$X = A\cos ax + B\sin ax \quad \therefore \quad X' = -Aa\sin ax + Ba\cos ax \quad \therefore$

$X'(0) = -Aa\sin(0) + Ba\cos(0) = 0 = Ba \quad \therefore \quad B = 0 \quad \therefore$

$X' = -Aa\sin ax \quad \therefore$

$X'(\pi) = -Aa\sin(a\pi) = 0 = A\sin(a\pi) \quad \therefore$

$a\pi = n\pi \quad \therefore \quad a = n \quad \therefore$

$\sqrt{\lambda} = n \quad \therefore \quad \lambda = n^2 \quad \therefore$

$X = A\cos(nx) \quad \therefore \quad X_n(x) = \cos(nx), \quad \lambda_n = n^2, \quad n = 0, 1, 2, 3, \dots \quad \therefore$

$Y'' - n^2 Y = 0 \quad \therefore \quad q^2 - n^2 = 0 \quad \therefore \quad q^2 = n^2 \quad \therefore \quad q = \pm n \quad \therefore$

$Y_n(y) = Ae^{ny} + Be^{-ny} \quad \therefore \quad Y_n'(0) = 0 \quad \therefore$

$Y_n' = Ane^{ny} - Bne^{-ny} \quad \therefore \quad Y_n'(0) = 0 = An - Bn = 0 = A - B \quad \therefore \quad A = B \quad \therefore$

$Y_n = A(e^{ny} + e^{-ny}) = 2A\frac{1}{2}(e^{ny} + e^{-ny}) = 2A\cosh(ny) = C\cosh(ny)$

$\therefore \quad u(x,y) = X(x)Y(y) = \sum_{n=0}^{\infty} C_n \cosh(ny)\cos(nx) \quad \therefore$

$u_y(x,y) = \sum_{n=0}^{\infty} C_n n \sinh(ny)\cos(nx) \quad \therefore$

$u_y(x,\pi) = f(x) = \sum_{n=0}^{\infty} C_n n \sinh(n\pi)\cos(nx) \quad \therefore$

$C_n n \sinh(n\pi) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx \quad \therefore$

$C_n = \frac{1}{\pi n \sinh(n\pi)}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{1}{\pi n \sinh(n\pi)}\int_{-\pi}^{\pi} f(s)\cos(ns)\,ds =$

$\frac{2}{\pi n \sinh(n\pi)}\int_{0}^{\pi} f(s)\cos(ns)\,ds \quad \therefore$

$u(x,y) = \sum_{n=0}^{\infty} \frac{2}{\pi n \sinh(n\pi)}\int_{0}^{\pi} f(s)\cos(ns)\,ds \cosh(ny)\cos(nx) \quad \therefore$

$u(x,y) = C_0 + \sum_{n=1}^{\infty} \frac{2\cosh(ny)\cos(nx)}{n\sinh(n\pi)}\int_{0}^{\pi} f(s)\cos(ns)\,ds, \quad C_0 \in \mathbb{R} \quad \therefore$

$u(x,y) = \sum_{n=0}^{\infty} C_n\cosh(ny)\cos(nx) \quad \therefore \quad u_y(x,y) = \sum_{n=0}^{\infty} nC_n\sinh(nx)\cos(nx) \quad \therefore$

at $y = \pi$: $u_y(x,0) = \sum_{n=0}^{\infty} B_n\cos(nx) = f(x), \quad B_n = n\sinh(n\pi)C_n \quad \therefore$

$B_0 = 0 \quad \therefore \quad \int_0^{\pi} f(x)\,dx = 0 \quad \therefore \quad C_0$ is arbitrary $\therefore$

for $n \neq 0$: $C_n = \frac{2}{n\sinh(n\pi)}\int_0^{\pi} f(x)\cos(nx)\,dx \quad \therefore$

$u(x,y) = C_0 + \sum_{n=1}^{\infty} \frac{2\cosh(ny)\cos(nx)}{n\sinh(n\pi)}\int_0^{\pi} f(s)\cos(ns)\,ds, \quad C_0 \in \mathbb{R}$

\ PP2021 / \ 1d / let $u = X(x)Y(y)$ ∴ $u_{xx} = X''Y$, $u_{yy} = XY''$ ∴

$X''Y + XY'' = 0$ ∴ $X''Y = -XY''$ ∴ $\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda = \text{const}$ ∴

$Y'' = \lambda Y$ ∴ $Y'' - \lambda Y = 0$, $X'' = -\lambda X$ ∴

$X'' + \lambda X = 0$, $X'(0) = 0$, $X'(\pi) = 0$ ∴

for $\lambda < 0$ ∴ $\lambda = -a^2 < 0$, $a \in \mathbb{R}$ ∴ $X'' - a^2 X = 0$ ∴ $q^2 - a^2 = 0$ ∴ $q = \pm a$

∴ $X = Ae^{ax} + Be^{-ax}$ ∴ $X' = Aae^{ax} - Bae^{-ax}$ ∴

$X'(0) = Aae^{0} - Bae^{-0} = Aa - Ba = 0 = A - B$ ∴ $-A = -B$ ∴

$X' = Aa(e^{ax} - e^{-ax})$ ∴ $X'(\pi) = Aa(e^{a\pi} - e^{-a\pi}) = 0 = A(e^{a\pi} - e^{-a\pi})$ ∴

$e^{a\pi} - e^{-a\pi} \neq 0$ ∴ $A = 0$ ∴ $B = 0$ ∴ $X = 0$

for $\lambda = 0$ ∴ $\cancel{q^2=0}$ ∴ $X'' = 0$ ∴ $X = Ax + B$ ∴ $X' = A$ ∴

A is arbitrary

for $\lambda > 0$: $\lambda = a^2 > 0$, $a \in \mathbb{R}$ ∴ $X'' + a^2 X = 0$ ∴ $q^2 + a^2 = 0$ ∴

$q = \pm ai$ ∴ $X = A\cos ax + B\sin ax$ ∴ $X' = -Aa\sin ax + Ba\cos ax$

∴ $X'(0) = -Aa\sin(0) + Ba\cos(0) = Ba = 0 = B$ ∴ $X' = -Aa\sin ax$ ∴

$X'(\pi) = -Aa\sin(a\pi) = 0 = A\sin(a\pi)$ ∴ $a\pi = n\pi$ ∴ $a = n$ ∴

$\sqrt{\lambda} = n$ ∴ $\lambda = n^2$ ∴ $X = A\cos(nx)$ ∴

$\lambda_n = n^2$, $X_n(x) = \cos(nx)$, $n = 0, 1, 2, 3, \ldots$ ∴

$Y'' - n^2 Y = 0$ ∴ $q^2 - n^2 = 0$ ∴ $q = \pm n$ ∴

$Y = Ae^{ny} + Be^{-ny}$ ∴ $Y' = Ane^{ny} - Bne^{-ny}$ ∴

$Y'(0) = 0 = Ane^{0} - Bne^{-0} = An - Bn = 0 = A - B$ ∴ $A = B$ ∴

$Y = Ae^{ny} + Ae^{-ny} = A(e^{ny} + e^{-ny}) = 2A\frac{1}{2}(e^{ny} + e^{-ny}) = 2A\cosh(ny) =$

$C\cosh(ny)$ ∴

$u(x,y) = X(x)Y(y) = \sum_{n=0}^{\infty} C_n \cosh(ny)\cos(nx) = C_0 + \sum_{n=1}^{\infty} C_n \cosh(ny)\cos(nx)$

∴ $u_y(x,y) = \sum_{n=1}^{\infty} nC_n \sinh(ny)\cos(nx)$ ∴

$u_y(x,\pi) = \sum_{n=1}^{\infty} nC_n \sinh(n\pi)\cos(nx)$ ∴

$nC_n \sinh(n\pi) = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi}\int_{0}^{\pi} f(x)\cos(nx)\,dx$ ∴

$C_n = \frac{2}{\pi n \sinh(n\pi)}\int_{0}^{\pi} f(x)\cos(nx)\,dx = \frac{2}{\pi n \sinh(n\pi)}\int_{0}^{\pi} f(s)\cos(ns)\,ds$ ∴

$u(x,y) = C_0 + \sum_{n=1}^{\infty} \frac{2\cosh(ny)\cos(nx)}{n\sinh(n\pi)} \int_0^{\pi} f(s)\cos(ns)\,ds \quad , C_0 \in \mathbb{R}$

2ai ✓ $\therefore a = 4x$, $2b = 0$, $c = -9x^2 \therefore$ $b = 0$ $\therefore -2b = 0 \therefore$

$b^2 - ac = 0^2 - 4x(-9x^2) = 36x^3 \therefore$

parabolic is $36x^3 = 0$ $\therefore$ $x^3 = 0$ $\therefore$ $x = 0$ $\therefore$

$x = 0$, $y \in \mathbb{R}$

hyperbolic is $36x^3 > 0 \therefore$ $x^3 > 0 \therefore$ $x > 0 \therefore$ $x > 0, y \in \mathbb{R}$

$\therefore$ elliptic is $x \neq 0$ and $x \ngtr 0 \therefore$ $x \neq 0 \therefore$ $x < 0 \therefore x < 0, y \in \mathbb{R}$

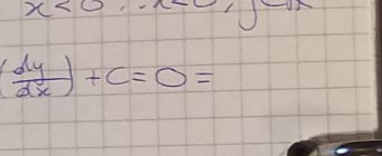
2aii ✓ hyperbolic $\therefore$ $x > 0$. $a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0 =$

$4x\left(\frac{dy}{dx}\right)^2 - 9x^2 = 0$ $\therefore$ $4x\left(\frac{dy}{dx}\right)^2 = 9x^2 \therefore$

$\left(\frac{dy}{dx}\right)^2 = \frac{9}{4}x$ $\therefore$ $\frac{dy}{dx} = \pm\sqrt{\frac{9}{4}x}$ $\therefore$ $\frac{dy}{dx} = \frac{3}{2}\sqrt{x}$, $\frac{dy}{dx} = -\frac{3}{2}\sqrt{x}$ $\therefore$

$y = \int \frac{3}{2}x^{1/2}\,dx = \frac{3}{2}x^{3/2}\left(\frac{2}{3}\right) + C_1 = x^{3/2} + C_1$, $y = \int -\frac{3}{2}x^{1/2}\,dx = -\frac{3}{2}x^{3/2}\left(\frac{2}{3}\right) = -x^{3/2} + C_2$

$\therefore$ $y - x^{3/2} = C_1$, $y + x^{3/2} = C_2$ are the characteristic curves $\therefore$



2ai ✓ $\therefore a = 4x, 2b = 0, c = -9x^2$ $\therefore b = 0, -2b = 0 \therefore$

$b^2 - ac = 0^2 - 4x(-9x^2) = 36x^3 \therefore$

parabolic is: $36x^3 = 0$ $\therefore x^3 = 0$ $\therefore x = 0$

hyperbolic is : $36x^3 > 0 \therefore x^3 > 0$ $\therefore x > 0$ $\therefore$

elliptic is : $x \neq 0$, $x \ngtr 0$ $\therefore x \neq 0 \therefore$ $x < 0$

2aii ✓ hyperbolic $\therefore x > 0$ $\therefore$ $a\left(\frac{dy}{dx}\right)^2 - 2b\frac{dy}{dx} + c = 0 =$

$4x\left(\frac{dy}{dx}\right)^2 - 9x^2 = 0$ $\therefore$ $4x\left(\frac{dy}{dx}\right)^2 = 9x^2 \therefore$

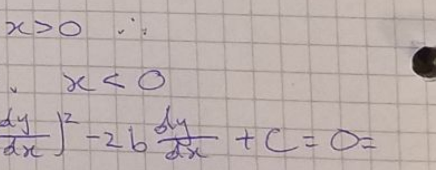
$\left(\frac{dy}{dx}\right)^2 = \frac{9}{4}x$ $\therefore$ $\frac{dy}{dx} = \pm\sqrt{\frac{9}{4}x}$ $\therefore$

$\frac{dy}{dx} = \frac{3}{2}x^{1/2}$, $\frac{dy}{dx} = -\frac{3}{2}x^{1/2}$ $\therefore$

$y = \int \frac{3}{2}x^{1/2}\,dx = \frac{3}{2}x^{3/2}\left(\frac{2}{3}\right) + C_1 = x^{3/2} + C_1$, $y = \int -\frac{3}{2}x^{1/2}\,dx = -\frac{3}{2}x^{3/2}\left(\frac{2}{3}\right) = -x^{3/2} + C_2$

$\therefore$ $y - x^{3/2} = C_1$, $y + x^{3/2} = C_2$

are the characteristic curves



PP2021 / 2b / $\therefore \xi_x = \frac{3}{2}x^{1/2}$, $\xi_y = 1$, $\eta_x = -\frac{3}{2}x^{1/2}$, $\eta_y = 1 \therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x = \frac{3}{2}x^{1/2}u_\xi - \frac{3}{2}x^{1/2}u_\eta \therefore$

$u_{xx} = \frac{\partial}{\partial x}\left(\frac{3}{2}x^{1/2}u_\xi - \frac{3}{2}x^{1/2}u_\eta\right) =$

$\frac{3}{2}\left(\frac{1}{2}\right)x^{-1/2}u_\xi + \frac{3}{2}x^{1/2}u_{\xi\xi}\xi_x + \frac{3}{2}x^{1/2}u_{\xi\eta}\eta_x - \frac{3}{2}\left(\frac{1}{2}\right)x^{-1/2}u_\eta - \frac{3}{2}x^{1/2}u_{\xi\eta}\xi_x - \frac{3}{2}x^{1/2}u_{\eta\eta}\eta_x =$

$\frac{3}{4}x^{-1/2}u_\xi + \frac{3}{2}x^{1/2}u_{\xi\xi}\left(\frac{3}{2}x^{1/2}\right) + \frac{3}{2}x^{1/2}u_{\xi\eta}\left(-\frac{3}{2}\right)x^{1/2} - \frac{3}{4}x^{-1/2}u_\eta - \frac{3}{2}x^{1/2}u_{\xi\eta}\left(\frac{3}{2}\right)x^{1/2} - \frac{3}{2}x^{1/2}u_{\eta\eta}\left(-\frac{3}{2}\right)x^{1/2}$

$= \frac{3}{4}x^{-1/2}u_\xi + \frac{9}{4}xu_{\xi\xi} - \frac{9}{4}xu_{\xi\eta} - \frac{3}{4}x^{-1/2}u_\eta - \frac{9}{4}xu_{\xi\eta} + \frac{9}{4}xu_{\eta\eta} =$

$\frac{3}{4}x^{-1/2}u_\xi - \frac{3}{4}x^{-1/2}u_\eta + \frac{9}{4}xu_{\xi\xi} - \frac{9}{2}xu_{\xi\eta} + \frac{9}{4}xu_{\eta\eta}$

$u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi + u_\eta \therefore$

$u_{yy} = u_{\xi\xi}\xi_y + u_{\xi\eta}\eta_y + u_{\xi\eta}\xi_y + u_{\eta\eta}\eta_y = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \therefore$

PDE:

$4x\left(\frac{3}{4}x^{-1/2}u_\xi - \frac{3}{4}x^{-1/2}u_\eta + \frac{9}{4}xu_{\xi\xi} - \frac{9}{2}xu_{\xi\eta} + \frac{9}{4}xu_{\eta\eta}\right) - 9x^2u_{yy} - 2\left(\frac{3}{2}x^{1/2}u_\xi - \frac{3}{2}x^{1/2}u_\eta\right) =$

$3x^{1/2}u_\xi - 3x^{1/2}u_\eta + 9x^2u_{\xi\xi} - 18x^2u_{\xi\eta} + 9x^2u_{\eta\eta} - 9x^2u_{yy} - 3x^{1/2}u_\xi + 3x^{1/2}u_\eta =$

$9x^2u_{\xi\xi} - 18x^2u_{\xi\eta} + 9x^2u_{\eta\eta} - 9x^2(u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}) =$

$9x^2u_{\xi\xi} - 18x^2u_{\xi\eta} + 9x^2u_{\eta\eta} - 9x^2u_{\xi\xi} - 18x^2u_{\xi\eta} - 9x^2u_{\eta\eta} =$

$-18x^2u_{\xi\eta} - 18x^2u_{\xi\eta} = 0 = -36x^2u_{\xi\eta} = 0 = x^2u_{\xi\eta} = 0 = u_{\xi\eta} = \frac{\partial}{\partial\xi}\left(\frac{\partial}{\partial\eta}u\right)$

$\therefore \frac{\partial}{\partial\eta}u = \hat{f}(\eta) \therefore$

$u = f(\eta) + g(\xi) = f(y - x^{3/2}) + g(y + x^{3/2}) \therefore$

$u(x,0) = 0 = f(0 - x^{3/2}) + g(0 + x^{3/2}) = f(-x^{3/2}) + g(x^{3/2}) \therefore$

$u_y = f'(y - x^{3/2}) + g'(y + x^{3/2}) \therefore$

$u_y(x,0) = f'(-x^{3/2}) + g'(x^{3/2}) = 6x^3 \therefore$

let $s = x^{3/2} \therefore s^2 = x^3 \therefore 6x^3 = 6s^2 \therefore$

$6s^2 = f'(-s) + g'(s) \therefore$

$2s^3 = -f(-s) + g(s) + A \therefore$ $A = \text{const} \therefore$

$0 = f(-s) + g(s) \therefore -f(-s) = g(s) \therefore$

$2s^3 = g(s) + g(s) + A = 2g(s) + A \therefore$

$g(s) = s^3 + B$, $B = \text{Constant} \therefore$

$g(y + x^{3/2}) = (y + x^{3/2})^3 + B \therefore$

$\therefore f(-s) = -g(s) = -s^3 - B = f(-(s))$ $\therefore$

$f(-(-(y-x^{3/2}))) = f(y-x^{3/2}) = -(-(y-x^{3/2}))^3 - B = (y-x^{3/2})^3 - B$ $\therefore$

$u(x,y) = f(y-x^{3/2}) + g(y+x^{3/2}) = (y-x^{3/2})^3 - B + (y+x^{3/2})^3 + B =$

$(y-x^{3/2})^3 + (y+x^{3/2})^3 = u(x,y) = 6x^3y + 2y^3$

\2c/ $x<0$ $\therefore$ elliptic $\therefore$ ξ, η ~~irrational~~ $\therefore$ complex

$\xi = y + ix^{3/2}$, $\eta = y - ix^{3/2}$ $\therefore$

$\alpha = \frac{\xi+\eta}{2} = \frac{y+ix^{3/2}+y-ix^{3/2}}{2} = \frac{2y}{2} = y = \alpha(x,y)$

$\beta = \frac{\xi-\eta}{2i} = \frac{y+ix^{3/2}-y+ix^{3/2}}{2i} = \frac{2ix^{3/2}}{2i} = x^{3/2} = \beta(x,y)$

\2c sol/ The $x<0$ region is where the equation is elliptic $\therefore$

The charac coords are complex $\therefore$

$\xi(x,y) = y - i(-x)^{3/2}$, $\eta(x,y) = y + i(-x)^{3/2} = \overline{\xi(x,y)}$ $\therefore$

$\therefore$ canonical coords chosen as real and imaginary parts $\therefore$

$\alpha = \mathrm{Re}(\xi(x,y)) = y$, $\beta = -\mathrm{Im}(\xi(x,y)) = -(-(-x)^{3/2}) = (-x)^{3/2}$

\3a/ $\frac{dQ}{dt} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} \frac{\partial u}{\partial t}\,dx = \int_{-\infty}^{\infty} -u - 2\frac{\partial u}{\partial x} + \frac{\partial^2 u}{\partial x^2}\,dx =$

$-\int_{-\infty}^{\infty} u\,dx + 2\int_{-\infty}^{\infty}\frac{\partial u}{\partial x}\,dx + \int_{-\infty}^{\infty}\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)dx =$

$= -Q + 2\left[u\right]_{x=-\infty}^{+\infty} + \left[\frac{\partial u}{\partial x}\right]_{x=-\infty}^{\infty} = -Q - 2\left[u(\infty,t) - u(-\infty,t)\right] + \left[\frac{\partial u(\infty,t)}{\partial x} - \frac{\partial u(-\infty,t)}{\partial x}\right]$

$= -Q - 2[0-0] + [0-0] = -Q$

\3b/ $\therefore$ $g(x,t) \to 0$ as $x \to \pm\infty$ or $t \to \infty$

$u(x,0) = 0$ $\therefore$ ~~tet~~

let $u = e^{px+qt}v(x,t)$ $\therefore$ $u_t = (qv + v_t)e^{px+qt}$,

$u_x = (pv + v_x)e^{px+qt}$ $\therefore$ $u_{xx} = (p^2v + pv_x + pv_x + v_{xx})e^{px+qt} = (p^2v + 2pv_x + v_{xx})e^{px+qt}$

$\therefore$ $u_t + u - 2u_x - u_{xx} =$

$(qv+v_t)e^{px+qt} + ve^{px+qt} + (-2pv - 2v_x)e^{px+qt} + (-p^2v - 2pv_x - v_{xx})e^{px+qt} =$

$\left[v(q+1-2p-p^2) + v_x(-2-2p) + v_t - v_{xx}\right]e^{px+qt}$ $\therefore$

let $-2-2p = 0$ $\therefore$ $-2p = 2$ $\therefore$ $p = -1$ $\therefore$

let $q+1-2p-p^2 = 0 = q+1-2(-1)-(-1)^2 = q+1+2-1 = q+2 = 0$ $\therefore$ $q = -2$ $\therefore$

\pp2021/ $u = ve^{-x-qt}$ $\therefore$ $u(\pm\infty,0) = \delta(x,t) = 0 \therefore$

$u(x,0) = 0 = v(x,0)e^{-x-q(0)} = v(x,0)e^{-x}$

$u(\pm\infty,0) = v(+\infty,0)e^{-\infty-q(0)} = \cancel{v(+\infty,0)}\ v(+\infty,0)e^{-\infty} = 0$

$u(-\infty,0) = v(-\infty,0)e^{\infty-q(0)} = 0 \nRightarrow \therefore v(\pm\infty,0) \therefore$

$-u+2u_x+u_{xx} =$

$-ve^{-x-4t} + (\cancel{-2}v+2v_x)e^{-x-4t} + (v-2v_x+v_{xx})e^{-x-4t} =$

$e^{-x-4t}\left[v(-1-2+1)+v_x(+2-2)\right.$

\3b sol/ let $u = e^{pt+qx}v(x,t) \therefore u_t = (pv+v_t)e^{pt+qx}$,

$u_x = (v_x+qv)e^{pt+qx} \therefore u_{xx} = (v_{xx}+qv_x+qv_x+q^2v)e^{pt+qx} = (v_{xx}+2qv_x+q^2v)e^{pt+qx}$

$u_t = v_te^{pt+qx} + pve^{pt+qx} \therefore$

$v_te^{pt+qx} = -pve^{pt+qx} - u + 2u_x + u_{xx} + \delta(x,t) =$

$e^{pt+qx}\left[-pv - v + 2(v_x+qv) + (v_{xx}+2qv_x+q^2v)e^{pt+qx}\right] + \delta(x,t) =$

$e^{pt+qx}\left[v(-p-1+2q+q^2) + v_x(2+2q) + v_{xx}\right] + \delta(x,t) = v_te^{pt+qx} \therefore$

$\cancel{2+2q=0}$ let $2+2q = 0 \therefore q = -1 \therefore$

let $-p-1+2q+q^2 = 0 = -p-1-2+1 = -p-2 \therefore p = -2 \therefore$

$v_te^{-2t-x} = e^{-2t-x}\left[v_{xx}\right] + \delta(x,t) \therefore$

$v_t = v_{xx} + e^{2t+x}\delta(x,t) \equiv v_{xx} + \tilde{\delta}(x,t)$

$u(x,0) = 0 = v(x,0)e^{-2(0)-x} = v(x,0)e^{-x} = 0 \therefore \quad v(x,0) = 0$

$\lim_{x\to\pm\infty} \delta(x,t) = 0 \therefore \lim_{x\to\pm\infty}\delta(x,0) = 0 \therefore$

as $x \to \pm\infty$: (Eq3b) $\to$ (Eq3a1) $\therefore \lim_{x\to\pm\infty} u(x,t) = 0 \therefore \lim_{x\to\pm\infty} u(x,0) = 0 \therefore$

$\lim_{x\to\pm\infty} u(x,0) = 0 = \lim_{x\to\pm\infty} v(x,0)e^{-2(0)-x} = \lim_{x\to\pm\infty} v(x,0)e^{-x} = \lim_{x\to\pm\infty} v(x,0)\lim_{x\to\pm\infty} e^{-x} = 0$

$\therefore \lim_{x\to\pm\infty} v(x,0) = 0 \equiv v(\pm\infty,0) \therefore$

$$v(x,t) = \int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} \tilde{\delta}(\xi,\tau)\, d\xi\, d\tau \therefore$$

$$u(x,t) = e^{-2t-x}v(x,t) = e^{-2t-x}\int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}} \tilde{\delta}(\xi,\tau)\, d\xi\, d\tau =$$

$$\int_0^t \int_{-\infty}^{\infty} \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-2t-x-\frac{(x-\xi)^2}{4(t-\tau)}} \delta(\xi,\tau) e^{2\tau+\xi}\, d\xi\, d\tau$$

$$G(x,t;\xi,\tau) = \frac{1}{\sqrt{4\pi(t-\tau)}} e^{-2t-x-\frac{(x-\xi)^2}{4(t-\tau)}+2\tau+\xi}$$

3a) let $Q(t) = \int_{-\infty}^{\infty} u(x,t)\,dx \therefore \frac{dQ}{dt} = \int_{-\infty}^{\infty} u_t\,dx = \int_{-\infty}^{\infty} -u + 2u_x + u_{xx} + \delta\,dx$

$\frac{dQ}{dt} = -Q + \int_{-\infty}^{\infty} \delta(x,t)\,dx$ ✓

4a) $\cancel{\begin{bmatrix} u_t \\ v_t \end{bmatrix} = \frac{d}{dt}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} u_{xx} \\ v_{xx} \end{bmatrix} + \begin{bmatrix} \Lambda(r) & -\Omega(r) \\ \Omega(r) & \Lambda(r) \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix}}$ $\therefore r^2 = u^2 + v^2 \therefore$

let $u(x,t) = r(x,t)\cos(\theta(x,t))$, $v(x,t) = r(x,t)\sin(\theta(x,t)) \therefore$

$r(x,t) = R(\xi)$, $\theta = \Theta(\xi) \therefore$

$u_t = r_t\cos\theta - r\sin(\theta)\theta_t$, $v_t = r_t\sin\theta + r\cos(\theta)\theta_t \therefore$

$u_x = r_x\cos\theta - r\sin(\theta)\theta_x$, $v_x = r_x\sin\theta + r\cos(\theta)\theta_x \therefore$

$u_{xx} = r_{xx}\cos\theta - r_x\sin(\theta)\theta_x - r_x\sin(\theta)\theta_x - r(\cos(\theta)\theta_x^2 - \sin(\theta)\theta_{xx}) =$

$r_{xx}\cos\theta - 2r_x\sin\theta\,\theta_x - r\cos(\theta)\theta_x^2 - r\sin(\theta)\theta_{xx} =$

$[r_{xx} - r\theta_x^2]\cos(\theta) + [-2r_x\theta_x - r\theta_{xx}]\sin(\theta) =$

$u_{xx} = [r_{xx} - r\theta_x^2]\cos(\theta) - [2r_x\theta_x + r\theta_{xx}]\sin(\theta)$

$v_{xx} = r_{xx}\sin\theta + r_x\cos(\theta)\theta_x + r_x\cos(\theta)\theta_x - r\sin(\theta)\theta_x^2 + r\cos(\theta)\theta_{xx} =$

$[r_{xx} - r\theta_x^2]\sin(\theta) + [2r_x\theta_x + r\theta_{xx}]\cos\theta \therefore$

$u_t = u_{xx} + \Lambda(r)u - \Omega(r)v \therefore$

$r_t\cos\theta - r\sin(\theta)\theta_t =$

$[r_{xx} - r\theta_x^2]\cos(\theta) - [2r_x\theta_x + r\theta_{xx}]\sin\theta + \Lambda(r)r\cos(\theta) - \Omega(r)r\sin\theta =$

$[r_{xx} - r\theta_x^2 + \Lambda(r)r]\cos\theta - [2r_x\theta_x + r\theta_{xx} + \Omega(r)r]\sin\theta \therefore$

$r_t = r_{xx} - r\theta_x^2 + \Lambda(r)r$, $r\theta_t = 2r_x\theta_x + r\theta_{xx} + \Omega(r)r$

$v_t = v_{xx} + \Omega(r)u + \Lambda(r)v \therefore$

$r_t\sin\theta + r\cos(\theta)\theta_t =$

$[r_{xx} - r\theta_x^2]\sin(\theta) + [2r_x\theta_x + r\theta_{xx}]\cos\theta + \Omega(r)r\cos(\theta) + \Lambda(r)r\sin(\theta) =$

$[r_{xx} - r\theta_x^2 + \Lambda(r)r]\sin(\theta) + [2r_x\theta_x + r\theta_{xx} + \Omega(r)r]\cos\theta \therefore$

$r_t = r_{xx} - r\theta_x^2 + \Lambda(r)r$, $r\theta_t = 2r_x\theta_x + r\theta_{xx} + \Omega(r)r \therefore \xi_x = 1$, $\xi_t = -c \therefore$

$r_t = R'(\xi)\xi_t = -cR'(\xi)$, $\theta_t = \Theta'(\xi)\xi_t = -c\Theta'(\xi)$,

$r_x = R'(\xi)\xi_x = R'(\xi)$, $\theta_x = \Theta'(\xi)\xi_x = \Theta'(\xi) \therefore$

$r_{xx} = R''(\xi)\xi_x = R''(\xi)$, $\theta_{xx} = \Theta''(\xi)\xi_x = \Theta''(\xi) \therefore$

$r_{xx} + r\theta_x^2 + \Lambda(r)r - r_t = 0$, $2r_x\theta_x + r\theta_{xx} + \Omega(r)r - r\theta_t \therefore$

$R''(\xi) + R(\xi)\Theta'(\xi)^2 + \Lambda(R(\xi))R(\xi) + cR'(\xi)$, PP2021/ ~~$R(\xi)$~~ $2r_x\theta_x + r\theta_{xx} + \Omega(r)r - r\theta_t$

$2R'(\xi)\Theta'(\xi) + R(\xi)\Theta''(\xi) + \Omega(R(\xi))R(\xi) + R(\xi)C\Theta'(\xi)$ ∴

$R'' + CR' - R(\Theta')^2 + R\Lambda(R) = 0$,

$R\Theta'' + CR\Theta' + 2R'\Theta' + R\Omega(R) = 0$

is the requested system of ODEs for $R(\xi)$, $\Theta(\xi)$

4b/ $R(\xi) = \text{const}$, $\Theta'(\xi) = \text{const}$ ∴

~~let~~ $R(\xi) = a = \text{const}$ $\Theta'(\xi) = \Theta' = k = \text{const}$ ∴

$R' = 0$, $\Theta'' = 0$ ∴, $R'' = 0$ ∴ $\Theta(\xi) = k\xi$ ∴

$0 + C(0) - ak^2 + a\Lambda(a) = 0 = -ak^2 + a\Lambda(a)$, ∴ $ak^2 = a\Lambda(a)$

$a(0) + Cak + 2(0)k + a\Omega(a) = 0 = Cak + a\Omega(a)$ ∴

$k^2 = \Lambda(a)$ ∴ $k = \pm\sqrt{\Lambda(a)}$ ∴

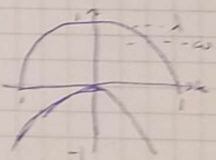
$Cak = -a\Omega(a)$ ∴ $Ck = -\Omega(a)$ ∴ $C = -\frac{\Omega(a)}{k} = -\frac{\Omega(a)}{\pm\sqrt{\Lambda(a)}}$ ∴

∴ let $\omega = Ck$ ∴ $\omega = -\frac{\Omega(a)}{k}k = -\Omega(a)$

∴ for suggested $\Lambda(r) = 1 - r^2$, $\Omega(r) = 1 - r^2$

$a < 1$, $k = \pm\sqrt{1-a^2} \in (-1, 1)$

$\omega = -1 + a^2$ ∴ $a(k) = \sqrt{1-k^2}$, $\omega(k) = -1 + 1 - k^2 = -k^2$



PP2020/ 2a/ $u_{xx}+3u_{xy}+2u_{yy}=-u_x-2u_y$ ∴

$a=1,\ 2b=3\ \therefore b=\frac{3}{2},\ c=2\ \therefore\quad b^2-ac=\left(\frac{3}{2}\right)^2-1(2)=0.25>0$ ∴ hyperbolic ∴

$a\left(\frac{dy}{dx}\right)^2-2b\left(\frac{dy}{dx}\right)+c=0=\left(\frac{dy}{dx}\right)^2-3\left(\frac{dy}{dx}\right)+2=0=$

$\left[\left(\frac{dy}{dx}\right)-1\right]\left[\left(\frac{dy}{dx}\right)-2\right]=0\ \therefore\ \frac{dy}{dx}=1,\quad \frac{dy}{dx}=2\ \therefore$

$\int 1\,dy=\int 1\,dx=y=x+c_1,\quad \int 1\,dy=\int 2\,dx=y=2x+c_2\ \therefore$

$c_1=y-x,\quad c_2=y-2x\ \therefore$

$\xi=y-x,\ \eta=y-2x\ \therefore$

$\xi_x=-1,\quad \xi_y=1,\quad \eta_x=-2,\quad \eta_y=1\ \therefore$

$u_x=u_\xi\xi_x+u_\eta\eta_x=-u_\xi-2u_\eta$

$u_y=u_\xi\xi_y+u_\eta\eta_y=u_\xi+u_\eta\ \therefore$

$u_{xy}=u_{\xi\xi}\xi_x+u_{\xi\eta}\xi_x+u_{\xi\eta}\eta_x+u_{\eta\eta}\eta_x=-u_{\xi\xi}-u_{\xi\eta}-2u_{\xi\eta}-2u_{\eta\eta}=$
$-u_{\xi\xi}-3u_{\xi\eta}-2u_{\eta\eta}$

$u_{xx}=-u_{\xi\xi}\xi_x-2u_{\xi\eta}\xi_x-u_{\xi\eta}\eta_x-2u_{\eta\eta}\eta_x=u_{\xi\xi}+4u_{\xi\eta}+4u_{\eta\eta}$

$u_{yy}=u_{\xi\xi}\xi_y+u_{\xi\eta}\xi_y+u_{\xi\eta}\eta_y+u_{\eta\eta}\eta_y=u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}\ \therefore$

PDE:

$u_{\xi\xi}+4u_{\xi\eta}+4u_{\eta\eta}-3u_{\xi\xi}-9u_{\xi\eta}-6u_{\eta\eta}+2u_{\xi\xi}+4u_{\xi\eta}+2u_{\eta\eta}-u_\xi-2u_\eta+$
$2u_\xi+2u_\eta=$

$4u_{\xi\eta}-9u_{\xi\eta}+4u_{\xi\eta}-u_\xi-2u_\eta+2u_\xi+2u_\eta=$

$-u_{\xi\eta}+u_\xi=0$

1a/ let $\xi=x+y\ \therefore\ \eta=x-y\ \therefore\quad \xi_x=1,\ \xi_y=1,\ \eta_x=1,\ \eta_y=-1$

$u_x=u_\xi\xi_x+u_\eta\eta_x=u_\xi+u_\eta,$

$u_y=u_\xi\xi_y+u_\eta\eta_y=u_\xi-u_\eta\ \therefore$

$u_x+u_y=u_\xi+u_\eta+u_\xi-u_\eta=2u_\xi$

$\xi-y=x\ \therefore\ \eta=\xi-y-y=\xi-2y\ \therefore\ 2y=\xi-\eta\ \therefore\ y=\frac{1}{2}\xi-\frac{1}{2}\eta$

$x=\xi-\frac{1}{2}\xi+\frac{1}{2}\eta=\frac{1}{2}\xi+\frac{1}{2}\eta\ \therefore$

PDE:

$2u_\xi-u=\cos(x-y)=\cos(\eta)\ \therefore$

$u_\xi-\frac{1}{2}u=\frac{1}{2}\cos(\eta)\ \therefore$

$IF = e^{\int -\frac{1}{2} d\xi} = e^{-\frac{1}{2}\xi}$ ∴

$\frac{\partial}{\partial \xi}\left(e^{-\frac{1}{2}\xi} u\right) = e^{-\frac{1}{2}\xi}\cos(\eta)$ ∴

$e^{-\frac{1}{2}\xi} u = -2e^{-\frac{1}{2}\xi}\cos(\eta) + f(\eta)$ ∴

$u = -2\cos(\eta) + e^{\frac{1}{2}\xi} f(\eta)$ ∴ $\frac{1}{2}\xi = \frac{1}{2}x + \frac{1}{2}y$ ∴

$u(x,y) = -2\cos(x-y) + e^{\frac{1}{2}x+\frac{1}{2}y} f(x-y)$

$u(x,0) = 0 = -2\cos(x) + e^{\frac{1}{2}x} f(x)$ ∴

$2\cos(x) = e^{\frac{1}{2}x} f(x)$ ∴ $2e^{-\frac{1}{2}x}\cos(x) = f(x)$ ∴

$f(s) = 2e^{-\frac{1}{2}s}\cos(s)$ ∴

$f(x-y) = 2e^{-\frac{1}{2}(x-y)}\cos(x-y) = 2e^{-\frac{1}{2}x+\frac{1}{2}y}\cos(x-y)$ ∴

$u(x,y) = -2\cos(x-y) + 2e^{+\frac{1}{2}x+\frac{1}{2}y} e^{-\frac{1}{2}x+\frac{1}{2}y}\cos(x-y) =$

$-2\cos(x-y) + 2e^{y}\cos(x-y) = 2\cos(x-y)(-1+e^{y})$

\16/ $u_{xx} - 2u_{xy} + (x^2+y^2)u_{yy} = x^2y^2 - yu_x - xu_y - xyu$ ∴

$a=1$, $2b=-2$ ∴ $b=-1$, $c = x^2+y^2$ ∴

$b^2 - ac = (-1)^2 - 1(x^2+y^2) = 1-(x^2+y^2)$ ∴

if $1-(x^2+y^2) = 0$ ∴ $1 = x^2+y^2$ then $y^2 = 1-x^2$ ∴

$y = \pm\sqrt{1-x^2}$ then parabolic

if $1-(x^2+y^2) > 0$ ∴ $1 > x^2+y^2$ ∴ $1-x^2 > y^2$ ∴

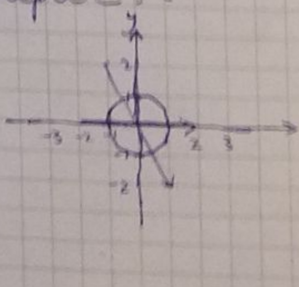
$-\sqrt{1-x^2} < y < \sqrt{1-x^2}$ ∴ inside the circle the hyperbolic

∴ if ~~y~~ $1-(x^2+y^2) < 0$ ∴ $1-x^2 < y^2$ ∴

$|y| > \sqrt{1-x^2}$ ∴ $y < -\sqrt{1-x^2}$ or $\sqrt{1-x^2} < y$ then eliptic ∴

out side the circle ∴

∴ $a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0 = \left(\frac{dy}{dx}\right)^2 + 2\left(\frac{dy}{dx}\right) + x^2+y^2 = 0$



∴ $\frac{dy}{dx} = \frac{-2 \pm \sqrt{4-4(x^2+y^2)}}{2} = -1 \pm \sqrt{1-(x^2+y^2)}$ ∴

at $(0,0)$: $\frac{dy}{dx} = -1 \pm \sqrt{1-(0^2+0^2)} = -1 \pm \sqrt{1}$

$= -1 \pm 1$ ∴ $\frac{dy}{dx} = -1+1$, $\frac{dy}{dx} = -1-1$ ∴

$\frac{dy}{dx} = 0$, $\frac{dy}{dx} = -2$

$\sqrt{PP2020}$ / 1c / let $u = X(x)T(t)$ $\therefore$ $u_t = XT'$, $u_{xx} = X''T$ $\therefore$

$XT' = X''T$ $\therefore$ $\frac{T'}{T} = \frac{X''}{X} = -\lambda = \text{Constant}$ $\therefore$

$T' = -\lambda T$, $X'' = -\lambda X$ $\therefore$

$X'' + \lambda X = 0$, $X(0) = 1$, $X'(1) = 1$ $\therefore$

$q^2 + \lambda = 0$ $\therefore$ $q^2 = -\lambda$ $\therefore$ $q = \pm\sqrt{-\lambda}$ $\therefore$

for $\lambda = 0$: $X = A + Bx$ $\therefore$ $X(0) = 1 = A + B(0) = A = 1$ $\therefore$

$X = 1 + Bx$ $\therefore$ $X' = B$ $\therefore$ $X'(1) = 1 = B$ $\therefore$ $\cancel{X = A + B}$ $X = 1 + x$

if $\lambda < 0$ $\therefore$ $-\lambda > 0$ $\therefore$ $X = Ae^{\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$ $\therefore$

$X(0) = 1 = Ae^{\sqrt{-\lambda}} + Be^{-\sqrt{-\lambda}}$,

$X' = A\sqrt{-\lambda}e^{\sqrt{-\lambda}x} - B\sqrt{-\lambda}e^{-\sqrt{-\lambda}x}$ $\therefore$

$X'(1) = A\sqrt{-\lambda}e^{\sqrt{-\lambda}} - B\sqrt{-\lambda}e^{-\sqrt{-\lambda}} = 1$

1c /
let $u + x = v$ $\therefore$ $u_t = v_t$, ~~u_{xx}~~ ~~$u_x = ...$~~

$v_x = u_x + 1$, $v_{xx} = u_{xx}$ $\therefore$

$u(0,t) = 1$ $\therefore$ $u(0,t) -$

1c / let $w = u + 1$ $\therefore$ $w - 1 = u$ $\therefore$ $u_t = w_t$, $u_x = w_x$ $\therefore$ $u_{xx} = w_{xx}$

$\therefore$ $u(0,t) = 1 = w(0,t) - 1$

1c / let $w = u - 1$ $\therefore$ $w + 1 = u$ $\therefore$ $w_t = u_t$, $w_x = u_x$, $w_{xx} = u_{xx}$ $\therefore$

$u(0,t) = 1 = w(0,t) + 1$ $\therefore$ $w(0,t) = 0$, $u_x(1,t) = 1 = w_x(1,t)$ $\therefore$

PDE: $u_t = u_{xx} = w_t = w_{xx}$ $\therefore$

let $w = X(x)T(t)$ $\therefore$ $w_t = XT'$, $w_{xx} = X''T$ $\therefore$

$XT' = X''T$ $\therefore$ $\cancel{\frac{X}{X''} = \frac{T}{T'}}$ $\frac{T'}{T} = \frac{X''}{X} = -\lambda = \text{Constant}$ $\therefore$

$T' = -\lambda T$, $X'' = -\lambda X$ $\therefore$

$X'' + \lambda X = 0$, $X(0) = 0$, $X_x(1) = 1 = X'(1)$ $\therefore$

$q^2 + \lambda = 0$ $\therefore$ $q^2 = -\lambda$ $\therefore$ $q = \pm\sqrt{-\lambda}$ $\therefore$

$\lambda = 0$: $X = A + Bx$ $\therefore$ $X(0) = 0 = A + 0 = A = 0$ $\therefore$ $X = Bx$ $\therefore$

$X' = B$ $\therefore$ $X'(1) = B = 1$ $\therefore$ $B = 1$

$\lambda < 0$: $\therefore$ $-\lambda > 0$ $\therefore$ $X = Ae^{+\sqrt{-\lambda}x} + Be^{-\sqrt{-\lambda}x}$ $\therefore$

$X(0) = 0 = A + B$ $\therefore$ $A = -B$ $\therefore$ $X = B(-e^{+\sqrt{-\lambda}x} + e^{-\sqrt{-\lambda}x})$ $\therefore$

$X' = B(-\sqrt{-\lambda}e^{+\sqrt{-\lambda}x} - \sqrt{-\lambda}e^{-\sqrt{-\lambda}x})$ $\therefore$

$X'(1) = 1 = B(-\sqrt{-\lambda}\, e^{+\sqrt{-\lambda}} - \sqrt{-\lambda}\, e^{-\sqrt{-\lambda}}) = -B\sqrt{-\lambda}\,(e^{\sqrt{-\lambda}} + e^{-\sqrt{-\lambda}})$

for $\lambda > 0 \therefore -\lambda < 0 \therefore q = \pm\sqrt{-\lambda} = \pm\sqrt{\lambda}\,i \therefore$

$X = A\cos(\sqrt{\lambda}\,x) + B\sin(\sqrt{\lambda}\,x) \therefore$

$X(0) = 0 = A + 0B = A \therefore X = B\sin(\sqrt{\lambda}\,x) \therefore$

$X' = \sqrt{\lambda}\, B\cos(\sqrt{\lambda}\,x) \therefore X'(1) = 1 = \sqrt{\lambda}\, B\cos(\sqrt{\lambda})$

$\setminus 1C/\ u(0,t) = 1,\ u_x(1,t) = 1 \therefore$ let $W = u - 1 ~~x~~ x \therefore$

$W_t = u_t,\ W_x = u_x - 1,\ W_{xx} = u_{xx}$

~~WTB~~

$\setminus 1C/\ u(0,t) = 1,\ u_x(t,t) = 1 \therefore$

let $W - 1 - x = u$

$\setminus 1C/\ u_t = u_{xx},\ u(0,t) = 1,\ \cancel{u_{xx}}\ u_x(1,t) = 1 \therefore$

if ~~u = 1 + x~~ $\int u_x\, dx = x,\ \int 1\, dx = x \therefore$

if $u = 1 + x$: $u_t = 0,\ u_x = 1,\ u_{xx} = 0 \therefore 0 = 0,$

$u(0,t) = 1 + 0 = 1,\ u_x(1,t) = 1 = 1 \therefore$

let ~~v~~ $u = v + 1 + x \therefore u_x = v_x + 1,\ u_{xx} = v_{xx},\ u_t = v_t \therefore$

PDE: $u_t = u_{xx} = v_t = v_{xx}$

$u(0,t) = v(0,t) + 1 = 1 \therefore v(0,t) = 0,$

$u_x(1,t) = v_x(1,t) + 1 = 1 \therefore v_x(1,t) = 0 \therefore$

let $v = X(x)T(t) \therefore v_t = XT',\ v_x = X'T \therefore v_{xx} = X''T$

$XT' = X''T \therefore \frac{T'}{T} = \frac{X''}{X} = -\lambda = \text{constant} \therefore$

$T' = -\lambda T,\ X'' = -\lambda X \therefore$

$X'' + \lambda X = 0 \therefore X(0) = 0,\ X'(1) = 0 \therefore$

$q^2 + \lambda = 0 \therefore q^2 = -\lambda \therefore q = \pm\sqrt{-\lambda} \therefore$

if $\lambda = 0$: $X = A + Bx \therefore X(0) = 0 = A + 0 = A = 0 \therefore X = Bx \therefore$

$X' = B \therefore X'(1) = 0 = B \therefore X = 0 \therefore$

if $\lambda < 0 \therefore -\lambda > 0 \therefore X = Ae^{\sqrt{-\lambda}\,x} + Be^{-\sqrt{-\lambda}\,x} \therefore$

$X(0) = 0 = A + B \therefore -B = A \therefore X = B(-e^{\sqrt{-\lambda}\,x} + e^{-\sqrt{-\lambda}\,x}) \therefore$

$X' = B(-\sqrt{-\lambda}\,e^{\sqrt{-\lambda}\,x} - \sqrt{-\lambda}\,e^{-\sqrt{-\lambda}\,x}) \therefore$

$X'(1) = B(-\sqrt{-\lambda}\,e^{\sqrt{-\lambda}} - \sqrt{-\lambda}\,e^{-\sqrt{-\lambda}}) = 0 = B \therefore X = 0$

\2020B/ if $\lambda > 0 \therefore -\lambda < 0 \therefore q = \pm\sqrt{-\lambda} = \pm\sqrt{\lambda}i \therefore$

$X = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \therefore$

$X(0) = 0 = A + 0 = A = 0 \quad \therefore X = B\sin(\sqrt{\lambda}x) \therefore$

$X' = \sqrt{\lambda}B\cos(\sqrt{\lambda}x) \therefore$

$X'(1) = 0 = \sqrt{\lambda}B\cos(\sqrt{\lambda}) = \sqrt{\lambda}B\cos(\sqrt{\lambda}) = 0 \therefore \cos(\sqrt{\lambda}) = 0 \therefore$

$\sqrt{\lambda} = n\pi + \frac{\pi}{2} = \frac{2n\pi}{2} + \frac{\pi}{2} = \frac{2n\pi+\pi}{2} = \frac{1}{2}(\pi)(2n+1) \quad n \in \{\mathbb{N}, 0\} \therefore n \in \mathbb{Z}_{\geq 0}$

$\sqrt{\lambda} = \pi(n+\frac{1}{2}) = (n+\frac{1}{2})\pi \therefore \lambda = (n+\frac{1}{2})^2\pi^2 \therefore$

$X = B\sin((n+\frac{1}{2})\pi x) \therefore$

$T' = -(n+\frac{1}{2})^2\pi^2 T \therefore \quad \frac{T'}{T} = -(n+\frac{1}{2})^2\pi^2 \therefore \int \frac{T'}{T}dt = \int -(n+\frac{1}{2})^2\pi^2 dt =$

$\ln|T| = -(n+\frac{1}{2})^2\pi^2 t + C_1 \quad \therefore |T| = C_2 e^{-(n+\frac{1}{2})^2\pi^2 t} \therefore$

$T = Be^{-(n+\frac{1}{2})^2\pi^2 t} \therefore$ by superposition:

$V = \sum_{n=0}^{\infty} B_n e^{-(n+\frac{1}{2})^2\pi^2 t}\sin((n+\frac{1}{2})\pi x) \therefore$

$u = 1+x+V = 1+x+\sum_{n=0}^{\infty} B_n e^{-(n+\frac{1}{2})^2\pi^2 t}\sin((n+\frac{1}{2})\pi x) \therefore$

$u(x,0) = 1+x+\sin(\frac{\pi}{2}x) = 1+x+\sum_{n=0}^{\infty} B_n\sin((n+\frac{1}{2})\pi x)$

$\therefore \sin(\frac{\pi}{2}x) = \sum_{n=0}^{\infty} B_n\sin((n+\frac{1}{2})\pi x) =$

$\sum_{n=0}^{0} B_n\sin((n+\frac{1}{2})\pi x) + \sum_{n=1}^{\infty} B_n\sin((n+\frac{1}{2})\pi x) =$

$B_0\sin(\frac{\pi}{2}x) + \sum_{n=1}^{\infty} B_n\sin((n+\frac{1}{2})\pi x) \therefore$

$B_n = 0 \; \forall n \in \mathbb{N}, \quad B_0 = 1 \therefore$

$u(x,t) = 1+x+\sum_{n=0}^{0} B_n e^{-(n+\frac{1}{2})^2\pi^2 t}\sin((n+\frac{1}{2})\pi x) =$

$1+x+e^{-(0+\frac{1}{2})^2\pi^2 t}\sin((0+\frac{1}{2})\pi x) = 1+x+e^{-\frac{1}{4}\pi^2 t}\sin(\frac{1}{2}\pi x)$

\1d/ $u_{xx}+u_{yy} = 0 \quad \therefore u = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{u(s,0)\,ds}{(x-s)^2+y^2} \therefore$

\1dso1/ $u_{xx}+u_{yy} = 0 \quad x \in (0,\infty), \; y \in (0,\infty) \quad u_x(0,y) = 0,$

$u(x,0) = \frac{1}{1+x^2}, \; u(x,y) \to 0$ as $x,y \to \infty \therefore$

\1d/ BC satisfied for even extension on $x \therefore x \in \mathbb{R} \therefore$

$u = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{u(s,0)}{(x-s)^2+y^2}ds$, BC $u(x,0) = \frac{1}{1+x^2} \therefore u(s,0) = \frac{1}{1+s^2} \therefore$

$\frac{u(s,0)}{(x-s)^2+y^2} = \frac{1}{1+s^2}\frac{1}{(x-s)^2+y^2} = \frac{1}{1+s^2}\frac{1}{(y^2+(x-s)^2)} = \frac{1}{(1+s^2)(y^2+(x-s)^2)}$

$\therefore \int_{-\infty}^{\infty} \frac{u(s,0)}{(x-s)^2+y^2}\,ds = \int_{-\infty}^{\infty}\frac{1}{(1+s^2)(y^2+(x-s)^2)}\,ds =$

$\int_{-\infty}^{\infty}\frac{1}{(1^2+(s-0)^2)(y^2+(x-s)^2)}\,ds = \int_{-\infty}^{\infty}\frac{1}{(1^2+(s-0)^2)(y^2+(s-x)^2)}\,ds =$

$\int_{-\infty}^{\infty}\frac{1}{(A^2+(s-a)^2)(B^2+(s-b)^2)}\,ds$ for $A=1,\ a=0,\ B=y,\ b=x$

$= \frac{\pi(A+B)}{AB((A+B)^2+(a-b)^2)} = \frac{\pi(1+y)}{1y((1+y)^2+(0-x)^2)} = \frac{\pi(1+y)}{y((1+y)^2+x^2)}$

$\therefore \frac{y}{\pi}\int \frac{u(s,0)}{(x-s)^2+y^2} = \frac{y}{\pi}\frac{\pi(1+y)}{y((1+y)^2+x^2)} = \frac{1+y}{(1+y)^2+x^2} = u(x,y) \therefore$

$u(1,1) = \frac{1+1}{(1+1)^2+1^2} = \frac{2}{5} = 0.4$ (4.s.f.)

\1d sol/ $u_{xx}+u_{yy}=0 \therefore x\in(0,\infty),\ y\in(0,\infty),\ u_x(0,y)=0$,

$u(x,0)=\frac{1}{1+x^2}$, $u(x,y)\to 0$ as $x,y\to\infty$ $\therefore$

$x=0$: $u_x(0,y)=0$ BC satisfied for even extension in x $\therefore$

PDE in $x\in\mathbb{R}$, $y>0$, BC $u(x,0)=\frac{1}{1+x^2}$ $\therefore$

$u(s,0)=\frac{1}{1+s^2}$ $\therefore$

$u(x,y)=\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{u(s,0)}{(x-s)^2+y^2}\,ds = \frac{y}{\pi}\int_{-\infty}^{\infty}\frac{1}{(1-s^2)(y^2+(x-s)^2)}\,ds =$

$\frac{y}{\pi}\frac{\pi(1+y)}{1y((1+y)^2+(0-x)^2)} = \frac{1+y}{(1+y)^2+x^2}$ $\therefore$

$u(1,1)=\frac{1+1}{(1+1)^2+1^2}=\frac{2}{4+1}=\frac{2}{5}=0.4$

\1e/ $u_{tt}-u_{xx}=0$, $x\in(-\infty,\infty)$, $t\in(0,\infty)$, $u(x,0)=\frac{1}{1+x^2}$, $u_t(x,0)=\frac{-2x}{(1+x^2)^2}$

~~u78~~ $\therefore u=f(x-t)+g(x+t)$ $\therefore$

$u(x,0)=\frac{1}{1+x^2}=f(x)+g(x)$,

$u_t(x,t)=-f'(x-t)+g'(x+t) \therefore u_t(x,0)=-f'(x)+g'(x)=\frac{-2x}{(1+x^2)^2}$

$\therefore \frac{1}{1+x^2}-f(x)=g(x) \therefore g'(x)=\frac{(1+x^2)(0)-1(2x)}{(1+x^2)^2}-f'(x)=\frac{-2x}{(1+x^2)^2}-f'(x)$

$\therefore -f'(x)+\frac{-2x}{(1+x^2)^2}-f'(x)=\frac{-2x}{(1+x^2)^2} \therefore -2f'(x)=0 \therefore f'(x)=0$ $\therefore$

$f(x)=0 \therefore g(x)=\frac{1}{1+x^2}-0=\frac{1}{1+x^2} \therefore g(x+t)=\frac{1}{1+(x+t)^2} \Rightarrow u(x,t)$ $\therefore$

\ff2020/ $U(t) = u(1,t) = \frac{1}{1+(1+t)^2} \therefore$

$U'(t) = \frac{-1(2(1+t)(1))}{(1+(1+t)^2)^2} = \frac{-2(1+t)}{(1+(1+t)^2)^2} = 0 \therefore$

$-2(1+t) = 0 = -2-2t \therefore 2 = -2t \therefore t = -1$ but $-1 \notin t \therefore$

$t \neq -1 \therefore \forall\, t \in [0, \infty) \therefore$

$U(0) = \frac{1}{1+(1+0)^2} = \frac{1}{1+1} = \frac{1}{2}$

~~U(t)~~ $\lim_{t\to\infty} U(t) = \lim_{t\to\infty} \frac{1}{1+(1+t)^2} = 0 \therefore$

$t_* = 0$, $U_* = \frac{1}{2}$

\1e sol/ $u_{tt} - u_{xx} = 0$, $x \in (-\infty, \infty)$, $t \in [0, \infty)$, $u(x,0) = \frac{1}{1+x^2}$

$u_t(x,0) = \frac{-2x}{(1+x^2)^2} \therefore$ ~~u(x,t)~~ G-S by D'Alembert's formula:

$u(x,t) = f(x-t) + g(x+t) \therefore$

$u_t(x,t) = -f'(x-t) + g'(x+t) \therefore$

$f(x) + g(x) = \frac{1}{1+x^2}$, $-f'(x) + g'(x) = \frac{-2x}{(1+x^2)^2} \therefore$

$f'(x) + g'(x) = \frac{-2x}{(1+x^2)^2} \therefore f'(x) = 0$, $g'(x) = \frac{-2x}{(1+x^2)^2} \therefore$

$f(x) = C$, $g(x) = \frac{1}{(1+x^2)} - C \therefore u(x,t) = \frac{1}{1+(x+t)^2}$

$\therefore \max_{t\geq 0} u(1,t) = \max_{t\geq 0} \frac{1}{1+(1+t)^2}$

$\therefore$ ~~$\frac{1}{1+t^2}$~~ $u(1,t) = \frac{1}{1+(1+t)^2}$ is monotonically decreasing $\therefore$

max at beginning of the interval $\therefore t_* = 0$, $U_* = \frac{1}{2}$

\2a/ $a = 1$, $2b = 3 \therefore b = \frac{3}{2}$, $c = 2 \therefore b^2 - ac = \left(\frac{3}{2}\right)^2 - 1(2) = \frac{1}{4} > 0 \therefore$

hyperbolic $\therefore a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = 0 = \left(\frac{dy}{dx}\right)^2 - 3\left(\frac{dy}{dx}\right) + 2 = 0 \therefore$

$\frac{dy}{dx} = \frac{3 \pm \sqrt{3^2 - 4(2)}}{2} = \frac{3}{2} \pm \frac{\sqrt{1}}{2} = \frac{3}{2} \pm \frac{1}{2} \therefore$

$\frac{dy}{dx} = 2$, $\frac{dy}{dx} = 1 \therefore \int 1\,dy = \int 2\,dx = y = 2x + C_1$,

$\int 1\,dy = \int 1\,dx = y = x + C_2 \therefore y - x = C_2$, $y - 2x = C_1 \therefore$

$\xi = y - x$, ~~$y-2x$~~ $\eta = y - 2x \therefore$ ~~ξ_x~~ $\xi = -x + y$, $\eta = -2x + y \therefore$

$\xi_x = -1$, $\xi_y = 1$, $\eta_x = -2$, $\eta_y = 1 \therefore$

$\eta + 2x = y$, $\xi = y - x = \eta + 2x - x = \eta + x \therefore \xi - \eta = x$, $\eta + 2(\xi - \eta) = 2\xi - \eta = y \therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x = -u_\xi - 2u_\eta \therefore$

$u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi + u_\eta \therefore$

$u_{xy} = u_{\xi\xi}\xi_x + u_{\xi\eta}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\eta}\eta_x = -u_{\xi\xi} - 3u_{\xi\eta} - 2u_{\eta\eta}$,

$u_{xx} = -u_{\xi\xi}\xi_x - 2u_{\xi\eta}\xi_x - u_{\xi\eta}\eta_x - 2u_{\eta\eta}\eta_x = u_{\xi\xi} + 4u_{\xi\eta} + 4u_{\eta\eta}$,

$u_{yy} = u_{\xi\xi}\xi_y + u_{\xi\eta}\xi_y + u_{\xi\eta}\eta_y + u_{\eta\eta}\eta_y = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$ $\therefore$

PDE:

$u_{\xi\xi} + 4u_{\xi\eta} + 4u_{\eta\eta} - 3u_{\xi\xi} - 9u_{\xi\eta} - 6u_{\eta\eta} + 2u_{\xi\xi} + 4u_{\xi\eta} + 2u_{\eta\eta} - u_\xi - 2u_\eta + 2u_\xi + 2u_\eta = 0 =$

$4u_{\xi\eta} - 9u_{\xi\eta} + 4u_{\xi\eta} - u_\xi + 2u_\xi = 0 = -u_{\xi\eta} + u_\xi = 0$

\2b/ $u = e^{\lambda\xi + \mu\eta}v$ $\therefore$ $u_\xi = \lambda e^{\lambda\xi+\mu\eta}v + e^{\lambda\xi+\mu\eta}v_\xi = e^{\lambda\xi+\mu\eta}(\lambda v + v_\xi)$

$\therefore$ $u_{\xi\eta} = u_{\eta\xi} = \mu e^{\lambda\xi+\mu\eta}(\xi v + v_\xi) + e^{\lambda\xi+\mu\eta}(\lambda v_\eta + v_{\xi\eta}) =$

$e^{\lambda\xi+\mu\eta}(\lambda\mu v + \mu v_\xi + \lambda v_\eta + v_{\xi\eta})$ $\therefore$

$e^{\lambda\xi+\mu\eta}\left[-(\lambda\mu v + \mu v_\xi + \lambda v_\eta + v_{\xi\eta}) + \lambda v + v_\xi\right] = 0 = \cancel{-\xi\mu v + \mu v_\xi + \xi v_\eta + v_{\xi\eta} + \xi}$

$-\lambda\mu v - \mu v_\xi - \lambda v_\eta - v_{\xi\eta} + \lambda v + v_\xi = (-\mu+1)v_\xi + (-\lambda\mu + \lambda)v - \lambda v_\eta - v_{\xi\eta} = 0$

$\therefore -\lambda = 0$ $\therefore$ $\lambda = 0$ $\therefore$

$(-\mu+1)v_\xi + 0v - v_{\xi\eta} = (-\mu+1)v_\xi - v_{\xi\eta} = 0$ $\therefore$ $-\mu+1 = 0$ $\therefore$ $\mu = 1$ $\therefore$

$-v_{\xi\eta} = 0 = v_{\xi\eta} = \frac{\partial}{\partial\xi}\left(\frac{\partial}{\partial\eta}v\right) = 0$ $\therefore$ $\frac{\partial}{\partial\eta}v = f(\eta)$ $\therefore$ $v = F(\eta) + g(\xi)$

$\therefore$ $e^{\lambda\xi+\mu\eta} = e^\eta$ $\therefore$ $u = e^\eta(F(\eta) + g(\xi)) = e^\eta F(\eta) + e^\eta g(\xi)$ $\therefore$

$u(x,y) = e^{-2x+y}F(-2x+y) + e^{-2x+y}g(-x+y)$ $\therefore$

\2bi/ $u(x,-x) = e^x = e^{-2x-x}F(-2x-x) + e^{-2x-x}g(-x-x) =$

$e^{-3x}F(-3x) + e^{-3x}g(-2x) = e^{-3x}(F(-3x) + g(-2x)) = e^x$ $\therefore$

$F(-3x) + g(-2x) = e^{4x}$ $\therefore$

$u_x = -2e^{-2x+y}F(-2x+y) - 2e^{-2x+y}F'(-2x+y) - 2e^{-2x+y}g(-x+y) - e^{-2x+y}g'(-x+y)$

$= e^{-2x+y}(-2F(-2x+y) - 2F'(-2x+y) - 2g(-x+y) - g'(-x+y))$,

$u_y = e^{-2x+y}F(-2x+y) + e^{-2x+y}F'(-2x+y) + e^{-2x+y}g(-x+y) + e^{-2x+y}g'(-x+y)$

$= e^{-2x+y}(F(-2x+y) + F'(-2x+y) + g(-x+y) + g'(-x+y))$ $\therefore$

$u_x(x,-x) = e^{-3x}(-2F(-3x) - 2F'(-3x) - 2g(-2x) - 2g'(-2x))$,

$u_y(x,-x) = e^{-3x}(F(-3x) + F'(-3x) + g(-2x) + g'(-2x))$ $\therefore$

$u_x(x,-x) + u_y(x,-x) = -e^x = e^{-3x}(-F(-3x) - F'(-3x) - g(-2x) - g'(-2x))$ $\therefore$

$e^{4x} = F(-3x) + F'(-3x) + g(-2x) + g'(-2x)$

\Pf2020/ $\therefore -3F'(-3x) - 2g'(-2x) = 4e^{4x}$

$g(-2x) = e^{4x} - F(-3x) \therefore -2g'(-2x) = 4e^{4x} + 3F'(-3x) \therefore$

$g'(-2x) = -2e^{4x} - \frac{3}{2}F'(-3x) \therefore$

$e^{4x} = F(-3x) + F'(-3x) + e^{4x} - F(-3x) - 2e^{4x} - \frac{3}{2}F'(-3x) \therefore$

$0 = -\frac{1}{2}F'(-3x) - 2e^{4x} \therefore 2e^{4x} = -\frac{1}{2}F'(-3x) \therefore -4e^{4x} = F'(-3x) \therefore$

$\cancel{F(-3x) = 2e^{4x} + c} \therefore \cancel{s = -3x} \therefore \cancel{-\frac{s}{3} = x} \therefore$

$\cancel{F(s) = -e^{4 \cdot \frac{-1}{3}s} + c = -e^{-\frac{4}{3}s} + c} \therefore$

$\cancel{g(-2x) = e^{4x} - e^{4x} - c}$

$\therefore F'(-3x) = -4e^{4x} \therefore$

$s = -3x \therefore -\frac{1}{3}s = x \therefore F'(s) = -4e^{4 \cdot \frac{-1}{3}s} = -4e^{-\frac{4}{3}s} \therefore$

$F(s) = 3e^{-\frac{4}{3}s} + c_1 = F(s) \therefore F(-3x) = 3e^{-\frac{4}{3}(-3x)} + c_1 = 3e^{4x} + c_1 \therefore$

$g(-2x) = e^{4x} - 3e^{4x} - c_1 = -2e^{4x} - c_1 \therefore$

$s = -2x \therefore -2s = 4x \therefore g(s) = -2e^{-2s} - c_1$

$g(-x+y) = -2e^{-2(-x+y)} - c_1 = -2e^{2x-2y} - c_1 \therefore$

$F(-2x+y) = 3e^{-\frac{4}{3}(-2x+y)} + c_1 = 3e^{\frac{8}{3}x - \frac{4}{3}y} + c_1 \therefore$

$u(x,y) = e^{-2x+y} 3 \cdot e^{\frac{8}{3}x - \frac{4}{3}y} + e^{-2x+y}(-2)e^{2x-2y} =$

$3e^{\frac{2}{3}x - \frac{1}{3}y} - 2e^{-y}$

\2bii/ $u(x,y) = e^{-2x+y}(F(-2x+y) + g(-x+y))$

$u(x,x) = e^{x} = e^{-x}(F(-x) + g(0)) \therefore e^{2x} = F(-x) + g(0)$

$u_x = e^{-2x+y}(-2F(-2x+y) - 2F'(-2x+y) - 2g(-x+y) - g'(-x+y))$

$u_y = e^{-2x+y}(F(-2x+y) + F'(-2x+y) + g(-x+y) + g'(-x+y)) \therefore$

$u_x(x,x) = e^{-x}(-2F(-x) - 2F'(-x) - 2g(0) - g'(0))$

$u_y(x,x) = e^{-x}(F(-x) + F'(-x) + g(0) + g'(0)) \therefore$

$u_x(x,x) - u_y(x,x) = -e^{x} = e^{-x}(-2F(-x) - 2F'(-x) - 2g(0) - g'(0) - F(-x) - F'(-x) - g(0) - g'(0))$

$= \therefore -e^{2x} = -3F(-x) - 3F'(-x) - 3g(0) - 2g'(0)$

$\therefore 2e^{2x} = -F'(x) + 0g'(0) = -F'(x) \therefore F'(x) = -2e^{2x}$

$g(0) = e^{2x} - F(-x) = \text{constant} \therefore g'(0) = 0 \therefore -2g'(0) = 0 \therefore$

$-e^{-2x} = -3F(-x) - 3F'(-x) - 3e^{2x} + 3F(-x) = -3F'(-x) - 3e^{2x}$

$-3F'(-x) = 3e^{2x} - e^{-2x}$ ∴

$F'(-x) = -e^{2x} + \frac{1}{3}e^{-2x}$ ∴ $s = -x$ ∴ $x = -s$ ∴

$F'(s) = -e^{-2s} + \frac{1}{3}e^{2s}$ ∴ $F(s) = \frac{1}{2}e^{-2s} + \frac{1}{6}e^{2s} + C_1$ ∴

$F(-3x) = \frac{1}{2}e^{6x} + \frac{1}{6}e^{-6x} + C_1$ ∴

$F(-x) = \frac{1}{2}e^{2x} + \frac{1}{6}e^{-2x} + C_1$ ∴

$g(0) = e^{2x} - F(-x) = e^{2x} - \frac{1}{2}e^{2x} - \frac{1}{6}e^{-2x} = \frac{1}{2}e^{2x} - \frac{1}{6}e^{-2x} - C_1$

which is impossible for $g(0)$ to be a function of x ∴

no solution exists

\2a sol/ $a=1, 2b=3, c=2$ ∴ $\left(\frac{dy}{dx}\right)^2 - 3\left(\frac{dy}{dx}\right) + 2 = 0$ ∴

$\left(\frac{dy}{dx} - 2\right)\left(\frac{dy}{dx} - 1\right) = 0$ ∴ $y = 2x + C$, $y = x + D$ ∴

$\xi = 2x - y$, $\eta = x - y$ ∴ $\xi_x = 2, \xi_y = -1$, $\eta_x = 1, \eta_y = -1$ ∴

$u_x = 2u_\xi + u_\eta$, $u_y = -u_\xi - u_\eta$,

$u_{xx} = 4u_{\xi\xi} + 4u_{\xi\eta} + u_{\eta\eta}$, $u_{yy} = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$, $u_{xy} = -2u_{\xi\xi} - 3u_{\xi\eta} - u_{\eta\eta}$

$u_{xx} + 3u_{xy} + 2u_{yy} = -u_{\xi\eta}$, $u_x + 2u_y = -u_\eta$ ∴

PDE: $-u_{\xi\eta} - u_\eta = 0$ ∴ $u_{\xi\eta} + u_\eta = 0$ is canonical form

$u_\xi = (v_\xi + \lambda v)e^{\lambda\xi + \mu\eta}$, $u_\eta = (v_\eta + \mu v)e^{\lambda\xi + \mu\eta}$,

$u_{\xi\eta} = (v_{\xi\eta} + \lambda v_\eta + \mu v_\xi + \lambda\mu v)e^{\lambda\xi + \mu\eta}$ ∴

$v_{\xi\eta} + (\lambda + 1)v_\eta + \mu v_\xi + (\lambda\mu + \mu)v = 0$ ∴ $\lambda = -1, \mu = 0$ ∴

$v_{\xi\eta} = 0$ ∴ $v = F(\xi) + G(\eta)$ $u = e^{-2x+y}(F(2x-y) + G(x-y))$ ∴

F, G arbitrary functions

\2bi sol/ ∴ $u_x + u_y = e^{-2x+y}(-F(2x-y) - G(x-y) + F'(2x-y))$ ∴

$F(3x) + G(2x) = e^{4x}$ ∴ $-F(3x) - G(2x) + F'(3x) = -e^{4x}$ ∴

$F = \text{constant}$, $G(2x) = e^{4x} - F$ ∴ $G(x) = e^{2x} - F$ ∴

$u = e^{2(x-y)}e^{-2x+y} = e^{-y}$

\2bii/ $u_x - u_y = [-3F(2x-y) - 3G(x-y) + 3F'(2x-y) + 2G'(x-y)]e^{-2x+y}$

∴ $F(x) + G(0) = 1$ ∴ $F(x) = 1 - G(0) = \text{constant}$,

$-3F(x) - 3G(0) + 3F'(x) + 2G'(0) = e^{2x}$ ∴ can not satisfy this

∴ system is inconsistent ∴ for this BCs, no solutions

\PP2020/ $u = (a(x,y), b(x,y))$

\3a/ $\therefore u_{xx} + u_{yy} = f(x,y)$ for $(x,y) \in D$,

$\vec{n} \cdot \nabla = (n_x \; n_y)(\partial_x \; \partial_y) = n_x \partial_x + n_y \partial_y \therefore$

$(n_x \partial_x + n_y \partial_y)(a(x,y) \; b(x,y)) =$

$n_x \frac{\partial}{\partial x} a(x,y) + n_y \frac{\partial}{\partial y} b(x,y)$

$\therefore$ let $W = u_1 - u_2$, $\nabla^2 u_1 = f(x,y)$, $\nabla^2 u_2 = f(x,y)$, $(x,y) \in D$,

$(\vec{n} \cdot \nabla) u_1 = g(x,y)$, $(\vec{n} \cdot \nabla) u_2 = g(x,y)$, $(x,y) \in \partial D \therefore$

$\nabla^2 W = \nabla^2 u_1 - \nabla^2 u_2 = 0$, $(x,y) \in D$,

$(\vec{n} \cdot \nabla) W = (\vec{n} \cdot \nabla) u_1 - (\vec{n} \cdot \nabla) u_2 = 0$, $(x,y) \in \partial D \therefore$

\3a sol/ let u_1 and u_2 be two different solutions of the problem.

let $U = u_1 - u_2 \therefore U_{xx} + U_{yy} = 0$ in D, $\frac{\partial U}{\partial \vec{n}} = 0$ on $\partial D \therefore$

$(\vec{n} \cdot \nabla) U = 0 \therefore \frac{\partial U}{\partial \vec{n}} = 0$ on ∂D.

Divergence theorem: $\iint_D \nabla \cdot \underline{A} \, dx\,dy = \oint_{\partial D} A \cdot \vec{n} \, ds$,

$\therefore$ let $\underline{A} = U \nabla U \therefore$

$$\iint_D \nabla \cdot (U \nabla U) dx\,dy = \oint_{\partial D} U \nabla U \cdot \vec{n} \, ds = \oint U \cdot \frac{\partial U}{\partial \vec{n}} \, ds,$$

$\nabla \cdot (\psi \underline{A}) = \nabla \psi \cdot \underline{A} + \psi \nabla \cdot \underline{A} \therefore$ for $\psi = U$, $\underline{A} = \nabla U$:

$\nabla \cdot (U \nabla U) = \nabla \cdot (U(\nabla U)) = \nabla U \cdot \nabla U + U \nabla \cdot \nabla U =$

$|\nabla U|^2 + U \nabla^2 U \therefore \nabla^2 U = 0$ in $D \therefore U \nabla^2 U = 0$ in $D \therefore$

$$\iint_D \nabla \cdot (U \nabla U) dx\,dy = \iint_D |\nabla U|^2 + U \nabla^2 U \, dx\,dy = \iint_D |\nabla U|^2 dx\,dy$$

$\therefore \iint_D |\nabla U|^2 dx\,dy = \oint_{\partial D} U \frac{\partial U}{\partial \vec{n}} ds \therefore \frac{\partial U}{\partial \vec{n}} = 0$ on $\partial D \therefore$

$U \frac{\partial U}{\partial \vec{n}} = 0$ on $\partial D \therefore \oint_{\partial D} U \frac{\partial U}{\partial \vec{n}} ds = 0 \therefore$

$\iint_D |\nabla U|^2 dx\,dy = 0 \therefore$ an area integral of non negative quantity $|\nabla U|^2$ is zero which is only possible when the integrand is zero everywhere, $\therefore U =$ constant $\therefore$ two arbitrary solutions u_1 and u_2 are different by a constant that is arbitrary $\therefore$ if u_1 is a solution, then $u_2 = u_1 + C$ is a solution $\forall C$.

Suppose $u(x,y)$ satisfies the given PDE and BC $\therefore$

by divergence theorem: $\iint_D f(x,y)\,dx\,dy = \iint_D \nabla^2 u(x,y)\,dx\,dy =$

$\oint_{\partial D} u(x,y)\cdot\vec{n}\,ds = \oint_{\partial D} \frac{\partial u}{\partial \vec{n}}\,ds = \oint_{\partial D} (\vec{n}\cdot\nabla)u\,ds = 0$

$\therefore$ Solution only if $\iint_D f(x,y)\,dx\,dy = 0$

\3b/ $\therefore$ seperating variables: $u(r,\theta) = R(r)Q(\theta)$ $\therefore$

$u_r = R'Q$, $u_{rr} = R''Q$, $u_{\theta\theta} = RQ''$ $\therefore$

$QR'' + \frac{Q}{r}R' + \frac{R}{r^2}Q'' = 0$ $\therefore$ $QR'' + \frac{Q}{r}R' = -\frac{R}{r^2}Q''$ $\therefore$

$r^2QR'' + rQR' = -RQ''$ $\therefore$ $r^2\frac{1}{R}QR'' + \frac{rQR'}{R} = -Q''$ $\therefore$

$\frac{r^2}{R}R'' + \frac{r}{R}R' = -\frac{1}{Q}Q'' = \lambda = \text{Constant}$ $\therefore$

$r^2R'' + rR' - \lambda R = 0$, $Q'' + \lambda Q = 0$

$\therefore$ $Q'' + \lambda Q = 0$ $\because$ θ is the polar angle, any function depending on it, should be 2π-periodic $\therefore$ for $Q(\theta)$: have a BVP with periodic BCs $\therefore$

$Q(0) = Q(2\pi)$, $Q'(0) = Q'(2\pi)$ $\therefore$

$\lambda = n^2$, $n \in \mathbb{Z}_{\geq 0}$, $\phi_0(\theta) = 1$, $\phi_n^{(1)}(\theta) = \cos(n\theta)$, $\phi_n^{(2)}(\theta) = \sin(n\theta)$, for $n > 0$ $\therefore$

$\lambda_n = n^2$, $n \in \mathbb{Z}_{\geq 0}$, $Q_0 = 1$, $Q_n^{(1)}(\theta) = \cos(n\theta)$, $Q_n^{(2)}(\theta) = \sin(n\theta)$ $\therefore$

for $r^2R'' + rR - \lambda R = 0$ let $R = r^n$ $\therefore$

$R = 1$, $R = \ln r$ for $n = 0$,

$R = r^n$, $R = r^{-n}$ for $n > 0$

requirement that the solution is bounded at $r = 0$ means we discard r^{-n} and $\ln r$ solutions $\therefore$

$u(r,\theta) = \frac{A_0}{2} + \sum_{n=1}^{\infty}(A_n\cos(n\theta) + B_n\sin(n\theta))r^n$ $\therefore$

$u_r(r,\theta) = \sum_{n=1}^{\infty}(nA_n\cos(n\theta) + nB_n\sin(n\theta))r^{n-1}$ $\therefore$ $u_r(1,\theta) = g(\theta)$ $\therefore$

at $r = 1$: $u_r(1,\theta) = g(\theta) = \sum_{n=1}^{\infty} n(A_n\cos(n\theta) + B_n\sin(n\theta))$ $\therefore$

$nA_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\cos(n\theta)\,d\theta$, $nB_n = \frac{1}{\pi}\int_0^{2\pi} g(\theta)\sin(n\theta)\,d\theta$, $n \geq 1$ $\therefore$

$\int_0^{2\pi} g(\theta)\,d\theta = 0$ $\because$ the constant in the Fourier series is the

\PP2020/ equation absent. The coefficient of $\frac{A_0}{2}$ remains arbitrary ∴ its not constrained by BC ∴

$$u(r,\theta) = \sum_{n=1}^{\infty} \frac{r^n}{n\pi} \int_0^{2\pi} g(\xi)\left[\cos(n\xi)\cos(n\theta) + \sin(n\xi)\sin(n\theta)\right] d\xi \therefore$$

$$u(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta \therefore$$

$$u(r,\theta) = C + \int_0^{2\pi} G(r,\theta-\xi)\, g(\xi)\, d\xi, \quad C = \frac{A_0}{2} \therefore$$

$$G(r,\phi) = \frac{1}{\pi}\sum_{n=1}^{\infty} \frac{r^n}{n}\cos(n\phi)$$

\4a/ ∴ $\lim_{x\to\infty} u(x,t) = 0$, $\lim_{x\to-\infty} u(x,t) = 0$ $\lim_{x\to\infty} u_x(x,t) = 0$, $\lim_{x\to-\infty} u_x(x,t) = 0$

$$\therefore \frac{\partial Q(t)}{\partial t} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} u(x,t)\,dx = \int_{-\infty}^{\infty} u_t(x,t)\,dx =$$

$$\int_{-\infty}^{\infty} 3u^2 u_x + u_{xx}\,dx = \int_{-\infty}^{\infty} 3\frac{\partial u}{\partial x}(u)^2\,dx + \int_{-\infty}^{\infty}\frac{\partial}{\partial x}(u_x)\,dx =$$

$$\left[\frac{3}{3}(u)^3\right]_{-\infty}^{\infty} + \left[u_x\right]_{-\infty}^{\infty} = \left[(u)^3\right]_{-\infty}^{\infty} + \left[u_x\right]_{-\infty}^{\infty} =$$

$$\left(\lim_{x\to\infty} u(x,t)\right)^3 - \left(\lim_{x\to-\infty} u(x,t)\right)^3 + \lim_{x\to\infty} u_x(x,t) - \lim_{x\to-\infty} u_x(x,t) =$$

$$0 - 0 + 0 - 0 = 0 = \frac{dQ(t)}{dt} \quad \therefore \quad Q(0) = Q(t) \ \forall t$$

$$\frac{\partial S(t)}{\partial t} = \frac{dS}{dt} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} \frac{u^2(x,t)}{2}\,dx = \int_{-\infty}^{\infty} \frac{1}{2}\frac{\partial}{\partial t}\left[(u(x,t))^2\right] dx =$$

$$\int_{-\infty}^{\infty} \frac{1}{2}(2)u(x,t)\,u_t(x,t)\,dx = \int_{-\infty}^{\infty} u u_t\,dx =$$

$$\int_{-\infty}^{\infty} (u)(3u^2 u_x + u_{xx})\,dx = \int_{-\infty}^{\infty} 3u^3 u_x + u_{xx}u\,dx = \int_{-\infty}^{\infty} 3u_x(u)^3\,dx + \int_{-\infty}^{\infty} \left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right)u\,dx$$

$$= \left[\frac{3}{4}(u)^4\right]_{-\infty}^{\infty} + \int_{-\infty}^{\infty}\left(\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right)\right)u\,dx =$$

$$\frac{3}{4}\left(\lim_{x\to\infty} u(x,t)\right)^4 - \frac{3}{4}\left(\lim_{x\to-\infty} u(x,t)\right)^4 + \int_{-\infty}^{\infty} u_{xx}u\,dx = \int_{-\infty}^{\infty} u_{xx}u\,dx = \int_{-\infty}^{\infty} u u_{xx}\,dx$$

$$= \left[u u_x\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_x u_x\,dx = -\int_{-\infty}^{\infty}(u_x)^2\,dx = \frac{dS}{dt} \therefore$$

$$(u_x)^2 \geq 0 \therefore \int_{-\infty}^{\infty}(u_x)^2\,dx \geq 0 \therefore -\int_{-\infty}^{\infty}(u_x)^2\,dx \leq 0 \therefore$$

$$\frac{dS}{dt} \leq 0$$

\4b/ let $u(x,t) = v(x+ct)$ $\therefore$ $u_t = cv'$, $u_x = v'$ $\therefore$ $u_{xx} = v''$ $\therefore$

PDE.

$cv' = 3(v)^2 v' + v''$ $\therefore$ $0 = 3v^2 v' - cv' + v'' = (3v^2 - c)v' + v''$ $\therefore$

$0 = (3v^2 + c) + \frac{v''}{v'}$, $v(\xi) > 0 \ \forall \xi \in \mathbb{R}$ $\therefore$ $v(x+ct) > 0 \ \forall (x+ct) \in \mathbb{R}$,

$v(\xi) \to 0$ as $\xi \to -\infty$ $\therefore$ $v(x+ct) \to 0$ as $x+ct \to -\infty$ $\therefore$ $x \to -\infty$,

$v'(\xi) \to 0$ as $\xi \to -\infty$ $\therefore$ $v'(x+ct) \to 0$ as ~~$x \to -\infty$~~ $x+ct \to -\infty$ $\therefore$ $x \to -\infty$

\4a sol/ $\frac{dQ}{dt} = \frac{d}{dt}\int_{-\infty}^{\infty} u(x,t)\,dx = \int_{-\infty}^{\infty} u_t(x,t)\,dt = \int_{-\infty}^{\infty} (3u^2 u_x + u_{xx})\,dx =$

$\left[u^3 + u_x\right]_{-\infty}^{\infty} = 0$,

$\frac{dS}{dt} = \int_{-\infty}^{\infty} \partial_t\left(\tfrac{1}{2}u^2\right)dx = \int_{-\infty}^{\infty} u u_t\,dx = \int_{-\infty}^{\infty} u(3u^2 u_x + u_{xx})\,dx =$

$3\int_{-\infty}^{\infty} u^3 u_x\,dx + \int_{-\infty}^{\infty} u u_{xx}\,dx = \left[u^4 + u u_x\right]_{-\infty}^{\infty} - \int_{-\infty}^{\infty} u_x^2\,dx - \int_{-\infty}^{\infty} u_x^2\,dx \leq 0$

\4b sol/ $cv' = 3v^2 v' + v''$ $\therefore$

$v'' + (3v^2 - c)v'$

$\therefore$ use reduction of order substitution $\frac{dv}{d\xi} = P(v) = p = v'$ $\therefore$

$\frac{d^2v}{d\xi^2} = \frac{d}{d\xi}\left(\frac{dv}{d\xi}\right) = \frac{d}{d\xi}(P(v)) = \frac{dP(v)}{dv}\frac{dv}{d\xi} = \frac{dP(v)}{dv}P(v) = P(v)\frac{dP(v)}{dv} = p\frac{dp}{dv} = v''$

$p\frac{dp}{dv} + p(3v^2 - c) = 0$

can be satisfied if $p = 0$ $\therefore$ $\frac{dv}{d\xi} = 0$ $\therefore$ $v = \text{Constant}$;

asymptotic conditions then require $v(\xi) \equiv 0$ but we need $v > 0$

$\therefore$ $\frac{dp}{dv} = c - 3v^2$ $\therefore$ $p = cv - v^3 + A = \frac{dv}{d\xi} = A + cv - v^3$, $A = \text{constant}$

$\therefore$ requires $p \to 0$ simultaneously with $v \to 0$ $\therefore$ must choose $A = 0$

$\therefore$ $p = \frac{dv}{d\xi} = cv - v^3$ $\therefore$ $\frac{\left(\frac{dv}{d\xi}\right)}{cv - v^3} = \frac{1}{v(c - v^2)}\frac{dv}{d\xi} = 1$ $\therefore$

$\int \frac{1}{v(c-v^2)}\,dv = \int 1\,d\xi = \xi + C_2$, $\therefore$ let $\sqrt{c} = \alpha$ $\therefore$

$\frac{1}{v(c-v^2)} = \frac{1}{v(\alpha+v)(\alpha-v)} = \frac{-1}{v(v+\alpha)(v-\alpha)}$ $\therefore$

$\xi + C_2 = \int \frac{1}{v(c-v^2)}\,dv = \int \frac{-1}{v(v+\alpha)(v-\alpha)}\,dv = \frac{1}{2\alpha^2}\int\left[\frac{2}{v} - \frac{1}{v+\alpha} - \frac{1}{v-\alpha}\right]dv$

$= \frac{1}{2\alpha^2}\ln\left|\frac{v^2}{v^2 - \alpha^2}\right| = \xi + \beta$ $\therefore$ $v = \frac{\alpha}{\sqrt{1 \mp e^{2\alpha^2(\beta - \xi)}}}$ $\therefore$ to be bounded $\forall \xi$

$u(x,t) = \left\{\frac{c}{1 + \exp[-2c(x+ct-\beta)]}\right\}^{1/2}$ $\therefore$ $u(-\infty, t) = \sqrt{c} > 0$ $\therefore$ $u(+\infty, t) \neq 0$ $\therefore$ doesn't satisfy assumptions for S, Q

VP2022 \1ai/ $\therefore (2xy, 1)\cdot(u_x, u_y) = 0 = (2xy, 1)\cdot\nabla u \therefore$

char curves satisfy

$\frac{dy}{dx} = \frac{1}{2xy} \therefore y\frac{dy}{dx} = \frac{1}{2x} \therefore$

$\int y\,dy = \int \frac{1}{2x}\,dx = \frac{1}{2}y^2 = \frac{1}{2}\int \frac{1}{x}\,dx = \frac{1}{2}\ln|x| + C_1 \therefore$

$y^2 = \ln|x| + C_2 \therefore$

$y^2 - \ln|x| = C_2$ is characteristic equation $\therefore$

$u(x,y) = f(y^2 - \ln|x|)$

$u(x,0) = \sin(x) = f(0^2 - \ln|x|) = f(-\ln|x|) \therefore$

let $s = -\ln|x| \therefore -s = \ln|x| \therefore e^{-s} = e^{\ln|x|} = |x| \therefore$

$x = \pm e^{-s} \therefore$

$f(s) = \sin(\pm e^{-s}) \therefore$

$f(y^2 - \ln|x|) = \sin(\pm(y^2 - \ln|x|)) = u(x,y)$

\1aii/ $u(0,y) = \cos(y) = f(y^2 - \ln|0|)$

which is impossible $\because \ln|0|$ is undefined $\therefore$

no solution exists $\therefore$

the BC is on the characteristic equation

\1b/ $2u_{xx} + 3u_{xt} + u_{tt} = au_{xx} + 2bu_{xt} + cu_{tt} \therefore$

$a = 2,\ 2b = 3 \therefore -2b = -3 \therefore b = \frac{3}{2},\ c = 1 \therefore$

$a\left(\frac{dx}{dt}\right)^2 - 2b\frac{dx}{dt} + c = 2\left(\frac{dx}{dt}\right)^2 - 3\frac{dx}{dt} + 1 = 0$

$\therefore b^2 - 4ac = \left(\frac{3}{2}\right)^2 - 4(2)1 = -4.75 < 0 \therefore$ equation is elliptic $\therefore$

$\frac{dx}{dt} = \frac{3 \pm \sqrt{9 - 4(2)1}}{2(2)} = \frac{3 \pm \sqrt{-4.75}}{4} = \frac{3}{4} \pm \frac{\sqrt{19}}{8}i \therefore$

$\int \frac{dx}{dt}\,dt = \int 1\,dx = x = \int \frac{3}{4} \pm \frac{\sqrt{19}}{8}i\,dt = \left(\frac{3}{4} \pm \frac{\sqrt{19}}{8}i\right)t + C_1 \therefore$

$x - \left(\frac{3}{4} \pm \frac{\sqrt{19}}{8}i\right)t = C_1 \therefore$

$\alpha = x - \left(\frac{3}{4} + \frac{\sqrt{19}}{8}i\right)t,\quad \beta = x + \left(\frac{3}{4} - \frac{\sqrt{19}}{8}i\right)t \therefore$

$Re(\alpha) = Re\left(x - \frac{3}{4}t - \frac{\sqrt{19}}{8}ti\right) = x - \frac{3}{4}t = \xi$

$Im(\alpha) = Im\left(x - \frac{3}{4}t - \frac{\sqrt{19}}{8}ti\right) = -\frac{\sqrt{19}}{8}t \therefore$

$-Im(\alpha) = \frac{\sqrt{19}}{8}t = \eta \therefore$

$\frac{8}{\sqrt{19}}\eta = t \therefore x = \xi + \frac{3}{4}t = \xi + \frac{3}{4}\times\frac{8}{\sqrt{19}}\eta = \frac{6}{\sqrt{19}}\eta + \xi \therefore$

$x - t = \frac{6}{\sqrt{19}}\eta + \xi - \frac{8}{\sqrt{19}}\eta = \frac{-2}{\sqrt{19}}\eta + \xi \therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x$

$\therefore \xi_x = 1, \quad \xi_t = -\frac{3}{4}, \quad \eta_x = 0, \quad \eta_t = \frac{\sqrt{19}}{8} \quad \therefore$

$u_x = u_\xi \quad \therefore$

$u_{xx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x = u_{\xi\xi}$

$u_{xt} = u_{\xi\xi}\xi_t + u_{\xi\eta}\eta_t = -\frac{3}{4}u_{\xi\xi} + \frac{\sqrt{19}}{8}u_{\xi\eta} \quad \therefore$

$u_t = u_\xi \xi_t + u_\eta \eta_t = -\frac{3}{4}u_\xi + \frac{\sqrt{19}}{8}u_\eta \quad \therefore$

$u_{tt} = -\frac{3}{4}u_{\xi\xi}\xi_t - \frac{3}{4}u_{\xi\eta}\eta_t + \frac{\sqrt{19}}{8}u_{\xi\eta}\xi_t + \frac{\sqrt{19}}{8}u_{\eta\eta}\eta_t =$

$\frac{9}{16}u_{\xi\xi} - \frac{3\sqrt{19}}{32}u_{\xi\eta} - \frac{3\sqrt{19}}{32}u_{\xi\eta} + \frac{19}{64}u_{\eta\eta} = \frac{9}{16}u_{\xi\xi} - \frac{3\sqrt{19}}{16}u_{\xi\eta} + \frac{19}{64}u_{\eta\eta} \quad \therefore$

$2u_{xx} + 3u_{xt} + u_{tt} =$

$2u_{\xi\xi} - \frac{9}{4}u_{\xi\xi} + \frac{3\sqrt{19}}{8}u_{\xi\eta} + \frac{9}{16}u_{\xi\xi} - \frac{3\sqrt{19}}{16}u_{\xi\eta} + \frac{19}{64}u_{\eta\eta} =$

$2u_{\xi\xi} - \frac{9}{4}u_{\xi\xi} + \frac{9}{16}u_{\xi\xi} + \frac{19}{64}u_{\eta\eta} =$

$\frac{5}{16}u_{\xi\xi} + \frac{19}{64}u_{\eta\eta} = 3\left(\frac{-2}{\sqrt{19}}\eta + \xi\right)^2 = 3\left[\frac{4}{19}\eta^2 + \xi^2 - \frac{4}{\sqrt{19}}\xi\eta\right] =$

$\frac{12}{19}\eta^2 + 3\xi^2 - \frac{12}{\sqrt{19}}\xi\eta = \frac{5}{16}u_{\xi\xi} + \frac{19}{64}u_{\eta\eta} \quad \therefore$

~~let~~

V a i $\therefore (2xy, 1)\cdot \nabla u = 0 \quad \therefore$ charac curves satisfy:

$\frac{dy}{dx} = \frac{1}{2xy} \quad \therefore \quad \int y\,dy = \frac{1}{2}\int \frac{1}{x}dx = \frac{1}{2}y^2 = \frac{1}{2}\ln|x| + C_1 \quad \therefore$

$y^2 = \ln|x| + C_2 \quad \therefore \quad e^{y^2} = e^{\ln|x| + C_2} = C_3 \cancel{x} \quad \therefore \quad \frac{e^{y^2}}{x} = C_3$ is charac curve $\therefore$

$u(x,y) = \tilde{f}\left(\frac{e^{y^2}}{x}\right) = f\left(\left(\frac{e^{y^2}}{x}\right)^{-1}\right) = f(xe^{-y^2})$

$u(x,0) = \sin(x) = f\left(\frac{e^{0^2}}{x}\right) = f\left(\frac{e^0}{x}\right) = f\left(\frac{1}{x}\right) = \sin(x) \quad \therefore$

~~$\frac{1}{x} = 8$~~ $\therefore$ let $s = \frac{1}{x} \quad \therefore \quad x = \frac{1}{s} \quad \therefore$

$f(s) = \sin\left(\frac{1}{s}\right) \quad \therefore \quad f\left(\frac{e^{y^2}}{x}\right) = \sin\left(\frac{x}{e^{y^2}}\right) = u(x,y) = \sin(xe^{-y^2})$

V a ii $u(0,y) = \cos(y) = f\left(\frac{e^{y^2}}{0}\right)$ which is impossible

$\therefore$ no solution exists $\therefore$ BC is on charac curve

$\therefore$ line $x = 0$ is one of the characteristics corres to $C = 0$

\PP2022/ \1b/ Try $u(x,t)=f(x+\alpha t)$ for the homog version of PDE $\therefore$

$u_x=f'(x+\alpha t)$, $u_t=\alpha f'(x+\alpha t)$, $u_{xx}=f''$, $u_{tt}=\alpha^2 f''$, $u_{xt}=\alpha f''$ $\therefore$

$2f''+3\alpha f''+\alpha^2 f''=0=(\alpha^2+3\alpha+2)f''=0=\alpha^2+3\alpha+2=$

$(\alpha+2)(\alpha+1)$ $\therefore$ $\alpha=-2$, $\alpha=-1$ $\therefore$

~~ff~~ $\underbrace{f=f(x-2t),\ f=f(x-t)}$ $\therefore$

~~u_{cf}~~ complementary funct: $u_{cf}=f(x-2t)+f(x-t)$ $\therefore$

~~for PI~~ $\therefore$ let ~~$\xi=x-2t$, $\eta=x-t$~~ $\xi=x-t$, $\eta=x-2t$ $\therefore$

~~u_x~~ $\xi_x=1$, $\xi_t=-1$, $\eta_x=1$, $\eta_t=-2$ $\therefore$

$u_x=u_\xi\xi_x+u_\eta\eta_x=u_\xi+u_\eta$,

$u_{xx}=u_{\xi\xi}\xi_x+u_{\xi\eta}\eta_x+u_{\xi\eta}\xi_x+u_{\eta\eta}\eta_x=u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}$ $\therefore$

$u_{xt}=u_{\xi\xi}\xi_t+u_{\xi\eta}\eta_t+u_{\xi\eta}\xi_t+u_{\eta\eta}\eta_t=-u_{\xi\xi}-2u_{\xi\eta}-u_{\xi\eta}-2u_{\eta\eta}=$

$-u_{\xi\xi}-3u_{\xi\eta}-2u_{\eta\eta}$,

~~u_{tt}~~ $u_t=u_\xi\xi_t+u_\eta\eta_t=-u_\xi-2u_\eta$

$u_{tt}=-u_{\xi\xi}\xi_t-u_{\xi\eta}\eta_t-2u_{\xi\eta}\xi_t-2u_{\eta\eta}\eta_t=u_{\xi\xi}+2u_{\xi\eta}+2u_{\xi\eta}+4u_{\eta\eta}=$

$u_{\xi\xi}+4u_{\xi\eta}+4u_{\eta\eta}$ $\therefore$

$\xi+t=x$ $\therefore$ $\eta-x=-2t=\eta-\xi-t=-2t$ $\therefore$ $\eta-\xi=-t$ $\therefore$ $t=\xi-\eta$ $\therefore$

$\xi=x+\eta-\xi$ $\therefore$ $2\xi=x+\eta$

$\xi+\xi-\eta=x=2\xi-\eta$ $\therefore$

$x-t=2\xi-\eta+\eta-\xi=\xi$ $\therefore$ PDE:

$2u_{xx}+3u_{xt}+u_{tt}=$

$2u_{\xi\xi}+4u_{\xi\eta}+2u_{\eta\eta}-3u_{\xi\xi}-9u_{\xi\eta}-6u_{\eta\eta}+u_{\xi\xi}+4u_{\xi\eta}+4u_{\eta\eta}=$

$(2-3+1)u_{\xi\xi}+(4-9+4)u_{\xi\eta}+(2-6+4)u_{\eta\eta}=$

$-u_{\xi\eta}=3(\xi)^2=3\xi^2$ $\therefore$

$u_{\xi\eta}=3\xi^2=\frac{\partial}{\partial\xi}\left(\frac{\partial}{\partial\eta}u\right)$ $\therefore$

$\frac{\partial}{\partial\eta}u=\xi^3+\tilde{h}(\eta)$ $\therefore$

$u=\xi^3\eta+h(\eta)+g(\xi)$ $\therefore$; h, g are arbit funcs

$u(x,t)=(x-t)^3(x-2t)+h(x-2t)+g(x-t)$

$\therefore u(x,x)=x=(x-x)^3(x-2x)+h(x-2x)+g(x-x)=0^3(-x)+h(-x)+g(0)=h(-x)+g(0)$

$u(2t,t)=t=(2t-t)^3(2t-2t)+h(2t-2t)+g(2t-t)=t^3(0)+h(0)+g(t)=h(0)+g(t)$ $\therefore$

$g(t) = t - h(0)$ ∴ let $h(0) = A = \text{constant}$ ∴

$g(s) = s - h(0) = s - A$ ∴

~~$g(x-2t) = x-2t-A$~~ $g(x-t) = x-t-A$ ∴

~~$h(-2x) =$~~ $g(0) = 0 - A = -A$ ∴

$h(-x) = x - g(0) = x + A$ ∴

let $s = -x$ ∴ $h(s) = -s + A$ ∴

$h(x-2t) = -x + 2t + A$ ∴

$u(x,t) = (x-2t)^3(x-t) - x + 2t + A + x - t - A =$

$(x-2t)^3(x-t) + t$ close enough

\1b/ let $u = \delta(x + \alpha t)$ be sol to homog eqn ∴

$u_{xx} = \delta''$, $u_{xt} = \alpha\delta''$, $u_{xx} = \alpha^2\delta''$ ∴

$2\delta'' + 3\alpha\delta'' + \alpha^2\delta'' = 0 = (\alpha^2 + 3\alpha + 2)\delta'' = 0 = \alpha^2 + 3\alpha + 2 = (\alpha+2)(\alpha+1) = 0$

∴ $\alpha = -1$, $\alpha = -2$ ∴

$\xi = x - t$, $\eta = x - 2t$

\1b/ $au_{xx} + 2bu_{xt} + cu_{tt} = 2u_{xx} + 3u_{xt} + u_{tt}$ ∴

$a = 2$, $2b = 3$ ∴ $b = \frac{3}{2}$ $-2b = -3$, $c = 1$ ∴

$D = b^2 - ac = \left(\frac{3}{2}\right)^2 - 2(1) = \frac{1}{4} > 0$ ∴ hyperbolic ∴

charac eqn: $a\left(\frac{dx}{dt}\right)^2 - 2b\frac{dx}{dt} + c = 0 = 2\left(\frac{dx}{dt}\right)^2 - 3\frac{dx}{dt} + 1 = 0$ ∴

$\frac{dx}{dt} = \frac{3 \pm \sqrt{9 - 4(2)(1)}}{2(2)} = \frac{3 \pm \sqrt{1}}{4} = \frac{3 \pm 1}{4}$ ∴ $\frac{dx}{dt} = 1$, $\frac{dx}{dt} = \frac{1}{2}$ ∴

$x = t + C_1$, $x = \frac{1}{2}t + C_2$ ∴ $2x = t + C_3$ ∴ $2x - t = C_3$, $x - t = C_1$

∴ charac coords: $\xi = 2x - t$, $\eta = x - t$

\1c/ let $u = ve^{pt}$ ∴ $u_t = (v_t + pv)e^{pt}$, $u_x = v_x e^{pt}$, $u_{xx} = v_{xx}e^{pt}$

∴ $0 = u_{xx} - \frac{u}{2} - u_t = v_{xx}e^{pt} - \frac{1}{2}ve^{pt} - (v_t + pv)e^{pt} =$

$e^{pt}\left(v_{xx} - \frac{1}{2}v - v_t - pv\right) =$ $0 = v_{xx} - \frac{1}{2}v - v_t - pv =$

$0 = v_{xx} - v_t + v\left(-\frac{1}{2} - p\right)$ ∴

let $-\frac{1}{2} - p = 0$ ∴ $p = -\frac{1}{2}$ ∴

$0 = v_{xx} - v_t$ ∴ $v_t = v_{xx}$

$u(x,0) = \psi(x) = v(x,0)e^{p(0)} = v(x,0)e^0 = v(x,0) = \psi(x) = \delta(x-2)$ ∴

PP2022 / $v_t - v_{xx} = 0,\quad v(x,0) = \psi(x)$ $\therefore$

$$v(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \psi(\xi)\, e^{-\frac{(x-\xi)^2}{4t}}\, d\xi$$

i) $\psi(x) = \delta(x-2)$ $\therefore$ $\psi(\xi) = \delta(\xi - 2)$ $\therefore$

$$u = v e^{pt} = u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{pt} \int_{-\infty}^{\infty} \psi(\xi)\, e^{-\frac{(x-\xi)^2}{4t}}\, d\xi =$$

$$\frac{1}{\sqrt{4\pi t}} e^{pt} \int_{-\infty}^{\infty} \delta(\xi-2)\, e^{-\frac{(x-\xi)^2}{4t}}\, d\xi = \frac{1}{\sqrt{4\pi t}} e^{pt} e^{-\frac{(x-2)^2}{4t}},\quad p = -\frac{1}{2}$$

$$\therefore P(t) = u(0,t) = \frac{1}{\sqrt{4\pi t}} e^{pt} e^{-\frac{(0-2)^2}{4t}} = \frac{1}{\sqrt{4\pi t}} e^{pt} e^{-\frac{1}{t}} = \frac{1}{2} \pi^{-1/2} t^{-1/2} e^{pt - \frac{1}{t}} \therefore$$

$$\ln P(t) = \ln\left(\tfrac{1}{2}\right) + \ln\left(\pi^{-1/2}\right) + \ln\left(t^{-1/2}\right) + pt - \frac{1}{t} = \ln\left(\tfrac{1}{2}\right) - \tfrac{1}{2}\ln(\pi) - \tfrac{1}{2}\ln t + pt - \frac{1}{t} \therefore$$

$$\left.\frac{d(\ln P(t))}{dt}\right|_{t=t_*} = -\frac{1}{2}\frac{1}{t} + p + \left.\frac{1}{t^2}\right|_{t=t_*} = 0 = -\frac{1}{2}\frac{1}{t_*} + p + \frac{1}{t_*^2} \therefore$$

$$0 = -\frac{1}{t_*} + 2p + \frac{2}{t_*^2} \therefore$$

$2pt_*^2 - t_* + 2 = 0$ $\therefore$ $2p = 2(-\tfrac{1}{2}) = -1$ $\therefore$

$-t_*^2 - t_* + 2 = 0$ $\therefore$ $t_*^2 + t_* - 2 = 0$

$$t_* = \frac{-1 \pm \sqrt{1^2 - 4(1)(-2)}}{2(1)} = \frac{-1 \pm \sqrt{9}}{2}$$

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{2}t} \int_{-\infty}^{\infty} \psi(\xi)\, e^{-\frac{(x-\xi)^2}{4t}}\, d\xi$$

$$u(x,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{2}t} e^{-\frac{(x-2)^2}{4t}} \therefore$$

$$P(t) = u(0,t) = \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{2}t - \frac{(0-2)^2}{4t}} = \frac{1}{\sqrt{4\pi t}} e^{-\frac{1}{2}t - \frac{1}{t}} \therefore$$

$$\ln P(t) = \ln\left(\frac{1}{\sqrt{4\pi}}\right) + \ln\left(t^{-1/2}\right) + -\frac{1}{2}t - \frac{1}{t} = \ln\left(\frac{1}{\sqrt{4\pi}}\right) - \frac{1}{2}\ln t - \frac{1}{2}t - \frac{1}{t} \therefore$$

$$\left.\frac{d\ln P(t)}{dt}\right|_{t=t_*} = -\frac{1}{2}\frac{1}{t} - \frac{1}{2} + \left.\frac{1}{t^2}\right|_{t=t_*} = -\frac{1}{2}\frac{1}{t_*} - \frac{1}{2} + \frac{1}{t_*^2} = 0 \therefore$$

$$\frac{1}{2}\frac{1}{t_*} + \frac{1}{2} - \frac{1}{t_*^2} = 0 = \frac{1}{2} + \frac{1}{2}\frac{1}{t_*} - \frac{1}{t_*^2} = 0 = 1 + \frac{1}{t_*} - \frac{2}{t_*^2}$$

$$t_*^2 - t_* - 2 = 0 \therefore t_* = \frac{1 \pm \sqrt{1^2 - 4(1)(-2)}}{2(1)} = \frac{1 \pm \sqrt{9}}{2} = \frac{1 \pm 3}{2} = -1, 2$$

$\therefore t_* > 0$ $\therefore$ $t_* = 2$ for Max $P(t)$ close enough

1d / Let $u = R(r)\Theta(\theta)$ $\therefore$ $u_r = R'\Theta$, $u_{rr} = R''\Theta$, $u_{\theta\theta} = R\Theta''$ $\therefore$

$$R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \therefore R''\Theta + \frac{1}{r}R'\Theta = -\frac{1}{r^2}R\Theta'' \therefore$$

$$R'' + \frac{1}{r}R' = -\frac{1}{r^2}R\frac{\Theta''}{\Theta} \therefore r^2R'' + rR' = -R\frac{\Theta''}{\Theta} \therefore$$

$$r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = +\lambda = \text{constant} \therefore$$

$r^2R'' = \lambda R \quad r^2R'' + rR' = \mp\lambda R \quad , \Theta'' = \mp\lambda\Theta \quad \therefore$

$r^2R'' + rR' \mp \lambda R = 0 \quad \therefore$

let $R = r^m \quad , \quad R' = mr^{m-1} \quad \therefore R'' = m(m-1)r^{m-2} = (m^2 - m)r^{m-2} \quad \therefore$

$r^2(m^2 - m)r^{m-2} + mr\,r^{m-1} \mp \lambda r^m = 0 = r^m[m^2 - m + m \mp \lambda] = r^m[m^2 \mp \lambda] = 0 \quad \therefore$

$\therefore m^2 - \lambda = 0 \quad \therefore \quad m^2 = \lambda \quad \therefore \quad m = \pm\sqrt{\lambda} \quad \therefore m = \sqrt{\lambda} \quad , m = -\sqrt{\lambda} \quad \therefore$

$R = Ar^{\sqrt{\lambda}} + Br^{-\sqrt{\lambda}} \quad ,$

$\Theta'' + \lambda\Theta = 0 \quad \therefore \quad \Theta(0) = \Theta(2\pi) \quad ,$

$\Theta'(0) = \Theta'(2\pi) \quad \therefore$

$\lambda = n^2 \quad , n = 0, 1, 2, \ldots \quad , \quad \phi_0(\theta) = 1 \quad , \phi_n^{(1)}(\theta) = \cos(n\theta) , \phi_n^{(2)}(\theta) = \sin(n\theta)$

for $n > 0$:

$R = Ar^{\sqrt{n^2}} + Br^{-\sqrt{n^2}} = Ar^n + Br^{-n} = R(r) \quad \therefore$

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} (Ar^n + Br^{-n})(\cos(n\theta) + \sin(n\theta))$$

~~\|d sol/~~ ~~for~~ for $m = \pm\sqrt{\lambda} = \pm\sqrt{n^2} = \pm n \quad \therefore m = n , m = -n$:

for $\lambda = 0$; $R(r) = C_1 r^0 + C_2 \ln(r) = C_1 + C_2 \ln(r)$

$\left|\lim_{r\to 0} u(r,\theta)\right| < \infty \quad \therefore \left|\lim_{r\to 0} R(r)\right| < \infty \quad \therefore \quad R(r) = C_1 \quad \therefore$

satisfy the condition $\lim_{r\to 0} R(r) = 0$ for $C_2 = 0$, arbit C_1

for $\lambda = n^2 > 0$: $\quad R(r) = C_1 r^n + C_2 r^{-n}$

which satisfy the condition at the centre is $C_2 = 0$, arbit C_1

$\therefore \left|\lim_{r\to 0} R(r)\right| < \infty = \left|\lim_{r\to 0} C_1 r^n + C_2 r^{-n}\right| = \left|\lim_{r\to 0} C_1 r^n + \frac{C_2}{r^n}\right| = \left|\lim_{r\to 0} C_1 r + \frac{C_2}{r}\right| \quad \therefore$

C_1 arbit $\quad , C_2 = 0 \quad \therefore$

$R(r) = C_1 r^n$

$\therefore$ for $\lambda = 0$: $Q = A_0 = \text{Const} \quad , \text{ for } \lambda = n^2 > 0 : Q = A_n\cos(n\theta) + B_n\sin(n\theta)$

$\therefore u = R(r)Q(\theta) \quad \therefore \quad |u(0,\theta)| < \infty \quad \therefore$

$$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^n[A_n\cos(n\theta) + B_n\sin(n\theta)]$$

$$u(1,\theta) = u(r=1,\theta) = A_0 + \sum_{n=1}^{\infty} (1)^n[A_n\cos(n\theta) + B_n\sin(n\theta)] =$$

$A_0 + \sum_{n=1}^{\infty} A_n\cos(n\theta) + B_n\sin(n\theta) = f(\theta) \quad \therefore \qquad \theta \in [0, 2\pi] \quad \therefore 2L = 2\pi \quad \therefore L = \pi \quad \therefore$

for $A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta$, $\quad A_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta$, $B_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta \quad \therefore$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta + \sum_{n=1}^{\infty} r^n\left[\frac{1}{\pi}\int_0^{2\pi} f(\xi)\cos(n\xi)\,d\xi\,\cos(n\theta) + \frac{1}{\pi}\int_0^{2\pi} f(\xi)\sin(n\xi)\,d\xi\,\sin(n\theta)\right]$$

\PP2022/ $\therefore$ at $r=0$: $r^n=0$ $\therefore$

$$u(\cancel{r=0,\theta}) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta + \sum_{n=1}^{\infty}(0) = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} e^{\cos(\theta)}|\sin(\theta)|\,d\theta =$$

$$\frac{1}{2\pi}\left[\int_0^{\pi} e^{\cos(\theta)}|\sin(\theta)|\,d\theta + \int_{\pi}^{2\pi} e^{\cos(\theta)}|\sin(\theta)|\,d\theta\right] =$$

$$\frac{1}{2\pi}\left[\int_0^{\pi} e^{\cos(\theta)}\sin(\theta)\,d\theta + \int_{\pi}^{2\pi} e^{\cos(\theta)}(-1)\sin(\theta)\,d\theta\right] =$$

$$\frac{1}{2\pi}\left[-\int_0^{\pi} -\sin(\theta)e^{\cos\theta}\,d\theta + \int_{\pi}^{2\pi} -\sin\theta\, e^{\cos\theta}\,d\theta\right] =$$

$$\frac{1}{2\pi}\left(-\left[e^{\cos\theta}\right]_{\theta=0}^{\pi} + \left[e^{\cos\theta}\right]_{\theta=\pi}^{2\pi}\right) =$$

$$\frac{1}{2\pi}\left(-\left[e^{\cos\pi}-e^{\cos 0}\right] + \left[e^{\cos 2\pi}-e^{\cos\pi}\right]\right) =$$

$$\frac{1}{2\pi}\left(-\left[e^{-1}-e^{1}\right] + \left[e^{1}-e^{-1}\right]\right) = \frac{1}{2\pi}\left(-e^{-1}+e+e-e^{-1}\right) = \frac{1}{2\pi}\left(2e^{1}-2e^{-1}\right) =$$

$$\left(\frac{2}{\pi}\right)\frac{1}{2}\left(e^{1}-e^{-1}\right) = \frac{2}{\pi}\sinh(1) = \frac{e-e^{-1}}{\pi} = \frac{e^2-1}{\pi e}$$

$$A_0 = \frac{1}{2\pi}\int_0^{2\pi} \cancel{\cos}\, e^{\cos\theta}|\sin\theta|\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{\cos\theta}|\sin\theta|\,d\theta = \frac{1}{\pi}\int_0^{\pi} e^{\cos\theta}\sin\theta\,d\theta\Big|_{x=\cos\theta} =$$

$$\frac{1}{\pi}\int_{-1}^{1} e^{x}\,dx = \frac{1}{\pi}\left(e-\frac{1}{e}\right) = \frac{e^2-1}{\pi e}$$

\2ai/ $u_{xx}-4yu_{yy}-2u_y=0$ $\therefore$

$u_{xx}-4yu_{yy} = au_{xx}+2bu_{xy}+cu_{yy}$ $\therefore$

$a=1$, $2b=0=-2b=b=0$ $\therefore$ $c=-4y$ $\therefore$

$D=b^2-ac=0^2-1(-4y)=4y$ $\therefore$

for $4y>0$ $\therefore$ $y>0$ is hyperbolic, eithe upper half plane

for $4y=0$ $\therefore$ $y=0$ is parabolic. on lines $y=0$.

~~so~~ $\therefore$ for $y<0$ equation is elliptic lower half plane

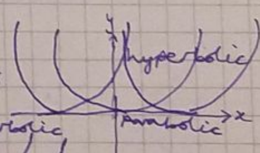
\2aii/ $\therefore$ $y>0$ $\therefore$ $a\left(\frac{dy}{dx}\right)^2-2b\left(\frac{dy}{dx}\right)+c=\left(\frac{dy}{dx}\right)^2-4y=0$ $\therefore$ $\left(\frac{dy}{dx}\right)^2=4y$ $\therefore$

$\frac{dy}{dx}=\pm 2y^{1/2}$ $\therefore$ $y^{-1/2}\frac{dy}{dx}=2$, $y^{-1/2}\frac{dy}{dx}=-2$ $\therefore$

$\int y^{-1/2}\,dy=\int 2\,dx = 2y^{1/2}=2x+C_1$, $\int y^{-1/2}\,dy=\int -2\,dx=2y^{1/2}=-2x+C_2$ $\therefore$

$y^{1/2}=x+C_3$, $y^{1/2}=-x+C_4$ $\therefore$

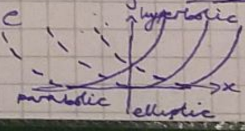
$y^{1/2}-x=C_3$, $y^{1/2}+x=C_4$ are charac curves



$\left(\frac{dy}{dx}\right)^2-4y=0$ $\therefore$ $y>0$: $\frac{dy}{dx}=\pm 2y^{1/2}$ $\therefore$ $\pm y^{1/2}=x-c$

$y=(x-c)^2$ is explicit charac curves $\therefore$

two families distinguished by sign of $y'(x)$



\2b/ $\therefore \xi_x = 1,\ \xi_y = -\frac{1}{2}y^{-1/2},\ \eta_x = 1,\ \eta_y = \frac{1}{2}y^{-1/2} \therefore$

$u_x = u_\xi \xi_x - u_\eta \eta_x = u_\xi - \frac{1}{2}y^{-1/2}u_\eta$,

$u_{xx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x - \frac{1}{2}y^{-1/2}u_{\xi\eta}\xi_x - \frac{1}{2}y^{-1/2}u_{\eta\eta}\eta_x$

$u_{\xi\xi}$

\2b/ $\therefore \xi_x = 1,\ \xi_y = -\frac{1}{2}y^{-1/2},\ \eta_x = 1,\ \eta_y = \frac{1}{2}y^{-1/2} \therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta \therefore$

$u_{xx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} \therefore$

$u_y = u_\xi \xi_y + u_\eta \eta_y = -\frac{1}{2}y^{-1/2}u_\xi + \frac{1}{2}y^{-1/2}u_\eta \therefore$

$u_{yy} = -\frac{1}{2}(-\frac{1}{2})y^{-3/2}u_\xi - \frac{1}{2}y^{-1/2}u_{\xi\xi}\xi_y - \frac{1}{2}y^{-1/2}u_{\xi\eta}\eta_y + \frac{1}{2}(-\frac{1}{2})y^{-3/2}u_\eta + \frac{1}{2}y^{-1/2}u_{\xi\eta}\xi_y + \frac{1}{2}y^{-1/2}u_{\eta\eta}\eta_y =$

$\frac{1}{4}y^{-3/2}u_\xi - \frac{1}{2}y^{-1/2}u_{\xi\xi}(-\frac{1}{2}y^{-1/2}) - \frac{1}{2}y^{-1/2}(\frac{1}{2})y^{-1/2}u_{\xi\eta} - \frac{1}{4}y^{-3/2}u_\eta + \frac{1}{2}y^{-1/2}u_{\xi\eta}(-\frac{1}{2}y^{-1/2}) + \frac{1}{2}y^{-1/2}u_{\eta\eta}\frac{1}{2}y^{-1/2} =$

$\frac{1}{4}y^{-3/2}u_\xi + \frac{1}{4}y^{-1}u_{\xi\xi} - \frac{1}{4}u_{\xi\eta} - \frac{1}{4}y^{-3/2}u_\eta - \frac{1}{4}y^{-1}u_{\xi\eta} + \frac{1}{4}y^{-1}u_{\eta\eta} =$

$\frac{1}{4}y^{-1}u_{\xi\xi} - \frac{1}{2}y^{-1}u_{\xi\eta} + \frac{1}{4}y^{-1}u_{\eta\eta} + \frac{1}{4}y^{-3/2}u_\xi - \frac{1}{4}y^{-3/2}u_\eta \therefore$

PDE: $u_{xx} - 4yu_{yy} - 2u_y =$

$\underline{u_{\xi\xi}} + 2u_{\xi\eta} + \underline{\underline{u_{\eta\eta}}} - \underline{u_{\xi\xi}} + 2u_{\xi\eta} - \underline{\underline{u_{\eta\eta}}} - y^{-1/2}u_\xi + y^{-1/2}u_\eta + y^{-1/2}u_\xi - y^{-1/2}u_\eta$

$= 2u_{\xi\eta} + 2u_{\xi\eta} = 4u_{\xi\eta} = 0 = u_{\xi\eta} = \frac{\partial}{\partial \xi}(\frac{\partial}{\partial \eta}u) = 0 \therefore$

$\frac{\partial}{\partial \eta}u = \tilde{f}(\eta) \therefore$

$u = f(\eta) + g(\xi) = u(x,y) = f(x+y^{1/2}) + g(x-y^{1/2})$

$\therefore u(0,y) = f(0+y^{1/2}) + g(0-y^{1/2}) = f(y^{1/2}) + g(-y^{1/2}) = 0$,

$u_x = f'(x+y^{1/2}) + g'(x-y^{1/2}) \therefore$

$u_x(0,y) = f'(y^{1/2}) + g'(-y^{1/2}) = 3y \therefore$

let $y^{1/2} = s \therefore\ y = s^2 \therefore\ -y^{1/2} = -s \therefore$

$f'(s) + g'(-s) = 3s^2 \therefore$

$f(s) - g(-s) = s^3 + A$, $A = \text{const} \therefore$

$f(s) + g(-s) = 0 \quad \therefore \quad f(s) = -g(-s) \therefore$

$f(s) + f(s) = 2f(s) = s^3 + A \quad \therefore \quad f(s) = \frac{1}{2}s^3 + B$, $B = \text{Const} \therefore$

$f(x+y^{1/2}) = \frac{1}{2}(x+y^{1/2})^3 + B \therefore$

$g(-s) = -f(s) = -\frac{1}{2}s^3 - B \quad \therefore \quad g(s) = -\frac{1}{2}(-s)^3 - B = \frac{1}{2}s^3 - B \therefore$

$g(x-y^{1/2}) = \frac{1}{2}(x-y^{1/2})^3 - B \therefore$

$u(x,y) = \frac{1}{2}(x+y^{1/2})^3 + B + \frac{1}{2}(x-y^{1/2})^3 - B = \frac{1}{2}(x+y^{1/2})^3 + \frac{1}{2}(x-y^{1/2})^3$

\PP2022/\1a/ The BVP has a solution and is uniquely defined for BC not on the characteristic curves ∴

i) $u(0,y)=0$ is BC on lines $x\neq 0$

$u_x(0,y)=3y$ is BC on lines $x=0$ ∴

uniquely defined for $x\neq 0$ X

\2a/ The BCs are set on the half line $x=0$, $y>0$ and its sol is unique in the domain that is covered by both families of charac curves crossing this half line.

∴ domain is defined by $y>x^2$

\4a/ ∴ $u_t(\pm\infty,t)=0$, $u_x(\pm\infty,t)=0$ ∴ $u_{tt}=u_{xx}-\sin u$ ∴

$$\frac{dE}{dt}=\frac{\partial E(t)}{\partial t}=\frac{\partial}{\partial t}\int_{-\infty}^{\infty}\frac{1}{2}u_t^2+\frac{1}{2}u_x^2-\cos u\,dx=$$

$$\frac{1}{2}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}(u_t^2)\,dx+\frac{1}{2}\int_{-\infty}^{\infty}\frac{\partial}{\partial t}(u_x^2)\,dx-\int_{-\infty}^{\infty}\frac{\partial}{\partial t}(\cos u)\,dx=$$

$$\frac{1}{2}\int_{-\infty}^{\infty}2u_tu_{tt}\,dx+\frac{1}{2}\int_{-\infty}^{\infty}2u_xu_{xt}\,dx-\int_{-\infty}^{\infty}-\sin(u)\,u_t\,dx=$$

$$\int_{-\infty}^{\infty}u_t(u_{xx}-\sin u)\,dx+\int_{-\infty}^{\infty}u_xu_{xt}\,dx+\int_{-\infty}^{\infty}\sin(u)\,u_t\,dx=$$

$$\int_{-\infty}^{\infty}u_{xx}u_t\,dx+\int_{-\infty}^{\infty}u_xu_{xt}\,dx-\int_{-\infty}^{\infty}\sin(u)\,u_t\,dx+\int_{-\infty}^{\infty}\sin(u)\,u_t\,dx=$$

$$\int_{-\infty}^{\infty}u_{xx}u_t+u_xu_{xt}\,dx=\int_{-\infty}^{\infty}\frac{\partial}{\partial x}(u_xu_t)\,dx=\left[u_xu_t\right]_{x=-\infty}^{\infty}=$$

$$\lim_{x\to\infty}u_x(x,t)u_t(x,t)-\lim_{x\to-\infty}u_x(x,t)u_t(x,t)=0-0=0$$

~~\3a/ ∴ $u(x,y,t)=\int_0^t\int\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}$ $\int_0^t\int_{-\infty}^{\infty}$~~

\4b/ The suggested sol is $u(x,t)=v(x-ct)$ ∴

$u_x=v'$, $u_{xx}=v''$, $u_t=-cv'$, $u_{tt}=c^2v''$ ∴

$c^2v''-v''+\sin v=0=v''(c^2-1)+\sin v$ ∴

$v''(c^2-1)=-\sin v$ ∴ $v''=\frac{-1}{c^2-1}\sin v$ ∴

$1-c^2=\frac{\sin v}{v''}$ ∴

$(1-c^2)v''=\sin v=\sin(v(\xi))=(1-c^2)v''(\xi)$

∴ let $v'(\xi)=\frac{dv(\xi)}{d\xi}=P(v)=P(v(\xi))$ ∴ $v''=v''(\xi)=\frac{dP(v)}{dv}v'=\frac{dP(v)}{dv}P(v)$ ∴

$(1-c^2)p(v)\frac{dp(v)}{dv} = \sin(v)$ $\therefore$

$\int(1-c^2)p(v)\frac{dp(v)}{dv}dv = \int\sin(v)dv = (1-c^2)\int p(v)dp(v) = (1-c^2)\frac{1}{2}(p(v))^2 = -\cos(v) + A$,

$A = \text{Const}$

$\therefore \frac{1}{2}(1-c^2)p^2 = A - \cos v = \frac{1}{2}(1-c^2)(p(v))^2 = \frac{1}{2}(1-c^2)\left(\frac{dv}{d\xi}\right)^2 = A - \cos(v(\xi))$

$\cancel{\int\frac{1}{2}(1-c^2)(p(v))}$ $\therefore$

$v'(\xi) \to 0$ as $\xi \to -\infty$, $v(\xi) \to 0$ as $\xi \to -\infty$ $\therefore$

as $\xi \to -\infty$: $v'(\xi) = \frac{dv}{d\xi} = p(v) \to 0$ $\therefore$

as $\xi \to -\infty$: $\frac{1}{2}(1-c^2)(0)^2 = A - \cos(0) = A - 1 = 0$ $\therefore A = 1$ $\therefore$

$\frac{1}{2}(1-c^2)\left(\frac{dv}{d\xi}\right)^2 = 1 - \cos(v(\xi))$ $\therefore$

$\left(\frac{dv}{d\xi}\right)^2 = \frac{2-2\cos(v(\xi))}{1-c^2}$ $\therefore$

$\frac{dv}{d\xi} = \pm\left(\frac{2-2\cos(v(\xi))}{1-c^2}\right)^{1/2} = \pm\left(\frac{2(1-\cos v)}{1-c^2}\right)^{1/2} = p = \pm\sqrt{2}\frac{1}{\sqrt{1-c^2}}(1-\cos v)^{1/2}$ $\therefore$

$\frac{1}{(1-\cos v)^{1/2}}\frac{dv}{d\xi} = \pm\left(\frac{2}{1-c^2}\right)^{1/2} = \frac{1}{\sqrt{1-\cos v}}\frac{dv}{d\xi}$ $\therefore$

$\int\frac{1}{\sqrt{1-\cos v}}\frac{dv}{d\xi}d\xi = \pm\int\left(\frac{2}{1-c^2}\right)^{1/2}d\xi = \int\frac{1}{\sqrt{1-\cos v}}dv = \sqrt{2}\ln|\tan(v/4)| + B = \pm\left(\frac{2}{1-c^2}\right)^{1/2}\xi$

$\therefore \sqrt{2}\ln|\tan(v/4)| = \pm\left(\frac{2}{1-c^2}\right)^{1/2}\xi - B = \pm\left(\frac{2}{1-c^2}\right)^{1/2}\xi - \pm\left(\frac{2}{1-c^2}\right)^{1/2}\left[\left(\frac{2}{1-c^2}\right)^{-1/2}B\right] =$

$\pm\left(\frac{2}{1-c^2}\right)^{1/2}\xi - \pm\left(\frac{2}{1-c^2}\right)^{1/2}[\beta] = \pm\left(\frac{2}{1-c^2}\right)^{1/2}(\xi-\beta) = 2^{1/2}\ln|\tan(v/4)|$ $\therefore$

$\ln|\tan(v/4)| = \pm\left(\frac{1}{1-c^2}\right)^{1/2}(\xi-\beta)2^{1/2}2^{-1/2} = \pm\left(\frac{1}{1-c^2}\right)^{1/2}(\xi-\beta)$ $\therefore$

$e^{\ln|\tan(v/4)|} = |\tan(v/4)| = e^{\pm\left(\frac{1}{1-c^2}\right)^{1/2}(\xi-\beta)}$ $\therefore$

$\tan\left(\frac{v}{4}\right) = \pm e^{\pm\left(\frac{1}{1-c^2}\right)^{1/2}(\xi-\beta)}$ $\therefore$

$v = 4\arctan\left(\pm e^{\pm\left(\frac{1}{1-c^2}\right)^{1/2}(\xi-\beta)}\right) = v(\xi)$, $\beta = \text{Const}$ $\qquad$ arctan $\therefore$

$\therefore v(\xi) \in (0, 2\pi)$ $\therefore v(\xi) > 0$ $\therefore$ $\quad$ $\arctan x$ is an odd func $\therefore$

$\therefore \pm e^{\pm\left(\frac{1}{1-c^2}\right)^{1/2}(\xi-\beta)} > 0$ $\therefore$ $e^x > 0$ $\therefore$ take positive sign: + sign $\therefore$

$v(\xi) = 4\arctan\left(e^{\pm\left(\frac{1}{1-c^2}\right)^{1/2}(\xi-\beta)}\right) = 4\arctan\left(e^{\pm\left(\frac{1}{1-c^2}\right)^{1/2}(x-ct-\beta)}\right) =$

$4\arctan\left(e^{\pm\frac{x-ct-\beta}{\sqrt{1-c^2}}}\right) = v(\xi) = v(x-ct) = u(x,t)$

\4c/ $u_{tt} - u_{xx} = -\sin u$ $\therefore$ $\quad$ $u_{tt} - u_{xx} = a u_{tt} + 2b u_{xt} + c u_{xx}$ $\therefore$

$a = 1$, $2b = 0 = 2b = b$, $c = -1$ $\therefore$ $\quad$ $b^2 - ac = 0^2 - 1(-1) = 1 > 0$ $\therefore$ hyperbolic $\therefore$

$a\left(\frac{dx}{dt}\right)^2 - 2b\left(\frac{dx}{dt}\right) + c = \left(\frac{dx}{dt}\right)^2 - 1 = 0$ $\therefore$ $\left(\frac{dx}{dt}\right)^2 = 1$ $\therefore$ $\frac{dx}{dt} = \pm 1$ $\therefore$

$\int 1dx = \int 1dt = x = t + C_1$, $\int 1dx = \int -1dt = x = -t + C_2$ $\therefore$ $C_1 = x - t$, $C_2 = x + t$ $\therefore$

\PP2020/ let $\xi = x - t$, $\eta = x + t$ ∴

$\xi_x = 1$, $\xi_t = -1$, $\eta_x = 1$, $\eta_t = 1$ ∴

iii) $u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$,

$u_{xx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$u_t = u_\xi \xi_t + u_\eta \eta_t = -u_\xi + u_\eta$ ∴

$u_{tt} = -u_{\xi\xi}\xi_t - u_{\xi\eta}\eta_t + u_{\xi\eta}\xi_t + u_{\eta\eta}\eta_t = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$ ∴

PDE: $u_{tt} - u_{xx} =$

$u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} - u_{\xi\xi} - 2u_{\xi\eta} - u_{\eta\eta} = -4u_{\xi\eta} = -\sin u$ ∴

$u_{\xi\eta} = \frac{1}{4}\sin u = \frac{\partial}{\partial\xi}\left(\frac{\partial}{\partial\eta}u\right)$

\4C Sol/ ∴ $\xi = x + t$, $\eta = x - t$ are charac coords sor the principle part ∴

canonical sorm $\tilde{u}_{\xi\eta} = \frac{1}{4}\sin(\tilde{u})$

∴ in canonical coords, $u(x,t) = \tilde{u}(\xi, \eta)$ ∴

$\xi - x = t$ ∴ $\eta = x - \xi + x = 2x - \xi$ ∴ $\eta + \xi = 2x$ ∴ $\frac{1}{2}\xi + \frac{1}{2}\eta = x$ ∴

$\xi - \frac{1}{2}\xi - \frac{1}{2}\eta = \frac{1}{2}\xi - \frac{1}{2}\eta = t$ ∴

$u(x = \frac{1}{2}\xi + \frac{1}{2}\eta = x,\ t = \frac{1}{2}\xi - \frac{1}{2}\eta = -x)$ ∴ $\frac{1}{2}\xi = \frac{1}{2}\eta - x$ ∴

$\frac{1}{2}\xi + \frac{1}{2}\eta = \frac{1}{2}\eta - x + \frac{1}{2}\eta = x$ ∴ $\eta - x = x$ ∴ $\eta = 2x$ ∴ $\frac{\eta}{2} = x$

$\frac{1}{2}\xi - x = -x$ ∴ $\frac{1}{2}\xi = 0 = \xi$ ∴

$\tilde{u}(\xi = 0, \frac{1}{2}\eta) = \phi(\eta/2)$

$u_t(x = x, t = -x) = \tilde{u}_t(\frac{1}{2}\xi + \frac{1}{2}\eta = x, \frac{1}{2}\xi - \frac{1}{2}\eta = -x)$ ∴

$\frac{1}{2}\xi = -x + \frac{1}{2}\eta$ ∴ $-x + \frac{1}{2}\eta + \frac{1}{2}\eta = x$ ∴ $\eta = 2x$ ∴ $\frac{\eta}{2} = x$ ∴

$\frac{1}{2}\xi - \frac{1}{2}\eta = -\frac{1}{2}\eta$ ∴ $\frac{1}{2}\xi = 0 = \xi$ ∴

$\tilde{u}_t(\xi = 0, \frac{\eta}{2}) = \psi(\eta/2)$ ∴ $\tilde{u}_t = \tilde{u}_\xi \xi_t + \tilde{u}_\eta \eta_t = \tilde{u}_\xi - \tilde{u}_\eta$ ∴

$\tilde{u}_\xi(\xi = 0, \frac{\eta}{2}) - \tilde{u}_\eta(\xi = 0, \frac{\eta}{2}) = \psi(\frac{\eta}{2})$ is BVP sor canonical coords ∴

$\tilde{u}_\eta(0, \frac{\eta}{2}) = \phi'(\frac{\eta}{2})\frac{1}{2} = \frac{1}{2}\phi'(\frac{\eta}{2})$ ∴

$\tilde{u}_\xi(0, \frac{\eta}{2}) = \psi(\frac{\eta}{2}) + \tilde{u}_\eta(0, \frac{\eta}{2}) = \psi(\frac{\eta}{2}) + \frac{1}{2}\phi'(\frac{\eta}{2})$ ∴

$\tilde{u}_{\xi\eta}(0, \eta) = \frac{1}{2}\psi'(\frac{\eta}{2}) + \frac{1}{4}\phi''(\frac{\eta}{2})$

and sor $\xi \neq 0$ the sunction $\tilde{u}$ satissies $\tilde{u}_{\xi\eta} = \frac{1}{4}\sin(\tilde{u})$ ∴

$\frac{1}{2}\psi'(\frac{\eta}{2}) + \frac{1}{4}\phi''(\frac{\eta}{2}) = \frac{1}{4}\sin(\phi(\eta/2))$ ∴ $2\psi'(\frac{\eta}{2}) + \phi''(\eta/2) = \sin(\phi(\eta/2))$

and sor $\frac{\eta}{2} = x$: $2\psi'(x) - \phi''(x) = \sin(\phi(x))$ is required solvability condition

\pp2022/\4C/ principle part: $u_{tt} - u_{xx} = 0 \therefore$

charac coords: $\xi = x+t,\ \eta = x-t \therefore$

~~ut~~ $\cancel{u_x = u_t}$ $\xi_x = 1,\ \xi_t = 1,\ \eta_x = 1,\ \eta_t = -1 \therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta \therefore$

$u_{xx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$u_t = u_\xi \xi_t + u_\eta \eta_t = u_\xi - u_\eta \therefore$

$u_{tt} = u_{\xi\xi}\xi_t + u_{\xi\eta}\eta_t - u_{\xi\eta}\xi_t - u_{\eta\eta}\eta_t = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} \therefore$

~~utx~~ $u_{tt} - u_{xx} = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} - u_{\xi\xi} - 2u_{\xi\eta} - u_{\eta\eta} =$

$-4u_{\xi\eta} = -\sin u \therefore$

$u_{\xi\eta} = \frac{1}{4}\sin u \therefore \quad \tilde{u}_{\xi\eta}(\xi,\eta) = \frac{1}{4}\sin \tilde{u}(\xi,\eta) \therefore$

$\cancel{u(x=x, t=-x) = u(}$ $\xi - x = t \therefore \eta = x - \xi + x = 2x - \xi \therefore \eta + \xi = 2x \therefore \frac{1}{2}\xi + \frac{1}{2}\eta = x$

$\therefore \xi - \frac{1}{2}\xi - \frac{1}{2}\eta = \frac{1}{2}\xi - \frac{1}{2}\eta = t \therefore$

$u(x=x, t=-x) = u(\frac{1}{2}\xi + \frac{1}{2}\eta = x, \frac{1}{2}\xi - \frac{1}{2}\eta = -x) = u(\frac{1}{2}\xi + \frac{1}{2}\eta = -\frac{1}{2}\xi + \frac{1}{2}\eta, \frac{1}{2}\xi - \frac{1}{2}\eta = -x) =$

$u(2\xi = 0, \frac{1}{2}\xi - \frac{1}{2}\eta = -x) = u(\xi = 0, \frac{1}{2}(0) - \frac{1}{2}\eta = -x) =$ ~~u(~~ $u(\xi = 0, -\frac{1}{2}\eta = -x) =$

$u(\xi = 0, \frac{1}{2}\eta = x) = \phi(\frac{1}{2}\xi + \frac{1}{2}\eta) = \phi(\frac{1}{2}(0) + \frac{1}{2}\eta) = \phi(\frac{1}{2}\eta) = u(\xi = 0, \eta = 2x) \therefore$

$\tilde{u}(\xi = 0, \eta) = \phi(\frac{1}{2}\eta)$

$u_t(x=x, t=-x) = u_t(\frac{1}{2}\xi + \frac{1}{2}\eta = x, \frac{1}{2}\xi - \frac{1}{2}\eta = -x) = u_t(\frac{1}{2}\xi + \frac{1}{2}\eta = x, -\frac{1}{2}\xi + \frac{1}{2}\eta = x) =$

$u_t(\frac{1}{2}\xi + \frac{1}{2}\eta = -\frac{1}{2}\xi + \frac{1}{2}\eta, -\frac{1}{2}\xi + \frac{1}{2}\eta = x) = u_t(\xi = 0, -\frac{1}{2}\xi + \frac{1}{2}\eta = x) =$

$u_t(\xi = 0, -\frac{1}{2}(0) + \frac{1}{2}\eta = x) = u_t(\xi = 0, \frac{1}{2}\eta = x) = u_t(\xi = 0, \eta = 2x) = \psi(x) =$

~~ψ~~ $\psi(\eta/2) \therefore \quad \tilde{u}_t(\xi = 0, \eta) = \psi(\eta/2) \therefore$

$\cancel{u_t(\xi = 0, \eta) = u}$ $u_t = u_\xi - u_\eta \therefore$

$\tilde{u}_\xi(\xi = 0, \eta) - \tilde{u}_\eta(\xi = 0, \eta) = \psi(\eta/2) \therefore$

$\tilde{u}_\xi(\xi = 0, \eta) = \psi(\eta/2) + \tilde{u}_\eta(\xi = 0, \eta) \cancel{= \phi(\eta/2)} \therefore$

$\tilde{u}_\eta(\xi = 0, \eta) = \frac{1}{2}\phi'(\frac{1}{2}\eta) \therefore$

$\tilde{u}_\xi(\xi = 0, \eta) = \psi(\eta/2) + \frac{1}{2}\phi'(\frac{1}{2}\eta) \therefore$

$\tilde{u}_{\xi\eta}(\xi = 0, \eta) = \frac{1}{2}\psi'(\eta/2) + \frac{1}{4}\phi''(\frac{1}{2}\eta) = \frac{1}{4}\sin(\tilde{u}(\xi = 0, \eta)) = \frac{1}{4}\sin(\phi(\frac{1}{2}\eta)) \therefore$

$\frac{1}{2}\psi'(\eta/2) + \frac{1}{4}\phi''(\frac{1}{2}\eta) = \frac{1}{4}\sin(\phi(\frac{1}{2}\eta)) \therefore 2\psi'(\eta/2) + \phi''(\frac{1}{2}\eta) = \sin(\phi(\frac{1}{2}\eta))$

$\therefore$ for $\frac{\eta}{2} = x$: $2\psi'(x) + \phi''(x) = \sin(\phi(x))$

PP2022 / 3a / w.l.o.g may consider the problem posed on the time interval $[0,T]$ for some T that can be chosen arbitrarily.

∴ due to condition that $G(x,y,t;\xi,\eta,\tau)\equiv 0$ for $t<\tau$, the expression for the putative solution can be replaced with an equivalent one with all fixed integration limits ∴

$$u(x,y,t)=\int_0^T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,y,t;\xi,\eta,\tau)f(\xi,\eta,\tau)\,d\xi\,d\eta\,d\tau \quad ∴$$

into PDE: $u_t-u_{xx}-u_{yy}=f(x,y,t)$ ∴

~~$u_t-u_{xx}-u_{yy}$~~ $(\partial_t-\partial_{xx}-\partial_{yy})\int_0^T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,y,t;\xi,\eta,\tau)f(\xi,\eta,\tau)\,d\xi\,d\eta\,d\tau$

$$=\int_0^T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\partial_t-\partial_{xx}-\partial_{yy})G(x,y,t;\xi,\eta,\tau)f(\xi,\eta,\tau)\,d\xi\,d\eta\,d\tau=$$

$$\int_0^T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\partial_tG-\partial_{xx}G-\partial_{yy}G)f(\xi,\eta,\tau)\,d\xi\,d\eta\,d\tau=$$

$$\int_0^T\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-\xi)\delta(y-\eta)\delta(t-\tau)f(\xi,\eta,\tau)\,d\xi\,d\eta\,d\tau=f(x,y,t)$$

∴ PDE is satisfied

for IC $u(x,y,0)=0$ use initial formulation ∴

$$u(x,0)=\int_0^0\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,y,t;\xi,\eta,\tau)f(\xi,\eta,\tau)\,d\xi\,d\eta\,d\tau=0$$

∴ IC is satisfied too

3a / let $u(x,y,t)=\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,y,t;\xi,\eta,\tau)f(\xi,\eta,\tau)\,d\xi\,d\eta\,d\tau$ ∴

$(\partial_t-\partial_{xx}-\partial_{yy})G=\delta(x-\xi)\delta(y-\eta)\delta(t-\tau)$ ∴

$$u_t-u_{xx}-u_{yy}=(\partial_t-\partial_{xx}-\partial_{yy})u=(\partial_t-\partial_{xx}-\partial_{yy})\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,y,t;\xi,\eta,\tau)f(\xi,\eta,\tau)\,d\xi\,d\eta\,d\tau$$

$$=\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}(\partial_t-\partial_{xx}-\partial_{yy})G(x,y,t;\xi,\eta,\tau)f(\xi,\eta,\tau)\,d\xi\,d\eta\,d\tau=$$

$$\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\delta(x-\xi)\delta(y-\eta)\delta(t-\tau)f(\xi,\eta,\tau)\,d\xi\,d\eta\,d\tau=f(x,y,t)$$

$$u(x,y,0)=\int_0^0\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}G(x,y,t;\xi,\eta,\tau)f(\xi,\eta,\tau)\,d\xi\,d\eta\,d\tau=0$$

∴ u satisfies PDE and IC

∵ $\delta(x-\xi)=\delta(-(-x+\xi))=\delta(-(\xi-x))=\delta(\xi-x)$

3b/ reformulate $G_t - G_{xx} - G_{yy} = \delta(x-\xi)\delta(y-\eta)\delta(t-\tau)$ by changing the independent variables

$\tilde{G}(\tilde{x},\tilde{y},\tilde{t}) = G(x,y,t)$, $\tilde{x} = x-\xi$, $\tilde{y} = y-\eta$, $\tilde{t} = t-\tau$

upon dropping the dependence on parameters ξ, η, τ:

$\tilde{G}_{\tilde{t}} - \tilde{G}_{\tilde{x}\tilde{x}} - \tilde{G}_{\tilde{y}\tilde{y}} = G_t - G_{xx} - G_{yy} = \delta(x-\xi)\delta(y-\eta)\delta(t-\tau) = \delta(\tilde{x})\delta(\tilde{y})\delta(\tilde{t})$;

$\tilde{G}(\tilde{x},\tilde{y},\tilde{t}) = 0$ for $\tilde{t} < 0$ $\because$ $\tilde{t} = t-\tau < 0$ $\therefore t < \tau$

$\therefore$ can now reinterpret this as an IVP for time domain $\tilde{t} > 0$, by integrating the equation over an interval $\tilde{t} \in [-\varepsilon, +\varepsilon]$ and taking the limit $\varepsilon \to 0$ for the first term in the LHS:

$\lim_{\varepsilon\to 0} \int_{-\varepsilon}^{\varepsilon} \tilde{G}_{\tilde{t}}(\tilde{x},\tilde{y},\tilde{t})\,d\tilde{t} = \lim_{\varepsilon\to 0}(\tilde{G}(\tilde{x},\tilde{y},\varepsilon) - \tilde{G}(\tilde{x},\tilde{y},-\varepsilon)) = \lim_{\varepsilon\to 0}\tilde{G}(\tilde{x},\tilde{y},\varepsilon) = \tilde{G}(\tilde{x},\tilde{y},0)$

where the lower limit substitution gives zero due to the condition $\tilde{G}(\tilde{x},\tilde{y},\tilde{t}) = 0$ for $\tilde{t} < 0$. The limits of the integrals of the second and the third terms in the LHS yield zero.

Finally integration of the RHS just eliminates the time delta-function. $\therefore$ in the domain $\tilde{t} > 0$ the RHS in

$\tilde{G}_{\tilde{t}} - \tilde{G}_{\tilde{x}\tilde{x}} - \tilde{G}_{\tilde{y}\tilde{y}} = \delta(x)\delta(\tilde{y})\delta(\tilde{t})$ $\quad \tilde{G}(\tilde{x},\tilde{y},\tilde{t}) = 0$ for $\tilde{t} < 0$

is identically zero. $\therefore$ IVP for this domain

$\tilde{G}_{\tilde{t}} - \tilde{G}_{\tilde{x}\tilde{x}} - \tilde{G}_{\tilde{y}\tilde{y}} = 0$, $\tilde{G}(\tilde{x},y,0) = \delta(\tilde{x})\delta(\tilde{y})$

This is exactly the IVP requested $\therefore$ identify the function Φ with $\tilde{G}$

3c/ under the assumption made, can write that $\psi(y,t)$ satisfies

$\psi_t(y,t) - \psi_{yy}(y,t) = 0$, $\psi(y,0) = \delta(y)$ $\therefore$ subbing the suggested solution into IVP PDE:

$LHS = (\partial_t - \partial_{xx} - \partial_{yy})\Phi(x,y,t) = \partial_t(\psi(x,t)\psi(y,t)) - \partial_{xx}(\psi(x,t)\psi(y,t)) - \partial_{yy}(\psi(x,t)\psi(y,t)) =$

$\psi_t(x,t)\psi(y,t) + \psi(x,t)\psi_t(y,t) - \psi_{xx}(x,t)\psi(y,t) - \psi(x,t)\psi_{yy}(y,t) =$

$\psi(y,t)[\psi_t(x,t) - \psi_{xx}(x,t)] + \psi(x,t)[\psi_t(y,t) - \psi_{yy}(y,t)] = \psi(y,t)[0] + \psi(x,t)[0] = 0 = RHS$

for IC: $\Phi(x,y,0) = \psi(x,0)\psi(y,0) = \delta(x)\delta(y)$ according to the formula:

$\Phi(x,t) = \frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}$ $\therefore$ ~~$\psi(x,y,t$~~ $\Phi(y,t) = \frac{1}{\sqrt{4\pi t}}e^{-y^2/4t}$ $\therefore \psi(x,y,t) = \Phi(x,t)\Phi(y,t) =$

$\frac{1}{\sqrt{4\pi t}}e^{-x^2/4t}\frac{1}{\sqrt{4\pi t}}e^{-y^2/4t} = \frac{1}{4\pi t}e^{-\frac{x^2+y^2}{4t}}$ $\therefore G(x,y,t;\xi,\eta,\tau) = \begin{cases} 0 & , t < \tau \\ \frac{1}{4\pi(t-\tau)}e^{-\frac{(x-\xi)^2+(y-\eta)^2}{4(t-\tau)}} & , t > \tau \end{cases}$

\ECM3708PP2018/ \1ai/ $\therefore (\cos y, -\cos x)\cdot\nabla u = 0 \therefore$ 1C2019

$\frac{dy}{dx} = \frac{-\cos x}{\cos y} \therefore$ ~~$\cos y \frac{dy}{dx}$~~ $\cos y = 0$ is special solution $\therefore$

$\cos y \frac{dy}{dx} = -\cos x \therefore \int \cos y \, dy = \int -\cos x \, dx =$ ~~[illegible]~~

$\sin y = -\sin x + C \therefore$ ~~[illegible]~~ $\sin y + \sin x = C \therefore$

$u(x,y) = f(\sin y + \sin x)$ f is arbitrary is GS $\therefore$

$u(x,0) = \sin^2 x = f(\sin(0) + \sin x) = f(\sin x) = (\sin x)^2 \therefore$

~~[illegible]~~ let $s = \sin x \therefore f(s) = s^2 \therefore$

$u(x,y) = f(\sin y + \sin x) = (\sin y + \sin x)^2$

\1aii/ $u(x,-x) = f(\sin(-x) + \sin x) = \sin^2 x = f(-\sin(x) + \sin(x)) = f(0) \therefore$
is impossible $\therefore$ no solution exists.

\1b/ let $s = x^2$, $t = y^3 \therefore$ ~~[illegible]~~ $s_x = 2x$, $t_y = 3y^2 \therefore$

$u_x = u_s s_x = 2x u_s$, $u_y = u_t t_y = 3y^2 u_t \therefore$

$u_{xx} = \partial_x(2x u_s) = 2u_s + 2x\partial_x u_s = 2u_s + 2x u_{ss} s_x = 2u_s + 4x^2 u_{ss}$,

$u_{yy} = \partial_y(3y^2 u_t) = 6y u_t + 3y^2 u_{tt} t_y = 6y u_t + 9y^4 u_{tt} \therefore$

into PDE:

$\frac{81}{x^2}(2u_s + 4x^2 u_{ss}) - \frac{16}{y^4}(6y u_t + 9y^4 u_{tt}) - \frac{81}{x^3}(2x u_s) + \frac{32}{y^5}(3y^2 u_t) =$

~~$162 \frac{1}{x^2}$~~ $162 \frac{1}{x^2} u_s + 324 u_{ss} - 96 \frac{1}{y^3} u_t - 144 u_{tt} - 182 \frac{1}{x^2} u_s + 96 \frac{1}{y^3} u_t =$

$324 u_{ss} - 144 u_{tt} = 0 \therefore$

let $u = f(\alpha s + t) \therefore u_{tt} = f''(\alpha s + t) = f''$, $u_{ss} = \alpha^2 f'' \therefore$

$324\alpha^2 f'' - 144 f'' = 0 \therefore 324\alpha^2 = 144 \therefore \alpha^2 = \frac{4}{9} \therefore \alpha_1 = \frac{2}{3}, \alpha_2 = -\frac{2}{3} \therefore$

$u = f(\frac{2}{3}s + t) + g(-\frac{2}{3}s + t) \therefore$

$u(x,t) = f(\frac{2}{3}x^2 + y^3) + g(-\frac{2}{3}x^2 + y^3)$

\1c/ let $u(x,t) = v(x,t)e^{\alpha t} \therefore u_t = v_t e^{\alpha t} + \alpha v e^{\alpha t}$,

$u_x = v_x e^{\alpha t} \therefore u_{xx} = v_{xx} e^{\alpha t} \therefore$

$u_t = v_t e^{\alpha t} + \alpha v e^{\alpha t} = v_t e^{\alpha t} + \alpha u = v_{xx} e^{\alpha t} - 2u \therefore$

let $\alpha = -2 \therefore \alpha u = -2u \therefore v_t e^{\alpha t} = v_{xx} e^{\alpha t} \therefore$

$V_t = V_{xx}$ $\therefore$ $u(x,0) = v(x,0)e^{\alpha(0)} = v(x,0) = \psi(x)$ $\therefore$

$v(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \psi(s) e^{-(x-s)^2/(4t)} ds = u(x,t)e^{-\alpha t} = e^{2t}u(x,t)$ $\therefore$

~~$u(x,t) = e^{-2t}\frac{1}{}$~~ $u(x,t) = e^{-2t}\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\psi(s)e^{-(x-s)^2/(4t)} dt$

$\therefore$ $\psi(x) = \delta(x+1)+\delta(x-1)$ $\therefore$ $\psi(s) = \delta(s+1)+\delta(s-1)$ $\therefore$

$u(x,t) = \frac{e^{-2t}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}(\delta(s+1)+\delta(s-1))e^{-(x-s)^2/(4t)} ds$ $\therefore$

at $x=0$:

$u(0,t)=p(t) = \frac{e^{-2t}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} (\delta(s+1)+\delta(s-1))e^{-(-s)^2/(4t)}ds =$

$\frac{e^{-2t}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}(\delta(s+1)+\delta(s-1))e^{-s^2/(4t)}ds$ $\therefore$

$t\ge 0$, $\therefore$ $e^{-2t}\le 1$, $\therefore$ $\max p(t) = p(0) = p(t_*)$

Vd/ $\therefore$ $5u_{xx}+8u_{xy}+5u_{y} = -30u_x - 6u_y - 90u$ $\therefore$

$a=5$, ~~$2b$~~ $-2b=8$ $\therefore$ $b=-4$, $c=5$ $\therefore$

$5(\frac{d^2y}{dx^2}) - 8\frac{dy}{dx}+5=0$ $\therefore$ $\frac{dy}{dx} = \frac{8\pm\sqrt{64-4(5)(5)}}{2(5)} = \cancel{\frac{8}{10}\pm\sqrt{}}$ $\therefore$

$2b=8$ $\therefore$ $b=4$ $\therefore$

$b^2-ac = 4^2-(5)(5) = -9<0$ $\therefore$

$5(\frac{d^2y}{dx^2}) - 8\frac{dy}{dx}+5=0$ $\therefore$ $\frac{dy}{dx} = \frac{8\pm\sqrt{64-4(5)(5)}}{2(5)} = \frac{4}{5} \pm \frac{\sqrt{-36}}{10} =$

$\frac{4}{5} \pm \frac{6}{10}i = \frac{4}{5}\pm\frac{3}{5} i$ $\therefore$

$\frac{dy}{dx} = \frac{4}{5}+\frac{3}{5} i$, $\frac{dy}{dx} = \frac{4}{5}-\frac{3}{5} i$ $\therefore$

$y = (\frac{4}{5}+\frac{3}{5} i)x + c_1$, $y=(\frac{4}{5}-\frac{3}{5} i)x + c_2$ $\therefore$

$c_1 = 5y-(4+3i)x$, $c_2 = 5y+(-4+3i)x$ $\therefore$ $c_1 = 5y+(-4-3i)x$ $\therefore$

can write characteristic coordinates as: $\alpha = 5y-4x$, $\beta = 3x$ $\therefore$

$\alpha_x = -4$, $\alpha_y = 5$, $\beta_x = 3$ $\therefore$

~~u_{xx} $u_y = \alpha_x\alpha_y$~~ $u_y = u_\alpha\alpha_y = 5u_\alpha$ $\therefore$ $u_{yy} = 5u_{\alpha\alpha}\alpha_y = 25u_{\alpha\alpha}$,

$u_x = u_\alpha\alpha_x + u_\beta\beta_x = -4u_\alpha + 3u_\beta$ $\therefore$

\ECM3708PP2018/ $u_{xx} = -4u_{\alpha\alpha}\alpha_x + 3u_{\alpha\beta}\alpha_x - 4u_{\alpha\beta}\beta_x + 3u_{\beta\beta}\beta_x =$

$16u_{\alpha\alpha} - 12u_{\alpha\beta} - 12u_{\alpha\beta} + 9u_{\beta\beta} = 16u_{\alpha\alpha} - 24u_{\alpha\beta} + 9u_{\beta\beta}$ ∴

$u_{xy} = 5u_{\alpha\alpha}\alpha_x + 5u_{\alpha\beta}\beta_x = -20u_{\alpha\alpha} + 15u_{\alpha\beta}$ ∴

into PDE:

$5(16u_{\alpha\alpha} - 24u_{\alpha\beta} + 9u_{\beta\beta}) + 8(-20u_{\alpha\alpha} + 15u_{\alpha\beta}) + 5(25u_{\alpha\alpha}) +$

$30(-4u_\alpha + 3u_\beta) + 6(5u_\alpha) + 90u =$

$80u_{\alpha\alpha} - 120u_{\alpha\beta} + 45u_{\beta\beta} - 160u_{\alpha\alpha} + 120u_{\alpha\beta} + 125u_{\alpha\alpha} - 120u_\alpha +$

$90u_\beta + 30u_\alpha + 90u =$

$45u_{\alpha\alpha} + 45u_{\beta\beta} - 90 + 90u_\beta + 90u = 0 =$

$u_{\alpha\alpha} + u_{\beta\beta} - 2 + 2u_\beta + 2u$ ∴

$u_{\alpha\alpha} + u_{\beta\beta} = 2u_\alpha - 2u_\beta - 2u$ ∴

let $u = e^{\lambda\alpha+\mu\beta}v$ ∴ $u_\alpha = \lambda e^{\lambda\alpha+\mu\beta}v + e^{\lambda\alpha+\mu\beta}v_\alpha = e^{\lambda\alpha+\mu\beta}(\lambda v + v_\alpha)$

$u_\beta = \mu e^{\lambda\alpha+\mu\beta}v + e^{\lambda\alpha+\mu\beta}v_\beta = e^{\lambda\alpha+\mu\beta}(\mu v + v_\beta)$,

$u_{\alpha\alpha} = \lambda e^{\lambda\alpha+\mu\beta}(\lambda v + v_\alpha) + e^{\lambda\alpha+\mu\beta}(\lambda v_\alpha + v_{\alpha\alpha}) = e^{\lambda\alpha+\mu\beta}(\lambda^2 v + 2\lambda v_\alpha + v_{\alpha\alpha})$,

$u_{\beta\beta} = \mu e^{\lambda\alpha+\mu\beta}(\mu v + v_\beta) + e^{\lambda\alpha+\mu\beta}(\mu v_\beta + v_{\beta\beta}) = e^{\lambda\alpha+\mu\beta}(\mu^2 v + 2\mu v_\beta + v_{\beta\beta})$ ∴

into equation:

$e^{\lambda\alpha+\mu\beta}(\lambda^2 v + 2\lambda v_\alpha + v_{\alpha\alpha}) + e^{\lambda\alpha+\mu\beta}(\mu^2 v + 2\mu v_\beta + v_{\beta\beta}) =$

$2e^{\lambda\alpha+\mu\beta}(\lambda v + v_\alpha) - 2e^{\lambda\alpha+\mu\beta}(\mu v + v_\beta) - 2e^{\lambda\alpha+\mu\beta}v$ ∴

$e^{\lambda\alpha+\mu\beta} > 0$ ∴

$\lambda^2 v + 2\lambda v_\alpha + v_{\alpha\alpha} + \mu^2 v + 2\mu v_\beta + v_{\beta\beta} = 2\lambda v + 2v_\alpha - 2\mu v - 2v_\beta - 2v$ ∴

$\lambda^2 v + 2\lambda v_\alpha + v_{\alpha\alpha} + \mu^2 v + 2\mu v_\beta + v_{\beta\beta} - 2\lambda v - 2v_\alpha + 2\mu v + 2v_\beta + 2v = 0 =$

$(\lambda^2 + \mu^2 - 2\lambda + 2\mu + 2)v + (2\lambda - 2)v_\alpha + (2\mu + 2)v_\beta + v_{\alpha\alpha} + v_{\beta\beta} = 0$ ∴

$2\lambda - 2 = 0$ ∴ $\lambda = 1$, $2\mu + 2 = 0$ ∴ $\mu = -1$ ∴

$\lambda^2 + \mu^2 - 2\lambda + 2\mu + 2 = 0$ ∴

$v_{\alpha\alpha} + v_{\beta\beta} = 0$

\1e/ let $u = X(x)Y(y)$ ∴ $u_{xx} = X''(x)Y(y)$, $u_{yy} = X(x)Y''(y)$ ∴

$X''(x)Y(y) + X(x)Y''(y) = 0$ ∴ $X''(x)Y(y) = -X(x)Y''(y)$ ∴ $\frac{X''(x)}{X} = \frac{-Y''}{Y} =$

$\lambda =$ Constant ∴

$X''(x) = \lambda X(x)$, $Y''(y) = -\lambda Y(y)$ $\therefore$

$X''(x) - \lambda X(x) = 0$ $\therefore$ $X(x) = e^{qx}$ $\therefore$

$q^2 - \lambda = 0$ $\therefore$ $q^2 = \lambda$ $\therefore$ $q = \pm\sqrt{\lambda}$, $X(0) = 0$, $X(1) = 0$ $\therefore$

$\lambda < 0$: $X(x) = A\sin(x) + B\cos(x)$ $\therefore$ $X(0) = 0 = 0A + 1B = B = 0$ $\therefore$

$X(x) = A\sin x$ $\therefore$ $X(1) = A\sin(1) = 0$ $\therefore$ $X(1) = 0 = A\sin(1)$ $\therefore A = 0$ $\therefore$

$\lambda = 0$: $X(x) = A + Bx$ $\therefore$ $X(0) = 0 = A + 0B = 0 = A$ $\therefore X = Bx$ $\therefore$

$X = Bx$ $\therefore$ $X(1) = 0 = B(1) = B = 0$

~~Sun~~ $\lambda > 0$. $X(x) = Ae^{\sqrt{\lambda}x} + Be^{-\sqrt{\lambda}x}$ $\therefore$ $X(0) = 0 = A(1) + B(1) = A + B = 0$

$\therefore A = -B$ $\therefore$ $X(x) = B(-e^{\sqrt{\lambda}x} + e^{-\sqrt{\lambda}x})$ $\therefore$

$X(1) = 0 = B(-e^{\sqrt{\lambda}1} + e^{-\sqrt{\lambda}}) = B(-e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}})$ $\therefore$

$-e^{\sqrt{\lambda}} + e^{-\sqrt{\lambda}} = 0$ $\therefore$ $e^{-\sqrt{\lambda}} = e^{\sqrt{\lambda}}$ $\therefore$ $-\sqrt{\lambda} = \sqrt{\lambda}$ $\therefore$

$0 = 2\sqrt{\lambda}$ $\therefore$ $\sqrt{\lambda} = 0$ $\therefore$ $\lambda = 0$

\2a/ $a = 1$, $b = 0$, $c = -1$ $\therefore$

$b^2 - ac = 0^2 - (1)(-1) = 0 + 1 = 1 > 0$ $\therefore$ hyperbolic equation $\therefore$

$a\left(\frac{dt}{dx}\right)^2 - 2b\left(\frac{dt}{dx}\right) + c = 0 = \left(\frac{dt}{dx}\right)^2 - 1 = 0$ $\therefore$ $\left(\frac{dt}{dx}\right)^2 = 1$ $\therefore$

$\frac{dt}{dx} = 1$, $\frac{dt}{dx} = -1$ $\therefore$ $t = x + C_1$, $t = -x + C_2$ $\therefore$

$t - x = C_1$, $t + x = C_2$ $\therefore$ $-(t - x) = -C_1 = C_1 = x - t$ $\therefore$

$\xi = x - t$, $\eta = x + t$ $\therefore$

$\xi_x = 1$, $\xi_t = -1$, $\eta_x = 1$, $\eta_t = 1$ $\therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$, $u_t = u_\xi \xi_t + u_\eta \eta_t = -u_\xi + u_\eta$ $\therefore$

$u_{xx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\xi_x + u_{\xi\eta}\eta_x + u_{\eta\eta}\eta_x = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$,

$u_{tt} = -u_{\xi\xi}\xi_t + u_{\xi\eta}\xi_t - u_{\xi\eta}\eta_t + u_{\eta\eta}\eta_t = +u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$ $\therefore$

into PDE:

$u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} - u_{\xi\xi} + 2u_{\xi\eta} - u_{\eta\eta} = 4u_{\xi\eta} = 0 = u_{\xi\eta} = \frac{\partial}{\partial\xi}(u_\eta)$

~~$u_\eta = 2f(\eta)$~~ . ~~$u = \ldots$~~ $u_\eta = F(\eta)$ $\therefore$ $u = \hat{F}(\eta) + \hat{g}(\xi)$

$\therefore$ $u = f(\xi) + g(\eta) = f(x - t) + g(x + t)$

\ENCM3708PP2018/ let u_1, u_2 be solutions $\therefore$

$u_{1_t} - u_{1_{xx}} = f(x,t)$, $u_1(x,0) = \phi(x)$, $u_1(0,t) = \psi_0(t)$, $u_1(L,t) = \psi_L(t)$

$u_{2_t} - u_{2_{xx}} = f(x,t)$, $u_2(x,0) = \phi(x)$, $u_2(0,t) = \psi_0(t)$, $u_2(L,t) = \psi_L(t)$

$\therefore$ let $\omega = u_1 - u_2 \doteq \omega(x,t) \therefore \omega_t - \omega_{xx} = 0$,

$\omega(x,0) = 0$, $\omega(0,t) = 0$ $\omega(L,t) = 0 \therefore$

$\omega_t = \omega_{xx} \therefore$ $\cancel{\omega = \frac{1}{\sqrt{4\pi t}} \int_{-\infty}^{\infty} \phi}$

\1a/ $\therefore (\cos y, -\cos x) \cdot \nabla u = 0 \therefore$

$\frac{dy}{dx} = \frac{-\cos x}{\cos y} \therefore \int \cos y \, dy = \int -\cos x \, dx \doteq \sin y = -\sin x + C_1 \therefore$

$C_1 = \sin y + \sin x \therefore$

$u(x,y) = f(\sin y + \sin x)$

\1ai/ $u(x,0) = \sin^2 x = f(\sin(0) + \sin x) = f(0 + \sin x) = f(\sin x) = \sin^2 x = (\sin x)^2$

$\therefore$ ~~for~~ $s = \sin x \therefore f(s) = s^2 \therefore$

$u(x,y) = f(\sin y + \sin x) = (\sin y + \sin x)^2$

\1aii/ $u(x,-x) =$ ~~f~~ $u(x = x, y = -x) = f(\sin(-x) + \sin x) =$

$f(-\sin x + \sin x) = f(0) = \sin^2 x$ which is impossible

$\therefore$ no solution exists $\because$ function is required to have different values for the same argument

\1b/ let $s = x^2$, $t = y^3 \therefore s_x = 2x$, $s_y = 0$, $t_x = 0$, $t_y = 3y^2 \therefore$

$u_x = u_s s_x + u_t t_x = 2x u_s \therefore$

$u_{xx} = 2u_s + 2x u_{ss} s_x + 2x u_{st} t_x = 2u_s + 4x^2 u_{ss}$

$u_y = u_s s_y + u_t t_y = 3y^2 u_t \therefore$

$u_{yy} = \cancel{3y}\ 6y u_t + 3y^2 u_{st} s_y + 3y^2 u_{tt} t_y = 6y u_t + 9y^4 u_{tt} \therefore$

$\frac{81}{x^2} u_{xx} - \frac{16}{y^4} u_{yy} - \frac{81}{x^3} u_x + \frac{32}{y^5} u_y =$

$\frac{81}{x^2} 2u_s + \frac{81}{x^2} 4x^2 u_{ss} - \frac{16}{y^4} 6y u_t - \frac{16}{y^4} 9y^4 u_{tt} - \frac{81}{x^3} 2x u_s + \frac{32}{y^5} 3y^2 u_t =$

$\frac{162}{x^2} u_s + 324 u_{ss} - \frac{96}{y^3} u_t - 144 u_{tt} - \frac{162}{x^2} u_s + \frac{96}{y^3} u_t =$

$324 u_{ss} - 144 u_{tt} = 0 = 9u_{ss} - 4u_{tt} = 0 \therefore$

let $u = f(s + \alpha t) \therefore u_{ss} = f''$, $u_{tt} = \alpha^2 f'' \therefore$

$9f'' - 4\alpha^2 f'' = 0 = (9-4\alpha^2)f''$ $\therefore 9-4\alpha^2 = 0 \therefore 9 = 4\alpha^2 \therefore \frac{9}{4} = \alpha^2 \therefore$

$\alpha = \pm\sqrt{\frac{9}{4}} = \pm\frac{3}{2} \therefore$ $\alpha = \frac{3}{2}$, $\alpha = -\frac{3}{2} \therefore$

$s + \alpha t \therefore$ $\tilde{\xi} = s + \frac{3}{2}t$, $\tilde{\eta} = s - \frac{3}{2}t \therefore$

$\xi = 2s+3t$, $\eta = 2s-3t \therefore$

$\cancel{u_\xi}\ \cancel{\xi_t =}\ \xi_s = 2$, $\xi_t = 3$, $\eta_s = 2$, $\eta_t = -3 \therefore$

$\cancel{u_\xi = u}$ $u_s = u_\xi \xi_s + u_\eta \eta_s = 2u_\xi + 2u_\eta$

$\therefore u_{ss} = 2u_{\xi\xi}\xi_s + 2u_{\xi\eta}\eta_s + 2u_{\xi\eta}\xi_s + 2u_{\eta\eta}\eta_s = 4u_{\xi\xi} + 8u_{\xi\eta} + 4u_{\eta\eta} \therefore$

$u_t = u_\xi \xi_t + u_\eta \eta_t = 3u_\xi - 3u_\eta \therefore$

$u_{tt} = 3u_{\xi\xi}\xi_t + 3u_{\xi\eta}\eta_t - 3u_{\xi\eta}\xi_t - 3u_{\eta\eta}\eta_t =$

$9u_{\xi\xi} + 18u_{\xi\eta} + 9u_{\eta\eta} \therefore$

$9u_{ss} - 4u_{tt} = 9(4u_{\xi\xi} + 8u_{\xi\eta} + 4u_{\eta\eta}) - 4(9u_{\xi\xi} + 18u_{\xi\eta} + 9u_{\eta\eta}) =$

$36u_{\xi\xi} + 72u_{\xi\eta} + 36u_{\eta\eta} - 36u_{\xi\xi} + 72u_{\xi\eta} - 36u_{\eta\eta} = 72u_{\xi\eta} = 0 = u_{\xi\eta} = \frac{\partial}{\partial\xi}\left(\frac{\partial}{\partial\eta}u\right)$

$\therefore \frac{\partial}{\partial\eta}u = \tilde{f}(\eta) \cancel{+ g(\xi)} \therefore$

$u = f(\eta) + g(\xi) \therefore$

$u(s,t) = f(2s+3t) + g(2s-3t) \therefore$

$u(x,t) = f(2x^2+3y^3) + g(2x^2-3y^3)$

\1c/ let $u = ve^{\alpha t} \therefore$ $u_t = (v_t + \alpha v)e^{\alpha t}$,

$\cancel{u_x = (v_x + \alpha v)}$ $u_x = v_x e^{\alpha t}$, $v_{xx} = v_{xx}e^{\alpha t} \therefore$

$u_t - u_{xx} + 2u = 0 \therefore$

$(v_t + \alpha v)e^{\alpha t} - v_{xx}e^{\alpha t} + 2ve^{\alpha t} = e^{\alpha t}(v_t + \alpha v - v_{xx} + 2v) = 0 = v_t + v(\alpha+2) - v_{xx} \therefore$

let $\alpha + 2 = 0 \therefore \alpha = -2 \therefore$

$u = ve^{-2t} \therefore$

$\cancel{u_{xx} - 2u =}$ $\therefore v_t - v_{xx} = 0 \therefore$ $v_t = v_{xx} \therefore$

$u(x,0) = \psi(x) = v(x, t=0)e^{-2(0)} = v(x,0) = \psi(x) \therefore$

$v(x,t) = \frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} \psi(s)e^{-(x-s)^2/(4t)}\,ds \therefore$

$u(x,t) = e^{-2t}v(x,t) = \frac{e^{-2t}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\psi(s)e^{-(x-s)^2/(4t)}\,ds$

$\psi(s) = \delta(s+1) + \delta(s-1) = \delta(s-(-1)) + \delta(s-1) \therefore$

$u(x,t) = \frac{e^{-2t}}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}(\delta(s-(-1)) + \delta(s-1))e^{-(x-s)^2/(4t)}\,ds =$

\PP2018/ $\frac{e^{-2t}}{\sqrt{4\pi t}}\left[\int_{-\infty}^{\infty}\delta(s-(-1))e^{-(x-s)^2/(4t)}ds+\int_{-\infty}^{\infty}\delta(s-1)e^{-(x-s)^2/(4t)}ds\right]=$

$\frac{e^{-2t}}{\sqrt{4\pi t}}\left[e^{-(x-(-1))^2/(4t)}+e^{-(x-1)^2/(4t)}\right]=\frac{e^{-2t}}{\sqrt{4\pi t}}\left[e^{-(x+1)^2/4t}+e^{-(x-1)^2/(4t)}\right]$ ∴

$P(t)=u(0,t)=u(x=0,t)=\frac{e^{-2t}}{\sqrt{4\pi t}}\left[e^{-\frac{1}{4t}}+e^{-\frac{1}{4t}}\right]=\frac{e^{-2t}}{\sqrt{\pi}t^{\frac{1}{2}}}e^{-\frac{1}{4t}}=\frac{1}{\sqrt{\pi}}t^{-\frac{1}{2}}e^{-2t-\frac{1}{4t}}$

$\therefore \ln P(t)=\ln\left(\frac{1}{\sqrt{\pi}}\right)+\ln\left(t^{-\frac{1}{2}}\right)-2t-\frac{1}{4t}=\ln\left(\frac{1}{\sqrt{\pi}}\right)-\frac{1}{2}\ln t-2t-\frac{1}{4t}$ ∴

$\frac{d\ln P(t)}{dt}=-\frac{1}{2}\frac{1}{t}-2+\frac{1}{4t^2}$

$\therefore \frac{d\ln P(t_*)}{dt}=-\frac{1}{2}\frac{1}{t_*}-2+\frac{1}{4t_*^2}=0=-\frac{1}{2}t_*-2t_*^2+\frac{1}{4}=0=2t_*^2+\frac{1}{2}t_*-\frac{1}{4}=0$ ∴

$t_*=\frac{-\frac{1}{2}\pm\sqrt{\frac{1}{4}-4(2)(-\frac{1}{4})}}{2(2)}=-\frac{1}{8}\pm\frac{\sqrt{\frac{9}{4}}}{4}=-\frac{1}{8}\pm\frac{3}{8}$ ∴

$t_*=-\frac{1}{2},\quad t_*=\frac{1}{4}$ ∴

$t_*>0 \quad \therefore t_*=\frac{1}{4}$

\1d/ $5u_{xx}+8u_{xy}+5u_{yy}=au_{xx}+2bu_{xy}+cu_{yy}$ ∴

$a=5,\ 2b=8 \therefore -2b=-8 \therefore b=4,\ c=5$ ∴

$b^2-ac=4^2-5(5)=16-25=-9<0$ ∴ elliptic ∴

$a\left(\frac{dy}{dx}\right)^2-2b\left(\frac{dy}{dx}\right)+c=5\left(\frac{dy}{dx}\right)^2-8\frac{dy}{dx}+5=0$ ∴

$\frac{dy}{dx}=\frac{8\pm\sqrt{64-4(5)(5)}}{2(5)}=\frac{8}{10}\pm\frac{\sqrt{-36}}{10}=\frac{8}{10}\pm\frac{6}{10}i$ ∴

$\frac{dy}{dx}=0.8+0.6i,\quad \frac{dy}{dx}=0.8-0.6i$ ∴

$\int 1dy=\int 0.8+0.6i\,dx=y=(0.8+0.6i)x+C_1,$

$y=(0.8-0.6i)x+C_2$ ∴

$y-(0.8+0.6i)x=C_1,\quad y-(0.8-0.6i)=C_2$ ∴

let $\xi=y-(0.8+0.6i)x,\quad \eta=y-(0.8-0.6i)x$ ∴

let ~~ξ ≠ β~~ $Re(\xi)=y-0.8x,\quad Im(\xi)=-0.6x$

$\tilde{\alpha}=y-\frac{4}{5}x,\quad \tilde{\beta}=-\frac{3}{5}x \therefore 5\tilde{\alpha}=5y-4x,\ -5\tilde{\beta}=+3x$ ∴

~~α = y~~ $\alpha=5y-4x,\ \beta=3x$ ∴

~~$u_x=$~~ $\alpha_x=-4,\ \alpha_y=5,\ \beta_x=3,\ \beta_y=0$ ∴

$u_x=u_\alpha\alpha_x+u_\beta\beta_x=-4u_\alpha+3u_\beta$

$u_{xx}=-4u_{\alpha\alpha}\alpha_x-4u_{\alpha\beta}\beta_x+3u_{\alpha\beta}\alpha_x+3u_{\beta\beta}\beta_x=16u_{\alpha\alpha}-24u_{\alpha\beta}+9u_{\beta\beta}$

$\cancel{u_\beta}$ $u_y = u_\alpha \alpha_y + u_\beta \beta_y = 5u_\alpha$ ∴

$u_{yy} = 5u_{\alpha\alpha}\alpha_y = 25u_{\alpha\alpha}$

$u_{xy} = 5u_{\alpha\alpha}\alpha_x + 5u_{\alpha\beta}\beta_x = -20u_{\alpha\alpha} + 15u_{\alpha\beta}$ ∴

$5u_{xx} + 8u_{xy} + 5u_{yy} + 30u_x + 6u_y + 90u =$

$5(16u_{\alpha\alpha}) + 5(-24)u_{\alpha\beta} + 5(9)u_{\beta\beta} + 8(-20u_{\alpha\alpha}) + 8(15u_{\alpha\beta}) + 5(25)u_{\alpha\alpha} + 30u_x + 6u_y + 90u =$

$45u_{\alpha\alpha} + (0)u_{\alpha\beta} + 45u_{\beta\beta} + 30(-4)u_\alpha + 30(3)u_\beta + 6(5)u_\alpha + 90u =$

$45u_{\alpha\alpha} + 45u_{\beta\beta} - 90u_\alpha + 90u_\beta + 90u = 0 = u_{\alpha\alpha} + u_{\beta\beta} - 2u_\alpha + 2u_\beta + 2u = 0$ ∴

$u_{\alpha\alpha} + u_{\beta\beta} = 2u_\alpha - 2u_\beta - 2u$

∴ let $u = ve^{\lambda\alpha + \mu\beta}$ ∴ ∴ let $e^{\lambda\alpha+\mu\beta} = g$ ∴

$u_\alpha = (v_\alpha + \lambda v)g$, $u_\beta = (v_\beta + \mu v)g$

$u_{\alpha\alpha} = (v_{\alpha\alpha} + 2\lambda v_\alpha + \lambda^2 v)g$, $u_{\beta\beta} = (v_{\beta\beta} + 2\mu v_\beta + \mu^2 v)g$ ∴

$[v_{\alpha\alpha} + 2\lambda v_\alpha + \lambda^2 v + v_{\beta\beta} + 2\mu v_\beta + \mu^2 v - 2v_\alpha - 2\lambda v + 2v_\beta + 2\mu v + 2v]g = 0$

$= [v_{\alpha\alpha} + v_\alpha(2\lambda - 2) + v_\beta(2\mu + 2) + v(\lambda^2 + \mu^2 - 2\lambda + 2\mu + 2) + v_{\beta\beta}] = 0$ ∴

let $2\lambda - 2 = 0$ ∴ $\lambda = 1$, let $2\mu + 2 = 0$ ∴ $\mu = -1$ ∴

$\lambda^2 + \mu^2 - 2\lambda + 2\mu + 2 = 1 + 1 - 2 - 2 + 2 = 0$ ∴

$v_{\alpha\alpha} + v_{\beta\beta} = 0$ ∴

\1e/ let $u = X(x)Y(y)$ ∴ $u_{xx} = X''Y$, $u_{yy} = XY''$ ∴

$X''Y + XY'' = 0$ ∴ $X(0) = 0$, $X(1) = 0$, $Y(1) = 0$, $Y(0) = f(0)$ ∴

$X''Y = -XY''$ ∴ $\frac{X''}{X} = -\frac{Y''}{Y} = -\lambda =$ Constant ∴

$X'' = -\lambda X$, $\cancel{Y''=\lambda}$ $Y'' = \lambda Y$ ∴

$X'' + \lambda X = 0$ ∴

for $\lambda = 0$: $X'' = 0$ ∴ $X' = A$ ∴ $X = Ax + B$ ∴

$X(0) = 0 = A(0) + B = B = 0$ ∴ $X = Ax$ ∴ $X(1) = 0 = A(1) = A = 0$ ∴ $X = 0$

~~for $\lambda < 0$ ∴ $\lambda = -q^2 < 0$ ∴ $X'' - q^2 X = 0$ ∴ $\lambda = -q^2 < 0$ ∴~~

~~$X'' - a^2 X = 0$ ∴ let $X = e^{qx}$ ∴ $q^2 e^{qx} - a^2 e^{qx} = 0$ ∴ $q^2 - a^2 = 0$ ∴ $q^2 = a^2$ ∴~~

~~$q = a$, $q = -a$ ∴ $q = \sqrt{\lambda}i$, $q = -\sqrt{\lambda}i$ ∴~~

~~$X = A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x)$ ∴~~

~~$X(0) = A\cos(0) + B\sin(0) = 0 = A$ ∴ $X = B\sin(\sqrt{\lambda}x)$ ∴~~

~~$X(1) = B\sin(\sqrt{\lambda}(1)) = B\sin(\sqrt{\lambda}) = 0$ ∴ $\sin(\sqrt{\lambda}) = 0 = \sin(n\pi)$ ∴ $\sqrt{\lambda} = n\pi$~~

√PP2018/ i. $\lambda = n^2\pi^2 \therefore$

$X = B\sin(n\pi x)$

i)) for $\lambda > 0$ i.e. $\lambda = a^2 > 0 \therefore \sqrt{\lambda} = \pm a \therefore$

$X'' + a^2 X = 0 \therefore$ let $X = e^{qx} \therefore X'' = q^2 e^{qx} \therefore$

$q^2 e^{qx} + a^2 e^{qx} = 0 = q^2 + a^2 \therefore q^2 = -a^2 \therefore \cancel{q = \pm\sqrt{a}}\ \ q = \pm a i \therefore$

$\cancel{X = Ae}\ \ X = A_2 e^{aix} + B_2 e^{-aix} = A\cos(ax) + B\sin(ax) =$

$A\cos(\sqrt{\lambda}x) + B\sin(\sqrt{\lambda}x) \therefore$

$X(0) = A = 0 \quad \therefore X = B\sin(\sqrt{\lambda}x) \therefore$

$X(1) = 0 = B\sin(\sqrt{\lambda}\,1) = B\sin(\sqrt{\lambda}) = 0 \therefore \sin(\sqrt{\lambda}) = 0 \therefore \sqrt{\lambda} = n\pi \therefore \lambda = n^2\pi^2,$
$n \in \mathbb{N}$

for $\lambda < 0 \therefore \lambda = -a^2 < 0 \therefore X = e^{qx} \therefore X'' = q^2 e^{qx} \therefore$

$q^2 e^{qx} - a^2 e^{qx} = 0 = q^2 - a^2 = 0 \therefore q^2 = a^2 \therefore q = \pm a \therefore$

$X = Ae^{ax} + Be^{-ax} \therefore X(0) = A + B = 0 \therefore B = -A \therefore$

$X = A(e^{ax} - e^{-ax}) \therefore X(1) = 0 = A(e^{a} - e^{-a}) = A = 0 \therefore X = 0 \therefore$

$\lambda = n^2\pi^2, \quad X = B\sin(\sqrt{\lambda}x) = B\sin(n\pi x) \therefore$

$Y'' - \lambda Y = 0 = Y'' - n^2\pi^2 Y = 0 \therefore$

$Y = e^{qy} \therefore \cancel{q}\ Y'' = q^2 e^{qy} \therefore q^2 e^{qy} - n^2\pi^2 e^{qy} = 0 = q^2 - n^2\pi^2 \therefore$

$q^2 = n^2\pi^2 \therefore q = \pm n\pi \therefore$

$\cancel{Y = A\cos}\ \ Y = Ae^{n\pi y} + Be^{-n\pi y} \therefore$

$\cancel{Y(0)}\ \ Y(1) = Ae^{n\pi} + Be^{-n\pi}\cancel{(1)} = 0 \therefore Be^{-n\pi} = -Ae^{n\pi} \therefore$

$B = -Ae^{n\pi}e^{n\pi} = -Ae^{2n\pi} \therefore$

$\cancel{Y = Ae^{2n y}}\ \ Y = Ae^{n\pi y} - Ae^{2n\pi}e^{-n\pi y} = Ae^{n\pi y} - Ae^{n\pi(2-y)} = A(e^{n\pi y} - e^{n\pi(2-y)}) \therefore$

$u(x,y) = X(x)Y(y) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)(e^{n\pi y} - e^{n\pi(2-y)}) \therefore$

$u(x,0) = f(x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)(e^{0} - e^{n\pi(2-0)}) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)(1 - e^{2n\pi}) = f(x)$

$\therefore \cancel{A_n = \frac{2}{1}\int_0^1 f(x)}\ \ A_n(1 - e^{2n\pi}) = \frac{2}{1}\int_0^1 f(x)\sin(n\pi x)\,dx \therefore$

$A_n = \frac{2}{1 - e^{2n\pi}}\int_0^1 f(x)\sin(n\pi x)\,dx = \frac{2}{1 - e^{2n\pi}}\int_0^1 f(\xi)\sin(n\pi\xi)\,d\xi \therefore$

$u(x,y) = \sum_{n=1}^{\infty} \frac{2(e^{n\pi y} - e^{n\pi(2-y)})}{1 - e^{2n\pi}}\int_0^1 f(\xi)\sin(n\pi\xi)\,d\xi\,\sin(n\pi x) \therefore$

for $f(x) = \sin(3\pi x) \therefore$

$f(x) = \sin(3\pi x) = \sum_{n=1}^{\infty} A_n \sin(n\pi x)(1-e^{2n\pi}) =$

$A_3 \sin(3\pi x)(1-e^{2n\pi}) + \sum_{n\neq 3}^{\infty} A_n \sin(n\pi x)(1-e^{2n\pi}) \therefore$

$A_n = 0 \ \forall n \neq 3 \ , \ A_3 \sin(3\pi x)(1-e^{2n\pi}) = \sin(3\pi x) \therefore$

$A_3 = \frac{1}{1-e^{2n\pi}} \therefore$

$u(x,y) = \frac{1}{1-e^{2n\pi}} \sin(3\pi x)(e^{n\pi y} - e^{n\pi(2-y)}) \therefore$

$u(x,0) = \frac{1}{1-e^{2n\pi}} \sin(3\pi x)(\cancel{1} - e^{2n\pi}) = \sin(3\pi x)$,

$u(0,y) = \frac{1}{1-e^{2n\pi}} \sin(0)(e^{n\pi y} - e^{n\pi(2-y)}) = \sin(3\pi y)$

$u(x,1) = \frac{1}{1-e^{2n\pi}} \sin(3\pi x)(e^{n\pi} - e^{n\pi}) = 0$,

$u(1,y) = \frac{1}{1-e^{2n\pi}} \sin(3\pi)(e^{n\pi y} - e^{n\pi(2-y)}) = 0$ as required

\2a/ $u_{xx} - u_{tt} = -u_{tt} + u_{xx} = a u_{tt} + 2b u_{tx} + c u_{xx} \therefore$

$a = -1 \ , \ 2b = 0 = -2b = b \ , \ c = 1 \therefore$

$b^2 - ac = 0^2 - (-1)(1) = 1 > 0 \therefore$ hyperbolic

$a\left(\frac{dx}{dt}\right)^2 - 2b\left(\frac{dx}{dt}\right) + c = -\left(\frac{dx}{dt}\right)^2 + 1 = 0 \therefore \quad 1 = \left(\frac{dx}{dt}\right)^2 \therefore$

$\frac{dx}{dt} = 1 \ , \ \frac{dx}{dt} = -1 \therefore \quad x = \int 1\,dt = t + c_1 \ , \ x = \int -dt = -t + c_2 \therefore$

$x - t = c_1 \ , \ x + t = c_2 \therefore$

$\xi = x - t \ , \ \eta = x + t \therefore$

$\xi_x = 1 \ , \ \xi_t = -1 \ , \ \eta_x = 1 \ , \ \eta_t = 1 \therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta \therefore$

$u_{xx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x + u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$

$u_t = u_\xi \xi_t + u_\eta \eta_t = -u_\xi + u_\eta \therefore$

$u_{tt} = -u_{\xi\xi}\xi_t - u_{\xi\eta}\eta_t + u_{\xi\eta}\xi_t + u_{\eta\eta}\eta_t = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta} \therefore$

$-u_{tt} + u_{xx} = -u_{\xi\xi} + 2u_{\xi\eta} - u_{\eta\eta} + u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} = 4u_{\xi\eta} = 0 = u_{\xi\eta} = \frac{\partial}{\partial\xi}\left(\frac{\partial}{\partial\eta} u\right)$

$\therefore \frac{\partial u}{\partial\eta} = f(\eta) \therefore$

$u = f(\eta) + g(\xi) = f(\xi) + g(\eta) = f(x-t) + g(x+t) = u(x,t)$

\2b/ $u_{xx} - u_{tt} = 0 \quad u(x,0) = \phi(x) \ , \ u_t(x,0) = \psi(x)$

$u(x,t) = f(x-t) + g(x+t) \therefore$

$u(x,0) = u(x,t=0) = f(x-0) + g(x+0) = f(x) + g(x) = \phi(x)$,

~~PP2018~~ $u_t(x,t) = -f'(x-t) + g'(x+t)$ ∴

$u_t(x,0) = u_t(x,t=0) = -f'(x-0) + g'(x+0) = -f'(x) + g'(x) = \psi(x)$

1) ∴ $g'(x) = \psi(x) + f'(x)$ ∴

$g(x) = \phi(x) - f(x)$ ∴

$g'(x) = \phi'(x) - f'(x)$ ∴

$g'(s) = \psi(s) + f'(s) = g'(s) = \phi'(s) - f'(s)$ ∴

$2f'(s) = \phi'(s) - \psi(s)$

∴ $\cancel{f'(s) = \phi}$ $f'(s) = \frac{1}{2}\phi'(s) - \frac{1}{2}\psi(s)$ ∴

$f(s) = \frac{1}{2}\phi(s) - \frac{1}{2}\int \psi(s)\,ds + A$, A = constant

∴ $\cancel{g(x)=}$ $g(s) = \phi(s) - f(s)$ ∴

$\cancel{g(s)=\phi}$ $g(s) = \frac{1}{2}\phi(s) + \frac{1}{2}\int \psi(s)\,ds - A$ ∴

$f(x-t) = \frac{1}{2}\phi(x-t) - \frac{1}{2}\int \psi(x-t)\,d(x-t) + A$,

$g(x+t) = \frac{1}{2}\phi(x+t) + \frac{1}{2}\int \psi(x+t)\,d(x+t) - A$ ∴

$u(x,t) = f(x-t) + g(x+t) = \frac{1}{2}\phi(x-t) - \frac{1}{2}\int \psi(x-t)\,d(x-t) + A + \frac{1}{2}\phi(x+t) + \frac{1}{2}\int \psi(x+t)\,d(x+t) - A =$

$\frac{1}{2}\phi(x-t) - \frac{1}{2}\int \psi(x-t)\,d(x-t) + \frac{1}{2}\phi(x+t) + \frac{1}{2}\int \psi(x+t)\,d(x+t)$

$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0$ $\quad u(\xi,0) = u(\xi,\eta=0) = \chi(\xi)$, $u_\eta(\xi,0) = u_\eta(\xi,\eta=0) = \omega(\xi)$

∴ $u_\eta = f'(\eta)$ ∴ $u = f(\eta) + g(\xi)$ ∴

$u(\xi,0) = \chi(\xi) = f(0) + g(\xi)$ ∴ let $f(0) = A$ = constant

$u_\eta(\xi,0) = f'(0) = B = \text{constant} = \omega(\xi)$ ∴

$g(\xi) = \chi(\xi) - A$ ∴ $u = f(\eta) + \chi(\xi) - A$ ∴

$u_\eta = f'(\eta)$ ∴

The ~~solution~~ BC are orthogonal and cannot be used to calculate an exact solution ∴ has either infinitely many or no solutions ∴

$u_{\xi\eta} = \frac{\partial}{\partial \xi}\left(\frac{\partial}{\partial \eta} u\right)$

2) ~~$u_\eta = f'$~~ depending on if $\omega(\xi)$ is independent of ξ or not ∴ if independent then infinite solutions and if not then no solutions

\2a/ $u = f(\xi) + g(\eta)$ ∴ $u(\xi, 0) = \rho(\xi)$, $u(0,\eta) = \sigma(\eta)$

$\rho(0) = \eta(0) = A$ ∴

$u(\xi, 0) = f(\xi) + g(0) = \rho(\xi) = f(\xi) + B$ ∴

$f(\xi) = \rho(\xi) - B$ ∴ $f(s) = \rho(s) - B$ ∴ $f(0) = \rho(0) - B = A - B$ ∴

$u(0,\eta) = \sigma(\eta) = f(0) + g(\eta) = A - B + g(\eta)$ ∴

$g(\eta) = \sigma(\eta) - A + B$ ∴

$g(s) = \sigma(s) - A + B$ ∴

$g(0) = \sigma(0) - A + B$ ∴

$\cancel{f(s)}$ ∴ $g(\eta) = \sigma(\eta) - A + B$,

$f(\xi) = \rho(\xi) - B$ ∴

$u = f(\xi) + g(\eta) = \rho(\xi) - B + \sigma(\xi) - A + B = \rho(\xi) + \sigma(\xi) - A$

\3a/ $u_t - u_{xx} = f(x,t)$ $u(x,0) = \phi(x)$, $u(0,t) = \psi_0(t)$, $u(L,t) = \psi_L(t)$

$x \in [0, L]$, $t \geq 0$ ∴

let u_1, u_2 be two solutions of the equation: $u_{1t} - u_{1xx} = f$, $u_1(x,0) = \phi(x)$

$u_1(0,t) = \psi_0(t)$, $u_1(L,t) = \psi_L(t)$

$u_{2t} - u_{2xx} = f$, $u_2(x,0) = \phi(x)$, $u_2(0,t) = \psi_0(t)$, $u_2(L,t) = \psi_L(t)$

∴ let $w = u_1 - u_2$ ∴

$w_t - w_{xx} = (u_1 - u_2)_t - (u_1 - u_2)_{xx} = u_{1t} - u_{2t} - u_{1xx} + u_{2xx} =$

$(u_{1t} - u_{1xx}) - (u_{2t} - u_{2xx}) = f - f = 0$ ∴ $w_t = w_{xx}$

$w(x,0) = u_1(x,0) - u_2(x,0) = \phi(x) - \phi(x) = 0$

$w(0,t) = u_1(0,t) - u_2(0,t) = \psi_0(t) - \psi_0(t) = 0$

$w(L,t) = u_1(L,t) - u_2(L,t) = \psi_L(t) - \psi_L(t) = 0$ ∴

∴ let $E(t) = \int_0^L w^2(x,t)\,dx$ ∴

$\frac{dE(t)}{dt} = \frac{\partial E(t)}{\partial t} = \frac{\partial}{\partial t}\int_0^L w^2(x,t)\,dx = \int_0^L \frac{\partial}{\partial t} w^2(x,t)\,dx = \int_0^L 2w\frac{\partial w}{\partial t}\,dx =$

$\int_0^L 2w w_{xx}\,dx = 2\int_0^L w w_{xx}\,dx = 2\Big[w w_x\Big]_0^L - 2\int_0^L w_x w_x\,dx =$

$\cancel{2[ww]}$ $2\Big[w(L,t)w_x(L,t) - w(0,t)w_x(0,t)\Big] - 2\int_0^L (w_x)^2\,dx =$

$2\Big[0\,w_x(L,t) - w_x(0,t)(0)\Big] - 2\int_0^L (w_x)^2\,dx = \cancel{-2\int} 0 - 2\int_0^L (w_x)^2\,dx = -2\int_0^L (w_x)^2\,dx = \frac{dE(t)}{dt}$

\PP2018/ $\therefore (w_x)^2 \geq 0 \quad \therefore \quad \int_0^L (w_x)^2\,dx \geq 0 \therefore$

$\frac{dE(t)}{dt} = -2\int_0^L (w_x)^2\,dx \leq 0 \therefore \quad \frac{dE(t)}{dt} \leq 0$

and $(w)^2 \geq 0 \therefore \quad \int_0^L w^2(x,t)\,dx \geq 0 \therefore$

$E(t) = \int_0^L w^2(x,t)\,dx \geq 0 \therefore \quad E(t) \geq 0 \therefore$

$E(0) = \int_0^L (w(0,t))^2\,dx = E(0) = \int_0^L (0)^2\,dx = 0$

$\therefore E(0) = 0$ but $\frac{dE(t)}{dt} \leq 0 \therefore E(t_1) \geq E(t_2)$ for $t_1 > t_2 \therefore$

$E(0) \geq E(t) \quad \forall t > 0$

but $E(t) \geq 0$ and $E(0) = 0 \geq E(t) \therefore$

$0 \geq E(t) \geq 0 \therefore \quad E(t) = 0 \therefore$

$E(t) = \int_0^L (w)^2\,dx = 0 \therefore \quad w^2 \geq 0 \quad \therefore w^2 = 0 \therefore$

$w = 0 \therefore \quad w = u_1 - u_2 = 0 \therefore$

$u_1 \equiv u_2 \quad \therefore u_1(x,t) \equiv u_2(x,t) \therefore$ $u(x,t)$ only has one solution

\3b/ $\|g(x)\| = \int_0^L |g(x)|^2\,dx \qquad u_t - u_{xx} = f(x,y)$.

$u(0,t) = \psi_0(t)$, $u(L,t) = \psi_L(t)$, $t \geq 0 \therefore$

$\|u_1(x,t) - u_2(x,t)\| = \int_0^L |(u_1(x,t) - u_2(x,t))|^2\,dx \leq \int_0^L (|u_1(x,t)| + |-u_2(x,t)|)^2\,dx$

$\|u_1(x,0) - u_2(x,0)\| = \int_0^L |u_1(x,0) - u_2(x,0)|^2\,dx \geq \int_0^L (||u_1(x,0)| - |u_2(x,0)||)^2\,dx$

$\therefore \|u_1(x,t) - u_2(x,t)\| \leq \int_0^L (|u_1(x,t)| + |u_2(x,t)|)^2\,dx =$

$\int_0^L |u_1(x,t)|^2 + 2|u_1(x,t)||u_2(x,t)| + |u_2(x,t)|^2\,dx =$

$\int_0^L u_1(x,t)^2 + 2|u_1(x,t)||u_2(x,t)| + u_2(x,t)^2\,dx$

~~let z =~~ $\cancel{\frac{d}{dt}}$ let $E(t) = \|u_1(x,t) - u_2(x,t)\| \therefore$

$\frac{dE(t)}{dt} = \frac{d}{dt}\|u_1(x,t) - u_2(x,t)\| = \frac{\partial}{\partial t}\int_0^L |u_1(x,t) - u_2(x,t)|^2\,dx =$

$\int_0^L \frac{\partial}{\partial t}|u_1(x,t) - u_2(x,t)|^2\,dx = \int_0^L 2|u_1(x,t) - u_2(x,t)|\,\frac{\partial}{\partial t}|u_1(x,t) - u_2(x,t)|\,dx =$

$2\int_0^L |u_1(x,t) - u_2(x,t)||u_{1t} - u_{2t}|\,dx = 2\int_0^L |u_1 - u_2||u_{1xx} + f - u_{2xx} - f|\,dx$

$= 2\int_0^L |u_1 - u_2|\,|u_{1xx} - u_{2xx}|\,dx$

\3b/ ~~HZ~~ at $x = 0$: $\|u_1(x,t) - u_2(x,t)\|\big|_{x=0} = \|u_1(0,t) - u_2(0,t)\| =$

$\int_0^L |u_1(0,t) - u_2(0,t)|^2\,dx = \int_0^L |\psi_0(t) - \psi_0(t)|^2\,dx = \int_0^L |0|^2\,dx = 0 \therefore$

$$||u_1(x,0) - u_2(x,0)|| = \int_0^L |u_1(x,0) - u_2(x,0)|^2 \, dx \geq 0 \quad \therefore$$

$$||u_1(0,t) - u_2(0,t)|| \leq ||u_1(x,0) - u_2(x,0)||$$

$$\therefore ||u_1(x,t) - u_2(x,t)|| \leq ||u_1(x,0) - u_2(x,0)||$$

3c $u_t - u_{xx} = 0$, $u(x,0) = \phi(x)$

$$u(x,t) = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-(x-s)^2/(4t)} \phi(s) \, ds$$

is $M(t) = \max_{x \in \mathbb{R}} u(x,t)$ $\therefore M(t) \leq M(0)$ $\forall t \geq 0 \therefore$

$$M(t) = \max_{x \in \mathbb{R}} u(x,t) = \max_{x \in \mathbb{R}} \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-(x-s)^2/(4t)} \phi(s) \, ds \quad \therefore$$

$$\frac{\partial}{\partial x} e^{-(x-s)^2/(4t)} = e^{-(x-s)^2/(4t)} \frac{\partial}{\partial x}\left[-(x-s)^2 \frac{1}{4t}\right] = e^{-(x-s)^2/(4t)} (-2)\frac{1}{4t}(x-s)'(1) =$$

$$-\frac{1}{2t}(x-s) e^{-(x-s)^2/(4t)} \leq 0 \text{ for } x \geq s$$

$\cancel{\max_{x \in \mathbb{R}} \frac{\partial}{\partial x}}$

$$\ln\left(e^{-(x-s)^2/(4t)}\right) = -\frac{(x-s)^2}{4t} = \ln(g(x)) \quad \therefore$$

$$\frac{\partial}{\partial x} \ln(g(x))\Big|_{x_*} = \frac{\partial}{\partial x}\left(-\frac{(x-s)^2}{4t}\right)\Big|_{x_*} = -2\frac{(x-s)'}{4t}(1)\Big|_{x_*} = -\frac{(x-s)}{2t}\Big|_{x_*} = -\frac{x_* - s}{2t} = 0$$

$$\therefore (x_* - s)(-1) = 0 = x_* - s \quad \therefore \quad x_* = s \quad \therefore$$

$$\max_{x \in \mathbb{R}} e^{-(x-s)^2/(4t)} \text{ for } x = x_* = s \therefore$$

$$M(t) = \max_{x \in \mathbb{R}} u(x,t) = \max_{x \in \mathbb{R}} \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-(x-s)^2/(4t)} \phi(s) \, ds =$$

$$\frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-(s-s)^2/(4t)} \phi(s) \, ds = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} e^{-\frac{0^2}{4t}} \phi(s) \, ds = \frac{1}{2\sqrt{\pi t}} \int_{-\infty}^{\infty} \phi(s) \, ds$$

$$\therefore M(0) = \frac{1}{2\sqrt{\pi(0)}} \int_{-\infty}^{\infty} \phi(s) \, ds = \frac{1}{2(0)} \int_{-\infty}^{\infty} \phi(s) \, ds = \frac{1}{0} \int_{-\infty}^{\infty} \phi(s) \, ds = \infty$$

$$\because \phi(s) \geq 0 \therefore \int_{-\infty}^{\infty} \phi(s) \, ds = \text{constant} \geq 0 \therefore$$

$$M(t) \leq M(0) = \infty$$

4a let $u(x,t) = r(x,t)\cos(\theta(x,t)) = r\cos\theta = u$,

$v(x,t) = r(x,t)\sin(\theta(x,t)) = r\sin\theta = v \therefore$

$$w_t = \frac{\partial}{\partial t}(re^{i\theta}) = r_t e^{i\theta} + r\theta_t i e^{i\theta}$$

$$w = re^{i\theta} = r\cos\theta + ir\sin\theta = u + iv$$

$\sqrt{PP2015}$ ∴ for real component of W:

$u_t = \partial_t(r\cos\theta) = r_t\cos\theta + -r\sin\theta\,\theta_t$

$0 = -w_t + w + (1+i\alpha)w_{xx} - (1+i\beta)|w|^2 w$

∴ $w = re^{i\theta}$ ∴ $w_t = r_t e^{i\theta} + re^{i\theta}\theta_t i = (r_t + r\theta_t i)e^{i\theta}$

$w_x = r_x e^{i\theta} + re^{i\theta}\theta_x i = (r_x + r\theta_x i)e^{i\theta}$ ∴

∴ $w_{xx} = (r_{x} + r_x\theta_x i + r\theta_{xx} i)e^{i\theta} + (r_x + r\theta_x i)e^{i\theta}\theta_x i =$

$(r_x + r_x\theta_x i + r\theta_{xx} i + r_x\theta_x i + r\theta_x^2(-1))e^{i\theta} =$

$(r_x + 2r_x\theta_x i + r\theta_{xx} i - r\theta_x^2)e^{i\theta}$ ∴

$w = r\cos\theta + ir\sin\theta$ ∴

$|w|^2 = \sqrt{(r\cos\theta)^2 + (r\sin\theta)^2}^2 = \sqrt{r^2(\cos^2\theta + \sin^2\theta)}^2 = r^2(1) = r^2$

∴ $|w|^2 w = r^2 r e^{i\theta} = r^3 e^{i\theta} = r^3\cos\theta + ir^3\sin\theta$ ∴

taking real components:

$\mathbb{R}\left[(1+i\alpha)(r_x + 2r_x\theta_x i + r\theta_{xx} i - r\theta_x^2)(\cos\theta + i\sin\theta)\right] = \mathbb{R}\left[(1+i\alpha)w_{xx}\right]$

$= \mathbb{R}\left[1(r_x\cos\theta - r\theta_x^2\cos\theta - 2r_x\theta_x\sin\theta - r\theta_{xx}\sin\theta) + \right.$

$\left. i\alpha(2r_x\theta_x i\cos\theta + r\theta_{xx} i\cos\theta + r_x i\sin\theta - r\theta_x^2 i\sin\theta)\right] =$

$\left[r_x\cos\theta - r\theta_x^2\cos\theta - 2r_x\theta_x\sin\theta - r\theta_{xx}\sin\theta - 2\alpha r_x\theta_x\cos\theta - \alpha r\theta_{xx}\cos\theta\right.$

$\left. - \alpha r_x\sin\theta + \alpha r\theta_x^2\sin\theta\right]$

$\mathbb{R}[w] = \mathbb{R}[r\cos\theta + ir\sin\theta] = r\cos\theta$

$\mathbb{R}\left[-(1+i\beta)|w|^2 w\right] = \mathbb{R}\left[-(1+i\beta)(r^3\cos\theta + ir^3\sin\theta)\right] =$

$\left[-r^3\cos\theta - (-1)\beta r^3\sin\theta\right] = -r^3\cos\theta + \beta r^3\sin\theta$ ∴

$\mathbb{R}[w_t] = \mathbb{R}\left[(r_t + r\theta_t i)e^{i\theta}\right] = \mathbb{R}\left[(r_t + r\theta_t i)(\cos\theta + i\sin\theta)\right] =$

$\left[r_t\cos\theta - r\theta_t\sin\theta\right]$ ∴

taking $\cos\theta$ component of equation:

$r_t = r + r_{xx} - r\theta_x^2 - 2\alpha r_x\theta_x - (\alpha r\theta_{xx}) - r^3$ ∴

$r_t = r + r_{xx} - r\theta_x^2 - \alpha(2r_x\theta_x + r\theta_{xx}) - r^3$

taking $\sin\theta$ component of equation:

$-r\theta_t = -r\theta_{xx} - 2r_x\theta_x - \alpha r_{xx} + \alpha r\theta_x^2 + \beta r^3$ ∴

$r\theta_t = r\theta_{xx} + 2r_x\theta_x + \alpha r_{xx} - \alpha r\theta_x^2 - \beta r^3$ ∴

$r\theta_t = r\theta_{xx} + 2r_x\theta_x + \alpha(r_{xx} - r\theta_x^2) - \beta r^3$

$\cancel{4}$b/ $r = R(\xi) = R(x-ct)$ $\quad \theta = \Theta(\xi) = \Theta(x-ct)$ $\therefore$

$\xi = x - ct$ $\therefore$ $\xi_x = 1$, $\xi_t = -c$ $\therefore$

$r_t = R'\xi_t = -cR'$, $\theta_t = \Theta'\xi_t = -c\Theta'$

$r_x = R'\xi_x = R'$, $\theta_x = \Theta'\xi_x = \Theta'$

$r_{xx} = R''\xi_x = R''$ $\quad \theta_{xx} = \Theta''\xi_x = \Theta''$ $\therefore$

$-r_t + r + r_{xx} - r\theta_x^2 - \alpha(2r_x\theta_x + r\theta_{xx}) - r^3 = 0$

$-r\theta_t + r\theta_{xx} + 2r_x\theta_x + \alpha(r_{xx} - r\theta_x^2) - \beta r^3 = 0$ $\therefore$

$cR' + R + R'' - R(\Theta')^2 - \alpha(2R'\cancel{\Theta'} + R\Theta'') - R^3 = 0$

$+cR\Theta' + R\Theta'' + 2R'\Theta' + \alpha(R'' - R(\Theta')^2 - \beta R^3 = 0$ for $R(\xi)$, $\Theta(\xi)$

$\cancel{4}$c/ $r_t = r + r_{xx} - r\theta_x^2 - \alpha(2r_x\theta_x + r\theta_{xx}) - r^3$

$r\theta_t = r\theta_{xx} + 2r_x\theta_x + \alpha(r_{xx} - r\theta_x^2) - \beta r^3$

$cR' + R + R'' - R(\Theta')^2 - \alpha(2R'\Theta' + R\Theta'') - R^3 = 0$

$cR\Theta' + R\Theta'' + 2R'\Theta' + \alpha(R'' - R(\Theta')^2 - \beta R^3 = 0$

$\therefore$ $R = \text{Constant}$, $\Theta' = \text{Constant}$

$\therefore$ let $R = a = \text{Constant}$, $\Theta' = \cancel{\text{const}}\ k = \text{Constant}$ $\therefore$

$R' = 0$, $\Theta'' = 0$ $\therefore$ $R'' = 0$ $\therefore$ $\Theta'(\xi) = k = \frac{d\Theta(\xi)}{d\xi}$ $\therefore$ $\Theta(\xi) = k\xi$ $\therefore$

$c(0) + a + 0 - a(k)^2 - \alpha(2(0)k + a(0)) - a^3 =$

$a - ak^2 - \alpha(0+0) - a^3 = a - ak^2 - a^3 = 0 = a(1 - k^2 - a^2) = 0 = 1 - k^2 - a^2$

$cak + a(0) + 2(0)k + \alpha(0 - a(k)^2) - a^3 = cak - \alpha a k^2 - a^3 = 0 =$

$a(ck - \alpha k^2 - a^2) = 0 = ck - \alpha k^2 - a^2$ $\therefore$

$k^2 = 1 - a^2$ $\therefore$ $k = \pm\sqrt{1-a^2}$ $\therefore$

$ck = \alpha k^2 + a^2$ $\therefore$ $\cancel{c = \alpha k + \alpha}$ $c = \alpha k + \frac{a^2}{k}$ $\therefore$

let $ck = \omega_1$ $\therefore$ $\omega_1 = \alpha k^2 + a^2 = \omega_1(k)$ $\therefore$

$a^2 = 1 - k^2 = a^2(k)$ $\therefore$ $a = a(k) = \pm\sqrt{1-k^2}$ $\therefore$ $a(k) > 0$ $\therefore$ $a(k) = \sqrt{1-k^2}$ $\therefore$

$w(x,t) = re^{i\theta} = R(\xi)e^{i\Theta(\xi)} = ae^{ik\xi} = a(k)e^{ik(x-ct)} =$

$a(k)e^{i(kx - i\frac{ck}{i}t)}$ $\therefore$ let $\frac{ck}{i}\cancel{t} = \omega(k) = \frac{1}{i}(\alpha k^2 - a^2) = -i(\alpha k^2 - a^2) =$

$-i(\alpha k^2 - k^2 - 1)$ $\therefore$

$\therefore$ $w(x,t) = a(k)e^{i(kx - i\omega(k)t)}$ for $\cancel{\phi}$ $-1 \le k \le 1$

$w - \dots$

\PP2017\1a/ $\therefore (xy, 1+y^2)\cdot(u_x, u_y) = 0 = (xy, 1+y^2)\cdot\nabla u \therefore$

$\frac{dy}{dx} = \frac{1+y^2}{xy} \therefore \quad \frac{y\,dy}{dx} = \frac{1+y^2}{x} \therefore \quad \frac{1}{2}\frac{2y}{1+y^2}\frac{dy}{dx} = \frac{1}{x} \therefore$

$\frac{1}{2}\int \frac{2y}{1+y^2}\,dy = \int \frac{1}{x}\,dx = \ln|x| + C_1 = \frac{1}{2}\ln|1+y^2| \therefore$

$\ln|1+y^2| = 2\ln|x| + C_2 = \ln|x^2| + C_2 = \ln x^2 + C_2 \therefore$

$|1+y^2| = e^{\ln x^2 + C_2} = C_3 e^{\ln x^2} = C_3 x^2 \therefore$

$1+y^2 = C_4 x^2 \quad \therefore \quad C_4 = \frac{1+y^2}{x^2} \quad \therefore u(x,y) = f\left(\frac{1+y^2}{x^2}\right) = g\left(\frac{x^2}{1+y^2}\right)$

\1ai/ $u(0,y) = y^2 = g\left(\frac{0^2}{1+y^2}\right) = g(0) = y^2$ which is impossible

$\therefore$ no solution exists $\because$ the BC is set on the characteristic curve

\1aii/ $u(x,0) = x^2 = g\left(\frac{x^2}{1+0^2}\right) = g(x^2) = x^2 \therefore$

$s = x^2 \therefore g(s) = s \therefore g\left(\frac{x^2}{1+y^2}\right) = \frac{x^2}{1+y^2} \therefore$

$u(x,y) = g\left(\frac{x^2}{1+y^2}\right) = \frac{x^2}{1+y^2}$

\1b/ let $x = \frac{1}{s}$, $y = \frac{1}{t} \quad \therefore \quad s = \frac{1}{x} = x^{-1}$, $t = \frac{1}{y} = y^{-1} \therefore$

$s_x = -x^{-2}$, $s_y = 0$, $t_y = -y^{-2}$, $t_x = 0 \quad \therefore$

$u_x = u_s s_x = -x^{-2}u_s \therefore$

$u_{xx} = 2x^{-3}u_s - x^{-2}u_{ss}s_x = 2x^{-3}u_s + x^{-4}u_{ss}$

$u_y = u_t t_y = -y^{-2}u_t \therefore$

$u_{yy} = 2y^{-3}u_t - y^{-2}u_{tt}t_y = 2y^{-3}u_t + y^{-4}u_{tt} \therefore$

$x^4u_{xx} + 2x^3u_x - 4y^4u_{yy} - 8y^3u_y =$

$2xu_s + u_{ss} - 2xu_s - 8yu_t - 4u_{tt} + 8yu_t = u_{ss} - 4u_{tt} = 0$

$\therefore$ let $u = f(\alpha s + t) \quad \therefore u_s = \alpha f'$, $u_{ss} = \alpha^2 f''$, $u_t = f'$, $u_{tt} = f'' \therefore$

$u_{ss} - 4u_{tt} = \alpha^2 f'' - 4f'' = 0 = (\alpha^2 - 4)f'' = 0 \quad \therefore \alpha^2 - 4 = 0 \therefore$

$\alpha^2 = 4 \therefore \quad \alpha = 2, \alpha = -2 \therefore$

$u = f(2s+t) + g(-2s+t) = u(x,t) = f\left(\frac{2}{x} + \frac{1}{y}\right) + g\left(\frac{-2}{x} + \frac{1}{y}\right)$

\1c/ $4u_{tt} - u_{xx} = 0 \quad x \in \mathbb{R}, t \geq 0, u(x,0) = 0, u_t(x,0) = \psi(x)$

$\therefore$ d'Alembert's formula: $u(x,t) = \frac{1}{2}[\phi(x+ct) + \phi(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct}\psi(s)\,ds$

$\therefore u(x,0) = \phi(x) = 0$, $u_t(x,0) = \psi(x) = \psi(x) \therefore$

$u_{tt} - \frac{1}{4}u_{xx} = u_{tt} - c^2u_{xx} = 0 \therefore c^2 = \frac{1}{4} \therefore c = \frac{1}{2} \therefore$

$\cancel{u(x,t) =}$ $\phi(x) = 0 \therefore \phi(x+ct) = 0, \quad \phi(x-ct) = 0 \therefore$

$$u(x,t) = \tfrac{1}{2}[0+0] + \frac{1}{2(\frac{1}{2})}\int_{x-\frac{1}{2}t}^{x+\frac{1}{2}t} \psi(s)\,ds = \int_{x-t/2}^{x+t/2} \psi(s)\,ds$$

1))

is $\psi(x)$ is odd then $\psi(s)$ is odd $\therefore$

$\psi(-s) = -\psi(s) \therefore \quad -\psi(-s) = \psi(s)$

$$u(x,t) = \int_0^{x+t/2} \psi(s)\,ds + \int_{x-t/2}^{0} \psi(s)\,ds \quad \therefore$$

$$u(0,t) = \int_0^{0+t/2} \psi(s)\,ds + \int_{0-t/2}^{0} \psi(s)\,ds = \int_0^{t/2} \psi(s)\,ds + \int_{-t/2}^{0} \psi(s)\,ds =$$

$$\cancel{\int_0^{t/2} \psi(s)\,ds + \int_{-t/2}^{0} \psi(s)\,ds = \int_0^{t/2} \psi(s)\,ds - \int_0^{t/2} \psi}$$

$\therefore$ let $\int_0^{t/2} \psi(s)\,ds = A(t) \therefore$

$$\int_{-t/2}^{0} \psi(s)\,ds = \int_{-t/2}^{0} -\psi(-s)\,ds = \int_{t/2}^{0} -\psi(s)\,ds \cancel{=}$$

$$\int_{-t/2}^{0} \psi(s)\,ds = -\int_0^{-t/2} \psi(s)\,ds = \cancel{\int_0^{t/2} -\psi(s)\,ds = \int_0^{t/2} (\psi(-s))\,ds = \int_0^{-t/2}}$$

$$-\int_0^{-t/2} \psi(s)\,ds = -\int_0^{t/2} \psi(s)\,ds = -A(t) \therefore$$

$$u(0,t) = \int_0^{t/2} \psi(s)\,ds - \int_0^{t/2} \psi(s)\,ds = A(t) - A(t) = 0$$

1d/ $u_t - Du_{xx} = 0$, $x \in [0, \infty)$, $u(x,0) = Q\,\delta(x-x_0)$, $u(0,t) = 0$

$u(0,t) = 0 \therefore$ do odd extension on $x \therefore$

$u_t - Du_{xx} = 0$, $x \in \mathbb{R}$, $u(0,t) = 0$,

$\cancel{u(x,0) = Q\,\delta(x-x_0) - Q\,\delta(-(x-x_0)) = Q\,\delta(x-x_0) - Q\,\delta(-x-x_0)}$

$\delta(x) = -\delta(-x)$

$u(x,0) = Q\,\delta(x-x_0) - Q\,\delta(-x-x_0) \quad \because \delta(x) = -\delta(x)$ for odd

$\delta(x-x_0) = -\delta(-(x-x_0)) = -\delta(-x+x_0) \quad \delta(-x) = \delta(x)$

$Q\,\delta(x-x_0) - Q(-x-x_0) \quad Q\,\delta(x-x_0) + Q\,\delta(x-x_0) \quad \Pi(-x-2) = \Pi(-(x+2)) = \Pi(x+2)$

$$u(x,0) = Q\,\delta(x-x_0) - Q\,\delta(-x-x_0) = Q\,\delta(x-x_0) - Q\,\delta(-(x+x_0)) = Q\,\delta(x-x_0) + Q\,\delta(x+x_0)$$

$= \psi(x,0) \therefore$

$u(s,0) = \psi(s) = Q\,\delta(s-x_0) + Q\,\delta(s+x_0) \quad \therefore Q\,\delta(s+x_0) = Q\,\delta(s-(-x_0)) \therefore$

$$u(x,t) = \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} u(s,0)\,e^{-(x-s)^2/(4Dt)}\,ds = \frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} \psi(s)\,e^{-(x-s)^2/(4Dt)}\,ds =$$

$$\frac{1}{\sqrt{4\pi Dt}}\int_{-\infty}^{\infty} [Q\,\delta(s-x_0) + Q\,\delta(s+x_0)]\,e^{-(x-s)^2/(4Dt)}\,ds =$$

PP2017/ $\frac{1}{\sqrt{4\pi D t}}\left[\int_{-\infty}^{\infty} Q\,\delta(\xi - x_0)\,e^{-(x-\xi)^2/(4Dt)}\,d\xi + \int_{-\infty}^{\infty} Q\,\delta(\xi-(-x_0))\,e^{-(x-\xi)^2/(4Dt)}\,d\xi\right]$

$$= \frac{1}{\sqrt{4\pi D t}}\left[Q e^{-(x-x_0)^2/(4Dt)} + Q e^{-(x-(-x_0))^2/(4Dt)}\right] =$$

$$\frac{1}{\sqrt{4\pi D t}}\left[Q e^{-(x-x_0)^2/(4Dt)} + Q e^{-(x+x_0)^2/(4Dt)}\right] =$$

$$\frac{Q}{\sqrt{4\pi D t}}\left[e^{-\frac{(x-x_0)^2}{4Dt}} + e^{-\frac{(x+x_0)^2}{4Dt}}\right] \therefore$$

$$u_x(x,t) = \frac{Q}{\sqrt{4\pi D t}}\left[-\frac{2(x-x_0)}{4Dt} e^{-\frac{(x-x_0)^2}{4Dt}} + \left(e^{-\frac{(x+x_0)^2}{4Dt}} \times \frac{-2(x+x_0)}{4Dt}\right)\right] =$$

$$\frac{Q}{4^{1/2}\pi^{1/2}D^{1/2}t^{1/2}}\,\frac{-2}{4Dt}\left[(x-x_0)e^{-\frac{(x-x_0)^2}{4Dt}} + (x+x_0)e^{-\frac{(x+x_0)^2}{4Dt}}\right] =$$

$$\frac{-2Q}{8\pi^{1/2}D^{3/2}t^{3/2}}\left[(x-x_0)e^{-\frac{(x-x_0)^2}{4Dt}} + (x+x_0)e^{-\frac{(x+x_0)^2}{4Dt}}\right] \therefore$$

$$\therefore u_x(0,t) = \frac{-2Q}{8\pi^{1/2}D^{3/2}t^{3/2}}\left[-x_0 e^{-\frac{x_0^2}{4Dt}} + x_0 e^{-\frac{x_0^2}{4Dt}}\right] =$$

$$-\frac{2Q}{8\pi^{1/2}D^{3/2}}t^{-3/2}\left[-x_0 e^{-\frac{x_0^2}{4Dt}} + x_0 e^{-\frac{x_0^2}{4Dt}}\right] \therefore$$

as $t \to \infty$: $u_x(0,t) \to \frac{-2Q}{8\pi^{1/2}D^{3/2}}t^{-3/2}\left[+x_0 e^{0} + x_0 e^{0}\right] =$

$\frac{-Qx_0}{2\pi^{1/2}D^{3/2}}t^{-3/2} \quad \therefore \quad \alpha = \frac{3}{2}, \quad C = \frac{-Qx_0}{2\pi^{1/2}D^{3/2}}$

1e/ let $u = R(r)\Theta(\theta) \therefore u_r = R'\Theta,\ u_{rr} = R''\Theta,\ u_\theta = R\Theta',$

$u_{\theta\theta} = R\Theta'' \therefore R''\Theta + \frac{1}{r}R'\Theta + \frac{1}{r^2}R\Theta'' = 0 \therefore$

$R''\Theta + \frac{1}{r}R'\Theta = -\frac{1}{r^2}R\Theta'' \therefore R'' + \frac{1}{r}R' = -\frac{1}{r^2}R\frac{\Theta''}{\Theta} \therefore$

$r^2R'' + rR' = -R\frac{\Theta''}{\Theta} \therefore r^2\frac{R''}{R} + r\frac{R'}{R} = -\frac{\Theta''}{\Theta} = \lambda =$ Constant

$\therefore$ ~~$r^2 \frac{R''}{R}$~~ $r^2R'' + rR' = \lambda R, \quad \Theta'' = -\Theta\lambda \therefore$

$\Theta'' + \Theta\lambda = 0, \quad \Theta(0) = \Theta(2\pi), \quad \Theta'(0) = \Theta'(2\pi) \therefore$

let $\lambda = 0 \therefore \Theta'' = 0 \therefore \Theta = A\theta + B \therefore \Theta(0) = B = \Theta(2\pi) = A2\pi + B$

$\therefore 2\pi A = 0 \therefore A = 0 \therefore \Theta = B$

For $\lambda > 0$: $\lambda = a^2 > 0 \therefore$ let ~~$\Theta = e^{\ }$~~ ~~$\Theta'' + \Theta$~~ $\Theta'' + a^2\Theta = 0 \therefore$

let $\Theta = e^{q\theta} \therefore \Theta'' = q^2 e^{q\theta} \therefore q^2 e^{q\theta} - a^2 e^{q\theta} = 0 = q^2 - a^2 \therefore$

$q^2 = a^2 \therefore q = \pm a \therefore \Theta = Ae^{a\theta} + Be^{-a\theta} \therefore$

$\Theta(0) = A + B = \Theta(2\pi) = Ae^{a2\pi} + Be^{-a2\pi} \therefore A(1 - e^{a2\pi}) = B(e^{-a2\pi} - 1) \therefore$

$\Theta' = a(Ae^{a\theta} - Be^{-a\theta}) \therefore \Theta'(0) = a(A - B) = \Theta'(2\pi) = a(Ae^{2\pi a} - Be^{-2\pi a})$

$\therefore A = B\dfrac{e^{-2\pi a}-1}{1-e^{2\pi a}} \therefore$

$B\dfrac{e^{-2\pi a}-1}{1-e^{2\pi a}} - B = B\dfrac{e^{-2\pi a}-1}{1-e^{2\pi a}}e^{2\pi a} - Be^{-2\pi a} \therefore$

$\therefore B\dfrac{e^{-2\pi a}-1}{(1-e^{2\pi a})}(1-e^{2\pi a}) = B(e^{-2\pi a}-1) = B(1-e^{-2\pi a}) = -B(e^{-2\pi a}-1)$

$\therefore 1 = -1$ but $1 \neq -1 \therefore B = 0 \therefore$

$A = 0 \times \dfrac{e^{-2\pi a}-1}{1-e^{2\pi a}} = 0 \therefore \Theta = 0$

For $\lambda > 0$: $\lambda = a^2 > 0 \therefore$ let $\Theta'' - a^2\Theta = 0 \therefore$ let $\Theta = e^{q\theta} \therefore$

$\Theta'' = q^2 e^{q\theta} \therefore \quad q^2 e^{q\theta} + a^2 e^{q\theta} = 0 = q^2 + a^2 \therefore q = \pm a i \therefore$

$\Theta = A\cos a\theta + B\sin a\theta \therefore$

$\Theta(0) = A(1) + B(0) = A = \Theta(2\pi) = A\cos(2\pi a) + B\sin(2\pi a) \therefore$

$\Theta'(\theta) = -Aa\sin a\theta + Ba\cos a\theta \therefore$

$\Theta'(0) = -Aa(0) + Ba(1) = Ba = \Theta'(2\pi) = -Aa\sin(2\pi a) + Ba\cos(2\pi a)$

$\therefore B = -A\sin(2\pi a) + B\cos(2\pi a) \therefore$

$A(1-\cos(2\pi a)) = B\sin(2\pi a) \therefore B = A\dfrac{1-\cos(2\pi a)}{\sin(2\pi a)} \therefore$

$B\dfrac{\sin(2\pi a)}{1-\cos(2\pi a)} = A \therefore$

$B = -\dfrac{B\sin(2\pi a)}{1-\cos(2\pi a)}\sin(2\pi a) + B\cos(2\pi a) \therefore$

$B\left[1 + \dfrac{\sin^2(2\pi a)}{1-\cos(2\pi a)} - \cos(2\pi a)\right] = 0 = B\left[\dfrac{1-\cos(2\pi a) + \sin^2(2\pi a) - \cos(2\pi a) + \cos^2(2\pi a)}{1-\cos(2\pi a)}\right]$

$= B\left[\dfrac{1-\cos(2\pi a)}{1-\cos(2\pi a)}\right] = B(1) = B = 0 \therefore A = 0$ X

1e/ $u = R\Theta \therefore \Theta'' = -\lambda\Theta$, $r^2R'' + rR' - \lambda R = 0 \therefore$

Let $R = r^m \therefore R' = mr^{m-1}$, $R'' = m(m-1)r^{m-2} = (m^2-m)r^{m-2} \therefore$

$r^2(m^2-m)r^{m-2} + rmr^{m-1} - \lambda r^m = (m^2 - m + m - \lambda)r^m = 0 = (m^2-\lambda)r^m \therefore$

$m^2 = \lambda \quad m = \pm\sqrt{\lambda} \therefore m = -\sqrt{\lambda}, m = +\sqrt{\lambda} \therefore$

$\Theta'' + \lambda\Theta = 0 \therefore \Theta(0) = \Theta(2\pi), \Theta'(0) = \Theta'(2\pi) \therefore$

$\lambda = n^2$, $n = 0, 1, 2, 3, \ldots$, $\phi_0(\theta) = 1$, $\phi_n^{(1)}(\theta) = \cos(n\theta)$, $\phi_n^{(2)}(\theta) = \sin(n\theta)$, $n > 0$

$\therefore m = -\sqrt{n^2} = -n$, $m = +\sqrt{\lambda} = +\sqrt{n^2} = n \therefore$

~~$u(r,t) = \sum_{n=1}^{\infty} A_n$~~ $\therefore R(r) = Ar^n + Br^{-n} \therefore \left|\lim_{r\to 0} R(r)\right| < \infty \therefore$

$B = 0 \therefore R(r) = Ar^n \therefore$

\PP 2017/ $\therefore u(r,\theta) = A + \sum_{n=1}^{\infty} A r^n(\cos(n\theta) + \sin(n\theta))$ X

$u(r,\theta) = \sum_{n=0}^{\infty} r^n(A_n\cos(n\theta) + B_n\sin(n\theta)) = A_0 + \sum_{n=1}^{\infty} r^n(A_n\cos(n\theta) + B_n\sin(n\theta))$

$\therefore u(a,\theta) = A_0 + \sum_{n=1}^{\infty} a^n(A_n\cos(n\theta) + B_n\sin(n\theta)) = \cos(\theta) = \cos(1\theta) \Rightarrow$

$A_0 + a^1 A_1\cos(\theta) + a^1 B_1\sin(\theta) + \sum_{n=2}^{\infty} a^n(A_n\cos(n\theta) + B_n\sin(n\theta)) \therefore$

$A_0 = 0$, $a^1 B_1 = 0 \therefore B_1 = 0$, $A_n = 0$, $B_n = 0$ for $n \geq 2 \therefore$

$r^a A_1 = 1 \therefore A_1 = \frac{1}{a^1} = \frac{1}{a} \therefore$

$u(r,\theta) = \sum_n r^n \frac{1}{a}\cos(\theta)$

\3i(a) for $u(0)=0$, $u(L)=0 \therefore u_{xx} = u_{tt} \therefore$

$\frac{d}{dt}\int_0^L u_t\,dx = \int_0^L \frac{\partial}{\partial t} u_t\,dx = \int_0^L u_{tt}\,dx = \int_0^L u_{xx}\,dx =$

$\int_0^L \frac{\partial}{\partial x} u_x\,dx = [u_x]_{x=0}^{L} = u_x(L,t) - u_x(0,t) \therefore$

$\therefore$ for $u(0)=0$, $u(L)=0$: $u_x(L,t) - u_x(0,t) \neq 0 \therefore$ doesn't hold

(b) for $u_x(0)=0$, $u_x(L)=0$. $u_x(L,t) - u_x(0,t) = 0 - 0 = 0 \therefore$ holds

\3ii/ $\frac{d}{dt}\int_0^L u_x\,dx = \int_0^L \frac{\partial}{\partial t} u_x\,dx = \int_0^L u_{xt} \neq 0$ for

(a) and for (b)

\3aiii/ $\frac{d}{dt}\int_0^L (\frac{1}{2}u_t^2 + \frac{1}{2}u_x^2)\,dx = \int_0^L \frac{\partial}{\partial t}(\frac{1}{2}u_t^2) + \frac{\partial}{\partial t}(\frac{1}{2}u_x^2)\,dx =$

$\int_0^L u_t u_{tt} + u_x u_{xt}\,dx = \int_0^L u_t u_{xx} + u_x u_{xt}\,dx = \int_0^L \frac{\partial}{\partial x}(u_t u_x)\,dx =$

$[u_t u_x]_0^L = u_t u_x(L,t) - u_t u_x(0,t)$

$\therefore$ for (a): $u_t u_x(L,t) - u_t u_x(0,t) \neq 0 \therefore$ doesn't hold

for (b): $u_t u_x(L,t) - u_t u_x(0,t) = u_t \times 0 - u_t \times 0 = 0 \therefore$ holds

\3b/ $u_{xx} - u_{tt}$, $x \in [0,L]$, $t \geq 0$;

let $u = f(\alpha x + t) \therefore u_{xx} = \alpha^2 f''$, $u_{tt} = f'' \therefore$

$\alpha^2 f'' - f'' = (\alpha^2 - 1)f'' = 0 \therefore \alpha^2 - 1 = 0 \therefore \alpha^2 = 1 \therefore \alpha = 1, \alpha = -1 \therefore$

$u = f(x+t) + g(-x+t) \therefore$

(a) $u(x,0)=\cos(3x)$, $u_t(x,0)=0$, $u_x(0,t)=0$, $u_x(\pi,t)$ ∴

$u_t = f'(x+t)+g'(-x+t)$ ∴

$u_t(x,0)=0=f(x)+g(-x)$,

$u_x = f'(x+t)-g'(-x+t)$ ∴

$u_x(0,t)=0=f'(t)-g'(t)$

$u(x,0)=\cos(3x)=f(x)+g(-x)$ ∴

$-3\sin(3x)=f'(x)-g'(-x)$

$u_x(\pi,t)=0=f(\pi+t)+g(-\pi+t)$

$-3\sin(0)=0=f'(0)-g'(0)$ ∴ $g'(0)=f'(0)=A$ ∴

$\cos(3s)=f(s)+g(-s)$ ∴

$0=f(\pi-\pi)+g(-\pi-\pi)=f(0)+g(-2\pi)=A+g(-2\pi)$ ∴

$g(-2\pi)=-A$

(b) $u=f(x+t)+g(-x+t)$ ∴

$u(x,0)=f(x)+g(-x)=\sin(3x)$

$f(t)+g(-t)=\sin(3t)$ ∴ $f(t)=\sin(3t)-g(-t)$ ∴

$u(0,t)=0=f(t)+g(t)=\sin(3t)+g(-t)+g(t)$

∴ for $u=f(x+t)+g(-x+t)=\sin(x+t)+\sin(-x+t)$

the ~~conserv~~ conservation laws do hold ∴

$u_t=\cos(x+t)+\cos(-x+t)$ ∴ $u_{tt}=-\sin(x+t)-\sin(x+t)$ ∴

$u_x=\cos(x+t)-\cos(-x+t)$ ∴ $u_{xx}=\cancel{\cos(x+t)}-\sin(x+t)-\sin(-x+t)$

∴ $u_{tt}-u_{xx}=-\sin(x+t)-\sin(x+t)+\sin(x+t)+\sin(-x+t)\equiv 0$ ∴

the conservation laws will always hold

2a) $u=U(\xi)$, $\xi=x-ct$ ∴ $u=U(x-ct)$ ∴ $\xi_x=1$, $\xi_t=-c$ ∴

$u_t=f(u)+u_{xx}$ ∴ $u_t=U'(\xi)\xi_t=U'(-c)=-cU'(\xi)=-cU'(x-ct)$

$u_x=U'(\xi)\xi_x=U'(\xi)=U'(x-ct)$ ∴ $u_{xx}=U''(\xi)\xi_x=U''(\xi)=U''(x-ct)$ ∴

$f(u)=f(U(\xi))=f(U(x-ct))$ ∴

$u_t=f(u)+u_{xx}$ ∴ $-cU'(\xi)=f(U(\xi))+U''(\xi)$ ∴ $-cU'(x-ct)=$

$f(U(x-ct))+U''(x-ct)$ ∴ $U''(\xi)+cU'(\xi)+f(U(\xi))=0$ ∴

\PP2017/ $\therefore$ let $\frac{dU}{d\xi} = U'(\xi) = P(U) \therefore \quad \frac{d}{d\xi} = P \therefore$

$U''(\xi) + cU'(\xi) = -f(U(\xi)) = -f(U) \therefore$

$\frac{d}{d\xi}(U'(\xi) + cU(\xi)) = -f(U) = \cancel{\frac{d}{d\xi}(U} \ \cancel{\frac{d}{d\xi}P}$

$\therefore \quad \frac{dP}{dU} = \frac{dP}{d\xi}\frac{d\xi}{dU} = \frac{dP(U)}{d\xi}\frac{d\xi}{dU} = \frac{d}{d\xi}\left(\frac{dU}{d\xi}\right)\frac{d\xi}{dU} =$

\2a/ let $u(x,t) = U(\xi)$, $\xi = x - ct \therefore \xi_x = 1$, $\xi_t = -c \therefore$

$u_x = U'(\xi)\xi_x = U'(\xi)(1) = U'(\xi)$

$\therefore u_{xx} = U''(\xi)\xi_x = U''(\xi)(1) = U''(\xi) \therefore$

$u_t = U'(\xi)\xi_t = U'(\xi)(-c) = -cU'(\xi) \therefore$

$f(u) = f(U(\xi)) \therefore$

$-cU'(\xi) = f(U(\xi)) + U''(\xi) \therefore$

$U''(\xi) + cU'(\xi) = -f(U(\xi)) = \frac{d^2}{d\xi^2}U(\xi) + c\frac{d}{d\xi}U(\xi) =$

$\frac{d}{d\xi}\left(\frac{d}{d\xi}U(\xi) + cU(\xi)\right)$

$\therefore$ let $P(U) = \frac{dU}{d\xi} \therefore \quad \frac{dP}{dU} = \frac{d}{dU}\left(\frac{dU}{d\xi}\right) \cancel{=}$

$\therefore P\left(\frac{dP}{dU} + c\right) = \frac{dU}{d\xi}\left(\frac{d}{dU}\left(\frac{dU}{d\xi}\right) + c\right)$

\2a/ $U''(\xi) + cU'(\xi) = -f(U(\xi)) = \frac{d}{d\xi}\left(\frac{dU}{d\xi} + cU\right)$

$\therefore$ let $P(U) = \frac{dU}{d\xi} \therefore P = \frac{d}{d\xi} \therefore \quad \frac{dP}{dU} = \frac{d}{dU}\left(\frac{d}{d\xi}\right) \cancel{=}$

$\therefore \frac{d}{dU} = \frac{d}{d\xi}\frac{d\xi}{dU} \qquad \therefore \frac{dU}{d\xi} = P(U) \therefore$ for $P(U)$: $P\left(\frac{dP}{dU} + c\right) = -f(U)$

\2b/ let $f(u) = u(u-a)(1-u) \quad 0 < a < 1$

$\therefore -f(U) = -U(U-a)(1-U) = -U(-U^2 + (1+a)U - a) = +U^2 \mp (1+a)U + a \therefore$

$\therefore P\left(\frac{dP}{dU} + c\right) = U^2 - (1-a)U + a = P(U)(P'(U) + c)$

\2a/ $u_t = -cU'(\xi)$, $u_{xx} = U''(\xi) \therefore \quad U''(\xi) + cU'(\xi) + f(U) = 0$

$\therefore$ let $\frac{dU}{d\xi} = P(U) \therefore \frac{d^2U}{d\xi^2} = \frac{dP(U)}{d\xi} = \frac{dP}{dU}\frac{dU}{d\xi} = P\frac{dP}{dU} \therefore$

$U''(\xi) + cU'(\xi) = -f(U) \therefore \left(P\frac{dP}{dU} + cP\right) = -f(U) = P\left(\frac{dP}{dU} + c\right)$

\2b/ $\therefore -f(U) = -U(U-a)(1-U) = -U(U - U^2 - a + aU) = -U^2 - (a+1)U + a$

is a cubic polynomial in U

$\therefore$ is $P(U)$ is a quadratic polynomial with $P\left(\frac{dP}{dU} + c\right)$ also being a cubic poly in U

$\therefore$ $P(U)$ is a divisor of $f(U)$ $\therefore$ the roots of P must be a Sub set of the roots of f as suggested

To find the such solutions we let $\{u_1, u_2, u_3\} = \{0, a, 1\}$.
and u_1, u_2 be the roots of $P(U)$ $\therefore$

$P(U) = B(U-u_1)(U-u_2)$ for $B \neq 0$.

$\therefore P(\frac{dP}{dU} + c) = -f(U)$ $\therefore$

$\frac{dP}{dU} = B(U-u_2) + B(U-u_1) = 2BU - B(u_1+u_2)$ $\therefore$

$B(U-u_1)(U-u_2)[2BU - B(u_1+u_2) + c] = U(U-a)(U-1) = (U-u_1)(U-u_2)(U-u_3)$

$\therefore B[2BU - B(u_1+u_2) + c] = U - u_3 = 2B^2U - B^2(u_1+u_2) + Bc$

$\therefore B^2(u_1+u_2) - Bc - u_3 = 2B^2U - U = (2B^2-1)U$ $\therefore$

$U - u_3 = 2B^2U - B^2(u_1+u_2) + Bc$ $\therefore$

let $\sigma = \pm 1$ $\therefore c = \sigma(u_1+u_2-2u_3)/\sqrt{2} = \pm(u_1+u_2-2u_3)/\sqrt{2}$, $B = \sigma/\sqrt{2} = \pm 1/\sqrt{2}$

$\therefore B^2 = \frac{1}{2}$ $\therefore$

$2\frac{1}{2}U - \frac{1}{2}(u_1+u_2) + \frac{\pm 1}{\sqrt{2}} \times \frac{\pm 1}{\sqrt{2}}(u_1+u_2-2u_3) = U - \frac{1}{2}u_1 - \frac{1}{2}u_2 + \frac{1}{2}(u_1+u_2-2u_3) =$

$U - \frac{1}{2}u_1 - \frac{1}{2}u_2 + \frac{1}{2}u_1 + \frac{1}{2}u_2 - u_3 = U - u_3$

for $u_2 = a$: $P(U) = \sigma U(U-1)/\sqrt{2} = \sigma(1-2a)\sqrt{2}$ $\therefore$

$\frac{dU}{d\xi} = P(U) = \sigma U(U-1)/\sqrt{2}$ $\therefore$

$\frac{\sigma}{\sqrt{2}}\int d\xi = \frac{\sigma}{\sqrt{2}}(\xi - \xi_0) = \int \frac{1}{U(U-1)} dU = \ln\left|\frac{U-1}{U}\right|$ $\therefore$

$\cancel{[1-\frac{1}{U}]}$ $\left|1 - \frac{1}{U}\right| = e^{\frac{\sigma}{\sqrt{2}}(\xi-\xi_0)} = \varepsilon$, $1 - \frac{1}{U} = \pm\varepsilon$ $\therefore \frac{1}{U} = 1 \pm \varepsilon$ $\therefore U = \frac{1}{1 \pm \varepsilon}$

$\therefore U(\xi) = \frac{1}{1 \pm e^{\sigma(\xi-\xi_0)/\sqrt{2}}}$

$\therefore$ to have $U(\xi)$ desired $\forall \xi$, need to choose $U(\xi) = \frac{1}{1 + e^{\sigma(\xi-\xi_0)/\sqrt{2}}}$

to have $U'(\xi) < 0$ need $\sigma = 1$

for $U(0) = \frac{1}{2}$ set $\xi_0 = 0$ $\therefore$ $U(\xi) = \cancel{\frac{1}{1+e^{(\xi-\xi}}}$ $\frac{1}{1+e^{(\xi-0)/\sqrt{2}}} = \frac{1}{1+e^{\xi/\sqrt{2}}}$ $\therefore$

$\therefore u(x,t) = \frac{1}{1+e^{((x-ct)/\sqrt{2}}}$, $c = (1-2a)/\sqrt{2}$

$\therefore c > 0$ for $a < \frac{1}{2}$

\2c/ $\therefore$ integrating $P(\frac{dP}{dU} + c) = -f(U)$ from u_1 to u_3:

$\int_{u_1}^{u_3} P\frac{dP}{dU}dU + c\int_{u_1}^{u_3} P\,dU = -\int_{u_1}^{u_3} f(U)\,dU$ $\therefore$ $\int_{u_1}^{u_3} P\frac{dP}{dU}dU = \left[\frac{1}{2}P^2\right]_{u_1}^{u_2} = \frac{1}{2}P^2(u_3) - \frac{1}{2}P^2(u_1)$

which vanishes $\because P(u_1) = P(u_3) = 0$ $\because P = U'(\xi) < 0$ $\therefore \int_{u_1}^{u_3} P\,dU < 0$ $\therefore$

$\text{Sign}(c) = \text{Sign}\int_{u_1}^{u_3} f(u)\,du$

\PP2017/ \4a/ $\therefore u(a,\theta) = f(\theta) = \frac{1}{2}A_0 + \sum_{n=1}^{\infty}[A_n\cos(n\theta) + B_n\sin(n\theta)]a^n \therefore$

$\cancel{\frac{1}{2}A_0 = \int_0^{2\pi} f(\theta)d\theta}$, $\theta \in [0, 2\pi] \therefore 2L = 2\pi \therefore \pi = L \therefore$

$\frac{1}{2}A_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta \quad A_n a^n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos(n\theta)\,d\theta,$

$B_n a^n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin(n\theta)\,d\theta \therefore$

$B_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\xi)\sin(n\xi)\,d\xi, \quad A_0 = \frac{1}{\pi}\int_0^{2\pi} f(\xi)\,d\xi,$

$A_n = \frac{1}{\pi a^n}\int_0^{2\pi} f(\xi)\cos(n\xi)\,d\xi \therefore \therefore$

$$u(r,\theta) = \frac{1}{\pi}\int_0^{2\pi} f(\xi)\,d\xi + \sum_{n=1}^{\infty}\frac{r^n}{\pi a^n}\left[\int_0^{2\pi} f(\xi)\cos(n\xi)\,d\xi\cos(n\theta) + \int_0^{2\pi} f(\xi)\sin(n\xi)\,d\xi\sin(n\theta)\right]$$

$$\frac{1}{\pi}\int_0^{2\pi} f(\xi)\,d\xi + \sum_{n=1}^{\infty}\frac{r^n}{\pi a^n}\left[\int_0^{2\pi} f(\xi)[\cos(n\xi)\cos(n\theta) + \sin(n\xi)\sin(n\theta)]\,d\xi\right] =$$

$$\frac{1}{\pi}\int_0^{2\pi} f(\xi)\,d\xi + \sum_{n=1}^{\infty}\frac{r^n}{\pi a^n}\int_0^{2\pi} f(\xi)(\cos(n(\theta-\xi)))\,d\xi =$$

$$\frac{1}{\pi}\int_0^{2\pi} f(\xi)\,d\xi + \int_0^{2\pi}\sum_{n=1}^{\infty}\frac{r^n}{\pi a^n} f(\xi)\cos(n(\theta-\xi))\,d\xi = \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{f(\xi)}{a^2-2ar\cos(\theta-\xi)+r^2}\,d\xi$$

$$u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left[1 + 2\sum_{n=1}^{\infty}\frac{r^n}{a^n}\cos(n(\theta-\xi))\right]d\xi =$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left[1 + \sum_{n=1}^{\infty}\frac{r^n}{a^n}e^{in(\theta-\xi)} + \sum_{n=1}^{\infty}\frac{r^n}{a^n}e^{-in(\theta-\xi)}\right]d\xi =$$

$$\frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left[1 + \frac{\frac{r}{a}e^{i(\theta-\xi)}}{1-\frac{r}{a}e^{i(\theta-\xi)}} + \frac{\frac{r}{a}e^{-i(\theta-\xi)}}{1-\frac{r}{a}e^{-i(\theta-\xi)}}\right] = \frac{1}{2\pi}\int_0^{2\pi} f(\xi)\left[\frac{a^2-r^2}{a^2-2ar\cos(\theta-\xi)+r^2}\right]d\xi$$

$$\frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{f(\xi)}{a^2-2ar\cos(\theta-\xi)+r^2}$$

\4b/ $u(r,\theta) = \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{f(\xi)}{a^2-2ar\cos(\theta-\xi)+r^2}\,d\xi \therefore \quad f(\theta) = \cos\theta \therefore f(\xi) = \cos\xi \therefore$

$u(r,\theta) = \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{\cos\xi}{a^2-2ar\cos(\theta-\xi)+r^2}\,d\xi \therefore$

~~let~~ \4c/ let $M = \max(|f(\theta)|) = \max|f(\theta)| \therefore$

$$|u(r,\theta)| = \left|\frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{f(\xi)}{a^2-2ar\cos(\theta-\xi)+r^2}\,d\xi\right| = \left|\frac{a^2-r^2}{2\pi}\right|\left|\int_0^{2\pi}\frac{f(\xi)}{a^2-2ar\cos(\theta-\xi)+r^2}\,d\xi\right| \leq$$

$\frac{a^2-r^2}{2\pi}\int_0^{2\pi}\left|\frac{f(\xi)}{a^2-2ar\cos(\theta-\xi)+r^2}\right|d\xi$ Since $a \geq r \geq 0 \therefore a^2 \geq r^2 \geq 0 \therefore a^2 - r^2 \geq 0$

$= \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{|f(\xi)|}{|a^2-2ar\cos(\theta-\xi)+r^2|}\,d\xi \leq \frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{\max|f(\xi)|}{a^2-r^2}\,d\xi = \therefore |\cos x| \leq 1$

$\frac{a^2-r^2}{2\pi}\int_0^{2\pi}\frac{M}{a^2-r^2}\,d\xi = \frac{a^2-r^2}{2\pi}\left[\frac{M}{a^2-r^2}\xi\right]_0^{2\pi} = \frac{a^2-r^2}{2\pi}\times\frac{2\pi M}{a^2-r^2} = M, \therefore |u(r,\theta)| \leq M \quad \forall\theta$

PP2016 / 1a / let $x=e^s$, $y=e^t$ ∴ $s=\ln x$, $t=\ln y$ ∴

$s_x=\frac{1}{x}=x^{-1}$, $t_y=\frac{1}{y}=y^{-1}$

~~$x^2u_{xx}-9$~~ ∴ $u_x=u_s s_x=\frac{1}{x}u_s=x^{-1}u_s$ ∴

$u_{xx}=-x^{-2}u_s+x^{-1}u_{ss}s_x=-x^{-2}u_s+x^{-2}u_{ss}$

$u_y=u_t t_y=y^{-1}u_t$ ∴

$u_{yy}=-y^{-2}u_t+y^{-1}u_{tt}t_y=-y^{-2}u_t+y^{-2}u_{tt}$ ∴

$x^2u_{xx}-9y^2u_{yy}+xu_x-9yu_y=$

~~u_s ...~~

$-u_s+u_{ss}+9u_t-9u_{tt}+u_s-9u_t=$

$u_{ss}-9u_{tt}=0$ ∴

let $f(\alpha s+t)=u$ ∴ $u_s=\alpha f'$ ∴ $u_{ss}=\alpha^2 f''$ ∴

$u_t=f'$, $u_{tt}=f''$ ∴

$\alpha^2 f''-9f''=(\alpha^2-9)f''=0$ ∴ $\alpha^2-9=0$ ∴ $\alpha^2=9$ ∴

$\alpha=3$, $\alpha=-3$ ∴

$u(s,t)=f(3s+t)+g(-3s+t)$ ∴

$u(x,t)=f(3\ln x+\ln y)+g(-3\ln x+\ln y)$ f, g arbitrary functions

1b/ $u_{xx}+4u_{xt}+3u_{tt}=3u_{tt}+4u_{xt}+u_{xx}=au_{tt}+2bu_{xt}+cu_{xx}$ ∴

$a=3$, $2b=4$, $c=1$ ∴ $-2b=-4$, $b=2$ ∴

$b^2-ac=2^2-3(1)=4-3=1>0$ ∴ hyperbolic ∴

$a\left(\frac{dx}{dt}\right)^2-2b\left(\frac{dx}{dt}\right)+c=3\left(\frac{dx}{dt}\right)^2-4\frac{dx}{dt}+1=0$ ∴

$\frac{dx}{dt}=\frac{4\pm\sqrt{16-4(3)(1)}}{2(3)}=\frac{4\pm\sqrt{4}}{6}=\frac{4\pm2}{6}$ ∴

$\frac{dx}{dt}=1$, $\frac{dx}{dt}=\frac{1}{3}$ ∴

$\int 1\,dx=\int 1\,dt=x=t+c_1$, $\int 1\,dx=\int\frac{1}{3}\,dt=x=\frac{1}{3}t+c_2$ ∴ $3x=t+c_3$ ∴

$x-t=c_1$, $3x-t=c_3$ ∴

let $\xi=x-t$, ~~$3x-t$~~ $\eta=3x-t$ ∴

$\xi_x=1$, $\xi_t=-1$, $\eta_x=3$, $\eta_t=-1$ ∴

$u_x=u_\xi\xi_x+u_\eta\eta_x=u_\xi+3u_\eta$ ∴

$u_{xx}=u_{\xi\xi}\xi_x+u_{\xi\eta}\eta_x+3u_{\eta\xi}\xi_x+3u_{\eta\eta}\eta_x=u_{\xi\xi}+6u_{\xi\eta}+9u_{\eta\eta}$

$u_t=u_\xi\xi_t+u_\eta\eta_t=-u_\xi-u_\eta$ ∴

1d/ let $u_1(x,t)$, $u_2(x,t)$ be two solutions of $u_t - a^2 u_{xx} = f(x,t)$

$\therefore u_1(x,0)=\phi(x)$, $u_1(0,t)=\psi_0(t)$, $u_1(L,t)=\psi_L(t)$

$u_2(0,t)=\psi_0(t)$, $u(L,t)=\psi_L(t)$, $u(x,0)=\phi(x)$ $\therefore$

$u_{1t}=a^2 u_{1xx}+f(x,t)$, $u_{2t}=a^2 u_{2xx}+f(x,t)$ $\therefore$

let ~~$E(t)=\int_0^\infty$~~ $w(x,t)=u_1-u_2$ $\therefore$

$w_t-a^2w_{xx}=u_{1t}-a^2u_{1xx}-u_{2t}+a^2u_{2xx}=f(x,t)-f(x,t)=0$,

$w(0,t)=u_1(0,t)-u_2(0,t)=\psi_0(t)-\psi_0(t)=0$ $\therefore$ by symmetry $\therefore$

$w(L,t)=0$, $w(x,0)=0$ $\therefore$ $w_t=a^2w_{xx}$

let ~~$E(t)=\int_0^\infty$~~ $E(t)=\int_0^L (w(x,t))^2 dx$ $\therefore$

$\therefore \frac{dE(t)}{dt}=\frac{\partial}{\partial t}\int_0^L (w(x,t))^2 dx=\int_0^L \frac{\partial}{\partial t}(w(x,t))^2 dx=$

$\int_0^L 2ww_t\,dx=\int_0^L 2w\,a^2w_{xx}\,dx=2a^2\int_0^L ww_{xx}\,dx=$

$2a^2\left([ww_x]_0^L-\int_0^L w_xw_x\,dx\right)=2a^2w_x\big(w(L,t)-w(0,t)\big)-2a^2\int_0^L (w_x)^2dx=$

$2a^2w_x(0-0)-2a^2\int_0^L (w_x)^2dx=-2a^2\int_0^L (w_x)^2dx\le 0$ $\therefore$

$E(t_1)\ge E(t_2)$ for ~~$t_1>t_2$~~ $t_1<t_2$ $\therefore$

$E(0)\ge E(t)$ $\forall t\ge 0$ $\therefore$

~~$E(0)=\int_0^L$~~ $E(t)=\int_0^L (w(x,t))^2dx\ge 0$ $\therefore$

$E(0)=\int_0^L (w(x,0))^2dx=\int_0^L 0^2dx=0$ $\therefore$

$E(0)=0\ge E(t)\ge 0$ $\therefore$

$E(t)=0=\int_0^L (w(x,t))^2dx$ $\therefore$ $(w(x,t))^2\ge 0$ $\therefore$ $(w(x,t))^2=0$ $\therefore$

$w(x,t)=u_1(x,t)-u_2(x,t)\equiv 0$ $\therefore$ $u_1(x,t)\equiv u_2(x,t)$ $\forall t\ge 0$ $x\in[0,L]$

1e/ let $u=R(r)\Theta(\theta)$ $\therefore$ $u_r=R'\Theta$ $\therefore$ $u_{rr}=R''\Theta$ $\therefore$ $u_\theta=R\Theta'$ $\therefore$

$u_{\theta\theta}=R\Theta''$ $\therefore$ $R''\Theta+\frac{1}{r}R'\Theta+\frac{1}{r^2}R\Theta''=0$ $\therefore$

$r^2R''\Theta+rR'\Theta+R\Theta''=0$ $\therefore$ $r^2R''\Theta+rR'\Theta=-R\Theta''$ $\therefore$

$r^2\frac{1}{R}R''\Theta+r\frac{1}{R}R'\Theta=-\Theta''$ $\therefore$

$r^2\frac{1}{R}R''+r\frac{1}{R}R'=\frac{\Theta''}{-\Theta}=\lambda=\text{constant}$ $\therefore$

$r^2R''+rR'=\lambda R$, $\Theta''=-\lambda\Theta$ $\therefore$

$\Theta''+\lambda\Theta=0$ $\therefore$

\PP2016/ $\Theta(0)=\Theta(2\pi)$, $\Theta'(0)=\Theta'(2\pi)$

for $\lambda=0$: $\Theta''+0\Theta=\Theta''=0 \therefore \Theta'=A \therefore$

$\Theta=A\theta+B \therefore \Theta(0)=A(0)+B=B=\Theta(2\pi)=A2\pi+B=B \therefore$

$2\pi A=0 \therefore A=0 \therefore \Theta=B$

for $\lambda<0$: $\lambda=-a^2<0 \therefore \Theta''-a^2\Theta=0 \therefore$ let $\Theta=e^{q\theta} \therefore \Theta''=q^2e^{q\theta} \therefore$

$q^2e^{q\theta}-a^2e^{q\theta}=0=q^2-a^2 \therefore q^2=a^2 \therefore q=a,\ q=-a \therefore$

$\Theta(\theta)=\tilde{A}e^{a\theta}+\tilde{B}e^{-a\theta}=A\cosh(a\theta)+B\sinh(a\theta) \therefore$

$\Theta(0)=A\cosh(0)+B\sinh(0)=A=\Theta(2\pi)=A\cosh(2\pi a)+B\sinh(2\pi a) \therefore$

$A(1-\cosh(2\pi a))=B\sinh(2\pi a) \therefore B=\dfrac{A(1-\cosh(2\pi a))}{\sinh(2\pi a)} \therefore$

$\Theta'(\theta)=aA\sinh(a\theta)+aB\cosh(a\theta)=$

$aA\left(\sinh(a\theta)+\dfrac{1-\cosh(2\pi a)}{\sinh(2\pi a)}\cosh(a\theta)\right) \therefore$

$\Theta'(0)=\Theta(2\pi)=\cancel{0}\ aA\left(\dfrac{1-\cosh(2\pi a)}{\sinh 2\pi a}\right)=$

$aA\left(\sinh(2\pi a)+\dfrac{1-\cosh(2\pi a)}{\sinh(2\pi a)}\cosh(2\pi a)\right)=$

$aA\left(\dfrac{-(\cosh^2(2\pi a)-\sinh^2(2\pi a))+1}{\sinh(2\pi a)}\right)=aA\left(\dfrac{-1+1}{\sinh(2\pi a)}\right)=aA(0)=0 \therefore$

$A=0 \therefore B=0 \therefore \Theta=0$

for $\lambda>0$: $\lambda=a^2>0 \therefore \Theta''+a^2\Theta=0 \therefore$ let $\Theta=e^{q\theta}$, $\Theta''=q^2e^{q\theta} \therefore$

$q^2e^{q\theta}+a^2e^{q\theta}=0=q^2+a^2 \therefore q^2=-a^2 \therefore \cancel{q=\pm}\ q=ai,\ q=-ai \therefore$

$\cancel{\Theta(\theta)=A\sinh-A\cos t}$ $\Theta(\theta)=A\cos(a\theta)+B\sin(a\theta) \therefore$

$\Theta'(\theta)=-aA\sin(a\theta)+aB\cos(a\theta) \therefore$

$\Theta(0)=\Theta(2\pi)=A(1)+B(0)=A=A\cos(2\pi a)+B\sin(2\pi a) \therefore$

$\Theta'(0)=\Theta'(2\pi)=-aA\sin(2\pi a)+aB\cos(2\pi a)=-aA(0)+aB(1)=aB \therefore$

$-A\sin(2\pi a)+B\cos(2\pi a)=B \therefore B(\cos(2\pi a)-1)=A\sin(2\pi a) \therefore$

$B=A\dfrac{\sin 2\pi a}{-1+\cos 2\pi a} \therefore$

$A(1-\cos 2\pi a)=B\sin(2\pi a)=A\dfrac{\sin^2(2\pi a)}{-1+\cos 2\pi a} \therefore$

$A(1-\cos 2\pi a)(-1+\cos 2\pi a)=A\sin^2(2\pi a)=A(-1-\cos^2 2\pi a+2\cos 2\pi a)$

$A(\sin^2 2\pi a+\cos^2 2\pi a)=A(1)=A=-A+2A\cos 2\pi a \therefore$

$2A=2A\cos 2\pi a \therefore A=\cos 2\pi a \therefore B\neq 0\neq A \therefore$

$\cancel{\sin(a\theta)=0}$ $\sin(2\pi a)=0 \therefore 2\pi a=n\pi$, $\cos(2\pi a)=0 \therefore$

$2\pi a=\dfrac{(n+1)\pi}{2} \therefore 4\pi a=n\pi+\pi \therefore 4a=n+1$

$\therefore a = n \therefore \sqrt{\lambda} = a^2 = n^2 \therefore \quad n = 0, 1, 2, 3, \dots \therefore$

$\phi_0(\theta) = 1, \quad \phi_n^{(1)}(\theta) = \cos(n\theta), \quad \phi_n^{(2)}(\theta) = \sin(n\theta)$ for $n > 0$

$\therefore r^2 R'' + rR' - \lambda R = 0 \therefore \quad \sqrt{\lambda} = n, \ \sqrt{\lambda} = -n \therefore$

let $R = r^m \therefore \ R' = mr^{m-1}, \ R'' = m(m-1) = (m^2 - m) r^{m-2} \therefore$

$r^2 (m^2 - m) r^{m-2} + r m r^{m-1} - \lambda r^m = (m^2 - m + m - \lambda) r^m = 0 = (m^2 - \lambda) r^m \therefore$

$m^2 - \lambda = 0 \therefore \ m^2 = \lambda \therefore \quad m = \pm\sqrt{\lambda} = \pm\sqrt{n^2} \therefore m = n, \ m = -n \therefore$

$R(r) = Ar^n + Br^{-n} \therefore$

$r \geq 0 \therefore \lim_{r\to\infty} u(r,\theta) = 0 \quad \therefore \lim_{r\to\infty} R(r) = \lim_{r\to\infty} Ar^n + Br^{-n} = 0 \therefore$

$A = 0 \therefore \quad R(r) = Br^{-n} \therefore$

$u(r,\theta) = A_0 + \sum_{n=1}^{\infty} r^{-n}(A_n\cos(n\theta) + B_n\sin(n\theta)) \therefore$

$\lim_{r\to\infty} (r,\theta) = 0 \quad \therefore \quad A_0 = u(r\to\infty, \theta) = 0 \quad \therefore$

$u(r,\theta) = \sum_{n=1}^{\infty} r^{-n}(A_n\cos(n\theta) + B_n\sin(n\theta)) \therefore$

$u_r(r,\theta) = \sum_{n=1}^{\infty} -nr^{-n-1}(A_n\cos(n\theta) + B_n\sin(n\theta)) \therefore$

$u_r(1,\theta) = \sum_{n=1}^{\infty} -n(1)^{-n-1}(A_n\cos(n\theta) + B_n\sin(n\theta)) =$

$\sum_{n=1}^{\infty}[-nA_n\cos(n\theta) - nB_n\sin(n\theta)] = \cos(7\theta) \therefore$

$\sum_{n\neq 7}[-nA_n\cos(n\theta)] + \sum_{n=1}^{\infty}[-nB_n\sin(n\theta)] - 7A_7\cos(7\theta) \therefore$

$A_n = 0 \ \forall n \neq 7, \quad B_n = 0 \ \forall n \therefore$

$-7A_7\cos(7\theta) = \cos(7\theta) \therefore -7A_7 = 1 \therefore \ A_7 = \frac{1}{-7} = -\frac{1}{7} \therefore$

$u(r,\theta) = \sum_{n=7}^{7} r^{-n}(A_n\cos(n\theta)) = r^{-7}A_7\cos(7\theta) = -\frac{1}{7}r^{-7}\cos(7\theta)$

\2a) $u_t + uu_x - u_{xx} = 0 \quad \therefore$ let $u = w_x, \quad w_t + \frac{1}{2}(w_x)^2 - w_{xx} = 0 \therefore$

$\therefore$ let $w = \alpha\ln\phi \therefore \quad w_x = \alpha\frac{1}{\phi}\phi_x = \alpha\phi^{-1}\phi_x \therefore$

$w_{xx} = \alpha(-1)\phi^{-2}\phi_x\phi_x + \alpha\phi^{-1}\phi_{xx} = -\alpha\phi^{-2}(\phi_x)^2 + \alpha\phi^{-1}\phi_{xx} \therefore$

$w_t = \alpha\phi^{-1}\phi_t \quad \therefore (w_x)^2 = (\alpha\phi^{-1}\phi_x)^2 = \alpha^2\phi^{-2}(\phi_x)^2 \therefore$

$w_t + \frac{1}{2}(w_x)^2 - w_{xx} = \alpha\phi^{-1}\phi_t + \frac{1}{2}\alpha^2\phi^{-2}(\phi_x)^2 + \alpha\phi^{-2}(\phi_x)^2 - \alpha\phi^{-1}\phi_{xx}$

$= \alpha\phi^{-1}\phi_t + (\frac{1}{2}\alpha^2 + \alpha)\phi^{-2}(\phi_x)^2 - \alpha\phi^{-1}\phi_{xx} = \cancel{\alpha\phi^{-1}(\phi_t - \phi_{xx})}$

$\alpha\phi^{-1}(\phi_t - \phi_{xx}) + (\frac{1}{2}\alpha^2 + \alpha)\phi^{-2}(\phi_x)^2$

\PP2016/ $\therefore$ let $u = w_x$ $\therefore$

$u_t = w_{xt}$, $u_x = w_{xx}$, $u_{xx} = w_{xxx}$ $\therefore$

$w_{xt} + w_x w_{xx} - w_{xxx} = 0 = \frac{d}{dx}\left(w_t + \frac{1}{2}(w_x)^2 - w_{xx}\right)$

$u = w_x$ $\therefore$

$\alpha \phi^{-1}(\phi_t - \phi_{xx}) + (\frac{1}{2}\alpha^2 + \alpha)\phi^{-2}(\phi_x)^2 = 0$ $\therefore$

let $\frac{1}{2}\alpha^2 + \alpha = 0$ $\therefore$ $\alpha \neq 0$ $\therefore$ $\frac{1}{2}\alpha - 1 = 0$ $\therefore$ $\frac{\alpha}{2} = 1$ $\therefore$ $\alpha = 2$ $\therefore$

$2\phi^{-1}(\phi_t - \phi_{xx}) = 0$ $\therefore$ $\cancel{\phi}$ $\phi \neq 0$ $\therefore$

$\phi_t - \phi_{xx} = 0$

$\therefore \frac{d}{dx}\left(w_t + \frac{1}{2}(w_x)^2 - w_{xx}\right) = 0$ $\therefore$

$u \neq 0$ $\therefore$ $u = w_x \neq 0$ $\therefore$ $w_t + \frac{1}{2}(w_x)^2 - w_{xx} = f(t)$

$\therefore$ $f(t) = 0$ $\therefore$

$w_t + \frac{1}{2}(w_x)^2 - w_{xx} = 0$

$\therefore$ $w = \alpha \ln \phi$ for $\alpha = 2$ $\therefore$ $\phi_t - \phi_{xx} = 0$

\2b/ $\phi_t - \phi_{xx} = 0$ $\therefore$ let $\phi = e^{Ax} f(t) + e^{Bx} g(t)$ $\therefore$

$\phi_t = e^{Ax} f'(t) + e^{Bx} g'(t)$ $\therefore$

$\phi_x = A e^{Ax} f(t) + B e^{Bx} g(t)$ $\therefore$

$\phi_{xx} = A^2 e^{Ax} f(t) + B^2 e^{Bx} g(t)$ $\therefore$

$\phi_t - \phi_{xx} = e^{Ax} f'(t) + e^{Bx} g'(t) - A^2 e^{Ax} f(t) - B^2 e^{Bx} g(t) =$

$e^{Ax}(f'(t) - A^2 f(t)) + e^{Bx}(g'(t) - B^2 g(t)) = 0$ $\therefore$

$f'(t) - A^2 f(t) = 0$ $\therefore$ $g'(t) - B^2 g(t) = 0$ $\therefore$

$\frac{f'(t)}{f(t)} = A^2$, $\frac{g'(t)}{g(t)} = B^2$ $\therefore$ $\int \frac{f'(t)}{f(t)} dt = \int A^2 dt = A^2 t = \ln f(t)$,

$\int \frac{g'(t)}{g(t)} dt = \int B^2 dt = \ln g(t) = B^2 t$ $\therefore$

$f(t) = e^{A^2 t}$, $g(t) = e^{B^2 t}$ $\therefore$

$\phi(x,t) = e^{Ax} e^{A^2 t} + e^{Bx} e^{B^2 t} = e^{Ax + A^2 t} + e^{Bx + B^2 t} = e^{A(x+At)} + e^{B(x+Bt)}$

$\therefore$ $w = 2\ln\phi = 2\ln\left(e^{A(x+At)} + e^{B(x+Bt)}\right)$ $\cancel{2}$ $\therefore$

$u = w_x = 2\frac{A e^{A(x+At)} + B e^{B(x+Bt)}}{e^{A(x+At)} + e^{B(x+Bt)}} = v(x+ct) = v(\xi)$ $\therefore$ $c = h(A,B)$

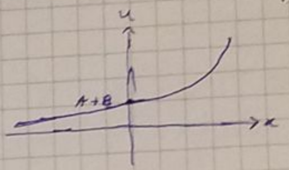
$\phi_t - \phi_{xx} = e^{Ax} f'(t) + e^{Bx} g'(t) - A^2 e^{Ax} f(t) - B^2 e^{Bx} g(t) = 0$ $\therefore$

$e^{Ax} f'(t) + e^{Bx} g'(t) = A^2 e^{Ax} f(t) + B^2 e^{Bx} g(t)$

2c/ $\therefore$ as $\lim_{x\to+\infty} u(x,t) = \lim_{x\to+\infty} 2\frac{Ae^{A(x+At)}+Be^{B(x+Bt)}}{e^{A(x+At)}+e^{B(x+Bt)}} =$

$\lim_{x\to\infty} 2\frac{A^2e^{A(x+At)}+B^2e^{B(x+Bt)}}{Ae^{A(x+At)}+Be^{B(x+Bt)}} = 2(A+B)$,

$\lim_{x\to-\infty} u(x,t) = \lim_{x\to-\infty} 2\frac{Ae^{A(x+At)}+Be^{B(x+Bt)}}{e^{A(x+At)}+e^{B(x+Bt)}} = 0$



$\therefore u(x,0) = 2\frac{Ae^{Ax}+Be^{Bx}}{e^{Ax}+e^{Bx}}$ $\therefore$

3a/ $\therefore au_{xx}+2bu_{xx}+cu_{yy} = -du_x - eu_y - fu$ $\therefore$

~~bz~~ is $b^2-ac=0$ then parabolic

is $b^2-ac>0$ then hyperbolic

$b^2-ac<0$ the elliptic

~~3b~~ $\therefore a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right)+c=0$

3b/ $\therefore u_{xx}+3u_{xy}+2u_{yy} = -2u_x-3u_y-u = au_{xx}+2bu_{xy}+cu_{yy}$ $\therefore$

$a=1,\ 2b=3,\ c=2$ $\therefore$ $-2b=-3,\ b=\frac{3}{2}$ $\therefore$

$b^2-ac = \left(\frac{3}{2}\right)^2-1(2) = \frac{1}{4}>0$ $\therefore$ hyperbolic

$\therefore 1\left(\frac{dy}{dx}\right)^2 - 3\left(\frac{dy}{dx}\right)+2=0$ $\therefore$

$\frac{dy}{dx} = \frac{3\pm\sqrt{9-4(1)(2)}}{2(1)} = \frac{3\pm1}{2}$ $\therefore \frac{dy}{dx}=2,\ \frac{dy}{dx}=1$ $\therefore$

$\int 1\,dy = \int 2\,dx = y = 2x+c_1$, $\int 1\,dy = \int 1\,dx = y = x+c_2$ $\therefore$

$-c_1 = c_3 = 2x-y$, $-c_2 = c_4 = x-y$ $\therefore$

let $\xi = 2x-y$, $\eta = x-y$

~~$u_x = 2t$~~ $\xi_x = 2,\ \xi_y=-1,\ \eta_x=1,\ \eta_y=-1$ $\therefore$

$u_x = u_\xi\xi_x + u_\eta\eta_x = 2u_\xi + u_\eta$

$u_{xx} = 2u_{\xi\xi}\xi_x + 2u_{\xi\eta}\eta_x + u_{\xi\eta}\xi_x + u_{\eta\eta}\eta_x = 4u_{\xi\xi}+4u_{\xi\eta}+u_{\eta\eta}$

$u_y = u_\xi\xi_y + u_\eta\eta_y = -u_\xi - u_\eta$

$u_{yy} = -u_{\xi\xi}\xi_y - u_{\xi\eta}\eta_y - u_{\xi\eta}\xi_y - u_{\eta\eta}\eta_y = u_{\xi\xi}+2u_{\xi\eta}+u_{\eta\eta}$.

$u_{xy} = 2u_{\xi\xi}\xi_y + u_{\xi\eta}\xi_y + 2u_{\xi\eta}\eta_y + u_{\eta\eta}\eta_y = -2u_{\xi\xi}-3u_{\xi\eta}-u_{\eta\eta}$ $\therefore$ into PDE

$4u_{\xi\xi}+4u_{\xi\eta}+u_{\eta\eta}-6u_{\xi\xi}-9u_{\xi\eta}-3u_{\eta\eta}+2u_{\xi\xi}+4u_{\xi\eta}+2u_{\eta\eta}+4u_\xi+2u_\eta-3u_\xi-3u_\eta+u = 0$

$= -u_{\xi\eta}+u_\xi-u_\eta+u=0$ is canonical form

\PP2016/ \3c/ let $u = v e^{\lambda\xi + \mu\eta} = v\, h(\xi,\eta) = vh \therefore$

$h_\xi = \lambda h, \quad h_\eta = \mu h \therefore$

$u_\xi = v_\xi h + v\lambda h = (v_\xi + \lambda v) h \therefore$

$u_\eta = (v_\eta + \mu v) h$

$u_{\xi\eta} = (v_{\xi\eta} + \lambda v_\eta) h + (v_\xi + \lambda v)\mu h = (v_{\xi\eta} + \lambda v_\eta + \mu v_\xi + \lambda\mu v) h \therefore$

into equation:

$-(v_{\xi\eta} + \lambda v_\eta + \mu v_\xi + \lambda\mu v) h + (v_\xi + \lambda v) h - (v_\eta + \mu v) h + vh =$

$[-v_{\xi\eta} - \lambda v_\eta - \mu v_\xi - \lambda\mu v + v_\xi + \lambda v - v_\eta - \mu v + v] h = 0 =$

$[-v_{\xi\eta} + (-\lambda - 1) v_\eta + (-\mu + 1) v_\xi + (-\lambda\mu + \lambda - \mu + 1) v] = 0 \therefore$

let $-\lambda - 1 = 0 \therefore \lambda = -1$ let $-\mu + 1 = 0 \therefore \mu = 1 \therefore$

$-\lambda\mu + \lambda - \mu + 1 = -(-1)1 - 1 - 1 + 1 = 1 - 1 = 0 \therefore$

$u_{yy} \therefore$

$-v_{\xi\eta} = 0 = v_{\xi\eta} = \frac{\partial}{\partial\xi}\left(\frac{\partial}{\partial\eta} v\right) = 0 \therefore$

$\therefore \frac{\partial}{\partial\eta} v = \tilde{f}(\eta) \therefore \quad v = f(\eta) + g(\xi) \therefore$

$u = vh = v e^{\lambda\xi + \mu\eta} = v e^{-\xi + \eta} \therefore \quad -(2x-y) + x - y = -2x + y + x - y = -x$

$\cancel{u(\xi,\eta) = f(\eta) e^{-\xi+\eta}}$ $\quad u(\xi,\eta) = (f(\eta) + g(\xi)) e^{-\xi+\eta}$

$\therefore u(x,y) = (f(x-y) + g(2x-y)) e^{-(2x-y) + x - y} =$

$(f(x-y) + g(2x-y)) e^{-x}$

\3d/ $u_t = \frac{\partial u}{\partial t} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} =$

$-\sinh(\theta) u_\xi + \cosh(\theta) u_\eta$, $\xi_t = -\sinh\theta$, $\eta_t = \cosh\theta \therefore$

$\cancel{u_{tt} = -\sinh(\theta) u_{\xi\xi}\xi_t - \sinh(\theta) u_{\xi\eta}}$

$u_{tt} = -\sinh(\theta) u_{\xi\xi}\xi_t - \sinh(\theta) u_{\xi\eta}\eta_t + \cosh(\theta) u_{\xi\eta}\xi_t + \cosh(\theta) u_{\eta\eta}\eta_t =$

$\sinh^2(\theta) u_{\xi\xi} - 2\sinh(\theta)\cosh(\theta) u_{\xi\eta} + \cosh^2(\theta) u_{\eta\eta}$

$\xi_x = \cosh\theta, \quad \eta_x = -\sinh\theta \therefore$

$u_x = u_\xi \xi_x + u_\eta \eta_x = \cosh(\theta) u_\xi - \sinh(\theta) u_\eta$

PDE:

$u_{xx} = \cosh(\theta) u_{\xi\xi}\xi_x + \cosh(\theta) u_{\xi\eta}\eta_x - \sinh(\theta) u_{\xi\eta}\xi_x - \sinh(\theta) u_{\eta\eta}\eta_x =$

+u $\cosh^2(\theta) u_{\xi\xi} - 2\sinh(\theta)\cosh(\theta) u_{\xi\eta} + \sinh^2(\theta) u_{\eta\eta} \therefore$

$u_{tt} - u_{xx} = \sinh^2(\theta) u_{\xi\xi} - 2\sinh(\theta)\cos(\theta) u_{\xi\eta} + \cosh^2(\theta) u_{\eta\eta} - \cosh^2(\theta) + 2\sinh(\theta)\cosh(\theta) u_{\xi\eta} - \sinh^2(\theta) u_{\eta\eta}$

$= -(\cosh^2(\theta) - \sinh^2(\theta)) u_{\xi\xi} + (\cosh^2(\theta) - \sinh^2(\theta)) u_{\eta\eta} = u_{\eta\eta} - u_{\xi\xi}$

which is the same equation with different variables $\therefore$
equation is invariant under given transformation

$\therefore \tanh\theta = w \therefore$ ~~tanh⁻¹t~~ $\operatorname{arctanh} w = \theta \therefore$

$\sinh\theta = \sinh(\operatorname{arctanh} w)$, $\cosh(\operatorname{arctanh} w) = \cosh\theta \therefore$

$\xi = x\cosh(\operatorname{arctanh} w) - t\sinh(\operatorname{arctanh} w)$

$\eta = -x\sinh(\operatorname{arctanh} w) + t\cosh(\operatorname{arctanh} w) \therefore u_{\eta\eta} - u_{\xi\xi} = 0$

$(x,t) \to (\xi, \eta)$

4a let $u = T(t)X(x) \therefore u_{xx} = X''(x)T = X''T \qquad u_t = XT' \therefore$

$XT' = X''T \therefore$

$\frac{X''}{X} = \frac{T'}{T} = -\lambda = \text{constant} \therefore$

$X'' = -\lambda X$, $T' = -\lambda T \therefore$

$X'' + \lambda X = 0 \therefore$ let $X = e^{qx} \therefore X'' = q^2 e^{qx} \therefore$

$q^2 e^{qx} + \lambda e^{qx} = 0 = q^2 + \lambda \therefore q^2 = -\lambda \therefore q = \pm\sqrt{\lambda}\, i$

$X = A_1 e^{\sqrt{\lambda} ix} + B_1 e^{-\sqrt{\lambda} ix} \therefore$

$T' + \lambda T = 0 = P(t)T' + \lambda P(t)T = \frac{d}{dt}(P(t)T) = P'(t)T + P(t)T'$

$\therefore \lambda P(t) = P'(t) \therefore \frac{P'(t)}{P(t)} = \lambda \therefore \int \frac{P'(t)}{P(t)}dt = \int \lambda\, dt = \ln P(t) = \lambda t \therefore P(t) = e^{\lambda t}$

$\therefore \frac{d}{dt}(e^{\lambda t}T(t)) = 0 \therefore e^{\lambda t}T(t) = C_1 \therefore T = C_1 e^{-\lambda t} \therefore$

$u(x,t) = X(x)T(t) = (A_1 e^{\sqrt{\lambda} i} + B_1 e^{-\sqrt{\lambda} i})C_1 e^{-\lambda t} =$

$(A_2 e^{\sqrt{\lambda} ix} + B_2 e^{-\sqrt{\lambda} ix})e^{-\lambda t}$

$\therefore$ let $\pm\sqrt{\lambda} = k \therefore \lambda = k^2$

$\therefore u(x,t) = (A_2 e^{\sqrt{\lambda} ix} + B_2 e^{-\sqrt{\lambda} ix})e^{-k^2 t} \therefore$

$u(x,t) = \int_{-\infty}^{\infty} (A_2 + B_2)e^{-ikx} e^{-k^2 t}\, dk = \int_{-\infty}^{\infty} A e^{-ikx} e^{-k^2 t}\, dt =$

$u(x \to \pm\infty, t) \to 0 \therefore$

$X(x \to \pm\infty) = 0 \therefore \lim_{x\to\infty} A_1 e^{\sqrt{\lambda} ix} + B e^{-\sqrt{\lambda} ix} = 0$

$\lim_{x\to-\infty} A_1 e^{\sqrt{\lambda} ix} + B e^{-\sqrt{\lambda} ix} = 0$

$\therefore A(k) = A_2 + B_2 = \text{constant} = A$

\PP2016/ \4b/ $u=\int_{-\infty}^{\infty} A(k) e^{ikx} e^{-k^2 t}\, dk$ ∴

$u(x,0)=f(x)=\int_{-\infty}^{\infty} A(k) e^{ikx} e^{-k^2(0)}\, dk \qquad e^{-k^2(0)}=e^0=1$ ∴

$u(x,0)=f(x)=\int_{-\infty}^{\infty} A(k) e^{ikx}\, dk$ ∴

$\frac{f(x)}{2\pi}=\frac{1}{2\pi}\int_{-\infty}^{\infty} A(k) e^{ikx}\, dk$ ∴ $A(k)=\int_{-\infty}^{\infty} f(s) e^{-iks}\, ds$ ∴

∴ $u(x,t)=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(s) e^{-iks}\, ds\, e^{ikx} e^{-k^2 t}\, dk=\int_{-\infty}^{\infty}\frac{f(s)}{2\sqrt{\pi t}} e^{-(x-s)^2/(4t)}\, ds$

\4c/ $v(0,t)=0$ ∴ do odd extension ∴

$v_t=v_{xx} \quad x\in\mathbb{R}$

for $x\geq 0$: $v(x,0)=\cancel{\phi}(x)=\delta(x-a)$

∴ for $x\leq 0$: $v(x,0)=\phi(x)=-\delta(-x-a)=-\delta(-(x+a))=\delta(x+a)$ ∴

$v(x,0)=\phi(x)=\begin{cases}\delta(x-a), & x\geq 0,\\ \delta(x-a), & x\leq 0.\end{cases}$ ∴ $v(x,0)=\phi(x)=\delta(x-a)+\delta(x+a)$ ∴

∴ $v(x,t)=\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}\phi(\xi) e^{-\frac{(x-\xi)^2}{4t}}\, d\xi=$

$\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty}(\delta(\xi-a)+\delta(\xi+a)) e^{-(x-\xi)^2/(4t)}\, d\xi=$

$\frac{1}{\sqrt{4\pi t}}\left[\int_{-\infty}^{\infty}\delta(\xi-a) e^{-(x-\xi)^2/(4t)}\, d\xi+\int_{-\infty}^{\infty}\delta(\xi-(-a)) e^{-(x-\xi)^2/(4t)}\, d\xi\right]=$

$\frac{1}{\sqrt{4\pi t}}\left[e^{-(x-a)^2/(4t)}+e^{-(x+a)^2/(4t)}\right]$

$v(x\to\infty, t)=0$ ∴

$v_x=\frac{1}{\sqrt{4\pi t}}\left[-\frac{2(x-a)}{4t} e^{-(x-a)^2/(4t)}-\frac{2(x+a)}{4t} e^{-(x-a)^2/(4t)}\right]$ ∴

$v_x(0,t)=\frac{1}{\sqrt{4\pi t}}\left[\frac{-2(0-a)}{4t} e^{-\frac{a^2}{4t}}+\frac{2a}{4t} e^{-\frac{a^2}{4t}}\right]=$

$\frac{4a}{4\times\sqrt{4\pi t}\, t} e^{-\frac{a^2}{4t}}=\frac{a}{2\sqrt{\pi t}\sqrt{t^2}} e^{-\frac{a^2}{4t}}=\frac{a}{2\sqrt{\pi t^3}} e^{-a^2/(4t)}$

\ / ~~is~~ $u_{tt}-u_{xx}=0$, $u=f(\alpha x+t)$ $u_{xx}=\alpha^2 f''$, $u_{tt}=f''$ ∴

$\cancel{\alpha^2 f}\ f''-\alpha^2 f''=(1-\alpha^2) f''=0$ ∴ $1-\alpha^2=0$ ∴

$\alpha^2=1$ ∴ $\alpha=1$, $\alpha=-1$ ∴ ~~to~~ for $u_{tt}-u_{xx}+\sin u=0$

let ~~xx~~ $\xi=x+t$, $\eta=-x+t$ ∴ $\xi_x=1$, $\xi_t=1$, $\eta_x=-1$, $\eta_t=1$ ∴

$u_x=u_\xi \xi_x+u_\eta \eta_x=u_\xi-u_\eta$, $u_t=u_\xi \xi_t+u_\eta \eta_t=u_\xi+u_\eta$ ∴

$u_{xx} = u_{\xi\xi}\xi_x + u_{\xi\eta}\eta_x - u_{\xi\eta}\xi_x - u_{\eta\eta}\eta_x = u_{\xi\xi} - 2u_{\xi\eta} + u_{\eta\eta}$

$u_{tt} = u_{\xi\xi}\xi_t + u_{\xi\eta}\eta_t + u_{\xi\eta}\xi_t + u_{\eta\eta}\eta_t = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta}$ ∴

$u_{tt} - u_{xx} \mp \sin u = u_{\xi\xi} + 2u_{\xi\eta} + u_{\eta\eta} - u_{\xi\xi} + 2u_{\xi\eta} - u_{\eta\eta} \mp \sin u =$

$4u_{\xi\eta} + \sin u = 0$ ∴

$u_{\xi\eta} = -\frac{1}{4}\sin u = \frac{\partial}{\partial\xi}\left(\frac{\partial}{\partial\eta}u\right)$

let $u = v(x - ct)$ ∴ $u_t = -cv_t$, $u_{tt} = c^2 v_{tt}$, $u_{xx} = v_{xx}$ ∴

$u_{tt} - u_{xx} + \sin u = c^2 v_{tt} - v_{xx} + \sin v =$

let $u = v(\xi) = v(x - ct)$ ∴ $\xi = x - ct$ ∴ $\xi_x = 1$, $\xi_t = -c$ ∴

$u_t = v'(\xi)\xi_t = -cv'(\xi)$ ∴ $u_{tt} = -cv'(\xi)\xi_t = c^2 v''$

$u_t = \frac{\partial}{\partial t}v(\xi) = \frac{\partial v(\xi)}{\partial\xi}\frac{\partial\xi}{\partial t} = -cv'(\xi)$ ∴ $u_{tt} = c^2 v''(\xi) = -cv''(\xi)\xi_t$

∴ $u_x = \frac{\partial v(\xi)}{\partial x} = \frac{\partial v(\xi)}{\partial\xi}\frac{\partial\xi}{\partial x} = v'(\xi)$ ∴ $u_{xx} = v''(\xi)\xi_x = v''(\xi)$ ∴

$c^2 v'' - v'' + \sin v = 0 = (c^2 - 1)v'' + \sin v$ ∴

$(1 - c^2)v'' = \sin v$ ∴

let $p(v) = \frac{dv}{d\xi}$ ∴ $v'' = \frac{d^2 v}{d\xi^2} = \frac{d}{d\xi}\frac{dv}{d\xi} = \frac{d}{d\xi}p(v) = \frac{\partial p(v)}{\partial v}\frac{\partial v}{\partial\xi} = \frac{\partial p(v)}{\partial v}p(v) = p(v)\frac{dp(v)}{dv}$

$(c^2 + 1)p(v)\frac{dp(v)}{dv} = \sin v$ ∴

$p(v)\frac{dp(v)}{dv} = \frac{1}{1-c^2}\sin v$ ∴ $\int p(v)\,d(p(v)) = \int \frac{1}{1-c^2}\sin v\,dv = \int p(v)\frac{dp(v)}{dv}dv$

$= -\frac{1}{1-c^2}\cos v + A$

$u_{xx} - 2u_{xy} + u_{yy} = au_{xx} + 2bu_{xy} + cu_{yy}$ ∴

$a = 1$, $-2 = 2b$, $c = 1$ ∴ $-2b = 2$, $b = -1$ ∴

$b^2 - ac = (-1)^2 - 1(1) = 1 - 1 = 0$ ∴ parabolic ∴

$a\left(\frac{dy}{dx}\right)^2 - 2b\left(\frac{dy}{dx}\right) + c = \left(\frac{dy}{dx}\right)^2 + 2\left(\frac{dy}{dx}\right) + 1 = 0$ ∴

$\frac{dy}{dx} = \frac{-2 \pm \sqrt{2^2 - 4(1)(1)}}{2(1)} = \frac{-2 \pm \sqrt{4-4}}{2} = \frac{-2 \pm 0}{2} = \frac{-2}{2} = -1$ ∴

$\int 1\,dy = \int 1\,dx = y = -x + c_1$ ∴ $y + x = c_1$ ∴

let $\xi = x + y$ ∴ let $\eta = x$

$5u_{ss} - u_{tt} = 0$ ∴ let $u = g(s + \alpha t)$ ∴ $u_{ss} = g''$, $u_{tt} = \alpha^2 g''$ ∴

$5g'' - \alpha^2 g'' = (5 - \alpha^2)g'' = 0$ ∴ $5 - \alpha^2 = 0$ ∴ $\alpha^2 = 5$ ∴ $\alpha = \sqrt{5}$, $\alpha = -\sqrt{5}$

$u_{xx} + u_{tt} = 0$ ∴ $a = 1$, $b = 0$, $c = 1$ ∴ $b^2 - ac < 0$ ∴ elliptic ∴

$\frac{d^2y}{dx^2} + 1 = 0$ ∴ $\frac{d^2y}{dx^2} = -1$ ∴ $\frac{dy}{dx} = \pm i$ ∴ $y = +ix + c_1$, $y = -ix + c_2$ ∴

$\backslash /$ $y = xi + c_1$, $y = -ix + c_2$ ∴ $\cancel{y =}$ $y - ix = c_1$, $y + ix = c_2$ ∴

let $\xi = y + ix$, $\eta = y - ix$ ∴ elliptic ∴

$Re(\xi) = Re(y + ix) = y$ ∴

$Re(\eta) = y$, $\cancel{Im(\xi) =}$ $Im(\xi) = Im(y + ix) = x$

$Im(y - ix) = -x$ ∴ $\cancel{I}$ $I(\xi) = I(y + ix) = x$ ∴

let $\mu = y$, $\nu = x$

$\cancel{u_\mu}$ $\mu_y = 1$, $\mu_x = 0$, $\nu_y = 0$, $\nu_x = 1$ ∴

$\cancel{u_\mu}$ $u_x = u_\mu \mu_y = u_\mu$

$\xi = x - y$, $\eta = x + y$ ∴ $\xi + y = x$ ∴ $\eta = \xi + y + y = \xi + 2y$ ∴ $-\xi + \eta = 2y$ ∴

$-\frac{1}{2}\xi + \frac{1}{2}\eta = y$ ∴ $\xi + -\frac{1}{2}\xi + \frac{1}{2}\eta = \frac{1}{2}\xi + \frac{1}{2}\eta = x$

$A\sin n\theta + B\cos n\theta$
$= A(\sin n\theta) + \cos n\theta)$

$\cosh ny = \dfrac{e^{ny} + e^{-ny}}{2}$

$(v)\frac{dA(v)}{dV}$

$C_1 = C_3 + C_4$ $\quad C_2 = C_3 + C_5$

$C_6\cosh(ny) + C_7 \sinh ny$

$\backslash /$ $u_x + u_y - u = \cos(x - y)$

let $\xi = x + y$ ∴ $\eta = x - y$ ∴

$\xi_x = 1$, $\xi_y = 1$, $\eta_x = 1$, $\eta_y = -1$ ∴

$u_x = u_\xi \xi_x + u_\eta \eta_x = u_\xi + u_\eta$

$u_y = u_\xi \xi_y + u_\eta \eta_y = u_\xi - u_\eta$ ∴

$u_x + u_y = u_\xi + u_\eta + u_\xi - u_\eta = 2u_\xi$

∴ $u_x + u_y - u = 2u_\xi - u = \cos(x - y)$ ∴

$u_\xi - \frac{1}{2}u = \frac{1}{2}\cos(x - y)$ ∴ $e^{\int -\frac{1}{2} d\xi} = e^{-\frac{1}{2}\xi}$ ∴

$\frac{\partial}{\partial \xi}(e^{-\frac{1}{2}\xi}u) = \frac{1}{2}e^{-\frac{1}{2}\xi}\cos(x - y) = \frac{1}{2}e^{-\frac{1}{2}\xi}\cos(\eta)$ ∴

$e^{-\frac{1}{2}\xi}u = -\frac{1}{4}e^{-\frac{1}{2}\xi}\cos(\eta) + f(\eta)$ ∴

$u = -\frac{1}{4}\cos(\eta) + f(\eta)e^{\frac{1}{2}\xi} = -\frac{1}{4}\cos(x - y) + f(x - y)e^{\frac{1}{2}x + \frac{1}{2}y}$

$u(x, 0) = 0 = -\frac{1}{4}\cos(x) + f(x)e^{-\frac{1}{2}x}$ ∴ $\frac{1}{4}\cos x = f(x)e^{-\frac{1}{2}x}$ ∴ $\frac{1}{4}e^{\frac{1}{2}x}\cos x = f(x)$ ∴

$u = -\frac{1}{4}\cos(x - y) + \frac{1}{4}e^{x-y}\cos(x - y)e^{\frac{1}{2}x + \frac{1}{2}y} = -\frac{1}{4}\cos(x - y) + \frac{1}{4}e^{\frac{3}{2}x + -\frac{1}{2}y}\cos(x - y)$