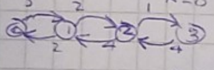
 equating probab flows gives the st st eqns

$3P_0 = 2P_1$, $2P_1 = 2P_2$, $2P_3 = P_2$

together $P_0 + P_1 + P_2 + P_3 = 1$ have the sol $P_0 = \frac{4}{19}$ $P_1 = \frac{6}{19}$, $P_2 = \frac{6}{19}$, $P_3 = \frac{3}{19}$

the expected number of broken down machines $\therefore$

$L_s = \sum_{k=0}^{3} kP_k = \frac{27}{19}$

 $\because$ the 2nd guy is only active if at least two machines are broken: $P_0 = \frac{16}{55} \quad \therefore L_s = \sum_{k=0}^{3} kP_k = \frac{57}{55}$

$4(6P_3) = 2P_1 = 24P_3 \therefore 12P_3 = P_1$

$3P_0 = 2(12P_3) = 24P_3 \therefore P_0 = 8P_3 \therefore P_0 + P_1 + P_2 + P_3 = 1 \therefore$

$8P_3 + 12P_3 + 6P_3 + P_3 = 1 = 27P_3 \therefore \frac{1}{27} = P_3 \therefore$

$6\frac{1}{27} = P_2 = \frac{2}{9} \therefore$

$12\frac{1}{27} = P_1 = \frac{4}{9} \therefore$

$8\frac{1}{27} = P_0 = \frac{8}{27}$ let X be number of machines out of action

$\therefore E[X] = (0)\frac{8}{27} + (1)\frac{4}{9} + (2)\frac{2}{9} + (3)\frac{1}{27} = 1$

with two engineer:

$$\begin{tikzcd} 0 \arrow[r, bend left, "\lambda_0"] & 1 \arrow[l, bend left, "\mu_1"] \arrow[r, bend left, "\lambda_1"] & 2 \arrow[l, bend left, "\mu_2"] \arrow[r, bend left, "\lambda_2"] & 3 \arrow[l, bend left, "\mu_3"] \end{tikzcd}$$

$\lambda_0 = 3,\ \lambda_1 = 2,\ \lambda_2 = 1$

$\mu_1 = 2 \quad \mu_2 = 4 \quad \mu_3 = 4 \therefore$

$\frac{dP_0}{dt} = -\lambda_0 P_0 + \mu_1 P_1 \therefore$ st st $\therefore \frac{dP_0}{dt} = 0 \therefore -\lambda_0 P_0 + \mu_1 P_1 = 0$,

$\frac{dP_3}{dt} = \lambda_2 P_2 - \mu_3 P_3 \therefore$ st st $\therefore \frac{dP_3}{dt} = 0 \therefore \lambda_2 P_2 - \mu_3 P_3 = 0 \therefore$

$\frac{dP_1}{dt} = \lambda_0 P_0 - \mu_1 P_1 - \lambda_1 P_1 + \mu_2 P_2 \therefore$ st st $\frac{dP_1}{dt} = 0 \therefore$

$\lambda_0 P_0 - \mu_1 P_1 - \lambda_1 P_1 + \mu_2 P_2 = 0 = -\lambda_1 P_1 + \mu_2 P_2$, $P_0 + P_1 + P_2 + P_3 = 1 \therefore$

$\mu_3 P_3 = \lambda_2 P_2 \therefore P_2 = \frac{\mu_3}{\lambda_2} P_3$,

$\lambda_1 P_1 = \mu_2 P_2 \therefore P_1 = \frac{\mu_2}{\lambda_1} P_2 = \frac{\mu_2 \mu_3}{\lambda_1 \lambda_2} P_3 \therefore$

$\lambda_0 P_0 = \mu_1 P_1 \therefore P_0 = \frac{\mu_1}{\lambda_0} P_1 = \frac{\mu_1 \mu_2 \mu_3}{\lambda_0 \lambda_1 \lambda_2} P_3 \therefore$

$\frac{\mu_1\mu_2\mu_3}{\lambda_0\lambda_1\lambda_2} P_3 + \frac{\mu_2\mu_3}{\lambda_1\lambda_2} P_3 + \frac{\mu_3}{\lambda_2} P_3 + P_3 = 1 = \left(\frac{\mu_1\mu_2\mu_3}{\lambda_0\lambda_1\lambda_2} + \frac{\mu_2\mu_3}{\lambda_1\lambda_2} + \frac{\mu_3}{\lambda_2} + 1\right) P_3 \therefore$

$P_3 = 1 \Big/ \left[\frac{\mu_1\mu_2\mu_3}{\lambda_0\lambda_1\lambda_2} + \frac{\mu_2\mu_3}{\lambda_1\lambda_2} + \frac{\mu_3}{\lambda_2} + 1\right] = 1 \Big/ \left[\frac{2\cdot 4\cdot 4}{3\cdot 2\cdot 1} + \frac{4\cdot 4}{2\cdot 1} + \frac{4}{1} + 1\right] = \frac{3}{55} = P_3 \therefore$

$P_2 = \frac{4}{1} \cdot \frac{3}{55} = \frac{12}{55}$

$P_1 = \frac{\mu_2}{\lambda_1} \cdot \frac{12}{55} = \frac{4}{2} \cdot \frac{12}{55} = \frac{24}{55}$

$P_0 = \frac{\mu_1}{\lambda_0} \cdot \frac{24}{55} = \frac{2}{3} \cdot \frac{24}{55} = \frac{16}{55} \therefore$

$E[X] = (0)\frac{16}{55} + (1)\frac{24}{55} + (2)\frac{12}{55} + (3)\frac{3}{55} = \frac{57}{55} \approx 1.04$ (3 s.f.)

\ redo of part i / $\lambda_0 = 3,\ \lambda_1 = 2,\ \lambda_2 = 1,\ \mu_1 = 2,\ \mu_2 = 2,\ \mu_3 = 2 \therefore$

$P_1 = \frac{\lambda_0}{\mu_1} P_0$, $P_2 = \frac{\lambda_1}{\mu_2} P_1 = \frac{\lambda_1 \lambda_0}{\mu_2 \mu_1} P_0$, $P_3 = \frac{\lambda_2}{\mu_3} P_2 = \frac{\lambda_2 \lambda_1 \lambda_0}{\mu_3 \mu_2 \mu_1} P_0 \therefore$

Sheet 3

$P_1(t) = \frac{1}{\lambda_0+\mu_1}\lambda_0 + C_2 e^{-(\lambda_0+\mu_1)t}$

$P_0(0) = 1 = P_0(t=0)$ $\therefore$

$P_0(0) = \frac{1}{\mu_1+\lambda_0}\mu_1 + C_1 e^0 = \frac{1}{\mu_1+\lambda_0}\mu_1 + C_1 = 1$ $\therefore$

$C_1 = 1 - \frac{1}{\mu_1+\lambda_0}\mu_1 = \frac{\mu_1+\lambda_0}{\mu_1+\lambda_0} + \frac{-\mu_1}{\mu_1+\lambda_0} = \frac{\lambda_0}{\mu_1+\lambda_0}$

$P_0(t) = \frac{1}{\mu_1+\lambda_0}\mu_1 + \frac{\lambda_0}{\mu_1+\lambda_0} e^{-(\mu_1+\lambda_0)t}$

$P_1(0) = P_1(t=0) = 1 - P_0(t=0) = 1 - 1 = 0$ $\therefore$

$P_1(0) = \frac{1}{\lambda_0+\mu_1}\lambda_0 + C_2 e^0 = 0 = \frac{1}{\lambda_0+\mu_1}\lambda_0 + C_2$ $\therefore$

$C_2 = -\frac{1}{\lambda_0+\mu_1}\lambda_0$ $\therefore$

$P_1(t) = \frac{1}{\lambda_0+\mu_1}\lambda_0 - \frac{1}{\lambda_0+\mu_1}\lambda_0 e^{-(\lambda_0+\mu_1)t} = \frac{1}{\lambda_0+\mu_1}\lambda_0\left(1 - e^{-(\lambda_0+\mu_1)t}\right)$

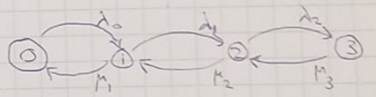
$\therefore \lim_{t\to\infty} P_0(t) = \lim_{t\to\infty}\left[\frac{1}{\mu_1+\lambda_0}\mu_1 + \frac{1}{\mu_1+\lambda_0}\lambda_0 e^{-(\mu_1+\lambda_0)t}\right] =$ $\{\lambda_0, \mu_1 \in \mathbb{R}_{\geq 0}\}$

$\frac{1}{\mu_1+\lambda_0}\mu_1 + \frac{1}{\mu_1+\lambda_0}\lambda_0(0) = \frac{1}{\mu_1+\lambda_0}\mu_1$

$\lim_{t\to\infty} P_1(t) = \cancel{\frac{1}{\lambda_0+\mu_1}\lambda_0}\ \lim_{t\to\infty}\frac{1}{\lambda_0+\mu_1}\lambda_0\left(1 - e^{-(\lambda_0+\mu_1)t}\right) = \frac{1}{\lambda_0+\mu_1}\lambda_0(1-0) = \frac{1}{\lambda_0+\mu_1}\lambda_0$

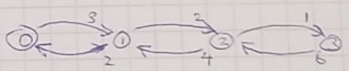
i

$\lambda = 1$, $\frac{1}{\mu} = \frac{1}{2}$ $\therefore$ $\mu = 2$



$\lambda_0 = 3$, $\lambda_1 = 2$, $\lambda_2 = 1$

$\mu_1 = 2$, $\mu_2 = 4$, $\mu_3 = 6$



$\frac{dP_0}{dt} = -\lambda_0 P_0 + \mu_1 P_1$; $P_0 + P_1 + P_2 + P_3 = 1$ $\therefore$

Steady State: $\frac{dP_0}{dt} = 0 \implies 0 = -\lambda_0 P_0 + \mu_1 P_1$ $\therefore$ $\cancel{\lambda_0 P_0 + \mu_2 P_1}$ $\therefore$ $\lambda_0 P_0 = \mu_1 P_1$ $\therefore$

$3P_0 = 2P_1$ steady state guaranteed because $N = 3$ $\therefore$

$\frac{dP_1}{dt} = +\lambda_0 P_0 - \mu_1 P_1 - \lambda_1 P_1 + \mu_2 P_2$ steady state $\frac{dP_1}{dt} = 0$ $\therefore$

$0 = +\lambda_0 P - \mu_1 P_1 - \lambda_1 P_1 + \mu_2 P_2 = -(-\lambda_0 P_0 + \mu_1 P_1) - \lambda_1 P_1 + \mu_2 P_2 = -0 - \lambda_1 P_1 + \mu_2 P_2 =$

$-\lambda_1 P_1 + \mu_2 P_2 = 0$ $\therefore$ by symmetry:

$\frac{dP_2}{dt} = \lambda_1 P_1 - \mu_2 P_2 - \lambda_2 P_2 + \mu_3 P_3$ $\therefore$ steady state $\frac{dP_2}{dt} = 0$ $\therefore$

$-0 - \lambda_2 P_2 + \mu_3 P_3 = 0 = -\lambda_2 P_2 + \mu_3 P_3$

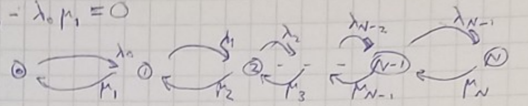
$\frac{dP_3}{dt} = \lambda_2 P_2 - \mu_3 P_3$ $\therefore$ $\frac{dP_1}{dt} = 0$ $\therefore$ $\lambda_2 P_2 - \mu_3 P_3 = 0$

$3P_0 = 2P_1$, $-2P_1 + 4P_2 = 0$ $\therefore$ $4P_2 = 2P_1$, $1P_2 - 6P_3 = 0$ $\therefore$ $P_2 = 6P_3$ $\therefore$

2 $\begin{bmatrix} \frac{dP_0}{dt} \\ \frac{dP_1}{dt} \end{bmatrix} = \begin{bmatrix} \mu_1 P_1(t) - \lambda_0 P_0(t) \\ \lambda_0 P_0(t) - \mu_1 P_1(t) \end{bmatrix} = \begin{bmatrix} -\lambda_0 & \mu_1 \\ \lambda_0 & -\mu_1 \end{bmatrix} \begin{bmatrix} P_0(t) \\ P_1(t) \end{bmatrix} \therefore$

$-\lambda_0(-\mu_1) - (\mu_1 \lambda_0) = +\lambda_0\mu_1 - \lambda_0\mu_1 = 0$

3 $\lambda = 10$



1 $\lambda = 1 \quad \mu = \frac{1}{2}$

2 retry $P_0(t) + P_1(t) = 1 \quad \therefore P_1(t) = 1 - P_0(t)$

$\frac{dP_0}{dt} = \mu_1(1 - P_0(t)) - \lambda P_0(t) = \mu_1 - \mu_1 P_0(t) - \lambda_0 P_0(t) = (-\mu_1 - \lambda_0) P_0 + \mu_1$

$\frac{d}{dt} P_0 + (\mu_1 + \lambda_0) P_0 = \mu_1 \quad \therefore h(t) P_0' + h(t)(\mu_1 + \lambda_0) P_0 = \mu_1 h(t)$

$h(t) P_0' + h'(t) P_0 = \frac{d}{dt}(h(t) P_0) = \frac{d}{dt}(h(t) P_0(t))$

$h'(t) = (\mu_1 + \lambda_0) h(t) \quad \therefore \quad \frac{h'(t)}{h(t)} = \mu_1 + \lambda_0 \quad \therefore \int \frac{h'(t)}{h(t)} dt = \int \mu_1 + \lambda_0 dt \therefore$

$\ln(h(t)) = \mu_1 t + \lambda_0 t \quad \therefore h(t) = e^{\mu_1 t + \lambda_0 t} \quad \therefore$

$\frac{d}{dt}(e^{\mu_1 t + \lambda_0 t} P_0(t)) = \mu_1$

$IF = e^{\int \mu_1 + \lambda_0 dt} = e^{\mu_1 t + \lambda_0 t}$

$\cancel{e^{\mu_1 t + \lambda_0 t} P_0(t) = \mu_1 t + C \quad P_0(t) = \mu_1 t (e^{\mu_1 t + \lambda_0 t})^{-1} = \mu_1 t e^{-\mu_1 t - \lambda_0 t}}$

$\frac{d}{dt}(e^{\mu_1 t + \lambda_0 t} P_0(t)) = h(t)\mu_1 = e^{\mu_1 t + \lambda_0 t}\mu_1 = e^{(\mu_1 + \lambda_0)t}\mu_1 \quad \therefore$

$e^{\mu_1 t + \lambda_0 t} P_0(t) = \int e^{(\mu_1 + \lambda_0)t}\mu_1 = \frac{1}{\mu_1 + \lambda_0}\mu_1 e^{(\mu_1 + \lambda_0)t} + C_1 \quad \therefore$

$P_0(t) = \frac{1}{\mu_1 + \lambda_0}\mu_1 + C_1 e^{-(\mu_1 + \lambda_0)t}$

$P_0(t) = 1 - P_1(t) \quad \therefore$

$\frac{dP_1}{dt} = \lambda_0(1 - P_1(t)) - \mu_1 P_1(t) = \lambda_0 - \lambda_0 P_1(t) - \mu_1 P_1(t) = \lambda_0 - (\lambda_0 + \mu_1) P_1(t) \quad \therefore$

$\frac{dP_1}{dt} + (\lambda_0 + \mu_1) P_1(t) = \lambda_0 \quad \therefore \quad IF = e^{\int \lambda_0 + \mu_1 dt} = e^{\lambda_0 t + \mu_1 t} = e^{(\lambda_0 + \mu_1)t} \quad \therefore$

$\frac{d}{dt}(e^{(\lambda_0 + \mu_1)t} P_1(t)) = \lambda_0 e^{(\lambda_0 + \mu_1)t} \quad \therefore e^{(\lambda_0 + \mu_1)t} P_1(t) = \int \lambda_0 e^{(\lambda_0 + \mu_1)t} dt = \cancel{\lambda_0}$

$\frac{1}{\lambda_0 + \mu_1}\lambda_0 e^{(\lambda_0 + \mu_1)t} + C_2 \quad \therefore$

$n = 1$

$$-(\lambda_0+\mu_1)t$$

$$\sum_{n=1}^{N} E(T^{(n)}) = \frac{N}{\lambda}\left(\frac{1}{N} + \frac{1}{N-1} + \dots + \frac{1}{2} + 1\right) = \frac{N}{\lambda}\sum_{k=1}^{N}\frac{1}{k} \approx \frac{N}{\lambda}\log(N) \text{ large } N$$

Sheet 3 ~~1i~~ $\lambda = 1+1+1 = 3$

$E(\text{fix machine}) = 0.5$ hours $= \frac{1}{\mu}$ $\therefore \mu = \frac{1}{0.5} = 2$ hours

Max number of machines broken is 3 $\therefore$ $N = 3$ and

$\lambda_n = 0$ for $n \geq 3$

$\lambda_0 = 3$ $\quad \lambda_1 = 2$ $\quad \lambda_2 = 1$

$$\begin{tikzcd}
0 \arrow[r, bend left, "\lambda_0"] & 1 \arrow[l, bend left, "\mu_1"] \arrow[r, bend left, "\lambda_1"] & 2 \arrow[l, bend left, "\mu_2"] \arrow[r, bend left, "\lambda_2"] & 3 \arrow[l, bend left, "\mu_3"]
\end{tikzcd}$$

$\mu_1 = 2$, $\mu_2 = 4$, $\mu_3 = 6$

$\cancel{\sum_{n=1}^{3}\frac{\lambda_2\lambda_1\lambda_0}{\mu_3\mu_2\mu_1}}$ $\quad \cancel{\sum_{n=1}^{3}\lambda_n}$ $\quad \sum_{n=1}^{1}\frac{\lambda_{1-1}}{\mu_1} = \frac{\lambda_0}{\mu_1} = \frac{3}{2}$

$$\sum_{n=2}^{2}\frac{\lambda_{2-1}\lambda_{2-2}}{\mu_2\mu_{2-1}} = \frac{\lambda_1\lambda_0}{\mu_2\mu_1} = \frac{2\cdot 3}{4\cdot 2} = \frac{6}{8} = \frac{3}{4}$$

$$\sum_{n=3}^{3}\frac{\lambda_{3-1}\lambda_{3-2}\lambda_{3-3}}{\mu_3\mu_{3-1}\mu_{3-2}} = \frac{\lambda_2\lambda_1\lambda_0}{\mu_3\mu_2\mu_1} = \frac{1\times 2\times 3}{6\times 4\times 2} = \frac{3}{2\times 3\times 4} = \frac{1}{8} \therefore$$

$$\sum_{n=1}^{3}\frac{\lambda_{n-1}\lambda_{n-2}\dots\lambda_0}{\mu_n\mu_{n-1}\dots\mu_1} = \frac{3}{2} + \frac{3}{4} + \frac{1}{8} = \frac{19}{8} \therefore$$

$$P_0 = \frac{1}{1+\left(\frac{19}{8}\right)} = \frac{8}{27} \therefore$$

$$\frac{\lambda_{3-1}\lambda_{3-2}\lambda_{3-3}}{\mu_3\mu_{3-1}\mu_{3-2}} = \frac{\lambda_2\lambda_1\lambda_0}{\mu_3\mu_2\mu_1} = \frac{1\times 2\times 3}{6\times 4\times 2} = \frac{1}{8} \therefore \quad P_n = \frac{1}{8}\left(\frac{8}{27}\right) = \frac{1}{27}$$

is Z probab of being in state n at $t = \infty$

$\lambda_n P_n = \mu_{n+1}P_{n+1}$ $\therefore$ $P_{n+1} = \frac{1}{27}$ $\therefore$ $\lambda_n\frac{1}{27} = \mu_{n+1}\frac{1}{27}$ $\therefore$

$\lambda_n = \mu_{n+1}$ $\therefore$ ~~M~~

if second engineer deployed if 2 or more machines broken:

$\lambda_0 = 3$, $\lambda_1 = 2$, $\lambda_2 = 1$

$\mu_1 = 2$ $\quad \frac{60\text{mins}}{2} = 30\text{mins}$ $\therefore$ $\frac{1}{\cancel{30}}$ $\frac{1}{0.5} = 2$ $\quad \therefore \mu_2 = 2$

$0.5\times 3 = 1.5$ $\quad \frac{1.5}{2} = 0.75$ $\quad \frac{1}{0.75} = 4/3 = \mu_3$ $\therefore$

~~1i~~ $N = 3$ $\therefore$ $1 \leq n \leq N-1 = 2$ $\therefore$ $1 \leq n \leq 2$ $\therefore$ $n = 1, 2$ $\therefore$

$$\sum_{n=1}^{n}\frac{\lambda_{n-1}\lambda_{n-2}\dots\lambda_0}{\mu_n\mu_{n-1}\dots\mu_1} = \frac{\lambda_{1-1}\cancel{\lambda_{1-2}}}{\mu_1} + \frac{\lambda_{2-1}\lambda_{2-2}}{\mu_2\mu_{2-1}} = \frac{\lambda_0}{\mu_1} + \frac{\lambda_1\lambda_0}{\mu_2\mu_1} = \frac{3}{2} + \frac{2\times 3}{2\times 2} = 3 \therefore$$

$P_0 = \frac{1}{1+3} = \frac{1}{4}$ $\therefore$ ~~λ_n~~ ~~μ_{n2}~~ ~~μ_{n2}~~ $\frac{\lambda_{n-1}\lambda_{n-2}}{\mu_n\mu_{n-1}}\frac{1}{4} = P_n$ $\therefore$

$\lambda_n P_n = \mu_{n+1}P_{n+1}$ $\therefore$ $\frac{\lambda_n\lambda_{n-1}\lambda_{n-2}}{\mu_n\mu_{n-1}}\frac{1}{4} = \mu_{n+1}\frac{\lambda_n\lambda_{n-1}}{\mu_{n+1}\mu_n}\frac{1}{4} = (\lambda_n\lambda_{n-1}/\mu_n)\left(\frac{1}{4}\right)$ $\therefore$ $\frac{\lambda_{n-2}}{\mu_{n-1}}$ $\quad \frac{3}{2} = 1.5 = \frac{\lambda_0}{\mu_1}$

transient. Let $i \in \Omega$ be any state, if $A = W \cup L$, with W as Z set of "winning" states that we would like to reach. & L as Z set of "losing" states that we would like to avoid. once we enter an absorbing state in A we consider Z game over as there's no escape.

\probab of reaching a winning state/ Let θ_i denote Z prob of reaching any state in W given that we start in state $i \in \Omega$. From Z law of total prob $\theta_i = P(\text{reach } W \text{ from } i)$ for $i \notin A$ is given by $P(\text{reach } W \text{ from } i) = \sum_{j \in \Omega} P(\text{reach } W \text{ from } j \mid i \to j) P(i \to j)$, where $P(i \to j) = T_{ij}$.

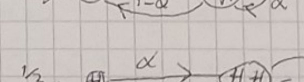
event reach W from j is indep of event $i \to j$ ($\because$ markov chain future states only depend on Z present state not Z history before it) & $\therefore P(\text{reach } W \text{ from } j \mid i \to j) = P(\text{reach } W \text{ from } j) = \theta_j$ $\therefore$ obtain Z simultaneous eqns $\theta_i = \sum_{j \in \Omega} T_{ij} \theta_j$

we need boundary conditions in order to solve eqn otherwise we get an infinite number of sols. we set $\theta_i = 0$ if $i \in L$ & $\theta_i = 1$ if $i \in W$ using algebraic means we then obtain θ_i for any given $i \in \Omega$

\expected waiting time to finish/ now let D_i denote Z expected time to finish ie Z expected time to enter A given $i \in \Omega$ note that $D_i = 0$ if $i \in A$, & $\therefore$ have already Z BCs on D_i if $i \notin A$ may find D_i $\therefore D_i = 1 + \sum_{j \in \Omega} T_{ij} D_j$ (by LoTP $\because$ let $Q(n, i)$ denote Z event of reaching an element of A starting in state i in exactly n steps. Then $P(Q(n,i)) = \sum_{j \in \Omega} P(Q(n,i) \mid i \to j) P(i \to j) = \sum_{j \in \Omega} P(Q(n-1, j)) T_{ij}$)

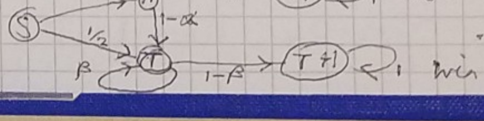
Ex/ two state Ex/ $\Omega = \{H, T\}$ find P_r(reaching TH)

win is TH lose is HH $\therefore$



$$\begin{array}{c|cc} & T & H \\ \hline T & \beta & 1-\beta \\ H & 1-\alpha & \alpha \end{array}$$

$\therefore$ if $\alpha = \frac{1}{2}, \beta = \frac{1}{2}$ coin

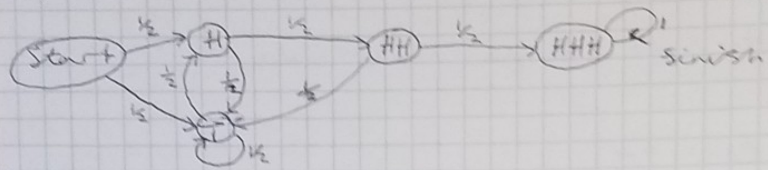


Week 10 / theory of runs /

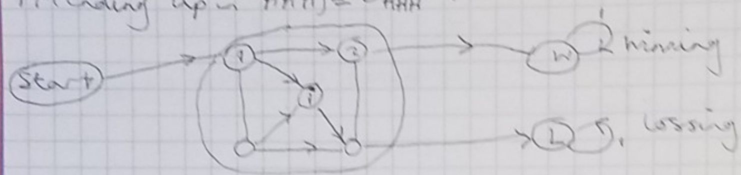
2-state system HTHTHHTT. . . WDWDWWDD. . .

simple random walk $S_{n+1} = S_n + Z_{n+1}$

Expected time to get HHH D_{HHH} (duration)



$Pr(\text{ending up in } HHH) = \theta_{HHH} = 1$



$\theta_i = P(\text{reaching } W \text{ eventually starting from } i)$
W_i

$\theta_W = 1$, $\theta_L = 1 - \theta_W = 0$

events B_{ij} $i \to j$ in 1 next step

$\Omega = B_{i1} \cup B_{i2} \cup B_{i3} \ldots \cup B_{im}$ (partition)

$\theta_i = P(W_i) = P(W_i \cap \Omega) =$ (law of total probability

$P_r(W_i \cap (B_{i1} \cup B_{i2} \cup B_{i3} \cup \ldots \cup B_{im})) =$

$P_r(W_i \cap B_{i1}) + P_r(W_i \cap B_{i2}) + \cancel{P_r(B_{i3})}\, P_r(W_i \cap B_{i3}) + \ldots + P_r(W_i \cap B_{im})$

$= P_r(W_1)P(B_{i1}) + P_r(W_2)P_r(B_{i2}) + P_r(W_3)P_r(B_{i3}) + \ldots + P(W_m)P(B_{im})$

$\theta_i = \sum_{j=1}^{m} [\theta_j P(i \to j)] = \sum_{j=1}^{m} [\theta_j T_{ij}]$

$\underline{\theta} = \underline{T}\,\underline{\theta}$ (vectors) (right eigvecs)

$\underline{P}_s = \underline{P}_s \underline{T}$ steady state left eigvecs

D_i = expected time to get to W or L state starting from i

$D_i = 1 + \sum_{j=1}^{m} T_{ij} D_j$ $\qquad \theta_i = \sum_{j=1}^{m} T_{ij} \theta_j$

~~Prop~~ prob θ_i of reaching a winning state, for 1 expected waiting time

D_i to finish (either win or lose) starting from state i

if Ω is a collection of states & there is a subset $A = \{a_1, a_2, \ldots, a_j\}$

of absorbing winning or losing states, with 1 remaining states

$S_n = \sum_{i=1}^{n} X_i$ ∴ $X_i = -1, +1$ ($P_r = \frac{1}{2}$, $P_r = \frac{1}{2}$) ..

$\frac{X_i + 1}{2} \sim Ber(\frac{1}{2})$ ∴ $\frac{X_i + 1}{2} = 0, 1$ ∴

$S_{n+1} = S_n + x_n$

recurrence properties of a simple random walk

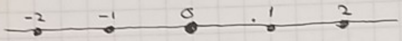
$S_n = X_1 + X_2 + \dots + X_n$

$P_r(X_i = +1) = \frac{1}{2}$ $\quad P_r(X_i = -1) = \frac{1}{2}$

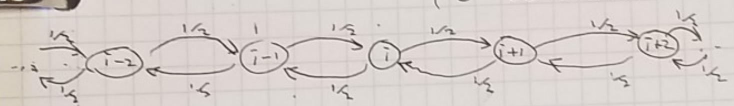
$S_{n+1} = S_n + X_{n+1}$ $\quad \{ ∴ S_{n+1} = X_1 + X_2 + \dots + X_n + X_{n+1} = S_n + X_{n+1} \}$

n=8 states in markov chain

$S_n \in \mathbb{Z}$



$T_{ij} = P_r(S_{n+1} = j \mid S_n = i) = \begin{cases} \frac{1}{2} & j = i+1 \text{ or } j = i-1 \\ 0 & \text{otherwise} \end{cases}$



. is transition diag

with T_{ij} is transition matrix

∴ $f_0 = \sum_{n>0} f_0^{(n)} = 1$ recurrent

is $\sum_{n>0} (T^n)_{00} = \infty$ recurrent

$P_r(S_{2k+1} = 0 \mid S_0 = 0) = 0$ $\quad k = 0, 1, 2, \dots$

$P_r(S_{2k} = 0 \mid S_0 = 0) = (T^{2k})_{00}$ $\quad$ {2k steps k are +1, k are -1
eg 6 steps: +++---, +-+-+-, ++--+- }

$= (\frac{1}{2})^{2k}\, {}^{2k}_{\ k}C = (\frac{1}{2})^{2k} \binom{2k}{k} =$

$(\frac{1}{2})^{2k} \left(\frac{(2k)!}{k!\,k!} \right)$

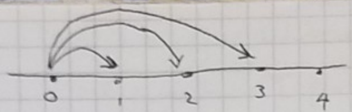
$t = \sum_{k=1}^{\infty} (\frac{1}{2})^{2k} \frac{(2k)!}{k!\,k!}$ $\quad$ stirling's approx $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$

$\frac{(2k)!}{k!\,k!} \sim C k^{-1/2}$ $\quad C = \frac{1}{\sqrt{\pi}}$

$S = \sum_{k=1}^{\infty} k^{-1/2} = \infty$ divergent $\left\{ \sum_{k=1}^{\infty} k^{-1/2} = \sum_{k=1}^{\infty} \frac{1}{k^{1/2}} > \sum_{k=1}^{\infty} \frac{1}{k} = \infty \right\}$

∴ $t = \sum_{k=1}^{\infty} (\frac{1}{2})^{2k} \frac{(2k)!}{k!\,k!} = \infty$ $\quad$ ∴ State 0 is recurrent

also its null recurrent



$t_i^{(1)} = \delta_i^{(1)}$

$t_i^{(2)} = \delta_i^{(1)} \delta_i^{(1)} + \delta_i^{(2)} = \delta_i^{(1)} t_i^{(1)} + \delta_i^{(2)}$

$t_i^{(3)} = \delta_i^{(1)} t_i^{(2)} + \delta_i^{(2)} t_i^{(1)} + \delta_i^{(3)}$

$t_i^{(4)} = \delta_i^{(1)} t_i^{(3)} + \delta_i^{(2)} t_i^{(2)} + \delta_i^{(3)} t_i^{(1)} + \delta_i^{(4)}$ $\therefore$

$t = \sum_{n>0}^{\infty} t_i^{(n)} = \left(\sum_{n>0}^{\infty} \delta_i^{(n)}\right)\left[1 + \sum_{n>0}^{\infty} t_i^{(n)}\right]$ $\therefore$ $t = \delta(1+t)$ $\therefore \delta = \frac{t}{1+t}$ $\therefore$ instead

$n_t \sim$ $\quad n_\delta \sim$

$\Rightarrow \delta < 1$ transient, if $t = \infty$: $\delta = 1$ recurrent

$\{ t = t^{(1)} + t_i^{(2)} + t_i^{(3)} + t_i^{(4)} + \cdots =$

$\delta_i^{(1)} + \delta_i^{(1)} t_i^{(1)} + \delta_i^{(2)} + \delta_i^{(1)} t_i^{(2)} + \delta_i^{(2)} t_i^{(1)} + \delta_i^{(3)} + \delta_i^{(1)} t_i^{(3)} + \delta_i^{(2)} t_i^{(2)} + \delta_i^{(3)} t_i^{(1)} + \delta_i^{(4)} + \cdots =$

$(\delta_i^{(1)} + \delta_i^{(2)} + \delta_i^{(3)} + \delta_i^{(4)} + \cdots) + (\delta_i^{(1)} t_i^{(1)} + \delta_i^{(1)} t_i^{(2)} + \delta_i^{(1)} t_i^{(3)} + \delta_i^{(2)} t_i^{(1)} + \delta_i^{(2)} t_i^{(2)} + \delta_i^{(3)} t_i^{(1)} + \cdots)$

$= \left(\sum_{n>0}^{\infty} \delta_i^{(n)}\right) + \left((\delta_i^{(1)} + \delta_i^{(2)} + \delta_i^{(3)} + \cdots) t_i^{(1)} + (\delta_i^{(1)} + \delta_i^{(2)} + \delta_i^{(3)} + \cdots) t_i^{(2)} + (\delta_i^{(1)} + \delta_i^{(2)} + \delta_i^{(3)} + \cdots) t_i^{(3)} + \cdots\right)$

$= \sum_{n>0}^{\infty} \delta_i^{(n)} + \left(\sum_{n>0}^{\infty} \delta_i^{(n)}\right) t_i^{(1)} + \left(\sum_{n>0}^{\infty} \delta_i^{(n)}\right) t_i^{(2)} + \left(\sum_{n>0}^{\infty} \delta_i^{(n)}\right) \delta_i^{(3)} + \cdots$

$\sum_{n>0}^{\infty} \delta_i^{(n)} + \left(\sum_{n>0}^{\infty} \delta_i^{(n)}\right)\left(t_i^{(1)} + t_i^{(2)} + t_i^{(3)} + \cdots\right) =$

$\sum_{n>0}^{\infty} \delta_i^{(n)} + \left(\sum_{n>0}^{\infty} \delta_i^{(n)}\right)\left(\sum_{n>0}^{\infty} t_i^{(n)}\right) = \left(\sum_{n>0}^{\infty} \delta_i^{(n)}\right)\left[1 + \sum_{n>0}^{\infty} t_i^{(n)}\right] \}$

\random walks/ $n = 0, 1, 2, 3, \ldots$ $S_n = \sum_{i=1}^{n} X_i$ X_i are iid r.v.s

$= X_1 + X_2 + \cdots X_n$ $\therefore S_{n+1} = S_n + X_{n+1}$ discrete time 1 dimen random walk (simple random walk)

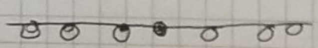
continuous time $\delta(t)$ wiener process

$\therefore E(S_n) = E[X_1 + X_2 + \cdots + X_n] = E(X_1) + E(X_2) + \cdots + E(X_n) = n\mu_x$

$var(S_n) = var(X_1 + X_2 + \cdots + X_n) = var(X_1) + var(X_2) + \cdots + var(X_n) = n\sigma_x^2$

$\therefore \lim_{n\to\infty} S_n \sim N(n\mu_x, n\sigma_x^2)$ $\therefore$

$\lim_{n\to\infty} \frac{S_n - n\mu_x}{\sqrt{n\sigma_x^2}} \sim N(0,1)$

 $S_0 = 0$ $\quad P_r(S_1 = 1) = \frac{1}{2}$ $\quad P_r(S_1 = -1) = \frac{1}{2}$ $\therefore$

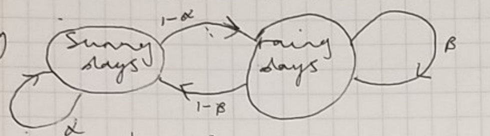
time dependent Markov chains (~~non~~nonhomog MC)

$p^{(n)} = p^{(n-1)}T = p^{(n-2)}T^2 = p^{(0)}T^n$ homog T is const

but: if $p^{(n)} = p^{(n-1)}T(n) = p^{(n-2)}T(n-1)T(n) =$

$p^{(0)}T(1)T(2) \cdots T(n)$

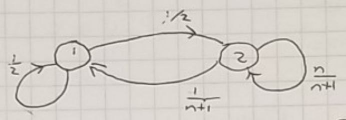
eg



$\therefore \beta$ is bigger in winter than summer

~~Ex~~ eg

$$T = \begin{array}{c c} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{n+1} & \frac{n}{n+1} \end{bmatrix} \end{array}$$



$\therefore f_1^{(1)} = \frac{1}{2}$ $\quad f_1^{(2)} = \frac{1}{2} \times \frac{1}{n+1}$ $\quad f_1^{(3)} = \frac{1}{2} \frac{n}{n+1} \frac{1}{n+1} = \frac{1}{2} \frac{3}{2+1} \frac{1}{3+1} \because n=3$

$f_1^{(4)} = \frac{1}{2} \frac{n}{n+1} \frac{n}{n+1} \frac{1}{n+1} = \frac{1}{2} \frac{2}{2+1} \frac{3}{3+1} \frac{1}{4+1} = \frac{1}{\cancel{2}} \frac{\cancel{2}}{\cancel{3}} \frac{3}{4} \frac{1}{5} \quad \because n=4 \quad \therefore$

$f_1^{(n)} = \frac{1}{n(n+1)} \quad \therefore$

$$f = \sum_{n>0}^{\infty} f_1^{(n)} = \sum_{n>0}^{\infty} \frac{1}{n(n+1)} = \sum_{n>0}^{\infty} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{k\to\infty} \sum_{n>0}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right) =$$

$$\lim_{k\to\infty} \left[\left(1 + \frac{1}{2} + \cdots \frac{1}{k}\right) - \left(\frac{1}{2} + \frac{1}{3} + \cdots \frac{1}{k+1}\right)\right] = \lim_{k\to\infty}\left(1 - \frac{1}{k+1}\right) = 1 - 0 = 1$$ $\therefore$ State 1 is ~~redundant~~ recurrent $\therefore$

mean recurrence: $\mu_1 = \sum_{n>0}^{\infty} n f_1^{(n)} = \sum_{n>0}^{\infty} \frac{n}{n(n+1)} = \sum_{n>0}^{\infty} \frac{1}{(n+1)} =$

$\frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots = -1 + \frac{1}{1} + \frac{1}{2} + \frac{1}{4} + \cdots = -1 + \sum_{n=1}^{\infty} \frac{1}{n}$ $\therefore$ {Harmonic series}

$\to \infty$ divergent $\therefore$

average recurrence time is infinite $\therefore$

state 1 is null recurrent

alternative view ~~on~~ recurrence

$f_i^{(n)}$ = probab of returning for Z 1st time to state i in n steps

$\sum_{n>0} f_i^{(n)} = 1$ recurrent

< 1 transient

$f_i^{(n)} = P_r(X_n = i \cap X_{n-1} \neq i \cap X_{n-2} \neq i \cap \cdots \cap X_1 \neq i \mid X_0 = i)$ $\therefore$ probab of being in state i after n steps starting from state i is:

$t_i^{(n)} = p(X_n = i \mid X_0 = i) \quad \therefore$

$p^{(n)} = p^{(n-1)}T = p^{(0)}T^n \quad \therefore \quad p(X_n = i \mid X_0 = i) \Rightarrow t_i^{(n)} = [T^n]_{ii}$

X_i as iid mean 0 & variance 1 ∴ steps sum $S_n = \sum_{i=1}^{n} X_i$

∴ $E[S_n] = 0$ $\quad var(S_n) = n$ ∴

$\frac{S_n}{\sqrt{n}} \to N(0,1)$ or $P\left(\frac{S_n}{\sqrt{n}} \in [a,b]\right) \to \int_a^b \frac{e^{-x^2/2}}{\sqrt{2\pi}}\,dx$ as $n \to \infty$

or take $X_i = 1$ probab p & -1 probabil $q = 1-q$ over each step

Mean displacement is $\mu = 2p-1$ & variance $\sigma^2 = 4p(1-p)$ ∴

S_n has mean $n(2p-1)$ ~~var~~ var $= 4np(1-p)$ as $n \to \infty$

∴ normal distri ∴ mean displacement $n\mu$ increases

while var ~~~~ scales like $n\sigma^2$ & the sd scales like $\sigma\sqrt{n}$ ∴

sd represents the typical displacent of the walk by time n

\random walk as markov chain/ ~~model~~ $S_n = \sum_{i \le n} X_i$

infinite states $\Omega = \mathbb{Z} = \{\dots, -2, -1, 0, 1, 2, \dots\}$

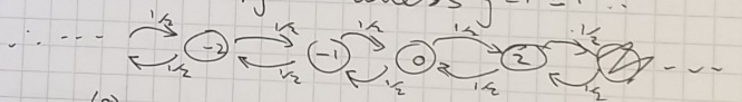
notice S_n is markov ∵ write $S_{n+1} = S_n + X_{n+1}$ & ∵

$P(S_n = l_n \mid S_{n-1} = l_{n-1}, \dots, l_0 = 0) = P(S_n = l_n \mid S_{n-1} = l_{n-1}) = T_{l_{n-1} l_n}$

∴ probab $\frac{1}{2}$ is $l_n = l_{n-1} \pm 1$ (0 otherwise

Mat T infinite demen but $T_{ij} = 0$ unless $j = i \pm 1$ ∴

$T_{i,i+1} = T_{i,i-1} = 1/2$ ∴



rec $p^{(0)}$ components $p_i^{(0)} = 0$ $\quad i \ne 0$ & $p_0^{(0)} = 1$

∴ $p^{(n+1)} = p^{(n)} T$: $\quad p_i^{(n+1)} = \sum_j T_{ji} p_j^{(n)} = \frac{1}{2} p_{i-1}^{(n)} + \frac{1}{2} p_{i+1}^{(n)}$ ∴

$p^{(1)} = (\dots, \frac{1}{2}, 0, \frac{1}{2}, \dots)$, $p^{(2)} = (\dots, 0, \frac{1}{4}, 0, \frac{1}{2}, 0, \frac{1}{4}, 0, \dots)$

\proposition 7.6/ walk start state 0 ∴ 0 is recurrent

& state 0 is null recurrent ∴ expected return time is infinite

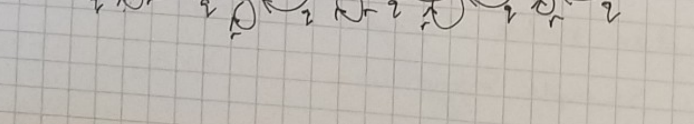
∵ $(T^n)_{00} = P(S_n = 0 \mid S_0 = 0)$ ∵

$P(S_{2k} = 0 \mid S_0 = 0) = \binom{2k}{k}\left(\frac{1}{2}\right)^{2k}$ ∴ ∵ $n! \sim \sqrt{2\pi n}\, n^n e^{-n}$:

$P(S_{2k} = 0 \mid S_0 = 0) \sim C k^{-1/2}$ const C ∴

$\{0\}$ is recurrent

eg



more likely too stay over Z next time step. but with probab one, eventually get ~~rejected~~ & sent to State 1 ∴ 1 & 2 are both recurrent ∴

state 1: $f_1^{(1)} = P(1 \mapsto 1) = \frac{1}{2}$ $\quad f_1^{(2)} = P(1 \mapsto 2 \mapsto 1) = \frac{1}{2} \times \frac{1}{3} = \frac{1}{6}$ ∴

$f_1^{(3)} = P(1 \mapsto 2 \mapsto 2 \mapsto 1) = \frac{1}{2} \times \frac{2}{3} \times \frac{1}{4} = \frac{1}{12}$

$f_1^{(4)} = P(1 \mapsto 2 \mapsto 2 \mapsto 2 \mapsto 1) = \frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \frac{1}{5} = 1/20$

∴ for $n \geq 1$: $f_1^{(n)} = P(1 \mapsto 2 \mapsto 2 \mapsto 2 \mapsto \cdots \mapsto 2 \mapsto 1) =$

$\frac{1}{2} \times \frac{2}{3} \times \frac{3}{4} \times \cdots \times \frac{n-1}{n} \times \frac{1}{n+1} = \frac{1}{n(n+1)}$ ∴

is $f_i = 1$ or $f_i < 1$ ∴ $f_1 = \sum_{n=1}^{\infty} f_1^{(n)} = \lim_{k \to \infty} \sum_{n=1}^{k} \left(\frac{1}{n} - \frac{1}{n+1}\right) = \lim_{k \to \infty} \left(1 - \frac{1}{k+1}\right) = 1$

∴ state 1 is recurrent. its null recurrent ∵

$\mu_1 = \sum_{n=1}^{\infty} n f_1^{(n)} = \lim_{k \to \infty} \sum_{n=1}^{k} \frac{1}{n+1} = \infty$

$n f_i^{(n)}$ can decay very slowly as $n \to \infty$ ∴ study is $\mu_i = \infty$ or $\mu_i < \infty$

state 2: $f_2^{(1)} = \frac{1}{2}$, $f_2^{(2)} = \frac{1}{4}$, ---, $f_2^{(n)} = \frac{1}{2^n}$ ∴

posi recurrent ∵ $\mu_2 < \infty$

$T(1) = \begin{pmatrix} 1 & 0 \\ 1/2 & 1/2 \end{pmatrix}$ & $\forall n \geq 2$: $T(n) = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$ ∴ $T(1)$: $f_1^{(1)} = 1$

∴ state 1 is recurrent but $T(n)$ $\forall n \geq 2$ see goes state 2 & stay forever (with probab 1) & probab of eventually return to 1 is 1 but def of recurrence not capture long run behaviour

T(n) doesnt preserve Z communications properties of markov chain: with $T(1)$, state 2 not accessible from state 1 but $T(n)$ $\forall n \geq 2$ find Z 2 is accessible from 1 is why usual def of recurrence fails ∴

T(n) preserves Z communication properties of markov chain ∴ Z usual def can be applied its possible to extend it to capture long run behaviour

Simple random walk / markov chain with infinite states

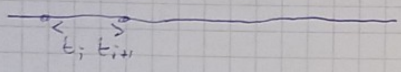
∴ simple random walk ∴ non markov chain

∴ realtime: $\mathbb{Z}$ simple symmetric: ∴ $X_i \in \{-1, +1\}$ each probab 1/2 start origin time zero revision / $M_x(t) = E(e^{tx})$ m.g.f

$G_x(\theta) = E(\theta^x)$ p.g.f

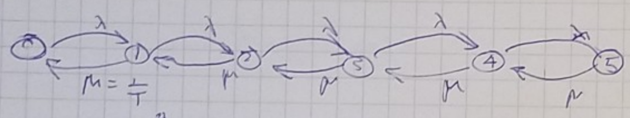
$G_x(e^t) = M_x(t)$

$E(x) = \frac{dM_x}{dt}\Big|_{t=0} = E(xe^{tx})$ $E(x^r) = \frac{d^r M_x}{dt^r}\Big|_{t=0}$



$t_{i+1} - t_i \sim Exp\left(\frac{1}{\lambda}\right)$

$N \sim Poi(\lambda)$



T is time served

$T = T_1 + T_2 + \dots + T_n + T$

$E(T) = E(N)\frac{1}{\mu}$

$E(S_n) = E(N)E(X)$

$S_n = X_1 + \dots + X_n$ can be $T = \frac{n+1}{\mu}$

$\lambda_{ess} = \sum_{n=0}^{4} \lambda_n P_n = \lambda(P_0 + P_1 + P_2 + P_3 + P_4) = \lambda(1 - P_5)$

$\lambda_0 P_0 + \lambda_1 P_1 + \lambda_2 P_2 + \dots + \lambda_5 P_5$

$(1-x)^{-1} = 1 + x + x^2 + \dots$ $\therefore \sum_{n=0}^{\infty} \rho^n = 1 + \rho + \rho^2 + \dots = (1-\rho)^{-1}$

$(1-x)^{-2} = 1 + 2x + 3x^2 + 4x^3 + \dots$

$\therefore \sum_{n=0}^{\infty} n\rho^n = \rho + 2\rho^2 + 3\rho^3 = \rho(1 + 2\rho + 3\rho^2 + \dots) = \rho(1-\rho)^{-2}$

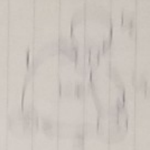
[illegible]

[illegible]

[illegible]

[illegible] steady state [illegible]

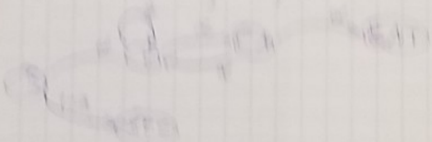
[illegible]



[illegible]

Expected time to get from start to end [illegible]

[illegible]



[illegible]

Discrete time $t \in \mathbb{Z}$

Continuous time $t \in \mathbb{R}$ $\quad Y(t)$

Discrete set of states $Y \in \Omega$

Branching process $N(t)$
Markov chain

Poisson process $N(t)$

Queuing process $N(t)$

Continuous set of states $Y \in \mathbb{R}$

random walk
$Y(t+1) = Y(t) + X(t)$
discrete rate exists

Wiener process (brownian process)
$Y(t) = \int^{t} x(t')\,dt'$

\Queuing theory/ : Coefficient of θ^{2k} where n can only be a multiple of two

$P_n = P_r(N=n) \qquad \frac{dP_0}{dt} = ? \qquad dP_1$

$\frac{dP_0}{dt} = -\lambda P_0 + \mu P_1 \Rightarrow P_1 = \frac{\lambda}{\mu} P_0$

$\frac{dP_1}{dt} = \lambda P_0 - \mu P_1 - \lambda P_1 + \mu P_2 \qquad P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$

$P_0 + P_1 + P_2 + \dots \qquad P_\infty = 1$

$P_0 \times \left[1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \dots\right] = 1$

A to B

$$\begin{array}{c} \\ \begin{bmatrix} 0.8 & 0.2 & 0 \\ 0.5 & 0 & 0.5 \\ 0.3 & 0.7 & 0 \end{bmatrix} \\ \begin{array}{ccc} \underbrace{\quad}_{A} & \underbrace{\quad}_{B} & \underbrace{\quad}_{C} \end{array} \end{array} \begin{array}{l}]A \\ ]B \\ ]C \end{array}$$

A to A is 0.8
B to A is 0.5

recurrent or transient states

$\lim_{n\to\infty} \frac{1}{n}\sum_{i=1}^{n} X_i \to E(X)$ LLN assume X_i are indep

$X_i = X^{n} \quad \frac{1}{n}\sum X_i = \underline{X} \neq E(X)$

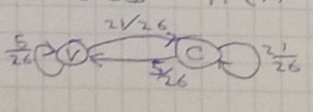
a small example $\quad$ v ccvcc vcvcccv

c	c	4
c	v	3
v	c	4
v	v	0

$$\begin{array}{c|cc} & V & C \\ \hline V & 0 & 1 \\ C & 0.43 & 0.57 \end{array}$$

is truly indep : States : $Pr(v) = \frac{5}{26} \qquad Pr(c) = \frac{21}{26}$

$Pr(vc|v)$

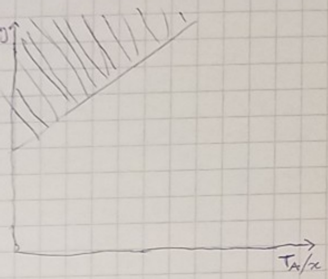


transition matrix $T = \begin{bmatrix} 0 & 1 \\ 3/7 & 4/7 \end{bmatrix}$

$$e^{-9}(1-e^{-10}) + 9e^{-9}(1-e^{-10}-10e^{-10}) + e^{-9}\frac{9^2}{2!}\left(1-e^{-10}-10e^{-10}-\frac{10^2}{2!}e^{-10}\right) \approx 0.0062$$

~~prob~~ the first banana is eaten at least 6 hours after the first apple?

- T_A is time that first apple eaten $T_B \sim$ time that first banana eaten



$$f_{T_A,T_B}(x,y) = (5e^{-5x})(3e^{-3y})$$

$$P(T_A \geq T_B + \tfrac{1}{4}) = \int_0^\infty\int_{x+\frac{1}{4}}^\infty (5e^{-5x})(3e^{-3y})\,dy\,dx =$$

$$\int_0^\infty dx\, 5e^{-5x}\int_{x+\frac{1}{4}}^\infty 3e^{-3y}\,dy = \int_0^\infty 5e^{-5x}\left[-e^{-3y}\right]_{x+\frac{1}{4}}^\infty dx =$$

$$\int_0^\infty 5e^{-5x}e^{-3x-\frac{3}{4}}\,dx = 5e^{-\frac{3}{4}}\int_0^\infty e^{-8x}\,dx = \frac{5}{8}e^{-3/4}\left[-e^{-8x}\right]_0^\infty = \frac{5}{8}e^{-3/4} \approx 0.295$$

- 6 hours is $\frac{1}{4}$ ~~of~~ a day $\therefore P(T_A \geq T_B + \frac{1}{4}) \approx 0.295$

is satisfied by:

① a a a b
② a a a a b
③ a a a a a b
④ a a a $\underbrace{\quad\quad}_{\text{n times}}$ b $\therefore p_A = \frac{5}{8}$, $p_B = \frac{3}{8}$

①: $p_A^3 p_B$ ② $p_A^4 p_B$ ③ $p_A^5 p_B$ ④ $p_A^3 p_A^n p_B$

$\therefore P(\geq 3 \text{ apples before banana}) = \sum_{n=0}^\infty p_A^3 p_A^n p_B = p_A^3 p_B \sum_{n=0}^\infty p_A^n = \frac{p_A^3 p_B}{1-p_A} =$

$$\frac{p_A^3(1-p_A)}{(1-p_A)} = p_A^3$$

14/ suppose exactly 1 banana is eaten between midday & midnight what time do you expect that the banana was eaten?

$$P(T_B \leq t \mid B_{1/2}=1) = \frac{P(\{T_B \leq t\} \cap \{B_{1/2}=1\})}{P(B_{1/2}=1)} \qquad P(B_{1/2}=1) = \tfrac{3}{2}e^{-3/2}$$

$P(\{T_B \leq t\} \cap \{B_{1/2}=1\}) = P(\text{one banana eaten in } [0,t] \text{ and none eaten in } [t,\tfrac{1}{2})) =$

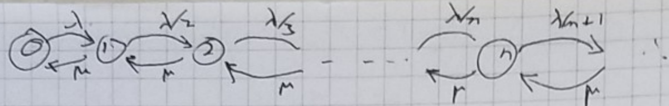
$$(3t e^{-3t})(e^{-3(\frac{1}{2}-t)}) = 3t\, e^{-3/2}$$

$$P(T_B \leq t \mid B_{1/2}=1) = \frac{3t e^{-3/2}}{\frac{3}{2}e^{-3/2}} = 2t \quad 0 \leq t \leq 1/2$$

conditional pdf $f_{T_B|B_{1/2}=1}(t)$

- $P(T_B = t \mid B_{1/2}=1) = 2$

$$E[T_B \mid B_{1/2}=1] = \int_0^{1/2} 2t\,dt = \tfrac{1}{4} \quad \text{(i.e. 6pm}$$

is  $\therefore$

in steady state : $\frac{\lambda}{n+1} P_n = \mu P_{n+1} \therefore P_n = \frac{\rho^n}{n!} P_0 \ \forall n \geq 1 \ \therefore \sum_{n\geq 0} P_0 = 1$

$\Big\{ \frac{\lambda}{0+1} P_0 = \mu P_1 \quad \frac{\lambda}{2} P_1 = \mu P_2 \quad \frac{\lambda}{3} P_2 = \mu P_3 \ \therefore \ \frac{\lambda}{n+1} P_n = \mu P_{n+1} \therefore \rho \frac{1}{n+1} P_n = P_{n+1} \therefore$

$\frac{\lambda}{\mu}\frac{1}{1} P_0 = \rho \frac{1}{1} P_0 = P_1 \quad \rho \frac{1}{2} P_1 = P_2 \quad \rho \frac{1}{3} P_2 = P_3 \therefore$

$P_2 = \rho \frac{1}{2} \rho \frac{1}{1} P_0 = \rho^2 \frac{1}{1\times 2} P_0 , \quad P_3 = \rho^3 \frac{1}{1\times 2\times 3} P_0 \quad \therefore P_n = \rho^n \frac{1}{n!} P_0 \quad \forall n \geq 1 \Big\}$

$\therefore \sum_{n\geq 0} P_n = 1 \therefore \sum_{n\geq 0} P_n = P_0 + P_1 + P_2 + \dots = P_0 + \rho \frac{1}{1!} P_0 + \rho^2 \frac{1}{2!} P_0 + \rho^3 \frac{1}{3!} P_0 + \dots$

$= \sum_{n\geq 0} \rho^n \frac{1}{n!} P_0 = P_0 \sum_{n\geq 0} \rho^n \frac{1}{n!} = 1 \quad \therefore \quad \frac{1}{P_0} = \sum_{n\geq 0} \frac{\rho^n}{n!} = e^{\rho} \therefore$

$P_0 = \frac{1}{e^{\rho}} = e^{-\rho} \quad \therefore \quad P_n = \rho^n \frac{1}{n!} P_0 = \rho^n \frac{1}{n!} e^{-\rho} = \frac{\rho^n}{n!} e^{-\rho} \therefore$

$L_s = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n \frac{\rho^n}{n!} e^{-\rho} = e^{-\rho} \sum_{n=0}^{\infty} \frac{\rho^n}{(n-1)!} = \sum_{n=0}^{\infty} n \rho^n \frac{1}{n!} P_0 = P_0 \sum_{n=0}^{\infty} \frac{\rho^n}{(n-1)!} =$

$L_s = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n \frac{\rho^n}{n!} P_0 = P_0 \sum_{n=0}^{\infty} n \frac{\rho^n}{n!} = P_0 \rho \frac{1}{\rho} \sum_{n=0}^{\infty} n \frac{\rho^n}{n!} =$

$P_0 \rho \sum_{n=0}^{\infty} n \rho^{-1} \frac{\rho^n}{n!} = P_0 \rho \left[0 + \rho^{-1} \frac{\rho^1}{1!} + 2\rho^{-1} \frac{\rho^2}{2!} + \dots \right] =$

$P_0 \rho \left(\left[\sum_{n=0}^{0} n \rho^{-1} \frac{\rho^n}{n!} \right] + \sum_{n=1}^{\infty} n \rho^{-1} \frac{\rho^n}{n!} \right) =$

$P_0 \rho \left(0 + \sum_{n=1}^{\infty} \frac{\rho^{n-1}}{(n-1)!} \right) = P_0 \rho \sum_{n=1}^{\infty} \frac{\rho^{n-1}}{(n-1)!} = P_0 \rho \sum_{n=0}^{\infty} \frac{\rho^n}{n!} = P_0 \rho e^{\rho} =$

$e^{-\rho} \rho e^{\rho} = \rho = \frac{\lambda}{\mu} = \frac{10}{15} = \frac{2}{3}$

$L_q = L_s -$ (Mean no being served) $= L_s - (1 - P_0) = L_s - 1 + P_0 =$

$L_q = \frac{2}{3} - 1 + P_0 = \frac{2}{3} - 1 + e^{-\rho} \approx -\frac{1}{3} + e^{2/3} \approx 0.18$

by little's thm: $\lambda_{eff} = \sum_{n=0}^{\infty} \left(\frac{\rho^n e^{-\rho}}{n!} \right) \frac{\lambda}{n+1} = \sum_{n=0}^{\infty} P_n \frac{\lambda}{n+1} = \sum_{n=0}^{\infty} \lambda_n P_n =$

$\sum_{n=0}^{\infty} \frac{\lambda}{n+1} P_n = \sum_{n=0}^{\infty} \left(\frac{\lambda}{n+1} \right) \left(\frac{\rho^n e^{-\rho}}{n!} \right) = \lambda e^{-\rho} \sum_{n=0}^{\infty} \frac{\rho^n}{(n+1)!} = \frac{1}{\rho} \lambda e^{-\rho} \sum_{n=0}^{\infty} \rho \frac{\rho^n}{(n+1)!}$

$= \frac{1}{\rho} \lambda e^{-\rho} \sum_{n=0}^{\infty} \frac{\rho^{n+1}}{(n+1)!} = \frac{1}{\rho} \lambda e^{-\rho} \left(-1 + 1 + \sum_{n=1}^{\infty} \frac{\rho^n}{n!} \right) =$

$\frac{1}{\rho} \lambda e^{-\rho} \left(-1 + \frac{\rho^0}{0!} + \sum_{n=1}^{\infty} \frac{\rho^n}{n!} \right) = \frac{1}{\rho} \lambda e^{-\rho} \left(-1 + \sum_{n=0}^{\infty} \frac{\rho^n}{n!} \right) = \frac{\lambda}{\rho} e^{-\rho} (e^{\rho} - 1)$

$= \frac{\lambda}{\rho} (1 - e^{-\rho}) = \frac{\lambda (1 - e^{-\rho})}{\rho} \therefore$ by littles thm:

$L_s = W_s \lambda_{eff} \therefore W_s = L_s \frac{1}{\lambda_{eff}} = \rho \frac{\rho}{\lambda (1 - e^{-\rho})} = \frac{\rho^2}{\lambda (1 - e^{-\rho})} \approx 5.48$ mins

$f_1^{(1)} = a_1 \quad f_1^{(2)} = (1-a_1)(1-b_2)$

$f_1^{(3)} = (1-a_1)(b_2)(1-b_3) \quad f_1^{(4)} = (1-a_1)b_2 b_3(1-b_4)$ ∴

for $n \geq 3$: $f_1^{(n)} = (1-a_1)b_2 b_3 \cdots b_{n-1}(1-b_n)$ ∴

Let $a_n = \frac{1}{n}$, $b_n = \frac{1}{n}$ ∴ $1-a_n = 1-\frac{1}{n} = \frac{n}{n} - \frac{1}{n} = \frac{n-1}{n}$ ∴ $1-b_n = \frac{n-1}{n}$ ∴

$f_1^{(1)} = \frac{1}{1} = 1$, $f_1^{(2)} = (1-1)(1-1) = 0$

$f_1^{(n)} = (1-1)b_2 b_3 \cdots b_{n-1}(1-\frac{1}{n}) = 0$ ✗ but it is possible for a_n, b_n ;

$\mu_1 = \sum_{n=1}^{\infty} n f_1^{(n)} = \infty$ ∴ State 1 is null recurrent & State 1 & 2 form a smallest possible subchain ∴ State 2 is also null recurrent

\6b/ $\tilde{p} = \tilde{p}T$ ∴ $p^{(1)} = p^{(0)}T(0)$ ∴ $T(n) = \begin{pmatrix} 1/2 & 1/2 \\ \frac{1}{n+1} & \frac{n}{n+1} \end{pmatrix}$ ∴

$1 - \frac{n}{n+1} = \frac{n+1-n}{n+1} = \frac{1}{n+1}$ ∴ $T(0) = \begin{pmatrix} 1/2 & 1/2 \\ 1 & 0 \end{pmatrix}$ ∴

as $n \to \infty$: $T(n) = \begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$ ∴ $\tilde{p} = (0,1)$ ∵ $\tilde{p} = \tilde{p}\begin{pmatrix} 1/2 & 1/2 \\ 0 & 1 \end{pmatrix}$ ∴

$\tilde{p} = (0,1)$ as $n \to \infty$ ∴ goes to state 2 in Z long run but escapes to state 1 with probab 1

\6c/ yes its possible ∵ $p^{(0)} = (\frac{1}{2}, \frac{1}{2})$, $p^{(1)} = p^{(0)}T(0)$ Z

$\tilde{p} = (0,1)$ ∴ $S_n = \frac{1}{n}\sum_{k=0}^{n-1} p^{(k)}$ $\tilde{p} = p^{(k)}$ as $k \to \infty$, $p^{(0)} = (\frac{1}{2}, \frac{1}{2})$, $\tilde{p} = (0,1)$ ∴

Let $p^{(n)} = (\gamma_n \ \delta_n)$ ∴ $\frac{1}{2} \leq \delta_n < 1$, $0 \leq \gamma_n \leq \frac{1}{2}$ ∴

$\sum_{k=0}^{n-1} \delta_n \notin \mathbb{R}$ ∴ divergent ∴ $\sum_{k=0}^{n-1} \delta_n = \infty$ as $n \to \infty$ ∴

$\begin{pmatrix} S_{n1} \\ S_{n2} \end{pmatrix} = \frac{1}{n}\sum_{k=0}^{n-1} p^{(k)} = \frac{1}{n}\sum_{k=0}^{n-1}\begin{pmatrix} \gamma_k \\ \delta_k \end{pmatrix} = \infty$ as $n \to \infty$ ∴ S_n doesn't exist as $n \to \infty$

\7/ eigval: $(1-\lambda)(2-\lambda)(3-\lambda)(\cdots)(N-1-\lambda)(N-1) = 0$ ∴

T is a 0 matrix apart from $T_{ii} = i$ ∴ for eigvals :

~~$|T_{ii} - \lambda I|$~~ $|T - \lambda I| = \prod_{i=1}^{N}(T_{ii} - \lambda) = \prod_{i=1}^{N}(i - \lambda) = 0$ ∴

eigvals : $\lambda = 1, 2, 3, \ldots, N-1, N$ ∴ $\lambda = n$ ∀ $n \leq N$, $n \in \mathbb{N}$

\Sheet 5/ \1/ have $X_n + Y_n = n$ desire: $W_n = X_n - Y_n$ of heads & tails. Z random variab W_n takes values $\{-3, -2, \ldots, 2, 3\}$ ∴ Z game stops is either $W_n = -3$ or $W_n = 3$. Z probab for W_n to increase is p, ∴ Z prob for W_n to decrease is $q = 1-p$ ∴

Setting $U_n = W_n + 3$: obtain a standard gamblers ruin probl & ∴

$X_n = Y_n + 3$ corres to $U_n = 6$ $\{W_n = X_n - Y_n = X_n - (X_n - 3) = 3$ ∴ $U_n = W_n + 3 = 3 + 3 = 6\}$

\Sheet 4/ 2 corres series is bounded by a geometric series $\therefore$ by similar arguments: states 2, 3 are posi recurrent

ii) to find steady state vec $\tilde{P} = (\tilde{P}_1, \tilde{P}_2, \tilde{P}) : \tilde{P}T = \tilde{P}$.

$\tilde{P}_1 + \tilde{P}_2 + \tilde{P}_3 = 1$ $\therefore$ $\begin{pmatrix} -1/2 & 0 & 1-\alpha \\ 1/2 & -1 & 0 \\ 0 & 1 & \alpha - 1 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ $\therefore$ $\tilde{P}_1 = 2\tilde{P}_2$, $\tilde{P}_2 = (1-\alpha)\tilde{P}_3$ $\therefore$

$P_1 = 1 - \tilde{P}_2 - \tilde{P}_3$:

steady state vec is : $\tilde{P} = \frac{1}{4-3\alpha}(2(1-\alpha), 1-\alpha, 1)$

\5b/ $f_1^{(1)} = \frac{1}{2}$, $f_1^{(2)} = 0$, $f_1^{(3)} = \frac{1}{2}(1-\alpha_3)$, $f_1^{(4)} = \frac{1}{2}\alpha_3(1-\alpha_4)$,

$f_1^{(5)} = \frac{1}{2}\alpha_3\alpha_4(1-\alpha_5)$, ..., $\therefore$ for $n \geq 5$: $f_1^{(n)} = \frac{1}{2}\alpha_3\alpha_4 \cdots \alpha_{n-1}(1-\alpha_n)$ $\therefore$

$\sum_{k=1}^{n} f_1^{(k)} = \frac{1}{2} + 0 + \frac{1}{2}(1-\alpha_3) + \frac{1}{2}\alpha_3(1-\alpha_4) + \frac{1}{2}\alpha_3\alpha_4(1-\alpha_5) + \dots + \frac{1}{2}\alpha_3\alpha_4 \cdots \alpha_{n-1}(1-\alpha_n) =$

$\left\{\frac{1}{2} + \frac{1}{2}\left(1 - \frac{3^3}{(4)^3}\right) + \frac{1}{2}\frac{3^3}{4^3}\left(1 - \frac{4^3}{5^3}\right) + \frac{1}{2}\frac{3^3}{4^3}\frac{4^3}{5^3}\left(1 - \frac{5^3}{6^3}\right) + \dots + \frac{1}{2}\frac{3^3}{4^3}\frac{4^3}{5^3} \cdots \frac{(n-1)^3}{n^3}\left(1 - \frac{n^3}{(n+1)^3}\right) = \right\}$

$\cancel{\frac{1}{2} + \frac{1}{2}\left(1 - \frac{3^3}{4^3}\right) \frac{1}{2}}$ $1 - \frac{1}{2} \times \frac{3^3}{4^3} \times \frac{4^3}{5^3} \times \frac{5^3}{6^3} \times \dots \times \frac{n^3}{(n+1)^3} = 1 - \frac{27}{2(n+1)^3}$ $\therefore$

$f_1 = \lim_{n\to\infty}\left(\sum_{k=1}^{\infty} f_1^{(k)}\right) = \lim_{n\to\infty}\left(1 - \frac{27}{2(n+1)^3}\right) = 1$. $\therefore$ State 1 is recurrent

$\therefore$ to show posi recurrence: $\cancel{nf_1^n =}$ we est: $nf_1^n = n\left(\frac{1}{2}\alpha_3 \cdots \alpha_{n-1}\right)(1-\alpha_n)$

$= n\left(\frac{27}{2n^3}\left(1 - \frac{n^3}{(n+1)^3}\right)\right) = \frac{27}{2n^3}\left(\frac{(n+1)^3 - n^3}{(n+1)^3}\right) \approx \frac{C}{n^3}$ $\left\{ 1 - \alpha_n = 1 - \frac{n^3}{(n+1)^3} \therefore n = n \therefore\right.$

$\left.\frac{1}{2}\alpha_3(\alpha_4)(\alpha_5)(\dots)(\alpha_{n-1}) = \frac{1}{2}\alpha_3\alpha_4\alpha_5 \cdots \alpha_{n-1} = \frac{1}{2}\frac{3^3}{4^3}\frac{4^3}{5^3}\frac{5^3}{6^3} \cdots \frac{(n-1)^3}{n^3} = \frac{1}{2}\frac{3^3}{n^3} = \frac{27}{2n^3}\right\}$

for some C $\therefore \sum_{n=0}^{\infty}\frac{1}{n^3} < \infty$ $\therefore$ $\mu_1 = \sum_{n=1}^{\infty} n f_1^{(n)} < \infty$ $\therefore$

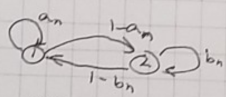
State 1 is posi recurrent

\6a/ for State 1: null recurrent is $\cancel{\ }$ $\mu_1 = \sum_{n=1}^{\infty} n f_1^{(n)} \leq \infty$

State 2: null recurrent is $\mu_2 = \sum_{n=1}^{\infty} n f_2^{(n)} \neq \infty$ $\therefore$

yes its possible $\therefore$

is $\cancel{f_1^{(n)} \geq}$ $\cancel{f}$



$\therefore$ $f_1^{(1)} = a_n$, $f_1^{(2)} = (1-a_n)(1-b_n)$

$f_1^{(3)} = (1-a_n)(b_n)(1-b_n)$ $\quad f_1^{(4)} = (1-a_n)(b_n)^2(1-b_n)$ $\therefore$

for $n \geq 3$: $f_1^{(n)} = (1-a_n)(b_n)^{n-2}(1-b_n)$ $\therefore$ Let $a_n = \frac{1}{n}$, $b_n = \frac{1}{n}$ $\therefore$

$\mu_1 = \sum_{n=1}^{\infty} n f_1^{(n)} = nf_1^{(1)} + nf_1^{(2)} + \sum_{n=3}^{\infty} nf_1^{(n)} =$

$n a_n + n(1-a_n)(1-b_n) + \sum_{n=3}^{\infty} n(1-a_n)(b_n)^{n-2}(1-b_n) =$

$n\frac{1}{n} + n\left(1 - \frac{1}{n}\right)\left(1 - \frac{1}{n}\right) + \sum_{n=3}^{\infty} n\left(1 - \frac{1}{n}\right)\left(\frac{1}{n}\right)^{n-2}\left(1 - \frac{1}{n}\right)$ X

$\tilde{P} = (\tilde{P}_S, \tilde{P}_R)$, $\tilde{P} = \tilde{P}T$ $T = \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix}$ $\therefore$

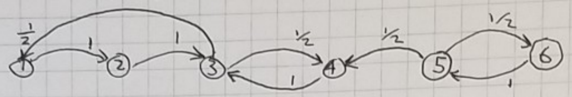
$(\tilde{P}_S, \tilde{P}_R) = (\tilde{P}_S, \tilde{P}_R)\begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} = (0.8\tilde{P}_S + 0.4\tilde{P}_R, \; 0.2\tilde{P}_S + 0.6\tilde{P}_R) = (\tilde{P}_S, \tilde{P}_R)$ $\therefore$

$0.8\tilde{P}_S + 0.4\tilde{P}_R = \tilde{P}_S$, $0.2\tilde{P}_S + 0.6\tilde{P}_R = \tilde{P}_R$ $\therefore$

$0.4\tilde{P}_R = 0.2\tilde{P}_S$, $0.2\tilde{P}_S = 0.4\tilde{P}_R$ $\therefore$ $2\tilde{P}_R = \tilde{P}_S$, $\tilde{P}_S = 2\tilde{P}_R$ $\therefore$

$P_R + P_S = 1$ $\therefore$ $\tilde{P}_R + \tilde{P}_S = 1$ $\therefore$ $\tilde{P}_R + 2\tilde{P}_R = 3\tilde{P}_R = 1$ $\therefore$ $\tilde{P}_R = \frac{1}{3}$ $\therefore$ $2\tilde{P}_R = 2\frac{1}{3} = \frac{2}{3}$

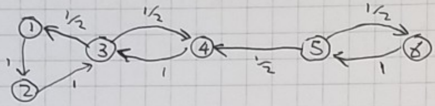
\2/



1, 2, 3, 4 is a subchain. 5, 6 is a subchain

5, 6 is transient, 1, 2, 3, 4 are recurrent

\2/

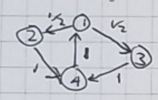


is ~~tradansition~~ transition diagram

identify two subchains

- subchain $\{1,2,3,4\}$ has aperiodic & posi recurrent states; subchain is ergodic
- subchain $\{5,6\}$ has states of period 2 which are transient

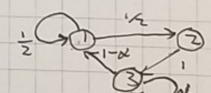
\3 sol/ .



$\therefore$ is possible $\because$ get to ② in 3, 6, 9, 12, ... steps $\therefore$ period of 3 $\therefore$ ② and ③ have a period of 3 and ① and ④ get to again always in 3 steps $\therefore$ period of 3

\4 sol/ no $\because$ of Z eignals of Z transition Mat

\5a/ const α: transition diag:



$\left\{ f_i = \sum_{n\geq 1} f_i^{(n)} = \frac{1}{2} + 0 + \sum_{n\geq 3} \frac{1}{2}\alpha^{n-3}(1-\alpha) = \frac{1}{2} + \frac{1}{2}(1-\alpha)\sum_{n\geq 3}\alpha^{n-3} = \frac{1}{2} + \frac{1}{2}(1-\alpha)\sum_{n\geq 0}\alpha^n = \frac{1}{2} + \frac{1}{2}(1-\alpha)\frac{1}{1-\alpha} = \frac{1}{2} + \frac{1}{2} = 1 \right\}$

$\therefore$ Let $f_i^{(n)} = P($first return to state i at time $n)$

$\therefore f_i = \sum_{n=1}^{\infty} f_i^{(n)}$, $\mu_i = \sum_{n=1}^{\infty} n f_i^{(n)}$ for recurrence & posi recurrence $\therefore$

for state i: $f_1^{(1)} = \frac{1}{2}$, $f_1^{(2)} = 0$, $f_1^{(3)} = \frac{1}{2}(1-\alpha)$, $f_1^{(4)} = \frac{1}{2}\alpha(1-\alpha)$,

$f_1^{(5)} = \frac{1}{2}\alpha^2(1-\alpha)$, ..., $f_1^{(n)} = \frac{1}{2}\alpha^{n-3}(1-\alpha)$. $\therefore$ $\Big\{ f_1 = \sum_{n=1}^{\infty}\frac{1}{2}\alpha^{n-3}(1-\alpha) =$

$\sum_{n=1}^{\infty}(\frac{1}{2}\alpha^{n-3} - \frac{1}{2}\alpha^{n-3}\alpha) = \frac{1}{2}(1-\alpha)\sum_{n=1}^{\infty}(\alpha^{n-3}) = \frac{1}{2}(1-\alpha)\frac{1}{\alpha^3}\sum_{n=1}^{\infty}\alpha^n = \frac{1}{2}(1-\alpha)\alpha^{-3}\frac{\alpha}{1-\alpha}$

$= \frac{1}{2}\alpha^{-3}\alpha = \frac{1}{2}\frac{1}{\alpha^2}\Big\}$ ✗ $f_1 = \sum_{n=1}^{\infty} f_1^{(n)} = \sum_{n=1}^{\infty}\frac{1}{2}\alpha^{n-3}(1-\alpha) = \frac{1}{2} + \frac{1}{2}(1-\alpha)\sum_{n=0}^{\infty}\alpha^n = 1$

& $\mu_1 < \infty$ $\Big\{ \mu_1 = \sum_{n=1}^{\infty} n f_1^{(n)} = \sum_{n=1}^{\infty} n\frac{1}{2}\alpha^{n-3}(1-\alpha) = \frac{1}{2}(1-\alpha)\sum_{n=1}^{\infty} n\alpha^n\alpha^{-3} =$

$\frac{1}{2}(1-\alpha)\alpha^{-3}\sum_{n=1}^{\infty} n\alpha^n = \frac{1}{2}(1-\alpha)\alpha^{-3}\alpha(1-\alpha)^{-2} = \frac{1}{2}\frac{1}{(1-\alpha)\alpha^2} < \infty$ for $|\alpha| < 1 \Big\}$ $\therefore$

Sheet 4 $Pr(\text{sunny at least once}) = 1 - Pr(\text{never sunny}) =$
$1 - Pr(R \to R \to R \to R) = 1 - 0.6^3 = 0.784$

1c/ if sunny on friday: $P(\text{sunny Sat \& Sund} \mid \text{sunny Fri}) =$
$Pr(S \to S \to S) = 0.8^2 = \cancel{0.512}\ 0.64$
if rain fri: $P(\text{sunny Sat \& Sund} \mid \text{rain fri}) =$
$Pr(R \to S \to S) = 0.4 \times 0.8 = 0.32$ $\therefore$
$P(\text{sunny Sat \& Sund}) = 0.64 + 0.32 = 0.96$ X 0.5504
$P(\underset{\text{wed}}{S} \to \underset{\text{thurs}}{?} \to \underset{\text{fri}}{?} \to \underset{\text{satur}}{S} \to \underset{\text{sund}}{S}) = \Big(\underset{\text{paths S to S in 3 steps}}{(T^3)_{SS}}\Big) \times \underset{\text{path S to S in 1 step}}{\big((T)_{SS}\big)}$
$= \Big[(T_{SS})^3 + (T_{SR} \times T_{RR} \times T_{RS}) + (T_{SR} \times T_{RS} \times T_{SS}) + (T_{SS} \times T_{SR} \times T_{RS})\Big] \times T_{SS} =$
$[0.8^3 + (0.2 \times 0.6 \times 0.4) + (0.2 \times 0.4 \times 0.8) + (0.8 \times 0.2 \times 0.4)] \times 0.8 =$
$0.688 \times 0.8 = 0.5504$

1d/ $P(\underset{\text{wed}}{S} \to \underset{\text{thurs}}{?} \to \underset{\text{fri}}{R}) =$

$\cancel{[(T_{SS} \times T_{SR}) + T_S}$ $P(S \text{ on weds}) \times (T^2)_{SR} + P(R \text{ on weds}) \times (T^2)_{RR} =$
$0.7[(T_{SS} \times T_{SR}) + (T_{SR} \times T_{RR})] + 0.3 \times [(T_{RS} \times T_{SR}) + (T_{RR} \times T_{RR})] =$

$0.7[(0.8 \times 0.2) + (0.2 \times 0.6)] + 0.3[(0.4 \times 0.2) + (0.6 \times 0.6)] =$
$0.7[0.28] + 0.3 \times [0.44] = 0.196 + 0.132 = 0.328$ ✓
wed is $(0.7, 0.3)$ $\therefore$ $(0.7, 0.3)T^2 = (0.7, 0.3)\begin{pmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{pmatrix} =$
$(0.672, 0.328)$ $\therefore 0.328$

the weather probab distri for wed is $(0.7, 0.3)$, a row vec of probabs:
$(0.7, 0.3)T^2 = (0.7, 0.3)\begin{pmatrix} 0.72 & 0.28 \\ 0.56 & 0.44 \end{pmatrix} = (0.672, 0.328)$ the second entry of this row vec is the probab for it to be raining on friday

1e/ $\cancel{\frac{dP_S}{dt} = -0.2P_S + 0.4P_R}$ X, $\frac{dP_R}{dt} = -0.4P_R + 0.6P_R + 0.2P_S = 0.2P_R + 0.2P_S$

$\frac{dP_S}{dt} = 0.8P_S - 0.2P_S + 0.4P_R = 0.6P_S + 0.4P_R$ $\therefore$ steady state, $\frac{dP_R}{dt} = \frac{dP_S}{dt} = 0$
$\therefore 0.2P_R + 0.2P_S = 0$ $\therefore 0.6P_S + 0.4P_R = 0$ $\therefore$ $0.2P_R = -0.2P_S$ $\therefore$ $P_R = -P_S$ $\therefore$
$0.6P_S + 0.4(-P_S) = 0.2P_S = 0$ $\therefore P_S = 0$ $\therefore P_R = 0$ X
Solve $\tilde{P} = \tilde{P}T$ $\therefore$ $\begin{bmatrix} -0.2 & 0.4 \\ 0.2 & -0.4 \end{bmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\therefore \tilde{P}_S = 2\tilde{P}_R$ $\therefore$ $\tilde{P}_S + \tilde{P}_R = 1$ $\therefore$
$2\tilde{P}_R + \tilde{P}_R = 3\tilde{P}_R = 1$ $\therefore \tilde{P}_R = \frac{1}{3}$ $\therefore$ $2\tilde{P}_R = 2\frac{1}{3} = \frac{2}{3} = \tilde{P}_S$ $\therefore$ $\tilde{P} = (\frac{2}{3}, \frac{1}{3})$

$\therefore P_n = 2\dfrac{\rho^n(1-\rho)}{1+\rho} \quad \forall n \geq 1 \therefore$

$L_s = \sum_{n\geq 0} n P_n = \cancel{\sum} \; 0 P_0 + \sum_{n\geq 1} n P_n = \sum_{n\geq 1} n\, 2\dfrac{\rho^n(1-\rho)}{1+\rho} = 2\dfrac{(1-\rho)}{1+\rho}\sum_{n\geq 1} n\rho^n =$

$\cancel{2\dfrac{1-\rho}{1+\rho}} \; 2\dfrac{\rho}{1-\rho^2} = 2\dfrac{\rho}{(1+\rho)(1-\rho)} = 2\dfrac{\rho(1-\rho)}{(1+\rho)(1-\rho)(1-\rho)} =$

$2\dfrac{(1-\rho)}{1+\rho}\dfrac{\rho}{(1-\rho)^2} = 2\dfrac{(1-\rho)}{1+\rho}\sum_{n\geq 1} n\rho^n = L_s \therefore \sum_{n\geq 1} n\rho^n = \dfrac{\rho}{(1-\rho)^2}$

$\therefore L_s = 2\dfrac{\rho}{1-\rho^2} = 2\dfrac{\left(\frac{\lambda}{2\mu}\right)}{\left(1-\left(\frac{\lambda}{2\mu}\right)^2\right)} \therefore L_s^{(1)} = \dfrac{\lambda}{1-\left(\frac{\lambda}{2}\right)^2} \therefore W_s^{(1)} = \dfrac{L_s^{(1)}}{\lambda} = \dfrac{4}{4-\lambda^2}$

\4b/ two M/M/1 systs $\dfrac{\lambda}{2}$, $\mu = 1 \therefore$ per line: $L_s^{(2)} = \dfrac{\rho}{1-\rho}$

$\rho = \dfrac{(\lambda/2)}{\mu} = \dfrac{(\lambda/2)}{1} = \dfrac{\lambda}{2} \therefore L_s^{(2)} = \dfrac{\lambda/2}{1-\lambda/2} \therefore$ for $|\rho| < 1 \therefore \left|\dfrac{\lambda}{2}\right| < 1 \therefore |\lambda| < 2$

$\therefore W_s^{(2)} = L_2^{(2)} \dfrac{\lambda}{2} = \dfrac{2}{2-\lambda}$

for $\lambda < 2$: $W_s^{(2)} - W_s^{(1)} = \dfrac{2}{2-\lambda} - \dfrac{4}{4-\lambda^2} = \dfrac{2\lambda}{4-\lambda^2} > 0 \therefore$ M/M/2 is more efficient

$\left\{ L_s^{(2)} = \sum_{n=0}^{\infty} n P_n \quad \dfrac{\lambda}{2}P_0 = \mu P_1 \therefore \rho P_0 = P_1 \quad \rho P_1 = P_2 \therefore \rho^2 P_0 = P_2 \therefore \rho^n P_0 = P_n\right.$

$\therefore \sum_{n\geq 0} P_n = 1 = P_0 + P_1 + P_2 + \cdots = \sum_{n\geq 0} P_0\rho^n = P_0 \sum_{n\geq 0}\rho^n = P_0\dfrac{1}{1-\rho} \therefore 1-\rho = P_0$

$\cancel{\sum_{n\geq 0}} \; L_s^{(2)} = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n\rho^n P_0 = P_0\sum_{n=0}^{\infty} n\rho^n \cancel{A} \therefore$

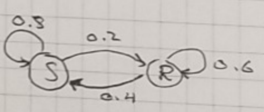
$\cancel{P_n = \rho^n \frac{1}{1-\rho}} \therefore P_n = \rho^n(1-\rho) \therefore L_s^{(2)} = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n\rho^n(1-\rho) =$

$(1-\rho)\sum_{n=0}^{\infty} n\rho^n = (1-\rho)\dfrac{\rho}{(1-\rho)^2} = \dfrac{\rho}{1-\rho} = \dfrac{\frac{\lambda}{2}}{1-\frac{\lambda}{2}}$

$\cancel{A} \quad \sum_{n\geq 1} n x^{n-1} = \dfrac{1}{x}\sum_{n\geq 1} n x^n = (1-x)^{-2}$

$\therefore \cancel{x} \sum_{n\geq 1}^{\infty} n x^n = x\dfrac{1}{x}\sum_{n\geq 1} n x^n = x(1-x)^{-2}$

\Sheet 4/ $T = \begin{array}{cc} & \begin{array}{cc} 1 & 2 \end{array} \\ \begin{array}{c} 1 \\ 2 \end{array} & \begin{pmatrix} 0.8 & 0.2 \\ 0.4 & 0.6 \end{pmatrix} \end{array}$



\1a/ pr(rain next | raining) = 0.6 $\therefore 0.6^2 = 0.36$

$P(R \to R \to R) = (0.6)^2 = 0.36$

\1b/ ~~P(rain on 1st day~~ or equals P(next rain is 1st or 2nd or 3rd day raining)

$= 0.6 + 0.4\times 0.6 + 0.4^2\times 0.6 = 0.936$ ✗

$0.6 + 0.4\times 0.2 + 0.4\times 0.8\times 0.2 = 0.744$ ✗

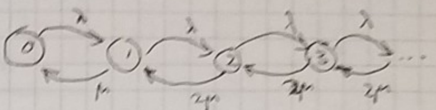
Sheet 3

$L_q = W_q \lambda_{ess}$ $\therefore W_q = \frac{L_q}{\lambda_{ess}} = \frac{0.18\,\rho}{\lambda(1-e^{-\rho})} =$

1.48 mins

4a

$M/M/2$ λ $\mu = 1$



$\therefore \mu + \mu = 2\mu = 1 + 1 = 2 \therefore$

need $\lambda < 2$ to have a steady state & $L_s^{(1)} = \frac{2\left(\frac{\lambda}{2\mu}\right)}{1-\left(\frac{\lambda}{2\mu}\right)^2} = \frac{\lambda}{1-\left(\frac{\lambda}{2}\right)^2}$

is $L_s^{(1)} = \sum_{n\geq 0} n P_n$ $\therefore \lambda P_0 = \mu P_1$,

$\lambda P_1 = 2\mu P_2$ $\quad \lambda P_2 = 2\mu P_3$ $\therefore$ $2\lambda P_0 = 2\mu P_1$ $\therefore$ $2\frac{\lambda}{2\mu} P_0 = P_1 = 2\rho P_0$ $\therefore$

$\frac{\lambda}{2\mu} P_1 = P_2 = \rho P_1$ $\quad \frac{\lambda}{2\mu} P_2 = \rho P_2 = P_3$ $\therefore \frac{\lambda}{2\mu} P_n = \rho P_n = P_{n+1}$ $\therefore$

$P_3 = \rho\rho P_1 = \rho^2 P_1$ $\therefore P_{n+1} = \rho^n P_1$ $\therefore P_3 = \rho^2 P_1 = 2\rho P_0 \rho^2 = 2\rho^3 P_0$ $\therefore$

$P_n = 2\rho^n P_0$ $\therefore \sum_{n\geq 0} P_n = 1$ $\therefore \sum_{n\geq 0} P_n = P_0 + P_1 + P_2 + \dots =$

$P_0 + 2\rho^1 P_0 + 2\rho^2 P_0 + 2\rho^3 P_0 + \dots$

$-P_0 + 2P_0 + 2\rho^1 P_0 + 2\rho^2 P_0 + 2\rho^3 P_0 + \dots = -P_0 + \sum_{n\geq 0} 2\rho^n P_0 = -P_0 + 2P_0 \sum_{n\geq 0} \rho^n =$

$-P_0 + 2P_0 \frac{1}{1-\rho} = 1 = P_0\left(-1 + 2\frac{1}{1-\rho}\right)$ $\therefore P_0 = \frac{1}{-1+2\frac{1}{1-\rho}}$ $\therefore$

$P_n = 2\rho^n P_0 = 2\rho^n \frac{1}{-1+2\left(\frac{1}{1-\rho}\right)}$ $\therefore$

$L_s = \sum_{n=0}^{\infty} n P_n = \sum_{n=0}^{\infty} n 2\rho^n \frac{1}{-1+2\left(\frac{1}{1-\rho}\right)} = 2\frac{1}{-1+2\left(\frac{1}{1-\rho}\right)} \sum_{n=0}^{\infty} n\rho^n$

$L_s^{(1)} = \frac{2\left(\frac{\lambda}{2\mu}\right)}{1-\left(\frac{\lambda}{2\mu}\right)^2}$ $\therefore W_s^{(1)} = L_s^{(1)} \frac{1}{\lambda} = \frac{L_s^{(1)}}{\lambda} = \frac{\lambda}{1-\left(\frac{\lambda}{2}\right)^2} \frac{1}{\lambda} = \frac{1}{1-\left(\frac{\lambda^2}{4}\right)} = \frac{4}{4-\lambda^2}$

$L_s^{(1)} = \frac{\lambda}{1-\left(\frac{\lambda}{2}\right)^2}$ $\because \mu = 1$ $\therefore$

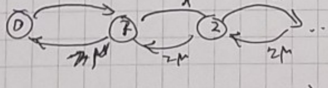
4b

$\mu = 1$ $\quad \lambda = \frac{1}{2}$ $\therefore$ $\lambda P_{01} = \mu P_{11}$ $\therefore \rho P_{01} = P_{11}$ $\therefore$

$\rho P_{11} = P_{21}$ $\therefore P_{21} = \rho^2 P_{01}$ $\therefore P_{n1} = \rho^n P_{01}$ $\therefore \sum_{n=0}^{\infty} P_{n1} = 1 = \sum_{n\geq 0} \rho^n P_{01} = P_{01} \sum_{n\geq 0} \rho^n$

$P_{01}\left(\frac{1}{1-\rho}\right) = 1$ $\therefore P_{01} = 1-\rho$ $\therefore$

4a Sol



$\lambda P_0 = \mu P_1$, $\lambda P_1 = 2\mu P_2$, $\lambda P_2 = 2\mu P_3$ $\therefore$

$\lambda P_n = 2\mu P_{n+1}$ $\forall n \geq 1$ $\therefore \rho = \frac{\lambda}{2\mu}$ $\therefore \frac{\lambda}{2\mu} P_n = P_{n+1}$ $\quad P_n = 2\rho^n P_0$ $\forall n \geq 1$ $\therefore$

$\sum_{n\geq 0} P_n = 1 = P_0 + P_1 + P_2 + P_3 + \dots = P_0 + 2\rho^1 P_0 + 2\rho^2 P_0 + 2\rho^3 P_0 + \dots = P_0 + 2P_0 \sum_{n\geq 1} \rho^n =$

$P_0\left(1 + 2\sum_{n\geq 1} \rho^n\right) = P_0\left(1 + 2\frac{\rho}{1-\rho}\right) = 1$ $\therefore P_0 = \frac{1}{1+2\frac{\rho}{1-\rho}}$ $\therefore P_0 = \frac{1-\rho}{1-\rho+2\rho} = \frac{1-\rho}{1+\rho}$ $\therefore$

$\theta_i = p\theta_{i+1} + q\theta_{i-1}$ this syst solve by trial sol $\theta_i = A\lambda^i$ Z solve for λ $\therefore$ get 2 vals for λ. When these vals of λ are distinct Z full sol is a linear combination of these trial sols for $0<i<N$:

$A\lambda^i = pA\lambda^{i+1} + qA\lambda^{i-1}$ $\therefore$ $A\lambda^{i-1}(p\lambda^2 - \lambda + q) = 0$ $\therefore$ $\lambda = 1$, $\lambda = q/p$ $\therefore$

$\because$ $p+q=1$ $\therefore$ assuming $p \neq q$: $\theta_i = A + B\rho^i$, $\rho = \frac{q}{p}$

$\theta_0 = 0$, $\theta_N = 1$ $\therefore$ $\theta_i = \frac{1-\rho^i}{1-\rho^N}$, when $\rho = \frac{q}{p} \neq 1$ $(0 \leq i \leq N)$

Z sol for $p = q = \frac{1}{2}$ can be found from this eqn using L'Hopital's rule or $\rho = 1+\varepsilon$ then limit of θ_i as $\varepsilon \to 0$ $\therefore$ $\theta_i = \frac{i}{N}$

when $N \to \infty$ these sols still make sense & taking Z limits:

$\theta_i = 0$ $(p \leq q)$, $\theta_i = 1 - \rho^i$ $(p > q)$

notice when $p = q = \frac{1}{2}$ (& N unbounded) we are certain (with prob $1 - \theta_i = 1$) to hit state zero $\therefore$ even in a fair game, a gambler will always eventually get ruined if they play against a player (or casino) with increasingly large reserves

to find Z expected waiting time to finish we solve Z syst

$D_i = 1 + pD_{i+1} + qD_{i-1}$ $(0 \leq i < N)$ $D_0 = D_N = 0$ $\therefore$

for $p = q = \frac{1}{2}$, Z sol is $D_i = i(N-i)$

remark: if Z states of Z syst take Z form $\Omega = \{a, a+s, a+2s, \ldots, b\}$, with $b - a = Ns$ then can convert Z probl to a standard gamblers ruin probl $\{0, 1, \ldots, N\}$ by Z sub $Y_n = (X_n - b)/s$ with $X_n \in \Omega$ denoting Z state of Z syst at time n. By taking $a \to \infty$ & $b \to \infty$ (with $s = 1$) recover Z random walk models, its possible to look at extended probls eg allowing transitions to Z same state etc

$\therefore$ try guessing $\theta_i = A\lambda^i$ $\therefore$ $A\lambda^i = pA\lambda^{i+1} + qA\lambda^{i-1}$ $\therefore$

$A\lambda^i[1 - p\lambda - q\lambda^{-1}] = 0$ $\therefore$ $A\lambda^{i-1}[\lambda - p\lambda^2 - q] = 0$ $\therefore$

$p\lambda^2 - \lambda + q = 0$ $\therefore$ $\lambda = \dfrac{+1 \pm \sqrt{1-2pq}}{2p} = 1, \rho = \dfrac{q}{p}$ $\therefore$

$\theta_i = A1^i + B\rho^i = A + B\rho^i = A + B\left(\frac{q}{p}\right)^i$

$\theta_N = 1$, $\theta_0 = 0$ $\therefore$

$A + B\rho^N = 1$, $A + B\rho^0 = 0 = A + B(1) + A + B = 0$ $\therefore$ $B = -A$ $\therefore$

$A(1-\rho^N) = 1$ $\therefore$ $A = (1-\rho^N)^{-1}$ $\therefore$ $B = -(1-\rho^N)^{-1}$ $\therefore$

$\theta_i = A + B\rho^i = A - A\rho^i = A(1-\rho^i) = (1-\rho^i)/(1-\rho^N) = \theta_i$

eg if $i = 10$, $N = 100$, $p = \frac{6}{10}$ $\therefore$ $q = \frac{4}{10}$ $\therefore$ $\rho = \frac{q}{p} = \frac{4}{6}$ $\therefore$

$\theta_{10} = (1-(\frac{4}{6})^{10})/(1-(\frac{4}{6})^{100})$ is prob of winning Z game $= 0.983(355)$

if $\rho \neq 1$

if $\rho = 1$ take limit $\therefore$ $\lim_{\rho \to 1}$? $\therefore$ $\rho = 1 + \varepsilon$ $\therefore$ $\lim_{\varepsilon \to 0}$ $\therefore$

$$\theta_i = \frac{1-(1-\varepsilon)^i}{1-(1+\varepsilon)^N} = \frac{1-(1+i\varepsilon+\frac{1}{2}i(i-1)\varepsilon^2+\frac{1}{3!}i(i-1)(i-2)\varepsilon^3+\dots)}{1-(1+N\varepsilon+\frac{1}{2}N(N-1)\varepsilon^2+\frac{1}{3!}N(N-1)(N-2)\varepsilon^3+\dots)}$$ $\therefore$

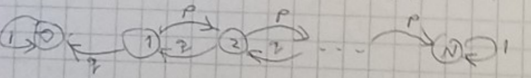
$\lim_{\varepsilon \to 0} \theta_i = \lim_{\varepsilon \to 0} \dfrac{1-(1-\varepsilon)^i}{1-(1+\varepsilon)^N} = \dfrac{i}{N}$ for $p = q = \frac{1}{2}$; $\rho = 1$

eg

Assume two players gambling against each other until either are ruined by losing their respective finite amounts of money $\therefore$ find prob of winning & Z expected duration of Z game.

$\therefore$ if I play start value £i & at each play of Z game I either win £1 with prob p or lose £1 with prob $q = 1-p$ $\therefore$ target is £N, $N > i$, game stops if I reach £0 or target of £N (other gambler is ruined if they started with only £$N-i$) what is prob of reaching target before zero? Z syst can be described by how much money I have $i \in \Omega = \{0, 1, 2, \dots, N\}$ with allowed transitions $(0 < i < N)$

$\{i\} \to \{i-1\} \cup \{i+1\}$, $T_{i,i+1} = p$, $T_{i,i-1} = q = 1-p$, absorbing set $A = \{0, N\}$



starting with £i, $0 < i < N$ want prob θ_i of reaching £N before reaching £0 (ruin) $\therefore$ BCs $= \theta_0 = 0$, $\theta_N = 1$ $0 < i < N$ $\therefore$

is $D_{HHH} = \cancel{0}\ 1$ $\therefore$ $D_{HH} = 1 + \frac{1}{2}D_T + \frac{1}{2}(1) = 1.5 + \frac{1}{2}D_T$ $\therefore$

$\frac{1}{2}D_T = 1 + \frac{1}{2}D_H$ $\therefore D_T = 2 + D_H$ $\therefore$

$D_H = 1 + 1 + \frac{1}{2}D_H + \frac{1}{2}D_{HH} = 2 + \frac{1}{2}D_H + \frac{1}{2}D_{HH}$ $\therefore$ $D_H = 4 + D_{HH}$ $\therefore$

$D_{HH} = 1.5 + 1 + \frac{1}{2}D_H = 2.5 + \frac{1}{2}D_H$ $\therefore$ $\cancel{D_H = 4 + 2.8}$ $D_H = 4 + 2.5 + \frac{1}{2}D_H$ $\therefore$

$\frac{1}{2}D_H = 6.5$ $\therefore D_H = 13$ $\therefore$

$D_T = 2 + D_H = 2 + 13 = 15$ $\therefore$

$D_S = 1 + \frac{1}{2}(15) + \frac{1}{2}(13) = 15$ }

{ is $D_{HHH} = 0$: $D_{HH} = \frac{1}{2}D_T + 1$ $\therefore$ $D_T = 2 + D_H$ $\therefore$ $D_H = 4 + D_{HH}$ $\therefore$

$D_{HH} = \frac{1}{2}(2 + D_H) + 1 = 1 + \frac{1}{2}D_H + 1 = 2 + \frac{1}{2}D_H = D_{HH}$ $\therefore$

$D_H = 4 + 2 + \frac{1}{2}D_H$ $\therefore \frac{1}{2}D_H = 6$ $\therefore D_H = 12$ $\therefore$

$D_T = 2 + 12 = 14$ $\therefore$ $D_S = 1 + 0.5(14) + \frac{1}{2}(12) = 14$ }

{ D_{HHH} is expected time to reach winning or losing state starting at HHH $\therefore$ $D_{HHH} = 0$ $\because$ your already there }

{ θ_i = P(reaching winning eventually starting from i) $\therefore$

θ_{HHH} = prob of winning eventually starting from HHH $\therefore \theta_{HHH} = 1$ }

\Ex8.3/ racing sequency HHH against TTHH) $\therefore$ state space:

$\Omega = \{S, T, H, HH, HHH, TT, TTH, TTHH\}$ all transition prob $\frac{1}{2}$ $\therefore$

$\{S\} \to \{T\} \cup \{H\}$, $\{T\} \to \{H\} \cup \{TT\}$, $\{H\} \to \{HH\} \cup \{T\}$,

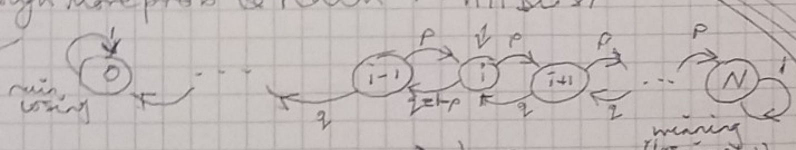
$\{HH\} \to \{HHH\} \cup \{T\}$, $\{TT\} \to \{TT\} \cup \{TTH\}$, ~~{TTH} → {T} ∪ {TTHH}~~

$\{TTH\} \to \{T\} \cup \{TTHH\}$. 2 absorbing states are

$A = \{HHH, TTHH\}$ $\therefore$ prob of reaching HHH before TTHH $\therefore$

transition matrix T_{ij} $\therefore$ BCs: $\theta_{HHH} = 1$, $\theta_{TTHH} = 0$ $\therefore$ $\theta_S = 5/12$ is prob of reaching HHH before TTHH is $5/12$. & expected time to reach already found as 14, expected to see TTHH is actually equal to 16 even though more prob to reach TTHH first $\therefore$

\Gambler's ruin/



prob (of getting to N before 0 starting from f i) $= \theta_i$

$\therefore$ $\theta_i = T_{i\,i+1}\theta_{i+1} + T_{i\,i-1}\theta_{i-1} = p\theta_{i+1} + q\theta_{i-1} = \theta_i$

$Pr(\text{reaching } TH \text{ starting from } S) = \theta_S$

$\theta_T, \theta_H, \theta_{TH} = 1, \theta_{HH} = 0$ ∴ $\underline{\theta} = T\underline{\theta}$ ∴ T:
$$\begin{array}{c|ccccc} & S & T & H & TH & HH \\ \hline S & & & & & \\ T & & & & & \\ H & & & & & \\ TH & & & & & \\ HH & & & & & \end{array}\quad \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \beta & 0 & 1-\beta & 0 \\ 0 & 1-\alpha & 0 & 0 & \alpha \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

∴ $\theta_S = \frac{1}{2}\theta_T + \frac{1}{2}\theta_H$, $\theta_T = \beta\theta_T + (1-\beta)\theta_{TH} = \beta\theta_T + (1-\beta)1 = \beta\theta_T + 1 - \beta$,

$\theta_H = (1-\alpha)\theta_T + \alpha\theta_{HH} = (1-\alpha)\theta_T + \alpha(0) = (1-\alpha)\theta_T$

$\theta_{TH} = 1\theta_{TH} = \theta_{TH} = 1$, $\theta_{HH} = 1\theta_{HH} = \theta_{HH} = 0$ ∴

$\theta_T = \beta\theta_T + 1 - \beta$ ∴ $\cancel{(1-\beta)}\ \theta_T - \beta\theta_T = (1-\beta)\theta_T = 1-\beta$ ∴ $\theta_T = 1$ ∴

$\theta_H = (1-\alpha)\theta_T = (1-\alpha)1 = 1-\alpha$ ∴

$\theta_S = \frac{1}{2}\theta_T + \frac{1}{2}\theta_H = \frac{1}{2}(\theta_T + \theta_H) = \frac{1}{2}(1+1-\alpha) = 1 - \frac{\alpha}{2}$ ∴

with coin: $\alpha = \frac{1}{2}$ ∴ $\theta_S = 1 - (\frac{1}{2})/2 = 1 - \frac{1}{4} = \frac{3}{4}$

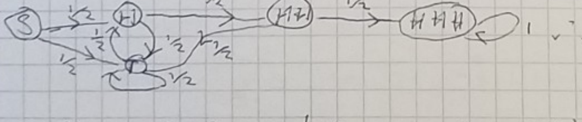
we have by def $D_i = \sum_{n=0}^{\infty} nP(Q(n,i))$ ∴ now aim for a relation between D_i & Z vals of D_j, where j is a state with $T_{ij} \neq 0$ ∴

$D_i = \sum_{n=1}^{\infty}\sum_{j\in\Omega} nP(Q(n-1,j))T_{ij} = \sum_{j\in\Omega}\sum_{n=0}^{\infty}(n+1)P(Q(n,j))T_{ij} =$

$\sum_{j\in\Omega}\Big(\underbrace{\sum_{n=0}^{\infty} nP(Q(n,j))}_{=D_j}\Big)T_{ij} + \sum_{j\in\Omega}\Big(\underbrace{\sum_{n=0}^{\infty} P(Q(n,j))}_{=1}\Big)T_{ij} =$

$\sum_{j\in\Omega} T_{ij} D_j + \sum_{j\in\Omega} T_{ij} = \sum_{j\in\Omega} T_{ij} D_j + 1$

\Ex 8.1/ \expected waiting time to see three heads in a row/ fair coin. determine expected number of tosses required to see three heads in a row ∴ most recent sequence of states $\Omega = \{S, T, H, HH, HHH\}$ & absorbing set $A = \{HHH\}$. Z state S denotes start & state HHH is Z final absorbing state ∴



∴ $T(\underline{\Omega})$: ordered as Ω
$$\begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

∴ $D_{HH} = 1 + \frac{1}{2}D_T + \frac{1}{2}D_{HHH}$, $D_H = 1 + \frac{1}{2}D_T + \frac{1}{2}D_{HH}$

$D_T = 1 + \frac{1}{2}D_T + \frac{1}{2}D_H$, $D_S = 1 + \frac{1}{2}D_T + \frac{1}{2}D_H$; $D_{HHH} = 0$ ∴

$D_S = 14$ & $D_T = 14$ ∵ going to T is equivalent to going back to S

{ is $D_{HHH} = 1$: $D_{HH} = 1 + \frac{1}{2}D_T + \frac{1}{2}(1) = 1.5 + \frac{1}{2}D_T$ ∴ $\cancel{D_T = 1 + \frac{1}{2}D_T + 0.75}$

$\frac{1}{2}D_T = 1 + \frac{1}{2}D_H$ ∴ $D_T = 2 + D_H$ ∴ $D_H = 1 + \cancel{1} + \frac{1}{2}D_H + \frac{1}{2}D_{HH}$ ∴ $\frac{1}{2}D_H = 2 + \frac{1}{2}D_{HH}$ ∴

$D_H = 4 + D_{HH}$ ∴

\1e / $G_Y(\theta) = G_{X_1}(\theta)G_{X_2}(\theta)G_{X_3}(\theta)G_{X_4}(\theta) = (G_X(\theta))^4 \therefore$

$E(Y) = G_Y'(\theta)|_{\theta=1} = \frac{d}{dx}\left((G(\theta))^4\right)|_{\theta=1} = 4(G_X(\theta))^3 G_X'(\theta)|_{\theta=1} \ldots$

$G_X'(\theta) = 0.5 + 0.6\theta^2 \therefore G_X'(\theta)|_{\theta=1} = 0.5 + 0.6(1)^2 = 1.1 \therefore$

$G_X(1) = 0.3 + 0.5(1) + 2(1)^3 = 2.8 \therefore$

$E(Y) = 4(2.8)^3 1.1 = 96.6$ ✗ (3 S.F.)

$G_Y(\theta) = (G_X(\theta))^4 = (0.3 + 0.5\theta + 0.2\theta^3)^4 =$

$(0.5\theta)^4 + 0.2\theta^3(0.5\theta)\frac{4!}{1!3!} + \ldots = \frac{37}{80}\theta^4 + \ldots \therefore$

$P(Y=4) = \cancel{0.463}\ \frac{37}{80} = 0.463$ (3 S.F.) ✗

\1e / $G_Y(\theta) = G_{X_1}(\theta)G_{X_2}(\theta)G_{X_3}(\theta) = (G_X(\theta))^3 \therefore$

$E(Y) = G_Y'(\theta)|_{\theta=1} = 3(G_X(\theta))^2 G_X'(\theta)|_{\theta=1}$

$G_X'(\theta)|_{\theta=1} = 0.5 + 0.6(1)^2 = 1.1$

$G_X(1) = 0.3 + 0.5(1) + 0.2(1)^3 = 1 \therefore$

$E(Y) = 3(1)^2 1.1 = 3.3$ ✓, $E(Y) = E(X_1) + E(X_2) + E(X_3) = 3E(X) = 3(1.1) = 3.3$

$G_Y(\theta) = (0.3 + 0.5\theta + 0.2\theta^3)^3 = 0.2\theta^3(0.5\theta)\frac{3!}{1!1!1!}(0.3) + \ldots = \frac{9}{50}\theta^4 + \ldots$

$\cancel{0.2\theta^3(0.3)^2\frac{3!}{1!2!} + (0.5\theta)^3 + \ldots} \therefore P(Y=4) = \frac{9}{50} = 0.18$ ✓

\1f / pt $E(Y) = 1 + E(X^2)\,4 = 1 + 4(0.3(0^2) + 0.5(1^2) + 0.2(3^2)) = 10.2$ ✓

$P(Y=4) = P(X_1^2 + X_2^2 + X_3^2 + X_4^2 = 3) = P(Y-1=3) \therefore$

$\cancel{G_Y(\theta)}\,(G_{X^2}(\theta))^4 = (0.3 + 0.5\theta^{1^2} + 0.2\theta^{3^2})^4 = (0.3 + 0.5\theta + 0.2\theta^9)^4 =$

$(0.5\theta)^3(0.3)\frac{4!}{3!1!} + \ldots = \frac{3}{20}\theta^3 + \ldots \therefore P(Y=4) = \frac{3}{20} = 0.15$ ✓

\2a / $G_X(\theta) = e^{\lambda(\theta^2-1)} = e^{\lambda\theta^2-\lambda} = e^{-\lambda}e^{\lambda\theta^2} = e^{-\lambda}\sum_{i=1}^{n}\frac{(\lambda\theta^2)^n}{n!} = e^{-\lambda}\sum_{i=0}^{n}\frac{\lambda^n\theta^{2n}}{n!} =$

$\cancel{e^{-\lambda}}\ e^{-\lambda}\left[1 + \frac{\lambda^1}{1!}\theta^{2(1)} + \frac{\lambda^2}{2!}\theta^{2(2)} + \frac{\lambda^3\theta^{2(3)}}{3!} + \frac{\lambda^4}{4!}\theta^{2(4)} + \ldots\right] =$

$e^{-\lambda} + e^{-\lambda}\lambda\theta^2 + \frac{1}{2}e^{-\lambda}\lambda^2\theta^4 + \frac{1}{6}e^{-\lambda}\lambda^3\theta^6 + \frac{1}{24}e^{-\lambda}\lambda^4\theta^8 + \ldots \therefore$

$\cancel{G_X(\theta) =}$ pt $f_X(k) = P(X=k) = \begin{cases} \frac{1}{n!}e^{-\lambda}\lambda^n, & k = 2n \\ 0, & k = 2n+1 \end{cases}$

\2a / $G_X(\theta) = e^{\lambda(\theta^2-1)} = e^{-\lambda}e^{\lambda\theta^2} = e^{-\lambda}\sum_{i=0}^{n}\frac{(\lambda\theta^2)^n}{n!} = \sum_{i=0}^{n}\frac{\lambda^n\theta^{2n}}{n!} =$

$e^{-\lambda}\left(1 + \lambda\theta^2 + \frac{(\lambda\theta^2)^2}{2!} + \ldots\right) = e^{-\lambda} + \lambda e^{-\lambda}\theta^2 + \frac{\lambda^2 e^{-\lambda}}{2!}\theta^4 + \ldots \therefore$

$F_X(k) = P(X=k) = \begin{cases} \frac{\lambda^{k/2}e^{-\lambda}}{(\frac{k}{2})!}, & k \text{ even} \\ 0, & k \text{ odd} \end{cases}$

CW1 $G_Y(\theta) = [0.5(2-\theta)^{-2} + 0.6(2-\theta)^{-4}](2-\theta)^{-2} =$

$0.5(2-\theta)^{-4} + 0.6(2-\theta)^{-6} = (0.5)(2^{-4})(1+(-\frac{\theta}{2}))^{-4} + 0.6(2^{-6})(1+(-\frac{\theta}{2}))^{-6} =$

$\cancel{0.5}\ \frac{1}{32}(1+(-\frac{\theta}{2}))^{-4} + \frac{3}{320}(1+(-\frac{\theta}{2}))^{-6} =$

$\frac{1}{32}\left[\frac{-4(-5)(-6)(-7)}{4!}(-\frac{\theta}{2})^4 + \cdots\right] + \frac{3}{32}\left[\frac{-6(-7)(-8)(-9)}{4!}(-\frac{\theta}{2})^4 + \cdots\right] =$

$\frac{1}{32}(35)\frac{1}{16}\theta^4 + \frac{3}{32}(128)(\frac{1}{16})\theta^4 + \cdots = \frac{413}{512}\theta^4 + \cdots \therefore$

$P(Y=4) = \frac{413}{512} = 0.807$ (3 S.F.) ✗

$G_Y = 0.3 + 0.5((2-\theta)^{-1}) + 0.2((2-\theta)^{-1})^3 = 0.3 + 0.5(2-\theta)^{-1} + 0.2(2-\theta)^{-3} =$

$0.3 + 0.5(2^{-1})(1+(-\frac{\theta}{2}))^{-1} + 0.2(2^{-3})(1+(-\frac{\theta}{2}))^{-3} =$

$0.3 + \frac{1}{4}(1+(-\frac{\theta}{2}))^{-1} + \frac{1}{40}(1+(-\frac{\theta}{2}))^{-3} =$

$\cancel{0.3 + \frac{1}{4}}$

$(\frac{1}{4})\frac{-1(-2)(-3)(-4)}{4!}(-\frac{1}{2})^4\theta^4 + (\frac{1}{40})\frac{-3(-4)(-5)(-6)}{4!}(-\frac{1}{2})^4(\theta^4) + \cdots =$

$\frac{5}{128}\theta^4 + \cdots \therefore$ ✓

$P(Y=4) = 0.0391$ (3 S.F.)

1d $G_Y(\theta) = G_X(G_W(\theta)) \therefore E(Y) = G_Y'(\theta)|_{\theta=1} = \frac{d}{d\theta}[G_X(G_W(\theta))]|_{\theta=1} =$

$G_X'(G_W(\theta))G_W'(\theta)|_{\theta=1} \therefore$

$G_W'(\theta) = -1(2-\theta)^{-2}(-1) = (2-\theta)^{-2} \therefore G_W'(\theta)|_{\theta=1} = (2-1)^{-2} = 1^{-2} = 1$

$G_X(G_W(\theta)) = 0.3 + 0.5((2-\theta)^{-1}) + 0.2((2-\theta)^{-1})^3 =$

$0.3 + 0.5(2-\theta)^{-1} + 0.2(2-\theta)^{-3} \therefore$

$G_X'(G_W(\theta)) = 0.5(-1)(2-\theta)^{-2}(-1) + 0.2(-3)(2-\theta)^{-4}(-1) = 0.5(2-\theta)^{-2} + 0.6(2-\theta)^{-4}$

$G_X'(G_W(\theta))|_{\theta=1} = 0.5(2-1)^{-2} + 0.6(2-1)^{-4} = 1.1 \therefore$

$E(Y) = 1.1 \times 1 = 1.1$

$G_Y(\theta) = 0.3 + 0.5(2-\theta)^{-1} + 0.2((2-\theta)^{-1})^3 = 0.3 + 0.5(2-\theta)^{-1} + 0.2(2-\theta)^{-3} =$

$0.3 + 0.5(2)^{-1}((1+(-\frac{\theta}{2})))^{-1} + 0.2(2)^{-3}(1+(-\frac{\theta}{2}))^{-3} =$

$\frac{1}{4}\left(\frac{-1(-2)(-3)(-4)}{4!}(-\frac{1}{2})^4\theta^4 + \cdots\right) + \frac{1}{40}\left(\frac{-3(-4)(-5)(-6)}{4!}(-\frac{1}{2})^4\theta^4 + \cdots\right)$

$\frac{1}{64}\theta^4 + \cdots + \frac{3}{128}\theta^4 + \cdots = \frac{5}{128}\theta^4 + \cdots \therefore$

$P(Y=4) = \frac{5}{128} = 0.0391$ (3 S.F.)

1c / $E(Y) = E(W+X) = E(W) + E(X)$ ∴

$E(X) = 0.3(0) + 0.5(1) + 0.2(3) = 1.1$

$E(W) = G_W'(\theta)\big|_{\theta=1} = \frac{d}{d\theta}\left[(2-\theta)^{-1}\right]\big|_{\theta=1} = -1(2-\theta)^{-2}(-1)\big|_{\theta=1} = (2-\theta)^{-2}\big|_{\theta=1} =$

$(2-1)^{-2} = 1^{-2} = 1 = E(W)$ ∴

$E(Y) = 1 + 1.1 = 2.1$ ✓

P($G_Y(\theta) = G_{X+W}(\theta) = G_X(\theta)G_W(\theta) = (0.3 + 0.5\theta + 0.2\theta^3)G_W(\theta)$ ∴

$G_W = (2-\theta)^{-1} = \left(2\left(1+\left(-\frac{\theta}{2}\right)\right)\right)^{-1} = \frac{1}{2}\left(1+\left(-\frac{\theta}{2}\right)\right)^{-1} =$

$\frac{1}{2}\left[\frac{1}{0!} + \frac{-1}{1!}\left(-\frac{\theta}{2}\right) + \frac{-1(-2)}{2!}\left(-\frac{\theta}{2}\right)^2 + \frac{-1(-2)(-3)}{3!}\left(-\frac{\theta}{2}\right)^3 + \frac{-1(-2)(-3)(-4)}{4!}\left(-\frac{\theta}{2}\right)^4 + \dots\right] =$

$\frac{1}{2}\left[1 + \frac{1}{2}\theta + \frac{1}{4}\theta^2 + \frac{1}{8}\theta^3 + \frac{1}{16}\theta^4 + \dots\right] = \frac{1}{2} + \frac{1}{4}\theta + \frac{1}{8}\theta^2 + \frac{1}{16}\theta^3 + \frac{1}{32}\theta^4 + \dots$ ∴

$G_Y = G_X(\theta)G_W(\theta) = (0.3 + 0.5\theta + 0.2\theta^3)\left(\frac{1}{2} + \frac{1}{4}\theta + \frac{1}{8}\theta^2 + \frac{1}{16}\theta^3 + \frac{1}{32}\theta^4 + \dots\right) =$

$0.3\left(\frac{1}{32}\theta^4\right) + 0.5\theta\left(\frac{1}{16}\theta^3\right) + 0.2\theta^3\left(\frac{1}{4}\theta\right) = \frac{29}{320}\theta^4$ ∴

$P(Y=4) = \frac{29}{320} = 0.0906$ (3 s.f.)

1b/ $E(Y) = E(1) + E(X_1^2) + E(X_2^2) + E(X_3^2) + E(X_4^2) = 1 + 4E(X^2)$ ∴

$E(X^2) = 0.3(0^2) + 0.5(1^2) + 0.2(3^2) = 2.1$ ∴

$E(Y) = 1 + 4(2.1) = 9.4$

$P(Y=4) = P(Y-1=3) = P(X_1^2 + X_2^2 + X_3^2 + X_4^2 = 3)$ ∴

$G_{X_1^2+X_2^2+X_3^2+X_4^2}(\theta) = G_{X_1^2}(\theta)G_{X_2^2}(\theta)G_{X_3^2}(\theta)G_{X_4^2}(\theta) = (G_{X^2}(\theta))^2$ ∴

$G_{X^2}(\theta) = \cancel{0.3 + 0.5\theta^4}\ 0.3(\theta^{0^2}) + 0.5\theta^{(1^2)} + 0.2(\theta^{(3^2)}) = 0.3\theta^0 + 0.5\theta^1 + 0.2\theta^9 = \cancel{0.3 + 0.5\theta + \dots}$

$0.3 + 0.5\theta + 0.2\theta^9$ ∴

$[G_{X^2}(\theta)]^4 = (0.3 + 0.5\theta + 0.2\theta^9)^4 =$

$0.3 \times (0.5\theta)^3\left(\frac{4!}{1!3!}\right) + \dots\ \cancel{= \frac{3}{5}\theta^3 + \dots} = 0.3(0.5)^3(4)\theta^3 + \dots = \frac{3}{20}\theta^3 + \dots$ ∴

$\cancel{P(Y=4) = \frac{3}{5} = 0.6}$ $P(Y=4) = \frac{3}{20} = 0.15$ ✓

1d) $E(Y) = \frac{d}{d\theta}[G_Y(\theta)]\big|_{\theta=1} = \frac{d}{d\theta}[G_X(G_W(\theta))]\big|_{\theta=1} = G_X'(G_W(\theta))G_W'(\theta)\big|_{\theta=1}$

∴ $G_W'(\theta) = -1(2-\theta)^{-2}(-1) = (2-\theta)^{-2}$,

$G_X(G_W(\theta)) = 0.3 + 0.5(2-\theta)^{-1} + 0.2[(2-\theta)^{-1}]^3 = 0.3 + 0.5(2-\theta)^{-1} + 0.2[(2-\theta)^{-3}]$ ∴

$G_X'(G_W(\theta)) = 0.5(-1)(2-\theta)^{-2}(-1) + 0.2(-3)(2-\theta)^{-4}(-1) =$

$0.5(2-\theta)^{-2} + 0.6(2-\theta)^{-4}$ ∴

$G_X'(G_W(\theta))G_W'(\theta)\big|_{\theta=1} = (0.5(2-1)^{-2} + 0.6(2-1)^{-4})(2-1)^{-2} = (0.5(1) + 0.6(1))(1) = 1.1$

∴ $E(Y) = 1.1$ ✓

\CW1 Sols/ $G_{W}(\theta)=(2-\theta)^{-1}$ $G_{X}(\theta)=0.3+0.5\theta+0.2\theta^{3}$

\1a/ $Y=X^2+3$ $E(Y)=E(X^2)+3+\left(\sum_j j^2 P(X=j)\right)+3=0.5+0.2\times 9+3=5.3$

$P(Y=4)=P(X=1)=0.5$

\1b/ $Y=2W$ $E(Y)=2E(W)=2G_W'(1)=2\left((2-\theta)^{-2}\right)\Big|_{\theta=1}=2$

$P(Y=4)=P(W=2)\equiv$ Coefficient of θ^2

$G_W(\theta)=(2-\theta)^{-1}=\frac{1}{2}+\frac{1}{4}+\frac{1}{8}\theta^2+\text{h.o.t}$ $\therefore P(Y=4)=\frac{1}{8}$

\1c/ $Y=W+X$, $G_Y(\theta)=G_{W+X}(\theta)=G_W(\theta)G_X(\theta)$

$E(Y)=E(W)+E(X)=1+1.1=2.1$

$P((Y)=4)=P(\{X=0\}\cap\{W=4\})+P(\{X=1\}\cap\{W=3\})+P(\{X=3\}\cap\{W=1\})=$

$0.3\left(\frac{1}{32}\right)+0.5\left(\frac{1}{16}\right)+0.2\left(\frac{1}{4}\right)=0.0906$

\1d/ $G_Y(\theta)=G_X(G_W(\theta))=0.3+0.5(2-\theta)^{-1}+0.2(2-\theta)^{-3}$ $\therefore$

$E(Y)=G_X'(1)G_W'(1)=1.1\times 1=1.1$

need to find coeff of θ^4: $G_Y(\theta)=$ ~~$0.3+\frac{0.5}{2}$~~

$0.3+\frac{0.5}{2}\left(1+\frac{\theta}{2}+\dots+\frac{\theta^4}{2^4}+\dots\right)+\frac{0.2}{2^3}\left(1+\frac{3\theta}{2}+\frac{3\cdot 4}{2!}\left(\frac{\theta}{2}\right)^2+\frac{3\cdot 4\cdot 5}{3!}\left(\frac{\theta}{2}\right)^3+\frac{3\cdot 4\cdot 5\cdot 6}{4!}\left(\frac{\theta}{2}\right)^4+\dots\right)$

$\therefore$ coeff of θ^4: $\frac{0.5}{2}\left(\frac{1}{2}\right)^4+\frac{0.2}{2^3}\cdot\frac{3\cdot 4\cdot 5\cdot 6}{4!}\left(\frac{1}{2}\right)^4=P(Y=4)=0.0391$

\CW1/ \1a/ $E(Y)=E(X^2+3)=E(X^2)+3$, $E(X^2)=0.3(0^2)+0.5(1^2)+0.2(3^2)=2.3$

$\therefore E(Y)=2.3+3=5.3$

$P(Y=4)=P(X^2+3=4)=P(X^2=1)=P(X=1)=0.5$

\1b/ $E(Y)=E(2W)=2E(W)$ $G_W'(\theta)=\frac{d}{d\theta}\left[(2-\theta)^{-1}\right]=-(2-\theta)^{-2}(-1)=(2-\theta)^{-2}$

$\therefore G_W'(1)=(2-1)^{-2}=(1)^{-2}=1=E(W)$ $\therefore$

$E(Y)=2(1)=2$,

$P(Y=4)=P(2W=4)=P(W=2)$ $\therefore$

$G_W(\theta)=-1+\frac{-1\cdot(-2)}{2!}\theta+\frac{-1\cdot(-2)(-3)}{3!}\theta^2+\dots$ $\therefore P(W=2)=1$ $\therefore$ $P(Y=4)=1$

$G_W(\theta)=(-2-\theta)^{-1}=\left(-2\left(1+\frac{\theta}{2}\right)\right)^{-1}=\left[-\frac{1}{2}\right]\left(1+\frac{\theta}{2}\right)^{-1}=\left[-\frac{1}{2}\right]\left(\frac{1}{0!}+\frac{(-1)}{1!}\frac{\theta}{2}+\frac{(-1)(-2)}{2!}\left(\frac{\theta}{2}\right)^2+\dots\right)=$

$\left[-\frac{1}{2}\right]\left(1-\frac{1}{2}\theta+\frac{1}{4}\theta^2+\dots\right)=-\frac{1}{2}+\frac{1}{4}\theta-\frac{1}{8}\theta^2+\dots$ $\therefore P(W=2)=\frac{1}{8}$ $\therefore$

$P(Y=4)=\frac{1}{8}$

$G_W(\theta)=(2-\theta)^{-1}=\left(2\left(1-\frac{\theta}{2}\right)\right)^{-1}=\frac{1}{2}\left(1+\left(-\frac{\theta}{2}\right)\right)^{-1}=\frac{1}{2}\left[\frac{1}{0!}+\frac{-1}{1!}\left(-\frac{\theta}{2}\right)+\frac{-1(-2)}{2!}\left(-\frac{\theta}{2}\right)^2+\dots\right]=$

$\frac{1}{2}\left[1+\frac{1}{2}\theta+\frac{1}{4}\theta+\dots\right]=\frac{1}{2}+\frac{1}{4}\theta+\frac{1}{8}\theta+\dots$ $\therefore P(Y=4)=P(2W=4)=P(W=2)=\frac{1}{8}=0.125$

∴ Z eqn satisfies $0=pD_{i+2}-D_i+qD_{i-1}$ ∴ let $D_i=k\lambda^i$ ∴ soltn homog part has Z form as ~~(0.2) ...~~ (0.1) where $\lambda_\pm$ are given by (0.2). to solve Z inhomog part, try particular soln of Z form $D_i^p=a+bi+ci^2+di^3$ ∴ subbing into (0.3) get:

$0=1-b(1-3p)+c(1+3p)-d(1-9p)+[3d(1+3p)-2c(1-3p)]i+3d(3p-1)i^2$

equating terms of order i^2 in Z above yields $d=0$, whilst for order i, we obtain $c=0$. ∴ Z particular soln is given by

$D_i^p=a+i/(1-3p)=a+i/(q-3p)$ ∴ by superposition, Z GS is:

$D_i=\frac{i}{q-2p}+A+B\lambda_+^i+C\lambda_-^i$ as before, Z consts A,B,C are found by applying Z BCs ∴ $A=(\lambda_-^N(1+N(1-\lambda_-))-\lambda_+^N(1+N(1-\lambda_+)))/(W(\lambda_-,\lambda_+))$

$B=(1-\lambda_-^N(1+N(1-\lambda_-)))/W(\lambda_-,\lambda_+)$

$C=(\lambda_+^N(1+N(1-\lambda_+))-1)/W(\lambda_-,\lambda_+)$ where

$W(\lambda_-,\lambda_+)=(2p-q)[(\lambda_+^N-1)(\lambda_-^{N+1}-1)-(\lambda_+^{N+1}-1)(\lambda_-^N-1)]$

$\sum_{n=0}^{\infty}\rho^n=(1-\rho)^{-1}=\frac{1}{1-\rho}$ $\quad\sum_{n=0}^{\infty}n\rho^n=\rho(1-\rho)^{-2}=\frac{\rho}{(1-\rho)^2}$ $\quad\sum_{n=0}^{\infty}\rho^n=\frac{1}{1-\rho}$

$\sum_{n=0}^{\infty}n\rho^n=\rho(1-\rho)^{-2}$ $\quad\sum_{n=0}^{\infty}\rho^n=\frac{1}{1-\rho}$ $\quad\sum_{n=0}^{\infty}n\rho^n=\frac{\rho}{(1-\rho)^2}$ $\quad\sum_{n=0}^{\infty}\rho^n=\frac{1}{1-\rho}$

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$M_x(t)=E(e^{tx})$ $\quad G_x(\theta)=E(\theta^x)$ $\quad G_x(e^t)=M_x(t)$ $\quad E(x)=\frac{dM_x}{dt}\Big|_{t=0}=E(xe^{tx})$

$E(x^r)=\frac{d^rM_x}{dt^r}\Big|_{t=0}$ $\quad M_x(t)=E(e^{tx})$ $\quad G_x(\theta)=E(\theta^x)$ $\quad G_x(e^t)=M_x(t)$ $\quad E(x)=\frac{dM_x}{dt}\Big|_{t=0}=E(xe^{tx})$

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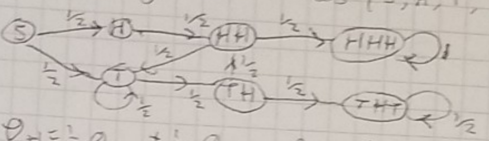
$\frac{d^rM_x}{dt^r}\Big|_{t=0}$ $\quad M_x(t)=E(e^{tx})$ $\quad G_x(\theta)=E(\theta^x)$ $\quad G_x(e^t)=M_x(t)$ $\quad E(x)=\frac{dM_x}{dt}\Big|_{t=0}=E(xe^{tx})$ $\quad E(x^r)=\frac{d^rM_x}{dt^r}\Big|_{t=0}$

\Sheet 5/ \4 Sol/ game has states $\{S, H, T, HH, TH, HHH, THT\}$

$\therefore$ transition diag:

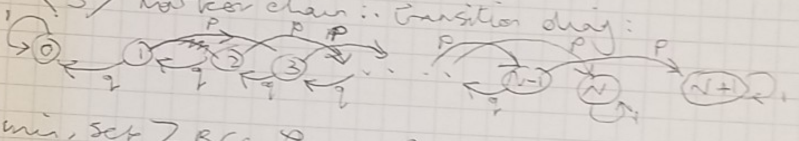


1)

$\theta_S = \frac{1}{2}\theta_H + \frac{1}{2}\theta_T$, $\theta_H = \frac{1}{2}\theta_{HH} + \frac{1}{2}\theta_T$, $\theta_T = \frac{1}{2}\theta_T + \frac{1}{2}\theta_{TH}$, $\therefore$ using $\theta_i = \sum_j \theta_j T_{ij}$ $\therefore$
$\theta_{HH} = \frac{1}{2}\theta_T + \frac{1}{2}\theta_{HHH}$, $\theta_{TH} = \frac{1}{2}\theta_{HH} + \frac{1}{2}\theta_{THT}$ $\therefore$ to find Z prob for HHH to win, set $\theta_{HHH} = 1$ & $\theta_{THT} = 0$, from which find $\theta_T = \frac{1}{3}$ & $\theta_H = \frac{1}{2}$ st $\theta_S = \frac{5}{12} < \frac{1}{2}$ meaning we are more likely to see sequence THT before Z sequence HHH.

using $D_i = 1 + \sum_j D_j T_{ij}$ $\therefore$ $D_S = 1 + \frac{1}{2}D_H + \frac{1}{2}D_T$, $D_H = 1 + \frac{1}{2}D_{HH} + \frac{1}{2}D_T$,
$D_T = 1 + \frac{1}{2}D_T + \frac{1}{2}D_{TH}$, $D_{HH} = 1 + \frac{1}{2}D_T + \frac{1}{2}D_{HHH}$, $D_{TH} = 1 + \frac{1}{2}D_{HH} + \frac{1}{2}D_{THT}$
$\therefore$ to find Z expected time to finish, set $D_{HHH} = D_{THT} = 0$
$\therefore$ find $D_T = \frac{14}{3}$ & $D_H = 5$ st $D_S = \frac{35}{6}$

\5/ markov chain $\therefore$ transition diag:



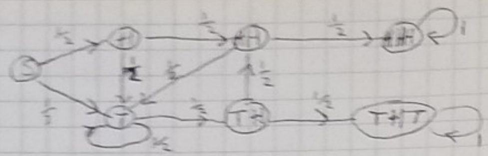
$\therefore$ to find Z prob to win, set Z BCs $\theta_0 = 0$, $\theta_N = \theta_{N+1} = 1$ & solve $\theta_i = \sum_j \theta_j T_{ij} = p\theta_{i+2} + q\theta_{i-1}$ $\therefore$
sub trial sol $\theta_i = k\lambda^i$ $\therefore$ $k\lambda^{i-1}(p\lambda^3 - \lambda + q) = 0$
$\{ k\lambda^i - pk\lambda^{i+2} - qk\lambda^{i-1} = 0 = k\lambda^{i-1}(\lambda - p\lambda^3 - q) = 0 = p\lambda^3 - \lambda + q$ $\therefore$
$\lambda = 1$: $p(1)^3 - 1 + q = p - (1-q) = p - (p) = 0$ $\therefore$
$p\lambda^3 - \lambda + q = 0 = (\lambda - 1)(p\lambda^2 - q + p\lambda + (-q+1)\lambda) = (\lambda-1)(p\lambda^2 + p\lambda - q)$, where we note $\lambda = 1$ solves Z eqn. Z GS is: $\theta_i = A + B\lambda_+^i + C\lambda_-^i$ where
$\lambda_\pm = -\frac{1}{2} \pm \frac{\sqrt{p(4-3p)}}{2p}$ are Z nontrivial roots of Z charac eqn. Z Consts A, B, C can be determined by applying Z BCs. $\therefore$ find

$$A = \frac{\lambda_+^N(\lambda_+ - 1) - \lambda_-^N(\lambda_- - 1)}{V(\lambda_-, \lambda_+)} \quad B = \frac{\lambda_-^N(\lambda_- - 1)}{V(\lambda_-, \lambda_+)} \quad C = \frac{\lambda_+^N(1 - \lambda_+)}{V(\lambda_-, \lambda_+)}$$ where

$V(\lambda_-, \lambda_+) = (\lambda_-^N - 1)(\lambda_+^{N+1} - 1) - (\lambda_-^{N+1} - 1)(\lambda_+^N - 1)$ combining these expressions, find Z prob of winning starting from E_i as θ_i =

$$\theta_i = (\lambda_-^N(\lambda_- - 1)(\lambda_+^i - 1) - \lambda_+^N(\lambda_+ - 1)(\lambda_-^i - 1)) / (V(\lambda_-, \lambda_+))$$ to find Z expected time until Z game ends, set $D_0 = D_N = D_{N+1} = 0$ & solve
$D_i = 1 + \sum_j D_j T_{ij} = 1 + pD_{i+2} + qD_{i-1}$ $\therefore$ by superposition: Z homog part

\4/



$\therefore \cancel{\Omega = \{S,}$

$$\Omega = \{S, H, HH, HHH, T, TH, THT\} \therefore \quad T = \begin{array}{c|ccccccc} & S & H & HH & HHH & T & TH & THT \\ \hline S & 0 & \frac12 & 0 & 0 & \frac12 & 0 & 0 \\ H & 0 & 0 & \frac12 & 0 & \frac12 & 0 & 0 \\ HH & 0 & 0 & 0 & \frac12 & \frac12 & 0 & 0 \\ HHH & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ T & 0 & 0 & 0 & 0 & \frac12 & \frac12 & 0 \\ TH & 0 & 0 & \frac12 & 0 & 0 & 0 & \frac12 \\ THT & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{array}$$

$\therefore$ ~~Θ~~ let D_i be number of steps for game to end from state i

let θ_i be prob to reach HHH from state i

$\therefore \theta_{HHH} = 1, \ \theta_{THT} = 0, \ \theta_S = \frac{1}{2}\theta_H + \frac{1}{2}\theta_T,$

$\theta_H = \frac{1}{2}\theta_{HH} + \frac{1}{2}\theta_T, \quad \theta_{HH} = \frac{1}{2}\theta_{HHH} + \frac{1}{2}\theta_T, \ \theta_T = \frac{1}{2}\theta_T + \frac{1}{2}\theta_{TH},$

$\theta_{TH} = \frac{1}{2}\theta_{HH} + \frac{1}{2}\theta_{THT} \ \therefore$

$\theta_{TH} = \frac{1}{2}\theta_{HH} + \frac{1}{2}(0) = \frac{1}{2}\theta_{HH},$

$\theta_{HH} = \frac{1}{2}(1) + \frac{1}{2}\theta_T = \frac{1}{2} + \frac{1}{2}\theta_T \ \therefore$

$\theta_{TH} = \frac{1}{2}\left(\frac{1}{2} + \frac{1}{2}\theta_T\right) = \frac{1}{4} + \frac{1}{4}\theta_T \ \therefore \ \theta_T = \frac{1}{2}\theta_T + \frac{1}{2}\left(\frac{1}{4} + \frac{1}{4}\theta_T\right) = \frac{1}{2}\theta_T + \frac{1}{8} + \frac{1}{8}\theta_T = \frac{1}{8} + \frac{5}{8}\theta_T$

$\therefore \frac{3}{8}\theta_T = \frac{1}{8} \therefore \theta_T = \frac{1}{3} \ \therefore$

$\theta_{HH} = \frac{1}{2} + \frac{1}{2}\left(\frac{1}{3}\right) = \frac{2}{3} \ \therefore \ \theta_H = \frac{1}{2}\left(\frac{2}{3}\right) + \frac{1}{2}\left(\frac{1}{3}\right) = \frac{1}{2} \ \therefore$

$\theta_S = \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2}\left(\frac{1}{3}\right) = \frac{5}{12}$ is Z prob of reaching HHH before THT $\therefore$ sequence THT is more likely to appear first $\because \theta_S < \frac{1}{2}$.

$D_{HHH} = 0, \ D_{THT} = 0 \ \therefore \Big\{ D_S = \frac{1}{2}D_H + \frac{1}{2}D_T, \ D_H = \frac{1}{2}D_{HH} + \frac{1}{2}D_T,$

$D_{HH} = \frac{1}{2}D_{HHH} + \frac{1}{2}D_T, \ D_T = \frac{1}{2}D_T + \frac{1}{2}D_{TH}, \ D_{TH} = \frac{1}{2}D_{HH} + \frac{1}{2}D_{THT} \ \therefore$

$D_{TH} = \frac{1}{2}D_{HH} + \frac{1}{2}(0) = \frac{1}{2}D_{HH}, \ D_{HH} = \frac{1}{2}(0) + \frac{1}{2}D_T = \frac{1}{2}D_T \ \therefore$

$D_{TH} = \frac{1}{2}\left(\frac{1}{2}D_T\right) = \frac{1}{4}D_T \ \therefore \ D_T = \frac{1}{2}D_T + \frac{1}{2}\left(\frac{1}{4}D_T\right) = \frac{1}{2}D_T + \frac{1}{8}D_T = \frac{5}{8}D_T \ \therefore$

$\frac{3}{8}D_T = 0 \therefore D_T = 0$ X $\Big\}$

$D_{HHH} = 0, \ D_{THT} = 0 \ \therefore \ D_S = 1 + \frac{1}{2}D_H + \frac{1}{2}D_T, \ D_H = 1 + \frac{1}{2}D_{HH} + \frac{1}{2}D_T,$

$D_{HH} = 1 + \frac{1}{2}D_{HHH} + \frac{1}{2}D_T, \ D_T = 1 + \frac{1}{2}D_T + \frac{1}{2}D_{TH}, \ D_{TH} = 1 + \frac{1}{2}D_{HH} + \frac{1}{2}D_{THT} \ \therefore$

$D_{TH} = 1 + \frac{1}{2}D_{HH}, \ D_{HH} = 1 + \frac{1}{2}D_T \ \therefore \ D_{TH} = 1 + \frac{1}{2}\left(1 + \frac{1}{2}D_T\right) = \frac{3}{2} + \frac{1}{4}D_T \ \therefore$

$D_T = 1 + \frac{1}{2}D_T + \frac{3}{4} + \frac{1}{8}D_T = \frac{7}{4} + \frac{5}{8}D_T \ \therefore \frac{3}{8}D_T = \frac{7}{4} \therefore D_T = \frac{14}{3} \ \therefore$

$D_{HH} = 1 + \frac{1}{2}\left(\frac{14}{3}\right) = \frac{10}{3} \ \therefore \ D_H = 1 + \frac{1}{2}\left(\frac{10}{3}\right) + \frac{1}{2}\left(\frac{14}{3}\right) = 5 \ \therefore$

$D_S = 1 + \frac{1}{2}(5) + \frac{1}{2}\left(\frac{14}{3}\right) = \frac{35}{6} \approx 5.83$ steps on average for game to end

Sheet 5

$\theta_i = q\theta_{i-1} + p\theta_{i+1} \therefore \theta_i = A + B\rho^i$ for $\rho = \frac{q}{p} \therefore \theta_0 = 0,\ \theta_5 = 1$

$\therefore A = -B \therefore \theta_i = B(-1+\rho^i) \therefore B = \frac{1}{-1+\rho^5} \therefore \theta_i = \frac{-1+\rho^i}{-1+\rho^5} = \frac{1-\rho^i}{1-\rho^5} \therefore$

$\theta_3 = \frac{1-\rho^3}{1-\rho^5} = \frac{1-(\frac{2}{3})^3}{1-(\frac{2}{3})^5} \approx 0.81$

3b/ this is equiv to a randomwalk with barriers $x \in \{\pm 2\} \therefore$ setting $W_n = X_n + 2$ transforms it into a standard gamblers ruin probl $\mathcal{E}$: $\theta_2 = \frac{1-\rho^2}{1-\rho^4} = \frac{1}{1+\rho^2} \approx 0.692$

$\{W_n = X_n + 2 \therefore W_n = \{0, 1, \dots, 4\} \therefore \theta_0 = 0,\ \theta_4 = 1 \therefore \theta_i = q\theta_{i-1} + p\theta_{i+1}$

$\therefore \theta_i = A + B\rho^i$ for $\rho = \frac{q}{p} \therefore A = -B \therefore \theta_i = \cancel{B(-B+}\ B(-1+\rho^i) \therefore$

$B = \frac{1}{-1+\rho^4} \therefore \theta_i = \frac{1-\rho^i}{1-\rho^4} \therefore \theta_2 = \frac{1-\rho^2}{1-\rho^4} = \frac{1-(\frac{2}{3})^2}{1-(\frac{2}{3})^4} = \frac{1}{1+\rho^2} = \frac{1}{1+(\frac{2}{3})^2} \approx 0.692$

3c/ this can be split in 3 cases:

(A): $\mathcal{Z}$ prob to reach state 1 before -3 is $P_A = \frac{1-\rho^3}{1-\rho^4} = \frac{57}{65}$

(B): then make a transition to state 0 $\because$ if we went to 2, we would go through state 1 again $\therefore$ to said $P_B = 0.4$.

(C): reach state -3 before 1, this is $\mathcal{Z}$ opposite of case (A) so $P_C = 1 - \frac{57}{65} = \frac{8}{65}$ $\therefore P = P_A P_B P_C \approx 0.0432$

$\{$ let $W_n = X_n + 3 \therefore W_n = \{0, 1, 2, 3, 4\} \because X_n = \{-3, -2, -1, 0, 1\} \therefore$

$\theta_i = p\theta_{i+1} + q\theta_{i-1} \therefore \theta_i = A\lambda^i \therefore \lambda = 1,\ \lambda = \frac{q}{p} \therefore$

$\theta_i = A + B\rho^i \quad \theta_0 = 0,\ \theta_4 = 1 \therefore A = -B \therefore \theta_i = B(-1+\rho^i) \therefore$

$B = \frac{1}{-1+\rho^4} \therefore \theta_i = \frac{1-\rho^i}{1-\rho^4} \therefore X$ starts at 0 $\therefore$

W starts at 3 $\therefore \theta_3 = \frac{1-\rho^3}{1-\rho^4} = \frac{1-(\frac{2}{3})^3}{1-(\frac{2}{3})^4} = \frac{57}{65} = P_A$ is state X_0 going to 1 before -3 $\therefore$

then X must go straight to state 0 $\therefore P_B = 0.4$

then X go to state -3 before 1 is opposite of going to state 1 before state -3 starting at $X_0 \therefore P_C = 1 - P_A = 1 - \frac{57}{65} = \frac{8}{65} \therefore$

$P = P_A P_B P_C = \frac{57}{65}(0.4)\frac{8}{65} \approx 0.0432$

$\theta_i = A + B\rho^i \quad \theta_0 = 0, \quad \theta_6 = 1 \therefore$

$\theta_0 = A + B\rho^0 = A + B = 0 \therefore A = -B \therefore \theta_i = -B + B\rho^i = B(-1+\rho^i) \therefore$

$\theta_6 = B(-1+\rho^6) = 1 \quad \therefore \cancel{B = \frac{1}{\rho}} \quad B = \frac{1}{-1+\rho^6} \therefore$

$\frac{1}{-1+\rho^6}(-1+\rho^i) = \theta_i = \frac{1-\rho^i}{1-\rho^6} \quad \therefore$

$\theta_3 = \frac{1-\rho^3}{1-\rho^6} = \frac{1-\rho^3}{(1-\rho^3)(1+\rho^3)} = \frac{1}{1+\rho^3}$ if $p \neq \frac{1}{2} \quad \therefore P = \frac{1}{1+\rho^3}$

if $p = \frac{1}{2}$: ~~if~~ $\rho = 1 \therefore \lim_{\rho \to 1} \left(\frac{1}{1+\rho^3}\right) = \theta_3 = \frac{1}{1+1^3} = \frac{1}{1+1} = \frac{1}{2} = P$ if $p = \frac{1}{2}$

\2/ let X_n = number of ball in box B. then $X_n = \{0, \dots, 15\}$ &
$X_0 = 10$. $\therefore$ have a standard gamber's ruin probl with
$P(X_n \text{ increases by } 1) = P(X_n \text{ decreases by } 1) = \frac{1}{2} \therefore$
$\theta_{10} = \frac{10}{15} = \frac{2}{3}$ for Z prob for box B to be full when box A is empty

\2 redo/ let X_n = number of balls in Box B, Y_n = balls in box A $\therefore$
$X_n + Y_n = 15$, $X_n = \{0, 1, 2, \dots, 15\}$ $\therefore$ let ~~$W_n = Z X_n Z Y_n$~~ $\therefore$
~~let~~ let Pr(number of ball in B reaching 15 before 0 starting at i) $= \theta_i$
$\therefore$ transition mat T $\therefore T_{i,i-1} = \frac{1}{2}, T_{i,i+1} = \frac{1}{2} \therefore$

$\theta_i = Pr(\text{going to } i-1 \text{ from } i)\theta_{i-1} + Pr(\text{going to } i+1 \text{ from } i)\theta_{i+1} = \frac{1}{2}\theta_{i-1} + \frac{1}{2}\theta_{i+1} = \theta_i$

$\therefore \theta_0 = 0, \theta_{15} = 1 \therefore$ let $\theta_i = A\lambda^i \therefore A\lambda^i + \frac{1}{2}A\lambda^{i-1} + \frac{1}{2}A\lambda^{i+1} \therefore \lambda = 1, \lambda = \frac{1/2}{1/2} = \rho \therefore$

$\therefore \theta_i = A + B\rho^i \therefore A = -B \therefore \theta_i = B(-1+\rho^i) \therefore$

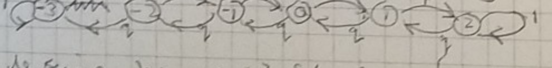
$B = \frac{1}{-1+\rho^{15}} \therefore \theta_i = \frac{-1+\rho^i}{-1+\rho^{15}} = \frac{1-\rho^i}{1-\rho^{15}} \therefore \theta_{10} = \frac{1-\rho^{10}}{1-\rho^{15}}$ ~~$= \frac{1+\rho^{10}}{(1-\rho^{10})(\rho^5)}$~~

for $\rho = \frac{q}{p} = \frac{1/2}{1/2} = 1 \quad \therefore \theta_{10} = \lim_{\rho \to 1} \frac{1-\rho^{10}}{1-\rho^{15}} = \lim_{\rho \to 1} \frac{\frac{d}{d\rho}(1-\rho^{10})}{\frac{d}{d\rho}(1-\rho^{15})} = \lim_{\rho \to 1} \frac{-10\rho^9}{-15\rho^{15}} =$

$\frac{-10(1)^9}{-15(1)^{15}} = \theta_{10} = \frac{10}{15} = \frac{2}{3}$ for Z probl for box B to be full when box A is empty

\3a/ transition mat $T_{i,i+1} = 0.6 \quad T_{i,i-1} = 0.4$

let X_n denote Z position of Z random walker in step n. with $p = 0.6$ &
$q = 0.4$. we have Z transition diag for Z random walker with absorbing barriers



$\therefore$ set $\rho = \frac{q}{p} = \frac{0.4}{0.6} = \frac{2}{3}$

define $W_n = X_n + 3$ st $W_n \in \{0, 1, \dots, 5\}$ & $W_0 = 3 \therefore \theta_3 = \frac{1-\rho^3}{1-\rho^5} \approx 0.81$

Sheet 5/ $U_n = 6$ $\therefore$ look for prob p to win $\therefore$

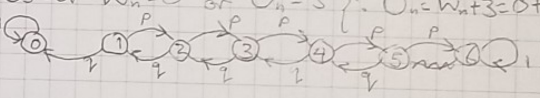
set $\rho = \frac{q}{p}$ $\because$ Z game starts at $W_n = 0$ or $U_n = 3$ $\{U_n = W_n + 3\}$

$\therefore$ (i): $P = \frac{1}{2}$ if $\rho = \frac{1}{2}$ $\{p = \frac{1}{2}$ $\therefore$ $q = 1 - p = \frac{1}{2}$ $\therefore \rho = \frac{q}{p} = \frac{1/2}{1/2} = 1\}$

(ii) $P = \frac{1-\rho^3}{1-\rho^6} = \frac{1}{1+\rho^3}$ if $p \neq \frac{1}{2}$ $\{p \neq \frac{1}{2}$ $\therefore$ $q = 1-p$ $\therefore$ $\rho = \frac{q}{p} =$
$\frac{1-p}{p}$ $\therefore$ $P = \frac{1-\rho^3}{1-\rho^6} = \frac{1-\rho^3}{(1-\rho^3)(1+\rho^3)} = \frac{1}{1+\rho^3}\}$

1/ have $X_n + Y_n = n$. desire Z difference $W_n = X_n - Y_n$ of heads & tails. Z r.v. W_n takes vals $\{-3, -2, \dots, 2, 3\} = \{-3, -2, -1, 0, 1, 2, 3\}$ $\because$ Z game stops if either $W_n = -3$ or $W_n = 3$, Z prob for W_n to increase is p, $\therefore$ Z prob for it to decrease is $q = 1-p$. setting $U_n = W_n + 3$: we obtain a standard gambler's ruin probl & $\because$ $X_n = Y_n + 3$ corresponds to $U_n = 6$ $\{\because W_n = X_n - Y_n$ $\therefore$ $U_n = W_n + 3 = X_n - (Y_n) + 3 = \cancel{X_n}(Y_n + 3) - (Y_n) + 3 = 3 + 3 = 6 = U_n$ eg: if $X_n = Y_n + 3$: $U_n = 6\}$, we look for prob P to win.

Set $\rho = \frac{q}{p}$ $\therefore$ $\because$ Z game starts at $W_n = 0$ or $U_n = 3$ $\{\because U_n = W_n + 3 = 0 + 3 = 3\}$

& $P = \frac{1-\rho^3}{1-\rho^6} = \frac{1}{1+\rho^3}$ if $p \neq \frac{1}{2}$ $\{$ 

$\theta_0 = 0$, $\theta_6 = 1$ $\therefore$ $\theta_i = p\theta_{i+1} + q\theta_{i-1}$ $\because$ Let this syst have transition mat with probs $T_{i,j}$ $\therefore$ $T_{i,i+1} = p$, $T_{i,i-1} = q$ $\therefore$ state i can go to $i-1$ or $i+1$

$\therefore$ P(getting to state 6 before 0 starting from i) $= \theta_i =$
P(going from i to $i-1$) P(getting to state 6 before 0 starting from $i-1$) +
P(going from i to $i+1$) P(getting to state 6 before 0 starting from $i+1$) =
P(going from i to $i-1$) θ_{i-1} + P(going from i to $i+1$) $\theta_{i+1} = T_{i,i-1}\theta_{i-1} + T_{i,i+1}\theta_{i+1} =$
$q\theta_{i-1} + p\theta_{i+1} = \theta_i$ $\therefore$ let $\theta_i = A\lambda^i$ $\therefore$ $\theta_{i-1} = A\lambda^{i-1}$, $\theta_{i+1} = A\lambda^{i+1}$ $\therefore$

$\theta_i - p\theta_{i+1} - q\theta_{i-1} = 0 = A\lambda^i - pA\lambda^{i+1} - qA\lambda^{i-1} = A[\lambda^{i-1}\lambda - p\lambda^{i-1}\lambda^2 - q\lambda^{i-1}1] =$
$A\lambda^{i-1}[\lambda - p\lambda^2 - q] = 0$ $\therefore$ $\lambda - p\lambda^2 - q = 0 = p\lambda^2 - \lambda - q = 0$ $\therefore$ $\lambda = \frac{+1 \pm \sqrt{1^2 - 4p(-q)}}{2p}$
$= \frac{1 \pm \sqrt{1 + 4pq}}{2p}$ $\therefore$ $p\lambda^2 - \lambda - q = 0$ $\therefore$ if $\lambda = 1$: $p(1)^2 - (1) - q = p - 1 - q = p - (1 - q) =$
$p - ((1) - q) = p - ((p+q) - q) = p - (p) = 0$ $\because$ $1 = p + q$, if $\lambda = \frac{q}{p} = \rho$
$p\lambda^2 - \lambda - q = 0$ $\therefore$ $\lambda_1 = 1$, $\lambda_2 = \frac{q}{p}$ $\therefore$ $\theta_i = A\lambda_1^i + B\lambda_2^i = A1^i + B\rho^i = A + B\rho^i$ $\therefore$

$G_{X+Y}(\theta) = G_X(\theta)\, G_Y(\theta) = e^{\lambda_X(\theta-1)} e^{\lambda_Y(\theta-1)} = e^{\lambda_X(\theta-1)+\lambda_Y(\theta-1)} = e^{(\lambda_X+\lambda_Y)(\theta-1)}$

is a poisson process with rate $\lambda_X+\lambda_Y$

∴ PGF of $X+Y$ agrees with poisson parameter $\lambda_X+\lambda_Y$

uniqueness of PGF

\1ci/ $P(V>1) = 1-P(V\le 1) = 1-P(X+Y\le 1)$

∵ $f_X(x) \sim \lambda_X e^{-\lambda_X x} = e^{-x}$, $f_Y(y) \sim \lambda_Y e^{-\lambda_Y y} = 3e^{-3y}$ ∴

∵ X, Y independent ∴ $f_V(t) = f_X(t) f_Y(t) = e^{-t}3e^{-3t}$ ∴

$\int_0^1 e^{-t}3e^{-3t}\,dt = \int_0^1 3e^{-4t}\,dt = 3\left[-\frac{1}{4}e^{-t}\right]_0^1 = -\frac{3}{4}\left[e^{-1}-e^{0}\right] = 0.474 = P(V\le 1)$

∴ $P(V>1) = 1-P(V\le 1) = 1-0.474 = 0.526$ (3 S.f)

\1cii/ $E(W) = G_W'(\theta)|_{\theta=1}$ ∴

$P(W<x) = 1-P(W\ge x) = 1-P(\min(X,Y)\ge x) = 1-P((X\cap Y)\ge x) =$

$1-P(X\ge x)P(Y\ge x) = 1-([1-P(X<x)][1-P(Y<x)])$ ∴

$P(X\ge x) = 1-P(X<x) = 1-\int_0^x e^{-t}\,dt = 1-\left[-e^{-t}\right]_0^x = 1+e^{-x}-e^{0} = e^{-x}$ ∴

$P(Y\ge x) = 1-P(Y<x) = 1-\int_0^x 3e^{-3t}\,dt = 1-\left[-\frac{3}{3}e^{-3t}\right]_0^x = 1+\frac{3}{3}e^{-3x}-\frac{3}{3} = e^{-3x}$

∴ $P(W<x) = 1-e^{-x}e^{-3x} = 1-e^{-4x}$ ∴

$f_W(x) = \frac{d}{dx}(1-e^{-4x}) = 4e^{-4x}$ ∴ $W\sim Exp(4)$ ∴

$E(W) = \frac{1}{\lambda_W} = \frac{1}{4}$

Let $U = X_1+X_2$ ∴ $E(U) = E(X_1)+E(X_2)$,

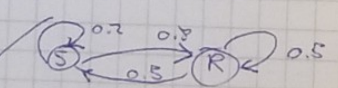
$E(X) = \frac{1}{\lambda_X} = \frac{1}{1} = 1$, $E(Y) = \frac{1}{\lambda_Y} = \frac{1}{3}$, $var(X) = \frac{1}{\lambda_X^2} = \frac{1}{1^2} = 1$, $var(Y) = \frac{1}{\lambda_Y^2} = \frac{1}{3^2} = \frac{1}{9}$

∴ $E(U) = 1+1 = 2$, $var(U) = var(X_1)+var(X_2) = 2$

$P(U<3) = Pr(\text{time to 2nd event}<3) = Pr(\text{At least two events in time } 3)$

$= 1-P(\text{one or no events in time } 3,\ \lambda_X = 1) =$

$1-(e^{-3}+3e^{-3}) = 0.801$

\3ai/ 

$P(S\to S\to S\to S|S) = (0.2)^3 = 0.008$

\3aii/ $P(\text{At least S on next } 3|R) = 1-Pr(R\to R\to R\to R|R) =$

$1-(0.5)^3 = 0.875$

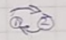
\3aiii/ Solve $\tilde{P} = \tilde{P}T$, $\tilde{P} = [P_1\ \ P_2]$, $P_1+P_2 = 1$ ∴ $\cancel{[P_1\ \ P_2]\begin{bmatrix}0.2 & 0.8\\ 0.5 & 0.5\end{bmatrix}}$

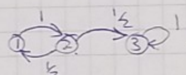
\EMCM3724PP2018/ \1di/ Let $f_{(i)}^{(n)}$ be the ~~probab~~ of being at state i after n steps after starting at it the recurrent is

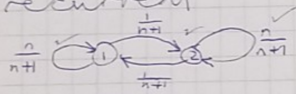
if $f_{(i)} = \sum_{n=1}^{\infty} f_{(i)}^{(n)} = 1$

\1di/ Let $f_i^{(n)}$ = prob ~~of~~ of first return to state i at time n (starting i initially) $\therefore$ let $f_i = \sum_{n=1}^{\infty} f_i^{(n)}$ $\therefore$ i is recurrent if $f_i = 1$

\1dii/ for $T = T(n)$ is transition matrix if $(T^n)_{ii} > 0$ for $n = pk$, $(T^n)_{ii} = 0$ otherwise where $k \in \mathbb{N}$, p is constant then state is periodic

A state i is periodic if $(T^n)_{ii} > 0$ for $n = kp$, $p \geq 1$, $k \in \mathbb{N}$, $(T^n)_{ii} = 0$ otherwise (p periodic) $\therefore$  T is transition matrix

\1diii/ no $\because$  ~~is~~ $\therefore$ periodic $\{1,2\}$ but ~~transient~~ $\therefore$ not recurrent

\1div/  $\therefore$ both states are null recurrent $\therefore$ yes

\2a/ Poi $\sim \frac{e^{-\cancel{x}}(\cancel{t})^x}{x!}$ $\therefore$ $t < T$ $r < k$ $\therefore$ $P(r \text{ in time } t \mid k \text{ in time } T) =$

$P(r \text{ in time } t \text{ and } k \text{ in time } T) / P(k \text{ in time } T) =$

$P(r \text{ in time } t \text{ and } (k-r) \text{ in time } T-t) / P(k \text{ in time } T) =$

$P(r \text{ in time } t) P(k-r \text{ in time } T-t) / P(k \text{ in time } T) =$

$P(N(t) = r) P(N(T-t) = k-r) / P(N(T) = k) =$

$$\frac{e^{\lambda t}(\lambda t)^r}{r!} \frac{e^{\lambda(T-t)}(\lambda(T-t))^{k-r}}{(k-r)!} \frac{k!}{e^{\lambda T}(\lambda T)^k} = \binom{k}{r} e^{\lambda t}(\lambda t)^r e^{\lambda(T-t)}(\lambda(T-t))^{k-r} \frac{1}{e^{\lambda T}(\lambda T)^k}$$

$$\binom{k}{r} e^{\lambda t}(\lambda^r) t^r e^{\lambda T - \lambda t} (\lambda T - \lambda t)^{k-r} \frac{1}{e^{\lambda T}(\lambda T)^k} =$$

$$\binom{k}{r} \cancel{e^{\lambda t}} \lambda^r t^r e^{\lambda T} e^{-\lambda t} (\lambda)^{k-r} (T-t)^{k-r} \cancel{e^{-\lambda T}} \lambda^{-k} T^{-k} =$$

$$\binom{k}{r} t^r e^{\lambda T} e^{-\lambda t} \lambda^{-r} (T-t)^k (T-t)^{-r} \lambda^{-k} T^{-k} =$$

$$\binom{k}{r} \left(\frac{t}{T}\right)^r \left(1 - \frac{t}{T}\right)^{k-r}$$

\2b/ Poi $\sim \frac{e^{-\lambda_x} \lambda_x^x}{x!} = e^{-\lambda_x} \frac{\lambda_x^x}{x!}$ $\therefore$ $e^{-\lambda_x}(1 + \frac{\lambda_x \theta}{1} + \frac{\lambda_x^2}{2!}\theta^2 + \frac{\lambda_x^3}{3!}\theta^3 + \dots)$

$= e^{-\lambda}\left(\sum_{n=0}^{\infty} \frac{(\lambda_x \theta)^n}{n!}\right) = e^{-\lambda_x} e^{\lambda_x \theta} = (e^{\lambda_x \theta - \lambda_x}) = e^{\lambda_x(\theta-1)}$ $\therefore$

$G_{T_x}(\theta) = e^{\lambda_x(\theta-1)}$, $G_{T_y}(\theta) = e^{\lambda_y(\theta-1)}$ $\therefore$

\Vbi/ $E(X) = G_X'(\theta)\big|_{\theta=1} = 0.3 + 0.4\theta + 1.2\theta^2\big|_{\theta=1} = 0.3 + 0.4 + 1.2 = 1.9 \therefore$

at gen 2: $E(S_2) = \mu_2 = 1.9^2 = 3.61$

\Vbii/ $G_{S_2}(\theta) = G_X(G_X(\theta)) = 0.1 + 0.3(0.1) + 0.2(0.1)^2 + 0.4(0.1)^3 + \dots$

$= 0.1324 + \dots \therefore$

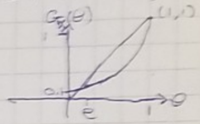
$P(S_2 = 0) = 0.132$ (3S.F.) $= e_2$

\Vbiii/ $E(X) = 1.9 > 1 \therefore e < 1 \therefore G_X(\theta) - \theta = 0 \therefore$

$0.1 + 0.3\theta - \theta + 0.2\theta^2 + 0.4\theta^3 = 0 = 0.1 - 0.7\theta + 0.2\theta^2 + 0.4\theta^3 =$

~~$(\theta - 1)$~~ $4\theta^3 + 2\theta^2 - 7\theta + 1 = 0 = (\theta - 1)(4\theta^2 + 6\theta - 1) = 0 \therefore$

$4\theta^2 + 6\theta - 1 \therefore e = \dfrac{-6 + \sqrt{36 - 4(4)(-1)}}{2(4)} = \dfrac{-3 + \sqrt{13}}{4} = 0.151$ (3 S.F.)



\Vbiv/ $\therefore G_Y(\theta) = (0.6 + 0.4\theta)^5 \therefore$

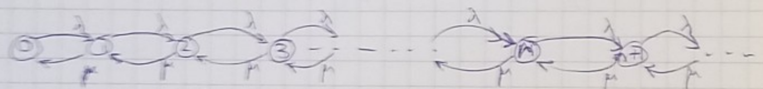
$E(Y) = G_Y'(\theta)\big|_{\theta=1} = 5(0.6 + 0.4\theta)^4\, 0.4\big|_{\theta=1} = 2 \therefore$

new mean at gen 2 is $\mu_{Y_2} = 2 \times 3.61 = 7.22$

new e_2 is $G_Y(e_2) = (0.6 + 0.4(e_2))^5 = (0.6 + 0.4(0.1324))^5 = 0.119$ (3S.F.)

new e is $\tilde{e} = G_Y(e) = (0.6 + 0.4(0.151))^5 = 0.126$ (3S.F.)

\Vci/



equate flows: $\lambda P_0 = \mu P_1 = P_1$, $\lambda P_1 = P_2$, $\lambda P_2 = P_3$, $\dots$ $\lambda P_n = P_{n+1} \therefore$

$\sum_{n=0}^{\infty} P_n = 1 \therefore \lambda P_0 = P_1,\ \lambda^2 P_0 = P_2,\ \lambda^n P_0 = P_n \therefore \sum_{n=0}^{\infty} \lambda^n P_0 = 1 = P_0 \sum_{n=1}^{\infty} \lambda^n =$

$P_0 \dfrac{1}{1-\lambda} \therefore 1 - \lambda = P_0\ 1 = P_0 \therefore P_n = \lambda^n(1-\lambda)$

\Vcii/ $M_X(t) = E(e^{tX}) = \sum_{n=0}^{\infty} e^{tn} P_n = \sum_{n=0}^{\infty} e^{tn}\lambda^n(1-\lambda) = (1-\lambda)\sum_{n=0}^{\infty} e^{tn}\lambda^n =$

$(1-\lambda)\sum_{n=0}^{\infty} (e^t)^n \lambda^n = (1-\lambda)\sum_{n=0}^{\infty} (e^t\lambda)^n = (1-\lambda)\dfrac{1}{1 - e^t\lambda} = \dfrac{1-\lambda}{1 - e^t\lambda} \therefore$

$M_X'(0) = E(X) = M_X'(t)\big|_{t=0} = \dfrac{d}{dt}\left[(1-\lambda)(1 - e^t\lambda)^{-1}\right] = (1-\lambda)(-1)(1 - e^t\lambda)^{-2}(-e^t\lambda)\big|_{t=0}$

$= (1-\lambda)e^t\lambda(1 - e^t\lambda)^{-2}\big|_{t=0} = (1-\lambda)(1)\lambda(1 - (1)\lambda)^{-2} = (1-\lambda)\lambda(1-\lambda)^{-2} = \dfrac{\lambda}{1-\lambda}$

$E(X^2) = M_X''(0) = M_X''(t)\big|_{t=0} = (1-\lambda)\lambda\left[e^t(1 - e^t\lambda)^{-2} + e^t(-2)(1 - e^t\lambda)^{-3}(-e^t\lambda)\right]\big|_{t=0}$

$(1-\lambda)\lambda\left[1(1 - 1\lambda)^{-2} + (1)(-2)(1 - (1)\lambda)^{-3}(-1\lambda)\right] = (1-\lambda)\lambda\left[(1-\lambda)^{-2} + 2\lambda(1-\lambda)^{-3}\right] =$

$\lambda(1-\lambda)^{-1} + 2\lambda^2(1-\lambda)^{-2} \therefore$

$(E(X))^2 = \dfrac{\lambda^2}{(1-\lambda)^2} \therefore \mathrm{Var}(X) = E(X^2) - (E(X))^2 = \dfrac{\lambda}{1-\lambda} + \dfrac{2\lambda^2}{(1-\lambda)^2} - \dfrac{\lambda^2}{(1-\lambda)^2} =$

$\dfrac{\lambda(1-\lambda)}{(1-\lambda)^2} + \dfrac{\lambda^2}{(1-\lambda)^2} = \dfrac{-\lambda^2 + \lambda + \lambda^2}{(1-\lambda)^2} = \dfrac{\lambda}{(1-\lambda)^2}$

\PP2018/

\1a sol/ find deriv of G_X at 1 is $E(X)$.

expand for only θ^3 coeff of $G_X(\theta)$ using combination formula

n ∴ coeff of θ^3 is $Pr(X=3)$

\1bi sol/ ~~pr~~ calc deriv of G_X then at $\theta=1$ is $E(X)$ ∴

Mean at gen 2 is μ_2 is $(E(X))^2$ ∵ $G_2(\theta)=G_X(G_X(\theta))$ ∴ $E(S_2)=G_2'(1)=G_X'(G_X(\theta))G_X'(\theta)|_{\theta=1}$
$=G_X'(G_X(1))G_X'(1)=G_X'(1)G_X'(1)=(G_X'(1))^2$

\1bii/ $G_{2}(\theta)=G_X(G_X(\theta))$ ∴ $e_2=G_{2}(0)$ for its coeff of θ^0 ∴

$G_X(G_X(0))$

\1biii/ ~~solve~~ $E(X)>1$ ∴ solve $\theta=G_X(\theta)$ ∴ $G_X(\theta)-\theta=0$ ∴

$\theta=1$ is always a factor ∴ factor out $\theta-1$ ∴ solve quadratic

for positive root is e ∴

at $\theta=0$. $G_X(\theta)=$ coeff(θ^0), at $\theta=1$; $G_X(\theta)=1$,

ultimate extinction no guaranteed ∴ $e<1$, crosses $y=x$ at $G_X(e)$

\1biv/ $G_Y(\theta)$ is binomial ∴ Bernouli to the 5 ∴ ~~...~~ Mean of Y is

$E(Y)=G_Y'(\theta)|_{\theta=1}$, new mean at gen 2 is: $E(Y)E(S_{X_2})$

∴ $E(S_{Y_1})=E(Y)G_X'(1)$, ∴ new ext(gen 2) $=G_Y(e_2)$,

new e is $\tilde{e}=G_Y(e)$ ∴ new extinction prob at gen 2 is old extinction

prob into $G_Y(\theta)$, new ultimate extinction prob is old prob in $G_Y(\theta)$

\1ci/ draw diag, equate flows ∴ find formula for P_n ∴

note other eqn of $\sum_{n=0}^{\infty} P_n$ then find P_0 as λ then P_n as λ

\1cii/ $M_X(t)=E(e^{tX})$ is sum of e^{tx} of each x with prob of x then

sub into formula then notice geo formula

\1ciii/ $E(X)=M_X'(0)$ ∴ plug in 0 into deriv wrt t ∴

$M_X''(0)=E(X^2)$ ∴ deriv again, plug 0 ∴ $var(X)=E(X^2)-(E(X))^2$

\1di/ def using $f_i^{(n)}$ notation

\1dii/ def using $(T^n)_{ii}$ notation describe transition mat.

\1diii/ transient states can be kept in but chance to leave only

100% as time → ∞ ∴

\1div/ show prob decreases of going to other as $n\to\infty$ & θ^1 as $n\to\infty$

$1-\lambda = P_0$,

$\lambda^n P_0 = P_n = \lambda^n(1-\lambda)$ for $n \geq 1$

\1c ii/ $G_X(\theta) = E(\theta^X) = P(X=0)\theta^0 + P(X=1)\theta^1 + P(X=2)\theta^2 + \dots =$

$1-\lambda + \lambda(1-\lambda)\theta + \lambda^2(1-\lambda)\theta^2 + \dots = \sum_{n=0}^{\infty} \lambda^n (1-\lambda)\theta^n$

$E(X) = G_X'(\theta)\big|_{\theta=1} = \frac{d}{d\theta}\left[\sum_{n=0}^{\infty} \lambda^n (1-\lambda)\theta^n\right]\big|_{\theta=1} =$

$\sum_{n=1}^{\infty} \lambda^n (1-\lambda) n \theta^{n-1}\big|_{\theta=1} = \sum_{n=1}^{\infty} \lambda^n (1-\lambda) n (1)^{n-1} = \sum_{n=1}^{\infty} \lambda^n (1-\lambda) n = (1-\lambda)\sum_{n=1}^{\infty} n\lambda^n =$

$(1-\lambda)\frac{\lambda}{(1-\lambda)^2} = \frac{\lambda}{1-\lambda}$ for $|\lambda| < 1$

$E(X^2) = G_X''(\theta)\big|_{\theta=1} = \sum_{n=2}^{\infty} \lambda^n (1-\lambda) n(n-1)\theta^{n-2}\big|_{\theta=1} = \sum_{n=2}^{\infty} \lambda^n (1-\lambda)(n^2 - n) =$

$\cancel{\sum_{n=1}^{\infty}} \sum_{n=2} \lambda^n (1-\lambda) n^2 - \sum_{n=2}^{\infty} \lambda^n (1-\lambda) n = (1-\lambda)\sum_{n=2}^{\infty} n^2\lambda^n - (1-\lambda)\sum_{n=2}^{\infty} n\lambda^n$

\1d i/ it is a state can always eventually be reached again
~~\1dii~~ even if it would take infinite time

\1d ii/ it is a state that can only be reached every kn steps when k is a fixed constant

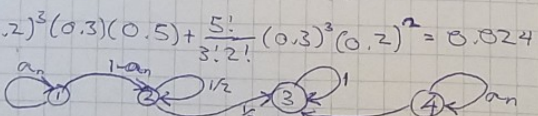
\1d iii/ yes because the state can be reached every kn steps for fixed k, then even as $n \to \infty$: the state can still be reached ∴ is recurrent.

\1d iv/ this Markov chain has no subchains ∴ with only two states, can only be at one of the two states ∴ both cannot be null recurrent, one must be positively recurrent ∴ no.

\1a sol/ $G_X(\theta) = (0.2 + 0.3\theta + 0.5\theta^2)^5$ $E(X) = G_X'(\theta)\big|_{\theta=1} = 5 \times 1.3 = 6.5$

$P(X=3) = \text{coeff } \theta^3 = \theta^2 \cdot \theta \cdot (\cdot)(\cdot)(\cdot) + \theta \cdot \theta \cdot \theta \cdot (-) \cdot (-) =$

$\frac{5!}{3!1!1!}(0.2)^3(0.3)(0.5) + \frac{5!}{3!2!}(0.3)^3(0.2)^2 = 0.0244 + 0.0108 = 0.0348$

\3b/ 

\1b sol/ $G_X'(\theta) = 0.3 + 0.4\theta + 1.2\theta^3$ ∴ $E(X) = G_X'(1) = 1.9$

mean at gen 2 : $\mu_2 = (1.9)^2 = 3.61$

\1b ii/ $G_2(\theta) = G_X(G_X(\theta))$ ∴ $e_2 = G_X(G_X(0)) = G_X(0.1) = 0.1 + 0.3(0.1) + 0.2(0.1)^2 + 0.4(0.1)^3$
$= 0.132$

\ECM3724PP2018 / \4a/ must be TTTT before any H $\therefore$ $(\frac{1}{4})^4 = \theta_{TTTT} = 0.00391$

\1a/ $E(X) = G_X'(\theta)\big|_{\theta=1} = 5(0.2+0.3\theta+0.5\theta^2)^4(0.3+1\theta)\big|_{\theta=1} =$

$5(0.2+0.3(1)+0.5(1)^2)^4(0.3+1(1)) = 6.5$

$G_X(\theta) = (0.3\theta)^3(0.2)^2 \frac{5!}{3!2!} + (0.5\theta^2)^1(0.3\theta)^3(0.2)^3 \frac{5!}{1!3!1!} + \dots =$

$\frac{27}{2500}\theta^3 + \frac{3}{125}\theta^3 + \dots = \frac{87}{2500}\theta^3 + \dots \therefore$

$P(X=3) = \frac{87}{2500} = 0.0348$ (3 S.F.)

\1bi/ $E(x) = G_X'(\theta)\big|_{\theta=1} = 0.3 + 0.4\theta + 1.2\theta^2\big|_{\theta=1} = 0.3 + 0.4(1) + 1.2(1)^2 = 1.9 \therefore$

$1 \times 1.9 = 1.9 \therefore$

$1.9 \times 1.9 = 3.61$ expected at Gen 2.

\1bii/ $P(S_1 = 0) = 0.1$ $\quad\therefore P(S_1 \neq 0) = 1 - 0.1 = 0.9 \therefore$

$P(S_2 = 0) = 0.1 + 0.9 \times 0.1 = 0.19$ X

$G_{S_2}(\theta) = G_X(G_X(\theta)) = 0.1 + 0.3(G_X(\theta)) + 0.2(G_X(\theta))^2 + 0.4(G_X(\theta))^3 =$

$0.1 + 0.3 \times 0.1 + 0.2 \times 0.1^2 + 0.4 \times 0.1^3 + \dots = 0.1324 + \dots \therefore$

$P(S_2 = 0) = 0.132$ (3 S.F.)

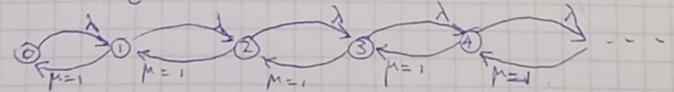
\1biii/ $\sum_{n=1}^{\infty}(S_n = 0) = \cancel{G_X(\theta) - 0} + \sum_{n=1}^{\infty} 0.1^n = \frac{0.1}{1-0.1} = \frac{1}{9} = 0.111$ (3 S.F.)

\1biv/ the probability of ultimate extinction doesn't change, the probabability of extintion at Gen 2 doesn't change.

Let $Y \sim Bin(5, 0.4)$ $\quad\therefore E(Y) = 5 \times 0.4 = 2 \therefore$

$E(X) = 2 \times 1.9 = 3.8$ X

\1ci/



$\therefore \frac{dP_0}{dt} = -\lambda P_0 + 1P_1$, $\quad \frac{dP_1}{dt} = \lambda P_0 - P_1 - \lambda P_1 + P_2$, $\quad \therefore \frac{dP_n}{dt} = \lambda P_{n-1} - P_n - \lambda P_n + P_{n+1}$

for $n > 0 \therefore$ steady state $\frac{dP_n}{dt} = 0 \therefore$

$-\lambda P_0 + P_1 = 0$, $\lambda P_0 - P_1 - \lambda P_1 + P_2 = 0$, $\lambda P_{n-1} - P_n - \lambda P_n + P_{n+1} = 0 \therefore$

$P_0 = \frac{1}{\lambda}P_1 \therefore -\lambda P_1 + P_2 = 0 \therefore P_1 = \frac{1}{\lambda}P_2 \therefore \lambda P_0 = P_1$, $\lambda P_1 = P_2 = \lambda^2 P_0 \therefore$

$\lambda^n P_0 = P_n$ for $n \geq 1 \therefore$ $P_0 + P_1 + P_2 + \dots = 1 = \sum_{n=0}^{\infty} P_n \therefore$

$P_0 + \lambda P_0 + \lambda^2 P_0 + \dots = 1 = P_0 + \sum_{n=1}^{\infty}\lambda^n P_0 = \sum_{n=0}^{\infty}\lambda^n P_0 = 1 = P_0 \sum_{n=0}^{\infty}\lambda^n \therefore$

for $|\lambda| < 1$: $1 = P_0 \frac{1}{1-\lambda}$ $\quad\therefore 1 - \lambda = P_0 \therefore$

\ECM3724PP2018 / \4a/ must be TTTF before any H ∴ $(\frac{1}{4})^4 = \theta_{TTTT} = 0.00391$

\1a/ $E(X) = G_X'(\theta)|_{\theta=1} = 5(0.2+0.3\theta+0.5\theta^2)^4(0.3+1\theta)|_{\theta=1} =$

$5(0.2+0.3(1)+0.5(1)^2)^4(0.3+1(1)) = 6.5$

$G_X(\theta) = (0.3\theta)^3(0.2)^2\frac{5!}{3!2!} + (0.5\theta^2)^1(0.3\theta)^1(0.2)^3\frac{5!}{1!3!1!} + \dots =$

$\frac{27}{2500}\theta^3 + \frac{3}{125}\theta^3 + \dots = \frac{87}{2500}\theta^3 + \dots$ ∴

$P(X=3) = \frac{87}{2500} = 0.0348$ (3 S.F.)

\1bi/ $E(X) = G_X'(\theta)|_{\theta=1} = 0.3 + 0.4\theta + 1.2\theta^2|_{\theta=1} = 0.3 + 0.4(1) + 1.2(1)^2 = 1.9$ ∴

$1 \times 1.9 = 1.9$ ∴

$1.9 \times 1.9 = 3.61$ expected at Gen 2.

\1bii/ $P(S_1=0) = 0.1$ ∴ $P(S_1 \neq 0) = 1 - 0.1 = 0.9$ ∴

$P(S_2=0) = 0.1 + 0.9\times0.1 = 0.19$ X

$G_{S_2}(\theta) = G_X(G_X(\theta)) = 0.1 + 0.3(G_X(\theta)) + 0.2(G_X(\theta))^2 + 0.4(G_X(\theta))^3 =$

$0.1 + 0.3\times0.1 + 0.2\times0.1^2 + 0.4\times0.1^3 + \dots = 0.1324 + \dots$ ∴

$P(S_2=0) = 0.132$ (3S.F.)

\1biii/ $\sum_{n=1}^{\infty}(S_n=0) = \cancel{G_X(\theta)-\theta} + \sum_{n=1}^{\infty}0.1^n = \frac{0.1}{1-0.1} = \frac{1}{9} = 0.111$ (3S.F.)

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the probabability of extintion at Gen 2 doesnt change.

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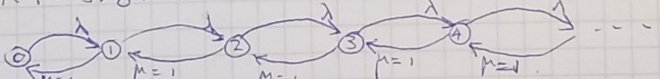
\1Ci/



∴ $\frac{dP_0}{dt} = -\lambda P_0 + 1P_1$, $\frac{dP_1}{dt} = \lambda P_0 - P_1 - \lambda P_1 + P_2$, ∴ $\frac{dP_n}{dt} = \lambda P_{n-1} - P_n - \lambda P_n + P_{n+1}$

for $n>0$ ∴ Steady state $\frac{dP_n}{dt} = 0$ ∴

$-\lambda P_0 + P_1 = 0$, $\lambda P_0 - P_1 - \lambda P_1 + P_2 = 0$, $\lambda P_{n-1} - P_n - \lambda P_n + P_{n+1} = 0$ ∴

$P_0 = \frac{1}{\lambda}P_1$ ∴ $-\lambda P_1 + P_2 = 0$ ∴ $P_1 = \frac{1}{\lambda}P_2$ ∴ $\lambda P_0 = P_1$, $\lambda P_1 = P_2 = \lambda^2 P_0$ ∴

$\lambda^n P_0 = P_n$ for $n \geq 1$ ∴ $P_0 + P_1 + P_2 + \dots = 1 = \sum_{n=0}^{\infty}P_n$ ∴

$P_0 + \lambda P_0 + \lambda^2 P_0 + \dots = 1 = P_0 + \sum_{n=1}^{\infty}\lambda^n P_0 = \sum_{n=0}^{\infty}\lambda^n P_0 = 1 = P_0\sum_{n=0}^{\infty}\lambda^n$ ∴

for $|\lambda|<1$: $1 = P_0\frac{1}{1-\lambda}$ ∴ $1-\lambda = P_0$ ∴

CW2 / 2a / $\lambda P_0 = \mu P_1$ $\quad \lambda P_1 = 2\mu P_2$, $\lambda P_n = 3\mu P_{n+1}$ for $n > 1$

$\therefore \frac{P_1}{P_0} = 3\rho$, $\frac{P_n}{P_0} = \frac{9}{2}\rho^n$ for $n > 1$ $\therefore$ if $\rho < 1$

$P_0\left(1 + 3\rho + \sum_{n>1} \frac{9}{2}\rho^n\right) = 1 \therefore P_0 = 2(1-\rho)(2+4\rho+3\rho^2)^{-1}$

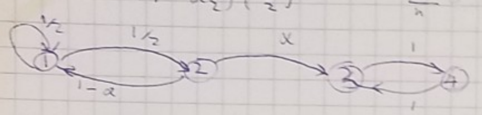
2b / $E(N) = \sum_{n=0}^{\infty} n P_n = 3\rho P_0 + \sum_{n>1}^{\infty} n \frac{9}{2}\rho^n P_0 = \left(3\rho + \frac{9}{2}\rho(1-\rho)^{-2} - 1 - \rho\right) P_0 =$

$(2+2\rho-\rho^2)\rho(1-\rho)^{-2}\frac{3P_0}{2} = \frac{3\rho}{1-\rho}\left(\frac{2+2\rho-\rho^2}{2+4\rho+3\rho^2}\right)$

2d / $\lambda = 3$, $\mu = 2$, $E(T) = E(N)\frac{1}{\lambda_{eff}}$, $\lambda_{eff} = \sum_{n>0} \lambda_n P_n$

$\lambda_n = \lambda \therefore \lambda_{eff} = \lambda \therefore E(T) = E(N)\frac{1}{\lambda}$:

2 / $f^{(n)} = (1-\alpha_2)\left(\frac{1}{2}\right)^{n-1}$ $\quad \sum_n f^{(n)} = (1-\alpha)\left(\frac{1}{2}\right)\left(1-\frac{1}{2}\right)^{-1} = 1-\alpha_2$



\2b/ $G_X(\theta) = e^{-\lambda} e^{\lambda\theta^2}$ $\therefore$

$G_{S_2} = G_X(G_X(\theta)) = e^{-\lambda} e^{\lambda(e^{-\lambda} e^{\lambda\theta^2})^2} = e^{-\lambda} e^{\lambda e^{-2\lambda} e^{2\lambda\theta^2}} =$

$e^{-\lambda} \exp\left(\lambda e^{-2\lambda} \sum_{n=0}^{\infty} \frac{(2\lambda\theta^2)^n}{n!}\right) = e^{-\lambda} \exp\left(\lambda e^{-2\lambda} (1 + 2\lambda\theta^2 + \dots)\right) =$

$e^{-\lambda} e^{\lambda e^{-2\lambda}} e^{2\lambda^2 e^{-2\lambda}\theta^2 + \dots} = e^{-\lambda+\lambda e^{-2\lambda}} \sum_{n=0}^{\infty} (2\lambda^2 e^{-2\lambda}\theta^2 + \dots)^n \frac{1}{n} =$

$e^{-\lambda+\lambda e^{-2\lambda}} \left[1 + 2\lambda^2 e^{-2\lambda}\theta^2 + \dots\right] = e^{-\lambda+\lambda e^{-2\lambda}} + 2\lambda^2 e^{-3\lambda+\lambda e^{-2\lambda}}\theta^2 + \dots$

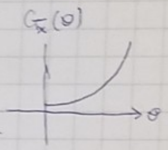
$\therefore P(S_2=0) = e^{-\lambda+\lambda e^{-2\lambda}}$ ✓, $P(S_2=1)=0$ ✓, $P(S_2=2) = 2\lambda^2 e^{-3\lambda+\lambda e^{-2\lambda}}$ ✓

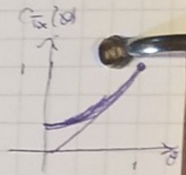
\2c/ Guaranteed is $E(X) < 1$ ✓ $\therefore$

$E(X) = G_X'(\theta)\big|_{\theta=1} = \frac{d}{d\theta}\left(e^{\lambda\theta^2-\lambda}\right)\big|_{\theta=1} = 2\lambda\theta e^{\lambda\theta^2-\lambda}\big|_{\theta=1} = 2\lambda(1)e^{\lambda(1)^2-\lambda} =$

$2\lambda e^0 = 2\lambda < 1$ ✓ $\therefore \lambda < \frac{1}{2}$ ✓ $\therefore$

not guaranteed for $\lambda \geq \frac{1}{2}$





\2di/ $\lambda = \frac{1}{4} < \frac{1}{2} \therefore G_X(\theta) = e^{\frac{1}{4}(\theta^2-1)} = e^{\frac{1}{4}\theta^2 - \frac{1}{4}}$ $\therefore$

\CW1/\2b/ $G_X(G_X(\theta)) = G_X(e^{\lambda(\theta^2-1)}) = \cancel{e^{\lambda\theta}} G_X(e^{\lambda\theta^2-\lambda}) = G_X(e^{-\lambda}e^{\lambda\theta^2})$

$= e^{-\lambda}e^{\lambda(e^{-\lambda}e^{\lambda\theta^2})^2} = e^{-\lambda}e^{\lambda(e^{-2\lambda}e^{2\lambda\theta^2})}$

$= e^{-\lambda} + \lambda e^{-\lambda}(e^{-\lambda}e^{\lambda\theta^2})^2 + \frac{\lambda^2 e^{-\lambda}}{2!}(e^{-\lambda}e^{\lambda\theta^2})^4 + \dots =$

$e^{-\lambda} + \lambda e^{-\lambda}(e^{-\lambda} + \lambda e^{-\lambda}\theta^2)^2 + \dots$

$P(S_2=0) = e^{-\lambda}$, $P(S_2=1) = 0$, $P(S_2=2) = \lambda e^{-\lambda}$

S_n = number of individuals at generation n

$G_{T_2}(\theta) = G_{T_X} \circ G_{T_X}(\theta) = G_{T_X}(G_{T_X}(\theta)) = G_{T_X}(e^{\lambda(\theta^2-1)})$

$G_{T_X}(\theta) = e^{\lambda(\theta^2-1)} = e^{\theta^2\lambda-\lambda} = e^{-\lambda}e^{\theta^2\lambda} = e^{-\lambda}\sum_{k=0}^{\infty}\frac{(\theta^2\lambda)^k}{k!} =$

$e^{-\lambda}\left[\frac{1}{0!}(\theta^2\lambda)^0 + \frac{1}{1!}(\theta^2\lambda)^1 + \frac{1}{2!}(\theta^2\lambda)^2 + \dots\right] =$

$e^{-\lambda}[1(1) + 1\lambda\theta^2 + \dots] = e^{-\lambda}[1 + \lambda\theta^2 + \dots] = e^{-\lambda} + \lambda e^{-\lambda}\theta^2 + \dots$

$G_{T_{S_2}}(\theta) = G_{T_2}(\theta) = G_{T_X}(G_{T_X}(\theta)) = G_{T_X}(e^{\lambda(\theta^2-1)}) = G_{T_X}(e^{-\lambda} + \lambda e^{-\lambda}\theta^2 + \dots) =$

$e^{\lambda([e^{-\lambda}+\lambda e^{-\lambda}\theta^2+\dots]^2-1)} = e^{-\lambda}e^{[e^{-\lambda}+\lambda e^{-\lambda}\theta^2+\dots]^2\lambda} =$

$e^{-\lambda}e^{[(e^{-\lambda})^2+2e^{-\lambda}\lambda e^{-\lambda}\theta^2+\dots]\lambda} = e^{-\lambda}e^{\lambda e^{-2\lambda}+[2\lambda e^{-2\lambda}\theta^2+\dots]\lambda} =$

$e^{-\lambda+\lambda e^{-2\lambda}}e^{2\lambda e^{-2\lambda}\theta^2+\dots} = e^{(-1+e^{-2\lambda})\lambda}\sum_{i=0}^{\infty}\frac{(2\lambda^2 e^{-2\lambda}\theta^2+\dots)^2}{k!} =$

$e^{(-1+e^{-2\lambda})\lambda}\left[\frac{1}{0!}(2\lambda^2 e^{-2\lambda}\theta^2+\dots)^0 + \frac{1}{1!}(2\lambda^2 e^{-2\lambda}\theta^2+\dots)^1 + \dots\right] =$

$e^{(-1+e^{-2\lambda})\lambda}[1(1) + 2\lambda^2 e^{-2\lambda}e^{-2\lambda}\theta^2 + \dots] = e^{(-1+e^{-2\lambda})\lambda} + 2\lambda^2 e^{(-3+e^{-2\lambda})\lambda}\theta^2 + \dots$

$G_{T_{S_2}}(\theta) = G_{T_2}(\theta) = G_{T_X}(G_{T_X}(\theta)) = G_{T_X}(e^{-\lambda} + \lambda e^{-\lambda\theta^2} + \dots) =$

$e^{-\lambda} + \lambda e^{-\lambda(G_{T_X}(\theta))^2} = e^{-\lambda} + \lambda e^{-\lambda(e^{-\lambda}+\lambda e^{-\lambda}\theta^2+\dots)^2}$

$G_{T_2}(\theta) = e^{+\lambda(e^{2\lambda(\theta^2-1)}-1)} = e^{-\lambda}e^{\lambda e^{2\lambda\theta^2-2\lambda}} = \cancel{e^{-\lambda}e^{2\lambda}}\ e^{-\lambda}e^{\lambda e^{-2\lambda}e^{2\lambda\theta^2}} =$

$e^{-\lambda}\exp(\lambda e^{-2\lambda}e^{2\lambda\theta^2}) = e^{-\lambda}\exp\left(\lambda e^{-2\lambda}\sum_{n=0}^{\infty}\frac{(2\lambda\theta^2)^n}{n!}\right) =$

$e^{-\lambda}\exp(\lambda e^{-2\lambda}[1 + 2\lambda\theta^2 + \dots]) = e^{-\lambda}\exp(\lambda e^{-2\lambda} + 2\lambda^2 e^{-2\lambda}\theta^2 + \dots) =$

$e^{-\lambda}e^{\lambda e^{-2\lambda}}\exp(2\lambda^2 e^{-2\lambda}\theta^2 + \dots) =$

$e^{-\lambda}e^{\lambda e^{-2\lambda}}\sum_{i=0}^{\infty}\frac{(2\lambda^2 e^{-2\lambda}\theta^2)^n}{n!} = e^{-\lambda}e^{\lambda e^{-2\lambda}}(1 + 2\lambda^2 e^{-2\lambda}\theta^2 + \dots) =$

$e^{-\lambda+\lambda e^{-2\lambda}} + 2\lambda^2 e^{-3\lambda+\lambda e^{-2\lambda}}\theta^2 + \dots$

$\therefore P(S_2=0) = e^{-\lambda+\lambda e^{-2\lambda}}$, $P(S_2=1) = 0$, $P(S_2=2) = 2\lambda^2 e^{-3\lambda+\lambda e^{-2\lambda}}$

\PP2021/~~EQ~~ |1ai/ $E(Y) = E(5X) = 5E(X)$

$G_X'(\theta) = 0.3 + 1\theta + 0.8\theta^3 \therefore G_X'(1) = E(X) = 0.3 + 1 + 0.8 = 2.1 \therefore$

$E(Y) = 5 \times 2.1 = 10.5$ ✓

$P(Y=5) = P(5X=5) = P(X=1) \therefore G_X(\theta) = 0.3\theta + \dots \therefore$

$P(X=1) = 0.3 = P(Y=5)$ ✓

(1aii) $E(Y) = E(X^2+1) = E(1) + E(X^2) = 1 + E(X^2)$

$G_X''(\theta) = 1 + 3 \times 0.8\theta^2 = 1 + 2.4\theta^2 \therefore$

$G_X''(1) = E(X^2) = 1 + 2.4 \times 1^2 = 3.4 \therefore$

$E(Y) = 3.4 + 1 = 4.4 \therefore$ ✗

$P(Y=5) = P(X^2+1=5) = P(X^2=4) \therefore$

$G_{X^2}(\theta) = 0.3\theta^{0^2} + 0.5\theta^{2^2} + 0.2\theta^{4^2} = 0.3\theta^0 + 0.5\theta^4 + 0.2\theta^{16} =$

$0.3 + 0.5\theta^4 + 0.2\theta^{16} \therefore$

$0.5 = P(X^2=4) = P(Y=5)$ ✓

(1aiii) $E(Y) = E(X_1 + X_2 + X_3) = E(X_1) + E(X_2) + E(X_3) =$

$E(X) + E(X) + E(X) = 3E(X) \therefore E(X) = 2.1 \therefore$

$E(Y) = 3 \times 2.1 = 6.3$ ✓

~~$P(Y=5)$~~ $G_Y(\theta) = G_{X_1+X_2+X_3}(\theta) = G_{X_1}(\theta)G_{X_2}(\theta)G_{X_3}(\theta) = (G_X(\theta))G_X(\theta)G_X(\theta) =$

$(G_X(\theta))^3 = (0.3\theta + 0.5\theta^2 + 0.2\theta^4)^3 =$

$(0.5\theta^2)^2(0.3\theta)^1 \frac{3!}{2! \times 1!} + \dots = 0.225\theta^5 + \dots \therefore$

$P(Y=5) = 0.225$ ✓ (3 s.f.)

\1aii redo/ $E(Y) = E(X^2+1) = E(X^2) + E(1) = E(X^2) + 1 \therefore$

$G_X'(\theta) = 0.3 + 1\theta + 0.8\theta^3 \therefore$

$G_X''(\theta) = 1 + 2.4\theta^2 \therefore E(X^2) = G_X''(1) = 1 + 2.4(1)^2 = 3.4$ ✗

$\therefore E(Y) = 3.4 + 1 = 4.4$ ✗

\1aii redo/ $G_X''(\theta) = G_X^{(2)}(\theta) \therefore$

$G_X''(\theta)|_{\theta=1} = G_X^{(2)}(\theta)|_{\theta=1} = G_X^{(2)}(1) = E[X(X-1)] = E[X(X-2+1)] = E(X^2 - X) =$

$E(X^2) - E(X) \therefore G_X''(\theta) = \frac{d}{d\theta}(0.3 + 1\theta + 0.8\theta^3) = 1 + 2.4\theta^2 \therefore$

$G_X''(1) = E(X^2) - E(X) = 1 + 2.4(1)^2 = 3.4 \therefore 3.4 + E(X) = E(X^2) = 3.4 + 2.1 = 5.5 \therefore$

$E(Y) = E(X^2+1) = E(X^2) + 1 = 5.5 + 1 = 6.5$ ✓

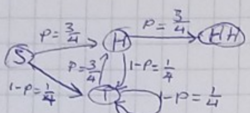
\4aii/ $P(S_n=0)=0$ if n odd

$P(S_n=0)=P(\underbrace{+++\dots+}_{k}\underbrace{---\dots-}_{k}) = \binom{2k}{k}\left(\frac{1}{2}\right)^{2k}$, $n=2k$

\4aiii/ $P(S_n=1)=0$ if n even,

$P(S_n=1)=P(\underbrace{+++\dots+}_{k+1}\underbrace{---\dots-}_{k}) = P(+\underbrace{---\dots-}_{k}\underbrace{+++\dots+}_{k}) =$

$\cancel{2k+1}$ $\binom{2k+1}{k}\left(\frac{1}{2}\right)^{2k+1}$ if $n=2k+1$

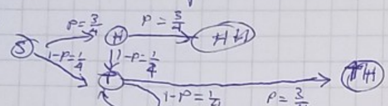
\4bi/ 

∴ solve $D_i = (\sum T_{ij} D_j) + 1$, D_i = Mean time to finish ∴ $D_{HH}=0$,

$D_S = 1+\frac{3}{4}D_H+\frac{1}{4}D_T$, $D_H=\frac{3}{4}D_{HH}+\frac{1}{4}D_T+1$, $D_T=\frac{3}{4}D_H+\frac{1}{4}D_T+1$,

notice $D_S=D_T$ ∴ $D_H=\frac{1}{4}D_T+1$, $D_T=\frac{3}{4}(\frac{1}{4}D_T+1)+\frac{1}{4}D_T+1=\frac{7}{16}D_T+\frac{7}{4}$

∴ $\frac{9}{16}D_T=\frac{7}{4}$ ∴ $D_T=\frac{28}{9}=D_S=3.11$ (3 s.f.)

\4bii/ 

∴ solve $\theta_i=\sum_j T_{ij}\theta_j$

θ_i = prob win from i ∴

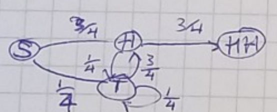
$\theta_{HH}=1$, $\theta_{TH}=0$ ∴ $\theta_S=\frac{3}{4}\theta_H+\frac{1}{4}\theta_T$, $\theta_H=\frac{3}{4}\theta_{HH}+\frac{1}{4}\theta_T$,

$\theta_T=\frac{3}{4}\theta_{TH}+\frac{1}{4}\theta_T=\frac{1}{4}\theta_T=\theta_T$ ∴ $\theta_T=0$ ∴ $\theta_S=\frac{3}{4}\theta_H$,

$\theta_H=\frac{3}{4}\theta_{HH}=\frac{3}{4}$ ∴ $\theta_S=\frac{3}{4}\cdot\frac{3}{4}=\frac{9}{16}=0.563$

need TTTT imediately. if H occurs at any point, then TTTT only reached if we see *TTT but * would a H

∴ $\theta_{TTTT}=\left(\frac{1}{4}\right)^4=0.00391$

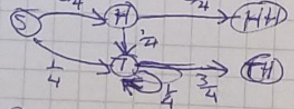
\4bi/ 

∴ $D_i=\sum_j T_{ij}D_j+1$ D_i is time to finish from i ∴

$D_S=1+\frac{3}{4}D_H+\frac{1}{4}D_T$, $D_H=\frac{3}{4}D_{HH}+\frac{1}{4}D_T+1$, $D_S=D_T$, $D_T=1+\frac{3}{4}D_H+\frac{1}{4}D_T$,

$D_{HH}=0$ ∴ $D_H=\frac{1}{4}D_T+1$ ∴ $D_T=1+\frac{3}{4}(\frac{1}{4}D_T+1)+\frac{1}{4}D_T=\frac{7}{16}D_T+\frac{7}{4}$ ∴

$\frac{9}{16}D_T=\frac{7}{4}$ ∴ $D_T=\frac{28}{9}=D_S=3.11$ (3 s.f.)

\4bii/ 

$\theta_i=\sum_j T_{ij}\theta_j$ θ_i is prob to win from i

∴ $\theta_{TH}=0$, $\theta_{HH}=1$ ∴

$\theta_S=\frac{3}{4}\theta_H+\frac{1}{4}\theta_T$ $\theta_H=\frac{1}{4}\theta_T+\frac{3}{4}\theta_{HH}=\frac{1}{4}\theta_T+\frac{3}{4}$, $\theta_T=\frac{1}{4}\theta_T+\frac{3}{4}\theta_{TH}=\frac{1}{4}\theta_T=\theta_T$ ∴

$\theta_T=0$ ∴ $\theta_H=\frac{1}{4}(0)+\frac{3}{4}$ ∴ $\theta_S=\frac{3}{4}\cdot\frac{3}{4}=\frac{9}{16}=0.563$ (3 s.f.)

\PP2018/ $\tilde{P} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ ∴ $\tilde{P} = T\tilde{P} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix}\begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} 0.2P_1+0.8P_2 \\ 0.5P_1+0.5P_2 \end{bmatrix}$

∴ $P_1 = 0.2P_1 + 0.8P_2$, $P_2 = 0.5P_1 + 0.5P_2$ ∴ ✗ ~~$P_1 = P_2$, P_1~~

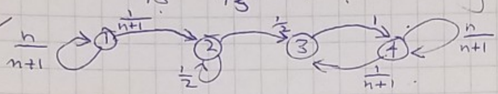
∴ $\tilde{P} = \tilde{P}T = [P_1 \quad P_2] = [P_1 \quad P_2]\begin{bmatrix} 0.2 & 0.8 \\ 0.5 & 0.5 \end{bmatrix} = [0.2P_1+0.5P_2 \quad 0.8P_1+0.5P_2]$

∴ $P_1 = 0.2P_1 + 0.5P_2$, $0.8P_1 + 0.5P_2 = P_2$ ∴ $1 = P_1 + P_2$ ∴ $P_1 = 1 - P_2$ ∴

$0.8(1-P_2) + 0.5P_2 = 0.8 - 0.8P_2 + 0.5P_2 = 0.8 - 0.3P_2 = P_2$ ∴ $0.8 = 1.3P_2$ ∴

$\frac{8}{13} = P_2$ ∴ $1 - \frac{8}{13} = \frac{5}{13} = P_1$. steady state

\3bi/



$P(\underbrace{1\to1\to1\to1\cdots}_{n\text{ times}}|1) = \frac{1}{2}\times\frac{2}{3}\times\frac{3}{4}\cdots\frac{n}{n+1} = \frac{1}{n+1} \to 0$ as $n\to\infty$

otherwise escape from ① ∴ $f_1 = \sum_{n=1}^{\infty} f_1^{(n)} < 1$ ∴ transient

\3bii/ ② is transient and $f_2^{(n)} = 0$ for $n > 1$ ∴ aperiodic

\3biii/ ④: $f_4^{(1)} = P(4\to4|4) = \frac{1}{1+1} = \frac{1}{2}$, $f_4^{(2)} = P(4\to3\to4|4) = \frac{1}{1+1}(1) = \frac{1}{2}$

$f_4^{(n)} = 0$ for ~~n≥~~ $n \geq 3$ ∴ $f_4 = \sum_{n=1}^{\infty} f_4^{(n)} = f_4^{(1)} + f_4^{(2)} + \sum_{n=3}^{\infty} f_4^{(n)} = f_4^{(1)} + f_4^{(2)} + \sum_{n=3}^{\infty} 0$

$= \frac{1}{2} + \frac{1}{2} = 1$ ∴ ④ is recurrent ∴

$\mu_4 = \sum_{n=0}^{\infty} n f_4^{(n)} = \sum_{n=0}^{2} n f_4^{(n)} + \sum_{n=3}^{\infty} n f_4^{(n)} = 1 f_4^{(1)} + 2 f_4^{(2)} + \sum_{n=3}^{\infty} n(0) = f_4^{(1)} + 2 f_4^{(2)} =$

$\frac{1}{2} + 2(\frac{1}{2}) = \frac{1}{2} + 1 = \frac{3}{2} < \infty$ ∴ ④ is positively recurrent ∴ yes

\3biv/ $\{1\}$, $\{2\}$ transient ∴ (not ergodic)

$P(3\to4\to3|3) = 1\times\frac{1}{2+1} = 1\times\frac{1}{3}$, $P(3\to4\to4\to3|3) = 1\times\frac{2}{2+1}\times\frac{1}{3+1} = 1\times\frac{2}{3}\times\frac{1}{4} = 1\times\frac{2}{3}\times\frac{1}{4}$

$P(3\to4\to4\to\cancel{4}\to3|3) = 1\times\frac{2}{2+1}\times\frac{3}{3+1}\times\frac{1}{4+1} = 1\times\frac{2}{3}\times\frac{3}{4}\times\frac{1}{5}$ etc ...

$f_3^{(n)} = P(3\to\underbrace{4\cdots\to4}_{n-2\text{ times}}\to3|3) = 1\times\frac{2}{3}\times\frac{3}{4}\times\frac{4}{5}\times\frac{5}{6}\times\frac{6}{7}\times\cdots\times\frac{n-1}{n}\times\frac{1}{n+1} = 2\cancel{\frac{1}{n}}\frac{2}{n}\times\frac{1}{n+1} =$

$\frac{2}{n(n+1)} = 2\frac{1}{n(n+1)} = 2\left(\frac{1}{n} - \frac{1}{n+1}\right)$ ∴

$f_3 = \sum_{n=1}^{\infty} f_3^{(n)} = \sum_{n=2}^{\infty} f_3^{(n)} = 2\cancel{\frac{1}{n}} \sum_{n=2}^{\infty} 2\left(\frac{1}{n} - \frac{1}{n+1}\right) = 2\left(\frac{1}{2} - 0\right) = 1$ telescoping

but $\mu_3 = \sum_{n=1}^{\infty} n f_3^{(n)} \approx \sum_{n=2}^{\infty} n2\left(\frac{1}{n} - \frac{1}{n+1}\right) \approx \sum \frac{1}{n} \approx \infty$ ∴

③ is null recurrent ∴ subchain $\{3, 4\}$ is not ergodic ∴ NO.

\4ai/ $P(S_{n+1} = \tilde{s} | S_n \cdots S_0 \text{ given}) = P(S_{n+1} = \tilde{s} | S_n \text{ given})$ ∴

$P(S_{n+1} = y | S_n = x) = \begin{cases} \frac{1}{2} & \text{if } y = x+1 \\ \frac{1}{2} & \text{if } y = x-1 \end{cases}$, ∀ state $\mathbb{Z}$

∴ $\{S_n\}$ forms Markov chain.