by part (a) $T_R|\lambda \sim \text{Gamma}(2,\lambda)$. The required probability is a posterior predictive probability

$$p(t_R \le 7 \mid T_p = 7) = \int_0^7 \int_0^\infty p(t_R|\lambda)\,\pi(\lambda|T_p=7)\,d\lambda\,dt_R =$$

$$\int_0^7 \int_0^\infty \lambda^2 t_R e^{-\lambda t_R} \frac{(\beta+7)^{\alpha+2}}{\Gamma(\alpha+2)} \lambda^{\alpha+1} e^{-\lambda(\beta+7)}\,d\lambda\,dT_R =$$

$$\int_0^7 \frac{t_R(\beta+7)^{\alpha+2}\Gamma(\alpha+4)}{\Gamma(\alpha+2)(t_R+\beta+7)^{\alpha+4}} \int_0^\infty \frac{(t_R+\beta+7)^{\alpha+4}}{\Gamma(\alpha+4)} \lambda^{\alpha+3} e^{-\lambda(t_R+\beta+7)}\,d\lambda\,dt_R =$$

$$\int_0^7 \frac{t_R(\beta+7)^{\alpha+2}\Gamma(\alpha+4)}{\Gamma(\alpha+2)(t_R+\beta+7)^{\alpha+4}}\,dt_R \quad , \quad \alpha=\beta=1$$

$$= \int_0^7 \frac{t_R(1+7)^{1+2}\Gamma(1+4)}{\Gamma(1+2)(t_R+1+7)^{1+4}}\,dt_R = \frac{8^3\,4!}{2!} \int_0^7 t_R \frac{1}{(t_R+8)^5}\,dt_R =$$

$$6144 \int_0^7 t_R (t_R+8)^{-5}\,dt_R = 6144\left(\left[t_R \frac{1}{-4}(t_R+8)^{-4}\right]_0^7 - \int_0^7 1\left(\frac{1}{-4}\right)(t_R+8)^{-4}\,dt_R\right) =$$

$$6144\left(\left[7\left(\frac{1}{-4}\right)(7+8)^{-4}\right] + \frac{1}{4}\left[-\frac{1}{3}(t_R+8)^{-3}\right]_0^7\right) =$$

$$6144\left(-\frac{7}{4(15^4)} + \frac{1}{4}\left[-\frac{1}{3(7+8)^3}\right]\right) = 0.636$$

\3a/ $y_{ij}$ is data layer $\theta$ form the process layer
the prior layer is $\sigma, \mu, \tau$ The rest of the parameters are not treated as random quantities

\3b/ by bayes theorem: $\pi(\mu|\theta,\sigma^2,\tau^2,y) \propto \pi(\mu) f(y,\theta,\sigma^2,\tau^2|\mu) \propto$

~~f π(μ)~~ $\pi(\mu) f(y|\mu,\theta,\sigma^2,\tau^2)\pi(\theta,\sigma^2,\tau^2|\mu) \propto$

$\pi(\mu|\theta,\sigma^2,\tau^2) f(y|,\mu,\theta,\sigma^2,\tau^2) \propto$

$\pi(\mu|\theta,\sigma^2,\tau^2) f(y|\theta,\sigma^2) \propto$

$\cancel{\pi(\mu|\tau^2) f(y|\theta,\sigma^2)}$ $\pi(\mu|\tau^2) f(y|\theta,\sigma^2)$ ∴

$y_{ij}|\theta_j,\sigma^2 \sim N(\theta_j,\sigma^2)$ ∴ $y_i|\theta,\sigma^2 \sim N(\theta,\sigma^2)$ ∴

$$f(y_i|\theta,\sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(y_i-\theta)^2}$$

$\cancel{\pi(\mu|\tau^2)}$ $\mu|\tau^2 \sim N(\mu_0,\tau_0^2)$ ∴ $\pi(\mu|\tau^2) = \frac{1}{\sqrt{2\pi}\,\tau_0} e^{-\frac{1}{2\tau_0^2}(\mu-\mu_0)^2}$ ∴

$\pi(\mu|\theta,\sigma^2,\tau^2,y) \propto \pi(\mu|\tau^2) f(y|\theta,\sigma^2) \propto$

$$\pi(\mu|\tau^2) \prod_{i=1}^n \left(\frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(y_i-\theta)^2}\right) \propto \pi(\mu|\tau^2)\sigma^{-n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n (y_i^2+\theta^2-2y_i\theta)}$$

$$\propto \pi(\mu|\tau^2)\sigma^{-n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^n(\theta^2-2y_i\theta)} \propto \pi(\mu|\tau^2)\sigma^{-n} e^{-\frac{1}{2\sigma^2}(n\theta^2-2n\bar{y}\theta)} \propto$$

$$\frac{1}{\sqrt{2\pi}\,\tau_0} e^{-\frac{1}{2\tau_0^2}(\mu-\mu_0)^2} \sigma^{-n} e^{-\frac{1}{2\sigma^2}(n\theta^2-2n\bar{y}\theta)} \propto e^{-\frac{1}{2\tau_0^2}\mu^2 + \frac{1}{2\tau_0^2}2\mu_0\mu} \sigma^{-n} e^{-\frac{1}{2\sigma^2}(n\theta^2-2n\bar{y}\theta)}$$

\PP2021/ $\propto e^{-\frac{1}{2\tau_0^2}\mu^2 + \frac{1}{2\tau_0^2}2\mu_0\mu - \frac{1}{2\sigma^2}n\theta^2 + \frac{1}{2\sigma^2}2n\bar{y}\theta} \propto$

$e^{-\frac{1}{2}\left[\mu^2\left(\frac{1}{\tau_0^2}\right) - \frac{2\mu_0}{\tau_0^2}\mu\right]} \propto e^{-\frac{1}{2}[\mu^2 a - 2b\mu]} \propto$

$e^{-\frac{1}{2}a\left[\mu^2 - \frac{2b}{a}\mu\right]} \propto e^{-\frac{1}{2a^{-1}}\left[\left(\mu - \frac{b}{a}\right)^2 - \frac{b^2}{a^2}\right]} \propto$

$e^{-\frac{1}{2a^{-1}}\left[\mu - \frac{b}{a}\right]^2}$ $\therefore$ proportional to a Normal density $\therefore$

$a = \frac{1}{\tau_0^2}$, $b = \frac{\mu_0}{\tau_0^2}$ $\therefore$ $a^{-1} = \tau_0^2$, $\frac{b}{a} = \frac{\mu_0}{\tau_0^2}\tau_0^2 = \mu_0$ $\therefore$

$\mu \mid \theta, \sigma^2, \tau^2, y \sim N(\mu_0, \tau_0^2)$

$\therefore \mu \mid \theta, \sigma^2, \tau^2, y \sim N\left(\frac{b}{a}, a^{-1}\right)$

\3b/ By Bayes: $\pi(\mu, \theta, \sigma^2, \tau^2, y) \propto \pi(\mu) f(y, \theta, \sigma^2, \tau^2 \mid \mu) \propto$

$\pi(\mu) f(y \mid \mu, \theta, \sigma^2, \tau^2) f(\theta, \sigma^2, \tau^2 \mid \mu) \propto$

$\pi(\mu \mid \theta, \sigma^2, \tau^2) f(y \mid \mu, \theta, \sigma^2, \tau^2) \propto$

$\pi(\mu \mid \theta, \sigma^2, \tau^2) f(y \mid \theta, \sigma^2) \propto$

$\pi(\mu) f(\theta, \sigma^2, \tau^2 \mid \mu) f(y \mid \theta, \sigma^2) \propto$

$\pi(\mu) f(\tau^2 \mid \theta, \sigma^2, \mu) f(\theta, \sigma^2 \mid \mu) f(y \mid \theta, \sigma^2)$

$\pi(\mu) f(\tau^2 \mid \mu) f(\theta, \sigma^2 \mid \mu) f(y \mid \theta, \sigma^2) \propto$

$\pi(\mu \mid \tau^2) f(\theta \mid \sigma^2, \mu) f(\sigma^2 \mid \mu) f(y \mid \theta, \sigma^2) \propto$

$\pi(\mu \mid \tau^2) f(\theta \mid \mu) f(\sigma^2) f(y \mid \theta, \sigma^2)$

\3b/ By Bayes: $\pi(\mu \mid \theta, \sigma^2, \tau^2, y) \propto \pi(\mu) f(y, \theta, \sigma^2 \tau^2 \mid \mu) \propto$

$\pi(\mu) f(y \mid \mu, \theta, \sigma^2, \tau^2) \pi(\theta, \sigma^2, \tau^2 \mid \mu) \propto$

$\pi(\mu \mid \theta, \sigma^2, \tau^2) f(y \mid \mu, \theta, \sigma^2, \tau^2) \propto \pi(\mu \mid \theta, \sigma^2, \tau^2) f(y \mid \theta, \sigma^2) \propto$

$\pi(\mu) f(y \mid \theta, \sigma^2) \pi(\sigma^2, \tau^2 \mid \theta, \mu) \pi(\theta \mid \mu)$

By Bayes:
$\pi(\mu \mid \theta, \sigma^2, \tau^2, y) \propto \pi(\mu) f(y, \theta, \sigma^2, \tau^2 \mid \mu) \propto$

$\pi(\mu) f(y \mid \theta, \sigma^2, \tau^2, \mu) \pi(\theta, \sigma^2, \tau^2 \mid \mu) \propto$

$\pi(\mu) f(y \mid \theta, \sigma^2, \tau^2, \mu) \pi(\theta \mid \sigma^2, \tau^2, \mu) \pi(\sigma^2, \tau^2 \mid \mu)$

$\mu \mid \tau^2 \sim N(\mu_0, \tau_0^2)$ $\therefore \pi(\mu \mid \tau^2, \sigma^2) = \frac{1}{\sqrt{2\pi}\,\tau_0} e^{-\frac{1}{2\tau_0^2}(\mu - \mu_0)^2}$

$\theta \mid \mu, \tau^2 \sim N(\mu, \tau^2)$ $\therefore$ $\pi(\theta \mid \mu, \tau^2) = \frac{1}{\sqrt{2\pi}\,\mu} e^{-\frac{1}{2\tau^2}(\theta - \mu)^2}$ $\pi(\theta \mid y) \propto \pi(\theta)\pi(y \mid \theta)$

By Bayes $\pi(\mu \mid \theta, \sigma^2, \tau^2, y) \propto \pi(\mu) \pi(y, \theta, \sigma^2, \tau^2 \mid \mu) \propto$

$\pi(\mu) \pi(y \mid \theta, \sigma^2, \tau^2, \mu) \pi(\theta, \sigma^2, \tau^2 \mid \mu) \propto$

$\pi(\mu \mid \theta, \sigma^2, \tau^2) \pi(y \mid \theta, \sigma^2, \tau^2, \mu)$

$\pi(\mu|\tau^2,\sigma^2) \propto \pi(\tau^2,\sigma^2|\mu)\pi(\mu)$ ,

$\cancel{\pi(\theta|\mu,\tau^2) \propto \pi(\theta,\sigma^2,\tau^2|\mu) \propto \pi(\tau^2,\sigma^2|\theta,\mu)\pi(\theta,\sigma^2|\mu)}$

$\cancel{\pi(\theta|\mu,\tau^2}$

$\pi(\theta,\sigma^2,\tau^2|\mu) \propto \pi(\theta|\sigma^2,\tau^2,\mu)\pi(\sigma^2,\tau^2|\mu)$

By Bayes

$\pi(\mu|\theta,\sigma^2,\tau^2,y) \propto \pi(\mu)\pi(\theta,\sigma^2,\tau^2,y|\mu) \propto$

$\pi(\mu)\pi(y|\theta,\sigma^2,\tau^2,\mu)\pi(\theta,\sigma^2,\tau^2|\mu) \propto$

$\pi(\mu)\pi(y|\theta,\sigma^2,\tau^2,\mu)\pi(\theta|\sigma^2,\tau^2,\mu)\pi(\sigma^2,\tau^2|\mu) \propto$

$\cancel{\pi}$

\3b/ By Bayes: $\pi(\mu|\theta,\sigma^2,\tau^2,y) \propto$

$\cancel{\pi(\mu)\propto}$ $\pi(\mu)\pi(y,\theta,\sigma^2,\tau^2|\mu) \propto$

$\pi(\mu)\pi(y|\theta,\sigma^2,\tau^2,\mu)\pi(\theta,\sigma^2,\tau^2|\mu) \propto$

$\pi(\mu)\pi(y|\theta,\sigma^2)\pi(\theta,\sigma^2,\tau^2|\mu) \propto$

$\pi(\mu)\pi(\theta,\sigma^2,\tau^2|\mu) \propto \pi(\mu)\pi(\theta|\sigma^2,\tau^2,\mu)\pi(\sigma^2,\tau^2|\mu) \propto$

$\pi(\mu)\pi(\sigma^2,\tau^2|\mu)\pi(\theta|\mu,\tau^2) \propto$

$\pi(\mu|\sigma^2,\tau^2)\pi(\theta|\mu,\tau^2) \propto \pi(\mu|\tau^2)\pi(\theta|\mu,\tau^2) \propto$

$\pi(\mu|\tau^2)\prod_{j=1}^{n}\pi(\theta_j|\mu,\tau^2)$ $\because$ $\cancel{\theta_j}$ $\theta_j|\mu,\tau^2 \sim N(\mu,\tau^2)$ $\therefore$

$\cancel{\mu|\tau^2}$ $\pi(\mu|\tau^2) = \frac{1}{\sqrt{2\pi}\,\tau_0}e^{-\frac{1}{2\tau_0^2}(\mu-\mu_0)^2}$

$\pi(\theta_j|\mu,\tau^2) = \frac{1}{\sqrt{2\pi}\,\tau}e^{-\frac{1}{2\tau^2}(\theta_j-\mu)^2}$ $\therefore$

$\pi(\mu|\theta,\sigma^2,\tau^2,y) \propto \pi(\mu|\tau^2)\prod_{j=1}^{n}\pi(\theta_j|\mu,\tau^2) \propto$

$e^{-\frac{1}{2\tau_0^2}(\mu-\mu_0)^2}\prod_{j=1}^{n}e^{-\frac{1}{2\tau^2}(\theta_j-\mu)^2} \propto e^{-\frac{1}{2}\left[\frac{1}{\tau_0^2}(\mu-\mu_0)^2+\frac{1}{\tau^2}\sum_{j=1}^{n}(\theta_j-\mu)^2\right]} \propto$

$e^{-\frac{1}{2}\left[\left(\frac{1}{\tau_0^2}+\frac{n}{\tau^2}\right)\mu^2-2\mu\left(\frac{\mu_0}{\tau_0^2}+\frac{n\bar{\theta}}{\tau^2}\right)\right]} \propto e^{-\frac{1}{2}[a\mu^2-2b\mu]} \propto e^{-\frac{1}{2}a\left[\mu^2-2\frac{b}{a}\mu\right]} \propto$

$e^{-\frac{1}{2a^{-1}}\left(\mu-\frac{b}{a}\right)^2}$ $\therefore$ $a=\frac{1}{\tau_0^2}+\frac{n}{\tau^2}$ , $b=\frac{\mu_0}{\tau_0^2}+\frac{n\bar{\theta}}{\tau^2}$

is proportional to a normal density $\therefore$

$\mu|\theta,\sigma^2,\tau^2,y \sim N\left(\frac{b}{a},a^{-1}\right) = N\left(\frac{\frac{\mu_0}{\tau_0^2}+\frac{n\bar{\theta}}{\tau^2}}{\frac{1}{\tau_0^2}+\frac{n}{\tau^2}}, \frac{1}{\frac{1}{\tau_0^2}+\frac{n}{\tau^2}}\right)$

\3c/ By Bayes:

$\pi(\tau^2|\mu,\theta,\sigma^2,y) \propto \pi(\tau^2)\pi(y,\mu,\theta,\sigma^2|\tau^2) \propto$

$\pi(\tau^2)\pi(y|\mu,\theta,\sigma^2,\tau^2)\pi(\mu,\theta,\sigma^2|\tau^2) \propto \pi(\tau^2)\pi(\mu,\theta,\sigma^2|\tau^2) \propto$

$\pi(\tau^2)\pi(\theta|\mu,\tau^2,\sigma^2)\pi(\mu,\sigma^2|\tau^2) \propto \pi(\tau^2)\pi(\theta|\mu,\tau^2)\pi(\mu,\sigma^2|\tau^2) \propto$

\PP2021/ $\pi(\tau^2)\pi(\theta|\mu,\tau^2)\pi(\mu|\sigma^2,\tau^2)\pi(\sigma^2|\tau^2) \propto$

$\pi(\tau^2)\pi(\theta|\mu,\tau^2)\pi(\mu|\tau^2)\pi(\sigma^2|\tau^2) \propto$

$\pi(\tau^2)\pi(\theta|\mu,\tau^2)\pi(\mu|\tau^2) \propto \pi(\tau^2)\left(\prod_{j=1}^{n}\pi(\theta_j|\mu,\tau^2)\right)\pi(\mu|\tau^2)$

$\therefore \cancel{\pi(\tau^2) = (\xi/2)^{\xi}}$

$$\pi(\tau^2) = p(\tau^2|\eta,\xi^2) = \frac{(\eta/2)^{\eta/2}}{\Gamma(\eta/2)}\xi^{\eta}(\tau^2)^{-(\eta/2+1)}e^{-\frac{\eta\xi^2}{2\tau^2}}$$

$$\pi(\theta_j|\mu,\tau^2) = \frac{1}{\sqrt{2\pi}\tau}e^{-\frac{1}{2\tau^2}(\theta_j-\mu)^2}$$

$$\pi(\mu|\tau^2) = \frac{1}{\sqrt{2\pi}\tau_0}e^{-\frac{1}{2\tau_0^2}(\mu-\mu_0)^2}$$

$\therefore \pi(\tau^2|\mu,\theta,\sigma^2,y) \propto \pi(\tau^2)\pi(\theta|\mu,\tau^2)\pi(\mu|\tau^2) \propto$

$\pi(\tau^2)\pi(\theta|\mu,\tau^2) \propto \cancel{\pi(\tau^2)\pi}\ \pi(\tau^2)\prod_{i=1}^{n}(\theta_j|\mu,\tau^2) \propto$

$(\tau^2)^{-(\frac{\eta}{2}+1)}e^{-\frac{\eta\xi^2}{2\tau^2}}\prod_{j=1}^{n}e^{-\frac{1}{2\tau^2}(\theta_j-\mu)^2}\tau^{-1} \propto$

$\cancel{e^{-2(\frac{\eta}{2}+1)}\eta\xi}$ $\tau^{-2(\frac{\eta}{2}+1)}e^{-\frac{\eta\xi^2}{2\tau^2}-\frac{1}{2\tau^2}\sum_{j=1}^{n}(\theta_j-\mu)^2}\tau^{-n} \propto$

$\tau^{-2(\frac{\eta}{2}+1)-n}e^{-\frac{1}{2\tau^2}\left[\eta\xi^2+\sum_{j=1}^{n}(\theta_j-\mu)^2\right]} \propto$

$\tau^{-2(\frac{\eta+n}{2}+1)}e^{-\frac{1}{2\tau^2}\left[\eta\xi^2+\sum_{j=1}^{n}(\theta_j-\mu)^2\right]} \propto$

$(\tau^2)^{-(\frac{\eta+n}{2}+1)}e^{-\frac{1}{2\tau^2}\left[\eta\xi^2+\sum_{j=1}^{n}(\theta_j-\mu)^2\right]} \propto (\tau^2)^{-(\frac{a_1}{2}+1)}e^{-\frac{1}{2\tau^2}[a_1 b_1^2]}$

is proportional to an inverse chi-squared density $\therefore$

$a_1 = \eta+n$, $a_1 b_1^2 = \eta\xi^2 + \sum_{j=1}^{n}(\theta_j-\mu)^2 = (\eta+n)b_1^2 \therefore$

$\cancel{\frac{1}{\eta+n}}$ $b_1^2 = \frac{1}{\eta+n}\left[\eta\xi^2+\sum_{j=1}^{n}(\theta_j-\mu)^2\right] \therefore$

$\tau^2|\mu,\theta,\sigma^2,y \sim \text{Inv-}\chi^2(a_1, b_1^2) = \text{Inv-}\chi^2\left(\eta+n, \frac{1}{\eta+n}\left[\eta\xi^2+\sum_{j=1}^{n}(\theta_j-\mu)^2\right]\right)$

\3b/ By Bayes $\pi(\mu|\theta,\sigma^2,\tau^2,y) \propto \cancel{\pi(y)}\ \pi(\mu)\pi(y,\theta,\sigma^2,\tau^2|\mu) \propto$

$\pi(\mu)\pi(y|\theta,\sigma^2,\tau^2,\mu)\pi(\theta,\sigma^2,\tau^2|\mu) \propto$

$\pi(\mu)\pi(y|\theta,\sigma^2)\pi(\theta,\sigma^2,\tau^2|\mu) \propto \pi(\mu)\pi(\theta,\sigma^2,\tau^2|\mu) \propto$

$\pi(\mu)\pi(\theta|\mu,\tau^2,\sigma^2)\pi(\sigma^2,\tau^2|\mu) \propto \pi(\mu)\pi(\theta|\mu,\tau^2)\pi(\sigma^2,\tau^2|\mu) \propto$

$\pi(\mu)\pi(\theta|\mu,\tau^2)\pi(\sigma^2|\tau^2,\mu)\pi(\tau^2|\mu) \propto \pi(\theta|\mu,\tau^2)\pi(\sigma^2)\pi(\mu)\pi(\tau^2|\mu)$

$\propto \pi(\mu)\pi(\tau^2|\mu)\pi(\theta|\mu,\tau^2) \propto \pi(\mu|\tau^2)\pi(\theta|\mu,\tau^2) \propto \pi(\mu|\tau^2)\prod_{j=1}^{n}\pi(\theta_j|\mu,\tau^2)$

$\pi(\mu|\tau^2) = \frac{1}{\sqrt{2\pi}\tau_0}e^{-\frac{1}{2\tau_0^2}(\mu-\mu_0)^2}$ $\pi(\theta_j|\mu,\tau^2) = \frac{1}{\sqrt{2\pi}\tau}e^{-\frac{1}{2\tau^2}(\theta_j-\mu)^2} \therefore$

$$\pi(\mu|\theta,\sigma^2,\tau^2,y) \propto \pi(\mu|\tau^2)\,\pi(\theta|\mu,\tau^2) \propto \pi(\mu|\tau^2)\prod_{j=1}^{J}\pi(\theta_j|\mu,\tau^2) \propto$$

$$\pi(\mu|\tau^2)\prod_{j=1}^{J} e^{-\frac{1}{2\tau^2}(\theta_j-\mu)^2} \propto \pi(\mu|\tau^2)\prod_{j=1}^{J} e^{-\frac{1}{2\tau^2}(\mu^2-2\theta_j\mu)} \propto$$

$$\pi(\mu|\tau^2)\, e^{-\frac{1}{2\tau^2}(J\mu^2-2J\bar{\theta}\mu)} \propto e^{-\frac{1}{2\tau_0^2}(\mu-\mu_0)^2}\, e^{-\frac{1}{2}\left(\frac{J}{\tau^2}\mu^2-2\frac{J\bar{\theta}}{\tau^2}\mu\right)}$$

$$\propto e^{-\frac{1}{2\tau_0^2}(\mu^2-2\mu_0\mu)}\, e^{-\frac{1}{2}\left(\frac{J}{\tau^2}\mu^2-2\frac{J\bar{\theta}}{\tau^2}\mu\right)} \propto$$

$$e^{-\frac{1}{2}\left[\frac{1}{\tau_0^2}\mu^2-2\frac{\mu_0}{\tau_0^2}\mu+\frac{J}{\tau^2}\mu^2-2\frac{J\bar{\theta}}{\tau^2}\mu\right]} \propto$$

$$e^{-\frac{1}{2}\left[\left(\frac{1}{\tau_0^2}+\frac{J}{\tau^2}\right)\mu^2-2\left(\frac{\mu_0}{\tau_0^2}+\frac{J\bar{\theta}}{\tau^2}\right)\mu\right]} \propto$$

$$e^{-\frac{1}{2}[a\mu^2-2b\mu]} \propto e^{-\frac{1}{2}a[\mu^2-2\frac{b}{a}\mu]} \propto e^{-\frac{1}{2a^{-1}}(\mu-\frac{b}{a})^2}$$

is proportional to a Normal density ∴

$$a = \frac{1}{\tau_0^2}+\frac{J}{\tau^2}, \quad b = \frac{\mu_0}{\tau_0^2}+\frac{J\bar{\theta}}{\tau^2} \;\therefore$$

$$a^{-1} = \frac{1}{\frac{1}{\tau_0^2}+\frac{J}{\tau^2}}, \quad \frac{b}{a} = \frac{\frac{\mu_0}{\tau_0^2}+\frac{J\bar{\theta}}{\tau^2}}{\frac{1}{\tau_0^2}+\frac{J}{\tau^2}} \;\therefore$$

$$\mu|\theta,\sigma^2,\tau^2,y \sim N\left(\frac{b}{a}, a^{-1}\right) = N\left(\frac{\frac{\mu_0}{\tau_0^2}+\frac{J\bar{\theta}}{\tau^2}}{\frac{1}{\tau_0^2}+\frac{J}{\tau^2}}, \frac{1}{\frac{1}{\tau_0^2}+\frac{J}{\tau^2}}\right)$$

\3 c/ $\pi(\tau^2|\mu,\theta,\sigma^2,y) \propto \pi(\theta|\mu,\tau^2)\,\pi(\tau^2)$ ∵ By Bayes:

$$\pi(\tau^2|\mu,\theta,\sigma^2,y) \propto \pi(\tau^2)\,\pi(\mu,\theta,\sigma^2,y|\tau^2) \propto$$

$$\pi(\tau^2)\,\pi(y|\tau^2,\mu,\theta,\sigma^2)\,\pi(\mu,\theta,\sigma^2|\tau^2) \propto \pi(\tau^2)\,\pi(y|\theta,\sigma^2)\,\pi(\mu,\theta,\sigma^2|\tau^2)$$

$$\propto \pi(\tau^2)\,\pi(\mu,\theta,\sigma^2|\tau^2) \propto \pi(\tau^2)\,\pi(\theta|\mu,\sigma^2,\tau^2)\,\pi(\mu,\sigma^2|\tau^2) \propto$$

$$\pi(\tau^2)\,\pi(\theta|\mu,\tau^2)\,\pi(\mu,\sigma^2|\tau^2) \propto$$

$$\pi(\tau^2)\,\pi(\theta|\mu,\tau^2)\,\pi(\mu|\tau^2,\sigma^2)\,\pi(\sigma^2|\tau^2) \propto$$

$$\pi(\tau^2)\,\pi(\theta|\mu,\tau^2)\,\pi(\mu|\tau^2)\,\pi(\sigma^2) \propto \pi(\tau^2)\,\pi(\theta|\mu,\tau^2) \propto \pi(\tau^2)\prod_{j=1}^{J}\pi(\theta_j|\mu,\tau^2)$$

$$\pi(\tau^2) = p(\tau^2|\eta,\xi^2) = \frac{(\eta/2)^{\eta/2}}{\Gamma(\eta/2)}\,\xi^{\eta}(\tau^2)^{-(\eta/2+1)}\,e^{-\frac{\eta\xi^2}{2\tau^2}},$$

$$\pi(\theta_j|\mu,\tau^2) \propto \frac{1}{\tau}e^{-\frac{1}{2\tau^2}(\theta_j-\mu)^2} \;\therefore$$

$$\pi(\tau^2|\mu,\theta,\sigma^2,y) \propto \pi(\tau^2)\prod_{j=1}^{J}\pi(\theta_j|\mu,\tau^2) \propto \pi(\tau^2)\prod_{j=1}^{J}\tau^{-1}e^{-\frac{1}{2\tau^2}(\theta_j^2+\mu^2-2\theta_j\mu)}$$

$$\propto \pi(\tau^2)\left(\prod_{j=1}^{J}e^{-\frac{1}{2\tau^2}(\theta_j-\mu)^2}\right)\tau^{-J} \propto \pi(\tau^2)\,e^{-\frac{1}{2\tau^2}\sum_{j=1}^{J}(\theta_j-\mu)^2}\,\tau^{-J} \propto$$

$$(\tau^2)^{-(\frac{\eta}{2}+1)}e^{-\frac{\eta\xi^2}{2\tau^2}}\,e^{-\frac{1}{2\tau^2}\sum_{j=1}^{J}(\theta_j-\mu)^2}\,\tau^{-\frac{2J}{2}} \propto \cancel{(\tau^2)^{-(\frac{\eta}{2}+1)}}$$

$$(\tau^2)^{-(\frac{\eta}{2}+1)}(\tau^2)^{-\frac{J}{2}}e^{-\frac{1}{2\tau^2}\left[\eta\xi^2+\sum_{j=1}^{J}(\theta_j-\mu)^2\right]} \propto (\tau^2)^{-(\frac{\eta}{2}+\frac{J}{2}+1)}e^{-\frac{1}{2\tau^2}\left[\eta\xi^2+\sum_{j=1}^{J}(\theta_j-\mu)^2\right]}$$

$$\propto (\tau^2)^{-(\frac{\eta+J}{2}+1)}e^{-\frac{1}{2\tau^2}\left[\eta\xi^2+\sum_{j=1}^{J}(\theta_j-\mu)^2\right]} \propto (\tau^2)^{-(\frac{c}{2}+1)}e^{-\frac{1}{2\tau^2}(cd^2)}$$

is proportional to an Inv-$\chi^2$ distribution ∴

$$c = \eta+J, \quad cd^2 = (\eta+J)d^2 = \eta\xi^2+\sum_{j=1}^{J}(\theta_j-\mu)^2 \;\therefore\; d^2 = \frac{1}{\eta+J}\left(\eta\xi^2+\sum_{j=1}^{J}(\theta_j-\mu)^2\right) \therefore$$

\PP2021/ $\tau^2 \mid \mu, \theta, \sigma^2, y \sim \text{Inv-}\chi^2(c, d^2) = \text{Inv-}\chi^2\left(\eta + J, \frac{1}{\eta+J}\left(\eta\xi^2 + \sum_{j=1}^{J}(\theta_j - \mu^2)\right)\right)$

\4a/ Let $T_E$ and $T_I$ be time Spend in E and $I_1$ class $\therefore$

$T_P = T_E + T_I \quad \therefore \quad T_I \mid \lambda \sim Exp(\lambda), \quad T_E \mid \lambda \sim Exp(\lambda) \therefore$

$f(T_P)(z) = \int_{-\infty}^{\infty} f_{T_E, T_I}(t, z-t)\,dt = \int_{-\infty}^{\infty} f_{T_E}(t) f_{T_I}(z-t)\,dt$ (independence

$= \int_{-\infty}^{\infty} \lambda e^{-\lambda t} \lambda e^{-\lambda(z-t)}\,dt = \int_{-\infty}^{\infty} \lambda^2 e^{-\lambda z}\,dt$

exponential distribution is 0 to $\infty$ $\therefore$

$t \geq 0$ but also $z - t \geq 0 \therefore$ $z \geq t \therefore t \leq z \therefore 0 \leq t \leq z \therefore$

$f(T_P)(z) = \int_{-\infty}^{\infty} \lambda^2 e^{-\lambda z}\,dt = \int_0^z \lambda^2 e^{-\lambda z}\,dt = \lambda^2 e^{-\lambda z}[t]_0^z =$

$\lambda^2 e^{-\lambda z}[z - 0] = \lambda^2 z e^{-\lambda z} = \frac{(\lambda)^2}{1} z^1 e^{-(\lambda) z} = \frac{\lambda^2}{1!} z^{2-1} e^{-(\lambda) z} = \frac{\lambda^2}{\Gamma(2)} z^{2-1} e^{-(\lambda) z}$

is the pdf of a Gamma$(2, \lambda)$ distribution

\4b/ $\therefore$ By Bayes $\pi(\lambda \mid y) \propto \pi(\lambda) f(y \mid \lambda) \therefore$

$\pi(\lambda) = \frac{\beta^\alpha}{\Gamma(\alpha)} \lambda^{\alpha-1} e^{-\beta\lambda} \quad \therefore f(y \mid \lambda) = f_{T_P}(T_P \mid \lambda) = \frac{\lambda^2}{\Gamma(2)} T_P^{2-1} e^{-\lambda T_P} = \lambda^2 T_P e^{-\lambda T_P}$

$\therefore$ ~~$\pi(\lambda)$~~ $f(y_i \mid \lambda) = f_{T_P}(T_{P_i} \mid \lambda) = \lambda^2 T_{P_i} e^{-\lambda T_{P_i}} \therefore$

$\pi(\lambda \mid y) \propto \pi(\lambda) \prod_{i=1}^{n} f_{T_P}(T_{P_i} \mid \lambda) \propto \pi(\lambda) \prod_{i=1}^{n} \lambda^2 e^{-\lambda T_{P_i}} \propto$

$\pi(\lambda) \lambda^{2n} e^{-\lambda \sum_{i=1}^{n} T_{P_i}} \propto$ ~~$\pi(\lambda) e$~~ $\pi(\lambda) \lambda^{2n} e^{-\lambda n \bar{T}_P} \propto$

$\lambda^{\alpha-1} e^{-\beta\lambda} \lambda^{2n} e^{-\lambda n \bar{T}_P} \propto \lambda^{(\alpha+2n)-1} e^{-(\beta + n\bar{T}_P)\lambda} \propto \lambda^{a-1} e^{-b\lambda}$

is proportional to a Gamma density for $a = \alpha + 2n$, $b = \beta + n\bar{T}_P$

$\therefore \lambda \mid y \sim$ Gamma$(a, b)$ $\therefore$ both the posterior and prior follow a Gamma distribution $\therefore$ they are conjugate

\4b sol / let $t_i$ for $i = 1, \ldots, n$ be exchangable observations of relevant time, by Bayes thearen. $\pi(\lambda \mid t) \propto \lambda^{\alpha-1} e^{-\beta\lambda} \prod_{i=1}^{n} \lambda^2 t_i e^{-\lambda t_i} \propto$ $\lambda^{\alpha+2n-1} e^{-\lambda(\beta + \sum_{i=1}^{n} t_i)}$ which is proportion to a Gamma density $\therefore$

$\lambda \mid t \sim$ Gamma$(\alpha + 2n, \beta + \sum_{i=1}^{n} t_i)$

\4c/ let $\lambda \sim$ Gamma$(1, 1)$ $\therefore$ $\pi(\lambda) = \frac{1^1}{\Gamma(1)} \lambda^{1-1} e^{-1\lambda} = \frac{1}{1} 1 e^{-\lambda} = e^{-\lambda}$

By Bayes: $\pi(\lambda \mid y) \propto \pi(\lambda) \pi(y \mid \lambda) \therefore p(y) = \int_{-\infty}^{\infty} \pi(\lambda) \pi(y \mid \lambda)\,d\lambda$,

$p(y \mid y) = \int_{-\infty}^{\infty} \pi(\lambda \mid y) \pi(y \mid \lambda)\,d\lambda$

$p(y) = \int_{-\infty}^{\infty} \pi(\lambda) f(y|\lambda)\,d\lambda = \int_0^{\infty} e^{-\lambda} f(y|\lambda)\,d\lambda = \int_0^{\infty} e^{-\lambda} f_{T_2+R}(y|\lambda)\,d\lambda$

\4CS07/ Let $T_R = T_2 + T_3$ $\therefore$ $T_R|\lambda \sim \mathrm{Gamma}(2,\lambda)$

$\therefore$ $T_{E+I_1} = 7 = T_p$ $\therefore$

$\lambda | T_p \sim \mathrm{Gamma}(\alpha + 2n, \beta + \sum_{i=1}^{n} t_i)$ $\therefore$

$\lambda \sim \mathrm{Gamma}(\alpha,\beta) = \mathrm{Gamma}(1,1)$ $\therefore \alpha = 1$, $\beta = 1$ $\therefore$

$n=1$, $t_1 = 7$ $\therefore$

$\lambda | T_p = 7 \sim \mathrm{Gamma}(1+2(1), 1+\sum_{i=1}^{1} t_i) =$

$\mathrm{Gamma}(3, 1+t_1) = \mathrm{Gamma}(3, 1+7) = \mathrm{Gamma}(3,8)$

$\therefore$ $p(t_R | T_p = 7) = \int_{-\infty}^{\infty} \pi(\lambda | T_p = 7)\pi(t_R|\lambda)\,d\lambda =$

$\therefore \pi(\lambda | T_p = 7) = \frac{8^3}{\Gamma(3)} \lambda^{3-1} e^{-8\lambda} = 256\lambda^2 e^{-8\lambda}$ $\therefore$

$p(t_R | T_p = 7) = \int_0^{\infty} \pi(\lambda | T_p = 7)\,\pi(t_R|\lambda)\,d\lambda =$

$T_R|\lambda \sim \mathrm{Gamma}(2,\lambda)$

$\int_0^{\infty} 256\lambda^2 e^{-8\lambda} \lambda e^{-\lambda t_R}\,d\lambda = 256\int_0^{\infty} \lambda^3 e^{-(8+t_R)\lambda}\,d\lambda =$

$256\int_0^{\infty} \lambda^{2-1} e^{-(t_R+8)\lambda}\,d\lambda$ $\therefore$

$\lambda^{2-1} e^{-(t_R+8)\lambda}$ is proportional to a density of $\mathrm{Gamma}(2, t_R+8)$

$\therefore p(t_R | T_p = 7) = 256 \frac{\Gamma(2)}{(t_R+8)^2} \int_0^{\infty} \frac{(t_R+8)^2}{\Gamma(2)} \lambda^{2-1} e^{-(t_R+8)\lambda}\,d\lambda =$

$\frac{256\Gamma(2)}{(t_R+8)^2} = \frac{256 \times 1!}{(t_R+8)^2} = \frac{256}{(t_R+8)^2}$ $\therefore$

$p(t_R \leq 7 | T_p = 7) = \int_0^7 \frac{256}{(t_R+8)^2}\,dt_R$

\4CS07/ Let $T_R = T_2 + T_3$ $\therefore$ $T_R|\lambda \sim \mathrm{Gamma}(2,\lambda)$ $\therefore$

$p(t_R|T_p) = \int_{-\infty}^{\infty} \pi(\lambda|T_p) p(t_R|\lambda)\,d\lambda$ $\therefore$

$p(t_R|\lambda) = \lambda^2 t_R e^{-\lambda t_R}$, $\lambda \sim \mathrm{Gamma}(\alpha=1, \beta=1)$ $\therefore$ $n=1$, $t_1 = 7$ $\therefore$

$\lambda | T_p = 7 \sim \mathrm{Gamma}(\alpha+2n, \beta+\sum_{i=1}^{n} t_i) = \mathrm{Gamma}(1+2(1), 1+7) =$

$\mathrm{Gamma}(3,8)$ $\therefore$ $\pi(\lambda|T_p=7) = \frac{8^3}{\Gamma(3)}\lambda^{3-1}e^{-8\lambda} = 256\lambda^2 e^{-8\lambda}$ $\therefore$

$p(t_R|T_p=7) = \int_{-\infty}^{\infty} \pi(\lambda|T_p=7)\,p(t_R|\lambda)\,d\lambda = \int_0^{\infty} 256\lambda^2 e^{-8\lambda} \lambda^2 t_R e^{-\lambda t_R}\,d\lambda =$

\PP2021/ $256 t_R \int_0^\infty \lambda^4 e^{-(t_R+8)\lambda}\, d\lambda = 256 t_R \int_0^\infty \lambda^{5-1} e^{-(t_R+8)\lambda}\, d\lambda$

$$256 t_R \frac{\Gamma(5)}{(t_R+8)^5} \int_0^\infty \frac{(t_R+8)^5}{\Gamma(5)} \lambda^{5-1} e^{-(t_R+8)\lambda}\, d\lambda =$$ pdf over domain

$$256 t_R \frac{4!}{(t_R+8)^5} \times (1) = 6144 \frac{t_R}{(t_R+8)^5} \therefore$$

$$p(t_R \leq 7 \mid T_p = 7) = \int_0^7 6144 \frac{t_R}{(t_R+8)^5}\, dt_R = 6144 \int_0^7 t_R (t_R+8)^{-5}\, dt_R =$$

$$6144\left(\left[t_R \frac{1}{-4}(t_R+8)^{-4}\right]_0^7 - \int_0^7 -\frac{1}{4}(t_R+8)^{-4}\, dt_R\right) =$$

$$6144\left(\left[7\left(-\tfrac{1}{4}\right)(7+8)^{-4} - 0\right] + \frac{1}{4}\left[\frac{1}{-3}(t_R+8)^{-3}\right]_0^7\right) = \frac{3577}{5625} = 0.636$$

$$\cancel{6144\left(-\frac{7}{450} + \frac{1}{2}\left[\frac{1}{-1}(7+8)^{-1} + \frac{1}{1}(0+8)^{-1}\right]\right) =}$$

$$\cancel{512\left(-\frac{7}{450} + \frac{1}{2}\left[-\frac{1}{15} + \frac{1}{8}\right]\right) = 512\left(-\frac{7}{450} + \frac{7}{240}\right) = 512\left(\frac{49}{3600}\right) =}$$

$$\cancel{\frac{1568}{225} = 6.968} \times$$

$\cancel{p(t_R \leq}$ $p(t_R \mid T_p = 7) = \int_0^\infty p(t_R \mid \lambda)\, \pi(\lambda \mid T_p = 7)\, d\lambda =$

$$\int_0^\infty \lambda^2 t_R e^{-\lambda t_R} \frac{(\beta+7)^{\alpha+2}}{\Gamma(\alpha+2)} \lambda^{\alpha+1} e^{-\lambda(\beta+7)}\, d\lambda =$$

$$\frac{t_R(\beta+7)^{\alpha+2}}{\Gamma(\alpha+2)} \int_0^\infty \lambda^{\alpha+3} e^{-(t_R+\beta+7)\lambda}\, d\lambda = \frac{t_R(\beta+7)^{\alpha+2}}{\Gamma(\alpha+2)} \int_0^\infty \lambda^{\alpha+4-1} e^{-(t_R+\beta+7)\lambda}\, d\lambda$$

$$= \frac{t_R(\beta+7)^{\alpha+2}\Gamma(\alpha+4)}{\Gamma(\alpha+2)(t_R+\beta+7)^{\alpha+4}} \int_0^\infty \frac{(t_R+\beta+7)^{\alpha+4}}{\Gamma(\alpha+4)} \lambda^{\alpha+4-1} e^{-(t_R+\beta+7)\lambda}\, d\lambda$$

$$= \frac{t_R(\beta+7)^{\alpha+2}\Gamma(\alpha+4)}{\Gamma(\alpha+2)(t_R+\beta+7)^{\alpha+4}}$$

... estimate is $F(z) \approx \frac{1}{N}\sum_{i=1}^{N} \mathbb{1}(y_i \leq z)$

\M 508 Mock~~2~~ M2022Mock/

\~~1~~a i/ It means that we have the same distribution for any subset of all of the y's $\therefore$ our beliefs do not depend on the labels. eg our beliefs do not change with shuffling the labels.

\1ai/ It means that we have the same distribution for any subset of all of the y's $\therefore$ our beliefs do not depend on the labels, eg our beliefs do not change with shuffling the labels.

\1aii/ ~~B~~ likelihood of $\mathbf{y}$ is: $L(\mathbf{y}|\theta)=p(\mathbf{y}|\theta)=\prod_{i=1}^{n} f(y_i|\theta)$

\1aii/ likelihood of $\mathbf{y}$ is: $p(\mathbf{y}|\theta)=\prod_{i=1}^{n} f(y_i|\theta)$

\1aiii/ A 90% credible interval is any interval $[a,b]$ such that
$p(a\leq\theta\leq b|\mathbf{y})=\int_a^b \pi(\theta|\mathbf{y})\,d\theta=0.90$

\1aiii/ A 90% credible interval is any interval $[a,b]$ $\therefore$
$p(a\leq\theta\leq b|\mathbf{y})=\int_a^b \pi(\theta|\mathbf{y})\,d\theta=0.90$

\1aiv/ The MAP estimate for $\theta$ is: $\hat{\theta}=\arg\max \pi(\theta|\mathbf{y})$

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\1av/ The maximum likelihood Estimate (MLE)

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\1a vi/ $F(z|\mathbf{y})=P(y\leq z|\mathbf{y})=\int_{-\infty}^{z}\int_{-\infty}^{\infty} p(y|\theta)\pi(\theta|\mathbf{y})\,d\theta\, dy$

\1avi/ The cumulative distribution function of $y|\mathbf{y}$, $F(z|\mathbf{y})$ is: $F(z|\mathbf{y})=P(y\leq z|\mathbf{y})=\int_{-\infty}^{z}\int_{-\infty}^{\infty} p(y|\theta)\pi(\theta|\mathbf{y})\,d\theta\, dy$

\1avii/ The procedure is:

A: For each MCMC sample, $\theta_1,\dots,\theta_N$ draw a single sample from $f(y|\theta_i)$ to give $y_1,\dots,y_N$

B: The monte carlo estimate is $F(z)\approx\frac{1}{N}\sum_{i=1}^{N}\mathbb{1}(y_i\leq z)$

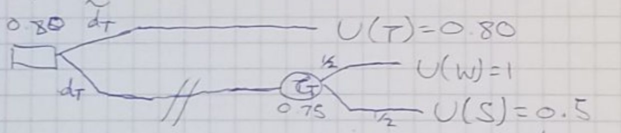
\1avii/ The procedure is: A: For each MCMC sample $\theta_1,\dots,\theta_N$ draw a single sample from $f(y|\theta_i)$ to give $y_1,\dots,y_N$
B: The Monte carlo estimate is $F(z)\approx\frac{1}{N}\sum_{i=1}^{N}\mathbb{1}(y_i\leq z)$

\1aviii/ The approx Monte Carlo error of this estimate based on the MCMC analysis ∴
Suppose the estimate were $\hat{p}$, then the Monte Carlo error !) is $\sqrt{\hat{p}(1-\hat{p})/N_{ess}}$ where $N_{ess}$ is the effective sample size of the MCMC sample and would be available from a good analysis

\1aviii/ Suppose the estimate were $\hat{p}$, then the Monte Carlo error is $\sqrt{\hat{p}(1-\hat{p})/N_{ess}}$ where $N_{ess}$ is the effective sample size of the MCMC sample and would be available from a good analysis

\1bi/ My preference ordering is $W^* > T^* > S^* > A$ ∴
using method in lectures: set $U(W)=1$, $U(A)=0$, and set $U(T)=0.8$, $U(S)=0.5$

\1bii/ let $d_T$ be the decision to trade ∴




$U(G)=U(W)P(W)+U(S)P(S) = 1\times\frac{1}{2}+\frac{1}{2}\times\frac{1}{2} = 0.75$ ∴
$U(T) > U(G)$ ∴
I should not trade

\2a/ ~~Consider~~ i: Generate $X^* \sim N(\hat{x}, \frac{1}{-H})$
ii: Generate $U \sim U(0,1)$
iii: If $Ug_u(X^*) = Ug(\hat{x})e^{\frac{H}{2}(X^*-\hat{x})^2} \leq g(X^*)$ accept $X^*$, else reject.

\2a/ i: Generate $X^* \sim N(\hat{x}, \frac{1}{-H})$
ii: Generate $U \sim U(0,1)$
iii: If $Ug_u(X^*) = Ug(\hat{x})e^{\frac{H}{2}(X^*-\hat{x})^2} \leq g(X^*)$ accept $X^*$, else reject.

\2b/ Generate $U_1, U_2$ from $Unif(0,1)$, let
$W = -2\ln(1-U_1)$, $\Theta = 2\pi U_2$, $Z_1 = \sqrt{W}\cos\Theta$, $Z_2 = \sqrt{W}\sin\Theta$,
then $Z_1, Z_2 \sim N(0,1)$

M2022 Mock / 2b / Generate $U_1, U_2$ from unif(0,1).

Let $W = -2\ln(1-U_1)$, $\Theta = 2\pi U_2$,

$Z_1 = \sqrt{W}\cos\Theta$, $Z_2 = \sqrt{W}\sin\Theta$.

Then $Z_1, Z_2 \sim N(0,1)$

2e / As the box-Muller algorithm uses pairs of uniform numbers to generate pairs of standard Normals, the questions demands mean we should use the first 2 uniforms to generate 2 standard normals, then the next 2 to perform 2 rejection sampling iterations and repeat

$W_1 = -2\ln(1-0.12) = 0.256$, $\Theta_1 = 2\pi(0.283) = 1.78$,

$Z_1 = -0.105$, $Z_2 = 0.494$

$\therefore$ The first iteration has $X^* = \sqrt{2}\,Z_1 = 0.147$

$U g_u(X^*) = 0.197 \times 4 \times e^{-\frac{1}{4}(0.147)^2} = 0.784 < g(X^*) = 3.98$

So accept $X^{(1)} = 0.147$. repeating $X^* = \sqrt{2}\,Z_2 = 0.699$

$U g_u(X^*) = 0.373 \times 4 \times e^{-\frac{1}{4}(0.699)^2} = 1.32 < g(X^*) = 3.51$

So accept $X^{(2)} = 0.699$

$W_2 = -2\ln(1-0.374) = 0.937$, $\Theta_2 = 2\pi(0.072) = 0.452$, $Z_3 = 0.871$,

$Z_4 = 0.423$

$X^* = \sqrt{2}\,Z_3 = 1.23$ $\quad U g_u(X^*) = 0.143 \times 4 \times e^{-\frac{1}{4}(1.23)^2} = 0.392 < g(X^*) = 2.49$,

So accept $X^{(3)} = 1.23$. Finally $X^* = \sqrt{2}\,Z_4 = 0.598$

$U g_u(X^*) = 0.733 \times 4 \times e^{-\frac{1}{4}(0.598)^2} = 2.68 < g(X^*) = 3.64$

So accept $X^{(4)} = 0.598$

2C / As the box-Muller algorithm uses pairs of uniform numbers to generate pairs of standard Normals, the question demands mean we should use the first 2 uniforms to generate 2 standard normals, then the next 2 perform 2 rejection sampling iterations and repeat.

$W_1 = -2\ln(1-0.12) = 0.256$, $\Theta = 2\pi(0.283) = 1.78$, $Z_1 = -0.105$, $Z_2 = 0.494$

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$U g_u(X^*) = 0.197 \times 4 \times e^{-\frac{1}{2}(0.147)^2} = 0.784 < g(X^*) = 3.98$

$\therefore$ accept $X^{(1)} = 0.147$ repeating $X^* = \sqrt{2}\, Z_2 = 0.699$

$U g_u(X^*) = 0.373 \cdot 4 \cdot e^{-\frac{1}{2}(0.699)^2} = 1.32 < g(X^*) = 3.51$

$\therefore$ accept $X^{(2)} = 0.699$

$W_2 = -2\ln(1-0.374) = 0.937$, $\theta_2 = 2\pi(0.072) = 0.452$, $Z_3 = 0.871$, $Z_4 = 0.423$

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$\therefore$ accept $X^{(3)} = 1.23$ $\quad X^* = \sqrt{2}\, Z_4 = 0.598$

$U g_u(X^*) = 0.733 \times 4 \times e^{-\frac{1}{2}(0.598)^2} = 2.68 < g(X^*) = 3.64$

$\therefore$ accept $X^{(4)} = 0.598$

\3a/ The normal random walk

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\3b/ i: Starting from some $X^{(0)}$ with $f(X^{(0)}) > 0$

ii: At iteration t generate $X^*$ from $N(X^{(t-1)}, \sigma^2)$

iii: Evaluate the Metropolis ratio $r = \frac{f(X^*)}{f(X^{(t-1)})}$

iv: Draw $U \sim \mathrm{unif}(0,1)$ and if $U \le r$ set $X^{(t)} = X^*$, else set $X^{(t)} = X^{(t-1)}$

\3b/ i: Starting from some $X^{(0)}$ with $f(X^{(0)}) > 0$

ii: At iteration t generate $X^*$ from $N(X^{(t-1)}, \sigma^2)$

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iv: Draw $U \sim \mathrm{unif}(0,1)$ and if $U \le r$ set $X^{(t)} = X^*$, else set $X^{(t)} = X^{(t-1)}$

\3c/ As with all normalising constants for target densities, they cancel out in the Metroloplis-Hastings ratio

\3c/ As with all normalising constants for target densities, they cancel out in the Metropolis-Hastings ratio

\3d/ let log Metropolis ratio is $\log r = 2(\ln X^* - \ln X^{(t-1)} + (X^{(t-1)} - X^*))$ which is more robust to compute $\therefore$

Iteration 1: $X^* = X^{(0)} + 1.909 = 3.409$, $r = 0.113 < U_1 (0.180)$ $\therefore$ reject $X^*$ and set $X^{(1)} = 1.5$

\M2022Mock/ Iteration 2: $X^* = X^{(1)} - 1.451 = 0.049$, $r = 0.019 < U_2(0.312)$, $\therefore$ reject $X^*$ and set $X^{(2)} = 1.5$

Iteration 3: $X^* = X^{(2)} + 1.279 = 2.779$, $r = 0.266 > U_3(0.312)$ $\therefore$ accept $X^*$ and set $X^{(3)} = 2.779$

\4a/ Coherence assumes that you do not prefer a given penalty if there is an available penalty that is certainly smaller

\4a/ Coherence assumes that you do not prefer a given penalty if there is an available penalty that is certainly smaller

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\4c/ An event is viewed as a random quantity with possible values 0 or 1 $\therefore$ the probability of an event is your expectation for that event. The definition of expectation represents a personal judgement amounting to preferring the random penalty L over any other of its type, $\therefore$ any individual's expectation could be distinct from that of any other. $\therefore$ This probability is entirely subjective

\4c/ An event is viewed as a random quantity with possible values 0 or 1 $\therefore$ the probability of an event is your expectation for that event. The definition of expectation represents a personal judgement amounting to preferring the random penalty L over any other of its type, $\therefore$ any individual's expectation could be distinct from that of any other. $\therefore$ This probability is entirely subjective

\4di/ The problematic words are ~~climate~~, change, rapid, probability, the. climate is defined symbolically ∴ ~~it is~~ the word is not a problem.

For rapid climate change to be an event, need to know what rapid and change mean. There are many ways to define these things properly, e.g. if $y_n$ is the climate today, then can say rapid climate change means $|y_r - y_n| > M$ for some threshold M or anything else. This wording could stay if there was a sub definition somewhere else, ~~pt~~ something precise is needed.

~~climate~~ The issue is that either climate will change rapidly or it wont ∴ it doesnt make sense to try to estimate the probability and that's not what Bayesian Models do. Instead, can find the probability via a subjective Bayesian Analysis

\4di/ climate is defined symbolically ∴ the word is not a problem. ∴ The problematic words are change, rapid, probability, the.

For rapid climate change to be an event, need to know what rapid and change mean. Something precise is needed, there are many ways to define these things properly, e.g. if $y_n$ is the climate today, then can say rapid climate change means $|y_r - y_n| > M$ for some threshold M or anything else. This wording could stay if there was a subdefinition somewhere else.

The issue is that either climate will change rapidly or it wont ∴ it doesnt make sense to try to estimate the probability, ~~and that's not what~~ ~~Instead~~ But, can find ~~the probability if~~ the probability via a subjective Bayesian Analysis. But this is not what Bayesian Models do.

\M2022Mock/ \4dii/ Must caresully explain and justisy the assumptions behind all layers os the model (data-process-prior). All prior judgements, in particular, should be justisied and transparent. A sensitivity analysis to show how sensitive the estimates are to the assumptions/beliess.

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ECM3741 Mock 2017

1ai To be coherent, with $E(X) = \bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i$ then as $n \to \infty$

$\therefore$ ~~suppose~~ $\frac{1}{n}\sum_{i=1}^{n} x_i \to E(X)$ with $k \neq 0$ and $X - \bar{x} = X - E(X)$ as $n \to \infty$ and $E(X^2) - E(X)^2 \to 0$ as $n \to \infty$

1aii Let $E(X) = \frac{1}{n}\sum_{i=1}^{n} x_i$ $\therefore$ with $\inf X \leq \sup X$ $\therefore$

$\sum_{i=1}^{n} \inf X = n \inf X \leq \sum_{i=1}^{n} x_i$ $\therefore$

$\frac{1}{n}\sum_{i=1}^{n} \inf X = \frac{1}{n} n \inf X \leq \cancel{\sum_{i=1}^{n}} = \inf X \leq \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x} = E(X)$

$\sum_{i=1}^{n} \sup X = n \sup X \geq \sum_{i=1}^{n} x_i$ $\therefore$

$\therefore$ $\frac{1}{n}\sum_{i=1}^{n} \sup X = \frac{1}{n} n \sup X \geq \frac{1}{n}\sum_{i=1}^{n} x_i = \bar{x} = E(X)$ $\therefore$

$\inf(X) \leq E(X) \leq \sup(X)$

1aiii $(A \wedge B) \vee (C \wedge D) = \sim((\sim(A \wedge B)) \wedge (\sim(C \wedge D)))$ $\therefore$

$P((A \wedge B) \vee (C \wedge D)) = P(\sim((\sim(A \wedge B)) \wedge (\sim(C \wedge D)))) =$

$1 - (P((\sim(A \wedge B)) \wedge (\sim(C \wedge D)))) = 1 - (P(\sim(A \wedge B))P(\sim(C \wedge D))) =$

$1 - ((1 - P(A \wedge B))(1 - P(C \wedge D))) =$

~~1-(1-P(~~ ~~1-((1-P(~~ $1 - ((1 - E(AB))(1 - E(CD))) =$

$1 - (1 - E(CD) - E(AB) + E(AB)E(CD)) =$

$\therefore$ $E(CD) + E(AB) - E(AB)E(CD) = 0.5 + 0.4 - 0.2 = 0.7$

1bi Set the best reward $r_k$ has having utility $U(r_k) = 1$ and for worst reward, $r_1$ has utility $U(r_1) = 0$ then

for all other rewards find the acceptable value of probability $P_i$ such $P_i U(r_k) + (1 - P_i)U(r_1) = P_i U(r_k) = U(r_i)$ $\therefore$

$\therefore$ $P_i r_k + (1 - P_i) r_1 = r_i$ for all the remaing rewards

1bii Let L, M, E, B be the cities respectively $\therefore$

$\therefore$ My preference is L, E, M, B $\therefore$ let

My $U(L) = 1$, $U(B) = 0$, $U(E) = 0.75$, $U(M) = 0.7$

1biii $\therefore$ expectation of the gamble is: $\frac{1}{5}U(L) + \frac{4}{5}U(M) =$

$\frac{1}{5}(1) + \frac{4}{5}(0.7) = 0.76$ $\therefore$ $0.76 > U(B) = 0$ $\therefore$ I should take the gamble

\1ci/ By Bayes theorem: $\pi(\theta|y) \propto \pi(\theta) f(y|\theta) \propto$
$\theta^{a-1} e^{-b\theta} f(y|\theta) \propto \theta^{a-1} e^{-b\theta} \prod_{i=1}^{n} f(y_i|\theta) \propto$
$\theta^{a-1} e^{-b\theta} \prod_{i=1}^{n} \theta y_i^{-\theta-1} \propto \theta^{a+n-1} e^{-b\theta} (\prod_{i=1}^{n} y_i)^{-\theta-1}$ ∴

let $a_1 = a+n$ ∴ $\theta|y \sim \text{Gamma}(a_1, b)$ ∴
$\pi(\theta|y) = \frac{b^{a_1}}{\Gamma(a_1)} \theta^{a_1-1} e^{-b\theta}$ ∵ $\pi(\theta|y) \propto \theta^{a_1-1} e^{-b\theta}$

\1cii/ if $n=0$, $a_1 = a$ ∴ $\pi(\theta|y) = \pi(\theta)$ ∴ the prior is conjugate

\1ciii/ it is proper ∵ $\theta$ is unbounded in the prior ∵ $\theta > 0$
∴ $f(y|\theta) \to 0$ as $\theta \to \infty$

\1di/ $(X_1 = 1, X_2 = 1) = 0.2$ ∴ $0.2(3) + 0.1(6) = 1.2$ ∴
let $k_1 \int_{-\infty}^{\infty} f(x_1, x_2) dx = 1$ ∴ $k_1 = \frac{1}{1.2} = \frac{5}{6}$ ∴

Sample from the distribution to find an expectation of the samples for each combination of $(x_1, x_2)$

\1dii/ you find which quantiles of the distribution the sampled values fall in then measuring what proportions of all the sample fall in each quantile and compare to the expected number for each quantile range which corresponds to the probability $q_n - q_{n-1}$ to check if the sample follows the expected distribution

\1diii/ $(x_1 = 3, x_2 = 3)$ ∴ 0.2 ∴ $0.441 > 0.2$ ∴ reject, $0.018 < 0.2$ ∴ accept, $0.304 > 0.2$ ∴ reject, $0.732 > 0.2$ ∴ reject ∴
$\frac{1+0+0+0}{4} = \frac{1}{4}$ ∴ 0.2 is $q_{0.25}$ ∴ 25% quantile ∴
$E(x_i) = 1.495/4 = 0.37375$ ∴ Sample 1: 0.1, 0.4, 0.5, 0.6 or
Sample 2: 0.05, 0.45, 0.55, 0.65

\2a/ $\frac{\lambda^x e^{-\lambda}}{x!}$ ∴ $P(x|\theta) = \frac{1}{x!} \theta^x e^{-\theta}$ ∴ $P(X_i | \lambda_i) = \frac{\lambda_i^{x_i} e^{-\lambda_i}}{x_i!}$
∴ $E(\theta^k) = \int_0^{\infty} \theta^k \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} d\theta = \frac{\beta^\alpha}{\Gamma(\alpha)} \int_0^{\infty} \theta^{\alpha+k-1} e^{-\beta\theta} d\theta$ ∴ let $\alpha_1 = \alpha + k$ ∴
$\frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha_1)}{\beta^{\alpha_1}} \int_0^{\infty} \frac{\beta^{\alpha_1}}{\Gamma(\alpha_1)} \theta^{\alpha_1} e^{-\beta\theta} d\theta = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha_1)}{\beta^{\alpha_1}}$ ∵ Gamma$(\alpha_1, \beta)$ distribution

ECM3741 Mock 2017 $\therefore E(\theta^k) = \frac{\beta^\alpha}{\Gamma(\alpha)} \frac{\Gamma(\alpha+k)}{\beta^{\alpha+k}} = \beta^{\alpha-\alpha-k} \frac{(\alpha+k-1)!}{\alpha!} =$

$\beta^{-k} \frac{(\alpha+k-1)(\alpha+k-2)\cdots(\alpha)(\alpha-1)\cdots}{\alpha(\alpha-1)\cdots} = \frac{(\alpha+k-1)(\alpha+k-2)\cdots\alpha}{\beta^k}$

$\therefore$ for $\alpha = 10$, $\beta = 0.1$ $\therefore \frac{(10+k-1)(10+k-2)\cdots(10)}{0.1^k} =$

$((9+k)(8+k)\cdots(10)) \times 0.1^{-k}$

3a $\therefore f(x) = \frac{1}{C}(2+x)(2-x)$ $\therefore \int_{-2}^{2} f(x)\,dx = 1 = \int_{-2}^{2} \frac{1}{C}(2+x)(2-x)\,dx$ $\therefore$

$\frac{1}{1/C} = \int_{-2}^{2} 4 - x^2\,dx = \left[4x - \frac{1}{3}x^3\right]_{-2}^{2} = 4(2-(-2)) - \frac{1}{3}(2^3 - (-2)^3) = \frac{32}{3}$ $\therefore$

$\frac{1}{C} = \frac{3}{32}$ $\therefore$ let $f(x) = \frac{3}{32}(2+x)(2-x)$ $\therefore \int_{-2}^{2} \frac{3}{32}(2+x)(2-x)\,dx = 1$ $\therefore$

$C = \frac{32}{3}$

3b $\therefore f_u(x) = \frac{1}{2-(-2)} = \frac{1}{4}$ $\therefore g(x) = (2+x)(2-x) = 4 - x^2$ $\therefore$

$\mathrm{Max}\, g(x) = 4 - 0^2 = 4$ $\therefore k f_u(x) = k\frac{1}{4} = 4$ $\therefore k = 16$

3c $g(x) = 4 - x^2$ $\therefore \ln(g(x)) = \ln(4 - x^2)$ $\therefore$

$\frac{d}{dx}(\ln g(x)) = \frac{d}{dx}(\ln(4-x^2)) = \frac{-2x}{4-x^2}$ $x \in [-2, 2]$ $\therefore 2 - x^2 \geq 0$ $\therefore$

$g(x)$ is log-concave $\therefore \frac{-2x}{4-x^2}$ is concave.

3d that the log of the modulus of all the samples then determine if the sample falls below or above the log of $g(x)$: $\ln(g(x))$ and if it falls below, accept it, else reject it. and alway reject any negative samples smaller than $-2$ with $\ln(g(x)) > \sup(\ln(f(x)))$

3e ~~ln(g(x))~~ $g(x) = 4 - x^2$, $\ln(g(x)) = \ln(4 - x^2)$, $g'(x) = -2x$,

$\frac{d}{dx}(\ln(g(x))) = \frac{-2x}{4-x^2}$ $\therefore \frac{c}{k} = \frac{32/3}{16} = \frac{2}{3}$ $\therefore R = \frac{4}{3\sqrt{\pi}} \approx 0.752$

3f $|-1.73| = 1.73$ $\therefore$ ~~-2(1.73)~~ $4 - (-1.73)^2 = 1.0071 > R$ $\therefore$ accept ~~reject~~

$4 - 0.255^2 = 3.93 > R$ $\therefore$ accept

~~0.654~~ $4 - 0.654^2 = 3.572 > R$ $\therefore$ accept

$4 - 0.338^2 = 3.89 > R$ $\therefore$ accept

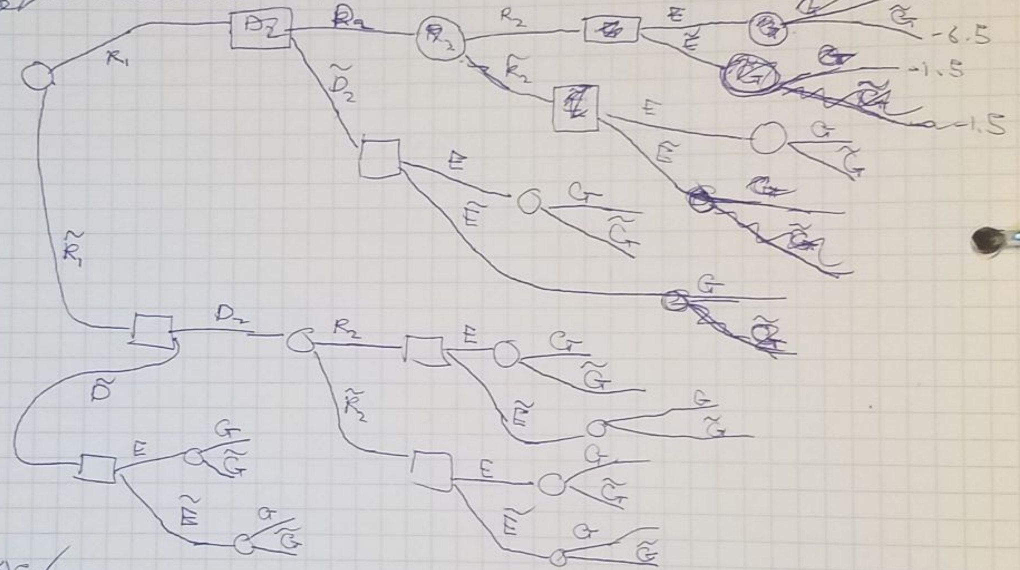
\4a/ with his original probabilities: $8(0.3)-5(0.7)=-1.1<0$ $\therefore$ he was acting optimally

\4b/ $D_2$ be decision to buy second survey

let $G$ be good buisness

let $R_1$ be first surveys predicts good buisness

~~let~~ let $E$ be decision to expand, let $R_2$ be 2nd survey predics good $\therefore$




\4c/ ~~P(~~ $P(R_1|G)=0.8$, $P(\tilde{R}_1|\tilde{G})=0.8$, $P(R_2|G)=0.9$, $P(\tilde{R}_2|\tilde{G})=0.9$

$P(\tilde{R}_1|G)=0.2=P(R_1|\tilde{G})$ $P(\tilde{R}_2|G)=0.1=P(R_2|\tilde{G})$

~~P(R₁)~~ $P(G)=0.3$ $P(G|R_1)=\frac{P(R_1|G)P(G)}{P(R_1)}$

$P(R_1)\neq P(R_1|\tilde{G})+P(R_1|\tilde{G})=1$

\4d/ if ~~bad~~ first survey shows good result then he should invest in 2nd, if 2nd shows good result he should expand, else he should not expand.

\4e contined/ ~~P(R₂)~~ $P(G|R_2)=\frac{P(R_2|G)P(G)}{P(R_2)}$

\1ai sol/ ~~To~~ be coherent, for this desinition of expectation, means that you do not have a preference for a given a given penalty if you have the option ~~of~~ another that is certainly smaller

\2a/ $\mathbb{E}(\theta^k) = \int_0^\infty \theta^k \frac{\beta^\alpha}{\Gamma(\alpha)} \theta^{\alpha-1} e^{-\beta\theta} d\theta = \frac{\beta^\alpha \Gamma(\alpha+k)}{\Gamma(\alpha)\beta^{\alpha+k}} \int_0^\infty \frac{\beta^{\alpha+k}}{\Gamma(\alpha+k)} \theta^{\alpha+k-1} e^{-\beta\theta} d\theta =$

$\frac{\beta^\alpha \Gamma(\alpha+k)}{\Gamma(\alpha)\beta^{\alpha+k}} = \frac{(\alpha+k-1)(\alpha+k-2)\cdots\alpha}{\beta^k}$ as Z integrand is that of a Gamma density & ∴ integrates to 1 & ∵ $\Gamma(t+1) = t\Gamma(t)$

\2c/ By Bayes theorem ∴ $\pi(\theta|y) \propto \pi(\theta) P(y|\theta) \propto$
$\theta^{\alpha-1} e^{-\beta\theta} \prod_{i=1}^n \theta^{y_i} e^{-\theta} \propto \theta^{\alpha+n\bar{y}-1} e^{-(\beta+n)\theta}$ which is proportional to a Gamma density ∴ $\theta|y \sim \text{Gamma}(\alpha+n\bar{y}, \beta+n)$ ~~[illegible]~~

\3a/ as $f(x)$ is a probab distri ∴ $\int_{-2}^2 g(x)dx = \int_{-2}^2 c f(x) dx =$ ~~[illegible]~~
$c\int_{-2}^2 f(x)dx = c(1) = c$ ∴ $c = \int_{-2}^2 (2+x)(2-x)dx = \left[4x - \frac{x^3}{3}\right]_{-2}^2 = 8 - \frac{8}{3} + 8 - \frac{8}{3} = \frac{32}{3}$

\3b/ Z max val of $g(x)$ is 4 at $x=0$ & $f_u(x) = \frac{1}{4}$ (Z unif distri on $[-2,2]$) ∴ require $g_u(x) = k f_u(x) \geq g(x) = c f(x)$ $\forall x$ ∴ accept rate $c/k$ is maximised ∴ $k \geq 16$ must hold else, at $x=0$ ∴ $g_u(x) < g(x)$ & Z acceptance rate is decreasing in $k$ ∴ $k=16$ is Z optimal $k$ & Z acceptance rate for this ~~proportional~~ proposal is $\frac{c}{k} = \frac{(32/3)}{16} = \frac{2}{3}$

\3c/ let $h(x) = \ln(g(x)) = \ln(4-x^2)$ ∴ $h'(x) = \frac{-2x}{4-x^2}$ ∴
$h''(x) = \frac{-2}{4-x^2} - \frac{4x^2}{(4-x^2)^2} < 0$ for $x \in [-2,2]$ ∴ $g(x)$ is log concave.

~~Let Z~~ \3d/ let $\hat{x} = \arg\max(g(x))$ & $H = \sup h''(x) < 0$, then, Z algorithm is to generate samples using $f_u(x)$ Z pdf of a normal distri $N(\hat{x}, \frac{1}{-H})$ with Z envelope $g_u(x) = e^{h(\hat{x})} e^{\frac{H}{2}(x-\hat{x})^2}$ ∴ $k$ ∴ $g_u(x) = k f_u(x) \leq g(x)$ is $k = e^{h(\hat{x})}\sqrt{\frac{2\pi}{-H}}$ in full, generate $X_u$ from $f_u(x)$, then generate $U \sim \text{Unif}(0,1)$, & accept $X_u$ if $U g_u(X_u) < g(X_u)$

\3e/ ∴ $h'''(x) = \frac{-12x}{(4-x^2)^2} - \frac{4x^2}{(4-x^2)^3}$ which is 0 if $x=0$ or if $12-3x^2+x=0$ ie if $x = \frac{1 \pm \sqrt{145}}{6}$ in Z interval $[-2,2]$, $h''(x)$ is maxed at $x=0$ & is $-\frac{1}{2} < 0$ there ∴ $H = -\frac{1}{2}$ & Z conditions of Z algorithm have been met, now $\hat{x} = 0$ & $g(\hat{x}) = 4$ ∴ $R = \frac{c}{k} = \frac{32/3}{4\sqrt{\frac{2\pi}{12}}} =$
$\frac{32/3}{4\sqrt{4\pi}} = \frac{4}{3\sqrt{\pi}}$

\3 8/ 1st shift & scale each $N(0,1)$ number st it has Z

Same distri as $f_u(x)$ : $X_u^{(1)} = 0 + (-1.73) \times \sqrt{\frac{1}{-(-\frac{1}{2})}} = -1.73 \times \sqrt{2} = 2.45$

& similarly $X_u^{(2)} = 0.255 \times \sqrt{2} = 0.361$ & $X_u^{(1)} \notin [-2,2]$ $\therefore g(X_u^{(1)}) = 0$ &

$X_u^{(1)}$ is rejected , $g_u(X_u^{(2)}) = 4e^{-\frac{1}{4}(0.361^2)} = 3.87$ &

$g(X_u^{(2)}) = (2+0.361)(2-0.361) = 3.87$ $\therefore U g_u(X_u^{(2)}) < g(X_u^{(2)})$ & $X_u^{(2)} = 0.361$

is accepted

\4a/ Yes. his utility for Z status quo was 0 & his utility for

Z gamble of opening a restaurant was $U(0.3 \times 8 + 0.7 \times (-5)) =$

$0.3 \times 8 - 0.7 \times 5 = -1.1 < 0$

\4b/ 1st desire decisions & outcomes:

let S be Z event Z new restaurant is profitable

Let $T_1$ event survey 1 predicts S

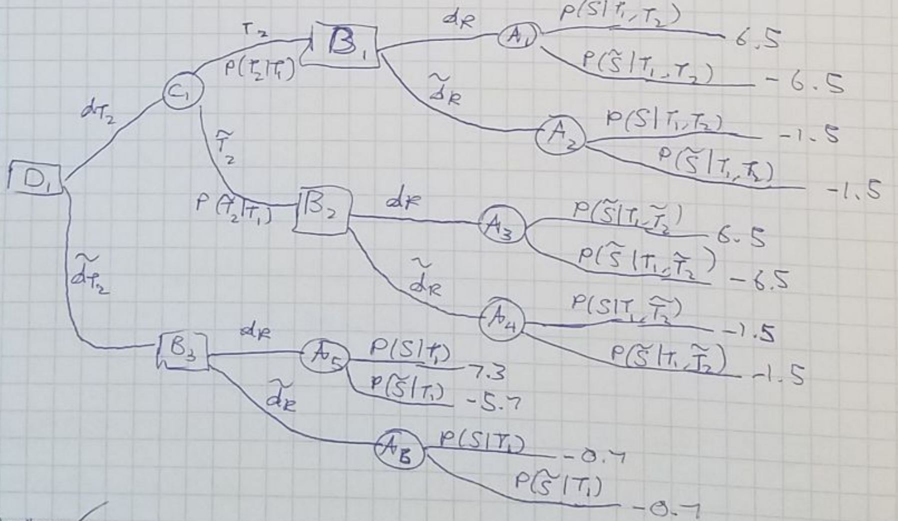
$T_2$ event survey 2 predicts S

$d_R$ decision to open $d_{T_1}$ be decision to pay for survey 1

$d_{T_2}$ be decision to pay for survey 2 $\therefore$

utilites are $U(S, d_{T_1}, d_{T_2}) = 8 - 0.7 - 0.8 = 6.5$ , $U(S, d_{T_1}, \tilde{d}_{T_2}) = 8 - 0.7 = 7.3$

$U(\tilde{S}, d_{T_1}, d_{T_2}) = -5 - 0.7 - 0.8 = -6.5$ , $U(\tilde{S}, d_{T_1}, \tilde{d}_{T_2}) = -5.7$




\4c/ Z probs : $P(S) = 0.3$ , $P(\tilde{S}) = 0.7$ , $P(T_1|S) = 0.8 = P(\tilde{T}_1|S)$ ,

$P(T_1|\tilde{S}) = 0.2 = P(\tilde{T}_1|S)$ & $P(T_2|S) = 0.9 = P(\tilde{T}_2|\tilde{S})$ , $P(T_2|\tilde{S}) = 0.1 = P(\tilde{T}_2|S)$

\2017 Mock/

$\therefore P(T_1)=P(T_1|S)P(S)+P(T_1|\tilde{S})P(\tilde{S})=0.3(0.3)+0.2\times0.7=0.38$

$P(T_2|T_1)=P(T_1,T_2)/P(T_1)$ , $P(T_2,T_1)=P(T_1,T_2|S)P(S)+P(T_1,T_2|\tilde{S})P(\tilde{S})=$

$P(T_2|S)P(T_1|S)P(S)+P(T_2|\tilde{S})P(T_1|\tilde{S})P(\tilde{S})=0.23$ $\because$ independence of

Surveys given true outcome $\therefore P(T_1,T_2|S)=P(T_1|S)P(T_2|S)$ $\therefore$

$P(T_2|T_1)=0.61$ $\therefore$

$P(\tilde{T}_2|T_1)=\frac{15}{38}=0.39$

$P(S|T_1)=\frac{P(T_1|S)P(S)}{P(T_1)}=\frac{12}{19}=0.63$

$P(\tilde{S}|T_1)=\frac{7}{19}=0.37$ , $P(S|T_1,T_2)=\frac{P(T_1,T_2|S)P(S)}{P(T_1,T_2)}=\frac{P(T_1|S)P(T_2|S)P(S)}{P(T_1,T_2)}=0.94$

$\therefore P(\tilde{S}|T_1,T_2)=0.06$ $\therefore$

$P(S|T_1,\tilde{T}_2)=0.16$ , $P(\tilde{S}|T_1,\tilde{T}_2)=0.84$ $\therefore$

$A_1=5.71$ $A_2=-1.5$ $A_3=-4.42$ , $A_4=-1.5$ $A_5=2.51$

$A_6=-0.7$ $\therefore B_1=5.71$ $(d_0)$ $B_2=-1.5$ $(\tilde{d}_0)$ $B_3=2.51$ $(d_0)$ $\therefore$

$C_1=2.86$ $\therefore$ $D_1=2.86$ $(d_{T_2})$ $\therefore$ al should ~~pay~~ 2nd survey

$d_{T_2}$ & if $T_2$ , should open , else do nothing & shouldnt open

\1ai/ To be coherent for this definition of expectation means that you do not have a preference for a given penalty if you have the option of another that is certainly smaller

\1aii/ ~~to~~ show violating $\inf X \le E(X) \le \sup X$ is incoherent :

if $E(X)<\inf X$ $\therefore -\inf X<-E(X)$ $\therefore X-\inf X<X-E(X)$ $\therefore$

$(X-\inf X)^2<(X-E(X))^2$ as $X-\inf X\ge0$ $\therefore$ $X-E(X)>0$

this implies you can reduce your penalty by changing $E(X)$ to $\inf(X)$ independent of $X$ . $\therefore$ by coherence $E(X)\ge\inf(X)$ $\therefore$

if $E(X)>\sup X$ $\therefore -\sup X>-E(X)$ $\therefore X-\sup X>X-E(X)$ $\therefore$

$(X-\sup X)^2<(X-E(X))^2$ $\because$ $X-\sup X<0$ $\therefore X-E(X)<0$ $\therefore$

both sides are negative $\therefore$ can reduce penalty by choosing $E(X)=\sup X$ , independent of $X$

\1aiii/ using duality of events and random quantities $\therefore$

as random quantity: $(A\wedge B)\vee(C\wedge D)=\sim((\sim(A\wedge B))\wedge(\sim(C\wedge D)))=$

$1-((\sim(A\wedge B))\wedge(\sim(C\wedge D)))=1-(1-AB)(1-CD)=1-(1-AB-CD+ABCD)=$

AB+CD-ABCD ∴

as probabilitys is the expectation of events, by linearity of expectation, ∴

$P((A\wedge B)\vee(C\wedge D)) = P(AB+CD-ABCD) = E(AB+CD-ABCD) =$

$E(AB)+E(CD)-E(ABCD) = 0.5+0.4-0.2 = 0.7$

\1aiii/ ~~ES~~ using the duality of events and random quantities to first write the event as a random quantity

$(A\wedge B)\vee(C\wedge D) = \sim((\sim(A\wedge B))\wedge(\sim(C\wedge D))) =$ ~~1-f~~

$1-((1-(AB))(1-(CD))) = 1-(1-AB)(1-CD) = 1-(1-AB-CD+ABCD) =$

$AB+CD-ABCD$

as probability is the expectation of events, by the linearity of expectation

$P((A\wedge B)\vee(C\wedge D)) = E(AB+CD-ABCD) =$

$E(AB)+E(CD)-E(ABCD) = 0.5+0.4-0.2 = 0.7$

\1bi/ Begin by establishing a preference ordering over the reward set: $r_1 \leq^* r_2 \leq^* \dots \leq^* r_k$

wlog: set $U(r_1)=0$, $U(r_k)=1$ ∴

For $j=2,\dots,k-1$ find $p_j$ ∴ $r_j \sim^* p_j r_k +_g (1-p_j) r_1$ ∴

$U(r_j)=p_j$

\1bi/ Begin by establishing a preference ordering over the reward set: $r_1 \leq^* r_2 \leq^* \dots \leq^* r_k$ ∴

wlog: set $U(r_1)=0$, $U(r_k)=1$ ∴

For $j=2,\dots,k-1$: find $p_j$ ∴ $r_j \sim^* p_j r_k +_g (1-p_j) r_1$ ∴

$U(r_j)=p_j$

\1bii/ ∴ $L >^* E >^* B >^* M$ ∴ $U(L)=1$, $U(M)=0$ ∴

let $U(E)=0.9$, $U(B)=0.4$

\1bii/ let $L >^* E >^* B >^* M$ ∴ $U(L)=1$, $U(M)=0$ ∴

let $U(E)=0.9$, $U(B)=0.4$

\1aii/ show violating $\inf X \le E(X) \le \sup X$ is incoherent

$\therefore$ if $E(X) < \inf X$ $\therefore -\inf X < -E(X)$ $\therefore X - \inf X < X - E(X)$ $\therefore$

$\Rightarrow (X-\inf X)^2 < (X-E(X))^2$ as $X - \inf X \ge 0$ $\therefore X - E(X) > 0$

this implies you can reduce you penalty by changing $E(X)$ to $\inf(X)$, independent of $X$. $\therefore$ by coherence $E(X) \ge \inf X$

$\therefore$ if $E(X) > \sup X$ $\therefore -\sup X > -E(X)$ $\therefore X - \sup X > X - E(X)$ $\therefore$

$(X - \sup X)^2 < (X - E(X))^2$ $\because$ $X - \sup X < 0$ $\therefore X - E(X) < 0$ $\therefore$

both sides are negative $\therefore$ can reduce penalty by choosing $E(X) = \sup X$, indep of $X$.

\1aiii/ use duality of events & random ~~variable~~ quantities $\therefore$

$\Rightarrow$ as random quantity: $(A \wedge B) \vee (C \wedge D) = ((A \wedge B) \wedge (C \wedge D)) =$

$1-(1-AB)(1-CC) = 1-(1-AB-CD+ABCD) = AB+CD-ABCD$ as

probability is the expectation of events, by the linearity of expectation

$\Rightarrow P((A \wedge B) \vee (C \wedge D)) = E[AB+CD-ABCD] = E(AB)+E(CD)-E(ABCD) = 0.5+0.4-0.2=0.7$

\1bi/ begin by establishing a preference ordering over the reward set: $r_1 \le^* r_2 \le^* \dots \le^* r_k$ wlog $\therefore$

set $U(r_1)=0$ & $U(r_k)=1$ $\therefore$ for $j=2,\dots,k-1$ find $p_j$:

$r_j \sim^* p_j r_k +_g (1-p_j) r_1$ $\therefore$ $U(r_j) = p_j$

\1bii/ $\therefore L \,{}^*\!\!> E \,{}^*\!\!> B \,{}^*\!\!> M$ $\therefore U(L)=1$, $U(M)=0$ & I set $U(E)=0.9$, $U(B)=0.4$.

\1biii/ gambles $G_1 = B$ for sure (with $U(G_1)=U(B)=0.4$) or

$G_2 = pL +_g (1-p)M$, $p=0.2$ $\therefore U(G_2)=0.2U(L)+0.8U(M)=0.2$ $\therefore$

$U(G_1) > U(G_2)$ $\therefore$ should accept Bristol weekend for sure

\1ci/ By Bayes theorem the posterior density for $\theta$ is

$\Rightarrow$ given $y$ is $\pi(\theta|y) \propto \pi(\theta) \prod_{i=1}^{n} \theta y_i^{-\theta-1} \propto \theta^{a-1} e^{-b\theta} \prod_{i=1}^{n} \theta y_i^{-\theta-1} \propto$

$\theta^{a+n-1} e^{-b\theta} \left(\prod_{i=1}^{n} y_i\right)^{-\theta} \propto \theta^{a+n-1} e^{-b\theta} e^{\ln\left(\left(\prod_{i=1}^{n} y_i\right)^{-\theta}\right)} \propto \theta^{a+n-1} e^{-b\theta} e^{-\theta \ln\left(\prod_{i=1}^{n} y_i\right)} \propto$

$\theta^{a+n-1} e^{-b\theta} e^{-\theta \sum_{i=1}^{n} \ln(y_i)} \propto \theta^{a+n-1} e^{-(b+\sum_{i=1}^{n} \ln(y_i))\theta}$ $\therefore$

Comparing to density of the Gamma$(a,b)$ prior shows it is proportional to a Gamma density $\therefore \theta | y \sim \text{Gamma}(a+n, b+\sum_{i=1}^{n} \ln y_i)$ with density ~~$\pi(\theta)$~~ $\therefore$ let $a+n = a_1$, $b_1 = b + \sum_{i=1}^{n} \ln y_i$ $\therefore$ density $\frac{b_1^{a_1}}{\Gamma(a_1)} \theta^{a_1 - 1} e^{-b_1 \theta}$ for $\theta > 0$

\1cii/ the prior and posterior distri are both Gamma, so Z prior is conjugate for Z pareto model

\1ciii/ Z prior distribution is a Gamma distribution & $\therefore$ is a valid probability ~~as~~ distri that integrates to 1 $\therefore$ Z prior is proper.

\1di/ Starting at values $X_1^{(0)}$ & $X_2^{(0)}$ for $t = 1, 2, \ldots$ set, sample $X_1^{(t)}, X_2^{(t)}$ by first sampling $X_1^{(t)}$ from $p(X_1 | X_2 = X_2^{(t-1)})$ & then sampling $X_2^{(t)}$ from $p(X_2 | X_1 = X_1^{(t)})$ Z joint probabs are proportional to those in Z table, & $\therefore$ each row & column sums to 0.4 $\therefore$ $p(X_1 = 1 | X_2 = 1) = \frac{p(X_1 = 1, X_2 = 1)}{p(X_2 = 1)} = \frac{0.2k}{0.4k} = 0.5$ &

$p(X_1 = 2 | X_2 = 1) = \frac{p(X_1 = 2, X_2 = 1)}{p(X_2 = 1)} = \frac{0.1k}{0.4k} = 0.25$ $\therefore$ due to symmetry of $X_1$ & $X_2$ $\therefore$ $p(X_1 = i | X_2 = j) = \begin{cases} 1/2 & i = j \\ 1/4 & i \neq j \end{cases}$ & same is true for $X_2 | X_1$

\1dii/ suppose X is discrete taking values $x_1, \ldots, x_m$ with probabs $p_1, \ldots, p_m$ respectively. to sample from X first define $F_j = p_1 + \ldots + p_j$ then $F^{-1}(p) = x_j$ for $p \in (F_{j-1}, F_j]$ $\therefore$ generate unif values on $[0,1]$ & treat these as p in Z above to set sample vals $X_U = F^{-1}(U)$

\1diii/ our sampler now runs as follows: $F(X_1^{(1)} | X_2^{(0)} = 3) =$
$(F_1, F_2, F_3) = (\frac{1}{4}, \frac{1}{2}, 1)$. $U_1 = 0.441$ $\therefore$ $X_1^{(1)} = 2$ $\therefore$
$F(X_2^{(1)} | X_1^{(1)} = 2) = (\frac{1}{4}, \frac{3}{4}, 1)$ $\therefore$ $U_2 = 0.018$ $\therefore$ $X_2^{(1)} = 1$
$F(X_1^{(2)} | X_2^{(1)} = 1) = (\frac{1}{2}, \frac{3}{4}, 1)$ $\therefore$ $U_3 = 0.304$ $\therefore$ $X_1^{(2)} = 1$
$F(X_2^{(2)} | X_1^{(2)} = 1) = (\frac{1}{2}, \frac{3}{4}, 1)$ $\therefore$ $U_4 = 0.732$ $\therefore$ $X_2^{(2)} = 2$
$\therefore$ Z 2 samples are $(2,1)$, $(1,2)$

\2017Mock/\1ai/ To be coherent, for this definition of expectation, means that you do not have a preference for a given penalty if you have the option of another that is certainly smaller

\1ai/ To be coherent, for this definition of expectation, means that you do not have a preference for a given penalty if you have the option of another that is certainly smaller

\1aii/ Show violating $\inf X \leq E(X) \leq \sup X$ is incoherent $\therefore$

if $E(X) < \inf X \therefore -\inf X < -E(X) \therefore X - \inf X < X - E(X) \therefore$

$(X - \inf X)^2 < (X - E(X))^2$ as $X - \inf X \geq 0 \therefore X - E(X) > 0 \therefore \because L = \left(\frac{X-\bar{x}}{k}\right)^2 \therefore$

implies can reduce your penalty by changing $E(X)$ to $\inf(X)$, independent of $X \therefore$ by coherence $E(X) \geq \inf(X) \therefore$

if $E(X) > \sup X \therefore -\sup X > -E(X) \therefore X - \sup X > X - E(X) \therefore (X - \sup X)^2 > (X - E(X))^2$

$\because X - \sup X < 0 \therefore X - E(X) < 0 \therefore$ both sides are negative $\therefore$ can reduce penalty by choosing $E(X) = \sup X$, independent of $X \therefore$ defying coherence $\therefore$ to be coherent: $X$ must be bounded: $\inf X \leq E(X) \leq \sup X$

\1biii/ gambles: $G_1 = B$ for sure (with $U(G_1) = U(B) = 0.4$) or $G_2 = pL +_g (1-p)M$, $P = 0.2 \therefore U(G_2) = 0.2U(L) + 0.8U(M) = 0.2 \therefore$ $U(G_1) > U(G_2) \therefore$ should accept Bristol weekend for sure

\1biii/ gambles: $G_1 = 1 \times B = B$ for sure, with $U(G_1) = U(B) = 0.4$. or $G_2 = PL +_g (1-P)M$, $P = \frac{1}{5} = 0.2 \therefore G_2 = 0.2L +_g 0.8M \therefore$

$U(G_2) = U(0.2L +_g 0.8M) = U(0.2L) + U(0.8M) = 0.2U(L) + 0.8U(M) =$

$0.2 \times 1 + 0.8 \times 0 = 0.2 \therefore \quad 0.4 > 0.2 \therefore$

$U(G_1) > U(G_2) \therefore$ Should accept Bristol weekend for sure

\1ci/ By Bayes theorem the posterior density for $\theta$ given $y$:

$\Rightarrow \pi(\theta|y) \propto \pi(\theta) f(y|\theta) \propto \pi(\theta) \prod_{i=1}^{n} f(y_i|\theta) \propto \pi(\theta) \prod_{i=1}^{n} \theta y_i^{-\theta-1} \propto$

$\pi(\theta) \prod_{i=1}^{n} \theta y_i^{-\theta} y_i^{-1} \propto \pi(\theta) \prod_{i=1}^{n} \theta y_i^{-\theta} \propto \pi(\theta)\theta^n \prod_{i=1}^{n} y_i^{-\theta} \propto \pi(\theta)\theta^n \left(\prod_{i=1}^{n} y_i\right)^{-\theta} \propto$

$\pi(\theta)\theta^n e^{\ln((\prod_{i=1}^{n} y_i)^{-\theta})} \propto \pi(\theta)\theta^n e^{-\theta \ln \prod_{i=1}^{n} y_i} \propto \pi(\theta)\theta^n e^{-\theta \sum_{i=1}^{n} \ln y_i} \propto$

$\frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}\theta^n e^{-\theta\sum_{i=1}^{n}\ln y_i} \propto \theta^{a-1}e^{-b\theta}\theta^n e^{-\theta\sum_{i=1}^{n}\ln y_i} \propto$

$\theta^{a+n-1}e^{-b\theta-\theta\sum_{i=1}^{n}\ln y_i} \propto \theta^{(a+n)-1}e^{-(b+\sum_{i=1}^{n}\ln y_i)\theta}$

Comparing this to the density of the Gamma$(a,b)$ prior shows that it is proportional to a Gamma density ∴

$\theta|y \sim \text{Gamma}(a+n, b+\sum_{i=1}^{n}\ln y_i)$

with density $\pi(\theta|y) = \frac{(b+\sum_{i=1}^{n}\ln y_i)^{a+n}}{\Gamma(a+n)}\theta^{a+n-1}e^{-(b+\sum_{i=1}^{n}\ln y_i)\theta}$ for $\theta > 0$

\1ci/ By Bayes theorem: the posterior density for $\theta$ given $y$:

$\pi(\theta|y) \propto \pi(\theta) f(y|\theta) \propto \pi(\theta)\prod_{i=1}^{n} f(y_i|\theta) \propto \pi(\theta)\prod_{i=1}^{n}\theta y_i^{-\theta-1} \propto$

$\pi(\theta)\prod_{i=1}^{n}\theta y_i^{-\theta}y_i^{-1} \propto \pi(\theta)\prod_{i=1}^{n}\theta y_i^{-\theta} \propto \pi(\theta)\theta^n\prod_{i=1}^{n}y_i^{-\theta} \propto \pi(\theta)\theta^n(\prod_{i=1}^{n}y_i)^{-\theta} \propto$

$\pi(\theta)\theta^n e^{\ln((\prod_{i=1}^{n}y_i)^{-\theta})} \propto \pi(\theta)\theta^n e^{-\theta\ln\prod_{i=1}^{n}y_i} \propto \pi(\theta)\theta^n e^{-\theta\sum_{i=1}^{n}\ln y_i} \propto$

$\frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}\theta^n e^{-\theta\sum_{i=1}^{n}\ln y_i} \propto \theta^{a-1}e^{-b\theta}\theta^n e^{-\theta\sum_{i=1}^{n}\ln y_i} \propto \theta^{a+n-1}e^{-b\theta-\theta\sum_{i=1}^{n}\ln y_i} \propto$

$\theta^{(a+n)-1}e^{-(b+\sum_{i=1}^{n}\ln y_i)\theta} \propto \theta^{a_1-1}e^{-b_1\theta}$ for $a_1 = a+n$, $b_1 = b+\sum_{i=1}^{n}\ln y_i$

∴ Comparing this to the density of the Gamma$(a,b)$ prior shows

$\theta|y \sim \text{Gamma}(a_1, b_1)$

with density $\pi(\theta|y) = \frac{b_1^{a_1}}{\Gamma(a_1)}\theta^{a_1-1}e^{-b_1\theta}$ for $\theta > 0$

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\2017Mock/ \1di/ Start at values $X_1^{(0)}$ and $X_2^{(0)}$.

for $t=1,2,\dots$ set sample $X_1^{(t)}, X_2^{(t)}$ by:

1) first sampling $X_1^{(t)}$ from $p(X_1 | x_2 = X_2^{(t-1)})$

eg ~~$X_1^{(1)}$ from~~ sample $X_1^{(1)}$ from $p(X_1 | x_2 = X_2^{(0)})$

then sampling $X_2^{(t)}$ from $p(X_2 | X_1 = X_1^{(t)})$

eg sample $X_2^{(1)}$ from $p(X_2 | X_1 = X_1^{(1)})$

question tells us the joint probabilities are proportional to those in the table, and each column and rows sums to 0.4: eg

$p(X_2=1) = k\left(p(X_2=1, X_1=1) + p(X_2=1, X_1=2) + p(X_2=1, X_1=3)\right) = k(0.2+0.1+0.1) = 0.4k$

∴ By ~~Bayes~~ conditional prob: $p(X_1=1, X_2=1) = 0.2k$ ∴

$p(X_1=1 | X_2=1) = \frac{p(X_1=1, X_2=1)}{p(X_2=1)} = \frac{0.2k}{0.4k} = 0.5$,

$p(X_1=2 | X_2=1) = \frac{p(X_1=2, X_2=1)}{p(X_2=1)} = \frac{0.1k}{0.4k} = 0.25 = p(X_1=3 | X_2=1)$ ∴

~~p(X,Z~~ due to symmetry: $p(X_1=i | X_2=j) = \begin{cases} \frac{1}{2} & i=j \\ 0.25 & i\neq j \end{cases}$

and same is true for $X_2 | X_1$ ∴ $p(X_2=i | X_1=j) = \begin{cases} \frac{1}{2} & i=j \\ \frac{1}{4} & i\neq j \end{cases}$

\1di/ Start at values $X_1^{(0)}, X_2^{(0)}$

for $t=1,2,\dots$ set samples $X_1^{(t)}, X_2^{(t)}$

by first sampling $X_1^{(t)}$ from $p(X_1 | X_2 = X_2^{(t-1)})$

then sampling $X_2^{(t)}$ from $p(X_2 | X_1 = X_1^{(t)})$

The joint probs are proportional to those in the table, & ∴ each row & column sums to 0.4:

$p(X_1=1 | X_2=1) = \frac{p(X_2=1, X_2=1)}{p(X_2=1)} = \frac{0.2k}{0.4k} = 0.5$

$p(X_1=2 | X_2=1) = \frac{p(X_2=2, X_2=1)}{p(X_2=1)} = \frac{0.1k}{0.4k} = 0.25$

∴ Due to symmetry of $X_1, X_2$: $p(X_1=i | X_2=j) = \begin{cases} 1/2 & i=j \\ 1/4 & i\neq j \end{cases}$

& same true for $X_2|X_1$: $p(X_2=i | X_1=j) = \begin{cases} 1/2 & i=j \\ 1/4 & i\neq j \end{cases}$

\1dii/ suppose $X$ is discrete taking values $x_1, \dots, x_m$ with probabilities $p_1, \dots, p_m$ respectively. To sample from $X$, first define $F_j = p_1 + \dots + p_j$ ∴ $F^{-1}(p) = x_j$ for $p \in (F_{j-1}, F_j]$ ∴, generate uniform values on $[0,1]$ and treat these as $p$ in the above to set sample values $X_U = F^{-1}(U)$

\1dii/ Suppose $X$ is discrete taking values $x_1, \dots, x_m$ with probs: $p_1, \dots, p_m$ respectively. To sample from $X$ first define $F_j = p_1 + \dots p_j$ then $F^{-1}(p) = x_j$ for $p \in (F_{j-1}, F_j]$ so generate uniform values on $[0,1]$ and treat these as $p$ in the above to get sample values $X_U = F^{-1}(U)$

\1diii/ our Gibbs sampler now runs as follows:

$F(X_1^{(1)} | X_2^{(0)} = 3) = (F_1, F_2, F_3) = (\frac{1}{4}, \frac{1}{4}+\frac{1}{4}, \frac{1}{4}+\frac{1}{4}+\frac{1}{2}) = (\frac{1}{4}, \frac{1}{2}, \frac{1}{2}+\frac{1}{2}) = (\frac{1}{4}, \frac{1}{2}, 1)$

$\therefore U_1 = 0.441 \therefore 0.441 < 0.5 = \frac{1}{2} \therefore \frac{1}{4} < U_1 < \frac{1}{2} \therefore X_1^{(1)} = 2 \therefore$

find $x_2^{(1)}$: $F(X_2^{(1)} | X_1^{(1)} = 2) = (\frac{1}{4}, \frac{1}{4}+\frac{1}{2}, \frac{1}{4}+\frac{1}{2}+\frac{1}{4}) = (\frac{1}{4}, \frac{3}{4}, \frac{3}{4}+\frac{1}{4}) = (\frac{1}{4}, \frac{3}{4}, 1)$

$\therefore U_2 = 0.018 \le 0.25 = \frac{1}{4} \therefore U_2 < \frac{1}{4} \therefore X_2^{(1)} = 1 \therefore$

find $X_1^{(2)}$: $F(X_1^{(2)} | X_2^{(1)} = 1) = (\frac{1}{2}, \frac{1}{2}+\frac{1}{4}, \frac{1}{2}+\frac{1}{4}+\frac{1}{4}) = (\frac{1}{2}, \frac{3}{4}, \frac{3}{4}+\frac{1}{4}) = (\frac{1}{2}, \frac{3}{4}, 1)$

$\therefore U_3 = 0.304 < \frac{1}{2} \therefore U_3 < \frac{1}{2} \therefore X_1^{(2)} = 1 \therefore$

find $x_2^{(2)}$: $F(X_2^{(2)} | X_1^{(2)} = 1) = (\frac{1}{2}, \frac{1}{2}+\frac{1}{4}, \frac{1}{2}+\frac{1}{4}+\frac{1}{4}) = (\frac{1}{2}, \frac{3}{4}, \frac{3}{4}+\frac{1}{4}) = (\frac{1}{2}, \frac{3}{4}, 1) \therefore$

$U_4 = 0.732 < 0.75 = \frac{3}{4} \therefore \frac{1}{2} < U_4 < \frac{3}{4} \therefore X_2^{(2)} = 2 \therefore$

two samples are $(X_1^{(1)}, X_2^{(1)}) = (2, 1)$, $(X_1^{(2)}, X_2^{(2)}) = (1, 2)$

\3/ As $f(x)$ is a probab distri $\therefore$ integrates to 1 over its domain $\therefore$

$\int_{-2}^{2} g(x)\,dx = \int_{-2}^{2} c f(x)\,dx = c\int_{-2}^{2} f(x)\,dx = c(1) = c$,

$c = \int_{-2}^{2} (2+x)(2-x)\,dx = \int_{-2}^{2} 4 - x^2\,dx = \left[4x - \frac{1}{3}x^3\right]_{-2}^{2} = 4(2-(-2)) - \frac{1}{3}(2^3 - (-2)^3) = \frac{32}{2}$

~~\3b/ $f_u(x)$ for $x \in [-2, 2]$ $\therefore \frac{1}{2-(-2)} = \frac{1}{2+2} = \frac{1}{4}$ $\therefore f_u(x) = \frac{1}{4}x$~~

~~$\therefore F_u(2) = \int_{-2}^{2} f_u(x)\,dx = \int_{-2}^{2} \frac{1}{4}x\,dx = \frac{1}{4}\int_{-2}^{2} x\,dx = \frac{1}{4}\left[\frac{1}{2}x^2\right]_{-2}^{2} =$ $\frac{1}{8}(2^2 - (-2)^2) =$~~

~~\3b/ $f_u(x)$ for $x \in [-2, 2]$ $\therefore \frac{1}{2-(-2)} = \frac{1}{2+2} = \frac{1}{4}$ $\therefore f_u(x) = \frac{1}{4}$ $\therefore$~~

~~$\text{Max} f(x) = \text{Max} \frac{1}{c} g(x) = \text{Max} \frac{2}{32} g(x) = \text{Max} \frac{2}{32}(4-x^2) =$ $\frac{2}{32}(4-x^2)\Big|_{x=0} = \frac{2}{32}(4-0^2) = \frac{2 \times 4}{32} = \frac{1}{4} = k$ $\therefore$~~

~~$g_u(x) = k f_u(x) = \frac{1}{4} \times \frac{1}{4} = \frac{1}{16}$ $\therefore$ acceptance rate $= \frac{c}{k} = \frac{(32/2)}{(1/16)} = 256$~~

~~$\frac{k}{c} = \frac{(1/16)}{(32/2)} = \frac{1}{256}$~~

\3b/ The max value of $g(x)$: $\text{Max}\, g(x) = \text{Max}(4-x^2) = (4-x^2)\Big|_{x=0} = 4$ $\therefore$ is 4 at $x = 0$

\2017 Mock/ $f_u(x)$ for $x \in [-2,2]$ and is a uniform distri $\therefore$

$f_u(x) = \frac{1}{2-(-2)} = \frac{1}{2+2} = \frac{1}{4}$ $\therefore$

~~The~~ $\cancel{\text{Max}\, g(x) = k}$ $g_u(x) = k f_u(x) = k\frac{1}{4}$,

$g_u \cancel{(x) \geq} \text{Max}\, g(x)$

\3b/ The maximum value of $g(x)$ is 4 at $x=0$

and $f_u(x) = \frac{1}{4}$ $\because$ its the uniform distribution on $[-2,2]$

require: $g_u(x) = K f_u(x) \geq g(x) = c f(x)$ $\forall x$ st acceptance rate

$\frac{c}{k}$ is maximised $\therefore k \geq 16$ must hold, else at $x=0$

$g_u(x) < g(x)$, and the acceptance rate is decreasing in k

$\therefore K = 16$ is optimal k $\therefore$

acceptance rate for this proposal is $c/k = (\frac{32}{3})/16 = \frac{2}{3}$

\3a/ $f(x)$ is a pdf $\therefore$ integrates to one $\therefore$

$\frac{1}{c} g(x) = f(x)$ $\therefore$ $f(x) \propto (2+x)(2-x) = g(x)$ $\therefore$ $f(x) \propto g(x)$ $\therefore$

$f(x) = \frac{1}{c} g(x)$ $\therefore$ $\int_{-2}^{2} f(x)\,dx = 1 = \int_{-2}^{2} \frac{1}{c} g(x)\,dx = \frac{1}{c}\int_{-2}^{2} g(x)\,dx =$

$\frac{1}{c}\int_{-2}^{2} (2+x)(2-x)\,dx = \frac{1}{c}\int_{-2}^{2} (4-x^2)\,dx = \frac{1}{c}\left[4x - \frac{1}{3}x^3\right]_{-2}^{2} = \frac{1}{c}\left[4(2-(-2)) - \frac{1}{3}(2^3-(-2)^3)\right] =$

$\frac{1}{c}\left[4(2+2) - \frac{1}{3}(8+8)\right] = \frac{1}{c}\left[4(4) - \frac{1}{3}(16)\right] = \frac{1}{c}\left[16 - \frac{16}{3}\right] = \frac{1}{c}\left[\frac{2}{3}(16)\right] = \frac{1}{c}\left(\frac{32}{3}\right) = 1$

$\therefore c = \frac{32}{3}$

\3b/ $f_u(x)$ is uniform distri for $x \in [-2,2]$ $\therefore$

$f_u(x) = \frac{1}{2-(-2)} = \frac{1}{2+2} = \frac{1}{4}$ $\therefore$

$g_u(x) = k f_u(x) = k\frac{1}{4}$ $\therefore$

$\text{Max}\, g(x) = \text{Max}(4-x^2) = (4-x^2)\big|_{x=0} = (4-0^2) = 4$ $\therefore$

acceptance rate $= \frac{c}{k}$ $\therefore$ want smallest k $\therefore$

$\cancel{g_u(x) \geq 4}$ $\therefore$ $g_u(x) \geq \text{Max}\, g(x) = 4$ $\therefore$ $g_u(x) \geq 4$ $\therefore$

$k\frac{1}{4} \geq 4$ $\therefore$ $k \geq 16$ $\therefore$ $\therefore$ $k = 16$ $\therefore$

acceptance rate $= \frac{c}{k} = \frac{32/3}{16} = \frac{2}{3}$

\3c/ let $h(x) = \ln g(x) = \ln(4-x^2)$ $\therefore$

$h'(x) = \frac{d}{dx} h(x) = \frac{d}{dx} \ln(4-x^2) = \frac{-2x}{4-x^2}$ $\therefore$

$h''(x) = \frac{d}{dx} h'(x) = \frac{d}{dx} \frac{-2x}{4-x^2} = \cancel{\frac{-2x(-2x) - (4-x^2)(-2) + 2x(-2x)}{(4-x^2)^2}} = \cancel{\frac{-8+2x^2-4x^2}{(4-x^2)^2}}$ z

$\frac{d}{dx}\left((-2x)(4-x^2)^{-1}\right) = \cancel{\frac{d}{dx}}\ -2(4-x^2)^{-1} - 2x(-1)(4-x^2)^{-2}(-2x) =$

$\frac{-2}{4-x^2} - \frac{4x^2}{(4-x^2)^2} = -\left(\frac{2}{4-x^2} + \frac{4x^2}{(4-x^2)^2}\right) < 0$ for $x \in [-2, 2]$ ie,

$h''(x) < 0 \therefore \frac{d^2}{dx^2}(\log(g(x))) < 0 \therefore g(x)$ is log concave

\3d/ let $\hat{x} = \text{argmax}(g(x))$ and $H = \sup h''(x) < 0 \therefore$ then, the algorithm is to generate samples using $f_u(x)$ the pdf of a Normal distri $N(\hat{x}, \frac{1}{-H})$ $\therefore h''(0) = -\frac{2}{4-0^2} - \frac{4(0)^2}{(4-0^2)^2} = \frac{-2}{4} = -\frac{1}{2} < 0$

argmax $g(x) = 4 \therefore$ ~~N~~ $\{$ maybe $N(4, \frac{1}{-(-\frac{1}{2})}) = N(4, \frac{1}{(\frac{1}{2})}) = N(4, 2)$

with the envelope $g_u(x) = e^{h(\hat{x})} e^{\frac{H}{2}(x-\hat{x})^2}$ st the $k$ st $g_u(x) = kf_u(x) \leq g(x)$ is $k = e^{h(\hat{x})}\sqrt{\frac{2\pi}{-H}}$

ie full we generate $X_u$ from $f_u(x)$, then generate Uniform$(0,1)$, and accept $X_u$ if $Ug_u(X_u) < g(X_u)$

\3e/ $\therefore h''(x) = \frac{d}{dx}\left(-\frac{2}{4-x^2} - \frac{4x^2}{(4-x^2)^2}\right) = \cancel{\frac{d}{dx}\left(\frac{-2x}{4-x}\right) - 2}$

$\frac{d}{dx}\left(-2(4-x^2)^{-1} - 4x^2(4-x^2)^{-2}\right) =$

$-2(4-x^2)^{-2}(-1)(-2x) - 4(2)x(4-x^2)^{-2} - 4x^2(-2)(4-x^2)^{-3}(-2x) = -\frac{12x}{(4-x^2)^2} - \frac{4x^2}{(4-x^2)^3}$

which is 0 if $x = 0$ or if $\cancel{12x(4-x^2) - 4x = 48x - 12x^3 - 4x}$

$12 - 3x^2 + x = 0$ ie $x = \frac{1 \pm \sqrt{145}}{6}$

in the interval $[-2, 2]$, $h''(x)$ is maximised at $x = 0$ and is $-\frac{1}{2} < 0$ there $\therefore \bar{H} = -\frac{1}{2}$ and the conditions of the alg have been met now $\hat{x} = 0$ and $g(\hat{x}) = 4$, so

$R = \frac{c}{k} = \frac{32/3}{4\sqrt{\frac{2\pi}{1/2}}} = \frac{32/3}{4\sqrt{4\pi}} = \frac{4}{3\sqrt{\pi}} \therefore e^{h(\hat{x})} e^{\frac{H}{2}(x-\hat{x})^2} = g_u(x) \therefore$

$k = e^{h(\hat{x})}\sqrt{\frac{2\pi}{-H}} = e^{h(0)}\sqrt{\frac{2\pi}{-(-\frac{1}{2})}} = e^{\ln(4+0)}\sqrt{\frac{2\pi}{\frac{1}{2}}} = e^{\ln 4}\sqrt{4\pi} = 4\sqrt{4\pi} = 8\sqrt{\pi} \therefore$

$R = \frac{c}{k} = \frac{32/3}{8\sqrt{\pi}} = \frac{4}{3\sqrt{\pi}}$

\3f/ first shift & scale each $N(0,1)$ number st it has the same distri as $f_u(x)$: $X_u^{(1)} = 0 + (-1.73) \times \sqrt{\frac{1}{-(-1/2)}} = -1.73 \times \sqrt{2} = 2.45$ and similarly $X_u^{(2)} = 0.255 \times \sqrt{2} = 0.361$

& $X_u^{(1)} \notin [-2, 2] \therefore g(X_u^{(1)}) = 0 \therefore X_u^{(1)}$ is rejected, $g_u(X_u^{(2)}) = 4e^{-\frac{1}{4}(0.361^2)} = 3.87$

& $g(X_u^{(2)}) = (2+0.361)(2-0.361) = 3.87 \therefore Ug_u(X_u^{(2)}) < g(X_u^{(2)})$ & $X_u^{(2)} = 0.381$ is accepted

~~PP2022~~ $\cancel{4b}$ By Bayes $\pi(\mu|y) \propto \pi(\mu) p(y_i|\mu)$

$\pi(\mu, \sigma^2)$

$\pi(\mu, \sigma^2|y) \propto \pi(\mu, \sigma^2) p(y_i|\mu, \sigma^2) = \pi(\mu|\sigma^2)\pi(\sigma^2) p(y_i|\mu, \sigma^2)$

$\therefore \sigma^2 \sim IG(a, b)$ $\therefore$

$\pi(\theta) = \frac{b^a}{\Gamma(a)} \left(\frac{1}{\theta}\right)^{a-1} \exp\left\{-\frac{b}{\theta}\right) = \frac{b^a}{\Gamma(a)} \left(\frac{1}{\theta}\right)^{a-1} e^{-\frac{b}{\theta}} = \frac{b^a}{\Gamma(a)} (\theta^{-1})^{a-1} e^{-b\theta^{-1}}$ $\therefore$

$\pi(\theta) = \frac{b^a}{\Gamma(a)} \theta^{-a+1} e^{-b\theta^{-1}}$ $\therefore$

~~Ar(p~~ is $x \sim N(\mu, \sigma^2)$ $\therefore$ $\pi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ $\therefore$

$\mu|\sigma^2 \sim N(\mu_0, \tau\sigma^2) = N(\mu_0, (\sqrt{\tau}\sigma)^2)$

$\pi(\mu|\sigma^2) = \frac{1}{\sqrt{2\pi}\sqrt{\tau}\sigma} e^{-\frac{1}{2\tau\sigma^2}(\mu-\mu_0)^2}$ $\therefore$

$\because$ $a, b, \tau, \mu_0, y_i$ are constants $\therefore$

By Bayes: $\pi(\theta|y) \propto \pi(\theta) f(y_i|\theta)$ $\therefore$

By Bayes theorem: $\pi(\mu, \sigma^2|y) \propto \pi(\mu, \sigma^2) f(y_i|\mu, \sigma^2) =$

$\pi(\mu|\sigma^2)\pi(\sigma^2) f(y_i|\mu, \sigma^2)$ $\therefore$

$f(y_i|\mu, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2}$ $\therefore$

$\pi(\mu, \sigma^2|y) \propto \pi(\mu|\sigma^2)\pi(\sigma^2) f(y_i|\mu, \sigma^2) \propto$

$\pi(\mu|\sigma^2)\pi(\sigma^2) \prod_{i=1}^{n} \left( \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2} \right. \propto$

$\pi(\mu|\sigma^2)\pi(\sigma^2) \prod_{i=1}^{n} \frac{1}{\sigma} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2} \propto$

$\pi(\mu|\sigma^2)\pi(\sigma^2) \sigma^{-n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2} \propto$

$\pi(\mu|\sigma^2)\pi(\sigma^2) \sigma^{-n} e^{-\frac{1}{2\sigma^2}\left[\sum_{i=n}^{n} y_i^2 - 2\mu\sum_{i=1}^{n} y_i + n\mu^2\right]} \propto$

$\pi(\mu|\sigma^2)\pi(\sigma^2) \sigma^{-n} e^{-\frac{1}{2\sigma^2}\left[\cancel{\;} - 2n\bar{y}\mu + n\mu^2\right]}$ $\cancel{\propto}$

$\cancel{\pi(\mu|\sigma^2)\frac{b^a}{\Gamma(a)}}$ $\pi(\sigma^2) = \frac{b^a}{\Gamma(a)} \left(\frac{1}{\sigma^2}\right)^{a+1} e^{-\frac{b}{\sigma^2}} = \frac{b^a}{\Gamma(a)} (\sigma^{-2})^{a+1} e^{-b\sigma^{-2}} =$

$\pi(\sigma^2) = \frac{b^a}{\Gamma(a)} \sigma^{-2a-2} e^{-b\sigma^{-2}}$ $\therefore$

$\pi(\mu, \sigma^2|y) \propto \pi(\mu|\sigma^2) \frac{b^a}{\Gamma(a)} \sigma^{-2a-2} e^{-b\sigma^{-2}} \sigma^{-n} e^{-\frac{n}{2\sigma^2}[-2\bar{y}\mu + \mu^2]} \propto$

$\cancel{\sigma^{-2a-2} e^{-b\sigma^{-2}} \sigma^{-n} e^{-\frac{n}{2\sigma^2}[-2\bar{y}\mu+\mu}}$

$\pi(\mu|\sigma^2) \sigma^{-2a-2} e^{-b\sigma^{-2}} \sigma^{-n} e^{-\frac{n}{2\sigma^2}[-2\bar{y}\mu + \mu^2]} \propto$

$\pi(\mu|\sigma^2) \sigma^{-2a-n-2} e^{-b\sigma^{-2} - \frac{n}{2\sigma^2}[-2\bar{y}\mu + \mu^2]} \propto$

$\frac{1}{\sqrt{2\pi}\sqrt{\tau}\sigma} e^{-\frac{1}{2\tau\sigma^2}(\mu-\mu_0)^2} \sigma^{-(2a+n+2)} e^{-\frac{1}{\sigma^2}\left[b + \frac{n}{2}(-2\bar{y}\mu + \mu^2)\right]} \propto$

$\sigma^{-1} e^{-\frac{1}{2\tau\sigma^2}(\mu^2 - 2\mu_0\mu + \mu_0)} \sigma^{-(2a+n+2)} e^{-\frac{1}{\sigma^2}\left[b + \frac{n}{2}(-2\bar{y}\mu + \mu^2)\right]} \propto$

$e^{-\frac{1}{\sigma^2}\frac{1}{2\tau}(\mu^2-2\mu_0\mu)}\sigma^{-(2a+n+3)}e^{-\frac{1}{\sigma^2}[b+\frac{n}{2}(-2\bar{y}\mu+\mu^2)]}\propto$

$\sigma^{-(2a+n+3)}e^{-\frac{1}{\sigma^2}\frac{1}{2\tau}(\mu^2-2\mu_0\mu)-\frac{1}{\sigma^2}[\frac{2b}{2}+\frac{n}{2}(-2\bar{y}\mu+\mu^2)]}\propto$

$\sigma^{-(2a+n+3)}e^{-\frac{1}{\sigma^2}[\frac{1}{2\tau}(\mu^2-2\mu_0\mu)+\frac{2b}{2}+\frac{n}{2}(-2\bar{y}\mu+\mu^2)]}\propto$

$\sigma^{-(2a+n+3)}e^{-\frac{1}{2\sigma^2}[\frac{1}{\tau}\mu^2-2\frac{1}{\tau}\mu_0\mu+2b+n\frac{1}{\tau}(-2\bar{y}\mu+\mu^2)]}\propto$

$\sigma^{-(2a+n+3)}e^{-\frac{1}{2\sigma^2}[\frac{1}{\tau}\mu^2-2\frac{1}{\tau}\mu_0\mu+2b-2n\frac{1}{\tau}\bar{y}\mu+n\frac{1}{\tau}\mu^2]}\propto$

$\sigma^{-(2a+n+3)}e^{-\frac{1}{2}\frac{1}{\sigma^2}[(\frac{1}{\tau}+n\frac{1}{\tau})\mu^2+(-2\frac{1}{\tau}\mu_0-2n\frac{1}{\tau}\bar{y})\mu+\frac{2b}{\sigma^2}]}\propto$

$\sigma^{-(2a+n+3)}e^{-\frac{1}{2}[(\frac{1}{\tau\sigma^2}+\frac{n}{\tau\sigma^2})\mu^2-2(\frac{\mu_0}{\tau\sigma^2}+\frac{n\bar{y}}{\tau\sigma^2})\mu+\frac{2b}{\sigma^2}]}\propto$

$\sigma^{-(2a+n+3)}e^{-\frac{1}{2}[\frac{1+n}{\tau\sigma^2}\mu^2-2(\frac{\mu_0+n\bar{y}}{\tau\sigma^2})\mu+\frac{2b}{\sigma^2}]}\propto$

$\sigma^{-(2a+n+3)}e^{-\frac{1}{2}[c\mu^2-2d\mu+g]}\propto$

$\sigma^{-(2a+n+3)}e^{-\frac{1}{2}[c\mu^2-2d\mu]}\propto$

$\sigma^{-(2a+n+3)}e^{-\frac{1}{2}c[\mu^2-\frac{2d}{c}\mu]}\propto e^{-\frac{1}{2c^{-1}}[(\mu-\frac{d}{c})^2-\frac{d^2}{c^2}]}\sigma^{-(2a+n+3)}$

$\propto\sigma^{-(2a+n+3)}e^{-\frac{1}{2c^{-1}}[(\mu-\frac{d}{c})^2]}$ ∴

~~πᵢ~~ $\mu,\sigma^2|y\sim N(\frac{d}{c},\frac{1}{c})$, ~~of~~ $c=\frac{1+n}{\tau\sigma^2}$, $d=\frac{\mu_0+n\bar{y}}{\tau\sigma^2}$ ∴

$\frac{1}{c}=\frac{\tau\sigma^2}{1+n}$, $\frac{d}{c}=\frac{(\mu_0+n\bar{y})\tau\sigma^2}{(1+n)\tau\sigma^2}=\frac{\mu_0+n\bar{y}}{1+n}$ ∴

the posterior is a normal distribution ∴

∴ both the posterior is conjugate to a ~~of~~ the normal prior

\4a/ ~~a seque~~ $y$ is exchangeable is the joint probability distribution of each subcollection on $n$ quantities is the same

~~Hgs~~ \ / $N\sim\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ $G\sim\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$, $B\sim\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$

$Poi\sim e^{-\lambda}\lambda^x\frac{1}{x!}$ $Exp\sim\lambda e^{-\lambda x}$, $unif\sim\frac{1}{b-a}$

$N\sim\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ $G\sim\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$ $B\sim\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$

$Exp\sim\lambda e^{-\lambda x}$, $unif\sim\frac{1}{b-a}$ $N\sim\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ $G\sim\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$ $poi\sim e^{-\lambda}\lambda^x\frac{1}{x!}$ $B\sim\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}x^{\alpha-1}(1-x)^{\beta-1}$

\1a/ By Bayes theorem: $\pi(\theta|y)\propto\pi(\theta)f(y_i|\theta)$ ∴ $\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$

let $\pi(\theta)=\lambda e^{-\lambda\theta}$, ∴ $\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ $\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}x^{a-1}(1-x)^{b-1}$ $\lambda e^{-\lambda x}$ $e^{-\lambda}\lambda^x\frac{1}{x!}$ $\frac{1}{b-a}$

$\pi(\theta|y)\propto\pi(\theta)f(y_i|\theta)\propto\lambda e^{-\lambda\theta}f(y_i|\theta)\propto e^{-\lambda\theta}f(y_i|\theta)\propto$

$e^{-\lambda\theta}\prod_{i=1}^{n}y_i^\theta e^{\theta(y_i-1)}\propto e^{-\lambda\theta}(\prod_{i=1}^{n}e^{\ln(y_i^\theta)})e^{\theta\sum_{i=1}^{n}(y_i-1)}\propto e^{-\lambda\theta}e^{\theta(-n+\sum_{i=1}^{n}y_i)}\prod_{i=1}^{n}e^{\theta\ln y_i}\propto$

$e^{-\lambda\theta}e^{\theta(-n+n\bar{y})}e^{\theta\sum_{i=1}^{n}\ln y_i}$ ~~$\propto e^{\theta t}$~~

$\propto e^{(-\lambda-n+n\bar{y}+\sum_{i=1}^{n}\ln y_i)\theta}\propto e^{-(\lambda+n-n\bar{y}-\sum_{i=1}^{n}\ln y_i)\theta}\propto e^{-\lambda_n\theta}$ for $\lambda_n=\lambda+n-n\bar{y}-\sum_{i=1}^{n}\ln y_i$

∴ ~~the~~ the posterior $\theta|y\sim Exp(\lambda_n)$ ∴

\PP2022/ both the posterior and the prior follow a exponential distribution $\therefore$ the posterior is conjugate to the exponential distribution prior

\1b/ the classical 95% confidence interval assumes a parameter is an unchanging constant with the interval being different each time it is calculated with the new interval enclosing the fixed parameter with a proportion of on average 95 out of 100 times

a 95% Bayesian credible interval assumes the parameter is subjective and not a fixed constant and assumes the calculated parameter will fall into the interval with a proportion of 95% of the time where the interval is fixed

\1ci/ $\therefore U(C) > U(B) > U(D) \quad \therefore U(C) > U(D) \therefore$

$C \overset{*}{>} B \overset{*}{>} D \quad \therefore \quad C \overset{*}{>} D \;\therefore$

desire utilities: $U(C) = 1$ and $U(D) = 0$

$\therefore$ desire $p$ such that $pC \overset{*}{\sim} B$ for him $\therefore$

he has no preference $\therefore$

~~$U(B) = 2pq$~~ $U(B) = p$

\1ci/ set $U(C) = 1$, $U(D) = 0$

find $p$ : such that $B \sim^* pC +_g (1-p)D$

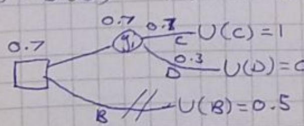
$\therefore$ set $U(B) = p$

~~Re~~ $\therefore$ he ~~might~~ ~~can set~~ lets $U(B) = 0.5$

\1cii/ for the gamble $g_1 \sim^* 0.7C +_g 0.3D \;\therefore$

its utility: $U(g_1) = U(0.7C + 0.3D) = U(0.7C) + U(0.3D)$
$= 0.7U(C) + 0.3U(D) = 0.7(1) + 0.3(0) = 0.7 \;\therefore\;$ $U(g_1) = 0.7 > 0.5 = U(B)$

$\therefore U(g_1) > U(B) \quad \therefore$ he should take the gamble $\therefore$



\ i c iii / rewards being coherently comparable

means if $r_i <^* r_j$, ~~$r_i$~~ $r_i \,{}^*\!\!> r_j$, $r_i \sim^* r_j$

only one can be true and one must be true

and if $r_i <^* r_j$ and $r_j <^* r_k$ then $r_i <^* r_k$

where $r$ are the rewards ∴

$C \,{}^*\!\!> B \,{}^*\!\!> D$ and $C \,{}^*\!\!> D$

and ~~not~~ none of the other possibilities are true ∴

his rewards are coherently comparable means $C \,{}^*\!\!> D$

and $C \,{}^*\!\!> B \,{}^*\!\!> D$ are true ∴ his rewards obey coherence

∴ there exists a solution to his descions

\ i c iv / $C \,{}^*\!\!> B \,{}^*\!\!> D$, $C \,{}^*\!\!> D$ ∴ $p < q$ ∴

for $pC +_g qB <^* pB +_g qC$ ∴

~~$pC \geq pC +_g qC + pB$~~. $pC +_g qB <^* qC +_g pB$ ∴

$pC +_g (1-p)B <^* (1-p)C +_g pB$ ∴

$p < q$ ∴ $qB \,{}^*\!\!> pB$,

$pC <^* qC$ ∴

~~$p(C +_g B -_g B$~~ ∴ $C \,{}^*\!\!> B$ ∴ $pC \,{}^*\!\!> pB$, $qC \,{}^*\!\!> qB$ ∴

$pC +_g (-pB) <^* qC +_g (-qB)$ ∴

$p(C +_g -B) <^* q(C +_g -B)$ ∴ $p < q$ ∴ $C +_g (-B) \,{}^*\!\!> 0$ ∴

$p(C +_g -B) <^* q(C +_g -B)$ holds true for $C \,{}^*\!\!> B$ ∴

$pC +_g qB <^* pB +_g qC$ holds with coherence

for $pD +_g qC <^* pC +_g qD$ ∴

~~$pD +_g$~~ $qC +_g (-qD) <^* pC +_g (-pD)$ ∴

$q(C +_g (-D)) <^* p(C +_g (-D))$ but $q > p$ and

$C \,{}^*\!\!> D$ ∴ $C +_g (-D) \,{}^*\!\!> 0$ ∴

$q(C +_g (-D)) \,{}^*\!\!> p(C +_g (-D))$ ∴

$pD +_g qC <^* pC +_g qD$ does not hold with coherence

$B <^* C$ ∴ $pB <^* pC$ ∴ $pB +_g qD <^* pC +_g qD$ ∴

~~$pB$~~ $pB +_g qD <^* pC +_g qD$ holds with coherence

\PP2022/\1di/ An events admits one of only two possible values True or False $\therefore$ let $A_1, \dots, A_n$ be mutually exclusive events $\therefore$ probability that the event that one of the $A_i$ happens is

$P(A) = P(A_1) + \dots + P(A_n)$ $\therefore$ $\because$ $P(A_i) = E(A_i)$ then $A_i$ can be viewed as random quantitees $\therefore$

by the event - random quantity ~~qua~~ duality, events can be viewed as random quantities taking values 1 or 0 and event $A$ such that $P(A) = E(A)$ as a random quantity

\1dii/ $\sim A = 1 - A$ $\therefore$ $\quad A \wedge B = AB$, $\quad A \vee B = \sim(\sim A \wedge \sim B) =$

$\sim((\sim A)(\sim B)) = \sim((1-A)(1-B)) = \sim(1 - A - B + AB) = 1 - (1 - A - B + AB) =$

$1 - 1 + A + B - AB = A + B - AB$

\1diii/ using the event - random quantity duality:

$P((A \wedge \sim(B \vee C)) \vee (A \wedge (B \vee C))) = P(\sim(\sim(A \wedge \sim(B \vee C)) \wedge \sim(A \wedge (B \vee C)))) =$

$P(\sim(\sim(A \wedge \sim\sim(\sim B \wedge \sim C)) \wedge \sim(A \wedge \sim(\sim B \wedge \sim C)))) =$

$P(\sim(\sim(A \wedge (\sim B \wedge \sim C)) \wedge \sim(A \wedge \sim(\sim B \wedge \sim C))))$

\1diii/ using event random quantity duality:

$(A \wedge \sim(B \vee C)) \vee (A \wedge (B \vee C)) =$

$(A \wedge \sim(\sim(\sim B \wedge \sim C))) \vee (A \wedge \sim(\sim B \wedge \sim C)) =$

$(A \wedge (\sim B \wedge \sim C)) \vee (A \wedge \sim(\sim B \wedge \sim C)) =$

$\sim[\sim(A \wedge (\sim B \wedge \sim C)) \wedge \sim(A \wedge \sim(\sim B \wedge \sim C))] =$

$\sim[\sim(A \wedge ((1-B) \wedge (1-C))) \wedge \sim(A \wedge \sim((1-B) \wedge (1-C)))] =$

$\sim[\sim(A \wedge ((1-B)(1-C))) \wedge \sim(A \wedge \sim((1-B)(1-C)))] =$

$\sim[\sim(A \wedge (1 - B - C + BC)) \wedge \sim(A \wedge \sim(1 - B - C + BC))] =$

$\cancel{\sim[\sim(AA}$ $\sim[\sim(A - AB - AC + ABC) \wedge \sim(A - AB - AC + ABC)]$

\1diii/ $\cancel{A \wedge B =}$ $A \vee B = \sim(\sim A \wedge \sim B)$ $\therefore$

by event - random quantity duality:

$(A \wedge \sim(B \vee C)) \vee (A \wedge (B \vee C)) =$

$\sim[\sim(A \wedge \sim(B \vee C)) \wedge \sim(A \wedge (B \vee C))] =$

$\sim[\sim(A \wedge \sim(\sim[\sim B \wedge \sim C])) \wedge \sim(A \wedge (\sim[\sim B \wedge \sim C]))] =$

$\sim[\sim(A \wedge [\sim B \wedge \sim C]) \wedge \sim(A \wedge (\sim[\sim B \wedge \sim C]))] =$

$$1-\left[\left(1-\left(A\left[(1-B)(1-C)\right]\right)\right)\left(1-\left(A\left(1-\left[(1-B)(1-C)\right]\right)\right)\right)\right]=$$

$$1-\left[\left(1-A\left[1-B-C+BC\right]\right)\left(1-A\left(1-\left[1-B-C+BC\right]\right)\right)\right]=$$

$$1-\left[(1-A+AB+AC-ABC)(1-A(B+C-BC))\right]=$$

$$1-\left[(1-A+AB+AC-ABC)(1-AB-AC+ABC)\right]=$$

$$1-\left[1-AB-AC+ABC-A+A^2B+A^2C-A^2BC+AB-A^2B^2\cancel{+A^2BC}-A^2BC+A^2B^2C+\right.$$
$$\left.AC-A^2BC-A^2C^2+A^2BC^2-ABC+A^2B^2C+A^2BC^2-A^2B^2C^2\right]=$$

$$1-\left[1-AB-AC+ABC-A+AB+AC-ABC+AB-AB-ABC+ABC+\right.$$
$$\left.AC-ABC-AC+ABC-ABC+ABC+ABC-ABC\right]=$$

$$1-\left[1-A\right]=1-1+A=A \therefore$$

$$P((A\wedge\sim(B\vee C))\vee(A\wedge(B\vee C)))=P(A)=0.7$$

\\(diii)/ by Event - random quantity duality $\therefore$

$$(A\wedge\sim(B\vee C))\vee(A\wedge(B\vee C))=$$

$$\cancel{\sim[\sim(A\wedge\sim[} \ \cancel{(A\wedge\sim(B\vee C}\ \sim\left[\sim(A\wedge\sim(B\vee C))\wedge\sim(A\wedge(B\vee C))\right]=$$

$$\cancel{\sim[\sim(A\wedge\sim(\sim B\wedge\sim C))\wedge\sim(A\wedge(\sim}$$

$$\sim\left[\sim(A\wedge\sim(\sim[\sim B\wedge\sim C]))\wedge\sim(A\wedge\sim(\sim B\wedge\sim C))\right]=$$

$$\sim\left[\sim(A\wedge[\sim B\wedge\sim C])\wedge\sim(A\wedge\sim(\sim B\wedge\sim C))\right]=$$

$$\sim\left[\sim(A\wedge(1-B)(1-C))\wedge\sim(A\wedge\sim(1-B)(1-C))\right]=$$

$$\sim\left[\sim(A\wedge(1-B-C+BC))\wedge\sim(A\wedge\sim(1-B-C+BC))\right]=$$

$$\sim\left[\sim(A-AB-AC+ABC)\wedge(A\wedge(B+C-BC))\right]=$$

$$\sim\left[(1-A+AB+AC-ABC)\wedge(AB+AC-ABC)\right]=$$

$$\cancel{\sim[(1-A+A}\ 1-\left[(1-A+AB+AC-ABC)(AB+AC-ABC)\right]=$$

$$1-\left[AB+AC-ABC-A^2B-A^2C+A^2BC+A^2B^2+A^2BC-A^2B^2C+A^2BC+A^2C^2-A^2BC^2-A^2B^2C-\right.$$
$$\left.(A^2BC^2)+A^2B^2C^2\right]=$$

$$1-\left[AB+AC-ABC-AB-AC+ABC+AB+ABC-ABC+ABC+AC-ABC-ABC-ABC+ABC\right]=$$

$$\cancel{1-AB-}\ 1-\left[AB+ABC+AC-ABC-ABC-ABC+ABC\right]=$$

$$1-\left[AB+AC-ABC\right]=1-AB-AC+ABC \therefore$$

$$P((A\wedge\sim(B\vee C))\vee(A\wedge(B\vee C)))=P(1-AB-AC+ABC)=$$

$$1-P(A)P(B)-P(A)P(C)+P(A)P(B)P(C)=1-0.7\times0.6-0.7\times0.3+0.7\times0.6\times0.3=$$

$$1-0.42-0.21+0.126=0.496$$

\PP2022/ \3a/ prior predictive $= P(y) = \int_{-\infty}^{\infty} P(y|\theta)\pi(\theta)d\theta$ $\therefore$

$\therefore$ for prior $\pi(\lambda)$, $\lambda > 0$,

prior predictive $= p(Y=y) = \int_0^y p(y|\lambda)\pi(\lambda)d\lambda = \int_0^y P(Y=y|\lambda)\pi(\lambda)d\lambda =$

$$\int_0^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda} d\lambda$$

\3b/ prior predictive integrate out the subjective assumptions of the prior parameter to give a predictive value of the data that is objective.
which will give a less subjective distribution of the data variable allowing a better assumption distribution to be made about the prior.

\3c/ $\therefore P(Y=y) = \int_0^{\infty} \frac{\lambda^y e^{-\lambda}}{y!} \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda} d\lambda =$

$$\frac{b^a}{y!\Gamma(a)} \int_0^{\infty} \lambda^y e^{-\lambda} \lambda^{a-1} e^{-b\lambda} d\lambda = \frac{b^a}{\Gamma(y+1)\Gamma(a)} \int_0^{\infty} \lambda^{y+a-1} e^{-\lambda-b\lambda} d\lambda =$$

$$\frac{b^a}{\Gamma(y+1)\Gamma(a)} \int_0^{\infty} \lambda^{(a+y)-1} e^{-(b+1)\lambda} d\lambda = \frac{b^a \Gamma(a+y)}{\Gamma(y+1)\Gamma(a)(b+1)^{a+y}} \int_0^{\infty} \frac{(b+1)^{a+y}}{\Gamma(a+y)} \lambda^{(a+y)-1} e^{-(b+1)\lambda} d\lambda$$

$\therefore$ let $b_n = b+1$, $a_n = a+y$ $\therefore$

$\frac{b_n^{a_n}}{\Gamma(a_n)} \lambda^{a_n-1} e^{-b_n\lambda} = \pi_{a_n, b_n}(\lambda)$ is a Gamma distribution domain $\lambda > 0$

$$\therefore P(Y=y) = \frac{b^a \Gamma(a+y)}{\Gamma(y+1)\Gamma(a)(b+1)^{a+y}} \int_0^{\infty} \frac{b_n^{a_n}}{\Gamma(a_n)} \lambda^{a_n-1} e^{-b_n\lambda} d\lambda =$$

$$\frac{b^a \Gamma(a+y)}{\Gamma(y+1)\Gamma(a)(b+1)^{a+y}} \times 1 = \frac{b^a \Gamma(a+y)}{\Gamma(y+1)\Gamma(a)(b+1)^{a+y}}$$

$\because$ integrating pdf over its domain $= 1$

\3d/ prior is $\pi(\lambda) = \frac{b^a}{\Gamma(a)} \lambda^{a-1} e^{-b\lambda}$ $\therefore$

as $b \to \infty$ : $\pi(\lambda) \to 0$ $\therefore$

$P(Y=y) = \int_0^{\infty} p(y|\lambda)\pi(\lambda)d\lambda \Rightarrow \int_0^{\infty} 0\, d\lambda = 0$ as $b \to \infty$

$$\therefore P(Y=y) = \frac{b^a \Gamma(a+y)}{\Gamma(y+1)\Gamma(a)(b+1)^{a+y}} = \frac{\Gamma(a+y)}{\Gamma(y+1)\Gamma(a)} \frac{b^a}{(b+1)^{a+y}} \therefore$$

$b+1 > b$, $a+y > a$ $\therefore (b+1)^{a+y} > b^a$ $\because b+1 > 1$ $\therefore$

$$\lim_{b\to\infty} \frac{b^a}{(b+1)^{a+y}} = \lim_{b\to\infty} \frac{ab^{a-1}}{(a+y)(b+1)^{a+y-1}} = 0 \therefore$$

$P(Y=y) \to 0$ as $b \to \infty$ $\therefore$ large enough $b$ encodes a prior of $P(Y=y) = 0$

Meaning any cases believed to be unlikely.

\3e/ prior is $P(Y=0)=2P(Y=1)$ . $P(Y=1)=2P(Y=2)$ ∴

$P(Y=0)=2\cdot 2P(Y=2)=4P(Y=2)$ ∴ $\frac{1}{2}P(Y=0)=P(Y=1)$, $\frac{1}{4}P(Y=2)=P(Y=0)$ ∴

is $P(Y=0)+P(Y=1)+P(Y=2)=P(Y=0)+\frac{1}{2}P(Y=0)+\frac{1}{4}P(Y=0)=\frac{7}{4}P(Y=0)$

. is $Y \sim \text{Geo}(\frac{1}{2})$ ∴ $P(Y=y)=(1-\frac{1}{2})^{y}\frac{1}{2}=(\frac{1}{2})^{y}\frac{1}{2}=(\frac{1}{2})^{y+1}$ ∴

$P(Y=y)=\int_0^\infty \pi(y|\lambda)\pi(\lambda)d\lambda=\int_0^\infty \frac{\lambda^y e^{-\lambda}}{y!}\pi(\lambda)d\lambda=$

$\int_0^\infty \frac{\lambda^y e^{-\lambda}}{y!}\frac{7}{4}P(Y=0)d\lambda$

\2a/ i) posterior $\pi(\phi|y)$ proposal distribution $q(\phi|\phi^{(t)})$

∴ draw a starting point $\phi^0$ for $\pi(\phi^0|y)>0$ for ~~some~~ $q_0(\phi)$ ∴

ii) at $t=1,2,\dots$ suppose our current $\phi^{t-1}$ is known, sample $\phi^*$ from a proposal distribution $q(\phi^*|\phi^{t-1})$

iii) ∴ compute Metropolis ratio $r(\phi^{t-1},\phi^*)=\frac{\pi(\phi^*|y)/q(\phi^*|\phi^{t-1})}{\pi(\phi^{t-1}|y)/q(\phi^{t-1}|\phi^*)}$

iv) ∴ set $\phi^t=\begin{cases}\phi^* & \text{with probability } p^*=\min(1,r)\\ \phi^{t-1} & \text{otherwise}\end{cases}$

\2b/ ∴ $r(\phi^*,\phi^{t-1})=r(\phi^{t-1},\phi^*)=\frac{\pi(\phi^*|y)\,q(\phi^*|\phi^{t-1})}{\pi(\phi^{t-1}|y)/q(\phi^{t-1}|\phi^*)}$

∴ ∴ $q(\phi^*|\phi^{(t-1)})$ such that $\phi^*|\phi^{(t-1)}\sim N(\phi^{(t-1)},\Sigma)$ ∴

$q(\phi^*|\phi^{(t-1)})=\frac{1}{\sqrt{2\pi}\sqrt{\Sigma}}e^{-\frac{1}{2\Sigma}(\phi^*-\phi^{(t-1)})^2}=\frac{1}{\sqrt{2\pi}\sqrt{\Sigma}}e^{-\frac{1}{2\Sigma}(\phi^{(t-1)}-\phi^*)^2}$

∴ ~~$\phi^{(t-1)}\sim$ At $\phi^{(t-1)}|\phi^*\sim N(\Sigma,\phi^{(t-1)})$~~ $\phi^{(t-1)}|\phi^*\sim N(\phi^*,\Sigma)$ ∴

$q(\phi^{(t-1)}|\phi^*)=\frac{1}{\sqrt{2\pi}\sqrt{\Sigma}}e^{-\frac{1}{2\Sigma}(\phi^{(t-1)}-\phi^*)^2}=q(\phi^*|\phi^{(t-1)})$

$q(\phi^*|\phi^{(t-1)})/q(\phi^{(t-1)}|\phi^*)=1$ ∴

$r(\phi^*,\phi^{(t-1)})=\frac{\pi(\phi^*|y)}{\pi(\phi^{(t-1)}|y)}$

\2c/ $y_{ij}|\theta_j,\sigma^2\sim N(\theta_j,\sigma^2)$ ∴ $\pi(y_{ij}|\theta_j,\sigma^2)=\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{1}{2\sigma^2}(y_{ij}-\theta_j)^2}$

$\theta_j|\mu,\tau^2\sim N(\mu,\tau^2)$ ∴ $\pi(\theta_j|\mu,\tau^2)=\frac{1}{\sqrt{2\pi}\tau}e^{-\frac{1}{2\tau^2}(\theta_j-\mu)^2}$

∴ $\log r(\phi^*,\phi^{(t-1)})=\log\left(\frac{\pi(\phi^*|y)}{\pi(\phi^{(t-1)}|y)}\right)=\log(\pi(\phi^*|y))-\log(\pi(\phi^{(t-1)}|y))$

By Bayes theorem $\pi(\theta|y)\propto\pi(\theta)P(y|\theta)$ ∴

$\pi(\phi|y)\propto\pi(\phi)P(y|\phi)$ ∴ $P(y|\phi)=P(y|\theta_1,\theta_2,\mu)$

\2d/ $\theta_1^{(0)}=\theta_2^{(0)}=\mu^{(0)}=0$ ∴ $\log r(\phi^*,\phi^{(t-1)})=\frac{1}{\sigma^2}\sum_{j=1}^{2}n_j\bar{y}_j\theta_j^*-\frac{1}{2}\sum_{j=1}^{2}(\frac{1}{\tau^2}+\frac{n_j}{\sigma^2})\theta_j^{*2}+\frac{1}{\tau^2}\sum_{j=1}^{2}\mu^*\theta_j^*-\frac{1}{\tau^2}\mu^{*2}$

\PP2022/ \1diii/ $(A \wedge \sim(B \vee C)) \vee (A \wedge (B \vee C)) =$

$$\sim[\sim(A \wedge \sim(B \vee C)) \wedge \sim(A \wedge (B \vee C))] =$$

$$\sim[\sim(A \wedge \sim[\sim(\sim B \wedge \sim C)]) \wedge \sim(A \wedge \sim(\sim B \wedge \sim C))] =$$

$$\sim[\sim(A \wedge \sim[\sim(1-B)(1-C)]) \wedge \sim(A \wedge \sim(1-B)(1-C))] =$$

$$\sim[\sim(A \wedge (1-B)(1-C)) \wedge \sim(A \wedge (\sim(1-B-C+BC)))] =$$

$$\sim[\sim(A \wedge (1-B-C+BC)) \wedge \sim(A \wedge (1-(1-B-C+BC)))] =$$

$$\sim[\sim(A \wedge (1-B-C+BC)) \wedge \sim(A \wedge (B+C-BC))] =$$

$$\sim[(\sim A(1-B-C+BC)) \wedge \sim(A(B+C-BC))] =$$

$$\sim[\sim(A-AB-AC+ABC) \wedge \sim(AB+AC-ABC)] =$$

$$\sim[(1-A+AB+AC-ABC) \wedge (1-AB-AC+ABC)] =$$

$$\sim[(1-A+AB+AC-ABC)(1-AB-AC+ABC)] =$$

$$\sim[1-AB-AC+ABC-A+A^2B+A^2C-A^2BC+(AB+AC-ABC)(1-AB-AC+ABC)] =$$

$$\sim[1-AB-AC+ABC-A+A^{\cancel{2}}B+AC-ABC+(AB+AC-ABC)(1-AB-AC+ABC)] =$$

$$\sim[1-A+(AB+AC-ABC)(1-AB-AC+ABC)] =$$

$$\sim[1-A+AB-A^2B^2-A^2BC+A^2B^2C+AC-A^2BC-A^2C^2+A^2BC^2-ABC+A^2B^2C+A^2BC^2-A^2B^2C^2] =$$

$$\sim[1-A+AB-AB-ABC+ABC+AC-ABC-AC+ABC-ABC+ABC+ABC-ABC] =$$

$$\sim[1-A] = 1-(1-A) = 1-1+A = A \therefore$$

$$P((A \wedge \sim(B \vee C)) \vee (A \wedge (B \vee C))) = P(A) = 0.7$$

\PP2020/ \1a i/ C: Means one and only one of the $A_i$ must happen for $i = 1, \ldots, n$

\1a ii/ $P(A) = P(A_1 \vee A_2 \vee \ldots \vee A_n) = P(A_1 + A_2 + \ldots A_n)$

using event, random quantity duality $= P(A_1) + P(A_2) + \ldots + P(A_n)$

by linearity of expectation $= \sum_{i=1}^{n} P(A_i) = 1$

\1a iii/ $A \vee B \vee C = (A \vee B) \vee (C) = \cancel{\sim A \vee B}$

$\sim(\sim(A \vee B) \wedge \sim C) = 1 - (\sim(A \vee B) \wedge \sim C) =$

$1 - (1 - (A \vee B))(1 - C) = 1 - (1 - (A \vee B))(1 - C) =$

$1 - (1 - (\sim(\sim A \wedge \sim B)))(1 - C) =$

$1 - (1 - (1 - (1 - A)(1 - B)))(1 - C) = 1 - ((1 - A)(1 - B))(1 - C) =$

$1 - ((1 - A - B + AB)(1 - C)) = 1 - (1 - A - B - C + AB + AC + BC - ABC) =$

$A + B + C - AB - AC - BC + ABC$ $\therefore$ by event, random quantity duality:

$P(A \vee B \vee C) = P(A + B + C - AB - AC - BC + ABC) =$

$P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$

$\therefore$ $P(A \wedge B) = P(AB) = 0.3$, $P(A \wedge C) = P(AC) = 0.2$ $P(B \wedge C) = P(BC) = 0.15$,

$P(A \wedge B \wedge C) = P(ABC) = 0.05$ $\therefore$

$P(A \vee B \vee C) = 0.6 + 0.4 + 0.3 - 0.3 - 0.2 - 0.15 + 0.05 = 0.6$

~~\1b/~~ $N \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ Gamma $\sim \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ Beta $\cancel{\frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}}$

Beta $\sim \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$ Exp $\sim \lambda e^{-\lambda x}$ poi $\sim \frac{\lambda^x e^{-\lambda x}}{x!}$ unif $\sim \frac{1}{b-a}$

$N \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ Gam $\sim \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ Beta $\sim \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

Exp $\sim \lambda e^{-\lambda x}$ ~~Poi~~ $\cancel{\sim \frac{\lambda^x e^{-\lambda}}{x!}}$ poi $\sim \frac{\lambda^x e^{-\lambda}}{x!}$ poi $\sim \frac{\lambda^x e^{-\lambda}}{x!}$ unif $\sim \frac{1}{b-a}$

$N \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ Gam $\sim \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ Beta $\sim \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

Exp $\sim \lambda e^{-\lambda x}$ poi $\sim \frac{e^{-\lambda}\lambda^x}{x!}$ Poi $\sim \frac{\lambda^x e^{-\lambda}}{x!}$ unif $\sim \frac{1}{b-a}$

$N \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ Gamma $\sim \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ Beta $\sim \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)} x^{\alpha-1}(1-x)^{\beta-1}$

Exp $\sim \lambda e^{-\lambda x}$ poi $\sim \frac{\lambda^x e^{-\lambda}}{x!}$ unif $\sim \frac{1}{b-a}$

\1b i/ By Bayes theorem $\pi(\theta|y) \propto \pi(\theta)P(y|\theta)$ $\therefore$ $\pi(\mu) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2}$

$\therefore \pi(\mu|y) \propto \pi(\mu)P(y|\mu) \propto \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2} \cancel{P(y|\mu)} \propto$

$e^{-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2} P(y|\mu) \propto e^{-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2} \prod_{i=1}^{n} P(y_i|\mu) \propto$

$$e^{-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2}\prod_{i=1}^{n}\frac{1}{\sqrt{2\pi}\,\sigma}\exp^{-\frac{1}{2\sigma^2}(y_i-\mu)^2}\propto$$

$$e^{-\frac{1}{2\sigma_0^2}(\mu-\mu_0)^2}\prod_{i=1}^{n}e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2}\propto$$

$$e^{-\frac{1}{2\sigma_0^2}(\mu^2-\mu_0^2-2\mu_0\mu)}\prod_{i=1}^{n}e^{-\frac{1}{2\sigma^2}(y_i^2+\mu^2-2y_i\mu)}\propto$$

$$e^{-\frac{1}{2\sigma_0^2}(\mu^2-2\mu_0\mu)}\prod_{i=1}^{n}e^{-\frac{1}{2\sigma^2}(\mu^2-2y_i\mu)}\propto$$

$$e^{-\frac{1}{2\sigma_0^2}(\mu^2-2\mu_0\mu)}e^{-\frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(-2y_i\mu)+n\mu^2\right)}\propto$$

$$e^{-\frac{1}{2\sigma_0^2}(\mu^2-2\mu_0\mu)}e^{-\frac{1}{2\sigma^2}(n\mu^2-2\mu\sum_{i=1}^{n}y_i)}\propto$$

$$e^{-\frac{1}{2}\left[\frac{1}{\sigma_0^2}(\mu^2-2\mu_0\mu)+\frac{1}{\sigma^2}(n\mu^2-2\mu\sum_{i=1}^{n}y_i)\right]}$$

$$\pi(\mu|y)=\frac{1}{\sqrt{2\pi\left(\frac{1}{\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}}\right)}}e^{-\frac{1}{2\left(\frac{1}{\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}}\right)}\left(\mu-\left(\frac{\frac{\mu_0}{\sigma_0^2}+\frac{n\bar{y}}{\sigma^2}}{\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}}\right)\right)^2}$$

$$\therefore\pi(\mu|y)\propto e^{-\frac{1}{2}\left[\frac{1}{\sigma_0^2}(\mu^2-2\mu_0\mu)+\frac{1}{\sigma^2}(n\mu^2-2n\mu\bar{y})\right]}\propto$$

$$e^{-\frac{1}{2}\left[\frac{1}{\sigma_0^2}\mu^2-\frac{2\mu_0}{\sigma_0^2}\mu+\frac{n}{\sigma^2}\mu^2-\frac{2n\bar{y}}{\sigma^2}\mu\right]}\propto$$

$$e^{-\frac{1}{2}\left[\left(\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}\right)\mu^2-2\left(\frac{\mu_0}{\sigma_0^2}+\frac{n\bar{y}}{\sigma^2}\right)\mu\right]}\propto e^{-\frac{1}{2}[a\mu^2-2b\mu]}\propto$$

$$\propto e^{-\frac{1}{2}a(\mu^2-\frac{2b}{a}\mu)}\propto e^{-\frac{1}{2}a\left[(\mu-\frac{b}{a})^2-\frac{b^2}{a^2}\right]}\propto e^{-\frac{1}{2}a\left[(\mu-\frac{b}{a})^2\right]}\propto e^{-\frac{1}{2\frac{1}{a}}(\mu-\frac{b}{a})^2}$$

$\therefore \mu|y\sim N(\frac{b}{a},\frac{1}{a})$, $a=\left(\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}\right)$, $b=\frac{\mu_0}{\sigma_0^2}+\frac{n\bar{y}}{\sigma^2}$ $\therefore$

$$\frac{1}{a}=\frac{1}{\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}},\quad \frac{b}{a}=\frac{\frac{\mu_0}{\sigma_0^2}+\frac{n\bar{y}}{\sigma^2}}{\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}}$$

\|bii\| a classical ~~credible~~ considence interval has the parameter of the interval as a fixed constant that is unknown to me with a 95% considence interval being a dissevent interval each time you calc it and it encapsulating the parameter 95% of the time.

credible interval is $\pi(a<\theta<b\,|y)=\int_a^b\pi(\theta|y)\,d\theta$ with the parameter not being a constant with the parameter ~~sath~~ being calced with in this ~~tale~~ credible interval 95% of the time

\|biii\| $0.95=\int_a^b\pi(\theta|y)\,d\theta$ $\quad\mu\sim N(3,1)$ $\therefore$ 10 observations from $N(\mu,2)$, $\mu\sim N(3,1)$ $\therefore$ mean(10 observed) $=5$ $\therefore$ let $\frac{b}{a}=\mu_n$, $\frac{1}{a}=\sigma_n^2$ $\therefore$ $\mu|y\sim N(\mu_n,\sigma_n^2)$ $\therefore$ a 95% credible interval is $\mu_n\pm\sigma_n Z_{0.975}$ where $Z_{0.975}=1.96$ $\therefore$ $n=10$, $\therefore\sigma_n^2=\frac{1}{\frac{1}{\sigma_0^2}+\frac{n}{\sigma^2}}=\frac{1}{\frac{1}{1}+\frac{10}{2}}=\frac{1}{6}$, $\mu_n=\frac{\frac{3}{1}+\frac{10(5)}{2}}{\frac{1}{1}+\frac{10}{2}}=\frac{14}{3}$ $\therefore$

\PP2020/ \1aiii/ $A \vee B \vee C = (A \vee B) \vee C = \sim[\sim(A \vee B) \wedge \sim C] =$

$\sim[\sim(\sim[\sim A \wedge \sim B]) \wedge \sim C] = \sim[[\sim A \wedge \sim B] \wedge \sim C] =$

$1 - [[\sim A \wedge \sim B] \wedge \sim C] = 1 - [[(1-A)(1-B)] \wedge \sim C] =$

$1 - [(1 - A - B + AB) \wedge \sim C] = 1 - [(1 - A - B + AB)(1-C)] =$

$1 - [1 - A - B + AB - C + AC + BC - ABC] =$

~~1 - 1 + A + B + C~~ $1 - 1 + A + B + C - AB - AC - BC + ABC =$

$A + B + C - AB - AC - BC + ABC$ by event - randomquantity duality

$\therefore P(A \vee B \vee C) = P(A + B + C - AB - AC - \cancel{BC} + ABC) =$

$P(A) + P(B) + P(C) - P(AB) - P(AC) - P(AC) + P(ABC) =$

$P(A) + P(B) + P(C) - P(A \wedge B) - P(A \wedge C) - P(B \wedge C) + P(A \wedge B \wedge C) =$

$0.5 + 0.4 + 0.3 - 0.3 - 0.2 - 0.15 + 0.05 = 0.6$

\1ai/ finite partion of events means one and only one of the $A_i$ must happen for $i = 1, \dots, n$

\1aii/ using event, random quantity duality : $P(A) = P(A_1 \vee A_2 \vee \dots \vee A_n) =$ $P(A_1 + A_2 + \dots A_n)$ because they form a partition

$= P(A_1) + P(A_2) + \dots + P(A_n)$ by linearity of expectation

$= \sum_{i=1}^{n} P(A_i) = 1$

\1bi/ By Bayes theorem $\pi(\mu | y) \propto \pi(\mu) p(y | \mu) \therefore$

$\mu \sim N(\mu_0, \sigma_0^2) \therefore \quad \pi(\mu) = \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{1}{2\sigma_0^2}(\mu - \mu_0)^2} \therefore$

$\pi(\mu | y) \propto \pi(\mu) P(y | \mu) \propto \pi(\mu) p(y | \mu, \sigma^2) \propto$

$\pi(\mu) \prod_{i=1}^{n} p(y_i | \mu, \sigma^2) \propto \pi(\mu) \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i - \mu)^2} \propto$

$\pi(\mu) \sigma^{-n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (y_i^2 - 2y_i\mu + \mu^2)} \propto \pi(\mu) \sigma^{-n} e^{-\frac{1}{2\sigma^2} \sum_{i=1}^{n} (-2y_i\mu + \mu^2)} \propto$

$\pi(\mu) \sigma^{-n} e^{-\frac{1}{2\sigma^2}(-2n\bar{y}\mu + n\mu^2)} \propto \frac{1}{\sqrt{2\pi}\sigma_0} e^{-\frac{1}{2\sigma_0^2}(\mu - \mu_0)^2} \sigma^{-n} e^{-\frac{1}{2\sigma^2}(-2n\bar{y}\mu + n\mu^2)} \propto$

$e^{-\frac{1}{2\sigma_0^2}(\mu^2 - 2\mu_0\mu + \mu_0)} e^{-\frac{1}{2}(-\frac{2n\bar{y}}{\sigma^2}\mu + \frac{n}{\sigma^2}\mu^2)} \propto$

$e^{-\frac{1}{2}(\frac{1}{\sigma_0^2}\mu^2 - \frac{2\mu_0}{\sigma_0^2}\mu) - \frac{1}{2}(-\frac{2n\bar{y}}{\sigma^2}\mu + \frac{n}{\sigma^2}\mu^2)} \propto e^{-\frac{1}{2}(\frac{1}{\sigma_0^2}\mu^2 - \frac{2\mu_0}{\sigma_0^2}\mu - \frac{2n\bar{y}}{\sigma^2}\mu + \frac{n}{\sigma^2}\mu^2)} \propto$

$e^{-\frac{1}{2}([\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}]\mu^2 - 2[\frac{\mu_0}{\sigma_0^2} + \frac{n\bar{y}}{\sigma^2}]\mu)} \propto e^{-\frac{1}{2}(a\mu^2 - 2b\mu)} \propto e^{-\frac{1}{2}a(\mu^2 - \frac{2b}{a}\mu)} \propto$

$e^{-\frac{1}{2}a((\mu - \frac{b}{a})^2 - \frac{b^2}{a^2})} \propto \cancel{e^{-\frac{1}{2}a\mu}} \; e^{-\frac{1}{2\frac{1}{a}}(\mu - \frac{b}{a})^2}$

which is proportion to a normal distribution of $N(\frac{b}{a}, \frac{1}{a}) \therefore$

$a = \frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}, \; b = \frac{\mu_0}{\sigma_0^2} + \frac{n\bar{y}}{\sigma^2} \therefore \quad \frac{b}{a} = \frac{\frac{\mu_0}{\sigma_0^2} + \frac{n\bar{y}}{\sigma^2}}{\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}}, \; \frac{1}{a} = \frac{1}{\sigma_0^2} + \frac{n}{\sigma^2} \therefore \mu | y \sim N(\frac{b}{a}, \frac{1}{a})$

\1bii/ $\therefore$ let $\frac{b}{a} = \mu_n$, $\frac{1}{a} = \sigma_n^2$ $\therefore$ $\mu | y \sim N(\mu_n, \sigma_n^2)$

$\therefore$ a 95% credible interval is $\mu_n \pm \sigma_n Z_{0.95} = \mu_n \pm 1.96\sigma_n$

$\therefore n = 10$, $\cancel{\sigma^2 = 2}$, $y | \mu, \sigma^2 \sim N(\mu, \sigma^2) = N(\mu, 2)$ $\therefore \sigma^2 = 2$,

$\mu \sim N(\mu_0, \sigma_0^2) = N(3, 1)$ $\therefore$ $\mu_0 = 3$, $\sigma_0^2 = 1$ $\therefore$

$\bar{y} = 5$ $\therefore$

$\cancel{\sigma_n = \frac{1}{\sigma_0^2} +}$ $\sigma_n = \frac{1}{\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}} = \frac{1}{\frac{1}{1} + \frac{10}{2}} = \frac{1}{6}$,

$$\mu_n = \frac{\frac{\mu_0}{\sigma_0^2} + \frac{n\bar{y}}{\sigma^2}}{\frac{1}{\sigma_0^2} + \frac{n}{\sigma^2}} = \left(\frac{\mu_0}{\sigma_0^2} + \frac{n\bar{y}}{\sigma^2}\right) \times \frac{1}{6} = 28 \times \frac{1}{6} = \frac{14}{3} \therefore$$

95% credible interval: $\mu_n \pm 1.96\sigma_n = \frac{14}{3} \pm 1.96 \times \frac{1}{6}$ $\therefore$

$\therefore$ $\cancel{\left(\frac{749}{150}, \frac{217}{50}\right)}$ $\left(\frac{217}{50}, \frac{749}{150}\right) = (4.34, 4.99)$ is the 95% credible interval for $\mu$

\1ci/ $d^* = \arg\max_d E[U(\theta, d)] = \arg\max_d \int_{-\infty}^{\infty} U(\theta, d)\pi(\theta)\,d\theta$

\1cii/ $E[U(\theta, d) | y] = \int_{-\infty}^{\infty} U(\theta, d)\pi(\theta|y)\,d\theta = \int_{-\infty}^{\infty} -(\theta - d)^2 \pi(\theta|y)\,d\theta =$

$-\int_{-\infty}^{\infty} (\theta^2 - 2d\theta + d^2)\pi(\theta|y)\,d\theta =$

$-\left(\int_{-\infty}^{\infty} \theta^2 \pi(\theta|y)\,d\theta - 2d\int_{-\infty}^{\infty} \theta\pi(\theta|y)\,d\theta + d^2\int_{-\infty}^{\infty} 1\,\pi(\theta|y)\,d\theta\right) =$

$-(E(\theta^2|y) - 2dE(\theta|y] + d^2E[1]) = -(E[\theta^2|y] - 2dE[\theta|y] + d^2)$

\1ciii/ $d^* = \arg\max_d E[U(\theta, d)]$

$E[U(\theta, d)] = -(E[\theta^2|y] - 2dE[\theta|y] + d^2)$ $\therefore$ is negative $\therefore$

$d^*$ for minimise $E[U(\theta, d)]$ $\therefore$

minimise $E[\theta^2|y] - 2dE[\theta|y] + d^2$ $\therefore$

$\cancel{\frac{dU}{dd^*}}$ $\therefore$ $\frac{dE(U)}{dd^*} = \frac{dE(U)}{dd}\Big|_{d=d^*} = \frac{d}{dd}\left(-E[\theta^2|y] - 2dE[\theta|y] + d^2\right)\Big|_{d=d^*}$

$= -2E[\theta|y] + 2d\,\Big|_{d=d^*} = -2E[\theta|y] + 2d^* = 0$ $\therefore$

$d^* = E[\theta|y]$

\4a By Bayes theorem: $\pi(\theta|y) \propto \pi(\theta)P(y|\theta) \propto \pi(\theta)P(y|\theta, k) \propto$

$\pi(\theta)\prod_{i=1}^{n} P(y_i|k, \theta) \propto \pi(\theta)\prod_{i=1}^{n}(k\theta^{-1}y_i^{k-1}e^{-y_i^k\frac{1}{\theta}} \propto \pi(\theta)\theta^{-n}e^{-\frac{1}{\theta}\sum_{i=1}^{n}y_i^k}\prod_{i=1}^{n}e^{\ln y_i^{k-1}} \propto$

$\pi(\theta)\theta^{-n}e^{-\frac{1}{\theta}\sum_{i=1}^{n}y_i^k} \propto \frac{b^a}{\Gamma(a)}(\theta^{-1})^{a+1}e^{-b\frac{1}{\theta}}\theta^{-n}e^{-\frac{1}{\theta}\sum_{i=1}^{n}y_i^k} \propto$

$\theta^{-a-1}\theta^{-n}e^{-b\frac{1}{\theta}}e^{-(\sum_{i=1}^{n}y_i^k)\frac{1}{\theta}} \propto \theta^{-(a+n+1)}e^{-(b+\sum_{i=1}^{n}y_i^k)\frac{1}{\theta}} \propto \left(\frac{1}{\theta}\right)^{(a+n)+1}e^{-\frac{b+\sum_{i=1}^{n}y_i^k}{\theta}}$

$\therefore$ let $a_n = a + n$, $b_n = b + \sum_{i=1}^{n} y_i^k$ $\therefore$ $\pi(\theta|y) \propto \left(\frac{1}{\theta}\right)^{a_n+1}e^{-\frac{b_n}{\theta}}$ $\therefore$

\PP2020/ $\pi(\theta|y)$ is proportional to a inverse gamma ~~distribution~~ density

$\therefore \pi(\theta|y) \simeq \frac{b_n^{a_n}}{\Gamma(a_n)}\left(\frac{1}{\theta}\right)^{a_n+1} e^{-\frac{b_n}{\theta}}$, $\theta \geq 0$ $\therefore \theta|y \sim$ inverse Gamma distribution

\4b/ $\log \pi(\theta|y) = \ln\left[\frac{b_n^{a_n}}{\Gamma(a_n)}\left(\frac{1}{\theta}\right)^{a_n+1} e^{-\frac{b_n}{\theta}}\right] =$

$\ln(b_n^{a_n}) - \ln(\Gamma(a_n)) + \ln\left(\left(\frac{1}{\theta}\right)^{a_n+1}\right) - \frac{b_n}{\theta} =$

$a_n \ln b_n - \ln(\Gamma(a_n)) + (a_n+1)\ln(\theta^{-1}) - b_n\theta^{-1} =$

$a_n \ln b_n - \ln(\Gamma(a_n)) + (-a_n-1)\ln(\theta) - b_n\theta^{-1}$ $\therefore$

$$\left.\frac{\partial \ln \pi(\theta|y)}{\partial \theta}\right|_{\theta=\hat{\theta}} = (-a_n-1)\frac{1}{\theta} + b_n\theta^{-2}\Big|_{\theta=\hat{\theta}} = (-a_n-1)\frac{1}{\hat{\theta}} + b_n\hat{\theta}^{-2} = 0$$

$\therefore \frac{1}{\hat{\theta}}(a_n+1) = b_n\frac{1}{\hat{\theta}^2}$ $\therefore$ $\hat{\theta}(a_n+1) = b_n$ $\therefore$

$\hat{\theta} = \frac{b_n}{a_n+1}$ is the MAP estimate for $\theta$.

\4c/ observed information $I(\hat{\theta}) = -\left[\frac{d^2}{d\theta^2}\log \pi(\theta|y)\right]_{\theta=\hat{\theta}}$ $\therefore$

$\theta|y \sim N(\hat{\theta}, I(\hat{\theta})^{-1})$ $\therefore$

$$\frac{\partial^2}{\partial\theta^2}\log \pi(\theta|y) = \frac{\partial}{\partial\theta}\left[(-a_n-1)\theta^{-1} + b_n\theta^{-2}\right] = (a_n+1)\theta^{-2} - 2b_n\theta^{-3} \therefore$$

$$I(\hat{\theta}) = -\left[(a_n+1)\frac{1}{\theta^2} - \frac{2b_n}{\theta^3}\right]\Big|_{\theta=\hat{\theta}} = \frac{-a_n-1}{\hat{\theta}^2} + \frac{2b_n}{\hat{\theta}^3} = \frac{(-a_n-1)\hat{\theta}+2b_n}{\hat{\theta}^3} \therefore$$

$$I(\hat{\theta})^{-1} = \frac{1}{I(\hat{\theta})} = \frac{\hat{\theta}^3}{(-a_n-1)\hat{\theta}+2b_n} \therefore \theta|y \sim N\left(\frac{b_n}{a_n+1}, -\frac{1}{3}\frac{b_n^2}{(a_n+1)^3}\right)$$

$$\theta|y \sim N\left(\frac{b_n}{a_n+1}, \frac{\hat{\theta}^3}{(-a_n-1)\hat{\theta}-2b_n}\right) = N\left(\frac{b_n}{a_n+1}, \frac{\left(\frac{b_n}{a_n+1}\right)^3}{(-a_n-1)\frac{b_n}{a_n+1}-2b_n}\right) = N\left(\frac{b_n}{a_n+1}, \frac{\left(\frac{b_n}{a_n+1}\right)^3}{-3b_n}\right)$$

\4d/ $a_n = a+n$, $b_n = b + \sum_{i=1}^{n} y_i^k$

$\therefore a_n \to \infty$ as $n \to \infty$, $b_n \to b + \sum_{i=1}^{\infty} y_i^k$ as $n \to \infty$ $\therefore$

$b_n \to \infty$ as $n \to \infty$ for $k \geq -1$ $\therefore k > 0 \therefore b_n \to \infty$ as $n \to \infty$ $\therefore$

$$\lim_{n\to\infty}\frac{b_n}{a_n+1} = \lim_{n\to\infty}\frac{b+\sum_{i=1}^{n} y_i^k}{a_0+n+1} = \lim_{n\to\infty}\frac{\frac{\partial}{\partial n}\sum_{i=1}^{n} y_i^k}{1} = \lim_{n\to\infty}\frac{\partial}{\partial n}\sum_{i=1}^{n} y_i^k = \infty \therefore$$

$\cancel{\frac{b_n^2}{(a_n+1)^3} = } \cancel{\lim_{n\to\infty}\frac{b_n^2}{(a_n+1)^3} \to \infty}$ $\therefore \lim_{n\to\infty} -\frac{1}{3}\frac{b_n^2}{(a_n+1)^3} = 0$ $\therefore$

$\theta|y \sim N(\infty, 0)$ as $n \to \infty$ $\therefore$

the normal approximation asymptotically tends to a mean of infinity with no variance for the posterior distribution of $\theta$ as n tends to infinity $\therefore$ the number of observations of $y_i$ tends to infinity as $n \to \infty$

3a/ By Bayes theorem: $\pi(\theta|\underline{k}) \propto \pi(\theta)P(\underline{k}|\theta) \propto \pi(\theta)P(\underline{k}|r,\theta) \propto$

$\pi(\theta)\prod_{i=1}^{n} P(k_i|r,\theta) \propto \pi(\theta)\prod_{i=1}^{n}\frac{\Gamma(k_i+r)}{\Gamma(k_i+1)\Gamma(r)}(1-\theta)^r\theta^{k_i} \propto$

$\pi(\theta)\prod_{i=1}^{n}(1-\theta)^r\theta^{k_i} \propto \pi(\theta)(1-\theta)^{nr}\theta^{n\bar{k}} \propto$

$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}(1-\theta)^{nr}\theta^{n\bar{k}} \propto \theta^{a-1+n\bar{k}}(1-\theta)^{b-1+nr} \propto$

$\theta^{(a+n\bar{k})-1}(1-\theta)^{(b+nr)-1}$ $\therefore$

let $a_n = a+n\bar{k}$, $b_n = b+nr$ $\therefore$

~~$\pi(\theta|\underline{k}) \propto \theta^{a_n-1}(1-\theta)^{b_n-1}$~~ $\therefore$ $\pi(\theta|\underline{k}) \propto \theta^{a_n-1}(1-\theta)^{b_n-1}$

$\pi(\theta|\underline{k})$ is proportional to a Beta distribution density $\therefore$

$\theta|k \sim \text{Beta}(a_n, b_n)$ $\therefore$

$\pi(\theta|\underline{k}) = \frac{\Gamma(a_n+b_n)}{\Gamma(a_n)\Gamma(b_n)}\theta^{a_n-1}(1-\theta)^{b_n-1}$ $\therefore$

the posterior Beta distribution is conjugate to the priors ~~for~~ of Beta family

3bi/ $\therefore n=10$, $\therefore \bar{y}=2$

$\therefore$ By Bayes theorem: $\pi(\theta|\underline{k}) \propto \pi(\theta)P(\underline{k}|\theta)$ $\therefore$

~~$\pi(\theta)$~~ $\theta \sim \text{Beta}(1,1)$ $\therefore$ $\pi(\theta) = \frac{\Gamma(1+1)}{\Gamma(1)\Gamma(1)}\theta^{1-1}(1-\theta)^{1-1} = \frac{1!}{0!0!}(1)(1) = 1$

$\therefore$ ~~$\pi$ $P(y) \sim$ Binomial( of 3 failures~~

play until all 3 lives lost $\therefore r=3$ $\therefore$

~~$\pi$ $P(k$~~ let $k_i$ be number of hits for ith ~~conste~~stent

$\therefore P(k_i|3,\theta) = \frac{\Gamma(k_i+3)}{\Gamma(k_i+1)\Gamma(3)}(1-\theta)^3\theta^{k_i}$ $\therefore$ By Bayes theorem

$\pi(\theta|\underline{k}) \propto \pi(\theta)P(\underline{k}|\theta) \propto \pi(\theta)P(\underline{k}|3,\theta) \propto 1P(\underline{k}|3,\theta) \propto$

$P(\underline{k}|3,\theta) \propto \prod_{i=1}^{n}P(k_i,3,\theta) \propto \prod_{i=1}^{n}\frac{\Gamma(k_i+3)}{\Gamma(k_i+1)\Gamma(3)}(1-\theta)^3\theta^{k_i} \propto$

~~$(1-\theta)^3\theta^{k_i}$~~ $\prod_{i=1}^{n}(1-\theta)^3\theta^{k_i} \propto (1-\theta)^{3n}\theta^{\frac{n}{n}\sum_{i=1}^{n}k_i} \propto (1-\theta)^{3n}\theta^{n\bar{k}}$ $\therefore$

$n=10$, $\bar{k}=2$ $\therefore$ $n\bar{k}=10\times 2=20$

$\pi(\theta|\underline{k}) \propto (1-\theta)^{3\times 10}\theta^{10\times 2} \propto (1-\theta)^{30}\theta^{20} \propto \theta^{21-1}(1-\theta)^{31-1}$ $\therefore$

$\pi(\theta|\underline{k})$ is proportional to a Beta distribution density $\therefore$

~~$\pi$~~ $\theta|\underline{k} \sim \text{Beta}(21,31)$ $\therefore$ $\pi(\theta|\underline{k}) = \frac{\Gamma(21+31)}{\Gamma(21)\Gamma(31)}\theta^{21-1}(1-\theta)^{31-1}$

$= \frac{\Gamma(52)}{\Gamma(21)\Gamma(31)}\theta^{20}(1-\theta)^{30}$

~~3bii/~~

PP2020 / 3bi / posterior predictive $= P(\tilde{y}|y) = \int_{-\infty}^{\infty} P(\tilde{y}|\theta)\pi(\theta|y)\,d\theta$ ∴

~~So~~ let $n=10$ $y_i$ be number of hits before lose

∴ $r=3$, ∴ $P(y_i|3,\theta) = \frac{\Gamma(y_i+3)}{\Gamma(y_i+1)\Gamma(3)}(1-\theta)^3\theta^{y_i}$ ∴ $\bar{y}=2$ ∴

∴ $\theta \sim Beta(1,1)$ ∴ $\pi(\theta)=1$, $0\le\theta\le1$ ∴

By Bayes theorem $\pi(\theta|y) \propto \pi(\theta)P(y|r=3,\theta)$ ∴

$\theta|y \sim Beta(a_n, b_n) = Beta(a+n\bar{y},\ b+nr)$ ∴

$a=1,\ b=1$, ∴

$\pi(\theta|y) = \frac{\Gamma(a_n+b_n)}{\Gamma(a_n)\Gamma(b_n)}\theta^{a_n-1}(1-\theta)^{b_n-1}$ ∴

$a+n\bar{y} = 1+10(2) = 21$, $b_n = 1+10\times3 = 31$ ∴

∴ $\pi(\theta|y) = \frac{\Gamma(21+31)}{\Gamma(21)\Gamma(31)}\theta^{21-1}(1-\theta)^{31-1}$, $0\le\theta\le1$ ∴

3bii

$P(\tilde{y}|y) = \int_0^1 \frac{\Gamma(\tilde{y}+3)}{\Gamma(\tilde{y}+1)\Gamma(3)}(1-\theta)^3\theta^{\tilde{y}}\frac{\Gamma(21+31)}{\Gamma(21)\Gamma(31)}\theta^{21-1}(1-\theta)^{31-1}d\theta =$

$\frac{\Gamma(\tilde{y}+3)\Gamma(52)}{\Gamma(\tilde{y}+1)\Gamma(3)\Gamma(21)\Gamma(31)}\int_0^1\theta^{(\tilde{y}+21)-1}(1-\theta)^{34-1}d\theta =$

$\frac{\Gamma(\tilde{y}+3)\Gamma(52)}{\Gamma(\tilde{y}+1)\Gamma(3)\Gamma(21)\Gamma(31)}\frac{\Gamma(\tilde{y}+21)\Gamma(34)}{\Gamma(\tilde{y}+21+34)}\int_0^1\frac{\Gamma(\tilde{y}+21+34)}{\Gamma(\tilde{y}+21)\Gamma(34)}\theta^{(\tilde{y}+21)-1}(1-\theta)^{34-1}d\theta =$

$\frac{\Gamma(\tilde{y}+3)\Gamma(52)\Gamma(\tilde{y}+21)\Gamma(34)}{\Gamma(\tilde{y}+1)\Gamma(3)\Gamma(21)\Gamma(31)\Gamma(\tilde{y}+21+34)} = \frac{\Gamma(\tilde{y}+3)\Gamma(52)\Gamma(\tilde{y}+21)\Gamma(34)}{\Gamma(\tilde{y}+1)\Gamma(3)\Gamma(21)\Gamma(31)\Gamma(\tilde{y}+55)}$

$= \frac{(\tilde{y}+2)!(51)!(\tilde{y}+20)!(33)!}{(\tilde{y})!(2!)(20!)(30!)(\tilde{y}+54)!} =$

$16368\ \frac{(\tilde{y}+2)!(51)!(\tilde{y}+20)!}{\tilde{y}!\,20!\,(\tilde{y}+54)!}$ ∴

$P(\tilde{y}=2|y) = 0.198$, $P(\tilde{y}=0|y) = 0.220$,

$P(\tilde{y}=1|y) = 0.252$ ∴

$P(\tilde{y}=0|y) + P(\tilde{y}=0|y) = P(\tilde{y}\le1|y) = 0.220+0.252 = 0.472 < 0.5$ ∴

$P(\tilde{y}\le2|y) = P(\tilde{y}\le1|y) + P(\tilde{y}=2) = 0.472+0.198 = 0.67 > 0.5$ ∴

Let needed hits to be 3 or more to get through to take out more then 50% of contestants.

\2a/ $\therefore \int_{-\infty}^{\infty} f(x)\,dx = 1 \therefore$

let the point ~~draw~~ from the area under $g(x) = c f(x)$ be $X_*$ $\therefore$

$X_*$ falls in a strip $[x, x+\Delta x]$ $\therefore$

$P(X_* \in [x, x+\Delta x]) \approx \Delta x\, c f(x)$ when $\Delta x$ is small $\therefore$

~~$f_{X_*}$ $P(X_* \in [x, x+$~~ $f_{X_*}(x) = F'_{X_*}(x) = \lim_{\Delta x \to 0} \frac{F_{X_*}(x+\Delta x) - F_{X_*}(x)}{\Delta x} =$

$\lim_{\Delta x \to 0} \frac{P(X_* \in [x, x+\Delta x])}{\Delta x} = f(x)$

for $g(x) = c f(x)$ $\therefore$ area is

$P(X_* \in [x, x+\Delta x]) = \frac{g(x)\Delta x}{c} = \frac{c f(x)\Delta x}{c} = f(x)\Delta x$ $\therefore$

$f_{X_*}(x) = F'_{X_*}(x) = \lim_{\Delta x \to 0} \frac{F_{X_*}(x+\Delta x) - F_{X_*}(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{P(X_* \in [x, x+\Delta x])}{\Delta x} =$

$\lim_{\Delta x \to 0} \frac{f(x)\Delta x}{\Delta x} = \lim_{\Delta x \to 0} f(x) = f(x)$

$\therefore$ The $x$ coordinate of a random point drawn ~~from~~ uniformly from the area under $g(x)$ has the same distribution as $X$

\2b/ $f(\theta) = \sin\theta \quad \theta \in [0, \frac{\pi}{2}] \quad \therefore g(\theta) = c f(\theta) = c\sin(\theta)$

let $f_u(\theta)$ be the pdf of the uniform distribution on $[0, \frac{\pi}{2}]$ and ~~k~~ $\therefore f(\theta) = \frac{1}{\frac{\pi}{2} - 0} = \frac{1}{(\frac{\pi}{2})} = \frac{2}{\pi}$ $\therefore$

let $g_u(\theta) = k f_u(\theta)$ on $[0, \frac{\pi}{2}]$ with $k$ st $g_u(\theta) \geq g(\theta) \quad \forall \theta \in [0, \frac{\pi}{2}]$ where $k$ maximises the acceptance rate whilst still leading to a valid rejection algorithm $\therefore$

~~$g(\theta)$~~ $\therefore \max f(\theta) = \max \sin\theta = \sin\theta\big|_{\theta = \frac{\pi}{2}} = \sin\frac{\pi}{2} = 1$ $\therefore$

$g_u(\theta) = \frac{2}{\pi} k \geq g(\theta) = c f(\theta) \quad \forall \theta$ st acceptance rate ~~is~~ $\frac{c}{k}$ is maximised

$\because \max f(\theta) = 1 \therefore \frac{2}{\pi} k \geq c(1) = c$

~~max~~

\2b/ let $f_u(\theta)$ be the pdf of the uniform distribution on $[0, \frac{\pi}{2}]$

$\therefore f_u(\theta) = \frac{1}{\frac{\pi}{2} - 0} = \frac{1}{(\frac{\pi}{2})} = \frac{2}{\pi}$ $\therefore$

$\max f(\theta) = \max \sin\theta = \sin\theta\big|_{\theta = \frac{\pi}{2}} = \sin\frac{\pi}{2} = 1$ $\therefore$ let $g(\theta) = c f(\theta)$ $\therefore$

let ~~$c f(\theta)$~~ $g_u(\theta) = k f_u(\theta) \geq g(\theta) = c f(\theta) \quad \forall \theta$ $\therefore$

\PP2020/ draw $x_u$ from $f_u(\theta)$ and $U \sim \mathrm{Unif}(0,1)$ ∴

accept $x_u$ if $U g_u(x_u) \le g(x_u)$ ∴

$g(\theta) = c f(\theta)$ ∵ $\int_0^{\pi/2} f(\theta)\,d\theta = 1$ ∴ $\int_0^{\pi/2} g(\theta)\,d\theta = \int_0^{\pi/2} c f(\theta)\,d\theta = c$ ∴

$g_u(\theta) = k f_u(\theta)$ ∵ $\int_0^{\pi/2} f_u(\theta) = 1$ ∴ $\int_0^{\pi/2} g_u(\theta) = \int_0^{\pi/2} k f_u(\theta)\,d\theta = k$ ∴

∴ acceptance rate $= \dfrac{\text{area under } g(\theta)}{\text{area under } g_u(\theta)} = \dfrac{c}{k}$

\2c/ acceptance rate $= \dfrac{\text{area under } g(\theta)}{\text{area under } g_u(\theta)} = \dfrac{c}{k}$

\2d/ let $f(x,y) \propto \dfrac{y}{x^2+y^2}$ ∴ let

$f(x,y) = c\dfrac{y}{x^2+y^2}$ ∴ $g(x,y) = \dfrac{y}{x^2+y^2}$ ∴

let $x^2+y^2 = r^2 \le 2$ ∵ $r > 0$ ∴ $r \le \sqrt{2}$ ∴

let $x = r\cos\theta$, $y = r\sin\theta$ ∴ $0 \le \theta \le \frac{\pi}{2}$ ∵ $x, y \ge 0$ ∴

Jacobian: ~~$\begin{vmatrix} \partial x/\partial x & \partial y/\partial y \\ \partial x/\partial\theta & \partial\theta/\partial y \end{vmatrix} =$~~ $J = \begin{vmatrix} \partial x/\partial r & \partial x/\partial\theta \\ \partial y/\partial r & \partial y/\partial\theta \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} =$

$r\cos^2\theta - (-r\sin^2\theta) = r(\cos^2\theta + r\sin^2\theta) = r$ ∴

$dx\,dy = J\,dr\,d\theta = r\,dr\,d\theta$ ∴

$f(x,y) = c\dfrac{y}{x^2+y^2} = c\dfrac{r\sin\theta}{r^2} = c\dfrac{\sin\theta}{r} = cf(r,\theta) = g(r,\theta)$ ∴

$\int_0^{\sqrt{2}}\int_0^{\pi/2} cf(r,\theta)\,\cancel{dr}\,d\theta\,\cancel{d\theta} = \int_0^{\pi/2}\int_0^{\sqrt{2}} c\dfrac{\sin\theta}{r}\,r\,dr\,d\theta =$

~~$c\sin\theta \int_0^{\pi/2} c\sin\theta\,dr\,d\theta =$~~ $\int_0^{\pi/2}\int_0^{\sqrt{2}} c\sin\theta\,dr\,d\theta = \int_0^{\pi/2} \sqrt{2}\,c\sin\theta\,d\theta =$

$\sqrt{2}\,c = 1$ ∴ $c = \dfrac{1}{\sqrt{2}}$ ∴

\2a/ $X \sim f(x)$, $g(x) = cf(x)$ $X_*$ a random ~~variable~~ point under $g$

∴ pdf of $X_*$ is: $f_{X_*}(x) = \lim_{\Delta x \to 0} \dfrac{F_{X_*}(x+\Delta x) - F_{X_*}(x)}{\Delta x} = \lim_{\Delta x \to 0} \dfrac{P(X_* \in [x, x+\Delta x])}{\Delta x}$

∵ area under $g(x) = c$ ∴ $P(X_* \in [x+\Delta x]) = f(x)\Delta x$ ∴

$f_{X_*}(x) = \lim_{\Delta x \to 0} \dfrac{f(x)\Delta x}{\Delta x} = \lim_{\Delta x \to 0} f(x) = f(x)$

\2a/ $X \sim f(x)$, $g(x) = cf(x)$ let $X_*$ be a random point under $g$ ∴

$\int_0^{\pi/2} f(\theta)\,d\theta = 1$ ∴ area under $g(x)$ is: $\int_0^{\pi/2} g(x)\,dx = \int_0^{\pi/2} cf(x)\,dx = c$ ∴

$P(X_* \in [x, x+\Delta x]) = \dfrac{g(x)\Delta x}{c} = \dfrac{cf(x)\Delta x}{c} = f(x)\Delta x$ ∴

$p(X_* \in [x, x+\Delta x]) = f(x)\Delta x \therefore$

$f_{X_*}(x) = F'_{X_*}(x) = \lim_{\Delta x \to 0} \frac{F_{X_*}(x+\Delta x) - F_{X_*}(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{p(X_* \in [x, x+\Delta x])}{\Delta x} =$

$\lim_{\Delta x \to 0} \frac{f(x)\Delta x}{\Delta x} = \lim_{\Delta x \to 0} f(x) = f(x) \therefore$

The x coordinate of a random point drawn uniformly from the area under $g(x)$ has the same distribution as X.

\2b/ let $f_u$ be the pdf of $Unif[0, \frac{\pi}{2}] \therefore$

$f_u(\theta) = \frac{1}{\frac{\pi}{2} - 0} = \frac{1}{(\frac{\pi}{2})} = \frac{2}{\pi} \therefore$

let $g_u(\theta) = K f_u(\theta)$, $K = \max f(\theta) = \max \sin\theta = \sin\theta \big|_{\theta = \frac{\pi}{2}} = \sin\frac{\pi}{2} = 1 \therefore$

$K = 1 \therefore$

sample $\theta_u$ from ~~$\theta_u$~~ $f_u$, and $U$ from $U \sim Unif(0,1)$

$\therefore$ accept $\theta_u$ is ~~$U g_u(\theta_u) \leq g(x_u)$~~ $U g_u(\theta_u) \leq g(\theta_u) = c f(\theta_u) = c \sin\theta_u$

accept $\theta_u$ is $U g_u(\theta_u) \leq K f_u(\theta_u) = \cancel{K} 1 f_u(\theta_u) = \cancel{8} f_u(\theta_u) = \frac{2}{\pi}$

\2c/ Acceptance rate $= \frac{\text{area under } g(\theta)}{\text{area under } g_u(\theta)} \therefore$

$\therefore$ area under $g(\theta)$ is: $\int_0^{\pi/2} g(\theta)\,d\theta = \int_0^{\pi/2} c f(\theta)\,d\theta = c$

area under ~~$g(\theta)$~~ $g_u(\theta)$ is: $\int_0^{\pi/2} g_u(\theta)\,d\theta = \int_0^{\pi/2} \cancel{g_u(\theta)} K f_u(\theta)\,d\theta = K = 1 \therefore$

acceptance rate $= \frac{c}{k} = \frac{c}{1} = c$

\2a/ $X \sim f(x)$, $g(x) = c f(x)$ let $X_*$ be a random point under $g \therefore$

$\int_{-\infty}^{\infty} f(x)\,dx = 1 \therefore$ area under $g(x) = \int_{-\infty}^{\infty} c f(x)\,dx = c\int_{-\infty}^{\infty} f(x)\,dx = c \therefore$

a strip under ~~$f(x)$~~ of width $\Delta x$ has area $f(x)\Delta x \therefore$

a small strip under $g(x)$ of width $\Delta x$ has area ~~$f$~~ $g(x)\Delta x = c f(x)\Delta x$

$\therefore p(X_* \in [x, x+\Delta x]) = \frac{\text{area of strip under } g(x) \text{ area } \Delta x}{\text{area of } g(x)} = \frac{c f(x)\Delta x}{c} = f(x)\Delta x$

$\therefore f_{X_*}(x) = F'_{X_*}(x) = \lim_{\Delta x \to 0} \frac{F_{X_*}(x+\Delta x) - F_{X_*}(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{p(X_* \in [x, x+\Delta x])}{\Delta x} =$

$\lim_{\Delta x \to 0} \frac{f(x)\Delta x}{\Delta x} = \lim_{\Delta x \to 0} f(x) = f(x) \therefore$ ~~The~~

The x coordinate of a random point drawn uniformly from the area under $g(x)$ has the same distribution as X

\PP2020/ \2b/ let $f_u(\theta)$ be pdf of ~~$f_u(\theta)$~~ ~~$\theta_u$~~ Unif$[0, \frac{\pi}{2}]$ ∴

$\therefore f_u(\theta) = \frac{1}{\frac{\pi}{2} - 0} = \frac{1}{\frac{\pi}{2}} = \frac{2}{\pi}$ ∴

$g_u(\theta) = k f_u(\theta) = k\frac{2}{\pi}$ ∴ $k = \max f(\theta) = \max \sin\theta = \sin\theta\big|_{\theta=\frac{\pi}{2}} = \sin\frac{\pi}{2} = 1$

∴ sample $\theta_u$ from $f_u$ and $U$ from Unif$(0,1)$ ∴

Accept $\theta_u$ if $U g_u(\theta_u) \le \frac{2}{\pi}$ ∴

\2c/ area under $f(x)$ is: $\int_0^{\pi/2} f(x)\,dx = 1$,

area under $g_u(x)$ is: $\int_0^{\pi/2} g_u(x)\,dx = \int_0^{\pi/2} k f_u(x) = k$ ~~$\ne 1$~~ ∴ ~~$\int_0^{\pi/2} \frac{2}{\pi}$~~

acceptance rate $= \dfrac{\text{area under } f(x)}{\text{area under } g_u(x)} = \frac{1}{\pi/2} = \frac{2}{\pi}$

\2d/ let $y = r\sin\theta$, $x = r\cos\theta$ ∴ $x^2+y^2 = r^2$ ∴ Jacobian: $J = r$ ∴

$f(r,\theta) \propto \frac{r\sin\theta}{r^2} = \frac{\sin\theta}{r}$ ∴ $f(r,\theta) = k\sin\theta$, $x, y \ge 0$ ∴ $\theta \in [0, \frac{\pi}{2}]$

$x^2+y^2 = r^2 \le 2$ ∴ $0 \le r \le \sqrt{2}$ ∴ $dx\,dy = \cancel{J}\,dr\,d\theta = dx\,dy = r\,dr\,d\theta$ ∴

$1 = \int_0^{\sqrt{2}}\int_0^{\pi/2} k\frac{\sin\theta}{r} r\,dr\,d\theta = \sqrt{2}\,k[-\cos\theta]_0^{\pi/2} = \sqrt{2}\,k$ ∴

$k = \frac{1}{\sqrt{2}}$ ∴ $f(r,\theta) = \frac{1}{\sqrt{2}}\frac{\sin\theta}{r}$ ∴

$f(\theta) = \int_0^{\sqrt{2}} \frac{1}{\sqrt{2}}\frac{\sin\theta}{r} r\,dr = \cancel{\sin\theta}\int_0^{\sqrt{2}} \frac{1}{\sqrt{2}}\sin\theta\,dr = \sin\theta$ ∴

$f(r,\theta) = f(r|\theta)f(\theta)$, $f(r|\theta) = \frac{1}{\sqrt{2}} = f(r)$ ∴

$r \sim$ Unif$(0, \sqrt{2})$ ∴

algorithm is: i: sample $r_*$ from Unif$(0, \sqrt{2}]$

ii sample $\theta_*$ from $f_u$, $f_u \sim$ Unif$[0, \frac{\pi}{2}]$ as in part (b)

∴ Accept $\theta_*$ if $U g_u(\theta_*) \le \frac{2}{\pi}$, $U$ sampled from Unif$(0,1)$

~~$g_u = \frac{1}{\sqrt{2}}(f_u)$~~ $g_u = \frac{1}{\sqrt{2}} f_u$

iii: if $\theta_*$ accepted $X_* = r_* \cos\theta_*$, $Y_* = r_* \sin\theta_*$

start PP2019

do 2021 ~~Q3~~ 3, ~~Q4~~ on recap video

~~do 2022 1st~~

\PP2019/\1a/ By Bayes: $\pi(\theta|\underline{y}) \propto \pi(\theta) p(\underline{y}|\theta) \propto \pi(\theta) \prod_{i=1}^{n} p(y_i|\theta) \propto$

$\pi(\theta) \prod_{i=1}^{n} \frac{1}{2\theta^3} y_i^2 e^{-y_i/\theta} \propto \pi(\theta) \prod_{i=1}^{n} \theta^{-3} e^{-y_i \frac{1}{\theta}} \propto \pi(\theta) \theta^{-3n} e^{-\frac{1}{\theta}\sum_{i=1}^{n} y_i} \propto$

$\pi(\theta) \theta^{-3n} e^{-n\frac{1}{\theta}\bar{y}} \propto \frac{\beta^\alpha}{\Gamma(\alpha)} \left(\frac{1}{\theta}\right)^{\alpha+1} e^{-\beta\frac{1}{\theta}} \theta^{-3n} e^{-n\bar{y}\frac{1}{\theta}} \propto$

$(\theta^{-1})^{\alpha+1} e^{-\beta\frac{1}{\theta} - n\bar{y}\frac{1}{\theta}} \theta^{-3n} \propto \left(\frac{1}{\theta}\right)^{\alpha+1} \left(\frac{1}{\theta}\right)^{3n} e^{-(\beta+n\bar{y})\frac{1}{\theta}} \propto$

$\left(\frac{1}{\theta}\right)^{(\alpha+3n)+1} e^{-(\beta+n\bar{y})\frac{1}{\theta}} \propto \left(\frac{1}{\theta}\right)^{a+1} e^{-b\frac{1}{\theta}}$

which is proportional to a inverse gamma density ∴

$a = \alpha + 3n$, $b = \beta + n\bar{y}$ ∴

~~π(θ)~~ $\theta|\underline{y} \sim IG(a, b)$ ∴

~~the~~ both the posterior and prior follow a inverse gamma distribution ∴ the posterior is conjugate to the prior ∴

∴ The inverse gamma distribution is its conjugate prior

\1bi/ probability can never be observed objectively and ∴ ~~ex~~ is always subjective on an individuals prior beliefs

\1bii/ Given a random quantity X, your expectation X, E(X) is that value $\bar{x}$ you would choose on the understanding that having chosen $\bar{x}$, you are committed to accepting any bet whatsoever with gain $c(X-\bar{x})$, where c is arbitrary and chosen by your opponent

\1bi/ The meaning of probability changes when we make it a measure of our own uncertainty, rather than a thing we measured in the world

\1biii/ ~~be~~ same $p(A = E(A)$ ∴ A is true or false ∴ $A=1$ or $A=0$

\1biv/ ~~pt~~ $A = A_1 + \dots + A_n$ ∴ $P(A) = E(A) = E(A_1 + \dots + A_n) =$

$E(A_1) + \dots + E(A_n) = p(A_1) + P(A_2) + \dots + P(A_n) = E(1) = 1$

\1bv/ ∴ for $n=2$ : $P(A) = p(A_1) + P(A_2) = 1 = P(A_1) + P(\tilde{A}_1)$ ∴

$P(\tilde{A}_1) = 1 - P(A)$ ~~so~~ ∴ for a partition $n=2$ ∴ $A, \tilde{A}$ ∵ $1 - P(A) = P(\tilde{A})$

\1ci/ suppose random quantity $X$ with cdf $F(x) = P(X \le x)$

for random quantity $F(X) = W$ ∴ $W \in [0,1]$ ∴

$P(W<0) = P(W>1) = 0$ for $w \in [0,1]$ :

$P(W \le w) = P(F(X) \le w) = P(X \le F^{-1}(w)) = P(F^{-1}(W) \le F^{-1}(w))$

$= P(F^{-1}(F(X)) \le F^{-1}(w)) = P(X \le F^{-1}(w)) = FF^{-1}(w) = w$ ∴

$P(W \le w) = w$ is cdf of $W$ ∴

pdf of $W$ is: $f_W(w) = \frac{\partial}{\partial w}(P(W \le w)) = \frac{\partial}{\partial w}(w) = 1$ ∴

pdf of $W$ is 1 ∴ $W$ is uniform on 1 ∴

to obtain a random value from the same distribution as $X$,

sample $U$ from $Unif(0,1)$ and compute $F^{-1}(U)$

\1ci/ you can generate random values from the discrete

distribution by: define $F_*^{-1}(p) = x_j$ , $p \in [F_{j-1}, F_j]$

then if $U \sim Unif(0,1)$ and $X_* = F_*^{-1}(U$

i) $X_*$ can only take values $x_1, \dots, x_m$

ii) $P(X_* = x_j) = P(U \in (F_{j-1}; F_j])$ $\quad F_j - F_{j-1} = p_j$ ∴

$X_*$ has same distribution as $X$ ∴ generate $U$ from $Unif(0,1)$ ∴

compute $F^{-1}(U)$ will generate values from the same distribution

as $X$ for $F(x) = P(X \le x)$

\1cii/ let $F(X) = W$ ∴ $P(W \le w) = P(F(X) \le w) = \cancel{P(W \le F^{-1}(w))} =$

$\cancel{P(F^{-1}(X) \le F^{-1}(w)) = P(F(F^{-1}(x) \le w) = P(X \le w) = F(w)}$ $F(w)$

let $F(X) = W$ ∴ $P(W \le w) = P(F(X) \le w) = P(X \le F^{-1}(w)) =$

$P(F^{-1}(W) \le F^{-1}(w)) = P(F^{-1}(F(X)) \le F^{-1}(w)) = \cancel{P(X \le w)}$ $P(X \le F^{-1}(w)) =$

$F(F^{-1}(w)) = w = P(W \le w)$ ∴

$f_W(w) = \frac{\partial}{\partial w} P(W \le w) = \frac{\partial}{\partial w} w = 1$ ∴ $W$ is uniform on $[0,1]$ ∴

∴ let $U$ from $Unif(0,1)$ ∴ $F^{-1}(U) = X$ samples values from $X$.

\1ciii/ $\theta = \frac{1}{2}$ ∴ $P(X=x) = (1-\frac{1}{2})^{x-1}\frac{1}{2} = (\frac{1}{2})^{x-1}\frac{1}{2} = (\frac{1}{2})^x$

∴ $F(x) = P(X \le x) = \sum_{n=1}^{x} (\frac{1}{2})^n = \sum_{n=0}^{x} (\frac{1}{2})^n - (\frac{1}{2})^0 = \frac{1-(\frac{1}{2})^{x+1}}{1-\frac{1}{2}} - 1 =$

$\frac{1-(\frac{1}{2})^{x+1}}{(\frac{1}{2})} - \frac{(\frac{1}{2})}{(\frac{1}{2})} = \frac{0.5 - 0.5(\frac{1}{2})^x}{(0.5)} = \frac{0.5}{0.5}(1-(\frac{1}{2})^x) = 1-(\frac{1}{2})^x = F$ ∴

$1 - F = (\frac{1}{2})^x$ ∴ $\ln(1-F) = \ln(\frac{1}{2})^x = x\ln(2^{-1}) = -x\ln(2)$ ∴ $x = \frac{-\ln(1-F)}{\ln(2)}$ ∴

\pp2019/ $\therefore F^{-1}(x) = \frac{-\ln(1-x)}{\ln(2)} \therefore$

$X = F^{-1}(U) \therefore \quad U_1, U_2, U_3, U_4, U_5 \therefore$

$X_1 = \frac{-\ln(1-0.649)}{\ln(2)} = 0.0968$

$X_2 = 2.52 \quad X_3 = F^{-1}(U_3) = 0.0923$

$X_4 = F^{-1}(U_4) = 4.78 \quad X_5 = F^{-1}(U_5) = F^{-1}(0.276) = \frac{-\ln(1-0.276)}{\ln(2)} = 0.466$

\1di/ a utility function $U$ assigns a real number $U(g)$ to each gamble $g$, subject to:

if $g_1 <^* g_2$ then $U(g_1) < U(g_2)$

if $g_1 \sim^* g_2$ then $U(g_1) = U(g_2)$

for any 2 rewards $r, s$ and $p \in [0,1]$:

$U(pr + (1-p)s) = pU(r) + (1-p)U(s)$

\1dii/ define $p_s$ for: $p_s R \sim^* s \therefore$

set $U(R) = 1$, $U(r) = 0 \therefore$

$U(s) = p_s$

\1dii/ set $U(R) = 1$, $U(r) = 0 \therefore$

define $p_s$ that $p_s R \sim^* S \therefore$

let $U(S) = p_s$

\2a/ By Bayes: $\pi(\theta|y) \propto \pi(\theta) f(y|\theta) \propto \pi(\theta) \prod_{i=1}^{n} (f(y_i|\theta)) \propto$
$\pi(\theta) \prod_{i=1}^{n} \theta \left(\frac{2}{y_i}\right)^\theta \propto \pi(\theta)\theta^n \prod_{i=1}^{n} e^{\ln\left(\frac{2}{y_i}\right)^\theta} \propto \pi(\theta)\theta^n \prod_{i=1}^{n} e^{\theta\ln\left(\frac{2}{y_i}\right)} \propto$
$\pi(\theta)\theta^n e^{\theta\sum_{i=1}^{n}\ln\left(\frac{2}{y_i}\right)} \propto \theta^{a-1}e^{-b\theta}\theta^n e^{-\left(-\sum_{i=1}^{n}\ln\left(\frac{2}{y_i}\right)\right)\theta} \propto$
$\theta^{(a+n)-1}e^{-b\theta-\left(-\sum_{i=1}^{n}\ln\left(\frac{2}{y_i}\right)\right)\theta} \propto \theta^{(a+n)-1}e^{-\left(b-\sum_{i=1}^{n}\ln\left(\frac{2}{y_i}\right)\right)\theta} \propto \theta^{\alpha-1}e^{-\beta\theta} \therefore$

~~is proportional~~ $\therefore$ the posterior is proportional to a Gamma density, for $\alpha = a+n$, $\beta = b - \sum_{i=1}^{n}\ln\left(\frac{2}{y_i}\right) \therefore$

$\theta|y \sim \text{Gamma}(\alpha,\beta) \therefore \pi(\theta|y) = \frac{\beta^\alpha}{\Gamma(\alpha)}\theta^{\alpha-1}e^{-\beta\theta} \quad \theta > 0$

\2b/ $\therefore \ln(\pi(\theta|y)) = \ln\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right) + \ln(\theta^{\alpha-1}) + \ln(e^{-\beta\theta}) = \ln\left(\frac{\beta^\alpha}{\Gamma(\alpha)}\right) + (\alpha-1)\ln(\theta) - \beta\theta$

$\therefore \frac{\partial\ln(\pi(\theta|y))}{\partial\theta} = \frac{\alpha-1}{\theta} - \beta \quad \therefore \frac{\alpha-1}{\hat\theta} - \beta = 0 \therefore \frac{\alpha-1}{\hat\theta} = \beta \therefore \frac{\alpha-1}{\beta} = \hat\theta$ is the MAP estimate for $\theta \therefore$

$\hat{\theta} = (\alpha-1)\frac{1}{\beta} = (a+n-1)\frac{1}{b-\sum_{i=1}^{n}\ln\left(\frac{2}{y_i}\right)}$

\2c / observed information $I(\hat{\theta}) = -\left[\frac{d^2}{d\theta^2}\ln\pi(\theta|y)\right]_{\theta=\hat{\theta}}$

$\theta|y \sim N(\hat{\theta}, I(\hat{\theta})^{-1}) \therefore$

$\frac{\partial}{\partial\theta^2}\ln\pi(\theta|y) = \frac{\partial}{\partial\theta}\left[\frac{\partial}{\partial\theta}\ln\pi(\theta|y)\right] = \frac{\partial}{\partial\theta}\left((\alpha-1)\theta^{-1}-\beta\right) = -(\alpha-1)\theta^{-2} \therefore$

$-\frac{\partial^2}{\partial\theta^2}\ln\pi(\theta|y) = (-1)(-1)(\alpha-1)\theta^{-2} = (\alpha-1)\theta^{-2} \therefore$

$-\frac{\partial^2}{\partial\theta^2}\ln\pi(\theta|y)\Big|_{\theta=\hat{\theta}} = (\alpha-1)\theta^{-2}\Big|_{\theta=\hat{\theta}} = (\alpha-1)\hat{\theta}^{-2} = I(\hat{\theta}) \therefore$

$I(\hat{\theta})^{-1} = \frac{1}{(\alpha-1)\hat{\theta}^{-2}} = \frac{\hat{\theta}^2}{\alpha-1} \therefore$

$\theta|y \sim N\left(\hat{\theta}, \frac{\hat{\theta}^2}{\alpha-1}\right)$

\2d/ $var(\theta|y) \approx \frac{\hat{\theta}^2}{\alpha-1} = \left[(a+n-1)\frac{1}{b-\sum_{i=1}^{n}\ln\left(\frac{2}{y_i}\right)}\right]^2\frac{1}{\alpha-1} \therefore$

$y>2 \therefore \quad \frac{2}{y_i}<1 \therefore \quad \ln\left(\frac{2}{y_i}\right)<0 \therefore \quad -\ln\left(\frac{2}{y_i}\right)>0 \therefore$

$\cancel{-\ln\left(\frac{2}{y_i}\right)}$ $-\sum_{i=1}^{n}\ln\left(\frac{2}{y_i}\right)>0 \therefore \quad b-\sum_{i=1}^{n}\ln\left(\frac{2}{y_i}\right)>0 \therefore$

$b-\sum_{i=1}^{n}\ln\left(\frac{2}{y_i}\right)\to\infty$ as $n\to\infty \therefore \quad \frac{1}{b-\sum_{i=1}^{n}\ln\left(\frac{2}{y_i}\right)}\to 0$ as $n\to\infty \therefore$

$\hat{\theta} = (a+n-1)\frac{1}{b-\sum_{i=1}^{n}\ln\left(\frac{2}{y_i}\right)} \to 0$ as $n\to\infty \therefore$ ~~this~~

$var(\theta|y)\to 0$ as $n\to\infty$ $\therefore$ the normal approx asymp tends to 0 meaning the posterior tends towards a fixed estimate with no variance

\4a/ By Bayes: ~~$\pi(\theta|y)$~~ $n=1 \therefore$ $\pi(\theta|y) = \pi(\theta|y) \propto \pi(\theta)p(y|\theta) \propto$
$\pi(\theta)\prod_{i=1}^{n}p(y_i|\theta) \propto \pi(\theta)p(y|\theta) \propto \pi(\theta_1,\theta_2)p(y|\theta) \propto 1\, p(y|\theta) \propto p(y|\theta)$
$\therefore$ the posterior is proportional to a bivariate normal density
$\therefore \theta|y = \theta|y_1 = \theta|y \sim N\left(\begin{pmatrix} y_1 \\ y_2 \end{pmatrix}, \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}\right)$

~~let~~ \4b/ let $Q(y_1, y_2, \theta_1, \theta_2) = [\theta_2-(y_2+\rho(\theta_1-y_1))]^2 + a(y_1,\theta_1)$
$= [\theta_2-(y_2+\rho\theta_1-\rho y_1)]^2 + a(y_1,\theta_1) = [\theta_2-y_2-\rho\theta_1+\rho y_1]^2 + a(y_1,\theta_1) =$
$((y_1-\theta_1)^2+(y_2-\theta_2)^2-2\rho(y_1-\theta_1)(y_2-\theta_2)) \therefore$

$\pi(\theta|y) = p(y|\theta) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{((y_1-\theta_1)^2+(y_2-\theta_2)^2-2\rho(y_1-\theta_1)(y_2-\theta_2))}{2(1-\rho^2)}\right\} =$

\PP2019/II $\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}Q(y_1, y_2, \theta_1, \theta_2)\right\} =$

$\frac{1}{2\pi\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[(\theta_2-(y_2+\rho(\theta_1-y_1)))^2+a(y,\theta_1)\right]\right.$

$\therefore (y_1-\theta_1)^2+(y_2-\theta_2)^2-2\rho(y_1-\theta_1)(y_2-\theta_2)=(\theta_2-(y_2+\rho(\theta_1-y_1)))^2+a(y,\theta_1)$

$=(\theta_2-(y_2+\rho(\theta_1-y_1)))^2+a(y_1,y_2,\theta_1)$

\4c/ By Bayes: $\pi(\theta_2|\theta_1,y) \propto \pi(\theta_2)\pi(\theta_1,y|\theta_2) \propto$

$\pi(\theta_2)\pi(y|\theta_1,\theta_2)\pi(\theta_1|\theta_2) \propto \pi(\theta_2|\theta_1)\pi(y|\theta_1,\theta_2)=\pi(\theta_2|\theta_1)\pi(y|\theta)$

$\therefore$ by symmetry: $\pi(\theta_1|\theta_2,y) \propto \pi(\theta_1|\theta_2)\pi(y|\theta)$

$\therefore \pi(\theta_1,\theta_2) \propto 1 \therefore$ ~~$\theta_1$ indep~~

observed information ~~$I(\hat{\theta}_2)$~~

$\ln(\pi(\theta|y)) = \ln\left(\frac{1}{2\pi\sqrt{1-\rho^2}}\right) - \frac{1}{2(1-\rho^2)}\left[(\theta_2-(y_2-\rho(\theta_1-y_1)))^2+a(y,\theta_1)\right] \therefore$

$\frac{\partial \ln\pi(\theta|y)}{\partial\theta_2}$

\4d/ Box-Muller Method is let $U_1, U_2 \sim Unif(0,1) \therefore$

$\theta = 2\pi U_1$, $R=\sqrt{-2\log U_2}$ $\therefore$

let $X_1 = R\cos\theta$, $X_2 = R\sin\theta$ $\therefore$ $X_1, X_2 \sim N(0,1)$

$\theta_1^{(0)} = \theta_2^{(0)} = 0$, $\rho = 0.8$, $y_1 = 3$, $y_2 = 2$ $\therefore$

$F(\theta_1^{(1)} | \theta_2^{(0)} = 0)$ $\therefore$

~~$\theta_1|\theta_2, y \sim N(3+0.8($~~ $\theta_1|\theta_2=0, y_1=3, y_2=2 \sim N(3+0.8(0-2), 1-0.8^2) = N(1.4, 0.36)$

$\therefore$