CH1/ $\therefore (1, x^2y)\cdot\nabla u = 0$ $\therefore$ Charac curves parallel to $(1, x^2y)$ $\therefore$

$\frac{dy}{dx} = \frac{x^2y}{1} = x^2y$ $\therefore$ $\frac{1}{y}\frac{dy}{dx} = x^2$ $\therefore$ $\int \frac{1}{y}\,dy = \int x^2\,dx = \ln|y| = \frac{1}{3}x^3 + C_1$ $\therefore$

$|y| = e^{\frac{1}{3}x^3 + C_1} = C_2 e^{\frac{1}{3}x^3}$ $\therefore$ $y = Ce^{\frac{1}{3}x^3}$ $\therefore$

$ye^{-\frac{1}{3}x^3} = C$ $\therefore$ $u(x,y) = f(ye^{-\frac{1}{3}x^3})$ $\therefore$ f arbitr $\therefore$

$u_x = y(-x^2)e^{-\frac{1}{3}x^3} f'$, $u_y = e^{-\frac{1}{3}x^3} f'$ $\therefore$

$u_x + x^2 y u_y = -yx^2 e^{-\frac{1}{3}x^3} f' + x^2 y e^{-\frac{1}{3}x^3} f' = 0 = 0$

4/ $\xi_x = a$, $\xi_y = b$, let $\xi = x - 2y$, $\eta = -2x - y$ $\therefore$

$\xi_x = 1$, $\xi_y = -2$, $\eta_x = -2$, $\eta_y = -1$ $\therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi - 2u_\eta$, $u_y = u_\xi \xi_y + u_\eta \eta_y = -2u_\xi - u_\eta$ $\therefore$

$u_x - 2u_y = u_\xi - 2u_\eta - 2[-2u_\xi - u_\eta] = u_\xi - 2u_\eta + 4u_\xi + 2u_\eta = 5u_\xi$ $\therefore$

$u_x - 2u_y + u = 5u_\xi + u = e^{5x}$ $\therefore$

$y = -2x - \eta$ $\therefore$ $\xi = x - 2(-2x - \eta) = x + 4x + 2\eta = 5x + 2\eta$ $\therefore$ $\xi - 2\eta = 5x$ $\therefore$

$\frac{1}{5}\xi - \frac{2}{5}\eta = x$ $\therefore$ $5u_\xi + u = e^{5(\frac{1}{5}\xi - \frac{2}{5}\eta)} = e^{\xi - 2\eta}$ $\therefore$

$u_\xi + \frac{1}{5}u = \frac{1}{5}e^{\xi - 2\eta}$ $\therefore$ $IF = e^{\int \frac{1}{5}d\xi} = e^{\frac{1}{5}\xi}$ $\therefore$

$\frac{\partial}{\partial \xi}(e^{\frac{1}{5}\xi}u) = \frac{1}{5}e^{\frac{1}{5}\xi}e^{\xi - 2\eta} = \frac{1}{5}e^{\frac{6}{5}\xi - 2\eta}$ $\therefore$

$e^{\frac{1}{5}\xi}u = (\frac{1}{5})\frac{5}{6}e^{\frac{6}{5}\xi - 2\eta} + f(\eta) = \frac{1}{6}e^{\frac{6}{5}\xi - 2\eta} + f(\eta)$ $\therefore$

$u = \frac{1}{6}e^{-\frac{1}{5}\xi}e^{\frac{6}{5}\xi - 2\eta} + e^{-\frac{1}{5}\xi}f(\eta) = \frac{1}{6}e^{\xi - 2\eta} + e^{-\frac{1}{5}\xi}f(\eta)$ $\therefore$

$u(x,y) = \frac{1}{6}e^{5x} + e^{-\frac{1}{5}(x-2y)}f(-2x-y) = \frac{1}{6}e^{5x} + e^{-\frac{1}{5}x + \frac{2}{5}y}f(-2x-y)$

5/ let $\varnothing = -1$: $1 + \varnothing = 1 - 1 = 0$ $\therefore$ $u_{xx} + \varnothing u_{yy} = 0$ $\therefore$

$u = f(\alpha x + y)$ $\therefore$ $u_x = \alpha f'$, $u_{xx} = \alpha f''$, $u_y = f'$, $u_{yy} = f''$ $\therefore$

$\alpha^2 f'' + \varnothing f'' = 0 = (\alpha^2 + \varnothing)f''$ $\therefore$ $\alpha^2 = -\varnothing$ $\therefore$ $\alpha = \pm\sqrt{-\varnothing} = \pm\sqrt{\varnothing}\,i$ $\therefore$

$u = f(+\sqrt{\varnothing}\,ix + y) + g(-\sqrt{\varnothing}\,ix + y)$

if $\varnothing = -1$: $u = f(\alpha x + y)$ $\therefore$ $u_{xx} = \alpha^2 f''$, $u_{yy} = f''$, $u_{xy} = \alpha f''$ $\therefore$

$\alpha^2 f'' - (1+\varnothing)\alpha f'' + \varnothing f'' = 0 = \alpha^2 - (1+\varnothing)\alpha + \varnothing$ $\therefore$

$\alpha = \frac{(1+\varnothing) \pm \sqrt{(1+\varnothing)^2 - 4(1)(\varnothing)}}{2(1)} = \frac{(1+\varnothing) \pm \sqrt{1 + 2\varnothing + \varnothing^2 - 4\varnothing}}{2} = \frac{(1+\varnothing) \pm \sqrt{\varnothing^2 - 2\varnothing + 1}}{2} =$

$\frac{(1+\varnothing) \pm \sqrt{(\varnothing - 1)^2}}{2} = \frac{1 + \varnothing \pm (\varnothing - 1)}{2}$ $\therefore$

$\alpha_1 = \frac{1+\sigma+(\sigma-1)}{2}$, $\alpha_2 = \frac{1+\sigma-(\sigma-1)}{2}$ ∴

$\alpha_1 = \frac{2\sigma}{2} = \sigma$, $\alpha_2 = \frac{1+\sigma-\sigma+1}{2} = \frac{2}{2} = 1$ ∴

$u = f(\sigma x+y) + g(\alpha x+y)$

\5 sol/ const coeffs ∴ $u = f(y+\alpha x)$ ∴ $[\alpha^2-(1+\sigma)\alpha+\sigma]f''=0$ ∴

$\alpha=1$, $\alpha=\sigma$ ∴ for $\sigma \neq 1$: $u = f(x+y)+g(\sigma x+y)$

for $\sigma=1$: $1+\sigma = 1+1=2$ ∴ $u_{xx}-2u_{xy}+u_{yy}=0$ ∴

let $\xi = x+y$ ∴ y is indep of $x+y$ ∴ let $\eta = y$ ∴

$\xi_x = 1$, $\xi_y = 1$, $\eta_x = 0$, $\eta_y = 1$ ∴

$u_x = u_\xi \xi_x = u_\xi$ ∴ $u_{xx} = u_{\xi\xi}\xi_x = u_{\xi\xi}$,

~~$u_{xy} = u_{\xi\eta}\eta_y = u_{\xi\eta}$~~ $u_{xy} = \partial_y(u_\xi) = u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y = u_{\xi\xi}+u_{\xi\eta}$,

~~$u_{\eta\eta}$~~ ~~$u_\eta =$~~ $u_y = \cancel{u_{\xi\xi}}\ u_\xi\xi_y + u_\eta\eta_y = u_\xi + u_\eta$,

$u_{yy} = u_{\xi\xi}\xi_y + u_{\xi\eta}\xi_y + u_{\xi\eta}\eta_y + u_{\eta\eta}\eta_y = u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}$ ∴

$u_{\xi\xi} - 2[u_{\xi\xi}+u_{\xi\eta}] + [u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}] =$

$u_{\xi\xi} - 2u_{\xi\xi} - 2u_{\xi\eta} + u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} = u_{\eta\eta} = 0$ ∴

$u_\eta = f(\xi)$ ∴ $u = \eta f(\xi) + F(\xi)$ ∴

$u(x,y) = y f(x+y) + F(x+y)$

\5 sol/ for $\sigma=1$ let $\xi = x+y$, $\eta = y$ ∴ $\xi_x = 1$, $\xi_y = 1$, $\eta_x = 0$, $\eta_y = 1$ ∴

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi$ ∴ $u_{xx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x = u_{\xi\xi}$,

$u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi + u_\eta$, $u_{xy} = u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y = u_{\xi\xi}+u_{\xi\eta}$

$u_{yy} = u_{\xi\xi}\xi_y + u_{\xi\eta}\xi_y + u_{\xi\eta}\eta_y + u_{\eta\eta}\eta_y = u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}$ ∴

~~$u(x,y) =$~~ $u_{\xi\xi} - 2(u_{\xi\xi}+u_{\xi\eta}) + u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} =$

$u_{\xi\xi} - 2u_{\xi\xi} - 2u_{\xi\eta} + u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} = u_{\eta\eta}$ ∴

$u_\eta = f(\xi)$ ∴ $u = \eta f(\xi) + F(\xi) = y f(x+y) + F(x+y)$

\6/ let $x = e^t$, $y = e^s$ ∴ $t = \ln x$, $s = \ln y$ ∴

$t_x = \frac{1}{x}$, $s_y = \frac{1}{y}$ ∴

$u_x = u_t t_x = \frac{1}{x}u_t$, ~~$u_{xx} = \partial_t(\frac{1}{x}u_t)t_x$~~ $u_{xx} = \partial_x(\frac{1}{x}u_t) = \partial_x(\frac{1}{x})u_t + \frac{1}{x}\partial_x u_t =$

$-\frac{1}{x^2}u_t + \frac{1}{x}u_{tt}t_x = -\frac{1}{x^2}u_t + \frac{1}{x^2}u_{tt}$,

\CW1/ $u_y = u_s s_y = \frac{1}{y} u_s \therefore u_{yy} = \partial_y(\frac{1}{y} u_s) = \partial_y(\frac{1}{y}) u_s + \frac{1}{y}\partial_y u_s =$

$-\frac{1}{y^2} u_s + \frac{1}{y} u_{ss} s_y = -\frac{1}{y^2} u_s + \frac{1}{y^2} u_{ss}$, $\therefore$ PDE:

$5x^2[-\frac{1}{x^2} u_t + \frac{1}{x^2} u_{tt}] - y^2[-\frac{1}{y^2} u_s + \frac{1}{y^2} u_{ss}] + 5x(\frac{1}{x} u_t) - y(\frac{1}{y} u_s) =$

$-5u_t + 5u_{tt} + u_s - u_{ss} + 5u_t - u_s = 5u_{tt} - u_{ss} = 0$

$\xi = 5t - s$, $\eta = -t - 5s$ $\therefore$

$\xi_t = 5$, $\xi_s = -1$, $\eta_t = -1$, $\eta_s = -5$ $\therefore$

$u_t = u_\xi \xi_t + u_\eta \eta_t = 5u_\xi - u_\eta$,

$u_{tt} = 5u_{\xi\xi}\xi_t - u_{\xi\eta}\xi_t + 5u_{\xi\eta}\eta_t - u_{\eta\eta}\eta_t = 25u_{\xi\xi} - 5u_{\xi\eta}\xi_t - 5u_{\xi\eta} + u_{\eta\eta}$

$u_s = u_\xi \xi_s + u_\eta \eta_s = -u_\xi - 5u_\eta$ $\therefore$

$u_{ss} = -u_{\xi\xi}\xi_s - 5u_{\xi\eta}\xi_s - u_{\xi\eta}\eta_s - 5u_{\eta\eta}\eta_s = u_{\xi\xi} + 5u_{\xi\eta} + 5u_{\xi\eta} + 25u_{\eta\eta} =$

$u_{ss} = u_{\xi\xi} + 10u_{\xi\eta} + 25u_{\eta\eta}$, $u_{tt} = 25u_{\xi\xi} - 10u_{\xi\eta} + u_{\eta\eta}$ $\therefore$

$5(25u_{\xi\xi} - 10u_{\xi\eta} + u_{\eta\eta}) - (u_{\xi\xi} + 10u_{\xi\eta} + 25u_{\eta\eta}) =$

$125u_{\xi\xi} - 50u_{\xi\eta} + 5u_{\eta\eta} - u_{\xi\xi} - 10u_{\xi\eta} - 25u_{\eta\eta}$

$5u_{tt} - u_{ss} = 0$ $\therefore$ ~~$u = f(x\xi + y)$~~ , $u = f(\alpha t + s)$ $\therefore$

$u_{tt} = \alpha^2 f''$, $u_{ss} = f''$, $5\alpha^2 f'' - f'' = 0 = (5\alpha^2 - 1) = 0$ $\therefore$

$\alpha^2 = \frac{1}{5}$, $\alpha = \pm\frac{1}{\sqrt{5}}$ $\therefore$ $u = f(\frac{1}{\sqrt{5}}t + s) + g(-\frac{1}{\sqrt{5}}t + s)$ $\therefore$

$u(x,y) = f(\frac{1}{\sqrt{5}}\ln(x) + \ln y) + g(-\frac{1}{\sqrt{5}}\ln(x) + \ln y)$

if $u = f(t + \alpha s)$: $u_{tt} = f''$, $u_{ss} = \alpha^2 f''$ $\therefore$

$f'' - \alpha^2 f'' = 0$ $\therefore$ $5 - \alpha^2 = 0$ $5 = \alpha^2$ $\therefore$ $\alpha_1 = \sqrt{5}$, $\alpha_2 = -\sqrt{5}$ $\therefore$

$u = f(t + \sqrt{5}s) + g(t - \sqrt{5}s)$ $\therefore$

$u(x,t) = f(\ln(x) + \sqrt{5}\ln y) + g(\ln(x) - \sqrt{5}\ln y)$ ✓

\2/ $X_x = \cos\theta$, $X_y = \sin\theta$, $Y_x = -\sin\theta$, $Y_y = \cos\theta$ $\therefore$

$u_x = u_X X_x + u_Y Y_x = \cos(\theta)u_X - \sin(\theta)u_Y$.

$u_{xx} = \cos(\theta)u_{XX}X_x - \sin(\theta)u_{XY}X_x + \cos(\theta)u_{XY}Y_x - \sin(\theta)u_{YY}Y_x =$

$\cos^2(\theta)u_{XX} - \sin\theta\cos(\theta)u_{XY} - \sin\theta\cos(\theta)u_{XY} + \sin^2(\theta)u_{YY} =$

$\cos^2(\theta)u_{XX} - 2\sin(\theta)\cos(\theta)u_{XY} + \sin^2(\theta)u_{YY}$

$u_y = u_X X_y + u_Y Y_y = \sin(\theta)u_X + \cos(\theta)u_Y$ $\therefore$

$u_{yy} = \sin(\theta)u_{XX}X_y + \cos(\theta)u_{XY}X_y + \sin(\theta)u_{XY}Y_y + \cos(\theta)u_{YY}Y_y =$

$\sin^2(\theta)u_{XX} + 2\sin(\theta)\cos(\theta)u_{XY} + \cos^2(\theta)u_{YY}$ $\therefore$

$u_{xx}+u_{yy} = \cos^2(\theta)u_{XX} - 2\sin(\theta)\cos(\theta)u_{XY} + \sin^2(\theta)u_{YY} + \sin^2(\theta)u_{XX} + 2\sin\theta\cos\theta u_{XY} + \cos^2(\theta)u_{YY}$

$= (\cos^2\theta+\sin^2\theta)u_{XX} + (\sin^2\theta+\cos^2\theta)u_{YY} = u_{XX}+u_{YY} = 0 \therefore$ $\xi \circlearrowleft y$

$u_X = 2X$, $u_{XX}=2$, $u_Y=-2Y$, $u_{YY}=-2 \therefore$

$u_{XX}+u_{YY} = 2-2=0$

$X^2 = (x\cos\theta+y\sin\theta)^2 = x^2\cos^2\theta + y^2\sin^2\theta + 2xy\sin\theta\cos\theta$,

$Y^2 = (-x\sin\theta+y\cos\theta)^2 = x^2\sin^2\theta + y^2\cos^2\theta - 2xy\sin\theta\cos\theta \therefore$

$u = X^2-Y^2 = x^2\cos^2\theta + y^2\sin^2\theta + 2xy\sin\theta\cos\theta - x^2\sin^2\theta - y^2\cos^2\theta + 2xy\sin\theta\cos\theta$

$(\cos^2\theta-\sin^2\theta)x^2 + (\sin^2\theta-\cos^2\theta)y^2 + 4xy\sin\theta\cos\theta \therefore$

$u_x = 2(\cos^2\theta-\sin^2\theta)x + 4y\sin\theta\cos\theta \therefore$

$u_{xx} = 2(\cos^2\theta-\sin^2\theta)$,

$u_y = 2(\sin^2\theta-\cos^2\theta)y + 4x\sin\theta\cos\theta \therefore$

$u_{yy} = 2(\sin^2\theta-\cos^2\theta) \therefore$

$u_{xx}+u_{yy} = 2\cos^2\theta - 2\sin^2\theta + 2\sin^2\theta - 2\cos^2\theta = 0 \therefore$

$u = (\cos^2\theta-\sin^2\theta)x^2 + (\sin^2\theta-\cos^2\theta)y^2 + 4xy\sin\theta\cos\theta$ is a sol of PDE

Sheet 4/ Mai/ let $\xi = x+a$, $\eta = y+b \therefore \xi_x = 1$, $\eta_y = 1 \therefore$

~~u_x~~ $u_x = u_\xi \xi_x = u_\xi$, $u_y = u_\eta \eta_y = u_\eta \therefore$

$u_{xx} = u_{\xi\xi}\xi_x = u_{\xi\xi}$, $u_{yy} = u_{\eta\eta}\eta_y = u_{\eta\eta} \therefore$ into PDE:

$u_{\xi\xi}+u_{\eta\eta} = 0 \therefore$ invariant to translation

let ~~$\xi = x\cos\theta$~~ $\xi = -x$, $\eta = y \therefore \xi_x = -1$, $\eta_y = 1 \therefore$

$u_x = u_\xi \xi_x = -u_\xi$, $u_y = u_\eta \eta_y = u_\eta \therefore u_{yy} = u_{\eta\eta}\eta_y = u_{\eta\eta}$,

$u_{xx} = -u_{\xi\xi}\xi_x = -u_{\xi\xi} \therefore$ into PDE: $u_{\xi\xi}+u_{\eta\eta} = 0 \therefore$ reflection invariant

$\xi = x\cos\theta + y\sin\theta$, $\eta = -x\sin\theta + y\cos\theta \therefore$

$\xi_x = -\cos\theta$, $\xi_y = \sin\theta$, $\eta_x = -\sin\theta$, $\eta_y = \cos\theta \therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x = -u_\xi\cos\theta - u_\eta\sin\theta$,

$u_y = u_\xi \xi_y + u_\eta \eta_y = \sin\theta u_\xi + \cos\theta u_\eta \therefore$

$u_{xx} = -\cos\theta u_{\xi\xi}\xi_x - \sin\theta u_{\xi\eta}\xi_x - \cos\theta u_{\xi\eta}\eta_x - \sin\theta u_{\eta\eta}\eta_x =$

$\cos^2\theta u_{\xi\xi} + 2\sin\theta\cos\theta u_{\xi\eta} + \sin^2\theta u_{\eta\eta}$

$u_{yy} = \sin\theta u_{\xi\xi}\xi_y + \cos\theta u_{\xi\eta}\xi_y + \sin\theta u_{\xi\eta}\eta_y + \cos\theta u_{\eta\eta}\eta_y =$

$\sin^2\theta u_{\xi\xi} + 2\sin\theta\cos\theta u_{\xi\eta} + \cos^2\theta u_{\eta\eta} \therefore$

\Sheet 4/ intro PDE:

$\cos^2\theta u_{\xi\xi} + 2\sin\theta\cos\theta u_{\xi\eta} + \sin^2\theta u_{\eta\eta} + \sin^2\theta u_{\xi\xi} - 2\sin\theta\cos\theta u_{\xi\eta} + \cos^2\theta u_{\eta\eta} =$

$(\cos^2\theta + \sin^2\theta)u_{\xi\xi} + (\cos^2\theta + \sin^2\theta)u_{\eta\eta} = u_{\xi\xi} + u_{\eta\eta} = 0$ ∴ invariant under rotation

divergence thm: $\int_D \nabla\cdot\vec{F}\,dV = \int_{\partial D}\vec{F}\cdot\vec{n}\,dS$, $\int_D \nabla\cdot\vec{F}\,dV = \int_{\partial D}\vec{F}\cdot\vec{n}\,dS$

$\int_D \nabla\cdot\vec{F}\,dV = \int_{\partial D}\vec{F}\cdot\vec{n}\,dS$ $\int_D \nabla\cdot\vec{F}\,dV = \int_{\partial D}\vec{F}\cdot\vec{n}\,dS$ $\int_D \nabla\cdot\vec{F}\,dV = \int_{\partial D}\vec{F}\cdot\vec{n}\,dS$

$\int_D \nabla\cdot\vec{F}\,dV = \int_{\partial D}\vec{F}\cdot\vec{n}\,dS$ $\int_D \nabla\cdot\vec{F}\,dV = \int_{\partial D}\vec{F}\cdot\vec{n}\,dS$ $\int_D \nabla\cdot\vec{F}\,dV = \int_{\partial D}\vec{F}\cdot\vec{n}\,dS$ $\int_D \nabla\cdot\vec{F}\,dV = \int_{\partial D}\vec{F}\cdot\vec{n}\,dS$

\2 Sol/ let ∃ 2 sols, set them to satisfying the IC,

let their difference be ω, find its IC,

then state divergence thm then plug it in with $\vec{F} = \omega\nabla\omega$

then use identity ~~$\nabla\cdot(\phi\nabla\omega) =$~~ $\nabla(\phi\underline{v}) = \phi\nabla\cdot\underline{v} + \underline{v}\cdot\nabla\phi$ ∴

$\nabla(\omega\nabla\omega) = \omega\nabla\cdot(\nabla\omega) + \nabla\omega\cdot\nabla\omega = \omega\nabla^2\omega + \nabla\omega\cdot\nabla\omega = \omega\nabla^2\omega + |\nabla\omega|^2$

∵ $\nabla\cdot\nabla\omega = \nabla^2\omega$ ∴ simplify show $\nabla\omega = 0$ $\forall D$ ∴ const

∴ $\omega = u_1 - u_2$ ∴ sol exists, unique up to an additive constant, the fact it may be a bit is verified by direct calculation, i.e. u_1 is a sol, $u_2 = u_1 + C$ is sol $\forall$ const C

\2/ ∵ let u_1, u_2 be sols ∴ $u_{1_{xx}} + u_{2_{yy}} + u_{3_{zz}} = f(x,y,z)$ on D,

$\frac{\partial u_1}{\partial \vec{n}} = g(x,y,z)$ on ∂D, $u_{2_{xx}} + u_{2_{yy}} + u_{3_{zz}} = f(x,y,z)$ on D,

$\frac{\partial u_2}{\partial \vec{n}} = g(x,y,z)$ on ∂D ∴

let $\omega = u_1 - u_2$ ∴ $\omega_{xx} + \omega_{yy} + \omega_{zz} = \nabla^2\omega = 0$ on D,

$\frac{\partial\omega}{\partial\vec{n}} = 0$ on ∂D ∴ $\int_D \nabla\cdot\vec{F}\,dV = \int_{\partial D}\vec{F}\cdot\vec{n}\,dS$ is divergence theorem

∴ let $\vec{F} = \omega\nabla\omega$ ∴ $\int_D \nabla\cdot\vec{F}\,dV = \int_D \nabla\cdot(\omega\nabla\omega)\,dV = \int_D \omega\nabla\cdot(\nabla\omega) + (\nabla\omega)\cdot\nabla\omega\,dV$

$\int_D \omega\nabla^2\omega + |\nabla\omega|^2\,dV = \int_D \omega(0) + |\nabla\omega|^2\,dV = \int_D |\nabla\omega|^2\,dV = \int_{\partial D}\vec{F}\cdot\vec{n}\,dS =$

$\int_{\partial D}\omega\nabla\omega\cdot\vec{n}\,dS = 0 = \int_D |\nabla\omega|^2\,dV$ ∴ $\nabla\omega = 0$ on D ∴ ω const ∴

$\omega = u_1 - u_2 =$ constant arbitrary $= C$ ∴ $u_1 = u_2 + C$ ∴ sol exists,

unique up to an additive constant ∴ if u_1 is a sol, $u_2 = u_1 + C$ is

a sol $\forall C$ ∵ $\int_{\partial D}\omega\nabla\omega\cdot\vec{n}\,dS = \int_{\partial D}\omega\frac{\partial\omega}{\partial\vec{n}}\,dS = \int_{\partial D}\omega(0)\,dS = 0$ ∵ $\nabla\omega\cdot\vec{n} = \frac{\partial\omega}{\partial\vec{n}}$

\Sheet 4 / \5/ $2u_{xx}+2u_{xy}+u_{yy}=-8u_x-2u_y-u$ ∴

$a=2,\ 2b=2\ \therefore b=1,\ c=1$ ∴

$b^2-ac=1^2-2(1)=1-2=-1<0$ ∴ elliptic ∴

$2\left(\frac{dy}{dx}\right)^2-2\left(\frac{dy}{dx}\right)+1=0$ ∴ $\frac{dy}{dx}=\frac{2\pm\sqrt{4-4(2)(1)}}{2(2)}=\frac{2\pm\sqrt{-4}}{4}$

$\frac{1}{2}\pm\frac{1}{2}i=\frac{1}{2}\pm\frac{1}{2}i$ ∴ $\frac{dy}{dx}=\frac{1}{2}+\frac{1}{2}i$, $\frac{dy}{dx}=\frac{1}{2}-\frac{1}{2}i$ ∴

$y=(\frac{1}{2}+\frac{1}{2}i)x+C_1$, $y=(\frac{1}{2}-\frac{1}{2}i)x+C_2$ ∴

$y+(-\frac{1}{2}-\frac{1}{2}i)x=C_1$, $y+(-\frac{1}{2}+\frac{1}{2}i)x=C_2$ ∴

$C_1=y-\frac{x}{2}$, $C_2=-\frac{x}{2}$ ∴ $\mu=y-\frac{x}{2}$, $\nu=-\frac{x}{2}$

\3 Sol / Since $\frac{dy}{dx}$ then gives ξ,η ∴ real form: $(\xi+\eta)/2$, $(\xi-\eta)/2$

\3/ $\frac{dy}{dx}=\frac{1}{2}\pm\frac{1}{2}i$ ∴ $y=(\frac{1}{2}+\frac{1}{2}i)x+C_1$, $y=(\frac{1}{2}-\frac{1}{2}i)x+C_2$

$y+(-\frac{1}{2}-\frac{1}{2}i)x=C_1$, $y+(-\frac{1}{2}+\frac{1}{2}i)x=C_2$ ∴ $\xi=y+(-\frac{1}{2}-\frac{1}{2}i)x$, $\eta=y+(-\frac{1}{2}+\frac{1}{2}i)x$

∴ real form: $(\xi+\eta)/2=y-\frac{x}{2}=\mu$, $(\xi-\eta)/2=-\frac{x}{2}=\nu$

\3b/ ∴ $\mu_x=-\frac{1}{2}$, $\mu_y=1$, $\nu_x=-\frac{1}{2}$, $\nu_y=0$ ∴

$u_x=u_\mu\mu_x+u_\nu\nu_x=-\frac{1}{2}u_\mu-\frac{1}{2}u_\nu$, $u_y=u_\mu\mu_y=u_\mu$,

$u_{xy}=u_{\mu\mu}\mu_x+u_{\mu\nu}\nu_x=-\frac{1}{2}u_{\mu\mu}-\frac{1}{2}u_{\mu\nu}$ ∴

$u_{yy}=u_{\mu\mu}\mu_y=u_{\mu\mu}$,

$u_{xx}=-\frac{1}{2}u_{\mu\mu}\mu_x-\frac{1}{2}u_{\mu\nu}\mu_x-\frac{1}{2}u_{\mu\nu}\nu_x-\frac{1}{2}u_{\nu\nu}\nu_x=\frac{1}{4}u_{\mu\mu}+\frac{1}{2}u_{\mu\nu}+\frac{1}{4}u_{\nu\nu}$ ∴

into PDE:

$2(\frac{1}{4}u_{\mu\mu}+\frac{1}{2}u_{\mu\nu}+\frac{1}{4}u_{\nu\nu})+2(-\frac{1}{2}u_{\mu\mu}-\frac{1}{2}u_{\mu\nu})+u_{\mu\mu}+8(-\frac{1}{2}u_\mu-\frac{1}{2}u_\nu)+$

$2u_\mu+u=0$ =

$\frac{1}{2}u_{\mu\mu}+u_{\mu\nu}+\frac{1}{2}u_{\nu\nu}-u_{\mu\mu}-u_{\mu\nu}+u_{\mu\mu}-3u_\mu-3u_\nu+2u_\mu+u=$

$\frac{1}{2}u_{\mu\mu}-u_\mu+\frac{1}{2}u_{\nu\nu}+u=3u_\mu+3u_\nu$ ∴ $\frac{1}{2}(u_{\mu\mu}+u_{\nu\nu})=u_\mu+3u_\nu-u$ ∴

$u_{\mu\mu}+u_{\nu\nu}=2(u_\mu+3u_\nu-u)$

\3c/ let $u=e^{\rho\mu+\sigma\nu}v$, $e^{\rho\mu+\sigma\nu}=t$ ∴ $u=tv$ ∴

$u_{\mu\mu}+u_{\nu\nu}-2u_\mu-6u_\nu+2u=0$ ∴

$u_\mu=\rho tv+tv_\mu=t(\rho v+v_\mu)$, $u_\nu=\sigma tv+tv_\nu=t(\sigma v+v_\nu)$,

$u_{\mu\mu}=\rho t(\rho v+v_\mu)+t(\rho v_\mu+v_{\mu\mu})=t(\rho^2v+2\rho v_\mu+v_{\mu\mu})$,

$u_{\nu\nu}=\sigma t(\sigma v+v_\nu)+t(\sigma v_\nu+v_{\nu\nu})=t(\sigma^2v+2\sigma v_\nu+v_{\nu\nu})$ ∴

Sheet 4 / $t(\rho^2 v + 2\rho v_\mu + v_{\mu\mu}) + t(\sigma^2 v + 2\sigma v_\nu + v_{\nu\nu}) - 2t(\rho v + v_\mu) - 6t(\sigma v + v_\nu) + 2v$
$= 0$ ∴ $t > 0$ ∴

1) $\rho^2 v + 2\rho v_\mu + v_{\mu\mu} + \sigma^2 v + 2\sigma v_\nu + v_{\nu\nu} - 2\rho v - 2v_\mu - 6\sigma v - 6v_\nu + 2v$
$= (\rho^2 + \sigma^2 - 2\rho - 6\sigma + 2)v + (2\rho - 2)v_\mu + (2\sigma - 6)v_\nu + v_{\mu\mu} + v_{\nu\nu}$
∴ let $2\rho - 2 = 0$ ∴ $\rho = 1$, $2\sigma - 6 = 0$ ∴ $\sigma = 3$ ∴
$\rho^2 + \sigma^2 - 2\rho - 6\sigma + 2 = 8$ ∴ $v_{\mu\mu} + v_{\nu\nu} = 8v$

4? / seperate u as variables then derivs into PDE then equal λ then as seperate ODE & BC ~~as~~ it then solve ODE of X & boundarys for $\lambda \le 0$, none, $\lambda = q^2 > 0$: $q = n\pi/a$, ∴ into Y ODE:
then apply BC ∴ u as sum ∴ IC:

4 / let $u = X(x)Y(y)$ ∴ $u_{xx} = X''Y$, $u_{yy} = XY''$ ∴
$X''(x)Y(y) + X(x)Y''(y) = 0$ ∴ $X''(x)Y = -XY''$ ∴
$\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda$ = constant ∴ $X'' = -\lambda X$, $Y'' = \lambda Y$ ∴

$X'' + \lambda X = 0$, $X(0) = 0$, $X(a) = 0$ ∴ $X = e^{px}$ ∴
$p^2 + \lambda = 0$ ∴ $p^2 = -\lambda$ ∴ $p = \pm\sqrt{-\lambda}$ ∴ $X = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$ ∴
$\lambda \le 0$: $X(0) = 0 = A + B$, ∴ $-B = +A$ ∴ $X = B(-e^{\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x})$ ∴
$X(a) = 0 = B(-e^{\sqrt{-\lambda}a} + e^{-\sqrt{-\lambda}a}) \ge B$ ∴ $B = 0$ ∴
no nontrivial solution for $\lambda \le 0$ ∴
for $\lambda = 0$: $X = A + Bx$ ∴ $X(0) = 0 = A + B(0) = A = 0$ ∴ $X = Bx$ ∴
$X(a) = 0 = Ba = B = 0$ ∴ no solutions ∴
for λ: $\lambda = q^2 > 0$; $\sqrt{-\lambda} = \sqrt{-q^2} = q$; ∴ $X = A\cos(qx) + B\sin(qx)$ ∴
$X(0) = 0 = A$ ∴ $X = B\sin(qx)$ ∴ $X(a) = 0 = B\sin(qa)$ ∴
$qa = n\pi$ ∴ $q = \frac{n\pi}{a}$ ∴ $q^2 = \frac{n^2\pi^2}{a^2} = \lambda$ ∴
~~X = B~~ $Y'' - \lambda Y = 0$ ∴ $Y'' - \frac{n^2\pi^2}{a^2}Y = 0$ ∴
$|u| < \infty$ as $y \to \infty$ ∴ $p^2 - \frac{n^2\pi^2}{a^2} = 0$ ∴ $p = \pm\frac{n\pi}{a}$ ∴
1) ~~Y(y) = Ase~~ $Y(y) = Ae^{\frac{n\pi}{a}y} + Be^{-\frac{n\pi}{a}y}$ ∴ as $y \to \infty$: $Y \ne \infty$ ∴
$A = 0$, $Y(y) = Be^{-\frac{n\pi}{a}y}$ ∴ super position:
$u = \sum_{n=1}^{\infty} B_n e^{-\frac{n\pi}{a}y} \sin\left(\frac{n\pi}{a}x\right)$ ∴

$u(x,0) = -u_0 x/a = \sum_{n=1}^{\infty} B_n e^0 \sin(\frac{n\pi}{a}x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{a}x)$

/4?/ into sources formula $B_n = \frac{1}{a}\int_{-a}^{a} \phi(x) \sin(\frac{n\pi}{a}x)\,dx$

to solve integral & simplify ∴ let $\frac{\pi}{2u_0} u = \Phi$ ∴

$e^{i\theta} = \cos\theta + i\sin\theta$ ∴ $\mathrm{Im}(e^{i\frac{n\pi}{a}x}) = \sin(\frac{n\pi}{a}x)$ ∴ $\Phi = \mathrm{Im}(\Psi)$ ∴

/4/ $-u_0 \frac{x}{a} = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{a}x)$ ∴ $B_n = \frac{1}{a}\int_{-a}^{a} \phi(x) \sin(\frac{n\pi}{a}x)\,dx =$

$\frac{1}{a}\int_{-a}^{a} -u_0 \frac{x}{a} \sin(\frac{n\pi}{a}x)\,dx = \frac{-2u_0}{a^2}\int_0^a x \sin(\frac{n\pi}{a}x)\,dx$ ∴

$\int_0^a x \sin(\frac{n\pi}{a}x)\,dx = \left[x(-1)\frac{a}{n\pi}\cos(\frac{n\pi}{a}x)\right]_0^a - \int_0^a \frac{-a}{n\pi}\cos(\frac{n\pi}{a}x)\,dx =$

$-\frac{a}{n\pi}\left[a\cos(\frac{n\pi}{a}a)\right] + \frac{a}{n\pi}\left[\frac{a}{n\pi}\sin(\frac{n\pi}{a}x)\right]_0^a =$

$-\frac{a^2}{n\pi}\cos(n\pi) = -\frac{a^2}{n\pi}(-1)^n$ ∴ $B_n = \frac{-2u_0}{a^2}\left(-\frac{a^2}{n\pi}\right)(-1)^n = \frac{2u_0}{n\pi}(-1)^n$

∴ $u = \sum_{n=1}^{\infty} \frac{2u_0}{n\pi}(-1)^n e^{-\frac{n\pi}{a}y} \sin(\frac{n\pi}{a}x) = \frac{2u_0}{\pi}\sum_{n=1}^{\infty} \frac{(-1)^n}{n} e^{-\frac{n\pi y}{a}} \sin(\frac{n\pi}{a}x)$

/5/ seperate u into $X(x)Y(y)$ ∴ $Y''/Y = -X''/X = k^2 > 0$ ∴ $Y'' - k^2 Y = 0$,

$X'' + k^2 X = 0$ ∴ write $X(x) =$, $Y(y) =$ solutions

∴ condition on Y to negate C_2 then combine X & Y as C_3 -- ∴

integrate out k by writing as integral ∴ BC on integral ∴

give C_3 as Fourier integral

/5/ $u = X(x)Y(y)$ ∴ $u_{xx} = X''(x)Y(y)$, $u_{yy} = X(x)Y''(y)$ ∴

$X''Y + XY'' = 0$ ∴ $X''Y = -XY''$ ∴ $\frac{X''}{X} = -\frac{Y''}{Y}$ ∴

$-\frac{X''}{X} = \frac{Y''}{Y} = k^2 > 0$ ∴ $X'' = -k^2 X$, $Y'' = k^2 Y$ ∴ $Y'' - k^2 Y = 0$,

$X'' + k^2 X = 0$ ∴ $p^2 + k^2 = 0$ ∴ $p = \pm\sqrt{k^2} = \pm|k|i$ ∴ $X = Ae^{|k|ix} + Be^{-|k|ix}$

∴ $p^2 = -k^2$ ∴ $p = \pm\sqrt{-k^2} = \pm\sqrt{k}\,i$ ∴ $X = Ce^{\sqrt{k^2}ix} = Ce^{kix}$,

$p^2 - k^2 = 0$ ∴ $p^2 = k^2$ ∴ $p = \pm|k|$ ∴ $Y = Ae^{|k|y} + Be^{-|k|y}$ ∴

$y \to \infty$ ∴ $u \to 0$ ∴ $A = 0$ ∴ $Y = Be^{-|k|y}$ ∴

$u = Ce^{kix} Be^{-|k|y} = C_3 e^{kix} e^{-|k|y}$ ∴

$u = \int_{-\infty}^{\infty} C_3 e^{kix} e^{-|k|y}\,dk$ ∴ $u(x,0) = f(x) = \int_{-\infty}^{\infty} C_3 e^{kix}\,dk$ ∴

$C_3(k) = \frac{1}{2\pi}\int_{-\infty}^{\infty} f(s) e^{kis}\,ds$

\Sheet 4/ \6/ seperate variables of u is R, Θ ∴ plug in, rearrange for $\frac{r^2}{R}$ for λ=const ∴ ODE of R, Θ ∴

1) Θ(θ) periodic 2π ∴ for $\sqrt{\lambda}$ & Θ sol ∴

solve for R ∴ solve for u ∴ BC on r=0, r finite ∴ u is sum

\6/ $u = R(r)\Theta(\theta)$ ∴ $u_{rr} = R''\Theta$, $u_r = R'\Theta$, $u_{\theta\theta} = R\Theta''$ ∴

$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0$ ∴ $R''\Theta + \frac{1}{r}R'\Theta = -\frac{1}{r^2}R\Theta''$ ∴

$r^2R''\Theta + rR'\Theta = -R\Theta''$ ∴ $\frac{r^2}{R}R'' + \frac{rR'}{R} = -\frac{\Theta''}{\Theta} = \lambda = \text{const}$ ∴

$\Theta'' = -\lambda\Theta$, $r^2R'' + rR' = \lambda R$ ∴ $r^2R'' + rR' - \lambda R = 0$,

$\Theta'' + \lambda\Theta = 0$ ∴ $\Theta(\theta) = \Theta(\theta + 2\pi)$ ∴

$p^2 + \lambda = 0$ ∴ $p^2 = -\lambda$ ∴ $p = \pm\sqrt{-\lambda} = \pm\sqrt{\lambda}\,i$ ∴ ∴

$\Theta = A\cos(\sqrt{\lambda}\theta) + B\sin(\sqrt{\lambda}\theta) = A\cos(\sqrt{\lambda}(\theta+2\pi)) + B\sin(\sqrt{\lambda}(\theta+2\pi)) =$

$A\cos(\sqrt{\lambda}\theta + 2\pi\sqrt{\lambda}) + B\sin(\sqrt{\lambda}\theta + 2\pi\sqrt{\lambda})$ ∴ $\sqrt{\lambda} = n$ ∴

let $R = r^m$ ∴ $R' = mr^{m-1}$ ∴ $R'' = m(m-1)r^{m-2} = (m^2-m)r^{m-2}$ ∴

$r^2R'' + rR - \lambda r^m = r^2(m^2-m)r^{m-2} + rmr^{m-1} - \lambda r^m = 0 = m^2 - m + m - \lambda =$

$m^2 - \lambda = 0$ ∴ $m^2 = \lambda$ ∴ $m = \pm\sqrt{\lambda}$ ∴ $R = Ar^{\sqrt{\lambda}} + Br^{-\sqrt{\lambda}}$ ∴

is $u = R\Theta$ ∴ $|R| < \infty$ as $r \to \infty$ ∴ $A = 0$ ∴ $m = \pm n$ ∴

$R = Ar^n + Br^{-n}$ ∴ $A = 0$ ∴ $R = Br^{-n}$ ∴

$u = Br^{-n}(A\cos(n\theta) + B\sin(n\theta)) = A_n\cos(n\theta)r^{-n} + B_n\sin(n\theta)r^{-n}$

for $n \geq 1$,

for $n = 0$: A_0 ∴ $u = A_0 + \sum_{n=1}^{\infty}(A_n\cos(n\theta)r^{-n} + B_n\sin(n\theta)r^{-n}) =$

$A_0 + \sum_{n=1}^{\infty}(A_n\cos(n\theta) + \sin(n\theta))r^{-n}$

\6/ put r = a BC then series A_0, A_n, B_n for $-\pi, \pi$ but its all 2π periodic

\6/ $u(a,\theta) = f(\theta) = A_0 + \sum_{n=1}^{\infty}(A_n\cos(n\theta) + B_n\sin(n\theta))a^{-n}$ ∴

$A_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\cancel{\cos(n\theta)}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\cancel{\cos(n\theta)}\,d\theta$

$A_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} f(\theta)\cos(n\theta)\,d\theta = \frac{a^n}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta$

$B_n = \frac{a^n}{\pi}\int_{-\pi}^{\pi} f(\theta)\sin(n\theta)\,d\theta = \frac{a^n}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta$

$\because a^n f(\theta) = \sum_{n=1}^{\infty} (A_n \cos(n\theta) + B_n \sin(n\theta)) + A_0 r^n \therefore$

$A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\,d\xi \qquad A_n = \frac{a^n}{\pi}\int_0^{2\pi} f(\xi)\cos(n\xi)\,d\xi,$

$B_n = \frac{a^n}{\pi}\int_0^{2\pi} f(\xi)\sin(n\xi)\,d\xi \quad \therefore$

$$u = \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\,d\xi + \sum_{n=1}^{\infty}\left(\frac{a^n}{\pi}\int_0^{2\pi} f(\xi)\cos(n\xi)\,d\xi\cos(n\theta) + \frac{a^n}{\pi}\int_0^{2\pi} f(\xi)\sin(n\xi)\,d\xi\sin(n\theta)\right) r^{-n}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\,d\xi + \sum_{n=1}^{\infty}\frac{a^n}{\pi r^n}\left(\int_0^{2\pi} f(\xi)\left[\cos(n\xi)\cos(n\theta) + \sin(n\xi)\sin(n\theta)\right]d\xi\right) =$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\xi)\,d\xi + \sum_{n=1}^{\infty}\frac{a^n}{\pi r^n}\int_0^{2\pi} f(\xi)\cos(n\xi - n\theta)\,d\xi =$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\xi)\,d\xi + \int_0^{2\pi} f(\xi)\sum_{n=1}^{\infty}\left(\frac{a^n}{\pi r^n}\cos(n(\xi-\theta))\right)d\xi =$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left[1 + 2\sum_{n=1}^{\infty}\frac{a^n}{r^n}\cos(n(\theta-\xi))\right]d\xi$$

∇/ $x \in \mathbb{R}$, $y \in [0,\infty)$ $\therefore u_{xx} + u_{yy}$; $u = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{f(s)}{(x-s)^2+y^2}\,ds$

$u(x,0) = \frac{1}{1+x^2} \therefore f(x) = \frac{1}{1+x^2} \quad \therefore f(s) = \frac{1}{1+s^2} \therefore$

$$\frac{f(s)}{(x-s)^2+y^2} = \frac{1}{(1+s)^2((x-s)^2+y^2)} = \frac{As+B}{1+s} + \frac{Cs+D}{(1+s)^2} + \frac{1}{((x-s)^2+y^2)} \therefore$$

$$\frac{\pi}{y}u = \int_{-\infty}^{\infty}\frac{1}{(1+s)^2((x-s)^2+y^2)}\,ds = \frac{\pi}{y}\,\frac{y+1}{x^2+(y+1)^2}$$

to verify by: sub. $\therefore$ derivs.

$\backslash 8/$ $u_{xx} = v_{xy}$, $u_{xy} = -v_{xx}$, $u_{yy} = -v_{xy}$ $\therefore$ LHS =

$u_{xx} + u_{yy} = RHS = v_{xy} + (-v_{xy}) = 0$,

$u_{xy} = v_{yy}$, $u_{xy} = -v_{xx} \therefore -u_{xy} = v_{xx} \therefore v_{xx} + v_{yy} = RHS = LHS = u_{xy} - u_{xy} = 0$

$\therefore$ is $v_1(x,y)$, $v_2(x,y)$ two sols for $v(x,y)$ $\therefore$

$w(x,y) = v_2(x,y) - v_1(x,y) \therefore \omega_x(x,y) = 0$, $\omega_y(x,y) = 0 \therefore$ in G $\therefore$

$\omega(x,y) = \text{const}$ in D $\therefore$ $\omega(\vec{x}_2) - \omega(\vec{x}_1) = \int_0^1 \nabla\omega(\vec{s}(t))\cdot d\vec{s}(t) = 0$

$\backslash 7/$ $u_{xx} + u_{yy} = 0$, $y = 0$: $u = \frac{1}{x^2+1}$, $u(x,\infty) = \lim_{y\to\infty}\frac{1}{(y+1)+\frac{x^2}{(y+1)}} = 0 \therefore$ BC satisfied

Sheet 5 $\because \delta(x-\xi)=0 \ \forall x\neq\xi \quad \therefore G(\xi^-)=G(\xi^+) \ \therefore$

$$\int_{\xi-\varepsilon}^{\xi+\varepsilon} G''(x)dx - a^2\int_{\xi-\varepsilon}^{\xi+\varepsilon} G(x)\,dx = \int_{\xi-\varepsilon}^{\xi+\varepsilon}\delta(x-\xi)dx \ \therefore$$

as $\varepsilon\to 0^+$ $\therefore \lim_{\varepsilon\to 0^+}\int_{\xi-\varepsilon}^{\xi+\varepsilon} G''(x)dx = \lim_{\varepsilon\to 0^+}\left[G'(\xi+\varepsilon)-G'(x-\varepsilon)\right] = G'(\xi^+)+G'(\xi^-)$,

$$\lim_{\varepsilon\to 0^+}\left|a^2\int_{\xi-\varepsilon}^{\xi+\varepsilon} G(x)dx\right| \le a^2\lim_{\varepsilon\to 0^+}\int_{\xi-\varepsilon}^{\xi+\varepsilon}|G(x)|dx \le a^2\lim_{\varepsilon\to 0^+}\left[\varepsilon \max_{x\in[\xi-\varepsilon,\xi+\varepsilon]}|G|=0\right],$$

$$\lim_{\varepsilon\to 0^+}\int_{\xi-\varepsilon}^{\xi+\varepsilon}\delta(x-\xi)dx = \lim_{\varepsilon\to 0^+} 1 = 1 \quad \therefore G'(\xi^+)-G'(\xi^-)=1 \ \therefore$$

$G_L(0)=0$, $G_R(\xi)-G_L(\xi)=0$, $G_R'(\xi)-G_L'(\xi)=1$, $G_R(l)=0$ $\therefore$

$G_L(x)=p\sinh(ax)$, $G_R(x)=q\sinh(a(l-x))$ $\therefore$

$G_L'(x)=ap\cosh(ax)$, $G_R'(x)=-aq\cosh(a(l-x))$ $\therefore$

$x=\xi$: $-\sinh(a\xi)p+\sinh(a(l-\xi))q=0$ $\therefore -a\cosh(a\xi)p-a\cosh(a(l-\xi))q=1$ $\therefore$

$p=-\dfrac{\sinh(a(l-\xi))}{2\sinh(al)}$, $q=-\dfrac{\sinh(a\xi)}{2\sinh(al)}$ $\therefore$

3a $G=G(x,t;\xi,\tau)$ $\therefore -\dfrac{\partial G}{\partial t}-a^2\dfrac{\partial^2 G}{\partial x^2}=\delta(x-\xi)\delta(t-\tau)$

$\dfrac{\partial G}{\partial \xi}(x=0)=\dfrac{\partial G}{\partial x}(x=l)=0$; $G(t=0)=0$ $\therefore$ for $t\in[\tau-\varepsilon,\tau-\varepsilon]$

$$\int_{\tau-\varepsilon}^{\tau+\varepsilon}\frac{\partial G}{\partial t}dt - a^2\int_{\tau-\varepsilon}^{\tau+\varepsilon}\frac{\partial^2 G}{\partial x^2}dt = \int_{\tau-\varepsilon}^{\tau+\varepsilon}\delta(x-\xi)\delta(t-\tau)dt \ \therefore$$

$$G(x,\tau+\varepsilon)-G(x,\tau-\varepsilon)-a^2\int_{\tau-\varepsilon}^{\tau+\varepsilon}\frac{\partial^2 G}{\partial x^2}dt=\delta(x-\xi)$$

for $x\in[0,l)$, $t\in[0,\tau-0]$ $\therefore G=0$ $\therefore$

for $t\in[\tau-\varepsilon,\tau-\varepsilon]$ already know $G(x,\tau-\varepsilon)=0$

for $A\neq\xi$, $B\neq\xi$ $\therefore \int_A^B\left[\int_{\tau-\varepsilon}^{\tau+\varepsilon}G_{Tx}dt\right]dx=\int_{\tau-\varepsilon}^{\tau+\varepsilon}\left[\int_A^B G_{Tx}dx\right]dt=$

$\int_{\tau-\varepsilon}^{\tau+\varepsilon}\left[G_x(B,t)-G_x(A,t)\right]dt \xrightarrow{\varepsilon\to 0} 0$

$\therefore G(x,\tau+0)=\delta(x-\xi)$ $\therefore$ $x\in[0,l]$, $t\in[\tau+0,\infty)$ $\therefore$

$\dfrac{\partial G}{\partial t}-a^2\dfrac{\partial^2 G}{\partial x^2}=0$ $\therefore G_x(0,t)=G_x(l,t)=0$ $\therefore G(t=\tau+0)=\delta(x-\xi)$

$G=T(t,\tau)X(x)$ $\therefore T'/(a^2T)=\dfrac{X''}{T}=-k^2$ $\therefore X''+k^2X=0$, $T'+k^2T=0$

$X(x)=A_n(\xi)\cos(n\pi x)/l$, $k^2=(n\pi)^2/l^2$, $n=0,1,2,\dots$

$T_n'-(n\pi)^2/l^2\,T_n=0$ $\therefore T_n(t,\tau)=B_n\tau e^{-a^2(n\pi)^2(t-\tau)/l^2}$ $\therefore$

$$G(x,t;\xi,\tau)=\sum_{n=0}^{\infty}C_n(\xi,\tau)e^{-a^2(n\pi)^2(t-\tau)/l^2}\cos(n\pi x/l) \ \therefore$$

$$\delta(x-\xi)=\sum_{n=0}^{\infty}C_n(\xi,\tau)\cos\left(\frac{n\pi x}{l}\right) \ \therefore \ C_n(\xi,\tau)=\frac{2}{Q_n l}\int_0^l\delta(x-\xi)\cos\left(\frac{n\pi x}{l}\right)dx=$$

h_n

$\frac{2}{Q_n l}\cos\left(\frac{n\pi\xi}{l}\right)$, $Q_0=2$, $Q_n=1$, $n\geq 1$ ∴

$$G(x,t;\xi,\tau)=\sum_{n=0}^{\infty}\frac{2}{Q_n l}\cos\left(\frac{n\pi\xi}{l}\right)\cos\left(\frac{n\pi x}{l}\right)e^{-(a\pi n/l)^2(t-\tau)}$$

/3b/ $$u(x,t)=\int_{\tau=0}^{t}\int_{\xi=0}^{l}[\omega\sin\omega\tau]\left[\sum_{n=0}^{n=\infty}\left(\frac{2}{Q_n l}\cos\left(\frac{n\pi\xi}{l}\right)\right)e^{-a^2(n\pi)^2(t-\tau)/l^2}\cos\left(\frac{n\pi x}{l}\right)\right]d\xi\, d\tau$$

∴ only $n=0$ gives non zero ∴ $$u(x,t)=\int_{\tau=0}^{t}\left(\int_{\xi=0}^{l}[\omega\sin\omega\tau]\left[\left(\frac{2}{Q_0 l}\right)\right]d\xi\right)d\tau=1-\cos\omega t$$

/4a/ let $u_t-a^2u_{xx}=f(x,t)$ ∴ $f(x,t)=p\left(1-\frac{x}{l}\right)e^{-\beta t}$ ∴

$$u(x,t)=\int_{\tau=0}^{t}\int_{\xi=0}^{l}f(x,t)G(x,t;\xi,\tau)\,d\xi\, d\tau \quad ∴$$

$\frac{\partial G}{\partial t}-a^2\frac{\partial^2 G}{\partial x^2}=\delta(x-\xi)\delta(t-\tau)$, $G(t=\tau-0)=0$, $G(x=0)=0$, $G(x=l)=0$

$t\in[\tau-0,\tau+0]$: $\int_{\tau-0}^{\tau+0}\frac{\partial^2 G}{\partial t}dt-a^2\int_{\tau-0}^{\tau+0}\frac{\partial^2 G}{\partial x^2}dt=\int_{\tau-0}^{\tau+0}\frac{\partial^2 G}{\partial x^2}dt=\int_{\tau-0}^{\tau+0}\delta(x-\xi)\delta(t-\tau)dt$

∴ $G(t<\tau)=0$, $G(t=\tau+0)=\delta(x-\xi)$

for $t>\tau$: $G_t-a^2G_{xx}=0$; $G(x=0)=G(x=l)=0$; $G(t=\tau-0)=\delta(x-\xi)$ ∴

~~$G=k$~~ $G=T(t,\tau)X(x)$ ∴ $T'(a^2T)=X''/X=-k^2$ ∴

$X''+k^2X=0$, $T'+k^2T=0$ ∴

$X(x)=A_n(\xi)\sin(n\pi x/l)$, $k^2=n^2\pi^2/l^2$, $n=1,2,3,\dots$ ∴

$T_n'-(n^2\pi^2/l^2)T_n=0 \rightarrow T_n(t,\tau)=B_n(\tau)e^{-a^2n^2\pi^2(t-\tau)/l^2}$ ∴

$$G(x,t;\xi,\tau)=\sum_{n=1}^{n=\infty}C_n(\xi,\tau)e^{-a^2n^2\pi^2(t-\tau)/l^2}\sin(n\pi x/l),$$

$$\delta(x-\xi)=\sum_{n=1}^{\infty}C_n(\xi,\tau)\sin\frac{n\pi x}{l},$$

$$C_n(\xi,\tau)=\frac{2}{l}\int_0^l\delta(x-\xi)\sin\left(\frac{n\pi x}{l}\right)dx=\frac{2}{l}\sin\left(\frac{n\pi\xi}{l}\right),\ n\geq 1 \quad ∴$$

$$G(x,t;\xi,\tau)=\frac{2}{l}\sum_{n=1}^{n=\infty}\sin\left(\frac{n\pi\xi}{l}\right)\sin\left(\frac{n\pi x}{l}\right)e^{-a^2(n\pi)^2(t-\tau)/l^2} \quad ∴$$

$$u(x,t)=\sum_{n=1}^{\infty}\frac{2\beta l^2\left[e^{-\beta t}-e^{-a^2(n\pi)^2t/l^2}\right]}{n\pi\left[a^2(n\pi)^2-\beta l^2\right]}\sin\left(\frac{n\pi x}{l}\right) \quad ∴$$

$$u(x,t)=\int_{\tau=0}^{t}\int_{\xi=0}^{l}\left[\beta\left(1-\frac{\xi}{l}\right)e^{-\beta\tau}\right]\left[\sum_{n=1}^{n=\infty}\left(\frac{2}{l}\sin\frac{n\pi\xi}{l}\right)e^{-a^2(n\pi)^2(t-\tau)/l^2}\sin\left(\frac{n\pi x}{l}\right)\right]d\xi\, d\tau$$

$$=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{l}\left(\frac{2\beta}{l}\right)\int_{\tau=0}^{t}\left[e^{-\beta\tau}e^{-a^2(n\pi)^2(t-\tau)/l^2}\right]d\tau\int_{\xi=0}^{l}\left[\sin\left(\frac{n\pi\xi}{l}\right)\left(1-\frac{\xi}{l}\right)\right]d\xi \quad \text{but}$$

$$\int_{\xi=0}^{l}\left[\sin\left(\frac{n\pi\xi}{l}\right)\left(1-\frac{\xi}{l}\right)\right]d\xi=\frac{l}{n\pi} \quad ∴$$

$$u(x,t)=\sum_{n=1}^{\infty}\sin\left(\frac{n\pi x}{l}\right)\left(\frac{2\beta}{l}\right)\int_{\tau=0}^{t}\left[e^{-\beta\tau}e^{-a^2(n\pi)^2(t-\tau)/l^2}\right]d\tau\left(\frac{l}{n\pi}\right)=$$

Sheet 5 / 1a / note that $\int_{-\infty}^{\infty} Q(x,a)\,dx = \int_{-\infty}^{\infty} P(x/a)\frac{dx}{a}\Big|_{x=az} =$

$\int_{-\infty}^{\infty} P(z)\,dz = 1$ $\therefore$ $\delta(x) = \delta(x)\int_{-\infty}^{\infty} Q(\xi - x, a)\,d\xi = \int_{-\infty}^{\infty} \delta(x) Q(\xi - x, a)\,d\xi$ $\therefore$

$\lim_{a\to 0}\int_{-\infty}^{\infty} (\delta(\xi) - \delta(x)) Q(\xi - x, a)\,d\xi = 0$ $\therefore$

$\lim_{a\to 0}\left|\int_{-\infty}^{\infty} (\delta(\xi) - \delta(x)) Q(\xi - x, a)\,d\xi\right| \le \lim_{a\to 0}\int_{-\infty}^{\infty} |\delta(\xi) - \delta(x)|\,|Q(\xi - x, a)|\,d\xi$

$\le k \lim_{a\to 0}\int_{-\infty}^{\infty} |\xi - x|\,|Q(\xi - x, a)|\,d\xi$ $\because$ as lipschitz continuity of $\delta(x)$

$= k \lim_{a\to 0}\int_{-\infty}^{\infty} |\xi - x|\left|P\left(\frac{\xi - x}{a}\right)\right|\frac{d\xi}{a}\Big|_{\xi = x + az} = k\lim_{a\to 0}\int_{-\infty}^{\infty} a|z|\,|P(z)|\,dz = kR\lim_{a\to 0}(a) = 0$

/2a/ $\frac{d^2y}{dx^2} - a^2 y = \delta(x)$ $\therefore$ GS: $y = Ae^{ax} + Be^{-ax}$ $\therefore$

$y(0) = 0$ $\therefore$ $A + B = 0$ $\therefore$ $A = -B = C \in \mathbb{R}$ $\therefore$

$y = C(e^{ax} + e^{-ax}) = \frac{1}{2}C\sinh(ax)$ $\therefore$ $C \neq 0$ $\therefore$ let $C = 2$ $\therefore$

$y_2 = \sinh(ax)$

$y(L) = 0$ $\therefore$ $Ae^{aL} + Be^{-aL} = 0$ $\therefore$ $A = -Ce^{-aL}$, $B = Ce^{aL}$, $C \in \mathbb{R}$ $\therefore$

$y = C(-e^{a(x-L)} + e^{a(L-x)}) = \frac{1}{2}C\sinh(a(L-x))$ $\therefore$ let $C = 2$ $\therefore$

$y_1(x) = \sinh(a(L-x))$ $\therefore$ $y_1(x) = \sinh(a(L-x))$, $y_2(x) = \sinh(ax)$ $\therefore$

~~W(x)~~ Wronskian: $W(x) = \begin{vmatrix} y_1(x) & y_2(x) \\ y_1'(x) & y_2'(x) \end{vmatrix} = \begin{vmatrix} \sinh(a(L-x)) & \sinh(ax) \\ -a\cosh(a(L-x)) & a\cosh(ax) \end{vmatrix}$

$= a(\sinh(a(L-x))\cosh(ax) + \cosh(a(L-x))\sinh(ax)) = a\sinh(aL)$ $\therefore$

$y(x) = -y_1(x)\left(C_1 + \int_0^{x} \frac{y_2(\xi)\delta(\xi)}{W(\xi)}\,d\xi\right) + y_2(x)\left(C_2 + \int_L^{x} \frac{y_1(\xi)\delta(\xi)}{W(\xi)}\,d\xi\right)$

$y_1(L) = 0$, $y_2(0) = 0$ $\therefore$

$y(0) = -y_1(0)\left(C_1 + \int_0^{0} \frac{y_2(\xi)\delta(\xi)}{W(\xi)}\,d\xi\right) + y_2(0)\left(C_2 + \int_L^{0} \frac{y_1(\xi)\delta(\xi)}{W(\xi)}\,d\xi\right) =$

$-C_1\sinh(aL) = 0$ $\therefore$ $C_1 = 0$ $\therefore$

$y(L) = -y_1(L)\left(C_1 + \int_0^{L} \frac{y_2(\xi)\delta(\xi)}{W(\xi)}\,d\xi\right) + y_2(L)\left(C_2 + \int_L^{L} \frac{y_1(\xi)\delta(\xi)}{W(\xi)}\,d\xi\right) =$

$C_2\sinh(aL) = 0$ $\therefore$ $C_2 = 0$ $\therefore$ $y(x) = -y_1(x)\int_0^{x} \frac{y_2(\xi)\delta(\xi)}{a\sinh(aL)}\,d\xi + y_2(x)\int_L^{x} \frac{y_1(\xi)\delta(\xi)}{a\sinh(aL)}\,d\xi$

$y(x) = -\int_0^{x} \frac{y_1(x)y_2(\xi)\delta(\xi)}{a\sinh(aL)}\,d\xi - \int_x^{L} \frac{y_2(x)y_1(\xi)\delta(\xi)}{a\sinh(aL)}\,d\xi = \int_0^{L} G(x,\xi)\delta(\xi)\,d\xi$

$G(x,\xi) = \begin{cases} (-\frac{1}{a}y_1(x)y_2(\xi)/\sinh(aL), & \xi \in (0,x) \\ -\frac{1}{a}y_2(x)y_1(\xi)/\sinh(aL), & \xi \in (x,L) \end{cases}$

$u_{xx}-u_{yy}=0 \therefore a=1,\ 2b=0,\ c=-1 \therefore b=0 \therefore -2b=0 \therefore$

2/Q. $b^2-ac=0^2-1(-1)=1>0 \quad \therefore$ hyperbolic $\therefore$

Cr $\left(\frac{dy}{dx}\right)^2-1=0 \quad \therefore \left(\frac{dy}{dx}\right)^2=1 \therefore \frac{dy}{dx}=\pm 1 \quad \therefore \frac{dy}{dx}=1,\ \frac{dy}{dx}=-1$

$\therefore y=x+C_1,\quad y=-x+C_2 \quad \therefore y-x=C_1,\ y+x=C_2 \therefore$

$\xi=y+x,\ \eta=y-x \therefore \xi=x+y,\ \eta=-x+y \quad \therefore$

$\therefore \xi_x=1,\ \xi_y=1,\ \eta_x=-1,\ \eta_y=1 \therefore$

$u_x=u_\xi\xi_x+u_\eta\eta_x=u_\xi-u_\eta \therefore$

$u_{xx}=u_{\xi\xi}\xi_x-u_{\xi\eta}\xi_x+u_{\xi\eta}\eta_x-u_{\eta\eta}\eta_x=u_{\xi\xi}-u_{\xi\eta}-u_{\xi\eta}+u_{\eta\eta}=$

$u_{\xi\xi}-2u_{\xi\eta}+u_{\eta\eta}$

$u_y=u_\xi\xi_y+u_\eta\eta_y=u_\xi+u_\eta \therefore$

$u_{yy}=u_{\xi\xi}\xi_y+u_{\xi\eta}\xi_y+u_{\xi\eta}\eta_y+u_{\eta\eta}\eta_y=u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta} \therefore$

$u_{xx}-u_{yy}=u_{\xi\xi}-2u_{\xi\eta}+u_{\eta\eta}-u_{\xi\xi}-2u_{\xi\eta}-u_{\eta\eta}=-4u_{\xi\eta}=0 \therefore$

$u_{\xi\eta}=0 \therefore$

$u_\eta=f(\eta) \quad \therefore u=f(\eta)+g(\xi) \therefore$

$u(x,y)=f(y-x)+g(y+x)$

$u_{xx}-u_{yy} \quad \therefore$ let $u=f(\alpha x+y) \therefore u_{xx}=\alpha^2 f'',\ u_{yy}=f'' \therefore$

$\alpha^2 f''-f''=0=(\alpha^2-1)f'' \therefore \alpha^2-1=0 \therefore \alpha^2=1 \therefore \alpha=1,-1 \therefore$

$u=f(x+y)+g(-x+y)$

2b/ $\therefore G(0,\xi)=-\frac{1}{a\sinh(aL)}\sinh(a\cdot 0)\sinh(a(L-\xi))=0$

$G(x,\xi)=-\frac{1}{a\sinh(aL)}\sinh(a(L-L))\sinh(a\xi)=0 \quad \therefore$ for $x\neq\xi \quad \therefore G(x)=C\sinh(ax)$

or $G(x)=C\sinh(a(L-x)) \quad \therefore G''-a^2G=0 \therefore$

$\lim_{\varepsilon\to 0^+}\int_{\xi-\varepsilon}^{\xi+\varepsilon}(G''-a^2G)\,dx=1 \quad \therefore \left|a^2\int_{\xi-\varepsilon}^{\xi+\varepsilon}G(x)\,dx\right|\le 2a^2\varepsilon\max_{x\in[\xi-\varepsilon,\xi+\varepsilon]}|G|\to 0$ as $\varepsilon\to 0$

$\therefore \int_{\xi-\varepsilon}^{\xi+\varepsilon}G''(x)\,dx=G'(\xi+\varepsilon)-G'(\xi-\varepsilon)=$

$\frac{1}{a\sinh(aL)}\left[a\cosh(a(\xi+\varepsilon))\sinh(a(L-\xi))-a\cosh(a(L-\xi+\varepsilon))\sinh(a\xi)\right]$

$\xrightarrow[\varepsilon\to 0]{} a\cosh(a\xi)\sinh(a(L-\xi))+\cosh(a(L-(\xi+\varepsilon)))\sinh(a\xi))/(a\sinh(aL))$

$=\frac{a\sinh(aL)}{a\sinh(aL)}=1$

Sheet 5 $\sum_{n=1}^{n=\infty} \sin\left(\frac{n\pi x}{L}\right)\left(\frac{2\beta}{n\pi}\right) e^{-a^2(n\pi)^2 t/L^2} \int_{t=0}^{t} e^{[a^2(n\pi)^2/L^2-\beta]\tau}\, d\tau =$

$\sum_{n=1}^{n=\infty} \sin\left(\frac{n\pi x}{L}\right)\left(\frac{2\beta}{n\pi}\right) e^{-a^2(n\pi)^2 t/L^2} \left[a^2(n\pi)^2/L^2-\beta\right]^{-1}\left[e^{[a^2(n\pi)^2/L^2-\beta]t}-1\right]$ ∴

$\therefore\ u(x,t)=\sum_{n=1}^{n=\infty} \sin\left(\frac{n\pi x}{L}\right)\left(\frac{2\beta}{n\pi}\right)\left[a^2(n\pi)^2/L^2-\beta\right]^{-1}\left[e^{-\beta t}-e^{-a^2(n\pi)^2 t/L^2}\right]$

5 odd extension ∴ $u_t - a^2 u_{xx} = 0$, $-\infty \le x \le \infty$ ∴

$u(x,0)=0$, $u(0,t)=0$ ∴

5.5.5 Greens function for $-\infty < x < \infty$ but $0 \le x < \infty$ ∴

extend problem to $-\infty < x < 0$ ∴ let $F(x,t) = f(x,t)$, $0 < x < \infty$,

$F(x,t) = -f(-x,t)$, $-\infty < x < 0$

which makes $F(x,t)$ an odd function of x. with this inhomogeneity being odd in x, the solution will also be odd in x ∴ the condition at $u(0,t)=0$ will automatically satisfied ∴

problem: $u_t - a^2 u_{xx} = F(x,t)$, $-\infty < x < \infty$, $u(x,t=0)=0$ ∴

$$u(x,t)=\int_{\tau=0}^{t}\int_{\xi=-\infty}^{\infty} F(\xi,\tau)\left[\frac{1}{2a\sqrt{\pi(t-\tau)}}\, e^{-(x-\xi)^2/(4a^2(t-\tau))}\right] d\xi\, d\tau \ ∴$$

Following the definition of function F, split the integral in ξ over $\mathbb{R}$ in two parts, $(-\infty,0]$, $[0,\infty)$ ∴

$$u(x,t)=\int_{\tau=0}^{t}\int_{\xi=-\infty}^{0}\left[-f(-\xi,\tau)\right]\frac{1}{2a\sqrt{\pi(t-\tau)}}\, e^{-(x-\xi)^2/(4a^2(t-\tau))}\, d\xi\, d\tau +$$

$$\int_{\tau=0}^{t}\int_{\xi=0}^{\infty} f(\xi,\tau)\frac{1}{2a\sqrt{\pi(t-\tau)}}\, e^{-(x-\xi)^2/(4a^2(t-\tau))}\, d\xi\, d\tau$$

odd extension ∴ $x \in \mathbb{R}$ ∴ $H(x,t) = f(x,t)$ for $0 \le x < \infty$

$H(x,t) = -f(-x,t)$ for $0 < x < \infty$

∴ let $\xi \to -\xi$ ∴ $u(x,t) = \int_{\tau}^{t}\int_{\xi=\infty}^{0}\left[-f(\xi,\tau)\right]\frac{1}{2a\sqrt{\pi(t-\tau)}}\, e^{-(x-\xi)^2/(4a^2(t-\tau))}\, d(-\xi)\, d\tau +$

$\int_{\tau=0}^{t}\int_{\xi=0}^{\infty} f(\xi,\tau)\frac{1}{2a\sqrt{\pi(t-\tau)}}\, e^{-(x-\xi)^2/(4a^2(t-\tau))}\, d\xi\, d\tau$ ∴

$$u(x,t) = -\int_{\tau=0}^{t}\int_{\xi=0}^{\infty} f(\xi,\tau)\frac{1}{2a\sqrt{\pi(t-\tau)}}\, e^{-(x+\xi)^2/(4a^2(t-\tau))}\, d\xi\, d\tau +$$

$$\int_{\tau=0}^{t}\int_{\xi=0}^{\infty} f(\xi,\tau)\frac{1}{2a\sqrt{\pi(t-\tau)}}\, e^{-(x-\xi)^2/(4a^2(t-\tau))}\, d\xi\, d\tau \ ∴$$

$$u(x,t)=\int_{\tau=0}^{t}\int_{\xi=0}^{\infty} f(\xi,\tau)\, G_T(x,t;\xi,\tau)\, d\xi\, d\tau \ ∴$$

the required Green's function for the problem on the half line is

$$G(x,t;\xi,\tau) = \frac{1}{2a\sqrt{\pi(t-\tau)}}\left[e^{-(x-\xi)^2/(4a^2(t-\tau))} - e^{-(x-\xi)^2/(4a^2(t-\tau))}\right]$$

\3a/ For two given scalar fields ~~the~~ ϕ, ψ the divergence theorem applied to the vector field $\phi\nabla\psi$ gives $\int_D \nabla\cdot(\phi\nabla\psi)\,dV = \cancel{\int_S \phi\nabla\psi\cdot d\underline{S}}$

$\cancel{\int_S (\phi\nabla\psi)\cdot}$ $\int_S \phi\nabla\psi\cdot\hat{\underline{n}}\,dS = \int_{\partial D}\phi\,\nabla\psi\cdot\hat{\underline{n}}\,dS = \int_{\partial D}\phi\nabla\psi\cdot\vec{n}\,dS = \int_{\partial D}\phi\frac{\partial\psi}{\partial n}\,dS$

using identity $\nabla\cdot(\phi\nabla\psi) = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi$ $\therefore$

$\nabla\cdot(\phi\nabla\psi) = \phi\nabla\cdot\nabla\psi + \nabla\psi\cdot\nabla\phi = \phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi$ $\therefore$

$\int_D \nabla\cdot(\phi\nabla\psi)\,dV - \int_D(\phi\nabla^2\psi + \nabla\phi\cdot\nabla\psi)\,dV = \int_{\partial D}\phi\frac{\partial\psi}{\partial n}\,dS$ $\therefore$

swapping the roles of ϕ, ψ: $\int_{\partial D}\psi\frac{\partial\phi}{\partial n}\,dS = \int_D(\psi\nabla^2\phi + \nabla\phi\cdot\nabla\psi)\,dV$ $\therefore$

$\int_{\partial D}\left(\phi\frac{\partial\psi}{\partial n} - \psi\frac{\partial\phi}{\partial n}\right)dS = \int_D(\phi\nabla^2\psi - \psi\nabla^2\phi)\,dV$

\3b/ for $\phi = u$, $\psi = G$: $\int_{\partial D}\left(u\frac{\partial G}{\partial n} - G\frac{\partial u}{\partial n}\right)dS = \int_D(u\nabla^2 G - G\nabla^2 u)\,dV$

, $G|_{\partial D} = 0$, $\nabla^2 G = \delta(\vec{r}-\vec{r}_0)$ $\therefore$

$u|_{\partial D} = g$, $\nabla^2 u = f$ $\therefore$ $\int_{\partial D}\left(g\frac{\partial G}{\partial n} - 0\cdot\frac{\partial u}{\partial n}\right)dS = \int_D(u\,\delta(\vec{r}-\vec{r}_0) - Gf)\,dV$ $\therefore$

$\int_D u(\vec{r})\delta(\vec{r}-\vec{r}_0)\,dV = u(\vec{r}_0)$ $\therefore$ $u(\vec{r}_0) = \int_{\partial D} g(\vec{r})\frac{\partial G(\vec{r},\vec{r}_0)}{\partial n}\,dS + \int_D G(\vec{r},\vec{r}_0)f(\vec{r})\,dV$

\7/ $\because$ $u(\vec{r}) = \int_{\mathbb{R}^2} G(\vec{r},\vec{r}\,')f(\vec{r}\,')$ $\therefore$ $H(\vec{r}) = G(\vec{r},\vec{0})$ $\therefore$

$G(\vec{r},\vec{r}\,') = H(\vec{r}-\vec{r}\,')$ $\therefore$ $\nabla^2 H(\vec{r}) = \delta(\vec{r})$ $\therefore$ expect

$|\nabla H(\vec{r})| \to 0$ as $|\vec{r}| \to \infty$ $\therefore$ may have solutions invariant with rotation

\Sheet 6/ \1a/ $\therefore$ $1 + P_x + t\,g'(P)P_x = P_x(1 + t\,g'(P))$ $\therefore$ $P_x = \frac{1}{1+t\,g'(P)}$ ✓

$0 = P_t + g(P) + t\,g'(P)P_t = P_t(1 + t\,g'(P)) + g(P)$ $\therefore$ $\frac{-g(P)}{1-t\,g'(P)} = P_t$

$\therefore$ $g(P)\,g_x = g(P)g'(P)\,P_x = g(P)g'(P)\frac{1}{1+t\,g'(P)} = g'(P)\frac{-g(P)}{1-t\,g'(P)}(-1) = -g_t$

$\therefore$ $g_t + g\,g_x = 0$ $\therefore$

$u(x,t) = g(P(x,t))$ $\therefore$ $u(x,0) = g(x) = g(P(x,t=0))$ ✓ $\therefore$

at $t=0$: $x = P + 0\,g(P) = P$ $\therefore$ $u(x,0) = g(P(x,t=0)) = g(x)$

\1a 5d/ $x + P + t\,g(P)$ $\therefore$ $1 = P_x + t\,g'(P)P_x$ $\therefore$ $P_x = \frac{1}{1+t\,g'(P)}$ $\therefore$

$0 = P_t + g(P) + t\,g'(P)P_t$ $\therefore$ $P_t = \frac{-g(P)}{(1+t\,g'(P))}$ $\therefore$

Sheet 6 $\therefore\ u = f(p(x,t))$, $u_t = f'(p)p_t = \dfrac{-f(p)f'(p)}{1+tf'(p)}$

$u_x = f'(p)p_x = \dfrac{f'(p)}{1+tf'(p)}$ $\therefore\ u_t + uu_x = f'(p)\dfrac{-f(p)}{1+tf'(p)} + \dfrac{f(p)f'(p)}{1+tf'(p)} = 0$

$\therefore$ let $t=0$ $\therefore$ $x = p + (0)f = p(x,0)$ $\therefore$ $u(x,0) = f(p(x,0)) = f(x)$

Star $f(x) = \sin(x)$:

$\therefore\ u_x = f'(p(x,t))p_x(x,t) = \dfrac{f'(p(x,t))}{1+tf'(p(x,t))} = \dfrac{\cos(p(x,t))}{1+t\cos(p(x,t))}$

$\therefore$ at $t \to 1-0$: denominator tends to zero

u_x diverges when

$\cos(p(x,t)) = -1$ $\therefore$ $p(x,t) = (2n+1)\pi$, $n \in \mathbb{Z}$ $\therefore$

$x + p + tf(p) = p + t\sin(p) = (2n+1)\pi + 1\cdot\sin((2n+1)\pi) = (2n+1)\pi$

2a) $u_t = t^{\alpha-1}\left[\alpha f(\xi) + \beta\xi f'(\xi)\right]$

$u_t = \alpha t^{\alpha-1} f + f'(\xi)\dfrac{\partial \xi}{\partial t} t^{\alpha} = \alpha t^{\alpha-1} f + f'(\xi)\beta x t^{\beta-1} t t^{\alpha-1} =$

$\alpha t^{\alpha-1} f + f'(\xi)\beta x t^{\beta} t^{\alpha-1} = \alpha t^{\alpha-1} f + f'(\xi)\beta\xi t^{\alpha-1} = t^{\alpha-1}\left[\alpha f + \beta\xi f'\right]$

$\therefore\ u_t = t^{\alpha-1}\left[\alpha f(\xi) + \beta f'(\xi)\right]$, $u_{tx} = t^{\alpha+\beta} f''$ $\therefore$ $u_x = t^{\alpha} f'(\xi)\xi_x = t^{\alpha} f'(\xi) t^{\beta}$ $\therefore$

$u_x = t^{\alpha+\beta} f'(\xi)$ $\therefore$ $u_{xx} = t^{\alpha+\beta} f''(\xi)\xi_x = t^{\alpha+\beta} f''(\xi) t^{\beta} = t^{\alpha+2\beta} f''(\xi)$

$\therefore$ PDE:

$t^{\alpha-1}\left[\alpha f(\xi) + \beta\xi f'(\xi)\right] = t^{\alpha+2\beta} f''(\xi)$ $\therefore$

if $\alpha - 1 = \alpha + 2\beta$ the dependence on t cancels out $\therefore$

$-1 = +2\beta$ $\therefore$ $\beta = -\frac{1}{2}$ $\therefore$ $u(x,t) = \cancel{t^{\alpha}}$

$\alpha f(\xi) + \xi f'(\xi) = f''(\xi)$ $\therefore$ $u(x,t) = t^{\alpha} f(\xi) = t^{\alpha} f(xt^{\beta}) = t^{\alpha} f(xt^{-1/2})$

$\therefore\ \int_{-\infty}^{\infty} |u(x,t)|\,dx < \infty$ $\therefore$ $\int_{-\infty}^{\infty} u(x,t)\,dx < \infty$ $\therefore$

$\frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\,dx = 0$ $\therefore$ $u(x,t) = t^{\alpha} f(xt^{-1/2})$ $\therefore$

$\int_{-\infty}^{\infty} u(x,t)\,dx = t^{\alpha}\int_{-\infty}^{\infty} f(xt^{-1/2})\,dx\Big|_{xt^{-1/2}=\xi} = t^{\alpha} t^{1/2}\int_{-\infty}^{\infty} f(\xi)\,d\xi = \text{const}$ $\therefore$ $\alpha = -\frac{1}{2}$ $\therefore$

$f'' + \frac{1}{2}\xi f' + \frac{1}{2} f = 0$ $\therefore$ $(\xi f'')' = \frac{d}{dx}(\xi f)' = \xi f' + f$ $\therefore$

$2\frac{df}{d\xi} + \frac{d}{dx}(\xi f) = 0$ $\therefore$ $2\frac{df}{d\xi} = A - \xi f$, $A = \text{const}$ $\therefore$

for $\int_{-\infty}^{\infty} |f(\xi)|\,d\xi < \infty$ need: $f(\pm\infty) = f'(\pm\infty) = 0$ $\therefore$ $A = 0$ $\therefore$

$f(\xi) = Be^{-\xi^2/4}$, $B = \text{const}$ $\therefore$ $u(x,t) = Bt^{-1/2} e^{-x^2/(4t)}$, $u(x,0) = B\delta(x)$

$\therefore u_{tt} = t^{\alpha-2}\left\{\alpha(\alpha-1)f + (2\alpha+\beta-1)\beta\xi f' + \beta^2\xi^2 f''\right\} \therefore \quad u_{tt} = u_{xx} \therefore$

$t^{\alpha-2}\left\{\alpha(\alpha-1)f + (2\alpha+\beta-1)\beta\xi f' + \beta^2\xi^2 f''\right\} = t^{\alpha+2\beta} f''(\xi) \therefore$

if $\alpha - 2 = \alpha + 2\beta$, dependence of t cancels out $\therefore \beta = -1 \therefore$

$u(x,t) = t^{\alpha} f(x\frac{1}{t}) \quad \therefore \quad \int_{-\infty}^{\infty} u(x,t)\,dx < \infty \therefore$

$\int_{-\infty}^{\infty} t^{\alpha} f(x\frac{1}{t})\,dx < \infty \quad \therefore \frac{d}{dt}\int_{-\infty}^{\infty} t^{\alpha} f(x\frac{1}{t})\,dx = 0 \therefore$

$\int_{-\infty}^{\infty} u(x,t)\,dx = t^{\alpha}\int_{-\infty}^{\infty} f(x\frac{1}{t})\,dx\Big|_{x\frac{1}{t}=\xi} = t^{\alpha-1}\int_{-\infty}^{\infty} f(\xi)\,d\xi = \text{const} \quad \therefore \alpha = -1 \therefore$

$(\xi^2-1)f'' + 4\xi f' + 2f = 0 \quad \therefore$

let $g(\xi) = (\xi^2-1)f(\xi) \quad \therefore \quad g'(\xi) = \frac{d}{d\xi}(\xi^2-1)f(\xi) = (2\xi)f(\xi) + (\xi^2-1)f'(\xi) \therefore$

$g''(\xi) = 2f(\xi) + 2\xi f'(\xi) + (2\xi)f'(\xi) + (\xi^2-1)f''(\xi) =$

$(\xi^2-1)f'' + 4\xi f' + 2f = 0 = g''(\xi) \therefore$

$g' = B \quad \therefore g(\xi) = A + B\xi = (\xi^2-1)f'' \quad \therefore \quad f'' = \frac{A+B\xi}{\xi^2-1} \quad \therefore$

non trivial solution unbounded at $\pm 1 = \xi$

2b $u = U(x-ct) \therefore \quad u_t = -cU'(x-ct)$, $u_{xx} = U''(x-ct) \therefore \quad \xi = x-ct \therefore$

$-cU'(x-ct) = U''(x-ct) \quad \therefore -cU'(\xi) = U''(\xi)$

$U(\xi) = A + Be^{-c\xi} \quad \therefore \forall c$ cannot have $\int_{-\infty}^{\infty}|u(x,t)|\,dx < \infty \quad \therefore$

$c^2U''(\xi) = U''(\xi) \quad \therefore$ for $c = \pm 1 : U(\xi)$ any twice differentiable function

3a let rescaled variables be X, T, U, related to the old variables through $X = Ax$, $T = Bt$, $u = CU \therefore$

$u_t = \frac{\partial u}{\partial U}\frac{\partial U}{\partial T}\frac{\partial T}{\partial t} = BC\frac{\partial U}{\partial T}$, $\frac{\partial u}{\partial x} = \frac{\partial u}{\partial U}\frac{\partial U}{\partial X}\frac{\partial X}{\partial x} = \frac{du}{dU}\frac{\partial U}{\partial X}\frac{\partial X}{\partial x} = AC\frac{\partial U}{\partial T}$

$\frac{\partial^3 u}{\partial x^3} = u_{xxx} = \frac{\partial u}{\partial U}\frac{\partial U}{\partial X^3}\left(\frac{dX}{dx}\right)^3 = A^3C\frac{\partial U}{\partial T} \quad \therefore$

$BCU_T + \alpha AC^2U_X + \beta A^3C\,U_{XXX} = 0 \quad \therefore U_T + U_X + U_{XXX} = 0 \quad \therefore$

$BC = \alpha AC^2 = \beta A^3C \quad \therefore \quad A = 1, B = \beta, C = \beta\frac{1}{\alpha}$

$\therefore x\lambda^{\rho} = X$, $t\lambda^{\sigma} = T$, $A = \lambda^{\rho}$, $B = \lambda^{\sigma}$, $C = \lambda \quad \therefore$

$U(X,T)$ satisfies same equation as $u(x,t) \therefore BC = AC^2 = A^3C \quad \therefore$

$\lambda^{\sigma+1} = \lambda^{\rho+2} = \lambda^{3\rho+1} \quad \therefore \rho = \frac{1}{2}, \sigma = \frac{3}{2} \quad \therefore$

$u(x,t) = \lambda U(x\lambda^{1/2}, t\lambda^{3/2}) \quad \therefore u(x,t) = 3c\,\text{sech}^2\left[\frac{\sqrt{c}}{2}(x-ct-\xi_0)\right] \therefore$

$U(X,T) = 3\,\text{sech}^2\left[\frac{1}{2}(X-T-\xi_0)\right]$, $c = \lambda$

3b $W(t) = \int_{-\infty}^{\infty}[2u^3 - u_x^2]\,dx \quad \therefore$

Sheet 6 $\frac{dW}{dt} = \int_{-\infty}^{\infty} (6u^2 u_t - 2u_x u_{xx})\,dx =$

$\int_{-\infty}^{\infty} -2u_x u_{xx}\,dx + \int_{-\infty}^{\infty} 6u^2 u_t\,dx = \left[-2u_x u_t\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} -2u_{xx} u_t\,dx + \int_{-\infty}^{\infty} 6u^2 u_t\,dx$

$= \left[-2u_x u_t\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty} (6u^2 + 2u_{xx}) u_t\,dx = \int_{-\infty}^{\infty} (6u^2 + 2u_{xx}) u_t\,dx =$ (PDE)

$= -\int_{-\infty}^{\infty} (6u^2 + 2u_{xx})(6uu_x + u_{xxx})\,dx =$

$-\int_{-\infty}^{\infty} 36u^3 u_x\,dx - \int_{-\infty}^{\infty} 6u^2 u_{xxx}\,dx - \int_{-\infty}^{\infty} -12uu_x u_{xx}\,dx - \int_{-\infty}^{\infty} 2u_{xx} u_{xxx}\,dx =$

$= -I_1 - I_2 - I_3 - I_4$, $I_1 = 36\int_{-\infty}^{\infty} u^3 u_x\,dx$, $I_2 = 12\int_{-\infty}^{\infty} uu_x u_{xx}\,dx$,

$I_3 = 6\int_{-\infty}^{\infty} u^2 u_{xxx}\,dx$, $I_4 = 2\int_{-\infty}^{\infty} u_{xx} u_{xxx}\,dx$ $\therefore$

$I_1 = 36\int_{-\infty}^{\infty} u_x (u)^3\,dx = 36\left[\frac{1}{4}u^4\right]_{-\infty}^{\infty} = 9\left[u^4\right]_{-\infty}^{\infty} = 0$

$I_4 = 2\int_{-\infty}^{\infty} u_{xx} u_{xxx}\,dx = 2\int_{-\infty}^{\infty} u_{xxx} u_{xx}\,dx = 2\int_{-\infty}^{\infty} \frac{d}{dx}(u_{xx})(u_{xx})\,dx = \left[(u_{xx})^2\right]_{-\infty}^{\infty}$

$= 2\left[u_{xx}^2\right]_{-\infty}^{\infty} = 2(0) = 0$

$I_3 = 6\int_{-\infty}^{\infty} (u)^2 u_{xxx}\,dx = 6\left[(u)^2 u_{xx}\right]_{-\infty}^{\infty} - 6\int_{-\infty}^{\infty} 2uu_x u_{xx}\,dx =$

$-12\int_{-\infty}^{\infty} uu_x u_{xx}\,dx = -12\left[uu_x u_x\right]_{-\infty}^{\infty} + 12\int_{-\infty}^{\infty} (u_x u_x + u_{xx} u_x) u_x\,dx$

$= 12\int_{-\infty}^{\infty} (u_x)^3 + u_{xx} u_x u_x\,dx = 12\int_{-\infty}^{\infty} (u_x)^3\,dx + 12\int_{-\infty}^{\infty} uu_x u_{xx}\,dx$ $\therefore$

$I_3 = 6\int_{-\infty}^{\infty} u^2 u_{xxx}\,dx = 6\left[u^2 u_x\right]_{-\infty}^{\infty} - 12\int_{-\infty}^{\infty} uu_x u_{xx}\,dx =$

$-12\int_{-\infty}^{\infty} uu_x u_{xx}\,dx = -I_2$ $\therefore$ $-I_2 - I_3 = -I_2 - (-I_2) = -I_2 + I_2 = 0$ $\therefore$

$\frac{dW}{dt} = 0 - I_2 + I_2 + 0 = 0$ $\therefore W(t) = \text{const}$

3c $\therefore$ for conservation of Q_0: $\frac{d}{dt}\int_{-\infty}^{\infty} u\,dx = \int_{-\infty}^{\infty} u_t\,dx =$

$-\int_{-\infty}^{\infty} (6uu_x + u_{xxx})\,dx = -\int_{-\infty}^{\infty} \partial_x(3(u)^2 + u_{xx})\,dx = -\left[3(u)^2 + u_{xx}\right]_{-\infty}^{\infty} = 0$

$\therefore$ for $n=1$: $Q_2 = -i\,\partial_x Q_1 + Q_0^2 = -u_{xx} - u^2 x - u^2 + \partial_x R_2$, $R_2 = -u_x$ $\therefore$

$\int_{-\infty}^{\infty} \partial_x R_2\,dx = 0$ $\therefore$

$\frac{d}{dt}\int_{-\infty}^{\infty} (-u^2)\,dx = -2\int_{-\infty}^{\infty} uu_t\,dx = 2\int_{-\infty}^{\infty} (uu_{xx} + u_{xxx})\,dx = \int_{-\infty}^{\infty} \partial_x((u)^2 + 2u_{xx})\,dx =$

$\left[(u)^2 + 2u_{xx}\right]_{-\infty}^{\infty} = 0$

For $n=2$: $Q_3 = -i\partial_x Q_2 - 2Q_0Q_1 = i(u_{xxx} + 2u(u_x)) - 2u(-iu_x) = iu_{xxx} + 4iuu_x$

$\therefore n=3 \therefore Q_4 = -i\partial_x Q_3 - 2Q_0Q_2 - Q_1^2 = 2u^3 - u_x^2 + \partial_x R_4$,

$R^4 = u_{xxx} + 6uu_x$

\4a/ $u_t = U'(\xi)\xi_t = -cU'(\xi)$, $u_x = U'(\xi)\xi_x = U'(\xi)$,

$u_{xx} = U''(\xi)\xi_x = U''(\xi) \therefore U''(\xi) + cU'(\xi) + f(U) = 0 \therefore$

$\frac{dU}{d\xi} = U' = P(U) \therefore \frac{d}{d\xi}\left(\frac{dU}{d\xi}\right) = \frac{d}{d\xi}(U') = U'' = \frac{d}{d\xi}(P(U)) = P'(U)U_\xi = P'(U)U'$

$P'(U)U' + cP(U) + f(U) = \frac{dP}{dU}P(U)\,cP(U) + f(U) = P\left(\frac{dP}{dU} + c\right) + f(U) = 0 \therefore$

$P\left(\frac{dP}{dU} + c\right) = -f(U) \therefore$

$\frac{dU}{d\xi} = P(U)$, $\frac{d^2U}{d\xi^2} = \frac{dP}{dU}\frac{dU}{d\xi} = P\frac{dP}{dU} \therefore P\left(\frac{dP}{dU} + c\right) = -f(U) \therefore$

\4b/ Cubic poly in U $\therefore$ if $P(U)$ is a quadratic then LHS also cubic $\therefore P(U)$ is a divisor of $f(U)$ $\therefore$ roots of P must be a subset of the roots of f $\therefore$ let $\{u_1, u_2, u_3\} = \{0, a, 1\}$,

u_1, u_2 are roots of $P(U)$ $\therefore P(U) = B(U-u_1)(U-u_2)$ for some $B \neq 0 \therefore$

$B(U-u_1)(U-u_2)[2BU - B(u_1+u_2) + c] = U(U-a)(U-1) =$

$(U-u_1)(U-u_2)(U-u_3) \therefore$

$2B^2U - B^2(u_1+u_2) + Bc = U - u_3 \therefore$ let $B = \sigma\frac{1}{\sqrt{2}} \therefore \sigma = \pm 1 \therefore$

$c = \sigma(u_1 + u_2 - 2u_3)\frac{1}{\sqrt{2}} \therefore$

for $u_2 = a$: $P(U) = \sigma U(U-1)\frac{1}{\sqrt{2}}$, $c = \sigma(1-2a)\frac{1}{\sqrt{2}} \therefore$

$\frac{dU}{d\xi} = P(U) = \sigma U(U-1)\frac{1}{\sqrt{2}} \therefore \sigma\frac{1}{\sqrt{2}}\int d\xi = \sigma\frac{1}{\sqrt{2}}(\xi - \xi_0) = \int\frac{1}{U(U-1)}dU =$

$\ln\left|\frac{U-1}{U}\right|$, ξ_0 is an arbitrary const $\therefore$

$\left|1 - \frac{1}{U}\right| = \exp\left[\frac{\sigma}{\sqrt{2}}(\xi - \xi_0)\right] = \varepsilon$, $1 - \frac{1}{U} = \pm\varepsilon$, $\frac{1}{U} = 1 \pm \varepsilon$, $U = \frac{1}{1\pm\varepsilon}$,

$U(\xi) = \frac{1}{1 \pm \exp(\sigma(\xi - \xi_0)\sqrt{2})}$ $\therefore U(\xi)\ \forall\xi$ choose + sign $\therefore$

$U'(\xi) < 0 \therefore \sigma = 1 \therefore U(0) = \frac{1}{2} \therefore \xi_0 = 0 \therefore u(x,t) = \frac{1}{1 + \exp((x-ct)\frac{1}{\sqrt{2}})}$,

$c = (1-2a)\frac{1}{\sqrt{2}} \therefore$ for $a < \frac{1}{2} \therefore 2a < 1 \therefore 0 < 1-2a \therefore c > 0$

\4c/ $\therefore$ now let $f(u)$ be any func, roots $u_1 < u_2 < u_3$ st $f(u) > 0$ for $u \in (-\infty, u_1) \cup (u_2, u_3)$, $f(u) < 0$ for $u \in (u_1, u_2) \cup (u_3, \infty)$

$\therefore$ if $u(x,t) = U(x-ct)$, $U(\infty) = u_1$, $U(-\infty) = u_3$, $\therefore$

\Sheet#6/ u_1, u_3 $\therefore \int_{u_1}^{u_3} P\frac{dP}{dU}dU + c\int_{u_1}^{u_3} P\,dU = -\int_{u_1}^{u_3} f(U)dU$ $\therefore$

Integrates to: $\frac{1}{2}P^2(u_3) - \frac{1}{2}P^2(u_1)$ $\therefore$ $P(u_1)=P(u_3)=0$ $\because$ $P=U'(\xi)<0$

$\mathrm{Sign}(c) = \mathrm{Sign}\left(\int_{u_1}^{u_3} f(u)du\right)$

\5a/ $r_t = r + r_{xx} - r\theta_x^2 - \alpha(2r_x\theta_x + r\theta_{xx}) - r^3$, $r\theta_t = r\theta_{xx} + 2r_x\theta_x + \alpha(r_{xx} - r\theta_x^2) - \beta r^3$

$\therefore$ $w = u + iv$, $\therefore$ $w = re^{i\theta}$ $\therefore$ $w_t = r_t e^{i\theta} + r\,i\theta_t e^{i\theta} = e^{i\theta}(r_t + ri\theta_t)$

$w_{xx} = [r_{xx} - r\theta_x^2 + i(2r_x\theta_x + r\theta_{xx})]e^{i\theta}$,

$(r_t + ir\theta_t)e^{i\theta} = [r + r_{xx} - r\theta_x^2 + i(2r_x\theta_x + r\theta_{xx}) + i\alpha(r_{xx} - r\theta_x^2) - \alpha(2r_x\theta_x + r\theta_{xx}) + r^3 - i\beta r^3]e^{i\theta}$ $\therefore$

common factor $e^{i\theta}$, seperating real, imaginary $\therefore$

$r_t = r + r_{xx} - r\theta_x^2 - \alpha(2r_x\theta_x + r\theta_{xx}) - r^3$, $r\theta_t = r\theta_{xx} + 2r_x\theta_x + \alpha(r_{xx} - r\theta_x^2) - \beta r^3$

\5b/ $r(x,t) = R(\xi)$, $\theta(x,t) = \Theta(\xi)$, $\xi = x - ct$ $\therefore$

$r_t = -cR'(\xi)$, $\theta_t = -c\Theta'(\xi)$, ~~$r_x = R'$~~

$r_x = R'(\xi)$, $\theta_x = \Theta'(\xi)$

$r_{xx} = R''(\xi)$, $\theta_{xx} = \Theta''(\xi)$ $\therefore$

$R'' + cR' + R - R(\Theta')^2 - \alpha(2R'\Theta' + R\Theta'') - R^3 = 0$,

$R\Theta'' + cR\Theta' + \alpha(R'' - R(\Theta')^2) - \beta R^3 = 0$ $\therefore$

\5c/ $\therefore$ let $\Theta' = k$, $R = a$ $\therefore$ $a - k^2a - a^3 = 0$, $cak - \alpha ak^2 - \beta a^3 = 0$ $\therefore$

$\because$ $a > 0$ $a = \sqrt{1-k^2}$ $\therefore$ $ck = \beta + (\alpha - \beta)k^2$ $\therefore$ $R = a$; $\Theta = k\xi$, $\xi = x - ct$.

into $w = Re^{i\theta}$ $\therefore$ $\omega = ck$ $\therefore$ $\omega = \beta + (\alpha - \beta)k^2$ $\therefore$ $a > 0$ $\therefore$ $k \in (-1, 1)$ $\therefore$

$a \in (0,1)$ $\therefore$ $\omega \in (\alpha, \beta) \cup (\beta, \alpha)$

\LN Ex2.7:/ $\therefore$ let $u = X(x)T(t)$ $\therefore$ ~~XT~~ $XT'' - C^2TX'' = 0$,

$\frac{T''}{C^2T} = \frac{X''}{X} = -\lambda$ $\therefore$ $T'' + C^2\lambda T = 0$, $X'' + \lambda X = 0$, $u_x(0,t) = 0$ $\therefore$ $X'(0) = 0$,

$u(L,t) + hu_x(L,t) = 0$, $u(x,0) = 0$, $u_t(x,0) = I\delta(x - x_0)$ $\therefore$

$X(L)T(t) + hX'(L)T(t) = 0 = X(L) + hX'(L) = 0$ $\therefore$

when $\lambda > 0$ $\therefore$ $a = \sqrt{\lambda} > 0$ $\therefore$ $X'' + a^2X = 0$ $\therefore$ $X(x) = A\cos ax + B\sin ax$ $\therefore$

$X'(0) = 0$ $\therefore$ $B = 0$ $\therefore$ $X = A\cos ax$ $\therefore$ $X' = -Aa\sin(ax)$ $\therefore$

$X(L) + hX'(L) = 0$ $\therefore$ $\cos aL - ha\sin(aL) = 0$ $\therefore$ $1 - ha\tan(La) = 0$ $\therefore$

$\tan(La) = \frac{1}{ha}$ $\therefore$ $\lambda_n = a_n^2$, $a = a_n$ $\therefore$ $0 < a_1 < a_2 < \dots < a_n < \dots$, $\lim_{n\to\infty} a_n = \infty$.

if $h>0$: $\alpha_n \in \left((n-1)\pi - \frac{\pi}{2}, (n-1)\pi + \frac{\pi}{2}\right)$, $\lim_{n\to\infty}\left[\alpha_n - (n-1)\pi\right] = 0$

$h<0$: $\alpha_n \in \left(n\pi - \frac{\pi}{2}, n\pi + \frac{\pi}{2}\right)$, $\lim_{n\to\infty}\left[\alpha_n - n\pi\right] = 0$

$\lambda_n = \alpha_n^2$, $\phi_n(x) = \cos(\alpha_n x)$

$T'' + c^2\alpha_n^2 T = 0 \therefore T(t) = A_n\cos(\alpha_n c t) + B_n\sin(\alpha_n c t)$

for $\lambda = 0$: $X'' = 0$, $X = A + Bx$, $X'(0) = B = 0$

$A = B = 0 \therefore \lambda = 0$

when $\lambda<0$: $p = \sqrt{-\lambda} > 0 \therefore X'' - p^2 X = 0 \therefore X(x) = A\cosh px + B\sinh px$,

$X'(0) = B = 0 \therefore X(x) = A\cosh px$, $X(L) + hX'(L) = A(\cosh pL + hp\sinh pL) = 0$

$\because p > 0$, $L > 0 \therefore h > 0 \therefore \cosh pL + hp\sinh(pL) > 0$, $A = 0$

$\lambda < 0 \therefore$ if $\lambda < 0$: $\cosh pL + hp\sinh pL$, $\tan pL = \frac{1}{(-h)p}$, if $L > 0$, $-h > 0$

$p = p_k(h, L) \therefore \lambda = -p_k^2 < 0 \therefore \phi_k(x) = \cosh p_k x \therefore T'' - c^2 p_k^2 T = 0$

$\therefore T(t) = A_k\cosh(p_k c t) + B_k\sinh(p_k c t)$

$u = \left\{\left[A_k\cosh(p_k c t) + B_k\sinh(p_k c t)\right]\cosh(p_k x)\right\} + \sum_{n=1}^{\infty}\left[A_n\cos(\alpha_n c t) + B_n\sin(\alpha_n c t)\right]\cos(\alpha_n x)$

where term in $\{\ \}$ is present for $h<0$, absent $h>0$

$\therefore u(x,0) = 0 \therefore \sum A_n\phi_n(x) = 0 \therefore A_n = 0 \therefore$

$u_t(x,0) = \sum \delta(x-x_0) \therefore cB_k p_k\phi_k(x) + \sum_{n=1}^{\infty} cB_n\alpha_n\phi_n(x) = \sum \delta(x-x_0)$

* only if $h<0$ terms, $\int_0^L \phi_n(x)\phi_m(x)\,dx = 0 \therefore$

for $m \neq n$ $\therefore B_n = \frac{\int_0^L \phi_n(s)\sum\delta(s-x_0)\,ds}{c\alpha_n\int_0^L \phi_n^2(s)\,ds} = \frac{\sum M_n}{c\alpha_n}\phi_n(x_0)$, $B_k = \frac{\sum M_k}{cp_k}\phi_k(x_0)$.

$M_n = \left(\int_0^L \phi_n^2(s)\,ds\right)^{-1} \therefore M_k = 2\left[L - \frac{h}{1+h^2p_k^2}\right]^{-1}$, $M_n = 2\left[L + \frac{h}{1+h^2\alpha_n^2}\right]^{-1} \therefore$

$u = \frac{\sum}{c}\left[\frac{M_k}{p_k}\cosh(p_k x_0)\sinh(p_k ct)\cosh(p_k x) + \sum_n \frac{M_n}{\alpha_n}\cos(\alpha_n x_0)\sin(\alpha_n ct)\cos(\alpha_n x)\right]$

p_k term for $h<0$, not $h>0$; p_k is of $hp\tanh(pL) = -1$,

α_n is of $h\tan(\alpha L) = 1$

\CW2 Q6/ $0 \le x \le \infty$ $u(x,0) = \phi = \begin{cases} 1, & a \le x \le b \\ 0 & \text{otherwise} \end{cases}$

Consider an odd extension i.e. the problem for the unknown function $v(x,t) \therefore v_t - Dv_{xx} = 0$, $x \in \mathbb{R}$, $v(x,0) = \phi(x) - \phi(-x) \therefore$

The equation is invariant wrt $x \to -x$, $v \to -v$, and the IC is odd $\therefore$ the solution is odd in x and $\therefore$ the solution of this

\CNZ Q6/ problem is odd and continuous, at $x=0$, have $v(0,t)=0 \ \forall t \geq 0$ $\therefore$ the restriction of $v(x,t)$ to $x \geq 0$ provides solution $u(x,t)$ to the original problem. $\therefore$

$$v(x,t) = \frac{1}{\sqrt{\pi D t}}\left[\int_{-\infty}^{\infty} \phi(s) e^{-(x-s)^2/(4Dt)}\,ds - \int_{-\infty}^{\infty} \phi(-s) e^{-(x+s)^2/(4Dt)}\,ds\right] \therefore$$

let $s = x + z\sqrt{4Dt}$ $\therefore$

$$\int_{-\infty}^{\infty} \phi(s) e^{-(s-x)^2/(4Dt)}\,ds = \int_a^b e^{-(s-x)^2/(4Dt)}\,ds = \sqrt{4Dt}\int_{(a-x)/\sqrt{4Dt}}^{(b-x)/\sqrt{4Dt}} e^{-z^2}\,dz =$$

$$\sqrt{\pi D t}\left[\operatorname{erf}\left(\frac{b-x}{\sqrt{4Dt}}\right) - \operatorname{erf}\left(\frac{a-x}{\sqrt{4Dt}}\right)\right] \therefore$$

use mapping $x \to -x$, $s \to s$ $\therefore$

$$v(x,t) = \frac{1}{2}\left[\operatorname{erf}\left(\frac{b-x}{\sqrt{4Dt}}\right) - \operatorname{erf}\left(\frac{a-x}{\sqrt{4Dt}}\right) - \operatorname{erf}\left(\frac{b+x}{\sqrt{4Dt}}\right) + \operatorname{erf}\left(\frac{a+x}{\sqrt{4Dt}}\right)\right] = u(x,t),\ x \geq 0$$

\Sheet 1/ \5a/ $a u_{xx} + 2b u_{xy} + c u_{yy} = 2u_{xx} + 5u_{xy} - 3u_{yy}$ $\therefore$

$a=2$, $2b=5$ $\therefore$ $-2b=5$, $b=\frac{5}{2}$, $c=-3$ $\therefore$

$b^2 - ac = \left(\frac{5}{2}\right)^2 - 2(-3) = \frac{49}{4} > 0$ $\therefore$ hyperbolic $\therefore$

$a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 2\left(\frac{dy}{dx}\right)^2 - 5\frac{dy}{dx} - 3 = 0$ $\therefore$

$\frac{dy}{dx} = \frac{5 \pm \sqrt{25 + 4(2)(3)}}{2(2)} = \frac{5 \pm 7}{4}$ $\therefore$ $\frac{dy}{dx} = -\frac{1}{2}$, $\frac{dy}{dx} = 3$ $\therefore$

$y = -\frac{1}{2}x + C_1$, $y = 3x + C_2$ $\therefore$ $2y = -x + C_3$ $\therefore$ $-y = -3x + C_4$ $\therefore$

$3x - y = C_4$, $x + 2y = C_3$ $\therefore$ are charac curves $\therefore$

let $\xi = 3x - y$, $\eta = x + 2y$ $\therefore$

$\xi_x = 3$, $\xi_y = -1$, $\eta_x = 1$, $\eta_y = 2$

$u_x = u_\xi \xi_x + u_\eta \eta_x = 3u_\xi + u_\eta$ $\therefore$

$u_{xx} = 3u_{\xi\xi}\xi_x + 3u_{\xi\eta}\eta_x + u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x = 9u_{\xi\xi} + 6u_{\xi\eta} + u_{\eta\eta}$,

$u_y = u_\xi \xi_y + u_\eta \eta_y = -u_\xi + 2u_\eta$ $\therefore$

$u_{yy} = -u_{\xi\xi}\xi_y - u_{\xi\eta}\eta_y + 2u_{\xi\eta}\xi_y + 2u_{\eta\eta}\eta_y = u_{\xi\xi} - 4u_{\xi\eta} + 4u_{\eta\eta}$,

$u_{xy} = -u_{\xi\xi}\xi_x - u_{\xi\eta}\eta_x + 2u_{\xi\eta}\xi_x + 2u_{\eta\eta}\eta_x = -3u_{\xi\xi} + 5u_{\xi\eta} + 2u_{\eta\eta}$ $\therefore$

$2u_{xx} + 5u_{xy} - 3u_{yy} =$

$18u_{\xi\xi} + 12u_{\xi\eta} + 2u_{\eta\eta} - 15u_{\xi\xi} + 25u_{\xi\eta} + 10u_{\eta\eta} - 3u_{\xi\xi} + 12u_{\xi\eta} - 12u_{\eta\eta} =$

$u_{\xi\xi}(18-15-3)+u_{\xi\eta}(12+25+12)+u_{\eta\eta}(2+10-12)=$

$49u_{\xi\eta}=0=u_{\xi\eta}=\frac{\partial}{\partial\xi}(\frac{\partial}{\partial\eta}u)=0\ \therefore$

$\frac{\partial}{\partial\eta}u=\tilde{f}(\eta)\ \therefore$

$u=f(\eta)+g(\xi)\ \therefore$

$u(x,y)=f(x+2y)+g(3x-y)$

\5b/ if $u_x-4u_y=0$: let $f(x+\alpha y)=u\ \therefore$

$u_x=f'$, $u_y=\alpha f'\ \therefore\ f'-4\alpha f'=0=(1-4\alpha)f'=0=1-4\alpha\ \therefore$

$4\alpha=1\ \therefore\ \alpha=\frac{1}{4}\ \therefore$

let $\tilde{\xi}=x+\frac{1}{4}y\ \therefore$ let $\xi=4x+y\ \therefore$ let $\eta=1x-4y=x-4y\ \therefore$

~~u_x~~ $\xi_x=4$, $\xi_y=1$, $\eta_x=1$, $\eta_y=-4\ \therefore$

$u_x=u_\xi\xi_x+u_\eta\eta_x=4u_\xi+u_\eta$,

$u_y=u_\xi\xi_y+u_\eta\eta_y=u_\xi-4u_\eta\ \therefore$

$u_x-4u_y-3u=4u_\xi+u_\eta-4u_\xi+16u_\eta-3u=$

$17u_\eta-3u=0=17h(\eta)u_\eta-3h(\eta)u=0=h(\eta)u_\eta-\frac{3}{17}h(\eta)u=0\ \therefore$

$h'(\eta)=-\frac{3}{17}h(\eta)\ \therefore\ \frac{h'(\eta)}{h(\eta)}=-\frac{3}{17}\ \therefore\ \ln(h(\eta))=-\frac{3}{17}\eta\ \therefore$

$h(\eta)=e^{-\frac{3}{17}\eta}\ \therefore$

$\frac{\partial}{\partial\eta}(e^{-\frac{3}{17}\eta}u)=0\ \therefore\ e^{-\frac{3}{17}\eta}u=f(\xi)\ \therefore$

~~$u=f(\xi)$~~ $u=e^{\frac{3}{17}\eta}f(\xi)=u(x,y)=e^{\frac{3}{17}(x-4y)}f(4x+y)$

\5b/ let ~~$\xi=x-4y$~~ $\therefore$ ~~$\tilde{\eta}=4x+y$~~ $\therefore\ \tilde{\xi}=x-4y$, $\tilde{\eta}=-4x-y$

$\therefore$ let $\xi=4x+y$, $\therefore$ let $\xi=4x+y$, $\eta=x-4y$

\5c/ let $au_{xx}+2bu_{xy}+cu_{yy}=x^2u_{xx}-4y^2u_{yy}\ \therefore$

$a=x^2$, $2b=0=-2b=b=0$, $c=-4y^2\ \therefore$

$b^2-ac=0^2-x^2(-4y^2)=4x^2y^2>0\ \therefore$ hyperbolic $\therefore$

$a(\frac{dy}{dx})^2-2b(\frac{dy}{dx})+c=x^2(\frac{dy}{dx})^2-4y^2=0\ \therefore$

$x^2(\frac{dy}{dx})^2=4y^2\ \therefore\ (\frac{dy}{dx})^2=\frac{4y^2}{x^2}\ \therefore\ \frac{dy}{dx}=\pm\frac{2y}{x}\ \therefore$

$\frac{1}{y}\frac{dy}{dx}=\pm 2\frac{1}{x}\ \therefore\ \int\frac{1}{y}dy=2\int\frac{1}{x}dx$, $\int\frac{1}{y}dy=-2\int\frac{1}{x}dx\ \therefore$

$\ln|y|=+2\ln|x|+C_1$, $\ln|y|=-2\ln|y|+C_2\ \therefore$

$e^{\ln|y|}=|y|=e^{2\ln|x|+C_1}=C_3(e^{\ln|x|^2})=C_3e^{\ln x^2}=C_3x^2\ \therefore\ y=\pm C_3x^2=C_4x^2$

$e^{\ln|y|}=e^{-2\ln|x|+C_2}=|y|=C_5e^{\ln|\frac{1}{x^2}|}=C_5e^{\ln\frac{1}{x^2}}=C_5\frac{1}{x^2}\ \therefore\ y=\pm C_5\frac{1}{x^2}=C_6\frac{1}{x^2}\ \therefore$

\Sheet 1/ $\frac{y}{x^2} = C_4$, $x^2 y = C_6$ $\therefore$ $\xi = \frac{y}{x^2}$, $\eta = x^2 y$ $\therefore$ $\xi = x^{-2} y$ $\therefore$

$\xi_x = -2x^{-3} y$, $\xi_y = x^{-2}$, $\eta_x = 2xy$, $\eta_y = x^2$ $\therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x = -2x^{-3} y u_\xi + 2xy u_\eta = u_x$,

$u_{xx} = \frac{\partial}{\partial x}(-2x^{-3} y u_\xi + 2xy u_\eta) =$

$6x^{-4} y u_\xi - 2x^{-3} y u_{\xi\xi} \xi_x - 2x^{-3} y u_{\xi\eta} \eta_x + 2y u_\eta + 2xy u_{\xi\eta} \xi_x + 2xy u_{\eta\eta} \eta_x =$

$6x^{-4} y u_\xi + 4x^{-6} y^2 u_{\xi\xi} - 4x^{-2} y^2 u_{\xi\eta} + 2y u_\eta - 4x^{-2} y^2 u_{\xi\eta} + 4x^2 y^2 u_{\eta\eta} =$

$6x^{-4} y u_\xi + 4x^{-6} y^2 u_{\xi\xi} - 8x^{-2} y^2 u_{\xi\eta} + 2y u_\eta + 4x^2 y^2 u_{\eta\eta} = u_{xx}$,

$u_y = u_\xi \xi_y + u_\eta \eta_y = x^{-2} u_\xi + x^2 u_\eta = u_y$

$u_{yy} = x^{-2} u_{\xi\xi} \xi_y + x^{-2} u_{\xi\eta} \eta_y + x^2 u_{\xi\eta} \xi_y + x^2 u_{\eta\eta} \eta_y =$

$x^{-4} u_{\xi\xi} + u_{\xi\eta} + u_{\xi\eta} + x^4 u_{\eta\eta} = x^{-4} u_{\xi\xi} + 2u_{\xi\eta} + x^4 u_{\eta\eta} = u_{yy}$ $\therefore$

$x^2 u_{xx} - 4y^2 u_{yy} + x u_x - 4y u_y =$

$6x^{-2} y u_\xi + 4x^{-4} y^2 u_{\xi\xi} - 8y^2 u_{\xi\eta} + 2x^2 y u_\eta + 4x^4 y^2 u_{\eta\eta} - 4x^{-4} y^2 u_{\xi\xi} - 8y^2 u_{\xi\eta} - 4x^4 y^2 u_{\eta\eta} + x u_x - 4y u_y =$

$6x^{-2} y u_\xi - 8y^2 u_{\xi\eta} + 2x^2 y u_\eta - 8y^2 u_{\xi\eta} + x u_x - 4y u_y =$

$6x^{-2} y u_\xi - 16y^2 u_{\xi\eta} + 2x^2 y u_\eta + x u_x - 4y u_y =$

$6x^{-2} y u_\xi - 16y^2 u_{\xi\eta} + 2x^2 y u_\eta - 2x^{-2} y u_\xi + 2x^2 y u_\eta - 4x^{-2} y u_\xi - 4x^2 y u_\eta =$

$u_\xi (6x^{-2} y - 2x^{-2} y - 4x^{-2} y) + u_\eta (2x^2 y + 2x^2 y - 4x^2 y) - 16y^2 u_{\xi\eta} =$

$-16y^2 u_{\xi\eta} = 0 = y^2 u_{\xi\eta} = u_{\xi\eta} = 0 = \frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \eta} u\right)$ $\therefore$

$\frac{\partial}{\partial \eta} u = \tilde{f}(\eta)$ $\therefore$

$u = f(\eta) + g(\xi) = u(x,y) = f(x^2 y) + g(x^{-2} y)$

\6/ $(x,t) \mapsto (\xi, \tau)$ $\therefore$ $u_x = u_\xi \xi_x + u_\tau \tau_x$ $\therefore$

$u_{xx} = u_\xi \xi_{xx} + u_{\xi\xi} \xi_x \xi_x + u_{\xi\tau} \xi_x \tau_x + \tau_{xx} u_\tau + u_{\xi\tau} \xi_x \tau_x + u_{\tau\tau} \tau_x \tau_x =$

$u_{\xi\xi} \xi_x^2 + u_{\xi\tau} \xi_x \tau_x + u_\xi \xi_{xx} + u_{\xi\tau} \xi_x \tau_x + u_{\tau\tau} \tau_x^2 + u_\tau \tau_{xx}$,

$u_t = u_\xi \xi_t + u_\tau \tau_t$ $\therefore$

$u_{tt} = u_{\xi\xi} \xi_t^2 + u_{\xi\tau} \xi_t \tau_t + u_\xi \xi_{tt} + u_{\xi\tau} \xi_t \tau_t + u_{\tau\tau} \tau_t^2 + u_\tau \tau_{tt}$

$= u_{tt} = u_{\xi\xi} \xi_t^2 + 2u_{\xi\tau} \xi_t \tau_t + u_\xi \xi_{tt} + u_{\tau\tau} \tau_t^2 + u_\tau \tau_{tt}$,

$u_{xx} = u_{\xi\xi} \xi_x^2 + 2u_{\xi\tau} \xi_x \tau_x + u_\xi \xi_{xx} + u_{\tau\tau} \tau_x^2 + u_\tau \tau_{xx}$ $\therefore$

into PDE: $LHS = u_{xx} - u_{tt} =$

$u_{\xi\xi} \xi_x^2 + u_\xi \xi_{xx} + 2u_{\xi\tau} \xi_x \tau_x + u_{\tau\tau} \tau_x^2 + u_\tau \tau_{xx} - (u_{\xi\xi} \xi_t^2 + u_\xi \xi_{tt} + 2u_{\xi\tau} \xi_t \tau_t + u_{\tau\tau} \tau_t^2 + u_\tau \tau_{tt}) =$

$\xi_x^2 u_{\xi\xi} + 2\xi_x\tau_x u_{\xi\tau} + \tau_x^2 u_{\tau\tau} - \xi_t^2 u_{\xi\xi} - 2\xi_t\tau_t u_{\xi\tau} - \tau_t^2 u_{\tau\tau} + \xi_{xx} u_\xi + \tau_{xx} u_\tau - \xi_{tt} u_\xi - \tau_{tt} u_\tau =$

$(\xi_x^2 - \xi_t^2) u_{\xi\xi} + 2(\xi_x\tau_x - \xi_t\tau_t) u_{\xi\tau} + (\tau_x^2 - \tau_t^2) u_{\tau\tau} + \xi_{xx} u_\xi + \tau_{xx} u_\tau - \xi_{tt} u_\xi - \tau_{tt} u_\tau =$

$a u_{xx} + 2b u_{xt} + c u_{tt} = \alpha u_{\xi\xi} + 2\beta u_{\xi\tau} + \gamma u_{\tau\tau} =$

$1 v_{\xi\xi} - 1 v_{\tau\tau} = 1 v_{\xi\xi} - 1 v_{\tau\tau} = 0 = RHS \therefore$

$\xi_x^2 - \xi_t^2 = 1$, $\xi_x\tau_x - \xi_t\tau_t = 0$, $\tau_x^2 - \tau_t^2 = -1 \therefore$

$\xi = Ax + Bt$, $\xi_x = A$, $\xi_t = B$, $\tau = Cx + Dt \therefore \tau_x = C$, $\tau_t = D \therefore$

$A^2 - B^2 = 1$, $AC - BD = 0$, $C^2 - D^2 = -1$ $\therefore B > 0 \therefore$

$A^2 = \cosh\phi$ $\therefore \cosh^2\phi - \sinh^2\phi = 1$ $\therefore$ $B = \sinh\phi$

$C > 0$, $D > 0 \therefore$ ~~is~~ $D = \cosh\psi \therefore D^2 = \cosh^2\psi \therefore D^2 - 1 = C^2 \therefore$

$D^2 - 1 = \cosh^2\psi - 1 = \sinh^2\psi = C^2 \therefore C = \sinh\psi$; $\psi > 0 \therefore$

$AC - BD = 0 \therefore AC = BD \therefore \frac{B}{A} = \frac{C}{D} = \frac{\sinh\phi}{\cosh\phi} = \frac{\sinh\psi}{\cosh\psi} \therefore$

$\tanh\phi = \tanh\psi \therefore \phi = \psi \therefore A = D = \cosh\phi$, $B = C = \sinh\phi$

\ sheet 2 / \ 6 / $v_{tt} - c^2 v_{xx} = 0$ $-\infty < x < 0$

$v(x,0) = A\cos kx$, $v_t(x,0) = -k\cos kx$ $v(0,t) = 0 \therefore$

to odd extension $\therefore$

$v_{tt} - c^2 v_{xx} = 0$ $x \in \mathbb{R}$ $\therefore$

$v(x,0) = A\cos kx - A\cos(k(-x)) = A\cos kx - A\cos(-kx) = A\cos kx + A\cos kx = 2A\cos kx$

$v_t(x,0) = -k\cos kx - k\cos(k(-x)) = -k\cos kx - k\cos(-kx) = -k\cos kx + k\cos kx = 0$

\ 6 sol / odd extension with ~~$\phi(x) = A\cos kx$, $2\psi(x) =$~~

$v(x,0) = \phi(x) = A\cos kx$, $v_t(x,0) = \psi(x) = -k\cos kx$ $-\infty < x < 0 \therefore$

$\phi_{odd}(x) = A\cos kx$, for $\psi_{odd}(x) = -k\cos kx$ for $x < 0$

$\therefore$ for $x > 0$: $\phi_{odd} = -A\cos(-kx) = A\cos kx$, $\psi_{odd} = +k\cos(-kx) = k\cos kx$ for $x > 0$

for $u(x,t)$: $u_{tt} - c^2 u_{xx} = 0$ $x \in \mathbb{R} \therefore$

$u(x,0) = \phi_{odd}(x) = \begin{cases} -A\cos kx, & x > 0, \\ A\cos kx, & x < 0. \end{cases}$ $u_t(x,0) = \psi_{odd}(x) = \begin{cases} k\cos kx, & x > 0, \\ -k\cos kx, & x < 0. \end{cases}$

u is odd $\therefore$ all odd functions vanish at 0 $\therefore u(0,t) = 0$ $\therefore$

$v(x,t) = u(x,t)$ for $-\infty < x < 0$, $t > 0 \therefore$ for $x < 0 \therefore$

$v(x,t) = u(x,t) = \frac{1}{2}\left[\phi_{odd}(x+ct) + \phi_{odd}(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \psi_{odd}(s)\,ds$ $\therefore$

for $x + ct < 0$ and $x - ct < 0 \therefore ct < -x$, $x < ct \therefore x < ct < -x \therefore ct < |x| \therefore$

\sheet2/ $\therefore$ in negative half line $\therefore$ $x<0$ $\therefore$

$\phi_{odd}(x) = A\cos kx$, $\psi_{odd}(x) = -k\cos kx$ $\therefore$

$\phi_{odd}(x+ct) = A\cos(k(x+ct))$, $\phi_{odd}(x-ct) = A\cos(k(x-ct))$ $\therefore$

$v(x,t) = \frac{1}{2}\left[A\cos(k(x+ct)) + A\cos(k(x-ct))\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} -k\cos ks \, ds$

$= \frac{A}{2}\left[\cos(k(x+ct)) + \cos(k(x-ct))\right] - \frac{1}{2c}\left[\sin(k(x+ct)) - \sin(k(x-ct))\right] =$

$A\cos(kx)\cos(kct) - \frac{1}{c}\cos(kx)\sin(kct)$

for $x+ct>0$, $x-ct<0$

$\because$ will ignore cos $x+ct>0$, $x-ct>0$ $\therefore$ only want $-\infty<x<0$

$\therefore$ for $x+ct>0, x-ct<0$ $\therefore$ $-x<ct$, $x<ct$ $\therefore$ $-x, x<ct$ $\therefore$

$|x|<ct$ $\therefore$

~~$\phi_{odd}(x) = A\cos kx$~~ $\therefore$ $\phi_{odd}(x+ct) = -A\cos k(x+ct)$

$\phi_{odd}(x-ct) = A\cos k(x-ct)$ $\psi(s>0) = k\cos ks$, $\psi(s<0) = -k\cos ks$ $\therefore$

$\psi(x+ct) = k\cos k(x+ct)$, $\psi(x-ct) = k\cos k(x-ct)$ $\therefore$

$v(x,t) = \frac{1}{2}\left[-A\cos k(x+ct) + A\cos k(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{0} -k\cos ks \, ds + \frac{1}{2c}\int_{0}^{x+ct} k\cos ks \, ds =$

$\frac{A}{2}\left[-\cos k(x+ct) + \cos k(x-ct)\right] + \frac{1}{2c}\left[\sin k(x-ct) + \sin k(x+ct)\right]$

$\therefore$ $v(x,t) = \begin{cases} v(x,t) , & |x|>ct, \\ v(x,t) , & |x|<ct. \end{cases}$ $v(0,t) = 0$

\sheet3/ \5/ $u_t - Du_{xx} = 0$ $x\in[0,\infty)$, $t\in[0,\infty)$ $\therefore$ $u(x,0) = Q\delta(x-x_0)$,

$u_x(0,t) = 0$ $\therefore$ do even extension $\therefore$

for $x\in\mathbb{R}$, $u_t - Du_{xx} = 0$, $u(x,0) = \phi(x)$ $\therefore$

$\phi(x) = Q\delta(x-x_0)$ for $x>0$, $\phi(x) = Q\delta(-x-x_0) = Q\delta(-(x+x_0)) =$

$Q\delta(x+x_0)$ for $x<0$ $\therefore$ $\therefore$ $\phi(x) = \begin{cases} Q\delta(x-x_0) , & x>0 \\ Q\delta(x+x_0) , & x<0 \end{cases}$ $\therefore$

$u(x,t) = \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} u(s,0)e^{-(x-s)^2/(4Dt)} \, ds =$

$\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty}\left[Q\delta(s-x_0) + Q\delta(s+x_0)\right]e^{-(x-s)^2/(4Dt)} \, ds =$

$\frac{1}{\sqrt{4\pi Dt}}\left[\int_{-\infty}^{\infty} Q\delta(s-x_0)e^{-(x-s)^2/(4Dt)} \, ds + \int_{-\infty}^{\infty} Q\delta(s-(-x_0))e^{-(x-s)^2/(4Dt)} \, ds\right] =$

$\frac{1}{\sqrt{4\pi Dt}}\left[Qe^{-(x-x_0)^2/(4Dt)} + Qe^{-(x+x_0)^2/(4Dt)}\right]$ for $x\geq 0$

\Sheet 5/ \5/ $u_t - a^2 u_{xx} = f(x,t)$, $0 \le x$ $0 \le t$ ∴ $a > 0$ ∴

$u(x,0) = 0$, $u(0,t) = 0$ ∴ use odd extension ∴

for $x \in \mathbb{R}$ $u_t - a^2 u_{xx} = f(x,t)$

for $x > 0$: $u(x,0) = \phi(x) = 0$

for $x < 0$: $u(x,0) = \phi(x) = -0\big|_{x=-x} = -0 = 0$ ∴ $\phi(x) = 0$ ∴

∴ let for $x > 0$ ∴ $F(x,t) = f(x,t)$ ∴

for $x < 0$: $F(x,t) = -f(-x,t)$ ∴ $F(x,t) = \begin{cases} f(x,t), & x > 0 \\ -f(-x,t), & x < 0 \end{cases}$

∴ $F(x,t)$ is odd ∴

$u_t - a^2 u_{xx} = F(x,t)$, $x \in \mathbb{R}$, $0 \le t$ $a > 0$, $u(0,t) = 0$, $u(x,0) = 0$ ∴

$$u(x,t) = \int_{\tau=0}^{t} \int_{\xi=-\infty}^{\infty} F(\xi,\tau) \left[\frac{1}{2a\sqrt{\pi(t-\tau)}} e^{-(x-\xi)^2/(4a^2(t-\tau))} \right] d\xi\, d\tau =$$

$$\int_{\tau=0}^{t} \int_{\xi=-\infty}^{0} F(\xi,\tau) \left[\frac{1}{2a\sqrt{\pi(t-\tau)}} e^{-(x-\xi)^2/(4a^2(t-\tau))} \right] d\xi\, d\tau + \int_{\tau=0}^{t} \int_{\xi=0}^{\infty} F(\xi,\tau) \left[\frac{e^{-(x-\xi)^2/(4a^2(t-\tau))}}{2a\sqrt{\pi(t-\tau)}} \right] d\xi\, d\tau =$$

$$\int_{\tau=0}^{t} \int_{\xi=-\infty}^{0} (-f(-\xi,\tau)) \left[\frac{e^{-(x-\xi)^2/(4a^2(t-\tau))}}{2a\sqrt{\pi(t-\tau)}} \right] d\xi\, d\tau + \int_{\tau=0}^{t} \int_{\xi=0}^{\infty} f(\xi,\tau) \left[\frac{e^{-(x-\xi)^2/(4a^2(t-\tau))}}{2a\sqrt{\pi(t-\tau)}} \right] d\xi\, d\tau$$

$$\therefore \int_{\xi=-\infty}^{0} (-f(-\xi,\tau))\, g(\xi)\, d\xi = \int_{\xi=\infty}^{0} -f(\xi,\tau)\, g(-\xi)\, d\xi = \int_{\xi=0}^{\infty} f(\xi,\tau)\, g(-\xi)\, d\xi$$

∴ $u(x,t) =$

$$\int_{\tau=0}^{t} \int_{\xi=0}^{\infty} f(\xi,\tau) \left[\frac{1}{2a\sqrt{\pi(t-\tau)}} \right] \left[-e^{-(x+\xi)^2/(4a^2(t-\tau))} + e^{-(x-\xi)^2/(4a^2(t-\tau))} \right] d\xi\, d\tau =$$

$$\int_{\tau=0}^{t} \int_{\xi=0}^{\infty} f(\xi,\tau)\, G_T(x,t;\xi,\tau)\, d\xi\, d\tau$$

\CW2/ \b/ $u_t - D u_{xx} = 0$, $0 \le$, $u(0,t) = 0$, $u(x,0) = \phi(x) = \begin{cases} 1 & a \le x \le b \\ 0, & \text{otherwise} \end{cases}$

∴ odd extension ∴ $b > a > 0$, $D > 0$ ∴

for $x > 0$: $u(x,0) = \phi(x) = 1$ for $a \le x \le b$

for $x < 0$: $u(x,0) = \phi(x) = -1\big|_{x=-x} = -1$ for $a \le -x \le b$ ∴ $-b \le x \le -a$ ∴

$u(x,0) = \phi(x) = \begin{cases} 1 & a \le x \le b \\ -1 & -b \le x \le -a \\ 0, & \text{otherwise} \end{cases}$ ∴ $u(x,0) = \phi(x) - \phi(-x)$ ∴

$u_t - D u_{xx} = 0$, $x \in \mathbb{R}$ ∴ $u(x,t) = \frac{1}{\sqrt{4\pi D t}} \int_{-\infty}^{\infty} [\phi(s) - \phi(-s)] e^{-(x-s)^2/(4Dt)}\, ds =$

$$\frac{1}{\sqrt{4\pi D t}} \left[\int_{-\infty}^{\infty} \phi(s) g(s)\, ds - \int_{-\infty}^{\infty} \phi(-s) g(s)\, ds \right] = \frac{1}{\sqrt{4\pi D t}} \left[\int_{a}^{b} \phi(s) g(s)\, ds - \int_{b}^{a} \phi(s) g(-s)\, ds \right]$$

19/ If $u_{xx}+u_{yy}=0$ for $x\in\mathbb{R}$, $y\geq 0$: $u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{u(s,0)}{(x-s)^2+y^2}\,ds$, $y>0$

$y''+\lambda y=0$; $y(a)=0$, $y(b)=0$ $(a<b)$:

$y_n(x)=\sin\left(\frac{n\pi(x-a)}{b-a}\right)$, $\lambda=\frac{n^2\pi^2}{(b-a)^2}$ $n\in\mathbb{N}$ ∴

$y''+\lambda y=0$ $y(0)=0$, $y(L)=0$ $(0<L)$:

$y_n(x)=\sin\left(\frac{n\pi}{L}x\right)$, $\lambda=\frac{n^2\pi^2}{L^2}$; $n\in\mathbb{N}$

V a/ let $\xi=x+y$, $\eta=x-y$ ∴ $\xi_x=1$, $\xi_y=1$, $\eta_x=1$, $\eta_y=-1$ ∴

$u_x=u_\xi\xi_x+u_\eta\eta_x=u_\xi+u_\eta$

$u_y=u_\xi\xi_y+u_\eta\eta_y=u_\xi-u_\eta$ ∴

$u_x+u_y=u_\xi+u_\eta+u_\xi-u_\eta=2u_\xi$ ∴ $\xi+\eta=x+y+x-y=2x$ ∴ $\frac{1}{2}\xi+\frac{1}{2}\eta=x$

PDE: $-\eta=-x+y$ ∴ $\xi-y=x+y-x+y=2y$ ∴ $x+y=\frac{1}{2}\xi+\frac{1}{2}\eta+\frac{1}{2}\xi-\frac{1}{2}\eta=\frac{1}{2}\xi-\frac{1}{2}\eta$

$2u_\xi+u=e^{x+y}=e^{\frac{1}{2}\xi-\frac{1}{2}\eta}$ ∴

$u_\xi+\frac{1}{2}u=\frac{1}{2}e^{\frac{1}{2}\xi-\frac{1}{2}\eta}$ ∴ $IF=e^{\int\frac{1}{2}d\xi}=e^{\frac{1}{2}\xi}$

$\frac{\partial}{\partial\xi}(e^{\frac{1}{2}\xi}u)=\frac{1}{2}e^{\frac{1}{2}\xi}e^{\frac{1}{2}\xi-\frac{1}{2}\eta}=\frac{1}{2}e^{\xi-\frac{1}{2}\eta}$ ∴

$e^{\frac{1}{2}\xi}u=\frac{1}{4}e^{\xi-\frac{1}{2}\eta}+f(\eta)$ ∴

$u=\frac{1}{4}e^{-\frac{1}{2}\xi}e^{\xi-\frac{1}{2}\eta}+e^{-\frac{1}{2}\xi}f(\eta)=u=\frac{1}{4}e^{\frac{1}{2}\xi-\frac{1}{2}\eta}+e^{-\frac{1}{2}\xi}f(\eta)$ ∴

$u=\frac{1}{4}e^{x+y}+e^{-\frac{1}{2}(x+y)}f(x-y)=u(x,y)$ ∴

$u(x,0)=0=\frac{1}{4}e^{x}+e^{-\frac{1}{2}x}(f(x-y))$ ∴

$-\frac{1}{4}e^{x}=e^{-\frac{1}{2}x}f(-y)$ ∴ $f(-y)=-\frac{1}{4}e^{\frac{3}{2}y}$ ∴

let $s=-y$ ∴ $-s=y$ ∴ $f(s)=-\frac{1}{4}e^{-\frac{3}{2}s}$ ∴

$f(x-y)=-\frac{1}{4}e^{-\frac{3}{2}(x-y)}=-\frac{1}{4}e^{-\frac{3}{2}x+\frac{3}{2}y}$ ∴

$u(x,y)=\frac{1}{4}e^{x+y}+e^{-\frac{1}{2}x-\frac{1}{2}y}(-\frac{1}{4})e^{-\frac{3}{2}x+\frac{3}{2}y}=\frac{1}{4}e^{x+y}-\frac{1}{4}e^{-2x+2y}$

V a/ $\xi=x-y$, $\eta=y$ ∴ $u_x=u_\xi$, $u_y=-u_\xi+u_\eta$ ∴

$x=\xi+\eta$, $y=\eta$ ∴ $u_\eta+u=e^{\xi+3\eta}$ ∴ $IF=e^{\int 1\,d\eta}=e^{\eta}$ ∴

$\frac{\partial}{\partial\eta}(e^{\eta}u)=e^{\xi+4\eta}$ ∴ $e^{\eta}u=\frac{1}{4}e^{\xi+4\eta}+A(\xi)$ ∴

$u=\frac{1}{4}e^{\xi+3\eta}+e^{-\eta}A(\xi)=\frac{1}{4}e^{x+3y}+e^{-y}A(x-y)$ ∴

$u(x,0)=0=\frac{1}{4}e^{x}+A(x)$ ∴ $A=-\frac{1}{4}e^{x}$ ∴ $A(x-y)=-\frac{1}{4}e^{x-y}$ ∴

$u = \frac{1}{4}e^{x+2y} + e^{-y}\left(-\frac{1}{4}\right)e^{x-y} = \frac{1}{4}\left(e^{x+2y} - e^{x-2y}\right) = \frac{1}{4}e^{x}\left(e^{2y} - e^{-2y}\right)$

\b/ $u_{xx} + 2(1-x^2)u_{xy} + y^2 u_{yy} = -(1-x)u_x - (1+x)u_y - yu$ ∴

$a = 1,\ 2b = 2(1-x^2)$ ∴ $b = 1-x^2$, $c = y^2$ ∴

$b^2 - ac = (1-x^2)^2 - 1y^2 = 1 - 2x^2 + x^4 - y^2 = (1-x^2)^2 - y^2$ ✓

∴ if $b^2 - ac = 0 = (1-x^2)^2 - y^2 = 0$ ∴ $y^2 = (1-x^2)^2$ ∴ $y = \pm(1-x^2)$

then parabolic,

if $b^2 - ac > 0$ ∴ $(1-x^2)^2 - y^2 > 0$ ∴ $(1-x^2)^2 > y^2$ ∴ $|y| < |1-x^2|$ ∴

$-(1-x^2) < y < 1-x^2$, then hyperbolic between $-(1-x^2)$, $(1-x^2)$

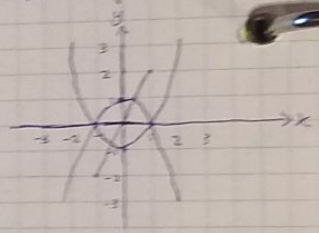
if $b^2 - ac < 0$ ∴ $(1-x^2)^2 - y^2 < 0$ ∴ $(1-x^2)^2 < y^2$ ∴

$y < -(1-x^2)$ or $(1-x^2) < y$ then eliptic ∴ $|y| > |1-x^2|$

$a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0$ ✓ ∴

$\left(\frac{dy}{dx}\right)^2 - 2(1-x^2)\left(\frac{dy}{dx}\right) + y^2 = 0$ ∴

$\frac{dy}{dx} = \frac{2(1-x^2) \pm \sqrt{4(1-x^2)^2 - 4y^2}}{2} = (1-x^2) \pm \sqrt{(1-x^2) - y^2}$



at $(0,0)$: $\frac{dy}{dx} = 1 \pm \sqrt{1} = 1 \pm 1 = 2, 0$ ∴

\c/ The $x=0$ BC can be satisfied by considering an even extension in x ie the same PDE ~~in~~ $x \in (-\infty, \infty)$, $t > 0$ and ICs

$u_t(x,0) = 0$, $u(x,0) = \Pi(x-2) + \Pi(x+2)$ ∴

The GS of the equation is given by D'Alemberts formula

$u(x,t) = f(x-t) + g(x+t)$

IC: $u_t(x,t) = 0$ implies $f(x) = g(x)$, then the other IC gives

$f(x) = g(x) = \frac{1}{2}u(x,0) = \frac{1}{2}\left(\Pi(x+2) + \Pi(x-2)\right)$ ∴

$u(x,t) = \frac{1}{2}\left(\Pi(x-t-2) + \Pi(x-t+2) + \Pi(x+t-2) + \Pi(x+t+2)\right)$

∴ $u(1,1) = \frac{1}{2}\left(\Pi(-2) + \Pi(2) + \Pi(0) + \Pi(4)\right) = \frac{1}{2}(0+0+1+0) = \frac{1}{2} = 0.5$

out of the four terms, ~~in the~~ solution, only two can be simultaneously nonzero, ∴ $\text{Max}(u(x,t)) = \frac{1}{2}(1+1) = 1$

\d/ Considering an even extension in x ie $x \in (-\infty, \infty)$, $t > 0$ and ~~B~~ BCs $u(x,\infty) = 0$, $u(x,0) = \Pi(x-2) + \Pi(x+2)$ ∴

is the GS of the equation from D'Alemberts formula:

\PP2019/ $\therefore u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{u(s,0)}{(x-s)^2+y^2}\,ds$, $y>0$ $\therefore$

$u(s,0) = \Pi(x-2) + \Pi(x+2)$ $\therefore$

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\Pi(x-2)+\Pi(x+2)}{(x-s)^2+y^2}\,ds$$

\1d sol/ The $x=0$ BC can be satisfied by considering an odd extension in x $\therefore$ The same PDE in $x\in(-\infty,\infty)$, $y>0$ and BC $u(x,0) = \Pi(x-2)-\Pi(x+2) = \begin{cases} 1, & x\in(1,3) \\ -1, & x\in(-3,-1) \\ 0, & \text{otherwise} \end{cases}$ $\therefore$

$$u(x,y) = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{u(s,0)}{(x-s)^2+y^2}\,ds = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\Pi(s-2)-\Pi(s+2)}{(x-s)^2+y^2}\,ds =$$

$$\frac{y}{\pi}\left[\int_{-\infty}^{-3} 0\,ds + \int_{-3}^{-1}\frac{-1}{(x-s)^2+y^2}\,ds + \int_{-1}^{1} 0\,ds + \int_{1}^{3}\frac{1}{(x-s)^2+y^2}\,ds + \int_{3}^{\infty} 0\,ds\right] =$$

$$\frac{y}{\pi}\left(\int_{1}^{3}\frac{1}{(x-s)^2+y^2}\,ds - \int_{-3}^{1}\frac{1}{(x-s)^2+y^2}\,ds\right)$$

let $I = \int \frac{1}{(x-s)^2+y^2}\,ds$ $\therefore$ let $s = x+yz$ $\therefore \frac{ds}{dz} = y$ $\therefore$ $ds = y\,dz$ $\therefore$

$-s = -x-yz$, $\frac{s-x}{y} = z$

$$I = \int\frac{1}{(-yz)^2+y^2}\,ds = \int\frac{1}{y^2z^2+y^2}\,ds = \int\frac{1}{y^2(z^2+1)}\,ds = \int\frac{1}{y^2(z^2+1)}\,y\,dz =$$

$$\frac{1}{y}\int\frac{1}{z^2+1}\,dz = \frac{1}{y}\arctan(z) = \frac{1}{y}\arctan\left(\frac{s-x}{y}\right) \therefore$$

$$u(x,y) = \frac{y}{\pi}\left(\frac{1}{y}\right)\left[\arctan\left(\frac{3-x}{y}\right) - \arctan\left(\frac{1-x}{y}\right) - \arctan\left(\frac{-1-x}{y}\right) + \arctan\left(\frac{-3-x}{y}\right)\right]$$

$$= \frac{1}{\pi}\left[\arctan\left(\frac{3-x}{y}\right) - \arctan\left(\frac{1-x}{y}\right) + \arctan\left(\frac{1+x}{y}\right) - \arctan\left(\frac{3+x}{y}\right)\right] \therefore$$

$$u(1,1) = \frac{1}{\pi}\left[\arctan(2) - \arctan(0) + \arctan(2) - \arctan(4)\right] =$$

$$\frac{1}{\pi}(2\arctan 2 - \arctan 4) \approx 0.2828$$

\1e/ consider dependence of $E(t) = \int_0^L u(x,t)\,dx$ on time $\therefore$

$$\frac{dE}{dt} = \int_0^L u_t(x,t)\,dx = \int_0^L u_{xx}(x,t)\,dx = \left[u_x\right]_{x=0}^{L} = u_x(L,t) - u_x(0,t) =$$

$0 - 0 = 0$ $\therefore$ $\frac{dE}{dt} = 0$ $\therefore$ $E(t) = E(0) = \int_0^L \phi(x)\,dx = \int_0^L u(x,t)\,dx$

\1etry/ let $F(t) = \int_0^L u^2(x,t)\,dx$ $\therefore$ $\frac{dF(t)}{dt} = \int_0^L 2u\,u_t\,dx =$

$2\int_0^L u u_{xx}\,dx = 2[u u_x]_0^L - 2\int_0^L u_x u_x\,dx =$

$2u[u_x(L,t) - u_x(0,t)] - 2\int_0^L u_x^2\,dx = 2(u)(0) - 2\int_0^L u_x^2\,dx =$

$-2\int_0^L u_x^2\,dx \le 0 \because (u_x)^2 \ge 0 \therefore$

$\frac{dF(t)}{dt} \le 0 \therefore t \ge 0 \therefore F(0) \ge F(t) \ \forall t \therefore$

$\int_0^L u^2(x,0)\,dx \ge \int_0^L u^2(x,t)\,dx \ \forall t \because \quad u(x,0) = \phi(x) \therefore u^2(x,0) = \phi^2(x)$

$\therefore \int_0^L \phi^2(x,0)\,dx \ge \int_0^L u^2(x,t)\,dt$

\ie sol / let $F(t) = \int_0^L u^2(x,t)\,dx \therefore \frac{dF}{dt} = 2\int_0^L u u_t\,dx = \int_0^L u u_{xx}\,dx =$

$2[u u_x]_{x=0}^L - \int_0^L u_x^2(x,t)\,dx = -\int_0^L u_x^2(x,t)\,dx \le 0 \therefore$

$F(t) \le F(0) = \int_0^L \phi^2(x)\,dx$

\2a/ $u_{xx} - y^2 u_{yy} = 2u_x - y u_y + 4 \therefore$

$a = 1,\ b = 0,\ c = -y^2 \because$

$b^2 - ac = 0 - 1(-y^2) = y^2 > 0 \therefore$ hyperbolic $\because \because y > 0 \therefore$

$a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0 = \left(\frac{dy}{dx}\right)^2 - y^2 = 0 \therefore \left(\frac{dy}{dx}\right)^2 = y^2 \therefore$

$\frac{dy}{dx} = \pm y \therefore \quad \frac{dy}{dx} = y,\ \frac{dy}{dx} = -y \therefore$

$\frac{1}{y}\frac{dy}{dx} = 1,\ \frac{1}{y}\frac{dy}{dx} = -1 \quad \because y > 0 \therefore$

$\int \frac{1}{y}\,dy = \int 1\,dx = x + C_1 = \ln|y| \quad , \ln|y| = \int \frac{1}{y}\,dy = \int -1\,dx = -x + D_1 \therefore$

$|y| = e^{x+C_1} = C_2 e^x \therefore y = C_3 e^x \quad , |y| = e^{-x+D_1} = D_2 e^{-x} \therefore y = D_3 e^{-x}$

$\therefore ye^{-x} = C_3 \ , ye^x = D_3 \quad \therefore \xi = ye^x \ , \eta = ye^{-x}$

\2b/ $\therefore \xi_x = ye^x \ , \xi_y = e^x \ , \eta_x = -ye^{-x} \ , \eta_y = e^{-x} \therefore$

$\cancel{u_x = u_\xi \xi_x = ye^x u_\xi} \quad u_x = u_\xi \xi_x + u_\eta \eta_x = ye^x u_\xi - ye^{-x} u_\eta$

$u_y = u_\xi \xi_y + u_\eta \eta_y = e^x u_\xi + e^{-x} u_\eta \therefore$

$u_{xx} = \cancel{ye} \ \frac{\partial}{\partial x}(u_x) = ye^x u_\xi + ye^x \frac{\partial}{\partial x} u_\xi + ye^{-x} u_\eta - ye^{-x} \frac{\partial}{\partial x} u_\eta =$

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$ye^{x}u_{\xi}+ye^{x}u_{\xi\xi}\xi_x+ye^{x}u_{\xi\eta}\eta_x+ye^{-x}u_{\eta}-ye^{-x}u_{\xi\eta}\xi_x-ye^{-x}u_{\eta\eta}\eta_x$

$u_{yy}=\frac{\partial}{\partial y}(u_y)=e^{x}u_{\xi\xi}\xi_y+e^{x}u_{\xi\eta}\eta_y+e^{-x}u_{\xi\eta}\xi_y+e^{-x}u_{\eta\eta}\eta_y=$

$e^{2x}u_{\xi\xi}+2u_{\xi\eta}+e^{-2x}u_{\eta\eta}$

$u_{xx}=ye^{x}u_{\xi}+y^2e^{2x}u_{\xi\xi}-y^2u_{\xi\eta}+ye^{-x}u_{\eta}-y^2u_{\xi\eta}+y^2e^{-2x}u_{\eta\eta}=$

$ye^{x}u_{\xi}+y^2e^{2x}u_{\xi\xi}-2y^2u_{\xi\eta}+ye^{-x}u_{\eta}+y^2e^{-2x}u_{\eta\eta}\ \therefore$

PDE:

$\cancel{ye^{x}u_{\xi}}+y^2e^{2x}u_{\xi\xi}-2y^2u_{\xi\eta}+ye^{-x}u_{\eta}+\cancel{y^2e^{-2x}u_{\eta\eta}}+$

$-\cancel{y^2e^{2x}u_{\xi\xi}}-2y^2u_{\xi\eta}-\cancel{y^2e^{-2x}u_{\eta\eta}}-2ye^{x}u_{\xi}+2ye^{-x}u_{\eta}+$

$\cancel{ye^{x}u_{\xi}}+e^{-x}u_{\eta}=4=$

$-2y^2u_{\xi\eta}+ye^{-x}u_{\eta}-2y^2u_{\xi\eta}+2ye^{-x}u_{\eta}+e^{-x}u_{\eta}=$

$-4y^2u_{\xi\eta}+3e^{-x}u_{\eta}+e^{-x}u_{\eta}=4=-4y^2u_{\xi\eta}+4e^{-x}u_{\eta}=4\ \therefore$

$-y^2u_{\xi\eta}+e^{-x}u_{\eta}=1\ \therefore$

$y^2u_{\xi\eta}=e^{-x}u_{\eta}-1=(ye^{x})(ye^{-x})u_{\xi\eta}=\xi\eta u_{\xi\eta}=$

$\eta u_{\eta}-1$

$\xi\eta u_{\xi\eta}-\eta u_{\eta}=-1\ \therefore\ \therefore$

$\forall\alpha\ u(\alpha\xi+\eta)\ \therefore\ u_{\eta}=u'',\ u_{\xi\eta}=\alpha u''$

Take $v=\eta u_{\eta}\ \therefore\ \xi\eta u_{\xi\eta}=\xi\eta\frac{\partial}{\partial\xi}(u_{\eta})=\xi\frac{\partial}{\partial\xi}(\eta u_{\eta})=\xi v_{\xi}\ \therefore$

$\xi\eta u_{\xi\eta}-\eta u_{\eta}=-1\ \therefore$

$\xi v_{\xi}-v=-1\ \therefore\ v_{\xi}-\frac{1}{\xi}v=-\frac{1}{\xi}\ \therefore\ IF=e^{\int-\frac{1}{\xi}d\xi}=e^{-\ln\xi}=e^{\ln(\xi^{-1})}=\frac{1}{\xi}$

$\therefore\ \frac{\partial}{\partial\xi}\left(\frac{1}{\xi}v\right)=-\frac{1}{\xi^2}=-\xi^{-2}\ \therefore\ \frac{1}{\xi}v=-\frac{1}{-1}\xi^{-1}=\frac{1}{\xi}+g(\eta)\ \therefore$

$\cancel{v=1=\eta u_{\eta}}\ \therefore\ \cancel{\frac{1}{\eta}=u_{\eta}\ \therefore\ \ln\eta+S(\xi)=u}\ \therefore$

$v=1+\xi g(\eta)=\eta u_{\eta}\ \therefore$

$u_{\eta}=\frac{1}{\eta}+\xi\frac{1}{\eta}g(\eta)=\frac{1}{\eta}+\xi\hat{g}(\eta)\ \therefore$

$u=\ln(\eta)+\xi C(\eta)+D(\xi)=\ln(ye^{-x})+ye^{-x}C(ye^{-x})+D(ye^{x})=$

$-x+\ln(y)+ye^{x}S(ye^{-x})+g(ye^{x})$

2c/ $u = -x + \ln y + ye^{x} f(ye^{-x}) + g(ye^{x})$ $\therefore$ $u(0,y) = \ln y + y^2$, $u_x(0,y) = -1$ $\therefore$

$u(0,y) = \ln y + y^2 = \ln y + y f(y) + g(y)$ $\therefore$ $y^2 = y f(y) + g(y)$

$u_x = -1 + ye^{x} f(ye^{-x}) - ye^{-x} ye^{x} f'(ye^{-x}) + ye^{x} g'(ye^{x}) =$

$-1 + ye^{x} f(ye^{-x}) - y^2 f'(ye^{-x}) + ye^{x} g'(ye^{x})$ $\therefore$

$u_x(0,y) = -1 = -1 + y f(y) - y^2 f'(y) + y g'(y)$ $\therefore$

$0 = y f(y) - y^2 f'(y) + y g'(y)$

$y^2 - y f(y) = g(y)$ $\therefore$ $g'(y) = 2y - f(y) - y f'(y)$ $\therefore$

$0 = y f(y) - y^2 f'(y) + 2y^2 - y f(y) - y^2 f'(y) = -2y^2 f'(y) + 2y^2 = 0$ $\therefore$

$2y^2 = 2y^2 f'(y)$ $\therefore$ $1 = f'(y)$ $\therefore$ $f(y) = y$ $\therefore$

$y^2 - yy = y^2 - y^2 = 0 = g(y)$ $\therefore$ $g(ye^{x}) = 0$, $f(ye^{-x}) = ye^{-x}$ $\therefore$

$u = -x + \ln y + ye^{x} ye^{-x} = -x + \ln y + y^2$

2c sol/ $u = -x + \ln y + ye^{x} f(ye^{-x}) + g(ye^{x})$ $u(0,y) = \ln y + y^2$, $u_x(0,y) = -1$

$\therefore u_x(0,y) = -1 + ye^{x} f(ye^{-x}) - y^2 f'(ye^{-x}) + ye^{x} g'(ye^{x})$ $\therefore$

$\ln y + y^2 = \ln y + y f(y) + g(y)$ $\therefore$ $-1 = -1 + y f(y) - y^2 f'(y) + y g'(y)$

$y^2 = y f(y) + g(y)$, $0 = y f(y) - y^2 f'(y) + y g'(y)$ $\therefore$

$g(y) = y^2 - y f(y)$ $\therefore$ $g'(y) = 2y - f(y) - y f'(y)$ $\therefore$

$y f(y) - y^2 f'(y) + y(2y - f(y) - y f'(y)) = 0$ $\therefore$ $2y^2 f'(y) = 2y^2$ $\therefore$

$f'(y) = 1$ $\therefore$ $f(y) = y + c$ $\therefore$ $g(y) = -cy$ $\therefore$ $u(x,y) = -x + \ln y + y^2$

4a/ $\rho = r^{-1}$, $\theta = \theta$ $\therefore$ $\rho_r = -r^{-2}$ $\theta_\theta = 1$ $\therefore$

$u_r = u_\rho \rho_r = -r^{-2} u_\rho$ $\therefore$ $u_{rr} = \frac{\partial}{\partial r}(-r^{-2} u_\rho) = 2r^{-3} u_\rho - r^{-2} u_{\rho\rho} \rho_r =$

$2r^{-3} u_\rho + r^{-4} u_{\rho\rho}$, $u_\theta = u_\theta \theta_\theta = u_\theta$, $u_{\theta\theta} = u_{\theta\theta} \theta_\theta = u_{\theta\theta}$ $\therefore$

The PDE:

$2r^{-3} u_\rho + r^{-4} u_{\rho\rho} + \frac{1}{r}(-r^{-2} u_\rho) + \frac{1}{r^2}(u_{\theta\theta}) =$

$2r^{-3} u_\rho + r^{-4} u_{\rho\rho} - r^{-3} u_\rho + \frac{1}{r^2} u_{\theta\theta} =$

$\frac{1}{r^3} u_\rho + \frac{1}{r^4} u_{\rho\rho} + \frac{1}{r^2} u_{\theta\theta} = 0$ $\therefore$ $u_\rho + \frac{1}{r} u_{\rho\rho} + r u_{\theta\theta} = 0$ $\therefore$

$u_\rho + r u_\rho + r^2 u_{\theta\theta} = 0$

4a sol/ let $u(r,\theta) = v(\rho,\theta)$ $\therefore$ $u_r = \rho_r v_\rho = -\frac{1}{r^2} v_\rho$ $\therefore$

$u_{rr} = \frac{2}{r^3} v_\rho - \frac{1}{r^2} \frac{\partial}{\partial r}(v_\rho) = \frac{2}{r^3} v_\rho + \frac{1}{r^4} v_{\rho\rho}$, $u_{\theta\theta} = v_{\theta\theta}$ $\therefore$

$\frac{1}{r^4} v_{\rho\rho} + \frac{2}{r^3} v_\rho - \frac{2}{r^3} v_\rho + \frac{1}{r^2} v_{\theta\theta} = \frac{1}{r^4} v_{\rho\rho} + \frac{1}{r^3} v_\rho + \frac{1}{r^2} v_{\theta\theta} = 0$ $\therefore$

PP2019/ $\therefore r = \frac{1}{\rho}$ $\therefore$ $\left(\frac{1}{1/\rho}\right)^4 V_{\rho\rho} + \frac{1}{(1/\rho)^3} V_\rho + \left(\frac{1}{1/\rho}\right)^2 V_{\phi\phi} = \rho^4 V_{\rho\rho} + \rho^3 V_\rho + \rho^2 V_{\phi\phi}$
$= 0 = V_{\rho\rho} + \frac{1}{\rho} V_\rho + \frac{1}{\rho^2} V_{\phi\phi} = 0$ $\therefore$ identical equation $\therefore$ invariant wrt change of variables

4b/ $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$, $r<1$, $u(1,\theta) = f(\theta)$, $|u(0,\theta)| < \infty$ $\therefore$

$$u = \frac{1-r^2}{2\pi} \int_0^{2\pi} \frac{f(\xi)}{1-2r\cos(\theta-\xi)+r^2}\, d\xi$$

$\therefore$ if $|u(\infty,\theta)| < \infty$: $V_{\rho\rho} + \frac{1}{\rho} V_\rho + \frac{1}{\rho^2} u_{\phi\phi}$ $\therefore$

let $v(r,\theta) = u(\frac{1}{r},\theta)$ $\therefore$ $v(r,\theta)$ satisfies the equation $r<1$, and reverse. and $u(r,\theta)$ satisfies conditions iff $v(r,\theta)$ satisfies conditions. $\therefore$ solution can be obtained from the solution of $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$, $u(1,\theta) = f(\theta)$, $|u(0,\theta)| < \infty$

by substituting $r \to \frac{1}{r}$ into $u = \frac{1-r^2}{2\pi} \int_0^{2\pi} \frac{f(\xi)}{1-2r\cos(\theta-\xi)+r^2}\, d\xi$

$$\therefore u = \frac{1-r^{-2}}{2\pi} \int_0^{2\pi} \frac{f(\xi)}{1-2r^{-1}\cos(\theta-\xi)+r^{-2}}\, d\xi = \frac{r^2-1}{2\pi} \int_0^{2\pi} \frac{f(\xi)}{r^2-2r\cos(\theta-\xi)+1}\, d\xi$$

4c/ $u_{rr} + \frac{1}{r} u_r + \frac{1}{r^2} u_{\theta\theta} = 0$, $r \geq 1$, $u(1,\theta) = \cos\theta$ $\therefore$

let $f(\xi) = \cos\xi$ $\therefore$ $u = \frac{r^2-1}{2\pi} \int_0^{2\pi} \frac{\cos(\xi)}{r^2-2r\cos(\theta-\xi)+1}\, d\xi$ $\therefore$

$$u = \frac{\cos\theta}{2\pi} \frac{r^2-1}{r^2+1} \int_0^{2\pi} \frac{\cos z}{1-2r\cos z/(r^2+1)}\, dz \quad \therefore \quad \alpha = \frac{2r}{r^2+1} \quad \therefore$$

$$u = \cos\theta \frac{r^2-1}{r^2+1} \frac{1-\sqrt{1-4r^2/(r^2+1)^2}}{\frac{2r}{r^2+1}\sqrt{1-\frac{4r^2}{(r^2+1)^2}}} \quad \therefore \because r>1 \therefore$$

$u = \frac{1}{r}\cos\theta$

$x = r\cos\theta$, $y = r\sin\theta$, $u_{xx} + u_{yy} = 0$, $u = \frac{1}{r}\cos\theta$ $\therefore$

$$u = \frac{\cos\theta}{r} = \frac{r\cos\theta}{r^2} = \frac{x}{r^2(\sin^2\theta + \cos^2\theta)} = \frac{x}{x^2+y^2} \quad \therefore$$

$$u_x = \frac{y^2-x^2}{(x+y)^2} \quad \therefore u_{xx} = \frac{2x(x^2-3y^2)}{(x^2+y^2)^3}$$

$$u_y = \frac{-2xy}{(x^2+y^2)^2}, \quad u_{yy} = \frac{2x(3y^2-x^2)}{(x^2+y^2)^3} \quad \therefore \quad u_{xx} + u_{yy} = 0$$

3a/ let $V = X(x)T(t)$ $\therefore$ $V_t = XT'$, $V_{xx} = X''T$ $\therefore$

$XT' = X''T$ $\therefore$ $\frac{T'}{T} = \frac{X''}{X} = +\lambda = \text{Constant}$ $\therefore$

$X'' = \lambda X$, $T' = \lambda T$ $\therefore$ $T' - \lambda T = 0$,

$X'' - \lambda X = 0$, $X(0) = 0$, $X(L) = 0$ $\therefore$

$q^2 - \lambda = 0$ $\therefore$ $q^2 = \lambda$ $\therefore$ $q = \pm\sqrt{\lambda}$ $\therefore$

$X = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$

$\therefore$ if $\lambda = 0$: $X = A + Bx$ $\therefore$

$X(0) = 0 = A + 0 = A = 0$ $\therefore X = Bx$ $\therefore$

$X(L) = 0 = BL = B = 0$ $\therefore$ no non trivial solutions

for $\lambda > 0$: $X = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$ $\therefore$

$X(0) = A + B = 0$ $\therefore$ $A = -B$ $\therefore$ $X = B(-e^{\sqrt{\lambda}x} + e^{-\sqrt{\lambda}x})$ $\therefore$

$X(L) = B(-e^{\sqrt{\lambda}L} + e^{-\sqrt{\lambda}L}) = 0 = B = 0$ $\therefore$ $X = 0$

for $\lambda < 0$: $X = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$ $\therefore$

$X(0) = A + 0B = A = 0$ $\therefore$ $X = B\sin(\sqrt{\lambda}x)$ $\therefore$

$X(L) = 0 = B\sin(\sqrt{\lambda}L)$ $\therefore \sqrt{\lambda}L = n\pi$ $n \in \mathbb{N}$ $\therefore$

$\sqrt{\lambda} = \frac{n\pi}{L}$ $\therefore$ $\lambda = \frac{n^2\pi^2}{L^2}$ $\therefore X = B\sin(\frac{n\pi}{L}x)$

$\therefore T' - \lambda T = 0 = T' - \frac{n^2\pi^2}{L^2}T = 0$ $\therefore$

$\frac{dT}{dt} = \frac{n^2\pi^2}{L^2}T$ $\therefore$ $\frac{1}{T}\frac{dT}{dt} = \frac{n^2\pi^2}{L^2}$ $\therefore \int \frac{1}{T}dT = \int \frac{n^2\pi^2}{L^2}dt =$

$\ln|T| = \frac{n^2\pi^2}{L^2}t + C_1$ $\therefore |T| = e^{\frac{n^2\pi^2}{L^2}t + C_1} = C_2 e^{\frac{n^2\pi^2}{L^2}t}$ $\therefore T = Be^{\frac{n^2\pi^2}{L^2}t}$ $\therefore$

$\therefore u = \sum_{n=1}^{\infty} B_n e^{\frac{n^2\pi^2}{L^2}t}\sin(\frac{n\pi}{L}x)$ by superposition $\therefore$

for $v(x,0) = \phi(x) = \sum_{n=1}^{\infty} B_n \sin(\frac{n\pi}{L}x)$ $\therefore$

$\cancel{B_n = \frac{1}{\phi}}$ $B_n = \frac{1}{L}\int_{-L}^{L}\phi(x)\sin(\frac{n\pi}{L}x)dx = \frac{2}{L}\int_0^L \phi(x)\sin(\frac{n\pi}{L}x)dx$

\3a redo/ let $v = X(x)T(t)$ $\therefore$ $v_{xx} = X''T$, $v_t = XT'$ $\therefore$

$XT' = X''T$ $\therefore$ $\frac{T'}{T} = \frac{X''}{X} = \lambda =$ constant $\therefore$ $T' = \lambda T$, $X'' = \lambda X$ $\therefore$

$T' - \lambda T = 0$, $X'' - \lambda X = 0$, $X(0) = 0$, $X(L) = 0$ $\therefore$

$q^2 - \lambda = 0$ $\therefore q^2 = \lambda$ $\therefore$ $q = \pm\sqrt{\lambda}$ $\therefore$

for $\lambda = 0$: $X = A + Bx$ $\therefore$ $X(0) = 0 = A$ $\therefore X = Bx$ $\therefore$

$X(L) = 0 = BL = B = 0$ $\therefore$ $X = 0$

for $\lambda > 0$: $X = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$ $\therefore$ $X(0) = 0 = A + B$ $\therefore$ $-B = A$ $\therefore$

$X = -Be^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x} = B(-e^{\sqrt{\lambda}x} + e^{-\sqrt{\lambda}x})$ $\therefore$

$X(L) = 0 = B(-e^{\sqrt{\lambda}L} + e^{-\sqrt{\lambda}L}) = B = 0$ $\therefore$ $X = 0$

for $\lambda < 0$: $X = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$ $\therefore$

$X(0) = 0 = A + 0B = A = 0$ $\therefore$

$X = B\sin(\sqrt{\lambda}\,x) \therefore X(L) = 0 = B\sin(\sqrt{\lambda}\,L) \therefore$

$\sqrt{\lambda} = \frac{n\pi}{L} \therefore \lambda = \frac{n^2\pi^2}{L^2}$, $X = B\sin(\frac{n\pi}{L}x) \therefore$

$T' = \frac{n^2\pi^2}{L^2}T \therefore \frac{T'}{T} = \frac{n^2\pi^2}{L^2} \therefore \int \frac{T'}{T}dt = \int \frac{n^2\pi^2}{L^2}dt =$

$\ln|T| = \frac{n^2\pi^2}{L^2}t + C_1 \therefore |T| = e^{+\frac{n^2\pi^2}{L^2}t + C_1} = C_2 e^{\frac{n^2\pi^2}{L^2}t} \therefore$

$T = Be^{\frac{n^2\pi^2}{L^2}t} \therefore$ by superposition:

$V = XT = \sum_{n=1}^{\infty} B_n e^{\frac{n^2\pi^2}{L^2}t}\sin(\frac{n\pi}{L}x)$

3a) $V = X(x)T(t) \therefore V_{xx} = X''T$, $V_t = XT' \therefore$

$XT' = X''T \therefore \frac{T'}{T} = \frac{X''}{X} = -\lambda = \text{constant} \therefore$

$T' = -\lambda T$, ~~$X'' - \lambda X = 0$~~ $\therefore X'' = -\lambda X \therefore X'' + \lambda X = 0$.

$X(0) = 0$, $X(L) = 0 \therefore$

$q^2 + \lambda = 0 \therefore q^2 = -\lambda \therefore q = \pm\sqrt{-\lambda}$

$\therefore$ For $\lambda = 0$: $X = A + Bx \therefore X(0) = 0 = A = 0 \therefore X = Bx \therefore$

$X(L) = BL = 0 = B = 0 \therefore X = 0$

For $\lambda < 0 \therefore -\lambda > 0 \therefore X = Ae^{+\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x} \therefore$

$X(0) = A + B = 0 \therefore A = -B \therefore X = B(-e^{+\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x}) \therefore$

$X(L) = 0 = B(-e^{\sqrt{-\lambda}L} + e^{-\sqrt{-\lambda}L}) = B = 0 \therefore X = 0$

For $\lambda > 0 \therefore -\lambda < 0 \therefore$ ~~$X = A\cos L$~~ ~~$X = A\sin(\sqrt{\lambda}x)$~~

$\pm\sqrt{-\lambda} = \pm\sqrt{\lambda}\,i = q \therefore X = A\sin(\sqrt{\lambda}x) + B\cos(\sqrt{\lambda}x) \therefore$

$X(0) = 0 = 0A + B = B = 0 \therefore X = A\sin(\sqrt{\lambda}x) \therefore$

$X(L) = 0 = A\sin(\sqrt{\lambda}L) \therefore \sqrt{\lambda}L = n\pi, n \in \mathbb{N} \therefore \sqrt{\lambda} = \frac{n\pi}{L} \therefore$

$\lambda = \frac{n^2\pi^2}{L^2} = \lambda_n \therefore X = A\sin(\frac{n\pi}{L}x) \therefore$

~~$T' = -\frac{n^2\pi^2}{L}$~~ $T' = -\frac{n^2\pi^2}{L^2}T \therefore \frac{T'}{T} = -\frac{n^2\pi^2}{L^2} \therefore \int \frac{T'}{T}dt = \int -\frac{n^2\pi^2}{L^2}dt$

$= \ln|T| = -\frac{n^2\pi^2}{L^2}t + C_1 \therefore |T| = e^{-\frac{n^2\pi^2}{L^2}t + C_1} = C_2 e^{-\frac{n^2\pi^2}{L^2}t} \therefore$

$T = Ae^{-\frac{n^2\pi^2}{L^2}t} \therefore$ by superposition:

$V = \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2}{L^2}t}\sin(\frac{n\pi}{L}x)$

\3a/ let $v = X(x)T(t)$ $\therefore X_t = XT'$, $X_{xx} = X''T$ $\therefore$

$XT' = X''T$ $\therefore$ $\frac{T'}{T} = \frac{X''}{X} = -\lambda = \text{constant}$ $\therefore$

$T' = -\lambda T$, $X'' = -\lambda X$ $\therefore$

$X'' + \lambda X = 0$, $X(0) = 0$, $X(l) = 0$ $\therefore$

$X_n(x) = A\sin\left(\frac{n\pi(x-0)}{l-0}\right) = A\sin\left(\frac{n\pi}{l}x\right)$, $\lambda_n = \frac{n^2\pi^2}{(l-0)^2} = \frac{n^2\pi^2}{l^2}$, $n \in \mathbb{N}$

$\therefore T' = -\lambda T = T' = -\frac{n^2\pi^2}{l^2}T$ $\therefore$ $\frac{T'}{T} = -\frac{n^2\pi^2}{l^2}$ $\therefore \int \frac{T'}{T}dt = \int -\frac{n^2\pi^2}{l^2}dt =$

$\cancel{\ln|T| = e^{-\frac{n^2\pi^2}{l^2}}}$ $\ln|T| = -\frac{n^2\pi^2}{l^2}t + C_1$ $\therefore$

$|T| = e^{-\frac{n^2\pi^2}{l^2}t + C_1} = C_2 e^{-\frac{n^2\pi^2}{l^2}t}$ $\therefore$

$T = Ae^{-\frac{n^2\pi^2}{l^2}t}$ $\therefore$ by superposition $\therefore$

$v = \sum_{n=1}^{\infty} A_n e^{-\frac{n^2\pi^2}{l^2}t}\sin\left(\frac{n\pi}{l}x\right)$ $\therefore$ $v(x,0) = \phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right)$ $\therefore$

$A_n = \frac{1}{l}\int_{-l}^{l}\phi(x)\sin\left(\frac{n\pi}{l}x\right)dx = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi}{l}x\right)dx$

$\therefore$ $\because$ the equation and BCs are not altered by $t \to t + \text{const}$, the new solution to the problem is obtained from a shift in time $\therefore$

$v(x,s) = \phi(s)$ $\therefore$ $v(x,t) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi}{l}x\right)e^{-\frac{n^2\pi^2}{l^2}(t-s)}$ $\therefore$

$v(x,s) = \phi(x) = \sum_{n=1}^{\infty} A_n \sin\left(\frac{n\pi x}{l}\right)$ $\therefore$

$A_n = \frac{1}{l}\int_{-l}^{l}\phi(x)\sin\left(\frac{n\pi}{l}x\right)dx = \frac{2}{l}\int_0^l \phi(x)\sin\left(\frac{n\pi}{l}x\right)dx$

\3b/ ~~let $v = u + \delta(x,t)$ $\Rightarrow$ to $u(x,t) = v(x,t) + \delta(x,t)$~~ $\therefore$

let ~~$w = u$~~ $w(x,t) = u(x,t) + \delta(x,t)$ $\therefore$ $w - \delta = u$ $\therefore$

~~w_t~~ $w_t - \delta_t = u_t$, $w_x - \delta_x = u_x$ $\therefore$ $w_{xx} - \delta_{xx} = u_{xx}$ $\therefore$

$u_t - u_{xx} = \delta$, $u_t - u_{xx} - \delta = 0$ $\therefore$ $u_t - u_{xx} - \delta = 0 =$

$w_t - \delta_t - w_{xx} + \delta_{xx} - \delta = 0$

\3b sol/ $\therefore$ LHS $= u_t = \frac{\partial}{\partial t}\int_0^t v(x,t;s)ds = v(x,t;t) + \int_0^t v_t(x,t,s)ds =$

$\frac{\partial}{\partial t}\int_0^t v(x,t;s)ds = v(x,t;t) + \int_0^t v_t(x,t;s)ds = \delta(x,t) + \int_0^t v_t(x,t;s)ds =$

$\delta(x,t) + \int_0^t v_{xx}(x,t;s)ds$,

PP 2019/ RHS $= u_{xx} + f(x,t) = f(x,t) \frac{\partial^2}{\partial x^2} \int_0^t v(x,t;s)\,ds =$

$f(x,t) + \int_0^t v_{xx}(x,t;s)\,ds = LHS$ $\therefore$ PDE is satisfied

BC: $u(0,t) = \int_0^t v(0,t;s)\,ds = \int_0^t 0\,ds = 0$

and $u(l,t) = \int_0^t v(l,t;s)\,ds = \int_0^t 0\,ds = 0$.

$u(x,0) = \int_0^0 v(x,0;s)\,ds = 0$ $\therefore$ BCs satisfied $\therefore$ Vistra Solution

3c $\therefore f(x,t) = \sin^3(x)$ $\therefore$

v is the solution with $v(x,s;s) = f(x,s) = \sin^3(x)$

$u = \int_0^t v(x,t;s)\,ds = \int_0^t \sin^3(x)\,ds = \int_0^t \frac{3}{4}\sin x - \frac{1}{4}\sin(3x)\,ds =$

$\left[-\frac{3}{4}\cos x + \frac{1}{12}\cos(3x)\right]_0^t = -\frac{3}{4}\left[\cos(t) + \frac{1}{12}\cos(3t) + \frac{3}{4}\cos(0) - \cos(0)\right] =$

$-\frac{3}{4}\cos(t) + \frac{1}{12}\cos(3t) - \frac{1}{4}$

3c $\therefore f(x,t) = \sin^3(x)$, $l = \pi$ $\therefore$ $v(x,s;s) = f(x,s) = \sin^3(x)$ $\therefore$

$u(x,t) = \int_0^t v(x,t;s)\,ds = \int_0^t \sin^3(x)\,ds = \left[\sin^3(x)\,s\right]_{s=0}^t =$

$\sin^3(x)\left[t - 0\right] = \sin^3(x)\,t$

3c Sol/ $\sin^3(x) = \frac{1}{4}(3\sin x - \sin(3x))$, $l = \pi$ $\therefore$

$v(x,t;0) = \frac{3}{4}\sin(x)e^{-t} - \frac{1}{4}\sin(3x)e^{-9t}$ $\therefore$

$v(x,t;s) = \frac{3}{4}\sin x\, e^{s-t} - \frac{1}{4}\sin(3x)e^{9s-9t}$ $\therefore$

$u(x,t) = \int_0^t \left[\frac{3}{4}\sin(x)e^{s-t} - \frac{1}{4}\sin(3x)e^{9s-9t}\right]ds =$

$\frac{3}{4}\sin x\,(1-e^{-t}) - \frac{1}{36}\sin(3x)\,(1-e^{-9t})$

6 remain

P 17 S 18 P 19 S 20 P 21 P 22 S 23 S 24 S 25 S 26 S 27 S 28 S 29

6 days till PDE 10 days till stochastic 13 days till fluids $\frac{13}{3} = 4.33$

21,22 25,26 27,28,29

21,22, 19 25,26,17,20 18, 27,28,29

PP2021

1a ∴ $(y, -x)\cdot(u_x, u_y) = 0 = (y, -x)\cdot\nabla u$ ∴ characteristic curves satisfy (rewrite equation)

$\left(\frac{dy}{dx}\right) = \frac{-x}{y}$ ∴ $y\frac{dy}{dx} = -x$ ∴ $y=0$ is special solution

∴ $\int y\,dy = \int -x\,dx = \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C_1$ ∴ $y^2 = -x^2 + 2C_1 = -x^2 + C_2$ ∴

$y^2 + x^2 = C_2$ ∴ $u(x,y) = f(y^2+x^2)$ ∴ f is an arbitrary function

1ai $u(x, \sqrt{4+x^2}) = x^2 = f((\sqrt{4+x^2})^2 + x^2) = f(4+x^2+x^2) = f(4+2x^2)$

∴ let $s = 4+2x^2$ ∴ $s-4 = 2x^2$ ∴ $\frac{s-4}{2} = x^2$ ∴

$f(s) = \frac{s-4}{2}$ ∴

$u(x,y) = f(y^2+x^2) = \frac{(y^2+x^2)-4}{2} = \frac{1}{2}y^2 + \frac{1}{2}x^2 - 2$

1aii $u(x, \sqrt{4-x^2}) = x^2 = f((\sqrt{4-x^2})^2 + x^2) = f(4-x^2+x^2) = f(4) = x^2$

∴ is impossible ∴ no solution exists ∴

the BC is on the characteristic curve

$\xi = y-x$, $\eta = y+x$

1b let $\xi = x-y$, $\eta = -x-y$ ∴

$\xi_x = 1$, $\xi_y = -1$, $\eta_x = -1$, $\eta_y = -1$

$\eta + x = -y$ ∴ $\xi = x + \eta + x$ ∴ $\xi - \eta = 2x$ ∴ $\frac{1}{2}\xi - \frac{1}{2}\eta = x$ ∴

$-\eta - \frac{1}{2}\xi + \frac{1}{2}\eta = -\frac{1}{2}\xi - \frac{1}{2}\eta = y$ ∴

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi - u_\eta$,

$u_y = u_\xi \xi_y + u_\eta \eta_y = -u_\xi - u_\eta$ ∴

$u_{xx} = u_{\xi\xi}\xi_x - u_{\xi\eta}\xi_x + u_{\xi\eta}\eta_x - u_{\eta\eta}\eta_x = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$

$u_{yy} = -u_{\xi\xi}\xi_y - u_{\xi\eta}\xi_y - u_{\xi\eta}\eta_y - u_{\eta\eta}\eta_y = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$ ∴

$u_{xx} - u_{yy} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} - u_{\xi\xi} - 2u_{\xi\eta} - u_{\eta\eta} =$

$-4u_{\xi\eta} = 4e^{2y} = 4e^{2(-\frac{1}{2}\xi - \frac{1}{2}\eta)} = 4e^{-\xi-\eta}$ ∴

$u_{\xi\eta} = -e^{-\xi-\eta} = \frac{\partial}{\partial\xi}\left(\frac{\partial}{\partial\eta}u\right)$ ∴

$\frac{\partial}{\partial\eta}u = e^{-\xi-\eta} + f(\eta)$ ∴ $-\xi-\eta = -x+y+x+y = 2y$ ∴

$u = -e^{-\xi-\eta} + f(\eta) + g(\xi)$ ∴ $\xi - \eta = x-y+x+y = 2x$

$u(x,y) = -e^{2y} + f(-x-y) + g(x-y)$ ∴

$u(x,x) = e^{2x} = -e^{2x} + f(-x-x) + g(x-x) = -e^{2x} + f(-2x) + g(0)$

$\cancel{u(x,x) = e^{2x}}$ $\therefore 2e^{2x} = f(-2x) + g(0)$

$u(x,-x) = e^{-2x} = -e^{2x} + f(-x+x) + g(x+x) = -e^{2x} + f(0) + g(2x)$

$\therefore g(0) = 2e^{2x} - f(-2x)$, $f(0) = e^{-2x} + e^{2x} - g(2x)$ $\therefore$

system is inconsistent $\therefore$ no solution exists.

\1b/ let $\xi = y - x$, $\eta = y + x$ $\therefore$

$\xi_x = -1$, $\xi_y = 1$, $\eta_x = 1$, $\eta_y = 1$ $\therefore$

$\eta - x = y$ $\therefore$ $\xi = \eta - x - x = \eta - 2x$ $\therefore 2x = -\xi + \eta$ $\therefore x = -\frac{1}{2}\xi + \frac{1}{2}\eta$ $\therefore$

$\eta + \frac{1}{2}\xi - \frac{1}{2}\eta = y = \frac{1}{2}\xi + \frac{1}{2}\eta$ $\therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x = -u_\xi + u_\eta$ $\therefore$

$u_{xx} = -u_{\xi\xi}\xi_x + u_{\xi\eta}\xi_x - u_{\xi\eta}\eta_x + u_{\eta\eta}\eta_x =$

$u_{\xi\xi} - u_{\xi\eta} - u_{\xi\eta} + u_{\eta\eta} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$

$u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi + u_\eta$ $\therefore$

$u_{yy} = u_{\xi\xi}\xi_y + u_{\xi\eta}\xi_y + u_{\xi\eta}\eta_y + u_{\eta\eta}\eta_y$

$\cancel{u_{yy} = u_\xi \xi_y + u_{\xi\eta}\eta_y + u_{\xi\eta}\xi_y}$

$u_{yy} = u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y + u_{\xi\eta}\xi_y + u_{\eta\eta}\eta_y = u_{\xi\xi} + u_{\xi\eta} + u_{\xi\eta} + u_{\eta\eta}$

$= u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$ $\therefore$

$u_{xx} - u_{yy} = 4e^{2y} = 4e^{2(\frac{1}{2}\xi + \frac{1}{2}\eta)} = 4e^{\xi+\eta} =$

$u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} - u_{\xi\xi} - 2u_{\xi\eta} - u_{\eta\eta} =$

$-4u_{\xi\eta} = 4e^{\xi+\eta}$ $\therefore$ $u_{\xi\eta} = e^{\xi+\eta} = \cancel{\frac{\partial}{\partial\xi}(u}\ \frac{\partial}{\partial\xi}\left(\left(\frac{\partial}{\partial\eta}u\right)\right)$

$\therefore \frac{\partial}{\partial\eta}u = e^{\xi+\eta} + F(\eta)$ $\therefore$

$u = e^{\xi+\eta} + f(\eta) + g(\xi)$ $\therefore$ $\xi + \eta = y - x + y + x = 2y$ $\therefore$

$\cancel{u(x,y) = e^{\xi+\eta}}$ $u(x,y) = e^{2y} + f(y+x) + g(y-x)$ $\therefore$

$u(x,x) = e^{2x} = e^{2x} + f(x+x) + g(x-x) = e^{2x} + f(2x) + g(0)$ $\therefore$

$0 = f(2x) + g(0)$

\PP2021/

\1b/ let $u(x,y) = v(\xi,\eta) \therefore \xi = x-y,\ \eta = x+y \therefore$

$\xi + y = x \quad \therefore \eta = \xi + y + y = \xi + 2y \therefore -\xi + \eta = 2y \therefore \frac{-\xi+\eta}{2} = y \therefore$

$\xi + \frac{-\xi+\eta}{2} = \frac{\xi+\eta}{2} = x \therefore$

$\xi_x = 1,\ \xi_y = -1,\ \eta_x = 1,\ \eta_y = +1 \therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta \therefore$

$u_{xx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\eta}\eta_x = u_{\xi\xi} + u_{\xi\eta} + u_{\xi\eta} + u_{\eta\eta} =$

$u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$u_y = u_\xi \xi_y + u_\eta \eta_y = -u_\xi + u_\eta$

$u_{yy} = -u_{\xi\xi}\xi_y + u_{\xi\eta}\xi_y - u_{\xi\eta}\eta_y + u_{\eta\eta}\eta_y = u_{\xi\xi} - u_{\xi\eta} - u_{\xi\eta} + u_{\eta\eta} =$

$u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} \therefore$

$u_{xx} - u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} - u_{\xi\xi} + 2u_{\xi\eta} - u_{\eta\eta} =$

$2u_{\xi\eta} + 2u_{\xi\eta} = 4e^{2y} = 4e^{2x - \frac{\xi+\eta}{2}} = 4e^{-\xi+\eta} = 4u_{\xi\eta} \therefore$

$u_{\xi\eta} = e^{-\xi+\eta} \therefore$

BCs: at $y = x$: $\xi = x - y = x - x = 0$, $\eta = x + y = x + x = 2x$,

at $y = -x$: $\xi = x - y = x - (-x) = 2x$, $\eta = x + y = x + (-x) = 0 \therefore$

$u_{\eta\eta}$

~~$v(0,\eta) =$~~ ~~$u(x,$~~

$\therefore$ at $y = x$: $2x = \eta$, $\xi = 0 \therefore v(\xi=0,\eta) = v(0,\eta) = e^{\eta}$,

at $y = -x$: $\xi = 2x \therefore -2x = -\xi$, $\eta = 0 \therefore v(\xi, \eta=0) = e^{-\xi} = v(\xi, 0) \therefore$

$\cancel{\int_0^{\xi} v_{\xi\eta}\, d\xi = \int_0^{\xi} e^{-\xi+\eta}\, d\xi = v_\eta = [-e^{-\xi+\eta}]}$

$\cancel{v_\eta(\xi,\eta) = \int_0^{\xi} v_{\xi\eta}\, d\xi + v_\eta(0,\eta) = \int_0^{\xi} e}$

$u_{\xi\eta} = e^{-\xi+\eta} \therefore$

$u_\eta = -e^{-\xi+\eta} + F(\eta) \therefore$

$u = -e^{-\xi+\eta} + F(\eta) + g(\xi) \therefore$

$u(x,x) = e^{2x} = u(x=x, y=x) \therefore u(\xi = \frac{\xi+\eta}{2}, \eta = \frac{-\xi+\eta}{2}) = e^{\xi+\eta} \therefore$

$u(x = x, y = -x) \therefore u(\xi = \frac{\xi+\eta}{2}, \eta = \frac{\xi-\eta}{2}) = e^{-\xi-\eta} \therefore$

$u = -e^{-\xi+\eta} + S(\eta) + g(\xi)$ ∴

at $y=x$: $\xi = x-y = x-x = 0$, $\eta = x+y = x+x = 2x$

at $y=-x$: $\xi = x-(-x) = 2x$, $\eta = x+y = x-x = 0$ ∴

∴ at $y=x$: $2x = \eta$, ∴ $u(\xi=0, \eta=2x) = e^{\eta} = u(0,\eta)$

∴ $u(0,\eta) = e^{\eta}$

at $y=-x$: $2x = \xi$ ∴ $-2x = -\xi$ ∴ $u(-\xi, \eta=0) = u(-\xi, 0) = e^{-\xi}$ ∴

~~$u(0,\eta) = -e^{-\xi+\eta} + S(\eta) + g(\xi) =$~~

~~$u(0,\eta) = e^{\eta} = -e^{-\xi+\eta} + S(\eta) + g(0)$~~

$u_{\xi\eta} = e^{\eta-\xi}$

BC: at $y=x$, $\xi = x-y$, $\eta = x+y$ ∴ $\xi = x-y = 0$, $\eta = x+y = 2x$,

at $y=-x$: $\xi = x-y = 2x$, $\eta = x+y = 0$ ∴

$v(0,\eta) = e^{\eta}$, $v(\xi,0) = e^{-\xi}$ ∴

$v_\eta(\xi,\eta) = \int_0^{\xi} v_{\xi\eta}\, d\xi + v_\eta(0,\eta) = \int_0^{\xi} e^{\eta-\xi'}\, d\xi' =$

$e^{\eta} + e^{\eta}\left[-e^{-\xi}\right]_0^{\xi} = e^{\eta} + e^{\eta}(1-e^{-\xi}) = 2e^{\eta} - e^{\eta-\xi}$ ∴

$v(\xi,\eta) = v(\xi,0) + \int_0^{\eta} v_\eta(\xi,\eta')\, d\eta' = e^{-\xi} + \int_0^{\eta} (2e^{\eta'} - e^{\eta'-\xi})\, d\eta' =$

$e^{-\xi} + 2\left[e^{\eta'}\right]_0^{\eta} - e^{-\xi}\left[e^{\eta'}\right]_0^{\eta} = e^{-\xi} + 2(e^{\eta}-1) - e^{-\xi}(e^{\eta}-1) =$

$2e^{-\xi} + 2e^{\eta} - e^{\eta-\xi} - 2$ ∴

in original coordinates

\ IC / let $u = v e^{pt+qx}$ ∴ $u_t = v_t e^{pt+qx} + v p e^{pt+qx} = e^{pt+qx}(pv + v_t)$

∴ $u_x = v_x e^{pt+qx} + qv e^{pt+qx} = e^{pt+qx}(qv + v_x)$ ∴

$u_{xx} = e^{pt+qx}(qv_x + v_{xx}) + q e^{pt+qx}(qv + v_x) = e^{pt+qx}(q^2 v + 2q v_x + v_{xx})$ ∴

PDE: $u_t - u_{xx} + 2u_x = 0$ ∴

$e^{pt+qx}\left[(pv + v_t) - (q^2 v + 2q v_x + v_{xx}) + 2(qv + v_x)\right] = 0 =$

$pv + v_t - q^2 v - 2q v_x - v_{xx} - 2qv - 2v_x = 0 =$

$v(p - q^2 - 2q) + (-2q - 2)v_x + v_t - v_{xx} = 0$ ∴

let $p - q^2 - 2q = 0$, $-2q - 2 = 0$ ∴ $-2 = 2q$ ∴ $-1 = q$ ∴ $p - 1 + 2 = 0 \Rightarrow p + 1$ ∴ $p = -1$

VP2021 $\therefore V_t - V_{xx} = 0 \quad \therefore V_t = V_{xx} \therefore$

$e^{pt+qx} = e^{-t-x} \therefore$

$u(x,0) = \varphi(x) = v(x,0)e^{-(0)-x} = v(x,0)e^{-x} \therefore$

$v(x,0) = \varphi(x)e^{x} \quad \therefore V_t = V_{xx} \therefore \quad v(x,0) = \varphi(x) = \varphi(x)e^{x} \therefore$

$$v(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \varphi(\xi)e^{-\frac{(x-\xi)^2}{4t}}\,d\xi = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \varphi(\xi)e^{\xi}e^{-\frac{(x-\xi)^2}{4t}}\,d\xi =$$

$$\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \varphi(\xi)e^{\xi-\frac{(x-\xi)^2}{4t}}\,d\xi \quad \therefore$$

$$u(x,t) = v(x,t)e^{pt+qx} = \frac{1}{\sqrt{4\pi t}}e^{pt+qx}\int_{-\infty}^{\infty} \varphi(\xi)e^{\xi-\frac{(x-\xi)^2}{4t}}\,d\xi =$$

$$\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \varphi(\xi)e^{pt+qx+\xi-\frac{(x-\xi)^2}{4t}}\,d\xi \therefore$$

Sor $\varphi(x) = \delta(x+1) \quad \therefore \varphi(\xi) = \delta(\xi+1) \therefore$

$$u(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \delta(\xi+1)e^{pt+qx+\xi-\frac{(x-\xi)^2}{4t}}\,d\xi \therefore$$

$$P(t) = u(0,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \delta(\xi+1)e^{pt+\xi-\frac{\xi^2}{4t}}\,d\xi \therefore$$

$$\frac{dP(t)}{dt} = -\frac{1}{2}\frac{1}{\sqrt{4\pi}}t^{-\frac{3}{2}}\int_{-\infty}^{\infty} \delta(\xi+1)e^{pt+\xi-\frac{\xi^2}{4t}}\,d\xi + \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \delta(\xi+1)\left(p+\frac{\xi^2}{4t^2}\right)e^{pt+\xi-\frac{\xi^2}{4t}}\,d\xi$$

$= 0 \quad \therefore \quad t = \frac{1}{2} \therefore$

$t_x = \frac{1}{2}$

Vb Sol $u_{\xi\eta} = e^{\eta-\xi}$, $u(x,x) = e^{2x}$, $u(x,-x) = e^{-2x}$

$\therefore u_{PI}(x,y) = -e^{2y} \therefore \quad u_{CF}(x,y) = f(x+y) + g(x-y) \therefore$

$\therefore u(x,y) = f(x+y) + g(x-y) - e^{2y} \quad \therefore$

$u(x,x) = e^{2x} = f(2x) + g(0) - e^{2x}$

$u(x,y) = -e^{2y} + f(x+y) + g(x-y) \therefore$

$u(x,x) = e^{2x} = -e^{2x} + f(x+x) + g(x-x) = -e^{2x} + f(2x) + g(0) \therefore$

$2e^{2x} - g(0) = f(2x) \therefore$

let $s = 2x \therefore \quad \frac{1}{2}s = x \therefore \quad 2e^{s} - g(0) = f(s) \therefore$

$f(0) = 2e^{0} - g(0) = 2 - g(0)$

$u(x,-x) = e^{-2x} = f(x+(-x)) + g(x-(-x)) - e^{-2x}$ $\therefore$

$2e^{-2x} = f(0) + g(2x)$ $\therefore$ $2e^{-2x} - f(0) = g(2x)$ $\therefore$

let $s = 2x$ $\therefore$ $g(s) = 2e^{-s} - (f(0))$ $\therefore$

$g(s) = 2e^{-s} - (2 - g(0)) = 2e^{-s} - 2 + g(0)$ $\therefore$

$g(0) = g(0)$

$g(s) = 2e^{-s} - 2 + g(0)$, $f(s) = 2e^{s} - g(0)$ $\therefore$

$u(x,y) = -e^{+2y} + f(x+y) + g(x-y)$ $\therefore$

$f(x+y) = 2e^{x+y} - g(0)$, $g(x-y) = 2e^{-x+y} - 2 + g(0)$ $\therefore$

$u(x,y) = -e^{2y} + 2e^{x+y} - g(0) + 2e^{-x+y} - 2 + g(0)$ $\therefore$

$u(x,y) = -e^{2y} - 2 + 2e^{x+y} + 2e^{-x+y}$ ✓

1e Sol ✓ $u(x,t) = v(x,t)e^{pt+qx}$ $\therefore$ $u_t = (v_t + pv)e^{pt+qx}$ $\therefore$

$u_x = (v_x + qv)e^{pt+qx}$ $\therefore$ $u_{xx} = (v_{xx} + 2qv_x + q^2 v)e^{pt+qx}$ $\therefore$

$v_t + pv = v_{xx} + (2q-2)v_x + (q^2 - 2q)v$, $u(x,0) = v(x,0)e^{qx} = \psi(x)$ $\therefore$

$v(x,0) = \psi(x)e^{-qx}$, $\therefore p = -1$, $q = 1$ $\therefore$

$v(x,0) = \psi(x)e^{-x} = \phi(x)$, $v_t = v_{xx}$ $\therefore$

$v(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \phi(\xi) e^{-(x-\xi)^2/(4t)} d\xi = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \psi(\xi) e^{-\xi} e^{-(x-\xi)^2/(4t)} d\xi$ $\therefore$

$e^{pt+qx} = e^{x-t}$ $\therefore$

$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{x-t} \int_{-\infty}^{\infty} \psi(\xi) e^{-\xi - (x-\xi)^2/(4t)} d\xi$ $\therefore$

$\psi(x) = \delta(x+1)$ $\therefore$ $\psi(\xi) = \delta(\xi+1)$ $\therefore$

$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{x-t} \int_{-\infty}^{\infty} \delta(\xi+1) e^{-\xi - (x-\xi)^2/(4t)} d\xi$ $\therefore$

$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{x-t} e^{1-(x+1)^2/(4t)}$ $\therefore$

$P(t) = u(0,t) = \frac{1}{\sqrt{4\pi t}} e^{-t} \int_{-\infty}^{\infty} \delta(\xi+1) e^{-\xi - \xi^2/(4t)} d\xi = \frac{1}{\sqrt{4\pi t}} e^{-t} e^{1 - \frac{1}{4t}} =$

$\frac{1}{\sqrt{4\pi t}} e^{1-t-\frac{1}{4t}} = P(t)$ $\therefore$

$\ln(P(t)) = \ln\left(\frac{1}{\sqrt{4\pi t}} e^{1-t-\frac{1}{4t}}\right) = \ln\left((4\pi t)^{-1/2}\right) + 1 - t - \frac{1}{4t} =$

$1 - \ln(2) - \frac{1}{2}\ln(\pi) - \frac{1}{2}\ln(t) - t - \frac{1}{4t}$ $\therefore$

$\frac{d(\ln(P))}{dt} = -\frac{1}{2t} - 1 + \frac{1}{4t^2} = 0$, Son $t = t_*$ $\therefore$ $-\frac{1}{2t_*} - 1 + \frac{1}{4t_*^2} = 0$ $\therefore$

PP2021 ✓ $-1-2t_*+\frac{1}{2t_*}=0 \therefore -2t_*-4t_*^2+1=0 \therefore$

$4t_*^2+2t_*-1=0 \therefore t_*=\frac{-2\pm\sqrt{4-4(4)(-1)}}{2(4)}=\frac{-2\pm\sqrt{20}}{8}=$

$-\frac{2}{8}\pm\frac{2\sqrt{5}}{8}=-\frac{1}{4}\pm\frac{\sqrt{5}}{4} \therefore t_*>0 \therefore$ positive root $\therefore$

$t_*=-\frac{1}{4}+\frac{\sqrt{5}}{4}=\frac{1}{1+\sqrt{5}}\approx 0.3090$

1d ✓ $u_{xx}+u_{yy}=0$, $u_x(0,y)=0$, $u_x(\pi,y)=0$, $u_y(x,0)=0$, $u_y(x,\pi)=\delta(x)$

$\therefore u=X(x)Y(y) \therefore u_x=X'Y$, $u_{xx}=X''Y$, $u_y=XY'$, $u_{yy}=XY'' \therefore$

$X''Y+XY'' \therefore X''Y=-XY'' \therefore$

$\frac{X''}{X}=-\frac{Y''}{Y}=-\lambda=$ constant $\therefore$

$X''=-\lambda X \therefore X''+\lambda X=0$, $Y''=\lambda Y$, $\therefore Y''-\lambda Y=0$

$X''+\lambda X=0$, ~~$u_x(0)$~~ $X'(0)=0$, $X'(\pi)=0 \therefore$

$q^2+\lambda=0 \therefore q^2=-\lambda \therefore q=\pm\sqrt{-\lambda} \therefore$

$\lambda=0: X=A+Bx \therefore X'=B \therefore X'(0)=0=B \therefore X=A$

$-\lambda>0 \therefore \lambda<0: X=Ae^{\sqrt{-\lambda}x}+Be^{-\sqrt{-\lambda}x} \therefore$

$X'=A\sqrt{-\lambda}e^{\sqrt{-\lambda}x}-B\sqrt{-\lambda}e^{-\sqrt{-\lambda}x} \therefore$

$X'(0)=0=A\sqrt{-\lambda}-B\sqrt{-\lambda}=A-B=0 \therefore A=B \therefore$

$X'=B\sqrt{-\lambda}(e^{\sqrt{-\lambda}x}-e^{-\sqrt{-\lambda}x}) \therefore$

$X'(\pi)=0=B\sqrt{-\lambda}(e^{\sqrt{-\lambda}\pi}-e^{-\sqrt{-\lambda}\pi})=B=0 \therefore X=0$

$-\lambda<0 \therefore \lambda>0 \therefore q=\pm\sqrt{-\lambda}=\pm\sqrt{\lambda}i \therefore$

$X=A\cos(\sqrt{\lambda}x)+B\sin(\sqrt{\lambda}x) \therefore$ ~~X'~~ $X'=-A\sqrt{\lambda}\sin(\sqrt{\lambda}x)+B\sqrt{\lambda}\cos(\sqrt{\lambda}x)$

$X'(0)=B\sqrt{\lambda}=0 \therefore B=0 \therefore$

$X'=-A\sqrt{\lambda}\sin(\sqrt{\lambda}x) \therefore$

$X'(\pi)=0=-A\sqrt{\lambda}\sin(\sqrt{\lambda}\pi)=A\sin(\sqrt{\lambda}\pi)=0 \therefore \sqrt{\lambda}\pi=n\pi \therefore$

$\sqrt{\lambda}=n$, $\lambda=n^2$, $n\in\mathbb{N} \therefore X=A\cos(nx)$,

$Y''-n^2Y=0 \therefore q^2-n^2=0 \therefore q^2=n^2 \therefore q=\pm n \therefore$

$Y=Ae^{+ny}+Be^{-ny} \therefore Y'(0)=0$, ~~$Y'(\pi)=\delta(x)$~~ $Y'(\pi)=1 \therefore$

$Y'=Ane^{ny}-Bne^{-ny} \therefore Y'(0)=0=An-Bn \therefore An=Bn \therefore A=B \therefore$

$Y'=An(e^{ny}-e^{-ny}) \therefore$ ~~$Y'(\pi)=\delta(x)=$~~

$Y'(\pi)=1=An(e^{n\pi}-e^{-n\pi})$ $\therefore$ $A=\frac{1}{n(e^{n\pi}-e^{-n\pi})}$ $\therefore$ $Y=(A)(e^{ny}+e^{-ny})$

$Y=\frac{1}{n(e^{n\pi}-e^{-n\pi})}(e^{ny}+e^{-ny})$ $\therefore$

$u(x,y)=A_0+\sum_{n=1}^{\infty}A_n\frac{1}{n(e^{n\pi}-e^{-n\pi})}(e^{ny}+e^{-ny})\cos(nx)$ $\therefore$

$\int_0^{\pi}g(x)dx=0$,

$u_y(x,y)=\sum_{n=1}^{\infty}A_n\frac{1}{n(e^{n\pi}-e^{-n\pi})}(ne^{ny}-ne^{-ny})\cos(nx)$ $\therefore$

$u_y(x,\pi)=g(x)=\sum_{n=1}^{\infty}A_n\frac{1}{n(e^{n\pi}-e^{-n\pi})}(ne^{n\pi}-ne^{-n\pi})\cos(nx)=$

$\sum_{n=1}^{\infty}A_n\frac{1}{n}\cos(nx)=g(x)$ $\therefore$

$\frac{A_n}{n}=\frac{1}{\pi}\int_{-\pi}^{\pi}g(x)\cos(nx)dx=\frac{2}{\pi}\int_0^{\pi}g(x)\cos(nx)dx=\frac{1}{n}A_n$,

$A_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}g(x)dx=\frac{1}{\pi}\int_0^{\pi}g(x)dx=0$ $\therefore$

$\therefore$ $u(x,y)=\sum_{n=1}^{\infty}\frac{A_n}{n}\frac{1}{(e^{n\pi}-e^{-n\pi})}(e^{ny}+e^{-ny})\cos(nx)=$

$\sum_{n=1}^{\infty}\frac{2}{\pi}\int_0^{\pi}g(\xi)\cos(n\xi)d\xi\frac{1}{(e^{n\pi}-e^{-n\pi})}(e^{ny}+e^{-ny})\cos(nx)=$

$\sum_{n=1}^{\infty}\frac{2}{\pi}\int_0^{\pi}g(\xi)\cos(n\xi)\cos(nx)d\xi\frac{1}{(e^{n\pi}-e^{-n\pi})}(e^{ny}+e^{-ny})$

\Id✓ $u=X(x)Y(y)$ $\therefore$ $u_{xx}=X''Y$, $u_{yy}=XY''$ $\therefore$

$X''Y+XY''=0$ $\therefore$ $X''Y=-XY''$ $\therefore$ $\frac{X''}{X}=-\frac{Y''}{Y}=-\lambda=$ Constant

$\therefore X''=-\lambda X$, $Y''=\lambda Y$ $\therefore$ $Y'(0)=0$, $Y'(\pi)=1$,

$X''+\lambda X=0$, $X'(0)=0$, $X'(\pi)=0$ $\therefore$

$X_n(x)=\cos(nx)$, $\lambda_n=n^2$, $n\in\mathbb{Z}_{\geq 0}$ $\therefore$

$Y''=n^2Y$ $\therefore$ $Y''-n^2Y=0$

\Id Sol✓ let $u=X(x)Y(y)$ $\therefore$ $-\frac{Y''}{Y}=\frac{X''}{X}=-\lambda=$ Constant $\therefore$

$Y''(y)-\lambda Y(y)=0$, $X''(x)+\lambda X(x)=0$, $X'(0)=0$, $X'(\pi)=0$ $\therefore$

$X(x)=X_n(x)=\cos(nx)$, $\lambda=\lambda_n=n^2$, $n\in\mathbb{Z}_{\geq 0}$ $\therefore$

$Y''-n^2Y=0$ $\therefore$ $q^2-n^2=0$ $\therefore$ $q^2=n^2$ $\therefore$ $q=\pm n$ $\therefore$

$Y=C_1e^{ny}+C_2e^{-ny}=Y_n(y)$, $Y_n'(0)=0$ $\therefore$

$Y_n'(y)=C_1ne^{ny}-C_2ne^{-ny}$ $\therefore$ $Y_n'(g)=0=C_1n-C_2n=C_1-C_2$ $\therefore$ $C_1=C_2$ $\therefore$

$)(e^{ny}+e^{-ny})$

\PP 2021/ $Y_n(y) = C_1(e^{ny}+e^{-ny}) = 2C_1\frac{1}{2}(e^{ny}+e^{-ny}) = 2C_1\cos(ny) = C\cosh(ny)$

$\therefore u(x,y) = \sum_{n=0}^{\infty} C_n\cosh(ny)\cos(nx)$ $\therefore$

$u_y(x,y) = \sum_{n=0}^{\infty} C_n n\sinh(ny)\cos(nx)$ $\therefore$

$u_y(x,\pi) = g(x) = \sum_{n=0}^{\infty} C_n n\sinh(n\pi)\cos(nx)$,

$u_y(x,0) = 0 = \sum_{n=0}^{\infty} C_n n\sinh(0)\cos(nx)$

$\therefore$ ~~$C_n = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\sinh(n\pi)\cos(nx)dx = \frac{2}{\pi}$~~

$C_n n\sinh(n\pi) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)dx = \frac{2}{\pi}\int_0^{\pi} g(x)\cos(nx)dx$ $\therefore$

$C_n = \frac{2}{\pi n\sinh(n\pi)}\int_0^{\pi} g(x)\cos(nx)dx = \frac{2}{\pi n\sinh(n\pi)}\int_0^{\pi} g(\xi)\cos(n\xi)d\xi$

$u(x,y) = C_0 + \sum_{n=1}^{\infty} C_n\cosh(ny)\cos(nx) =$

$C_0 + \sum_{n=1}^{\infty}\frac{2\cosh(ny)\cos(nx)}{n\sinh(n\pi)}\int_0^{\pi} g(s)\cos(ns)ds$, $C_0 \in \mathbb{R}$

\1d retry/ let $u = X(x)Y(y)$ $\therefore$ $u_{xx} = X''Y$, $u_{yy} = XY''$

$\therefore X''Y + XY'' = 0$ $\therefore$ $X''Y = -XY''$ $\therefore$ $\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda = \text{constant}$ $\therefore$

$Y'' = \lambda Y$ $\therefore$ $Y'' - \lambda Y = 0$, $X'' = -\lambda X$ $\therefore$

$X'' + \lambda X = 0$, $X'(0) = 0$, $X'(\pi) = 0$ $\therefore$

$X_n(x) = \cos(nx)$, $\lambda_n = n^2$, $n \in \mathbb{Z}_{\geq 0}$ $\therefore$

$Y'' - n^2 Y = 0$ $\therefore$ $q^2 - n^2 = 0$ $\therefore$ $q^2 = n^2$ $\therefore$ $q = \pm n$ $\therefore$

$Y_n(y) = Ae^{ny} + Be^{-ny}$ $\therefore$ $Y_n'(0) = 0$ $\therefore$

$Y_n' = Ane^{ny} - Bne^{-ny}$ $\therefore$ $Y_n'(0) = 0 = An - Bn = 0 = A - B$ $\therefore A = B$ $\therefore$

$Y_n(y) = Ae^{ny} + Ae^{-ny} = A(e^{ny}+e^{-ny}) = 2A\frac{1}{2}(e^{ny}+e^{-ny}) = 2A\cosh(ny) =$

$C\cosh(ny)$ $\therefore$

$u(x,y) = X(x)Y(y) = \sum_{n=0}^{\infty} C_n\cosh(ny)\cos(nx)$ $\therefore$

$u_y(x,y) = \sum_{n=0}^{\infty} C_n n\sinh(ny)\cos(nx)$ $\therefore$

$u_y(x,\pi) = g(x) = \sum_{n=0}^{\infty} C_n n\sinh(n\pi)\cos(nx)$ $\therefore$

$C_n n\sinh(n\pi) = \frac{1}{\pi}\int_{-\pi}^{\pi} g(x)\cos(nx)dx$ $\therefore C_n = \frac{1}{\pi n\sinh(n\pi)}\int_{-\pi}^{\pi} g(x)\cos(nx)dx =$

$C_1 = C_2$ $\therefore$

$\frac{2}{\pi n \sinh(n\pi)} \int_{-\pi}^{\pi} S(s)\cos(ns)\,ds = \frac{2}{\pi n \sinh(n\pi)} \int_{0}^{\pi} S(s)\cos(ns)\,ds$

$u(x,y) = \sum_{n=0}^{\infty} \frac{2\cosh(ny)\cos(nx)}{\pi n \sinh(n\pi)} \int_0^{\pi} S(s)\cos(ns)\,ds$

$u(x,y) = C_0 + \sum_{n=1}^{\infty} \frac{2\cosh(ny)\cos(nx)}{\pi n \sinh(n\pi)} \int_0^{\pi} S(s)\cos(ns)\,ds$, $C_0 \in \mathbb{R}$

$\therefore u(x,y) = \sum_{n=0}^{\infty} C_n \cosh(ny)\cos(nx)$ $\therefore u_y(x,y) = \sum_{n=0}^{\infty} n C_n \sinh(nx)\cos(nx)$

$\therefore$ at $y=\pi$: $u_y(x,\pi) = \sum_{n=0}^{\infty} B_n \cos(nx) = S(x)$, $B_n = n\sinh(n\pi) C_n$ $\therefore$

$B_0 = 0$ $\therefore \int_0^{\pi} S(x)\,dx = 0$ $\therefore$ C_0 is arbitrary $\therefore$

for $n \neq 0$: $C_n = \frac{2}{n\sinh(n\pi)} \int_0^{\pi} S(x)\cos(nx)\,dx$ $\therefore$

$u(x,y) = C_0 + \sum_{n=1}^{\infty} \frac{2\cosh(ny)\cos(nx)}{n\sinh(n\pi)} \int_0^{\pi} S(s)\cos(ns)\,ds$, $C_0 \in \mathbb{R}$

2a) $\therefore a = 4x$, $2b = 0$, $c = -9x^2$ $\therefore b = 0$ $\therefore -2b = 0$ $\therefore$

$b^2 - ac = 0^2 - 4x(-9x^2) = 36x^3$ $\therefore$

hyperbolic: parabolic is $36x^3 = 0$ $\therefore x^3 = 0$ $\therefore x = 0$ $\therefore$

$x = 0, y \in \mathbb{R}$

hyperbolic is $36x^3 > 0$ $\therefore x^3 > 0$ $\therefore x > 0$, $\therefore x > 0, y \in \mathbb{R}$

$\therefore$ elliptic is $x \neq 0$ and $x \not> 0$ $\therefore x \not> 0$ $\therefore x \leq 0$ $\therefore x \leq 0, y \in \mathbb{R}$

b) hyperbolic $\therefore x > 0$ $\therefore a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0 =$

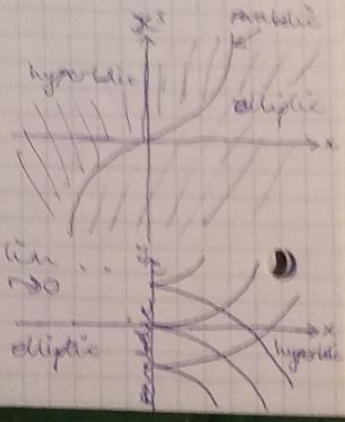
$4x\left(\frac{dy}{dx}\right)^2 - 9x^3 = 0$ $\therefore 4x\left(\frac{dy}{dx}\right)^2 = 9x^3$ $\therefore$

$\left(\frac{dy}{dx}\right)^2 = \frac{9}{4}x$ $\therefore \frac{dy}{dx} = \pm\sqrt{\frac{9}{4}x}$ $\therefore$ ~~$\frac{dy}{dx} = +\sqrt{\frac{9}{4}}$~~ $\frac{dy}{dx} = +\frac{3}{2}\sqrt{x}$, $\frac{dy}{dx} = -\frac{3}{2}\sqrt{x}$

$\therefore y = \int \frac{3}{2}x^{1/2}\,dx = \frac{3}{2}x^{3/2}\left(\frac{2}{3}\right) + C_1 = x^{3/2} + C_1$, $y = \int -\frac{3}{2}x^{1/2}\,dx = -\frac{3}{2}x^{3/2}\left(\frac{2}{3}\right) = -x^{3/2} + C_2$

$= -x^{3/2} + C_2$ $\therefore y - x^{3/2} = C_1$, $y + x^{3/2} = C_2$ $\therefore$

are the characteristic curves



$X(r) = Ae^{\sqrt{-1}r} + Be^{\sqrt{-1}r}$

$|X(0)| < \infty$ $X(0) = A + B$ $\therefore |X(0)| = |A+B| < \infty$

$|u(0,0)| < \infty$ $u(r,0) = \frac{2}{\pi}\sin^3\left(\frac{\pi r}{R}\right)$ $\therefore u(0,0) = \lim_{r\to 0}$ $\therefore$

$\therefore u(0,t) = 0$

$\therefore$ parabolic when $x=0$, elliptic when $x<0$, hyperbolic when $x>0$

PP2021

2b) let $\xi = y + x^{3/2}$, $\eta = y - x^{3/2}$, $x > 0$ ∴

$\xi_x = \frac{3}{2}x^{1/2}$, $\xi_y = 1$, $\eta_x = -\frac{3}{2}x^{1/2}$, $\eta_y = 1$ ∴

$u_x = u_\xi \xi_x + u_\eta \eta_x = \frac{3}{2}x^{1/2}u_\xi - \frac{3}{2}x^{1/2}u_\eta$

$u_{xx} = \frac{\partial}{\partial x}\left(\frac{3}{2}x^{1/2}u_\xi\right) - \frac{3}{2}\frac{\partial}{\partial x}\left(x^{1/2}u_\eta\right) = \frac{3}{4}x^{-1/2}u_\xi + \frac{3}{2}x^{1/2}u_{\xi\xi}\xi_x - \frac{3}{4}x^{-1/2}u_\eta - \frac{3}{2}x^{1/2}u_{\eta\eta}\eta_x$

$= \frac{3}{4}x^{-1/2}u_\xi + \frac{9}{4}x u_{\xi\xi} - \frac{3}{4}x^{-1/2}u_\eta + \frac{9}{4}x u_{\eta\eta}$,

$u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi + u_\eta$ ∴

~~$u_{yy} = u_{\xi\xi}\xi_y + u_{\eta\eta}\eta_y$~~

$u_{yy} = u_{\xi\xi}\xi_y + u_{\xi\eta}\xi_y + u_{\xi\eta}\eta_y + u_{\eta\eta}\eta_y = u_{\xi\xi} + u_{\xi\eta} + u_{\xi\eta} + u_{\eta\eta} =$

$u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$ ∴

PDE : $4xu_{xx} - 9x^2 u_{yy} - 2u_x = 0 =$

$4x\left(\frac{3}{4}x^{-1/2}u_\xi + \frac{9}{4}x u_{\xi\xi} - \frac{3}{4}x^{-1/2}u_\eta + \frac{9}{4}x u_{\eta\eta}\right) - 9x^2\left(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}\right) - 2\left(\frac{3}{2}x^{1/2}u_\xi - \frac{3}{2}x^{1/2}u_\eta\right) = 0 =$

~~$3xu_\xi$~~

$3x^{1/2}u_\xi + 9x^2 u_{\xi\xi} - 3x^{1/2}u_\eta + 9x^2 u_{\eta\eta} - 9x^2 u_{\xi\xi} - 18x^2 u_{\xi\eta} - 9x^2 u_{\eta\eta} + -3x^{1/2}u_\xi + 3x^{1/2}u_\eta = 0 =$

$-18x^2 u_{\xi\eta} = 0 = x^2 u_{\xi\eta}$ ∴ $x > 0$ ∴ $u_{\xi\eta} = 0 = \frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \eta}u\right)$

∴ $\frac{\partial u}{\partial \eta} = f(\eta)$ ∴ $u = g(\eta) + g(\xi)$ ∴

$u(x,0) = 0$, $u_y(x,0) = 6x^3$ ∴

~~$u(x,0) = 0 = g$~~ $u(x,y) = g(y - x^{3/2}) + g(y + x^{3/2})$ ∴

$u(x,0) = 0 = g(-x^{3/2}) + g(x^{3/2})$, ~~let~~

$u_y(x,0) = g'(y - x^{3/2}) + g'(y + x^{3/2})$ ∴

$u_y(x,0) = 6x^3 = g'(-x^{3/2}) + g'(x^{3/2})$

let $x^{+3/2} = s$ ∴ $0 = g(-s) + g(s)$ ∴ $0 = -g'(s) + g'(s)$ ∴

$g'(s) = g'(s)$ ∴ $g'(-s) = g'(-s)$ ∴

$(x^{3/2})^2 = s^2 = x^3$ ∴ $s^{2/3} = x$ ∴ ∴

$6s = g'(-s) + g'(s)$ ∴ $6s = g'(-s) + g'(s)$ ∴

$3s^2 = -g(-s) + g(s)$ ∴ $0 = g(-s) + g(s)$

$u(x,y) = f(y-x^{3/2}) + g(y+x^{3/2})$ $\therefore$, $u(x,0)=0$, $u_y(x,0) = 6x^3$ $\therefore$

$u(x,0) = 0 = f(-x^{3/2}) + g(x^{3/2})$ $\therefore$

let $s = x^{3/2}$ $\therefore$

$0 = f(-s) + g(s)$,

$u_y(x,y) = f'(y-x^{3/2}) + g'(y+x^{3/2})$ $\therefore$

$u_y(x,0) = 6x^3 = f'(-x^{3/2}) + g'(x^{3/2})$ $\therefore$

$s^2 = x^3$ $\therefore$

$6s^2 = f'(-s) + g'(s)$ $\therefore$

$2s^3 = -f(-s) + g(s) + C$ $\therefore$

$-f(-s) = g(s)$ $\therefore$

$2s^3 = g(s) + g(s) + C = 2g(s) + C_1$ $\therefore$ $g(s) = s^3 - C_2$ $\therefore$

$f(-s) = -g(s) = -s^3 + C_2$ $\therefore$

$f(s) = -(-s)^3 + C_2 = s^3 + C_2$ $\therefore$

$\cancel{f(s) + g(s}$ $f(y-x^{3/2}) = (y-x^{3/2})^3 + C_2$,

$g(y+x^{3/2}) = (y+x^{3/2})^3 - C_2$ $\therefore$

$u(x,y) = (y-x^{3/2})^3 + C_2 + (y+x^{3/2})^3 - C_2 = (y-x^{3/2})^3 + (y+x^{3/2})^3 = u(x,y)$

$= 6x^3y + 2y^3$

\2c/ $a = 4x$, $2b = 0$, $b = 0$, $c = -9x^2$ $\therefore$

$\cancel{\left(\frac{dy}{dx}\right)^2}$ $\therefore$ $x < 0$ $\therefore$ $4x\left(\frac{dy}{dx}\right)^2 - 9x^2 = 0 =$

$x\left(4\left(\frac{dy}{dx}\right)^2 - 9x\right) = 0 = x\left(2\frac{dy}{dx} - 3x^{1/2}\right)\left(2\frac{dy}{dx} + 3x^{1/2}\right) = 0$ $\because x < 0$ $\therefore$

$\cancel{x} \; 2\frac{dy}{dx} - 3x^{1/2} = 0$, $2\frac{dy}{dx} + 3x^{1/2} = 0$ $\therefore$

$2y - 2x^{3/2} = C_1$, $2y + 2x^{3/2} = C_2$ $\therefore$

$y - x^{3/2} = C_3$, $y + x^{3/2} = C_4$ $\therefore$

$\alpha(x,y) = y - x^{3/2}$, $\beta(x,y) = y + x^{3/2}$

\2e/ The $x<0$ region is where the equation is elliptic, so the ~~equation is~~ characteristic coordinates are complex $\therefore$

when $x>0$ the coordinates were $\xi = y + x^{3/2}$, $\eta = y - x^{3/2}$ $\therefore$

for $x<0$: $\xi = y + (-x)^{3/2}$, $\eta = y - (-x)^{3/2}$ $\therefore$ choose canonical coordinates from real and imaginary parts $\therefore$

$\int_{-\infty}^{\infty} \quad e^{-\frac{(x-\xi)^2}{}} \quad \ldots$

\PP2021/ $\therefore \eta = y - i(-x)^{3/2} = \overline{\xi(x,y)} \therefore$

$Re(\xi(x,y)) = Re(y + i(-x)^{3/2}) = y = \alpha$,

$Im(\xi(x,y)) = \cancel{(-x)^{3/2}}\ Im(y + i(-x)^{3/2}) = (-x)^{3/2} = \beta$

$\xi = y + x^{3/2}$, $\eta = y - x^{3/2}$ for $x > 0 \therefore$

for $x < 0$: $\xi(x,y) = y - i(-x)^{3/2}$, $\eta(x,y) = y + i(-x)^{3/2} = \overline{\xi(x,y)} \therefore$

$\alpha = Re(\xi(x,y)) = Re(y - i(-x)^{3/2}) = y = \alpha(x,y)$,

$\beta = \beta(x,y) = -Im(\xi(x,y) = -Im(y - i(-x)^{3/2}) = -(-(-x)^{3/2}) = (-x)^{3/2}$

\3a/ $\lim_{x\to\infty} u(x,t) = 0$, $\lim_{x\to-\infty} u(x,t) = 0$, $\lim_{x\to\infty} u_x(x,t) = 0$, $\lim_{x\to-\infty} u_x(x,t) = 0$

$u_t = -u + 2u_x + u_{xx} \therefore$

$\frac{d}{dt}Q(t) = \frac{\partial}{\partial t}Q(t) = \frac{dQ}{dt} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}\,dx =$

$\int_{-\infty}^{\infty} -u + 2u_x + u_{xx}\,dx = -\int_{-\infty}^{\infty} u\,dx + 2\int_{-\infty}^{\infty} \frac{\partial u}{\partial x}\,dx + \int_{-\infty}^{\infty} \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)dx =$

$-\int_{-\infty}^{\infty} u\,dx + 2\left[u(x,t)\right]_{-\infty}^{\infty} + \left[u_x(x,t)\right]_{-\infty}^{\infty} =$

$-\int_{-\infty}^{\infty} u\,dx + 2\lim_{x\to\infty} u(x,t) - 2\lim_{x\to-\infty} u(x,t) + \lim_{x\to\infty} u_x(x,t) - \lim_{x\to-\infty} u_x(x,t) =$

$-\int_{-\infty}^{\infty} u\,dx + 2(0) - 2(0) + 0 - 0 = -\int_{-\infty}^{\infty} u\,dx = -\int_{-\infty}^{\infty} u(x,t)\,dx = -Q$

\3b/ $\therefore \lim_{x\to\pm\infty} f(x,t) = 0 \therefore$ $u(\pm\infty, 0) = 0 \therefore$, $u(x,0) = 0$, $\therefore$

let $u(x,t) = h(x,t)\,v(x,t) \therefore$

$v_t(x,t) = v_{xx}(x,t) + f(x,t)$, $v(x,0) = 0$, $u(\pm\infty, 0) = 0$

$v(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} f(\xi,\tau)\,d\xi\,d\tau \therefore$

$u(x,t) = h(x,t) \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} f(\xi,\tau)\,d\xi\,d\tau =$

$\int_0^t \int_{-\infty}^{\infty} h(x,t) \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} f(\xi,\tau)\,d\xi\,d\tau \therefore$

$G(x,t;\xi,\tau) = \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} h(x,t)$

$r_t = -cR'(\xi)$, $\theta_t = -c\Theta'(\xi)$

$r_x = +R'(\xi)$, $\theta_x = \Theta'(\xi)$,

$r_{xx} = R''(\xi)$, $\theta_{xx} = \Theta''(\xi)$ $\therefore$

$R'' + cR' - R(\Theta')^2 + R\Lambda(R) = 0$,

$R\Theta'' + cR\Theta' + 2R'\Theta' + R\Omega(R) = 0$ is the requested system of ODEs

for $R(\xi)$, $\Theta(\xi)$

set $\Theta' = k$, $R = a$ $\therefore$

$-ak^2 + a\Lambda(a) = 0$, $cak + a\Omega(a) = 0$ $\therefore$

$ak^2 = a\Lambda(a)$ $\therefore$ $k^2 = \Lambda(a)$ $\therefore$ $k = \pm\sqrt{\Lambda(a)}$ $\therefore$

$cak = -a\Omega(a)$ $\therefore$ $c = \frac{-\Omega(a)}{k}$ $\therefore$

$R = a$, $\Theta = k\xi$, $\xi = x - ct$ into $u = R\cos\Theta$, $v = R\sin\Theta$, we get the required formula for the periodic travelling wave solutions where in addition to $k(a)$ already identified, have $\omega = ck$, $\therefore$

$\omega = -\Omega(a)$, $\Lambda(r) = 1 - r^2$, $-\Omega(r) = 1 - r^2$ $\therefore$ $a < 1$, $k = \pm\sqrt{1-a^2} \in (-1, 1)$,

$\omega = -1 + a^2$, $\therefore$ $\omega(k) = \sqrt{1-k^2}$, $\omega(k) = -k^2$

Vc/ let $u(x,t) = Ve^{pt+qx}$ $\therefore$ $u_t = V_t e^{pt+qx} + Vpe^{pt+qx} = (V_t + pV)e^{pt+qx}$

$\therefore$ $u_x = (V_x + qV)e^{pt+qx}$, $u_{xx} = (V_{xx} + qV_x)e^{pt+qx} + (V_x + qV)qe^{pt+qx} =$

$(q^2V + 2qV_x + V_{xx})e^{pt+qx}$ $\therefore$ $u_t - u_{xx} + 2u_x = 0$ $\therefore$

$(V_t + pV)e^{pt+qx} - (q^2V + 2qV_x + V_{xx})e^{pt+qx} + 2(V_x + qV)e^{pt+qx} = 0$ $\therefore$

$V_t + pV - (q^2V + 2qV_x + V_{xx}) + 2(V_x + qV) = 0$ $\therefore$

$V_t + pV - q^2V - 2qV_x - V_{xx} + 2V_x + 2qV = 0$ $\therefore$

$V_t + V(p - q^2 + 2q) + V_x(-2q + 2) - V_{xx} = 0$

$\therefore$ $-2q + 2 = 0$ $\therefore$ $2 = 2q$ $\therefore$ $q = 1$ $\therefore$

$p - q^2 + 2q = 0 = p - 1^2 + 2(1) = p - 1 + 2 = p + 1 = 0$ $\therefore$ $p = -1$ $\therefore$

$u = Ve^{-t+x}$ $\therefore$

$V_t - V_{xx} = 0$ $\therefore$ $V_t = V_{xx}$, $u(x,0) = \psi(x) = V(x,0)e^{-(0)+x} = V(x,0)e^{-x}$ $\therefore$

$\psi(x)e^{x} = V(x,0) = \phi(x)$ $\therefore$ $\phi(\xi) = \psi(\xi)e^{\xi}$

$V(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\phi(\xi)e^{-\frac{(x-\xi)^2}{4t}}\,d\xi = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\psi(\xi)e^{\xi}e^{-\frac{(x-\xi)^2}{4t}}\,d\xi$ $\therefore$

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$u(x,t) = Ve^{-t+x} = e^{-t+x}V = \frac{1}{\sqrt{4\pi t}} e^{-t+x} \int_{-\infty}^{\infty} \psi(\xi) e^{-\xi} e^{-\frac{(x-\xi)^2}{4t}} d\xi$

$\cancel{= \frac{1}{\sqrt{4\pi t}}}$ $\psi(x) = \delta(x+1) \therefore \psi(\xi) = \delta(\xi+1) \therefore$

$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-t+x} \int_{-\infty}^{\infty} \delta(\xi+1) e^{\xi} e^{-\frac{(x-\xi)^2}{4t}} d\xi$

$\int_{-\infty}^{\infty} \delta(x) dx = 1 \qquad \int_{-\infty}^{\infty} \delta(x-\xi) dx = 1$

$\int_{-\infty}^{\infty} \psi(x) \delta(x-\xi) dx = \psi(\xi)$

$\int_{-\infty}^{\infty} \psi(\xi) \delta(\xi - x) d\xi = \psi(x)$

$\int_{-\infty}^{\infty} \psi(\xi) \delta(\xi - 1) d\xi = \psi(1)$

$\delta(x) = \begin{cases} \pm\infty, & x = 0 \\ 0, & x \neq 0 \end{cases} \qquad \int_{-\infty}^{\infty} \delta(x) dx = 1, \quad \int_{-\infty}^{\infty} \delta(x) \delta(x-\xi) dx = \delta(\xi)$

$\int_{-\infty}^{\infty}$

$u(x,t) = \int_{-\infty}^{\infty} \psi(\xi) e^{-\xi+x} d\xi \qquad \psi(x) = \delta(x+1)$

$\psi(\xi) = \delta(\xi+1) \qquad \delta(x+1) = \begin{cases} +\infty, & x = -1 \\ 0, & x \neq -1 \end{cases}$

$\delta(x - (-1))$

$\xi = -1$

$\delta(x - \xi)$

$\delta((+1)$

$\int_{-\infty}^{\infty} \cancel{\psi(\xi)} \cdot \psi(\xi+1) e^{-\xi+x} dx$

$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-t+x} \int_{-\infty}^{\infty} \psi(\xi) e^{\xi} e^{-\frac{(x-\xi)^2}{4t}} d\xi \therefore \psi(x) = \delta(x+1) \therefore$

$\psi(\xi) = \delta(\xi+1) \therefore u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-t+x} \int_{-\infty}^{\infty} \delta(\xi+1) e^{\xi} e^{-\frac{(x-\xi)^2}{4t}} d\xi =$

$\frac{1}{\sqrt{4\pi t}} e^{-t+x} \int_{-\infty}^{\infty} \delta(\xi - (-1)) e^{\xi} e^{-\frac{(x-\xi)^2}{4t}} d\xi =$

$e^{-1} e^{-\frac{(x+1)^2}{4t}} \times \cancel{u(x,t)} \frac{1}{\sqrt{4\pi t}} e^{-t+x} = u(x,t) \therefore P(t) = u(0,t) =$

$P(t) = u(0,t) = e^{-1} e^{-\frac{1}{4t}} \frac{1}{\sqrt{4\pi t}} e^{-t} \doteq \frac{1}{\sqrt{4\pi t}} e$

$\ln(P(t)) = \ln\left(\frac{1}{\sqrt{4\pi t}} e^{-1} e^{-\frac{1}{4t}} e^{-t}\right) \doteq 1 - \ln 2 - \frac{1}{2}\ln\pi - \frac{1}{2}\ln t - t - \frac{1}{4t}$;

$\frac{d\ln(P)}{dt} = -\frac{1}{2t} - 1 + \frac{1}{4t^2} = 0$; $t = t_* \neq$ ∴

$t_* = \frac{\sqrt{5}-1}{4} = \frac{1}{1+\sqrt{5}} \approx 0.309$

1a i) rewrite equation: $(y, -x)\cdot(u_x, u_y) = 0 = (y, -x)\cdot\nabla u$ ∴

characteristic curves satisfy: $\frac{dy}{dx} = \frac{-x}{y}$ ∴ ~~$y=0$ is a~~

$y=0$ is a special solution ∴ $y\frac{dy}{dx} = -x$ ∴

$\int y \frac{dy}{dx}\, dx = \int -x\, dx = \int y\, dy = \frac{1}{2}y^2 = -\frac{1}{2}x^2 + C_1$ ∴

$y^2 = -x^2 + C_2$ ∴ $y^2 + x^2 = C_2$ ∴

$u(x,y) = f(y^2+x^2)$, f is an arbitrary function

∴ $u(x, \sqrt{4+x^2}) = x^2 = f((\sqrt{4+x^2})^2 + x^2) = f(4+x^2+x^2) = f(4+2x^2)$ ∴

let $s = 4+2x^2$ ∴ $s-4 = 2x^2$ ∴ $\frac{s-4}{2} = x^2$ ∴

$f(s) = \frac{s-4}{2}$ ∴

$u(x,y) = f(y^2+x^2) = \frac{(y^2+x^2)-4}{2} = \frac{1}{2}y^2 + \frac{1}{2}x^2 - 2$

1a ii) $u(x, \sqrt{4-x^2}) = x^2 = f((\sqrt{4-x^2})^2 + x^2) = f(4-x^2+x^2) = f(4) = x^2$

∴ is impossible ∴ no solution exists ∴

the BC is on the characteristic curve

1a i) rewrite equation: $(y, -x)\cdot(u_x, u_y) = 0 = (y, -x)\cdot\nabla u$ ∴

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$y^2 = -x^2 + C_2$ ∴

$y^2 + x^2 = C_2$ ∴ $u(x,y) = f(y^2+x^2)$, f is an arbitrary func

∴ $u(x, \sqrt{4+x^2}) = x^2 = f((\sqrt{4+x^2})^2 + x^2) = f(4+x^2+x^2) = f(4+2x^2)$ ∴

let $s = 4+2x^2$ ∴ $s-4 = 2x^2$ ∴ $\frac{s-4}{2} = x^2$ ∴

$f(s) = \frac{s-4}{2}$ ∴

PP2021 $u(x,y) = f(y^2+x^2) = \frac{(y^2+x^2)-4}{2} = \frac{1}{2}y^2 + \frac{1}{2}x^2 - 2$

1a ii $u(x,\sqrt{4-x^2}) = x^2 = f((\sqrt{4-x^2})^2 + x^2) = f(4-x^2+x^2) = f(4) = x^2$ ∴

1) is impossible ∵ function cannot have dissent outputs for the same input ∴ no solutions exist.

1b let $\xi = x-y$, $\tilde{\eta} = -x-y$ ∴ $\eta = x+y$ ∴

$\xi_x = 1$, $\xi_y = -1$, $\eta_x = 1$, $\eta_y = 1$ ∴

$\xi + y = x$ ∴ $\eta = \xi + y + y = \xi + 2y$ ∴ $\frac{\eta - \xi}{2} = y$ ∴, $\xi + \frac{\eta-\xi}{2} = x = \frac{1}{2}\xi + \frac{1}{2}\eta = x$

∴ $u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$,

$u_y = u_\xi \xi_y + u_\eta \eta_y = -u_\xi + u_\eta$ ∴

$u_{xx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\xi}\xi_x + u_{\eta\eta}\eta_x =$

$u_{\xi\xi} + u_{\xi\eta} + u_{\xi\eta} + u_{\eta\eta} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$,

$u_{yy} = -u_{\xi\xi}\xi_y - u_{\xi\eta}\eta_y + u_{\eta\xi}\xi_y + u_{\eta\eta}\eta_y =$

$u_{\xi\xi} - u_{\xi\eta} - u_{\xi\eta} + u_{\eta\eta} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$ ∴

$\eta - \xi = 2y$ ∴ $\xi + \eta = 2x$ ∴

$u_{xx} - u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} - u_{\xi\xi} + 2u_{\xi\eta} - u_{\eta\eta} =$

$4u_{\xi\eta} = 4e^{2y} = 4e^{\eta-\xi}$ ∴

$u_{\xi\eta} = e^{\eta-\xi}$ ∴

$u(x,x) = e^{2x}$, $u(x,-x) = e^{-2x}$ ∴ $\frac{\partial}{\partial\xi}(\frac{\partial}{\partial\eta}u) = e^{\eta-\xi}$ ∴

$\frac{\partial}{\partial\eta}u = -e^{\eta-\xi} + F(\eta)$ ∴

$u = -e^{\eta-\xi} + f(\eta) + g(\xi) = u(\xi,\eta)$

~~$u(x,x) = e^{2x} =$~~ $u(x=x, y=x) = e^{2x} = e^{\xi+\eta}$ ∴

$u(x,y) = -e^{2y} + f(x+y) + g(x-y)$ ∴

$u(x,x) = e^{2x} = u(x=x, y=x) = -e^{2x} + f(x+x) + g(x-x) = -e^{2x} + f(2x) + g(0)$

∴ $2e^{2x} = f(2x) + g(0)$,

$u(x,-x) = e^{-2x} = u(x=x, y=-x) = -e^{-2x} + f(x-x) + g(x+x) = -e^{-2x} + f(0) + g(2x)$

∴ $2e^{-2x} = f(0) + g(2x)$ ∴

let $s = 2x$ ∴

$2e^{s} = f(s) + g(0)$, $2e^{-s} = f(0) + g(s)$

$\therefore\ f(s) = 2e^{s} - g(0)$, $2e^{-0} = f(0) + g(0) = 2$ $\therefore$

~~let~~ $f(0) = \text{const}$ $\therefore$ let $f(0) = A = \text{const}$ $\therefore$

$2 = A + g(0)$ $\therefore$ $2 - A = g(0)$ $\therefore$

$f(s) = 2e^{s} - 2 + A$ $\therefore$

$2e^{-s} - f(0) = g(s)$ $\therefore$ $g(s) = 2e^{-s} - A$ $\therefore$

$A = 2e^{-s} - g(s)$ $\therefore$

$\cancel{f(s) = 2e^{s} - 2 + 2e^{-s} - g(s)}$ $\therefore$

$\cancel{f(s) + g(s) = 2e^{s} - 2e^{-s} - 2}$ $\therefore$

$f(x+y) = 2e^{x+y} - 2 + A$,

$g(x-y) = 2e^{-x+y} - A$ $\therefore$

$f(x+y) + g(x-y) = 2e^{x+y} + 2e^{-x+y} - 2$ $\therefore$

$u(x,y) = -e^{2y} + 2e^{x+y} + 2e^{-x+y} - 2$

Vb / let $\xi = x - y$ $\therefore$ $\eta = -x - y$ $\therefore$ $\eta = x + y$ $\therefore$

$\xi_x = 1$, $\xi_y = -1$, $\eta_x = 1$, $\eta_y = 1$ $\therefore$

$\xi + y = x$ $\therefore$ $\eta = \xi + y + y = \xi + 2y$ $\therefore$ $2y = \eta - \xi$ $\therefore$ $y = \frac{1}{2}\eta - \frac{1}{2}\xi$ $\therefore$

$x = \xi + \frac{1}{2}\eta - \frac{1}{2}\xi = \frac{1}{2}\xi + \frac{1}{2}\eta$ $\therefore$ $2x = \xi + \eta$ $\therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$,

$u_y = u_\xi \xi_y + u_\eta \eta_y = -u_\xi + u_\eta$ $\therefore$

$u_{xx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x = u_{\xi\xi} + u_{\xi\eta} + u_{\xi\eta} + u_{\eta\eta} =$
$u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$,

$u_{yy} = -u_{\xi\xi}\xi_y - u_{\xi\eta}\eta_y + u_{\xi\eta}\xi_y + u_{\eta\eta}\eta_y = u_{\xi\xi} - u_{\xi\eta} - u_{\xi\eta} + u_{\eta\eta} =$
$u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$ $\therefore$

$u_{xx} - u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} - u_{\xi\xi} + 2u_{\xi\eta} - u_{\eta\eta} = 4u_{\xi\eta} = 4e^{2y} =$
$4e^{\eta - \xi}$ $\therefore$

$u_{\xi\eta} = e^{\eta - \xi} = \frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \eta} u\right)$ $\therefore$

$u_\eta = \frac{\partial}{\partial \eta} u = -e^{\eta - \xi} + \delta(\eta)$ $\therefore$

$u = -e^{\eta - \xi} + f(\eta) + g(\xi)$ $\therefore$

$u(x,y) = -e^{2y} + f(x+y) + g(x-y)$ $\therefore$

$u(x,x) = e^{2x} = u(x+x, y-x) = -e^{2x} + f(x+x) + g(x-x) = -e^{2x} + f(2x) + g(0)$ $\therefore$

$f(s) = \ldots$ 2

PP2021 / $f(2x) = 2e^{2x} - g(0)$.

$u(x,-x) = e^{-2x} = u(x=x, y=-x) = -e^{-2x} + f(x-x) + g(x+x) =$

$-e^{-2x} + f(0) + g(2x)$ $\therefore$ $2e^{-2x} = f(0) + g(2x)$

let $f(0) = A = \text{const}$ $\therefore$

$\cancel{-e^{-2x} + A}$ $2e^{-2x} = A + g(2x)$ $\therefore$

$2e^{-2(0)} = 2e^0 = 2 = A + g(2\times 0) = \cancel{g(0)}$ $A + g(0)$ $\therefore$

$-g(0) = A - 2$ $\therefore$

$f(2x) = 2e^{2x} + A - 2$ $\therefore$

$f(s) = 2e^{s} + A - 2$ $\therefore$

$g(2x) = 2e^{-2x} - A$ $\therefore$

$g(s) = 2e^{-s} - A$ $\therefore$

$f(x+y) = 2e^{x+y} + A - 2$,

$g(x-y) = 2e^{-x+y} - A$ $\therefore$

$f(x+y) + g(x-y) = 2e^{x+y} + A - 2 + 2e^{-x+y} - A = 2e^{x+y} + 2e^{-x+y} \neq 2$ $\therefore$

$u(x,y) = -e^{2y} + 2e^{x+y} + 2e^{-x+y} - 2$ ✓

VIC / let $u = ve^{pt+qx}$ $\therefore$ $u_t = (v_t + pv)e^{pt+qx}$,

$u_x = (v_x + qv)e^{pt+qx}$ $\therefore$ $u_{xx} = (v_{xx} + qv_x + qv_x + q^2 v)e^{pt+qx} = (v_{xx} + 2qv_x + q^2 v)e^{pt+qx}$

$\therefore$ $(v_t + pv)e^{pt+qx} = (v_{xx} + 2qv_x + q^2 v)e^{pt+qx} - 2(v_x + qv)e^{pt+qx}$ $\therefore$

$v_t + pv = v_{xx} + 2qv_x + q^2 v - 2v_x - 2qv = v_{xx} + (2q-2)v_x + (q^2 - 2q)v$,

$u(x,0) = v(x,0)e^{p(0)+qx} = v(x,0)e^{qx} = \psi(x)$ $\therefore$

$v(x,0) = \psi(x)e^{-qx}$ $\therefore$

~~let 2q-2~~ $v_t = v_{xx} + (2q-2)v_x + (q^2 - 2q - p)v$ $\therefore$

let $2q - 2 = 0$ $\therefore$ $q = 1$ $\therefore$ $q^2 - 2q - p = 1^2 - 2(1) - p = 1 - 2 - p = -1 - p$ $\therefore$

let $q^2 - 2q - p = 0$ $\therefore$ $-1 - p = 0$ $\therefore$ $p = -1$ $\therefore$

$v_t = v_{xx}$, $v(x,0) = \psi(x)e^{-x} = \phi(x)$ $\therefore$ $\phi(\xi) = \psi(\xi)e^{-\xi}$ $\therefore$

$v(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \phi(\xi) e^{-(x-\xi)^2/(4t)} d\xi = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \psi(\xi) e^{-\xi} e^{-(x-\xi)^2/(4t)} d\xi$

$\therefore$ $e^{pt+qx} = e^{-t+x}$ $\therefore$ $\psi(x) = \delta(x+1)$ $\therefore$ $\psi(\xi) = \delta(\xi+1) = \delta(\xi - (-1))$ $\therefore$

$u(x,t) = \frac{1}{\sqrt{4\pi t}} \left(e^{-t+x}\right) \int_{-\infty}^{\infty} \psi(\xi) e^{-\xi - (x-\xi)^2/(4t)} d\xi =$

$$u(x,t)=\frac{1}{\sqrt{4\pi t}}e^{-t+x}\int_{-\infty}^{\infty}\delta(\xi-(-1))e^{-\xi-(x-\xi)^2/(4t)}\,d\xi =$$

$$u(x,t)=\frac{1}{\sqrt{4\pi t}}e^{x-t}\,e^{-(-1)-(x-(-1))^2/(4t)}$$

$$= u(x,t)=\frac{1}{\sqrt{4\pi t}}e^{x-t}e^{1-(x+1)^2/(4t)} \quad \therefore$$

$$P(t)=u(0,t)=\frac{1}{\sqrt{4\pi t}}e^{-t}e^{1-(0+1)^2/(4t)}=\frac{1}{\sqrt{4\pi t}}e^{-t}e^{1-\frac{1}{4t}}=$$

$$P(t)=\frac{1}{\sqrt{4\pi t}}e^{1-t-\frac{1}{4t}} \quad \therefore$$

$$\ln P(t)=\ln\left(\frac{1}{\sqrt{4\pi t}}\right)+1-t-\frac{1}{4t}=-\frac{1}{2}\left(\ln(4)+\ln(\pi)+\ln(t)\right)+1-t-\frac{1}{4t}=$$

$$1-\ln 2-\frac{1}{2}\ln\pi-\frac{1}{2}\ln(t)-t-\frac{1}{4t} \quad \therefore$$

$$\frac{d\ln P(t)}{dt}=-\frac{1}{2t}-1+\frac{1}{4t^2}=0 \quad \cancel{\text{for } t=t_*}$$

$$\therefore \text{ for } t=t_*: -\frac{1}{2t_*}-1+\frac{1}{4t_*^2}=0 \quad \therefore$$

$$-\frac{2}{t_*}-4+\frac{1}{t_*^2}=0 \quad \therefore \quad +4+\frac{2}{t_*}-\frac{1}{t_*^2}=0 \quad \therefore$$

$$4t_*^2+2t_*-1=0 \quad \therefore$$

$$t_*=\frac{-2\pm\sqrt{4-4(4)(-1)}}{2(4)}=\frac{-2\pm\sqrt{20}}{8}=-\frac{2}{8}\pm\frac{2\sqrt{5}}{8}=-\frac{1}{4}\pm\frac{\sqrt{5}}{4} \quad \therefore$$

$t_*>0 \quad \therefore$ positive root $\therefore$

$$t_*=-\frac{1}{4}+\frac{\sqrt{5}}{4}\approx 0.309 \quad (3\text{ s.f.})$$

VC / let $u=ve^{pt+qx} \quad \therefore \quad u_t=(v_t+pv)e^{pt+qx}, \quad u_x=(v_x+qv)e^{pt+qx}$

$$\therefore u_{xx}=(v_{xx}+qv_x+qv_x+q^2v)e^{pt+qx}=(v_{xx}+2qv_x+q^2v)e^{pt+qx} \quad \therefore$$

$$(v_t+pv)e^{pt+qx}=(v_{xx}+2qv_x+q^2v)e^{pt+qx}-2(v_x+qv)e^{pt+qx} \quad \therefore$$

$$v_t+pv=v_{xx}+2qv_x+q^2v-2v_x-2qv \quad \therefore$$

$$v_t=v_{xx}+2qv_x+q^2v-2v_x-2qv-pv \quad \therefore$$

$$v_t=v_{xx}+(2q-2)v_x+(q^2-2q-p)v \quad \therefore$$

let $2q-2=0 \quad \therefore q=1$,

let $q^2-2q-p=0=1-2-p=-1-p \quad \therefore \quad p=-1 \quad \therefore$

$v_t=v_{xx}$,

$$u(x,0)=ve^{-1(0)+x}=ve^{x}=\psi(x) \quad \therefore$$

$$v=\psi(x)e^{-x}=\phi(x)$$

$$\therefore \phi(\xi)=\psi(\xi)e^{-\xi} \quad \therefore$$

$$v(x,t)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\phi(\xi)e^{-(x-\xi)^2/(4t)}\,d\xi=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\psi(\xi)e^{-\xi}e^{-(x-\xi)^2/(4t)}\,d\xi$$

\PP2021/ $\therefore\ \varphi(x) = \delta(x+1) \quad \therefore\ \varphi(\xi) = \delta(\xi+1) = \delta(\xi-(-1)) \ \therefore$

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-t+x} \int_{-\infty}^{\infty} \varphi(\xi) e^{-\xi-(x-\xi)^2/(4t)} d\xi =$$

$$\frac{1}{\sqrt{4\pi t}} e^{-t+x} \int_{-\infty}^{\infty} \delta(\xi-(-1)) e^{-\xi-(x-\xi)^2/(4t)} d\xi =$$

$$\frac{1}{\sqrt{4\pi t}} e^{-t+x} e^{-(-1)-(x-(-1))^2/(4t)} =$$

$$\frac{1}{\sqrt{4\pi t}} e^{x-t} e^{1-(x+1)^2/(4t)} \ \therefore$$

$$P(t) = u(0,t) = \frac{1}{\sqrt{4\pi t}} e^{-t} e^{1-1^2/(4t)} = \frac{1}{\sqrt{4\pi t}} e^{-t+1-\frac{1}{4t}} \ \therefore$$

$$\ln P(t) = \ln\left((4\pi t)^{-1/2}\right) - t + 1 - \frac{1}{4t} = -\frac{1}{2}(\ln 4 + \ln\pi + \ln t) - t + 1 - \frac{1}{4t} =$$

$$-\ln 2 - \frac{1}{2}\ln\pi - \frac{1}{2}\ln t - t + 1 - \frac{1}{4t} \ \therefore$$

$$\frac{d\ln P(t)}{dt} = -\frac{1}{2t} - 1 + \frac{1}{4t^2} \ \therefore$$

For $t = t_*$: $\quad -\frac{1}{2t_*} - 1 + \frac{1}{4t_*^2} = 0 \ \therefore$

$$\frac{1}{2t_*} + 1 - \frac{1}{4t_*^2} = 0 \quad \therefore \quad \frac{2}{t_*} + 4 - \frac{1}{t_*^2} = 0 \ \therefore$$

$$4t_*^2 + 2t_* - 1 = 0 \ \therefore$$

$$t_* = \frac{-2 \pm \sqrt{4 - 4(4)(-1)}}{2(4)} = \frac{-2 \pm \sqrt{20}}{8} = -\frac{2}{8} \pm \frac{2\sqrt{5}}{8} = -\frac{1}{4} \pm \frac{\sqrt{5}}{4} \ \therefore$$

$t_* > 0 \ \therefore$ positive root $\therefore$

$$t_* = -\frac{1}{4} + \frac{\sqrt{5}}{4} \approx 0.309 \ (3\ s.f.)$$

\1d/ let $u = X(x)Y(y) \ \therefore \quad u_{xx} = X''Y, \ u_{yy} = XY'' \ \therefore$

$X''Y + XY'' = 0 \ \therefore \quad X''Y = -XY'' \quad \therefore \quad \frac{X''}{X} = -\frac{Y''}{Y} = -\lambda = \text{constant} \ \therefore$

$Y'' = \lambda Y \quad \therefore \quad Y'' - \lambda Y = 0, \quad X'' = -\lambda X \ \therefore$

$X'' + \lambda X = 0, \quad X'(0) = 0, \quad X'(\pi) = 0 \ \therefore$

For $\lambda < 0 \ \therefore \ \lambda = -a^2 < 0, \ a \in \mathbb{R} \ \therefore \quad X'' - a^2 X = 0 \ \therefore$

$X = Ae^{ax} + Be^{-ax} \ \therefore$

$X' = Aae^{ax} - Bae^{-ax} \ \therefore$

$X'(0) = Aa - Ba = 0 = A - B \ \therefore \quad -B = -A \ \therefore \quad X' = Aa(e^{ax} - e^{-ax}) \ \therefore$

$X'(\pi) = 0 = Aa(e^{a\pi} - e^{-a\pi}) \quad \therefore \quad A = 0 \ \therefore \quad B = 0 \ \therefore \quad X = 0$

For $\lambda = 0 \ \therefore \quad Ax + B = X \quad \therefore \quad X' = A \quad \therefore \quad A$ is arbitrary

For $\lambda > 0 \ \therefore \ \lambda = a^2 > 0, \ a \in \mathbb{R} \ \therefore \quad X'' + a^2 X = 0 \ \therefore$