$$\frac{dP_3}{dt} = 3P_2 - 3\mu P_3 - 3P_3 + 3\mu P_4 = 0 = -3P_3 + 3\mu P_4$$

$$\frac{dP_4}{dt} = 3P_3 - 3\mu P_4 - 3P_4 + 3\mu P_5 = 0 = -3P_4 + 3\mu P_4 \therefore$$

$$3\frac{1}{\mu}P_0 = P_1 \quad 3P_1 = 2\mu P_2 \quad \therefore \quad \frac{3}{2}\frac{1}{\mu}P_1 = P_2 = \frac{3}{2}\frac{1}{\mu}3\frac{1}{\mu}P_0 = \frac{9}{2}\frac{1}{\mu^2}P_0$$

$$3P_2 = 3\mu P_3 \quad \therefore \quad \frac{1}{\mu}P_2 = P_3 \quad \therefore \quad \cancel{\frac{9}{2}\frac{1}{\mu}} \frac{1}{\mu}\frac{9}{2}\frac{1}{\mu^2} = \frac{9}{2}\frac{1}{\mu^3}P_0 = P_3 \quad \therefore$$

$$\frac{9}{2}\frac{1}{\mu^n}P_0 = P_n \text{ for } n \geq 2 \quad \therefore \quad \sum_{n=0}^{\infty} P_n = 1 \quad \therefore$$

$$\sum_{n=0}^{\infty} \rho^n = \frac{1}{1-\rho} \quad \therefore \quad \sum_{n=0}^{\infty}\left(\frac{1}{\mu}\right)^n = \frac{1}{1-\left(\frac{1}{\mu}\right)} \quad \therefore$$

$$P_0 + P_1 + \sum_{n=2}^{\infty} P_n = 1 = P + \frac{3}{\mu}P_0 + \sum_{n=2}^{\infty}\frac{9}{2}\left(\frac{1}{\mu}\right)^n P_0 = P_0\left(1 + \frac{3}{\mu} + \frac{9}{2}\cdot\sum_{n=2}^{\infty}\left(\frac{1}{\mu}\right)^n\right.$$

$$\sum_{n=0}^{\infty}\rho^n = \frac{1}{1-\rho}, \quad \sum_{n=0}^{\infty} n\rho^n = \frac{\rho}{(1-\rho)^2} \qquad \sum_{n=1}^{\infty}\rho^n = \frac{\rho}{1-\rho} \quad \therefore \quad \sum_{n=1}^{\infty}\left(\frac{1}{\mu}\right)^n = \frac{\left(\frac{1}{\mu}\right)}{\left(1-\left(\frac{1}{\mu}\right)\right)}$$

$$\therefore \sum_{n=2}^{\infty}\left(\frac{1}{\mu}\right)^n = \frac{\left(\frac{1}{\mu}\right)}{1-\frac{1}{\mu}} - \frac{1}{\mu} = \frac{1}{\mu-1} - \frac{1}{\mu} = \frac{\mu}{\mu(\mu-1)} - \frac{\mu-1}{\mu(\mu-1)} = \frac{1}{\mu(\mu-1)} \quad \therefore$$

$$\cancel{P_0} \Rightarrow 1 = P_0\left(1 + \frac{3}{\mu} + \frac{9}{2}\sum_{n=2}^{\infty}\left(\frac{1}{\mu}\right)^n\right) = P_0\left(1 + \frac{3}{\mu} + \frac{9}{2}\frac{1}{\mu(\mu-1)}\right) =$$

$$P_0\left(\frac{\mu(\mu-1)}{\mu(\mu-1)} + \frac{3(\mu-1)}{\mu(\mu-1)} + \frac{\left(\frac{9}{2}\right)}{\mu(\mu-1)}\right) = P_0\left(\frac{\mu^2 - \mu + 3\mu - 3 + \frac{9}{2}}{\mu(\mu-1)}\right) = P_0\left(\frac{\mu^2 + 2\mu + \frac{3}{2}}{\mu(\mu-1)}\right)$$

$$\therefore \frac{\mu(\mu-1)}{\mu^2 + 2\mu + \frac{3}{2}} = P_0 = \frac{2\mu(\mu-1)}{2\mu^2 + 4\mu + 3} \qquad \therefore \quad P_1 = \frac{6(\mu-1)}{2\mu^2 + 4\mu + 3}$$

$$3\frac{\mu-1}{\mu^2 + 2\mu + \frac{3}{2}} = P_1, \quad \frac{9}{2}\frac{1}{\mu^n}\frac{\mu(\mu-1)}{\mu^2 + 2\mu + \frac{3}{2}} = \frac{9}{2}\frac{1}{\mu^{n-1}}\frac{\mu-1}{\mu^2 + 2\mu + \frac{3}{2}} = P_n \text{ for } n \geq 2$$

$\therefore \dfrac{9(\mu-1)}{\mu^{n-1}(2\mu^2 + 4\mu + 3)}$ steady state exists for $3\mu > \lambda = 3 \therefore \mu > 1$

\2 a iii/

$$G_X(\theta) = P(X=0) + P(X=1)\theta + P(X=2)\theta^2 + \ldots = \sum_{n=0}^{\infty} P_n\theta^n =$$

$$\frac{2\mu(\mu-1)}{2\mu^2 + 4\mu + 3}\left(1 + \frac{3\theta}{\mu} + \frac{9}{2}\sum_{n=2}^{\infty}\frac{\theta^n}{\mu^n}\right) \quad \therefore$$

$$G_X'(\theta) = \frac{2\mu(\mu-1)}{2\mu^2 + 4\mu + 3}\left(\frac{3}{\mu} + \frac{9}{2\mu}\sum_{n=2}^{\infty} n\frac{\theta^{n-1}}{\mu^{n-1}}\right) = \frac{2\mu(\mu-1)}{2\mu^2 + 4\mu + 3}\left(\frac{3}{\mu} + \frac{9}{2\mu}\sum_{n=1}^{\infty}\frac{(n+1)\theta^n}{\mu^n}\right)$$

$$\therefore G_X'(1) = \frac{2\mu(\mu-1)}{2\mu^2 + 4\mu + 3}\left(\frac{3}{\mu} + \frac{9}{2\mu}\sum_{n=1}^{\infty}\frac{n+1}{\mu^n}\right) = \frac{2\mu(\mu-1)}{2\mu^2 + 4\mu + 3}\left(\frac{3}{\mu} + \frac{9}{2\mu}\frac{(2\mu-1)}{(\mu-1)^2}\right) =$$

$$\frac{2\mu(\mu-1)}{2\mu^2 + 4\mu + 3}\left(\frac{6(\mu^2 - 2\mu + 1) + 18\mu - 9}{2\mu(\mu-1)^2}\right) = \frac{6\mu^2 - 12\mu + 6 + 18\mu - 9}{(\mu-1)(2\mu^2 + 4\mu + 3)} = \frac{3(2\mu^2 + 2\mu - 1)}{(\mu-1)(2\mu^2 + 4\mu + 3)}$$

$= L_S \equiv$ expected system size

$\therefore$ for $\mu = 1.2$: $L_S = 6.01$ (3 s.f.)

\PP2021/ $\cancel{e =} \; \cancel{e = -\alpha+1} \; \therefore \; \alpha < \frac{1}{3} \therefore \; 3\alpha < 1 \; \therefore 3\alpha - 1 < 0 \therefore$

$-(3\alpha-1) > 0 \; \therefore \; \frac{-\alpha+1+(3\alpha-1)}{2-4\alpha} < \frac{-\alpha+1-(3\alpha-1)}{2-4\alpha} \therefore$

$$\theta = \frac{-\alpha+1-3\alpha+1}{2-4\alpha} = \frac{2-4\alpha}{2-4\alpha} = 1 \therefore$$

$$e = \frac{-\alpha+1+(3\alpha-1)}{2-4\alpha} = \frac{-\alpha+1+3\alpha-1}{2-4\alpha} = \frac{2\alpha}{2-4\alpha} = \frac{\alpha}{1-2\alpha}$$

\1div/ Probability of extinction for $\alpha < \frac{1}{3}$ given by: solving

$G_X(\theta) = \theta \; \therefore \; \alpha + (\alpha-1)\theta + (1-2\alpha)\theta^2 = 0$

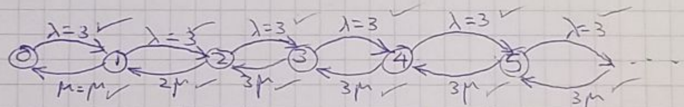
$2(1-2\alpha)\theta_{\pm} = 1-\alpha \pm \sqrt{\alpha^2-2\alpha+1-4\alpha(1-2\alpha)} = 1-\alpha \pm \sqrt{\alpha^2-2\alpha+1-4\alpha+8\alpha^2} =$

$1-\alpha \pm \sqrt{9\alpha^2-6\alpha+1} = 1-\alpha \pm \sqrt{(3\alpha-1)^2} = 1-\alpha \pm (3\alpha-1) \therefore$

$\cancel{\theta_+ = 1-\alpha+3\alpha-1}$ $\theta_+ = \frac{1-\alpha+3\alpha}{2(1-2\alpha)} = \frac{\alpha}{1-2\alpha}$, $\theta_{-1} = \frac{1-\alpha-3\alpha+1}{2(1-2\alpha)} = 1$.

$\therefore$ for $\alpha < \frac{1}{3}$, θ_+ is minimal root

\2ai/



M/M/3 queue with infinite capacity with $\lambda = 3$, $\mu > 0$

\2aii/ steady state: $3P_0 = \mu P_1$, $3P_1 + \mu P_1 = 3P_0 + 2\mu P_2$,

$3P_2 + 2\mu P_2 = 3P_1 + 3\mu P_3$, $3P_3 + 3\mu P_3 = 3P_2 + 3\mu P_4 = 3P_3(1+\mu) = 3(P_2 + \mu P_4)$

$\therefore P_3(1+\mu) = P_2 + \mu P_4 \; \therefore \; (1+\mu)P_3 - P_2 = \mu P_4 \therefore$

$\frac{-1}{\mu}P_2 + \frac{1}{\mu}(1+\mu)P_3 = P_4 \therefore$

$3\frac{1}{\mu}P_0 = P_1$ $\quad (3+\mu)P_1 = 3P_0 + 2\mu P_2 \; \therefore -3P_0 + (3+\mu)P_1 = 2\mu P_2 \therefore \frac{-3}{2\mu}P_0 + \frac{(3+\mu)}{2\mu}P_1 = P_2$

$-3P_1 + (3+2\mu)P_2 = 3\mu P_3 \therefore -\frac{1}{\mu}P_1 + \frac{(3+2\mu)}{3\mu}P_2 = P_3$

$3P_5 + 3\mu P_5 = 3P_4 + 3\mu P_6 \therefore$

$P_5 + \mu P_5 = (1+\mu)P_5 = P_4 + \mu P_6 \therefore -P_4 + (1+\mu)P_5 = \mu P_6 \therefore \cancel{-\frac{1}{\mu}P_4 + \frac{1}{\mu}(1+\mu)}$

$-\frac{1}{\mu}P_4 + \frac{1}{\mu}(1+\mu)P_5 = P_6 \therefore$

$-\frac{1}{\mu}P_{n-2} + \frac{1}{\mu}(1+\mu)P_{n-1} = P_n$ for $n \geq 4$, $\sum_{n=0}^{\infty} P_n = 1 \therefore$

$\cancel{-3P_2}$ $\frac{-3}{2\mu}P_0 + \frac{(3+\mu)}{2\mu}3\frac{1}{\mu}P_0 = \frac{-3}{2\mu}P_0 + \frac{9+3\mu}{2\mu^2}P_0$

\2aii/ $\frac{dP_0}{dt} = -3P_0 + \mu P_1 = 0$, $\frac{dP_1}{dt} = -\mu P_1 + 3P_0 - 3P_1 + 2\mu P_2 = 0 = -3P_1 + 2\mu P_2$

$\frac{dP_2}{dt} = 3P_1 - 2\mu P_2 - 3P_2 + 3\mu P_3 = 0 = -3P_2 + 3\mu P_3$,

\1 c ii/: $P(X_4=2 \mid X_0=2) = P(2\to2\to2\to2\to2) +$
$P(2\to1\to2\to1\to2) + P(2\to1\to2\to2\to2) + P(2\to2\to1\to2\to2) +$
$P(2\to2\to2\to1\to2) =$
$P(2\to2\to2\to2\to2) + P(2\to1\to2\to1\to2) + 3P(2\to1\to2\to2\to2) =$
$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times1\times\frac{1}{2}\times1 + 3\left(\frac{1}{2}\times1\times\frac{1}{2}\times\frac{1}{2}\right) = \frac{1}{16} + \frac{1}{4} + \frac{3}{8} = \frac{11}{16} = .688$ (3 S.F.)

\1 d i/ valid is $G_X(1) = 1 = \alpha + \alpha(1) + (1-2\alpha)(1)^2 = \alpha + \alpha + 1 - 2\alpha =$
$2\alpha + 1 - 2\alpha = 1$
$P(S_0=0)=\alpha$, $P(S_0=1)=\alpha$ $P(S_0=2)=1-2\alpha$ ∴
$\alpha \geq 0$, $1-2\alpha \geq 0$ ∴ $1 \geq 2\alpha$ ∴ $\frac{1}{2} \geq \alpha$ ∴
$\alpha \in [0, \frac{1}{2}]$ must be true

(1 d ii) $E(X) = G_X'(1) = G_X'(\theta)|_{\theta=1} = \alpha + 2(1-2\alpha)\theta|_{\theta=1} = \alpha + (2-4\alpha)1 =$
$2 - 3\alpha$ ∴
$E(S_n) = (E(X))^n = (2-3\alpha)^n$

\1 d iii/ Extinct guaranteed is $E(X) < \frac{1}{2}$ ∴ X
$e=1$ is $2-3\alpha < \frac{1}{2}$ ∴ $\frac{3}{2} < 3\alpha$ ∴
$\frac{1}{2} < \alpha$ but
$\alpha \in [0, \frac{1}{2}]$ ∴ guaranteed extinction is impossible X

~~\1 d iv~~ \1 d iii/ $e=1$ is $E(X) < 1$ ∴ $2-3\alpha < 1$ ∴
$1 < 3\alpha$ ∴ $\alpha > \frac{1}{3}$ extinction is guaranteed
\1 d iii/ Guaranteed extinction is $G_X'(1) \leq 1$ ∴ $2-3\alpha \leq 1$ ∴ $3\alpha \geq 1$ ∴
$\alpha \geq \frac{1}{3}$

\1 d iv/ ∴ $\alpha < \frac{1}{3}$ ∴ $G_X(\theta) - \theta = 0 = \alpha + \alpha\theta + (1-2\alpha)\theta^2 - \theta =$
$\alpha + (\alpha-1)\theta + (1-2\alpha)\theta^2 =$ $(1-2\alpha)\theta^2 + (\alpha-1)\theta + \alpha = 0$ ∴
$\theta = \frac{-\alpha+1 \pm \sqrt{(\alpha-1)^2 - 4(1-2\alpha)\alpha}}{2(1-2\alpha)} = e =$
$\frac{-\alpha+1 \pm \sqrt{\alpha^2 - 2\alpha + 1 - 4(\alpha - 2\alpha^2)}}{2-4\alpha} = \frac{-\alpha+1 \pm \sqrt{\alpha^2 - 2\alpha + 1 - 4\alpha + 8\alpha^2}}{2-4\alpha} =$
$\frac{-\alpha+1 \pm \sqrt{9\alpha^2 - 6\alpha + 1}}{2-4\alpha} = \frac{-\alpha+1 \pm \sqrt{(3\alpha-1)(3\alpha-1)}}{2-4\alpha} = \frac{-\alpha+1 \pm (3\alpha-1)}{2-4\alpha}$ ∴

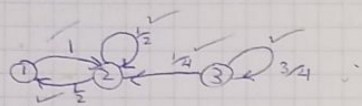
\PP2021/ $\therefore M_x''(t) = \left(\mu + \frac{2t}{\lambda^2(\lambda^2-t^2)}\right) M_x'(t) + \left(\frac{\lambda^2(\lambda^2-t^2)2 - 2t\lambda^2(-2t)}{(\lambda^2(\lambda^2-t^2))^2}\right) M_x(t)$

$\therefore E(x^2) = M_x''(0) = \left(\mu + \frac{2(0)}{\lambda^2(\lambda^2-0^2)}\right) M_x'(0) + \left(\frac{\lambda^2(\lambda^2-0^2)2 - 2(0)\lambda^2(-2(0))}{(\lambda^2(\lambda^2-0^2))^2}\right) M_x(0)$

$= (\mu + 0)\mu + \frac{2\lambda^2(\lambda^2) - 0}{(\lambda^2(\lambda^2))^2}(1) = \mu^2 + \frac{2\lambda^4}{\lambda^8} = \mu^2 + \frac{2}{\lambda^4} \therefore$

$\text{var}(x) = E(x^2) - (E(x))^2 = \mu^2 + \frac{2}{\lambda^4} - \mu^2 = \frac{2}{\lambda^4}$ X $\frac{2}{\lambda^2}$

\1ci/



∴

① and ② is a irroducable subchain

\1cii/ A: ① and ② is a subchain $\therefore P(X_3=1 \mid X_0=1) =$

$P(X_1=2, X_2=2, X_3=1 \mid X_0=1) = 1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(1\to2\to2\to1 \mid 1)$

B: $P(X_3=2 \mid X_0=1) =$

$P(X_1=2, X_2=1, X_3=2 \mid X_0=1) + P(X_1=2, X_2=2, X_3=2 \mid X_0=1) =$

$P(1\to2\to1\to2 \mid 1) + P(1\to2\to2\to2 \mid 1) =$

$1 \times \frac{1}{2} \times 1 + 1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

C: $P(X_3=3 \mid X_0=1) = P(1\to2\to2\to3 \mid 1) =$

$1 \times \frac{1}{2} \times 0 = 0$ $\because$ 1 and 2 is a subchain

D: $P(X_4=2 \mid X_0=2) =$

$P(2\to1\to2\to1\to2 \mid 2) + P(2\to1\to2\to2\to2 \mid 2) + P(2\to2\to2\to2\to2 \mid 2)$

$= \frac{1}{2}\times1\times\frac{1}{2}\times1 + \frac{1}{2}\times1\times\frac{1}{2}\times\frac{1}{2} + \frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{7}{16} = 0.4375$ (35.5)

X

\1ciii/ steady state $\tilde{P} = (P_1, P_2, P_3) = (P_1 \; P_2 \; P_3)$

$\tilde{P} = \tilde{P}T = (P_1 \; P_2 \; P_3) = (P_1 \; P_2 \; P_3)\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix} =$

$(\frac{1}{2}P_2 \quad P_1 + \frac{1}{2}P_2 + \frac{1}{4}P_3 \quad \frac{3}{4}P_3) \therefore$

$P_1 = \frac{1}{2}P_2$, $P_2 = P_1 + \frac{1}{2}P_2 + \frac{1}{4}P_3$, $P_3 = \frac{3}{4}P_3 \therefore \frac{1}{4}P_3 = 0 \therefore P_3 = 0 \therefore$

$P_2 = \frac{1}{2}P_2 + \frac{1}{2}P_2 + \frac{1}{4}(0) = P_2 = P_2$,

$P_1 + P_2 = 1 \therefore P_1 = 1 - P_2 \therefore 1 - P_2 = \frac{1}{2}P_2 \therefore 1 = \frac{3}{2}P_2 \therefore \frac{2}{3} = P_2 \therefore$

$\frac{1}{2} \times \frac{2}{3} = P_1 = \frac{1}{3} \therefore \tilde{P} = (\frac{1}{3}, \frac{2}{3}, 0)$

~~M2~~ (1bi) $M_x(t) = E(e^{tx}) = \int_{-\infty}^{\infty} e^{tx} f_x(x)\,dx =$

$$\int_{-\infty}^{\infty} e^{tx}\frac{\lambda}{2}e^{-\lambda|x-\mu|}\,dx = \frac{\lambda}{2}\int_{-\infty}^{\infty} e^{tx}e^{-\lambda|x-\mu|}\,dx =$$

$$\frac{\lambda}{2}\left[\int_{-\infty}^{\mu} e^{tx}e^{-\lambda|x-\mu|}\,dx + \int_{\mu}^{\infty} e^{tx}e^{-\lambda|x-\mu|}\,dx\right] =$$

$$\frac{\lambda}{2}\left[+\int_{-\infty}^{\mu} e^{tx}e^{-(\lambda)(-(x-\mu))}\,dx + \int_{\mu}^{\infty} e^{tx}e^{-\lambda(x-\mu)}\,dx\right] =$$

$$\frac{\lambda}{2}\left[\int_{-\infty}^{\mu} e^{tx}e^{\lambda x-\lambda\mu}\,dx + \int_{\mu}^{\infty} e^{tx}e^{-\lambda x+\lambda\mu}\,dx\right] =$$

$$\frac{\lambda}{2}\left[\int_{-\infty}^{\mu} e^{(t+\lambda)x-\lambda\mu}\,dx + \int_{\mu}^{\infty} e^{(t-\lambda)x+\lambda\mu}\,dx\right] =$$

$$\frac{\lambda}{2}\left[\frac{1}{t+\lambda}e^{(t+\lambda)x-\lambda\mu}\right]_{-\infty}^{\mu} + \frac{\lambda}{2}\left[\frac{1}{t-\lambda}e^{(t-\lambda)x+\lambda\mu}\right]_{\mu}^{\infty} = \quad \text{for } |t|<\lambda$$

$$\frac{\lambda}{2}\left[\frac{1}{t+\lambda}e^{(t+\lambda)\mu-\lambda\mu} - 0\right] + \frac{\lambda}{2}\left[-\frac{1}{t-\lambda}e^{(t-\lambda)\mu+\lambda\mu} + 0\right] =$$

$$\frac{\lambda}{2}\left[\frac{1}{t+\lambda}e^{-\lambda\mu}e^{t\mu}e^{\lambda\mu} + -\frac{1}{t-\lambda}e^{\lambda\mu}e^{-\lambda\mu}e^{t\mu}\right] =$$

$$\frac{\lambda}{2}\left[\frac{1}{t+\lambda}e^{t\mu} - \frac{1}{t-\lambda}e^{t\mu}\right] = \left[\frac{1}{t+\lambda} - \frac{1}{t-\lambda}\right]\frac{\lambda}{2}e^{t\mu} = \frac{t-\lambda-(t+\lambda)}{(t+\lambda)(t-\lambda)}\frac{\lambda}{2}e^{t\mu} =$$

$$\frac{-2\lambda}{t^2-\lambda^2}\frac{\lambda}{2}e^{t\mu} = \frac{-\lambda^2}{t^2-\lambda^2}e^{\mu t} = \frac{\lambda^2}{\lambda^2-t^2}e^{\mu t} = \frac{1}{1-\left(\frac{t}{\lambda}\right)^2}e^{\mu t}$$

(1bii) $\therefore E(X) = M_x(t)\big|_{\theta=0} = \frac{1}{1-\left(\frac{0}{\lambda}\right)^2}e^{\mu(0)} = \frac{1}{1-0^2}e^0 = \frac{1}{1}e^0 = 1(1) = 1$

$E(X^2) = M_x'(t)\big|_{\theta=0} \therefore M_x'(t) = \mu\frac{1}{1-\left(\frac{t}{\lambda}\right)^2}e^{\mu t} + \frac{-\lambda^2(-2t)}{(\lambda^2-t^2)^2}e^{\mu t} \therefore$

$E(X^2) = \mu\frac{1}{1-\left(\frac{0}{\lambda}\right)^2}e^0 + \frac{-\lambda^2(0)}{(\lambda^2)^2}e^0 = \mu\frac{1}{1}1 + 0 = \mu \therefore$

$var(X) = E(X^2) - (E(X))^2 = \mu - 1^2 = \mu - 1$

(1bii retry) $E(X) = M_x'(0) = M_x'(t)\big|_{t=0} = \mu M_x(t) + \frac{-\lambda(-2t)}{(\lambda^2-t^2)^2}e^{\mu t}\Big|_{t=0}$

$= \mu M_x(t) + \frac{2t}{\lambda(\lambda^2-t^2)}\frac{\lambda^2}{(\lambda^2-t^2)}e^{\mu t}\Big|_{t=0} = \mu M_x(t) + \frac{2t}{\lambda^2(\lambda^2-t^2)}M_x(t)\Big|_{t=0} =$

$\cancel{\mu\frac{\lambda^2}{\lambda^2-0^2}e^{\mu(0)} + \frac{2(0)}{\lambda^2(\lambda^2-0^2)}}\left(\mu + \frac{2t}{\lambda^2(\lambda^2-t^2)}\right)M_x(t)\Big|_{t=0} = \left(\left(\mu + \frac{2(0)}{\lambda^2(\lambda^2-0^2)}\right)\frac{\lambda^2}{\lambda^2-0^2}e^{\mu(0)}\right) = \mu \therefore$

\PP2021/ $\therefore M_x''(t) = \left(\mu + \frac{2t}{\lambda^2(\lambda^2-t^2)}\right) M_x'(t) + \left(\frac{\lambda^2(\lambda^2-t^2)2 - 2t\lambda^2(-2t)}{(\lambda^2(\lambda^2-t^2))^2}\right) M_x(t)$

$\therefore E(X^2) = M_x''(0) = \left(\mu + \frac{2(0)}{\lambda^2(\lambda^2-0^2)}\right) M_x'(0) + \left(\frac{\lambda^2(\lambda^2-0^2)2 - 2(0)\lambda^2(-2(0))}{(\lambda^2(\lambda^2-0^2))^2}\right) M_x(0)$

$= (\mu+0)\mu + \frac{2\lambda^2(\lambda^2) - 0}{(\lambda^2(\lambda^2))^2}(1) = \mu^2 + \frac{2\lambda^4}{\lambda^8} = \mu^2 + \frac{2}{\lambda^4} \therefore$

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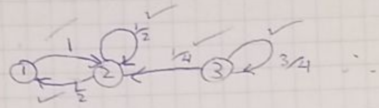
\1ci/  $\therefore$

① and ② is a irroducable subchain

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$P(X_1=2, X_2=2, X_3=1 \mid X_0=1) = 1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} = P(1 \to 2 \to 2 \to 1 \mid 1)$

B: $P(X_3=2 \mid X_0=1) =$

$P(X_1=2, X_2=1, X_3=2 \mid X_0=1) + P(X_1=2, X_2=2, X_3=2 \mid X_0=1) =$

$P(1 \to 2 \to 1 \to 2 \mid 1) + P(1 \to 2 \to 2 \to 2 \mid 1) =$

$1 \times \frac{1}{2} \times 1 + 1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

C: $P(X_3=3 \mid X_0=1) = P(1 \to 2 \to 2 \to 3 \mid 1) =$

$1 \times \frac{1}{2} \times 0 = 0 \because$ 1 and 2 is a subchain

D: $P(X_4=2 \mid X_0=2) =$

$P(2 \to 1 \to 2 \to 1 \to 2 \mid 2) + P(2 \to 1 \to 2 \to 2 \to 2 \mid 2) + P(2 \to 2 \to 2 \to 2 \to 2 \mid 2)$

$= \frac{1}{2} \times 1 \times \frac{1}{2} \times 1 + \frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{7}{16} = 0.4375$ (35.5) X

\1ciii/ steady state $\tilde{P} = (P_1, P_2, P_3) = (P_1 \quad P_2 \quad P_3)$

$\tilde{P} = \tilde{P}T = (P_1 \quad P_2 \quad P_3) = (P_1 \quad P_2 \quad P_3)\begin{pmatrix} 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{pmatrix} =$

$(\frac{1}{2}P_2 \quad P_1 + \frac{1}{2}P_2 + \frac{1}{4}P_3 \quad \frac{3}{4}P_3) \therefore$

$P_1 = \frac{1}{2}P_2$, $P_2 = P_1 + \frac{1}{2}P_2 + \frac{1}{4}P_3$, $P_3 = \frac{3}{4}P_3 \therefore \frac{1}{4}P_3 = 0 \therefore P_3 = 0 \therefore$

$P_2 = \frac{1}{2}P_2 + \frac{1}{2}P_2 + \frac{1}{4}(0) = P_2 = P_2$,

$P_1 + P_2 = 1 \therefore P_1 = 1 - P_2 \therefore 1 - P_2 = \frac{1}{2}P_2 \therefore 1 = \frac{3}{2}P_2 \therefore \frac{2}{3} = P_2 \therefore$

$\frac{1}{2} \times \frac{2}{3} = P_1 = \frac{1}{3} \therefore \tilde{P} = (\frac{1}{3}, \frac{2}{3}, 0)$

(1b i) $M_x(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx =$

$\int_{-\infty}^{\infty} e^{tx}\frac{\lambda}{2}e^{-\lambda|x-\mu|}\,dx = \frac{\lambda}{2}\int_{-\infty}^{\infty} e^{tx}e^{-\lambda|x-\mu|}\,dx =$

$\frac{\lambda}{2}\left[\int_{-\infty}^{\mu} e^{tx}e^{-\lambda|x-\mu|}\,dx + \int_{\mu}^{\infty} e^{tx}e^{-\lambda|x-\mu|}\,dx\right]=$

$\frac{\lambda}{2}\left[\int_{-\infty}^{\mu} e^{tx}e^{-\lambda(-(x-\mu))}\,dx + \int_{\mu}^{\infty} e^{tx}e^{-\lambda(x-\mu)}\,dx\right]=$

$\frac{\lambda}{2}\left[\int_{-\infty}^{\mu} e^{tx}e^{\lambda x-\lambda\mu}\,dx + \int_{\mu}^{\infty} e^{tx}e^{-\lambda x+\lambda\mu}\,dx\right]=$

$\frac{\lambda}{2}\left[\int_{-\infty}^{\mu} e^{(t+\lambda)x-\lambda\mu}\,dx + \int_{\mu}^{\infty} e^{(t-\lambda)x+\lambda\mu}\,dx\right]=$

$\frac{\lambda}{2}\left[\frac{1}{t+\lambda}e^{(t+\lambda)x-\lambda\mu}\right]_{-\infty}^{\mu} + \frac{\lambda}{2}\left[\frac{1}{t-\lambda}e^{(t-\lambda)x+\lambda\mu}\right]_{\mu}^{\infty} =$ for $|t|<\lambda$

$\frac{\lambda}{2}\left[\frac{1}{t+\lambda}e^{(t+\lambda)\mu-\lambda\mu} - 0\right] + \frac{\lambda}{2}\left[\frac{1}{t-\lambda}e^{(t-\lambda)\mu+\lambda\mu} + 0\right]=$

$\frac{\lambda}{2}\left[\frac{1}{t+\lambda}e^{-\lambda\mu}e^{t\mu}e^{\lambda\mu} + -\frac{1}{t-\lambda}e^{\lambda\mu}e^{-\lambda\mu}e^{t\mu}\right]=$

$\frac{\lambda}{2}\left[\frac{1}{t+\lambda}e^{t\mu} - \frac{1}{t-\lambda}e^{t\mu}\right] = \left[\frac{1}{t+\lambda}-\frac{1}{t-\lambda}\right]\frac{\lambda}{2}e^{t\mu} = \frac{t-\lambda-(t+\lambda)}{(t+\lambda)(t-\lambda)}\frac{\lambda}{2}e^{t\mu}=$

$\frac{-2\lambda}{t^2-\lambda^2}\frac{\lambda}{2}e^{t\mu} = \frac{-\lambda^2}{t^2-\lambda^2}e^{\mu t} = \frac{\lambda^2}{\lambda^2-t^2}e^{\mu t} = \frac{1}{1-(\frac{t}{\lambda})^2}e^{\mu t}$

(1b ii) $\therefore E(X) = M_x(t)\big|_{t=0} = \frac{1}{1-(\frac{0}{\lambda})^2}e^{\mu(0)} = \frac{1}{1-0^2}e^0 = \frac{1}{1}e^0 = 1(1) = 1$

$E(X^2) = M_x'(t)\big|_{t=0} \therefore M_x'(t) = \mu\frac{1}{1-(\frac{t}{\lambda})^2}e^{\mu t} + \frac{-\lambda^2(-2t)}{(\lambda^2-t^2)^2}e^{\mu t} \therefore$

$E(X^2) = \mu\frac{1}{1-(\frac{0}{\lambda})^2}e^0 + \frac{-\lambda^2(0)}{(\lambda^2)^2}e^0 = \mu\frac{1}{1}1+0 = \mu \therefore$

$var(X) = E(X^2)-(E(X))^2 = \mu - 1^2 = \mu - 1$

(1b ii retry) $E(X) = M_x'(0) = M_x'(t)\big|_{t=0} = \mu M_x(t) + \frac{-\lambda(-2t)}{(\lambda^2-t^2)^2}e^{\mu t}\Big|_{t=0}$

$= \mu M_x(t) + \frac{2t}{\lambda(\lambda^2-t^2)}\frac{\lambda^2}{(\lambda^2-t^2)}e^{\mu t}\Big|_{t=0} = \mu M_x(t) + \frac{2t}{\lambda^2(\lambda^2-t^2)}M_x(t)\Big|_{t=0} =$

$\cancel{\mu\frac{\lambda^2}{\lambda^2-0^2}e^{\mu(0)} + \frac{2(0)}{\lambda^2(\lambda^2-0^2)}}$ $\left(\mu+\frac{2t}{\lambda^2(\lambda^2-t^2)}\right)M_x(t)\Big|_{t=0} = \left(\left(\mu+\frac{2(0)}{\lambda^2(\lambda^2-0^2)}\right)\frac{\lambda^2}{\lambda^2-0^2}e^{\mu(0)}\right) = \mu \therefore$

\PP2021/ $\theta_i = A\lambda_+^i + B\lambda_-^i = A(1)^i + B\left(\frac{p}{q}\right)^i = A + B\left(\frac{p}{q}\right)^i = A + B\rho^i$, $\rho = \frac{p}{q}$

$\therefore \theta_0 = 0 = A + B\rho^0 = A + B = 0 \therefore A = -B \therefore$

$\theta_i = -B + B\rho^i = B(-1+\rho^i) \therefore$

$\theta_N = 1 = B(-1+\rho^N) \therefore \quad B = \frac{1}{-1+\rho^N} = \frac{1}{\rho^N - 1} \therefore$

~~A = 2B~~ $\theta_i = \frac{1}{\rho^N - 1}(-1+\rho^i) = \frac{1}{\rho^N-1}(\rho^i - 1) = \frac{\rho^i - 1}{\rho^N - 1} = \frac{1-\rho^i}{1-\rho^N}$

$i = 1, \dots, N-1 \therefore$

let E_n denote the number of England wins, A_n denote the number of Australia wins, $Z_n = E_n - A_n$,

Z_n has values in $\{-5, -4, \dots, +4, 5\} \therefore$

let $U_n = Z_n + 5 \therefore U_n \in \{0, 1, \dots, 10\} \therefore$

England wins is $U_n = 10 \therefore$

P_E for start at $U_n = 5 \therefore P_E = \frac{1-\rho^5}{1-\rho^{10}} = \frac{1-\rho^5}{(1+\rho^5)(1-\rho^5)} = \frac{1}{1+\rho^5} =$

$\frac{1}{1+\frac{p^5}{q^5}} = \frac{q^5}{q^5+p^5} = \frac{q^5}{p^5+q^5}$

\4aii/ $D_i = 1 + qD_{i+1} + 3qD_i + pD_{i-1}$, $D_N = 0$

\4aiii/ stopping time given by $D_i = 1 + \sum_j T_{ij} D_j \therefore$

$D_i = 1 + qD_{i+1} + 3qD_i + pD_{i-1} \therefore$

$0 = 1 + qD_{i+1} + (3q-1)D_i + pD_{i-1} \therefore$

let $D_i = k\lambda^i \therefore$ let for homogeneous part: D_i^h solve it $\therefore$

for: $0 = qD_{i+1} + (3q-1)D_i + pD_{i-1} \therefore$

$0 = qD_{i+1}^h + (3q-1)D_i^h + pD_{i-1}^h$

let $D_i^h = k\lambda^i \therefore$

$0 = qk\lambda^{i+1} + (3q-1)k\lambda^i + pk\lambda^{i-1} = k\lambda^{i-1}(q\lambda^2 + (3q-1)\lambda + p) = 0 \therefore$

$\lambda_\pm = \left[-(3q-1) \pm ((3q-1)^2 - 4qp)^{1/2}\right]/(2q) \therefore q + 3q + p = 1 \therefore p = 1-4q \therefore$

$\lambda_+ = 1, \quad \lambda = \frac{p}{q} = \rho \therefore \quad D_i^h = A(1)^i + B\rho^i = A + B\rho^i \therefore$

inhomogeneous part: $-1 = qD_{i+1} + (3q-1)D_i + pD_{i-1} \therefore$

let $D_i^p = a + bi + ci^2 \therefore$

$-1 = aq + bq(i+1) + cq(i+1)^2 + 3aq + 3q + 3bqi + 3cqi^2 - a - bi - ci^2 + ap + bp(i-1) + cp(i-1)^2 =$

iterate to get $E(T_r)=\frac{n}{(n-r)\lambda}$ ∴

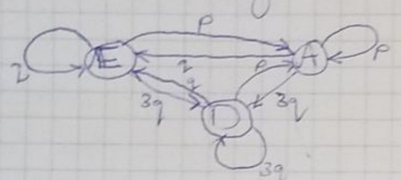
expected time to have all flavours ordered is

$E(T_1)+E(T_2)+\dots+E(T_n)=$

$\frac{n}{\lambda}\left(\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\dots 1\right)=\frac{n}{\lambda}\sum_{k=1}^{n}\frac{1}{k}\sim\frac{n\log n}{k}$ ∴

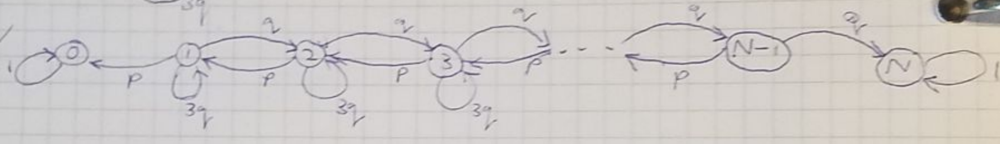
this diverges as $n\to\infty$ ∴ no finite window is sufficiently large to expect to see all flavour orders

\4ai/ E is England win D is draw, A is Australia win ∴



X its a random walk to stop till win

\4ai/



\4aii/ have $\theta_0=0$, $\theta_N=1$, $\theta_i=q\theta_{i+1}+p\theta_{i-1}+3q\theta_i$ ∴

$q+p+3q=1=4q+p$ ∴ $p=1-4q$ ∴

$\theta_i=q\theta_{i+1}+(1-4q)\theta_{i-1}+3q\theta_i$ ∴

$0=q\theta_{i+1}+(1-4q)\theta_{i-1}+(3q-1)\theta_i$ ∴

Ansatz ∴ let $\theta_i=k\lambda^i$ ∴ $\theta_{i+1}=k\lambda^{i+1}$, $\theta_{i-1}=k\lambda^{i-1}$ ∴

$\theta_i=k\lambda\lambda^{i-1}$, $\theta_{i+1}=k\lambda^2\lambda^{i-1}$ ∴

$0=qk\lambda^{i+1}+(3q-1)k\lambda^i+(1-4q)k\lambda^{i-1}=$

~~$k\lambda^{i-1}(q\lambda^2+(3q-1)$~~ $0=k\lambda^{i-1}(q\lambda^2+(3q-1)\lambda+(1-4q))=0$ ∴

solve for λ: ∴ $\lambda_\pm=\frac{1-3q\pm((3q-1)^2-4q(1-4q))^{1/2}}{2q}=$

$[1-3q\pm(9q^2-6q+1-4q+16q^2)^{1/2}]/(2q)=$

$[1-3q\pm\sqrt{25q^2-10q+1}]/(2q)=[1-3q\pm\sqrt{(5q-1)^2}]/(2q)=$

$[1-3q\pm(5q-1)]/(2q)$ ∴

$\lambda_+=\frac{1-3q+5q-1}{2q}=\frac{2q}{2q}=1$, $\lambda_-=\frac{1-3q-(5q-1)}{2q}=\frac{1-3q-5q+1}{2q}=$

$\frac{2-8q}{2q}=\frac{1-4q}{q}=\frac{p}{q}$ ∴

\PP2021/ $\frac{2^2 e^{-2}}{2!} \cdot \frac{4^3 e^{-4}}{3!} e^{-2} + \frac{2e^{-2} \cdot 4^2 e^{-4}}{2!} \cdot 2e^{-2} + e^{-2} \cdot 4e^{-4} \cdot \frac{2^2 e^{-2}}{2!}$

$= 0.0206$ (3 s.f.)

\3ci/ 20 mins $= \frac{1}{3}$ hour $= t$ $\quad e^{-\lambda t}(\lambda t)^x \frac{1}{x!}$

orders for beer occur at $\lambda_{bear} = 15$ p/h, $\lambda_{cola} = 5$ p/h, $\lambda_{water} = 2$ p/h

probability over 20 mins of recieving at least 5 bear, 2 cola + 1 water orders $\therefore$ use independence let Beer, cola, water orders $\equiv B, C, W \therefore$

$P(B \geq 5, C \geq 2, W \geq 1) = P(B \geq 5) P(C \geq 2) P(W \geq 1)$ by independence

$= (1 - P(B \leq 4))(1 - P(C \leq 1))(1 - P(W = 0)) =$

$\left(1 - \left[e^{-15 \times \frac{1}{3}} (15 \times \tfrac{1}{3})^0 \frac{1}{0!} + e^{-15 \times \frac{1}{3}} (15 \times \tfrac{1}{3})^1 \frac{1}{1!} + e^{-15 \times \frac{1}{3}} (15 \times \tfrac{1}{3})^2 \frac{1}{2!} + e^{-15 \times \frac{1}{3}} (15 \times \tfrac{1}{3})^3 \frac{1}{3!} + e^{-15 \times \frac{1}{3}} (\tfrac{15}{4})^4 \frac{1}{4!}\right]\right) \times$

$\left(1 - \left[e^{-5 \times \frac{1}{3}} (5 \times \tfrac{1}{3})^0 \frac{1}{0!} + e^{-5 \times \frac{1}{3}} (5 \times \tfrac{1}{3})^1 \frac{1}{1!}\right]\right) \times \left(1 - \left[e^{-2 \times \frac{1}{3}} (2 \times \tfrac{1}{3})^0 \frac{1}{0!}\right]\right) =$

$\left(1 - e^{-5}\left[5^0 \frac{1}{0!} + 5^1 \frac{1}{1!} + 5^2 \frac{1}{2!} + 5^3 \frac{1}{3!} + 5^4 \frac{1}{4!}\right]\right) \times \left(1 - e^{-\frac{5}{3}}\left((\tfrac{5}{3})^0 \frac{1}{0!} + (\tfrac{5}{3})^1 \frac{1}{1!}\right)\right)\left(1 - e^{-\frac{2}{3}}\right) =$

$0.5595 \times 0.4963 \times 0.4866 = 0.135$

\3cii/ orders are independent $\therefore \frac{\lambda_{cola}}{\lambda} = \frac{\lambda_{cola}}{\lambda_B + \lambda_{cola} + \lambda_W} =$

$\frac{5}{15+5+2} = P(\text{next order is cola}) = \frac{5}{22} = 0.227$ (3 s.f.)

\3ciiSol/ Expected arrival time of poisson process is given by exponential distribution $\therefore$ ~~Joint density: f_C~~

merge beer + water processes: $Y = W + B \sim Poi(17) \therefore$

Joint density: $f_{C,Y}(x,y) = (5e^{-5x})(17e^{-17y}) \therefore$

$P(T_C < T_Y) = \int_0^\infty \int_0^y f_{C,Y}(x,y)\,dx\,dy = \int_0^\infty \int_0^y (5e^{-5x})(17e^{-17y})\,dx\,dy =$

$\int_0^\infty 17e^{-17y} \int_0^y (5e^{-5x})\,dx\,dy = \int_0^\infty 17e^{-17y}\left[-e^{-5x}\right]_0^y dy = 17\int_0^\infty e^{-17y}(1 - e^{-5y})\,dy =$

$17\int_0^\infty e^{-17y} - e^{-22y}\,dy = 17\left[-\frac{1}{17}e^{-17y} + \frac{1}{22}e^{-22y}\right]_0^\infty = \left[\frac{17}{22}e^{-22y} - e^{-17y}\right]_0^\infty =$

$\frac{17}{22}\left[0 - e^0\right] - \left[0 - e^0\right] = 1 - \frac{17}{22} = \frac{5}{22} = 0.227$

\3d/ Ice cream orders at λ per hour. Label order of flavours by indexing set $i = 1, \ldots, n$ $\therefore$ Expected arrival time of first order λ^{-1} ie $E(T_1) = \frac{1}{\lambda}$ $\therefore$

start by assuming n finite $\therefore$

\3a Sol/ $P(N(t)=n \mid N(T)=k) = \dfrac{P(N(t)=r \cap N(T)=n)}{P(N(T)=n)} =$

$P(N(t)=r)\,P(N(T-r)=n-r) / P(N(t)=n) =$

$\left((\lambda t)^r e^{-\lambda t}/r!\right)\left(\dfrac{\lambda(T-t)^{n-r}}{(n-r)!} e^{-\lambda(T-t)}\right) \Big/ \left[(\lambda T)^n e^{-\lambda T}/n!\right] =$

$\dfrac{n!}{r!(n-r)!}\left(\dfrac{t}{T}\right)^r\left(1-\dfrac{t}{T}\right)^{n-r} \sim \text{Binomial}\left(\dfrac{t}{T}, n\right)$

\3bi/ P(hot dog is next order) $= 0.1$

\3bii/ $\lambda = 20 \therefore \frac{1}{20} = 0.05$ hours $= 3$ minutes till expect next order $= E(\text{arrival})$

\3biii/ ~~$=P(\ldots) = 1-P(\ldots)$~~ prob 5 pizza orders in 30 mins

rate of pizza orders : $\lambda_{pizza} = 0.4 \times 20 = 8$ per hour $\therefore$

$P(N_{pizza}(\frac{1}{2})=5) = e^{-\lambda_{pizza} t}\frac{1}{x!}(\lambda_{pizza} t)^x = e^{-8(\frac{1}{2})}\frac{1}{5!}(8 \times 0.5)^5 =$

$= e^{-4}\frac{1}{5!}(4)^5 = 0.156$ (3 s.f.)

\3biv/ observations are independent $\therefore$

P(2 hotdog orders in 2 hours) $= P(N_{dog}(2)=2) = \dfrac{e^{-0.1\times20\times2}(0.1\times20\times2)^2}{2!}$

$= 0.1465 \therefore \therefore$

let $N_{(a,b]}$ be the number of hdog orders over the interval $(a,b]$ $\therefore N_{(0,1]} \sim Poi(2) \therefore N_{(1,2]} \sim Poiss(2) \therefore N_{(2,4]} \sim Poiss(4) \therefore$

LoTP:

$P(N_{(0,2]}=2 \text{ and } N_{(1,4]}=3) = P(N_{(0,1]} + N_{(1,2]} = 2 \text{ and } N_{(1,2]} + N_{(2,4]} = 3) =$

$\sum_{k=0}^{\infty} P(N_{(0,1]} + N_{(1,2]} = 2 \text{ and } N_{(1,2]} + N_{(2,4]} = 3 \mid N_{(1,2]} = k)\,P(N_{(1,2]} = k) =$

$P(N_{(0,1]}=2, N_{(2,4]}=3 \mid N_{(1,2]}=0)\,P(N_{(1,2]}=0) +$

$P(N_{(0,1]}=1, N_{(2,4]}=2 \mid N_{(1,2]}=1)\,P(N_{(1,2]}=1) +$

$P(N_{(0,1]}=0, N_{(2,4]}=1 \mid N_{(1,2]}=2)\,P(N_{(1,2]}=2) =$

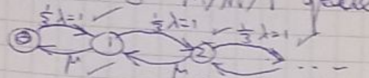
$P(N_{(0,1]}=2, N_{(2,4)}=3)\,P(N_{(1,2]}=0) + P(N_{(0,1]}=1, N_{(2,4]}=2)\,P(N_{(1,2]}=1) +$

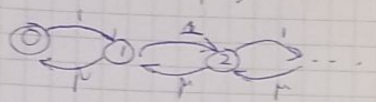
$P(N_{(0,1]}=0, N_{(2,4]}=1)\,P(N_{(1,2]}=2) =$ by independence

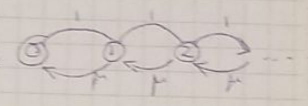
$P(N_{(0,1]}=2)\,P(N_{(2,4]}=3)\,P(N_{(1,2]}=0) + P(N_{(0,1]}=1)\,P(N_{(2,4]}=2)\,P(N_{(1,2]}=1) +$

$P(N_{(0,1]}=0)\,P(N_{(2,4]}=1)\,P(N_{(1,2]}=2) =$

\PP2020 1/ ∴ 3 M/M/1 queues with infinite capacity, $\frac{1}{3}\lambda = 1$, $\mu > 0$

Server1:  for all 3 queues

Server2: 

Server3: 

∴ $\rho = \frac{1}{\mu}$ ∴ for a standard M/M/1 queue, expected system size of each queue is $L_s = \frac{\rho}{1-\rho}$, $\rho = \frac{1}{\mu}$ ∴

$$L_s^i = \frac{\frac{1}{\mu}}{1 - \frac{1}{\mu}} = \frac{1}{\mu - 1}$$ ∴

overall system size is $L_s = \sum_{i=1}^{3} L_s^i = \frac{3}{\mu - 1}$ ∴

By Littles law, expected waiting time per queue is

$W_s^i = L_s^i = \frac{1}{\mu - 1}$ ∴

For the M/M/3 queue get: $W_s = \frac{L_s}{3} = \frac{2\mu^2 + 2\mu - 1}{(\mu-1)(2\mu^2 + 4\mu + 3)}$ ∴

$W_s^i - W_s = \frac{1}{\mu - 1}\left(1 - \frac{2\mu^2 + 2\mu - 1}{2\mu^2 + 4\mu + 3}\right)$ & ∴ $1 - \frac{2\mu^2 + 2\mu - 1}{2\mu^2 + 4\mu + 3} > 0$ ∴

M/M/3 queue leads to shorter mean waiting times is term in bracket is positive ∴ ∵ $1 - \frac{2\mu^2 + 2\mu - 1}{2\mu^2 + 4\mu + 3} = \frac{2\mu^2 + 4\mu + 3 - 2\mu^2 - 2\mu + 1}{2\mu^2 + 4\mu + 3}$ ∴

∵ $\mu > 0$, denominator is positive ∴ $2\mu + 4 = 2(\mu + 2) > 0$ ∴

M/M/3 queue more efficient than 3 × M/M/1 queues

or $\frac{W_s^i}{W_s} = \frac{2\mu^2 + 4\mu + 3}{2\mu^2 + 2\mu - 1} = 1 + \frac{2\mu + 4}{2\mu^2 + 2\mu - 1} > 1$ ∴ $W_s^i > W_s$

\3 a/ P(k orders in time t | n orders in time T) =

P((k orders in time t) ∩ (n orders in time T)) / P(n orders in time T) =

P(k orders in time t) P((n−k) orders in time (T−t)) / P(n orders in time T)

∵ $\lambda t \sim Poi$ ∴ $e^{-\lambda t}\frac{(\lambda t)^x}{x!}$ ∴

$$= e^{-\lambda t}(\lambda t)^k \frac{1}{k!} e^{-\lambda(T-t)} \frac{1}{(n-k)!}(\lambda(T-t))^{n-k} \frac{n!}{1} \frac{1}{e^{-\lambda T}} \frac{1}{(\lambda T)^n} =$$

$$\binom{n}{k} e^{-\lambda t}(\lambda t)^k e^{-\lambda T + \lambda t}(\lambda(T-t))^n (\lambda(T-t))^{-k} e^{\lambda T}(\lambda T)^{-n}$$

$\binom{n}{k}\left(\frac{t}{T}\right)^k\left(1 - \frac{t}{T}\right)^{n-k}$ ∴ follows a binomial $B(n, p)$, $p = \frac{t}{T}$

$$\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\left(\frac{3}{\mu}+\frac{9}{2\mu}\left(\frac{2\mu-1}{(\mu-1)^2}\right)\right)=$$

$$\frac{\cancel{2\mu(\mu-1)}}{2\mu^2+4\mu+3}\left(\frac{6(\mu-1)^2+18\mu-9}{\cancel{2\mu(\mu-1)^2}}\right)=\frac{1}{2\mu^2+4\mu+3}\left(\frac{6\mu^2+6-12\mu+18\mu-9}{\cancel{2\mu^2+4\mu}\ (\mu-1)}\right)=$$

$$\frac{3(2\mu^2+2\mu-1)}{(\mu-1)(2\mu^2+4\mu+3)}\equiv L_s\equiv \text{expected system size} \therefore$$

for $\mu=1.2$: $L_s=6.01$ (3 s.f.)

\2aiii/ $G_X(\theta)=P(X=0)+P(X=1)\theta+P(X=2)\theta^2+\ldots=\sum_{n=0}^{\infty}P_n\theta^n=$

$$P_0+P_1\theta+\sum_{n=2}^{\infty}P_n\theta^n=\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}+\frac{3}{\mu}\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\theta+\sum_{n=2}^{\infty}\frac{9}{2}\frac{1}{\mu^n}\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\theta^n=$$

$$\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\left(1+\frac{3\theta}{\mu}+\frac{9}{2}\sum_{n=2}^{\infty}\frac{\theta^n}{\mu^n}\right) \therefore$$

$$E(X)=G_X'(1)=G_X'(\theta)\Big|_{\theta=1}=\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\left(\frac{3}{\mu}+\frac{9}{2}\sum_{n=2}^{\infty}n\frac{\theta^{n-1}}{\mu^n}\right)\Big|_{\theta=1}=$$

$$\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\left(\frac{3}{\mu}+\frac{9}{2}\sum_{n=2}^{\infty}n\frac{1}{\mu^n}\right)$$

$$\therefore \sum_{n=2}^{\infty}n\frac{1}{\mu^n}=\sum_{n=2}^{\infty}n\left(\frac{1}{\mu}\right)^n=\frac{\left(\frac{1}{\mu}\right)}{\left(1-\left(\frac{1}{\mu}\right)\right)^2}-\frac{1}{\mu}=\frac{\mu}{(\mu-1)^2}-\frac{(\mu-1)^2}{\mu(\mu-1)^2}=\frac{\mu^2-\mu^2-1+2\mu}{\mu(\mu-1)^2}=$$

$$\frac{2\mu-1}{\mu(\mu-1)^2} \therefore$$

$$E(X)=\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\left(\frac{3}{\mu}+\frac{9}{2}\frac{2\mu-1}{\mu(\mu-1)^2}\right)=$$

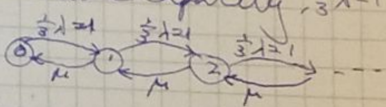
$$\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\left(\frac{3\times2(\mu-1)^2}{2\mu(\mu-1)^2}+\frac{18\mu-9}{2\mu(\mu-1)^2}\right)=\frac{\cancel{2\mu(\mu-1)}}{2\mu^2+4\mu+3}\left(\frac{6(\mu^2+1-2\mu)+18\mu-9}{\cancel{2\mu(\mu-1)^2}}\right)=$$

$$\frac{6\mu^2+6-12\mu+18\mu-9}{(\mu-1)(2\mu^2+4\mu+3)}=\cancel{3(2\mu^2+2\mu-1)}\ \frac{6\mu^2+6\mu-3}{(\mu-1)(2\mu^2+4\mu+3)}=$$

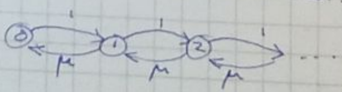
$$\frac{3(2\mu^2+2\mu-1)}{(\mu-1)(2\mu^2+4\mu+3)}=L_s=\text{expected system size}$$

$\therefore$ for $\mu=1.2$: $L_s=6.01$ (3 s.f.)

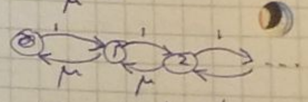
\2bi/ $\therefore$ 3 M/M/1 queues with infinite capacity, $\frac{1}{3}\lambda=1$, $\mu>0$ for all 3 queues $\therefore$ Server 1:



Server 2:



Server 3:



$\therefore \rho=\frac{1}{\mu}$ $\therefore$ for a standard M/M/1 queues, expected system size ~~is~~ of each queue is $L_s=\frac{\rho}{1-\rho}$, $\rho=\frac{1}{\mu}$

PP2021 $3P_3=3\mu P_4$, $3P_4=3\mu P_5$ $\therefore$

$$\frac{9}{2}\frac{1}{\mu^n}P_0=P_n \quad \text{for } n\geq 2 \quad \therefore \sum_{n=0}^{\infty}P_n=1 \quad \therefore$$

$$P_0+P_1+\sum_{n=2}^{\infty}P_2=1=P_0+\frac{3}{\mu}P_0+\sum_{n=2}^{\infty}\frac{9}{2}\frac{1}{\mu^n}P_0=P_0\left(1+\frac{3}{\mu}+\frac{9}{2}\sum_{n=2}^{\infty}\left(\frac{1}{\mu}\right)^n\right) \therefore$$

$$\sum_{n=0}^{\infty}\rho^n=\frac{1}{1-\rho} \quad \therefore \quad \sum_{n=1}^{\infty}\rho^n=\frac{1}{1-\rho}-1=\frac{1}{1-\rho}-\frac{1-\rho}{1-\rho}=\frac{\rho}{1-\rho} \quad \therefore$$

$$\sum_{n=2}^{\infty}\rho^n=\frac{\rho}{1-\rho}-\rho \quad \therefore$$

$$\sum_{n=2}^{\infty}\left(\frac{1}{\mu}\right)^n=\frac{\left(\frac{1}{\mu}\right)}{1-\left(\frac{1}{\mu}\right)}-\frac{1}{\mu}=\frac{1}{\mu-1}-\frac{1}{\mu}=\frac{\mu}{\mu(\mu-1)}-\frac{\mu-1}{\mu(\mu-1)}=\frac{1}{\mu(\mu-1)} \quad \therefore$$

$$1=P_0\left(1+\frac{3}{\mu}+\frac{9}{2}\sum_{n=2}^{\infty}\left(\frac{1}{\mu}\right)^n\right)=P_0\left(1+\frac{3}{\mu}+\frac{9}{2}\frac{1}{\mu(\mu-1)}\right)=$$

$$P_0\left(\frac{\mu(\mu-1)}{\mu(\mu-1)}+\frac{3(\mu-1)}{\mu(\mu-1)}+\frac{\left(\frac{9}{2}\right)}{\mu(\mu-1)}\right)=P_0\left(\frac{\mu^2-\mu+3\mu-3+\frac{9}{2}}{\mu(\mu-1)}\right)=P_0\left(\frac{\mu^2+2\mu+\frac{3}{2}}{\mu(\mu-1)}\right) \therefore$$

$$\frac{\mu(\mu-1)}{\mu^2+2\mu+\frac{3}{2}}=P_0=\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3} \quad \therefore$$

$$P_1=3\frac{1}{\mu}P_0=\frac{3}{\mu}\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}=\frac{6(\mu-1)}{2\mu^2+4\mu+3},$$

$$P_n=\frac{9}{2}\frac{1}{\mu^n}P_0=\frac{9}{2}\frac{1}{\mu^n}\frac{\mu(\mu-1)}{\mu^2+2\mu+\frac{3}{2}}=\frac{9(\mu-1)}{\mu^{n-1}(2\mu^2+4\mu+3)} \quad \text{for } n\geq 2$$

steady state exists for $3\mu>\lambda=3 \therefore \mu>1$

2aiii $G_X(\theta)=P(X=0)+P(X=1)\theta+P(X=2)\theta^2+\ldots=\sum_{n=0}^{\infty}P_n\theta^n=$

$$P_0+P_1\theta+\sum_{n=2}^{\infty}P_n\theta^n=\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}+\frac{3}{\mu}\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\theta+\sum_{n=0}^{\infty}\frac{9}{2}\frac{1}{\mu^n}\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\theta^n=$$

$$\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\left(1+\frac{3\theta}{\mu}+\frac{9}{2}\sum_{n=2}^{\infty}\frac{\theta^n}{\mu^n}\right) \therefore$$

$$G_X'(\theta)=\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\left(\frac{3}{\mu}+\frac{9}{2}\sum_{n=2}^{\infty}n\frac{\theta^{n-1}}{\mu^n}\right)=\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\left(\frac{3}{\mu}+\frac{9}{2\mu}\sum_{n=2}^{\infty}n\left(\frac{\theta}{\mu}\right)^{n-1}\right)=$$

$$\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\left(\frac{3}{\mu}+\frac{9}{2\mu}\sum_{n=1}^{\infty}(n+1)\left(\frac{\theta}{\mu}\right)^n\right) \therefore$$

$$E(X)=G_X'(1)=\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\left(\frac{3}{\mu}+\frac{9}{2\mu}\sum_{n=1}^{\infty}\frac{n+1}{\mu^n}\right)=\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\left(\frac{3}{\mu}+\frac{9\mu}{2\mu}\sum_{n=1}^{\infty}\frac{n+1}{\mu^{n+1}}\right)=$$

$$\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\left(\frac{3}{\mu}+\frac{9}{2}\sum_{n=2}^{\infty}n\left(\frac{1}{\mu}\right)^n\right)=$$

$$\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\left(\frac{3}{\mu}+\frac{9}{2}\left(\sum_{n=1}^{\infty}n\left(\frac{1}{\mu}\right)^n-\frac{1}{\mu}\right)\right)=\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\left(\frac{3}{\mu}+\frac{9}{2}\left(\frac{\left(\frac{1}{\mu}\right)}{\left(1-\left(\frac{1}{\mu}\right)\right)^2}-\frac{1}{\mu}\right)\right)=$$

$$\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\left(\frac{3}{\mu}+\frac{9}{2}\left(\frac{\mu}{(\mu-1)^2}\times\frac{\mu}{\mu}-\frac{(\mu-1)^2}{\mu(\mu-1)^2}\right)\right)=\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3}\left(\frac{3}{\mu}+\frac{9}{2\mu}\left(\frac{\mu^2-\mu^2-1+2\mu}{(\mu-1)^2}\right)\right)=$$

\PP2021/ \1Ci/ A: 1 and 2 is a subchain ∴

$P(X_3=1 \mid X_0=1) = P(X_1=2, X_2=2, X_3=1 \mid X_0=1) = P(1\to2\to2\to1 \mid 1) =$

$1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

B: $P(X_3=2 \mid X_0=1) = P(X_1=2, X_2=1, X_3=2 \mid X_0=1) + P(X_1=2, X_2=2, X_3=2 \mid X_0=1) =$

$P(1\to2\to1\to2 \mid 1) + P(1\to2\to2\to2 \mid 1) =$

$1 \times \frac{1}{2} \times 1 + 1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2} + \frac{1}{4} = \frac{3}{4}$

C: $P(X_3=3 \mid X_0=1) = P(1\to\cancel{1}2\to2\to3 \mid 1) =$

$1 \times \frac{1}{2} \times 0 = 0$ ∵ 1 and 2 is a subchain

D: $P(X_4=2 \mid X_0=2) =$

$P(2\to1\to2\to1\to2 \mid 2) + P(2\to\ \to2\to2\to2 \mid 2) + P(2\to2\to2\to2\to2 \mid 2) =$

$\frac{1}{2} \times 1 \times \frac{1}{2} \times 1 + \frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} = \frac{7}{16}$

~~\1Cii/~~ D: $P(X_4=2 \mid X_0=2) =$

$P(2\to2\to2\to2\to2) + P(2\to1\to2\to1\to2) + 3P(2\to1\to2\to2\to2) =$

$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times 1 \times \frac{1}{2} \times 1 + 3(\frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{2}) = \frac{1}{16} + \frac{1}{4} + \frac{3}{8} = \frac{11}{16} = 0.688$ (3S.F.)

\1Cii/ A: 1 and 2 is a subchain ∴

$P(X_3=1 \mid X_0=1) = P(X_1=2, X_2=2, X_3=1 \mid X_0=1) = P(1\to2\to2\to1 \mid 1) =$

$1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$

B: $P(X_3=2 \mid X_0=1) = P(1\to2\to1\to2 \mid 1) + P(1\to2\to2\to2 \mid 1) =$

$1 \times \frac{1}{2} \times 1 + 1 \times \frac{1}{2} \times \frac{1}{2} = \frac{1}{2} \times \frac{1}{4} = \frac{3}{4}$

C: $P(X_3=3 \mid X_0=1) = P(1\to2\to2\to3 \mid 1) = 1 \times \frac{1}{2} \times 0 = 0$ ∵

1 and 2 is a subchain

D: $P(X_4=2 \mid X_0=2) =$

$P(2\to2\to2\to2\to2) + P(2\to1\to2\to1\to2) + 3P(2\to1\to2\to2\to2) =$

$\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} + \frac{1}{2} \times 1 \times \frac{1}{2} \times 1 + 3(\frac{1}{2} \times 1 \times \frac{1}{2} \times \frac{1}{2}) = \frac{1}{16} + \frac{1}{4} + \frac{3}{8} = \frac{11}{16} = 0.688$ (3S.F.)

\1Ciii/ Steady state $\tilde{P} = (P_1, P_2, P_3) = (P_1\ \ P_2\ \ P_3)$ ∴

$\cancel{\tilde{P} = \tilde{P}I}$ $\tilde{P} = \tilde{P}T = (P_1\ \ P_2\ \ P_3) = (P_1\ \ P_2\ \ P_3)\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix} =$

$(\frac{1}{2}P_2\ \ \ P_1 + \frac{1}{2}P_2 + \frac{1}{4}P_3\ \ \ \frac{3}{4}P_3)$ ∴

$P_1 = \frac{1}{2}P_2$, $P_2 = P_1 + \frac{1}{2}P_2 + \frac{1}{4}P_3$, $P_3 = \frac{3}{4}P_3$ ∴

$\frac{1}{4}P_3 = 0$ ∴ $P_3 = 0$ ∴

$$\frac{t-\lambda-(t+\lambda)}{(t+\lambda)(t-\lambda)}\frac{\lambda}{2}e^{t\mu}=\frac{-2\lambda}{t^2-\lambda^2}\frac{\lambda}{2}e^{t\mu}=\frac{-\lambda^2}{t^2-\lambda^2}e^{t\mu}=$$

$$\frac{\lambda^2}{\lambda^2-t^2}e^{t\mu}=\frac{1}{1-\left(\frac{t}{\lambda}\right)^2}e^{t\mu}$$

Vbii/ $E(x)=M_x'(0)=M_x'(t)\big|_{t=0}=\frac{\lambda^2}{\lambda^2-t^2}\mu e^{t\mu}\big|_{t=0}=\frac{\lambda^2}{\lambda^2-0^2}\mu e^{(0)\mu}=\frac{\lambda^2}{\lambda^2}\mu e^0$

$=1\mu(1)=\mu$

$var(x)=\cancel{E}\, M_x''(0)=M_x''(t)\big|_{t=0}$

Vbii/ $E(x)=M_x'(0)=M_x'(t)\big|_{t=0}=\frac{\lambda^2}{\lambda^2-t^2}\mu e^{t\mu}+\frac{(\lambda^2-t^2)(0)-\lambda^2(-2t)}{(\lambda^2-t^2)^2}e^{t\mu}\big|_{t=0}$

$=\frac{2\lambda^2 t}{(\lambda^2-t^2)^2}e^{t\mu}+\frac{\lambda^2}{\lambda^2-t^2}\mu e^{t\mu}\big|_{t=0}=\frac{0}{(\lambda^2-0^2)^2}e^0+\frac{\lambda^2}{\lambda^2-0^2}\mu e^0=$

$0+\frac{\lambda^2}{\lambda^2}\mu=\mu$ ∴

$\cancel{var(x)}\, M_x''(0)=M_x''(t)\big|_{t=0}=\cancel{(\lambda^2-t^2)2\lambda^2-2\lambda^2 t2(\lambda^2-t^2)(-2t)}$

$$\frac{(\lambda^2-t^2)^2 2\lambda^2-2\lambda^2 t2(\lambda^2-t^2)(-2t)}{(\lambda^2-t^2)^4}e^{t\mu}+\frac{2\lambda^2 t}{(\lambda^2-t^2)^2}\mu e^{\mu t}+-\frac{\lambda^2(-2t)}{(\lambda^2-t^2)^2}\mu e^{\mu t}+\frac{\lambda^2}{\lambda^2-t^2}\mu^2 e^{\mu t}\Big|_{t=0}$$

$$=\frac{(\lambda^2-0)^2}{(\lambda^2-0)^4}2\lambda^2 e^0+0-0+\frac{\lambda^2}{\lambda^2-0}\mu^2 e^0=$$

$$\frac{2\lambda^4\lambda^2}{\lambda^8}+\mu^2=\frac{2\lambda^6}{\lambda^8}+\mu^2=\frac{2}{\lambda^2}+\mu^2=E(x^2)$$ ∴

$(E(x))^2=\mu^2$ ∴

$var(x)=E(x^2)-(E(x))^2=\frac{2}{\lambda^2}+\mu^2-\mu^2=\frac{2}{\lambda^2}$

Vbii/ $E(x)=M_x'(0)=M_x'(t)\big|_{t=0}=\frac{\lambda^2}{\lambda^2-t^2}\mu e^{\mu t}+\frac{(\lambda^2-t^2)(0)-\lambda^2(-2t)}{(\lambda^2-t^2)^2}e^{\mu t}\big|_{t=0}$

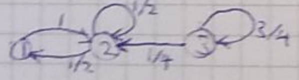
$=\frac{2\lambda^2 t}{(\lambda^2-t^2)^2}e^{t\mu}+\frac{\lambda^2}{\lambda^2-t^2}\mu e^{\mu t}\big|_{t=0}=\frac{0}{(\lambda^2-0)^2}e^0+\frac{\lambda^2}{\lambda^2-0^2}\mu e^0=0+\frac{\lambda^2}{\lambda^2}\mu=\mu$ ∴

$E(x^2)=M_x''(0)=M_x''(t)\big|_{t=0}=$

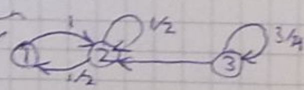
$$(\lambda^2-t^2)^2 2\lambda^2-2\lambda^2 t2(\lambda^2-t^2)(-2t)\frac{e^{t\mu}}{(\lambda^2-t^2)^4}+\frac{2\lambda^2 t\mu e^{\mu t}}{(\lambda^2-t^2)^2}-\frac{\lambda^2(-2t)\mu e^{\mu t}}{(\lambda^2-t^2)^2}+\frac{\lambda^2}{\lambda^2-t^2}\mu^2 e^{\mu t}\Big|_{t=0}=$$

$$\frac{(\lambda^2-0)^2}{(\lambda^2-0)^4}2\lambda^2 e^0+0-0+\frac{\lambda^2}{\lambda^2-0}\mu^2 e^0=\frac{2\lambda^4\lambda^2}{\lambda^8}+\mu^2=\frac{2}{\lambda^2}+\mu^2$$ ∴

$var(x)=E(x^2)-E(x)^2=\frac{2}{\lambda^2}+\mu^2-(\mu)^2=\frac{2}{\lambda^2}+\mu^2-\mu^2=\frac{2}{\lambda^2}$

Vci/  1 and 2 is an irreducable

Subchain

Vci/  ∴ 1 and 2 is a irreducable

Subchain

\ pp2021 / P4 $G_{T_Y}(\theta) = G_{T_{X_1+X_2+X_3}}(\theta) = G_{T_{X_1}}(\theta)G_{T_{X_2}}(\theta)G_{T_{X_3}}(\theta) = G_{T_X}(\theta)G_{T_X}(\theta)G_{T_X}(\theta) =$

$(G_{T_X}(\theta))^3 = (0.3\theta + 0.5\theta^2 + 0.2\theta^4)^3 =$

$(0.3\theta)^1(0.5\theta^2)^2 \frac{3!}{1!\times 2!} + \dots = 0.225\theta^5 + \dots$

$P(Y=5) = 0.225$ (3 s.f.)

\ 1bi / $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx = \int_{-\infty}^{\infty} e^{tx} \frac{\lambda}{2} e^{-\lambda|x-\mu|}\,dx =$

$\frac{\lambda}{2}\int_{-\infty}^{\infty} e^{tx} e^{-\lambda|x-\mu|}\,dx = \frac{\lambda}{2}\left[\int_{-\infty}^{\mu} e^{tx} e^{-\lambda(-(x-\mu))}\,dx + \int_{\mu}^{\infty} e^{tx} e^{-\lambda(x-\mu)}\,dx\right] =$

$\frac{\lambda}{2}\left[\int_{-\infty}^{\mu} e^{tx+\lambda x-\lambda\mu}\,dx + \int_{\mu}^{\infty} e^{tx-\lambda x+\lambda\mu}\,dx\right] =$

$\frac{\lambda}{2}\left[\int_{-\infty}^{\mu} e^{(t+\lambda)x-\lambda\mu}\,dx + \int_{\mu}^{\infty} e^{(t-\lambda)x+\lambda\mu}\,dx\right] =$

$\frac{\lambda}{2}\left[\frac{1}{t+\lambda} e^{(t+\lambda)x-\lambda\mu}\right]_{-\infty}^{\mu} + \frac{\lambda}{2}\left[\frac{1}{t-\lambda} e^{(t-\lambda)x+\lambda\mu}\right]_{\mu}^{\infty} =$

$\frac{\lambda}{2}\left[\frac{1}{t+\lambda} e^{(t+\lambda)\mu-\lambda\mu} - 0\right] + \frac{\lambda}{2}\left[\cancel{\frac{1}{t-\lambda} e^{(t-\lambda)}}\ 0 - \frac{1}{t-\lambda} e^{(t-\lambda)\mu+\lambda\mu}\right]$ for $|t|<\lambda$

$= \frac{\lambda}{2}\left[\frac{1}{t+\lambda} e^{-\lambda\mu} e^{t\mu} e^{\lambda\mu} - \frac{1}{t-\lambda} e^{\lambda\mu} e^{-\lambda\mu} e^{t\mu}\right] =$

$\frac{\lambda}{2}\left[\frac{1}{t+\lambda} e^{t\mu} - \frac{1}{t-\lambda} e^{t\mu}\right] = \left[\frac{1}{t+\lambda} - \frac{1}{t-\lambda}\right]\frac{\lambda}{2} e^{t\mu} = \frac{t-\lambda-(t+\lambda)}{(t+\lambda)(t-\lambda)} \frac{\lambda}{2} e^{t\mu} =$

$\frac{-2\lambda}{t^2-\lambda^2} \frac{\lambda}{2} e^{t\mu} = \frac{-\lambda^2}{t^2-\lambda^2} e^{\mu t} = \frac{\lambda^2}{\lambda^2-t^2} e^{\mu t} = \frac{1}{1-\left(\frac{t}{\lambda}\right)^2} e^{\mu t}$

\ 1bi / $M_X(t) = E(e^{tX}) = \int_{-\infty}^{\infty} e^{tx} f_X(x)\,dx = \int_{-\infty}^{\infty} e^{tx} \frac{1}{2} e^{-\lambda|x-\mu|}\,dx =$

$\frac{\lambda}{2}\int_{-\infty}^{\infty} e^{tx} e^{-\lambda|x-\mu|}\,dx = \frac{\lambda}{2}\left[\int_{-\infty}^{\mu} e^{tx} e^{-\lambda(-(x-\mu))}\,dx + \int_{\mu}^{\infty} e^{tx} e^{-\lambda(x-\mu)}\,dx\right] =$

$\frac{\lambda}{2}\left[\int_{-\infty}^{\mu} e^{tx} e^{\lambda x-\lambda\mu}\,dx + \int_{\mu}^{\infty} e^{tx} e^{-\lambda x+\lambda\mu}\,dx\right] =$

$\frac{\lambda}{2}\left[\int_{-\infty}^{\mu} e^{(t+\lambda)x-\lambda\mu}\,dx + \int_{\mu}^{\infty} e^{(t-\lambda)x+\lambda\mu}\,dx\right] =$

$\frac{\lambda}{2}\left[\frac{1}{t+\lambda} e^{(t+\lambda)x-\lambda\mu}\right]_{-\infty}^{\mu} + \frac{\lambda}{2}\left[\frac{1}{t-\lambda} e^{(t-\lambda)x+\lambda\mu}\right]_{\mu}^{\infty} =$

$\frac{\lambda}{2}\left[\frac{1}{t+\lambda} e^{(t+\lambda)\mu-\lambda\mu} - 0\right] + \frac{\lambda}{2}\left[0 - \frac{1}{t-\lambda} e^{(t-\lambda)\mu+\lambda\mu}\right]$ for $|t|<\lambda$

$= \frac{\lambda}{2}\left[\frac{1}{t+\lambda} e^{\mu t+\lambda\mu-\lambda\mu} - \frac{1}{t-\lambda} e^{\mu t-\lambda\mu+\lambda\mu}\right] = \frac{\lambda}{2}\left[\frac{1}{t+\lambda} e^{t\mu} - \frac{1}{t-\lambda} e^{t\mu}\right] =$

$\left[\frac{1}{t+\lambda} - \frac{1}{t-\lambda}\right]\frac{\lambda}{2} e^{t\mu} =$

\1a i/ $E(Y)=E(5X)=5E(X)$ ∴

$G_X'(\theta)=0.3+1\theta+0.8\theta^3$ ∴

$G_X'(1)=0.3+1(1)+0.8(1)=2.1=E(X)$ ∴

$E(Y)=5(2.1)=10.5$

\1a i/ $E(Y)=E(5X)=5E(X)$ ∴

$G_X'(\theta)=0.3+1\theta+0.8\theta^3$ ∴

$E(X)=G_X'(1)=0.3+1(1)+0.8(1)=2.1$ ∴

$E(Y)=5(2.1)=10.5$

$P(Y=5)=P(5X=5)=P(X=1)$ ∴

$G_X(\theta)=0.3\theta+\ldots$ ∴

$0.3=P(X=1)=P(Y=5)$

\1a ii/ $P(Y=5)=P(X^2+1=5)=P(X^2=4)$ ∴

$G_{X^2}(\theta)=0.3\theta^{1^2}+0.5\theta^{2^2}+0.2\theta^{4^2}=0.3\theta^1+0.5\theta^4+0.2\theta^{16}=0.3\theta+0.5\theta^4+0.2\theta^{16}$

∴ $P(Y=5)=P(X^2=4)=0.5$

$E(Y)=E(X^2+1)=E(X^2)+1$ ∴

$\cancel{E(Y)}$ $E(X^2)=0.3(1^2)+0.5(2^2)+0.2(4^2)=5.5$ ∴

$E(Y)=E(X^2)+1=5.5+1=6.5$

\1a iii/ $E(Y)=E(X_1+X_2+X_3)=E(X_1)+E(X_2)+E(X_3)=$

$E(X)+E(X)+E(X)=3E(X)$ ∴

$E(X)=0.3(1)+0.5(2)+0.2(4)=2.1$ ∴

$E(Y)=3E(X)=3(2.1)=6.3$

$G_Y(\theta)=\cancel{G_{X_1}}\ G_{X_1+X_2+X_3}(\theta)=G_{X_1}(\theta)G_{X_2}(\theta)G_{X_3}(\theta)=G_X(\theta)G_X(\theta)G_X(\theta)=$

$(G_X(\theta))^3=(0.3\theta+0.5\theta^2+0.2\theta^4)^3=$

$(0.5\theta^2)^2(0.3\theta)^1\frac{3!}{2!\times1!}+\ldots=0.225\theta^5+\ldots$ ∴

$\cancel{P(Y)}$ $P(Y=5)=0.225$ (3 s.f.)

\1a iii/ $E(Y)=E(X_1+X_2+X_3)=E(X_1)+E(X_2)+E(X_3)=$

$E(X)+E(X)+E(X)=3E(X)$ ∴

$E(X)=0.3(1)+0.5(2)+0.2(4)=2.1$ ∴

$E(Y)=3\times2.1=6.3$

\PP2021/ Expected time to finish is unchanged ∴

$$D_5 = \frac{5(p^5+q^3)-10q^5}{(p-q)(p^5+q^5)}$$

or: $\theta_0 = 0$, $\theta_N = 1$: $\theta_i = p\theta_{i+1} + 3q\theta_i + q\theta_{i-1}$, $p = 1-4q$ ∴

let $\theta_i = k\lambda^i$ ∴ $0 = k\lambda^{i-1}(p\lambda^2 + (3q-1)\lambda + q)$ ∴

$2p\lambda_\pm = 1-3q \pm (1-6q+9q^2-4(1-4q)q)^{1/2} = 1-3q \pm (1-5q)$ ∴

$\lambda_+ = \frac{2-8q}{2p} = \frac{1-4q}{2p} = \frac{1-4q}{1-4q} = 1$, $\lambda_- = \frac{2q}{2p} = \frac{q}{p}$ ∴

$\theta_i = A + B\tilde{p}^i$, $\tilde{p} = \frac{q}{p}$ ∴

$\theta_0 = 0$ ∴ $A = -B$ ∴ $\theta_N = 1$ ∴ $A + B\tilde{p}^N = 1$ ∴

$B(\tilde{p}^N - 1) = 1$ ∴ $B = \frac{1}{\tilde{p}^N - 1}$ ∴ $A = \frac{1}{-\tilde{p}^N - 1}$ ∴

$\theta_i = \frac{1-\tilde{p}^i}{1-\tilde{p}^N}$, $i = 1, \dots, N-1$, and similar for D

\1aii/ $E(X^2) = \sum_i P(X^2 = i) \times i^2$ ∴

$G_{X^2}(\theta) = 0.3\theta^{1^2} + 0.5\theta^{2^2} + 0.2\theta^{4^2} = 0.3\theta + 0.5\theta^4 + 0.2\theta^{16}$ ∴

$E(X^2) = 0.3(1^2) + 0.5(2^2) + 0.2(4^2) = 0.3 + 2 + 3.2 = 5.5$,

$G'_{X^2}(\theta)|_{\theta=1} = 0.3 + 0.5(4)\theta^3 + 0.2(16)\theta^{15}|_{\theta=1} = 0.3 + 2(1)^3 + 3.2(1)^{15} =$

$5.5 = E(X^2)$

\PP2021/

\1ai/ $E(Y) = E(5X) = 5E(X)$

$G'_X(\theta) = 0.3 + 1\theta + 0.8\theta^3$ ∴ $G'_X(1) = E(X) = 0.3 + 1 + 0.8 = 2.1$ ∴

$E(Y) = 5 \times 2.1 = 10.5$

$P(Y=5) = P(5X=5) = P(X=1)$ ∴ $G_X(\theta) = 0.3\theta + \dots$ ∴

$P(X=1) = 0.3 = P(Y=5)$

\1aii/ $P(Y=5) = P((X^2+1)=5) = P(X^2=4)$ ∴

$G_{X^2}(\theta) = 0.3\theta^{0^2} + 0.5\theta^{2^2} + 0.2\theta^{4^2} = 0.3\theta^0 + 0.5\theta^4 + 0.2\theta^{16} = 0.3 + 0.5\theta^4 + 0.2\theta^{16}$

∴ $P(X^2=4) = 0.5 = P(Y=5)$

$E(Y) = E(X^2+1) = E(X^2)+1$ ∴ $E(X^2) = 0.3(1^2) + 0.5(2^2) + 0.2(4^2) = 5.5$ ∴

$E(Y) = 5.5 + 1 = 6.5$

$-1 = aq + bqi + bq + cqi^2 + 2cqi + cq + 3aq + 3bqi + 3cqi^2 - a - bi - ci^2 + a(1-4q) +$
$b(1-4q)i - b(1-4q) + c(1-4q)i^2 - 2c(1-4q)i + c(1-4q) =$

$-1 = 2cqi + cq \cancel{+} - b + 4bq - 2ci + 8cqi + c - 4cq =$

$\cancel{-1 = bq + cq - b + 4bq + c - 4cq = b(5q-1) + c(1-3q)}$ ∴

Coefficients:

$i^0: -1 = bq + cq - b + 4bq + c - 4cq = b(5q-1) + c(1-3q)$

$i^1: 0 = 2cq - 2c + 8cq = 2c(5q-1)$ ∴ $c = 0$ ∴

$-1 = b(5q-1)$ ∴ $b = \frac{1}{1-5q} = \frac{1}{1-4q-q} = \frac{1}{p-q}$

$i^2: 0 = 0$ ∴ a can be absorbed into A ∴

Check: $0 = 1 - 5q + q(i+1) + 3qi - i + p(i-1) = 0$ ∴

~~Solution~~ PS: particular solutions: $D_i^P = \frac{i}{p-q} = \frac{i}{1-5q}$

GS: $D_i = \frac{i}{p-q} + A + B\rho^i = \frac{i}{p-q} + A + B\frac{p^i}{q^i}$ ∴

$D_N = 0$, $D_0 = 0$ ∴

at $i=0$: $0 = 0 + A + B$ ∴ $A = -B$ ∴

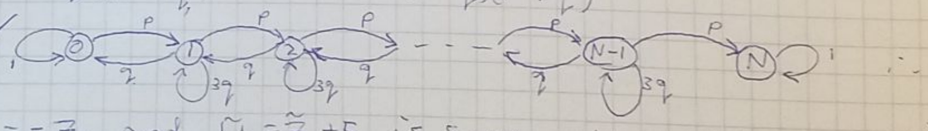
at $i=N$: $0 = \frac{N}{p-q} + A + B\frac{p^N}{q^N} = \frac{N}{p-q} + B\left(\frac{p^N}{q^N} - 1\right) = \frac{N}{p-q} + B\left(\frac{p^N - q^N}{q^N}\right)$ ∴

$B = \frac{N}{(q-p)}\frac{q^N}{(p^N - q^N)}$ ∴

$D_i = \frac{i}{p-q} + \frac{Nq^N(1-\rho^i)}{(p-q)(p^N-q^N)} = \frac{i(p^N - q^N) + Nq^N(1-p^iq^{-i})}{(p-q)(p^N-q^N)} = \frac{i(p^N-q^N) + Nq^N - Np^iq^{N-i}}{(p-q)(p^N-q^N)}$

∴ set $i=5$, $N=10$ ∴ $D_5 = \frac{5(p^{10}-q^{10}) + 10q^5(q^5-p^5)}{(p-q)(p^5-q^5)(p^5+q^5)} =$

$\frac{5(p^5-q^5)(p^5+q^5) - 10(p^5-q^5)q^5}{(p-q)(p^5-q^5)(p^5+q^5)} = \frac{5(p^5+q^5) - 10q^5}{(p-q)(p^5+q^5)}$

\4b/ 

∴

Let $\tilde{Z}_n = -Z_n$ and $a_n = \tilde{Z}_n + 5$ is same problem as before but with England winning corresponding to $N=0$ ∴

P(England win in new system) = P(Australia win in old system) =
$1 - P(\text{England win in old system}) = 1 - \frac{q^5}{p^5+q^5} = \frac{p^5}{p^5+q^5}$

\PP2021/ $\frac{2^2e^{-2}}{2!}\cdot\frac{4^3e^{-4}}{3!}e^{-2}+\frac{2e^{-2}4^2e^{-4}}{2!}\cdot 2e^{-2}+e^{-2}\cdot 4e^{-4}\cdot\frac{2^2e^{-2}}{2!}=0.0206\ (3\,S.F.)$

\3Ci/ 20 mins $=\frac{1}{3}$ hour $=t$ $\quad e^{-\lambda t}(\lambda t)^x\frac{1}{x!}$ ∴

orders for beer occur at $\lambda_{beer}=15$ p/h, $\lambda_{cola}=5$ p/h, $\lambda_{water}=2$ p/h

probability over 20 mins of recieving at least 5 beer, 2 cola, 1 water orders ∴ use independence

∴ let Beer, cola, water orders = B, C, W ∴

$P(B\geq 5, C\geq 2, W\geq 1)=P(B\geq 5)P(C\geq 2)P(W\geq 1)$ by independence

$=(1-P(B\leq 4))(1-P(C\leq 1))(1-P(W=0))=$

$(1-[e^{-15\times\frac{1}{3}}(15\times\frac{1}{3})^0\frac{1}{0!}+e^{-15\times\frac{1}{3}}(15\times\frac{1}{3})^1\frac{1}{1!}+e^{-15\times\frac{1}{3}}(15\times\frac{1}{3})^2\frac{1}{2!}+e^{-15\times\frac{1}{3}}(15\times\frac{1}{3})\frac{1}{3!}+e^{-15\times\frac{1}{3}}(\frac{15}{3})^4\frac{1}{4!})]\times$

$(1-[e^{-5\times\frac{1}{3}}(5\times\frac{1}{3})^0\frac{1}{0!}+e^{-5\times\frac{1}{3}}(5\times\frac{1}{3})^1\frac{1}{1!})\times(1-[e^{-2\times\frac{1}{3}}(2\times\frac{1}{3})^0\frac{1}{0!}])=$

$(1-e^{-5}[5^0\frac{1}{0!}+5^1\frac{1}{1!}+5^2\frac{1}{2!}+5^3\frac{1}{3!}+5^4\frac{1}{4!}])\times(1-e^{-\frac{5}{3}}((\frac{5}{3})^0\frac{1}{0!}+(\frac{5}{3})^1\frac{1}{1!}))(1-e^{-2/3}]=$

$0.5595\times 0.4963\times 0.4866=0.135\ (3\,S.F.)$

\3Cii/ Expected arrival time of poisson process is given by exponential distribution ∴

Merge beer + water processes ∴ $Y=W+B\sim Poi(15+2)=Poi(17)$ ∴

Joint density $f_{C,Y}(x,y)=(5e^{-5x})(17e^{-17y})$ ∴

$P(T_C<T_Y)=\int_0^\infty\int_0^y f_{C,Y}(x,y)=\int_0^\infty\int_0^y(5e^{-5x})(17e^{-17y})\,dx\,dy=$

$\int_0^\infty 17e^{-17y}\int_0^y 5e^{-5x}\,dx\,dy=\int_0^\infty 17e^{-17y}\left[-e^{-5x}\right]_0^y dy=17\int_0^\infty e^{-17y}(1-e^{-5y})\,dy=$

$17\int_0^\infty e^{-17y}-e^{-22y}\,dy=17\left[-\frac{1}{17}e^{-17y}+\frac{1}{22}e^{-22y}\right]_0^\infty=\left[\frac{17}{22}e^{-22y}-e^{-17y}\right]_0^\infty=$

$\frac{17}{22}[0-e^0]-[0-e^0]=1-\frac{17}{22}=\frac{5}{22}=0.227$

\3d/ Ice cream orders at λ per hour label orders of flavours by indexing set $i=1,\dots,n$ ∴ Expected arrival time of first order λ^{-1} ie $E(T_1)=\frac{1}{\lambda}$ ∴

start by assuming n finite ∴ Iterate to get $E(T_r)=\frac{n}{(n-r)\lambda}$ ∴

expected time to have all flavours ordered is

$E(T_1)+E(T_2)+\dots+E(T_n)=\frac{n}{\lambda}\left(\frac{1}{n}+\frac{1}{n-1}+\frac{1}{n-2}+\dots+1\right)=\frac{n}{\lambda}\sum_{k=1}^{n}\frac{1}{k}\sim\frac{n\log n}{k}$ ∴

this diverges as $n\to\infty$ ∴ no finite window is sufficiently large to expect to see all flavours orders start by assuming n finite ∴

iterate to get $E(T_r) = \frac{n}{(n-r)\lambda}$ ∴

expected time to have all flavours ordered is

$E(T_1) + E(T_2) + \dots + E(T_n) =$

$\frac{n}{\lambda}\left(\frac{1}{n} + \frac{1}{n-1} + \frac{1}{n-2} + \dots + 1\right) = \frac{n}{\lambda}\sum_{k=1}^{n}\frac{1}{k} \sim \frac{n\log n}{k}$ ∴

this diverges as $n \to \infty$ ∴ no finite window is sufficiently large to expect to see all flavour orders

\3a/ $P(N(t)=n \mid N(T)=k) = P(N(t)=r \cap N(T) \leq n)/P(N(T)=n) =$

$P(N(t)=r)P(N(T-r)=n-r)/P(N(t)=n) =$

$((\lambda t)^r e^{-\lambda t}/r!)\left(\frac{\lambda(T-t)^{n-r}}{(n-r)!}e^{-\lambda(T-t)}\right)/[(\lambda T)^n e^{-\lambda T}/n!] =$

$\frac{n!}{r!(n-r)!}\left(\frac{t}{T}\right)^r\left(1-\frac{t}{T}\right)^{n-r} \sim \text{Binomial}\left(\frac{t}{T}, n\right)$

\3bi/ P(hot dog is next order) = 0.1

\3bii/ $\lambda = 20$ ∴ $\frac{1}{20} = 0.05$ hrs = 3 minutes expected next order = E(arrival)

\3biii/ P(5 pizza orders in 30 mins | rate of pizza orders $= \lambda_{pizza} = 0.4 \times 20 = 8$ per hour) $= P(N_{pizza}(\frac{1}{2}) = 5) =$

$e^{-\lambda_{pizza}t}\frac{1}{x!}(\lambda_{pizza}t)^x = e^{-8(\frac{1}{2})}\frac{1}{5!}(8 \times 0.5)^5 = e^{-4}\frac{1}{5!}(4)^5 = 0.156$ (3 s.f.)

\3biv/ let $N_{(a,b]}$ be the number of dog orders over the interval $(a,b]$ ∴ $N_{(0,1]} \sim Poi(2)$ ∴ $N_{(1,2]} \sim Poi(2)$, $N_{(2,4]} \sim Poi(4)$ ∴ LoTP:

$P(N_{(0,2]}=2 \cap N_{(1,4]}=3) = P(N_{(0,1]}+N_{(1,2]}=2 \cap N_{(1,2]}+N_{(2,4]}=3) =$

$\sum_{k=0}^{\infty} P(N_{(0,1]}+N_{(1,2]}=2 \cap N_{(1,2]}+N_{(2,4]}=3 \mid N_{(1,2]}=k)P(N_{(1,2]}=k) =$

$P(N_{(0,1]}=2, N_{(2,4]}=3 \mid N_{(1,2]}=0)P(N_{(1,2]}=0) +$

$P(N_{(0,1]}=1, N_{(2,4]}=2 \mid N_{(1,2]}=1)P(N_{(1,2]}=1) +$

$P(N_{(0,1]}=0, N_{(2,4]}=1 \mid N_{(1,2]}=2)P(N_{(1,2]}=2) =$

$P(N_{(0,1]}=2, N_{(2,4]}=3)P(N_{(1,2]}=0) + P(N_{(0,1]}=1, N_{(2,4]}=2)P(N_{(1,2]}=1) +$

$P(N_{(0,1]}=0, N_{(2,4]}=2)P(N_{(1,2]}=2) =$ by independence

$P(N_{(0,1]}=2)P(N_{(2,4]}=3)P(N_{(1,2]}=0) + P(N_{(0,1]}=1)P(N_{(2,4]}=2)P(N_{(1,2]}=1) +$

$P(N_{(0,1]}=0)P(N_{(2,4]}=1)P(N_{(1,2]}=2) =$

$$\therefore \frac{W_s^i}{W_s} = \frac{1}{\mu-1} \div \left(\frac{2\mu^2+2\mu-1}{(\mu-1)(2\mu^2+4\mu+3)}\right) = \frac{(\mu-1)(2\mu^2+4\mu+3)}{(\mu-1)(2\mu^2+2\mu-1)} = \frac{2\mu^2+4\mu+3}{2\mu^2+2\mu-1}$$

$$\therefore (2\mu^2+4\mu+3)-(2\mu^2+2\mu-1) = 2\mu^2+4\mu+3-2\mu^2-2\mu+1 = 2\mu+4 = 2(\mu+2) > 0 \therefore$$

$$\frac{W_s^i}{W_s} = \frac{2\mu^2+4\mu+3}{2\mu^2+2\mu-1} > 0 \therefore$$

M/M/3 queue is more efficient than 3 × M/M/1 queues

\3a/ $P(N(t)=n \mid N(T)=k) = \frac{P(N(t)=r \cap N(T)=n)}{P(N(T)=n)} =$

$P(N(t)=r)P(N(T-r)=n-r)/P(N(t)=n) =$

$((\lambda t)^r e^{-\lambda t}/r!)\left(\frac{\lambda(T-t)^{n-r}}{(n-r)!}e^{-\lambda(T-t)}\right)/\left[(\lambda T)^n e^{-\lambda T}/n!\right] =$

$\frac{n!}{r!(n-r)!}\left(\frac{t}{T}\right)^r\left(1-\frac{t}{T}\right)^{n-r} \sim \text{Binomial}\left(\frac{t}{T}, n\right)$

\3bi/ P(hot dog is next order) = 0.1

\3bii/ $\lambda = 20 \therefore \frac{1}{20} = 0.05$ hours = 3 minutes until ~~next~~ expect next order = E(arrival)

\3biii/ P(5 pizza orders in 30 mins | rate of pizza orders $= \lambda_{pizza} = 0.4 \times 20 = 8$ per hour) $= P(N_{pizza}(\frac{1}{2})=5) =$

$e^{-\lambda_{pizza}t}\frac{1}{x!}(\lambda_{pizza}t)^x = e^{-8(\frac{1}{2})}\frac{1}{5!}(8\times 0.5)^5 = e^{-4}\frac{1}{5!}(4)^5 = 0.156$ (3 s.f.)

\3biv/ ~~observations are independent~~ ∴

~~P(2 hotdog orders in 2 hours) = P(N_dog(2)=2) =~~

~~e^(−0.1×20×2)(0.1×20×2)^2/2! = 0.147 (3 s.f.)~~

\3biv/ let $N_{(a,b]}$ be the number of dog orders ~~before~~ over the interval $(a,b]$ $\therefore N_{(0,1]} \sim Poi(2) \therefore N_{(1,2]} \sim Poi(2)$, $N_{(2,4]} \sim Poi(4)$ ∴ LOTP ∴

$P(N_{(0,2]} = 2 \cap N_{(1,4]} = 3) = P(N_{(0,1]} + N_{(1,2]} = 2 \cap N_{(1,2]} + N_{(2,4]} = 3) =$

$\sum_{k=0}^{\infty} P(N_{(0,1]} + N_{(1,2]} = 2 \cap N_{(1,2]} + N_{(2,4]} = 3 \mid N_{(1,2]} = k)P(N_{(1,2]} = k) =$

$P(N_{(0,1]} = 2, N_{(2,4]} = 3 \mid N_{(1,2]} = 0)P(N_{(1,2]} = 0) +$

$P(N_{(0,1]} = 1, N_{(2,4]} = 2 \mid N_{(1,2]} = 1)P(N_{(1,2]} = 1) +$

$P(N_{(0,1]} = 0, N_{(2,4]} = 1 \mid N_{(1,2]} = 2)P(N_{(1,2]} = 2) =$

\PP2021/ $P(N_{(0,1]}=2, N_{(2,4]}=3)P(N_{(1,2]}=0)+P(N_{(0,1]}=1, N_{(2,4]}=2)P(N_{(1,2]}=1)+$
$P(N_{(0,1]}=0, N_{(2,4]}=1)P(N_{(1,2]}=2) =$ by independence

$P(N_{(0,1]}=2)P(N_{(2,4]}=3)P(N_{(1,2]}=0)+P(N_{(0,1]}=1)P(N_{(2,4]}=2)P(N_{(1,2]}=1)+$
$P(N_{(0,1]}=0)P(N_{(2,4]}=1)P(N_{(1,2]}=2)=$
$\frac{2^2 e^{-2}}{2!}\cdot\frac{4^3 e^{-4}}{3!}e^{-2}+\frac{2e^{-2}\cdot 4^2 e^{-4}}{2!}\cdot 2e^{-2}+e^{-2}\cdot 4e^{-4}\cdot\frac{2^2 e^{-2}}{2!}\approx 0.0206$ (3 S.F.)

\3Ci/ 20 mins $=\frac{1}{3}$ hour $=t$ $\quad e^{-\lambda t}(\lambda t)^x\frac{1}{x!}$
orders for beer occur at $\lambda_{beer}=15$ p/h, $\lambda_{cola}=5$ p/h, $\lambda_{water}=2$ p/h
probability over 20 mins of recieving at least 5 beer, 2 cola, 1 water orders $\therefore$ use independence
$\therefore$ let Beer, Cola, water (orders $=B, C, W$ $\therefore$
$P(B\geq 5, C\geq 2, W\geq 1)=P(B\geq 5)P(C\geq 2)P(W\geq 1)$ by independence
$=(1-P(B\leq 4))(1-P(C\leq 1))(1-P(W=0))=$
$(1-[e^{-15\times\frac{1}{3}}(15\times\frac{1}{3})^0\frac{1}{0!}+e^{-15\times\frac{1}{3}}(15\times\frac{1}{3})^1\frac{1}{1!}+e^{-15\times\frac{1}{3}}(15\times\frac{1}{3})^2\frac{1}{2!}+e^{-15\times\frac{1}{3}}(15\times\frac{1}{3})\frac{1}{3!}+e^{-15\times\frac{1}{3}}(\frac{15}{3})^4\frac{1}{4!}])\times$
$(1-[e^{-5\times\frac{1}{3}}(5\times\frac{1}{3})^0\frac{1}{0!}+e^{-5\times\frac{1}{3}}(5\times\frac{1}{3})^1\frac{1}{1!}])\times(1-[e^{-2\times\frac{1}{3}}(2\times\frac{1}{3})^0\frac{1}{0!}])=$
$(1-e^{-5}[5^0\frac{1}{0!}+5^1\frac{1}{1!}+5^2\frac{1}{2!}+5^3\frac{1}{3!}+5^4\frac{1}{4!}])\times(1-e^{-\frac{5}{3}}((\frac{5}{3})^0\frac{1}{0!}+(\frac{5}{3})^1\frac{1}{1!}))(1-e^{-2/3})=$
$0.5595\times 0.4963\times 0.4866=0.135$ (3 S.F.)

\3Cii/ Expected arrival time of poisson process is given by exponential distribution.
Merge beer + water processes $\therefore$ $Y=W+B\sim Poi(15+2)=Poi(17)$ $\therefore$
Joint density: $f_{C,Y}(x,y)=(5e^{-5x})(17e^{-17y})$ $\therefore$
$P(T_C<T_Y)=\int_0^\infty\int_0^y f_{C,Y}(x,y)=\int_0^\infty\int_0^y(5e^{-5x})(17e^{-17y})\,dx\,dy=$
$\int_0^\infty 17e^{-17y}\int_0^y 5e^{-5x}\,dx\,dy=\int_0^\infty 17e^{-17y}[-e^{-5x}]_0^y\,dy=17\int_0^\infty e^{-17y}(1-e^{-5y})\,dy=$
$17\int_0^\infty e^{-17y}-e^{-22y}\,dy=17[-\frac{1}{17}e^{-17y}+\frac{1}{22}e^{-22y}]_0^\infty=[\frac{17}{22}e^{-22y}-e^{-17y}]_0^\infty=$
$\frac{17}{22}[0-e^0]-[0-e^0]=1-\frac{17}{22}=\frac{5}{22}=0.227$

\3d/ Ice cream orders at λ per hour. label order of ~~&~~ flavours by indexing set $i=1,\ldots,n$ $\therefore$ Expected ~~arrival~~ arrival time of first order λ^{-1} ie $E(T_1)=\frac{1}{\lambda}$ $\therefore$

\PP2021/ \2bii/ $\therefore L_s^i = \frac{\frac{1}{\mu}}{1-\frac{1}{\mu}} = \frac{1}{\mu-1} \therefore$

overall system size is $L_s = \sum_{i=1}^{3} L_s^i = \frac{3}{\mu-1} \therefore$

By littles law, expected waiting time per qeue is

$W_s^i = L_s^i = \frac{1}{\mu-1} \therefore$

For the M/M/3 qeue get: $W_s = \frac{L_s}{3} = \frac{2\mu^2+2\mu-1}{(\mu-1)(2\mu^2+4\mu+3)} \therefore$

$$W_s^i - W_s = \frac{1}{\mu-1} - \frac{2\mu^2+2\mu-1}{(\mu-1)(2\mu^2+4\mu+3)} = \frac{1}{\mu-1}\left(1 - \frac{2\mu^2+2\mu-1}{2\mu^2+4\mu+3}\right) \therefore$$

$\frac{2\mu^2+2\mu-1}{2\mu^2+4\mu+3} < 1 \therefore$ $\cancel{1-\frac{2\mu^2+2\mu-1}{}}$ $1 - \frac{2\mu^2+2\mu-1}{2\mu^2+4\mu+3} > 0 \therefore$

$W_s^i > W_s \therefore$ M/M/3 queue leads to shorter mean waiting times is term in bracket is positive $\therefore$ $\because$

$1 - \frac{2\mu^2+2\mu-1}{2\mu^2+4\mu+3} = \frac{2\mu^2+4\mu+3-2\mu^2-2\mu+1}{2\mu^2+4\mu+3} \therefore$ $\because \mu>0$, denominator is positive $\therefore$ $2\mu+4 = 2(\mu+2) > 0 \therefore$

M/M/3 queue is more efficient then $3\times$ M/M/1 queues

$\therefore$ $\cancel{\frac{W_s^i}{W_s} = \left(\frac{2\mu^2+2\mu-1}{(\mu-1)(2\mu^2+4\mu+3)}\right)^{-1}}$ $\frac{W_s^i}{W_s} = \frac{1}{\mu-1} \div \left(\frac{2\mu^2+2\mu-1}{(\mu-1)(2\mu^2+4\mu+3)}\right) = \frac{2\mu^2+4\mu+3}{2\mu^2+2\mu-1}$

$= \frac{2\mu^2+2\mu-1}{2\mu^2+2\mu-1} + \frac{2\mu+4}{2\mu^2+2\mu-1} = 1 + \frac{2\mu+4}{2\mu^2+2\mu-1} > 1 \therefore W_s^i > W_s$

\2bi/ $\therefore 3\times$ M/M/1 queues with infinite capacity, $\frac{1}{3}\lambda = 1$, $\mu > 0$

for all 3 queues $\therefore$ server 1:

$$\begin{tikzcd} 0 \arrow[r, bend left, "1"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "1"] & 2 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "1"] & \cdots \arrow[l, bend left, "\mu"] \end{tikzcd}$$

server 2:

$$\begin{tikzcd} 0 \arrow[r, bend left, "1"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "1"] & 2 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "1"] & \cdots \arrow[l, bend left, "\mu"] \end{tikzcd}$$

server 3:

$$\begin{tikzcd} 0 \arrow[r, bend left, "1"] & 1 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "1"] & 2 \arrow[l, bend left, "\mu"] \arrow[r, bend left, "1"] & \cdots \arrow[l, bend left, "\mu"] \end{tikzcd}$$

\2bii/ $\therefore \rho = \frac{1}{\mu} \therefore$ for a standard M/M/1 queues, expected system size of each queue is $L_s = \frac{\rho}{1-\rho}$, $\rho = \frac{1}{\mu} \therefore$

$L_s^i = \frac{\frac{1}{\mu}}{1-\frac{1}{\mu}} = \frac{1}{\mu-1} \therefore$

overall system size is $L_s = \sum_{i=1}^{3} L_s^i = \frac{3}{\mu-1} \therefore$

By littles law, expected waiting time per queue is

$$W_s^i = L_s^i \frac{1}{\lambda_{eff}} = L_s^i \frac{1}{\lambda_n} = L_s^i \frac{1}{\lambda} = L_s^i \frac{1}{1} = L_s^i = \frac{1}{\mu-1} \therefore$$

For the M/M/3: ~~queue leads to shorter W_s =~~

$$W_s = L_s \frac{1}{\lambda_{eff}} = L_s \frac{1}{\lambda_n} = L_s \frac{1}{\lambda} = L_s \frac{1}{3} = \frac{3(2\mu^2+2\mu-1)}{(\mu-1)(2\mu^2+4\mu+3)} \times \frac{1}{3} = \frac{2\mu^2+2\mu-1}{(2\mu^2+4\mu+3)(\mu-1)}$$

$3\frac{1}{\mu}P_0 = P_1 \therefore$

$3P_1 = 2\mu P_2 \therefore \frac{3}{2}\frac{1}{\mu}P_1 = P_2 = \frac{3}{2}\frac{1}{\mu}3\frac{1}{\mu}P_0 = \frac{9}{2}\frac{1}{\mu^2}P_0 \therefore$

$3P_2 = 3\mu P_3 \therefore \frac{1}{\mu}P_2 = P_3 \therefore \frac{1}{\mu}\frac{9}{2}\frac{1}{\mu^2}P_0 = \frac{9}{2}\frac{1}{\mu^3}P_0 = P_3 \therefore$

$3P_3 = 3\mu P_4,\ 3P_4 = 3\mu P_5 \therefore$

$\frac{9}{2}\frac{1}{\mu^n}P_0 = P_n$ for $n \geq 2 \therefore \sum_{n=0}^{\infty} P_n = 1 \therefore$

$\sum_{n=0}^{\infty} \rho^n = \frac{1}{1-\rho} \therefore \sum_{n=0}^{\infty}\left(\frac{1}{\mu}\right)^n = \frac{1}{1-\left(\frac{1}{\mu}\right)} \therefore$

$P_0 + P_1 + \sum_{n=2}^{\infty} P_n = 1 = P_0 + \frac{3}{\mu}P_0 + \sum_{n=2}^{\infty}\frac{9}{2}\frac{1}{\mu^n}P_0 = P_0\left(1+\frac{3}{\mu}+\frac{9}{2}\sum_{n=2}^{\infty}\left(\frac{1}{\mu}\right)^n\right)$

$\therefore \sum_{n=0}^{\infty}\rho^n = \frac{1}{1-\rho} \therefore \sum_{n=1}^{\infty}\rho^n = \frac{1}{1-\rho} - 1 = \frac{1}{1-\rho} - \frac{1-\rho}{1-\rho} = \frac{\rho}{1-\rho} \therefore$

$\sum_{n=2}^{\infty}\rho^n = \frac{\rho}{1-\rho} - \rho \therefore$

$\sum_{n=2}^{\infty}\left(\frac{1}{\mu}\right)^n = \frac{\left(\frac{1}{\mu}\right)}{1-\left(\frac{1}{\mu}\right)} - \frac{1}{\mu} = \frac{1}{\mu-1} - \frac{1}{\mu} = \frac{\mu}{\mu(\mu-1)} - \frac{\mu-1}{\mu(\mu-1)} = \frac{1}{\mu(\mu-1)} \therefore$

$1 = P_0\left(1+\frac{3}{\mu}+\frac{9}{2}\sum_{n=2}^{\infty}\left(\frac{1}{\mu}\right)^n\right) = P_0\left(1+\frac{3}{\mu}+\frac{9}{2}\frac{1}{\mu(\mu-1)}\right) =$

$P_0\left(\frac{\mu(\mu-1)}{\mu(\mu-1)} + \frac{3(\mu-1)}{\mu(\mu-1)} + \frac{\left(\frac{9}{2}\right)}{\mu(\mu-1)}\right) = P_0\left(\frac{\mu^2-\mu+3\mu-3+\frac{9}{2}}{\mu(\mu-1)}\right) = P_0\left(\frac{\mu^2+2\mu+\frac{3}{2}}{\mu(\mu-1)}\right) \therefore$

$\frac{\mu(\mu-1)}{\mu^2+2\mu+\frac{3}{2}} = P_0 = \frac{2\mu(\mu-1)}{2\mu^2+4\mu+3} \therefore$

$3\frac{1}{\mu}P_0 = P_1 = \frac{6(\mu-1)}{2\mu^2+4\mu+3},$

$P_n = \frac{9}{2}\frac{1}{\mu^n}P_0 = \frac{9}{2}\frac{1}{\mu^n}\frac{2\mu(\mu-1)}{2\mu^2+4\mu+3} = \cancel{+8}\,\frac{9}{2}\frac{1}{\mu^{n-1}}\frac{\cancel{18}(\mu-1)}{\mu^2+2\mu+\frac{3}{2}} = \frac{9(\mu-1)}{\mu^{n-1}(2\mu^2+4\mu+3)}$

for $n \geq 2$

Steady state exists for $3\mu > \lambda = 3 \therefore \mu > 1$

\2aii/ Steady state: $\cancel{3P_0 = \mu P_1, 2}$ $\frac{dP_0}{dt} = -3P_0 + \mu P_1 = 0,$

$\frac{dP_1}{dt} = -\mu P_1 + 3P_0 - 3P_1 + 2\mu P_2 = 0 = -3P_1 + 2\mu P_2,$

$\frac{dP_2}{dt} = 3P_1 - 2\mu P_2 - 3P_2 + 3\mu P_3 = -3P_2 + 3\mu P_3 = 0 \therefore$

$\frac{dP_3}{dt} = 3P_2 - 3\mu P_3 - 3P_3 + 3\mu P_4 = 0 = -3P_3 + 3\mu P_4 \therefore$

$3\frac{1}{\mu}P_0 = P_1 \therefore$

$3P_1 = 2\mu P_2 \therefore \frac{3}{2}\frac{1}{\mu}P_1 = P_2 = \frac{3}{2}\frac{1}{\mu}3\frac{1}{\mu}P_0 = \frac{9}{2}\frac{1}{\mu^2}P_0 \therefore$

$3P_2 = 3\mu P_3 \therefore \frac{1}{\mu}P_2 = P_3 = \frac{1}{\mu}\frac{9}{2}\frac{1}{\mu^2}P_0 = \frac{9}{2}\frac{1}{\mu^3}P_0 \therefore$

\PP2021/ \Idiv/ $\therefore \alpha < \frac{1}{3}$ $\therefore$ ~~B~~ Probab of extinction for $\alpha < \frac{1}{3}$ given by: solving $G_X(\theta) = \theta \therefore \alpha + (\alpha - 1)\theta + (1-2\alpha)\theta^2 = 0 \therefore$

$$2(1-2\alpha)\theta_{\pm} = 1 - \alpha \pm \sqrt{\alpha^2 - 2\alpha + 1 - 4\alpha(1-2\alpha)} = 1 - \alpha \pm \sqrt{\alpha^2 - 2\alpha + 1 - 4\alpha + 8\alpha^2} =$$

$$1 - \alpha \pm \sqrt{9\alpha^2 - 6\alpha + 1} = 1 - \alpha \pm \sqrt{(3\alpha \mp 1)^2} = 1 - \alpha \pm (3\alpha - 1) \therefore$$

$$\theta_+ = \frac{1 - \alpha + 3\alpha - 1}{2(1-2\alpha)} = \frac{2\alpha}{2(1-2\alpha)} = \frac{\alpha}{1-2\alpha}, \quad \theta_- = \frac{1 - \alpha - 3\alpha + 1}{2(1-2\alpha)} = \frac{2 - 4\alpha}{2(1-2\alpha)} = \frac{2(1-2\alpha)}{2(1-2\alpha)} = 1$$

$\therefore$ for $\alpha < \frac{1}{3}$, θ_+ is the minimal root, $\theta_+ = \frac{\alpha}{1-2\alpha}$

\Idiv/ $\therefore \alpha < \frac{1}{3}$ $\therefore$ Probability of extinction for $\alpha < \frac{1}{3}$ given by: Solving $G_X(\theta) = \theta \therefore$ ~~G_X~~

$$G_X(\theta) - \theta = 0 = \alpha + \alpha\theta + (1-2\alpha)\theta^2 - \theta = \alpha + (\alpha - 1)\theta + (1-2\alpha)\theta^2 =$$

$$(1-2\alpha)\theta^2 + (\alpha - 1)\theta + \alpha = 0 \therefore$$

$$2(1-2\alpha)\theta_{\pm} = -(\alpha - 1) \pm [(\alpha-1)^2 - 4(1-2\alpha)\alpha]^{1/2} =$$

$$1 - \alpha \pm [\alpha^2 - 2\alpha + 1 - 4\alpha + 8\alpha^2]^{1/2} = 1 - \alpha \pm [9\alpha^2 - 6\alpha + 1]^{1/2} =$$

$$1 - \alpha \pm [(3\alpha - 1)^2]^{1/2} = 1 - \alpha \pm (3\alpha \mp 1) \therefore$$

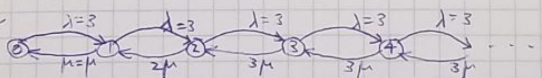
$$2(1-2\alpha)\theta_+ = 1 - \alpha + (3\alpha \mp 1) = 1 - \alpha + 3\alpha \mp 1 = 2\alpha \therefore$$

$$(1-2\alpha)\theta_+ = \alpha \therefore \theta_+ = \frac{\alpha}{1-2\alpha},$$

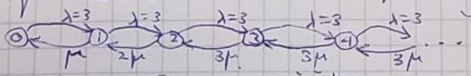
$$2(1-2\alpha)\theta_- = 1 - \alpha - (3\alpha - 1) = 1 - \alpha - 3\alpha + 1 = 2 - 4\alpha = 2(1-2\alpha) \therefore$$

$$(1-2\alpha)\theta_- = 1 - 2\alpha \therefore \theta_- = 1 \therefore$$

for $\alpha < \frac{1}{3}$, θ_+ is minimal root; $\theta_+ = \frac{\alpha}{1-2\alpha}$

\2ai/ 

M/M/3 queue with infinite capacity with $\lambda = 3$, $\mu > 0$

\2ai/ 

M/M/3 queue with infinite capacity

\2aii/ $\frac{dP_0}{dt} = -3P_0 + \mu P_1 = 0$,

$$\frac{dP_1}{dt} = -\mu P_1 + 3P_0 - 3P_1 + 2\mu P_2 = 0 = -3P_1 + 2\mu P_2$$

$$\frac{dP_2}{dt} = 3P_1 - 2\mu P_2 - 3P_2 + 3\mu P_3 = 0 = -3P_2 + 3\mu P_3$$

$$\frac{dP_3}{dt} = 3P_2 - 3\mu P_3 - 3P_3 + 3\mu P_4 = 0 = -3P_3 + 3\mu P_4$$

$$\frac{dP_4}{dt} = 3P_3 - 3\mu P_4 - 3P_4 + 3\mu P_5 = 0 = -3P_4 + 3\mu P_4 \therefore$$

$P_2 = \frac{1}{2}P_2 + \frac{1}{2}P_2 + \frac{1}{4}(0) = P_2 = P_2$,

$1 = P_1 + P_2 + P_3 = P_1 + P_2 = 1 \therefore$ $1 - P_2 = P_1 \therefore$

$1 - P_2 = \frac{1}{2}P_2 \therefore 1 = \frac{3}{2}P_2 \therefore$ $\cancel{\frac{2}{3}P}$ $\frac{2}{3} = P_2 \therefore$

$1 - \frac{2}{3} = P_1 = \frac{1}{3} \therefore$

$\tilde{p} = (\frac{1}{3}, \frac{2}{3}, 0)$

\1ciii/ Steady state $\tilde{p} = (P_1 \; P_2 \; P_3) = \tilde{p}T = (P_1 \; P_2 \; P_3)\begin{bmatrix} 0 & 1 & 0 \\ 1/2 & 1/2 & 0 \\ 0 & 1/4 & 3/4 \end{bmatrix}$

$= (\frac{1}{2}P_2 \quad P_1 + \frac{1}{2}P_2 + \frac{1}{4}P_3 \quad \frac{3}{4}P_3) \therefore$

$\cancel{P_3}$ $\cancel{\frac{1}{2}}$ $P_1 = \frac{1}{2}P_2$, $P_2 = P_1 + \frac{1}{2}P_2 + \frac{1}{4}P_3$, $P_3 = \frac{3}{4}P_3 \therefore$

$\frac{1}{4}P_3 = 0 \therefore P_3 = 0 \therefore$

$1 = P_1 + P_2 + P_3 = P_1 + P_2 \therefore$ $1 - P_2 = P_1 \therefore$

$P_1 = 1 - P_2 = \frac{1}{2}P_2 \therefore$ $1 = \frac{3}{2}P_2 \therefore \frac{2}{3} = P_2 \therefore$

$\frac{1}{2} \times P_2 = \frac{1}{2} \times \frac{2}{3} = P_1 = \frac{1}{3} \therefore$

$\tilde{p} = (\frac{1}{3}, \frac{2}{3}, 0)$

\1di/ valid if $G_X(1) = 1 = \alpha + \alpha(1) + (1-2\alpha)(1)^2 = \alpha + \alpha + 1 - 2\alpha = 2\alpha + 1 - 2\alpha = 1$

$P(S_0 = 0) = \alpha$, $P(S_0 = 1) = \alpha$, $P(S_0 = 2) = 1 - 2\alpha \therefore$

$\alpha \geq 0$, $1 - 2\alpha \geq 0 \therefore 1 \geq 2\alpha \therefore \frac{1}{2} \geq \alpha \therefore$ $\alpha \in [0, \frac{1}{2}]$

\1di/ $P(S_0 = 2) = 1 - 2\alpha \therefore$ valid if: $P(S_0 = 0) = \alpha \geq 0$,

$1 - 2\alpha \geq 0 \therefore \frac{1}{2} \geq \alpha \therefore \alpha \in [0, \frac{1}{2}]$

\1dii/ $E(X) = G_X'(1) = G_X'(\theta)|_{\theta=1} = \alpha + 2(1-2\alpha)\theta|_{\theta=1} = \alpha + (2 - 4\alpha)1 = 2 - 3\alpha \therefore$

$E(S_n) = E(X)]^n = (2 - 3\alpha)^n$

\1dii/ $E(X) = G_X'(1) = G_X'(\theta)|_{\theta=1} = \alpha + 2(1-2\alpha)\theta|_{\theta=1} = \alpha + 2(1-2\alpha)1 =$

$\alpha + 2 - 4\alpha = 2 - 3\alpha \therefore$

$E(S_n) = (E(X))^n = (2 - 3\alpha)^n$

\1diii/ Guaranteed extinction if $G_X'(1) \leq 1 \therefore 2 - 3\alpha \leq 1 \therefore$

$3\alpha \geq 1 \therefore \alpha \geq \frac{1}{3}$

\1diii/ guaranteed extinction, $e = 1$ if $G_X'(1) \leq 1 \therefore 2 - 3\alpha \leq 1 \therefore$

$\frac{1}{3} \leq \alpha$

1bi $\frac{0.5}{0.5+0.1+0.4} = \frac{\lambda_Y}{\lambda_Y+\lambda_R+\lambda_G} = \frac{0.5}{1} = 0.5 = P(\text{next bus is Yellow})$

1bii $\frac{1}{10}$ hours $= 0.1$ hours $= 0.1 \times 60$ Minutes $= 6$ Minutes

1biii $\therefore \lambda_G = 0.4 \times 10 = 4$ per hour $\therefore \lambda_G t \sim Poi \therefore$

$G \sim \frac{e^{-\lambda t}(\lambda t)^x}{x!} \therefore$ 30 Minutes $= 0.5$ hours $\therefore$

$P(N_G(t=0.5)=2) = e^{-4\times0.5}(4\times0.5)^2 \frac{1}{2!} = 0.271$ (3 S.F.)

1biv Poisson observations are time independent $\therefore$

$\lambda_Y = 0.5\times10 = 5$ per hour $\therefore$

$Y \sim e^{-\lambda_Y t}(\lambda_Y t)^x \frac{1}{x!} \therefore$

P(3rd Yellow bus in 30 Minutes to 1 hour) =

$P(\text{Yellow bus in 30 Minutes}) = P(N_Y(t=0.5)=1) = e^{-5\times0.5}(5\times0.5)^1 \frac{1}{1!}$

$= 0.205$ (3 S.F.)

1bv 15 minutes $= \frac{1}{4}$ hour $\therefore$

$P(N_G(t=\frac{1}{4})=2) + P(N_G(\frac{1}{4})=3) + P(N_G(\frac{1}{4})=4) + P(N_G(\frac{1}{4})=5) + P(N_G(\frac{1}{4})=6)$

$= e^{-4\times\frac{1}{4}}\left[(4\times\tfrac{1}{4})^2\frac{1}{2!} + (4\times\tfrac{1}{4})^3\frac{1}{3!} + (4\times\tfrac{1}{4})^4\frac{1}{4!} + (4\times\tfrac{1}{4})^5\frac{1}{5!} + (4\times\tfrac{1}{4})^6\frac{1}{6!}\right] =$

$e^{-1}\left[1^2\frac{1}{2!} + 1^3\frac{1}{3!} + 1\frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!}\right] = 0.264$ (3 S.F.)

1biv sol/ cause gamma distribution $\therefore (P(\theta))_{\text{seg}} \int_{\frac{1}{2}}^{1} g_3(x)\,dx$

or $P(\text{3rd yellow} \in (\frac{1}{2},1)) = P(\text{3 yellow} \le 1 \text{ hour}) - P(\text{3rd yellow} \le \frac{1}{2} \text{ hour}) =$

$P(N_Y(1)\ge3) - P(N_Y(\frac{1}{2})\ge3) = (1-P(N_Y(1)\le2)) - (1-P(N_Y(\frac{1}{2})\le2)) =$

$P(N_Y(\frac{1}{2})\le2) - P(N_Y(1)\le2) =$

$e^{-5\times\frac{1}{2}}\left[(5\times\tfrac{1}{2})^0\frac{1}{0!} + (5\times\tfrac{1}{2})^1\frac{1}{1!} + (5\times\tfrac{1}{2})^2\frac{1}{2!}\right] - e^{-5\times1}\left[(5\times1)^0\frac{1}{0!} + (5\times1)^1\frac{1}{1!} + (5\times1)^2\frac{1}{2!}\right] =$

$e^{-\frac{5}{2}}\left(1+\frac{5}{2}+(\tfrac{5}{2})^2\tfrac{1}{2}\right] - e^{-5}(1+5+5^2\cdot\tfrac{1}{2}) = 0.4191$

1bv if N arrivals in time T then for a time $s<T$, N: of arrivals is given by Binomial $B(N,\frac{t}{T})$ distribution $\therefore Bin(6,\frac{1}{4})$

$P(X=2) = \binom{6}{2}(\tfrac{1}{4})^2(\tfrac{1}{4})^{6-2} = \left(\frac{6!}{2!(6-2)!}\right)(\tfrac{1}{4})^2(\tfrac{1}{4})^4 = 0.297$

\PP2015/

$X \in \{0, 2, 4\}$ ∴

\1ai/ $(X-2)^2 = X^2+4-4X$ ∴

$Y=(X-2)^2 \in \{4, 0, 4\}$ ∴

$\cancel{E(X^2)}$ ∴ $E(Y) = E(X^2)+4-4E(X)$ ∴ $P(Y=0)=P(X=2)=0.2$

$E(X) = 0.5\times(0)+0.2(2)+0.3(4) = 1.6$

$E(X^2) = 0.5\times(0)+0.2\times 2^2+0.3(4^2) = 5.6$ ∴

$E(Y) = 5.6+4-4\times 1.6 = 3.2$

$P(Y=4) = P(X^2+4-4X=4) = P(X^2-4X=0) = P(X(X-4)=0)$

$P(X=0)+P(X=4) = 0.5+0.3 = 0.8$

\1aii/ $E(Y) = G_Y'(\theta)|_{\theta=1} = \frac{d}{d\theta}(G_X(G_X(\theta)))|_{\theta=1} =$

$G_X'(G_X(\theta))G_X'(\theta)|_{\theta=1}$ ∴

$G_X'(\theta) = 0.4\theta+1.2\theta^3$ ∴

$G_X'(\theta)|_{\theta=1} = 0.4(1)+1.2(1)^3 = 1.6$ ∴

$E(Y) = 1.6\,G_X'(G_X(\theta))|_{\theta=1}$ ∴ $G_X(\theta)|_{\theta=1} = 0.5+0.2(1)^2+0.3(1)^4 = 1$ ∴

$E(Y) = 1.6\,G_X'(1) = 1.6\times 1.6 = 2.56$

∴ $G_Y(\theta) = G_X(G_X(\theta)) = G_X(0.5+0.2\theta^2+0.3\theta^4) =$

$0.5+0.2(0.5+0.2\theta^2+0.3\theta^4)^2+0.3(0.5+0.2\theta^2+0.3\theta^4)^4 =$

$0.2\left[(0.2\theta^2)^2\frac{2!}{2!}+(0.5)^1(0.3\theta^4)^1\frac{2!}{1!1!}\right]+0.3\left[(0.5)^3(0.3\theta^4)^1\frac{4!}{3!1!}+(0.5)^2(0.2\theta^2)^2\frac{4!}{2!2!}\right]$

$+\dots$

$=0.2\left[0.04\theta^4+0.3\theta^4\right]+0.3\left[0.15\theta^4+0.06\theta^4\right]+\dots = 0.131\theta^4+\dots$

$P(Y=4) = 0.131$

$\cancel{E(Z)}$ \1aiii/ $E(Y) = E(X_1)+E(X_2) = E(X)+E(X) = 2E(X)$ ∴

$E(X) = 1.6$ ∴ $E(Y) = 2\times 1.6 = 3.2$

$G_Y(\theta) = G_{X_1+X_2}(\theta) = G_{X_1}(\theta)G_{X_2}(\theta) = G_X(\theta)G_X(\theta) = (G_X(\theta))^2 =$

$(0.5+0.2\theta^2+0.3\theta^4)^2 = (0.2\theta^2)^2\frac{2!}{2!}+(0.3\theta^4)^1(0.5)^1\frac{2!}{1!\times 1!}$

$(0.04+0.3)\theta^4 = 0.34\theta^4+\dots$ ∴ $P(Y) = 0.34$

\1aii/ $G_Y(\theta) = G_X(G_X(\theta)) = G_X(0.5+0.2\theta^2+0.3\theta^4) =$

$0.5+0.2(0.5+0.2\theta^2+0.3\theta^4)^2+0.3(0.5+0.2\theta^2+0.3\theta^4)^4+\dots =$

$\left[0.2\left[0.2^2+0.3^1\times 0.5^1\times\frac{2!}{1!1!}\right]+0.3\left(0.5^3\times 0.3^1\times\frac{4!}{3!\times 1!}\right)\right]\theta^4 = 0.113\theta^4+\dots$ ∴ $P(Y=4) = 0.113$

\2aii/ $N \sim Poi(3)$ ∴ $g_N(n) = \frac{e^{-3}3^n}{n!}$ ∴

$G_N(\theta) = e^{3(\theta-1)}$ ∴

$G_Y = G_N(G_X(\theta)) = G_N(0.5+0.5\theta^2) = e^{3(0.5+0.5\theta^2-1)} = e^{3(0.5\theta^2-0.5)}$

$= e^{1.5\theta^2-1.5} = e^{-1.5}e^{1.5\theta^2} = e^{-1.5}\sum_{k=0}^{\infty}\frac{(1.5\theta^2)^k}{k!} =$

$e^{-1.5}\left[\frac{(1.5\theta^2)^0}{0!} + \frac{(1.5\theta^2)^1}{1!} + \dots\right] =$

$e^{-1.5}\left[1+\frac{1.5}{1}\theta^2+\dots\right] = e^{-1.5}+1.5e^{-1.5}\theta^2+\dots$

∴ $P(Y=2) = \cancel{\theta}\, 1.5e^{-1.5} = 0.335$ (3 s.f.) X $0.251 = P(Y=4)$

∴ $G_N(\theta)=e^{3(\theta-1)}$, $G_X(\theta)=0.5+0.5\theta^2$ ∴

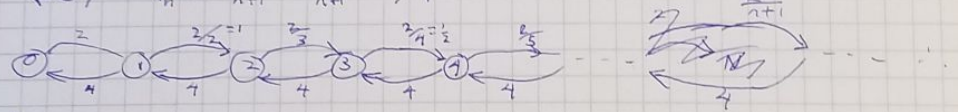
$G_Y(\theta) = G_N(G_X(\theta)) = e^{3(G_X(\theta)-1)} = e^{3(0.5+0.5\theta^2-1)} =$

$e^{3(-0.5+0.5\theta^2)} = e^{-1.5+1.5\theta^2} = e^{-1.5}e^{1.5\theta^2} = e^{-1.5}\sum_{k=0}^{\infty}\frac{(1.5\theta^2)^k}{k!} =$

$e^{-1.5}\left[\frac{(1.5\theta^2)^2}{2!}\right]+\dots = e^{-1.5}\frac{1.5^2}{2}\theta^4+\dots = \frac{9}{8}e^{-1.5}\theta^4+\dots$ ∴

$P(Y=4)=\frac{9}{8}e^{-1.5} = 0.251$ (3 s.f.)

\2b/ $\lambda_n = 2\frac{1}{n+1} = \frac{2}{n+1}$, $\mu = 4$ ∴



$\sum_{n=0}^{\infty} P_n = 0$, steady state.

\1cii/ P(3 bus in $\frac{1}{3}$ hr to $\frac{1}{2}$ hr) =

P(3rd bus in $\frac{1}{2}$ hr) − P(3rd bus in $\frac{1}{3}$ hr) =

$P(B\geq 3$ in $\frac{1}{2}$ hr$) - P(B \geq 3$ in $\frac{1}{3}$ hr$) =$

$1 - P(B<3$ in $\frac{1}{2}$ hr$) - (\cancel{1} - P(B<3$ in $\frac{1}{3}$ hr$)) =$

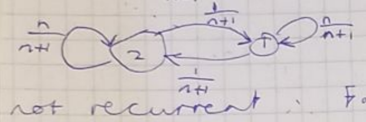
$P(B<3$ in $\frac{1}{3}$ hr$) - P(B<3$ in $\frac{1}{2}$ hr$) = P(B\leq 2$ in $\frac{1}{3}$ hr$) - P(B\leq 2$ in $\frac{1}{2}$ hr$) =$

$e^{-6(1/3)}\left[(6\times\frac{1}{3})^0\frac{1}{0!} + (6\times\frac{1}{3})^1\frac{1}{1!} + (6\times\frac{1}{3})^2\frac{1}{2!}\right] - e^{-6\times\frac{1}{2}}\left[(6\times\frac{1}{2})^0\frac{1}{0!}+(6\times\frac{1}{3})^1\frac{1}{1!}+(6\times\frac{1}{3})^2\frac{1}{2!}\right] =$

$0.676676 - 0.42319 = 0.253$ (3 s.f.) X 0.146 but correct method

\PP2017/ \1ei/ recurrent if it is sure to return to itself eventually ∴ $f_i^{(n)}$ is prob of first return ∴

if $f_i = \sum_{n=1}^{\infty} f_i^{(n)} = 1$ is recurrent ∴



∴ ① is an absorbing state but is not recurrent ∴ False X

an absorbing state is always recurrent

\1cii/ periodic state is period $k>1$ $(T^n)_{ii} = 0$ for $n = k, 2k, 3k, \dots$

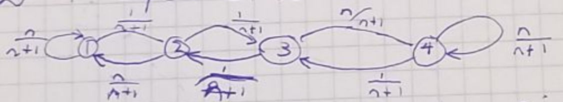
but $(T^n)_{ii} > 0$ $\forall n = k, 2k, 3k$ ∴ ~~Fal~~ True ∴

for $m \in \mathbb{N}$ ∴ $\forall n = mk$ ∴ for $\lim_{n\to\infty} (T^n)_{ii} > 0$ $\forall n = mk$ as $m \to 0$ X

False ∴ a periodic state is not always ~~transient~~ recurrent

\1eiii/ False because a recurrent state can be null recurrent ∴ expected time to first return can be infinite

\1eiv/



∴ True ∴

①, ④ are recurrent, ②, ③ are transient ∴

for prob of first return $f_i^{(n)}$: $\sum_{n=1}^{\infty} f_i = f_i < 1$ for $i = 2, 3$

\2ai/ $G_Y(\theta) = G_{X_1+X_2+\dots+X_N}(\theta) = G_{X_1}(\theta) G_{X_2}(\theta) \cdots G_{X_N}(\theta)$ by independence

$= \underbrace{G_{X}(\theta) G_{X}(\theta) \cdots G_X(\theta)}_{n \text{ times}} = (G_X(\theta))^N$ by independence

$G_Y(\theta) = E_Y(\theta^Y) = E_N[E_Y(\theta^Y | N)] = \sum_{k=0}^{\infty} E_Y(\theta^Z | N=k)] P(N=k) =$

$\sum_{k=0}^{\infty} E_Y[\theta^{X_1} \theta^{X_2} \cdots \theta^{X_k}] P(N=k) = \sum_{k=0}^{\infty} E_X(\theta^{X_1}) E_X(\theta^{X_2}) \cdots E_X(\theta^{X_k}) P(N=k) =$

$\sum_{k=0}^{\infty} (G_X(\theta))^k P(N=k) = G_N(G_X(\theta))$

$G_Y(\theta) = E(\theta^Y) = E_N[E(\theta^Y | N)] = \sum_{k=0}^{\infty} E(\theta^Y | N=k) P(N=k) =$

$\sum_{k=0}^{\infty} E(\theta^{X_1 X_2 X_3 \cdots X_N}) ~~P(N=k))~~ P(N=k) = \sum_{k=0}^{\infty} E(\theta^{X_1 X_2 \cdots X_k}) P(N=k) =$

$\sum_{k=0}^{\infty} E(\theta^{X_1} \theta^{X_2} \cdots \theta^{X_k}) P(N=k) = \sum_{k=0}^{\infty} E(\theta^{X_1}) E(\theta^{X_2}) \cdots E(\theta^{X_k}) P(N=k)$ by independence

$= \sum_{k=0}^{\infty} \underbrace{E(\theta^X) E(\theta^X) \cdots E(\theta^X)}_{k \text{ times}} P(N=k) = \sum_{k=0}^{\infty} (E(\theta^X))^k P(N=k) = \sum_{k=0}^{\infty} (G_X(\theta))^k P(N=k)$

$= G_N(G_X(\theta))$

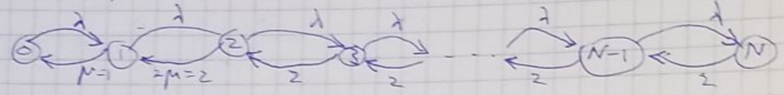
\1Cii/ $P(N(t=\frac{30}{60})=2) \mid (N(t=\frac{20}{60})=1)) = \frac{P(N(t=\frac{1}{2})=2), (N(t=\frac{1}{3})=1))}{P(N(t=\frac{1}{3})=1)}$

$P(N(t=\frac{20}{60})=1) \mid N(t=\frac{30}{60})=2) =$

$P(N(t=\frac{1}{3})=1, N(t=\frac{1}{2})=2) / P(N(t=\frac{1}{2})=2) =$

$P(N(t=\frac{1}{3}=1, N(t=\frac{1}{6})=1 / P(N(t=\frac{1}{2})=2) =$

$$e^{-6\times\frac{1}{3}}(6\times\tfrac{1}{3})^1\frac{1}{1!} \times e^{-\frac{1}{6}\times 6}(6\times\tfrac{1}{6})^1\frac{1}{1!}\frac{1}{e^{-(6\times\frac{1}{2})}(6\times\frac{1}{2})^2\frac{1}{2!}} = \frac{4}{9} = 0.444 \times 0.146$$

\1d/ 

$\sum_{n=0}^{N} P_n = 1$, $\frac{dP_0}{dt} = -\lambda P_0 + 1P_1 = 0$ for steady state $\therefore$

$P_0 = P_1$ $\therefore$ $\frac{dP_1}{dt} = -\lambda P_1 + 2P_2 - (-\lambda P_0 + P_1) = 0 = -\lambda P_1 + 2P_2$ $\therefore$

$+\frac{1}{2}\lambda P_1 = P_2 = \frac{1}{2}\lambda P_0$ $\therefore$

$\frac{dP_2}{dt} = -\lambda P_2 + 2P_3 - (-\lambda P_1 + 2P_2) = 0 = -\lambda P_2 + 2P_3$ $\therefore$

$P_3 = \frac{1}{2}\lambda P_2 = \frac{1}{2}\lambda\frac{\lambda}{2}P_0 = (\frac{1}{2})^2\lambda^2 P_0 = (\frac{\lambda}{2})^2 P_0$ $\therefore$

for $n \geq 2$ $P_n = (\frac{\lambda}{2})^{n-1} P_0$ $\therefore$

$$\sum_{n=0}^{N} P_n = 1 = P_0 + P_1 + \sum_{n=2}^{N} P_n = P_0 + P_0 + \sum_{n=2}^{N}\left(\frac{\lambda}{2}\right)^{n-1} P_0 = 2P_0 + P_0\sum_{n=1}^{N-1}\left(\frac{\lambda}{2}\right)^n =$$

$$P_0\left(\frac{1-(\frac{\lambda}{2})^N}{1-\frac{\lambda}{2}} - 1\right) + 2P_0 = P_0\left(\frac{1-(\frac{\lambda}{2})^N}{1-\frac{\lambda}{2}} + 1\right) = 1 = P_0\left(\frac{1-(\frac{\lambda}{2})^N + 1 - \frac{\lambda}{2}}{1-\frac{\lambda}{2}}\right) =$$

$$\left(\frac{2-\frac{\lambda}{2}-(\frac{\lambda}{2})^N}{1-\frac{\lambda}{2}}\right)P_0 = 1 \therefore P_0 = \frac{1-\frac{\lambda}{2}}{2-\frac{\lambda}{2}-(\frac{\lambda}{2})^N} = P_1 = \frac{2-\lambda}{4-\lambda-2(\frac{\lambda}{2})^N}$$

$$P_n = \left(\frac{\lambda}{2}\right)^{n-1}\frac{2-\lambda}{4-\lambda-2(\frac{\lambda}{2})^N} \text{ for } n \geq 2$$

for $\lambda = 2$: $\lim_{\lambda\to 2} P_0 = \lim_{\lambda\to 2}\frac{1-\frac{\lambda}{2}}{2-\frac{\lambda}{2}-(\frac{\lambda}{2})^N} =$

$\therefore \frac{N+0}{2} = \frac{N}{2} = 1 P_{\frac{N}{2}}$, $P_n = 0$ for $n \neq \frac{N}{2}$

$\therefore$ let $\rho = \frac{\lambda}{2\mu} = \frac{\lambda}{2(1)} = \frac{\lambda}{2}$ $\therefore$ $P_0 = \frac{1-\rho}{1+\rho-2\rho^{N+1}} = \frac{1-\frac{\lambda}{2}}{1+\frac{\lambda}{2}-2(\frac{\lambda}{2})^{N+1}} =$

$\frac{1-\frac{\lambda}{2}}{1+\frac{\lambda}{2}-2(\frac{\lambda}{2})^N(\frac{\lambda}{2})}$, $P_n = 2\rho^n P_0 = 2(\frac{\lambda}{2})^n P_0$

if $\rho = 1$: $P_0 = \frac{1}{2N+1}$, etc

\PP2017/ \1biii/ $G_X(e) - e = 0 = (2-e)^{-2} - e = \frac{1}{(2-e)^2} - \frac{e(2-e)^2}{(2-e)^2} = 0$

$= 1 - e(2-e)^2 = 0 = -1 + e(4 + e^2 - 4e) =$

$\Rightarrow e^3 - 4e^2 + \cancel{4}e - 1 = (e-1)(e^2 + 1 - 3e) = e^3 + e - 3e^2 - e^2 - 1 + 3e =$

$e^3 - 4e^2 + \cancel{4}e - 1 \therefore$

$e_3 = 1,\ e^2 + 1 - 3e = e^2 - 3e + 1 = 0 \therefore$

$e_\pm = \frac{3 \pm \sqrt{9 - 4(1)(1)}}{2(1)} = \frac{3 \pm \sqrt{5}}{2}$, $e_- = \frac{3-\sqrt{5}}{2} = 0.382$, $e_+ = \frac{3+\sqrt{5}}{2} = 2.62$

$\therefore e = e_- = 0.382$ (3 s.f.) $= \frac{3-\sqrt{5}}{2}$

\1biv/ for (i): $5 \times 8 = 40$

for (ii) $P(S_2 = 0) = 0.327$

$\cancel{P\neq}$ $e = \frac{3-\sqrt{5}}{2} = 0.382$ X

\1biv/ new mean $= 5 \times 8 = 40$

new $e_2 = (\text{old } e_2)^5 = (0.327)^5 = 0.00374$ (3 s.f.)

new $e = (\text{old } e)^5 = (0.382)^5 = 0.00813$ (3 s.f.)

\1ci/ $\lambda = 6$ per hour $\therefore$ $\frac{1}{\lambda} = \frac{\text{hour}}{6} = 60\text{mins}/6 =$

$= 10$ mins per bus $\therefore$

E(time for 3 buses) $= 3 \times 10 = 30$ minutes

\1cii/ $P(N(t) = n) = \frac{e^{-\lambda t}(\lambda t)^n}{n!}$ $\because$ poisson processes independent $\therefore$

$P(N(t = \tfrac{1}{6}) = 1) = e^{-6 \times \frac{1}{6}} (6 \times \tfrac{1}{6})^1 \frac{1}{1!} = e^{-1} (1)^1 1 = e^{-1} = 0.368$ (3 s.f.) X

\1ciii/ $P((N(t = \tfrac{1}{3}) = 3) \mid (N(t=1) = 5)) = \frac{P(N(t=\frac{1}{3})=3),\ (N(t=1)=5))}{P(N(t=1)=5)} =$

$P(N(t = \tfrac{1}{3}) = 3)\ P(N(t=1)=5) / P(N(t=1)=5) =$

$P(N(t = \tfrac{1}{3}) = 3) = e^{-6(\frac{1}{3})} (6 \times \tfrac{1}{3})^3 \frac{1}{3!} = e^{-2}\, 2^3\, \frac{1}{6} = 0.180$ (3 s.f.) X

\1ciii/ use Binomial $\sim Bin(5, \frac{20}{60}) = Bin(5, \frac{1}{3}) \therefore$

$\binom{5}{3} (\tfrac{1}{3})^3 (1 - \tfrac{1}{3})^{5-3} = \frac{5!}{3!(5-3)!} (\tfrac{1}{3})^3 (1 - \tfrac{1}{3})^2 = \frac{40}{243} = 0.165$ (3 s.f.)

\1cii/ P(2nd bus arrives in 20 mins to 30) =

P(2nd bus by 30 mins) − P(2nd bus by 20 mins) =

$P(N(t = \tfrac{30}{60}) = 2) - P(N(t = \tfrac{20}{60}) = 2) = P(N(t = \tfrac{1}{2}) = 2) - P(N(t = \tfrac{1}{3}) = 2) =$

$e^{-6 \times \frac{1}{2}} (6 \times \tfrac{1}{2})^2 \frac{1}{2!} - e^{-6 \times \frac{1}{3}} (6 \times \tfrac{1}{3})^2 \frac{1}{2!} = 0.224 - 0.271 = -0.0466$ X 0.146

1ai $E(Y)=E(X^2+1)=E(X^2)+E(1)=E(X^2)+1$

$E(X)=G_X'(1)=G_X'(\theta)\big|_{\theta=1}=2(1)e^{2(\theta-1)}\big|_{\theta=1}=2e^{2(1-1)}=2e^{2(0)}=2e^0=2(1)=2$ ∴

$E(X(X-1))=E(X^2-X)=E(X^2)-E(X)=G''(1)=G''(\theta)\big|_{\theta=1}=$

$2(2(1))e^{2(\theta-1)}\big|_{\theta=1}=4e^{2(\theta-1)}\big|_{\theta=1}=4e^{2(1-1)}=4e^{2(0)}=4e^0=4(1)=4$

$E(X^2)=4+E(X)=4+2=6$ ∴

$E(Y)=6+1=7$ ✓

$P(Y=2)=P((X^2+1)=2)=P(X^2=1)=P(X=1)$

$\therefore G_X(\theta)=e^{2\theta-2}=e^{-2}e^{2\theta}=e^{-2}\sum_{n=0}^{\infty}\frac{(2\theta)^n}{n!}=e^{-2}\left[\frac{(2\theta)^0}{0!}+\frac{(2\theta)^1}{1!}+\dots\right]=$

$e^{-2}\left[\frac{1}{1}+(2\theta)^1+\dots\right]=e^{-2}\theta^0+2e^{-2}\theta^1+\dots$ ∴ ✓

$P(Y=2)=P(X=1)=2e^{-2}=0.271$ (3S.F.) ✓

1aii $E(Y)=E(X_1+X_2)=E(X_1)+E(X_2)=E(X)+E(X)=2E(X)=2(2)=4$

$G_Y(\theta)=G_{X_1+X_2}(\theta)=G_{X_1}(\theta)G_{X_2}(\theta)=G_{X}(\theta)G_X(\theta)=(G_X(\theta))^2=$

$(e^{2(\theta-1)})^2=e^{2\cdot2(\theta-1)}=e^{4(\theta-1)}=e^{-4}e^{4\theta}=e^{-4}\sum_{n=0}^{\infty}\frac{(4\theta)^n}{n!}=e^{-4}\left[\frac{(4\theta)^0}{0!}+\frac{(4\theta)^1}{1!}+\frac{(4\theta)^2}{2!}\dots\right]$

$=e^{-4}\left[\frac{1}{1}+4\theta+\frac{4^2\theta^2}{2}+\dots\right]=e^{-4}\left[1\theta^0+4\theta^1+8\theta^2+\dots\right]=e^{-4}\theta^0+4e^{-4}\theta+8e^{-4}\theta^2+\dots$ ✓

$\therefore P(Y=2)=8e^{-4}=0.147$ (3S.F.)

1bi $E(X)=G_X'(\theta)\big|_{\theta=1}=-2(2-\theta)^{-3}(-1)\big|_{\theta=1}=2(2-1)^{-3}=2(1)^{-3}=2$ ∴

$E(S_3)=2^3=8$ ✓

1bii $G_X(G_X(\theta))\big|_{\theta=0}=(2-(2-\theta)^{-2})^{-2}\big|_{\theta=0}=(2-(2-0)^{-2})^{-2}=$

$(2-2^{-2})^{-2}=(2-0.25)^{-2}=1.75^{-2}=\frac{16}{49}=P(S_2=0)=0.327$ (3S.F.)

1biii $E(X)=2>1$ ∴ $e<1$ ∴

$G_X(\theta)-\theta=(2-\theta)^{-2}-\theta=\frac{1}{(2-\theta)^2}-\theta=\frac{1}{(2-\theta)^2}-\frac{\theta(2-\theta)^2}{(2-\theta)^2}=\frac{1-\theta(4+\theta^2-4\theta)}{(2-\theta)^2}=$

$\frac{1-4\theta-\theta^3+4\theta^2}{(2-\theta)^2}=0=-\theta^3+4\theta^2-4\theta+1=0=\theta^3+4\theta^2+4\theta-1$

$=(\theta-1)(\theta^2+1+5\theta$

$G_X(\theta)-\theta=0=(2-\theta)^{-2}-\theta=\frac{1}{(2-\theta)^2}-\frac{\theta(2-\theta)^2}{(2-\theta)^2}=0=1-\theta(2-\theta)^2=0=$

$1+\theta(4+\theta^2-4\theta)=\theta^3-4\theta^2+4\theta-1=(\theta-1)(\theta^2+1-3\theta)=0$ ∴

$\theta_\pm=\frac{-1\pm\sqrt{1-4(1)(-3)}}{2(1)}=\frac{-1\pm\sqrt{13}}{2}$ ∴ $e>0$ ∴

$e=\frac{-1+\sqrt{13}}{2}=1.30$ ✗ $e=\frac{3-\sqrt{5}}{2}$

\pp2017/ 1ai/ $E(Y) = E(X^2) + 1$,

$G_X''(\theta)|_{\theta=1} = E(X^2) - E(X)$ ∴

$G_X'(\theta) = 2e^{2(\theta-1)}$ ∴ $G'' = 4e^{2(\theta-1)}$

∴ $G'(\theta)|_{\theta=1} = 2e^{2(1-1)} = 2e^{2(0)} = 2$,

$G''(1) = 4e^{2(1-1)} = 4$ ∴

$4 + 2 = E(X^2) = 6$, ∴ $E(Y) = 6 + 1 = 7$

$P(Y=2) = P(X^2+1=2) = P(X^2=1) = P(X=1)$

∴ $G_X(\theta) = e^{-2}e^{2\theta} = e^{-2}\sum_{n=0}^{\infty}\frac{1}{n!}(2\theta)^n = e^{-2}\frac{1}{1!}2\theta + \dots = 2e^{-2}\theta + \dots$

∴ $P(Y=2) = 2e^{-2} = 0.271$

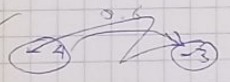
\1aii/ $G_Y(\theta) = G_{X_1+X_2}(\theta) = G_{X_1}(\theta)G_{X_2}(\theta) = [G_X(\theta)]^2 = (e^{2(\theta-1)})^2 =$

$e^{4(\theta-1)} = e^{-4}e^{4\theta}$ ∴

$G_Y'(\theta)|_{\theta=1} = e^{-4}4e^{4\theta}|_{\theta=1} = e^{-4}4e^{4} = 4$

$P(Y=2)$ ∴ $G_Y(\theta) = e^{-4}\sum_{n=0}^{\infty}\frac{1}{n!}(4\theta)^n = e^{-4}\frac{1}{2!}4^2\theta^2 + \dots = 8e^{-4}\theta^2 + \dots$

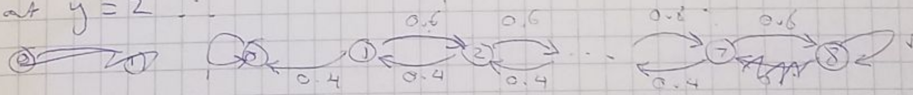
$P(Y=2) = 8e^{-4} = 0.147$

\4ai/ 

$x \in \{-4, -3, \dots, 3, 4\}$ ∴

let $y = x + 4$ ∴ $y \in \{0, 1, \dots, 7, 8\}$ ∴

start at $y = 2$ ∴



$\theta_i = A\lambda^i$ ∴ $\theta_i = 0.6\theta_{i+1} + 0.4\theta_{i-1}$ ∴

$0.6\theta_{i+1} - \theta_i + 0.4\theta_{i-1} = 0$ ∴

$0.6A\lambda^{i+1} - A\lambda^i + 0.4A\lambda^{i-1} = A\lambda^{i-1}(0.6\lambda^2 - \lambda + 0.4) = 0$ ∴

$0.6\lambda^2 - \lambda + 0.4 = 0$ $\lambda = 1$,

$\lambda = \frac{1 - \sqrt{1 - 4(0.6)0.4}}{2(0.6)} = \frac{2}{3}$ ∴

$\theta_i = A + B\left(\frac{2}{3}\right)^i$ ∴ $\theta_0 = 0$, $\theta_8 = 1$ ∴

\pp2021/\4aiii/ Stopping time given by $D_i = 1 + \sum_j r_{ij} D_{ij}$ $\therefore$

$D_i = 1 + qD_{i+1} + 3qD_i + pD_{i-1}$ $\therefore$

$\therefore\ 0 = 1 + qD_{i+1} + (3q-1)D_i + pD_{i-1}$ $\therefore$

let $D_i = k\lambda^i$ $\therefore$ let for homogeneous part. D_i^h solve it $\therefore$

for $0 = qD_{i+1} + (3q-1)D_i + pD_{i-1}$ $\therefore$

$0 = qD_{i+1}^h + (3q-1)D_i^h + pD_{i-1}^h$

let $D_i^h = k\lambda^i$ $\therefore$

$0 = qk\lambda^{i+1} + (3q-1)k\lambda^i + pk\lambda^{i-1} = k\lambda^{i-1}(q\lambda^2 + (3q-1)\lambda + p) = 0$ $\therefore$

$\lambda_\pm = [-(3q-1) \pm ((3q-1)^2 - 4qp)^{1/2}]/(2q)$ $\therefore$ $q + 3q + p = 1$ $\therefore p = 1 - 4q$ $\therefore$

$\lambda_+ = 1$, $\lambda_- = \frac{p}{q} = \rho$ $\therefore$ $D_i^h = A(1)^i + B\rho^i = A + B\rho^i$ $\therefore$

$\therefore$ inhomogeneous part: $-1 = qD_{i+1} + (3q-1)D_i + pD_{i-1}$ $\therefore$

let $D_i^p = a + bi + ci^2$ $\therefore$

$-1 = aq + bq(i+1) + cq(i+1)^2 + 3aq + 3q + 3bqi + 3cqi^2 - a - bi - ci^2 + ap + bp(i-1) + cp(i-1)^2 =$

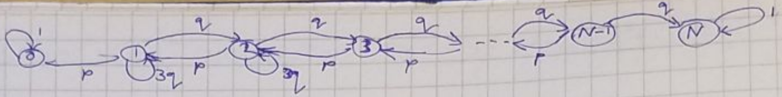
$-1 = aq + bqi + bq + cqi^2 + 2cqi + cq + 3aq + 3bqi + 3cqi^2 - a - bi - ci^2 + a(1-4q) +$

$b(1-4q)i - b(1-4q) + c(1-4q)i^2 - 2c(1-4q)i + c(1-4q) =$

$-1 = 2cqi + cq - b + 4bq - 2ci + 8cqi + c - 4cq$ $\therefore$

coefficients:

i^0: $-1 = bq + cq - b + 4bq + c - 4cq = b(5q-1) + c(1-3q)$

\4ai/ 

\4aii/ have $\theta_0=0$, $\theta_N=1$, $\theta_i=q\theta_{i+1}+p\theta_{i-1}+3q\theta_i$ ∴

$q+p+3q=1=4q+p$ ∴ $p=1-4q$ ∴

$\theta_i=q\theta_{i+1}+(1-4q)\theta_{i-1}+3q\theta_i$ ∴

$0=q\theta_{i+1}+(1-4q)\theta_{i-1}+(3q-1)\theta_i$ ∴

Ansatz ∴ let $\theta_i=k\lambda^i$ ∴ $\theta_{i+1}=k\lambda^{i+1}$, $\theta_{i-1}=k\lambda^{i-1}$ ∴

$\theta_i=k\lambda\lambda^{i-1}$, $\theta_{i+1}=k\lambda^2\lambda^{i-1}$ ∴

$0=qk\lambda^{i+1}+(3q-1)k\lambda^i+(1-4q)k\lambda^{i-1}=k\lambda^{i-1}(q\lambda^2+(3q-1)\lambda+(1-4q))$ ∴

Solve for λ: $\lambda_\pm=(1-3q\pm((3q-1)^2-4q(1-4q))^{1/2})/(2q)=$

$[1-3q\pm(9q^2-6q+1-4q+16q^2)^{1/2}]/(2q)=$

$[1-3q\pm\sqrt{25q^2-10q+1}]/(2q)=[1-3q\pm\sqrt{(5q-1)^2}]/(2q)=$

$[1-3q\pm(5q-1)]/(2q)$ ∴

$\lambda_+=\frac{1-3q+5q-1}{2q}=\frac{2q}{2q}=1$, $\lambda_-=\frac{1-3q-(5q-1)}{2q}=\frac{1-3q-5q+1}{2q}=$

$\frac{2-8q}{2q}=\frac{1-4q}{q}=\frac{p}{q}$ ∴

$\theta_i=A\lambda_+^i+B\lambda_-^i=A(1)^i+B(\frac{p}{q})^i=A+B(\frac{p}{q})^i=A+B\rho^i$, $\rho=\frac{p}{q}$ ∴

$\theta_0=0=A+B\rho^0=A+B=0$ ∴ $A=-B$ ∴

$\theta_i=-B+B\rho^i=B(-1+\rho^i)$ ∴

$\theta_N=1=B(-1+\rho^N)$ ∴ $B=\frac{1}{-1+\rho^N}=\frac{1}{\rho^N-1}$ ∴

$\theta_i=\frac{1}{\rho^N-1}(-1+\rho^i)=\frac{1}{\rho^N-1}(\rho^i-1)=\frac{\rho^i-1}{\rho^N-1}=\frac{1-\rho^i}{1-\rho^N}$; $i=1,\dots,N$ ∴

let E_n denote the number of England wins, A_n denote the number of Australians wins, $Z_n=E_n-A_n$

Z_n has values in $\{-5,-4,\dots,4,5\}$ ∴

let $U_n=Z_n+5$ ∴ $U_n\in\{0,1,\dots,10\}$ ∴

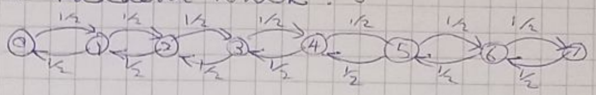
England wins is $U_n=10$ ∴

P_E for start at $U_n=5$ ∴ $P_E=\frac{1-\rho^5}{1-\rho^{10}}=\frac{1-\rho^5}{(1+\rho^5)(1-\rho^5)}=\frac{1}{1+\rho^5}=$

$\frac{1}{1+\frac{p^5}{q^5}}=\frac{q^5}{q^5+p^5}=\frac{q^5}{p^5+q^5}$

\PP2015/
\4a/ forms a random walk ∴

let $U = \cancel{A}$



Start at S ∴ θ_i is Pr(winning from state i) ∴

$\theta_i = \frac{1}{2}\theta_{i-1} + \frac{1}{2}\theta_{i+1}$ ∴

let $\theta_i = A\lambda^i$ ∴ $\frac{1}{2}\theta_{i-1} + \frac{1}{2}\theta_{i+1} - \theta_i = 0$ ∴

$\frac{1}{2}A\lambda^{i-1} + \frac{1}{2}A\lambda^{i+1} - A\lambda^i = A\lambda^{i-1}(\frac{1}{2} + \frac{1}{2}\lambda^2 + \lambda) = 0$ ∴

$\frac{1}{2}\lambda^2 + \lambda + \frac{1}{2} = 0 = \lambda^2 + 2\lambda + 1 = (\lambda+1)(\lambda+1) = 0$ ∴

$\lambda = -1$

$\theta_i = A$

\4a Sol/ $X_n = N^\circ$ Heads, $T_n = N^\circ$ of tails ∴

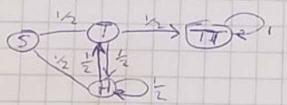
let $W_n = Y_n - X_n$ ∴ $\{-2, -1, 0, 1, \ldots, 5\}$, $W_0 = 0$ ∴

let $U_n = W_n + 2 \in \{0, 1, 2, \ldots, 7\}$ ∴ $U_0 = 2$ ∴ random walk ∴

∴ Game stops is reach state 7, starting at 2 ∴

$\sum_{n=0}^{N} P_n = 1 = \sum_{n=0}^{7} P_n = 1$ probability is $\frac{i}{N} = \frac{2}{7}$ (state i, destination N)

\4b/



$D_i = \sum_j T_{ij} D_j + 1$ ∴

$D_{TT} = 0$ ∴ $D_T = \frac{1}{2}D_{TT} + \frac{1}{2}D_H + 1 = \frac{1}{2}D_H + 1 = D_T$

$D_S = \frac{1}{2}D_T + \frac{1}{2}D_H + 1$ ∴ $D_H = 1 + \frac{1}{2}D_H + \frac{1}{2}D_T = 1 + \frac{1}{2}D_H + \frac{1}{4}D_H + \frac{1}{2} = \frac{3}{2} + \frac{3}{4}D_H$

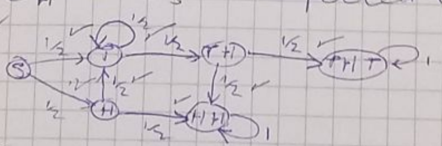
∴ $\frac{1}{4}D_H = \frac{3}{2}$ ∴ $D_H = 6$

∴ $\frac{1}{2}(6) + 1 = D_T = 4$ ∴

$D_S = 1 + \frac{1}{2}(4) + \frac{1}{2}(6) = 6$

$D_T = 2$, $D_H = 4 = D_S$ ∴ expected time = 4

\4c/



$\theta_i = P$(winning from start i)

$\theta_{THT} = 1$, $\theta_{HH} = 0$, $\theta_S = \frac{1}{2}\theta_T + \frac{1}{2}\theta_H$,

$\theta_T = \frac{1}{2}\theta_{TH} + \frac{1}{2}\theta_T$ ∴ $\frac{1}{2}\theta_T = \frac{1}{2}\theta_{TH}$ ∴ $\theta_T = \theta_{TH}$

$\theta_{TH} = \frac{1}{2}\theta_{HH} + \frac{1}{2}\theta_{THT} = \frac{1}{2}(0) + \frac{1}{2}(1) = \frac{1}{2}$ ∴ $\theta_T = \frac{1}{2}$ ∴

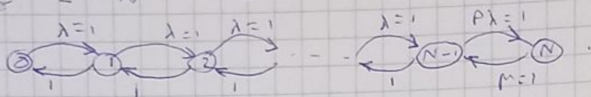
$\theta_H = \frac{1}{2}\theta_T + \frac{1}{2}\theta_{HH} = \frac{1}{2}\theta_T = \frac{1}{4}$ ∴ $\theta_S = \frac{1}{2}(\frac{1}{2}) + \frac{1}{2}(\frac{1}{4}) = \frac{3}{8}$

$$W_s = L_s \frac{1}{\lambda_{eff}} = \frac{\lambda - (N+1)\lambda^{N+1} + N\lambda^{N+2}}{(1-\lambda)(1-\lambda)^{N+1}\lambda} \frac{1}{\lambda} \frac{1-\lambda^{N+1}}{1-\lambda^N}$$

for $\lambda = 1$: $L_s = \frac{N}{2}$, $\lambda_{eff} = \frac{N}{N+1}$ $\therefore$

$$W_s = L_s \frac{1}{\lambda_{eff}} = \frac{N}{2} \frac{N+1}{N} = \frac{N+1}{2}$$

\3d /



$\frac{dP_0}{dt} = -\lambda P_0 + P_1 = 0 \therefore \lambda P_0 = P_1$

$\frac{dP_1}{dt} = \lambda P_0 - P_1 - \lambda P_1 + P_2 = 0 = -\lambda P_1 + P_2 \quad \therefore P_2 = \lambda P_1 = \lambda^2 P_0 \therefore$

$P_n = \lambda^n P_0$, $n \leq N-2 \therefore \sum_{n=0}^{N} P_n = 1$, $\lambda P_{N-2} = P_{N-1} \therefore$

$\frac{dP_{N-1}}{dt} = \lambda P_{N-2} - P_{N-1} - \rho\lambda P_{N-1} + P_N = 0 = -\rho\lambda P_{N-1} + P_N \therefore P_N = \rho\lambda P_{N-1} = \rho\lambda\lambda^{N-1} P_0$

$\therefore \sum_{n=0}^{N} P_n = \sum_{n=0}^{N-1} P_n + P_N = 1 = \sum_{n=0}^{N-1} \lambda^n P_0 + \rho\lambda P_{N-1} =$

$P_0 \frac{1-\lambda^N}{1-\lambda} + \rho\lambda\lambda^{N-1} P_0 = P_0 \left(\frac{1-\lambda^N}{1-\lambda} + \rho\lambda^N\right) = 1 = P_0 \left(\frac{1-\lambda^N}{1-\lambda} + \frac{\rho\lambda^N - \rho\lambda^{N+1}}{1-\lambda}\right) =$

$P_0 \left(\frac{1+(\rho-1)\lambda^N - \rho\lambda^{N+1}}{1-\lambda}\right) = 1 \quad \therefore P_0 = \frac{1-\lambda}{1+(\rho-1)\lambda^N - \rho\lambda^{N+1}} \therefore$

$P_n = \lambda^n \frac{1-\lambda}{1+(\rho-1)\lambda^N - \rho\lambda^{N+1}}$, $n < N$,

$$P_N = \rho\lambda^N \frac{1-\lambda}{1+(\rho-1)\lambda^N \rho\lambda^{N+1}}$$

\3d / $\lambda = 1$:

$\therefore P_0 = P_n \ \forall n \leq N-1$, $\rho P_{N-1} = P_N \therefore$

$\sum P_i = 1 = \sum_{n=0}^{N-1} P_n + \rho P_{N-1} = \sum_{n=0}^{N-1} P_0 + \rho P_{N-1} = \sum_{n=0}^{N-1} P_0 + \rho P_0 = 1 = (N)P_0 + \rho P_0 =$

$P_0 (N+\rho) = 1 \therefore \quad P_0 = \frac{1}{N+\rho} = P_n, n \neq N, P_N = \rho \frac{1}{1+\rho}$

$L_s = \sum_{n=0}^{N} nP_n = \sum_{n=0}^{N-1} nP_n + N\rho \frac{1}{1+\rho} = \sum_{n=0}^{N-1} n\left(\frac{1}{N+\rho}\right) + N\rho \frac{1}{1+\rho} = \frac{1}{2}(N-1)N \frac{1}{N+\rho} + \frac{N\rho}{1+\rho} =$

$\frac{N(N-1+2\rho)}{2(N+\rho)}$

$\lambda_{eff} = (P_0 + \dots + P_{N-2}) + \rho P_{N-1} + 0 P_N \quad \therefore \sum_{n=0}^{N} P_n = 1 = P_0 + P_1 + \dots + P_{N-2} + P_{N-1} + P_N$

$\lambda_{eff} = (1 - P_{N-1} - P_{N-2}) + \rho P_{N-1} = 1 - \frac{(1-\rho)}{N+\rho} - \frac{1}{N+\rho} = \frac{N+\rho+\rho-2}{N+\rho} = \frac{N+2\rho-2}{N+\rho}$

$\therefore W_s = \frac{L_s}{\lambda_{eff}} = \frac{N(N-1+2\rho)}{2(N+2\rho-2)}$

\1a/ if $\lambda = 1$: $P_n = P_0$ ∴ $\sum_{n=0}^{N} P_n = 1 = \sum_{n=0}^{N} P_0 = \cancel{NP_0} \sum_{n=1}^{N+1} P_0 = (N+1)P_0$ ∴

$P_0 = P_n = \frac{1}{N+1}$

\1b/ when $\lambda = 1$: $L_s = \sum_{n=0}^{N} \lambda P_n = \sum_{n=0}^{N} n \frac{1}{N+1} = \frac{1}{N+1} \sum_{n=0}^{N} n =$

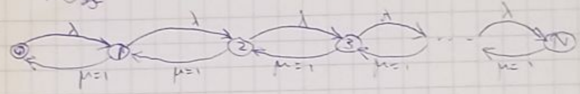
$0 + \frac{1}{N+1} \sum_{n=1}^{N} n = \frac{1}{N+1} \frac{1}{2}N(N+1) = \frac{1}{2}N$ ✓ , $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$

\1c/

$\lambda_{eff} = \sum_{n=0}^{N} \lambda_n P_n = \sum_{n=0}^{N-1} \lambda_n P_n$ ∴ if $\lambda = 1$:

$\lambda_{eff} = \sum_{n=0}^{N-1} \lambda_n P_n = \sum_{n=0}^{N-1} 1 P_n = \sum_{n=0}^{N-1} P_0 = P_0 \sum_{n=0}^{N-1} 1 = P_0 \sum_{n=1}^{N} 1 = P_0 N = \frac{N}{N+1}$ ✓

∴ $W_s = \frac{L_s}{\lambda_{eff}} = \frac{1}{2}N \frac{N+1}{N} = \frac{1}{2}(N+1) = \frac{1}{2}N + \frac{1}{2}$ for $\lambda = 1$

\3a/ 

$\sum_{n=0}^{N} P_n = 1$ ∴

$\frac{dP_0}{dt} = -\lambda P_0 + \mu P_1 = -\lambda P_0 + \mu P_1 = -\lambda P_0 + P_1 = 0$ ∴ $\lambda P_0 = P_1$ ∴

$\frac{dP_1}{dt} = \lambda P_0 - \mu P_1 - \lambda P_1 + P_2 = 0 = -\lambda P_1 + P_2$ ∴ $P_2 - \lambda P_1 = \lambda^2 P_0$ ∴

$P_n = \lambda^n P_0$ ∴ $\sum_{n=0}^{N} P_n = 1 = \sum_{n=0}^{N} \lambda^n P_0 = P_0 \sum_{n=0}^{N} \lambda^n = P_0 \frac{1-\lambda^{N+1}}{1-\lambda}$ ∴

$\frac{1-\lambda}{1-\lambda^{N+1}} = P_0$ ∴ $P_n = \lambda^n P_0 = \lambda^n \frac{1-\lambda}{1-\lambda^{N+1}}$ ∴

for $\lambda = 1$: $P_0 = P_n$ ∴ $\sum_{n=0}^{N} P_n = 1 = \sum_{n=0}^{N} P_0 = P_0 \sum_{n=1}^{N+1} 1 = (N+1)P_0$ ∴ $\frac{1}{N+1} = P_0$

\3b/ $G_X(\theta) = \sum_{n=0}^{N} \theta^n P_n = \sum_{n=0}^{N} \theta^n \lambda^n P_0 = \cancel{P_0 \sum_{n=0}^{N} (\lambda\theta} \; P_0 \sum_{n=0}^{N} (\lambda\theta)^n = P_0 \frac{1-(\lambda\theta)^{N+1}}{1-\lambda\theta}$

$= \frac{1-\lambda}{1-\lambda^{N+1}} \frac{1-(\lambda\theta)^{N+1}}{1-\lambda\theta}$ ∴

$G_X'(\theta)\big|_{\theta=1} = P_0 \frac{d}{d\theta}\left(\frac{1-\lambda^{N+1}\theta^{N+1}}{1-\lambda\theta}\right)\Big|_{\theta=1} = P_0 \frac{(1-\lambda\theta)(\lambda^{N+1}(N+1)\theta^N) - (1-\lambda^{N+1}\theta^{N+1})(-\lambda)}{(1-\lambda\theta)^2}\Big|_{\theta=1}$

$\cancel{P_0 \frac{1-\lambda(\lambda^{N+1}(N+1)) + (1-\lambda^{N+1})\lambda}{(1-\lambda)^2} = \frac{1-(N+1)\lambda^{N+2} + \lambda - \lambda^{N+2}}{(1-\lambda)(1-\lambda)^{N+1}}} = P_0 \frac{(1-\lambda)(N+1)\lambda^{N+1} + (1-\lambda^{N+1})\lambda}{(1-\lambda)^2}$

$= \frac{\lambda - (N+1)\lambda^{N+1} + N\lambda^{N+2}}{(1-\lambda)(1-\lambda)^{N+1}} = L_s = E(X)$

when $\lambda = 1$: $L_s = \sum_{n=0}^{N} \lambda P_n = \sum_{n=0}^{N} n \frac{1}{N+1} = \frac{1}{N+1} \sum_{n=0}^{N} n = 0 + \frac{1}{N+1} \sum_{n=1}^{N} n = \frac{1}{N+1} \frac{1}{2}N(N+1) = \frac{N}{2}$

\1c/ $\lambda_{eff} = \sum_{n=0}^{N} \lambda_n P_n = \sum_{n=0}^{N-1} \lambda_n P_n + 0P_N = \sum_{n=0}^{N-1} \lambda_n P_0 = \cancel{N} \sum_{n=0}^{N-1} P_0 = NP_0 = \frac{N}{N+1}$

∴ for $\lambda = 1$: $\lambda_{eff} = \frac{N}{N+1}$

∴ $\lambda_{eff} = \sum_{n=0}^{N} \lambda_n P_n = \sum_{n=0}^{N-1} \lambda P_n + 0P_N = \lambda \sum_{n=0}^{N-1} P_n = \lambda(1-P_N) = \lambda\left(1 - \frac{1-\lambda}{1-\lambda^{N+1}}\lambda^N\right) = \lambda\left(\frac{1-\lambda^N}{1-\lambda^{N+1}}\right)$

2e ∴ let $Y \sim B(n,p) = B(4, 0.5)$ ∴ $G_Y(\theta) = (0.5+0.5\theta)^4$ ∴

∴ $E(Y) = 4(0.5+0.5\theta)^3(0.5)\big|_{\theta=1} = 4(1)^3 0.5 = 2$ ∴

$2\lambda^n$ = would be expected number at generation $n \neq E(S_n)$

∴ $E(S_1) = 2\lambda$ ∴ $e<1$ if $2\lambda > 1$ ∴ $\lambda > \frac{1}{2}$,

$\tilde{e}_3$ = probability of extinction at time 3 $= G_Y(e_3) = G_Y(0.193) =$

$(0.5+0.5\times 0.193)^4 = 0.127$ (3 S.F.)

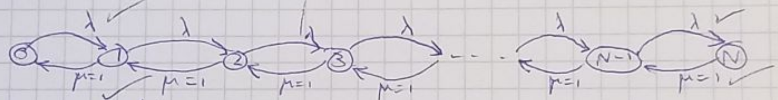
2e let $Y \sim Bin(n, p) = B(4, 0.5)$ ∴ $G_Y(\theta) = (0.5+0.5\theta)^4$ ∴

$E(Y) = 4(0.5+0.5\theta)^3 0.5\big|_{\theta=1} = 4(1)^3 0.5 = 2$ ∴

$E(S_n) = E(Y)E(S_{X_n}) = 2\lambda^n$ ∴

$E(S_1) = 2\lambda$ ∴ $e<1$ if $2\lambda > 1$ ∴ $\lambda > 1$, ∴

$\tilde{e}_3 = G_Y(e_3) = (0.5+0.5e_3)^4 = (0.5+0.5\times 0.193)^4 = 0.127$ (3 S.F.)

3a 

$\sum_{n=0}^{N} P_n = 1$ ∴ $\frac{dP_0}{dt} = -\lambda P_0 + \mu P_1 = -\lambda P_0 + P_1 = 0$ ∴ $\lambda P_0 = P_1$ ∴

$\frac{dP_1}{dt} = \lambda P_0 - P_1 - \lambda P_1 + P_2 = 0 = -\lambda P_1 + P_2$ ∴ $P_2 = \lambda P_1 = \lambda^2 P_0$ ∴

$P_n = (\lambda)^n P_0$ ∴ $\sum_{n=0}^{N} P_n = 1 = \sum_{n=0}^{N} \lambda^n P_0 = P_0 \sum_{n=0}^{N} \lambda^n = P_0 \frac{1-(\lambda)^{N+1}}{1-\lambda}$ ∴

3b

$\cancel{P_N} \cancel{(\lambda)^n \frac{1-(\lambda)^N}{1-\lambda} P_0 = P_n}$ $\frac{1-\lambda}{1-\lambda^{N+1}} = P_0$ ∴ $P_n = \frac{1-\lambda}{1-\lambda^{N+1}}\lambda^n$ ∴

$E(x) = \sum_{n=0}^{N} nP_n = \sum_{n=0}^{N} \frac{1-\lambda}{1-\lambda^N}\lambda^n n = \frac{1-\lambda}{1-\lambda^N}\sum_{n=0}^{\infty} n\lambda^n = \frac{1-\lambda}{1-\lambda^N}\frac{\lambda}{(1-\lambda)^2}\lambda^N$

3b Mean of N^o in system: PGF:

$G_X(\theta) = \sum_{n=0}^{N}\theta^n P_n = \sum_{n=0}^{N}\theta^n \frac{1-\lambda}{1-\lambda^{N+1}}\lambda^n = \frac{1-\lambda}{1-\lambda^{N+1}}\sum_{n=0}^{N}(\lambda\theta)^n =$

$\frac{1-\lambda}{1-\lambda^{N+1}}\frac{1-(\lambda\theta)^{N+1}}{1-\lambda\theta}$ ∴

$G_X'(\theta)\big|_{\theta=1} = \cancel{\frac{1-\lambda}{1-\lambda}} \frac{1-\lambda}{1-\lambda^{N+1}}\frac{d}{d\theta}\left(\frac{1-(\lambda\theta)^{N+1}}{1-\lambda\theta}\right)\Big|_{\theta=1} = \frac{\lambda-(N+1)\lambda^{N+1}+N\lambda^{N+2}}{(1-\lambda)(1-\lambda)^{N+1}} = L_S$

3c $\lambda_{eff} = \sum_{n=0}^{N}\lambda_n P_n = \sum_{n=0}^{N}\lambda P_n = \lambda\sum_{n=0}^{N}P_n = \lambda(1) = \lambda$ ∴

$W_S = L_S\frac{1}{\lambda_{eff}} = L_S\frac{1}{\lambda} = \frac{\lambda-(N+1)\lambda^{N+1}+N\lambda^{N+2}}{(1-\lambda)(1-\lambda)^{N+1}\lambda}$ X $\lambda_N = 0$ ∴

$\lambda_{eff} = \sum_{n=0}^{N}\lambda_n P_n = \sum_{n=0}^{N-1}\lambda_n P_n - \lambda_N P_N = \sum_{n=0}^{N-1}\lambda P_n - (0)P_N = \lambda\sum_{n=0}^{N-1}P_n = \lambda(1-P_N) =$

$\lambda\left(1-\frac{1-\lambda}{1-\lambda^{N+1}}\lambda^N\right)$

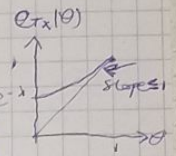
\PP2015/ $G_X(G_X(G_X(\theta))) = G_{S_3}(\theta) = G_X(e^{2e^{-2}-2}) = e^{2(2e^{-2}-2-1)} =$

$e^{2(2e^{-2}-3)} = e^{4e^{-2}-6} \simeq 4.26\times10^{-3} = 0.00426$ (3 S.F.) X

$e_0 = 0 \therefore \quad G_X(e_0) = G_X(0) = e_1 = e^{-2} \therefore$

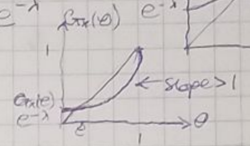
$e_2 = G_X(e_1) = \cancel{e^{-2}} = G_X(e^{-2}) = e^{2(e^{-2}-1)} = 0.177 \therefore$

$e_3 = G_X(e_2) = G_X(0.177) = e^{2(0.177-1)} = 0.193$ (3 S.F.)



\2e/ for $e=1 \therefore \lambda \leq 1 \therefore G_X(\theta) = e^{+\lambda(\theta-1)} = e^{-\lambda}$

for $e<1 \therefore \lambda > 1 \therefore G_X(\theta) = e^{-\lambda} \quad \therefore$



\2d/ $G_X(\theta) = e^{-\lambda}e^{\lambda\theta} = e^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda\theta)^n}{n!} =$

$e^{-\lambda}\left[\cancel{1+\lambda} \frac{(\lambda\theta)^0}{0!} + \frac{(\lambda\theta)^1}{1!} + \dots\right] = e^{-\lambda}\left[1+\dots+\lambda\theta\right] = e^{-\lambda} + \lambda e^{-\lambda}\theta + \dots \therefore$

$P(S_1=1) = \lambda e^{-\lambda}$ ✓

$G_X(G_X(\theta)) = G_X(e^{-\lambda} + \lambda e^{-\lambda}\theta + \dots) = e^{-\lambda}\left[\sum_{n=0}^{\infty}\frac{1}{n!}(e^{-\lambda} + \lambda e^{-\lambda}\theta)^n + \dots\right]$

$= e^{-\lambda}\left[\frac{1}{1!}(e^{-\lambda} + \lambda e^{-\lambda}\theta)^1 + \dots\right] =$

$e^{-\lambda}e^{\lambda[e^{-\lambda}+\lambda e^{-\lambda}\theta+\dots]}$ //

$G_X(\theta) = e^{\lambda(\theta-1)} \therefore G_X(G_X(\theta)) = e^{\lambda(\exp[\lambda(\theta-1)]-1)} = e^{-\lambda}e^{\lambda\exp[\lambda(\theta-1)]} =$

$e^{-\lambda}\sum_{n=0}^{\infty}\frac{1}{n!}(\lambda\exp[\lambda(\theta-1)])^n \cancel{= e^{-\lambda}}$

$= e^{-\lambda}e^{\lambda e^{\lambda(\theta-1)}} = e^{-\lambda}e^{\lambda e^{\lambda\theta-\lambda}} = e^{-\lambda}e^{\lambda e^{-\lambda}e^{\lambda\theta}} =$

$e^{-\lambda}e^{\lambda e^{-\lambda}(1+\lambda\theta+\dots)} = e^{-\lambda}e^{\lambda e^{-\lambda}+\lambda^2e^{-\lambda}\theta+\dots} =$

$e^{-\lambda}e^{\lambda e^{-\lambda}}e^{\lambda^2e^{-\lambda}\theta} = e^{-\lambda}e^{\lambda e^{-\lambda}}\left[\lambda^2e^{-\lambda}\theta+\dots\right] \therefore$

$P(S_2=1) = \cancel{e^{-\lambda}e^{\lambda e}}\ e^{-\lambda}e^{\lambda e^{-\lambda}}\lambda^2e^{-\lambda} = \lambda^2e^{-2\lambda}e^{\lambda e^{-\lambda}} \therefore$

$P(S_1=1) \cancel{=} \therefore G_X(\theta) = e^{\lambda(\theta-1)} = e^{+\lambda\theta-\lambda} = e^{-\lambda}e^{\lambda\theta} = e^{-\lambda}\left[1+\lambda\theta+\dots\right]$

$= e^{-\lambda} + \lambda e^{-\lambda}\theta + \dots \therefore P(S_1=1) = \lambda e^{-\lambda}$

$G_X(G_X(\theta)) = e^{-\lambda}e^{\lambda(e^{-\lambda}e^{\lambda\theta})} = e^{-\lambda}e^{\lambda e^{-\lambda}e^{\lambda\theta}} =$

$e^{-\lambda}e^{\lambda e^{-\lambda}[1+\lambda\theta+\dots]} = e^{-\lambda}e^{\lambda e^{-\lambda}+\lambda^2e^{-\lambda}\theta} = \cancel{e^{-\lambda}e^{\lambda}\exp}$

$e^{-\lambda}\exp[\lambda e^{-\lambda}]\exp[\lambda^2e^{-\lambda}\theta] = e^{-\lambda}\exp[\lambda e^{-\lambda}]\left[1+\lambda^2e^{-\lambda}\theta+\dots\right] =$

$e^{-\lambda}\exp[\lambda e^{-\lambda}]\lambda^2e^{-\lambda}\theta+\dots = \lambda^2e^{-2\lambda}\exp[\lambda e^{-\lambda}]\theta+\dots \therefore$

$\lambda^2e^{-2\lambda}\exp[\lambda e^{-\lambda}] = P(S_2=1)$

\1 d i C:/ then let $\sum_{n=1}^{\infty} f_i^{(n)} = 1$ then recurrent ∴ if
$\mu_i = \sum_{n=1}^{\infty} n f_i^{(n)} < \infty$ then state is positively recurrent

\1 d i D:/ let $\sum_{n=1}^{\infty} f_i^{(n)} = f_i = 1$ ∴ recurrent ∴
if $\sum_{n=1}^{\infty} n f_i^{(n)} = \mu_i = \infty$ then null recurrent state ;

\1 d ii /

∴ $\{1, 2\}$ is an irreducible subchain
ergodic subchain

for state 3: let probability for first return : ~~f_n~~ $f_3^{(n)}$ ∴

$f_3 = \sum_{n=1}^{\infty} f_3^{(n)} = \frac{1}{2} + \sum_{n=2}^{\infty} f_3^{(n)}$ ∴ $f_3^{(1)} = \frac{1}{2}$, $f_3^{(n)} = 0 \ \forall n \geq 2$ ∴

$f_3 = \frac{1}{2} + \sum_{n=2}^{\infty} 0 = \frac{1}{2} < \infty$ ∴ $\{3\}$ is transient

for state 2: $f_2^{(1)} = 0$, $f_2^{(2)} = 1 \times \frac{2}{3} = \frac{2}{3}$,

$f_2^{(3)} = 1 \times \frac{1}{3} \times \frac{2}{3} = \frac{1}{3} \times \frac{2}{3}$, $f_2^{(4)} = 1 \times \frac{1}{3} \times \frac{1}{3} \times \frac{2}{3} = (\frac{1}{3})^2 \times \frac{2}{3}$ ∴

for $n \geq 2$: $f_2^{(n)} = (\frac{1}{3})^{n-2} \times \frac{2}{3}$ ∴

$f_2 = \sum_{n=1}^{\infty} f_2^{(n)} = 0 + \sum_{n=2}^{\infty} (\frac{1}{3})^{n-2} (\frac{2}{3}) = \frac{2}{3} \sum_{n=2}^{\infty} (\frac{1}{3})^{n-2} = \frac{2}{3} \sum_{n=0}^{\infty} (\frac{1}{3})^n =$

$\frac{2}{3} \frac{1}{1-(\frac{1}{3})} = 1$, $n f_2^{(n)}$ bounded by an exponentially decaying term

~~f_2~~ $\mu_2 = \sum_{n=1}^{\infty} n f_2^{(n)} = \sum_{n=2}^{\infty} n (\frac{1}{3})^{n-2} (\frac{2}{3}) = \frac{2}{3} \sum_{n=2}^{\infty} n (\frac{1}{3})^n (\frac{1}{3})^{-2} =$

$6 \left[\sum_{n=2}^{\infty} n (\frac{1}{3})^n \right] = 6 \left[\sum_{n=0}^{\infty} n (\frac{1}{3})^n - (0)(\frac{1}{3})^0 - (1)(\frac{1}{3})^1 \right] = 6 \left[\frac{(1/3)}{(1-\frac{1}{3})^2} - \frac{1}{3} \right]$

$= \frac{5}{2} = 2.5 < \infty$ ∴ positively recurrent $\{2\}$ is

\2a/ $G_X(\theta) = e^{\lambda(\theta-1)} = e^{\lambda\theta} e^{-\lambda} = e^{-\lambda} e^{\lambda\theta}$ ∴

∴ $G_X'(\theta) = e^{-\lambda} \lambda e^{\lambda\theta}$ ∴ $G_X'(\theta) = \lambda e^{\lambda(\theta-1)}$ ∴

$G_X'(\theta)|_{\theta=1} = E(X) = \lambda e^{\lambda(1-1)} = \lambda e^0 = \lambda$ ∴

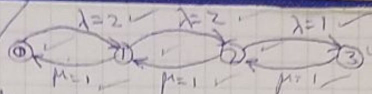
$E(S_n) = \lambda^n$ for generation n

$e = 1$ if $E(X) \leq 1$ ∴ $e = 1$ if $\lambda \leq 1$ ∴

$e < 1$ if $\lambda > 1$ e is ultimate extinction probability

\2b/ $G_X(\theta) = e^{-2} e^{2\theta} = e^{-2} \sum_{n=0}^{\infty} \frac{1}{n!} (2\theta)^n = e^{-2} \frac{1}{0!} (2\theta)^0$ ~~$= e^{-2}$~~ $+ \dots =$
$e^{-2} + \dots$ ∴ $G_X(G_X(\theta)) = e^{-2} + \dots$ X

$G_X(0) = e^{2(0-1)} = e^{-2}$ ∴ $G_X(G_X(0)) = e^{2(e^{-2}-1)} = e^{2e^{-2}-2}$ ∴

PP 2015 1Ci  M/M/1 queue with finite capacity

i) $\frac{dP_0}{dt} = -2P_0 + P_1 = 0 \quad \therefore 2P_0 = P_1$

$\frac{dP_1}{dt} = 2P_0 - P_1 - 2P_1 + P_2 = 0 = -2P_1 + P_2 \quad \therefore \quad 2P_1 = P_2 = 2(2P_0) = 2^2 P_0 = P_2 = 4P_0$

$\frac{dP_2}{dt} = 2P_1 - P_2 - 1P_2 + P_3 = -P_2 + P_3 = 0 \quad \therefore P_3 = P_2 = 4P_0$

$\frac{dP_3}{dt} = P_2 - P_3 = 0 \quad , P_0 + P_1 + P_2 + P_3 = 1$

$P_0 + 2P_0 + 4P_0 + 4P_0 = 1 = 11P_0 \quad \therefore \frac{1}{11} = P_0 \quad \therefore P_1 = \frac{2}{11}, P_2 = \frac{4}{11}, P_3 = \frac{4}{11}$

1Cii $E(X) = \sum_{n=0}^{3} nP_n = 0P_0 + 1P_1 + 2P_2 + 3P_3 = \frac{2}{11} + 2\left(\frac{4}{11}\right) + 3\left(\frac{4}{11}\right) = 2 = L_s$

$L_q = \sum_{n=1}^{3} (n-1)P_n = P_2 + 2P_3$

$L_q = \sum_{n\geq 1} (n-1)P_n = 0P_1 + 1P_2 + 2P_3 = 1\left(\frac{4}{11}\right) + 2\left(\frac{4}{11}\right) = \frac{12}{11}$

Cii $L_s = \sum_{n=0}^{3} nP_n = 0P_0 = 1P_1 + 2P_2 + 3P_3 = \frac{2}{11} + 2\times\frac{4}{11} + 3\times\frac{4}{11} = 2$

$L_q = \sum_{n=1}^{3} (n-1)P_0 = 0P_1 + 1P_2 + 2P_3 = \frac{4}{11} + 2\times\frac{4}{11} = \frac{12}{11}$

1Ciii $\lambda_{ess} = \sum_{n=0}^{3} \lambda_n P_n = 2P_0 + 2P_1 + 1P_2 + 0P_3 = 2\left(\frac{1}{11}\right) + 2\left(\frac{2}{11}\right) + \frac{4}{11} = \frac{10}{11} \therefore$

$W_q = L_q \frac{1}{\lambda_{ess}} = \frac{12}{11} \times \frac{1}{(10/11)} = \frac{6}{5}$ is expected waiting time in queue $= 1.2$

expected waiting time is system is $W_s = L_s \frac{1}{\lambda_{ess}} = 2 \times \frac{1}{(10/11)} = \frac{11}{5} = 2.2$

1di ~~A~~ An absorbing state is an irreducible subchain of one state. Absorbing is $T_{ii} = 1$, $T_{ij} = 0 \ \forall i \neq j$

A state $i \in E$ having $T_{ii} = 1$ and $T_{ij} = 0$ for $i \neq j$ ie once the system arrives at such state it can no longer escape from it

1diB: as $n \to \infty$: For state i is transient is:

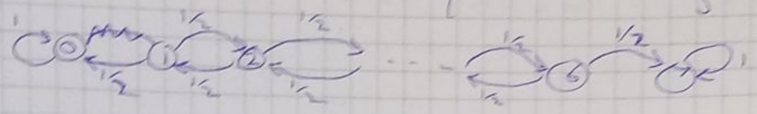
$T_{ji} \to 0 \quad \forall$ states $j \therefore$ is steps to get back to state j is n

~~then~~ is ~~f~~ ~~f_n~~ $f_n^{(j)}$ then $\sum_{n=1}^{\infty} f_n^{(j)} < 1$ then transient

let $f_i^{(n)}$ denote the probability of first return to state i then

) is $\sum_{n=1}^{\infty} f_i^{(n)} < 1$ then state i is transient

$\therefore \sum_{n=1}^{\infty} f_i^{(n)} = f_i$ is probability of eventual return let $U = Y_n + X_n \therefore \quad U \in \{0, 1, 2, \dots, 7\} \therefore$



let $\theta_i = P(\text{winning from state } i) \therefore$

$\theta_7 = 1 \quad , \theta_0 = 0 \therefore \quad$ starting at $i = 2 \therefore$

$\theta_{i+1} = \frac{1}{2}\theta_i + \frac{1}{2}\theta_{i+2} \quad \therefore \theta_i = \frac{1}{2}\theta_{i-1} + \frac{1}{2}\theta_{i+1} \therefore$

let $\theta_i = A\lambda^i \quad \therefore \theta_{i-1} = A\lambda^{i-1} \quad , \theta_{i+1} = A\lambda^{i+1} = A\lambda\lambda^{i} = A\lambda^2\lambda^{i-1}$

$A\lambda^i = A\lambda\lambda^{i-1} = \frac{1}{2}A\lambda^{i-1} + \frac{1}{2}A\lambda^2\lambda^{i-1} = A\lambda^{i-1}\left(\frac{1}{2} + \frac{1}{2}\lambda^2\right) \therefore$

$0 = A\lambda^{i-1}\left(\frac{1}{2} + \frac{1}{2}\lambda^2\right) - A\lambda^{i-1}\lambda = A\lambda^{i-1}\left(\frac{1}{2}\lambda^2 - \lambda + \frac{1}{2}\right) = 0 = \frac{1}{2}\lambda^2 - \lambda + \frac{1}{2} =$

$0 = \lambda^2 - 2\lambda + 1 = (\lambda - 1)(\lambda - 1) = (\lambda - 1)^2 \quad \therefore \lambda = 1 \therefore$

$\theta_i = A(1)^i + Ai(1)^i = A + Ai = A(1 + i) \therefore$

$\theta_0 = 0 = A(1 + i)$ ~~X~~ $\theta_7 = 1 = A(1 + 7) = A(8) \quad \therefore A = \frac{1}{8} \therefore \theta_i = \frac{1+i}{8}$

$\theta_i = \frac{1}{2}\theta_{i-1} + \frac{1}{2}\theta_{i+1} \therefore$

let $\theta_i = A\lambda^i \therefore \quad 0 = \frac{1}{2}\theta_{i+1} - \theta_i + \frac{1}{2}\theta_{i-1}$

$\sum_{n=0}^{N} P_n = 1 = \sum_{n=0}^{7} P_n = 1 =$

let $p = \frac{1}{2} \therefore \quad \lambda = 1, \rho = \frac{p}{q} = \frac{p}{1-p} \therefore$

$\theta_i = A(1)^i + B(\rho)^i = A + B\rho^i \therefore$

$\theta_0 = 0 = A + B\rho^0 = A + B = 0 \therefore -B = A \therefore$

$\theta_i = -B + B\rho^i = B(-1 + \rho^i) \therefore \quad \theta_7 = 1 = B(-1 + \rho^7) \quad \therefore B = \frac{1}{-1+\rho^7} \therefore$

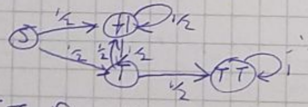
$\theta_i = \frac{-1+\rho^i}{-1+\rho^7} \therefore \quad \rho = \frac{p}{1-p} \quad \therefore \lim_{p \to \frac{1}{2}} \rho = \lim_{p \to \frac{1}{2}} \frac{p}{1-p} = \frac{1/2}{1-1/2} = \frac{1/2}{1/2} = 1 \therefore$

$\lim_{p \to \frac{1}{2}} \theta_i = \lim_{\rho \to 1} \theta_i = \lim_{\rho \to 1} \frac{-1+\rho^i}{-1+\rho^7} = \lim_{\rho \to 1}\left(\frac{i\rho^{i-1}}{7\rho^6}\right) = \frac{i(1)^{i-1}}{7(1)^6} = \frac{i(1)}{7(1)} = \frac{i}{7}$

$\therefore \theta_i = \frac{i}{7} \therefore \quad$ starting at $i = 2 \therefore$

$\theta_2 = \frac{2}{7} = P(Y_n = X_n + 5$ when tosses stopped$)$

\46/



$\therefore$ Expected tosses to win from i:

$D_i = 1 + \sum_j T_{ij} D_j \therefore \quad D_{TT} = 0 \therefore$

$D_T = 1 + \frac{1}{2}D_{TT} + \frac{1}{2}D_H = 1 + \frac{1}{2}D_H = D_T \qquad D_H = 1 + \frac{1}{2}D_H + \frac{1}{2}D_T = \frac{1}{1} + \frac{1}{2}D_H + \frac{1}{2} + \frac{1}{4}D_H$

$= \frac{3}{2} + \frac{3}{4}D_H = D_H \therefore \frac{1}{4}D_H = \frac{3}{2} \therefore D_H = 6 \therefore 1 + \frac{1}{2}(6) = D_T = 1 + 3 = 4 \therefore$

\PP2015/ $W_s = L_s \frac{1}{\lambda_{eff}} = \frac{\lambda-(N+1)\lambda^{N+1}+N\lambda^{N+2}}{(1-\lambda)(1-\lambda)^{N+1}\lambda} \frac{1}{\lambda}\left(\frac{1-\lambda^{N+1}}{1-\lambda^{N}}\right)$

$= \frac{-N\lambda^{N+1}+N\lambda^{N+2}-\lambda^{N+1}+\lambda}{(1-\lambda)^2} \frac{1-\lambda^{N+1}}{\lambda-\lambda^{N+1}} = \frac{\lambda-(N+1)\lambda^{N+1}+N\lambda^{N+2}}{(1-\lambda)(1-\lambda)^{N+1}\lambda} \frac{1-\lambda^{N+1}}{\lambda-\lambda^{N+1}}$

\3a/ for $\lambda = 1 = \mu$ steady state : $P_n = \frac{1}{N+1}$ ∴

$E(n)$ ∴ $G_n(\theta) = E(\theta^n) = \sum_{n=0}^{N} P_n\theta^n = \frac{1}{N+1}\sum_{n=0}^{N}\theta^n = \frac{1}{N+1}\frac{1-\theta^{N+1}}{1-\theta}$ ∴

$L_s = E(n) = G_n'(\theta)\big|_{\theta=1} = \frac{1}{N+1}\sum_{n=0}^{N} n\theta^{n-1}\big|_{\theta=1} = \frac{1}{N+1}\sum_{n=1}^{N} n\theta^{n-1}\big|_{\theta=1} =$

$\frac{1}{N+1}\sum_{n=1}^{N} n = \frac{1}{N+1}\frac{1}{2}N(N+1) = \frac{N}{2}$

$\lambda_{eff} = \sum_{n=0}^{N}\lambda_n P_n = \sum_{n=0}^{N-1} P_n\lambda_n = \sum_{n=0}^{N-1} P_n = \sum_{n=0}^{N-1}\frac{1}{N+1} = \frac{1}{N+1}\sum_{n=0}^{N-1} 1 = \frac{1}{N+1}N = \frac{N}{N+1}$ ∴

E(waiting in system) $= W_s = L_s\frac{1}{\lambda_{eff}} = \frac{N}{2}\frac{N+1}{N} = \frac{N+1}{2}$

$\lambda_{eff} = \sum_{n=0}^{N}\lambda_n P_n = \sum_{n=0}^{N-1} P_n\lambda_n = \sum_{n=0}^{N-1} P_n = \sum_{n=0}^{N} P_n - P_N = 1 - P_N = 1 - \frac{1}{N+1} = \frac{N+1-1}{N+1} = \frac{N}{N+1}$

\3d/ $\sum_{n=0}^{N} P_n = 1$ ∴ $P_1 = P_0 = P_2 = \dots P_{N-1}$ ∴ steady state :

$\sum_{n=0}^{N} P_n = 1 = \sum_{n=0}^{N-1} P_n + P_N = P_N + \sum_{n=0}^{N-1} P_0 = (N-1)P_0 + P_N$ ∴ $1-(N-1)P_0 = P_N$ ∴

$\frac{dP_N}{dt} = \rho P_{N-1} - P_N = 0$ ∴ $P_N = \rho P_{N-1} = \rho P_0 = 1-(N-1)P_0$ ∴

$\rho P_0 + (N-1)P_0 = P_0(\rho+N-1) = 1$ ∴ $P_0 = \frac{1}{\rho+N-1} = P_n$ for $n \le N-1$

$P_N = \rho P_{N-1} = \frac{\rho}{\rho+N-1}$ ∴

$L_s = E(\text{System Size}) = \sum_{n=0}^{N} nP_n = \sum_{n=0}^{N-1} nP_n + NP_N =$

$\sum_{n=0}^{N-1}\frac{1}{\rho+N-1}n + N\frac{\rho}{\rho+N-1} = \frac{1}{\rho+N-1}\sum_{n=0}^{N-1} n + \frac{N\rho}{\rho+N-1} = \frac{1}{\rho+N-1}\frac{1}{2}(N-1)N + \frac{N\rho}{\rho+N-1} =$

$\frac{\frac{1}{2}N^2 - \frac{1}{2}N + N\rho}{\rho+N-1} = \frac{N^2-N+2N\rho}{2\rho+2N-2}$ ∴

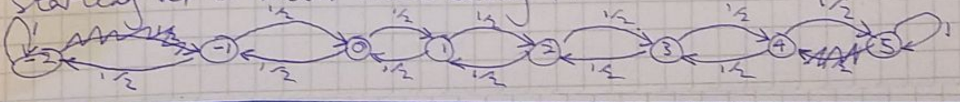
$\lambda_{eff} = \sum_{n=0}^{N}\lambda_n P_n = \sum_{n=0}^{N-1}\lambda_n P_n = \sum_{n=0}^{N-2}\lambda_n P_n + \lambda_{N-1}P_{N-1} = \sum_{n=0}^{N-2} P_n + \rho\frac{1}{\rho+N-1} =$

$\sum_{n=0}^{N-2}\frac{1}{\rho+N-1} + \frac{\rho}{\rho+N-1} = \frac{N-1}{\rho+N-1} + \frac{\rho}{\rho+N-1} = \frac{\rho+N-1}{\rho+N-1} = 1$ ∴

E(waiting time in system) $= W_s = L_s\frac{1}{\lambda_{eff}} = L_s\frac{1}{1} = L_s = \frac{N^2+2N\rho-N}{2N+2\rho-2}$

\4a/ it is a random walk $P(H) = P(T) = 0.5$ ∴

starting at 0 with Tails being +1 ∴



∴

\PP2015/ \2a/ $E(X) = G_X'(\theta)\big|_{\theta=1} = \frac{d}{d\theta} e^{\lambda(\theta-1)}\big|_{\theta=1} = \lambda e^{\lambda(\theta-1)}\big|_{\theta=1} =$

$\lambda e^{\lambda(1-1)} = \lambda e^0 = \lambda$ $\therefore$

$E(S_n) = \lambda^n$ $\therefore$

$e<1$ for $\lambda \geqslant 1$ e is ultimate extinction

\2b/ $P(S_3=0) = G_{S_3}(0) = G_X(G_X(G_X(\theta)))\big|_{\theta=0}$ $\therefore$

$G_X(0) = e^{\lambda(\theta-1)}\big|_{\theta=0} = e^{\lambda(-1)} = e^{-\lambda}$ $\therefore$

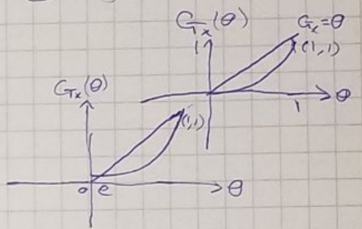
$P(S_3=0) = G_X(e^{\lambda(e^{-\lambda}-1)}) = G_X(e^{\lambda e^{-\lambda}-\lambda}) = G_X(e^{-\lambda}e^{\lambda e^{-\lambda}}) =$

$e^{\lambda(e^{-\lambda}e^{\lambda e^{-\lambda}}-1)} = e^{\lambda e^{-\lambda}e^{\lambda e^{-\lambda}}-\lambda}$

$\lambda=2$: $e^{2e^{-2}e^{2e^{-2}}-2} = 0.193$ (3.s.f.) $= e_3$

\2c/ for $e=1$:

for $e<1$:



\2d/ $G_X(\theta) = e^{\lambda(\theta-1)} = e^{\lambda\theta-\lambda} = e^{-\lambda}e^{\lambda\theta} =$

$e^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda\theta)^n}{n!} = e^{-\lambda}\left[\frac{(\lambda\theta)^1}{1!}\right] + \dots = e^{-\lambda}\lambda\theta^1 + \dots = \lambda e^{-\lambda}\theta + \dots$ $\therefore$

$P(S_1=1) = \lambda e^{-\lambda}$

$G_X(G_X(\theta)) = G_X(e^{-\lambda}e^{\lambda\theta}) = e^{-\lambda}e^{\lambda(e^{-\lambda}e^{\lambda\theta})} = e^{-\lambda}e^{\lambda e^{-\lambda}e^{\lambda\theta}} = \cancel{e^{-\lambda}}$

$\cancel{e^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda e^{-\lambda}e^{\lambda\theta})^n}{n!}}$ $e^{-\lambda}e^{\lambda e^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda\theta)^n}{n!}} = e^{-\lambda}e^{\lambda e^{-\lambda}(1+\lambda\theta+\frac{\lambda^2}{2}\theta^2+\dots)} =$

$e^{-\lambda}e^{\lambda e^{-\lambda}+\lambda^2e^{-\lambda}\theta+\frac{\lambda^3}{2}e^{-\lambda}\theta^2+\dots} =$

$e^{-\lambda}e^{\lambda e^{-\lambda}}e^{\lambda^2e^{-\lambda}\theta}e^{\frac{\lambda^3}{2}e^{-\lambda}\theta^2}(\dots) = e^{-\lambda}e^{\lambda e^{-\lambda}}e^{1+\lambda\theta+\frac{\lambda^2}{2}\theta^2+\dots} =$

$e^{-\lambda}e^{\lambda e^{-\lambda}}e^{1+\lambda\theta+\dots} = e^{-\lambda}e^{\lambda e^{-\lambda}}$

$\cancel{PR}$ $G_{S_2}(\theta) = e^{-\lambda}e^{\lambda e^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda\theta)^n}{n!}} = e^{-\lambda}e^{\lambda e^{-\lambda}(1+\lambda\theta)} = e^{-\lambda}e^{\lambda e^{-\lambda}+\lambda^2e^{-\lambda}\theta} =$

$(e^{-\lambda}e^{\lambda e^{-\lambda}})e^{\lambda^2e^{-\lambda}\theta} = (e^{-\lambda}e^{\lambda e^{-\lambda}})\sum_{n=0}^{\infty}\frac{(\lambda^2e^{-\lambda}\theta)^n}{n!} =$

$(e^{-\lambda}e^{\lambda e^{-\lambda}}\frac{1}{1!}\lambda^2e^{-\lambda}\theta + \dots = \lambda^2e^{-2\lambda}e^{\lambda e^{-\lambda}}\theta + \dots$ $\therefore$

$P(S_2=1) = \lambda^2e^{-2\lambda}e^{\lambda e^{-\lambda}}$

\2e/ EZ let $N \sim Bin(n,p) = Bin(4, 0.5)$ $\therefore$ $G_N(\theta) = (1-p+p\theta)^4 =$

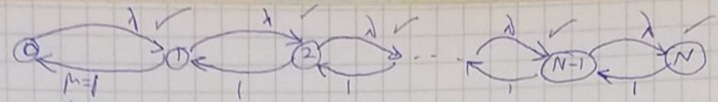
$(1-0.5+0.5\theta)^4 = (0.5+0.5\theta)^4$ $\therefore$

$E(\text{new}S_n) = G_N(E(\text{old}S_n)) = G_N(\lambda^n) = (0.5+0.5\lambda^n)^4$ X

$\text{new}e_3 = G_N(\text{old}e_3) = (0.5+0.5e_3)^4 = (0.5+0.5(0.193))^4 = 0.127$ (3s.f.)

\2e sol/ $E(\text{new}S_n) = E(N)E(\text{old}S_n) = 0.5\times4\times\lambda^n = 2\lambda^n$

range of λ for ultimate extinction is unchanged

\3a/ 

Steady State: $\sum_{n=0}^{N} P_n = 1 \therefore \quad \frac{dP_0}{dt} = +\mu P_1 - \lambda P_0 = 1P_1 - \lambda P_0 = 0 \therefore$

$P_1 = \lambda P_0 \quad \therefore \quad \frac{dP_1}{dt} = 1P_2 - \lambda P_1 - (1P_1 - \lambda P_0) = 0 = P_2 - \lambda P_1 \therefore$

$P_2 = \lambda P_1 = \lambda \lambda P_0 = \lambda^2 P_0 \therefore$

$P_n = \lambda^n P_0 \therefore$

$\sum_{n=0}^{N} \lambda^n P_0 = 1 = P_0 \sum_{n=0}^{N} \lambda^n = P_0 \frac{1-\lambda^{N+1}}{1-\lambda} \quad \therefore \quad \frac{1-\lambda}{1-\lambda^{N+1}} = P_0 \therefore$

$P_n = \lambda^n \frac{1-\lambda}{1-\lambda^{N+1}} \qquad n \leq N$

\3b/ mean number of customers in system $= E(n) = \sum_{n=0}^{N} n P_n =$

$\sum_{n=0}^{N} n \lambda^n \frac{1-\lambda}{1-\lambda^{N+1}} = \frac{1-\lambda}{1-\lambda^{N+1}} \sum_{n=0}^{N} n\lambda^n = \frac{1-\lambda}{1-\lambda^{N+1}} \frac{\lambda}{(1-\lambda)^2}$ ✗

\3b/ use PGF: $G_n(\theta) = E(\theta^n) = \sum_{n=0}^{N} P_n \theta^n =$

$\sum_{n=0}^{N} \lambda^n \frac{1-\lambda}{1-\lambda^{N+1}} \theta^n = \frac{1-\lambda}{1-\lambda^{N+1}} \sum_{n=0}^{N} \lambda^n \theta^n = \frac{1-\lambda}{1-\lambda^{N+1}} \sum_{n=0}^{N} (\lambda\theta)^n = \frac{1-\lambda}{1-\lambda^{N+1}} \frac{1-(\lambda\theta)^{N+1}}{1-\lambda\theta} \therefore$

mean customers $= E(n) = G_n'(\theta)\big|_{\theta=1} = \frac{d}{d\theta}\left[(1-(\lambda\theta)^{N+1})(1-\lambda\theta)^{-1}\right]\big|_{\theta=1} =$

$(-(N+1)(\lambda\theta)^N(\lambda))(1-\lambda\theta)^{-1} + (1-(\lambda\theta)^{N+1})(-1(1-\lambda\theta)^{-2}(-\lambda))\big|_{\theta=1} =$

$(-(N+1)\lambda^N \lambda)(1-\lambda)^{-1} + (1-\lambda^{N+1})(-1(1-\lambda)^{-2}(-\lambda)) =$

$\frac{-(N+1)\lambda^{N+1}}{1-\lambda} + \frac{\lambda(1-\lambda^{N+1})}{(1-\lambda)^{+2}} = \frac{+(N\lambda^{N+1}+\lambda^{N+1})(-1+\lambda) + \lambda - \lambda^{N+2}}{(1-\lambda)^2} =$

$(-N\lambda^{N+1} + N\lambda^{N+2} - \lambda^{N+1} + \lambda^{N+2} + \lambda - \lambda^{N+2})/(1-\lambda)^2 =$

$\frac{-N\lambda^{N+1} + N\lambda^{N+2} - \lambda^{N+1} + \lambda}{(1-\lambda)^2} = L_s$

\3c/ effective arrival rate $= \lambda_{eff} = \sum_{n=0}^{N} P_n \lambda_n = \sum_{n=0}^{N-1} P_n \lambda_n = \sum_{n=0}^{N-1} P_n \lambda =$

$\lambda \sum_{n=0}^{N-1} \lambda^n \frac{1-\lambda}{1-\lambda^{N+1}} = \frac{\lambda-\lambda^2}{1-\lambda^{N+1}} \frac{1-\lambda^N}{1-\lambda} = \frac{\lambda(1-\lambda^N)}{1-\lambda^{N+1}} = \frac{\lambda-\lambda^{N+1}}{1-\lambda^{N+1}} \therefore$

E(Waiting time of system) $= W_s = L_s \frac{1}{\lambda_{eff}} = \frac{-N\lambda^{N+1} + N\lambda^{N+2} - \lambda^{N+1} + \lambda}{(1-\lambda)^2} \frac{1-\lambda^{N+1}}{\lambda-\lambda^{N+1}}$

$\lambda_{eff} = \sum_{n=0}^{N-1} P_n \lambda = \lambda \sum_{n=0}^{N-1} P_n = \lambda(P_0+P_1+P_2+\dots+P_{N-1}) = \lambda(P_0+P_1+\dots+P_N - P_N) =$

$\lambda((P_0+P_1+\dots+P_N) - P_N) = \lambda((1)-P_N) = \lambda(1-P_N) = \lambda\left(1-\lambda^N \frac{1-\lambda}{1-\lambda^{N+1}}\right) =$

$\lambda\left(\frac{1-\lambda^{N+1}}{1-\lambda^{N+1}} + \frac{-\lambda^N+\lambda^{N+1}}{1-\lambda^{N+1}}\right) = \lambda\left(\frac{1-\lambda^N}{1-\lambda^{N+1}}\right) \therefore$

$P_0 + 2P_0 + 4P_0 + 4P_0 = 1 = 11P_0 \therefore \frac{1}{11} = P_0$

$\therefore P_1 = \frac{2}{11}, P_2 = P_3 = \frac{4}{11}$

\1c ii/ Expected system size $= E(\text{size}) = E(n) =$

$\sum_{i=0}^{3} nP_n = 0P_0 + 1P_1 + 2P_2 + 3P_3 = 1\left(\frac{2}{11}\right) + 2\left(\frac{4}{11}\right) + 3\left(\frac{4}{11}\right) = 2 = L_S \therefore$

Expected queue size $= E(\text{queue}) = E(n_q) =$

$\sum_{n=2}^{3} nP_n = 2P_2 + 3P_3 = 2\left(\frac{4}{11}\right) + 3\left(\frac{4}{11}\right) = \frac{20}{11}$ ✗

$L_q = \sum_{n>1}^{3} \cancel{n}P_n = (2-1)P_2 + (3-1)P_3 = 1P_2 + 2P_3 = \left(\frac{4}{11}\right) + 2\left(\frac{4}{11}\right) = \frac{3 \times 4}{11} = \frac{12}{11}$

\1c iii/ $E(\text{time in shop}) = L_S \frac{1}{\lambda_{eff}} \therefore$

$\sum_{n=0}^{\infty} \lambda_n P_n = 2P_0 + 2P_1 + 1P_2 + 0P_3 = \lambda_{eff} = 2\left(\frac{1}{11}\right) + 2\left(\frac{2}{11}\right) + 1\left(\frac{4}{11}\right) = \frac{10}{11} \therefore$

$E(\text{time in shop}) = W_S = L_S \frac{1}{10/11} = 2 \times \frac{11}{10} = \frac{11}{5} = 2.2$

$E(\text{time in queue}) = L_q \frac{1}{\lambda_{eff}} = \frac{12}{11} \times \frac{1}{10/11} = \frac{6}{5} = 1.2$

\1d i/ A: an absorbing state is $(P_{ii}) = 1$ and $T_{ij} = 0$ for $i \neq j$

B: transient state is for $p(\text{of first return}) = f_i^{(n)} \therefore$

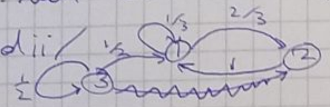
$p(\text{of eventual return}) = \sum_{n=1}^{\infty} f_i^{(n)} = f_i < 1 \therefore$ transient

C: a positively recurrent state: $f_i = \sum_{n=1}^{\infty} f_i^{(n)} = 1$ and

$\mu_i = \sum_{n=1}^{\infty} n f_i^{(n)} < \infty$

D: null recurrent: $f_i = 1$ and $\mu_i = \sum_{n=1}^{\infty} n f_i^{(n)} = \infty$

\1d ii/



$\therefore \{1, 2\}$ is a irreducible subchain

$\therefore P(\text{first return in time } n) = f_i^{(n)} \therefore$

$\sum_{n=0}^{\infty} f_3^{(n)} = \frac{1}{2} + \sum_{n=1}^{\infty} f_3^{(n)} = \frac{1}{2} + \sum_{n=1}^{\infty} 0 = \frac{1}{2} + 0 = \frac{1}{2} < 1 = f_3 \therefore$

3 is transient

for 2: $f_2 = \sum_{n=0}^{\infty} f_2^{(n)} = \sum_{n=1}^{\infty} 0 + 1 \times \frac{2}{3} + 1 \times \frac{1}{3} \times \frac{2}{3} + 1 \times \left(\frac{1}{3}\right)^2 \times \frac{2}{3} + \ldots =$

$\frac{2}{3} \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n = \frac{2}{3} \frac{1}{1 - \frac{1}{3}} = 1 \therefore$ is recurrent $\therefore$

$\mu_2 = \sum_{n=0}^{\infty} n f_2^{(n)} = 0 + 2 \times 1 \times \frac{2}{3} + 3 \times 1 \times \left(\frac{1}{3}\right) \times \frac{2}{3} + \ldots = \sum_{n=2}^{\infty} n \times 1 \times \left(\frac{1}{3}\right)^{n-2} \left(\frac{2}{3}\right) = \frac{2}{3} \left(\frac{1}{3}\right)^{-2} \sum_{n=2}^{\infty} n \left(\frac{1}{3}\right)^n =$

$6 \frac{(1/3)}{(1 - 1/3)^2} - \frac{1}{3} = \frac{9}{2} - \frac{1}{3} = \frac{25}{6} < \infty \therefore$ positively recurrent

PP2015/ 11biv/ $P(3\text{rd } Y \text{ bus in } \tfrac{1}{2}\text{hr to } 1\text{hr}) =$

$P(3\ Y \text{ buses in } 1\text{hr}) - P(3\ Y \text{ buses in } 30 \text{ mins}) =$

$P(N_Y(t=1)=3) - P(N_Y(t=\tfrac{1}{2})=3)$

11biv/ Poisson observations are time independent ∴

$\lambda_Y = 0.5\times10 = 5$ per hr ∴ $Y \sim e^{-\lambda_Y t}(\lambda_Y t)^x \frac{1}{x!}$ ∴

$P(3\text{rd } Y \text{ bus in } 30 \text{ mins to } 1\text{hr}) = P(3\ Y \text{ bus in } 30 \text{ mins}) =$

$P(N_Y(t=\tfrac{1}{2})=3) = e^{-5\times\frac{1}{2}}(5\times\tfrac{1}{2})^3\frac{1}{3!} = 0.214$ (3 s.f.)

11biv/ $P(3\text{rd } Y \in (\tfrac{1}{2},1)) = P(3Y \le 1) - P(3Y \le \tfrac{1}{2}) =$

$P(3\ Y \text{ or more} \le 1) - P(3\ Y \text{ or more} \le \tfrac{1}{2}) =$

$P(Y \ge 3 \text{ in } t \le 1) - P(Y \ge 3 \text{ in } t \le \tfrac{1}{2}) =$

$1 - P(Y<3 \text{ in } t\le1) - (1 - P(Y<3 \text{ in } t \le \tfrac{1}{2})) =$

$1 - P(Y\le2 \text{ in } t\le1) - (1 - P(Y\le2 \text{ in } t \le \tfrac{1}{2})) =$

$P(Y\le2 \text{ in } t\le\tfrac{1}{2}) - P(Y\le2 \text{ in } t\le1) = P(N_Y(\tfrac{1}{2})\le2) - P(N_Y(1)\le2) =$

$e^{-5\times\frac{1}{2}}\left((5\times\tfrac{1}{2})^0\frac{1}{0!} + (5\times\tfrac{1}{2})^1\frac{1}{1!} + (5\times\tfrac{1}{2})^2\frac{1}{2!}\right) - e^{-5\times1}\left((5)^0\frac{1}{0!} + (5)^1\frac{1}{1!} + (5)^2\frac{1}{2!}\right) =$

$0.544 - 0.125 = 0.419$

11bv/ $P(2\ G \text{ in } \tfrac{15}{60}\text{hr} \mid 6\ G \text{ in } 1\text{hr}) = \dfrac{P(2\ G \text{ in } \frac{1}{4}\text{hr},\ 6\ G \text{ in } 1\text{hr})}{P(6\ G \text{ in } 1\text{hr})} =$

$P(2\ G \text{ in } \tfrac{1}{4}\text{hr},\ 4\ G \text{ in } \tfrac{3}{4}\text{hr}) / P(6G \text{ in } 1\text{hr}) =$

$P(2\ G \text{ in } \tfrac{1}{4}\text{hr})P(4\ G \text{ in } \tfrac{3}{4}\text{hr}) / P(6G \text{ in } 1\text{hr}) =$

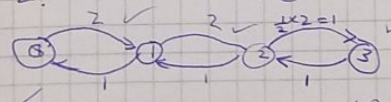
$P(N_G(t=\tfrac{1}{4})=2)P(N_G(t=\tfrac{3}{4})=4) / P(N_G(t=1)=6) \sim Bin(6, \tfrac{15}{60}) = Bin(6,\tfrac{1}{4})$

∴ $P(2\ G \text{ in } 15 \text{ mins} \mid 6\ G \text{ in } 1\text{hr}) = \binom{6}{2}(\tfrac{1}{4})^2(1-\tfrac{1}{4})^{6-2} =$

$\frac{6!}{2!\,4!}(\tfrac{1}{4})^2(\tfrac{3}{4})^4 = \frac{1215}{4096} = 0.297$ (3 s.f.)

∴ if N observations in time T then for a time $t<T$, N: # arrivals is given by $Bin(N, \frac{t}{T})$ distr ∴ $P(X=2) = \binom{6}{2}(\tfrac{1}{4})^2(\tfrac{3}{4})^4 = 0.297$

$e^{-6(\frac{1}{4})}(6\times\tfrac{1}{4})^2\frac{1}{2!}\,e^{-6(\frac{3}{4})}(6\times\tfrac{3}{4})^4\frac{1}{4!}\ \dfrac{1}{e^{-6(1)}(6)^6\frac{1}{6!}} = \frac{1215}{4096} = 0.297$

11ci/ 

is M/M/1 queue with finite capacity

ii) 11cii/ steady state: $\frac{dP_0}{dt} = P_1 - 2P_0 = 0$ ∴ $P_1 = 2P_0$

$\frac{dP_1}{dt} = P_2 - 2P_1 - (P_1 - 2P_0) = 0 = P_2 - 2P_1$ ∴ $P_2 = 2P_1 = 4P_0$, $\sum_{n=0}^{3} P_n = 1$

$\frac{dP_2}{dt} = P_3 - P_2 - (P_2 - 2P_1) = 0 = P_3 - P_2$ ∴ $P_3 = P_2 = 4P_0$ ∴

\1ai/ $Y = X^2 + 4 - 4X$ $\therefore$

$E(X) = 0.5(0) + 0.2(2) + 0.3(4) = 1.6$

$E(Y) = E(X^2) + 4 - 4E(X)$

$E(X^2) = 0.5(0^2) + 0.2(2^2) + 0.3(4^2) = 5.8$ $\therefore$

$E(Y) = 5.6 + 4 - 4 \times 1.6 = 3.2$

$P(Y=4) = P((X-2)^2 = 4) = P(X-2=2) = P(X=4) = 0.3$ X

$P((X-2)^2 = 4) = P(X-2 = \pm 2) = P(X-2 = +2) + P(X-2 = -2) =$

$P(X=4) + P(X=0) = 0.3 + 0.5 = 0.8$

\1aii/ $E(Y) = G_Y'(\theta)|_{\theta=1} = \frac{d}{d\theta} G_X(G_X(\theta))\ |_{\theta=1} =$

$G_X'(G_X(\theta))\,G_X'(\theta)|_{\theta=1} = G_X'(G_X(1))\,G_X'(1) = G_X'(1)\,G_X'(1) =$

$(G_X'(1))^2 = (E(X))^2 = 1.6^2 = 2.56$

~~For~~ $P(Y=4) = \cancel{P(}$: $G_Y = G_X(G_X(\theta)) = G_X(0.5 + 0.2\theta^2 + 0.3\theta^4) =$

~~eg~~ $0.5 + 0.2(0.5 + 0.2\theta^2 + 0.3\theta^4)^2 + 0.3(0.5 + 0.2\theta^2 + 0.3\theta^4)^4 =$

$\cancel{0.2[(0.2\theta^2)^2\ 0.2}$

$0.2\left[(0.2\theta^2)^2 \frac{2!}{2!} + (0.3\theta^4)^1(0.5)^1 \frac{2!}{1!1!}\right] + 0.3\left[(0.5)^3(0.3\theta^4)^1 \frac{4!}{3!1!} + (0.2\theta^2)^2(0.5)^2 \frac{4!}{2!2!}\right] + \dots$

$= 0.2[0.04\theta^4 + 0.3\theta^4] + 0.3[0.15\theta^4 + 0.06\theta^4] + \dots = 0.131\theta^4 + \dots$ $\therefore$

$P(Y=4) = 0.131$

\1aiii/ $E(Y) = E(X_1 + X_2) = E(X_1) + E(X_2) = E(X_1) + E(X_1) = 2E(X_1) =$

$2 \times 1.6 = 3.2$

$G_Y(\theta) = G_{X_1+X_2}(\theta) = G_{X_1}(\theta)\,G_{X_2}(\theta) = G_X(\theta)\,G_X(\theta) = (G_X(\theta))^2 =$

$(0.5 + 0.2\theta^2 + 0.3\theta^4)^2 = (0.2\theta^2)^2 \frac{2!}{2!} + (0.3\theta^4)^1 (0.5)^1 \frac{2!}{1!1!} + \dots =$

$0.04\theta^4 + 0.3\theta^4 + \dots = 0.34\theta^4 + \dots$ $\therefore$

$P(Y=4) = 0.34$

\1bi/ $\frac{\lambda_Y}{\lambda} = \left(\frac{10}{0.5 \times 10}\right)^{-1} = \frac{0.5}{1} = 0.5 = P(\text{next bus is yellow})$

$\therefore \lambda = 10$, $\lambda_Y = 0.5 \times 10 = 5$

\1bii/ 10 buses per hour

E(time between buses) $= \frac{1}{10}$ hours $= \frac{1}{10} \times 60$ mins $= 6$ minutes

\1biii/ $\lambda_G = 0.4 \times 10 = 4$ per hr $\therefore P(N_G(t = \frac{1}{2}) = 2) = e^{-4(\frac{1}{2})}(4 \times \frac{1}{2})^2 \frac{1}{2!} =$

$e^{-2}(2)^2 \frac{1}{2!} = 0.271$ (3 s.f.)