MTH3007 FLUID DYNAMICS

practicals thursdays after reading on learn enable organise

read fundamentals — for each weeks tile goto top for module discussion form to ask questions

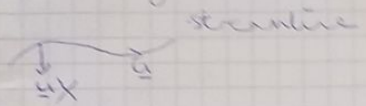
provisional 22nd Oct 28th Nov

important formula sheet

fluid dynamics read lecture notes — green is activity try them yourselves

red boxes is problems try solving them & at Z top is a link to her solution

streamlines tangents are their velocities



equation (2.16) is navier stokes eqn

if its in a box you need to remember it

$\underline{c} = (c_1, c_2, c_3)$ is suffix notation

vector — 3 scalars

tuesday strongly recommended about it all coming together as an orientation

thursday is an optional help session about anything

goto module discussion form add subject & ask question

all sessions appear in recap recordings tile on ele

fluid is anything that flows

matter can be devided into 3 categories: solids, liquids, gases — fluids

not intrested in fluids like toothpaste

intrested in air, water

space $\underline{x} = (x, y, z) = x\underline{i} + y\underline{j} + z\underline{k}$

time t velocity: $\underline{u}(x, t) = (u, v, w) = u(x, y, z, t)\underline{i} + v(x, y, z, t)\underline{j} + w(x, y, z, t)\underline{k}$

pressure $p = p(x, y, z, t)$ density $\rho = \rho(x, y, z, t)$

p & ρ are scalars

• Z x-component of Z velocity is a scalar denoted by u
• Z velocity, denoted by $\mathbf{u}$, $\vec{u}$, or $\underline{u}$
spherical polar coords (r, θ, ϕ) θ is from z-axis
cylindrical polar coords (R, ϕ, z) ϕ is from x-axis

flow is <u>steady</u> if velocity independent of time ie $\underline{u}=\underline{u}(\underline{x})$ only flow is <u>unsteady</u> if $\underline{u}=\underline{u}(\underline{x},t)$
stagnation point $(\underline{x}^*)$ is location $\underline{u}(\underline{x}^*)=\underline{0}$
flow $\underline{u}=(y, 1-z^2, x)$ has stagnation pt at $x^*=(0,0,1)$

in fluid dynamics pressure p gives rise to a force $-\nabla p$
is directed from regions of high to low pressure
∇p is perpen to Z level surfaces $p=\text{const}$

\fact 2.1/ low pressure cold air high pressure hot air

conservation of mass, rate of change of mass in fixed volume V
a small volume of fluid δV at location $\underline{x}$ has mass ~~δM~~
$\delta M = \delta V \rho(x,t)$

Z <u>flux</u> of a quantity through a surface is Z rate of Z quantity through Z surface (amount of quantity that flows through Z surface per unit time) quantity can be mass, heat, particles, magnetic fields.
volume V of fluid surrounded by a smooth surface S fixed in space fluid flows with velocity $\underline{u}$ through Z vector surface area element $\underline{dS} = \hat{n}\,dS$ $\hat{n}$ is unit outwards normal
Z volume that is pushed out through Z area dS in time dt is approx $(\hat{n}\cdot\underline{u})\,dS\,dt$

\fact 2.2/ volume out increased by dS & dt & $\underline{u}$

$\therefore$ Z volume flux (rate volume crosses dS) is $(\hat{n}\cdot\underline{u})dS = \underline{u}\cdot d\underline{S}$

& Z mass flux: $\rho\underline{u}\cdot d\underline{S}$

$\therefore$ Z total mass flux through whole surface is $\int_S \rho\underline{u}\cdot d\underline{S}$

mass inside fixed volume is $\int_V \rho\, dV$ which is a pos quantity

Z rate of change of mass in V is $\therefore \frac{d}{dt}\int_V \rho\, dV$ & Z time derivative can be taken inside Z volume integral

continuity eqn can be written: $\frac{\partial\rho}{\partial t} + \frac{\partial(\rho u)}{\partial x} + \frac{\partial(\rho v)}{\partial y} + \frac{\partial(\rho w)}{\partial z} = 0$

~~also~~ $\nabla\cdot(\rho\underline{u}) = \rho\nabla\cdot\underline{u} + \underline{u}\cdot\nabla\rho$ $\therefore$

Z continuity eqn: $\frac{D\rho}{Dt} + \rho\nabla\cdot\underline{u} = 0$ operator $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\nabla$ is Z material (or Lagrangian) derivative

$D\rho/Dt$ is Z rate of change of density following Z fluid motion

it can be nonzero because of changes in pressure, temperature or composition

incompressible fluids don't expand or contract & $\therefore D\rho/Dt = 0$

$\therefore$ incompressible form of continuity eqn: $\nabla\cdot\underline{u} = 0$

2D vector fields visualised by drawing Z vector a sequence of pts with length & direction of Z arrow denoting Z magnitude & direction of Z vect

A streamline is an instantaneous curve drawn in Z fluid that is tangent to Z velocity vec at each pt along Z curve

let curve C represented parametrically by endpts $\underline{r}(s) = (x(s), y(s), z(s))$ for certain range of params $s\,(s_1 < s < s_2)$ Z requirement tangent: $\frac{d\underline{r}}{ds} = \left(\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}\right)$ is parallel to Z velocity $\underline{u}$ $\therefore \frac{d\underline{r}}{ds} = \lambda\underline{u}$

λ is const of proportionality

streamlines are sols to $\frac{dx}{ds} = u(x(s), y(s), z(s))$,

$\frac{dy}{ds} = v(x(s), y(s), z(s))$, $\frac{dz}{ds} = w(x(s), y(s), z(s))$ where absorbed const λ into param s (taken $\lambda = 1$ without generality loss) streamline eqns: $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$ assuming $u, v, w \neq 0$

or $\frac{dy}{dx} = \frac{v}{u}$, $\frac{dz}{dy} = \frac{w}{v}$, etc $\frac{w}{u} = \frac{dz}{dx}$

\Ex 2.4.1 : finding streamlines / $\underline{u} = (\alpha y, -\alpha x, 0)$ α is const

Method 1: $\frac{dx}{\alpha y} = \frac{dy}{-\alpha x}$ $\therefore \int -\alpha x \, dx = \int \alpha y \, dy$

$\Big\{ -\frac{\alpha x}{dy} = \frac{\alpha y}{dx}$ $\therefore \int \frac{-\alpha x}{dy} dx = \int \frac{\alpha y}{dx} dx = \int \alpha y$ $\therefore$

$\int \frac{-\alpha x}{dy} dx \, dy = \int \alpha y \, dy = \int -\alpha x \, dx \Big\}$ $\therefore -\frac{\alpha x^2}{2} = \frac{\alpha y^2}{2} + C$

also $dz = 0$ since $w = 0$ $\therefore z = z_0$, a constant $\therefore$

streamlines circles $x^2 + y^2 = \text{const}$ in z plane $z = z_0$ 'rigid body rotation'

Method 2: $\frac{dx}{ds} = \alpha y$ $\frac{dy}{ds} = -\alpha x$ $\frac{dz}{ds} = 0$ $\therefore z = z_0$

const $z = z_0$ $\therefore \frac{d^2x}{ds^2} = \alpha \frac{dy}{ds} = -\alpha^2 x$ $\therefore \frac{d^2x}{ds^2} + \alpha^2 x = 0$

$\therefore$ Sol: $x = e^{ps}$ $\therefore p^2 + \alpha^2 = 0$ ie $p = \pm i\alpha$ $\therefore$ sols

$x(s) = A\cos(\alpha s) + B\sin(\alpha s)$ consts A & B $\therefore$

$\frac{dx}{ds} = \alpha y$ $\therefore y(s) = \frac{1}{\alpha}[-A\alpha\sin(\alpha s) + B\alpha\cos(\alpha s)]$ $\therefore$

$x(s) = A\cos(\alpha s) + B\sin(\alpha s)$ $y(s) = -A\sin(\alpha s) + B\cos(\alpha s)$ $z(s) = z_0$

$\therefore$ curves $x^2(s) + y^2(s) = A^2 + B^2 = \text{const}$ ie circles

set $t = \text{const}$ for streamlines

streamlines constantly changing apart from steady motion

only at stagnation pts $\underline{u} = 0$ can streamlines cross

particle path (pathline) $\underline{r}(\underline{x}_0, t)$ initial pos $\underline{x}_0$

velocity at pt $\underline{x}$ at t is $\underline{u}(\underline{x}, t)$

$\frac{dr}{dt} = u(r, t)$, $r = (x(t), y(t), z(t))$ $\therefore$

$\frac{dx}{dt} = u = \frac{dy}{dt} = v$ $\frac{dz}{dt} = w$

\Ex 2.5.1 / fluid velocity $u(\underline{x}, t) = (\alpha y, -\alpha(x - bt), 0)$ α & b const

pathlines: $\frac{dx}{dt} = \alpha y$ $\frac{dy}{dt} = -\alpha(x - bt)$ $\frac{dz}{dt} = 0$ $\therefore$

$$\frac{d^2x}{dt^2} = a\frac{dy}{dt} = -a^2(x - bt) \quad \& \; z = z_0$$

$\frac{d^2x}{dt^2} + a^2x = 0 \therefore \quad x_h(t) = A\cos(at) + B\sin(at)$

$\frac{d^2x}{dt^2} + a^2x = a^2bt$ is $x_p(t) = Ct \quad C = b \quad \therefore$

$x(t) = x_h + x_p = A\cos(at) + B\sin(at) + bt$

$\therefore \frac{dx}{dt} = ay \quad \therefore y(t) = -A\sin(at) + B\cos(at) + \frac{b}{a}$

at $t = 0$: $(x_0, y_0, z_0) \quad A = x_0, \; B = y_0 - b/a \quad \therefore$

pathline: $x(t) = x_0\cos(at) + (y_0 - b/a)\sin(at) + bt$

$y(t) = -x_0\sin(at) + (y_0 - b/a)\cos(at) + b/a, \quad z(t) = z_0 \quad \therefore$

$(x - bt)^2 + (y - b/a)^2 = x_0^2 + (y_0 - b/a)^2$ ie circular paths radius $\sqrt{x_0^2 + (y_0 - b/a)^2}$ centre $(bt, b/a, z_0)$ cycloid

when $b = 0$ flow is steady, circles centred $(0, 0, z_0)$ radius $\sqrt{x_0^2 + y_0^2}$ ie same as streamlines

$\underline{F} = m\underline{a}$

$\underline{x}$ at t moves $\underline{x} + \delta\underline{x}$ at $t + \delta t \quad \therefore \underline{x} = (x_1, x_2, x_3)$ & $\delta\underline{x} =$
$\delta\underline{x} = (\delta x_1, \delta x_2, \delta x_3) \quad \therefore$

change in velocity is $\underline{u}(\underline{x} + \delta\underline{x}, t + \delta t) - \underline{u}(x, t) \simeq$

$\underline{u}(x, t) + \delta x_1 \frac{\partial \underline{u}}{\partial x_1} + \delta x_2 \frac{\partial \underline{u}}{\partial x_2} + \delta x_3 \frac{\partial \underline{u}}{\partial x_3} + \delta t \frac{\partial \underline{u}}{\partial t} - \underline{u}(x, t)$

which is taylor with quadratic & beyond ignored $\therefore$

divide by δt & let $\delta t \to 0 \therefore$ acceleration:

$$a(\underline{x}, t) = \lim_{\delta t \to 0} \left(\frac{\delta x_1}{\delta t}\frac{\partial \underline{u}}{\partial x_1} + \frac{\delta x_2}{\delta t}\frac{\partial \underline{u}}{\partial x_2} + \frac{\delta x_3}{\delta t}\frac{\partial \underline{u}}{\partial x_3} + \frac{\partial \underline{u}}{\partial t} \right) \quad \therefore$$

$\delta x_1 / \delta t \to u_1$

$$\underline{a}(\underline{x}, t) = u_1\frac{\partial \underline{u}}{\partial x_1} + u_2\frac{\partial \underline{u}}{\partial x_2} + u_3\frac{\partial \underline{u}}{\partial x_3} + \frac{\partial \underline{u}}{\partial t} = \underline{u}\cdot\nabla\underline{u} + \frac{\partial \underline{u}}{\partial t} = \frac{D\underline{u}}{Dt}$$

steady $\frac{\partial \underline{u}}{\partial t} = 0 \quad \therefore \underline{u} = \underline{u}(\underline{x})$ only still acceleration by $\underline{u}\cdot\nabla\underline{u}$

volume δV & mass $\delta M = \rho\,\delta V$ experience:

- acceleration $\frac{D\underline{u}}{Dt} = \frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}$
- pressure force $\delta V(-\nabla p)$
- gravitational force $\delta M\underline{g} = \rho\,\delta V\underline{g}$
- viscous force $\delta V\mu\nabla^2\underline{u}$

$\underset{\sim}{F} = m\underline{a} \therefore \rho\left(\frac{\partial u}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho \underline{g} + \mu\nabla^2\underline{u}$

μ is coeff of viscosity

viscous term $\mu\nabla^2\underline{u}$ is resistence to being sheared

can add:

· Coriolis term $\underline{\Omega}\times\underline{u}$ for earth's rotation

· Lorentz term $(\nabla\times\underline{B})\times\underline{B}$ for magnetic fields

$\rho = \rho(p, T)$ T temperature

· take density ρ const $\therefore \frac{\partial \rho}{\partial t} = 0$ & $\nabla\rho = 0$

∴ const density incompressible flow ~~$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right)$~~

$\rho\left(\frac{\partial u}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho\underline{g} + \mu\nabla^2\underline{u}$, $\nabla\cdot\underline{u} = 0$

non linear through $\underline{u}\cdot\nabla\underline{u}$ term

$\underline{c} = \underline{a} + \underline{b}$ $\quad \underline{c} = (c_1, c_2, c_3) \therefore \quad c_i = a_i + b_i$ for $i = 1, 2, 3$

suffix i is a free suffix is arbit ∴ equally write

$c_j = a_j + b_j$

a scalar product: $\underline{a}\cdot\underline{b} = a_1b_1 + a_2b_2 + a_3b_3 = \sum_{j=1}^{3} a_j b_j$

$\therefore \underline{a}\cdot\underline{b} = a_j b_j$

a repeated suffix is a dummy suffix & can't appear more than twice

Kronecker delta δ_{ij}: $\quad \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$

$$\therefore I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \delta_{11} & \delta_{12} & \delta_{13} \\ \delta_{21} & \delta_{22} & \delta_{23} \\ \delta_{31} & \delta_{32} & \delta_{33} \end{bmatrix}$$

δ_{ij} is symmetric ie $\delta_{ij} = \delta_{ji}$

alternating tensor ε_{ijk}:

$$\varepsilon_{ijk} = \begin{cases} +1 & \text{if } (i,j,k) = (1,2,3), (3,1,2), (2,3,1) \\ -1 & \text{if } (i,j,k) = (3,2,1), (2,1,3), (1,3,2) \\ 0 & \text{if any } i, j, k \text{ are equal} \end{cases}$$

has $3^3 = 27$ elements but all but 6 are zero:

$\varepsilon_{123} = \varepsilon_{231} = \varepsilon_{312} = 1$, $\varepsilon_{132} = \varepsilon_{213} = \varepsilon_{321} = -1$

$\varepsilon_{ijk} = \varepsilon_{kij} = \varepsilon_{jki}$

$\varepsilon_{ijk} = -\varepsilon_{jik}$

$\underline{(a \times b)_i = \varepsilon_{ijk} a_j b_k}$ j & k repeated so must be summed over

$\underline{a} \cdot (\underline{b} \times \underline{c}) = a_i (\underline{b} \times \underline{c})_i = a_i \varepsilon_{ijk} b_j c_k = \varepsilon_{ijk} a_i b_j c_k$

$\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$ i appears twice on left ∴ summed over ∴ not present on right

cartesian coords (x, y, z) as (x_1, x_2, x_3) ∴

- gradient of scalar field $\nabla\phi = \left(\frac{\partial\phi}{\partial x_1}, \frac{\partial\phi}{\partial x_2}, \frac{\partial\phi}{\partial x_3}\right)$

∴ in suffix notation write the i^{th} component of $\nabla\phi$ as

$[\nabla\phi]_i = \frac{\partial\phi}{\partial x_i}$

Sometimes i^{th} component of operator $\nabla_i = \frac{\partial}{\partial x_i}$

- the divergence of a vec field $\underline{u}$ is $\nabla \cdot \underline{u} = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_3}$

∴ suffix notation: $\nabla \cdot \underline{u} = \frac{\partial u_i}{\partial x_i}$

∴ $\nabla \times \underline{u}$ is $[\nabla \times \underline{u}]_i = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}$

- the laplacian $\nabla^2 = \frac{\partial^2}{\partial x_i \partial x_i}$ ∴

for ϕ: $\nabla^2\phi = \frac{\partial^2\phi}{\partial x_i \partial x_i}$

for vec field $\underline{F}$: $[\nabla^2 \underline{F}]_j = \frac{\partial^2 F_j}{\partial x_i \partial x_i}$

∴ $(\underline{F} \cdot \nabla)\phi = F_i \frac{\partial\phi}{\partial x_i}$ & $[(\underline{F} \cdot \nabla)\underline{F}]_j = F_i \frac{\partial F_j}{\partial x_i}$

vec $\underline{c} = (c_1, c_2, c_3)$

x comp: c_1 y comp c_2 z comp: c_3

i^{th} component c_i $(i = 1, 2, 3)$

$\underline{u}$, p-pressure, ρ-density

u_i ✓ p_i ✗, ρ_i ✗

$\partial_t \underline{u} = 3\underline{v} + a\underline{u}$ underlining vecs $\underline{u}$ $(\vec{u}, \boldsymbol{u})$

i^{th} comp: $\partial_t u_i = \frac{\partial}{\partial t} u_i = 3v_i + a u_i$

products $\underline{a} \cdot \underline{b} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} \cdot \begin{pmatrix} b_1 \\ b_2 \\ b_3 \end{pmatrix} = a_1 b_1 + a_2 b_2 + a_3 b_3$ vec notn

$= \sum_{i=1}^{3} a_i b_i = a_i b_i$ Summation convention applies

$\partial_t \underline{u} = (\underline{a} \cdot \underline{b}) \underline{u}$

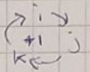
$\partial_t u_j = a_i b_i u_j$ ← anything other than i (already repeated)

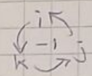
j^{th} comp of eqn

kronecker delta δ_{ij} (3x3 Matrix) $= 1$ when $i = j$, 0 otherwise

$\delta_{ij} a_j = a_i$ (Since its 0 unless $i = j$)

alternating tensor ε_{ijk}





$\therefore$ $\varepsilon_{ijk} = +1$ when i, j, k clockwise

$\varepsilon_{ijk} = -1$ ~~when~~ when i, j, k anticlockwise

$\varepsilon_{ijk} = 0$ otherwise

$(\underline{a} \times \underline{b})_i = \varepsilon_{ijk} a_j b_k$ ← def 2.19

ex simplify $\delta_{ij} \varepsilon_{ijk} \sigma_k = \varepsilon_{iik} \sigma_k$ by def of δ_{ij}
$= 0$

$\varepsilon_{ijk} \varepsilon_{ilm} = \delta\delta - \delta\delta$

$\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta - \delta\delta$ middle middle

$\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta\delta$ middle middle endend

$\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$ middle middle endend - endmid midend

Differentiation operators $\nabla f = \left(\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z} \right) = \left(\frac{\partial f}{\partial x_1}, \frac{\partial f}{\partial x_2}, \frac{\partial f}{\partial x_3} \right)$

$(\nabla f)_i = \frac{\partial f}{\partial x_i}$

$\nabla_i = \frac{\partial}{\partial x_i}$ i^{th} component of Z gradient operator

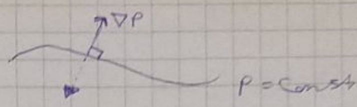
$\nabla \cdot \underline{u} = \nabla_i u_i = \frac{\partial u_i}{\partial x_i} = \left(\sum_{i=1}^{3} \frac{\partial u_i}{\partial x_i} \right)$

$(\nabla \times \underline{u})_i = \varepsilon_{ijk} \nabla_j u_k = \varepsilon_{ijk} \frac{\partial u_k}{\partial x_j}$

$\underline{F} \cdot \nabla = F_i \frac{\partial}{\partial x_i}$ $\quad (\underline{F} \cdot \nabla) \phi = F_i \frac{\partial \phi}{\partial x_i}$

$\cancel{(\underline{F} \cdot \nabla) \underline{G} = F_i \frac{\partial}{\partial x_i} G_i}$ $\quad [(\underline{F} \cdot \nabla) \underline{G}]_j = F_i \frac{\partial}{\partial x_i} G_j$

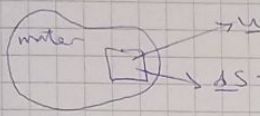
Pressure (page 5)



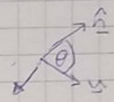
high pressure calm weather

low pressure - rain & storms

mass = $\int \rho \, dV$



$d\underline{S} = dS\,\hat{n}$ - unit normal to surf



$(\hat{n} \cdot \underline{u})\, dS\, dt$ $\qquad \hat{n} \cdot \underline{u} = \underbrace{|\hat{n}|}_{=1}|\underline{u}|\cos\theta$

Week 2 ✓ $\qquad y = 0 \qquad z + 2 = 0 \therefore z = -2 \qquad x - 1 = 0 \therefore x = 1 \therefore$

$(1, 0, -2)$ ✓

on streamline velocity vec is tangent to

streamlines $\frac{dx}{ds} = u \qquad \frac{dy}{ds} = v \qquad \frac{dz}{ds} = w \quad \therefore \quad ds = \frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

Method 2: (sep of variables not poss) $\frac{dx}{ds} = y \Rightarrow \frac{d^2x}{ds^2} = \frac{dy}{ds} = -x \quad (a=1)$

$\therefore \quad \frac{d^2x}{ds^2} + x = 0$

```
>>xMax = 1 ; yMax = xMax;
>>[x, y] = meshgrid(-xMax : 0.1 : xMax, -yMax : 0.1 : yMax);
>>u = y; v = -x; psi = (x.^2 + y.^2);
>>figure (1)
>>contour(x, y, psi, 'linewidth', 1)
>>xlabel x, ylabel y, colorbar
>>axis equal
>> hold on
>> quiver(x, y, u, v, 'linewidth', 2)
```

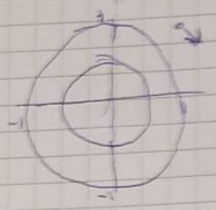
$\underline{u} = (y, -x, 0) \qquad \underline{u} = \nabla \times \psi \underline{k} = (\partial_y \psi, -\partial_x \psi, 0) \qquad \psi$ stream func

x comp $\Rightarrow \frac{\partial \psi}{\partial y} = y \Rightarrow \psi = \frac{y^2}{2} + f(x) \Rightarrow \frac{\partial \psi}{\partial x} = f'$

y comp $\quad -\frac{\partial \psi}{\partial x} = -x \Leftarrow \quad \therefore f'(x) \Rightarrow f = \frac{x^2}{2} + c \Rightarrow$

$\psi = \frac{y^2+x^2}{2} + \text{const}$ streamlines are const ie $x^2+y^2 = \text{const}$

$x \in [-x_{max}, x_{max}] \in [-1, 1]$ with $\Delta x = 0.1$



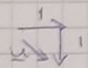
plot $x^2+y^2 = c_1$ $c_1 = 0.1$

$x^2+y^2 = c_2$ $c_2 = 0.5$

arrows increases further away from origin

@ $(x, y) = (1, 1)$

$(u, v) = (y, -x) = (1, -1)$



acceleration of a fluid element is the lagrangian derivative of the velocity

particle acceleration $\underline{a} = \partial_t \underline{u} + \underline{u} \cdot \nabla \underline{u} = \frac{D\underline{u}}{Dt}$ lagrandian derivative

steady flows $\partial_t \underline{u} = \underline{0}$

suffix convention - a repeated suffix must be summed over

$a_i b_i c_j \therefore \sum_{i=1}^{3} (a_i b_i c_j) + u_i \ldots$

$\partial_t \rho + \nabla \cdot (\rho \underline{u}) = 0$ Mass continuity eqn

$\frac{D\rho}{Dt} = 0$ the fluid is incompressible $\Rightarrow \nabla \cdot \underline{u} = 0$

if $\nabla \cdot \underline{u} \neq 0$ the fluid is compressible

week 2 exact sols/

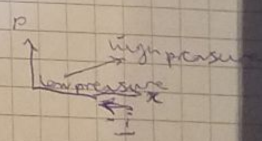
μ is viscosity

$$\underbrace{\rho \underbrace{\left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u}\right)}_{\text{acceleration}} = \underbrace{-\nabla p + \rho \underline{g} + \mu \nabla^2 \underline{u}}_{\text{forces}}}$$

Navier Stokes eqn

not on formula sheet - remember

a fluid moves because forces act on it - gravity, viscosity, pressure, gradients



pressure gradient: fluid moves according to $f_b - \nabla p$

$\nabla p = \left(\frac{dp}{dx}, 0, 0\right)$ $\frac{dp}{dx} > 0$ $-\nabla p = -\frac{dp}{dx} \underline{i}$

solve N-S in simplified setting $\underline{u} = u\underline{i} \Rightarrow \underline{u} \cdot \nabla = u\underline{i} \cdot \left(\underline{i}\frac{\partial}{\partial x} + \underline{j}\frac{\partial}{\partial y} + \underline{k}\frac{\partial}{\partial z}\right) = u\frac{\partial}{\partial x}$

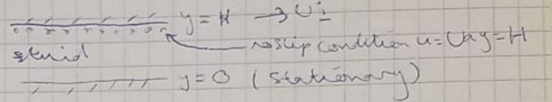
$\therefore \underline{u} \cdot \nabla \underline{u} = u \frac{\partial}{\partial x}(u\underline{i}) = \underline{i}\, u \frac{\partial u}{\partial x}$ $\underline{i} = (1, 0, 0)$

$\therefore$ if u is indep of x then $\underline{u} \cdot \nabla \underline{u} = \underline{0}$

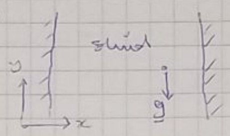
\ ~~problem~~ 2.14 / Hydrostatic support - find pressure $p(x,y,t)$

$\underline{u} = \underline{0}$ take $\downarrow \underline{g}$

\Ex 2.15/ Couette flow

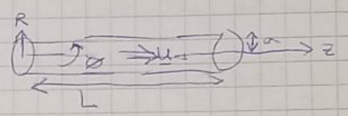


\ ~~Ex~~ 2.16/ poiseuille flow



find $\underline{u} = (0, 0, u(x))$ from Navier stokes

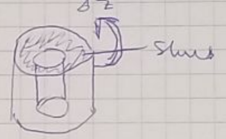
\Section 2.8.3/ Hagen-poiseuille flow



cylindrical polars (R, θ, z)

$\underline{u} = w(R)\,\hat{\underline{z}}$ $\quad p_L < p_0 \quad \therefore \frac{\partial p}{\partial z} \neq 0 \quad 0 = -\nabla p + \mu \nabla^2 \underline{u}$

\2.8.5/ rotating cylinder



\exact sols of Z Navier Stokes eqn/ exact sols of navier Stokes eqns Z categories: those with Z nonlinear term $\underline{u}\cdot\nabla\underline{u}$ vanishing (eg Couette flow, poiseuille flow) & those which it doesn't (eg stagnation pt flow, flow in Convergent & divergent channels, flow over a porous wall)

\plane couette flow/ consider Z case of incompressible unidirectional flow in Z x-direction with $\underline{u} = u\underline{i}$ then

$\nabla \cdot \underline{u} = 0 \Rightarrow \frac{\partial u}{\partial x} = 0$ & $\therefore$ $u = u(y, z, t)$ only &

~~for~~ $\underline{u}\cdot\nabla = (u, 0, 0)\cdot\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) = u\frac{\partial}{\partial x}$ & $\therefore$

$(\underline{u}\cdot\nabla)\underline{u} = u\frac{\partial}{\partial x}(u\underline{i}) = u\frac{\partial u}{\partial x}\underline{i} = \underline{0}$ $\therefore$ Z nonlinear term vanishes

in addition, if u doesn't depend on z ie $\underline{u} = u(y,t)\underline{i}$ &

take $\underline{g} = (0, -g, 0)$ then Z x & y components of Z Navier-Stokes eqn (2.16) become $\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2} \qquad 0 = -\frac{\partial p}{\partial y} - \rho g$

\moving fluid/ if $\underline{u} = u(y,t)\underline{i} \neq \underline{0}$. Z y-component of Z Navier stokes eqn still gives $p = -\rho g y + f(x,t)$ & $\therefore$ it looks

at though $\partial p/\partial x$ could be a func of x & t but the
x-component of the Navier Stokes eqn be:gives
$\rho \frac{\partial u}{\partial t} - \mu \frac{\partial^2 u}{\partial y^2} = -\frac{\partial p}{\partial x}$ where separated variables st

LHS depends of y & t only while RHS is potentially a func of x. this cannot be so for an equality & $\therefore$ $\partial p/\partial x$ must be indep of x $\therefore$ set $-\frac{\partial p}{\partial x} = G(t)$ where $G(t)$ is the streamwise pressure gradient (in same direction of $\underline{u}$)

1.8.3 Hagen-poiseuille flow / neglect body forces ($\underline{g}=\underline{0}$) & consider flow in a long cylindrical pipe of radius a & length L that is driven by a pressure gradient
$P(z=0)=P_0$, $P(z=L)=P_L$, $(P_L<P_0)$ where P_0 & P_L are consts
& we are using cylindrical polar coords (R,ϕ,z) with z pointing along the pipe axis
assume that the flow is steady & $\underline{u}=w(R)\hat{\underline{z}}$
the Navier Stokes eqn reduces to (show $\underline{u}\cdot\nabla\underline{u}=\underline{0}$)
$0=-\nabla p+\mu\nabla^2\underline{u}$, the z-component of which gives
$0=-\frac{\partial p}{\partial z}+\mu(\nabla^2\underline{u})_z$ $\therefore$
implies $\frac{\partial p}{\partial z}$ must be indep of z: $\frac{\partial p}{\partial z}=f(R)$ $\therefore$

using boundary conditions: $\frac{\partial p}{\partial z}=\frac{P_L-P_0}{L}$ $(=-G)$
to calc $(\nabla^2\underline{u})_z$ in cylindrical coords use
$\nabla\times(\nabla\times\underline{u})=\nabla(\nabla\cdot\underline{u})-\nabla^2\underline{u}$ with $\underline{u}=w(R)\hat{\underline{z}}$ $\therefore$
$\nabla\cdot\underline{u}=\frac{1}{R}\frac{\partial}{\partial R}(Ru_R)+\frac{1}{R}\frac{\partial u_\phi}{\partial\phi}+\frac{\partial u_z}{\partial z}=0$.
cause $u_R=u_\phi=0$ & $u_z=w(R)$ & $\nabla\times\underline{u}=\frac{1}{R}\begin{vmatrix}\hat{R} & R\hat{\phi} & \hat{z}\\ \partial/\partial R & \partial/\partial\phi & \partial/\partial z\\ u_R & Ru_\phi & u_z\end{vmatrix}=$

$\frac{1}{R}\begin{vmatrix}\hat{R} & R\hat{\phi} & \hat{z}\\ \partial/\partial R & \partial/\partial\phi & \partial/\partial z\\ 0 & 0 & w(R)\end{vmatrix}=-\frac{dw}{dR}\hat{\underline{\phi}}$ st:

$$\nabla\times(\nabla\times\underline{u}) = \frac{1}{R}\begin{vmatrix}\hat{R} & R\hat{\phi} & \hat{z}\\ \partial/\partial R & \partial/\partial\phi & \partial/\partial z\\ 0 & -R\frac{dw}{dR} & 0\end{vmatrix} = -\frac{1}{R}\frac{d}{dR}\left(R\frac{dw}{dR}\right)\hat{z} \therefore$$

$$(\nabla^2\underline{u})_z = \frac{1}{R}\frac{d}{dR}\left(R\frac{dw}{dR}\right) \therefore$$

$$\frac{\mu}{R}\frac{d}{dR}\left(R\frac{dw}{dR}\right) = \frac{\partial P}{\partial z} = -G \quad \left(= \frac{P_L - P_0}{L}\right) \therefore$$

$$w = -\frac{GR^2}{4\mu} + A\log R + B \quad \text{is General Sol}$$

note: always need 2 boundary conditions:

- <u>no slip</u>: fluid at boundary always moves at same velocity as boundary
- <u>Regularity</u>: eliminate unphysical terms by an appropriate choice of consts
- <u>free surface</u>: absence or neglect of forces on a surf gives rise to a <u>no stress boundary condition</u> $\mu\frac{\partial u}{\partial y} = 0$

\Couette flow between rotating cylinders/ flow flow between 2 concentric rotating cylinders by: $R=R_1$ & $R=R_2>R_1$ in cylindrical polar coords (R,ϕ,z) they rotate with const angular velocities $\underline{\Omega}_1 = \Omega_1\hat{z}$ & $\underline{\Omega}_2 = \Omega_2\hat{z}$, neglect gravity

flow of form $\underline{u} = v(R)\hat{\phi}$

automatically satisfies $\nabla\cdot\underline{u} = \frac{1}{R}\frac{\partial}{\partial R}(Ru_R) + \frac{1}{R}\frac{\partial u_\phi}{\partial\phi} + \frac{\partial u_z}{\partial z} = 0$

~~Stokes ...~~ Navier-Stokes eqn:

$$\rho\left(\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \mu\nabla^2\underline{u}$$

in ~~cylindrical~~ cylindrical polar coords

for 2 Laplacian ~~of~~ of vec in curvilinear coords use identity:

$$\nabla\times(\nabla\times\underline{u}) = \nabla(\nabla\cdot\underline{u}) - \nabla^2\underline{u}$$

flow incompressible $\therefore$ $\nabla^2\underline{u} = -\nabla\times(\nabla\times\underline{u})$ $\therefore$

$$\nabla\times\underline{u} = \frac{1}{R}\begin{vmatrix}\hat{R} & R\hat{\phi} & \hat{z}\\ \partial_R & \partial_\phi & \partial_z\\ 0 & Rv(R) & 0\end{vmatrix} = \frac{1}{R}\frac{\partial}{\partial R}(Rv)\hat{z} \therefore$$

$$\nabla\times(\nabla\times\underline{u}) = \frac{1}{R}\begin{vmatrix}\hat{R} & R\hat{\phi} & \hat{z}\\ \partial_R & \partial_\phi & \partial_z\\ 0 & 0 & \frac{1}{R}\frac{\partial}{\partial R}(Rv)\end{vmatrix} = -\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial}{\partial R}(Rv)\right)\hat{\phi} \therefore$$

$\nabla^2 \underline{u} = \frac{\partial}{\partial R}\left(v' + \frac{v}{R}\right)\hat{\phi} = \left(v'' + \frac{v'}{R} - \frac{v}{R^2}\right)\hat{\phi}$

R component of the Navier-Stokes eqn is then $\rho\left(0 - \frac{V^2}{R}\right) = -\frac{\partial p}{\partial R}$

while ϕ component: $\rho(0+0) = -\frac{1}{R}\frac{\partial p}{\partial \phi} + \mu\left(v'' + \frac{v'}{R} - \frac{v}{R^2}\right)$

if assume p is indep of ϕ:

Solve: $v'' + \frac{v'}{R} - \frac{v}{R^2} = 0$ take $v = R^m$ $\therefore v' = mR^{m-1}$ ect $\therefore$

$m(m-1) + m - 1 = 0 \Rightarrow m = \pm 1 \therefore$

General sol is: $V = AR + \frac{B}{R}$

\special cases/ 1) if $\Omega_1 = \Omega_2 = \Omega$ find: $B = 0$ & $A = \Omega$ $\therefore V = R\Omega$

corresponds to solid body rotation

2) if $A = 0$ (ie $\Omega_2 = R_1^2\Omega_1/R_2^2$) & set $B = \frac{\Gamma}{2\pi}$ then $V = \frac{\Gamma}{2\pi R}$

is the flow of a line vortex at the origin, with circulation

$\oint_C \underline{u}\cdot d\underline{r} = \int_{\phi=0}^{2\pi} \underline{u}\cdot(R\,d\phi\,\hat{\phi}) = \int_{\phi=0}^{2\pi} \frac{\Gamma}{2\pi R} R\,d\phi = \Gamma$

\pressure/ justify: p is indep of ϕ

note: from ϕ-compon of Navier eqn gives $\partial p/\partial\phi$ indep of ϕ $\therefore$

$\frac{\partial p}{\partial \phi} = P_0(R) \Rightarrow p = P_0(R)\phi + P_1(R)$

but p must be single valued as $\phi \to \phi + 2\pi$ & $\therefore P_0(R) = 0$ ie

$p = P_1(R)$ indep of ϕ

the radial compon of Navier eqn can integrate to give

$p = \rho\int \frac{V^2}{R}\,dR$ so if $A = \Omega$ (case 1) have $p = \rho\Omega^2R^2/2 + C$

(C is const) & $\therefore$ the pressure increases outwards

\week 2 help session/

1) $u = (y, z^3 - 1, x^2 - 2)$ stagnation pts $\underline{u} = 0$ $y = 0, z^3 = 1$ $\therefore z = 1$

$x^2 = 2$ $\therefore x = \pm\sqrt{2}$ $(+\sqrt{2}, 0, 1)$ & $(-\sqrt{2}, 0, 1)$

\2/ material derivative $\frac{\partial}{\partial t} + \underline{u}\cdot\nabla$ $(\underline{u}\cdot\nabla \neq \nabla\cdot\underline{u})$

\3/ i^{th} compon of $\underline{u}\times\underline{v}$: $(\underline{u}\times\underline{v})_i = \varepsilon_{ijk}u_j v_k$ ε_{kji} ε_{kij}

$\therefore \varepsilon_{kij}$

\4/ mass continuity for incompressible:

$\nabla \cdot \vec{u} = 0$

$\partial_t \rho + \nabla \cdot (\rho \underline{u}) = 0$ mass conty

$\partial_t \rho + \rho \nabla \cdot \underline{u} + \underline{u} \cdot \nabla \rho = 0$ (formula sheet)

$\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u} = 0$

incompressible fluid $\frac{D\rho}{Dt} = 0 \therefore \nabla \cdot \underline{u} = 0$

\5/ RHS of navier stokes $\quad -\nabla p + \rho \vec{g} + \mu \nabla^2 \vec{u}$

$\rho \frac{D\vec{u}}{Dt} = -\nabla p + \rho g + \mu \nabla^2 \underline{u}$

\6/ pathlines $\vec{r}(t)$:

$\frac{dr}{dt} = \underline{u} \quad \therefore \frac{dr}{dt} = \underline{u}$

\7/ $\varepsilon_{ijk} \varepsilon_{ilm} = \delta_{jl} \delta_{km} - \delta_{jm} \delta_{kl}$

so ~~midmid~~ endend, midend endmid

\8/ $u_i v_i + \frac{\partial a_j}{\partial x_j} + 3$ reads

$\therefore u_i v_i + \frac{\partial a_j}{\partial x_j} + 3 = \sum_{i=1}^{3} u_i v_i + \sum_{j=1}^{3} \frac{\partial a_j}{\partial x_j} + 3 =$

$(u_1 v_1 + u_2 v_2 + u_3 v_3) + \left(\frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3}\right) + 3 = \underline{u} \cdot \underline{v} + \nabla \cdot \underline{a} + 3$

\9/ j^{th} compon $(\vec{u} \cdot \nabla)\vec{u}$:

$[\underline{u} \cdot \nabla \underline{u}]_j = [(\underline{u} \cdot \nabla)\underline{u}]_j = (\underline{u} \cdot \nabla) u_j \quad \{\text{since } (\underline{u} \cdot \nabla) \text{ is scalar}\}$

$= u_i \frac{\partial}{\partial x_i} u_j \quad \{$only 2 i's or j's in an expression at most

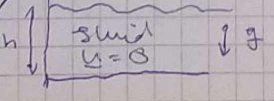
$u_i \frac{\partial u_j}{\partial x_i}$

\10/ δ_{ii}: $\quad \delta_{ii} = \sum_{i=1}^{3} \delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 1 + 1 + 1 = 3$

\exact sols/ $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u}\right) = -\nabla p + \rho g + \mu \nabla^2 \underline{u}$

some ~~pos~~ assumption of the question like ignore gravity or steady flow? that lets you throw away terms that allows an exact sol

\ex 2.14/

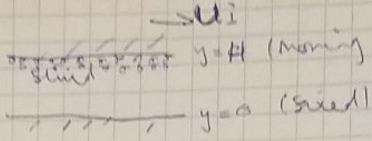


solve for $p(\underline{x}, t)$

pressure increases with depth

- weight of Z fluid above

\Ex 2.15/ \plane couette flow/



find $\underline{u} = u(y)\,\underline{i}$

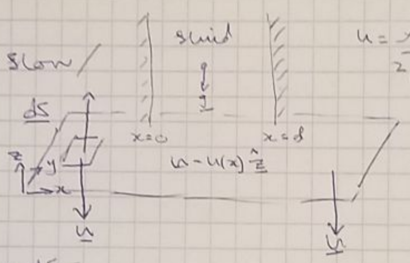
no de[illegible] in x (place on x axis doesn't matter) ∴ expect u to be indep of x

but place on y axis does matter since boundaries in y

$u = \frac{U}{H} y$

\Ex 2.16 poiseuille flow/

\activity 2.17/



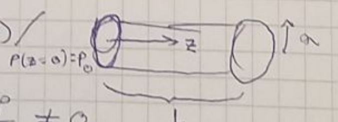
$u = \frac{\rho g}{2\mu}(x-d)x < 0$

mass flux $= \cancel{\rho \hat{z} \int} \;\rho \int \underline{u} \cdot \underline{dS} =$

$\rho \int u \hat{\underline{z}} \cdot dx\,dy\,\hat{\underline{z}} = \rho \int_S u \, dx\,dy$

per unit width in y: $\int_{x=0}^{d} u\,dx \to -\frac{\rho g d^3}{12\mu}$ {need to draw diagram to show is correct sign or not so must draw diagram!}

\Hagen-poiseuille flow (2.8.2)/



$p_L = p(z = L)$

$\underline{u} = w(R)\,\hat{\underline{z}}$ find $w(R)$ $\qquad \frac{dp}{dz} \neq 0$

(2.21) $0 = -\frac{dp}{dz} + \mu(\nabla^2 \underline{u})_z \quad \therefore \frac{dp}{dz} = \underbrace{\mu(\nabla^2 \underline{u})_z}_{\text{func of } R \text{ only}}$

∴ left handside cannot be a func of z

$\therefore \frac{\partial p}{\partial z} = f(R) \quad \therefore \quad p = f(R)z + c_1$

∴ apply boundary conds ∴ $p(z=0) = p_0 = c_1 \quad \therefore \quad c_1 = p_0 \quad \therefore$

$p(z=L) = p_L = f(R) \cdot L + p_0 \quad \therefore \quad f(R) = \frac{p_L - p_0}{L} \qquad \therefore \quad p = \frac{(p_L - p_0)}{L} z + p_0$

common mistake: $\underline{u} = w(R)\,\hat{\underline{z}} \quad \therefore \nabla^2 \underline{u} = \nabla^2 (w(R)\,\hat{\underline{z}}) = \nabla^2 w(R)\,\hat{\underline{z}} \to \frac{d^2 w}{dR^2}\hat{\underline{z}}$ ✗ use: $\nabla^2 \underline{u} \to \quad \nabla \times (\nabla \times \underline{u}) = \nabla(\nabla \cdot \underline{u}) - \nabla^2 \underline{u}$

use: $(\nabla^2 \underline{u})_z = \frac{1}{R}\frac{d}{dR}\left(R\frac{dw}{dR}\right) = \frac{1}{R}\left\{R\frac{d^2w}{dR} + \frac{dw}{dR} \cdot 1\right\} = \frac{d^2w}{dR^2} + \frac{1}{R}\frac{dw}{dR}$

\Ex 2.20/ given angular velocity $\Omega_1 = \Omega_1 \hat{\underline{z}}$

linear velocity $\underline{u} = V(R)\,\hat{\underline{\phi}}$

$\rightarrow V = AR + \frac{B}{R}$

$$\underline{u} = \underline{\Omega} \times \underline{r} = \begin{vmatrix} \hat{\underline{R}} & \hat{\underline{\phi}} & \hat{\underline{z}} \\ 0 & 0 & \Omega \\ R & 0 & z \end{vmatrix}$$

Cartesians $\underline{r} = x\underline{i} + y\underline{j} + z\underline{k}$

$\therefore$ Cylindrical Coords:

(Common mistake)

$\underline{r} = R\hat{\underline{R}} + \phi\,\hat{\underline{\phi}} + z\hat{\underline{z}}$ X

X

$z\hat{\underline{z}}$

$\rightarrow R\hat{\underline{R}} + z\hat{\underline{z}} = \underline{r}$

Spherical polars: $\underline{r} \neq r\hat{\underline{r}} + \theta\hat{\underline{\theta}} + \phi\hat{\underline{\phi}}$ X

X

\week 3/ exact sols of N-S: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho\underline{g} + \mu\nabla^2\underline{u}$

- either $\underline{u} = 0$ (fluid not moving) - solve for the pressure p given ρ)
- or, forces act to drive the flow $\underline{u} \neq 0$

static fluid $\therefore$ pressure increases with depth

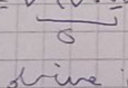
no slip boundary condition implies the fluid at the boundary moves with same speed & direction as the boundary condition

a flow in cylindrical polar coords has zero divergence & zero curl $\therefore$ the viscous term is zero (see Hagen poiseuille flow $\therefore$

$\underbrace{\nabla\times(\nabla\times\underline{u})}_{0} = \nabla\underbrace{(\nabla\cdot\underline{u})}_{0} - \underbrace{\nabla^2\underline{u}}$ $\therefore$

$\therefore$ viscous term is zero $\therefore$ fluid is static unless forces drive it $\therefore$ boundary condition that moves

- gravity

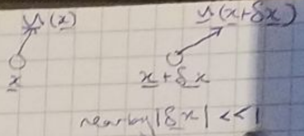


- imposed pressure gradient (2.8.3)

P_L (fluid $\rightarrow$) P_R $\frac{dp}{dz} \neq 0$

in solving the PDE we need boundary conditions/initial conditions

Week 3 theory ① deformation Mat



$\underline{u}(\underline{x}+\delta\underline{x}) = \underline{u}(\underline{x}) + \delta\underline{u}$ - small correction ∴ nearby $|\delta\underline{x}| << 1$

$\delta\underline{u} = \underline{u}(\underline{x}+\delta\underline{x}) - \underline{u}(\underline{x})$

what is $\underline{u}(\underline{x}+\delta\underline{x})$? taylor series $f(x+h) \approx f(x) + h\frac{df}{dx} + \frac{h^2}{2}\frac{d^2f}{dx^2} + \cdots$

$f(x+h, y+g) = f(x,y) + \frac{\partial f}{\partial x}h + \frac{\partial f}{\partial y}g + \cdots$ ∴

$\underline{u}(\underline{x}+\delta\underline{x}) \approx \underline{u}(\underline{x}) + \frac{\partial \underline{u}}{\partial x_1}\delta x_1 + \frac{\partial \underline{u}}{\partial x_2}\delta x_2 + \frac{\partial \underline{u}}{\partial x_3}\delta x_3$ ∴

$\delta\underline{u} \approx \sum_{j=1}^{3}\delta x_j \frac{\partial \underline{u}}{\partial x_j} = \delta x_j \frac{\partial \underline{u}}{\partial x_j}$

Consider the i^{th} component of $\delta\underline{u}$ ∴ $\delta u_i = \delta x_j \frac{\partial u_i}{\partial x_j}$ ∴

$D_{ij} = \frac{\partial u_i}{\partial x_j}$ deformation Matrix

Break D_{ij} into 2 parts.

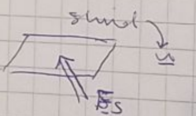
e_{ij} - represents stretching/squashing/straining

ξ_{ij} - rotation

② want to derive N-S eqn/ $\underline{F} = m\underline{a} = \frac{d}{dt}\underline{P}$ ∴ $\frac{dP}{dt} = \int_{V(t)} \rho \frac{D\underline{u}}{Dt}\, dV$ = forces acting

i) Body forces eg gravity $\underline{F}_g = \int_V \rho \underline{g}\, dV$

ii) surface forces $\underline{F}_S = \int_S \underline{\Sigma}\, dS$



∴ $\frac{dP}{dt} = \underline{F}_g + \underline{F}_s \rightarrow (2.23)$ Momentum Balance eqn

its given $\Sigma_i = \sigma_{ij}\hat{n}_j$ is the unit normal to the surf

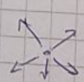
stress tensor

step 3: given for a fluid at rest $\underline{\Sigma} = -p\underline{n}$ ∴ $\Sigma_i = -pn_i$ ∴

$\sigma_{ij}n_j = -pn_i$ ∴ is $\sigma_{ij} = -p\delta_{ij}$ ∴ ~~$\sigma_{ij} = -p\delta$~~ ~~$\sigma_{ij}n_j = p\delta_{ij}n_j$~~

$\sigma_{ij}n_j = p\delta_{ij}n_j = -pn_i$

is $\underline{u}(x_2, x_1, 0)$ ∴ $\nabla\cdot\underline{u} > 0$

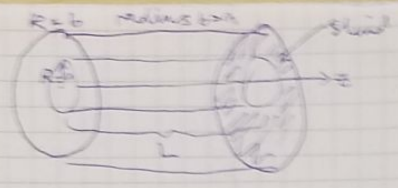


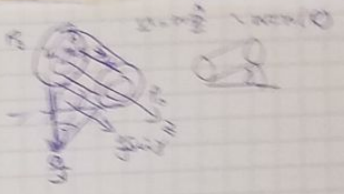
$\nabla\cdot\underline{u} < 0$

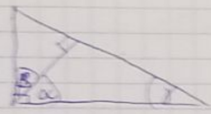
$\nabla\times\underline{u}$ ~rotation $\underline{u}$

Ex sheet 2

1a







$\therefore \alpha = \pi - \frac{\pi}{2} - \gamma = \frac{\pi}{2} - \gamma \quad \therefore \beta + \alpha = \frac{\pi}{2}$

$\beta = \frac{\pi}{2} - \alpha = \frac{\pi}{2} - \left(\pi - \frac{\pi}{2} - \gamma\right) = \gamma$

Navier-Stokes $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho \underline{g} + \mu\nabla^2\underline{u}$

$\underline{u} = w(R)\hat{\underline{z}} \quad \therefore \left(\frac{\partial \underline{u}}{\partial t}\right)_z = \frac{\partial}{\partial t} w(R) = 0$ since w is independent of time

$(\underline{u}\cdot\nabla\underline{u})_z$: $\underline{u}\cdot\nabla = w\hat{\underline{z}}\cdot\left(\hat{\underline{R}}\frac{\partial}{\partial R} + \frac{1}{R}\hat{\underline{\phi}}\frac{\partial}{\partial\phi} + \hat{\underline{z}}\frac{\partial}{\partial z}\right) = w\frac{\partial}{\partial z} \quad \therefore$

$\underline{u}\cdot\nabla\underline{u} = w\frac{\partial}{\partial z}(w\hat{\underline{z}}) = w\frac{\partial w}{\partial z}\hat{\underline{z}} = 0$ since $w = w(R)$ only

$(\nabla p)_z = \frac{\partial p}{\partial z} \qquad (\rho \underline{g})_z = \rho(\underline{g})_z = \rho g \sin\gamma$

$(\mu\nabla^2\underline{u})_z = \mu(\nabla^2\underline{u})_z \quad \therefore \quad (\nabla^2\underline{u})_z = \frac{1}{R}\frac{d}{dR}\left(R\frac{dw}{dR}\right)$ from §2.8.3

z-component N-S: $\rho(0+0) = -\frac{\partial p}{\partial z} + \rho g \sin\gamma + \frac{\mu}{R}\frac{d}{dR}\left(R\frac{dw}{dR}\right) = 0$

1b $\frac{\partial p}{\partial z} = \underbrace{\rho g \sin\gamma}_{\text{const}} + \underbrace{\frac{\mu}{R}\frac{d}{dR}\left(R\frac{dw}{dR}\right)}_{\text{func of } R} \qquad w = w(R) \quad \therefore$

$\frac{\partial p}{\partial z}$ is independent of z (~~ie the pressure gradient along the cylinder is zero~~)

$\frac{\partial p}{\partial z} = c \quad \therefore \quad p = cz + d \quad \therefore p(z=0) = p_0 = c(0) + d = d \quad \therefore \quad d = p_0$

$p(z=L) = p_0 = cL + d = cL + p_0 \quad \therefore \quad cL = 0 \quad \therefore \quad c = 0 \quad \therefore$

$\frac{\partial p}{\partial z} = 0 \quad \therefore$ the pressure gradient along the cylinder is zero

1c $\underbrace{-\frac{\partial p}{\partial z}}_{=0} + \rho g \sin\gamma + \frac{\mu}{R}\frac{d}{dR}\left(R\frac{dw}{dR}\right) = 0 \quad \therefore$

$\frac{d}{dR}\left(R\frac{dw}{dR}\right) = -\rho g \sin\gamma \frac{R}{\mu} \quad \therefore \quad R\frac{dw}{dR} = -\rho\frac{g\sin\gamma}{\mu}\frac{R^2}{2} + A$ (A const) $\therefore$

$\frac{dw}{dR} = -\frac{\rho g \sin\gamma R}{2\mu} + \frac{A}{R} \quad \therefore$

$$w = -\frac{\rho g \sin\gamma}{2\mu}\frac{R^2}{2} + A\ln R + B \qquad (B \text{ const})$$

1d/ No-slip $w(R=a)=0 \quad \therefore \quad -\frac{\rho g \sin\gamma}{2\mu}\frac{a^2}{2} + A\ln a + B = 0$

$w(R=b)=0 \quad \therefore \quad -\frac{\rho g \sin\gamma\, b^2}{2\mu\;\;2} + A\ln b + B = 0 \quad \therefore$ solve 2 sim eqns ~~for~~

A & B $\therefore \quad A = \frac{\rho g (a^2-b^2)\sin\gamma}{4\mu\ln(a/b)} \qquad B = \frac{\rho g a^2 \sin\gamma}{4\mu} - A\ln a$

use $p\ln q = \ln(q^p)$ & $\ln p + \ln q = \ln(pq)$

2a/ $\frac{\partial v}{\partial t} = \nu\left(\frac{\partial^2 v}{\partial R^2} + \frac{1}{R}\frac{\partial v}{\partial R} + -\frac{v}{R^2}\right)$ separable sols $v = f(R)g(t)$

$\frac{dv}{dt} = f(R)\frac{dg}{dt} \qquad \frac{dv}{dR} = \frac{df}{dR}g \qquad \frac{d^2v}{dR^2} = \frac{d^2f}{dR^2}g \quad \therefore$ sub $\therefore$

$$f\frac{dg}{dt} = \nu\left(g\frac{d^2f}{dR^2} + \frac{1}{R}g\frac{df}{dR} - \frac{1}{R^2}fg\right)$$

take $g = e^{-int} \quad \therefore \frac{dg}{dt} = e^{-int}(-in) = -ing \quad \therefore$ sub $\therefore$

$-fin = -\nu\left(f'' + \frac{1}{R}f' - \frac{1}{R^2}f\right) \qquad \div$ by $g \qquad \therefore \times\frac{R^2}{\nu}$:

$R^2 f'' + Rf' + (\alpha^2R^2 - 1)f = 0$ where $\alpha^2 = in/\nu$

now change of vars $S = \alpha R \quad \therefore dS = \alpha\, dR \qquad \therefore \frac{d}{dR} = \alpha\frac{d}{dS} \quad \therefore$

$\frac{d^2}{dR^2} = \alpha^2\frac{d^2}{dS^2} \quad \therefore \quad R^2 f'' \Rightarrow R^2\alpha^2\frac{d^2f}{dS^2} \quad \therefore$

$R^2\alpha^2\frac{d^2f}{dS^2} + R\alpha\frac{df}{dS} + (S^2-1)f = 0 \quad \therefore \quad \cancel{S^2 d^2 f}$

$$S^2\frac{d^2f}{dS^2} + S\frac{df}{dS} + (S^2-1)f = 0$$

$\therefore$ sols: $\left.\begin{matrix} f = J_1 \\ f = Y_1 \end{matrix}\right\}$ general sol $\approx$ $\cancel{f = \&}$ $\boxed{f = AJ_1 + BY_1}$

week 3/ at fixed time t, z velocity at two nearby pts, x & $\underline{x} + \delta\underline{x}$ $\therefore$

$\underline{u}(\underline{x}+\delta\underline{x}, t) = \underline{u}(\underline{x}, t) + \delta\underline{u}$ where $\delta\underline{u} = \underline{u}(\underline{x}+\delta\underline{x}, t) - \underline{u}(\underline{x}, t) =$

$\underline{u}(\underline{x}, t) + \delta x_1\frac{\partial \underline{u}}{\partial x_1} + \delta x_2\frac{\partial \underline{u}}{\partial x_2} + \delta x_3\frac{\partial \underline{u}}{\partial x_3} + \dots - \underline{u}(\underline{x}, t) \approx \delta x_j\frac{\partial \underline{u}}{\partial x_j}$ $\therefore$ for small $\delta\underline{x}$ z two pts have approx z same velocity. z relative motion between z two pts has components $\delta u_i = \delta x_j\frac{\partial u_i}{\partial x_j}$ $\therefore$ in matrix:

$$\begin{bmatrix}\delta u_1\\ \delta u_2\\ \delta u_3\end{bmatrix} = \begin{bmatrix}\partial u_1/\partial x_1 & \partial u_1/\partial x_2 & \partial u_1/\partial x_3\\ \partial u_2/\partial x_1 & \partial u_2/\partial x_2 & \partial u_2/\partial x_3\\ \partial u_3/\partial x_1 & \partial u_3/\partial x_2 & \partial u_3/\partial x_3\end{bmatrix}\begin{bmatrix}\delta x_1\\ \delta x_2\\ \delta x_3\end{bmatrix}$$ Z <u>deformation matrix</u> D, components

$D_{ij} = \partial u_i/\partial x_j$ (known as Z velocity gradient tensor) may be decomposed into its symmetric & anti-symmetric parts $D_{ij} = e_{ij} + \xi_{ij}$ where

$e_{ij} = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right]$ $\quad \xi = \frac{1}{2}\left[\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right]$ it can be shown that e_{ij} represents a pure straining motion (ie a stretching & squashing motion in mutually perpen directions but with no overall rotation), while ξ_{ij} represents a local rigid body rotation.

Z <u>trace</u> of Z matr e_{ij} is $e_{ii} = \partial u_i/\partial x_i = \nabla \cdot \underline{u}$

& is Z <u>divergence</u> of Z flow, it is zero for Z incompressible flows considered in this model.

Consider a <u>material volume</u> $V(t)$ that contains fluid particles & moves with Z flow. Z total momentum of fluid in $V(t)$ is $\underline{P} = \int_{V(t)} \rho \underline{u}\, dV$ since both integrand & Z domain of integration are funcs of time, have: $\frac{d\underline{P}}{dt} = \int_V \rho \frac{D\underline{u}}{Dt} dV$ where on Z RHS have Z material deriv. Z time rate of change of momentum equals Z sum of Z forces acting.

Z total force of gravity g on V is $\underline{F}_g = \int_V \rho \underline{g}\, dV$

gravity is an example of a body force. these are forces that are exerted from outside Z fluid & are felt at each location in Z fluid

there are also surface forces. let $\Sigma(\underline{x}, t, \hat{\underline{n}})\, dS$ be Z force on a surf elem dS (has outward unit normal $\hat{\underline{n}}$ & is located at $\underline{x}$ at time t) from fluid outside V. Z total surface force is $\underline{F}_S = \int_S \Sigma(\underline{x}, t, \hat{\underline{n}})\, dS$ & it arises from pressure & viscous forces

equating $d\underline{P}/dt = \underline{F}_g + F_S$ gives $\int_V \rho \frac{D\underline{u}}{Dt} dV = \int_V \rho \underline{g}\, dV + \int_S \Sigma(\underline{x}, t, \hat{\underline{n}})\, dS$ is Z momentum balance eqn.

derivation of NS eqn using momentum balance eqn

<u>step</u> 1: $\Sigma(\underline{x}, t, \hat{\underline{n}})\, dS$ is Z force on a surf elem dS, unit normal $\hat{\underline{n}}$ at $(\underline{x}, t)$ by Z fluid towards which $\hat{\underline{n}}$ is directed. By considering Z force balance on a small tetrahedral fluid elem shown: $\Sigma_i(\underline{x}, t, \hat{\underline{n}}) = \sigma_{ij}(\underline{x}, t)\hat{n}_j$ where Z matr σ_{ij} (at each x & t) is called Z <u>stress tensor</u>

<u>shown</u>: σ_{ij} is <u>symmetric</u> ie $\sigma_{ij} = \sigma_{ji}$

__step 2:__ sub expression sor Z stess tensor into Z momentum balance eqn & use Z generalised divergence thm to: $\rho \frac{Du_i}{Dt} = \rho\left(\frac{\partial u_i}{\partial t} + \underline{u}\cdot\nabla u_i\right) = \rho g_i + \frac{\partial \sigma_{ij}}{\partial x_j}$

__step 3:__ Z stress tensor includes pressure & viscous sorces, is Z sluid is at __rest__ then $\Sigma = -p\,\square$ $\therefore \sigma_{ij}\hat{n}_j = -p\hat{n}_i$ $\therefore \sigma_{ij} = -p\delta_{ij}$ sor more general motion:

$p = -\frac{1}{3}\sigma_{ii}$ & set $\sigma_{ij} = -p\delta_{ij} + d_{ij}$

__step 4:__ Z essects os viscosity are contained in d_{ij} & it is natural to link to e_{ij} which captures straining motion. assuming Z sluid is isotropic (ie has no preserred direction) means Z only relationship (sor slows with $\nabla\cdot\underline{u} = 0$) is $d_{ij} = 2\mu e_{ij}$

__step 5:__ use expression sor d_{ij} in Z momentum balance eqn to derive NS eqn

$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho\underline{g} + \mu\nabla^2\underline{u}$ with $\nabla\cdot\underline{u} = 0$

sor const gravity $\underline{g}$ can write $\underline{g} = \nabla(\underline{g}\cdot\underline{x})$

& absorb it into a modisied pressure P st $-\nabla p + \rho\underline{g} = -\nabla P$ with

$P = p - \rho\underline{g}\cdot\underline{x}$

units os measuring physical quantities are os two types:

- Fundamental units: standard reserences ie centimetres
- Derived units: obtained using dess os sundamental units eg speed is ratio os distance & time

can use cgs system os units (cm / g)

SI (mks) system (m, kg)

is Z unit os length is decreased by a sactor os L & Z unit os time is decreased by a sactor os T then Z new unit os velocity is a sactor LT^{-1} times smaller than Z original unit. so Z numeric vals os all velocities are increased by a sac os LT^{-1}. Z change in numerical vals os a physical quantity when one system os units is change into another is determined by its __dimension__. Z dimension os Z quantity ϕ is denoted by $[\phi]$ so sor density $[\rho] = [ML^{-3}]$ & sor speed $[u] = LT^{-1}$

Z dimension os a quantity is dissesent in dissesent systems os units (eg in LFT class (where F stands for force) $[\nu] = L^{-4} F T^{2}$). Quantities whose numerical vals are identical in different systems of units are called dimensionless (have $\neq$ dimension 1)

dimensions of both sides of an eqn describing physical behaviours must be identical (otherwise equality would hold in one system of units but not another) $\therefore$

for $\underline{F} = m\underline{a}$ $\therefore [F] = [m][a] = LMT^{-2}$

for incompressible fluid dynamics, all physical quantities that we will be interested in can be expressed in terms of a power-law monomial that is: $[\phi] = L^{\alpha} M^{\beta} T^{\gamma}$

where α, β, γ are consts eg z dimensions of pressure p (force per unit area) is $[p] = L^{-1} M T^{-2}$

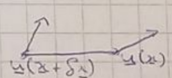
\Def 3.1/ Quantities $a_1, a_2, \ldots a_k$ have <u>indep</u> <u>dimensions</u> if none of z quantities have a dimension that can be represented as z product of powers of z dimensions of z others

note: none of $a_1, a_2, \ldots a_k$ can be dimensionless (since 1 can be written as z product of z zeroth power of all of z other dimensions).

Motion of an incompressible fluid with velocity $\underline{u}$ between two horizontal boundaries. z boundaries are a distance L apart; & z upper one moves with speed U to z right. study flow change for density z viscosity $\therefore$ write z governing eqns in terms of dimensionless variables $\therefore$ z effect of changing z param ρ & μ is seperated from z effect of changes to z units. define z dimensionless variables

$\underline{x}' = \frac{\underline{x}}{L}$ $\quad \underline{u}' = \frac{\underline{u}}{U}$ $\quad t' = \frac{t}{L/U}$ $\quad p' = \frac{p}{\rho U^2}$

z deformation matrix measures z relative motion at nearby pts



$\delta u_i = \frac{\partial u_i}{\partial x_j} \delta x_j$ $\quad e_{ij}$ symmetric part (ξ_{ij} antisymmetric part)

NS eqn derived by equating z total forces acting on z fluid to z rate of change of momentum

forces acting = rate of change of momentum

force comes in two types - body forces (externally imposed) gravity...

- surf forces (viscous force)

Dimensions of a quantity - units L length, T time, M mass

$[F]=[m][a]=M[LT^{-2}]=MLT^{-2}=ML/T^2$

can non-dimensionalise NS eqn $\rightarrow \mu = 1/R_e$ $R_e = UL/\nu$ $\nu = \mu/\rho$

i)
mathematically use what we know about dimensional analysis to simplify a PDE (NS) into an ODE

ii) physically - certain physical behaviours are "similar" to other physical behaviours

Law of singularity - Reynolds

2 flows with the same reynolds number are similar

use: Buckingham-Π thm:

(3.1): $y = f(\underbrace{a_1, a_2, \dots, a_k}_{k}, \underbrace{b_1, b_2, \dots, b_m}_{m})$ total params $= n = k+m$

simplified $\Pi = F(\underbrace{\Pi_1, \Pi_2, \dots, \Pi_m}_{m \text{ params}})$

1)
y is the quantity of interest (eg flow speed $u = |\underline{u}|$)

2)
$a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_m$ are the governing params $\leftarrow$ quantities that the sol depends on eg x, y, z, t, μ

3)
$i = 1, 2, \dots, k$ a_i have indep dimension L, T, M

b_j $(j = 1, 2, \dots, m)$ have a dependent dimension $\therefore$

the b_j can be written as products of the a_i's

Ex/ if y is flow speed $[y] = LT^{-1}$ $a_1 := x$ $[a_1] = L$ $a_2 = \Omega t$ $[a_2] = T$ ...

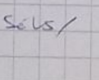
Def/ $\Pi = \frac{y}{a_1^{\alpha} a_2^{\beta} a_3^{\gamma} \cdots a_k^{\chi}}$ st $[y] = [a_1^{\alpha} a_2^{\beta} a_3^{\gamma} \cdots a_k^{\chi}]$ $\alpha, \beta, \gamma, \dots \chi$ are consts

$u = y = LT^{-1}$ $[a_1] = L$ $[a_2] = T$ $[y] = [a_1^1][a_2^{-1}]$ $\therefore \alpha = 1$ $\therefore \beta = -1$ $\therefore$

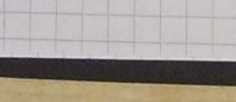
$\Pi = \frac{u}{a_1^1 a_2^{-1}}$

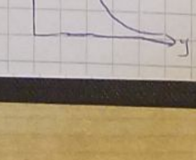
$\Pi_1 = \frac{b_1}{a_1^{p} a_2^{q} a_3^{r} \cdots a_k^{s}}$ where $p, q, r, \dots, s$ are consts st $[b_1] = [a_1^{p} a_2^{q} a_3^{r} \cdots a_k^{s}]$

$\Pi_2 =$ $\Pi_3 =$

prob 3.6/ 

period is 1/time

3.4 Similarity Sols/ 



week 4 / in study: aim is often to relationships between quantities that characterise z behaviour under study ie determine y in z functional form

$y = f(a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_m)$

y is physical quantity of interest (eg flow speed) & its $n = k + m$ arguments $a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_m$ are assumed to be given (eg viscosity)

z a's & b's are called z <u>governing params</u> z $a_1, a_2, \dots, a_k$ have <u>indep dimensions</u> z $b_1, b_2, \dots, b_m$ have dimensions that can be written in terms of z dimensions of z a's $\therefore [b_i] = [a_1]^{p_i} \dots [a_k]^{r_i}$ $i = 1, 2, \dots m$

if z dimensions of all of z governing params are indep have $m = 0$

if all of z governing params are dimensionless have $k = 0$

know z dimension of z quantity of interest can be: $[y] = [a_1]^p \dots [a_k]^r$

& ∴ can introduce z dimensionless params

$$\Pi = \frac{y}{a_1^p \dots a_k^r}$$

$$\Pi_1 = \frac{b_1}{a_1^{p_1} \dots a_k^{r_1}}, \quad \dots, \quad \Pi_i = \frac{b_i}{a_1^{p_i} \dots a_k^{r_i}}, \quad \dots, \quad \Pi_m = \frac{b_m}{a_1^{p_m} \dots a_k^{r_m}}$$

with z exponents $(p, r, p_1, r_1, \dots, p_m, r_m)$ being chosen st $\Pi, \Pi_1, \dots, \Pi_m$ are dimensionless

thus in dimensional analysis:

thm 3.5 - Buckingham-Π / a physical relationship between a dimensional quantity (y) & several dimensional governing params $(a_1, a_2, \dots, a_k, b_1, b_2, \dots, b_m)$ can be rewritten as a relationship betw a dimensionless param (Π) & several dimensionless products of z governing params $(\Pi_1, \Pi_2, \dots \Pi_m)$. z number of dimensionless products (m) is z total number of governing params (n) minus z num of governing params with indep dimensions (k)

initially a fluid is at rest above an infinite stationary horizontal plate given by $y = 0$ for $t > 0$ z plate is set in motion with velocity $W\mathbf{i}$

neglect gravity & assume z pressure p tends to a const val p_0 as $y \to \infty$

require to find z velocity of z fluid flow at later times in fact, z dependence of u on W & ν can be made more simple in this case. Suppose we implement z change of variables $u' = u/W$ (note both u & W have dimensions of velocity ∴ this ratio makes physical sense) & ∴ now u' is dimensionless. Subing into eqns (3.3) & z associated boundary & initial conditions yields z new probl $\frac{\partial u'}{\partial t} = \nu \frac{\partial^2 u'}{\partial y^2}$ $u'(y=0, t=0)=0$, $u'(y=0, t>0)=1$, $u'(y \to \infty, t \geq 0) \to \infty$ in which dependence on W has dropped out

∴ from exercise can deduce z original velocity takes form $u = WF(\eta)$, & $\eta = \frac{y}{\sqrt{4\nu t}}$ η is z similarity variable. note: introduced z extra factor of 4 for algebraic convenience but on-consts are dimensionless & ∴ don't change z dimension of a quantity when introduced

z error func defined as: $\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(-s^2)\, ds$

& z complementary error func: $\text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty \exp(-s^2)\, ds$

note: $\text{erf}(x=0) = 0$ & $\text{erfc}(x \to \infty) = 0$ also shown:

$\text{erf}(x) + \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_0^\infty \exp(-s^2)\, ds = 1$ ∴ $\text{erf}(x \to \infty) = 1$ & $\text{erfc}(0) = 1$

\ Activity 2.21 / show $\underline{\omega} = 2(\xi_{32}, \xi_{13}, \xi_{21})$ $\underline{x} = (x, y, z) = (x_1, x_2, x_3)$

$$\underline{\omega} = \nabla \times \underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial x_1 & \partial y x_2 & \partial x_3 \\ u_1 & u_2 & u_3 \end{vmatrix} = \underline{i}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right) + \underline{j}(\quad) + \underline{k}(\quad)$$

also $\xi_{ij} = \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ $\xi_{32} = \frac{1}{2}\left(\frac{\partial u_3}{\partial x_2} - \frac{\partial u_2}{\partial x_3}\right)$ ∴ $\omega_1 = 2\xi_{32}$

prove $\xi_{ij} = -\frac{1}{2}\varepsilon_{ijk}\omega_k$ take rhs: $\varepsilon_{ijk}\omega_k = \varepsilon_{ijk}(\nabla \times \underline{u})_k = \varepsilon_{ijk}\varepsilon_{klm}\frac{\partial}{\partial x_l}u_m$

$= \varepsilon_{kij}\varepsilon_{klm}\frac{\partial u_m}{\partial x_l} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\frac{\partial u_m}{\partial x_l} =$

$\delta_{il}\delta_{jm}\frac{\partial u_m}{\partial x_l} - \delta_{im}\delta_{jl}\frac{\partial u_m}{\partial x_l} = \frac{\partial u_j}{\partial x_i} - \frac{\partial u_i}{\partial x_j} = -2\xi_{ij}$ ∴

$-\frac{1}{2}\varepsilon_{ijk}\omega_k = \xi_{ij}$

\Activity 2.24/ $\sigma_{ij} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$ for trace sum 2 diagonal

$\sigma_{ij} = -p\delta_{ij} + d_{ij}$

trace of σ is $\sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{ii} \left(= \sum_{i=1}^{3} \sigma_{ii} \right)$

$\therefore \sigma_{ii} = -p\delta_{ii} + d_{ii}$ (setting $i=j$ in 2 def of σ) $\{\delta_{ii} = \delta_{11} + \delta_{22} + \delta_{33} = 3\}$

$= -p \cdot 3 + d_{ii} = -(-\frac{1}{3}\sigma_{ii})3 + d_{ii} = \sigma_{ii} + d_{ii} = \sigma_{ii} \therefore d_{ii} = 0$

\Activity 3.2/ proof of ρ, $\underline{u}$, $\underline{F}$ have indep dimensn

$\therefore$ assume they're dependent & aim for a contradictn

$[\underline{F}] = [\rho]^q [\underline{u}]^r \quad \therefore [m][a] = [\rho]^q [\underline{u}]^r \therefore$

$[m][a] = M\frac{L}{T^2} = \left(\frac{M}{L^3}\right)^q \left(\frac{L}{T}\right)^r = M^q L^{r-3q} T^{-r} \therefore$

$M: 1 = q$

$L: 1 = r - 3q = r - 3 \quad \therefore r = 4$ contradiction

$T: -2 = -r \quad \therefore r = 2$ $\times$ $\perp$ $\therefore$ they are indep

\Activity 3.3/ $[\underline{x}'] = \frac{[\underline{x}]}{L} = \frac{L}{L} = 1 \quad \therefore \underline{x}'$ is dimensnless

from NS eqn: $\cdots + [\rho \underline{u} \cdot \nabla \underline{u}] = [-\nabla p] + \cdots \quad \{\because \cdots + \rho\underline{u}\cdot\nabla\underline{u} = -\nabla p + \cdots\}$

$\therefore [\rho] \frac{U^2}{L} = \frac{1}{L}[p] \therefore \quad [p] = [\rho]U^2$ (*) $\therefore$

$[p'] = \left[\frac{p}{\rho U^2}\right] = \frac{[\rho]U^2}{[\rho]U^2}$ from (*) $= 1 \therefore p'$ is dimensnless

$\left\{ \nabla = \left(\frac{d}{dx}, \frac{d}{dy}, \frac{d}{dz}\right) \sim \frac{1}{L} \quad [\underline{x}] = L \therefore [-\nabla p] = \frac{1}{L}[p] \right\}$

<u>Buckingham - Π thm</u>: $y = f(\underbrace{a_1, a_2, \ldots, a_k}_{\text{indep dimensn}}, \underbrace{b_1, b_2, \ldots, b_m}_{\text{dependent dimensn}})$ (3.1)

$\therefore \Pi = F(\Pi_1, \Pi_2, \ldots, \Pi_m) \therefore$ fewer variables

\Activity 3.10/ from (3.4): $\partial_t u' = \nu \partial_{yy} u'$ $y=0$ $\rightarrow W_i$

$u' = \frac{u}{W} \quad \therefore$ • Quantity of interest $y = u' \quad \therefore [u'] = \left[\frac{u}{W}\right] = 1$ (ie u' is dimensnless)

$\therefore$ • Expect $u' = u'(\underbrace{y, t, \nu}_{3 \text{ params} = k+m=n})$ {can guess from eqn} $\therefore$

$[y]=L \quad [t]=T \quad \left\{ \partial_t u \sim \nu \partial_{xx} u \quad \therefore \frac{1}{T} \sim \nu / L^2 \quad \therefore [\nu] = L^2 T^{-1} \right\} \quad [\nu] = L^2 T^{-1}$

$\therefore a_1 = y \quad a_2 = t \quad \therefore \qquad b_1 = \nu$

Buckingham-Π $\therefore \Pi = F(\Pi_1, \Pi_2, \ldots, \Pi_m)$ $\therefore k=2, m=1 \therefore n=3$

$\therefore \Pi = F(\Pi_1)$ ie 1 dimensionless variable

$\Pi = y = u' \quad \Pi_1 = \frac{b_1}{a_1^\alpha a_2^\beta \cdots a_k^\gamma} \quad \therefore \quad [b_1] = [\nu] = L^2 T^{-1} \quad \therefore$ require:

$\cancel{y^\alpha t^\beta}$ $[y^\alpha t^\beta] = L^2 T^{-1} = L^\alpha T^\beta \quad \therefore \quad \alpha = 2, \beta = -1 \quad \therefore$

$\Pi_1 = \frac{\nu}{y^2 t^{-1}} = \frac{\nu t}{y^2}$ is 2 required dimensionless variable $\therefore$

$u' = F\left(\frac{\nu t}{y^2}\right)$ ← this should be η

Consider $F(x) = x^4 + x^2 + 2 \qquad G(x^2) = (x^2)^2 + (x^2) + 2$ &

$H(x^{-1}) = (x^{-1})^{-4} + (x^{-1})^{-2} + 2$

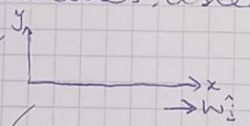
$\therefore$ have $u' = F\left(\frac{\nu t}{y^2}\right) = G\left(\frac{y^2}{\nu t}\right) = H\left(\frac{y}{\sqrt{\nu t}}\right) = J\left(\frac{y}{\sqrt{4 \nu t}}\right) \quad \therefore$

$u' = J(\eta)$ where $\eta = \frac{y}{\sqrt{4 \nu t}}$

\last week/ some problems can be simplified so that they no longer depend on many variables, instead just a few ratios

\problem 3.8/

\Rayleigh problem Stokes first problem/



$\rightarrow$ eqn (3.3) $u = u(y, t, \nu, w) \ldots$

$u = wF(\eta)$

$\eta = \frac{y}{\sqrt{4 \nu t}}$

eqn for $F(\eta)$ $F'' + 2\eta F' = 0$

revise need to do solving ODE

(i) $ay'' + by' + cy = 0$ seek sol^n $y = e^{px}$ → characteristic polynomial for p $y = Ae^{r_1 x} + Be^{r_2 x}$

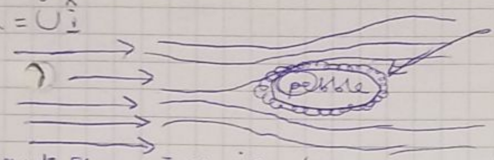
(ii) seperation of variables

\this week/ use dimensional analy & similarly to study an example "Boundary layers"

- what are boundary layers? (layers (typically thin) near to a boundary)
- why do they exist? Because 2 moving surrounding fluid has to adjust to setting 2 boundary cond

imagine a pebble on otherwise clear riverbed

$\underline{u} = U\hat{\underline{i}}$



near to z object z flow adjusts to z BC

~~boundary~~ boundary layer

strong gradients $\underline{\omega} \neq \underline{0}$

strong viscous effects $\mu\nabla^2\underline{u} \neq 0$

outerflow is unif (constant speed guide direction)

$\omega = \nabla \times \underline{0}$ (irrotational) viscous effects can be neglect $\mu\nabla^2\underline{u}$

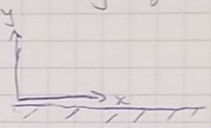
kelvin vorticity thm: $\int \underline{\omega} \cdot d\underline{s}$;) conserved (see vorticity dynamics) late

thickness is known as z <u>boundary</u> <u>layer</u> <u>width</u>

- there ~~isn't~~ a formula for the boundary layer width

\ several examples /

- §4.1 Stokes 2nd probl



$\longleftrightarrow W\cos(\Omega t)\hat{\underline{i}} = \mathbb{R}[We^{i\Omega t}]$

→ fluid inherits z oscillatory time dependence

<u>Revise</u> $e^{it} = \cos t + i\sin t \therefore u = \mathbb{R}[We^{i\Omega t}]$ $\frac{\partial u}{\partial t} = \mathbb{R}[Wi\Omega e^{i\Omega t}] =$

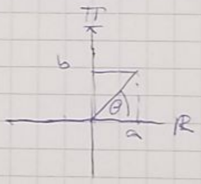
$\mathbb{R}[iW\Omega(\cos\Omega t + i\sin\Omega t)] = \mathbb{R}[W\Omega(i\cos\Omega t - \sin\Omega t)] = -W\Omega\sin\Omega t$

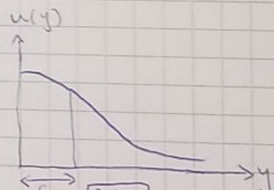
$e^{it} = \cos t + i\sin t$

$(a+ib)(c+id) = \dots$

$z = a+ib = re^{i\theta}$ $z^{1/2} = r^{1/2}e^{i\theta/2}$

$(i = \sqrt{-1})$



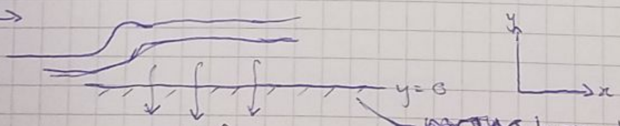


$\delta = \sqrt{\frac{2\nu}{\Omega}}$

"boundary layer width"

§4.4 use z vorticity eqn rather than Navier-Stokes

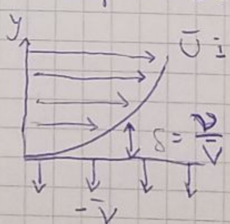
$\bar{U}\hat{\underline{i}}$



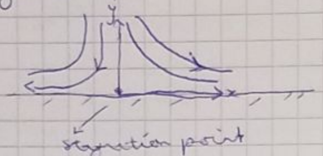
$-\bar{V}\hat{\underline{j}}$

~~porous~~ boundary ~~condition~~ (householes)

find $\underline{u} = (u(y), -V, 0)$



§ 4.4 stagnation point flow (Hiemann flow)



2D flow $\underline{u} = (u, v, 0) = \nabla \times (\psi \underline{k})$ $\qquad \underline{\nabla} \cdot \underline{u} = 0$

$$= \begin{vmatrix} \partial_x & \partial_y & \partial_z \\ 0 & 0 & \psi \end{vmatrix}$$

$$= \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0\right) \qquad \psi = \psi(x, y)$$

$\therefore \rightarrow$ 3rd order ODE for $F(\eta)$

\Week 5/

Boundary layers are thin layers adjacent to Z surf of a body in which strong viscous forces exist. this is despite Re^{-1} being small & is because large velocity gradients develop. Z large velocity gradients are a result of Z no-slip boundary conditn which reduces Z flow velocity to zero (on a stationary boundary) - vorticity is generated in these layers

Consider Z same geometry as in stokes's first probl, but now assume that Z boundary executes sinusoidal oscillations parallel to itself with frequency Ω ie Z velocity of Z boundary is $W\cos(\Omega t)\underline{i} = \mathbb{R}[W\exp(i\Omega t)]\underline{i}$ where $\mathbb{R}$ stands for real part. we are interested in Z periodic fluid flow in Z region $y \geq 0$ that occurs after all transients have died down.

as for Z first probl we expect Z resulting fluid flow to have velocity $\underline{u} = (u(y,t), 0, 0)$ with $u(y=0,t) = W\cos(\Omega t)$ & $u \to 0$ as $y \to \infty$, Z first conditn representing Z no-slip conditn & Z second quiescent fluid at infinity). Again Z governing eqn is $\frac{\partial u}{\partial t} = \nu \frac{\partial^2 u}{\partial y^2}$ in Z steady state we expect that Z flow variables will inherit Z same temporal periodicity as Z boundary motion $\therefore$ we will consider a separable sol of Z form

$u(y,t) = \mathbb{R}[\hat{u}(y)\exp(i\Omega t)]$

have seen how Z Navier Stokes eqn $\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\nabla P + \underline{g} + \nu \nabla^2 \underline{u}$ where $P = p/\rho$ & $\nu = \mu/\rho$ reduces to a diffusion eqn in some cases for ex, for time-dependent flow in one direction we have met (see §3.4.1) $\frac{\partial u}{\partial t} + \nu \frac{\partial^2 u}{\partial y^2}$ this may be thought of as diffusion of momentum $\rho \underline{u}$ through Z effect of viscosity.

for more complicated flows this idea is of limited use cause Z pressure term also transfers momentum. More useful is Z eqn for Z vorticity $\underline{\omega} = \nabla \times \underline{u}$ which is obtained by taking Z curl of Z navier-Stokes eqn & reads $\frac{\partial \underline{\omega}}{\partial t} + \underline{u} \cdot \nabla \underline{\omega} =$ $\underline{\omega} \cdot \nabla \underline{u} + \nu \nabla^2 \underline{\omega}$, $\nabla \cdot \underline{u} = 0$ or $\frac{\partial \underline{\omega}}{\partial t} = \nabla \times (\underline{u} \times \underline{\omega}) + \nu \nabla^2 \underline{\omega}$

in additn to Z terms that constitute Z diffusn eqn, Z other terms represent transport of vorticity by Z fluid flow

Consider a fluid flow whose velocity vector coincides with Z y-axis & impinges on a solid plane boundary $y=0$. this flow was first considered by Hiemenz (1911) & is called <u>Hiemenz flow</u>. (Note that Z boundary could be curved such as Z surf of a circular cylinder, provided that Z region under considern is compared with Z radius of curvature.)

Solve Z steady vorticity eqn $u \frac{\partial \omega}{\partial x} + v \frac{\partial \omega}{\partial y} = \nu \nabla^2 \omega$ in terms of Z streamfunc $\psi(x,y)$ where $\underline{u} = \nabla \times (\psi \underline{k})$

given that at large distances from Z boundary Z flow takes Z form of a stagnatn pt flow $\psi \approx \frac{x(y-y_0)}{\tau}$ as $y \to \infty$

where τ is a const with dimensns of time T & y_0 is a const that is to be found as part of Z sol $\therefore$ as $y \to \infty$

$u = \frac{\partial \psi}{\partial y} \approx \frac{x}{\tau}$, $v = -\frac{\partial \psi}{\partial x} \approx -\frac{y-y_0}{\tau}$, $\omega \to 0$

also $\psi \approx x(y-y_0)/\tau$ as $y \to \infty$ is equiv to $F(\eta) \approx \frac{\delta}{\nu \tau}(y-y_0) = \frac{y-y_0}{\delta}$ as $y \to \infty$ which is Z same as $F(\eta) \approx \eta - \eta_0$, $\eta_0 = \frac{y_0}{\delta}$ $\therefore$

& impose $F'(\eta) \to 1$, $F(\eta) \approx \eta - \eta_0$ as $\eta \to \infty$

- can only make limited analytical progress with probl since $(F'^2 - FF'')' = 2F'F'' - F'F'' - FF''' = F'F'' - FF'''$ can integrate Z governing ODE $F'F'' - FF''' = F''''$ to give $F'^2 - FF'' = F''' + c$

where c is a const using $F' \to 1$, $F'' \to 0$ & $F''' \to 0$ as $\eta \to \infty$

we obtain $c=1$ & $\therefore$ $F'^2 - FF''' = F''' + 1$

a numerical sol of this third order ODE, subject to $F(0)= $))

$F'(0)=0$ & $F' \to 1$ as $\eta \to \infty$ was given by Hiemenz (1911) & gives

$\eta_0 \approx 0.65$ z corres $y_0 = \eta_0 \delta$ is called z <u>displacement</u> <u>thickness</u>

GFN 4 for phd in fluid dynamics

Week 7

review - week 1 $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \mu\nabla^2\underline{u} + \rho\underline{g} + \dots$ Navier-Stokes

fluid density

Continuity eqn: $\partial_t \rho + \nabla\cdot(\rho\underline{u}) = 0$ $\qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\nabla$

$\therefore \partial_t \rho + \rho\nabla\cdot\underline{u} + \underline{u}\cdot\nabla\rho = 0$

$\underbrace{\frac{D\rho}{Dt}}_{=0} + \rho\nabla\cdot\underline{u} = 0$

for incompressible flow $= \nabla\cdot\underline{u} = 0$

in week 2: Simplification in order to solve N-S + incompressible flow (a) ^

$\partial_t \underline{u} = 0$ Steady $\qquad \underline{u}\cdot\nabla\underline{u} = \underline{0}$ $\qquad \underline{u} = \underline{u}\,\underline{i}$

- fluid only moves if forces drive it (pressure diver, gravity, boundary motion, ...)

1 - Determine z contrasts using · Boundary conditions · regularity conditions

in week 3: Derivation of Navier stokes

Dimensional analysis $[x]=L$ $\quad [t]=T$ $\quad [\text{Mass}]=M$

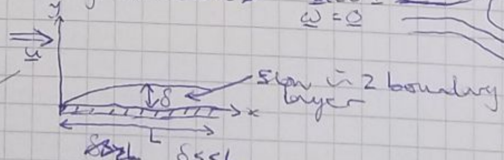
$[\rho] = \left[\frac{\text{mass}}{\text{volume}}\right] = ML^{-3}$

Change of variables $\underline{x}' = \frac{\underline{x}}{L}$ $\quad \cancel{\frac{t}{T}}$ $\quad t' = \frac{t}{T} \Rightarrow$ ~~$\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p + Re$~~
(problem 3.4)

$\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p + Re^{-1}\nabla^2\underline{u}$ where $Re = \frac{UL}{\nu}$ $\quad (\nu = \mu/\rho)$

week 4: Buckingham-π thm - form special ratios to reduce z number of variables in a problem (PDE $\to$ ODE)

week 5: Boundary layers - thin layers near to an object where z fluid velocity adjusts to z boundary conditions



Boundary layer approx

Boundary layer is thin $\qquad \delta \ll L$

$\delta = \sqrt{4\nu t}$ Stokes 1st eqn $\qquad \delta = \sqrt{4\nu/\Omega}$ oscillatory plate

Cases for ex, for ...

$\delta = \sqrt{\nu T}$ stagnation point

simplified eqn: use 2d cartesian geometry & use the fact that the thickness (y) << length (x) → neglect cartesian terms in Navier Stokes eqn

Ex/ $100+1 = 99+2$ $\therefore$ $100 \simeq 99$ ← apparent equals

x component of N-S $\underline{u} = (u, v, 0)$ $\quad \partial_t u + \underbrace{u\partial_x u + v\partial_y u}_{\underline{u}\cdot\nabla \underline{u}} = -\frac{1}{\rho}\partial_x p + \nu\partial_{xx} u + \nu\partial_{yy} u$

$\nu \frac{\partial^2 u}{\partial x^2} \sim \frac{\nu U}{L^2}$ using $\frac{d}{dx} \sim \frac{1}{L}$ {since length of x axis is L (graphed)}

$\nu \frac{\partial^2 u}{\partial y^2} \sim \frac{\nu U}{\delta^2}$ using $\frac{d}{dy} \sim \frac{1}{\delta}$ {since hight of y axis is δ from diagram}

but $\delta << L \Rightarrow \frac{1}{\delta} >> \frac{1}{L}$ $\therefore$ $\frac{\nu U}{\delta^2} >> \frac{\nu U}{L^2}$

& we reject $\nu\partial_{xx} u$ compared with $\nu\partial_{yy} u$

y comp: $\Rightarrow \frac{\partial p}{\partial y} = 0$ flow is steady $\Rightarrow u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$

is the steady boundary layer eqn (4.2)

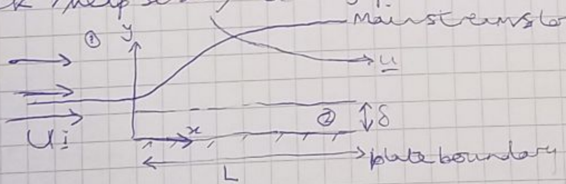
3 unknowns, 1 eqn? but can be solved since 2d $\therefore$

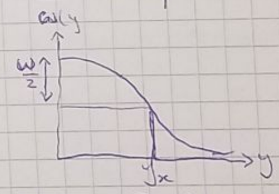
$\underline{u} = \nabla \times (\psi \underline{k}) = (\partial_y \psi, -\partial_x \psi, 0)$ $\therefore$ $(u, v, 0) \to \psi$

determine $p(x)$ using the mainstream flow (probl 4.1)

→ Blasius boundary layer

week 7 help sess / scaling pressure for input to BL eqn





$\therefore$ at ① the pressure varies on large length scale determined by the mainstream flow

$\partial_t \omega = \nu \partial_{yy} \omega$

$\omega = Ae^{-\nu k^2 y t}$

$\to t^*$ $\quad \to L$

at ② pressure approx over BL (probl 4.8 $\frac{\partial p}{\partial y} = 0$)

BL eqn: $u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = -\frac{1}{\rho}\frac{dp}{dx} + \nu\frac{\partial^2 u}{\partial y^2}$

just outside the BL theres little variation in y {$\therefore$ neglect terms}

$$u\frac{\partial u}{\partial x} = -\frac{1}{\rho}\frac{dp}{dx} \Rightarrow \frac{\partial}{\partial x}\left(\frac{u^2}{2}\right) = -\frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) \Rightarrow$$

$$\frac{u^2}{2} + c = -\frac{p}{\rho} \Rightarrow p = -\frac{\rho u^2}{2} + c' \quad \text{(pg 71)}$$

(4.2) → eqn for ψ

Bernoulli's eqn $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla(p + \Pi)$ $\quad \rho \underline{g} = -\nabla\Pi \quad \Pi = \rho g z$

or identity vii with $\underline{u} = \underline{v}$: $\underline{u}\cdot\nabla\underline{u} = \nabla\left(\frac{|\underline{u}|^2}{2}\right) - \underline{u}\times\underline{\omega}$ $\quad (\underline{\omega} = \nabla\times\underline{u}) \therefore$

N-S $\Rightarrow \rho(\partial_t \underline{u} - \underline{u}\times\underline{\omega}) = -\nabla(p + \Pi + \frac{1}{2}\rho|\underline{u}|^2)$

$\partial_t \underline{u} + \nabla H = \underline{u}\times\underline{\omega}$ Bernoulli's eqn

{Bernoulli's eqn $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla(p + \Pi) + \mu\nabla^2\underline{u}$ ignore $\because$ inviscid case}

$\therefore H = \frac{p}{\rho} + \frac{\Pi}{\rho} + \frac{1}{2}|\underline{u}|^2$ Bernoulli func

for irrotational flow $\underline{\omega} = \nabla\times\underline{u} = \underline{0}$ (Mainstream $\underline{u} = \nabla\phi$

if Z flow is steady $\partial_t \underline{u} = \underline{0}$

$\therefore$ Bernoulli eqn is $\nabla\left(\frac{p}{\rho} + \frac{\Pi}{\rho} + \frac{1}{2}|\underline{u}|^2\right) = 0 \therefore$

$H = \frac{p}{\rho} + \frac{\Pi}{\rho} + \frac{1}{2}|\underline{u}|^2 = \text{const}$

in Z absence of gravity $H = \frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2 = \text{const}$ $(p \gg \Pi)$

\quiz/ \3/ mainstream flow streamfunc $\psi = xy^2 + 2y$

as Z adjacent boundary layer, Z negative pressure gradient $G = -\frac{dp}{dx}$ is:

$\underline{u} = (u, v, 0) \quad |\underline{u}|^2 = u^2 + v^2 \approx u^2$

$\underline{u} = \nabla(\psi\,\underline{k}) = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0\right) \Rightarrow u = \frac{\partial\psi}{\partial y} \therefore \quad u = 2xy + 2 \therefore$

$p = -\frac{1}{2}\rho|\underline{u}|^2 = -\frac{1}{2}\rho(2xy+2)^2$

but y is small in Z boundary layer $\therefore 2xy + 2 \approx 2$

$\therefore p \approx -\frac{1}{2}\rho\cdot 4 \quad \therefore G = -\frac{dp}{dx} \approx 0$

~~recall: Z ith component~~ \skin friction / shear stress/

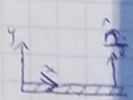
Recall: Z ith component of force on a surface area element dS

with unit normal $\hat{\underline{n}}$ is $\sum_{i \neq n} \sigma_{ij}\hat{n}_j$ where $\sigma_{ij} = \underset{\text{fluid pressure}}{-p\,\delta_{ij}} + \underset{\text{viscosity}}{\mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)}$

is Z stress tensor $\therefore \quad \sigma = \begin{pmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{pmatrix}$

"normal stresses" are Z diagonal components

"stress stresses" are Z off diagonal elements



pressure constitutes a normal stress

viscous part of Z sheer stress is $\sum_i^{\text{viscous}} = \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\hat{n}_j$

Blasius probl Z boundary is Z flat plate $y = 0$

$\therefore$ any contribution is from $j = 2$ $\quad \sum_i^{\text{viscous}} = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

$i = 1 \quad \sum_1^{\text{viscous}} = \mu\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = \mu\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right)$

$i=2$ $\tau_2^{\text{viscous}} = \mu\left(\frac{\partial v}{\partial y} + \frac{\partial v}{\partial y}\right) = 2\mu \frac{\partial v}{\partial y}$

emf on $y=0$ $\underline{u} = (\underbrace{\partial_y \psi}_{u}, \underbrace{-\partial_x \psi}_{v}, 0)$

can be shown $\frac{\partial v}{\partial x}\Big|_{y=0} = \frac{\partial v}{\partial y}\Big|_{y=0} = 0$

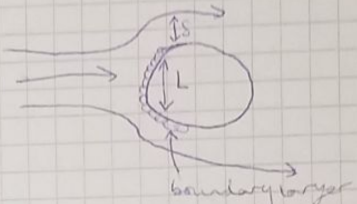
$\therefore$ only remaining contributions to 2 shear stress is from

$\mu \frac{\partial u}{\partial y}\Big|_{y=0} = \mu \frac{\partial}{\partial y}\left(\frac{\partial \psi}{\partial y}\right)\Big|_{y=0}$ (problem 4.15)

\week 8 in person lecture/

Boundary layer review: $\mu\nabla^2\underline{u}$ is normally viscous term or $R_e^{-1}\nabla^2 \underline{u}$

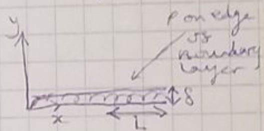
assume $\delta \ll L$



(2d) $\Rightarrow$ $\underline{u} = \nabla \times (\psi \underline{k}) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0\right)$

$\nabla \cdot \underline{u} = 0$

\problem 4.7/ $\frac{\partial \psi}{\partial y} \sim \frac{\psi}{\delta}$ $\quad \frac{\partial \psi}{\partial x} \sim \frac{\psi}{L}$



$\frac{\frac{\partial \psi}{\partial y}}{\frac{\partial \psi}{\partial x}} \sim \frac{\psi/\delta}{\psi/L} = \frac{L}{\delta} \gg 1 \Rightarrow \frac{\partial \psi}{\partial y} \gg \frac{\partial \psi}{\partial x} \Rightarrow u \gg v$

\problem 4.8/ Compare sizes of terms in Navier-Stokes

$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{G}{\rho} + \nu\frac{\partial^2 u}{\partial y^2}$ $\quad G = -\frac{dp}{dx}$ $\quad \frac{\mu}{\rho}\nabla^2 u = \underbrace{\frac{\mu}{\rho}}_{\nu}\left(\underbrace{\frac{\partial^2 u}{\partial x^2}}_{\text{small} \therefore \text{neglected}} + \frac{\partial^2 u}{\partial y^2}\right)$ (eqn 4.2)

$p \approx -\frac{1}{2}\rho u^2 + c$ $\quad u$ is 2 mainstream flow (uniform...)

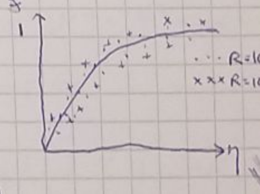
\Activity 4.10/ mainstream flow $\underline{u} = U\underline{i}$ where U is const $\therefore$

$p \approx -\frac{1}{2}\rho U^2 + \text{const}$ $\quad \therefore \frac{dp}{dx} = 0$ since ρ, U are consts $\therefore$

$G = -\frac{dp}{dx} = 0$

\problem 4.13/4.14/ $f''' + \frac{1}{2}ff'' = 0$ $\quad f'(0) = 0$ $\quad f(0) = 0$ $\quad f' \to 1$ far from 2 plate

hard ~~numerically~~ analytically but can solve numerically $\therefore$



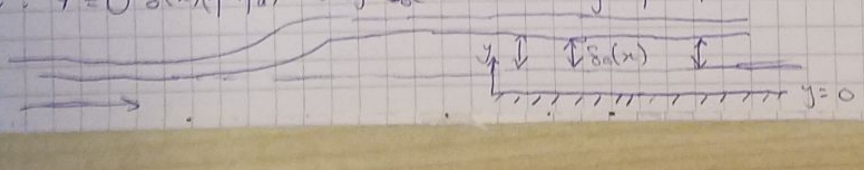
$f' = g = \frac{u}{U}$ $\quad \eta = \frac{y}{\delta(x)}$ is example of a similarity soln

~~Dis~~

Displacement thickness / $\psi = U\delta(x)f(\eta)$ (problem 4.13)

$\frac{df}{d\eta} \to 1$ as $\eta \to \infty$ $\Rightarrow$ $f = \eta - \text{const}$ as $\eta \to \infty$
$= \eta - \eta_0$ say

$\therefore \psi = U\delta(x)(\eta - \eta_0) = U(y - \delta_0(x))$ since $y = \eta\delta$ (problem 4.13) $\quad \delta_0(x) = \eta\delta(x)$



this week: Stokes flow (slow flow) low reynolds num behaviour ∴

viscous terms are important

again we compare sizes of terms → simpler eqn (stokes eqn)

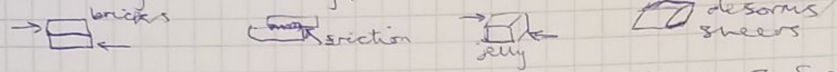
there is just one length scale L, flow speed U, $\nu = \mu/\rho$ ← kinematic viscosity

$R_e = \frac{UL}{\nu} \ll 1$ eg swimming of bacteria 10^{-6} m , earths atmosphere 40,000 km

viscosity μ $\qquad \rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \ldots + \mu\nabla^2\underline{u}$

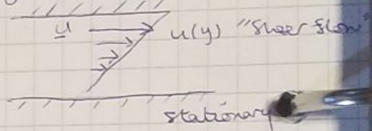
Dynamic viscosity (viscosity) μ is a measure of the fluids resistance

to flow



a force is needed to maintain the sheer - the strength of the force is a

measure of viscosity



newton's law of viscosity sheer stress

$\tau = \mu \frac{\partial u}{\partial y}$ $\qquad$ τ = force per unit area "stress"

units milli pascal s: water 1, whole milk 2, olive oil 56, honey 10^4,

peanut butter 10^6

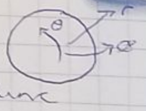
\Stokes flow problem 5.1 / comparing sizes of terms in NS → provided $R_e \ll 1$

$\underbrace{0 = -\nabla p + \mu\nabla^2\underline{u}}$ $\quad \nabla\cdot\underline{u} = 0$ is Stokes eqn

\Simplified geometries / 2D cartesian flow $\underline{u} = \nabla\times(\psi\underline{k}) \to \nabla^4\psi = \nabla^2(\nabla^2\psi) = 0$

"biharmonic eqn" (eqn 5.2)

- spherical polar coords $\underline{u} = u(r,\theta,t)\hat{\underline{r}} + v(r,\theta,t)\hat{\underline{\theta}}$



$\nabla\cdot\underline{u} = 0$ ∴ ~~$\underline{u} = \nabla$~~ $\underline{u} = \nabla\times\left(\frac{\psi(r,\theta,t)}{r\sin\theta}\hat{\underline{\phi}}\right)$ ψ is the stokes stream func

$\to E^4\psi = E^2(E^2\psi) = 0$ (eqn 5.7) $\quad \underbrace{E^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)}$

where E^2 is given in (eqn 5.5) is on formula sheet

\Week 8 help sess / stokes eqn or slow flow eqns

\Activity 3.3 / $\underline{x}' = \frac{\underline{x}}{L} \Rightarrow \partial x' = \frac{1}{L}\partial x \Rightarrow \frac{\partial}{\partial x} = \frac{1}{L}\frac{\partial}{\partial x'}$

→ Navier Stokes: $\partial_t\underline{u}' + \underline{u}'\cdot\nabla\underline{u}' = -\nabla p' + \frac{1}{R_e}\nabla^2\underline{u}'$

$\partial_t\underline{u} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\underline{u}$ $\qquad \frac{\text{nonlinear}}{\text{diffusion}} = \frac{\underline{u}\cdot\nabla\underline{u}}{\mu/\rho\,\nabla^2\underline{u}} \sim \frac{U^2/L}{\mu/\rho\; U/L^2} =$

$\frac{UL}{\mu/\rho} = \frac{UL}{\nu} = R_e$

planar flow: $\underline{u} = (u, v, 0)$ incompressible $\underline{u} = \nabla \times (\psi \underline{\hat{k}}) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0\right)$

$\{\nabla \cdot \underline{u} = 0\}$

Meridional flow: $\underline{u} = (r, \theta, t)\hat{\underline{r}} + v(r, \theta, t)\hat{\underline{\theta}}$ incompressible $\nabla \cdot \underline{u} = 0$

$\underline{u} = \nabla \times \left(\frac{\psi}{r\sin\theta}\hat{\underline{\phi}}\right) \rightarrow \{\nabla^4 \psi = 0\} \quad E^4 \psi = 0$

Activity 5.4 $\underline{u} = \nabla \times \left(\frac{\psi}{r\sin\theta}\hat{\underline{\phi}}\right) = \frac{1}{r^2 \sin\theta}\begin{vmatrix} \hat{r} & r\hat{\underline{\theta}} & r\sin\theta\,\hat{\underline{\phi}} \\ \partial_r & \partial_\theta & \partial_\phi \\ 0 & 0 & r\sin\theta \frac{\psi}{r\sin\theta} \end{vmatrix} =$

$\frac{1}{r^2\sin\theta}\left[\hat{\underline{r}}\frac{\partial \psi}{\partial \theta} - r\hat{\underline{\theta}}\frac{\partial \psi}{\partial r} + 0\hat{\underline{\phi}}\right] \Longrightarrow u = \frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta} \quad v = -\frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r}$

$\underline{\omega} = \nabla \times \underline{u} = \left(\nabla \times \nabla \times \left(\frac{\psi}{r\sin\theta}\hat{\underline{\phi}}\right)\right) = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\,\hat{\underline{\phi}} \\ \partial_r & \partial_\theta & \partial_\phi \\ \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} & -\frac{r}{r\sin\theta}\frac{\partial \psi}{\partial r} & r\sin\theta\,(0) \end{vmatrix} =$

$\frac{1}{r^2\sin\theta}\left[\hat{\underline{r}}\left(0 - \frac{\partial}{\partial \phi}(rv)\right) - r\hat{\underline{\theta}}\left(\frac{\partial 0}{\partial r} - \frac{\partial u}{\partial \phi}\right) + r\sin\theta\,\hat{\underline{\phi}}\left(\frac{\partial}{\partial r}(rv) - \frac{\partial}{\partial \theta}u\right)\right] =$

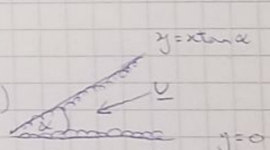
$\frac{1}{r^2\sin\theta} r\sin\theta\,\hat{\underline{\phi}}\left(\frac{\partial}{\partial r}(rv) - \frac{\partial u}{\partial \theta}\right) \therefore$

$\frac{\partial}{\partial r}(rv) = \frac{\partial}{\partial r}\left(-\frac{r}{r\sin\theta}\frac{\partial \psi}{\partial r}\right) = -\frac{1}{\sin\theta}\frac{\partial^2 \psi}{\partial r^2}$

$\frac{\partial u}{\partial \theta} = \frac{\partial}{\partial \theta}\left(\frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta}\right) = \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \theta}\right) \therefore$

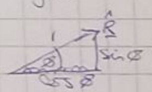
$\omega = \frac{1}{r}\hat{\underline{\phi}}\left[-\frac{1}{\sin\theta}\frac{\partial^2 \psi}{\partial r^2} - \frac{1}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \theta}\right)\right] = -\frac{1}{r\sin\theta}\hat{\underline{\phi}}\left[\frac{\partial^2 \psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta}\frac{\partial \psi}{\partial \theta}\right)\right]$

$= -\frac{1}{r\sin\theta}\hat{\underline{\phi}} E^2 \psi$



Sheet 7 1a mainstream flow $\underline{U} = U\hat{\underline{R}}$ in polar coords (R, ϕ)

$\therefore \nabla \cdot \underline{U} = \frac{1}{R}\frac{\partial}{\partial R}(RU) + \frac{1}{R}\frac{\partial 0}{\partial \phi} = 0 \Rightarrow RU = \text{const}$



$\therefore U = \frac{c}{R}$ take $c = -Q \therefore U = -\frac{Q}{R}$ with $Q > 0$

1b (py > 1:) $U\frac{\partial U}{\partial x} = -\frac{1}{\rho}\frac{dp}{dx}$ $\quad \underline{U} = U\hat{\underline{R}} = -\frac{Q}{R}(\cos\phi\,\underline{i} + \sin\phi\,\underline{j})$

$= -\frac{Q}{\sqrt{x^2+y^2}}(\cos\phi\,\underline{i} + \sin\phi\,\underline{j}) \approx -\frac{Q}{\sqrt{x^2}}(1\underline{i} + 0\underline{j})$ {when y & ϕ is small}

$= -\frac{Q}{x}\underline{i} \therefore U = -\frac{Q}{x} \therefore \frac{\partial U}{\partial x} = \frac{\partial}{\partial x}(-Qx^{-1}) = \frac{Q}{x^2} \therefore$

$U\frac{\partial U}{\partial x} = -\frac{Q}{x}\left(\frac{Q}{x^2}\right) = -\frac{1}{\rho}\frac{dp}{dx} \therefore \frac{1}{\rho}\frac{dp}{dx} = \frac{Q}{x^3}$

1d when $y = 0 \therefore \eta = \frac{y}{\delta} = \frac{0}{\delta} = 0 \therefore$ solid boundary $\therefore$ no-slip condition

$\therefore$ tangential velocity satisfied $\therefore u = 0 \therefore u = -\frac{Q}{x}F' \therefore u = \frac{Q}{x}F'(\eta = 0) \therefore$

$F'(0) = F'(\eta = 0) = 0$ $\quad v$ is automatically satisfied

$\eta = \frac{y}{\delta(x)}$ as $\eta \to \infty$ we require $\underline{u} = \underline{U}$ $\quad u = -\frac{Q}{x}F' = U = -\frac{Q}{x} \therefore F'(\infty) = 1$

$u = -\frac{Q}{x} F'$ $\therefore \frac{du}{dy} = 0 \Rightarrow -\frac{Q}{x} F'' = 0$ as $\eta \to \infty \to F''(\infty) = 0$

\week 9 lecture/

\Menti/ \1/ Z reynolds number measures Z ratio of?

Z inertia terms of Z viscous terms

$\rho(\partial_t \underline{u} + \underline{u} \cdot \nabla \underline{u}) = -\nabla p + \mu \nabla^2 \underline{u} + \dots + \rho \underline{f}$

$Re = \frac{\text{inertia}}{\text{viscous}} = \frac{\rho \underline{u} \cdot \nabla \underline{u}}{\mu \nabla^2 \underline{u}} \to Re = \frac{UL}{\nu}$ $\quad \nu = \mu/\rho$

$Re \ll 1$ inertia $\ll$ viscous forces

NS $\to 0 = -\nabla p + \mu \nabla^2 \underline{u}$ stokes eqn

\2/ which of Z following is an ex of ~~a~~ flow in Z stokes regime?

Swimming bacteria - stokes regime

\3/ Z stokes stream func is introduced in order to?

Automatically satisfy Z incompressibility condition

$\underline{u} = \nabla \times \left(\frac{\psi}{r \sin\theta} \hat{\underline{\phi}} \right)$ $\quad \nabla \cdot \underline{u} = 0$

$\to E^4 \psi = 0$

[planer case $\underline{u} = \nabla \times \psi \underline{k} \Rightarrow \nabla^4 \psi = 0$]

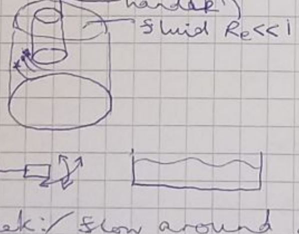
\5.4/ uniqueness of stokes flow - reversibility

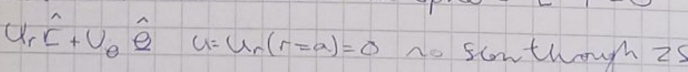
consider sol $\underline{u}_1(\underline{x})$ $\quad 0 = -\nabla p_1 + \mu \nabla^2 \underline{u}_1$ $\quad \underline{u}_1 = f_1(x)$ on S

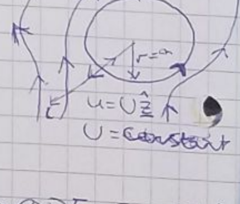
now change Z BC to $\underline{u}_1 = -f_1(\underline{x})$ on S $\quad -\underline{u}_1 = f_1(\underline{x})$ on S

& $\underline{u}_1$ satisfies Z stokes eqn $0 = -\nabla p + \mu \nabla^2 (-\underline{u}_1) = -\nabla p - \mu \nabla^2 \underline{u}_1$

$= -\underbrace{(\nabla p + \mu \nabla^2 \underline{u}_1)}_{=0 \text{ if } p = -p_1}$ $\quad -\underline{u}_1, -p_1$ is a sol with BC $\underline{u}_1 = -f_1(\underline{x})$ on S







\this week:/ flow around a sphere $E^4 \psi = 0$

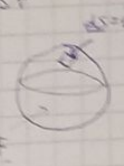
$\underline{u} = u_r \hat{\underline{r}} + u_\theta \hat{\underline{\theta}}$ $\quad u = u_r(r=a) = 0$ no flow through Z sphere

$u_\theta(r=a) = v = 0$ $\quad$ no-slip

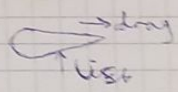
$\psi(r,\theta) = f(r)\sin^2\theta$ ~~Sepp~~ separable sol $\quad$ sub into $E^4 \psi = 0 \Rightarrow$ ODE for $f(r)$ -

coeffs are func of r   seek sols $f(r) = r^{\alpha}$ $\alpha = \text{const}$

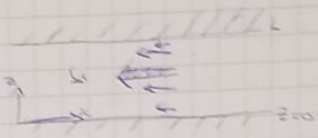
$\rightarrow$ find α $\rightarrow \psi(r, \theta)$



? find the force F using stress tensor - Stokes drag



\Coursework1 sols/ \1a/



$\nabla p = \frac{dp}{dx}\,\underline{i}$ $\frac{dp}{dx} > 0$

\1b/ $m\underline{a} = \underline{F}$ $\underline{a} = \partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u} = \frac{D\underline{u}}{Dt}$

$\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \cancel{\rho \underline{g}} + \mu \nabla^2 \underline{u}$

? pressure gradient driving the flow, it acts in the neg x-direction

since $\underline{F} = -\nabla p = -\frac{dp}{dx}\underline{i} = -ve\,\underline{i}$ since $\frac{dp}{dx} > 0$ $\therefore$ fluid expected to

move in neg x-direction

steady, i.e. not a func of time

$\underline{u} = u\underline{i}$ where $u < 0$ on which coords x, y, z (t) does u depend?

no special locations in x $\therefore$ indep of x

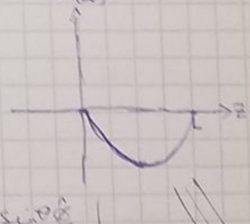
expect u to depend on z since there are boundaries in z where

no-slip applies

\1c/ $\underline{u} = u(z)\,\underline{i}$ $\cancel{\partial_t \underline{u}} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p + \frac{\mu}{\rho}\nabla^2\underline{u}$

$\underline{u}\cdot\nabla\underline{u} = (\underline{u}\cdot\nabla)\underline{u}$ $\underline{u}\cdot\nabla = u(z)\underline{i}\cdot(\underline{i}\,\partial_x + \underline{j}\,\partial_y + \underline{k}\,\partial_z) = u(z)\frac{\partial}{\partial x}$

$(\underline{u}\cdot\nabla)\underline{u} = u\frac{\partial}{\partial x}(u(z)\underline{i}) = \underline{i}\,u\frac{\partial u(z)}{\partial x} = \underline{0}$



x comp: $-\frac{1}{\rho}\frac{dp}{dx} + \frac{\mu}{\rho}\frac{d^2u}{dz^2} = 0$ integrate twice to find

$u(z) = \frac{1}{\mu}\frac{dp}{dx}\frac{z}{2}(z - L)$ $z \in (0, L]$ $u(z=0) = 0$ $u(z=L) = 0$

$u < 0 \therefore u = (-ve)\,\underline{i}$

\2a/ $\underline{u} = u_r\hat{\underline{r}} + u_\theta\hat{\underline{\theta}} + u_\phi\hat{\underline{\phi}}$ $\underline{\omega} = \nabla\times\underline{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\,\hat{\underline{\phi}} \\ \partial_r & \partial_\theta & \partial_\phi \\ u_r & u_\theta & r\sin\theta\,u_\phi \end{vmatrix} =$

$$\frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\,\hat{\underline{\phi}} \\ \partial/\partial r & \partial/\partial\theta & \partial/\partial\phi \\ r^2\cos\theta & r\frac{1}{r} & r\sin\theta\frac{1}{\sin\theta} \end{vmatrix} =$$

$$\frac{1}{r^2\sin\theta}\left\{\hat{\underline{r}}\left(\frac{\partial}{\partial\theta}r - \frac{\partial}{\partial\phi}1\right) - r\hat{\underline{\theta}}\left(\frac{\partial r}{\partial r} - \frac{\partial}{\partial\phi}(r^2\cos\theta)\right) + r\sin\theta\,\hat{\underline{\phi}}\left(\frac{\partial 1}{\partial r} - \frac{\partial}{\partial\theta}(r^2\cos\theta)\right)\right\}$$

$$= \frac{1}{r^2\sin\theta}\left\{0\,\hat{\underline{r}} - r\hat{\underline{\theta}}\,1 + r\sin\theta\,\hat{\underline{\phi}}(-r^2)(-\sin\theta)\right\} = -\frac{1}{r\sin\theta}\hat{\underline{\theta}} + r\sin\theta\,\hat{\underline{\phi}}.$$

2b/ $\underline{u} = R\cos\phi\,\hat{R} + \sin\phi\,\hat{\phi}$ cylindrical polars $\underline{u}\cdot\nabla\underline{u} = (\underline{u}\cdot\nabla)\underline{u}$

$$(\underline{u}\cdot\nabla) = \begin{pmatrix}R\cos\phi\\ \sin\phi\\ 0\end{pmatrix}\cdot\begin{pmatrix}\partial/\partial R\\ \frac{1}{R}\partial/\partial\phi\\ \partial/\partial z\end{pmatrix} = R\cos\phi\frac{\partial}{\partial R} + \frac{\sin\phi}{R}\frac{\partial}{\partial\phi} + 0 \quad \therefore$$

$$(\underline{u}\cdot\nabla)\underline{u} = \left[R\cos\phi\frac{\partial}{\partial R} + \frac{\sin\phi}{R}\right](R\cos\phi\,\hat{R} + \sin\phi\,\hat{\phi})$$

Consider $R\cos\phi\frac{\partial}{\partial R}(R\cos\phi\,\hat{R}) = R\cos^2\phi\frac{\partial}{\partial R}(R\hat{R}) = R\cos^2\phi\,\hat{R}\frac{\partial R}{\partial R} = R\cos^2\phi\,\hat{R}$

Consider $\frac{\sin\phi}{R}\frac{\partial}{\partial\phi}(R\cos\phi\,\hat{R}) = \frac{R\sin\phi}{R}\frac{\partial}{\partial\phi}(\cos\phi\,\hat{R}) =$

$\sin\phi\left[\cos\phi\frac{\partial\hat{R}}{\partial\phi} + \hat{R}\frac{\partial}{\partial\phi}\cos\phi\right] = \sin\phi\left[\cos\phi\,\hat{\phi} + \hat{R}(-\sin\phi)\right]$

3a/ $\underline{u} = \nabla\times\left(\frac{\Psi}{R}\hat{\phi}\right)$ stagnation points $u = 0$

$$\underline{u} = \frac{1}{R}\begin{vmatrix}\hat{R} & R\hat{\phi} & \hat{z}\\ \partial_R & \partial_\phi & \partial_z\\ 0 & R\frac{\Psi}{R} & 0\end{vmatrix} = \frac{1}{R}\left(-\hat{R}\frac{\partial\Psi}{\partial z} - R\hat{\phi}\cdot 0 + \hat{z}\frac{\partial\Psi}{\partial R}\right) \therefore \quad u = \nabla\times\left(\frac{\Psi}{R}\hat{\phi}\right) = u\hat{R} + w\hat{z}$$

$\therefore u = -\frac{1}{R}\frac{\partial\Psi}{\partial z} \quad w = \frac{1}{R}\frac{\partial\Psi}{\partial R} \therefore$

$u = -\frac{1}{R}\frac{\partial}{\partial z}\left(-\frac{R^2}{10}(a^2-R^2-z^2)\right) = \frac{R}{10}(-2z) = -\frac{zR}{5}$

$w = \frac{1}{R}\frac{\partial}{\partial R}\left(-\frac{R^2}{10}(a^2-R^2-z^2)\right) = \frac{1}{5}(2R^2-a^2+z^2)$

$\underline{u} = 0 \Rightarrow z = 0$ or $R = 0$

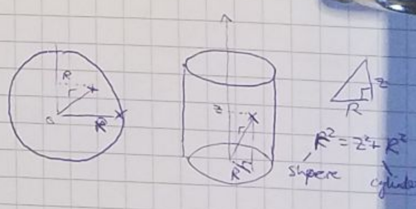
$z = 0 \Rightarrow w = 0$ when $2R^2 = a^2 \Rightarrow R = \pm\sqrt{\frac{a^2}{2}} = +\frac{a}{\sqrt{2}} \therefore$

stagnation pt $\left(\frac{a}{\sqrt{2}}, \phi, 0\right)$

$R = 0 \Rightarrow w = 0$ when $\frac{1}{5}(-a^2+z^2) = 0 \Rightarrow z = \pm a$

$\therefore$ stagnation pts $(0, \phi, a)$, $(0, \phi, -a)$

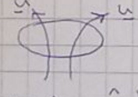
1b/ $|\underline{u}|^2 = u^2 + w^2 = \frac{z^2R^2}{25} + \frac{1}{25}(2R^2-a^2+z^2)^2$

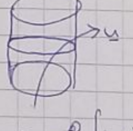


$\therefore$ use $z^2 + R^2 = a^2 \Rightarrow$ sub $|\underline{u}|^2 = \frac{R^2a^2}{25}$

Maximal $R = a$ ie equator

1c/

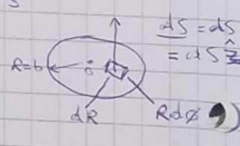




Mass flux $\rho\int_S \underline{u}\cdot d\underline{S}$ density = $\frac{\text{mass}}{\text{volume}}$

$= \rho\int_S (u\hat{R} + w\hat{z})\cdot(R\,dR\,d\phi\,\hat{z}) = \rho\int wR\,dR\,d\phi =$

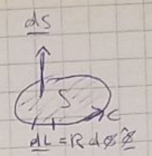
$d\underline{S} = dS\,\hat{n} = dS\,\hat{z}$



$\rho\int_S \frac{1}{R}\frac{\partial\Psi}{\partial R}R\,dR\,d\phi = \rho\,2\pi\int_S\frac{\partial\Psi}{\partial R}dR = \rho\,2\pi\left[\Psi\right]_{R=0}^{b}$

Sub for $\Psi \Rightarrow \pi\frac{\rho}{5}b^2(b^2+z^2-a^2)$

Mass flux $= \rho \int_S \underline{u} \cdot \underline{dS} = \int \rho \nabla \times \underbrace{\left(\frac{\psi \rho}{R} \hat{\underline{\phi}} \right)}_{\underline{F}} \cdot \underline{dS} =$



$\int_C \frac{\psi}{R} \rho \hat{\underline{\phi}} \cdot \underline{dL}$ by Stokes thm

$= \int_C \frac{\psi \rho}{R} \hat{\underline{\phi}} \cdot R d\phi \hat{\underline{\phi}} = \int_C \psi \rho \, d\phi = 2\pi \psi \rho$ on $R = b$

\4a/ incompressible is $\nabla \cdot \underline{u} = 0$

\4b/ $\rho \big(\underbrace{\cancel{\frac{\partial \underline{u}}{\partial t}}}_{\text{flow is steady}} + \underline{u} \cdot \nabla \underline{u} \big) = -\nabla p + \cancel{\rho \underline{g}} + \underbrace{\cancel{\mu \nabla^2 \underline{u}}}_{\text{inviscid}}$ $\therefore$

$\rho \underline{u} \cdot \nabla \underline{u} = -\nabla p = \rho (\underline{u} \cdot \nabla) \underline{u} = -\nabla p$

$\underline{u} \cdot \nabla \underline{u} = E(x\underline{i} + y\underline{j})$

x comp: $\rho E^2 x = -\frac{\partial p}{\partial x} \Rightarrow p = -\rho \frac{E^2 x^2}{2} + f(y)$

y comp: $\rho E^2 y = -\frac{\partial p}{\partial y} \Rightarrow \frac{\partial p}{\partial y} = \frac{df}{dy} \Rightarrow f = \ldots \Rightarrow$

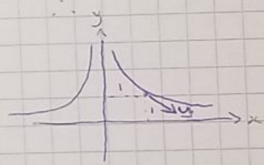
$p = -\frac{\rho E^2}{2}(x^2 + y^2) + \underbrace{P_0}_{\text{constant}}$

\4c/ p = constant $\Rightarrow x^2 + y^2 = c$ ie semi-circles

\4d/ streamlines $\frac{dx}{u} = \frac{dy}{v} = \cancel{\frac{dz}{w}}$ $\quad \frac{dx}{Ex} = \frac{dy}{-Ey}$ $\Rightarrow$ ~~$c + \ln|x| = -\ln|y|$~~

$c + \ln|x| = -\ln|y| = \ln|y|^{-1}$ $\therefore e^{c + \ln|x|} = e^{\ln|y|^{-1}}$ $\therefore e^c e^{\ln|x|} = |y|^{-1}$ $\therefore$

$A|x| = \frac{1}{|y|}$ $\quad A = e^c$ $\quad \therefore x \in (-\infty, \infty)$ $\quad y > 0$ $\therefore$



\4e/ $(x, y) = (1, 1) \Rightarrow \underline{u} = E(x, -y) = E(1, 1)$

\4f/ $y = 0$ $\quad \therefore v = -Ey = 0$ on $y = 0$

Solid, stationary

no slip on stationary boundary $\Rightarrow u = 0$ $\quad u = Ex$ on $y = 0$?

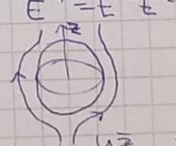
Stokes flow - approx when inertia << viscous tension in N-S

- $R_e = \frac{UL}{\nu} << 1$ $\quad$ kinematic viscosity $\nu = \mu / \rho$ $\quad 0 = -\nabla p + \mu \nabla^2 \underline{u}$

2d (2D?) • $\underline{u} = \nabla \times \psi \underline{k}$ Cartesian $\underline{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0 \right) \to \nabla^4 \psi = 0$

• $\underline{u} = \nabla \times \left(\frac{\psi}{r \sin\theta} \hat{\underline{\phi}} \right) \to E^4 \psi = 0$ $\quad E^4 = E^2 E^2$ where E^2 is given

\Ex - flow around a sphere/



$\hat{\underline{z}} = (0, 0, 1) = \underline{k}$

\Act 5.16 / $\psi = \frac{Ua^2}{4}\left[\frac{a}{r} - \frac{3r}{a} + \frac{2r^2}{a^2}\right]\sin^2\theta$ (probl 5.15)

\probl 5.3 / $u = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = \frac{Ua^2}{4}\left[\frac{a}{r} - \frac{3r}{a} + \frac{2r^2}{a^2}\right]\frac{2\sin\theta\cos\theta}{r^2\sin\theta}$

$v = -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -\frac{1}{r\sin\theta}\frac{Ua^2}{4}\sin^2\theta\left[-\frac{a}{r^2} - \frac{3}{a} + \frac{4r}{a^2}\right]$

u is $\hat{\underline{r}}$ comp of $\underline{u}$ ~~&~~ $v = \hat{\underline{\theta}}$ comp of $\underline{u}$

$\underline{u} = \nabla \times \psi \underline{k}$

$\underline{u} = \nabla \times \frac{\psi}{r^2\sin\theta}\hat{\underline{\phi}}$ $\qquad \underline{u} = \nabla \times \psi\hat{\underline{z}}$

\Act 5.18 / given $t_i = \sigma_{ij}\hat{n}_j$ ← stress tensor, ← unit normal

$= \left[-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]\hat{n}_j$

required show ith comp of $\underline{t}$ is t_i;

$\underline{t} = -p\hat{\underline{n}} + 2\mu(\hat{\underline{n}}\cdot\nabla)\underline{u} + \mu\hat{\underline{n}}\times(\nabla\times\underline{u})$

• The ith comp of $-p\hat{\underline{n}} = -(p\hat{\underline{n}})_i = -p\hat{n}_i$

while $-p\delta_{ij}\hat{n}_j = -p\hat{n}_i$ since $\delta_{ij}a_j = a_i$

• $\mu\frac{\partial u_i}{\partial x_j}\hat{n}_j = \mu\hat{n}_j\frac{\partial u_i}{\partial x_j} = \mu[\hat{\underline{n}}\cdot\nabla\underline{u}]_i \quad \nabla_j = \frac{\partial}{\partial x_j}$

• Consider $[\hat{\underline{n}}\times(\nabla\times\underline{u})]_i = \varepsilon_{ijk}\hat{n}_j(\nabla\times\underline{u})_k = \varepsilon_{ijk}\hat{n}_j\varepsilon_{klm}\frac{\partial}{\partial x_l}u_m =$

$\varepsilon_{kij}\varepsilon_{klm}\hat{n}_j\frac{\partial u_m}{\partial x_l} = (\delta_{il}\delta_{jm} - \delta_{im}\delta_{jl})\hat{n}_j\frac{\partial u_m}{\partial x_l}$ eqn (2.20)

$\{\varepsilon_{kij}\varepsilon_{klm} = \delta_{il}\delta_{jm} - \delta_{im}\delta_{jm}\}$

$= \delta_{il}\delta_{jm}\hat{n}_j\frac{\partial u_m}{\partial x_l} - \delta_{im}\delta_{jl}\hat{n}_j\frac{\partial u_m}{\partial x_l}$

$= n_j\frac{\partial u_j}{\partial x_i} - \hat{n}_j\frac{\partial u_i}{\partial x_j} \quad \therefore \quad n_j\frac{\partial u_j}{\partial x_j} + [\hat{\underline{n}}\times(\nabla\times\underline{u})]_i \therefore$

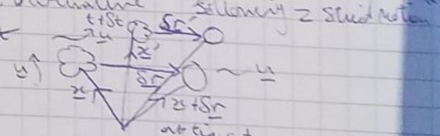
Sub back into t_i $\quad t_i = -p\hat{\underline{n}}_i + \mu[\hat{\underline{n}}\cdot\nabla\underline{u}]_i + \mu[\hat{\underline{n}}\cdot\nabla\underline{u}]_i + \mu[\hat{\underline{n}}\times(\nabla\times\underline{u})]_i$

\week 9 vorticity Dynamics / Vorticity $\underline{\omega} = \nabla\times\underline{u}$

∴ eqn for $\underline{\omega}$ is formed by taking the curl of N-S (sheet 1)

$\frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\nabla\underline{u} + \nu\nabla^2\underline{\omega} \qquad \frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\nabla$ "material derivative" "following the fluid motion"

↑ vortex stretching term $\qquad$ ↑ viscous term



∴ 2D $\omega\cdot\nabla\underline{u} = 0$ (probl 6.3)

\probl 6.7 / $\frac{D}{Dt}(\underline{\delta r}) = \underline{\delta r}\cdot\nabla\underline{u}$ inviscid case

<u>Kelvin's vorticity thm</u>: (Kelvin's circulation thm) (probl 6.9)

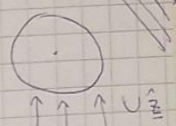
$\int_{S(t)}\underline{\omega}\cdot\underline{dS} = \int_{S(t)}(\nabla\times\underline{u})\cdot\underline{dS} = \int_{C(t)}\underline{u}\cdot\underline{dr}$ by Stokes thm $= \Gamma$ circulation

$\frac{D\Gamma}{Dt} = 0 \therefore \Gamma = \text{const} \qquad \frac{D\Gamma}{Dt} = \frac{D}{Dt}\int_{C(t)}\underline{u}\cdot\underline{dr} = \int_{C(t)}\frac{D}{Dt}(\underline{u}\cdot\underline{dr}) = \cancel{\int_{C(t)}\frac{D u}{}}$

$\int_{C(t)} \frac{D\underline{u}}{Dt} \cdot d\underline{r} + \underline{u} \cdot \frac{D}{Dt}(d\underline{r})$

N-S   prob 6.7

Week 10 tutorial / quiz Q3 / $\underline{u} = u\hat{\underline{r}} + v\hat{\underline{\theta}} + w\hat{\underline{\phi}}$



$\therefore \underline{u} = u\hat{\underline{r}} + v\hat{\underline{\theta}} \to U\hat{\underline{z}}$ at large r (§5.3)

from formula sheet: $\hat{\underline{r}} = \frac{\underline{r}}{r} = \sin\theta\cos\phi\,\hat{\underline{i}} + \sin\theta\sin\phi\,\hat{\underline{j}} + \cos\theta\,\hat{\underline{k}}$

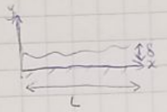
$\hat{\underline{\theta}} = \cos\theta\cos\phi\,\hat{\underline{i}} + \cos\theta\sin\phi\,\hat{\underline{j}} - \sin\theta\,\hat{\underline{k}}$ $\therefore$ compute $\cos\theta\,\hat{\underline{r}} - \sin\theta\,\hat{\underline{\theta}} =$

$\cos^2\theta\,\underline{k} + \sin^2\underline{k} = \underline{k} = \hat{\underline{z}}$.

$\underline{u} = u\hat{\underline{r}} + v\hat{\underline{\theta}} \to U(\cos\theta\,\hat{\underline{r}} - \sin\theta\,\hat{\underline{\theta}})$ at large r $\therefore$

$u = U\cos\theta$, $v = -U\sin\theta$

• Dimensions? §4.5



eqn (4.2) by equating 2 sizes of 2

terms in 2 eqn

$\underline{u} = \nabla \times (\psi\underline{k}) = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0\right)$ $\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}\left(\frac{\partial\psi}{\partial y}\right) \sim \frac{1}{L}\frac{1}{\delta}\psi$

$[u] = \left[\frac{\partial\psi}{\partial y}\right]$ $\frac{L}{T} = \left[\frac{\partial}{\partial y}\right][\psi] = \frac{1}{\text{length}}[\psi]$. $\Rightarrow [\psi] = L^2T^{-1}$

$\rho\left(\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho\underline{g} + \mu\nabla^2\underline{u}$

§3.1 $[\phi] = L^\alpha M^\beta T^\gamma$ α, β, γ consts

dimension   physical quantity of interest u, ρ, p, g

Suffix notation: identity (i) exam formula sheet:

$\underline{u}\cdot(\underline{v}\times\underline{w}) = \underline{v}\cdot(\underline{w}\times\underline{u})$ $\underline{a}\cdot\underline{b} = a_ib_i = \sum_{i=1}^{3} a_ib_i$

$(\underline{a}\times\underline{b})_i = \varepsilon_{ijk}a_jb_k$

$\underline{u}\cdot(\underline{v}\times\underline{w}) = u_i(\underline{v}\times\underline{w})_i = u_i\varepsilon_{ijk}v_jw_k = v_j\varepsilon_{ijk}w_ku_i = v_j(\varepsilon_{ijk}w_ku_i) =$

$v_j(\varepsilon_{jki}w_ku_i) = v_j(\underline{w}\times\underline{u})_j = \underline{v}\cdot(\underline{w}\times\underline{u})$

identity (iii) $\nabla\cdot(\phi\underline{v}) = \nabla_i(\phi\underline{v})_i = \frac{\partial}{\partial x_i}(\phi v_i) = \phi\frac{\partial v_i}{\partial x_i} + v_i\frac{\partial\phi}{\partial x_i} =$

$\phi\nabla\cdot v + \underline{v}\cdot\nabla\phi$ $\varepsilon_{ijk}\varepsilon_{ilm} = \delta_{jl}\delta_{km} - \delta_{jm}\delta_{kl}$

identity (viii) $\nabla\times(\nabla\times\underline{u}) = \nabla(\nabla\cdot\underline{u}) - \nabla^2\underline{u}$

consider $[\nabla\times(\nabla\times\underline{u})]_i = \varepsilon_{ijk}\nabla_j(\nabla\times\underline{u})_k = \varepsilon_{ijk}\frac{\partial}{\partial x_j}(\varepsilon_{kmn}\nabla_m u_n) =$

$\varepsilon_{ijk}\varepsilon_{kmn}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_m}u_n\right) = \varepsilon_{kij}\varepsilon_{kmn}\frac{\partial}{\partial x_j}\left(\frac{\partial}{\partial x_m}u_n\right) = (\delta_{im}\delta_{jn} - \delta_{in}\delta_{jm})\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_m}u_n$ (2.20)

$\delta_{im}\delta_{jn}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_m}u_n - \delta_{in}\delta_{jm}\frac{\partial}{\partial x_j}\frac{\partial u_n}{\partial x_m}$ $\delta_{ij} = 1$ if $i = j$, $= 0$ otherwise

$= \delta_{ii}\delta_{jj}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_i}u_j - \delta_{ii}\delta_{jj}\frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}u_i = \frac{\partial}{\partial x_i}\frac{\partial}{\partial x_j}u_j - \frac{\partial}{\partial x_j}\frac{\partial}{\partial x_j}u_i$

$= \frac{\partial}{\partial x_i}(\nabla\cdot\underline{u}) - \nabla^2u_i = [\nabla(\nabla\cdot\underline{u}) - \nabla^2\underline{u}]_i$

Week 7 lecture notes: Z boundary layer eqns:

~~Bou-dar~~
Boundarylayers are thin fluid layers near to Z surface of a body in which viscous effects are important. Consider flow around an object of typical scale L. Far from Z object Z flow around an object of typical scale L. Far from Z object Z flow is uniform, $\underline{u} = U\underline{i}$ say. From U, L & Z ~~kinet~~ kinematic viscosity $\nu = \mu/\rho$ (which has dimensions L^2T^{-1}) we may construct Z dimensionless quantity $R_e = \frac{UL}{\nu}$ called Z Reynolds numb (prob 3.4). When $R_e >> 1$ viscous effects can generally be neglected, except in thin boundary layers adjacent to Z body & in Z downstream wake. aim to develop a theory that simplifies Z Navier-Stokes eqn to approx these thin layers. Key idea is $\frac{\text{thickness of boundary layer}}{\text{Lateral scale of boundary layer}} = \frac{\delta}{L} << 1$ we will use x as Z coords along Z boundary layer & y as Z coord perpendicular to Z layer.

It remains to determine $G(x)$: as we increase y & leave Z boundary layer we have $u\frac{\partial u}{\partial x} \approx \frac{G}{\rho} = -\frac{1}{\rho}\frac{\partial p}{\partial x} \Rightarrow p \approx -\frac{1}{2}\rho u^2 + C$ ∴ knowing Z velocity component $u(x)$ just outside Z boundary layer gives Z pressure $p(x)$ to be applied in (4.2).

Blasius obtained an exact sol to Z steady boundary layer eqn for a uniform flow (or stream) $\underline{u} = (U, 0)$ (where U is a const) over a semi-infinite plate given by $y = 0$ for $x > 0$. Downstream from Z tip $(0,0)$ of Z plate we seek a sol of Z steady boundary layer eqn (4.2)

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \frac{G}{\rho} + \nu\frac{\partial^2 u}{\partial y^2} \quad \text{with } u = \frac{\partial \psi}{\partial y}, \quad v = -\frac{\partial \psi}{\partial x}$$

numerical sols can be compared with Z series sols that was derived by Blasius

Def: • skin friction (prob 4.15)

• Drag force - acting on Z surface is obtained by integrating wrt x Z shear stress, ie up to Z ~~pt x~~ drag force is $F_D = \int_0^x \tau_0(\xi)\, d\xi$

• Displacement thickness: for large η it is found that $f(\eta) \approx \eta - \eta_0$ & ∴ $\psi \approx U\delta(x)[\eta - \eta_0] = U[y - \eta_0 \delta(x)] = U[y - \delta_0(x)]$ where $\delta_0(x)$ is called Z displacement thickness. It is Z amount by which a streamline in Z mainstream as $x \to -\infty$ is displaced for $x>0$.

Week 8 notes: Z Stokes limit incompressible, viscous flow satisfies Z Navier-Stokes eqn $\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p + \nu\nabla^2\underline{u}$, $\nabla\cdot\underline{u}=0$, $\nu = \mu/\rho$, plus associated boundary conditions (eg no-slip).

Suppose have a fluid flow of magnitude U on a lengthscale L, giving a turnover timescale $T=L/U$ $\therefore$ $u \sim U$, $\frac{\partial}{\partial t} = \frac{1}{T} = \frac{U}{L}$, $\nabla \sim \frac{1}{L}$

note: Z reynolds number may be small by virtue of: • large ν (eg treacle, honey) • small U (slow flow) • small L (eg flow on tiny scales: bacteria, thin films)

planar flow: consider Z two dimensional planar flow

$\underline{u} = (u(x,y,t), v(x,y,t), 0)$, $\nabla\cdot\underline{u}=0$ we introduced a streamfunc $\psi(x,y,t)$ with $\underline{u} = \nabla\times(\psi\underline{k}) = (\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0)$. then $\nabla\cdot\underline{u}=0$ is satisfied automatically & Z Stokes eqn reads $-\nabla p + \mu\nabla^2(\nabla\times(\psi\underline{k}))=0$

Meridional flow: $\underline{u} = u(r,\theta,t)\hat{\underline{r}} + v(r,\theta,t)\hat{\underline{\theta}}$ in spherical polar coords (r,θ,ϕ). can satisfy Z incompressibility condition $\nabla\cdot\underline{u}=0$ automatically by introducing Z Stokes streamfunc $\psi = \psi(r,\theta,t)$ with $\underline{u} = \nabla\times\left(\frac{\psi}{r\sin\theta}\hat{\underline{\phi}}\right)$ (5.3)

Uniqueness of Stokes Flow suppose solve Stokes eqns $0 = -\nabla p + \mu\nabla^2\underline{u}$, $\nabla\cdot\underline{u}=0$ in Z volume V with Z boundary condition $\underline{u} = \underline{G}(\underline{x})$ on Z surface S given. Z sol is unique proved in a series of steps.

if we create a Stokes flow by, say, moving boundaries & then reverse Z motion Z whole fluid flow reverses. Z sol $\underline{u}(\underline{x})$ for $\underline{G}(\underline{x})$ on S gives a sol $-\underline{u}(\underline{x})$ for $-\underline{G}(\underline{x})$ on S. this fails to be true for Z Navier-Stokes eqn, but in Z Stokes regime fluid flows are reversible.

Week 9 notes: flow around a sphere consider low Reynolds number, axisymmetric flow around a sphere of radius a, centre Z origin.

• Far from Z sphere Z flow is uniform $\underline{u} \approx U\hat{\underline{z}}$ nearer to Z sphere we use spherical polar coords (r,θ,ϕ) & write Z velocity in terms of Z Stokes streamfunc $\psi(r,\theta)$ $\underline{u} = \nabla\times\left(\frac{\psi}{r\sin\theta}\hat{\underline{\phi}}\right) = u\hat{\underline{r}} + v\hat{\underline{\theta}}$ (5.8) as shown in §5.3 we are required to solve eqn (5.7) $E^4\psi = 0$ (5.9)

Boundy Conditions on Z spherical surface we require Z no slip conditions

$u = v = 0 \Rightarrow \frac{\partial\psi}{\partial\theta} = \frac{\partial\psi}{\partial r}$ on $r = a$.

Force on Z sphere arises from pressure & viscous forces. we recall that considering a small element of Z surface dS with unit normal $\hat{\underline{n}}$, Z i^{th} component of Z force on Z surface element is

$dF_i = \sigma_{ij}\hat{n}_j\, dS = \left[-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]\hat{n}_j\, dS$ where σ_{ij} is Z Stress tensor. ∴ need to find Z pressure p.

returning to Z force, & ignoring for Z moment Z surface area element dS, consider $t_i = \sigma_{ij}\hat{n}_j = \left[-p\delta_{ij} + \mu\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right)\right]\hat{n}_j$

\week11 lec/ recap: vorticity Dynamics $\underline{\omega} = \nabla\times\underline{u}$

take Z curl of N-S $\frac{\partial\underline{\omega}}{\partial t} = \nabla\times(\underline{u}\times\underline{\omega}) + \nu\nabla^2\underline{\omega}$

incompressible flow $\nabla\cdot\underline{u} = 0$ then identity vi: $\nabla\cdot\underline{\omega} = \nabla\cdot(\nabla\times\underline{u}) \equiv 0$

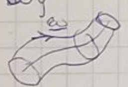
$\frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\nabla\underline{u} + \nu\nabla^2\underline{\omega}$ material derivative = following Z fluid ~~motion~~

inviscid 2D case: $\frac{D\omega}{Dt} = 0$ ∴ $\omega=1$, $\omega=0$, $t=0$ → $\omega=1$, $\omega=0$ at later times

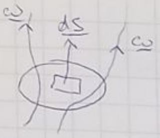
vortex line $\frac{\partial \underline{r}}{\partial s} = \underline{\omega}$ $\frac{\partial x}{\partial s} = \omega_x$ $\frac{\partial y}{\partial s} = \omega_y$ $\frac{\partial z}{\partial s} = \omega_z$

$\frac{\partial x}{\omega_x} = \frac{\partial y}{\omega_y} = \frac{\partial z}{\omega_y}$ vortex lines

\Act 6.6/



strength of tube $= \int_S \underline{\omega}\cdot d\underline{S}$

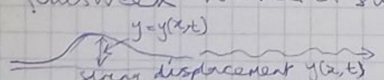


$= \int_S (\nabla\times\underline{u})\cdot d\underline{S} = \int_C \underline{u}\cdot d\underline{r}$ {by Stokes thm} $= \Gamma$ circulation

$\frac{D}{Dt}(d\underline{r}) = d\underline{r}\cdot\nabla\underline{u}$ material element $d\underline{r}$

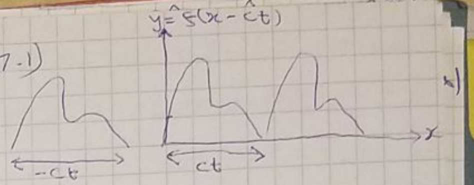
\Kelvin's vorticity thm:/ $\nu = 0$ $\frac{D}{Dt}\Gamma = 0$ $\Gamma = \oint_C \underline{u}\cdot d\underline{r}$ ∴ const forcing Z fluid motion

\this week: waves in fluids./ (sound waves in compressible fluids)



$\underline{F} = m\underline{a} \Longrightarrow \frac{\partial^2 y}{\partial t^2} = c^2\frac{\partial^2 y}{\partial x^2}$ 1 dimension (1D) wave eqn

$c = \sqrt{T/\rho}$ T is how tight ρ is string density d'Alembert's sol $y = f(x-ct) + g(x+ct)$ (prob 7.1)



take f & g to be periodic funcs eg

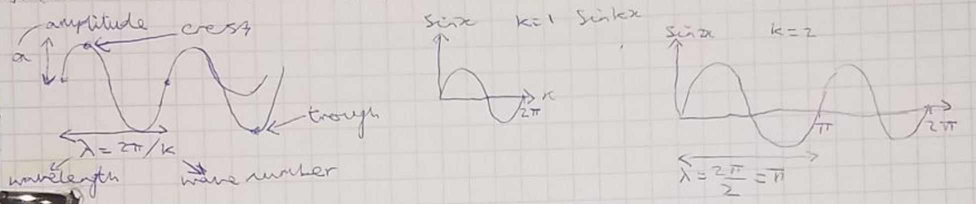
i) $f = \sin(x - ct)$ $\quad g = \cos(x + ct)$

$e^{ikx} = \cos kx + i\sin kx \qquad \frac{d}{dx}(e^{ikx}) = e^{ikx}\, ik$

$(e^{ax})^3 e^{bx} = e^{3ax+bx}$ easier than $\cos^3(ax)\sin(bx)$

take the real part $\mathbb{R}(e^{ikx}) = \mathbb{R}(\cos kx + i\sin kx) = \cos kx$

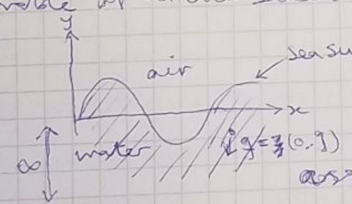
$\mathbb{R}\left(\frac{d}{dx}(e^{ikx})\right) = \mathbb{R}(ik e^{ikx}) = \mathbb{R}(ik(\cos kx + i\sin kx)) = -k\sin x = \frac{d}{dx}(\mathbb{R}(e^{ikx}))$ ✓



3d wave eqn $\frac{\partial^2 \beta}{\partial t^2} = c^2 \nabla^2 \beta \qquad \beta = a e^{i(\underline{k}\cdot \underline{r} - \omega t)} \qquad \underline{r} = (x, y, t) \quad \underline{k} = (k_1, k_2, k_3)$

$\beta = a e^{i(k_1 x + k_2 y + k_3 z - \omega t)}$ wave propagates along $\underline{k}$ $\nearrow \underline{k}$ (fixed bit)

problem at an interface of air & water (works in the sea)



$y = \eta(x,t)$ consider 2D domain (no dependence on z) with infinite depth for simplicity

assume the fluid is irrotational & inviscid - write $\underline{u} = \nabla\phi$

η & ϕ are unknown $\quad \therefore \quad u = \nabla\phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, 0\right) \quad \nabla\cdot\underline{u} = 0 \Rightarrow$

$\boxed{\nabla^2\phi = 0}$

and boundary conditions: to be satisfied on $y = \eta(x,t)$

(i) kinematic cond - surface moves with the flow $F(x,y,t) = y - \eta(x,t)$:

$\frac{DF}{Dt} = 0 \quad \frac{dF}{dt} + u\cdot\nabla F = 0 \longrightarrow \boxed{\text{eqn for } \eta(x,t)}$

(ii) Dynamic condition $\partial_t \underline{u} + \nabla H = \underline{u} \times \underline{\xi} \qquad \underline{\xi} = \nabla \times \underline{u}$ vorticity

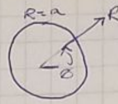
$H = \frac{p}{\rho} + \frac{1}{2}|\underline{u}|^2 + \underset{\text{Bernoulli's func}}{\Pi} \qquad \Pi = \rho g y$ on the interface $y = \eta \qquad \underline{u} = (u, v, 0)$

$\rightarrow \boxed{\frac{\partial\phi}{\partial t} + \frac{1}{2}(u^2+v^2)^2 + g\eta = 0 \text{ on } y = \eta}$ (prob 7.7)

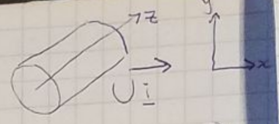
- Simplify by assuming small amplitude waves & linearise the eqns

solve to find wavelike sols

\ 1 week 11 tutorial / \ week 8 sheet / $\underline{U\,i} \rightarrow$



\ 1 a /



N-S $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \rho\underline{g} + \mu\nabla^2\underline{u}$

for $R_e = \frac{UL}{\nu} \ll 1$ $\{\nu = \mu/\rho\}$ then inertia << viscous terms ∴

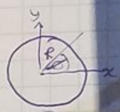
$0 = -\nabla p + \mu^2 \nabla \underline{u}$ Stokes eqn / slow flow eqn (5.1)

$\left(\frac{\partial^2}{\partial R^2} + \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R^2}\frac{\partial^2}{\partial^2\phi^2}\right)^2 \psi = 0$ ψ stream func $\underline{u} = \nabla \times (\psi \hat{\underline{z}})$

\ 1 b / $\underline{u} = \underline{0}$ on $R = a$ ∴ $\underline{u} = \nabla \times (\psi\hat{\underline{z}}) = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & R\cdot 0 & \psi \end{vmatrix} =$

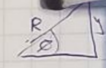
$\frac{1}{R}\left\{\hat{\underline{R}}\frac{\partial\psi}{\partial\phi} - R\hat{\underline{\phi}}\frac{\partial\psi}{\partial R} + \hat{\underline{z}}\cdot 0\right\}$ ∴ $\cancel{u_R = \frac{1}{R}\left|\frac{\partial\psi}{\partial\phi}\right|}$ $u_R = \frac{1}{R}\frac{\partial\psi}{\partial\phi}\Big|_{R=a} = 0$ & $u_\phi = -\frac{\partial\psi}{\partial R}\Big|_{R=a} = 0$

if it doesnt talk about gravity you can probably ignore it



at large R, $\underline{u} \rightarrow U\underline{i} = U$ from 2 exam formula: $\hat{\underline{R}} = \cos\phi\,\underline{i} + \sin\phi\,\underline{j}$

$\hat{\underline{\phi}} = -\sin\phi\,\underline{i} + \cos\phi\,\underline{j}$ ∴ $\cos\phi\,\hat{\underline{R}} - \sin\phi\,\hat{\underline{\phi}} = \underline{i}$ ∴



$\underline{u} \rightarrow U(\cos\phi\,\hat{\underline{R}} - \sin\phi\,\hat{\underline{\phi}})$ at large R

$\cos\phi = \frac{x}{R}$
$\sin\phi = \frac{y}{R}$

$u_R = U\cos\phi = \frac{1}{R}\frac{\partial\psi}{\partial\phi}$ ①, $u_\phi = -U\sin\phi = -\frac{\partial\psi}{\partial R}$ ② ∴

integrate ② wrt R: $\psi = \int U\sin\phi\, dR = U\sin\phi\, R + g(\phi)$ ∴

Diff wrt ϕ: $\frac{\partial\psi}{\partial\phi} = UR\cos\phi + \frac{dg}{d\phi} = RU\cos\phi$ by ① ∴

$\frac{dg}{d\phi} = 0$ ∴ g = constant ∴ $\psi = U\sin\phi + C \longrightarrow UR\sin\phi$ at large R

\ 1 c / consider $\psi = f(R)\sin\phi$ $f(R) = R^m$ ∴ $\nabla^4\psi = \nabla^2(\nabla^2\psi)$ ∴

$\nabla^2\psi = \frac{\partial^2}{\partial R^2}\psi + \frac{1}{R}\frac{\partial\psi}{\partial R} + \frac{1}{R^2}\frac{\partial^2\psi}{\partial\phi^2}$ ∴ $\frac{\partial\psi}{\partial R} = f'\sin\phi$ $f' = \frac{df}{dr}$ ∴

$\frac{1}{R}\frac{\partial\psi}{\partial R} = f'\frac{\sin\phi}{R}$ ∴ $\frac{\partial^2\psi}{\partial R^2} = \frac{\partial}{\partial R}\left(\frac{\partial\psi}{\partial R}\right) = \frac{\partial}{\partial R}(f'\sin\phi) = \sin\phi\, f''$

$\frac{\partial^2\psi}{\partial\phi^2} = \frac{\partial}{\partial\phi}\left(\frac{\partial\psi}{\partial\phi}\right) = \frac{\partial}{\partial\phi}(f\cos\phi) = -f\cos\phi$ ∴

$\nabla^2\psi = \sin\phi\, f'' + \frac{f'}{R}\sin\phi + \frac{1}{R^2}(-f\sin\phi) = \sin\phi\left[f'' + \frac{f'}{R} - \frac{f}{R^2}\right]$; $f' = mR^{m-1}$, $f'' = m(m-1)R^{m-2}$

$\sin\phi\left[m(m-1)R^{m-2} + mR^{m-1-1} - R^{m-2}\right] = \sin\phi R^{m-2}\left[m(m-1) + m - 1\right] = \sin\phi R^{m-2}(m^2-1) = \nabla^2\psi$

require $\nabla^4\psi = \nabla^2[\nabla^2\psi] = \nabla^2\left[\sin\phi R^{m-2}(m^2-1)\right] = (m^2-1)\nabla^2\left[\sin\phi R^{m-2}\right]$ ∴

set $m-2 = p$ ∴ $\nabla^2\left[\sin\phi R^p\right] = \sin\phi R^{p-2}(p^2-1)$ ∴

$\nabla^4\psi = (m^2-1)\nabla^2\left[\sin\phi R^{m-2}\right] = (m^2-1)\sin\phi R^{m-2-2}\left((m-2)^2-1\right) =$

$\sin\phi R^{m-4}(m^2-1)((m-2)^2-1)$ {using similar form to 2 sol above} ∴

$\nabla^4\psi = 0 \Rightarrow (m^2-1)\sin\phi R^{m-4}\underbrace{(m^2+3-4m)}^{(m-3)(m-1)} = 0$ ∴ $m = \pm 1$ or $m = 3, 1$ ∴

$f(R) = AR^1 + BR^{-1} + CR^3 + DR\ln R$ {since repeated m=1 ∴ $y'' + ay' + by = 0$ $y = e^{px} \rightarrow$

$p^2 + ap + b = 0$ ∴ $p = p_1$ or $p = p_2$ $y = Ae^{p_1 x} + Be^{p_2 x}$ is repeated $p_1 = p_2$: $y = Ae^{p_1 x} + Bxe^{p_1 x}$ ∴

using variation of params: given a sol $y_1(x)$, find a second sol by writing $y = y_1(x)u(x) \rightarrow$ ODE for $u(x)$ ... find $y'' + ay' + by = 0 \therefore u(x) = x$

$\therefore$ do similar for $f(R)$ $\}$

\ii d/ expect $\psi = UR\sin\phi$ at large R

$= (AR + \frac{B}{R} + CR^3 + DR\ln R)\sin\phi$ $\quad\therefore AR + \frac{B}{R} + CR^3 + DR\ln R = UR$

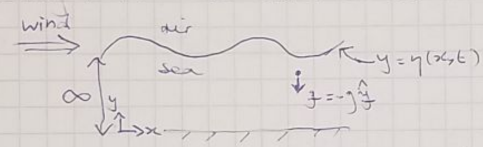
$\therefore$ at large R: $\frac{B}{R} \rightarrow 0$ & $AR << CR^3, DR\ln R$ $\therefore$

so as $R \rightarrow \infty$ R^3 & $R\ln R >> R$ $\therefore$ must set $C = D = 0$ & take $A = U$

$\therefore \psi = (UR + \frac{B}{R})\sin\phi$ & boundary conditions $\frac{\partial\psi}{\partial\phi} = 0$ on $R = a$ & $\frac{\partial\psi}{\partial R} = 0$ on $R = a$

$\therefore$ these are satisfied for different B $\therefore$ ✗

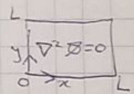
considering waves at Z surface of an infinitely deep ocean



simplicity: 2D geometry (2 dimensional) $\underline{u} = (u, v, 0)$, incompressible,

invisid $\nu = 0$. irrotational $\underline{u} = \nabla\phi$

$\nabla \cdot \underline{u} = 0 \Rightarrow \nabla \cdot \nabla\phi = 0 \Rightarrow \nabla^2\phi = 0$ Satisfies Laplaces eqn



what happen on $y = \eta$?

\Activity 7.2/ $\beta = \Re[Ae^{ik(x-ct)}] = \Re[A\{\cos k(x-ct) + i\sin k(x-ct)\}]$ $\quad \theta = x - ct$

$= \Re[A\cos k\theta + iA\sin k\theta] = A\cos k\theta$ where A is $\mathbb{R}$

for $A \in \mathbb{C}$, write $A = A_R + iA_I$ & sub in (*)

$\beta = \Re[(A_R + iA_I)[\cos k\theta + i\sin\theta]] = \Re[(A_R\cos k\theta - A_I\sin k\theta) + i(A_I\cos k\theta + A_R\sin k\theta)]$

$= A_R\cos k\theta - A_I\sin k\theta$ $\therefore$ want $A_R = 0$, $A_I = -1$ $\therefore$ $A = A_R + iA_I = -i$

what happens on $y = \eta$? ① <u>kinematic condition</u> $\frac{D}{Dt}(y - \eta) = 0$ particle on Z surface moves with Z flow

<u>alternatively</u>: Z normal velocity of a particle on Z surface is Z normal velocity of Z surface $y_p = \eta(x_p, t)$ (x_p, y_p) position of Z fluid parcel

for a small motion $y_p + \delta y_p = \eta(x_p + \delta x_p, t + \delta t) \approx \eta(x_p, t) + \frac{\partial\eta}{\partial x}\delta x_p + \frac{\partial\eta}{\partial t}\delta t + \ldots$ (taylor expansion)

$\{y_p = \eta(x_p, t) \ldots\} \Rightarrow \frac{\delta y_p}{\delta t} \approx \frac{\partial\eta}{\partial x}\frac{\delta x_p}{\delta t} + \frac{\partial\eta}{\partial t} \Rightarrow v \approx \frac{\partial\eta}{\partial x}u + \frac{\partial\eta}{\partial t}$ on $y = \eta$ (7.6)

2 hour exam in May

② Dynamic condition - pressure is continuous across the free surface

N-S: $\overbrace{\nabla(\partial_t \phi + H)}^{\partial_t \underline{u}} = 0 \qquad H = \frac{1}{2}|\underline{u}|^2 + \frac{p}{\rho} + gy \implies$

$\partial_t \phi + H = f(t)$ ⊛ how is $f(t)$ arbitrary? Consider $\phi \to \phi' = \phi + \int^t f(s)\,ds$

note that $\underline{u} = \nabla\phi$ doesn't change

then $\partial_t \phi \to \partial_t \phi' = \partial_t \phi + \frac{\partial}{\partial t}\int^t f(s)\,ds = \partial_t \phi + f(t)$

$\therefore$ eqn ⊛ becomes $\partial_t \phi + f(t) + H = f(t)$ Choose $f(t) = \frac{p}{\rho}$ i.e. $f(t)$

doesn't matter so this makes it simple

LN: chose $f(t) = \frac{p}{\rho}$ at $y = \eta$ $\therefore f(t) = \frac{P_{atmospheric}}{\rho} = \frac{P_{atm}}{\rho}$ {prob 7.7}

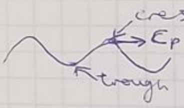
$\implies \underbrace{\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2+v^2) + g\eta = 0 \text{ on } y = \eta}$ (prob 7.7) (7.9)

Activity 7.9/ $\frac{\partial \phi}{\partial t} + \frac{1}{2}(u^2+v^2) + g\eta = 0$ on $y = \eta$ (u^2+v^2) small $\therefore$ neglected

use the fact that η is small $\left.\frac{\partial \phi}{\partial t}\right|_{y=\eta} \approx \left.\frac{\partial \phi}{\partial t}\right|_{y=0} + \frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial t}\right)\eta + \frac{\partial^2}{\partial y^2}\left(\frac{\partial \phi}{\partial t}\right)\frac{\eta^2}{2} + \dots$

$\{ f(a+h) \approx f(a) + h f'(a) + \frac{h^2}{2} f''(a) + \dots \}$ $\therefore$

$\left.\frac{\partial \phi}{\partial t}\right|_{y=\eta} \approx \left.\frac{\partial \phi}{\partial t}\right|_{y=0}$ when linearised $\left.\frac{\partial \phi}{\partial t}\right|_{y=0} + g\eta = 0$ (7.11)



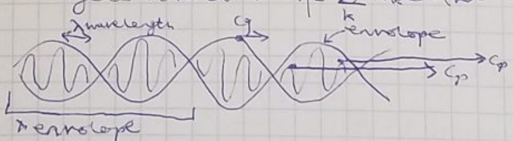
(prob 7.10) $\to \underset{\text{frequency}}{\omega} = \pm\sqrt{g\underset{\text{wavenumber}}{k}}$ dispersion relation

there are two different wave speeds: phase speed: $c_p = \frac{\omega}{k} = \frac{\pm\sqrt{g}}{\sqrt{k}} = c_p$

group speed: $c_g = \frac{d\omega}{dk}$ if $\omega = ak$ then $c_p = \frac{\omega}{k} = a = \frac{d\omega}{dk} = c_g$

$\eta = A\cos(kx - \omega t)$ general relation $\eta = \sum_k A_k \cos(kx - \omega t)$ (fourier cosine series)

wave packet

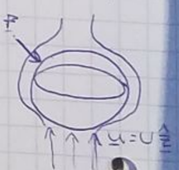


dispersive waves $c_p \neq c_g$

Week 12 help session/

Activity 5.16/ stokes flow / slow flow $Re \ll 1$ $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \mu\nabla^2\underline{u}$

$\implies -\nabla p + \mu\nabla^2\underline{u} = 0$



Cartesian $\underline{u} = \nabla \times (\psi\hat{\underline{z}})$ $\hat{\underline{z}} = \underline{k} = (0,0,1)$

Spherical $\underline{u} = \nabla \times \left(\frac{\psi}{r\sin\theta}\hat{\underline{\phi}}\right)$ (5.8)

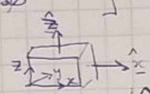
Stokes eqn: Cartesian $\to \nabla^4\psi = 0$ Spherical $\to E^4\psi = 0$

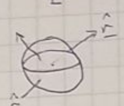
from (prob 5.15) $\psi = \frac{Ua^2}{4}\left[\frac{a}{r} - \frac{3r}{a} + \frac{2r^2}{a^2}\right]\sin^2\theta$

$\to \underline{u} = u\hat{\underline{r}} + v\hat{\underline{\theta}} + \cancel{(w\hat{\underline{\phi}})}$

Activity 5.20 $\hat{\underline{n}}\times(\nabla\times\underline{u})$ spherical polars $\nabla\times\underline{u} = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\hat{\underline{\phi}} \\ \partial_r & \partial_\theta & \partial_\phi \\ u_r & ru_\theta & r\sin\theta u_\phi \end{vmatrix}$ a)

$$= \frac{1}{r^2\sin\theta}\left\{\hat{\underline{r}}\left[\frac{\partial}{\partial\theta}(r\sin\theta u_\phi) - \frac{\partial}{\partial\phi}(ru_\theta)\right] - r\hat{\underline{\theta}}\left[\frac{\partial}{\partial r}[r\sin\theta u_\phi] - \frac{\partial u_r}{\partial\phi}\right] + r\sin\theta\hat{\underline{\phi}}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial}{\partial\theta}u_r\right]\right\}$$





$$= \underline{\omega} = \omega_r\hat{\underline{r}} + \omega_\theta\hat{\underline{\theta}} + \omega_\phi\hat{\underline{\phi}}$$

$\therefore \hat{\underline{n}}\times(\nabla\times\underline{u}) = \hat{\underline{r}}\times(\nabla\times\underline{u})$ ← normal to the sphere $= \begin{vmatrix} \hat{\underline{r}} & \hat{\underline{\theta}} & \hat{\underline{\phi}} \\ 1 & 0 & 0 \\ \omega_r & \omega_\theta & \omega_\phi \end{vmatrix} =$

$$\hat{\underline{r}}\cdot 0 - \hat{\underline{\theta}}\omega_\phi + \hat{\underline{\phi}}\omega_\theta = -\omega_\phi\hat{\underline{\theta}} + \omega_\theta\hat{\underline{\phi}}$$

$$= -\frac{1}{r^2\sin\theta}r\sin\theta\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right]\hat{\underline{\theta}} + \frac{1}{r^2\sin\theta}(-r)\left[\frac{\partial}{\partial r}(r\sin\theta u_\phi) - \frac{\partial u_r}{\partial\phi}\right]\hat{\underline{\phi}} =$$

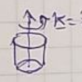
$$-\frac{1}{r}\left[\frac{\partial}{\partial r}(ru_\theta) - \frac{\partial u_r}{\partial\theta}\right]\hat{\underline{\theta}} - \frac{1}{r\sin\theta}\left[\frac{\partial}{\partial r}(r\sin\theta u_\phi) - \frac{\partial}{\partial\phi}u_r\right]\hat{\underline{\phi}}$$ ✓ (5.10)

Activity 6.2

Vorticity Dynamics $\underset{\text{vorticity}}{\underline{\omega}} = \nabla\times\underline{u}$ ← linear velocity

angular velocity $\underline{\Omega}$:

$\underline{u} = \underline{\Omega} = \underline{r}$ $\quad \underline{r} = (x, y, z)$ $\quad \underline{r} = r\hat{\underline{r}}$ sphere $\therefore$ $\underline{\Omega} = \Omega\underline{k}$ (speed of rotation; axis of rotation), $\Omega = \text{const}$




$\underline{k} = \hat{\underline{z}}$

$$\underline{u} = \underline{\Omega}\times\underline{r} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ 0 & 0 & \Omega \\ x & y & z \end{vmatrix} = \underline{i}(-\Omega y) - \underline{j}(-\Omega x) + \underline{k}\cdot 0$$

$$= \Omega(-y, x, 0)$$

$$\underline{\omega} = \nabla\times\underline{u} = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \partial_x & \partial_y & \partial_z \\ -\Omega y & \Omega x & 0 \end{vmatrix} = \Omega[\underline{i}0 - 0\underline{j} + \underline{k}(1+1)] = 2\Omega\underline{k} = 2\underline{\Omega}$$

Week 11 sheet waves 2 $\beta = ae^{i(\underline{k}\cdot\underline{r} - \omega t)}$ $\quad a = \text{const}$ $\quad \underline{k} = (k_1, k_2, k_3)$

$\underline{r} = x\hat{\underline{x}} + y\hat{\underline{y}} + z\hat{\underline{z}}$ $\qquad \frac{\partial^2\beta}{\partial t^2} = c\nabla^2\beta$ $\qquad$ note $\frac{\partial\beta}{\partial t} = a\frac{\partial}{\partial t}(e^{i(\,)}) = ae^{i(\,)}(-i\omega) = -i\omega\beta$

$$\therefore \frac{\partial^2\beta}{\partial t^2} = \frac{\partial}{\partial t}(-i\omega\beta) = -i\omega\frac{\partial\beta}{\partial t} = (-i\omega)^2\beta = -\omega^2\beta$$

$\nabla^2\beta = \frac{\partial^2\beta}{\partial x^2} + \frac{\partial^2\beta}{\partial y^2} + \frac{\partial^2\beta}{\partial z^2}$ $\qquad$ consider: $\frac{\partial\beta}{\partial x} = ae^{i(\,)}\frac{\partial}{\partial x}\Big[i(\overbrace{k_1x + k_2y + k_3z}^{\underline{k}\cdot\underline{r}} - \omega t)\Big]$

$$= ae^{i(\,)}ik_1 = ik_1\beta \quad \therefore \quad \frac{\partial^2\beta}{\partial x^2} = \frac{\partial}{\partial x}(ik_1\beta) = ik_1\frac{\partial\beta}{\partial x}\,(ik_1)^2\beta = -k_1^2\beta$$

similarly: $\frac{\partial^2\beta}{\partial y^2} = -k_2^2\beta$ & $\frac{\partial^2\beta}{\partial z^2} = -k_3^2\beta$ $\quad \therefore \nabla^2\beta = (-k_1^2 - k_2^2 - k_3^2)\beta = -k^2\beta$,

$k = |\underline{k}|$ $\quad\therefore$ Sub back into wave eqn: $-\omega^2\beta = c^2(-k^2\beta) \Rightarrow$

$\omega^2 = c^2k^2 \Rightarrow \underset{\text{frequency}}{\omega} = \pm c\underset{\text{wave number}}{k}$ is the dispersion relation

phase speed $c_p = \frac{\omega}{k} = \pm c$ $\qquad$ group speed $c_g = \frac{\partial\omega}{\partial k} = \pm c$ $\quad\therefore c_p = c_g \;\therefore$

not dispersive

\Week 10 Notes/ Vorticity Dynamics. NS-eqn $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla \underline{u}\right) =$
$-\nabla p + \rho \underline{g} + \mu \nabla^2 \underline{u} = \rho \frac{D}{Dt}\underline{u} = \rho\left(\frac{\partial}{\partial t} + \underline{u}\cdot\nabla\right)\underline{u}$

if write $\underline{g} = -\nabla \Pi$ for some gravitational potential Π (e.g. in cartesian coords if $g = (0, 0, -g)$ then $\Pi = gz$) then have

$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\nabla P + \nu\nabla^2\underline{u}$, $P = p/\rho + \Pi$ & $\nu = \mu/\rho$ is Z kinematic viscosity

written $\frac{\partial \underline{u}}{\partial t} = \underline{u}\times\underline{\omega} - \nabla\left(P + \frac{1}{2}|\underline{u}|^2\right) - \nu\nabla\times\underline{\omega}$, $\nabla\cdot\underline{u} = 0$ (6.1) $\because$

$\underline{u}\times\underline{\omega} = \underline{u}\times(\nabla\times\underline{u}) = \nabla\left(\frac{1}{2}|\underline{u}|^2\right) - \underline{u}\cdot\nabla\underline{u}$ & $\nabla\times\underline{\omega} = \nabla(\nabla\cdot\underline{u}) - \nabla^2\underline{u}$

then use identities formula

Recall Z deformation Matrix $D_{ij} = \frac{\partial u_i}{\partial x_j} = e_{ij} - \xi_{ij}$

in Z vorticity eqn, Z term $\underline{\omega}\cdot\nabla\underline{u}$ has ith component $(\underline{\omega}\cdot\nabla\underline{u})_i = \underline{\omega}\cdot\nabla u_i =$
$\omega_j \frac{\partial u_i}{\partial x_j} = \omega_j D_{ij}$ & $\therefore$ it is Z deformation of fluid elements that gives this source or sink term.

· Z vorticity eqn looks compact but we need $\underline{u}$ to solve it, & we only have $\underline{\omega} = \nabla\times\underline{u}$. Sols involves a 'Biot Savart' integral

$\underline{u}(x,t) = -\frac{1}{4\pi}\int_{V'} \frac{(x - x')\times\underline{\omega}(\underline{x}', t)}{|x - x'|^3}\, dV'$ $\therefore$ $\underline{u}$ is not locally related to $\underline{\omega}$. this is Z penalty for eliminating preassure (which itself is non local) earlier.

If $\nu = 0$ then $D\underline{\omega}/Dt = 0$ & vorticity 'follows fluid particles' in particular, if $\underline{\omega} = 0$ initially it means so for all times. Some numerical codes for inviscid fluid dynamics are based on such 'contour dynamics'

If $\nu \neq 0$ Z viscous term can both destroy & create vorticity, such as in boundary layers

\Inviscid vorticity dynamics in 3D/ here consider inviscid flow (ie $\nu = 0$) in three dimensions with a view towards understanding basic processes. this has some relevance to many flows with large Reynolds number $R_e = UL/\nu$

$\therefore$ consider $\frac{D\underline{\omega}}{Dt} = \frac{\partial \underline{\omega}}{\partial t} + \underline{u}\cdot\nabla\underline{\omega} = \underline{\omega}\cdot\nabla\underline{u}$

\Def 6.4/ at a fixed time t, a <u>vortex</u> line is a curve which is everywhere tangent to Z local vorticity vector such a curve is