\Sheet 5/ sol iff $\int_{0}^{1} f(x)\,dx = 0$ we can directly check that those solvability & sol results given by Z Fredholm alternative are correct. Z non homog IE is of finite rank 2 we can solve it using Z standard techniques. introducing $P_1 = \int_{-1}^{1} y(t)\,dt$ & $P_2 = \int_{-1}^{1} (3t^2-1)\,y(t)\,dt$ ∴

$y(x) = \frac{1}{2}P_1 + P_2 x + f(x)$ ∴ $P_1 = \int_{-1}^{1} f(x)\,dx + P_1$, $P_2 = \int_{-1}^{1} (3x^2-1) f(x)\,dx$

∴ $\int_{-1}^{1} f(x)\,dx = 0$ is a necessary cond for Z ~~first~~ eqn to hold & Z 2nd gives $P_2 = \int_{-1}^{1} (3x^2-1) f(x)\,dx$. ∄ no conds on P_1, it is ∴ an arbit const param (representing Z sol of Z homog eqn) ∴

$y(x) = f(x) + 3x \int_{-1}^{1} t^2 f(t)\,dt + P_1$

\3/ $\lambda y - Ky = f$, $\|y\|_\infty \leq C \|f\|_\infty$ ∴

$\lambda y(x) - \int_{a}^{b} k_1(x) k_2(t) y(t)\,dt = f(x)$ ∴ $\lambda \neq \int_{a}^{b} k_1(t) k_2(t)\,dt$ eqn has exactly one sol $y(x) = \frac{1}{\lambda}\left(f(x) + \frac{\int_a^b f(t) k_2(t)\,dt}{\lambda - \int_a^b k_1(t) k_2(t)\,dt} k_1(x) \right)$ ∴

$$|y(x)| \leq \frac{1}{\lambda}\left(|f(x)| + \frac{\left|\int_a^b f(t) k_2(t)\,dt\right|}{\left|\lambda - \int_a^b k_1(t) k_2(t)\,dt\right|} |k_1(x)| \right) \leq$$

$$\frac{1}{\lambda}\left(\|f\|_\infty + \frac{\|f\|_\infty \int_a^b |k_2(t)|\,dt}{\left|\lambda - \int_a^b k_1(t) k_2(t)\,dt\right|} \|k_1\|_\infty \right) ∴$$

$C = \frac{1}{|\lambda|}\left(1 + \frac{\|k_1\|_\infty \|k_2\|_1}{\left|\lambda - \int_a^b k(t,t)\,dt\right|} \right)$, where $\|k_2\|_1 = \int_a^b |k_2(t)|\,dt$

\4/ $y - Ky = f$ $k(x,t) = \frac{1}{2} + x(3t^2-1)$ $x \in [-1, 1]$ ∴

adjoint homog case: $Z(x) - K^* Z(x) = 0$ ∴

$K^*(x,t) = \frac{1}{2} t(3x^2-1)$ ∴

$Z(x) - \int_{-1}^{1} \frac{1}{2} Z(t)\,dt - (3x^2-1) \int_{-1}^{1} t\,Z(t)\,dt = 0$

∴ $Z(x) = \frac{1}{2}P_1 + (3x^2-1)P_2$ ∴ $P_1 = \int_{-1}^{1} Z(t)\,dt$, $P_2 = \int_{-1}^{1} t\,Z(t)\,dt$ ∴

$P_1 = \int_{-1}^{1} \frac{1}{2}P_1 + (3t^2-1)P_2\,dt$, $P_2 = \int_{-1}^{1} t\left(\frac{1}{2}P_1 + (3t^2-1)P_2\right) dt$ ∴

$P_1 = P_1$, $P_1 \in \mathbb{R}$, $P_2 = 0$ ∴ $Z(x) = \frac{1}{2}P_1$

\ Week 7 sheet / \1a/ $(Ku)(x) = \int_0^a \cos(x-t)\,u(t)\,dt$

$\therefore \|K\|_\infty = \max_{x\in[0,a]} \int_0^a |\cos(x-t)|\,dt = \max_{x\in[0,a]} \int_{x-a}^{x} |\cos u|\,du = \int_0^a |\cos t|\,dt$

\1b/ $(Ku)(x) = \int_0^1 e^{xt} u(t)\,dt \;\therefore\; \|K\|_\infty = \max_{x\in[0,1]} \int_0^1 |e^{xt}|\,dt = \max_{x\in[0,1]} \int_0^1 e^{xt}\,dt =$

$\max_{x\in[0,1]} \left[\frac{e^x-1}{x}\right]$ note Z formula also applies when $x=0 \;\therefore$

Z func $\frac{e^x-1}{x}$ is increasing $\therefore$ its max on $[0,1]$ is e', Z norm of K

\1c/ $(Ku)(x) = \int_{-1}^1 xt\,u(t)\,dt \;\therefore\; \|K\|_\infty = \max_{x\in[-1,1]} \int_{-1}^1 |xt|\,dt = 2\max_{x\in[-1,1]} \int_0^1 |x|t\,dt$

$= \max_{x\in[-1,1]} |x| = 1$

\2/ $k(x,t) = \cos(x+t) \quad h(x,t) = \sin(x+t)$

kernel of KH is: $\int_0^1 k(x,s)h(s,t)\,ds = \int_0^1 \cos(x+s)\sin(s+t)\,dS =$

$\Big\{ \frac{1}{2}\int_0^1 \sin(x-t) + \sin(x+2s+t)\,ds = \frac{1}{2}\sin(x-t) + \frac{1}{2}\int_0^1 \sin(2s+x+t)\,ds =$

$\frac{1}{2}\sin(x-t) - \frac{1}{4}\left[\cos(2s+x+t)\right]_0^1 \Big\}$

$\int_0^1 k(x,s)h(s,t)\,ds = \int_0^1 \cos(x+s)\sin(s+t)\,ds = \frac{1}{2}\sin 1 \sin(x+t+1) + \frac{1}{2}\sin(x-t)$

for HK: $\int_0^1 h(x,s)k(s,t)\,ds = \int_0^1 \sin(x+s)\cos(s+t)\,dS =$

$\frac{1}{2}\sin 1 \sin(x+t+1) - \frac{1}{2}\sin(x-t)$, Z two operators are symmetric but do not compute

\3a/ $y(x) - \int_0^1 xt^3 y(t)\,dt = x^2 \quad 0 \le x \le 1 \;\therefore$

$\Big\{ y(x) - x\int_0^1 t^3 y(t)\,dt = x^2 = y(x) - xP \quad , P = \int_0^1 t^3 y(t)\,dt \;\therefore$

$x^2 + xP = y(x) \;\therefore\; y(t) = t^2 + Pt \;\therefore\; P = \int_0^1 t^3 y(t)\,dt = \int_0^1 t^3(t^2+Pt)\,dt =$

$\int_0^1 t^5 + Pt^4\,dt = \left[\frac{1}{6}t^6 + \frac{1}{5}Pt^5\right]_0^1 = \frac{1}{6}(1^6 - 0^6) + \frac{1}{5}P(1^5 - 0^5) = \frac{1}{6} + \frac{1}{5}P = P \;\therefore$

$\frac{4}{5}P = \frac{1}{6} \;\therefore\; P = \frac{5}{24} \;\therefore$

$y(x) = x^2 + \frac{5}{24}x$

\3b/ $y(x) - \int_0^1 xt^3 y(t)\,dt = x^2 \;\therefore\; \cancel{y_0 = x^2} \;\therefore\; y(x) = x^2 + \int_0^1 xt^3 y(t)\,dt \;\therefore$

$y_0(x) = x^2 \quad \therefore \quad y_1(x) = x^2 + x\int_0^1 t^3 y_0(t)\,dt = x^2 + x\int_0^1 t^3 t^2\,dt = x^2 + x\int_0^1 t^5\,dt =$

Week 7 Sheet / $x^2 + \frac{1}{6}x = y_1(x) \therefore y_1(t) = t^2 + \frac{1}{6}t \therefore$

$y_2 = x^2 + x\int_0^1 t^3 (y_1(t))\,dt = x^2 + x\int_0^1 t^3\left(t^2 + \frac{t}{6}\right)dt = x^2 + \frac{x}{5} \therefore y_2(t) = t^2 + \frac{t}{5}$

$\therefore y_3(x) = x^2 + x\int_0^1 t^3 y_2(t)\,dt = x^2 + x\int_0^1 t^3\left(t^2 + \frac{t}{5}\right) = x^2 + \frac{31}{150}x$

3c / $\|K\|_\infty = \frac{1}{4}$ $\left\{ \max_{x\in[0,1]} \int_0^1 |xt^3|\,dt = \max_{x\in[0,1]} |x| \int_0^1 t^3\,dt = \right.$

$\left. \max_{x\in[0,1]} |x| \left[\frac{1}{4}t^4\right]_0^1 = \max_{x\in[0,1]} |x| \frac{1}{4} = \frac{1}{4}\times 1 = \frac{1}{4} \right\}$ $\therefore$ Z error for y_3 is

bounded by $\frac{1}{3\cdot 4^3} = \frac{1}{192}$. Z actual error is $\frac{1}{500}$ at its max

$\left\{ \|y - y_n\|_\infty \le \frac{1}{|\lambda| - \|K\|_\infty}\left(\frac{\|K\|_\infty^{n+1}}{|\lambda|^{n+1}}\right)\|f\|_\infty \therefore \right.$

$\|y - y_3\|_\infty \le \frac{1}{|\lambda| - \|K\|_\infty}\left(\frac{\|K\|_\infty^4}{|\lambda|^{n+1}}\right)\|f\|_\infty \therefore \|f\|_\infty = \max_{x\in[0,1]} x^2 = 1^2 = 1$

$\left. \lambda = 1 \therefore |\lambda| = 1 \therefore \|y - y_3\|_\infty \le \frac{1}{1 - \left(\frac{1}{4}\right)}\left(\frac{\left(\frac{1}{4}\right)^4}{1}\right)1 = \frac{4}{3}\times\left(\frac{1}{4}\right)^4 = \frac{1}{3\times 4^3} = \frac{1}{192} \right\}$

1d / solving Z resolvent kernel of Z integral operator is

$r = \frac{5}{4}xt^3$ & Z sol of Z integral eqn with $f = x^2$ is ~~$y(x)$~~

$y(x) = x^2 + \frac{5}{24}x \therefore y(x) = f(x) + \frac{5}{4}x\int_0^1 t^3 f(t)\,dt$

1d / $\left\{ y(x) = x^2 + \int_0^1 xt^3 y(t)\,dt = f(x) + x\int_0^1 t^3 y(t)\,dt \right.$

$\left. = f(x) + \int_0^1 xt^3 y(t)\,dt = y(x) \right\} \therefore$

$y(x) = f(x) + \frac{5}{4}x\int_0^1 t^3 f(t)\,dt \therefore r = \frac{5}{4}xt^3 \therefore$

$P_2 = \int_0^1 t^3 f(t)\,dt \therefore y(x) = f(x) + \frac{5}{4}xP_2 \therefore y(t) = f(t) + \frac{5}{4}tP_2 \therefore$

~~$P_2 = \int_0^1 t^3$~~ ~~$y(t) - f(t) =$~~ $y(t) - \frac{5}{4}P_2 t = f(t) \therefore x^2 = f(x) \therefore f(t) = t^2 \therefore$

$y(t) = t^2 + \frac{5}{4}tP_2 \therefore P_2 = \int_0^1 t^3\left[t^2\right]dt = \int_0^1 t^5\,dt = \left[\frac{1}{6}t^6\right]_0^1 = \frac{1}{6} \therefore$

$y(x) = x^2 + \frac{5}{4}x \times \frac{1}{6} = x^2 + \frac{5}{24}x$

1d / $y(x) = x^2 + \int_0^1 xt^3 y(t)\,dt = x^2 + x\int_0^1 t^3 y(t)\,dt = x^2 + xP_2 \therefore$

$P_2 = \int_0^1 t^3 y(t)\,dt \therefore y(t) = t^2 + tP_2 \therefore$ Let $y(x) = f(x) + \frac{5}{4}x\int_0^1 t^3 f(t)\,dt$

$\therefore r = \frac{5}{4}xt^3 \therefore P_3 = \int_0^1 t^3 f(t)\,dt \therefore f(x) = x^2 \therefore y(x) = x^2 + \frac{5}{4}x\int_0^1 t^3 (t^2)\,dt$

$= x^2 + \frac{5}{4}x\int_0^1 t^5\,dt = x^2 + \frac{5}{4}x\left[\frac{1}{6}t^6\right]_0^1 = x^2 + \frac{5}{4}x\left[\frac{1}{6}\right] = x^2 + \frac{5}{24}x$

4a/ $\int_0^1 |k(x,t)|\,dt = \int_0^x |x-t|\,dt + \int_x^1 |0|\,dt = \int_0^x x-t\,dt + 0 =$

$\left[xt - \tfrac{1}{2}t^2\right]_0^x = xx - \tfrac{1}{2}x^2 = x^2 - \tfrac{1}{2}x^2 = \tfrac{1}{2}x^2$ but $x \in [0,1]$ ∴

$x_1^2 > x_2^2$ for $x_1 > x_2$ ∴ $\max_{x\in[0,1]}(\tfrac{1}{2}x^2) = \tfrac{1}{2}\times 1^2 = \tfrac{1}{2}$ ∴

$\tfrac{1}{2}x^2 \le \tfrac{1}{2}$ for $x \in [0,1]$ ∴ $\int_0^1 |k(x,t)|\,dt \le \tfrac{1}{2}$

4a sol/ rewrite eqn as volterra IE: $y(x) - \int_0^x (x-t)y(t)\,dt = f(x)$

∴ $\int_0^1 |k(x,t)|\,dt = \int_0^x (x-t)\,dt = -\tfrac{1}{2}\left[(x-t)^2\right]_0^x = \frac{x^2}{2} \le \frac{1}{2} \quad \forall x \in [0,1]$

4b/ find $y_0(x) = x$ $\{ y - Ky = x$ ∴ $y(x) - \int_0^1 k(x,t)y(t)\,dt = x$

$y(x) - \int_0^x (x-t)y(t)\,dt = y(x) - x\int_0^x y(t)\,dt + \int_0^x t\,y(t)\,dt \}$

$y - Ky = x$ ∴ $y_0(t) = t$ ∴ $y(x) = x + \int_0^x k(x,t)y(t)\,dt = x + \int_0^x (x-t)y(t)\,dt$ ∴

$y_1(x) = x + \int_0^x (x-t)y_0(t)\,dt = x + \int_0^x (x-t)t\,dt = x + \frac{1}{3!}x^3$ ∴ $y_1(t) = t + \frac{1}{3!}t^3$ ∴

$y_2(x) = x + \int_0^x (x-t)y_1(t)\,dt = x + \int_0^x (x-t)(t + \frac{1}{3!}t^3)\,dt = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5$ ∴

claim $y_n(x) = x\sum_{k=0}^{n} \frac{x^{2k}}{(2k+1)!}$ ∴ $y_{n+1}(x) = x + \int_0^x (x-t)y_n(t)\,dt =$

$x + x\int_0^x y_n(t)\,dt - \int_0^x t\,y_n(t)\,dt = x + x\int_0^x \sum_{k=0}^{n} \frac{t^{2k+1}}{(2k+1)!})\,dt - \int_0^x \sum_{k=0}^{n} \frac{t^{2k+2}}{(2k+1)!})\,dt$

$= x + x\sum_{k=0}^{n} \frac{x^{2k+2}}{(2k+2)!} - \sum_{k=0}^{n} \frac{x^{2k+3}}{(2k+3)\cdot(2k+1)!} =$

$x + x\sum_{k=0}^{n} \frac{x^{2k}}{(2k+1)!}\left(\frac{1}{2k+2} - \frac{1}{2k+3}\right) = x + x\sum_{k=1}^{n+1} \frac{x^{2k}}{(2k+1)!}$ note the exact sol

is $y(x) = \sinh x$ ∴ y is also the sol of $y'' - y = f''$ with

ICs $y(0) = f(0)$, $y'(0) = f'(0)$ ∴ $y'' - y = 0$, $y(0) = 0$, $y'(0) = 1$ ∴

the GS is $y(x) = A\cosh x + B\sinh x$ ∴ BC: $y(x) = \sinh x$

Sheet 8/ 5/ $y(x) - \lambda\int_0^x \sin(x-t)y(t)\,dt = f(x)$ $x \ge 0$ $\lambda \ge 0$ for

Q5a/ solve & give the sol for $0 \le \lambda 1$, $\lambda = 1$, $\lambda > 1$

5b/ give the sol $f(x) = x$ $\lambda = 1$ & $\lambda = 5$

5a sol/ Laplace: $\hat{y}(s) - \lambda\hat{y}(s)\,\mathcal{L}(\sin(x)) = \hat{f}(s)$ $\mathcal{L}(\lambda\int_0^x \sin(x-t)y(t)\,dt)$ LT22

$\hat{y}(s) - \lambda\hat{y}(s)\frac{1}{s^2+1} = \hat{f}(s)$ ∴ $\hat{y}(s) = \left(\frac{s^2+1}{s^2+1-\lambda}\right)\hat{f}(s)$

$\hat{y}(s) = \left(\frac{s^2+1+\lambda-\lambda}{s^2+1-\lambda}\right)\hat{f}(s) = \left(1 + \frac{\lambda}{s^2+1-\lambda}\right)\hat{f}(s)$ ∴

$y(x) = y(x) + e^{x} \int e^{-x} y(x)\,dx + c$ [illegible] I.C. $y(0) = y(0)$

\4/b $y(x) = x$ ∴ $y(x) = x + e^{x} \int e^{-x} x \, dx$ [illegible]

$y(x) = -1 + e^{x}$

using Neuman iterations: $g(x) = x$ ∴ $y_0 = x$ ∴ $\lambda = 1$

using $\frac{1}{\lambda} \sum_{n=0}^{\infty} \frac{K^n g}{\lambda^n} \to y = ...$

$(Kg)(x) = \int_0^x t \, dt = \frac{1}{2} x^2$ $\quad K(Kg(x)) = \int_0^x \frac{1}{2} t^2 \, dt = \frac{1}{6} x^3$

$K(K(Kg(x))) = \int_0^x \frac{1}{6} t^3 \, dt = \left[\frac{1}{24} t^4\right]_0^x = \frac{1}{24} x^4$

$K^n g = \frac{x^n}{n!}$ $\quad y_n = \sum_{n=1}^{\infty} \frac{x^n}{n!}$ $\quad e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \sum_{n=1}^{\infty} \frac{x^n}{n!}$

$y_n = \sum_{n=1}^{\infty} \frac{x^n}{n!} = e^{x} - 1$

\3/ $y(x) - \int_0^x e^{-t} y(t)\,dt = 1$ Neumann method:

$y_0 = \frac{f}{\lambda} = 1$ $\quad y_1 = \frac{f}{\lambda} + \frac{1}{\lambda} K y_0 = 1 + \int_0^x e^{-t} \cdot 1 \, dt = 1 + (1 - e^{-x})$

$y_n = \frac{f}{\lambda} + \frac{1}{\lambda} \int_0^x e^{-t} y_{n-1}(t)\,dt$

$y_2 = 1 + \int_0^x e^{-t}(1 + (1 - e^{-t}))\,dt = 1 + (1 - e^{-x}) + \frac{1}{2}(1 - e^{-x})^2$

$y_n(x) = 1 + \sum_{j=1}^{n} \frac{1}{j!}(1 - e^{-x})^j$ ∴ use induction, goal: $y_{n+1} = 1 + \sum_{j=1}^{n+1} \frac{1}{j!}(1 - e^{-x})^j$

$y_{n+1} = 1 + \int_0^x e^{-t}\left[1 + \sum_{j=1}^{n} \frac{1}{j!}(1 - e^{-t})^j\right] dt =$

$1 + (1 - e^{-x}) + \sum_{j=1}^{n} \frac{1}{(j+1)!}\left[(1 - e^{-t})^{j+1}\right]_0^x =$

$1 + \sum_{j=1}^{n+1} \frac{1}{j!}(1 - e^{-x})^j$ as we expect for our goal

using index shift $y_{n+1} = 1 + (1 - e^{-x}) + \sum_{j=1}^{n} \frac{1}{(j+1)!}\{(1 - e^{-x})^{j+1}\}$

$= 1 + (1 - e^{-x}) + \sum_{j=2}^{n+1}\left(\frac{1}{j!}(1 - e^{-x})^j\right)$

$y_n(x) = 1 + \sum_{j=1}^{n} \frac{1}{j!}(1 - e^{-x})^j = \exp(1 - \exp(-x)) = e^{1 - e^{-x}}$

\Week 7 Sheet/ \4a/ rewrite as volterra IE:

$$y(x) - \int_0^x (x-t)y(t)\,dt = f(x) \quad \therefore \int_0^1 |k(x,t)|\,dt = \int_0^x (x-t)\,dt = -\frac{1}{2}\left[(x-t)^2\right]_0^x =$$

$$\frac{x^2}{2} \leq 1 \quad \forall x \in [0,1]$$

\4b/ $y_0(x) = x \quad \therefore y_1(x) = x + \int_0^x (x-t)y_0(t)\,dt = x + \int_0^x (x-t)t\,dt = x + \frac{1}{3!}x^3 \quad \therefore$

$$y_2(x) = x + \int_0^x (x-t)y_1(t)\,dt = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5 \quad \therefore$$

Claim $y_n(x) = \sum_{k=0}^{n} \frac{x^{2k+1}}{(2k+1)!} = x\sum_{k=0}^{n} \frac{x^{2k}}{(2k+1)!} \quad \therefore y_n(t) = \sum_{k=0}^{n} \frac{t^{2k+1}}{(2k+1)!}$

$$y_{n+1} = x + \int_0^x (x-t)y_n(t)\,dt = x + x\int_0^x y_n(t)\,dt - \int_0^x t\,y_n(t)\,dt =$$

$$x + x\int_0^x \sum_{k=0}^{n} \frac{t^{2k+1}}{(2k+1)!}\,dt - \int_0^x t\sum_{k=0}^{n} \frac{t^{2k+1}}{(2k+1)!}\,dt =$$

$$x + x\int_0^x \sum_{k=0}^{n} \frac{t^{2k+1}}{(2k+1)!}\,dt - \int_0^x \sum_{k=0}^{n} \frac{t^{2k+2}}{(2k+1)!}\,dt =$$

$$x + x\sum_{k=0}^{n}\int_0^x \frac{t^{2k+1}}{(2k+1)!}\,dt - \sum_{k=0}^{n}\int_0^x \frac{t^{2k+2}}{(2k+1)!}\,dt =$$

$$x + x\sum_{k=0}^{n}\left[\frac{t^{2k+2}}{(2k+2)(2k+1)!}\right]_0^x - \sum_{k=0}^{n}\left[\frac{t^{2k+3}}{(2k+3)(2k+1)!}\right]_0^x =$$

$$x + x\sum_{k=0}^{n} \frac{x^{2k+2}}{(2k+2)(2k+1)!} - \sum_{k=0}^{n} \frac{x^{2k+3}}{(2k+3)(2k+1)!} =$$

$$x + x\sum_{k=0}^{n} \frac{x^{2k+2}}{(2k+2)!} - \sum_{k=0}^{n} \frac{x^{2k+3}}{(2k+3)(2k+1)!} =$$

$$x + \sum_{k=0}^{n} \frac{x^{2k+3}}{(2k+2)!} - \sum_{k=0}^{n} \frac{x^{2k+3}}{(2k+3)(2k+1)!} = x + x\sum_{k=0}^{n} \frac{x^{2k+2}}{(2k+2)!} - x\sum_{k=0}^{n} \frac{x^{2k+2}}{(2k+3)(2k+1)!} =$$

$$x + x\sum_{k=0}^{n}\left(\frac{x^{2k+2}}{(2k+2)!} - \frac{x^{2k+2}}{(2k+3)(2k+1)!}\right) = x + x\sum_{k=0}^{n}\left(x^{2k+2}\left(\frac{1}{(2k+2)(2k+1)!} - \frac{1}{(2k+3)(2k+1)!}\right)\right)$$

$$= x + x\sum_{k=0}^{n} \frac{x^{2k+2}}{(2k+1)!}\left(\frac{1}{2k+2} - \frac{1}{2k+3}\right) =$$

$$x + x\left[\frac{x^2}{1!}\left(\frac{1}{2} - \frac{1}{3}\right) + \frac{x^4}{3!}\left(\frac{1}{4} - \frac{1}{5}\right) + \frac{x^6}{5!}\left(\frac{1}{6} - \frac{1}{7}\right) + \dots + \frac{x^{2n}}{(2n-1)!}\left(\frac{1}{2n} - \frac{1}{2n+1}\right) + \frac{x^{2n+2}}{(2k+1)!}\left(\frac{1}{2n+2} - \frac{1}{2n+3}\right)\right]$$

$$= x + x\left[\frac{x^2}{1!}\left(\frac{1}{2\times 3}\right) + \frac{x^4}{3!}\left(\frac{1}{4\times 5}\right) + \frac{x^6}{5!}\left(\frac{1}{6\times 7}\right) + \dots + \frac{x^{2n}}{(2n-1)!}\left(\frac{1}{2n\times(2n+1)}\right) + \frac{x^{2n+2}}{(2n+1)!}\left(\frac{1}{(2n+2)(2n+3)}\right)\right] =$$

$$x + x\left[\frac{x^2}{3!} + \frac{x^4}{5!} + \frac{x^6}{7!} + \dots + \frac{x^{2n}}{(2n+1)!} + \frac{x^{2n+2}}{(2n+3)!}\right] = x + x\sum_{k=1}^{n+1} \frac{x^{2k}}{(2k+1)!}$$

note the exact sol is $y(x) = \sinh x$. this can be found easily

$\therefore$ y is also the sol of $y'' - y = f''$ with ICs $y(0) = f(0)$, $y'(0) = f'(0)$ $\therefore$

$y'' - y = 0$, $y(0) = 0$, $y'(0) = 1$ $\therefore$ GS: $y(x) = A\cosh x + B\sinh x$ $\therefore$ BC: $y(x) = \sinh x$

\Sheet 8/ $y(x) = f(x) + \mathcal{L}^{-1}\Big(\underbrace{\frac{\lambda}{s^2+1-\lambda}}_{\hat{g}(s)}\hat{f}(s)\Big)$

$y(x) = f(x) + \int^x f(t)\,g(x-t)\,dt$ (used LT22)

$\mathcal{L}^{-1}(\hat{g}(s)) = \mathcal{L}^{-1}\left(\frac{\lambda}{s^2+1-\lambda}\right)$ our inverse depends on λ

- $\lambda = 1$: $\hat{g}(s) = \frac{1}{s^2}$ $\therefore g(x) = x$
- $\lambda < 1$: $\hat{g}(s) = \frac{\lambda}{s^2+\underbrace{1-\lambda}_{posit}} = \cancel{\lambda g}\ \frac{\lambda}{\sqrt{1-\lambda}}\frac{\sqrt{1-\lambda}}{s^2+1-\lambda}$ $\therefore g(x) = \frac{\lambda}{\sqrt{1-\lambda}}\sin(\sqrt{1-\lambda}\,x)$

$\lambda > 1$: $\hat{g}(s) = \frac{\lambda}{s^2+\underbrace{1-\lambda}_{\to\ neg^n}}$ $\therefore g(x) = \frac{\lambda}{\sqrt{\lambda-1}}\sinh(\sqrt{\lambda-1}\,x)$ $\therefore$

$y(x) = f(x) + \int_0^x f(t)\,g_\lambda(x-t)\,dt$

$$g_\lambda(x-t) = \begin{cases} x-t & \lambda = 1 \\ \frac{\lambda}{\sqrt{1-\lambda}}\sin(\sqrt{1-\lambda}(x-t)) & \lambda < 1 \\ \frac{\lambda}{\sqrt{1-\lambda}}\sinh(\sqrt{1-\lambda}\,(x-t)) & \lambda > 1 \end{cases}$$

\5b/ $f(x) = x$, $\lambda = 1$: $y(x) = x + \int_0^x t(x-t)\,dt = x + \int_0^x tx - t^2\,dt$

$y(x) = x + \frac{1}{6}x^3$

$\lambda = 5$: $y(x) = x + \int_0^x t\,\frac{\lambda}{\sqrt{\lambda-1}}\sinh(\sqrt{\lambda-1}(x-t))\,dt$

$y(x) = x + \int_0^x t \cdot \frac{5}{2}\sinh(2(x-t))\,dt$ $u = t$ $u' = 1$ $v' = \sinh(2(x-t))$, ~~$v = \frac{1}{2}$~~

$v = -\frac{1}{2}\cosh(2(x-t))$ $\therefore y(x) = x + \frac{5}{2}\left[-\frac{1}{2}t\cosh(2(x-t))\right]_0^x + \frac{5}{4}\int_0^x \cosh(2(x-t))\,dt$

$y(x) = x - \frac{5}{4}x + \frac{-5}{8}\left[\sinh(2(x-t))\right]_0^x = x - \frac{5}{4}x - \frac{5}{8}\sinh(2x)$

$y(x) = -\frac{1}{4}x + \frac{5}{8}\sinh(2x)$

\Sheet 9 ~~1~~/ $(Ky)(x) = \int_0^x y(t)\,dt$

\Q 1a/ solve $y - (Ky)(x) = g$

\Q 1b/ $g(x) = x$ solve analytical & Neumann method

$y(x) - \int_0^x y(t)\,dt = g(x)$ $y'(x) - y(x) = g'(x)$ $y(0) = g(0)$

using IF: $e^{\int -1\,dx} = e^{-x}$

$\frac{d}{dx}(y(x)e^{-x}) = e^{-x}g'(x)$ $\therefore$ $y(x)e^{-x} = \int_0^x e^{-x}g'(x)\,dx$

IBP $u = e^{-x}$ $u' = -e^{-x}$, $v' = g'$, $v = g$ $\therefore$

$y(x)e^{-x} = \left[e^{-x}g(x)\right] - \int_0 -e^{-x}g\,dx$ $\therefore$ $y(x) = g(x) + \frac{1}{e^{-x}}\int e^{-x}g\,dx$ $\therefore$

$\therefore \quad \lambda = 1 \therefore \quad y_0 = f = 1 \therefore$

Ex5

$y(x) - \frac{1}{80}\int_0^{\pi}(\sin x + e^t)\,y(t)\,dt = 1 \quad 0 \le x \le \pi \therefore$

$y(x) = 1 + \frac{1}{80}\int_0^{\pi}(\sin x + e^t)\,y(t)\,dt = 1 + Ky \therefore$

$y_1 = 1 + Ky_0 = 1 + K1 \therefore \quad y_2 = 1 + Ky_2 \therefore$

$\left\{ y_1(x) = 1 + \frac{1}{80}\int_0^{\pi}(\sin x + e^t)\,1\,dt = 1 + \frac{1}{80}\left[t\sin x + e^t\right]_0^{\pi} = \right.$

$\left. 1 + \frac{1}{80}\pi\sin x + \frac{e^{\pi} - e^0}{80} = 1 + \frac{1}{80}e^{\pi} - \frac{1}{80} = \left(\frac{79 + e^{\pi}}{80}\right) + \frac{\pi}{80}\sin x \right\} \therefore$

$y_2 = 1 + \frac{1}{80}\int_0^{\pi}(\sin x + e^t)\,y_1(t)\,dt = 1 + \frac{1}{80}\int_0^{\pi}(\sin x + e^t)\left[\frac{79 + e^{\pi}}{80} + \frac{\pi}{80}\sin x\right]dt =$

$1 + \frac{1}{80^2}\left[(79 + e^{\pi})(e^{\pi} - 1) + \frac{\pi}{2}(e^{\pi} - 1)\right] + \frac{(81\pi + \pi e^{\pi})}{80^2}\sin x$

$\left\{ \max_{x \in [0,\pi]} \frac{1}{80}\int_0^{\pi}|\sin x + e^t|\,dt = \max_{x \in [0,\pi]} \frac{1}{80}\int_0^{\pi}|\sin x| + e^t\,dt = \right.$

$\left. \max_{x \in [0,\pi]} \frac{1}{80}\left[t|\sin x| + e^t\right]_0^{\pi} = \frac{1}{80}\left[t \times 1 + e^t\right]_0^{\pi} = \frac{1}{80}\left[\pi + e^{\pi} - 1\right] = \frac{\pi - 1 + e^{\pi}}{80} \right\}$

$\therefore \|K\|_{\infty} = \frac{\pi - 1 + e^{\pi}}{80} \quad \therefore$ error is:

$\|y - y_n\|_{\infty} \le \frac{1}{|\lambda| - \|K\|_{\infty}}\left(\frac{\|K\|_{\infty}^{n+1}}{|\lambda|^{n+1}}\right)\|f\|_{\infty} \therefore$

$\|y - y_2\|_{\infty} \le \frac{1}{|\lambda| - \|K\|_{\infty}}\left(\frac{\|K\|^3}{|\lambda|^3}\right)\|f\|_{\infty} = \frac{1}{|1| - \|K\|_{\infty}}\left(\frac{\|K\|^3}{|1|^3}\right)1 =$

$\frac{1}{1 - \|K\|_{\infty}}\|K\|_{\infty}^3 = \frac{1}{1 - \left(\frac{\pi - 1 + e^{\pi}}{80}\right)}\left(\frac{\pi - 1 + e^{\pi}}{80}\right)^3 = \frac{(\pi - 1 + e^{\pi})^3}{80^2} = 0.0481767$

note we calc Z exact sol from usual techniques. Z eqn is of rank 2

$\therefore P_1 = \int_0^{\pi} y(t)\,dt, \quad P_2 = \int_0^{\pi} e^t y(t)\,dt \therefore \quad y(x) = \frac{80 + P_2}{80} + \frac{P_1}{80}\sin x \therefore$

$y(t) = \frac{80 + P_2}{80} + \frac{P_1}{80}\sin t \therefore \quad P_1 = \int_0^1 y(t)\,dt = \int_0^1 \frac{80 + P_2}{80} + \frac{P_1}{80}\sin t\,dt$

$P_2 = \int_0^{\pi} e^t y(t)\,dt = \int_0^{\pi} e^t\left(\frac{80 + P_2}{80} + \frac{P_1}{80}\sin t\right)dt \therefore$

$78P_1 - \pi P_2 = 80\pi, \quad (e^{\pi} + 1)\frac{P_1}{2} - (81 - e^{\pi})P_2 = 80(1 - e^{\pi}) \therefore$

$P_1 = 1.5884, \quad P_2 = 30.94 \therefore \quad y(x) = 1.387 + 0.0196\sin x$

6a

kernel of K_pL_q is: $k_p(x,t) = xt^p \therefore k_p(x,s) = xs^p$. (2)

$L_q(x,t) = x^q t \therefore L_q(s,t) = s^q t \therefore K_pL_q$ is: $\int_0^1 k_p(x,s)\,L_q(s,t)\,ds =$

$\int_0^1 xs^p s^q t\,ds = \int_0^1 xt\,s^{p+q}\,ds = \left[\frac{xt\,s^{p+q+1}}{p+q+1}\right]_0^1 = \frac{xt}{p+q+1} \therefore$

Week 7 Sheet $l_q(x,s) = x^q s$, $k_p(s,t) = s t^p$ ∴

$L_q K_p$: $\int_0^1 l_q(x,s) k_p(s,t) ds = \int_0^1 x^q s s t^p ds = \int_0^1 x^q t^p s^2 ds = \left[\frac{1}{3} x^q t^p s^3\right]_0^1 =$

$\frac{1}{3} x^q t^p$ ∴ Z operators K_p & L_q commute when $p = q = 1$ that is when they are equal. Z adjoint of K_p has kernel $x^p t$ that is $K_p^* = L_p$ so they only commute when $p = 1$, when they are symmetric

6b we know Z norm of Z composition of two operators is smaller or equal to Z product of their norms ;

$\|K_p\|_\infty = \max_{x\in[0,1]} \int_0^1 |x t^p| dt = \int_0^1 t^p dt = \left[\frac{1}{p-1} t^{p+1}\right]_0^1 = \frac{1}{p+1}$

$\|L_q\|_\infty = \max_{x\in[0,1]} \int_0^1 |x^q t| dt = \int_0^1 1^q t dt = \int_0^1 t dt = \left[\frac{1}{2} t^2\right]_0^1 = \frac{1}{2}$ $\forall q \geq 0$

note Z max we need to calc in Z General formula for Z norms always occur at $x = 1$

$\|K_p L_q\|_\infty = \left\{\max_{x\in[0,1]} \int_0^1 |x s^p s^q t| ds\right\} X$

$\left\{\max_{x\in[0,1]} \int_0^1 |x t^p x^q t| dt\right\}$ $\left\{\max_{x\in[0,1]} \int_0^1 |x s^p s^q t| ds = \int_0^1 s^p s^q ds = \frac{1}{p+q+1}\right\} X$

$\|K_p L_q\|_\infty = \frac{1}{2(p+q+1)} \leq \frac{1}{2} \frac{1}{p+1} = \|L_q\|_\infty \|K_p\|_\infty$

$\|L_q K_p\|_\infty = \frac{1}{3(p+1)}$ $\left\{\int_0^1 |x^q s s t^p| ds = \frac{1}{3} x^q t^p\right\} X$

$\|L_q K_p\|_\infty = \frac{1}{3(p+1)} \leq \frac{1}{2} \frac{1}{p+1} = \|L_q\|_\infty \|K_p\|_\infty$

6c to calc Z inverse of $I - K_p$ need to solve Fredholm I.E

$y(x) - \int_0^1 x t^p y(t) dt = f(x)$ ∴ $y(x) - x \int_0^1 t^p y(t) dt = f(x)$ ∴

Let $a = \int_0^1 t^p y(t) dt$ ∴ $y(x) = f(x) + ax$ ∴ $y(t) = f(t) + at$ ∴

$a = \int_0^1 t^p y(t) dt = \int_0^1 t^p (f(t) + at) dt = \frac{a}{p+2} + \int_0^1 t^p f(t) dt$ ∴

$y(x) = f(x) + x \int_0^1 t^p y(t) dt$ ∴ $y(x) = f(x) + ax$ ∴

$a - \frac{a}{p+2} = \int_0^1 t^p f(t) dt = \frac{(p+2)a}{p+2} - \frac{a}{p+2} = \frac{pa + 2a - a}{p+2} = \frac{pa + a}{p+2} = \frac{(p+1)a}{p+2}$ ∴

$a = \frac{p+2}{p+1} \int_0^1 t^p f(t) dt$ ∴ $y(x) = f(x) + \left(\frac{p+2}{p+1}\right) x \int_0^1 t^p f(t) dt = (I - K_p)^{-1} f$

Its norm is estid by $\|(I-K_p)^{-1}\|_\infty \le \frac{1}{1-\|K_p\|_\infty} = \frac{1}{1-\frac{1}{p+1}} = 1+\frac{1}{p}$

its actual value can be calced to

$\|(I-K_p)^{-1}\|_\infty = 1+\frac{p+2}{(p+1)^2} \le 1+\frac{1}{p}$

\7a/ Let $k(x,t)=\sum_{j=1}^{n} k_1^{(j)}(x)\,k_2^{(j)}(t)$ $\therefore$ $k^*(x,t)=\sum_{j=1}^{n} k_2^{(j)}(x)\,k_1^{(j)}(t)$

Z matrix M^* corres to K^* has (i,j) entry

$(M^*)_{ij} = \int_a^b k_1^{(i)}(t)\,k_2^{(j)}(t)\,dt = M_{ji} = (M^T)_{ij}$

\7b/ $\forall y,z\in C[a,b]$: $\int_a^b (Ky)(x)\,z(x)\,dt = \int_a^b\int_a^b k(x,t)\,y(t)\,z(x)\,dt\,dx$

$= \int_a^b y(t)\left[\int_a^b k(x,t)\,z(x)\,dx\right]dt = \int_a^b y(t)\left[\int_a^b k^*(t,x)\,z(x)\,dx\right]dt =$

$\int_a^b y(t)\,(K^*z)(t)\,dt = \int_a^b (K^*z)(x)\,y(x)\,dx$

\Week 8 Sheet/ \1/ $y'(x)-e^{-x}y(x)=2x$, $y(0)=1$

$y\in C'[0,1]$ $\therefore$ ~~y~~ a cont sol of Z IE is diffable $\therefore$

$y(x)=1+x^2+\int_0^x e^{-t}y(t)\,dt$ $\therefore$ diffing Z IE sid :

$y'(x)=2x+e^{-x}y(x)$ $\therefore$ $y'(x)-e^{-x}y(x)=2x$ IC : $y(0)=1$ $\therefore$

Z converse holds by integrate of deriv eqn, taking into account

Z initial val to fix Z integration const :

$\int_0^x y'(t)\,dt - \int_0^x e^{-x}y(t)\,dt = \int_0^x 2t\,dt$ $\therefore$

$y(x)-y(0)-\int_0^x e^{-x}y(t)\,dt = x^2$ & $y(0)=1$ $\therefore$ $y(x)-\int_0^x e^{-t}y(t)\,dt = 1+x^2$

Laplace: $\mathcal{L}\left(\int_0^x e^{-t}y(t)\,dt\right) = \hat{y}(s+1)$ $\therefore$

$\mathcal{L}\left(y(x)-\int_0^x e^{-t}y(t)\,dt\right) = \mathcal{L}(1+x^2) = \frac{1}{s}+\frac{2}{s^3} = \frac{s^2+2}{s^3} = \hat{y}(s)-\hat{y}(s+1)$ $\therefore$

for ODE : integrating factor h : $h'(x) = -e^{-x}h(x)$ $\therefore$ $h(x)=e^{e^{-x}}$ $\therefore$

$(hy)'(x)=2x\,h(x)$ $\therefore$ $\cancel{y(x)=e^{1-e^{-x}}+2\int_0^x te^{e^{-t}}\,dt}$

$y(x)=e^{1-e^{-x}}+2\int_0^x te^{e^{-t}(1-e^{(t-x)})}\,dt$

\2a/ $y(x)+\int_\pi^x \frac{y(t)}{t}\,dt = 2+\cos x$, $x\ge\pi$ $\therefore$ deriv:

$\left\{ y'(x)+\frac{y(x)}{x} = -\sin x \;\therefore\; xy'(x)+y(x) = -x\sin x \right\}$ $\therefore$ $xy'(x)+y(x) = -x\sin x$

\Week 8 Sheet/ IC: $y(\pi)=2-1=1$ $\therefore$

integrating factor: $\therefore$ $y(x)=\cos x-\frac{\sin x}{x}+\frac{2\pi}{x}$

\2 b/ $y(x)-\int_0^{\pi} t\,y(t)\,dt=x$ $x\geq 0$ $\therefore$ deriv:

IV: $y(0)=0$ $\therefore$ IF: $y(x)=e^{x^2/2}\int_0^x e^{-t^2/2}\,dt=\int_0^x e^{\frac{x^2-t^2}{2}}\,dt$

cant integrate $e^{-t^2/2}$

\2b/ deriv: $y'(x)-xy(x)=1$ $\therefore$ $IF=e^{\int -x\,dx}=e^{-\frac{1}{2}x^2}$ $\therefore$

$\frac{d}{dx}\left(e^{-\frac{1}{2}x^2}y(x)\right)=e^{-x^2/2}$ $\therefore$ $x\geq 0$, $y(0)=0$ $\therefore$

$e^{-\frac{1}{2}x^2}y(x)=\int_0^x e^{-t^2/2}\,dt$ $\therefore$ $y(x)=e^{\frac{1}{2}x^2}\int_0^x e^{-t^2/2}\,dt=\int_0^x e^{\frac{x^2-t^2}{2}}\,dt$

\3/ $\int_0^x e^{x-t}y(t)\,dt=f(x)$ $x\geq 0$, $f(0)=0$

probl has a sol $\because$ $f(0)=0$ $\therefore$ deriv:

$f'(x)=y(x)+\int_0^x e^{x-t}y(t)\,dt=y(x)+f(x)$

$\{$ $\int_0^x e^x e^{-t}y(t)\,dt=f(x)=e^x\int_0^x e^{-t}y(t)\,dt$ $\therefore$

$f'(x)=e^x\int_0^x e^{-t}y(t)\,dt+e^x e^{-x}y(x)=f(x)+y(x)\}$ $\therefore$

$y(x)=f'(x)-f(x)$

Laplace: $Y(p)=(p-1)F(p)=pF(p)-f(0)-F(p)$

$\mathcal{L}\left(\int_0^x e^{x-t}y(t)\,dt\right)=\mathcal{L}(f(x))=\mathcal{L}(e^x)\mathcal{L}(y(t))=\frac{1}{p-1}Y(p)=F(p)$ $\therefore$

$Y(p)=(p-1)F(p)=pF(p)-F(p)=pF(p)-0-F(p)=pF(p)-f(0)-F(p)$

$\therefore$ $y(x)$ is always defined & is equal to Z inverse transform of $Y(p)$

$\therefore$ $\mathcal{L}^{-1}(Y(p))=\mathcal{L}^{-1}(pF(p)-f(0)-F(p))=y(x)=\mathcal{L}^{-1}(pF(p)-f(0))-\mathcal{L}^{-1}(F(p))=$

$f'(x)-f(x)=y(x)$

\4 a/ $\lambda y(x)-\int_0^x k(x-t)y(t)\,dt=f(x)$ Let $\hat{f}(s)$, $\hat{k}(s)$ be Z Laplace transforms of f & k respectively $\therefore$ $\{\mathcal{L}(\lambda y(x)-\int_0^x k(x-t)y(t)\,dt)=$

$\mathcal{L}(f(x))=\hat{f}(s)=\lambda\hat{y}(s)-\mathcal{L}(k)\mathcal{L}(y(t))=\lambda\hat{y}(s)-\hat{k}(s)\hat{y}(s)=\hat{y}(s)(\lambda-\hat{k}(s))=\hat{f}(s)\}$

$\therefore$ $\hat{y}(s)=\frac{\hat{f}(s)}{\lambda-\hat{k}(s)}$ $\because$ $\lambda\hat{y}(s)=\hat{f}(s)+\hat{k}(s)\hat{y}(s)$ $\therefore$ $\hat{y}(s)=\frac{\hat{f}(s)}{\lambda-\hat{k}(s)}$

\4b/ $\mathcal{L}(f(x))=\mathcal{L}(e^x)=\frac{1}{s-1}$ $\hat{f}(s)=\frac{1}{s-1}$, $\hat{k}(s)=\frac{1}{s^2+1}$ $\therefore$

$k(x) = \sin x \quad \therefore \quad \mathcal{L}(\sin x) = \frac{1}{s^2+1} \quad \therefore \quad \lambda = \frac{1}{2} \quad \therefore$

$\hat{y}(s) = \frac{\hat{f}(s)}{\lambda - \hat{k}(s)} = \frac{2(s^2+1)}{(s-1)^2(s+1)} \Rightarrow \frac{1}{s+1} + \frac{1}{s-1} + \frac{2}{(s-1)^2}$

$y(x) = e^{-x} + e^{x} + 2xe^{x} = 2\cosh x + 2xe^{x}$

$\{\because \mathcal{L}^{-1}(\hat{y}(s)) = \mathcal{L}^{-1}\left(\frac{1}{s+1} + \frac{1}{s-1} + \frac{2}{(s-1)^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + 2\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right)$

$= e^{-x} + e^{x} + 2e^{x}x = y(x)\}$

\4c/ $f(x) = 1 - e^{x^2/2} \quad \therefore$ Z Sure f grows too fast to have a Laplace transform

\4d/ deriv: $\lambda = 0$, $f(x) = 1 - e^{x^2/2}$ $\quad k = \sin x \quad \therefore -\int_0^x \sin(x-t)y(t)\,dt = 1 - e^{x^2/2}$

$\therefore e^{x^2/2} - 1 = \int_0^x \sin(x-t)y(t)\,dt \quad \therefore$

$\frac{d}{dx}(e^{x^2/2} - 1) = \frac{d}{dx}\left(\int_0^x \sin(x-t)y(t)\,dt\right) = \cancel{2x}$

$xe^{x^2/2} = \frac{d}{dx}\left(\int_0^x (\sin x\cos t - \cos x \sin t)y(t)\,dt\right) =$

$\frac{d}{dx}\left(\sin x \int_0^x \cos t\, y(t)\,dt\right) + \frac{d}{dx}\left(-\cos x \int_0^x \sin t\, y(t)\,dt\right) =$

$\cos x \int_0^x \cos t\, y(t)\,dt + \sin x \cos x\, y(x) + \sin x \int_0^x \sin t\, y(t)\,dt - \cos x \sin x\, y(x) =$

$\cos x \int_0^x \cos t\, y(t)\,dt + \sin x \int_0^x \sin t\, y(t)\,dt =$

$\int_0^x (\cos(x-t))y(t)\,dt = xe^{x^2/2} \quad \therefore$ deriv:

$\frac{d}{dx}(xe^{x^2/2}) = \frac{d}{dx}\left(\cos x \int_0^x \cos t\, y(t)\,dt\right) + \frac{d}{dx}\left(\sin x \int_0^x \sin t\, y(t)\,dt\right) =$

$e^{x^2/2} + x^2 e^{x^2/2} =$

$-\sin x \int_0^x \cos t\, y(t)\,dt + \cos x \cos x\, y(x) + \cos x \int_0^x \sin t\, y(t)\,dt + \sin x \sin x\, y(x) =$

$-\int_0^x (\sin x \cos t - \cos x \sin t)y(t)\,dt + y(x) = -\int_0^x \sin(x-t)\,y(t)\,dt + y(x) =$

$1 - e^{x^2/2} + y(x) = e^{x^2/2} + x^2 e^{x^2/2} = (1-x^2)e^{x^2/2} \quad \therefore$

$y(x) = (1-x^2)e^{x^2/2} - 1 + e^{x^2/2} = \cancel{(2)} (2-x^2)e^{x^2/2} - 1$ ✓

\4d/ $\lambda y(x) - \int_0^x k(x-t)\,y(t)\,dt = f(x) \quad \lambda = 0$, $f(x) = 1 - e^{x^2/2}$ $\quad k(x) = \sin x$

$\therefore -\int_0^x k(x-t)y(t)\,dt = 1 - e^{x^2/2} = -\int_0^x \sin(x-t)y(t)\,dt \quad \therefore$

$e^{x^2/2} - 1 = \int_0^x \sin(x-t)\,y(t)\,dt \quad \therefore \quad xe^{x^2/2} = \int_0^x \cos(x-t)\,y(t)\,dt \quad \therefore$

$-\int_0^x \sin(x-t)\,y(t)\,dt + y(x) = (1-x^2)e^{x^2/2} = 1 - e^{x^2/2} + y(x) = (1-x^2)e^{x^2/2} \quad \therefore y(x) = (2-x^2)e^{x^2/2} - 1$

Week 8 Sheet / 5a / $y(x) - \lambda\int_0^x \sin(x-t)\,y(t)\,dt = f(x)$

$\therefore$ deriv: $y'(x) - \lambda \frac{d}{dx}\left(\sin x\int_0^x \cos t\, y(t)\,dt\right) - \lambda\frac{d}{dx}\left(-\cos x\int_0^x \sin t\, y(t)\,dt\right) = f'(x)$

$= y'(x) - \lambda\cos x\int_0^x \cos t\, y(t)\,dt - \lambda\sin x\cos x\, y(x) - \lambda\sin x\int_0^x \sin t\, y(t)\,dt + \lambda\cos x\sin x\, y(x) = f'(x)$

$= y'(x) - \lambda\cos x\int_0^x \cos t\, y(t)\,dt - \lambda\sin x\int_0^x \sin t\, y(t)\,dt = f'(x)$ $\therefore$ deriv:

$y''(x) + \lambda\sin x\int_0^x \cos t\, y(t)\,dt - \lambda\cos x\cos x\, y(x) - \lambda\cos x\int_0^x \sin t\, y(t)\,dt - \lambda\sin x\sin x\, y(x) = f''(x)$

$\therefore$ $y''(x) + \lambda\sin x\int_0^x \cos t\, y(t)\,dt - \lambda y(x) - \lambda\cos x\int_0^x \sin t\, y(t)\,dt = f''(x)$ $\therefore$

$y''(x) + \lambda\int_0^x (\sin x\cos t - \cos x\sin t)\,y(t)\,dt =$

$y''(x) + \lambda\int_0^x \sin(x-t)\,y(t)\,dt = f''(x)$ $\therefore$ $y(x) - f(x) = \lambda\int_0^x \sin(x-t)\,y(t)\,dt$ $\therefore$

$y''(x) + y(x) - f(x) = f''(x)$

5a redo / $\mathcal{L}\left(y(x) - \lambda\int_0^x \sin(x-t)\,y(t)\,dt\right) = \mathcal{L}(f(x)) =$

$\hat{y}(s) - \lambda\mathcal{L}\left[\int_0^x \sin(x-t)\,y(t)\,dt\right] = \hat{f}(s) = \hat{y}(s) - \lambda\,\mathcal{L}(\sin x)\,\mathcal{L}(y(x)) =$

$\hat{y}(s) - \lambda\frac{1}{s^2+1}\hat{y}(s) = \hat{f}(s) = \hat{y}(s)\left(1 - \frac{\lambda}{s^2+1}\right) = \hat{y}(s)\left(\frac{s^2+1-\lambda}{s^2+1}\right) = \hat{f}(s)$ $\therefore$

$\hat{y}(s) = \frac{s^2+1}{s^2+(1-\lambda)}\hat{f}(s)$ $\therefore$

for $\lambda = 1$: $\hat{y}(s) = \frac{s^2+1}{s^2}\hat{f}(s)$

5a sol / Laplace: $\hat{y}(s) = \frac{1+s^2}{1+s^2-\lambda}\hat{f}(s) = \left[1 + \frac{\lambda}{1+s^2-\lambda}\right]\hat{f}(s)$ $\therefore$

$\Rightarrow$ $y(x) = f(x) + \int_0^x g_\lambda(x-t)\,f(t)\,dt$

$$g_\lambda(x) = \begin{cases} \frac{\lambda}{\sqrt{1-\lambda}}\sinh(\sqrt{1-\lambda}\,x) & , \lambda < 1 \\ x & \\ \frac{\lambda}{\sqrt{\lambda-1}}\sinh(\sqrt{\lambda-1}\,x) & \lambda > 1 \end{cases}$$

5a sol / $y(x) - \lambda\int_0^x \sin(x-t)\,y(t)\,dt = f(x)$ $x \geq 0$ $\lambda \geq 0$ $\therefore$

Laplace: $\hat{y}(s) - \lambda\hat{y}(s)\,\mathcal{L}(\sin(x)) = \hat{f}(s) = \hat{y}(s) - \lambda\hat{y}(s)\frac{1}{s^2+1}$ $\therefore$

$\hat{y}(s) = \left(\frac{s^2+1}{s^2+1-\lambda}\right)\hat{f}(s) = \left(\frac{s^2+1-\lambda+\lambda}{s^2+1-\lambda}\right)\hat{f}(s) = \left(1 + \frac{\lambda}{s^2+1-\lambda}\right)\hat{f}(s) =$

$\hat{f}(s) + \frac{\lambda}{s^2+1-\lambda}\hat{f}(s) = \hat{y}(s)$ $\therefore$ $\hat{g}(s) = \frac{\lambda}{s^2+1-\lambda}$ $\therefore$

$y(x) = f(x) + \mathcal{L}^{-1}\left(\frac{\lambda}{s^2+1-\lambda}\hat{f}(s)\right)$ $\therefore$

$y(x) = f(x) + \int_0^x f(t)\,g(x-t)\,dt$ by L+22 $\therefore$

$\mathcal{L}^{-1}(\hat{g}(s)) = \mathcal{L}^{-1}\left(\frac{\lambda}{s^2+1-\lambda}\right)$ which depends on λ ∴

for $\lambda = 1$: $\hat{g}(s) = \frac{1}{s^2}$ ∴ $g(x) = x$

for $\lambda < 1$: $\hat{g}(s) = \frac{\lambda}{s^2+(1-\lambda)} = \lambda \frac{1}{s^2+(1-\lambda)} = \frac{\lambda}{\sqrt{1-\lambda}} \frac{(\sqrt{1-\lambda})}{s^2+(1-\lambda)}$ ∴

~~g(x)~~ $1-\lambda$ is positive ∴ $g(x) = \frac{\lambda}{\sqrt{1-\lambda}} \sin(\sqrt{1-\lambda}\, x)$

for $\lambda > 1$: $\hat{g}(s) = \frac{\lambda}{\sqrt{1-\lambda}} \frac{(\sqrt{1-\lambda})}{s^2+(1-\lambda)}$ $1-\lambda$ is negative ∴

~~$g(x) = \frac{\lambda}{\sqrt{1-\lambda}} \sinh(\sqrt{\lambda-1}$~~ $g(x) = \frac{\lambda}{\sqrt{\lambda-1}} \sinh(\sqrt{\lambda-1}\, x)$ ∴

$y(x) = f(x) + \int_0^x f(t) g_\lambda(x-t)\, dt$ ∴ $g_\lambda(x-t) = \begin{cases} x-t & \lambda = 1 \\ \frac{\lambda}{\sqrt{1-\lambda}} \sin(\sqrt{1-\lambda}(x-t)) & \lambda < 1 \\ \frac{\lambda}{\sqrt{1-\lambda}} \sinh(\sqrt{1-\lambda}(x-t)) & \lambda > 1 \end{cases}$

∴ $g_\lambda(x) = \begin{cases} \frac{\lambda}{\sqrt{1-\lambda}} \sin(\sqrt{1-\lambda}\, x), & \lambda < 1 \\ x, & \lambda = 1 \\ \frac{\lambda}{\sqrt{\lambda-1}} \sinh(\sqrt{\lambda-1}\, x), & \lambda > 1 \end{cases}$

\5b sol/ $f(x) = x$, $\lambda = 1$ ∴ $y(x) = x + \int_0^x t(x-t)\,dt = x + \int_0^x tx - t^2\, dt$ ∴

$y(x) = x + \frac{1}{6}x^3$

$\lambda = 5$: $y(x) = x + \int_0^x t \frac{\lambda}{\sqrt{\lambda-1}} \sinh(\sqrt{\lambda-1}(x-t))\,dt = x + \int_0^t t \cdot \frac{5}{2} \sinh(2(x-t))\, dt$

~~I.B.P~~ $u = t$, $u' = 1$ $v' = \sinh(2(x-t))$ ∴ $v = -\frac{1}{2}\cosh(2(x-t))$ ∴

$y(x) = x + \frac{5}{2}\left[-\frac{1}{2} t \cosh(2(x-t))\right]_0^x + \frac{5}{4}\int_0^x \cosh(2(x-t))\,dt$ ∴

$y(x) = x - \frac{5}{4}x + \frac{-5}{8}\left[\sinh(2(x-t))\right]_0^x = x - \frac{5}{4}x - \frac{5}{8}\sinh(2x)$ ∴

$y(x) = -\frac{1}{4}x + \frac{5}{8}\sinh(2x)$

\5b sol/ $f(x) = x$, $\hat{f}(s) = s^{-2}$ ∴ $\lambda = 1$ $y(x) = x + \int_0^x (x-t)t\,dt = \frac{1}{6}x^3 + x$.

$\lambda = 5$: $y(x) = x + \int_0^x \frac{5}{2}\sinh(2x-2t)\,t\,dt = -\frac{1}{4}x + \frac{5}{8}\sinh(2x)$

deriv: $y'(x) = f'(x) + \lambda\int_0^x \cos(x-t)y(t)\,dt$ ∴

$y''(x) = f''(x) + \lambda y(x) - \lambda\int_0^x \sinh(x-t)\,y(t)\,dt = \cancel{f''(x) + \lambda y(x)} -$

$f''(x) + \lambda y(x) - (y(x) - f(x))$ ∴ $y'' + (1-\lambda)y = f + f''$ IC $y(0) = f(0)$,

~~&~~ $y'(0) = f'(0)$

\6/ Laplace: $\hat{g}(s) = \frac{\hat{f}(s)}{\lambda - \hat{k}(s)}$ ∴ if $\lambda \neq 0$:

$(\deg(\lambda - \hat{k}(s)) = \deg(\lambda) = 0$ ∴ $\deg(\hat{g}) = \deg(\hat{f})$

when $\lambda = 0$: $\deg(\hat{g}) = \deg(\hat{f}) - \deg(\hat{k}(s))$

Week 9 sheet 1a $y(x) - \int_0^x y(t)\,dt = g(x)$ ∴

$y'(x) = g'(x) + y(x)$ $\{$deriv: $y'(x) - y(x) = g'(x)$ ∴ $y'(x) = g'(x) + y(x)$ $\}$

IV: $y(0) = g(0)$ $\{$at $x = 0$: $y(0) - \int_0^0 y(t)\,dt = g(0) = y(0) - 0 = y(0) = g(0)\}$

∴ $\{y'(x) - y(x) = g'(x)$ ∴ $IF = e^{+\int -1\,dx} = e^{-x}$ ∴

$\frac{d}{dx}(e^{-x}y(x)) = e^{-x}g'(x)$ ∴ $e^{-x}y(x) = \int_0^x e^{-t}g'(t)\,dt = [e^{-t}g(t)]_0^x - \int_0^x -e^{-t}g(t)\,dt\}$

$y'(x) = g'(x) + y(x)$ $y(0) = g(0)$ ∴ $-e^x(e^{-x}y(x)) = y(x) - y'(x) = -g'(x)$

$\{$∴ $y'(x) - y(x) = g'(x)$ ∴ $-g'(x) = y(x) - y'(x)$ ∴

$\frac{d}{dx}(e^{-x}y(x)) = -e^{-x}y(x) + e^{-x}y'(x)$ ∴ $-e^x\frac{d}{dx}(e^{-x}y(x)) = -e^x[-e^{-x}y(x) + e^{-x}y'(x)]$

$= -e^x(-e^{-x})y(x) - e^xe^{-x}y'(x) = y(x) - y'(x)$ ∴

$-e^x\frac{d}{dx}(e^{-x}y(x)) = y(x) - y'(x)\}$ ∴

$\{y(x) - y'(x) = -e^x\frac{d}{dx}(e^{-x}y(x)) = -g'(x)\}$ ∴

$\frac{d}{dx}(e^{-x}y(x)) = -e^{-x}(-1)g'(x) = e^{-x}g'(x)$ ∴

$\{e^{-x}y(x) = \int e^{-x}g'(x)\,dx$ ∴ $[e^{-x}y(x)]_0^x = \int_0^x e^{-t}g'(t)\,dt =$

$e^{-x}y(x) - e^{-0}y(0) = e^{-x}y(x) - y(0) = e^{-x}y(x) - g(0) = \int_0^x e^{-t}g'(t)\,dt$ ∴

$e^{-x}y(x) = g(0) + \int_0^x e^{-t}g'(t)\,dt$ ∴ $y(x) = g(0)e^x + e^x\int_0^x e^{-t}g'(t)\,dt =$

$g(0)e^x + \int_0^x e^xe^{-t}g'(t)\,dt = y(x) = g(0)e^x + \int_0^x e^{x-t}g'(t)\,dt$

1b when $g(x) = x$: $(Kg)(x) = \frac{x^2}{2}$ ∴ $(K^ng)(x) = \frac{x^n}{n!}$ ∴

$\sum_{n=1}^{\infty}\frac{x^n}{n!} = e^x - 1$ which is the true sol found from Z exact formula

$y(x) = \int_0^x e^{x-t}\,dt = e^x - 1$

$\{g(x) = x$ ∴ $y_0(x) = x$ $\}$

1a $y(x) - \int_0^x y(t)\,dt = g(x)$ ∴ $y'(x) - y(x) = g'(x)$ $y(0) = g(0)$ ∴

IF: $e^{\int -1\,dx} = e^{-x}$ ∴ $\frac{d}{dx}(y(x)e^{-x}) = e^{-x}g'(x)$ ∴

$y(x)e^{-x} = \int_0^x e^{-x}g'(x)\,dx$ ∴ IBP: $u = e^{-x}$, $u' = -e^{-x}$, $v' = g'$, $v = g$

∴ $y(x)e^{-x} = [e^{-x}g(x)] - \int -e^{-x}g(x)\,dx$ ∴ $y(x) = g(x) + e^x\int e^{-x}y(x)\,dx + C$

IC: $y(0) = g(0)$

\1b/ $g(x) = x \therefore y(x) = x + e^{x}\int e^{-x} x\,dx$ IBP:

$y(x) = -1 + e^{x} \therefore$

using neumann iterations: $g(x) = x \therefore y_0 = x \therefore \lambda = 1 \therefore$

using $\frac{1}{\lambda}\sum_{n=0}^{\infty}\frac{k^n g}{\lambda^n} \to g = x$:

$(Kg)(x) = \int_0^x t\,dt = \frac{1}{2}x^2 \therefore K(Kg(x)) = \int_0^x \frac{1}{2}t^2\,dt = \frac{1}{6}x^3 \therefore$

$K(K(Kg(x))) = \int_0^x \frac{1}{6}t^3\,dt = \left[\frac{1}{24}x^4\right]_0^x = \frac{1}{24}x^4 \therefore$

$K^n g = \frac{x^n}{n!} \therefore y_n = \sum_{n=1}^{\infty}\frac{x^n}{n!} \therefore e^{x} = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + \sum_{n=1}^{\infty}\frac{x^n}{n!} \therefore$

$y_n = \sum_{n=1}^{\infty}\frac{x^n}{n!} = e^{x} - 1$

\1b/ $y = Ky + g \therefore y(x) = g(x) + \int_0^x y(t)\,dt \therefore$

~~y_0~~ $g(x) = x \therefore y(x) = x + \int_0^x y(t)\,dt \therefore y_0 = x \therefore$

$y_1 = x + \int_0^x t\,dt = x + \left[\frac{1}{2}t^2\right]_0^x = x + \frac{1}{2}x^2 \therefore$

$y_2 = x + \int_0^x y_1\,dt = x + \int_0^x t + \frac{1}{2}t^2\,dt = x + \left[\frac{1}{2}t^2 + \frac{1}{6}t^3\right]_0^x = x + \frac{1}{2}x^2 + \frac{1}{6}x^3$

$\therefore$ ~~$y_n = x + \frac{1}{2}$~~ $y_n = \sum_{n=1}^{\infty}\frac{x^n}{n!}$ $e^{x} = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + \sum_{n=1}^{\infty}\frac{x^n}{n!} \therefore$

$y_n + 1 = 1 + \sum_{n=1}^{\infty}\frac{x^n}{n!} = e^{x} \therefore y_n = e^{x} - 1$

\2/ {at $x = 0$: $\int_0^0 \cos(0-t)\,y(t)\,dt = \cosh(0) - C = 0 = 1 - C \therefore C = 1$}

know $\cosh(x) - C$ must be 0 at $x = 0 \therefore C = 1 \therefore$

Laplace: $\frac{s}{s^2-1} = \frac{1}{s} + \frac{s}{s^2+1}\hat{y}(s)$ {$\mathcal{L}\left(\int_0^x \cos(x-t)\,y(t)\,dt\right) = \mathcal{L}(\cosh x - 1) =$

$\mathcal{L}(\cos(x))\,\mathcal{L}(y(x)) = \mathcal{L}(\cosh x) - \mathcal{L}(1) = \frac{s}{s^2-1} - \frac{1}{s} = \frac{s}{s^2+1}\hat{y}(s) \therefore$

$\hat{y}(s)\frac{s}{s^2+1} - \frac{1}{s} = \frac{s}{s^2-1} \therefore \hat{y}(s) = \frac{s^2+1}{s^2(s^2-1)} = -\frac{1}{s^2} + \frac{1}{s-1} - \frac{1}{s+1} \therefore$

$y(x) = -x + 2\sinh(x)$ {$\mathcal{L}(\hat{y}(s)) = \mathcal{L}^{-1}\left(-\frac{1}{s^2} + \frac{1}{s-1} - \frac{1}{s+1}\right) = y(x) = -x + e^{x} - e^{-x} =$

$-x + 2 \times \frac{1}{2}(e^{x} - e^{-x}) = -x + 2\sinh x$

deriv: $y'(x) = 2\cosh x - 1$ $y(0) = 0 \therefore y(x) = -x + 2\sinh x$

Week 9 Sheet $\left\{ \frac{d}{dx}\left(\int_0^x \cos(x-t)\,y(t)\,dt\right) = \frac{d}{dx}(\cosh x - 1) \therefore \right.$

$\frac{d}{dx}\left(\cos x \int_0^x \cos t\, y(t)\,dt\right) + \frac{d}{dx}\left(+\sin x \int_0^x \sin t\, y(t)\,dt\right) = \sinh x =$

$-\sin x \int_0^x \cos t\, y(t)\,dt + \cos x \cos x\, y(x) + \cos x \int_0^x \sin t\, y(t)\,dt + \sin x \sin x\, y(x)$

$= -\sin x \int_0^x \cos t\, y(t)\,dt + \cos x \int_0^x \sin t\, y(t)\,dt + y(x) = \sinh x \therefore$

deriv:

$-\cos x \int_0^x \cos t\, y(t)\,dt - \sin x \cos x\, y(x) - \sin x \int_0^x \sin t\, y(t) + \cos x \sin x\, y(x) + y'(x)$

$= -\int_0^x (\cos x \cos t + \sin x \sin t)\, y(t)\,dt + y'(x) = \sinh x = -\int_0^x \cos(x-t)\,y(t)\,dt + y'(x) =$

$= \cancel{\cosh x} - \cosh x + 1 + y'(x) = \sinh x \therefore$

$y'(x) = \sinh x + \cosh x - 1 \therefore$

$y(x) = \cosh x + \sinh(x) - x + C_2 \qquad y(0) = 0 \therefore$

$0 = 1 + \cancel{1}0 - 0 + C_2$

3 $\left\{ f(x) = 1 \therefore \; y_0(x) = 1 \therefore \; y_0(t) = 1 \therefore \; y(x) = 1 + \int_0^x e^{-t} y(t)\,dt \therefore \right.$

$y_1(x) = 1 + \int_0^x e^{-t} \cdot 1\,dt = \left[-e^{-t}\right]_0^x + 1 = -e^{-x} + e^0 = -e^{-x} + 1 + 1 \therefore y_1(t) = -e^{-t} + 2$

$\therefore y_2(x) = 1 + \int_0^x e^{-t} y_1(t)\,dt = 1 + \int_0^x e^{-t}(-e^{-t} + 2)\,dt = 1 + \int_0^x -e^{-2t} + 2e^{-t}\,dt$

$= 1 + \left[\frac{1}{2}e^{-2t} - 2e^{-t}\right]_0^x = 1 + \frac{1}{2}e^{-2x} - \frac{1}{2} - 2e^{-x} - 2 = -\frac{3}{2} + \frac{1}{2}e^{-2x} - 2e^{-x} \Big\}$

$y_0(x) = 1 \therefore \quad y_n(x): \quad y_{n+1}(x) = 1 + \int_0^x e^{-t} y_n(t)\,dt \therefore$

$y_1(x) = 1 + \int_0^x e^{-t}\,dt = 1 + (1 - e^{-x})$

$y_2(x) = 1 + \int_0^x e^{-t}\left[1 + (1 - e^{-t})\right]dt = 1 + (1 - e^{-x}) + \frac{1}{2}(1 - e^{-x})^2 \therefore$

assuming $y_n(x) = 1 + \sum_{j=1}^{n} \frac{1}{j!}(1 - e^{-t})^j \therefore$

$y_{n+1}(x) = 1 + \int_0^x e^{-t}\left[1 + \sum_{j=1}^{n} \frac{1}{j!}(1 - e^{-t})^j\right]dt =$

$1 + \int_0^x e^{-t}\,dt + \sum_{j=1}^{n} \frac{1}{j!} \int_0^x e^{-t}(1 - e^{-t})^j\,dt =$

$1 + (1 - e^{-x}) + \sum_{j=1}^{n} \frac{1}{(j+1)!}\left[(1 - e^{-t})^{j+1}\right]_0^x = 1 + \sum_{j=1}^{n+1} \frac{1}{j!}(1 - e^{-x})^j \therefore$

deriv: $y'(x) = e^{-x} y(x)$ IV: $y(0) = 1 \therefore y(x) = e^{1 - e^{-x}} = \exp(1 - e^{-x}) \therefore$

they're equal of order n

$\{$ deriv: $y'(x)-e^{-x}y(x)=0 \therefore y'(x)=e^{-x}y(x)$

~~for~~ $x=0$: $y(0)-\int_0^0 e^{-t}y(t)\,dt=1=y(0)-0=y(0)=1 \therefore$

$\frac{y'(x)}{y(x)}=e^{-x} \therefore \int\frac{y'(x)}{y(x)}dx=\int e^{-x}dx=\ln|y(x)|=e^{x}+c_2 \therefore$

$\ln|y(0)|=\ln|1|=0=e^0+c_2=1+c_2 \therefore c_2=-1 \therefore$

$\therefore$ ~~ln~~ $e^{\ln|y(x)|}=|y(x)|=e^{x+c_2}\cancel{=}e^x+e^{c_2}=c_3e^x \therefore$

$y(x)=c_4e^x \therefore y(0)=1=c_4e^0=c_4=1 \therefore y(x)=e^x$ X $\}$

$\{ y'(x)=e^{-x}y(x) \quad y(0)=1 \therefore \int\frac{y'(x)}{y(x)}dx=\int e^{-x}dx \therefore$

~~++~~ $\ln|y(x)|=-e^{-x}+c_2 \therefore |y(x)|=e^{-e^{-x}+c_2} \therefore$

$y(x)=Ae^{-e^{-x}} \therefore y(0)=1=Ae^{-e^{-0}}=Ae^{-1}=1 \therefore A=e \therefore$

$y(x)=\cancel{e^{-e}}\,ee^{-e^{-x}}=e^{(1-e^{-x})}$

Week 10 Sheet

1a/ $y(x)-\int_0^1 a(x)y(t)+a(x)y(t)^4\,dt=0$

$\therefore y(x)=\int_0^1 a(x)y(t)+a(x)y(t)^4\,dt=a(x)\int_0^1 y(t)\,dt+a(x)\int_0^1 y(t)^4\,dt=y(x) \therefore$

$P_1=\int_0^1 y(t)\,dt$, $P_2=\int_0^1 y(t)^4\,dt \therefore y(x)=a(x)P_1+a(x)P_2 \therefore$

$y(t)=a(t)P_1+a(t)P_2 \therefore$

$P_1=\int_0^1 a(t)P_1+a(t)P_2\,dt=(P_1+P_2)\int_0^1 a(t)\,dt$

$P_2=\int_0^1 (a(t)P_1+a(t)P_2)^4\,dt=\int_0^1 a(t)^4(P_1+P_2)^4\,dt$

1a Sol/ let $P=\int_0^1 (y(x)+y(x)^4)\,dx \therefore y(x)=Pa(x) \therefore y(t)=Pa(t) \therefore$

~~$P=P\int$~~ $P=\int_0^1 (Pa(x)+(Pa(x))^4)\,dx=P\int_0^1 a(x)\,dx+P^4\int_0^1 a(x)^4\,dx=P \therefore$

if $\int_0^1 a(x)\,dx=1$: $\{ P=P+P^4\int_0^1 a(x)^4\,dx \therefore 0=P^4\int_0^1 a(x)^4\,dx \therefore P=0$

$\because \int_0^1 a(x)\,dx=1 \therefore \int_0^1 a(x)^4\,dx\neq 0 \}$

$P=P\int_0^1 a(x)\,dx+P^4\int_0^1 a(x)^4\,dx$ & $\therefore$, if $\int_0^1 a(x)\,dx=1$, $P=0$ $(\because \int_0^1 a(x)^4\,dx\neq 0)$.

if $\int_0^1 a(x)\,dx\neq 1 \therefore P-P\int_0^1 a(x)\,dx=P^4\int_0^1 a(x)^4\,dx=P\left(1-\int_0^1 a(x)\,dx\right) \therefore$

$P^3=\frac{1-\int_0^1 a(x)\,dx}{\int_0^1 a(x)^4\,dx} \therefore P=\left(\frac{1-\int_0^1 a(x)\,dx}{\int_0^1 a(x)^4\,dx}\right)^{1/3} \therefore$

Week 10 Sheet / if $\int_0^1 a(x)\,dx = 1$: $p = \left(\frac{1-1}{\int_0^1 a(x)^4\,dx}\right)^{1/3} = \left(\frac{0}{\int_0^1 a(x)^4\,dx}\right)^{1/3} =$

$0^{1/3} = 0 \therefore y(x) = 0\,a(x) = 0 \therefore$

$y \equiv 0$ is always a Sol.

when $\int_0^1 a(x)\,dx \neq 1$: $y(x) = \left(\frac{1-\int_0^1 a(x)\,dx}{\int_0^1 a(x)^4\,dx}\right)^{1/3} a(x)$ we have a nontrivial Sol

1b Sol / $\left\{ y(x) = \int_0^1 x\,y(t) + x^2 t\,y(t)^2\,dt = x\int_0^1 y(t) + x^2\int_0^1 t\,y(t)^2\,dt \right\} \therefore$

Let $P_1 = \int_0^1 y(x)\,dx$, $P_2 = \int_0^1 x\,y(x)^2\,dx$ & $\therefore y(x) = P_1 x + P_2 x^2$ & $\therefore$

$y(t) = P_1 t + P_2 t^2 \therefore$

$P_1 = \int_0^1 y(x)\,dx = \int_0^1 P_1 x + P_2 x^2\,dx = P_1\int_0^1 x\,dx + P_2\int_0^1 x^2\,dx = P_1\left[\frac{1}{2}x^2\right]_0^1 + P_2\left[\frac{1}{3}x^3\right]_0^1 =$

$\frac{1}{2}(1^2 - 0^2) + P_2\frac{1}{3}(1^3 - 0^3) = \frac{1}{2}P_1 + \frac{1}{3}P_2 = P_1$,

$P_2 = \int_0^1 x\,y(x)^2\,dx = \int_0^1 x(P_1 x + P_2 x^2)^2\,dx = \int_0^1 x(P_1^2 x^2 + P_2^2 x^4 + 2P_1P_2 x^3)\,dx =$

$\int_0^1 P_1^2 x^3 + P_2^2 x^5 + 2P_1P_2 x^4\,dx$ $\left[\frac{1}{4}P_1^2 x^4 + \frac{2}{5}P_1P_2 x^5 + \frac{1}{6}P_2^2\right]_0^1 = \frac{1}{4}P_1^2 + \frac{2}{5}P_1P_2 + \frac{1}{6}P_2^2 = P_2$

$\therefore P_1 = \frac{2}{3}P_2 \quad \therefore P_2 = \frac{49}{90}P_2^2 \quad \left\{ \therefore 0 = \frac{49}{90}P_2^2 - P_2 = P_2\left(\frac{49}{90}P_2 - 1\right) = 0 \right\}$

$\therefore P_2 = 0,\ P_2 = \frac{90}{49} \therefore$

if $P_2 = 0$: $P_1 = 0$, if $P_2 = \frac{90}{49}$: $P_1 = \frac{60}{49}$ $\therefore y(x) = P_1 x + P_2 x^2 \therefore$

Z Sols are: $y \equiv 0$ or $y(x) = \frac{30}{49}x(2+3x)$

1c Sol / $\lambda y(x) = \int_{-\pi}^{\pi} \cos(t)y(t) + \sin(x)y(t)^3\,dt \therefore y(x) = \frac{1}{\lambda}\int_{-\pi}^{\pi}\cos(t)y(t) + \sin(x)y(t)^3\,dt \therefore$

$\therefore \lambda y(x) = \int_{-\pi}^{\pi}\cos(t)y(t)\,dt + \sin(x)\int_{-\pi}^{\pi} y(t)^3\,dt \therefore$

$a = \int_{-\pi}^{\pi}\cos(t)y(t)\,dt$, $b = \int_{-\pi}^{\pi} y(t)^3\,dt \therefore \lambda y(x) = a + b\sin x \therefore y(t) = \frac{1}{\lambda}(a + b\sin t)$

$\therefore a = \int_{-\pi}^{\pi}\cos(t)\frac{1}{\lambda}(a + b\sin t)\,dt \therefore \lambda a = \int_{-\pi}^{\pi}\cos(t)(a + b\sin t)\,dt =$

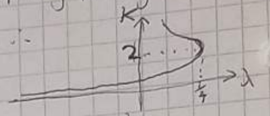
$a\int_{-\pi}^{\pi}\cos t\,dt + b\int_{-\pi}^{\pi}\cos(t)\sin(t)\,dt = 0 = \lambda a$, &

$b = \int_{-\pi}^{\pi}\left(\frac{1}{\lambda}(a + b\sin t)\right)^3 dt \therefore \lambda^3 b = \int_{-\pi}^{\pi}(a + b\sin t)^3\,dt =$

$\int_{-\pi}^{\pi} a^3 + 3a^2 b\sin t + 3ab^2\sin^2(t) + b^3\sin^3(t)\,dt = 2\pi a^3 + 3\pi ab^2 = b\lambda^3$ & $\therefore$

if $\lambda = 0$: $a = b = 0$, & if $\lambda \neq 0$: $a = b = 0$ $\therefore y \equiv 0$ is Z only Sol $\forall \lambda$

\2 Sol/ From 2 eqn, its clear y must be a const: $y = p$ say & ∴



2 eqn becomes $p = 1 + \lambda p^2$ $\left\{ y(x) = 1 + \lambda \int_0^1 y(t)^2 dt \right.$ ∴

for $1 + \lambda \int_0^1 y(t)^2 dt = p$ ∴ $y(x) = p$ ∴ $y(t) = p$ ∴

$p = 1 + \lambda \int_0^1 p^2 dt = 1 + \lambda p^2 [t]_0^1 = 1 + \lambda p^2 = p \}$ ∴ $\{ \lambda p^2 - p + 1 = 0$ ∴

$p = \frac{1 \pm \sqrt{1 - 4\lambda}}{2\lambda}$ } that is, have two branches paramed by λ: $P_{\pm}(\lambda) = \frac{1 \pm \sqrt{1-4\lambda}}{2\lambda}$

\3 Sol/ $y(x) = \int_0^1 \frac{1}{2} y(t) + \lambda x y(t)^2 dt = \frac{1}{2} \int_0^1 y(t) + \lambda x \int_0^1 y(t)^2 dt$ ∴

$a = \int_0^1 y(t) dt$, $b = \int_0^1 y(t)^2 dt$ ∴ $y(x) = \frac{1}{2} a + \lambda b x$ ∴ $y(t) = \frac{1}{2} a + \lambda b t$ ∴

get 2 syst: $a = \frac{1}{2} a + \frac{1}{2} \lambda b$, $b = \frac{1}{4} a^2 + \frac{1}{2} \lambda a b + \frac{1}{3} \lambda^2 b^2$

$\{ a = \int_0^1 y(t) dt = \int_0^1 \frac{1}{2} a + \lambda b t \, dt = \frac{1}{2} a + \frac{1}{2} \lambda b = a$

$b = \int_0^1 y(t)^2 dt = \int_0^1 (\frac{1}{2} a + \lambda b t)^2 dt = \int_0^1 \frac{1}{4} a^2 + \lambda^2 b^2 t^2 + \lambda a b t \, dt = \frac{1}{4} a^2 + \frac{1}{2} \lambda a b + \frac{1}{3} \lambda^2 b^2 = b \}$ ∴

then its sols are either $a = b = 0$ or paramed by $\lambda \neq 0$ via

$a(\lambda) = \frac{12}{13\lambda}$ & $b(\lambda) = \frac{12}{13\lambda^2}$ ∴ ~~2 two sols~~ there are two sols:

$y(x) = 0 \ \forall \lambda$ & $y(x) = \frac{6}{13\lambda}(1 + 2x)$, $\lambda \neq 0$

\4 Sol/ $\phi(x) = \int_0^x \frac{1+t^2}{1+\phi(t)^2} dt$ ∴ $\phi(x) - \int_0^x \frac{1+t^2}{1+\phi(t)^2} dt = 0$ ∴

deriv: $\phi'(x) - \frac{1+x^2}{1+\phi(x)^2} = 0$ & $\phi(0) - \int_0^0 \frac{1+t^2}{1+\phi(t)^2} dt = \phi(0) = 0$

is a seperable eqn with sol: $\phi + \frac{1}{3}\phi^3 = x + \frac{1}{3} x^3$ $\left\{ \phi'(x) = \frac{1+x^2}{1+\phi(x)^2} \right.$ ∴

$\phi'(x)(1 + \phi(x)^2) = 1 + x^2$ ∴ $\int \phi'(x)(1 + (\phi(x))^2) dx = \int 1 + x^2 dx =$

$x + \frac{1}{3}x^3 + C = \int \phi'(x) + \phi'(x)(\phi(x))^2 dx = \phi(x) + \frac{1}{3}\phi(x)^3 = x + \frac{1}{3}x^3 + C$ ∴

$\phi(0) + \frac{1}{3}\phi(0)^3 = 0 + \frac{1}{3}(0)^3 + C = 0 + \frac{1}{3}(0)^3 = 0 = 0 + C = C = 0$ ∴

$\phi + \frac{1}{3}\phi^3 = x + \frac{1}{3}x^3$ } using I C find

$(\phi - x) = \frac{1}{3}(x^3 - \phi) = \frac{1}{3}(x - \phi)(\phi^2 + x\phi + x^2)$ $\left\{ \phi + \frac{1}{3}\phi^3 = x + \frac{1}{3}x^3 \right.$ ∴

$\phi - x = \frac{1}{3}x^3 - \frac{1}{3}\phi^3 = \frac{1}{3}(x^3 - \phi^3) = \frac{1}{3}(x - \phi)(x^2 + \phi^2 + x\phi) = \frac{1}{3}(x - \phi)(\phi^2 + x\phi + x^2) \}$

∴ there is a sol $\phi(x) = x$ &, dividing by $(\phi - x)$ otherwise no other sols

\5 Sol/ $\phi(x) - \int_{-1}^1 (x+t)^2 \phi(t)^2 dt = 0$ ∴ $\phi(x) = \int_{-1}^1 (x^2 + 2xt + t^2) \phi(t)^2 dt =$

for $x^2 \int_{-1}^1 \phi(t)^2 dt + 2x \int_{-1}^1 t \phi(t)^2 dt + \int_{-1}^1 t^2 \phi(t)^2 dt = \phi(x)$

Week 10 Sheet / $\phi(x) - \int_{-1}^{1} (x+t)^2 \phi(t)^2 \, dt = 0$ is a nonlinear hammerstein eqn, $\phi(x) - x^2 \int_{-1}^{1} \phi(t)^2 dt - 2x \int_{-1}^{1} t\phi(t)^2 dt - \int_{-1}^{1} t^2 \phi(t)^2 dt = 0$

$\therefore$ if $\phi(x) = ax^2 + 2bx + c \quad \therefore \phi(t) = at^2 + bt + c \quad \therefore$

$$a = \int_{-1}^{1} \phi(t)^2 dt = \int_{-1}^{1} (at^2+bt+c)^2 dt = \int_{-1}^{1} a^2t^4 + c^2 + b^2t^2 + 2act^2 + 2abt^3 + 2bct \, dt$$

$$= \left[\frac{1}{5}a^2t^5 + c^2t + \frac{1}{3}b^2t^3 + \frac{2}{3}act^3 + \frac{1}{2}abt^4 + bct^2\right]_{-1}^{1} = \frac{2}{5}a^2 + \frac{2}{3}ac + \frac{2}{3}ac + \frac{8}{3}b^2 + 2c^2$$

$$b = \int_{-1}^{1} t\phi(t)^2 dt = \int_{-1}^{1} t(at^2+bt+c)^2 dt = \frac{8}{5}ab + \frac{8}{3}bc$$

$$c = \int_{-1}^{1} t^2 \phi(t)^2 dt = \int_{-1}^{1} t^2(at^2+2bt+c)^2 dt = \frac{2}{7}a^2 + \frac{8}{5}b^2 + \frac{2}{3}c^2 + \frac{4}{5}ac \quad \therefore$$

Solving this nonlinear syst for a, b, c, sub into $\phi(x) = ax^2 + 2bx + c$ for sol

6 sol / $\phi(x) - \int_0^x \frac{t\phi(t)}{1+\phi(t)^2} dt = 0$

$$\frac{\partial}{\partial x} \int_{g(x)}^{f(x)} k(x,t) y(t) = k(x,x)y(x) + \int \frac{\partial k(x,t)}{\partial x} y(t) dt \quad \therefore$$

$$\frac{\partial}{\partial x} \int_0^x \frac{t\phi(t)}{1+\phi(t)^2} dt \qquad \frac{d}{dx} \int_{g(x)}^{h(x)} f(t) dt = f(h(x))h'(x) - f(g(x))g'(x)$$

$$\frac{\partial}{\partial x} \int_0^x h(t) dt = h(x) = h(x)\frac{d}{dx}x - h(0)\frac{d}{dx}(0) = h(x)(1) - h(0)(0) = h(x) \quad \therefore$$

$$\frac{\partial}{\partial x} \int_0^x \frac{t\phi(t)}{1+\phi(t)^2} dt = \frac{x\phi(x)}{1+\phi(x)^2}(1) - \frac{0\phi(0)}{1+\phi(0)^2} = \frac{x\phi(x)}{1+\phi(x)^2}$$

$\left\{ \text{if } \phi \equiv 0 \quad \phi - \int_0^x \frac{t(0)}{1+0^2} dt = 0 - \int_0^x 0 \, dt = 0 = 0 \right\} \therefore$ obviously $\phi \equiv 0$ is a sol. it is also z only sol starting at $\phi(0) = 0$

$\left\{ \phi(0) - \int_0^0 \frac{t\phi(0)}{1+(\phi(0))^2} dt = 0 = \phi(0) - 0 = \phi(0) = 0 \right\} \therefore$ Z operator

$(K\phi)(x) = \int_0^x \frac{t\phi(t)}{1+\phi(t)^2} dt$ satisfies $|(K\phi)(x)| \leq \frac{1}{2}|x|^2 \|\phi\|_\infty$ & $\therefore$

~~$\|K\phi\|_\infty \leq$~~ $\|\phi\|_\infty = \|K\phi\|_\infty \leq \frac{1}{2}\|\phi\|_\infty$ on $[0,1]$ which is correct only for $\phi(x) \equiv 0$

$\left\{ |(K\phi)(x)| = \left|\int_0^x \frac{t\phi(t)}{1+\phi(t)^2} dt\right| \leq \frac{1}{2}|x|^2 \|\phi\|_\infty \qquad \phi(x) = \int_0^x \frac{t\phi(t)}{1+\phi(t)^2} dt = (K\phi) \quad \therefore \right.$

$\|\phi\|_\infty = \|K\phi\|_\infty \qquad |K\phi| \leq \frac{1}{2}|x|^2 \|\phi\|_\infty \quad \therefore$

$\|\phi\|_\infty = \|K\phi\|_\infty \leq \frac{1}{2}\|\phi\|_\infty \quad \therefore \|\phi\|_\infty \leq \frac{1}{2}\|\phi\|_\infty$ on $[0,1]$ which is only correct for $\phi(x) \equiv 0$

$$\frac{\partial}{\partial x} \int_{g(x)}^{h(x)} k(x,t) y(t) = k(x,x)y(x) + \int_{g(x)}^{h(x)} \left(\frac{\partial k(x,t)}{\partial x}\right) y(t) dt,$$

$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))\,h'(x) - f(g(x))\,g'(x)$ $\quad \frac{\partial}{\partial x}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))\,h'(x) - f(g(x))\,g'(x)$

$\frac{\partial}{\partial x}\int_{g(x)}^{h(x)} k(x,t)y(t)\,dt = k(x,x)y(x) + \int_{g(x)}^{h(x)} \frac{\partial k(x,t)}{\partial x} y(t)\,dt$ $\quad \frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$

$\frac{d}{dx}\int_{g(x)}^{h(x)} k(x,t)y(t)\,dt = k(x,x)y(x) + \int_{g(x)}^{h(x)} \frac{\partial k(x,t)}{\partial x} y(t)\,dt$ $\quad \frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$

$\frac{d}{dx}\int_{g(x)}^{h(x)} k(x,t)y(t)\,dt = k(x,x)y(x) + \int_{g(x)}^{h(x)} (\partial k(x,t)/\partial x)\, y(t)\,dt$ $\quad \frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$

$\frac{d}{dx}\int_{g(x)}^{h(x)} k(x,t)y(t)\,dt = k(x,x)y(x) + \int_{g(x)}^{h(x)} (\partial k(x,t)/\partial x)\, y(t)\,dt$ $\quad \frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$

$\frac{d}{dx}\int_{g(x)}^{h(x)} k(x,t)y(t)\,dt = k(x,x)y(x) + \int_{g(x)}^{h(x)} (\partial k(x,t)/\partial x)\, y(t)\,dt$ $\quad \frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$

\7 sol/ let y be a sol of Z eqn $\therefore K = a - 2\int_0^1 y(x)^2\,dx$ is some real number

& $\therefore$ y satisfies Z BVP: $\ddot{y}(x) + Ky(x) = 0 \quad y(0) = y(1) = 0 \therefore$

{ ~~$y = e^{qx}$~~ $\therefore$ ~~y~~ $\quad y = T(t)X(x) \therefore \ddot{y} = T''(t)X(t) \therefore$

$T''(t)X(x) + KT(t)X(x) = 0 \therefore \quad T''(t)X(x) = -KT(t)X(x) \therefore$

$\frac{T''(t)}{T(t)} = -K\frac{X(x)}{X(x)} = -K \quad \therefore T''(t) = -KT(t) \therefore \quad T''(t) + KT(t) \therefore$

$e^{qt} = T \therefore \quad T'' = q^2 e^{qt} \therefore q^2 + K = 0 \therefore q^2 = -K \therefore q = \pm i\sqrt{K} \therefore$

$T(t) = A_2 e^{i\sqrt{K}t} + B_2 e^{-i\sqrt{K}t} = A\cos(\sqrt{K}t) + B\sin(\sqrt{K}t)$ } M

$y(x) = y \therefore \quad \ddot{y} = y'' = \ddot{y}(x) \therefore \quad q^2 + K = 0 \quad \therefore$

$y(x) = A\cos(\sqrt{K}x) + B\sin(\sqrt{K}x) \quad y(0) = y(1) = 0 \therefore A = 0 \quad \therefore$

$y(x) = B\sin(\sqrt{K}x) \quad \therefore y(1) = B\sin(\sqrt{K}) = 0 \therefore \sqrt{K} = n\pi \therefore$

$K = (n\pi)^2$ } $\therefore$ ~~only~~ $\ddot{y}(x) + Ky(x) = 0$, $y(0) = y(1) = 0$, $\therefore$ only nonzero sols when $K = (n\pi)^2$ $\quad n \in \mathbb{N}$, Z eigvals of Z prob $\therefore$

$y(x) = A\sin(n\pi x)$ (Z eigfunc) & $\therefore$: sol is of $\{y(t) = A\sin(n\pi t)\}$

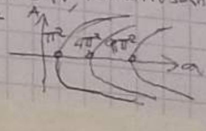
$(n\pi)^2 = a - 2\int_0^1 (A\sin(n\pi t))^2\,dt = a - 2\int_0^1 A^2\sin^2(n\pi x)\,dx$

{ $\cos(2x) = \cos^2 x - \sin^2 x = 1 - \sin^2 x - \sin^2 x = 1 - 2\sin^2 x \therefore \sin^2 x = \frac{1}{2} - \frac{1}{2}\cos(2x)$

$-\cos(2x)\frac{1}{2} + \frac{1}{2} = 1 - \cos^2(x)$, $\therefore \cos^2(x) = \frac{1}{2} + \cos(2x)$ }

$(n\pi)^2 = a - 2\int_0^1 A^2\sin^2(n\pi x)\,dx = a - 2A^2\int_0^1 \frac{1}{2} - \cancel{\frac{1}{2}\cos(n\pi x)\,dx}\ \frac{1}{2}\cos(2n\pi x)\,dx$

$= a - 2A^2\frac{1}{2}[1 - 0] = a - A^2 = (n\pi)^2 \therefore a - (n\pi)^2 = A^2$, $\forall n \in \mathbb{N}$, which is an infinite family of parabolas for $n=1$: $a = 0$: $-\pi^2 = A^2 \notin \mathbb{R} \therefore a > (n\pi)^2$



$\|y-y_n\|_\infty \le \frac{1}{|\lambda|-\|K\|_\infty}\left(\frac{\|K\|_\infty^{n+1}}{|\lambda|^{n+1}}\right)\|f\|_\infty$ $\|y-y_n\|_\infty \le \frac{1}{|\lambda|-\|K\|_\infty}\left(\frac{\|K\|_\infty^{n+1}}{|\lambda|^{n+1}}\right)\|f\|_\infty$

$\|y-y_n\|_\infty \le \frac{1}{|\lambda|-\|K\|_\infty}\left(\frac{\|K\|_\infty^{n+1}}{|\lambda|^{n+1}}\right)\|f\|_\infty$ $\|y-y_n\| \le \frac{1}{|\lambda|-\|K\|_\infty}\left(\frac{\|K\|_\infty}{|\lambda|^{n+1}}\right)^{n+1}\|f\|_\infty$

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$\|y\|_\infty \le \frac{1}{|\lambda|-\|K\|_\infty}\left(\frac{\|K\|_\infty}{|\lambda|}\right)^{n+1}\|f\|_\infty$ $\|y-y_n\|_\infty \le \frac{1}{|\lambda|-\|K\|_\infty}\left(\frac{\|K\|_\infty}{|\lambda|}\right)^{n+1}\|f\|_\infty$

$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$ $\frac{d}{dx}\int_{g(x)}^{h(x)} k(x,t)y(t)\,dt = k(x,x)y(x) + \int_{g(x)}^{h(x)}(dk(x,t)/dx)y(t)\,dt$

$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$ $\frac{d}{dx}\int_{g(x)}^{h(x)} k(x,t)y(t)\,dt = k(x,x)y(x) + \int_{g(x)}^{h(x)}(dk(x,t)/dx)y(t)\,dt$

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$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x)$ $\frac{\partial}{\partial x}\int_0^x f(t)\,dt = f(x)$ $\frac{\partial}{\partial x}\int_0^x f(t)\,dt = f(x)$

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$\|y-y_n\|_\infty \le \frac{1}{|\lambda|-\|K\|_\infty}\left(\frac{\|K\|_\infty}{|\lambda|}\right)^{n+1}\|f\|_\infty$ $\|y-y_n\|_\infty \le \frac{1}{|\lambda|-\|K\|_\infty}\left(\frac{\|K\|_\infty}{|\lambda|}\right)^{n+1}\|f\|_\infty$

LT14: $F(t+T)=F(t)$ $f(s) = \frac{1}{1-e^{-sT}}\int_0^T e^{-st}F(t)\,dt$ $\mathcal{L}(1) = \frac{1}{s}$ $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$

$\mathcal{L}(e^{at}) = \frac{1}{s-a}$ $\mathcal{L}(\sin \omega t) = \frac{\omega}{s^2+\omega^2}$ $\therefore$ $\mathcal{L}(\sinh \omega t) = \frac{\omega}{s^2-\omega^2}$

$\mathcal{L}(\cos \omega t) = \frac{s}{s^2+\omega^2}$ $\mathcal{L}(\cosh \omega t) = \frac{s}{s^2-\omega^2}$ $\mathcal{L}(F(at)) = \frac{1}{a} f\left(\frac{s}{a}\right)$

$\mathcal{L}(F'(t)) = s f(s) - F(0)$ $\mathcal{L}(F''(t)) = s^2 f(s) - sF(0) - F'(0)$ $\mathcal{L}(t^n F(t)) = (-1)^n f^{(n)}(s)$

$\mathcal{L}(e^{at}F(t)) = f(s-a)$ $\mathcal{L}(H(t-a)) = \frac{e^{-as}}{s}$ $\mathcal{L}\left(\int_0^t F(u)G(t-u)\,du\right) = f(s)g(s)$

$\mathcal{L}(F(t+T)) = F(t) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)dt$ $\quad \mathcal{L}(t^2)(s) = \int_0^\infty e^{-st} t^2 dt$

$f(s) = \int_0^\infty e^{-st} F(t)dt \quad f(s) = \int_0^\infty e^{-st} F(t)dt \quad f(s) = \int_0^\infty e^{-st} F(t)dt$

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$f(s) = \int_0^\infty e^{-st} F(t)dt \quad f(s) = \int_0^\infty e^{-st} F(t)dt \quad f(s) = \int_0^\infty e^{-st} F(t)dt \quad f(s) = \int_0^\infty e^{-st} F(t)dt$

\Sheet 1/

\1/ $F(t) = t^2 \quad f(s) = \mathcal{L}(F(t)) = \int_0^\infty e^{-st} F(t)dt = \mathcal{L}(F)(s) =$

$\mathcal{L}(F(t)) = \int_0^\infty e^{-st} t^2 dt = \left[\frac{1}{-s} t^2 e^{-st}\right]_0^\infty - \int_0^\infty 2t \frac{1}{-s} e^{-st} dt =$

$-\frac{1}{s}\left[\lim_{t\to\infty}(t^2 e^{-st}) - (0)^2 e^{-s(0)}\right] - \left[\frac{1}{s^2} e^{-st} 2t\right]_0^\infty + \int_0^\infty 2 \frac{1}{s^2} e^{-st} dt =$

$-\frac{1}{s}[0-0] - \frac{1}{s^2}\left[\lim_{t\to\infty}(2te^{-st}) - 2(0)e^{-s(0)}\right] + \left[2 \frac{1}{-s^3} e^{-st}\right]_0^\infty =$

$0 - \frac{1}{s^2}[0-0] - 2\frac{1}{s^3}\left[\lim_{t\to\infty}(e^{-st}) - e^{-s(0)}\right] = -2\frac{1}{s^3}[0-1] = \frac{2}{s^3}$

\2i/ $\cosh(at) = \frac{e^{at}+e^{-at}}{2} \quad \mathcal{L}(\cosh(at)) = \frac{1}{2}\mathcal{L}(e^{at}) + \frac{1}{2}\mathcal{L}(e^{-at}) =$

$\frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a} = \frac{1}{2}\frac{s+a}{s^2-a^2} + \frac{1}{2}\frac{s-a}{s^2-a^2} = \frac{1}{2}\frac{s+a+s-a}{s^2-a^2} = \frac{s}{s^2-a^2}$

\2ii/ $\mathcal{L}(\sinh(at)) = \mathcal{L}(\frac{1}{2}e^{at} - \frac{1}{2}e^{-at}) = \frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at}) =$

$\frac{1}{2}\frac{1}{s-a} - \frac{1}{2}\frac{1}{s+a} = \frac{1}{2}\frac{s+a}{s^2-a^2} - \frac{1}{2}\frac{s-a}{s^2-a^2} = \frac{1}{s^2-a^2}\left(\frac{1}{2}s + \frac{1}{2}a - \frac{1}{2}s + \frac{1}{2}a\right) =$

$\frac{1}{s^2-a^2}(a) = \frac{a}{s^2-a^2}$

\3i/ $F(t) = t^2\cos(t) \quad \therefore \quad \mathcal{L}(\cos(t)) = \frac{s}{s^2+1} \quad \therefore$

$\mathcal{L}(t^2\cos(t)) = (-1)^2\mathcal{L}(\cos(t))^{(2)} = 1\left(\frac{s}{s^2+1}\right)^{(2)} = \frac{d^2}{ds^2}\left[s(s^2+1)^{-1}\right] = \therefore$

$(s^2+1) + s(2s) = s^2+1+2s^2 = 3s^2+1 = \frac{d}{ds}(s(s^2+1)) \therefore$ ✗

$\frac{d^2}{ds^2}(s(s^2+1)) = 6s$ ✗

$\mathcal{L}(t^2\cos(t)) = \frac{d^2}{ds^2}\left[\frac{s}{s^2+1}\right] = \frac{d}{ds}\left[\frac{(s^2+1)1 - s(2s)}{(s^2+1)^2}\right] = \frac{d}{ds}\left[\frac{-s^2+1}{(s^2+1)^2}\right] =$

$\left((s^2+1)^2(-2s) - (-s^2+1)2(s^2+1)2s\right)/(s^2+1)^4 =$

$\left((s^2+1)(-2s) - (-s^2+1)4s\right)/(s^2+1)^3 = (-2s^3 - 2s + 4s^3 - 4s)/(s^2+1)^3 = \frac{2s^3-6s}{(s+1)^3}$

Sheet 1 / 3 ii / $F(t) e^{2t} \therefore \mathcal{L}(e^{2t} t^2) = \mathcal{L}(t^2 e^{2t}) = (-1)^2 \mathcal{L}(e^{2t})^{(2)} =$

$1\left(\frac{1}{s-2}\right)^{(2)} = \left(\frac{1}{s-2}\right)^{(2)} = \frac{d^2}{ds^2}\left(\frac{1}{s-2}\right) = \frac{d}{ds}\left[\frac{(s-2)(0)-1(1)}{(s-2)^2}\right] = \frac{d}{ds}\left[\frac{-1}{(s-2)^2}\right] =$

$\left((s-2)^2(0)-(-1)2(s-2)(1)\right)/(s-2)^4 = \frac{2(s-2)}{(s-2)^4} = \frac{2}{(s-2)^3}$

3 iii / $\cos^3(t) = \cos(t)(1-\sin^2(t)) \therefore F(t) = \cos^3(t) \therefore F'(t) = \frac{d}{dt}\cos^3(t) =$

$-3\cos^2(t)\sin(t) = -3\sin(t)(1-\sin^2(t)) = -3\sin(t) + 3\sin^3(t) \therefore G(t) = \sin^3(t) \therefore$

$G'(t) = 3\sin^2(t)\cos(t) = 3\cos(t)(1-\cos^2(t)) = 3\cos(t) - 3\cos^3(t) \therefore$

$\mathcal{L}(G'(t)) = s g(s) - G(0) = s\mathcal{L}(G(s)) - \sin^3(0) = s\mathcal{L}(\sin^3(t)) - 0 = s\mathcal{L}(\sin^3(t))$

$g(s) = \mathcal{L}(G(t)) \therefore F(t) = \cos^3(t) \therefore F'(t) = -3\cos^2(t)\sin(t) = -3\sin(t)(1-\sin^2(t)) =$

$-3\sin(t) + 3\sin^3(t) \therefore \mathcal{L}(F'(t)) = s f(s) - F(0) \therefore f(s) = \mathcal{L}(\cos^3(t)) \therefore$

$\mathcal{L}(F'(t)) = \mathcal{L}(-3\sin(t) + 3\sin^3(t)) = -3\mathcal{L}(\sin(t)) + 3\mathcal{L}(\sin^3(t)) = s f(s) - \cos^3(0) = s f(s) - 1$

3 iii redo / $F(t) = \cos^3(t) \therefore F'(t) = -3\cos^2(t)\sin(t) = -3\sin(t) + 3\sin^3(t) \therefore$

$\mathcal{L}(F'(t)) = s f(s) - F(0) = s\mathcal{L}(F(t)) - F(0) = s\mathcal{L}(\cos^3(t)) - 1 \therefore$

$s\mathcal{L}(\cos^3(t)) - 1 = -3\mathcal{L}(\sin t) + 3\mathcal{L}(\sin^3(t))$ (*)

$\hat{F}(t) = \sin^3(t) \therefore \hat{F}'(t) = 3\cos(t) - 3\cos^3(t) \therefore \mathcal{L}((\sin^3(t))') = \mathcal{L}(3\sin^2(t)\cos t) =$

$\mathcal{L}(3\cos t - 3\cos^3 t) = 3\mathcal{L}(\cos t) - 3\mathcal{L}(\cos^3 t) = s f(s) - s^3(0) = s\mathcal{L}(\sin^3(t)) \therefore$

$\mathcal{L}(\sin^3(t)) = \frac{3}{s}\mathcal{L}(\cos(t)) - \frac{3}{s}\mathcal{L}(\cos^3(t)) \therefore \mathcal{L}(\sin^3(t)) = 3\mathcal{L}(\cos t) - 3\mathcal{L}(\cos^3(t))$ ☺

$\therefore \mathcal{L}(F'(t)) = s f(s) \therefore f(s) = \frac{\mathcal{L}(F'(t))}{s} \therefore$ ☺ into (*) $\therefore s\mathcal{L}(\cos^3(t)) - 1 =$

$-3\mathcal{L}(\sin t) + 3\mathcal{L}(\sin^3(t)) = -3\mathcal{L}(\sin t) + 3\left(\frac{3}{s}\mathcal{L}(\cos t) - \frac{3}{s}\mathcal{L}(\cos^3(t))\right) =$

$-3\mathcal{L}(\sin t) + \frac{9}{s}\mathcal{L}(\cos t) - \frac{9}{s}\mathcal{L}(\cos^3 t) \therefore (s + \frac{9}{s})\mathcal{L}(\cos^3 t) =$

$-3\mathcal{L}(\sin t) + \frac{9}{s}\mathcal{L}(\cos t) + 1 = \frac{-3}{s^2-1} + \frac{9}{s^2+1} + 1 \therefore \mathcal{L}(\cos^3(t)) = \left(\frac{s^2+7}{s^2+1}\right)\left(\frac{s}{s^2+9}\right) = \frac{s(s^2+7)}{(s^2+1)(s^2+9)}$

3 iii / is $\cos^2 t = \frac{1}{2} + \frac{1}{2}\cos(2t) \because \cos(2t) = \cos(t+t) = \cos t\cos t - \sin t \sin t =$

$\cos^2 t - \sin^2(t) = \cos^2 t - (\sin^2 t) = \cos^2 t - (1-\cos^2 t) = -1 + 2\cos^2 t \therefore$

$\cos 2t = \cos^2 t - \sin^2 t = \cos^2 t - (\sin^2 t) = \cos^2 t - (1-\cos^2 t) =$

$\cos^2 t - 1 + \cos^2 t = 2\cos^2(t) - 1 = \cos(2t) \therefore 2\cos^2(t) = 1 + \cos(2t) \therefore$

$\cos^2(t) = \frac{1}{2} + \frac{1}{2}\cos(2t) = 1 - \sin^2(t) \therefore \sin^2(t) = \frac{1}{2} - \frac{1}{2}\cos(2t)$

- $F(t) = \cos^3(t) = \cos(t)(\cos^2(t)) = \cos(t)(1-\sin^2(t)) = \cos t - \cos(t)\sin^2(t)$

$\therefore F'(t) = 3\cos^2(t)(-\sin t) = -3\cos^2(t)\sin t = -3(1-\sin^2 t)\sin t = -3\sin t + 3\sin^3 t$

$\therefore \mathcal{L}(F'(t)) = s f(s) - F(0) = s\mathcal{L}(\cos^3 t) - \cos^3(0) = 3\mathcal{L}(\cos^3 t) - 1$

$F'(t) = -3\sin t + 3\sin^3 t \therefore \mathcal{L}(F'(t)) = -3\mathcal{L}(\sin t) + 3\mathcal{L}(\sin^3 t) = s\mathcal{L}(\cos^3(t)) - 1$

Let $\hat{F}(t) = \sin^3(t) \therefore \hat{F}'(t) = \frac{d}{dt}\sin^3(t) = 3\sin^2(t)\cos(t) = 3\cos(t) - 3\cos^3(t)$

$\therefore \mathcal{L}(\hat{F}'(t)) = s\hat{f}(s) - \hat{F}(0) = s\mathcal{L}(\sin^3(t)) - \sin^3(0) = s\mathcal{L}(\sin^3(t)) \therefore$

$\frac{1}{s}\mathcal{L}(\hat{F}'(t)) = \mathcal{L}(\sin^3(t)) \therefore$

$\frac{1}{s}\mathcal{L}(\hat{F}'(t)) = \frac{1}{s}\mathcal{L}(3\cos(t) - 3\cos^3(t)) = \frac{3}{s}\mathcal{L}(\cos t) - \frac{3}{s}\mathcal{L}(\cos^3(t)) = \mathcal{L}(\sin^3(t)) \therefore$

$\therefore s\mathcal{L}(\cos^3(t)) - 1 = -3\mathcal{L}(\sin t) + 3\mathcal{L}(\sin^3(t)) =$

$-3\mathcal{L}(\sin t) + 3\left[\frac{3}{s}\mathcal{L}(\cos t) - \frac{3}{s}\mathcal{L}(\cos^3(t))\right] \therefore$

$\mathcal{L}(\cos^3(t)) = \frac{1}{s} - \frac{3}{s}\mathcal{L}(\sin t) + \frac{9}{s^2}\mathcal{L}(\cos t) - \frac{9}{s^2}\mathcal{L}(\cos^3(t)) \therefore$

$\left(s + \frac{9}{s}\right)\mathcal{L}(\cos^3(t)) = -3\mathcal{L}(\sin t) + \frac{9}{s}\mathcal{L}(\cos t) + 1 = -\frac{3}{s^2+1} + \frac{9}{s^2+1} + 1 \therefore$

$\mathcal{L}(\cos^3(t)) = \frac{1}{s + \left(\frac{9}{s}\right)}\left[-\frac{3}{s^2+1} + \frac{9}{s^2+1} + 1\right] = \left(\frac{s^2+7}{s^2+1}\right)\left(\frac{s}{s^2+9}\right) = \frac{s(s^2+7)}{(s^2+1)(s^2+9)}$

\3vi/ $F(t) = e^{-t}\cos(2t) \therefore \mathcal{L}(F(t)) = \mathcal{L}(e^{-t}\cos(2t)) \therefore$

$\mathcal{L}(\cos(2t)) = \frac{s}{s^2+2^2} = \frac{s}{s^2+4} = \hat{f}(s) \therefore \mathcal{L}(F(t)) = \mathcal{L}(e^{-t}\cos(2t)) = \hat{f}(s+1) =$

$\frac{(s+1)}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$ ✓

\3v/ $\cosh(2\theta) = 1 + 2\sinh^2(\theta) \therefore \mathcal{L}(\sinh^2(t)) = \mathcal{L}\left(\frac{1}{2}\cosh(2t) - \frac{1}{2}\right) =$

$\frac{1}{2}\mathcal{L}(\cosh(2t)) - \frac{1}{2}\mathcal{L}(1) = \frac{1}{2}\frac{s}{s^2-2^2} - \frac{1}{2}\frac{1}{s} = \frac{s}{2(s^2-4)} - \frac{1}{2s} = \frac{2}{s(s^2-4)}$

\3vii/ ~~$F(t) = e^{-t}\cos(2t)$~~ $F(t) = t^2 e^{3t}$ $\quad \mathcal{L}(t^2 e^{3t}) = \mathcal{L}(e^{3t}t^2) \therefore$

$\mathcal{L}(t^2) = \frac{2!}{s^{2+1}} = \frac{2}{s^3} = \hat{f}(s) \therefore \mathcal{L}(e^{3t}t^2) = \hat{f}(s-3) = \frac{2}{(s-3)^3}$

\4/ $\frac{d}{dt}\cosh(3t) = 3\sinh(3t) \therefore \mathcal{L}(\sinh at) = \frac{a}{s^2-a^2} \quad \mathcal{L}(\cosh(at)) = \frac{s}{s^2-a^2}$

$\therefore \mathcal{L}\left(\frac{d}{dt}\cosh(3t)\right) = s\mathcal{L}(\cosh(3t)) - \cosh(3(0)) =$

$s\frac{s}{s^2-3^2} - \cos(0) = s\frac{s}{s^2-9} - 1 = \cancel{3\frac{3}{s^2-9}} = s\frac{s}{s^2-9} - \frac{s^2-9}{s^2-9} = \frac{9}{s^2-9}$

$\mathcal{L}(3\sinh 3t) = 3\frac{3}{s^2-3^2} = 3\frac{3}{s^2-9} = \frac{9}{s^2-9}$

~~xi~~ mock sols/ $\ddot{x}(t) - 2\dot{x}(t) + 2x(t) = 2e^t \quad x(0) = 0 \quad \dot{x}(0) = 1 \therefore$

$\mathcal{L}(\ddot{x}(t)) = s^2 f(s) - s\cancel{x}(0) - \dot{x}(0) = s^2 f(s) - s(0) - 1 = s^2 f(s) = s^2 x(s) - 1 \therefore$

$\mathcal{L}(x(t)) = x(s) \therefore$ ~~$\mathcal{L}(\dot{x}$~~ $f(s) = \mathcal{L}(F(t)) = \mathcal{L}(x(t)) = x(s) \therefore$

$\mathcal{L}(\dot{x}(t)) = s f(s) - F(0) = s x(s) - x(0) = s x(s) - 0 = s x(s) \therefore$

$\mathcal{L}(\ddot{x}(t) - 2\dot{x}(t) + 2x(t)) = \mathcal{L}(2e^t) = \mathcal{L}(\ddot{x}(t)) - 2\mathcal{L}(\dot{x}(t)) + 2\mathcal{L}(x(t)) = 2\mathcal{L}(e^t) =$

$s^2 x(s) - 1 - 2s x(s) + 2x(s) = 2\frac{1}{s-1} = x(s)(s^2 - 2s + 2) - 1 \therefore \frac{2}{s-1} + 1 = \frac{2+s-1}{s-1} = \frac{s+1}{s-1} \therefore$

1) Mini mock sols / $\frac{s+1}{s-1} = x(s)(s^2-2s+2) \therefore x(s) = \frac{s+1}{(s-1)(s^2-2s+2)} =$

$\frac{A}{s-1} + \frac{Bs+C}{s^2-2s+2} \therefore s+1 = A(s^2-2s+2) + (Bs+C)(s-1) \therefore$

$\{ s=-1: -1+1=0 = A((-1)^2-2(-1)+2) + (Bs+C)(0) = A(1+2+2) = 0 = 5A \}$

$s=1: 1+1=2 = A(1^2-2(1)+2) + (Bs+C)(0) = A(1-2+2) = 2 = A(1) \therefore A=2 \therefore$

$s+1 = 2s^2-4s+4+Bs^2-Bs+Cs-C \therefore$

$s^2: 0s^2 = 2s^2+Bs^2 \therefore 0 = 2+B \therefore B=-2$

$s^0: 1s^0 = 4-C = 1 \therefore 3 = C \therefore x(s) = \frac{2}{s-1} + \frac{-2s+3}{s^2-2s+2} =$

$2\frac{1}{s-1} + \frac{-2s+3}{(s-1)^2+2-1} = 2\frac{1}{s-1} + \frac{-2s+3}{(s-1)^2+1} \therefore$

$\frac{-2s+3}{(s-1)^2+1} = \frac{-2(s-1)+3-2}{(s-1)^2+1} = \frac{-2(s-1)+1}{(s-1)^2+1} = -2\frac{s-1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1} \therefore$

$x(s) = \frac{2}{s-1} - 2\frac{s-1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1} \therefore$

$x(t) = \mathcal{L}^{-1}(x(s)) = 2\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) - 2\mathcal{L}^{-1}\left(\frac{s-1}{(s-1)^2+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2+1}\right) =$

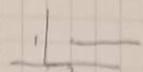
$2e^{1t} - 2\mathcal{L}^{-1}\left(\frac{(s-1)}{(s-1)^2+1}\right) + e^{1t}\mathcal{L}^{-1}\left(\frac{1}{s^2+1^2}\right) = 2e^t - 2\mathcal{L}^{-1}\left(\frac{(s-1)}{(s-1)^2+1^2}\right) + e^t\sin(1t) =$

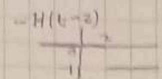
(by LT4, LT5, LT13)

$2e^t - 2e^t\cos(1t) + e^t\sin(t) = 2e^t - 2e^t\cos(t) + e^t\sin(t) = x(t)$

2/ $F(t) = \begin{cases} t & , t<2 \\ 8-3t & , 2 \le t < 3 \\ t-4 & , 3 \le t < 4 \\ 0 & , 4 \le t \end{cases} \therefore H(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases} \therefore H(t-a) = \begin{cases} 1 & , t \ge a \\ 0 & , t < a \end{cases} \therefore$

for $t<2$: $F(t) = t$ $\quad H(t-2) = \begin{cases} 1 & , t \ge 2 \\ 0 & t < 2 \end{cases}$:





$\therefore 0 \le t < \infty \therefore$ for $t<2$: $F(t) = t \therefore F(t) = (H(t) - H(t-2))t \therefore$

$\forall t$:

$F(t) = t(H(t)-H(t-2)) + (8-3t)(H(t-2)-H(t-3)) + (t-4)(H(t-3)-H(t-4)) + 0(H(t-4)) =$

$tH(t) - tH(t-2) + (8-3t)H(t-2) - (8-3t)H(t-3) + (t-4)H(t-3) - (t-4)H(t-4) =$

$tH(t) + (8-4t)H(t-2) + (-8+3t+t-4)H(t-3) - (t-4)H(t-4) =$

$tH(t) + 4(2-t)H(t-2) + (-12+4t)H(t-3) - (t-4)H(t-4) =$

$tH(t) - 4(t-2)H(t-2) + 4(t-3)H(t-3) - (t-4)H(t-4) = F(t) \therefore$ LT17: $a = 0, 2, 3, 4$

$\therefore \mathcal{L}(F(t)) = \mathcal{L}(tH(t)) - 4\mathcal{L}((t-2)H(t-2)) + 4\mathcal{L}((t-3)H(t-3)) - \mathcal{L}((t-4)H(t-4)) =$

$e^{-0s}\mathcal{L}(t) - 4e^{-2s}\mathcal{L}(t) + 4e^{-3s}\mathcal{L}(t) - e^{-4s}\mathcal{L}(t) =$

$1\frac{1!}{s^{1+1}} - 4e^{-2s}\frac{1}{s^2} + 4e^{-3s}\frac{1}{s^2} - e^{-4s}\frac{1}{s^2} = \frac{1}{s^2}\left[1 - 4e^{-2s} + 4e^{-3s} - e^{-4s}\right] = \mathcal{L}(F(t))$

$\Big\{$ eg $F(t)=\begin{cases} 0 & 0<t<3 \\ 3t^2 & t\geq 3 \end{cases}$ $\therefore F(t) = 0\,(H(t) - H(t-3)) + 3t^2\,(H(t-3)) = 3t^2 H(t-3)$

$\therefore \mathcal{L}(F(t)) = \int_0^{\infty} e^{-st} F(t)\,dt = \int_3^{\infty} e^{-st}\,3t^2\,dt \Big\}$

\3a/ $\lambda y(x) - \int_0^{2\pi} \cos(x-t)\,y(t)\,dt = f(x)$ $\therefore$ Fredholm $\therefore$ p-method $\therefore$

$\cos(x-t) = \cos(x)\cos(t) + \sin(x)\sin(t)$ $\therefore \int_0^{2\pi} \cos(x-t)\,y(t)\,dt =$

$\int_0^{2\pi} (\cos(x)\cos(t) + \sin(x)\sin(t))\,y(t)\,dt = \cos(x)\int_0^{2\pi} \cos(t)\,y(t)\,dt + \sin(x)\int_0^{2\pi} \sin(t)\,y(t)\,dt$ $\therefore$

$\lambda y(x) - \cos(x)\int_0^{2\pi} \cos(t)\,y(t)\,dt - \sin(x)\int_0^{2\pi} \sin(t)\,y(t)\,dt = f(x)$ $\therefore$

$\int_0^{2\pi} \cos(t)\,y(t)\,dt = P_1$, $\int_0^{2\pi} \sin(t)\,y(t)\,dt = P_2$ $\therefore$

$\lambda y(x) - \cos(x) P_1 - \sin(x) P_2 = f(x)$ $\therefore y(x) = \frac{1}{\lambda} f(x) + \frac{1}{\lambda}\cos(x) P_1 + \sin(x)\frac{1}{\lambda} P_2$

$= \frac{1}{\lambda}\left[f(x) + \cos(x) P_1 + \sin(x) P_2\right]$ $\therefore y(t) = \frac{1}{\lambda}\left[f(t) + \cos(t) P_1 + \sin(t) P_2\right]$ $\therefore$

$P_1 = \int_0^{2\pi} \cos(t)\frac{1}{\lambda}\left[f(t) + \cos(t) P_1 + \sin(t) P_2\right] dt =$

$\frac{1}{\lambda}\int_0^{2\pi} \cos(t) f(t)\,dt + \frac{1}{\lambda} P_1 \int_0^{2\pi} \cos^2(t)\,dt + \frac{1}{\lambda} P_2 \frac{1}{2}\int_0^{2\pi} 2\sin(t)\cos(t)\,dt$ $\therefore$

$\frac{1}{\lambda} P_1 \int_0^{2\pi} \cos^2(t)\,dt = \frac{1}{\lambda} P_1 \int_0^{2\pi} \frac{1}{2} + \frac{1}{2}\cos(2t)\,dt = \frac{1}{\lambda} P_1 \left[\frac{1}{2}t + \frac{1}{4}\sin(2t)\right]_0^{2\pi} =$

$\frac{1}{\lambda} P_1 \left[\frac{1}{2}(2\pi - 0) + \frac{1}{4}(\sin(4\pi) - \sin(0))\right] = \frac{1}{\lambda}\pi P_1$,

$\int_0^{2\pi} 2\sin(t)\cos(t)\,dt = \int_0^{2\pi} \sin(2t)\,dt = \left[-\frac{1}{2}\cos(2t)\right]_0^{2\pi} = -\frac{1}{2}\left[\cos(4\pi) - \cos(0)\right] = \frac{-1}{2}(1-1) = 0$

$\therefore P_1 = \frac{1}{\lambda}\left(\int_0^{2\pi} \cos(t) f(t)\,dt + \pi P_1\right)$ $\therefore (\lambda - \pi) P_1 = \int_0^{2\pi} \cos(t) f(t)\,dt$ $\therefore$

$P_1 = \frac{1}{\lambda - \pi}\int_0^{2\pi} \cos(t) f(t)\,dt$

for P_2: $P_2 = \frac{1}{\lambda}\left(\int_0^{2\pi} \sin(t)\left(f(t) + \cos(t) P_1 + \sin(t) P_2\right) dt\right) =$

$\frac{1}{\lambda}\left(\int_0^{2\pi} \sin(t) f(t)\,dt + P_1 \frac{1}{2}\int_0^{2\pi} 2\sin(t)\cos(t)\,dt + P_2 \int_0^{2\pi} \sin^2(t)\,dt\right)$ $\therefore$

$\int_0^{2\pi} 2\sin(t)\cos(t)\,dt = \int_0^{2\pi} \sin(2t)\,dt = \left[-\frac{1}{2}\cos(2t)\right]_0^{2\pi} = -\frac{1}{2}\left[\cos(4\pi) - \cos(0)\right] = -\frac{1}{2}(1-1) = 0$,

$\int_0^{2\pi} \sin^2(t)\,dt = \int_0^{2\pi} \frac{1}{2} - \frac{1}{2}\cos(2t)\,dt = \left[\frac{1}{2}t - \frac{1}{4}\sin(2t)\right]_0^{2\pi} = \frac{1}{2}(2\pi - 0) - \frac{1}{4}\left[\sin(4\pi) - \sin(0)\right] =$

$\pi - \frac{1}{4}(0 - 0) = \pi$ $\therefore$

$P_2 = \frac{1}{\lambda}\left(\int_0^{2\pi} \sin(t) f(t)\,dt + \pi P_2\right)$ $\therefore (\lambda - \pi) P_2 = \int_0^{2\pi} \sin(t) f(t)\,dt$ $\therefore$

$P_2 = \frac{1}{\lambda - \pi}\int_0^{2\pi} \sin(t) f(t)\,dt$ $\therefore$

$\lambda \neq \pi$ for a sol to exist

$$\frac{4}{\sqrt{3}}\frac{\sqrt{3}}{s^2-(\sqrt{3})^2}-\frac{1}{s^2+1}=\frac{4}{s^2-3}-\frac{1}{s^2+1} \quad \times$$

$$f(s)=\frac{4}{s-3}-\frac{1}{s+1} \therefore F(t)=\mathcal{L}^{-1}(f(s))=4\mathcal{L}\left(\frac{1}{s-3}\right)-\mathcal{L}\left(\frac{1}{s-(-1)}\right)=$$

$$4e^{3t}-e^{-t} \checkmark \qquad \cancel{\mathcal{L}^{-1}\left(\frac{3(s-1)}{(s}\right.} \quad \mathcal{L}^{-1}\left(\frac{3s+7}{s^2-2s-3}\right)=\mathcal{L}^{-1}\left(\frac{3s+7}{(s-1)^2-3-1}\right)=$$

$$\mathcal{L}^{-1}\left(\frac{3s+7}{(s-1)^2-4}\right)=\mathcal{L}^{-1}\left(\frac{3(s-1)+3+7}{(s-1)^2-4}\right)=\mathcal{L}^{-1}\left(\frac{3(s-1)+10}{(s-1)^2-4}\right)=3\mathcal{L}^{-1}\left(\frac{(s-1)}{(s-1)^2-2^2}\right)+5\mathcal{L}^{-1}\left(\frac{2}{(s-1)^2-2^2}\right)$$

$$3\cos(2t)e^t+5\sinh(2t)e^t=4e^{3t}-e^{-t}=F(t)$$

\\iv/ $f(s)=\dfrac{4s+12}{s^2+8s+16} \therefore \mathcal{L}^{-1}\left(\dfrac{4s+12}{s^2+8s+16}\right)=\mathcal{L}^{-1}\left(\dfrac{4s+12}{(s+4)^2}\right) \therefore$

$$\frac{4s+12}{(s+4)^2}=\frac{A}{s+4}+\frac{B}{(s+4)^2} \therefore 4s+12=A(s+4)+B \therefore$$

$$s=-4: \ 4(-4)+12=A(0)+B=-16+12=-4=B, \quad 4s=As \therefore A=4 \therefore$$

$$F(t)=\cancel{\mathcal{L}^{-1}(2/2} \quad 4\mathcal{L}^{-1}\left(\frac{1}{s+4}\right)-4\mathcal{L}^{-1}\left(\frac{1}{(s+4)^2}\right)=4e^{-4t}-4te^{-4t}=4e^{-4t}(1-t)$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s-a)^n}\right)=e^{at}\mathcal{L}^{-1}\left(\frac{1}{s^n}\right)=e^{at}\left(\frac{t^{n-1}}{(n-1)!}\right)$$

\\ix/ $f(s)=\dfrac{s^2}{s^3+a^3} \quad a\in\mathbb{R} \therefore f(s)=\dfrac{s^2}{(s+a)(s^2+a^2-as)}=\dfrac{A}{s+a}+\dfrac{Bs+C}{s^2-as+a^2}$

$$\therefore s^2=A(s^2-as+a^2)+(Bs+c)(s+a) \therefore A=\frac{1}{3}, B=\frac{2}{3}, C=-\frac{a}{3} \therefore$$

$$f(s)=\frac{1/3}{s+a}+\frac{(2/3)s-a/3}{s^2-as+a^2}=\frac{1/3}{s+a}+\frac{2}{3}\frac{s-a/2}{(s-\frac{a}{2})^2+\frac{3}{4}a^2} \therefore F(t)=\frac{1}{3}e^{-at}+\frac{2}{3}e^{(a/2)t}\cos\left(\frac{\sqrt{3}}{2}at\right)$$

\\x/ $f(s)=\dfrac{(s+1)e^{-\pi s}}{s^2+2s+1} \therefore \dfrac{s+1}{s^2+2s+1}=\dfrac{s+1}{(s+1)^2}=\dfrac{1}{s+1} \therefore \mathcal{L}^{-1}\left(\dfrac{1}{s+1}\right)=e^{-t} \therefore$

$$\mathcal{L}(F(t-a)H(t-a))=e^{-as}f(s) \therefore F(t)=H(t-\pi)e^{\pi-t}=H(t-\pi)e^{-(t-\pi)} \therefore$$

$$F(t-a)H(t-a)=\mathcal{L}^{-1}(e^{-as}f(s)) \therefore \hat{f}(s)=\frac{s+1}{s^2+2s+1}=\frac{1}{s+1} \therefore \mathcal{L}^{-1}\left(e^{-\pi s}\frac{s+1}{s^2+2s+1}\right)=$$

$$\mathcal{L}^{-1}\left(e^{-as}\frac{1}{s+1}\right)=\mathcal{L}^{-1}(e^{-as}\hat{f}(s)), \ \mathcal{L}^{-1}(\hat{f}(s))=\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)=e^{-t} \therefore$$

$$\mathcal{L}^{-1}(f(s))=\mathcal{L}^{-1}\left(\frac{1}{s+1}e^{-as}\right) \neq \mathcal{L}(F(t))=\mathcal{L}(F(t-a)H(t-a))=e^{-as}f(s)=e^{-\pi s}\frac{1}{s+1}=$$

$$e^{-\pi s}\frac{s+1}{s^2+2s+1} \therefore F(t)=\mathcal{L}^{-1}\left(\frac{1}{s+1}\right)=e^{-t} \therefore F(t-a)=F(t-\pi)=e^{-(t-\pi)}=e^{-t+\pi} \therefore$$

$$\mathcal{L}^{-1}\left(e^{-\pi s}\frac{s+1}{s^2+2s+1}\right)=F(t-a)H(t-a)=e^{-t+\pi}H(t-a)=F(t)=e^{-t+\pi}H(t-\pi)$$

$$\hat{f}(s)=\frac{s+1}{s^2+2s+1} \therefore \hat{F}(t)=e^{-t} \therefore \mathcal{L}^{-1}\left(e^{-\pi s}\frac{s+1}{s^2+2s+1}\right)=\hat{F}(t-\pi)H(t-\pi)=e^{-(t-\pi)}H(t-\pi)=$$

$$e^{\pi-t}H(t-\pi)$$

\\xi/ $f(s)=\ln\left(\dfrac{s^2+1}{s(s+3)}\right) \therefore \mathcal{L}(t^nF(t))=(-1)^nf^{(n)}(s) \quad \mathcal{L}(t^1F(t))=-1f^{(1)}(s) \therefore$

$$\mathcal{L}(tF(t))=-f'(s) \therefore F(t)=-\frac{\mathcal{L}^{-1}(f'(s))}{t} \therefore f'(s)=\frac{3s^2-2s+3}{s(s^2+1)(s+3)}=\frac{A}{s}+\frac{Bs+C}{s^2+1}+\frac{D}{s+3} \therefore$$

$$A=-1, \ B=2, \ C=0, \ D=-1 \therefore$$

\Sheet 2/ $f'(s) = -\frac{1}{s} + \frac{2s}{s^2+1} - \frac{1}{s+3}$ ∴ $\mathcal{L}^{-1}(f'(s)) = -1 - e^{-3t} + 2\cos t$ ∴

$F(t) = -\frac{\mathcal{L}^{-1}(f'(s))}{t} = \frac{1}{t}(1 + e^{-3t} - 2\cos t)$

\2/ periodic $T = 1$ & $(1, 1)$ ∴ $f(t) = t$ ∴

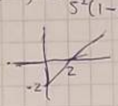
$f(s) = \mathcal{L}(F(t)) = \frac{1}{1-e^{-s(1)}}\int_0^1 e^{-st}t\,dt = \frac{1}{1-e^{-s}}\int_0^1 te^{-st}\,dt =$

$\frac{1}{1-e^{-s}}\left(\left[t\frac{1}{-s}e^{-st}\right]_0^1 - \int_0^1 \frac{1}{-s}e^{-st}\,dt\right) =$

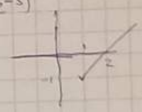
$\frac{1}{1-e^{-s}}\left(\left[1\frac{1}{-s}e^{-s} - 0\right] - \left[\frac{1}{s^2}e^{-st}\right]_0^1\right) = \frac{1}{1-e^{-s}}\left(\frac{1}{-s}e^{-s} - \frac{1}{s^2}e^{-s} + \frac{1}{s^2}\right) =$

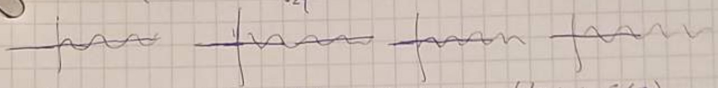
$\frac{1}{s(1-e^{-s})}\left(\frac{1}{s} - e^{-s} - \frac{1}{s}e^{-s}\right) = \frac{1}{s^2(1-e^{-s})}(1 - se^{-s} - e^{-s}) = \frac{1 - e^{-s}(1+s)}{s^2(1-e^{-s})}$

3i/ $H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$



$H(t-1) = \begin{cases} 1 & t \geq 1 \\ 0 & t \not\geq 1 \end{cases}$ ∴





\4i/ $F'' - 4F' + 3F = 1$ $F(0) = F'(0) = 0$ ∴ $\mathcal{L}(F(t)) = f(s)$ ∴

$\mathcal{L}(F'(t)) = sf(s)$ & $\mathcal{L}(F''(t)) = s^2 f(s)$ ∴ $\mathcal{L}(F'' - 4F' + 3F) = \mathcal{L}(1) =$

$\mathcal{L}(F''(t)) - 4\mathcal{L}(F'(t)) + 3\mathcal{L}(F(t)) = s^2 f(s) - 4sf(s) + 3f(s) = \frac{1}{s} = (s^2 + -4s + 3)f(s)$ ∴

$f(s) = \frac{1}{s}\frac{1}{s^2 - 4s + 3} = \frac{1}{s(s-3)(s-1)} = f(s) = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-1}$ ∴

$1 = A(s-3)(s-1) + B(s(s-1)) + Cs(s-3)$ ∴ $s = 0: A = \frac{1}{3}$, $s = 3: B = \frac{1}{6}$

$s = 1: C = -\frac{1}{2}$ ∴ $f(s) = \frac{1}{3s} + \frac{1}{6(s-3)} - \frac{1}{2(s-1)}$ ∴

$F(t) = \mathcal{L}^{-1}(f(s)) = \frac{1}{3}\mathcal{L}(s) + \frac{1}{6}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) =$

$\frac{1}{3} + \frac{1}{6}e^{3t} - \frac{1}{2}e^{t} = F(t)$

\4ii/ $F'' - 2aF' + a^2F = 0$ $F'(0) = 1$, $F(0) = 0$ $a \in \mathbb{R}$ ∴ $\mathcal{L}(F) = \mathcal{L}(F(t)) = f(s)$ ∴

$\mathcal{L}(F'') = \mathcal{L}(F''(t)) = s^2 f(s) - sF(0) - F'(0) = s^2 f(s) - s(0) - 1 = s^2 f(s) - 1$ ∴

$\mathcal{L}(F') = \mathcal{L}(F'(t)) = sf(s) - F(0) = sf(s) - 0 = sf(s)$ ∴

$\mathcal{L}(F'' - 2aF' + a^2F) = \mathcal{L}(0) = 0 = \mathcal{L}(F'') - 2a\mathcal{L}(F') + a^2\mathcal{L}(F) =$

$s^2 f(s) - 1 - 2a(sf(s)) + a^2 f(s) = 0 = [s^2 - 2as + a^2]f(s) - 1$ ∴

$\frac{1}{s^2 - 2as + a^2} = f(s) = \frac{1}{(s-a)^2} = \frac{A}{s-a} + \frac{B}{(s-a)^2}$ ∴ $1 = A(s-a) + B$ ∴

$s = a$: $1 = A(0) + B = B = 1$, s': $0s' = As'$ ∴ $A = 0$ ∴

$f(s) = \frac{1}{(s-a)^2}$ ∴ $\mathcal{L}^{-1}(f(s)) = F(t) = F = \mathcal{L}^{-1}\left(\frac{1}{(s-a)^2}\right) = e^{at}t' = e^{at}t = F(t)$ ✓

Seminar work 2/3 / 3b/ $y(x) = \frac{1}{\lambda}\left[f(x) + \cos(x)\frac{1}{\lambda-\pi}\int_0^{2\pi}\cos(t)f(t)dt + \frac{\sin(x)}{\lambda-\pi}\int_0^{2\pi}\sin(t)f(t)dt\right]$

$= \frac{1}{\lambda}\left(f(x) + \frac{1}{\lambda-\pi}\int_0^{2\pi}(\cos(x)\cos(t) + \sin(x)\sin(t))f(t)\,dt\right) =$

$y(x) = \frac{1}{\lambda}\left(f(x) + \frac{1}{\lambda-\pi}\int_0^{2\pi}\cos(x-t)f(t)dt\right] \quad \lambda \neq \pi$

3b/ $\therefore y(x) = \frac{1}{\lambda}\left(f(x) + \int_0^{2\pi}\frac{1}{\lambda-\pi}\cos(x-t)f(t)dt\right) \therefore$

$\lambda y(x) = \int_0^{2\pi}\frac{1}{\lambda-\pi}\cos(x-t)f(t)dt + f(x) = f(x) + \int_0^{2\pi}\frac{1}{\lambda-\pi}\cos(x-t)f(t)dt$

$\therefore \lambda y = (1+K)f$

$(K f)(x) = \int_0^{2\pi}\frac{1}{\lambda-\pi}\cos(x-t)f(t)dt \therefore$

the resolvent kernel is $r(x,t) = \frac{1}{\lambda-\pi}\cos(x-t) \therefore$

$y = \frac{1}{\lambda}(f + R_\lambda f)$

$\|y\|_\infty \leq C\|f\|_\infty, \quad C = \frac{1}{|\lambda|}(1 + (b-a)\|r_\lambda\|_\infty)$

$\|K\|_\infty = \max_{x\in[a,b]}\int_a^b|k(x,t)|dt \quad \|K\|_\infty = \max_{x\in[a,b]}\int_a^b|k(x,t)|dt \quad \|K\|_\infty = \max_{x\in[a,b]}\int_a^b|k(x,t)|dt$

$\|K\|_\infty = \max_{x\in[a,b]}\int_a^b|k(x,t)|dt \quad \|K\|_\infty = \max_{x\in[a,b]}\int_a^b|k(x,t)|dt \quad \|K\|_\infty = \max_{x\in[a,b]}\int_a^b|k(x,t)|dt$

Sheet 2/ i/ $f(s) = \frac{6s}{s^2-16} \quad \mathcal{L}(\cosh(at)) = \frac{s}{s^2-a^2} \therefore$

$\mathcal{L}^{-1}(f(s)) = \mathcal{L}^{-1}\left(\frac{6s}{s^2-16}\right) = 6\mathcal{L}^{-1}\left(\frac{s}{s^2-4^2}\right) = 6\cosh(4t)$

ii/ $f(s) = \frac{1}{s^3} \therefore F(t) = \frac{1}{6}t^2$ by LT2

iii/ $f(s) = \frac{4}{s-2} \therefore F(t) = 4e^{2t}$ by LT1, LT3

iiii/ $f(s) = \frac{1}{s^2+9} \therefore F(t) = \frac{1}{3}\sin(3t)$

iv/ ~~$f(s) = \frac{6s}{s^2-16}$~~ $f(s) = \frac{3s+7}{s^2-2s-3} \therefore$

$(s-3)(s+1) = s^2-3s+s-3 = s^2-2s+3 \therefore \quad f(s) = \frac{3s+7}{(s-3)(s+1)} = \frac{A}{s-3} + \frac{B}{s+1} \therefore$

$3s+7 = A(s+1) + B(s-3) \therefore$

$3(-1)+7 = A(0) + B(-1-3) = -3+7 = -4B = 4 \therefore B = -1$

$3(3)+7 = A(3+1) + B(0) = 4A = 9+7 = 16 \therefore A = 4 \therefore f(s) = \frac{4}{s-3} - \frac{1}{s+1} =$

$\frac{4}{\sqrt{3}}\frac{\sqrt{3}}{s-(\sqrt{3})^2} - \frac{1}{s+1^2} \therefore F(t) = \mathcal{L}^{-1}(s) = \frac{4}{\sqrt{3}}\mathcal{L}^{-1}\left(\frac{\sqrt{3}}{s-(\sqrt{3})^2}\right) - \mathcal{L}^{-1}\left(\frac{1}{s+1^2}\right) =$

$\frac{4}{\sqrt{3}}\sinh(\sqrt{3}t) - \sin(t) \therefore \mathcal{L}\left(\frac{4}{\sqrt{3}}\sinh(\sqrt{3}t) - \sin(t)\right) = \frac{4}{\sqrt{3}}\mathcal{L}(\sinh(\sqrt{3}t)) - \mathcal{L}(\sin(t))$

\4ii/ Laplace: $s^2 y(s) - 1 - 2as\,y(s) + a^2 y(s) = 0 \therefore y(s) = \frac{1}{(s-a)^2} \therefore Y(t) = te^{at}$

\4iii/ $F'' - 4F' + 3F = 0 \quad F(0) = 1, F'(0) = 1 \quad$ let $\mathcal{L}(F) = \mathcal{L}(F(t)) = f(s)$

$\mathcal{L}(F') = \mathcal{L}(F'(t)) = s f(s) - F(0) = s f(s) - 1 \therefore$

$\mathcal{L}(F'') = \mathcal{L}(F''(t)) = s^2 f(s) - sF(0) - F'(0) = s^2 f(s) - s(1) - 1 = s^2 f(s) - s - 1 \therefore$

$\mathcal{L}(F'' - 4F' + 3F) = \mathcal{L}(0) = 0 = \mathcal{L}(F'') - 4\mathcal{L}(F') + 3\mathcal{L}(F) =$

$s^2 f(s) - s - 1 - 4(s f(s) - 1) + 3 f(s) =$

$s^2 f(s) - s - 1 - 4s f(s) + 4 + 3 f(s) = 0 = [s^2 - 4s + 3] f(s) - s + 3 \therefore$

$(s^2 - 4s + 3) f(s) = s - 3 \therefore f(s) = \frac{s-3}{(s-1)(s-3)} = \frac{1}{s-1} = f(s) \therefore$

$\mathcal{L}^{-1}(f(s)) = F(t) = \mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^{1t} = e^t = F(t)$

\5/ $f(s) = \frac{1}{s^4+1} = \frac{1}{(s^2+\sqrt{2}s+1)(s^2-\sqrt{2}s+1)} = \frac{As+B}{(s^2+\sqrt{2}s+1)} + \frac{Cs+D}{(s^2-\sqrt{2}s+1)} =$

$\frac{\frac{1}{2\sqrt{2}}s + \frac{1}{2}}{s^2+\sqrt{2}s+1} + \frac{\frac{1}{2} - \frac{1}{2\sqrt{2}}s}{s^2-\sqrt{2}s+1} = \frac{s+\sqrt{2}}{2\sqrt{2}(s^2+\sqrt{2}s+1)} - \frac{s-\sqrt{2}}{2\sqrt{2}(s^2-\sqrt{2}s+1)} =$

$\frac{1}{2\sqrt{2}} \cdot \frac{s+\frac{\sqrt{2}}{2}}{(s+\frac{\sqrt{2}}{2})^2+\frac{1}{2}} + \frac{1}{4} \cdot \frac{1}{(s+\frac{\sqrt{2}}{2})^2+\frac{1}{2}} - \frac{1}{2\sqrt{2}} \cdot \frac{s-\frac{\sqrt{2}}{2}}{(s-\frac{\sqrt{2}}{2})^2+\frac{1}{2}} + \frac{1}{4} \cdot \frac{1}{(s-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}$

inverse laplace:

$F(t) = \frac{1}{2\sqrt{2}} e^{-\frac{t}{\sqrt{2}}} \cos\left(\sqrt{\tfrac{1}{2}}\,t\right) + \frac{1}{4}\sqrt{2}\, e^{-\frac{t}{\sqrt{2}}} \sin\left(\sqrt{\tfrac{1}{2}}\,t\right) - \frac{1}{2\sqrt{2}} e^{\frac{t}{\sqrt{2}}} \cos\left(\sqrt{\tfrac{1}{2}}\,t\right) + \frac{1}{4}\sqrt{2}\, e^{\frac{t}{\sqrt{2}}} \sin\left(\sqrt{\tfrac{1}{2}}\,t\right)$

$\|y - y_n\|_\infty \le \frac{1}{|\lambda| - \|K\|_\infty} \left(\frac{\|K\|_\infty}{|\lambda|}\right)^{n+1} \|f\|_\infty \quad \|y - y_n\|_\infty \le \frac{1}{|\lambda| - \|K\|_\infty} \left(\frac{\|K\|_\infty}{|\lambda|}\right)^{n+1} \|f\|_\infty$

$\frac{d}{dx}\int_{g(x)}^{h(x)} k(x,t)y(t)\,dt = k(x,x)y(x) + \int_{g(x)}^{h(x)} (\partial k(x,t)/\partial x)\, y(t)\,dt \quad \frac{d}{dx}\int_{g(x)}^{h(x)} k(x,t)y(t)\,dt = k(x,x)y(x) + \int_{g(x)}^{h(x)} \frac{\partial k(x,t)}{\partial x} y(t)\,dt$

$\frac{d}{dx}\int_{g(x)}^{h(x)} k(x,t)y(t)\,dt = k(x,x)y(x) + \int_{g(x)}^{h(x)} \frac{\partial k(x,t)}{\partial x} y(t)\,dt \quad \frac{d}{dx}\int_{g(x)}^{h(x)} k(x,t)y(t)\,dt = k(x,x)y(x) + \int_{g(x)}^{h(x)} \frac{\partial k(x,t)}{\partial x} y(t)\,dt$

$\frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x) \quad \frac{d}{dx}\int_{g(x)}^{h(x)} f(t)\,dt = f(h(x))h'(x) - f(g(x))g'(x) \quad \frac{d}{dx}\int_{0}^{x} f(t)\,dt = f(x)$

\Sheet 3/ \1i/ $\mathcal{L}\left(\int_0^t F(u)G(t-u)\,du\right) = f(s)g(s) \therefore \mathcal{L}^{-1}(f(s)g(s)) = \int_0^t F(u)G(t-u)\,du$

$f(s) = \frac{s}{(s^2+\omega^2)^2} \quad f(s) = g(s)k(s) \quad \therefore \frac{s}{(s^2+\omega^2)^2} = \frac{s}{s^2+\omega^2} \cdot \frac{1}{s^2+\omega^2} \therefore g(s) = \frac{s}{s^2+\omega^2}$,

$k(s) = \frac{1}{s^2+\omega^2} \therefore G(t) = \mathcal{L}^{-1}(g(s)) = \mathcal{L}^{-1}\left(\frac{s}{s^2+\omega^2}\right) = \cos(\omega t), \quad K(t) = \mathcal{L}^{-1}(k(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) =$

$\frac{1}{\omega}\mathcal{L}^{-1}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{1}{\omega}\sin(\omega t) \quad \because g(s)k(s) = \mathcal{L}(G(t) * K(t)) \therefore \mathcal{L}^{-1}(g(s)k(s)) = G(t) * K(t)$

$\therefore \mathcal{L}^{-1}(g(s)k(s)) = G(t) * K(t) = \mathcal{L}^{-1}\left(\frac{s}{(s^2+\omega^2)^2}\right) = \mathcal{L}^{-1}\left(\frac{s}{s^2+\omega^2} \cdot \frac{1}{s^2+\omega^2}\right) =$

$\mathcal{L}^{-1}(g(s)) * \mathcal{L}^{-1}(k(s)) = \mathcal{L}^{-1}\left(\frac{s}{s^2+\omega^2}\right) * \mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \cos(\omega t) * \frac{1}{\omega}\sin(\omega t) =$

$\int_0^t \cos(\omega u)\frac{1}{\omega}\sin(\omega(t-u))\,du = \int_0^t \frac{1}{\omega}\cos(\omega u)\sin(\omega(t-u))\,du = \frac{1}{\omega}\int_0^t \cos(\omega u)\sin(\omega t - \omega u)\,du =$

Sheet 3 $\frac{1}{\omega}\int_0^t \frac{1}{2}\left(\sin(\omega u+\omega t-\omega u) - \sin(\omega u-(\omega t-\omega u))\right)du =$

$\frac{1}{2\omega}\int_0^t \sin(\omega t) - \sin(2\omega u-\omega t)\,du = \frac{1}{2\omega}\left[\sin(\omega t)u + \frac{1}{2\omega}\cos(2\omega u-\omega t)\right]_0^t =$

$\frac{1}{2\omega}\left[\sin(\omega t)[t-0] + \frac{1}{2\omega}\cos(2\omega t-\omega t) - \frac{1}{2\omega}\cos(2\omega(0)-\omega t)\right] =$

$\frac{1}{2\omega}\left[\sin(\omega t)t + \frac{1}{2\omega}\cos(\omega t) - \frac{1}{2\omega}\cos(-\omega t)\right] = \frac{1}{2\omega}\left[t\sin(\omega t) + \frac{1}{2\omega}\cos(\omega t) - \frac{1}{2\omega}\cos(\omega t)\right] =$

$\frac{1}{2\omega}t\sin(\omega t)$ ✓

ii $F(s) = \frac{1}{s^2(s+1)^2} = \frac{1}{s^2}\frac{1}{(s+1)^2} \therefore \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = \mathcal{L}^{-1}\left(\frac{1!}{s^{1+1}}\right) = t^1 = t$

$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) = e^{-t}\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = e^{-t}t = te^{-t} \therefore$

$\mathcal{L}^{-1}\left(\frac{1}{s^2(s+1)^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\cdot\frac{1}{(s+1)^2}\right) = \mathcal{L}^{-1}(g(s)k(s)) = G(t) * K(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) * \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) =$

$t * te^{-t} = \int_0^t u(t-u)e^{-(t-u)}du = \int_0^t ute^{u-t} - u^2e^{u-t}du \therefore$

$\int_0^t ute^{u-t}du = \left[ut\frac{1}{1}e^{u-t}\right]_0^t - \int_0^t t\frac{1}{1}e^{u-t}du = \left[tte^{t-t} - 0\right] - \left[te^{u-t}\right]_0^t =$

$t^2e^0 - te^{t-t} \cancel{\;} = t^2 - t + te^{-t}$

$\int_0^t u^2e^{u-t}du = \left[u^2e^{u-t}\right]_0^t - \int_0^t 2ue^{u-t}du = t^2e^0 - 0 - \left[2ue^{u-t}\right]_0^t + \int_0^t 2e^{u-t}dt =$

$t^2 - 2te^0 + 0 + \left[2e^{u-t}\right]_0^t = t^2 - 2t + 2e^0 - 2e^{-t} = t^2 - 2t + 2 - 2e^{-t} \therefore$

$F(t) = \int_0^t ute^{u-t} - u^2e^{u-t}du = t^2 - t - t^2 + 2t - 2 + 2e^{-t} \cancel{\;} 2e^{-t} + t - 2 \cancel{+} e^{-t}(t+2) + t - 2$

$= 2e^{-t} + te^{-t} + t - 2 = e^{-t}(t+2) + t - 2$

$\int_0^t ute^{u-t} - u^2e^{u-t}du = \int_0^t ute^{u-t}du + \int_0^t -u^2e^{u-t}du \therefore$

$\int_0^t ute^{u-t}du = \left[ute^{u-t}\right]_0^t - \int_0^t te^{u-t}du = tte^{t-t} - \left[te^{u-t}\right]_0^t = t^2 - te^{t-t} + te^{0-t} = t^2 - t + te^{-t}$

$\int_0^t -u^2e^{u-t}du = \left[-u^2e^{u-t}\right]_0^t - \int_0^t -2ue^{u-t}du = -t^2e^{t-t} - \left[-2ue^{u-t}\right]_0^t + \int_0^t -2e^{u-t}du =$

$-t^2 + 2te^{t-t} + \left[-2e^{u-t}\right]_0^t = -t^2 + 2t - 2e^{t-t} + 2e^{0-t} = -t^2 + 2t - 2 + 2e^{-t} \therefore$

$\int_0^t ute^{u-t} - u^2e^{u-t}du = \underline{t^2} - \underline{\underline{t}} + te^{-t} - \underline{t^2} + \underline{\underline{2t}} - 2 + 2e^{-t} = t + te^{-t} - 2 + 2e^{-t} = e^{-t}(t+2) + t - 2$

iii $F(s) = \frac{1}{(s+1)^2(s^2+4)} = \frac{1}{(s+1)^2}\cdot\frac{1}{(s^2+4)} \therefore \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) = te^{-t}$

$\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = \mathcal{L}^{-1}\left(\frac{1}{2}\frac{2}{s^2+4}\right) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right) = \frac{1}{2}\sin(2t) \therefore \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2(s^2+4)}\right) =$

$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) * \mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = te^{-t} * \frac{1}{2}\sin(2t) = \frac{1}{2}\int_0^t ue^{-u}\sin(2(t-u))du = \frac{1}{2}\int_0^t ue^{-u}\sin(2t-2u)du$

$= 2I \therefore I = \int_0^t ue^{-u}\sin(2t-2u)du$ & $J = \int_0^t e^{-u}\sin(2t-2u)du =$

$\left[\frac{1}{2}e^{-u}\cos(2t-2u)\right]_0^t + \left[-\frac{1}{2}e^{-u}\sin(2t-2u)\right]_0^t - \frac{1}{2}J = J + \frac{2}{5}e^{-t} - \frac{2}{5}\cos(2t) + \frac{1}{5}\sin(2t) \therefore$

$U = ue^{-u}, V' = \sin(2t-2u) \therefore I = \left[\frac{1}{2}ue^{-u}\cos(2t-2u)\right]_0^t + \frac{1}{4}\left[(e^{-u} - ue^{-u})\sin(2t-2u)\right]_0^t +$

$\frac{1}{4}\int_0^t(2e^{-u} - ue^{-u})\sin(2t-2u)du = \frac{te^{-t}}{2} - \frac{1}{4}\sin(2t) + \frac{1}{2}J + \frac{1}{4}I \therefore$

$I = \frac{2}{5}te^{-t} + \frac{4e^{-2}}{(s-1)(s-2)} \cancel{= e^{-2s}} \cancel{\frac{1}{s-1}} \frac{1}{s-2} - \frac{4}{25}\cos(2t) - \frac{3}{25}\sin(2t)$

\1iv/ $\mathcal{L}^{-1}\left(\frac{e^{-3s}}{(s-1)(s-2)}\right) = \mathcal{L}^{-1}\left(e^{-3s}\frac{1}{(s-1)(s-2)}\right) = \mathcal{L}^{-1}\left(e^{-as}\hat{g}(s)\right) = \hat{F}(t-a)H(t-a)$ ∴
$\hat{F}(t-3)H(t-3)$ ∴

$\hat{g}(s) = \frac{1}{(s-1)(s-2)} = \frac{A}{s-1} + \frac{B}{s-2}$ ∴ $1 = A(s-2) + B(s-1)$ ∴ $1 = A(0) + B(2-1) = B = 1$

$1 = A(1-2) + B(0) = -A = 1$ ∴ $A = -1$ ∴ $\hat{g}(s) = -\frac{1}{s-1} + \frac{1}{s-2}$ ∴

$\hat{F}(t) = \mathcal{L}^{-1}(\hat{g}(s)) = -\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) + \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{t}1 + e^{2t}1 = e^{t} + e^{2t}$ ∴

$\mathcal{L}^{-1}[g(s)] = \hat{F}(t-3)H(t-3) = (e^{t-3} + e^{2(t-3)})H(t-3)$

\2a/ Fred nonlinear nonhomog 2nd kind

\2b/ 2nd kind linear volter homog

\2c/ 1st kind nonhomog linear Fred

\2d/ 2nd kind Fred linear nonhomog

\Sheet 4/ {1a/ $y(x) - \int_0^1 x^2 t^2 y(t)\,dt = 1$ let $p = \int_0^1 t^2 y(t)\,dt$ ∴

$y(x) = 1 + \int_0^1 x^2 t^2 y(t)\,dt = 1 + px^2$ ∴ $y(t) = 1 + pt^2$ ∴

$p = \int_0^1 t^2 y(t)\,dt = \int_0^1 t^2(1+pt^2)\,dt = \int_0^1 t^2 + pt^4\,dt \left[\frac{1}{3}t^3 + p\frac{1}{5}t^5\right]_0^1 = \frac{1}{3}(1^3 - 0^3) + p\frac{1}{5}(1^5 - 0^5) =$

$\frac{1}{3} - p\frac{1}{5} = p$ ∴ $\frac{1}{3} = \frac{4}{5}p$ ∴ $p = \frac{5}{12}$ ∴ $y(x) = 1 + \frac{5}{12}x^2$

\1b/ $y(x) - \int_0^1 xt\,y(t)\,dt = \sin x$ ∴ let $p = \int_0^1 t\,y(t)\,dt$ ∴ $y(x) = \sin(x) + x\int_0^1 t\,y(t)\,dt = \sin(x) + px$

∴ $y(t) = \sin(t) + pt$ ∴ $p = \int_0^1 t\,y(t)\,dt = \int_0^1 t(\sin(t) + pt)\,dt = \int_0^1 t\sin(t)\,dt + p\int_0^1 t^2\,dt =$

$\sin(1) - \cos(1) + \frac{p}{3} = p$ ∴ $p = \frac{3}{2}(\sin(1) - \cos(1))$ ∴ $y(x) = \sin(x) + \frac{3}{2}(\sin(1) - \cos(1))x$

\1c/ $y(x) - \int_0^1 \ln\left(\frac{x}{t}\right) y(t)\,dt = 1 = y(x) - \int_0^1 (\ln(x) - \ln(t))\,y(t)\,dt = y(x) - \ln(x)\int_0^1 y(t)\,dt + \int_0^1 \ln(t)y(t)\,dt = 1$

∴ let $P_1 = \int_0^1 y(t)\,dt$, $P_2 = \int_0^1 \ln(t)\,y(t)\,dt$ ∴ $y(x) = 1 + P_1\ln(x) - P_2$ ∴ $y(t) = 1 + P_1\ln(t) - P_2$

∴ $y(x) - \ln(x)P_1 + P_2 = 1$ ∴ $y(x) - \ln(x)\int_0^1 1 + P_1\ln(t) - P_2\,dt + \int_0^1 \ln(t)(1 + P_1\ln(t) - P_2)\,dt = 1$ ∴

$P_1 = \int_0^1 y(t)\,dt = \int_0^1 1 + P_1\ln(t) - P_2\,dt = \int_0^1 1\,dt + \int_0^1 P_1\ln(t)\,dt - \int_0^1 P_2\,dt = 1 + P_1\int_0^1 \ln(t)\,dt - P_2$

$P_2 = \int_0^1 \ln(t)\,y(t)\,dt = \int_0^1 \ln(t)(1 + P_1\ln(t) - P_2)\,dt = \int_0^1 \ln(t)\,dt + P_1\int_0^1 (\ln t)^2\,dt - P_2\int_0^1 \ln(t)\,dt$ ∴

$\int_0^1 \ln(t)\,dt = [t\ln t - t]_0^1 = -1$ & $\int_0^1 (\ln t)^2\,dt = [t(\ln t)^2]_0^1 - 2\int_0^1 \ln(t) = -2(-1) = 2$ ∴

$P_1 = 1 - P_1 - P_2$, $P_2 = -1 + 2P_1 + P_2$ ∴ $P_1 = \frac{1}{2}$, $P_2 = 0$ ∴

$y(x) = 1 + \ln(\sqrt{x}) = \ln(e\sqrt{x})$

\2a/ $y(x) - 2\int_0^x \cos(x-t)\,y(t)\,dt = 1$ ∴ Laplace: $\mathcal{L}(y(x)) = \hat{y}(s)$ ∴ $\mathcal{L}\left(y(x) - 2\int_0^x \cos(x-t)\,y(t)\,dt\right) =$

$\mathcal{L}(1) = \frac{1}{s} = \mathcal{L}(y(x)) - 2\mathcal{L}\left(\int_0^x \cos(x-t)\,y(t)\,dt\right) = \hat{y}(s) - 2\mathcal{L}\left(\int_0^x \cos(x-u)\,y(u)\,du\right) =$

$\hat{y}(s) - 2\mathcal{L}\left(\int_0^x y(u)\cos(x-u)\,du\right) = \hat{y}(s) - 2\mathcal{L}\left(\int_0^x F(u)G(x-u)\,du\right) = 2\hat{f}(s)\hat{g}(s) + \hat{y}(s) = \hat{y}(s) - 2\hat{y}(s)\hat{g}(s)$

$= \hat{y}(s) - 2\hat{y}(s)\mathcal{L}(\cos(x)) = \hat{y}(s) - 2\hat{y}(s)\frac{s}{s^2+1} = \hat{y}(s)\left(1 - \frac{2s}{s^2+1}\right) = \frac{1}{s}$ ∴

Sheet 4/ $\hat{y}(s)=\frac{1}{s}\frac{1}{1-\left(\frac{2s}{s^2+1}\right)}=\frac{1}{s}\frac{1}{\left(\frac{s^2-2s+1}{s^2+1}\right)}=\frac{s^2+1}{s(s-1)^2}=\frac{A}{s}+\frac{B}{s-1}+\frac{C}{(s-1)^2}\therefore$

$s^2+1=A(s-1)^2+B(s)(s-1)^2+C(s)\therefore$

$1^2+1=2=A(0)+B(0)+1C=C=2$

$0^2+1=A(0-1)^2+B(0)+2(0)=1+1A=A=1,$

$0s^3+Bs^3\therefore B=0\therefore\ \hat{y}(s)=\frac{1}{s}+2\frac{1}{(s-1)^2}\therefore \mathcal{L}^{-1}(\hat{y}(s))=y(x)=\mathcal{L}^{-1}\left(\frac{1}{s}\right)+2\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right)$

$=1+2e^{x}\mathcal{L}^{-1}\left(\frac{1!}{s^{1+1}}\right)=1+2e^{x}x^1=1+2xe^{x}=y(x)$

by deriv: $\frac{d}{dx}\int_{g(x)}^{h(x)}k(x,t)y(t)dt=k(x,x)y(x)+\int_{g(x)}^{h(x)}\frac{\partial k(x,t)}{\partial x}y(t)dt\therefore$

$\frac{d}{dx}(y(x))-2\frac{d}{dx}\left(\int_0^x\cos(x-t)y(t)dt\right)=\frac{d}{dx}(1)=0=$

$y'(x)-2\cos(x-x)y(x)-2\int_0^x-\sin(x-t)y(t)dt=y'(x)-2y(x)+2\int_0^x\sin(x-t)y(t)dt=0\therefore$

$\frac{d}{dx}(y'(x))-2\frac{dy(x)}{dx}+2\frac{d}{dx}\left(\int_0^x\sin(x-t)y(t)dt\right)=\frac{d}{dx}(0)=0=$

$y''(x)-2y'(x)+2\sin(x-x)y(x)+2\int_0^x\cos(x-t)y(t)dt=$

$y''(x)-2y'(x)+2\int_0^x\cos(x-t)y(t)dt=0\therefore\quad y(x)-1=2\int_0^x\cos(x-t)y(t)dt\therefore$

$y''(x)-2y'(x)+y(x)-1=0\therefore y''(x)-2y'(x)+y(x)=1\therefore q^2-2q+q=0=(q-1)^2\therefore q=1\therefore$

$Ae^{x}+Bxe^{x}=y_{CF}(x)\quad, y(0)-2\int_0^0\cos(0-t)y(t)dt=1=y(0)-0=y(0)=1$

$y'(0)-2y(0)+2\int_0^0\sin(0-t)y(t)dt=0=y'(0)-2(1)+0=y'(0)-2\therefore\ y'(0)=2\therefore$

$\therefore Ae^{x}+Bxe^{x}+y_{PI}(x)=y(x)\therefore$

let $y_{PI}(x)=ax+b\therefore\ y'_{PI}=a,\ y''_{PI}=0\therefore$

$0-2a+ax+b=1\quad .\ a=0\therefore -2a+b=1=b\quad\therefore\ y_{PI}(x)=0x+b=1\therefore$

$Ae^{x}+Bxe^{x}+1=y(x)\quad\therefore Ae^{0}+0+1=1=A+1\therefore\ A=0\ \therefore y(x)=Bxe^{x}+1\therefore$

$y'(x)=Be^{x}+Bxe^{x}\therefore 2=B+0=B\therefore\ y(x)=2xe^{x}+1$ ✓

2b/ $y(x)\mp\int_0^x\sin(x-t)y(t)dt=\sin x\quad\therefore \varepsilon=\pm1\therefore y(x)-\varepsilon\int_0^x\sin(x-t)y(t)dt=\sin x\therefore$

~~deriv~~ $y(x)-\sin x=\varepsilon\int_0^x\sin(x-t)y(t)dt\therefore$ deriv:

$y'(x)-\varepsilon\sin(x-x)y(x)-\varepsilon\int_0^x\cos(x-t)y(t)dt=\cos x=y'(x)-\varepsilon\int_0^x\cos(x-t)y(t)dt\therefore$

$y''(x)-\varepsilon\cos(x-x)y(x)-\varepsilon\int_0^x-\sin(x-t)y(t)dt=-\sin x=y''(x)-\varepsilon y(x)+\varepsilon\int_0^x\sin(x-t)y(t)dt$

$=y''(x)-\varepsilon y(x)+\varepsilon y(x)-\sin x=-\sin x\therefore\ y''(x)-\varepsilon y(x)+y(x)=0=y''(x)+y(x)(1-\varepsilon)$

$\therefore y(0)=0,\ y'(0)=1\quad\therefore$ when $\varepsilon=1$: $y''(x)=0\therefore q=0\therefore Ae^{0}+Bxe^{0}=A+Bx$

$\therefore 0=A+0=A\therefore y(x)=Bx\therefore B=1\therefore y(x)=1x=x$, when $\varepsilon=-1$: $y''(x)+2y(x)=0\therefore$

$q^2=-2\therefore y(x)=A\cos(\sqrt{2}x)+B\sin(\sqrt{2}x)\therefore A=0,\ \sqrt{2}B=1\therefore B=\frac{1}{\sqrt{2}}\therefore y(x)=\frac{1}{\sqrt{2}}\sin(\sqrt{2}x)$ ✓

Laplace: $\mathcal{L}(y(x)) - \varepsilon\mathcal{L}\left(\int_0^x \sin(x-t)y(t)\,dt\right) = \mathcal{L}(\sin x) = \frac{1}{s^2+1} =$

$\hat{y}(s) - \varepsilon\mathcal{L}\left(\int_0^x y(u)\sin(x-u)\,du\right) = \hat{y}(s) - \varepsilon\mathcal{L}\left(\int_0^x F(u)G(x-u)\,du\right) = \hat{y}(s) - \varepsilon f(s)g(s) =$

$\hat{y}(s) - \varepsilon\hat{y}(s)g(s) = \hat{y}(s) - \varepsilon\hat{y}(s)\mathcal{L}(\sin x) = \hat{y}(s) - \varepsilon\hat{y}(s)\frac{1}{s^2+1} = \frac{1}{s^2+1} \therefore$

$\hat{y}(s) = \frac{1}{s^2+1} + \varepsilon\frac{1}{s^2+1}\hat{y}(s) \therefore \quad \hat{y}(s) = \frac{1}{s^2+(1-\varepsilon)} \therefore, \ \hat{y}(s) = \frac{1}{s^2+2} \therefore$

when $\varepsilon = 1$: $\hat{y}(s) = \frac{1}{s^2}$, when $\varepsilon = -1$

$\mathcal{L}^{-1}(\hat{y}(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = y(x) = x$ for $\varepsilon = 1$,

when $\varepsilon = -1$: $\mathcal{L}^{-1}(\hat{y}(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^2+2}\right) = \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left(\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right) = \frac{1}{\sqrt{2}}\sin(\sqrt{2}x) = y(x)$

\2e/ $y(x) + \int_0^x \cos(x-t)y(t)\,dt = \cos x \quad \therefore -y(x) + \cos(x) = \int_0^x \cos(x-t)y(t)\,dt \therefore$

$y'(x) + \cos(x-x)y(x) + \int_0^x -\sin(x-t)y(t)\,dt = -\sin x = y'(x) + y(x) + \int_0^x -\sin(x-t)y(t)\,dt$

$\therefore y''(x) + y'(x) = \sin(x-x)y(x) - \int_0^x \cos(x-t)y(t)\,dt =$

$y''(x) + y'(x) + y(x) - \cos(x) = -\cos x \therefore y''(x) - y'(x) + y(x) = 0 \therefore$

$q^2 + q + 1 = 0 \quad \therefore \quad q = \frac{-1 \pm \sqrt{1-4(1)(1)}}{2(1)} = \frac{-1}{2} \pm \frac{\sqrt{-3}}{2} = -\frac{1}{2} \pm \frac{\sqrt{3}}{2}i \therefore$

$y(x) = A_2 e^{-\frac{1}{2}x + \frac{\sqrt{3}}{2}ix} + B_2 e^{-\frac{1}{2}x - \frac{\sqrt{3}}{2}ix} = e^{-\frac{1}{2}x}\left(A_2 e^{\frac{\sqrt{3}}{2}ix} + B_2 e^{-\frac{\sqrt{3}}{2}ix}\right) =$

$e^{-\frac{1}{2}x}\left(A\cos\left(\frac{\sqrt{3}}{2}x\right) + B\sin\left(\frac{\sqrt{3}}{2}x\right)\right) \therefore \ y(0) = 1, \ y'(0) + 1 + 0 = 0 \therefore y'(0) = -1 \therefore$

$y(0) = A = 1 \quad \therefore \quad y(x) = e^{-\frac{1}{2}x}\left(\cos\left(\frac{\sqrt{3}}{2}x\right) + B\sin\left(\frac{\sqrt{3}}{2}x\right)\right) \therefore$

$y'(x) = -\frac{1}{2}e^{-\frac{1}{2}x}\left(\cos\left(\frac{\sqrt{3}}{2}x\right) + B\sin\left(\frac{\sqrt{3}}{2}x\right)\right) + e^{-\frac{1}{2}x}\left(-\frac{\sqrt{3}}{2}\sin\left(\frac{\sqrt{3}}{2}x\right) + \frac{\sqrt{3}}{2}B\cos\left(\frac{\sqrt{3}}{2}x\right)\right) \therefore$

$y'(0) = -\frac{1}{2} + 1\left(+\frac{\sqrt{3}}{2}\right)B = -1 \therefore \frac{\sqrt{3}}{2}B = -\frac{1}{2} \therefore B = -\frac{1}{\sqrt{3}} \therefore$

$y(x) = e^{-\frac{1}{2}x}\left(\cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}x\right)\right)$

Laplace: $\mathcal{L}(y(x)) + \mathcal{L}\left(\int_0^x y(t)\cos(x-t)\,dt\right) = \mathcal{L}(\cos x) = \frac{s}{s^2+1} =$

$\hat{y}(s) + \hat{y}(s)\mathcal{L}(\cos x) = \hat{y}(s) + \hat{y}(s)\frac{s}{s^2+1} = \hat{y}(s)\left(1 + \frac{s}{s^2+1}\right) = \hat{y}(s)\left(\frac{s^2+s+1}{s^2+1}\right) = \frac{s}{s^2+1} \therefore$

$\hat{y}(s) = \frac{s}{s^2+s+1} = \frac{s}{(s+\frac{1}{2})^2+1-\frac{1}{4}} = \frac{s}{(s+\frac{1}{2})^2+\frac{3}{4}} = \frac{s+\frac{1}{2}-\frac{1}{2}}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} = \frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} - \frac{1}{2}\frac{1}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}$

$= \frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} - \frac{1}{\sqrt{3}}\frac{(\frac{\sqrt{3}}{2})}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2} \therefore$

$y(x) = \mathcal{L}^{-1}(y(s)) = \mathcal{L}^{-1}\left(\frac{(s+\frac{1}{2})}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right) - \frac{1}{\sqrt{3}}\mathcal{L}^{-1}\left(\frac{(\frac{\sqrt{3}}{2})}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right) =$

$e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{1}{\sqrt{3}}e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x\right) = e^{-\frac{1}{2}x}\left(\cos\left(\frac{\sqrt{3}}{2}x\right) - \frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}x\right)\right) = y(x)$

\3/ $\int_0^x f(x)\,dx = \frac{1}{k}f(x)x \quad \times \quad kxf(x) = \int_0^x f(t)\,dt \therefore$ diff:

$\frac{d}{dx}(kxf(x)) = \frac{d}{dx}\left(\int_0^x f(t)\,dt\right) = kf(x) + kxf'(x) = f(x)\frac{dx}{dx} - f(0)\frac{d}{dx}(0) = f(x) \therefore$

$kxf'(x) = f(x) - k(f(x)) = f(x)(1-k) \therefore \frac{f'(x)}{f(x)} = \frac{k}{1-k}x \therefore \int\frac{f'(x)}{f(x)}\,dx = \int\frac{k}{1-k}\frac{1}{x}\,dx =$

$\ln|f(x)| = \frac{k}{1-k}\ln|x| + C = \ln\left(|x|^{\frac{k}{1-k}}\right) + C \quad \therefore |f(x)| = C_2|x|^{\frac{k}{1-k}} \quad \therefore f(x) = C_3 x^{\frac{k}{1-k}}$

Sheet 4 / 4 / $\lambda y(x) - \int_0^1 (x-t)y(t)\,dt = 1 = \lambda y(x) - x\int_0^1 y(t)\,dt + \int_0^1 t\,y(t)\,dt = 1$

$P_1 = \int_0^1 y(t)\,dt$, $P_2 = \int_0^1 t\,y(t)\,dt$ $\therefore$ $\lambda y(x) = 1 + P_1 x - P_2$ $\therefore$ $\lambda y(t) = 1 + P_1 t + P_2$ $\therefore$

$y(t) = \frac{1}{\lambda} + \frac{1}{\lambda}P_1 + \frac{1}{\lambda}P_2$ $\therefore$ $P_1 = \int_0^1 y(t)\,dt = \int_0^1 \frac{1}{\lambda} + \frac{1}{\lambda}P_1 t - \frac{1}{\lambda}P_2\,dt$ $\therefore$

$\lambda P_1 = \int_0^1 1 + P_1 t - P_2\,dt = \left[t + \frac{1}{2}P_1 t^2 - P_2 t\right]_0^1 = \lambda P_1 = 1 + \frac{1}{2}P_1 - P_2$,

$P_2 = \int_0^1 t\,y(t)\,dt = \int_0^1 t\,\frac{1}{\lambda}(1 + P_1 t + P_2)\,dt$ $\therefore$ $\lambda P_2 = \int_0^1 t + P_1 t^2 + P_2 t\,dt = \left[\frac{1}{2}t^2 + \frac{1}{3}P_1 t^3 + \frac{1}{2}P_2 t^2\right]_0^1 =$

$\lambda P_2 = \frac{1}{2} + \frac{1}{3}P_1 - \frac{1}{2}P_2$ $\therefore$ $\quad \lambda P_1 = 1 + \frac{1}{2}P_1 - P_2$, $\lambda P_2 = \frac{1}{2} + \frac{1}{3}P_1 - \frac{1}{2}P_2$ $\therefore$

$\lambda \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \lambda P_1 \\ \lambda P_2 \end{bmatrix} = \lambda P = \begin{bmatrix} 1 + \frac{1}{2}P_1 - P_2 \\ \frac{1}{2} + \frac{1}{3}P_1 - \frac{1}{2}P_2 \end{bmatrix}$ $\therefore$ $\begin{bmatrix} 1 \\ 1/2 \end{bmatrix} = \begin{bmatrix} (\lambda - \frac{1}{2})P_1 + P_2 \\ -1/3 + (\lambda + \frac{1}{2})P_2 \end{bmatrix} = \begin{bmatrix} (\lambda - \frac{1}{2}) & 1 \\ -1/3 & \lambda + \frac{1}{2} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} =$

$\begin{bmatrix} \lambda - \frac{1}{2} & 1 \\ -\frac{1}{3} & \lambda + \frac{1}{2} \end{bmatrix} P = A(\lambda)P$ $\therefore$ syst has a unique sol $\forall \lambda \in \mathbb{R}$ $\therefore$

$\det(A(\lambda)) = \begin{vmatrix} \lambda - \frac{1}{2} & 1 \\ -\frac{1}{3} & \lambda + \frac{1}{2} \end{vmatrix} = (\lambda - \frac{1}{2})(\lambda + \frac{1}{2}) - \frac{1}{3}(-1) = \lambda^2 - \frac{1}{4} + \frac{1}{3} = \lambda^2 + \frac{1}{12} \neq 0$ $\therefore$

$\lambda \in \mathbb{R}$ $\therefore$ $\lambda^2 \geq 0$ $\therefore$ $\lambda^2 + \frac{1}{12} \geq \frac{1}{12} > 0$ $\therefore$ $\lambda^2 + \frac{1}{12} \neq 0$ $\therefore$

Z Sol is: $(A(\lambda))^{-1}\begin{bmatrix} 1 \\ 1/2 \end{bmatrix} = \frac{1}{\lambda^2 + \frac{1}{12}} \begin{bmatrix} \lambda + \frac{1}{2} & -1 \\ \frac{1}{3} & \lambda - \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ 1/2 \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix}$ $\therefore$

$P_1 = \frac{12\lambda}{12\lambda^2 + 1}$, $P_2 = \frac{6\lambda + 1}{12\lambda^2 + 1}$ $\therefore$ $\lambda y(x) = 1 + P_1 x + P_2$ $\therefore$ when $\lambda \neq 0$:

$y(x) = 1 + P_1 x + P_2 = 1 + \frac{12\lambda}{12\lambda^2 + 1}x + \frac{6\lambda^2 + 1}{12\lambda^2 + 1} = \frac{12x + 12\lambda - 6}{12\lambda^2 + 1}$

when $\lambda = 0$: $1 = -\int_0^1 (x-t)y(t)\,dt$ $\therefore$ add any sol to homog eqn to particular sol

y_p $\therefore$ $\int_0^1 (x-t)\,y(t)\,dt = x\int_0^1 y(t)\,dt - \int_0^1 t\,y(t)\,dt = 0 = x\hat{P}_1 + \hat{P}_2$ $\therefore$

compare coeffs: $\int_0^1 y(t)\,dt = 0$, $\int_0^1 t\,y(t)\,dt = 0$ $\therefore$

$y(x) = a + bx + x^2 g(x)$ cont $g \in C[0,1]$ $\therefore$ $y(t) = a + bt + t^2 g(t)$ $\therefore$

$x\int_0^1 a + bt + t^2 g(t)\,dt - \int_0^1 t(a + bt + t^2 g(t))\,dt = x\int_0^1 a + bt + t^2 g(t)\,dt - \int_0^1 at + bt^2 + t^3 g(t)\,dt = 0$

$y_p(x) = 12x - 6$, $y(x) = a + bx + x^2 g(x)$ $\therefore$

$x\int_0^1 y(t)\,dt - \int_0^1 t\,y(t)\,dt = 0$ $\therefore$ $a = 2\int_0^1 (3t^3 - 2t^2)g(t)\,dt$, $b = 6\int_0^1 (t^2 - 2t^3)g(t)\,dt$ $\therefore$

$\int_0^1 a + bt\,dt = \left[at + \frac{1}{2}bt^2\right]_0^1 = a + \frac{1}{2}b$ $\therefore$ $\int_0^1 at + bt^2\,dt = \left[\frac{1}{2}at^2 + \frac{1}{3}bt^3\right]_0^1 = \frac{1}{2}a + \frac{1}{3}b$ $\therefore$

$ax + \frac{1}{2}bx + x\int_0^1 t^2 g(t)\,dt - \frac{1}{2}a - \frac{1}{3}b - \int_0^1 t^3 g(t)\,dt = 0$

$a = \int_0^1 (6t^3 - 4t^2)g(t)\,dt$; $b = \int_0^1 (6t^2 - 12t^3)g(t)\,dt$

$y_p(t) = 12t - 6$ $\therefore$ $-\int_0^1 t\,y(t)\,dt = -\int_0^1 t(12t - 6)\,dt = \int_0^1 6t - 12t^2\,dt$

$x\int_0^1 y(t)\,dt - \int_0^1 t\,y(t)\,dt = 0$ $\therefore$ $\int_0^1 y(t)\,dt = 0$, $\int_0^1 t\,y(t)\,dt = 0$ $\therefore$

$y(x) = a + bx + x^2 f(x) \quad \therefore y(t) = a + bt + t^2 f(t) \therefore$

$\int_0^1 a + bt + t^2 f(t) \cancel{dt} = 0 = \int_0^1 a + bt\,dt + \int_0^1 t^2 f(t)\,dt = \cancel{[at}$

$\left[at + \frac{1}{2}bt^2\right]_0^1 + \int_0^1 t^2 f(t)\,dt = a + \frac{1}{2}b + \int_0^1 t^2 f(t)\,dt = 0$

$\int_0^1 t(a + bt + t^2 f(t))\,dt = 0 = \int_0^1 at + bt^2\,dt + \int_0^1 t^3 f(t)\,dt =$

$\left[\frac{1}{2}at^2 + \frac{1}{3}bt^3\right]_0^1 + \int_0^1 t^3 f(t)\,dt = \frac{1}{2}a + \frac{1}{3}b + \int_0^1 t^3 f(t)\,dt = 0 \therefore$

$a = -\frac{1}{2}b - \int_0^1 t^2 f(t)\,dt \therefore$

$\frac{1}{2}\left(-\frac{1}{2}b - \int_0^1 t^2 f(t)\,dt\right) + \frac{1}{3}b + \int_0^1 t^3 f(t)\,dt = 0 =$

$-\frac{1}{4}b - \frac{1}{2}\int_0^1 t^2 f(t)\,dt + \frac{1}{3}b + \int_0^1 t^3 f(t)\,dt = 0 = \frac{1}{12}b + \int_0^1 \left(t^3 - \frac{1}{2}t^2\right) f(t)\,dt \therefore$

$b = 6\int_0^1 (t^2 - 2t^3) f(t)\,dt \therefore$

$a = -\frac{1}{2} \times 6\int_0^1 (t^2 - 2t^3) f(t)\,dt - \int_0^1 t^2 f(t)\,dt = a = 2\int_0^1 (3t^2 - 2t^2) f(t)\,dt$

$\therefore$ GS for $\lambda = 0$:

$y(x) = 12x - 6 + x^2 f(x) + 2\int_0^1 (3t^2 - 2t^2) f(t)\,dt + 6x\int_0^1 (t^2 - 3t) f(t)\,dt$

for any choice of $f \in C[0,1]$

\5 sol/ $k(x,t) = g(x,t)\ln|x-t| \therefore g$ is bounded : $|k(x,t)| = |g(x,t)\ln|x-t|| \leq$

$|g(x,t)||\ln|x-t|| \leq C|\ln|x-t|| = C|\ln|x-t||\frac{|x-t|^\beta}{|x-t|^\beta} = C\frac{|\ln|x-t|||x-t|^\beta}{|x-t|^\beta}$

$\therefore$ L'Hospital's rule shows numerator is bounded as $t \to \infty \therefore$

$\lim_{y\to 0} y^\beta \ln(y) = 0$ for $y > 0,\ 0 < \beta < 1 \therefore \lim_{y\to 0} y^\beta \ln(y) = \lim_{y\to 0}\frac{\ln y}{y^{-\beta}} = \lim_{y\to 0}\frac{1/y}{-y^{-\beta-1}} =$

$-\lim_{y\to 0} y^\beta = 0 \therefore C\frac{|\ln|x-t|||x-t|^\beta}{|x-t|^\beta} \therefore \lim_{t\to x} C|\ln|x-t|||x-t|^\beta = C\lim_{t\to x}|\ln|x-t|||x-t|^\beta$

$= C\lim_{y\to 0}\ln(y)y^\beta = 0 \therefore \lim_{t\to x}|\ln|x-t|||x-t|^\beta = 0 \therefore$

$|x-t|^\beta \ln|x-t|$ is bounded near $x = t \therefore g(x,t)\ln|x-t|$ is weakly singular

\6/ $y''(x) - y(x) - \int_0^1 e^{x-t}y(t)\,dt = 0 \quad y(0) = y_0,\ y(1) = y_1 \therefore$

$y''(x) - y(x) = e^x\int_0^1 e^{-t}y(t)\,dt \therefore \int_0^1 e^{-t}y(t)\,dt = p \therefore$

$y''(x) - y(x) = e^x p \quad \therefore \cancel{y} \; q^2 - 1 = 0 \quad \therefore q = 1, -1 \therefore \; y_{CF} = Ae^x + Be^{-x} \therefore$

$axe^x + be^x = y \therefore y'' = ae^x + axe^x + be^x \therefore$

$ae^x + axe^x + be^x - axe^x - be^x = e^x p = ae^x \quad \therefore p = a \therefore$

$y(x) = Ae^x + Be^{-x} \quad \therefore y(0) = y_0,\ y(1) = y_1 \therefore 3p - 4A - 2B(1 - e^{-2}) = 0,\ A + B = y_0 \therefore$

$ep + 2eA + 2e^{-1}B = 2y_1 \therefore p = \frac{1}{e(2e^2-1)}(ey_0(e^2-3) + y_1(e^2+1)) \quad A = \frac{1}{2(2e^2-1)}(-y_0(e^2+2) + 3ey_1)$

$B = \frac{e(5ey_0 - 3y_1)}{2(2e^2-1)} \therefore y(x) = \frac{1}{2(2e^2-1)}\left[e^x(-y_0(e^2+2) + 3ey_1) + e^{1-x}(5ey_0 - 3y_1) + xe^{x-1}(ey_0(e^2-3) + y_1(e^2+1))\right]$

\Sheet 5/ \1a/ $y(x)-\int_{-1}^{1}(\frac{1}{2}+xt)y(t)dt=f(x)$ ∴ let $a=\int_{-1}^{1}y(t)dt$, $b=\int_{-1}^{1}ty(t)dt$ ∴

$y(x)=\frac{1}{2}a+bx$ ∴ Z IE is rank 2 ∴

$y(x)+\int_{-1}^{1}-\frac{1}{2}y(t)dt+\int_{-1}^{1}-xty(t)dt=f(x)$ ∴

$y(x)-\frac{1}{2}\int_{-1}^{1}y(t)dt-x\int_{-1}^{1}ty(t)dt=f(x)$ ∴

$y(x)-\frac{1}{2}a-xb=f(x)$ ∴

$y(x)=\frac{1}{2}a-xb$ ∴ $y(t)=\frac{1}{2}a-tb$ ∴

$a=\int_{-1}^{1}y(t)dt=\int_{-1}^{1}\frac{1}{2}a-tb\,dt=\left[\frac{1}{2}at-\frac{1}{2}t^2b\right]_{-1}^{1}=$

$\frac{1}{2}a(1--1)-\frac{1}{2}(1^2-(-1)^2)b=\frac{1}{2}a(2)-\frac{1}{2}(1-1)b=a$ ∴ a is arbit

$b=\int_{-1}^{1}ty(t)dt=\int_{-1}^{1}t(\frac{1}{2}a-tb)dt=\int_{-1}^{1}\frac{1}{2}at-bt^2dt=\left[\frac{1}{4}at^2-\frac{1}{3}bt^3\right]_{-1}^{1}=$

$\frac{1}{4}a(1^2-(-1)^2)-\frac{1}{3}b(1^3-(-1)^3)=\frac{1}{4}a(1-1)-\frac{1}{3}b(1--1)=-\frac{1}{3}b(1+1)=-\frac{2}{3}b$ ∴ $\frac{5}{3}b=0$ ∴ $b=0$ ∴

Z const funcs are sols of Z homog eqns Z ∴ we are in Z 2nd alternative.

\1b/ Z adjoint eqn is Z same, ∴ Z const funcs are Z sols of Z homog adjoint eqn.

\1c/ from (b) know Z IE has a sol if $\int_{-1}^{1}f(t)z(t)dt=c\int_{-1}^{1}f(t)dt=0$ for

$f_1(x)=3x-1$: $\int_{-1}^{1}(3x-1)dx=-2\neq 0$ ∴ ∄ no sols to Z IE, ~~when Z $f_2(x)$~~

when $f_2(x)=3x^2-1$ ∴ $\int_{-1}^{1}f_2(x)dx=\int_{-1}^{1}3x^2-1\,dx=0$ ∴ ∃ infinite number of sols, we know f_2 is a particular sol, only need to add Z sols of Z homog eqn: $y(x)=f_2(x)+c=3x^2-1+c$ for $c\in\mathbb{R}$

\2a/ $k(x,t)=1+\cos(\pi x)\cos(\pi t)$ $C([0,1])$ $y=Ky$ ∴ let

$y(x)=\int_0^1 k(x,t)y(t)dt=\int_0^1 y(t)dt+\cos(\pi x)\int_0^1\cos(\pi t)y(t)dt=y(x)$ ∴

$a=\int_0^1 y(t)dt$, $b=\int_0^1 y(t)\cos(\pi t)dt$ ∴

$y(x)=a+\cos(\pi x)b$ ∴ $y(t)=a+\cos(\pi t)b$ ∴

$a=\int_0^1 y(t)dt=\int_0^1 a+b\cos(\pi t)dt=a$ ∴ a is arbit

$b=\int_0^1 y(t)\cos(\pi t)dt=\int_0^1(a+\cos(\pi t)b)\cos(\pi t)dt=\frac{b}{2}$ ∴ $b=0$ ∴

Z const funcs $y\equiv a$ are sols of Z homog IE

\2b/ $k(x,t)=1+\cos(\pi x)\cos(\pi t)$ ∴ $k(t,x)=1+\cos(\pi t)\cos(\pi x)=$

$1+\cos(\pi x)\cos(\pi t)=k(x,t)$ ∴ Z IE is self adjoint ∴ $k^*=k$ ∴ Z const funcs are also Z sols of Z homog adjoint eqn ∴

2c/ need $\int_0^1 f(t)=0$ $\therefore$ for $f(x)=x^2$: $f(t)=t^2$ $\therefore$

$\int_0^1 f(t)\,dt=\int_0^1 t^2\,dt=\left[\frac{1}{3}t^3\right]_0^1=\frac{1}{3}\neq 0$ $\therefore$ $f(x)=x^2$ doesn't meet this condition

2d/ $y=f+Ky$ $\cancel{k(x,t)=}$ $\therefore$ $k(x,t)=1+\cos(\pi x)\cos(\pi t)$ $\therefore$

$a=\int_0^1 y(t)\,dt$, $b=\int_0^1 y(t)\cos(\pi t)\,dt$.

$y(x)=\int_0^1 (f(t)+a+b\cos\pi t)\cos(\pi t)\,dt=\int_0^1 f(t)\cos(\pi t)\,dt=\frac{b}{2}$ $\therefore$

$y(x)=f(x)+\int_0^1 (1+\cos(\pi x)\cos(\pi t))y(t)\,dt=$

$f(x)+\int_0^1 y(t)\,dt+\cos(\pi x)\int_0^1 \cos(\pi t)y(t)\,dt=f(x)+a+\cos(\pi x)b=y(x)$ $\therefore$

$y(t)=f(t)+a+\cos(\pi t)b$ $\therefore$

$a=\int_0^1 y(t)\,dt=\int_0^1 (f(t)+a+b\cos(\pi t))\,dt=\int_0^1 f(t)\,dt+a=a$,

$b=\int_0^1 \cos(\pi t)y(t)\,dt=\int_0^1 (f(t)+a+b\cos(\pi t))\cos(\pi t)\,dt=\int_0^1 f(t)\cos(\pi t)\,dt=\frac{b}{2}$

$b=2\int_0^1 f(t)\cos(\pi t)\,dt$ $\therefore$ Z final form for y is

$y(x)=f(x)+2\left(\int_0^1 f(t)\cos(\pi t)\,dt\right)\cos(\pi x)+C$, C is arbit const

3a/ $x\in[0,1]$ $\therefore$ $\int_0^1 |k(x,t)|\,dt=\int_0^x |k(x,t)|\,dt+\int_x^1 |k(x,t)|\,dt=$

$\int_0^x (1-x)\,dt+\int_x^1 (1-t)\,dt=\frac{1}{2}(1-x^2)$ $\therefore$ $\max\left|\int_0^1 k(x,t)\,dt\right|=\max\left|\frac{1}{2}(1-x^2)\right|=\frac{1}{2}\max|(1-x^2)|$

$=\frac{1}{2}\times 1=\frac{1}{2}<1$ $\therefore$ $\|k\|_\infty=\frac{1}{2}<1$ $\therefore$ $\int_0^1 |k(x,t)|\,dt<1$

3b/ $\cancel{y(x)=}$ $y-Ky=1$ $\therefore$ Z IE : $y(x)-\int_0^1 y(t)\,dt+x\int_0^x y(t)\,dt+\int_x^1 t\,y(t)\,dt=f(x)$

$\therefore$ $y(x)-\int_0^1 k(x,t)y(t)\,dt=y(x)-\int_0^x (1-x)y(t)\,dt-\int_x^1 (1-t)y(t)\,dt=$

$y(x)-\int_0^x y(t)\,dt+x\int_0^x y(t)\,dt-\int_x^1 y(t)\,dt+\int_x^1 t\,y(t)\,dt=$

$y(x)-\int_0^1 y(t)\,dt+x\int_0^x y(t)\,dt+\int_x^1 t\,y(t)\,dt=f(x)$ $\therefore$ deriv:

$y'(x)+\int_0^x y(t)\,dt+x\,y(x)-x\,y(x)=f'(x)=y'(x)+\int_0^x y(t)\,dt$,

$y(0)-\int_0^1 y(t)\,dt+\int_0^1 t\,y(t)\,dt=f(0)$, $y(0)=f(0)+\int_0^1 (1-t)y(t)\,dt$

$f'(0)=y'(0)$ $\therefore$ deriv:

$y''(x)+y(x)=f''(x)$ $\therefore$ for $f(x)=1$: $y''(x)+y(x)=0$, $y(0)=1+\int_0^1 (1-t)y(t)\,dt$ &

$y'(0)=0$ $\therefore$ $q^2+1=0$ $\therefore$ $q^2=-1$ $\therefore$ $q=\pm i$ $\therefore$ $y(x)=A\cos x+B\sin x$ $\therefore$

$y'(x)=-A\sin x+B\cos x$ $\therefore$ $y'(0)=B=0$ $\therefore$ $y(x)=A\cos x$ $\therefore$

$A=1+\int_0^1 (1-t)A\sin t\,dt=$ $A=1+A\sin(1)-A\sin(1)+A\int_0^1 \sin(t)\,dt=$

$1-A\cos(1)+A$ $\therefore$ $A=\frac{1}{\cos(1)}$ $\therefore$ $y(x)=\frac{\cos x}{\cos(1)}$ $\therefore$

~~3z~~

\Sheet 5\ \3c\ $f(x)=x \therefore f'(x)=1 \therefore f''(x)=0 \therefore y''(x)+y(x)=0 \therefore$

IC: $y(0)=\int_0^1 (1-t)y(t)\,dt$, $y'(x)+\int_0^x y(t)\,dt = f'(x) \therefore y'(0)=1 \therefore$

$y''(x)+y(x)=f''(x)=0=y''(x)+y(x)=0 \therefore$

$y(x)=A\cos x+B\sin x \therefore$

$y'(x)=-A\sin x+B\cos x \therefore y'(0)=-A(0)+B=B=1 \therefore$

$y(x)=A\cos x+\sin x \therefore y(t)=A\cos t+\sin t \therefore$

$y(0)=A(1)+0=A=\int_0^1 (1-t)(A\cos t+\sin t)\,dt=\int_0^1 A\cos t+\sin t\,dt+\int_0^1 -At\cos t-t\sin t\,dt$

$\therefore A=\frac{1-\sin(1)}{\cos(1)} \therefore y(x)=\left(\frac{1-\sin(1)}{\cos(1)}\right)\cos x+\sin x$

\4\ $y-Ky=f$ $k(x,t)=\frac{1}{2}+x(3t^2-1) \therefore y(x)-\int_{-1}^1 \left(\frac{1}{2}+x(3t^2-1)\right)y(t)\,dt=f(x)$

st. interested in Z sols of Z corres homog eqns, if a homog eqn has non trivial sols, we know that sols of Z adjoint homog eqn exist iff f is orthogonal to Z sols of Z adjoint homog eqn $\therefore$

homog eqn: $y(x)-\int_{-1}^1 \left[\frac{1}{2}+x(3t^2-1)\right]y(t)\,dt=0$ this eqn is of finite rank 2 $\therefore$

$y(x)-\int_{-1}^1 \frac{1}{2}y(t)+3xt^2 y(t) - xy(t)\,dt=y(x)-\left(\frac{1}{2}-x\right)\int_{-1}^1 y(t)\,dt-3x\int_{-1}^1 t^2y(t)\,dt=0 \therefore$

Let $P_1=\int_{-1}^1 y(t)\,dt$, $P_2=\int_{-1}^1 (3t^2-1)y(t)\,dt \therefore y(x)=\frac{1}{2}P_1+P_2x \therefore y(t)=\frac{1}{2}P_1+P_2t \therefore$

$P_1=\int_{-1}^1 \frac{1}{2}P_1+P_2t\,dt=P_1$, $P_2=\int_{-1}^1 (3t^2-1)\left(\frac{1}{2}P_1+P_2x\right)dt=0 \therefore y(x)=\frac{1}{2}P_1 \therefore$

$\exists$ nontrivial sols of Z eqn which are $y(x)=\frac{1}{2}P_1$, P_1 is arbit const $\therefore$ Z adjoint eqn has kernel $\cancel{k^*(x,t)=\frac{1}{2}t(3x^2-1)}$ $\therefore k^*(x,t)=\frac{1}{2}+t(3x^2-1) \therefore$ using Z same techniques: can calc that $y^*=K^*y^*$ has Z sol $y^*(x)=1 \therefore$

$\{ y^*(x)=\int_{-1}^1 \left[\frac{1}{2}+t(3x^2-1)\right]y^*(t)\,dt \therefore P_1^*=\int_{-1}^1 y(t)\,dt$, $P_2^*=\int_{-1}^1 ty(t)\,dt \therefore$

$y^*(x)=\frac{1}{2}P_1^*+(3x^2-1)P_2^* \therefore y^*(t)=\frac{1}{2}P_1^*+(3t^2-1)P_2^* \therefore P_1^*=\int_{-1}^1 \frac{1}{2}P_1^*+3P_2^*t^2-P_2^*\,dt=$

$\left[\frac{1}{2}P_1^*t+P_2^*t^3-P_2^*t\right]_{-1}^1=P_1^*+2P_2^*-2P_2^*=P_1^*$, $P_2^*=\int_{-1}^1 t\left(\frac{1}{2}P_1^*+3P_2^*t^2-P_2^*\right)dt=\int_{-1}^1 \frac{1}{2}P_1^*t+3P_2^*t^3-P_2^*t\,dt$

$=\left[\frac{1}{4}P_1^*t^2+\frac{3}{4}P_2^*t^4-\frac{1}{2}P_2^*t^2\right]_{-1}^1=\frac{1}{4}P_1^*(0)+\frac{3}{4}P_2^*(0)-\frac{1}{2}P_2^*(0)=0 \therefore$

$y^*(x)=\frac{1}{2}P_1^*$ P_1^* is arbit const $\therefore$ Choose $P_1^*=2 \therefore y^*(x)=\frac{1}{2}(2)=1 \therefore$

$y(x)=\frac{1}{2}P_1$, $y^*(x)=1$, $y(x)-\int_{-1}^1 \left[\frac{1}{2}+x(3t^2-1)\right]y(t)\,dt=f(x) \therefore$

$y(x)=f(x)+\int_{-1}^1 \left[\frac{1}{2}+x(3t^2-1)\right]y(t)\,dt \therefore$ Z homog eqn has non trivial sols, & Z adjoint homog eqn has sols $y^*(x)=1 \therefore f$ is orthogonal to Z $y^*(x)=1 \therefore$

$\int_{-1}^1 f(x)\cdot 1\,dx=\int_{-1}^1 f(x)=0 \therefore$ we can check those solvability & sol results given by Z Fredholm alternative are correct. Z non homog IE is of

Stationary rank 2 $\therefore$ can solve $\because$ $P_1 = \int_{-1}^{1} y(t)\,dt$, $P_2 = \int_{-1}^{1}(3t^2-1)y(t)\,dt$ $\therefore$

$y(x) = \frac{1}{2}P_1 + P_2 x + f(x)$ $\therefore$ $y(t) = \frac{1}{2}P_1 + P_2 t + f(t)$ $\therefore$

$P_1 = \int_{-1}^{1} y(t)\,dt = \int_{-1}^{1} \frac{1}{2}P_1 + P_2 t + f(t)\,dt = \left[\frac{1}{2}P_1 t + \frac{1}{2}P_2 t^2\right]_{-1}^{1} + \int_{-1}^{1} f(t)\,dt = P_1 + \int_{-1}^{1} f(t)\,dt = P_1 + \int_{-1}^{1} f(x)\,dx$ $\therefore$

$P_2 = \int_{-1}^{1}(3t^2-1)y(t)\,dt = \int_{-1}^{1}(3t^2-1)\left(\frac{1}{2}P_1 + P_2 t + f(t)\right)dt = \int_{-1}^{1}\frac{3}{2}P_1 t^2 + 3P_2 t^3 - \frac{1}{2}P_1 - P_2 t\,dt + \int_{-1}^{1}(3t^2-1)f(t)\,dt$

$= \left[\frac{1}{2}P_1 t^3 + \frac{3}{4}P_2 t^4 - \frac{1}{2}P_1 t - \frac{1}{2}P_2 t^2\right]_{-1}^{1} + \int_{-1}^{1}(3t^2-1)f(t)\,dt = P_1 + \frac{3}{4}P_2(0) - P_1 - \frac{1}{2}P_2(0) + \int_{-1}^{1}(3t^2-1)f(t)\,dt$

$= \int_{-1}^{1}(3t^2-1)f(t)\,dt = \int_{-1}^{1}(3x^2-1)f(x)\,dx$ $\therefore$

$P_1 = \int_{-1}^{1} f(x)\,dx + P_1$, $P_2 = \int_{-1}^{1}(3x^2-1)f(x)\,dx$ $\therefore$

See $\int_{-1}^{1} f(x)\,dx = 0$ is a necessary condition for the first eqn to hold, & the 2nd: $P_2 = \int_{-1}^{1}(3x^2-1)f(x)\,dx$ $\because$ $\exists$ no conds on P_1 $\therefore$ is an arbit const param (representing the sol of the homog eqn) $\therefore$

$y(x) = \frac{1}{2}P_1 + P_2 x + f(x) = P_1 + P_2 x + f(x) = P_1 + x\int_{-1}^{1}(3t^2-1)f(t)\,dt + f(x) =$

$P_1 + 3x\int_{-1}^{1} t^2 f(t)\,dt - x\int_{-1}^{1} f(t)\,dt + f(x) = P_1 + 3x\int_{-1}^{1} t^2 f(t)\,dt - x(0) + f(x) = f(x) + 3x\int_{-1}^{1} t^2 f(t)\,dt + P_1$

\5/ Separable $\therefore$ $\lambda y(x) - \int_a^b k_1(x)k_2(t)y(t)\,dt = f(x) = \lambda y(x) - k_1(x)\int_a^b k_2(t)y(t)\,dt$

$\therefore$ $\lambda \neq \int_a^b k_1(t)k_2(t)\,dt$ has exactly one sol $y(x) = \frac{1}{\lambda}\left(f(x) + \frac{\int_a^b f(t)k_2(t)\,dt}{\lambda - \int_a^b k_1(t)k_2(t)\,dt}k_1(x)\right)$ $\because$

$\because$ $\lambda y(x) = f(x) + k_1(x)\int_a^b k_2(t)y(t)\,dt$ $\therefore$ $y(x) = \frac{1}{\lambda}f(x) + \frac{1}{\lambda}k_1(x)\int_a^b k_2(t)y(t)\,dt$ $\therefore$ $P_1 = \int_a^b k_2(t)y(t)\,dt$ $\therefore$

$y(t) = \frac{1}{\lambda}f(t) + \frac{1}{\lambda}k_1(t)P_1$ $\therefore$ $P_1 = \int_a^b k_2(t)\left[\frac{1}{\lambda}f(t) + \frac{1}{\lambda}k_1(t)P_1\right]dt =$

$\frac{1}{\lambda}\int_a^b k_2(t)f(t)\,dt + \frac{P_1}{\lambda}\int_a^b k_1(t)k_2(t)\,dt$ $\therefore$ $P_1\left(1 - \frac{1}{\lambda}\int_a^b k_1(t)k_2(t)\,dt\right) = \frac{1}{\lambda}\int_a^b k_2(t)f(t)\,dt$ $\therefore$

$P_1\left(\lambda - \int_a^b k_1(t)k_2(t)\,dt\right) = \int_a^b k_2(t)f(t)\,dt$ $\therefore$ $P_1 = \frac{\int_a^b k_2(t)f(t)\,dt}{\lambda - \int_a^b k_1(t)k_2(t)\,dt}$ $\therefore$

$y(x) = \frac{1}{\lambda}\left(f(x) + \frac{\int_a^b k_2(t)f(t)\,dt}{\lambda - \int_a^b k_1(t)k_2(t)\,dt}k_1(x)\right)$ $\therefore$

$|y(x)| = \left|\frac{1}{\lambda}\left(f(x) + \frac{\int_a^b k_2(t)f(t)\,dt}{\lambda - \int_a^b k_1(t)k_2(t)\,dt}k_1(x)\right)\right| \leq \frac{1}{\lambda}\left(|f(x)| + \left|\frac{\int_a^b k_2(t)f(t)\,dt}{\lambda - \int_a^b k_1(t)k_2(t)\,dt}k_1(x)\right| =\right.$

$\frac{1}{\lambda}\left(|f(x)| + \frac{|\int_a^b k_2(t)f(t)\,dt|}{|\lambda - \int_a^b k_1(t)k_2(t)\,dt|}|k_1(x)|\right) \leq \frac{1}{\lambda}\left(\|f(x)\|_\infty + \frac{\|f\|_\infty\int_a^b |k_2(t)|\,dt}{|\lambda - \int_a^b k_1(t)k_2(t)\,dt|}\|k_1\|_\infty\right) =$

$\cancel{\frac{1}{\lambda}\left(1 + \int_a^b |k_2(t)|\,dt\right.}$ $\frac{1}{\lambda}\left(1 + \frac{\int_a^b |k_2(t)|\,dt}{|\lambda - \int_a^b k_1(t)k_2(t)\,dt|}\|k_1\|_\infty\right)\|f\|_\infty = C\|f\|_\infty$ $\therefore$

$C = \frac{1}{|\lambda|}\left(1 + \frac{\|k_1\|_\infty\|k_2\|_1}{|\lambda - \int_a^b k_1(t)k_2(t)\,dt|}\right) = \frac{1}{|\lambda|}\left(1 + \frac{\|k_1\|\|k_2\|_1}{|\lambda - \int_a^b k(t,t)\,dt|}\right)$, $\|k_2\|_1 = \int_a^b |k_2(t)|\,dt$

\Week 7 Sheet/ $\|k\|_\infty = \max_{x\in[a,b]}\int_a^b |k(x,t)|\,dt$ $\|k\|_\infty = \max_{x\in[a,b]}\int_a^b |k(x,t)|\,dt$

$\|k\|_\infty = \max_{x\in[a,b]}\int_a^b |k(x,t)|\,dt$ $\|k\|_\infty = \max_{x\in[a,b]}\int_a^b |k(x,t)|\,dt$ $\max_{x\in[a,b]}\int_a^b |k(x,t)|\,dt$ $\|k\|_\infty = \max_{x\in[a,b]}\int_a^b |k(x,t)|\,dt$

\Week 7 sheet/ \1a/ $\|K\|_\infty = \max_{x\in[a,b]} \int_a^b |k(x,t)|dt \therefore k(x,t)=\cos(x-t) \therefore$

$\|K\|_\infty = \max_{x\in[0,a]} \int_0^a |\cos(x-t)|dt = \max_{x\in[0,a]} \int_{x-a}^{x} |\cos u|\,du = \int_0^a |\cos t|\,dt$

$u = x-t \therefore \frac{du}{dt} = -1 \therefore -du = dt \therefore t=0 \to x-0=x=u,\ t=a \to x-a=u \therefore$

$\int_x^{x-a} |\cos u|(-1)du = \int_{x-a}^{x} |\cos u|du \quad \max_{x\in[0,a]} \int_{x-a}^{x} |\cos u|du = \int_{a-a}^{a} |\cos u|du = \int_0^a |\cos u|du$

\1b/ $(Ku)(x) = \int_0^1 e^{xt}u(t)dt \therefore \|K\|_\infty = \max_{x\in[0,1]} \int_0^1 |e^{xt}|dt = \max_{x\in[0,1]} \int_0^1 e^{xt}\,dt =$

$\max_{x\in[0,1]} \left[\frac{e^x-1}{x}\right]$ & Z formula also applies when $x=0 \therefore$

Z func $\frac{e^x-1}{x}$ is increasing $\therefore$ its max on $[0,1]$ is e^1, Z norm of K

\1c/ $(Ku)(x) = \int_{-1}^1 xtu(t)dt \therefore \|K\|_\infty = \max_{x\in[-1,1]} \int_{-1}^1 |xt|dt = 2\max_{x\in[-1,1]} \int_0^1 |x|t\,dt =$

$2\max_{x\in[-1,1]} |x| \int_0^1 t\,dt = 2\max_{x\in[-1,1]} |x| \left[\frac{1}{2}t^2\right]_0^1 = \max_{x\in[-1,1]} |x| = 1$

\2/ $k(x,t) = \cos(x+t) \quad h(x,t) = \sin(x+t)$ kernel of KH is $\int_0^1 k(x,s)h(s,t)ds =$

$\int_0^1 \cos(x+s)\sin(s+t)ds = \frac{1}{2}\int_0^1 \sin(x-t) + \sin(x+2s+t)ds = \frac{1}{2}\sin(x-t) + \frac{1}{2}\int_0^1 \sin(2s+x+t)ds =$

$\frac{1}{2}\sin(x-t) - \frac{1}{4}\left[\cos(2s+x+t)\right]_0^1$

$\int_0^1 k(x,s)h(s,t)\,ds = \int_0^1 \cos(x+s)\sin(s+t)ds = \frac{1}{2}\sin(1)\sin(x+t+1) + \frac{1}{2}\sin(x-t)$

for $HK: \int_0^1 h(x,s)k(s,t)ds = \int_0^1 \sin(x+s)\cos(s+t)\,ds =$

$\frac{1}{2}\sin(1)\sin(x+t+1) - \frac{1}{2}\sin(x-t)$, Z ~~two~~ operators are symmetric but don't compute

\3a/ $y(x) - \int_0^1 xt^3y(t)dt = x^2 \quad 0\le x\le 1 \therefore y(x) - x\int_0^1 t^3y(t)\,dt = x^2 = y(x) - xp,\ p = \int_0^1 t^3y(t)dt \therefore$

$x^2 + xp = y(x) \therefore y(t) = t^2 + pt \therefore p = \int_0^1 t^3y(t)dt = \int_0^1 t^3(t^2+pt)dt = \frac{1}{6}$ ~~$(1-0)$~~ $=$

$\frac{1}{6} + \frac{1}{5}p = p \therefore p = \frac{5}{24} \therefore \quad y(x) = x^2 + \frac{5}{24}x$

\3b/ $y(x) - \int_0^1 xt^3y(t)dt = x^2 \therefore y(x) = x^2 + \int_0^1 xt^3y(t)dt \therefore y_0(x) = x^2 \therefore$

$y_1(x) = x^2 + x\int_0^1 t^3y_0(t)dt = x^2 + x\int_0^1 t^3t^2dt = x^2 + \frac{1}{6}x = y_1(x) \therefore y_1(t) = t^2 + \frac{1}{6}t \therefore$

$y_2 = x^2 + x\int_0^1 t^3(y_1(t))dt = x^2 + x\int_0^1 t^3\left(t^2 + \frac{t}{6}\right)dt = x^2 + \frac{x}{5} = y_2(x) \therefore y_2(t) = t^2 + \frac{t}{5} \therefore$

$y_3(x) = x^2 + x\int_0^1 t^3y_2(t)dt = \int_0^1 t^3\left(t^2 + \frac{t}{5}\right)dt = x^2 + \frac{31}{150}x$

\3c/ $\|K\|_\infty = \frac{1}{4} \because \|K\|_\infty = \max_{x\in[0,1]} \int_0^1 |xt^3|dt = \max_{x\in[0,1]} |x|\int_0^1 t^3\,dt = \max_{x\in[0,1]} |x|\frac{1}{4} = \frac{1}{4} \therefore$

Z error for y_3 is: $\|y-y_3\|_\infty \le \frac{1}{|\lambda| - \|K\|_\infty}\left(\frac{\|K\|_\infty}{|\lambda|}\right)^{3+1}\|f\|_\infty \therefore \|f\|_\infty = \max_{x\in[0,1]} x^2 = 1^2 = 1 \therefore$

$\lambda = 1 \therefore |\lambda| = 1 \therefore \|y-y_3\|_\infty \le \frac{1}{1-\frac{1}{4}}\left(\frac{(\frac{1}{4})}{1}\right)^4 1 = \frac{1}{3\times 4^3} = \frac{1}{192}$

\3d/ let $y - Ky = f(x) \therefore y(x) = f(x) + x\int_0^1 t^3y(t)dt \therefore p = \int_0^1 t^3y(t)dt \therefore$

$y(x) = f(x) + xp \therefore y(t) = f(t) + tp \therefore$ ~~$y(x)$~~ $p = \int_0^1 t^3(f(t) + pt)\,dt = \int_0^1 t^3f(t)dt + p\int_0^1 t^4dt$

$\therefore p = \int_0^1 t^3f(t)dt + p\left[\frac{1}{5}t^5\right]_0^1 = \int_0^1 t^3f(t)dt + \frac{1}{5}p \therefore \frac{5}{4}\int_0^1 t^3f(t)\,dt = p \therefore$

$y(x) = f(x) + \frac{5}{4}x\int_0^1 t^3 f(t)\,dt = f(x) + \int_0^1 (\frac{5}{4}xt^3) f(t)\,dt$ ∴ resolvent kernel is $r = \frac{5}{4}xt^3$

∴ when $f(x) = x^2$ ∴ $y(x) = x^2 + \int_0^1 (\frac{5}{4}xt^3)t^2\,dt = x^2 + \frac{5}{4}x\int_0^1 t^5\,dt =$

$x^2 + \frac{5}{4}x\left[\frac{1}{6}x^6\right]_0^1 \; x^2 + \frac{5}{4}\cdot\frac{1}{6}x = x^2 + \frac{5}{24}x$

\4/ $\int_0^1 |k(x,t)|dt = \int_0^x |k(x,t)|dt + \int_x^1 |k(x,t)|dt = \int_0^x |x-t|dt + \int_x^1 |0|\,dt = \int_0^x x-t\,dt =$

$\left[xt - \frac{1}{2}t^2\right]_0^x = x^2 - \frac{1}{2}x^2 = \frac{1}{2}x^2$ but $x \in [0,1]$ ∴ $x_1^2 > x_2^2$ for $x_1 > x_2$ ∴

$\max_{x\in[0,1]}(\frac{1}{2}x^2) = \frac{1}{2}(1)^2 = \frac{1}{2} = \frac{1}{2}$ ∴ $\frac{1}{2}x^2 \le \frac{1}{2}$ for $x \in [0,1]$ ∴ $\int_0^1 |k(x,t)|dt \le \frac{1}{2}$

\4b/ $y_0(x) = 0$ ∴ $y - Ky = x$ ∴ $y(x) - \int_0^1 k(x,t)y(t)dt = f(x)$ ∴

$\int_0^1 |k(x,t)|dt = \int_0^x (x-t)dt = -\frac{1}{2}\left[(x-t)^2\right]_0^x = \frac{x^2}{2} \le \frac{1}{2} \;\; \forall x \in [0,1]$

\4bredo/ $y_0(x) = x$ ∴ $y - Ky = x$ ∴ $y(x) - \int_0^1 k(x,t)y(t)dt = x$ ∴

$y(x) - \int_0^x (x-t)y(t)dt = y(x) - x\int_0^x y(t)dt + \int_0^x t\,y(t)dt$ ∴ $y - Ky = x$ ∴ $y_0(x) = x$ ∴ $y_0(t) = t$ ∴

$y(x) = x + \int_0^1 k(x,t)\,y(t)dt = x + \int_0^1 (x-t)y(t)dt$ ∴

$y_1(x) = x + \int_0^x (x-t)y_0(t)dt = x + \int_0^x (x-t)\,t\,dt = x + \frac{1}{3!}x^3$ ∴ $y_1(t) = t + \frac{1}{3!}t^3$ ∴

$y_2(x) = x + \int_0^x (x-t)y_1(t)dt = x + \int_0^x (x-t)(t + \frac{1}{3!}t^3)dt = x + \frac{1}{3!}x^3 + \frac{1}{5!}x^5$ ∴

if $y_n(x) = x\sum_{k=0}^{n} \frac{x^{2k}}{(2k+1)!}$ ∴ $y_{n+1}(x) = x + \int_0^x (x-t)y_n(t)dt =$

$x + x\int_0^x y_n(t)\,dt - \int_0^x t\,y_n(t)dt\} = x + x\int_0^x \sum_{k=0}^{n} \frac{t^{2k+1}}{(2k+1)!}dt - \int_0^x \sum_{k=0}^{n} \frac{t^{2k+2}}{(2k+1)!}dt =$

$x + x\sum_{k=0}^{n} \frac{x^{2k+2}}{(2k+2)!} - \sum_{k=0}^{n} \frac{x^{2k+3}}{(2k+3)(2k+1)!} = x + x\sum_{k=0}^{n} \frac{x^{2k}}{(2k+1)!}\left(\frac{1}{2k+2} - \frac{1}{2k+3}\right) =$

$x + x\sum_{k=0}^{n} \frac{x^{2k}}{(2k+1)!}\left(\frac{1}{(2k+2)(2k+3)}\right) = x + x\sum_{k=0}^{n} \frac{x^{2k}}{(2k+3)!} = x + x\sum_{k=1}^{n+1} \frac{x^{2k}}{(2k+1)!}$ note the exact sol is

$y(x) = \sinh x$ ∵ y is also the sol of $y'' - y = f''$ with ICs $y(0) = f(0)$, $y'(0) = f'(0)$ ∴

$y'' - y = 0$, $y(0) = 0$, $y'(0) = 1$ ∴ the GS is $y(x) = A\cosh x + B\sinh x$ ∴ BC: $y(x) = \sinh x$

~~[illegible]~~/ for 5/ $x + x\sum_{k=0}^{n} \frac{x^{2k+2}}{(2k+1)!}\left(\frac{1}{2k+2} - \frac{1}{2k+3}\right) = x + x\sum_{k=0}^{n} \frac{x^{2k+2}}{(2k+1)!}\left(\frac{2k+3-2k-2}{(2k+2)(2k+3)}\right) =$

$x + x\sum_{k=0}^{n} \frac{x^{2k+2}}{(2k+1)!}\left(\frac{1}{(2k+2)(2k+3)}\right) = x + x\sum_{k=0}^{n} \frac{x^{2k+2}}{(2k+3)!} = x + x\sum_{k=1}^{n+1} \frac{x^{2(k-1)+2}}{(2(k-1)+3)!} = x + x\sum_{k=1}^{n+1} \frac{x^{2k}}{(2k+1)!}$

\5/ $\lambda = 1$ ∴ $y_0 = f = 1$, ∴ $y_1 = 1 + K1$ & $y_2 = 1 + Ky_1$ ∴

$y_1(x) = \left(\frac{79 + e^{\pi}}{80}\right) + \frac{\pi}{80}\sin x$ ∴ $y_2(x) = 1 + \frac{1}{80}\left[(79 + e^{\pi})(e^{\pi} - 1) + \frac{\pi}{2}(e^{\pi} + 1)\right] + \frac{(81\pi + \pi e^{\pi})}{80^2}\sin x$

∴ $\|K\|_\infty = \max_{x\in[0,\pi]} \left|\int_0^{\pi} |\sin x + e^t|\,dt \frac{1}{80}\right| = \frac{1}{80} \times \max_{x\in[0,\pi]} \int_0^{\pi} |\sin x| + e^t\,dt = \max_{x\in[0,\pi]} \frac{1}{80}\left[t|\sin x| + e^t\right]_0^{\pi} =$

$\frac{1}{80}\left[t \times 1 + e^t\right]_0^{\pi} = \frac{1}{80}\left[\pi + e^{\pi} - 1\right] = \frac{\pi - 1 + e^{\pi}}{80}$ ∴ $\|K\|_\infty = \frac{\pi - 1 + e^{\pi}}{80}$ ∴ error is:

$\|y - y_n\|_\infty \le \frac{1}{|\lambda| - \|K\|_\infty}\left(\frac{\|K\|_\infty}{|\lambda|}\right)^{n+1}\|f\|_\infty$ ∴ $\|y - y_2\|_\infty \le \frac{1}{|\lambda| - \|K\|_\infty}\left(\frac{\|K\|_\infty}{|\lambda|}\right)^3\|f\|_\infty =$

$\frac{1}{|1| - \|K\|_\infty}\left(\frac{\|K\|_\infty}{|1|}\right)^3 = \frac{1}{1 - \|K\|_\infty}\|K\|_\infty^3 = \frac{1}{1 - \left(\frac{\pi - 1 + e^{\pi}}{80}\right)}\left(\frac{\pi - 1 + e^{\pi}}{80}\right)^3 = \frac{(\pi - 1 + e^{\pi})^3}{80} = 0.0461467$ note

the eqn is rank 2 ∴ $P_1 = \int_0^{\pi} y(t)dt$ & $P_2 = \int_0^{\pi} e^t y(t)dt$ ∴ $y(x) = \frac{80 + P_2}{80} + \frac{P_1}{80}\sin x$ ∴ $78 - \pi P_2 = 80$ ∴

$(e^{\pi} - 1)\frac{P_1}{2} - (81 - e^{\pi})P_2 = 80(1 - e^{\pi})$ ∴ $P_1 = 1.5684$, $P_2 = 30.94$ ∴ $y(x) = 1.387 + 0.0196\sin x$

\Week 7 Sheet/ \6a/ kernel of $K_p L_q$ is : $k_p(x,t)=xt^p \therefore k_p(x,s)=xs^p \therefore$ $L_q(x,t)=x^q t \therefore L_q(s,t)=s^q t \therefore K_p L_q$ is $:\int_0^1 k_p(x,s) L_q(s,t)ds = \int_0^1 x s^p s^q t\, ds = \int_0^1 xt\, s^{p+q} ds =$
$\left[\frac{xts^{p+q+1}}{p+q+1}\right]_0^1 = \frac{xt}{p+q+1} \therefore$
$L_q=(x,s)=x^q s$, $k_p(s,t)=st^p \therefore$ $L_q K_p$: $\int_0^1 L_q(x,s)k_p(s,t)\,ds = \int_0^1 x^q s s t^p ds = \int_0^1 x^q t^p s^2 ds = \left[\frac{1}{3}x^q t^p s^3\right]_0^1 =$
$\frac{1}{3}x^q t^p$ $\therefore$ Z operators K_p & L_q commute when $p=q=1$, when theyre equal. Z adjoint of K_p has kernel $x^p t$ that is $K_p^* = L_p$ so they only commute when $p=1$ when theyre are symmetric

\6b/ know Z norm of Z composition of two operators is smaller or equal to Z product of their norms : $\|K_p\|_\infty = \max_{x\in[0,1]}\int_0^1 |xt^p|dt = \int_0^1 t^p\, dt = \left[\frac{1}{p+1}t^{p+1}\right]_0^1 = \frac{1}{p+1}$
$\|L_q\|_\infty = \max_{x\in[0,1]}\int_0^1 |x^q t|dt = \int_0^1 1^q t\, dt = \int_0^1 t\,dt = \left[\frac{1}{2}t^2\right]_0^1 = \frac{1}{2}\ \forall q \geq 0$ note Z max we need to calc in Z GS for Z norms always occure at $x=1 \therefore$
$\|K_p L_q\|_\infty = \frac{1}{2(p+q+1)} \leq \frac{1}{2}\frac{1}{p+1} = \|L_q\|_\infty \|K_p\|_\infty$ & $\|L_q K_p\|_\infty = \frac{1}{3(p+1)} \therefore$
$\|L_q K_q\|_\infty = \frac{1}{3(p+1)} \leq \frac{1}{2}\frac{1}{p+1} = \|L_q\|_\infty \|K_p\|_\infty$

\6c/ to calc Z inverse of $I-K_p$ need to solve Z Fredholm IE $\therefore$
$y(x) - \int_0^1 xt^p y(t)dt = f(x) \therefore y(x) - x\int_0^1 t^p y(t)dt = f(x) \therefore$ let $a = \int_0^1 t^p y(t)dt \therefore$
$y(x) = f(x) + ax \therefore y(t) = f(t) + at \therefore a = \int_0^1 t^p y(t)\,dt = \int_0^1 t^p (f(t)+at)\,dt =$
$\frac{a}{p+2} + \int_0^1 t^p f(t)dt \therefore y(x) = f(x) + x\int_0^1 t^p y(t)dt \therefore y(x) = f(x) + ax \therefore$
$a - \frac{a}{p+2} = \int_0^1 t^p f(t) = \frac{(p+2)a}{p+2} - \frac{a}{p+2} = \frac{pa+2a-a}{p+2} = \frac{(p+1)a}{p+2} \therefore$
$a = \frac{p+2}{p+1}\int_0^1 t^p f(t)dt \therefore y(x) = f(x) + \left(\frac{p+2}{p+1}\right)x\int_0^1 t^p f(t)\,dt = (I-K_p)^{-1}f$
its norm is estied by $\|(I-K_p)^{-1}\|_\infty \leq \frac{1}{1-\|K_p\|_\infty} \leq \cancel{\frac{1}{1-\|K_p\|}}\ \frac{1}{1-\frac{1}{p+1}} = 1 + \frac{1}{p}$
its actual: $\|(I-K_p)^{-1}\|_\infty = 1 + \frac{p+2}{(p+1)^2} \leq 1 + \frac{1}{p}$

\7a/ let $k(x,t) = \sum_{j=1}^n k_1^{(j)}(x)k_2^{(j)}(t) \therefore k^*(x,t) = \sum_{j=1}^n k_2^{(j)}(x)k_1^{(j)}(t)$ Z mat M^* corres to K^* has (i,j) entry $(M^*)_{ij} = \int_a^b k_1^{(i)}(t)k_2^{(j)}(t)dt = M_{ji} = (M^T)_{ji}$

\7b/ $\forall y, z \in C[a,b]$: $\int_a^b (Ky)(x)\, z(x)dt = \int_a^b\int_a^b k(x,t)y(t)\, z(x)dt\,dx =$
$\int_a^b y(t)\left[\int_a^b k(x,t)z(x)dx\right]dt = \int_a^b y(t)\left[\int_a^b k^*(t,x)z(x)dx\right]dt = \int_a^b y(t)(K^*z)(t)dt =$
$\int_a^b (K^*z)(x)y(x)dx$

\Week 8 Sheet/ $y'(x) - e^{-x}y(x) = 2x \therefore y(0)=1 \therefore y \in C'[0,1] \therefore$ a cont sol of Z IE is dissable $\therefore y(x) = 1 + x^2 + \int_0^x e^{-t}y(t)dt \therefore$ deriv Z IE:
$y'(x) = 2x + e^{-x}y(x) + \int_0^x 0\,y(t)\,dt = y'(x) = 2x + e^{-x}y(x) \therefore y'(x) - e^{-x}y(x) = 2x$ IC: $y(0)=1$.

Z converse holds by integrate of deriv eqn, taking into account Z initial vals to fix Z integration const: $\int_0^x y'(t)\,dt - \int_0^x e^{-t}y(t)\,dt = \int_0^x 2t\,dt$ ∴

$y(x) - y(0) - \int_0^x e^{-t}y(t)\,dt = x^2$ & $y(0)=1$ ∴ $y(x) - \int_0^x e^{-t}y(t)\,dt = 1+x^2$

Laplace: $\mathcal{L}\left(\int_0^x e^{-t}y(t)\,dt\right) = \hat{y}(s+1)$ ∴

$\mathcal{L}\left(y(x) - \int_0^x e^{-t}y(t)\,dt\right) = \mathcal{L}(1+x^2) = \frac{1}{s} + \frac{2}{s^3} = \frac{s^2+2}{s^3} = \hat{y}(s) - \hat{y}(s+1)$ ∴

$\mathcal{L}\left(y(x) - \int_0^x e^{-t}y(t)\,dt\right) = \mathcal{L}(1+x^2) = \frac{1}{s} + \frac{2}{s^3} = \frac{s^2+2}{s^3} = \hat{y}(s) - \hat{y}(s+1)$ ∴ for ODE:

integrating factor h: $h'(x) = -e^{-x}h(x)$ ∴ $h(x) = e^{e^{-x}}$ ∴ $(hy)'(x) = 2x\,h(x)$ ∴

$y(x) = e^{1-e^{-x}} + 2\int_0^x t\,e^{e^{-t}(1-e^{(t-x)})}\,dt$

\2a/ $y(x) = \int_\pi^x \frac{y(t)}{t}\,dt = 2 + \cos x$, $x \geq \pi$ ∴ deriv:

$y'(x) + \frac{y(x)}{x} = -\sin x$ ∴ $xy'(x) + y(x) = -x\sin x$ ∴

$xy'(x) + y(x) = -x\sin x$ ∴ IC: $y(\pi) = 2-1 = 1$ ∴

$y(x) = \cos x - \frac{\sin x}{x} + \frac{2\pi}{x}$ ∴ $q(x)xy'(x) + q(x)y(x) = -x\,q(x)\sin x$ ∴

$\frac{d}{dx}(q(x)y(x)) = q(x)y'(x) + q'(x)y(x) = q(x)y'(x) + q(x)\frac{1}{x}y(x) = -q(x)\sin x$ ∴

$q(x)\frac{1}{x} = q'(x)$ ∴ $\frac{1}{x} = \frac{q'(x)}{q(x)}$ ∴ $\ln|x| = \ln|q(x)|$ ∴ $x = q(x)$ ∴

$\frac{d}{dx}(xy(x)) = -x\sin x$ ∴ $xy(x) = x\cos x - \int \cos x\,dx = x\cos x - \sin x + C$ ∴

$\pi y(\pi) = \pi\cos\pi - \sin\pi + C = \pi = -\pi + C$ ∴ $2\pi = C$ ∴

$xy(x) = x\cos x - \sin x + 2\pi$ ∴ $y(x) = \cos x - \frac{\sin x}{x} + \frac{2\pi}{x}$

\2b/ $y(x) - \int_0^x t\,y(t)\,dt = x$ ∴ deriv: $y'(x) - xy(x) = 1$ ∴ IF $= e^{\int -x\,dx} = e^{-\frac{1}{2}x^2}$ ∴

$\frac{d}{dx}\left(e^{-\frac{1}{2}x^2}y(x)\right) = e^{-x^2/2}$ ∴ $x \geq 0$, $y(0) = 0$ ∴ $e^{-\frac{1}{2}x^2}y(x) = \int_0^x e^{-t^2/2}\,dt$ ∴

$y(x) = e^{\frac{1}{2}x^2}\int_0^x e^{-t^2/2}\,dt = \int_0^x e^{\frac{x^2-t^2}{2}}\,dt$

\3/ $\int_0^x e^{x-t}y(t)\,dt = f(x)$, $x \geq 0$, $f(0) = 0$ probl has a sol ∵ $f(0) = 0$ ∴ deriv:

$f'(x) = e^{x-x}y(x) + \int_0^x e^{x-t}y(t)\,dt = y(x) + \int_0^x e^{x-t}y(t)\,dt = y(x) + f(x) = f'(x)$ ∴

$y(x) = f'(x) - f(x)$

Laplace: $\mathcal{L}\left(\int_0^x e^{x-t}y(t)\,dt\right) = \mathcal{L}(f(x)) = \mathcal{L}(e^x)\mathcal{L}(y(x)) = \mathcal{L}\left(\int_0^x y(t)e^{x-t}\,dt\right) = \mathcal{L}\left(\int_0^x y(u)e^{x-u}\,du\right)$

$= \frac{1}{p-1}Y(p) = F(p)$ ∴ $Y(p) = (p-1)F(p) = pF(p) - F(p) = pF(p) - 0 - F(p) = pF(p) - f(0) - F(p)$ ∴

$y(x)$ is always desired & is equal to Z inverse transform of $Y(p)$ ∴

$\mathcal{L}^{-1}(Y(p)) = \mathcal{L}^{-1}(pF(p) - f(0) - F(p)) = y(x) = \mathcal{L}^{-1}(pF(p) - f(0)) - \mathcal{L}^{-1}(F(p)) = f'(x) - f(x) = y(x)$

\4a/ $\lambda y(x) - \int_0^x k(x-t)y(t)\,dt = f(x)$ ∴ let $\hat{f}(s)$, $\hat{k}(s)$ be Z Laplaces of f & k ∴

$\mathcal{L}\left(\lambda y(x) - \int_0^x k(x-t)y(t)\,dt\right) = \mathcal{L}(f(x)) = \hat{f}(s)$ ∴ $\lambda\hat{y}(s) - \mathcal{L}(k(x))\mathcal{L}(y(x)) = \lambda\hat{y}(s) - \hat{k}(s)\hat{y}(s) =$

$(e^{\pi}-1)\frac{1}{2}(8)-e)/2 = 8a(1-e)$ ...

Week 8 Sheet / $\hat{y}(s)(\lambda-\hat{k}(s))=\hat{f}(s)$ ∴ $\hat{y}(s)=\frac{\hat{f}(s)}{\lambda-\hat{k}(s)}$

$\lambda\hat{y}(s)=\hat{f}(s)+\hat{k}(s)\hat{y}(s)$ ∴ $\hat{y}(s)=\frac{\hat{f}(s)}{\lambda-\hat{k}(s)}$

4b / $\mathcal{L}(f(x))=\hat{f}(s)=\mathcal{L}(e^{x})=\frac{1}{s-1}$

$\mathcal{L}(k(x))=\hat{k}(s)=\mathcal{L}(\sin x)=\frac{1}{s^2+1}$ ∴ $\lambda=\frac{1}{2}$ ∴

~~$\hat{y}(s)=2$ $\frac{(\frac{1}{s-1})}{(\frac{1}{2}-\frac{1}{s^2+1})}=2(s^2+1)$~~ $\hat{y}(s)=\frac{\hat{f}(s)}{\lambda-\hat{k}(s)}=\frac{(\frac{1}{s-1})}{\frac{1}{2}-\frac{1}{s^2+1}}=\frac{2(s^2+1)}{(s-1)^2(s+1)}=\frac{1}{s+1}+\frac{1}{s-1}+\frac{2}{(s-1)^2}$

∴ $\mathcal{L}^{-1}(\hat{y}(s))=y(x)=e^{-x}+e^{x}+2xe^{x}=\mathcal{L}^{-1}(\frac{1}{s+1})+\mathcal{L}^{-1}(\frac{1}{s-1})+\mathcal{L}^{-1}(\frac{2}{(s-1)^2})=2\cosh x+2xe^{x}$

4c / $f(x)=1-e^{x^2/2}$ ∴ f grows too fast to have a laplace transform

4d / $\lambda=0$, $f(x)=1-e^{x^2/2}$ $k=\sin x$ ∴ $-\int_0^x \sin(x-t)y(t)\,dt=1-e^{x^2/2}$ ∴

taking the derivative: $-\sin(x-x)y(x)-\int_0^x \cos(x-t)y(t)\,dt=-xe^{x^2/2}$

$=-\int_0^x \cos(x-t)y(t)\,dt=-xe^{\frac{x^2}{2}}$ ∴ taking the derivative again:

$-\cos(x-x)y(x)-\int_0^x \sin(x-t)y(t)\,dt=-y(x)+1-e^{\frac{x^2}{2}}=-e^{\frac{x^2}{2}}-x^2e^{\frac{x^2}{2}}$ ∴

$y(x)=(x^2+2)e^{\frac{x^2}{2}}-1$

5a / $y(x)-\lambda\int_0^x \sin(x-t)y(t)\,dt=f(x)$ $x\geq 0$ $\lambda\geq 0$ ∴

Laplace: $\hat{y}(s)-\lambda\hat{y}(s)\mathcal{L}(\sin(x))=\hat{f}(s)=\hat{y}(s)-\lambda\hat{y}(s)\frac{1}{s^2+1}$ ∴

$\hat{y}(s)=\left(\frac{s^2+1}{s^2+1-\lambda}\right)\hat{f}(s)=\left(\frac{s^2+1-\lambda+\lambda}{s^2+1-\lambda}\right)\hat{f}(s)=\left(1+\frac{\lambda}{s^2+1-\lambda}\right)\hat{f}(s)=\hat{f}(s)+\frac{\lambda}{s^2+1-\lambda}\hat{f}(s)=\hat{y}(s)$ ∴

$\hat{g}(s)=\frac{\lambda}{s^2+1-\lambda}$ ∴ $y(x)=f(x)+\mathcal{L}^{-1}\left(\frac{\lambda}{s^2+1-\lambda}\hat{f}(s)\right)$ ∴

$y(x)=f(x)+\int_0^x f(t)g(x-t)\,dt$ by ZT22 ∴

$\mathcal{L}^{-1}(\hat{g}(s))=\mathcal{L}^{-1}\left(\frac{\lambda}{s^2+1-\lambda}\right)$ which depends on λ ∴

for $\lambda=1$: $\hat{g}(s)=\frac{1}{s^2}$ ∴ $g(x)=x$

for $\lambda<1$: $\hat{g}(s)=\frac{\lambda}{s^2+(1-\lambda)}=\lambda\frac{1}{s^2+(1-\lambda)}=\frac{\lambda}{\sqrt{1-\lambda}}\frac{(\sqrt{1-\lambda})}{s^2+(\sqrt{1-\lambda})^2}$ ∴ $1-\lambda$ is posi ∴

$g(x)=\frac{\lambda}{\sqrt{1-\lambda}}\sin(\sqrt{1-\lambda}\,x)$

for $\lambda>1$: $\hat{g}(s)=\frac{\lambda}{\sqrt{1-\lambda}}\frac{(\sqrt{1-\lambda})}{s^2+(\sqrt{1-\lambda})^2}$ ∴ $1-\lambda$ is nega ∴ $g(x)=\frac{\lambda}{\sqrt{\lambda-1}}\sinh(\sqrt{\lambda-1}\,x)$ ∴

$y(x)=f(x)+\int_0^x f(t)g_\lambda(x-t)\,dt$, ~~$g_\lambda(x-t)$~~ $g_\lambda(x)=\begin{cases}\frac{\lambda}{\sqrt{1-\lambda}}\sin(\sqrt{1-\lambda}\,x), & \lambda<1\\ x, & \lambda=1\\ \frac{\lambda}{\sqrt{1-\lambda}}\sinh(\sqrt{1-\lambda}\,x), & \lambda>1\end{cases}$ ∴

$g_\lambda(x-t)=\begin{cases}\frac{\lambda}{\sqrt{\lambda-1}}\sin(\sqrt{1-\lambda}(x-t)), & \lambda<1\\ x-t, & \lambda=1\\ \frac{\lambda}{\sqrt{1-\lambda}}\sinh(\sqrt{1-\lambda}(x-t)), & \lambda>1\end{cases}$

5b / $f(x)=x$, $\hat{f}(s)=s^{-2}$ ∴ $\lambda=1$ ∴ $y(x)=x+\int_0^x (x-t)t\,dt=\frac{1}{6}x^3+x$

$\lambda=5$ $y(x)=x+\int_0^x \frac{5}{2}\sinh(2x-2t)t\,dt=-\frac{1}{4}x+\frac{5}{8}\sinh(-2x)$ deriv: $y''(x)=$

$f''(x)+\lambda y(x)-\lambda\int_0^x \sinh(x-t)y(t)\,dt=f''(x)+\lambda y(x)-(y(x)-f(x))$ ∴ $y''+(1-\lambda)y=f+f''$: IC $y(0)=f(0)$

$y'(0)=f'(0)$

\6/ Laplace: $\hat{y}(s) = \frac{\hat{g}(s)}{\lambda - \hat{k}(s)}$ ∴ is $\lambda \neq 0$: $(deg(\lambda - \hat{k}(s)) = deg(\lambda) = 0$ ∴ $deg(\hat{y}) = deg(\hat{g})$

when $\lambda = 0$: $deg(\hat{y}) = deg(\hat{g}) - deg(k(s))$

\Week 9 sheet/ \1a/ $y(x) - \int_0^x y(t)\,dt = g(x)$ ∴ $y'(x) = g'(x) + y(x)$

∴ deriv: $y'(x) - y(x) = g'(x)$ ∴ $y'(x) = g'(x) + y(x)$ ∴

IC: $y(0) = g(0)$ ∴ at $x=0$: $y(0) - \int_0^0 y(t)\,dt = g(0) = y(0) - 0 = y(0) = g(0)$ ∴

$y'(x) - y(x) = g'(x)$ ∴ $IF = e^{\int -1\,dx} = e^{-x}$ ∴ $\frac{d}{dx}(e^{-x}y(x)) = e^{-x}g'(x)$ ∴

$e^{-x}y(x) = \int_0^x e^{-t}g'(t)\,dt = [e^{-t}g(t)]_0^x - \int_0^x -e^{-t}g(t)\,dt$ ∴

$y'(x) = g'(x) + y(x)$ $\quad y(0) = g(0)$ ∴ $e^{-x}(e^{-x}y(x)) = y(x) - y'(x) = -g'(x)$ ∴

$y'(x) - y(x) = g'(x)$ ∴ $-g'(x) = y(x) - y'(x)$ ∴ $\frac{d}{dx}(e^{-x}y(x)) = e^{-x}y(x) + e^{-x}y'(x)$ ∴

$-e^{x}\frac{d}{dx}(e^{-x}y(x)) = -e^{x}[-e^{-x}y(x) + e^{-x}y'(x)] = -e^{x}(-e^{-x})y(x) - e^{x}e^{-x}y'(x) = y(x) - y'(x)$ ∴

$-e^{x}\frac{d}{dx}(e^{-x}y(x)) = y(x) - y'(x)$ ∴ $y(x) - y'(x) = -e^{x}\frac{d}{dx}(e^{-x}y(x)) = -g'(x)$ ∴

$\frac{d}{dx}(e^{-x}y(x)) = -e^{-x}(-1)g'(x) = e^{-x}g'(x)$ ∴

$e^{-x}y(x) = \int e^{-x}g'(x)\,dx$ ∴ $[e^{-x}y(x)]_0^x = \int_0^x e^{-t}g'(t)\,dt = e^{-x}y(x) - e^{-0}y(0) =$

$e^{-x}y(x) - y(0) = e^{-x}y(0) - g(0) = \int_0^x e^{-t}g'(t)\,dt$ ∴

$e^{-x}y(x) = g(0) + \int_0^x e^{-t}g'(t)\,dt$ ∴ $y(x) = g(0)e^{x} + e^{x}\int_0^x e^{-t}g'(t)\,dt =$

$g(0)e^{x} + \int_0^x e^{x}e^{-t}g'(t)\,dt$ ∴ $y(x) = g(0)e^{x} + e^{x}\int_0^x e^{-t}g'(t)\,dt =$

$y(x) = g(0)e^{x} + \int_0^x e^{x-t}g'(t)\,dt$

\1b/ when $g(x) = x$ ∴ $(Ky)(x) = \frac{x^2}{2}$ ∴ $(K^n g)(x) = \frac{x^n}{n!}$ ∴ $\sum_{n=1}^{\infty}\frac{x^n}{n!} = e^{x} - 1$ which is Z

true sol found from Z exact formula $y(x) = \int_0^x e^{x-t}\,dt = e^{x} - 1$ ∴

$g(x) = x$ ∴ $y_0(x) = 0$

\1a/ $y(x) - \int_0^x y(t)\,dt = g(x)$ ∴ $y'(x) - y(x) = g'(x)$ $\quad y(0) = g(0)$ ∴ IF: $e^{\int -1\,dx} = e^{-x}$ ∴

$\frac{d}{dx}(y(x)e^{-x}) = e^{-x}g'(x)$ ∴ $y'(x)e^{-x} = \int_0^x e^{-x}g'(x)\,dx$ ∴ IBP: $u = e^{-x}$, $u' = -e^{-x}$, $v' = g'$,

$v = g$ ∴ $y(x)e^{-x} = [e^{-x}g(x)] - \int -e^{-x}g(x)\,dx$ ∴ $y(x) = g(x) + e^{x}\int e^{-x}y(x)\,dx + c$

IC: $y(0) = g(0)$ ∴

\1b/ $y = Ky + g$ ∴ $y(x) = g(x) + \int_0^x y(t)\,dt$ ∴ $g(x) = x$ ∴ $y(x) = x + \int_0^x y(t)\,dt$ ∴ $y_0 = x$ ∴

$y_1 = x + \int_0^x t\,dt = x + [\frac{1}{2}t^2]_0^x = x + \frac{1}{2}x^2$ ∴ $y_2 = x + \int_0^x y_1(t)\,dt = x + \int_0^x t + \frac{1}{2}t^2\,dt = x + [\frac{1}{2}t^2 + \frac{1}{6}t^3]_0^x =$

$x + \frac{1}{2}x^2 + \frac{1}{6}x^3$ ∴ $y_n = \sum_{n=1}^{\infty}\frac{x^n}{n!}$ $\quad e^{x} = \sum_{n=0}^{\infty}\frac{x^n}{n!} = 1 + \sum_{n=1}^{\infty}\frac{x^n}{n!}$ ∴

$y_n + 1 = 1 + \sum_{n=1}^{\infty}\frac{x^n}{n!} = e^{x}$ ∴ $y_n = e^{x} - 1$

Week 9 Sheet

1/2 at $x=0$: $\int_0^0 \cos(0-t)y(t)\,dt = \cosh(0) - c = 0 = 1 - c \therefore c = 1$

$\therefore$ Laplace: $\frac{s}{s^2-1} = \frac{1}{s} + \frac{s}{s^2+1}\hat{y}(s) \quad \therefore \mathcal{L}\left(\int_0^x \cos(x-t)y(t)\,dt\right) = \mathcal{L}(\cosh x - 1) =$

$\mathcal{L}(\cos(x))\mathcal{L}(y(x)) = \mathcal{L}(\cosh x) - \mathcal{L}(1) = \frac{s}{s^2-1} - \frac{1}{s} = \frac{s}{s^2+1}\hat{y}(s) \therefore$

$\hat{y}(s)\frac{s}{s^2+1} - \frac{1}{s} = \frac{s}{s^2-1} \quad \therefore \hat{y}(s) = \frac{s^2+1}{s^2(s^2-1)} = -\frac{1}{s^2} + \frac{1}{s-1} - \frac{1}{s+1} \therefore$

$y(x) = -x + 2\sinh(x) \quad \therefore \mathcal{L}^{-1}(\hat{y}(s)) = \mathcal{L}^{-1}\left(-\frac{1}{s^2} + \frac{1}{s-1} - \frac{1}{s+1}\right) = y(x) = -x + e^x - e^{-x} =$

$-x + 2(\frac{1}{2})(e^x - e^{-x}) - x + 2\sinh x = -x + 2\sinh x$

deriv: $y'(x) = 2\cosh x - 1 \quad y(0) = 0 \therefore \quad y(x) = -x + 2\sinh x$

$\int_0^x \cos(x-t)y(t)\,dt = \cosh(x) - 1 \quad \therefore$ deriv:

$\cos(x-x)y(x) + \int_0^x -\sin(x-t)y(t)\,dt = y(x) - \int_0^x \sin(x-t)y(t)\,dt = \sinh(x) \quad \therefore$ deriv:

$y'(x) - \int_0^x \cos(x-t)y(t)\,dt - \sin(x-x)\,y(x) = y'(x) - \int_0^x \cos(x-t)y(t)\,dt =$

$y'(x) - (\cosh(x) - 1) = \cosh(x) = y'(x) - \cosh(x) + 1 \quad \therefore$

$y'(x) = 2\cosh(x) - 1 \quad \therefore \quad y(x) = \int 2\cosh(x) - 1\,dx = 2\sinh(x) - x + C_2 \therefore$

$y'(0) = 2\cosh(0) - 1 = 2(1) - 1 = 1 \quad \therefore$

$y(t) = 2\sinh(t) - t + C_2 \therefore \quad \int_0^x \cos(x-t)(2\sinh(t) - t + C_2)\,dt = \cosh(x) - 1$

$y(0) = 0 \therefore \quad y(x) = 2\sinh(x) - x + C_2 \therefore$

$y(0) = 0 = 2\sinh(0) - 0 + C_2 = C_2 = 0 \therefore \qquad y(x) = 2\sinh(x) - x$

3/ $g(x) = 1 \quad \therefore y_0(x) = 1 \therefore \quad y_0(t) = 1 \therefore y(x) = 1 + \int_0^x e^{-t}y(t)\,dt \therefore$

$y_1(x) = 1 + \int_0^x e^{-t}\,1\,dt = \left[-e^{-t}\right]_0^x + 1 = -e^{-x} + e^0 = -e^{-x} + 1 + 1 \therefore y_1(t) = -e^{-t} + 2 \therefore$

$y_2(x) = 1 + \int_0^x e^{-t}y_1(t)\,dt = 1 + \int_0^x e^{-t}(-e^{-t} + 2)\,dt = 1 + \int_0^x -e^{-2t} + 2e^{-t}\,dt = 1 + \left[\frac{1}{2}e^{-2t} - 2e^{-t}\right]_0^x$

$= 1 + \frac{1}{2}e^{-2x} - \frac{1}{2} - 2e^{-x} - 2 = -\frac{3}{2} + \frac{1}{2}e^{-2x} - 2e^{-x}$

$y_{n+1}(x) = 1 + \int_0^x e^{-t}y_n(t)\,dt \quad \therefore \quad y_2(x) = 1 + \int_0^x e^{-t}\left[1 + (1 - e^{-t})\right]dt = 1 + (1 - e^{-x}) + \frac{1}{2}(1 - e^{-x})^2 \therefore$

$y_n(x) = 1 + \sum_{j=1}^{n} \frac{1}{j!}(1 - e^{-x})^j \quad \therefore y_{n+1}(x) = 1 + \int_0^x e^{-t}\left[1 + \sum_{j=1}^{n} \frac{1}{j!}(1 - e^{-t})^j\right]dt =$

$1 + \int_0^x e^{-t}\,dt + \sum_{j=1}^{n} \frac{1}{j!}\int_0^x e^{-t}(1 - e^{-t})^j\,dt = 1 + (1 - e^{-x}) + \sum_{j=1}^{n} \frac{1}{(j+1)!}\left[(1 - e^{-t})^{j+1}\right]_0^x = 1 + \sum_{j=1}^{n+1} \frac{1}{j}(1 - e^{-x})^j$

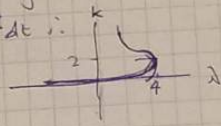
$\therefore$ deriv: $y'(x) = e^{-x}y(x)$ IC: $y(0) = 1 \therefore \quad y(x) = e^{1-e^{-x}}$ they equal to order n

deriv: $y'(x) - e^{-x}y(x) = 0 \qquad \therefore y'(x) = e^{-x}y(x)$ for $x=0$: $y(0) - \int_0^0 e^{-t}y(t)\,dt = 1 = y(0) \therefore$

$\frac{y'(x)}{y(x)} = e^{-x} \quad \therefore \quad \ln|y(x)| = e^{x} + c_2 \quad \therefore \quad |y| = e^{-e^{-x} + c_2} = Ae^{-e^{-x}} \therefore$

$y(0) = 1 = Ae^{-e^{-0}} = Ae^{-1} = 1 \therefore \quad A = e \therefore \quad y(x) = e\,e^{-e^{-x}} = e^{(1 - e^{-x})}$

\Week 10 Sheet / from 2 eqn its clear y must



be a const : $y=p$ say & $\therefore$ 2 eqn ; $p=1+\lambda p^2$ $\therefore$ $y(x)=1+\lambda\int_0^1 y(t)^2\,dt$ $\therefore$

$1+\lambda\int_0^1 y(t)^2\,dt=p$ $\therefore y(x)=p$ $\therefore y(t)=p$ $\therefore$

$p=1+\lambda\int_0^1 p^2\,dt=1+\lambda p^2[t]_0^1=1+\lambda p^2=p$ $\therefore$

$\lambda p^2-p+1=0$ $\therefore$ $p=\frac{1\pm\sqrt{1-4\lambda}}{2\lambda}$ have 2 branches paramed by $p_\pm(\lambda)=\frac{1\pm\sqrt{1-4\lambda}}{2\lambda}$

\3/ $y(x)=\int_0^1 \frac{1}{2}y(t)+\lambda x y(t)^2\,dt=\frac{1}{2}\int_0^1 y(t)+\lambda x\int_0^1 y(t)^2\,dt$ $\therefore$ $a=\int_0^1 y(t)\,dt$, $b=\int_0^1 y(t)^2\,dt$ $\therefore$

$y(x)=\frac{1}{2}a+\lambda bx$ $\therefore y(t)=\frac{1}{2}a+\lambda bt$ $\therefore$ get 2 syst : $a=\frac{1}{2}a+\frac{1}{2}\lambda b$, $b=\frac{1}{4}a^2+\frac{1}{2}\lambda ab+\frac{1}{3}\lambda^2b^2$

$\therefore a=\int_0^1 y(t)\,dt=\int_0^1 \frac{1}{2}a+\lambda bt\,dt=\frac{1}{2}a+\frac{1}{2}\lambda b=a$

$b=\int_0^1 y(t)^2\,dt=\int_0^1(\frac{1}{2}a+\lambda bt)^2\,dt=\int_0^1\frac{1}{4}a^2+\lambda^2b^2t^2+\lambda abt\,dt=\frac{1}{2}a^2+\frac{1}{2}\lambda at+\frac{1}{3}\lambda^2b^2=b$ $\therefore$

sols are either $a=b=0$ or paramed by $\lambda\neq0$ via $a(\lambda)=\frac{12}{13\lambda}$ & $b(\lambda)=\frac{12}{13\lambda^2}$ $\therefore$

two sols: $y(x)=0\ \forall\lambda$ & $y(x)=\frac{6}{13\lambda}(1+2x)$, $\lambda\neq0$

\4/ $\phi=\int_0^x\frac{1+t^2}{1+\phi(t)^2}\,dt$ $\therefore$ $\phi(x)-\int_0^x\frac{1+t^2}{1+\phi(t)^2}\,dt=0$ $\therefore$ deriv:

$\phi'(x)-\frac{1+x^2}{1+\phi(x)^2}=0$ & $\phi(0)-\int_0^0\frac{1+t^2}{1+\phi(t)^2}\,dt=\phi(0)=0$ is a seperable eqn with sol:

$\phi+\frac{1}{3}\phi^3=x+\frac{1}{3}x^3$ $\therefore\phi'(x)=\frac{1+x^2}{1+\phi(x)^2}$ $\therefore$ $\phi'(x)(1+\phi(x)^2)=1+x^2$ $\therefore$ $\int\phi'(x)(1+\phi(x)^2)\,dx=\int 1+x^2\,dx$

$=x+\frac{1}{3}x^3+c=\int\phi'(x)+\phi'(x)(\phi(x))^2\,dx=\phi(x)+\frac{1}{3}\phi(x)^3=x+\frac{1}{3}x^3+c$ $\therefore$

$\phi(0)+\frac{1}{3}\phi(0)^3=0+\frac{1}{3}(0)^3+c=0+\frac{1}{3}(0)^3=0=0+c=c=0$ $\therefore$

$\phi(x)+\frac{1}{3}\phi^3=x+\frac{1}{3}x^3$ $\therefore$ IC : $(\phi-x)=\frac{1}{3}(x^3-\phi)=\frac{1}{3}(x-\phi)(\phi^2+x\phi+x^2)$ $\therefore\phi+\frac{1}{3}\phi^3=x+\frac{1}{3}x^3$

$\therefore\phi-x=\frac{1}{3}x^3-\frac{1}{3}\phi^3=\frac{1}{3}(x^3-\phi^3)=\frac{1}{3}(x-\phi)(x^2+\phi^2+x\phi)=\frac{1}{3}(x-\phi)(\phi^2+x\phi+x^2)$ $\therefore$

$\exists$ a sol $\phi(x)=x$ & dividing by $(\phi-x)$ otherwise no other sols.

\5 sol/ $\phi(x)-\int_{-1}^1(x+t)^2\phi(t)^2\,dt=0$ $\therefore\phi(x)=\int_{-1}^1(x^2+2xt+t^2)\phi(t)^2\,dt=$

$x^2\int_{-1}^1\phi(t)^2\,dt+2x\int_{-1}^1 t\phi(t)^2\,dt+\int_{-1}^1 t^2\phi(t)^2\,dt=\phi(x)$.

$\phi(x)-\int_{-1}^1(x+t)^2\phi(t)^2\,dt=0$ is a nonlinear hammerstein eqn

$\phi(x)-x^2\int_{-1}^1\phi(t)^2\,dt-2x\int_{-1}^1 t\phi(t)^2\,dt-\int_{-1}^1 t^2\phi(t)^2\,dt=0$ $\therefore$ is $\phi(x)=ax^2+2bx+c$ $\therefore$

$\phi(t)=at^2+bt+c$ $\therefore$ $a=\int_{-1}^1\phi(t)^2\,dt=\int_{-1}^1(at^2+bt+c)^2\,dt=\cancel{\int_{-1}^1 a^2t^4+c^2+b^2t^2+2act^2+2abt^3}$

$\int_{-1}^1 a^2t^4+c^2+b^2t^2+2act^2+2abt^3+2bct\,dt=\frac{2}{5}a^2+\frac{2}{3}ac+\frac{2}{3}ac+\frac{8}{3}b^2+2c^2$

$b=\int_{-1}^1 t\phi(t)^2\,dt=\int_{-1}^1 t(at^2+bt+c)^2\,dt=\frac{8}{5}ab+\frac{8}{3}bc$

$c=\int_{-1}^1 t^2\phi(t)^2\,dt=\int_{-1}^1 t^2(at^2+2bt+c)^2\,dt=\frac{2}{7}a^2+\frac{8}{5}b^2+\frac{2}{3}c^2+\frac{4}{5}ac$ $\therefore$ solving this

nonlinear syst for a,b,c sub into $\phi(x)=ax^2+2bx+c$ for sol

\Week10sheet/ \1a/ $y(x)-\int_0^1 a(x)y(t)+a(x)y(t)^4\,dt=0$ ∴

$y(x)=\int_0^1 a(x)y(t)+a(x)y(t)^4\,dt=a(x)\int_0^1 y(t)\,dt+a(x)\int_0^1 y(t)^4\,dt=y(x)$ ∴

$P_1=\int_0^1 y(t)\,dt$, $P_2=\int_0^1 y(t)^4\,dt$ ∴ $y(x)=a(x)P_1+a(x)P_2$ ∴ $y(t)=a(t)P_1+a(t)P_2$ ∴

$P_1=\int_0^1 a(t)P_1+a(t)P_2\,dt=(P_1+P_2)\int_0^1 a(t)\,dt$,

$P_2=\int_0^1 (a(t)P_1+a(t)P_2)^4\,dt=\int_0^1 a(t)^4(P_1+P_2)\,dt$ ~~x~~

\1a soln/ let $P=\int_0^1 (y(x)+y(x)^4)\,dx$ ∴ $y(x)=Pa(x)$ ∴ $y(t)=Pa(t)$ ∴

$P=\int_0^1 (Pa(x)+(Pa(x))^4)\,dx=P\int_0^1 a(x)\,dx+P^4\int_0^1 a(x)^4\,dx=P$ ∴

if $\int_0^1 a(x)\,dx=1$ $P=P+P^4\int_0^1 a(x)^4\,dx$ ∴ $0=P^4\int_0^1 a(x)^4\,dx$ ∴ $P=0$ ∴

$\int_0^1 a(x)\,dx=1$ ∴ $\int_0^1 a(x)^4\,dx\neq 0$

$P=P\int_0^1 a(x)\,dx+P^4\int_0^1 a(x)^4\,dx$ & if $\int_0^1 a(x)\,dx=1$, $P=0$ ∵ $\int_0^1 a(x)^4\,dx\neq 0$ ∴

if $\int_0^1 a(x)\,dx\neq 1$ ∴ $P-P\int_0^1 a(x)\,dx=P^4\int_0^1 a(x)^4\,dx=P\left(1-\int_0^1 a(x)\,dx\right)$ ∴

$P^3=\dfrac{1-\int_0^1 a(x)\,dx}{\int_0^1 a(x)^4\,dx}$ ∴ $P=\left(\dfrac{1-\int_0^1 a(x)\,dx}{\int_0^1 a(x)^4\,dx}\right)^{1/3}$ ∴ if $\int_0^1 a(x)\,dx=1$: $P=\left(\dfrac{1-1}{\int_0^1 a(x)^4\,dx}\right)^{1/3}=0$ ∴

$y(x)=0a(x)=0$ ∴ $y=0$ is always a sol

when $\int_0^1 a(x)\,dx\neq 1$: $y(x)=\left(\dfrac{1-\int_0^1 a(x)\,dx}{\int_0^1 a(x)^4\,dx}\right)^{1/3}a(x)$ have a non trivial sol

\1b/ $y(x)=\int_0^1 xy(t)+x^2ty(t)^2\,dt=x\int_0^1 y(t)+x^2\int_0^1 ty(t)^2\,dt$ ∴ let $P_1=\int_0^1 y(x)\,dx$ ∴

$P_2=\int_0^1 xy(x)^2\,dt$ & ∴ $y(x)=P_1x+P_2x^2$ & ∴ $y(t)=P_1t+P_2t^2$ ∴

$P_1=\int_0^1 y(x)\,dx=\int_0^1 P_1x+P_2x^2\,dx=P_1\int_0^1 x\,dx+P_2\int_0^1 x^2\,dx=P_1\left[\frac{1}{2}x^2\right]_0^1+P_2\left[\frac{1}{3}x^3\right]_0^1=$

$P_1\frac{1}{2}(1^2-0^2)+P_2\frac{1}{3}(1^3-0^3)=\frac{1}{2}P_1+\frac{1}{3}P_2=P_1$, $P_2=\int_0^1 xy(x)^2\,dx=\int_0^1 x(P_1x+P_2x^2)^2\,dx=$

$\int_0^1 x(P_1^2x^2+P_2^2x^4+2P_1P_2x^3)\,dx=\int_0^1 P_1^2x^3+P_2^2x^5+2P_1P_2x^4\,dx=\left[\frac{1}{4}P_1^2x^4+\frac{2}{5}P_1P_2x^5+\frac{1}{6}P_2^2\right]_0^1=$

$\frac{1}{4}P_1^2+\frac{2}{5}P_1P_2+\frac{1}{6}P_2^2=P_2$ ∴ $P_1=\frac{2}{3}P_2$ ∴ $P_2=\frac{49}{90}P_2^2$ ∴ $0=\frac{49}{90}P_2^2-P_2=\left(\frac{49}{90}P_2-1\right)=0$ ∴

$P_2=0$, $P_2=\frac{90}{49}$ ∴ if $P_2=0$: $P_1=0$ if $P_2=\frac{90}{49}$: $P_1=\frac{60}{49}$ ∴ $y(x)=P_1x+P_2x^2$ ∴

2 sols are: $y=0$ or $y(x)=\frac{30}{49}(x)(2+3x)$

\1c/ $\lambda y(x)=\int_{-\pi}^{\pi}\cos(t)y(t)+\sin(x)y(x)^3\,dt$ $y(x)=\frac{1}{\lambda}\int_{-\pi}^{\pi}\cos(t)y(t)+\sin(x)y(t)^3\,dt$ ∴

$\lambda y(x)=\int_{-\pi}^{\pi}\cos(t)y(t)\,dt+\sin(x)\int_{-\pi}^{\pi}y(t)^3\,dt$ ∴ $a=\int_{-\pi}^{\pi}\cos(t)y(t)\,dt$, $b=\int_{-\pi}^{\pi}y(t)^3\,dt$ ∴

$\lambda y(x)=a+b\sin x$ ∴ $y(t)=\frac{1}{\lambda}(a+b\sin t)$ ∴ $a\int_{-\pi}^{\pi}\cos t\,dt+b\int_{-\pi}^{\pi}\cos(t)\sin(t)\,dt=0=\lambda a$ ∴

$b=\int_{-\pi}^{\pi}\left(\frac{1}{\lambda}(a+b\sin t)\right)^3dt$ ∴ $\lambda^3b=\int_{-\pi}^{\pi}(a+b\sin t)^3\,dt=\int_{-\pi}^{\pi}a^3+3a^2b\sin t+3ab^2\sin^2(t)+b^3\sin^3(t)\,dt=$

$2\pi a^3+3\pi ab^2=b\lambda^3$ ∴ if $\lambda=0$: $a=b=0$, if $\lambda\neq 0$: $a=b=0$ ∴ $y=0$ is 2 only

soln ∀ λ

\6/ $\phi(x) - \int_0^x \frac{t\phi(t)}{1+\phi(t)^2}\,dt = 0 \therefore \frac{\partial}{\partial x}\int_0^x \frac{t\phi(t)}{1+\phi(t)^2}\,dt = \frac{x\phi(x)}{1+\phi(x)^2}(1) - 0 = \frac{x\phi(x)}{1+\phi(x)^2} \therefore$

is $\phi = 0$: $0 - \int_0^x \frac{t(0)}{1+0^2}\,dt = 0 - 0 = 0 \therefore \phi \equiv 0$ is a sol it is also Z only sol satisfies

starting at $\phi(0) = 0$; Z operator $(K\phi)(x) = \int_0^x \frac{t\phi(t)}{1+\phi(t)^2}\,dt$ which is correct only

$|(K\phi)(x)| \le \frac{1}{2}|x|^2\|\phi\|_\infty \therefore \|\phi\|_\infty = \|K\phi\|_\infty \le \frac{1}{2}\|\phi\|_\infty$ on $[0,1]$ $\phi(x) = \int_0^x \frac{t\phi(t)}{1+\phi(t)^2}\,dt = (K\phi) \therefore$

for $\phi(x) = 0 \because |(K\phi)(x)| = \left|\int_0^x \frac{t\phi(t)}{1+\phi(t)^2}\,dt\right| \le \frac{1}{2}|x|^2\|\phi\|_\infty$

$\|\phi\|_\infty = \|K\phi\|_\infty \quad |K\phi| \le \frac{1}{2}|x|^2\|\phi\|_\infty \therefore \|\phi\|_\infty = \|K\phi\|_\infty \le \frac{1}{2}\|\phi\|_\infty \therefore \|\phi\|_\infty \le \frac{1}{2}\|\phi\|_\infty$

on $[0,1]$ which is only correct for $\phi(x) = 0$

\7/ let y be a sol of Z eqn $\therefore$ $k = a - 2\int_0^1 y(x)^2\,dx$ is some real number $\therefore$ y satisfies

Z BVP : $\ddot{y}(x) + Ky(x) = 0 \quad y(0) = y(1) = 0 \therefore y = T(t)X(x) \therefore \ddot{y} = T''(t)X(t) \therefore$

$T''(t)X(x) + KT(t)X(t)X(x) = 0 \therefore T''(t)X(x) = -KT(t)X(x) \therefore \frac{T''(t)}{T(t)} = -K\frac{X(x)}{X(x)} = -K$

$T''(t) = -KT(t) \therefore T''(t) + KT(t) \therefore \cancel{\frac{T''(t)}{T(t)} = -K}$ $e^{qt} = T \therefore T'' = q^2e^{qt} \therefore q^2 + K = 0 \therefore$

$q^2 = -K \therefore q = \pm i\sqrt{K} \therefore T(t) = A_2e^{i\sqrt{K}t} + B_2e^{-i\sqrt{K}t} = A\cos(\sqrt{K}t) + B\sin(\sqrt{K}t) \therefore$

$y(x) = y \therefore \ddot{y} = y'' = \ddot{y}(x) \therefore q^2 + K = 0 \therefore y(x) = A\cos(\sqrt{K}x) + B\sin(\sqrt{K}x) \therefore y(0) = y(1) = 0 \therefore A = 0$

$\therefore y(x) = B\sin(\sqrt{K}x) \therefore y(1) = B\sin(\sqrt{K}) = 0 \therefore \sqrt{K} = n\pi \therefore K = (n\pi)^2 \therefore$

$\ddot{y}(x) + Ky(x) = 0 \quad \therefore y(0) = y(1) = 0 \therefore A = 0 \therefore y(x) = B\sin(\sqrt{K}x) \therefore y(1) = B\sin(\sqrt{K}) = 0 \therefore$

only nonzero sols when $K = (n\pi)^2$ $n \in \mathbb{N}$, Z eigvals of Z probl $\therefore$

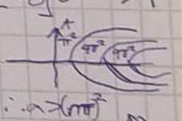
$y(x) = A\sin(n\pi x)$ (Z eigfunc) $\therefore$: sol is $y(t) = A\sin(n\pi t) \therefore$

$(n\pi)^2 = a - 2\int_0^1 (A\sin(n\pi t))^2\,dt = a - 2\int_0^1 A^2\sin^2(n\pi x)\,dx \therefore$

$\cos(2x) = \cos^2 x - \sin^2 x \therefore \cos^2(x) = \frac{1}{2} + \frac{1}{2}\cos(2x)$, $\sin^2(x) = \frac{1}{2} - \frac{1}{2}\cos(2x) \therefore$

$(n\pi)^2 = a - 2\int_0^1 A^2\sin^2(n\pi x)\,dx = a - 2A^2\int_0^1 \frac{1}{2} - \frac{1}{2}\cos(2n\pi x)\,dx = a - 2A^2\frac{1}{2}[1 - 0] =$

$a - A^2 = (n\pi)^2 \therefore a - (n\pi)^2 = A^2$



$\forall n \in \mathbb{N}$ which is a infinite family of parabolas for $n = 1$: $a = 0$: $-\pi^2 = A^2 \notin \mathbb{R} \therefore a > (n\pi)^2$

$\frac{1}{\sin\theta} = \operatorname{cosec}\theta \qquad \frac{1}{\cos\theta} = \sec\theta \qquad \frac{1}{\tan\theta} = \cot\theta$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \qquad \sin 2\theta = 2\sin\theta\cos\theta$

$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \qquad \cos 2\theta = \cos^2\theta - \sin^2\theta$

$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta} \qquad \cos 2\theta = 2\cos^2\theta - 1 \therefore \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos 2\theta$

$\cos 2\theta = 1 - 2\sin^2\theta \therefore \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$

$\tan(2\theta) = \frac{2\tan\theta}{1 - \tan^2\theta} \qquad \tan^2\theta + 1 = \sec^2\theta \qquad 1 + \cot^2\theta = \operatorname{cosec}^2\theta \therefore$

$\tan^2\theta + 1 = \frac{1}{\cos^2\theta} \qquad 1 + \frac{1}{\tan^2\theta} = \frac{1}{\sin^2\theta}$

(4ai) (4aii)

3042 integrals 2021 PP

1ai $\therefore \quad y(x) - \int_1^x \frac{y(t)}{t}\,dt = 2 \therefore$ the integral equation is volterra of the 2nd kind and linear and non homogeneous.

and the kernel is seperable.

1aii taking the derivative:

$\frac{y(x)}{x} + \int_1^x \frac{\partial}{\partial x}(t)\, y(t)\,dt = y'(x) = \frac{y(x)}{x} + \int_1^x (0)\, y(t)\,dt = \frac{y(x)}{x} = y'(x) \therefore$

at $x=1$: $\quad \int_1^1 \frac{y(t)}{t}\,dt = 0 = y(1) - 2 \quad \therefore y(1) = 2 \therefore$

$y'(x) - \frac{1}{x}y(x) = 0 \quad \therefore IF = e^{\int -\frac{1}{x}dx} = e^{-\ln x} = e^{\ln x^{-1}} = x^{-1} = \frac{1}{x} \therefore$

$\frac{d}{dx}\left(\frac{1}{x}y(x)\right) = 0 \quad \therefore \quad \int 0\,dx = \frac{1}{x}y(x) = C_1 \therefore \quad x \geq 1 \therefore x \neq 0 \therefore$

$y(x) = xC_1 \therefore \quad IC: \; y(1) = 2 = (1)C_1 = C_1 = 2 \therefore$

$y(x) = 2x$

1bi the equation is linear and non homogeneous, and Fredholm of the second kind.

1bii $\therefore \; y(x) = x^2 + \int_{-\pi}^{\pi} \cos(t)\sin(x)y(t) + x^2\sin(t)y(t)\,dt =$

$\cancel{\int_{-\pi}^{\pi}} x^2 + \sin(x)\int_{-\pi}^{\pi}\cos(t)y(t)\,dt + x^2\int_{-\pi}^{\pi}\sin(t)y(t)\,dt \therefore$

let $P_1 = \int_{-\pi}^{\pi}\cos(t)y(t)\,dt$, $\quad P_2 = \int_{-\pi}^{\pi}\sin(t)y(t)\,dt \therefore$

$y(x) = x^2 + \sin(x)P_1 + x^2P_2 \therefore \quad y(t) = t^2 + \sin(t)P_1 + t^2P_2 =$

$t^2(1+P_2) + \sin(t)P_1 \therefore$

$P_1 = \int_{-\pi}^{\pi}\cos(t)y(t)\,dt = \int_{-\pi}^{\pi}\cos(t)\left(t^2(1+P_2) + \sin(t)P_1\right)dt =$

$(1+P_2)\int_{-\pi}^{\pi} t^2\cos(t)\,dt + P_1\int_{-\pi}^{\pi}\sin(t)\cos(t)\,dt \therefore$

$\sin(t)\cos(t) = \frac{1}{2}(2)\sin(t)\cos(t) = \frac{1}{2}\sin(2t) \therefore$

$\int_{-\pi}^{\pi}\sin(t)\cos(t) = \frac{1}{2}\int_{-\pi}^{\pi}\sin(2t)\,dt = \frac{1}{2}\left[-\frac{1}{2}\cos(2t)\right]_{-\pi}^{\pi} =$

$-\frac{1}{4}\left[\cos(2\pi) - \cos(-2\pi)\right] = -\frac{1}{4}\left[1-1\right] = 0$

$\int_{-\pi}^{\pi} t^2\cos(t)\,dt = \left[t^2\sin(t)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2t\sin(t)\,dt =$

$$\left[\pi^2 \sin\pi - (-\pi)^2 \sin(-\pi)\right] - \int_{-\pi}^{\pi} 2t\sin(t)\,dt = 0 - 0 - \int_{-\pi}^{\pi} 2t\sin(t)\,dt = -2\int_{-\pi}^{\pi} t\sin(t)\,dt$$

$$= -2\left[t(-1)\cos(t)\right]_{-\pi}^{\pi} - 2(-1)\int_{-\pi}^{\pi}(-1)\cos(t)\,dt =$$

$$2\left[\pi\cos(\pi) - (-\pi)\cos(-\pi)\right] - 2\int_{-\pi}^{\pi}\cos(t)\,dt =$$

$$2\left[\pi(-1) + \pi(-1)\right] - 2\left[\sin(t)\right]_{-\pi}^{\pi} = 2\left[-2\pi\right] - 2\left[\sin(\pi) - \sin(-\pi)\right] =$$

$$-4\pi - 2\left[0 - 0\right] = -4\pi \therefore$$

$$P_1 = (1+P_2)(-4\pi) \cancel{+ \sin(t)} \quad P_1(0) = P_1 = (1+P_2)(-4\pi)$$

$$P_2 = \int_{-\pi}^{\pi}\sin(t)y(t)\,dt = \int_{-\pi}^{\pi}\sin(t)\left(t^2 + \sin(t)P_1 + t^2P_2\right)dt = \int_{-\pi}^{\pi}\sin(t)\left(t^2(1+P_2) + \sin(t)P_1\right)dt$$

$$= (1+P_2)\int_{-\pi}^{\pi} t^2\sin(t)\,dt + P_1\int_{-\pi}^{\pi}\sin^2(t)\,dt \therefore$$

$$\sin^2(t) = \tfrac{1}{2} - \tfrac{1}{2}\cos(2t) \therefore$$

$$\int_{-\pi}^{\pi}\sin^2(t)\,dt = \int_{-\pi}^{\pi}\tfrac{1}{2} - \tfrac{1}{2}\cos(2t)\,dt = \left[\tfrac{1}{2}t - \tfrac{1}{4}\sin(2t)\right]_{-\pi}^{\pi} =$$

$$\tfrac{1}{2}\left[\pi - (-\pi)\right] - \tfrac{1}{4}\left[\sin(2\pi) - \sin(-2\pi)\right] = \tfrac{1}{2}\left[\pi + \pi\right] - \tfrac{1}{4}(0 - 0) = \tfrac{1}{2}(2\pi) = \pi,$$

$$\int_{-\pi}^{\pi} t^2\sin(t)\,dt = \left[t^2(-1)\cos(t)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2t(-1)\cos(t)\,dt =$$

$$-1\left[\pi^2\cos(\pi) - \pi^2\cos(-\pi)\right] + 2\int_{-\pi}^{\pi} t\cos(t)\,dt =$$

$$-1\left[\pi^2(-1) - \pi^2(-1)\right] + 2\left[t\sin(t)\right]_{-\pi}^{\pi} - 2\int_{-\pi}^{\pi}\sin(t)\,dt =$$

$$-1\left[\pi^2 - \pi^2\right] + 2\left[\pi\sin(\pi) - (-\pi)\sin(-\pi)\right] - 2\left[-\cos(t)\right]_{-\pi}^{\pi}\,dt =$$

$$0 + 2\left[0 - 0\right] + 2\left[\cos(t)\right]_{-\pi}^{\pi} = 2\left[\cos(\pi) - \cos(-\pi)\right] =$$

$$\cancel{2[-1} \quad 2\left[-1 - (-1)\right] = 2\left[1 - 1\right] = 0 \therefore$$

$$P_2 = (1+P_2)(0) + P_1(\pi) = \pi P_1 = P_2 \therefore$$

$$P_1 = (-4\pi)(1+P_2) = (-4\pi)(1+\pi P_1) = -4\pi - 4\pi^2 P_1 = P_1 \therefore$$

$$-4\pi = P_1 + 4\pi^2 P_1 = P_1(1+4\pi^2) \therefore \quad P_1 = \frac{-4\pi}{1+4\pi^2} \therefore \quad P_2 = \pi P_1 = \frac{-4\pi^2}{1+4\pi^2} \therefore$$

$$y(x) = x^2 + \sin(x)P_1 + x^2P_2 = x^2 + \frac{-4\pi}{1+4\pi^2}\sin(x) + \frac{-4\pi^2}{1+4\pi^2}x^2$$

\1c/ $\mathcal{L}(F(t)) = f(s) = \mathcal{L}\left(e^t\cos(t) + t\sin(2t)\right) =$

$$\mathcal{L}(e^t\cos(t)) + \mathcal{L}(t\sin(2t)) \therefore$$

$$\mathcal{L}(e^t\cos(t)) = \mathcal{L}(e^{at}\hat{F}(t)) = \hat{f}(s-a), \quad a=1, \quad \hat{F}(t) = \cos(t) \therefore$$

$$\hat{f}(s) = \mathcal{L}(\hat{F}(t)) = \mathcal{L}(\cos(t)) = \frac{s}{s^2+1} \therefore \quad \hat{f}(s-1) = \frac{s-1}{(s-1)^2+1} \therefore$$

\3042 PP2021/ $\mathcal{L}(e^{t}\cos(t)) = \frac{s-1}{(s-1)^2+1}$,

$\mathcal{L}(t\sin(2t)) = \mathcal{L}(t^n \overset{*}{F}(t)) = (-1)^n \overset{*}{f}{}^{(n)}(s)$, $n=1$,

$\sin(2t) = \overset{*}{F}(t)$ $\therefore$ $\cancel{\overset{*}{f}{}^{(n)}(s)}$ $\therefore$ $\overset{*}{f}(s) = \mathcal{L}(\overset{*}{F}(t)) = \mathcal{L}(\sin(2t)) = \frac{2}{s^2+4}$ $\therefore$

$\overset{*}{f}{}^{(1)}(s) = \frac{d}{ds}\left(\frac{2}{s^2+4}\right) = \cancel{2(2s)}\ \frac{(s^2+4)(0)-2(2s)}{(s^2+4)^2} = \frac{-4s}{(s^2+4)^2}$ $\therefore$

$\mathcal{L}(t\sin(2t)) = (-1)^1 \frac{-4s}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2}$ $\therefore$

$\mathcal{L}(F(t)) = f(s) = \frac{s-1}{(s-1)^2+1} + \frac{4s}{(s^2+4)^2}$

\1di/ $F(t) = \mathcal{L}^{-1}(f(s)) = \mathcal{L}^{-1}\left(\frac{1}{(s+3)^2(s-2)}\right)$ $\therefore$

$f(s) = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s-2} = \frac{1}{(s+3)^2(s-2)}$ $\therefore$

$1 = A(s+3)(s-2) + B(s-2) + C(s+3)^2$ $\therefore$

at $s=2$: $1 = A(2+3)(0) + B(0) + C(2+3)^2 = 0+0+C(5)^2 = 25C = 1$ $\therefore$ $C = \frac{1}{25}$

at $s=-3$: $1 = A(0)(-3-2) + B(-3-2) + C(0)^2 = 0 + B(-5) + 0 = 1 = -5B$ $\therefore$ $-\frac{1}{5} = B$

coefficient of s^2: $0s^2 = As^2 + \frac{1}{25}s^2$ $\therefore$ $0 = A + \frac{1}{25}$ $\therefore$ $-\frac{1}{25} = A$ $\therefore$

$f(s) = -\frac{1}{25}\frac{1}{s+3} - \frac{1}{5}\frac{1}{(s+3)^2} + \frac{1}{25}\frac{1}{s-2}$ $\therefore$

$\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = e^{-3t}$, $\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t}$,

$\mathcal{L}^{-1}\left(\frac{1}{(s+3)^2}\right) = \mathcal{L}^{-1}\left(\frac{1!}{(s+3)^{1+1}}\right) = t^1 e^{-3t} = te^{-3t}$

$F(t) = \mathcal{L}^{-1}(f(s)) = -\frac{1}{25}e^{-3t} - \frac{1}{5}te^{-3t} + \frac{1}{25}e^{2t} =$

$-\frac{1}{5}e^{-3t}\left(\frac{1}{5}+t\right) + \frac{1}{25}e^{2t}$

\1dii/ ~~LkF~~ let $\mathcal{L}[F] = \mathcal{L}(F(t)) = f(s)$ $\therefore$

$\mathcal{L}[F'(t)] = sf(s) - F(0) = sf(s) - 0 = sf(s)$ $\therefore$

$\mathcal{L}(F''(t)) = s^2 f(s) - sF(0) - F'(0) = s^2 f(s) - s(0) - 0 = s^2 f(s)$ $\therefore$

$\mathcal{L}(F'' + F' \cancel{+} 6F) = \mathcal{L}(e^{-3t}) = \frac{1}{s+3} = s^2 f(s) + sf(s) \cancel{+} 6f(s) =$

$f(s)(s^2+s-6) = f(s)(s+3)(s-2)$ $\therefore$

$f(s) = \frac{1}{(s+3)(s+3)(s-2)} = \frac{1}{(s+3)^2(s-2)}$ $\therefore$

$F(t) = \mathcal{L}^{-1}(f(s)) = \mathcal{L}^{-1}\left(\frac{1}{(s+3)^2(s\cancel{+}-2)}\right) = -\frac{1}{5}e^{-3t}\left(\frac{1}{5}+t\right) + \frac{1}{25}e^{2t}$

2 a) it is Fredholm, nonlinear, non homogeneous, of the second kind $\therefore$

$y(x) = 2 + \int_1^2 y(t)\,dt - 2x\int_1^2 \frac{1}{(y(t))^2}\,dt \therefore$

let $P_1 = \int_1^2 y(t)\,dt$, $P_2 = \int_1^2 \frac{1}{(y(t))^2}\,dt \therefore$

$y(x) = 2 + P_1 - 2xP_2 \therefore y(t) = (2+P_1) - 2tP_2 = (2+P_1) - 2P_2 t$

$(y(t))^2 = ((2+P_1) - 2P_2 t)^2 = ((2+P_1) - 2P_2 t)((2+P_1) - 2P_2 t)$

$\cancel{(2+P_1)^2} = (2+P_1)^2 - 2P_2(2+P_1)(2)t + 4P_2 t^2 = (2+P_1)^2 - 4P_2(2+P_1)t + 4P_2 t^2 \therefore$

$P_1 = \int_1^2 y(t)\,dt = \int_1^2 (2+P_1) - 2P_2 t\,dt = \left[(2+P_1)t - 1P_2 t^2\right]_1^2 =$

$(2+P_1)[2-1] - 1P_2[2^2 - (1)^2] = (2+P_1)1 - P_2[4-1] = 2 + P_1 - 3P_2 = P_1 \therefore$

$\cancel{2+P_1}$ $2 - 3P_2 = 0 \therefore 2 = 3P_2 \therefore \frac{2}{3} = P_2 \therefore$

$y(t) = (2+P_1) - 2\left(\frac{2}{3}\right)t = (2+P_1) - \frac{4}{3}t \therefore$

$(y(t))^2 = \left((2+P_1) - \frac{4}{3}t\right)^2 = (2+P_1)^2 + \frac{16}{9}t^2 - \frac{8}{3}(2+P_1)t$

~~y(t) = (2+P_1)~~ let $2 + P_1 = b \therefore$

$y(t) = b - \frac{4}{3}t = -\frac{4}{3}t + b = -\frac{4}{3}\left(t - \frac{3}{4}b\right) \therefore$

$(y(t))^2 = \left(-\frac{4}{3}\left(t - \frac{3}{4}b\right)\right)^2 = \frac{16}{9}\left(t - \frac{3}{4}b\right)^2 \therefore$

$P_2 = \frac{2}{3} = \int_1^2 \frac{1}{(y(t))^2}\,dt = \int_1^2 \frac{1}{\frac{16}{9}\left(t - \frac{3}{4}b\right)^2}\,dt = \frac{9}{16}\int_1^2 \frac{1}{\left(t - \frac{3}{4}(2+P_1)\right)^2}\,dt = \frac{2}{3} \therefore$

let $P_1 = 0 \therefore$

$y(x) = 2 + 0 - 2x\left(\frac{2}{3}\right) = 2 - \frac{4}{3}x$

2 b) $f(s) = \frac{1}{(s^2+a^2)(s^2+b^2)} = \frac{1}{(s^2+a^2)} \cdot \frac{1}{(s^2+b^2)} = \hat{f}(s) \cdot \hat{g}(s) \therefore$

$\hat{f}(s) = \frac{1}{(s^2+a^2)} \therefore \hat{F}(t) = \mathcal{L}^{-1}(\hat{f}(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) =$

$\frac{1}{a}\sin(at)$,

$\hat{g}(s) = \frac{1}{s^2+b^2} \therefore \hat{G}(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+b^2}\right) = \frac{1}{b}\sin(bt) \therefore$

$F(t) = \mathcal{L}^{-1}(f(s)) = \int_0^t \hat{F}(u)\hat{G}(t-u)\,du = \int_0^t \frac{1}{a}\sin(au)\,\frac{1}{b}\sin(bt-bu)\,du$

$\therefore$ ~~sin(bu)~~ $= \frac{1}{ab}\int_0^t \sin(au)\sin(bt-bu)\,du \therefore$

\3072PP2021/ $\sin(au)\sin(bt-bu) = \sin(au+bt-bu) + \sin(au-(bt-bu))$ X

$\cancel{=\sin(bt)} = \sin((a-b)u+bt) + \sin((a+b)u-bt) \therefore$

$F(t) = \frac{1}{ab}\int_0^t \sin((a-b)u+bt) + \sin((a+b)u-bt)\,du =$

$\frac{1}{ab}\left[\frac{-1}{a-b}\cos((a-b)u+bt) + \frac{-1}{a+b}\cos((a+b)u-bt)\right]_{u=0}^{t} =$

$\frac{1}{ab}\left(\frac{-1}{a-b}\left[\cos((a-b)t+bt) - \cos((a-b)(0)+bt)\right] + \frac{-1}{a+b}\left[\cos((a+b)t-bt) - \cos((a+b)(0)-bt)\right]\right) =$

$\frac{1}{ab}\left(\frac{-1}{a-b}\left[\cos(at) - \cos(bt)\right] + \frac{-1}{a+b}\left[\cos(at) - \cos(-bt)\right]\right) =$

$\frac{-1}{ab(a-b)}\cos(at) + \frac{1}{ab(a-b)}\cos(bt) + \frac{-1}{ab(a+b)}\cos(at) + \frac{1}{ab(a+b)}\cos(bt) =$

$\cos(at)\left(\frac{-1}{ab(a-b)} + \frac{-1}{ab(a+b)}\right) + \cos(bt)\left(\frac{1}{ab(a-b)} + \frac{1}{ab(a+b)}\right) =$

$\cos(at)\left(\frac{-a-b-a+b}{ab(a-b)(a+b)}\right) + \cos(bt)\left(\frac{(a+b+a-b)}{ab(a-b)(a+b)}\right) =$

$\cancel{\left(\frac{-2a}{}\right.}$ $\frac{-2}{b(a-b)(a+b)}\cos(at) + \frac{2\cos(bt)}{b(a-b)(a+b)} = \frac{2}{b(a-b)(a+b)}\left(\cos(bt) - \cos(at)\right)$

X

\2ci/ $\therefore$ $\cancel{Kz}$ $Kz = (Kz)(x) = \int_0^1 e^{x+t} z(t)\,dt \quad \therefore$

$z=1: \quad Kz = (Kz)(x) = \int_0^1 e^{x+t}\,1\,dt = \int_0^1 e^{x+t}\,dt = e^x\int_0^1 e^t\,dt =$

$e^x\left[e^t\right]_0^1 = e^x\left[e^1 - e^0\right] = e^x\left[e^1 - 1\right] = (e-1)e^x = (e-1)\left(\frac{e^2-1}{2}\right)^0 e^x =$

$(e-1)\left(\frac{e^2-1}{2}\right)^{1-1} e^x$

$\therefore K^2 z = K(Kz) = \int_0^1 e^{x+t}((Kz)(t))\,dt = e^x\int_0^1 e^t\left((e-1)e^t\right)dt =$

$(e-1)e^x\int_0^1 e^{2t}\,dt = (e-1)e^x\left[\frac{1}{2}e^{2t}\right]_0^1 = (e-1)e^x\,\frac{1}{2}\left[e^2 - e^0\right] =$

$(e-1)e^x\,\frac{1}{2}(e^2-1)^1 = (e-1)\left(\frac{e^2-1}{2}\right)^1 e^x = (e-1)\left(\frac{e^2-1}{2}\right)^{2-1} e^x$

\2cii/ for $n=1$: $K^1 z = (e-1)\left(\frac{e^2-1}{2}\right)^{1-1} e^x$ is true $\therefore$

assuming true for $n=k$: $K^k z = (e-1)\left(\frac{e^2-1}{2}\right)^{k-1} e^x \quad \therefore$

for $n=k+1$: $K^{n+1} z = e^x\int_0^1 e^t\left[K^k z(t)\right]dt = e^x\int_0^1 e^t(e-1)\left(\frac{e^2-1}{2}\right)^{k-1} e^t\,dt =$

$e^x(e-1)\left(\frac{e^2-1}{2}\right)^{k-1}\int_0^1 e^{2t}\,dt = e^x(e-1)\left(\frac{e^2-1}{2}\right)^{k-1}\left[\frac{1}{2}e^{2t}\right]_0^1 =$

$(e-1)\left(\frac{e^2-1}{2}\right)^{k-1}\left(\frac{e^2-1}{2}\right)e^x=(e-1)\left(\frac{e^2-1}{2}\right)^{(k+1)-1}e^x$ ∴

is true for $n=1$ and
is true for $n=k$ then

$K^n z=(e-1)\left(\frac{e^2-1}{2}\right)^{n-1}e^x$ is true for $n=k+1$ ∴

$K^n z=(e-1)\left(\frac{e^2-1}{2}\right)^{n-1}e^x$ is true $\forall n\in\mathbb{N}$.

3ai $y_0(x)=1$ ∴ $y_0(t)=1$ ∴ $\cancel{y_1(x)=1+\int_0^x t\, y}$

$y_n(x)=1+\int_0^x t\,y_{n-1}(t)\,dt$ ∴ $y_1(x)=1+\int_0^x t\,y_0(t)\,dt=1+\int_0^x t(1)\,dt=$

$1+\int_0^x t\,dt=1+\left[\frac{1}{2}t^2\right]_0^x=1+\frac{1}{2}\left[x^2-0^2\right]=1+\frac{x^2}{2}=$

$\frac{1}{0!}\left(\frac{x^2}{2}\right)^0+\frac{1}{1!}\left(\frac{x^2}{2}\right)^1=\sum_{j=0}^{1}\frac{1}{j!}\left(\frac{x^2}{2}\right)^j=y_1(x)$ ∴

is true for $n=1$ ∴

assuming true for $n=k$: $y_k(x)=\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{x^2}{2}\right)^j$ ∴

for $n=k+1$: $y_{k+1}(x)=\int_0^x t\,y_k(t)\,dt=\int_0^x t\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{t^2}{2}\right)^j dt=$

$\cancel{\sum_{j=0}^{k}\int_0^x}=\int_0^x t\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{1}{2}\right)^j t^{2j}\,dt=\cancel{\int_0^x t\sum_{j=0}^{k}}$

$\left[t\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{1}{2}\right)^j\frac{1}{2j+1}t^{2j+1}\right]_0^x-\int\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{1}{2}\right)^j\frac{1}{2j+1}t^{2j+1}\,dt=$

$x\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{1}{2}\right)^j\frac{1}{2j+1}x^{2j+1}-\left[\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{1}{2}\right)^j\frac{1}{2j+1}\frac{1}{2j+2}t^{2j+2}\right]_0^x=$

$\int_0^x\sum_{j=0}^{k}\frac{1}{j!}t\left(\frac{t^2}{2}\right)^j dt=\left[\sum_{j=0}^{k}\frac{1}{j!}\frac{1}{j+1}\left(\frac{t^2}{2}\right)^{j+1}\right]_0^x=$

$\left[\sum_{j=0}^{k}\frac{1}{(j+1)!}\left(\frac{t^2}{2}\right)^{j+1}\right]_0^x=\left[\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{t^2}{2}\right)^j+\frac{1}{(k+1)!}\left(\frac{t^2}{2}\right)^{k+1}\right]_0^1=$

$\left[\sum_{j=0}^{k+1}\frac{1}{j!}\left(\frac{t^2}{2}\right)^j\right]_0^x=\sum_{j=0}^{k+1}\frac{1}{j!}\left(\frac{x^2}{2}\right)=y_{k+1}(x)$ ∴

is true for $n=1$ and is true for $n=k$, its true for $n=k+1$

∴ is true $\forall n\in\mathbb{N}$ ∴ $y_n(x)=\sum_{j=0}^{n}\frac{1}{j!}\left(\frac{x^2}{2}\right)^j$

3aii ∴ $y'(x)=xy(x)+\int_0^x y(t)\,dt$ ∴

$y''(x)=y(x)+xy'(x)+y(x)$ and at $x=0$: $y(0)=1$,

$y'(0)=0+0=0$: $y'(0)$ are I C ∴

$y''(x)-xy'(x)=2y(x)$

\3042PP2021/

\3b i/ $\lambda = 0 \therefore f(x) + \int_0^x t\,y(t)\,dt = 0 \therefore$

$f'(x) + xy(x) + \int_0^x y(t)\,dt = 0 \therefore$ X

$f''(x) + y(x) + xy'(x) + y(x) = 0 \therefore$

at $x = 0$ $f(0) + 0 = 0 = f(0) \therefore f(0) = 0$ is the condition

$2y(x) + xy'(x) = -f''(x) \therefore$

$y'(x) = \frac{2}{x} y(x) = -\frac{1}{x} f''(x) \therefore$

IF: $e^{\int \frac{2}{x} dx} = e^{2\ln x} = e^{\ln x^2} = x^2 \therefore$

$\frac{d}{dx}(x^2 y(x)) = -\frac{x^2}{x} f'(x) = -x f'(x) \therefore$

$y(x) = \frac{1}{x^2}\int_0^x -t f'(t)\,dt$?? X

\3b ii/ $\lambda y'(x) = f(x) + xy(x) + \int_0^x y(t)\,dt \therefore$ X

$\lambda y''(x) = f'(x) + y(x) + xy'(x) + y(x) \therefore$

?? both solutions are equal X

\3c/ $\cos x = e^{-x x} y(x) + \int_0^x -t e^{-xt} y(t)\,dt =$

$e^{-x^2} y(x) + \int_0^x -t e^{-xt} y(t)\,dt \therefore$

$e^{-x^2} y(x) = \cos(x) + \int_0^x e^{-xt}\, t\, y(t)\,dt \therefore$

$y(x) = e^{x^2}\cos(x) + e^{x^2}\int_0^x e^{-xt}\, t\, y(t)\,dt \therefore$

$y(x) = e^{x^2}\cos(x) + \int_0^x e^{x^2 - xt}\, t\, y(t)\,dt = e^{x^2}\cos x + \int_0^x e^{x(x-t)}\, t\, y(t)\,dt$

$\therefore$ (Q 3c) has a unique solution.

\4a i/ $\|k\|_\infty = \max_{x\in[0,2\pi]} \int_0^{2\pi} \left|\frac{\cos(x-t)}{6+5\sin(x-t)}\right| dt = \max_{x\in[0,2\pi]} \int_0^{2\pi} \left|\frac{1}{6+5\sin(x-t)}\right| |\cos(x-t)|\,dt$

$|\cos(x-t)| \le 1,\ 0 \le |\sin(x-t)| \le 1 \therefore \left|\frac{\cos(x-t)}{6+5\sin(x-t)}\right| < \infty \therefore$ bounded

$\therefore \|k\|_\infty \le \infty \therefore$ bounded $\therefore$

$\max_{x\in[0,2\pi]}\left|\frac{\cos(x-t)}{6+5\sin(x-t)}\right| = \frac{1}{6+5(0)} = \frac{1}{6} \therefore \|k\|_\infty \le 2\pi$ X

\4a ii/ $\cos(x-t) = \cos x \cos t + \sin x \sin t \therefore$

$$\|k\|_\infty = \max_{x\in[0,2\pi]} \int_0^{2\pi} \left| \frac{\cos(x-t)}{6+5\sin(x-t)} \right| dt = \max_{x\in[0,2\pi]} \left| \int_0^{2\pi} -\frac{5}{5}\frac{\cos(x-t)}{6+5\sin(x-t)}\, dt \right| =$$

$$\max_{x\in[0,2\pi]} \frac{1}{5} \left| \int_0^{2\pi} \frac{-5\cos(x-t)}{6+5\sin(x-t)}\, dt \right| = \max_{x\in[0,2\pi]} \frac{1}{5} \left| \Big[\ln|6+5\sin(x-t)| \Big]_0^{2\pi} \right| =$$

$$\frac{1}{5}\ln\Big| \ln|6+5\sin(x-2\pi)| - \ln|6+5\sin(x)| \Big| = \ln(11)$$

\4a / for $f(x)=0$ ∴ $3y(x) = \int_0^{2\pi} \frac{\cos(x-t)}{6+5\sin(x-t)} y(t)\,dt$

$3y(x) = (ky)(x)$ ∴ $\|3y(x)\|_\infty = 3\|y(x)\|_\infty = \|k\|_\infty = \ln(11)$ ∴

$\|y(x)\|_\infty = \frac{1}{3}\ln(11)$ ∴ the homogeneous equation of the given equation has a unique solution ∵ $\|y(x)\|_\infty < \infty$ ∴ by the Fredholm Alternative: the given equation has ~~unique~~ one unique, continuous solution $\forall f \in C^0[0,2\pi]$

\4c / $\|f\|_\infty = 4$ ∴ $\|y-y_n\|_\infty \le \frac{1}{|\lambda|-\|k\|_\infty}\left(\frac{\|k\|_\infty}{|\lambda|}\right)^{n+1}\|f\|_\infty$ ∴

∴ $\lambda = 3$ ∴ $|\lambda| = 3$, $\|k\|_\infty = \ln(11)$ ∴

$\|y-y_n\|_\infty = 10^{-2} = 0.01 \le \frac{1}{3-\ln(11)}\left(\frac{\ln(11)}{3}\right)^{n+1} 4$ ∴

$1.505262\times10^{-3} = 0.001505262 \le \left(\frac{\ln(11)}{3}\right)^{n+1} = \left(\frac{\ln(11)}{3}\right)^{n}\left(\frac{\ln(11)}{3}\right)$ ∴

$0.00188323 \le \left(\frac{\ln(11)}{3}\right)^n$ ∴

$\ln(0.00188323) \le \ln\left(\left(\frac{\ln(11)}{3}\right)^n\right) = n\ln\left(\frac{\ln(11)}{3}\right)$ ∴

∴ $\frac{-6.274768}{\ln\left(\frac{\ln(11)}{3}\right)} \le n$ ∴ $28.0097 \le n$ ∴ $n = 29$ iterations

\1a sol / 2 integral eqn $\int_1^x \frac{y(t)}{t}\,dt = y(x) - 2 \quad x \ge 1$ is a linear volterra eqn of second kind with continuous, seperable kernel. to solve it, can differentiate it ∴ $y'(x) = \frac{y(x)}{x}$ this is a seperable 1st order ODE with general sol $y(x) = Cx$

for 2 integral eqn, $y(1) = 2$ ∴ sol: $y(x) = 2x$

\1b / 2 eqn is a linear, non homog, second kind Fredholm eqn, with cont & rank 2 kernel.

let $a = \int_{-\pi}^{\pi} \cos(t)\, y(t)\,dt$, $b = \int_{-\pi}^{\pi} \sin(t)\, y(t)\,dt$ ∴

\3042 PP2021/ $\cancel{a = \int_{-\pi}^{\pi} \cos(t)(1+b)}$ $y(x) = (1-b)x^2 + a\sin(x)$ ∴

$a = \int_{-\pi}^{\pi} \cos(t)\left((1+b)t^2 + a\sin(t)\right)dt = (1+b)\int_{-\pi}^{\pi} t^2\cos(t)\,dt + a\int_{-\pi}^{\pi}\cos(t)\sin(t)\,dt$

& $b = \int_{-\pi}^{\pi} \sin(t)\left((1+b)t^2 + a\sin t\right)dt = (1+b)\int_{-\pi}^{\pi} t^2\sin(t)\,dt + a\int_{-\pi}^{\pi}(\sin(t))^2\,dt$

∴ note: $\int_{-\pi}^{\pi}\cos(t)\sin(t)\,dt = \int_{-\pi}^{\pi} t^2\sin(t)\,dt = 0$ ∵ Z integrands are odd functions,

$\int_{-\pi}^{\pi}(\sin(t))^2\,dt = \pi$, $\int_{-\pi}^{\pi} t^2\cos(t)\,dt = \left[t^2\sin t\right]_{-\pi}^{\pi} - 2\int_{-\pi}^{\pi} t\sin t\,dt =$

$-2\left[-t\cos t\right]_{-\pi}^{\pi} - 2\int_{-\pi}^{\pi}\cos(t)\,dt = -4\pi$ ∴

$a = -4\pi(1+b)$, $b = \pi a$ ∴

$a = \frac{-4\pi}{1+4\pi^2}$, $b = \frac{-4\pi^2}{1+4\pi^2}$ ∴ $y(x) = (1+b)x^2 + a\sin(x) = \frac{x^2 - 4\pi\sin(x)}{1+4\pi^2}$

\e/ $F(t) = e^t\cos(t) + t\sin(2t)$ ∴ Laplace:

$\mathcal{L}(F(t)) = \frac{s-1}{(s-1)^2+1} + \frac{4s}{(s^2+4)^2}$ using first shift thm for Z 1st one & LT12 for Z 2nd

\di/ $f(s) = \frac{1}{(s+3)^2(s-2)}$ ∴ $f(s) = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s-2}$ ∴

$1 = A(s+3)(s-2) + B(s-2) + C(s+3)^2$ ∴ $B = -\frac{1}{5}$, $C = \frac{1}{25}$, $A = -\frac{1}{25}$ ∴

$F(t) = -\frac{1}{25}e^{-3t} - \frac{t}{5}e^{-3t} + \frac{1}{25}e^{2t}$ by first shift thm

\dii/ ∴ $s^2 f(s) + s f(s) - 6 f(s) = \frac{1}{s+3}$ by Laplace of each term of ODE ∴ $f(s) = \frac{1}{(s+3)^2(s-2)}$ ∴ $F(t)$ is that of Z one given in part (i)

\2a/ $y(x) = 2 + \int_1^2 \left(y(t) - \frac{2x}{y^2(t)}\right)dt$ this is a nonlinear Hammerstein finite rank IE of Z 2nd kind ∴ $y(x) = 2 + \underbrace{\int_1^2 y(t)\,dt}_{P_1} - 2x\underbrace{\int_1^2 \frac{dt}{y^2(t)}}_{P_2}$

$y(x) = 2 + P_1 - 2xP_2$ ∴ $P_1 = \int_1^2 (2 + P_1 - 2tP_2)\,dt$ ①

$P_2 = \int_1^2 \frac{dt}{(2 + P_1 - 2tP_2)^2}$ ② ∴

from ① $P_1 = 2 + P_1 - 3P_2$ ∴ $P_2 = \frac{2}{3}$

from ② $\cancel{\frac{2}{3} = \frac{3}{4}\left(2 + P_1 - \frac{8}{3}\right)}$ $\frac{2}{3} = \int_1^2 \frac{dt}{(2 + P_1 - \frac{4}{3}t)^2}$ $= -\frac{3}{4}\int_1^2 -\frac{4}{3}\left(2 + P_1 - \frac{4}{3}t\right)^{-2}dt$

$= -\frac{3}{4}\left[\frac{1}{-1}\left(2 + P_1 - \frac{4}{3}t\right)^{-1}\right]_1^2 = \left[\frac{3}{4}\left(2 + P_1 - \frac{4}{3}t\right)^{-1}\right]_1^2$ ∴

$\frac{2}{3} = \frac{3}{4}\left(2+p_1-\frac{8}{3}\right)^{-1} - \frac{3}{4}\left(2+p_1-\frac{4}{3}\right)^{-1}$ $\therefore$ $\frac{8}{9} = \frac{1}{p_1-\frac{2}{3}} - \frac{1}{p_1+\frac{2}{3}}$ $\therefore$

$\cancel{p_1^2 = 7^2}$ $p_1^2 - \frac{4}{9} = \frac{36}{24}$ $\therefore$ $p_1 = \pm\sqrt{\frac{35}{18}}$ $\therefore$ $\exists$ two sols:

$y(x) = 2 \pm \sqrt{\frac{35}{18}} - \frac{4x}{3}$

2b find Z inverse laplace transform of each part:

$\mathcal{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin(at)$, $\mathcal{L}^{-1}\left(\frac{1}{s^2+b^2}\right) = \frac{1}{b}\sin(bt)$ $\therefore$ from Z convolution thm.

$\mathcal{L}^{-1}\left(\frac{1}{(s^2+a^2)(s^2+b^2)}\right) = \frac{1}{a}\sin(at) * \frac{1}{b}\sin(bt) = \frac{1}{ab}\int_0^t \sin(au)\sin(bt-bu)\,du$

$\therefore$ $\because$ $\sin(a)\sin(b) = (\cos(a-b) - \cos(a+b))\frac{1}{2}$.

$\sin(au)\sin(bt-bu) = (\cos(au-bt+bu) - \cos(au+bt-bu))\frac{1}{2} =$

$(\cos((a+b)u-bt) - \cos((a-b)u+bt))\frac{1}{2}$ $\therefore$

$\mathcal{L}^{-1}(f(s)) = \cancel{\frac{1}{ab}\int_0^t \frac{1}{2}(\cos(a-b)-\cos(a+b))\,dt} =$

$\frac{1}{ab}\int_0^t \frac{1}{2}(\cos(au-bt+bu) - \cos(au+bt-bu))\,du =$

$\frac{1}{2ab}\int_0^t \cos(au-bt+bu) - \cos(au+bt-bu)\,du =$

$\frac{1}{2ab}\left[\frac{1}{a+b}\sin(au-bt+bu) - \frac{1}{a-b}\sin(au+bt-bu)\right]_{u=0}^{t} =$

$\frac{1}{2ab}\left[\frac{1}{a+b}(\sin(at) - \sin(-bt)) - \frac{1}{a-b}(\sin(at) - \sin(bt))\right] =$

$\frac{1}{2ab}\left[\frac{1}{(a+b)(a-b)}(a-b)(\sin(at)+\sin(bt)) - \frac{1}{(a+b)(a-b)}(a+b)(\sin(at)-\sin(bt))\right] =$

$\frac{1}{2ab}\left(\frac{1}{a^2-b^2}\right)\left[a\sin(at) + a\sin(bt) - b\sin(at) - b\sin(bt) - a\sin(at) + a\sin(bt) - b\sin(at) + b\sin(bt)\right]$

$= \frac{1}{2ab(a^2-b^2)}\left[2a\sin(bt) - 2b\sin(at)\right] = \frac{1}{ab(a^2-b^2)}(a\sin(bt) - b\sin(at))$

2c let $z \equiv 1$ $\therefore$ $(Kz)(x) = e^x\int_0^1 e^t\,dt = e^x(e-1)$ $\therefore$

$(K^2z)(x) = e^x\int_0^1 e^t e^t(e-1)\,dt = (e-1)e^x\frac{e^2-1}{2}$ $\therefore$

by induction $(K^{n+1}z)(x) = e^x\int_0^1 e^t(K^nz)(t)\,dt = e^x\int_0^1 e^t(e-1)e^t\left(\frac{e^2-1}{2}\right)^{n-1}dt$

$(e-1)e^x\left(\frac{e^2-1}{2}\right)^n$ base clause also verified

3ai Z neumann series is $y(x) = \sum_{j=0}^{\infty} \tilde{y}_j = \lim_{n\to\infty}\sum_{j=0}^{n}\tilde{y}_j$ where

$\tilde{y}_j = \frac{1}{\lambda^{j+1}}K^j f = K\tilde{y}_{j-1}$ $\therefore$ $\tilde{y}_0 = \frac{f}{\lambda}$ to prove that $\tilde{y}_j = \frac{1}{j!}\left(\frac{x^2}{2}\right)^j$,

$\because$ $f = 1$, $\lambda = 1$ $\therefore$ $\tilde{y}_0 = 1$ $\therefore$ for Z inductive step:

$\tilde{y}_{j+1}(x) = (K\tilde{y}_j)(x) = \int_0^x t\,y_j(t)\,dt = \frac{1}{j!}\int_0^x \frac{t^{2j+1}}{2^j}\,dt = \int_0^x t\,\frac{1}{j!}\left(\frac{x^2}{2}\right)^j dt =$

\3042 PP2021/ $\frac{1}{j!}\int_0^x \frac{t^{2j+1}}{2^j}\,dt = \frac{1}{j!\,2^j}\int_0^x t^{2j+1}\,dt =$

$\frac{1}{j!\,2^j}\left[\frac{1}{2j+2}t^{2j+2}\right]_0^x = \frac{1}{(j!)(j+1)(2)(2^j)}\left[x^{2(j+1)}\right] = \frac{1}{(j+1)!}\frac{1}{2^{j+1}}\left[x^2\right]^{j+1} =$

$\frac{1}{(j+1)!}\frac{x^{2(j+1)}}{2^{j+1}} = \frac{1}{(j+1)!}\left(\frac{x^2}{2}\right)^{j+1} = \tilde{y}_{j+1}(x)$ $\therefore$ $\tilde{y}_j = \frac{1}{j!}\left(\frac{x^2}{2}\right)^j$ with

base clause verified

\3a ii/ differentiate Z eqn: $y'(x) = 0 + xy(x) + \int_0^x 0\,y(t)\,dt =$
$0 + xy(x) + \int_0^x 0\,dt = 0 + xy(x) + 0 = y'(x) = xy(x)$ $\therefore$ $y'(x) = xy(x)$ with
$y(0) = 1 + 0 = 1$ $\therefore$ $y(0) = 1$ $\therefore$ $\frac{y'(x)}{y(x)} = x$ $\therefore$ $\ln(y) = \frac{x^2}{2} + \ln(c)$ $\therefore$
$\ln(1) = \frac{0^2}{2} + \ln(c) = 0 = 0 + \ln(c) = 0 = \ln(c)$ $\therefore$ $\ln(y) = \frac{x^2}{2}$ $\therefore$ $y(x) = e^{\frac{x^2}{2}}$
$\therefore$ we see that Z result in part (i) is Z taylor series of $y(x) = e^{x^2/2}$

\3b i/ $\lambda = 0$ $\therefore$ $\int_0^x t\,y(t)\,dt = -f(x)$ $\therefore$ $-f(0) = \int_0^0 t\,y(t)\,dt = 0 = f(0)$ $\therefore$
$f(0) = 0$ $\therefore$ differentiating, have: $-f'(x) = xy(x) + \int_0^x 0\,y(t)\,dt = xy(x)$
$\therefore$ $xy(x) = -f'(x)$ ie Z sol is $y(x) = -\frac{f'(x)}{x}$ $\therefore$ for Z sol to be cont,
need $\frac{f'}{x} \in C[0,\infty)$, particularly $f'(0) = 0$

\3b ii/ $\lambda \neq 0$ $\lambda y(x) = f(x) + \int_0^x t\,y(t)\,dt$, $x \geq 0$ $\therefore$
$\lambda y'(x) = f'(x) + xy(x)$ $\therefore$ $y'(x) - \frac{x}{\lambda}y(x) = \frac{f'(x)}{\lambda}$, $\lambda y(0) = f(x) + 0$ $\therefore$
$y(0) = \frac{f(0)}{\lambda}$ $\therefore$ $IF = e^{\int -\frac{x}{\lambda}dx} = e^{-\frac{1}{2\lambda}x^2}$:
$\frac{d}{dx}\left(e^{-\frac{1}{2\lambda}x^2}y(x)\right) = \frac{1}{\lambda}e^{-\frac{1}{2\lambda}x^2}f'(x)$ $\therefore$

$e^{-\frac{1}{2\lambda}x^2}y(x) = \int_0^x \frac{1}{\lambda}e^{-\frac{1}{2\lambda}t^2}f'(t)\,dt + y(0) = \int_0^x \frac{1}{\lambda}e^{-\frac{1}{2\lambda}t^2}f'(t)\,dt + \frac{f(0)}{\lambda}$ $\therefore$

$y(x) = \frac{f(0)}{\lambda}e^{x^2/(2\lambda)} + \frac{1}{\lambda}e^{\frac{x^2}{2\lambda}}\int_0^x e^{-\frac{1}{2\lambda}t^2}f'(t)\,dt =$

$y(x) = \frac{f(0)}{\lambda}e^{\frac{x^2}{2\lambda}} + \frac{1}{\lambda}\int_0^x e^{\frac{1}{2\lambda}(x^2-t^2)}f'(t)\,dt$ $\therefore$ integration by parts:

$y(x) = \frac{f(0)}{\lambda}e^{\frac{x^2}{2\lambda}} + \frac{1}{\lambda}\left[e^{\frac{1}{2\lambda}(x^2-t^2)}f(t)\right]_{t=0}^x - \frac{1}{\lambda}\int_0^x -2t\frac{1}{2\lambda}e^{\frac{1}{2\lambda}(x^2-t^2)}f(t)\,dt =$

$\frac{f(0)}{\lambda}e^{\frac{x^2}{2\lambda}} + \frac{1}{\lambda}f(x) - \frac{1}{\lambda}e^{\frac{1}{2\lambda}x^2}f(0) + \frac{1}{\lambda^2}\int_0^x t\,e^{\frac{1}{2\lambda}(x^2-t^2)}f(t)\,dt =$

$y(x) = \frac{f(x)}{\lambda} + \frac{1}{\lambda^2}\int_0^x t\,e^{\frac{x^2-t^2}{2\lambda}}f(t)\,dt$

\3c/ Z 1st kind integral eqn is equivalent to Z 2nd kind probl obtained by differentiating ($\because k(x,x)=e^{-x^2}\neq 0$ & $f(0)=0$ $\therefore$ $k(x,t)=e^{-xt}$, $\sin(0)=f(0)=0$) $\therefore$

$\cos x = e^{-xx}\cancel{y(x)}\int_0^x \cancel{-t}e^{-xt}y(t)\,dt = e^{-x^2}y(x)+\int_0^x(-t)e^{-xt}y(t)\,dt$ $\cancel{=}$ $\cancel{=}\int_0^x +te^{-xt}y(t)\,dt$ $\therefore$ $e^{-x^2}y(x)=\cos x+\int_0^x te^{-xt}y(t)\,dt$ $\therefore$

$y(x)=e^{x^2}\cos(x)+e^{x^2}\int_0^x te^{-xt}y(t)\,dt=y(x)=e^{x^2}\cos(x)+\int_0^x e^{x(x-t)}t\,y(t)\,dt$

this 2nd kind volterra eqn has a unique sol & by equivalence, Z original eqn has a unique sol

\4ai/ $(Ky)(x)=\int_0^{2\pi}\frac{\cos(x\cancel{-t}-t)}{6+5\sin(x-t)}y(t)\,dt$

$k(x,t)=\frac{\cos(x-t)}{6+5\sin(x-t)}$ $\therefore$ ~~sup~~ $\sup(\cos(x-t))=1$,

$\inf(\cancel{\sin}(x-t))=-1$ $\therefore$ $\sup(k(x,t))=\sup\left(\frac{\cos(x-t)}{6+5\sin(x-t)}\right)=$

$\frac{\sup(\cos(x-t))}{\inf(6+5\sin(x-t))}=\cancel{\sup}\frac{1}{6+5\inf(\sin(x-t))}=\frac{1}{6+5(-1)}=\frac{1}{6-5}=\frac{1}{1}=1$

$\therefore$ $\cancel{\|\|_\infty}$ $\|y\|_\infty=\sup\{|y(x)|:x\in[a,b]\}$

$|(Ky)(x)|\le\int_0^{2\pi}\|k\|_\infty\|y\|_\infty\,dt$ $\therefore$ $\|k\|_\infty=\frac{1}{6-5}$ $\therefore$

$|(Ky)(x)|\le\int_0^{2\pi}\left\|\frac{\cos(x-t)}{6+5\sin(x-t)}\right\|_\infty|y(t)|\,dt=\int_0^{2\pi}\frac{1}{6-5}|y(t)|\,dt=$

$\int_0^{2\pi}1|y(t)|\,dt\le|y(t)|\int_0^{2\pi}1\,dt=|y(t)|\,[t]_0^{2\pi}=|y(t)|(2\pi-0)=$

$|y(t)|2\pi$ $\therefore$ $|Ky(x)|\le|y(t)|2\pi$ $\therefore$

$\|k\|_\infty=1$ $\therefore$ $\|K\|_\infty\le\|k\|_\infty 2\pi=2\pi$ $\therefore$ $\|K\|_\infty\le 2\pi$

\4aii/ $\|K\|_\infty=\max_{x\in[0,2\pi]}\int_0^{2\pi}\left|\frac{\cos(x-t)}{6+5\sin(x-t)}\right|dt$ $\therefore$ let $u=t-x$ $\therefore$

$du=dt$ $\therefore$ $t=0\to u=0-x=-x$, $t=2\pi\to u=2\pi-x$ $\therefore$

$\|K\|_\infty=\max_{x\in[0,2\pi]}\int_{-x}^{2\pi-x}\left|\frac{\cos u}{6+5\sin u}\right|du=\int_{-0}^{2\pi-0}\left|\frac{\cos u}{6+5\sin u}\right|du=\int_0^{2\pi}\left|\frac{\cos u}{6+5\sin u}\right|du$

$\therefore$ $\cancel{\|K\|_\infty}$ {let $u=(x-t)$ $\therefore$ $du=-dt$ $\therefore$ $-du=dt$ $\therefore$ $t=0\to x-0=x$, $t=2\pi\to x-2\pi=u$

$\therefore\|K\|_\infty=\max_{x\in[0,2\pi]}\int_0^{2\pi}|k(x,t)|\,dt=\max_{x\in[0,2\pi]}\int_0^{2\pi}\left|\frac{\cos(x-t)}{6+5\sin(x-t)}\right|dt=\max_{x\in[0,2\pi]}\int_x^{x-2\pi}\left|\frac{\cos u}{6+5\sin u}\right|(-1)\,du$

$=\max_{x\in[0,2\pi]}\int_{x-2\pi}^{x}\left|\frac{\cos u}{6+5\sin u}\right|du=\int_{2\pi-2\pi}^{2\pi}\left|\frac{\cos u}{6+5\sin u}\right|du=\int_0^{2\pi}\left|\frac{\cos u}{6+5\sin u}\right|du=$

30+2 pp2021 / $\int_0^{2\pi} \frac{|\cos u|}{|6+5\sin u|}\,du = \int_0^{2\pi} \frac{|\cos(u)|}{6-5\sin u}\,du =$ (due to 2π-periodicity)

$2\int_{-\frac{\pi}{2}}^{\pi/2} \frac{\cos u}{6-5\sin(u)}\,du = \frac{2}{-5}\int_{-\frac{\pi}{2}}^{\pi/2} \frac{-5\cos u}{6-5\sin u}\,du = \frac{2}{-5}\left[\ln|6-5\sin u|\right]_{-\pi/2}^{\pi/2}$

$-\frac{2}{5}\left[\ln|6-5\sin\frac{\pi}{2}| - \ln|6-5\sin(-\frac{\pi}{2})|\right] =$

$-\frac{2}{5}\left[\ln|6-5(1)| - \ln|6-5(-1)|\right] = -\frac{2}{5}\left[\ln(1)-\ln(11)\right] = \frac{2}{5}\ln(11) \neq \ln(11)$ X

4 a ii / $\|k\|_\infty = \max_{x\in[0,2\pi]} \int_0^{2\pi} \left|\frac{\cos(x-t)}{6+5\sin(x-t)}\right| dt$ ∴ let $u=t-x$ ∴ $du=dt$ ∴

$t=0 \to u=0-x=-x$, $t=2\pi \to 2\pi-x=u$ ∴ $-u=-(t-x)=-t+x=x-t$ ∴

$\|k\|_\infty = \max_{x\in[0,2\pi]} \int_{-x}^{2\pi-x} \left|\frac{\cos(-u)}{6+5\sin(-u)}\right| du = \int_{-0}^{2\pi-0} \left|\frac{\cos(-u)}{6+5\sin(-u)}\right| du = \int_0^{2\pi} \left|\frac{\cos(-u)}{6+5\sin(-u)}\right| du$

$\int_0^{2\pi} \left|\frac{\cos(u)}{6+5\sin(-u)}\right| du = \int_0^{2\pi} \left|\frac{\cos(u)}{6-5\sin(u)}\right| du = \int_0^{2\pi} \frac{|\cos(u)|}{|6-5\sin(u)|}\,du$

∴ $6-5\sin u > 0$ ∴ $|6-5\sin(u)| = 6-5\sin(u)$ ∴

$\|k\|_\infty = \int_0^{2\pi} \frac{|\cos(u)|}{6-5\sin(u)}\,du =$ (due to 2π periodicity)

$2\int_{-\frac{\pi}{2}}^{\pi/2} \frac{\cos u}{6-5\sin u}\,du = \left[-\ln|6-5\sin(u)|\right]_{-\pi/2}^{\pi/2} = \ln(11)-\ln(1) = \ln(11) \approx 2.399$

4 b / ∵ $\|k\|_\infty = 2.399$, Z eqn $\lambda y - Ky = \delta$ is uniquely solvable for $|\lambda| > \|k\|_\infty \approx 2.399$ & here, $\lambda = 3$ ∴ have a unique cont sol by Z Fredholm alternative

4 c / $\|y-y_n\|_\infty \leq \frac{1}{|\lambda|-\|k\|_\infty}\left(\frac{\|k\|_\infty}{|\lambda|}\right)^{n+1}\|\delta\|_\infty = \frac{1}{3-2.397}\left(\frac{2.397}{3}\right)^{n+1} 4 = \Delta$

∴ $\Delta < 10^{-2}$ is sufficient for $\|y-y_n\|_\infty$ to be less than 10^{-2} ∴

$\left(\frac{2.397}{3}\right)^{n+1} < \frac{(3-2.397)}{400}$ ∴ $(n+1)\ln(0.799) < \ln(0.0015075)$ ∴

$n+1 > 28.955$ ∴ $n > 28$

2 a / nonlinear nonhomogeneous Fredholm of the second kind, with kernel of finite rank ∴

$y(x) = 2 + \underbrace{\int_1^2 y(t)\,dt}_{P_1} - 2x\underbrace{\int_1^1 \frac{1}{(y(t))^2}\,dt}_{P_2}$ ∴

$y(x) = 2 + P_1 - 2xP_2 = (2+P_1) - 2P_2 x$ ∴ $y(t) = (2+P_1) - 2P_2 t$ ∴

$P_1 = \int_1^2 y(t)\,dt = \int_1^2 (2+P_1) - 2P_2 t\,dt = \left[(2+P_1)t - P_2 t^2\right]_1^2 = (2+P_1)(2-1) - P_2(2^2-1^2) =$

$2+P_1-3P_2 = P_1$ ∴ $2 = 3P_2$ ∴ $\frac{2}{3} = P_2$ ∴

$y(t) = (2+P_1) - \frac{4}{3}t \quad \therefore \quad y(x) = (2+P_1) - \frac{4}{3}t \quad \therefore$

$P_2 = \int_1^2 \frac{1}{(y(t))^2}\,dt = \int_1^2 (y(t))^{-2}\,dt = \int_1^2 \left((2+P_1) - \frac{4}{3}t\right)^{-2} dt = 2$

$\& \quad \frac{d}{dt}\left[\left((2+P_1) - \frac{4}{3}t\right)^{-1}\right] = -1\left(-\frac{4}{3}\right)\left((2+P_1) - \frac{4}{3}t\right)^{-2} = \frac{4}{3}\left((2+P_1 - \frac{4}{3}t\right)^{-2} \quad \therefore$

$\therefore \cancel{y(\ )}\, P_2 = \int_1^2 \frac{1}{\left((2+P_1) - \frac{4}{3}t\right)^2}\,dt = -\frac{3}{4}\int_1^2 \frac{\left(-\frac{4}{3}\right)}{\left((2+P_1) - \frac{4}{3}t\right)^2}\,dt =$

$-\frac{3}{4}\int_1^2 \left(-\frac{4}{3}\right)\left((2+P_1) - \frac{4}{3}t\right)^{-2} dt = -\frac{3}{4}\left[\frac{1}{-1}\left((2+P_1) - \frac{4}{3}t\right)^{-1}\right]_1^2 =$

$\frac{3}{4}\left[\frac{1}{\left((2+P_1) - \frac{4}{3}t\right)}\right]_1^2 = \frac{3}{4}\left[\frac{1}{2+P_1 - \frac{4}{3}(2)} - \frac{1}{2+P_1 - \frac{4}{3}(1)}\right] = \frac{3}{4}\left[\frac{1}{P_1 - \frac{2}{3}} - \frac{1}{P_1 + \frac{2}{3}}\right] =$

$\frac{2}{3} \quad \therefore \quad \frac{1}{P_1 - \frac{2}{3}} - \frac{1}{P_1 + \frac{2}{3}} = \frac{8}{9} \doteq$

$\frac{P_1 + \frac{2}{3}}{\left(P_1 - \frac{2}{3}\right)\left(P_1 + \frac{2}{3}\right)} - \frac{P_1 - \frac{2}{3}}{\left(P_1 - \frac{2}{3}\right)\left(P_1 + \frac{2}{3}\right)} = \frac{8}{9} = \frac{\left(\frac{4}{3}\right)}{P_1^2 - \frac{4}{9}} \quad \therefore$

$P_1^2 - \frac{4}{9} = \frac{3}{2} \quad \therefore P_1^2 = \frac{35}{18} \quad \therefore \quad P_1 = \pm\sqrt{\frac{35}{18}} = \pm\frac{\sqrt{70}}{6} \quad \therefore$

$y(x) = 2 \pm \frac{\sqrt{70}}{6} - \frac{4}{3}t$

\2b/ $f(s) = \frac{1}{(s^2+a^2)} \cdot \frac{1}{(s^2+b^2)} \quad \therefore \quad \mathcal{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\sin(at)$,

$\mathcal{L}^{-1}\left(\frac{1}{s^2+b^2}\right) = \frac{1}{b}\sin(bt) \quad \therefore \quad \mathcal{L}^{-1}(f(s)) = f(t) = \frac{1}{a}\sin(at) * \frac{1}{b}\sin(bt) =$

$\int_0^t \frac{1}{a}\sin(au)\frac{1}{b}\sin(b(t-u))\,du = \frac{1}{ab}\int_0^t \sin(au)\sin(bt - bu)\,du \quad \therefore$

$\sin(A)\sin(B) = \cos(A-B) - \cos(A+B) \quad \therefore \quad \cancel{X}\ \sin(A)\sin B = \frac{1}{2}\left[\cos(A-B) - \cos(A+B)\right]$

$\mathcal{L}^{-1}(f(s)) = \frac{1}{2ab}\int_0^t \cos(au - bt + bu) - \cos(au + bt - bu)\,du =$

$\frac{1}{2ab}\left[\frac{1}{a+b}\sin(au - bt + bu) - \frac{1}{a-b}\sin(au + bt - bu)\right]_{u=0}^{t} =$

$\frac{1}{2ab}\left[\frac{1}{a+b}\left(\sin(at) - \sin(-bt)\right) - \frac{1}{a-b}\left(\sin(at) - \sin(bt)\right)\right] = \cancel{X}$

$\cancel{\frac{1}{ab}} \quad \frac{1}{ab(a^2-b^2)}\left(a\sin(bt) - b\sin(at)\right)$

\3bi/ $\lambda = 0: \quad f(x) + \int_0^x t\,y(t)\,dt = 0 \quad \therefore f(0) = 0 \quad \therefore$

$f'(x) = x\,y(x) = 0 \quad \therefore \quad y(x) = \frac{f'(x)}{x}$

\3bii/ $\cancel{\lambda y} \quad \lambda y(0) = f(0) \quad \therefore \quad y(0) = \frac{f(0)}{\lambda} \quad \therefore$

$\lambda y'(x) = f'(x) + x\,y(x) \quad \therefore \quad y'(x) - \frac{1}{\lambda}x\,y(x) = \frac{f'(x)}{\lambda} \quad \therefore$

\30+2FP2021/ $IF = e^{\int -\frac{1}{\lambda}x\,dx} = \frac{1}{\lambda}e^{-\frac{1}{2\lambda}x^2}$ $\therefore$ $\frac{d}{dx}\left(e^{-\frac{1}{2\lambda}x^2}y(x)\right) = \frac{1}{\lambda}e^{-\frac{1}{2\lambda}x^2}f'(x)$

$\therefore$ $e^{-\frac{1}{2\lambda}x^2}y(x) = \int \frac{1}{\lambda}e^{-\frac{1}{2\lambda}x^2}f'(x)\,dx + C$ $\therefore$ at $x=0$:

$$e^0 y(0) = y(0) = \frac{f(0)}{\lambda} = 0 + C = C = \frac{f(0)}{\lambda} \therefore$$

$\cancel{\int 0}$ $e^{-\frac{1}{2\lambda}x^2}y(x) = \int \frac{1}{\lambda}e^{-\frac{1}{2\lambda}x^2}f'(x)\,dx + \frac{f(0)}{\lambda}$ $\therefore$

$y(x) = e^{\frac{1}{2\lambda}x^2}\int \frac{1}{\lambda}e^{-\frac{1}{2\lambda}x^2}f'(x)\,dx + \frac{f(0)}{\lambda}e^{\frac{1}{2\lambda}x^2}$ $\therefore$

is $\int e^{-\frac{1}{2\lambda}x^2}f'(x) = \left[\frac{1}{\lambda}e^{-\frac{1}{2\lambda}x^2}f(x)\right] - \int -\frac{1}{\lambda^2}xe^{-\frac{1}{2\lambda}x^2}f(x)\,dx =$

$\frac{1}{\lambda}e^{-\frac{1}{2\lambda}x^2}f(x) + \int \frac{1}{\lambda^2}xe^{-\frac{1}{2\lambda}x^2}f(x)\,dx$ $\therefore$

$$\frac{d}{dx}\left(e^{-\frac{1}{2\lambda}x^2}y(x)\right) = \frac{1}{\lambda}e^{-\frac{1}{2\lambda}x^2}f'(x) \therefore$$

$e^{-\frac{1}{2\lambda}x^2}y(x) = \int_0^x \frac{1}{\lambda}e^{-\frac{1}{2\lambda}x^2}f'(\cancel{x})\,d\cancel{x} = \int_0^x \frac{1}{\lambda}e^{-\frac{1}{2\lambda}t^2}f'(t)\,dt + C$ $\therefore$

at $x=0$: $e^0 y(0) = y(0) = \frac{f(0)}{\lambda} = 0 + C = C = \frac{f(0)}{\lambda}$ $\therefore$

$e^{-\frac{1}{2\lambda}x^2}y(x) = \int_0^x \frac{1}{\lambda}e^{-\frac{1}{2\lambda}t^2}f'(t)\,dt + \frac{f(0)}{\lambda}$ $\therefore$

$y(x) = e^{\frac{1}{2\lambda}x^2}\int_0^x \frac{1}{\lambda}e^{-\frac{1}{2\lambda}t^2}f'(t)\,dt + \frac{f(0)}{\lambda}e^{\frac{1}{2\lambda}x^2}$ $\therefore$

$y(x) = \int_0^x \frac{1}{\lambda}e^{\frac{1}{2\lambda}(x^2-t^2)}f'(t)\,dt + \frac{f(0)}{\lambda}e^{\frac{1}{2\lambda}x^2}$ $\therefore$

$y(x) = \left[\frac{1}{\lambda}e^{\frac{1}{2\lambda}(x^2-t^2)}f(t)\right]_{t=0}^{x} - \int -\frac{1}{\lambda^2}t\,e^{\frac{1}{2\lambda}(x^2-t^2)}f(t)\,dt + \frac{f(0)}{\lambda}e^{\frac{1}{2\lambda}x^2}$

$= \cancel{e^0 f}\ \frac{1}{\lambda}f(x) - \frac{1}{\lambda}e^{\frac{1}{2\lambda}x^2}f(0) + \int \frac{1}{\lambda^2}e^{\frac{1}{2\lambda}(x^2-t^2)}t\,f(t)\,dt + \frac{1}{\lambda}e^{\frac{1}{2\lambda}x^2}f(0) =$

$y(x) = \frac{1}{\lambda}f(x) + \int \frac{1}{\lambda^2}e^{\frac{1}{2\lambda}(x^2-t^2)}t\,f(t)\,dt$

\3042PP2020/

\1a/ let $\mathcal{L}(F(t)) = f(s) = \mathcal{L}(2t^2) - \mathcal{L}((t+1)e^{-t}) + \mathcal{L}((\sinh(t))^2)$ ∴

- $\mathcal{L}(2t^2) = 2\mathcal{L}(t^2) = 2\ \frac{2!}{s^{2+1}} = 2\frac{2}{s^3} = \frac{4}{s^3}$

$\mathcal{L}((t+1)e^{-t}) = \mathcal{L}(e^{-t}t) + \mathcal{L}(e^{-t})$ ∴ $\frac{1!}{(s+1)^2} = \frac{1}{(s+1)^2} = \mathcal{L}(e^{-t}t)$,

$\mathcal{L}(e^{-t}) = \frac{1}{s-(-1)} = \frac{1}{s+1}$ ∴ $\mathcal{L}((t+1)e^{-t}) = \frac{1}{(s+1)^2} + \frac{1}{s+1}$,

$(\sinh(t))^2 = \left(\frac{1}{2}(e^t - e^{-t})\right)^2$ &

let $\hat{F}(t) = (\sinh(t))^2$ ∴ $\hat{F}'(t) = 2\sinh(t)\cosh(t)$ &

∴ $(\sinh(t))^2 = \frac{1}{4}(e^{2t} + e^{-2t} - 2)$ ∴

$\mathcal{L}(\sinh(t)^2) = \frac{1}{4}\mathcal{L}(e^{2t}) + \frac{1}{4}\mathcal{L}(e^{-2t}) - \frac{1}{2}\mathcal{L}(1) = \frac{1}{4}\frac{1}{s-2} + \frac{1}{4}\frac{1}{s+2} - \frac{1}{2}\frac{1}{s}$

∴ $\mathcal{L}(F(t)) = \frac{4}{s^3} + \frac{1}{(s+1)^2} + \frac{1}{s+1} + \frac{1}{4(s-2)} + \frac{1}{4(s+2)} - \frac{1}{2s}$

\1b/ let $\mathcal{L}^{-1}(f(s)) = F(t)$ ∴ $f(s) = \frac{5}{s^2} \times \frac{1}{(2-s)^2}$ ∴

$\frac{1}{(2-s)^2} = \frac{1}{(-1(s-2))^2} = \frac{1}{(s-2)^2}$ ∴ $f(s) = \frac{5}{s^2} \times \frac{1}{(s-2)^2}$ ∴

$\mathcal{L}^{-1}(f(s)) = \mathcal{L}^{-1}\left(\frac{5}{s^2} \times \frac{1}{(s-2)^2}\right) = \mathcal{L}^{-1}\left(\frac{5}{s^2}\right) * \mathcal{L}^{-1}\left(\frac{1}{(s-2)^2}\right)$ &

∴ $\mathcal{L}^{-1}\left(\frac{5}{s^2}\right) = 5\mathcal{L}^{-1}\left(\frac{1!}{s^{1+1}}\right) = 5t^1 = 5t$

$\mathcal{L}^{-1}\left(\frac{1}{(s-2)^2}\right) = \mathcal{L}^{-1}\left(\frac{1!}{(s-2)^{1+1}}\right) = e^{2t}t$ ∴

$\mathcal{L}^{-1}(f(s)) = 5t * e^{2t}t = e^{2t}t * 5t = \int_0^t e^{2u}u\ 5(t-u)\,du =$

$5\int_0^t t\,ue^{2u}\,du - \int_0^t u^2e^{2u}\,du = 5t\int_0^t ue^{2u}\,du - \int_0^t u^2e^{2u}\,du$ ∴

$\int_0^t ue^{2u}\,du = \left[\frac{1}{2}ue^{2u}\right]_{u=0}^t - \int_0^t \frac{1}{2}e^{2u}\,du =$

$\left[\frac{1}{2}te^{2t} - \frac{1}{2}(0)e^{2(0)}\right] - \frac{1}{2}\left[\frac{1}{2}e^{2u}\right]_{u=0}^t = \frac{1}{2}te^{2t} - 0 - \frac{1}{4}\left[e^{2t} - e^{2(0)}\right] =$

$\frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + \frac{1}{4}$ ∴

- $\int_0^t u^2e^{2u}\,du = \left[\frac{1}{2}u^2e^{2u}\right]_{u=0}^t - \int_0^t \frac{1}{2}2ue^{2u}\,du =$

$\frac{1}{2}t^2e^{2t} - \frac{1}{2}(0)^2e^{2(0)} - \int_0^t ue^{2u}\,du =$

$\frac{1}{2}t^2e^{2t} - \left[\frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + \frac{1}{4}\right] = \frac{1}{2}t^2e^{2t} - \frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} - \frac{1}{4}$ $\therefore$

$\mathcal{L}^{-1}(g(s)) = 5t\left[\frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + \frac{1}{4}\right] - \left[\frac{1}{2}t^2e^{2t} - \frac{1}{2}te^{2t} + \frac{1}{4}e^{2t} - \frac{1}{4}\right] =$

$\frac{5}{2}t^2e^{2t} - \frac{5}{4}te^{2t} + \frac{5}{4}t - \frac{1}{2}t^2e^{2t} + \frac{1}{2}te^{2t} - \frac{1}{4}e^{2t} + \frac{1}{4} =$

$2t^2e^{2t} - \frac{3}{4}te^{2t} + \frac{5}{4}t - \frac{1}{4}e^{2t} + \frac{1}{4}$

\1ci/ a kernel is weakly singular if the $k(x,t)$ has: ~~$\lim_{x \to t} k(x,t) \notin \mathbb{R}$~~ ~~lim~~ $k(t,t) \notin \mathbb{R}$ so the kernel is not continuous at $x=t$ but $\lim_{x \to t} k(x,t) < \lim_{x \to t}\left(\frac{1}{x-t}\right)$

\1cii/ for A: its kernel $k_A(x,t) = \cos(x-t) = \cos(x)\cos(t) + \sin(x)\sin(t)$ $\therefore$ $\cos(x-t)$ is continuous every where $\therefore$ the kernel is not weakly singular.

~~\1c~~ for B: its kernel $k_B(x,t) = \frac{1}{(x-t)^{1/2}}$ $\therefore$ $\frac{1}{(x-t)^{1/2}} < \frac{1}{x-t}$ and $k_B(t,t) = \frac{1}{(t-t)^{1/2}} = \frac{1}{0^{1/2}} = \frac{1}{0} \notin \mathbb{R}$ $\therefore$ the kernel is not continuous and not strongly singular $\therefore$ it is weakly singular.

for C: its kernel $k_C(x,t) = \cos(x-t)|x-t|^{-1/2} = \cos(x-t)\frac{1}{|x-t|^{1/2}}$ $\therefore$ $k_C(t,t) \notin \mathbb{R}$ $\therefore$ kernel is not continuous and $-1 < \cos(x-t) < 1$; $\frac{1}{|x-t|^{1/2}} = \frac{1}{\sqrt{|x-t|}} < \frac{1}{x-t}$ $\therefore$ kernel is not strongly singular $\therefore$ the kernel is weakly singular.

\1d/ The equation is: linear, volterra, nonhomogeneous, of the second kind. $\therefore$ let $\mathcal{L}(y(x)) = \hat{y}(s)$ $\therefore$

$y(x) = 2 + \int_0^x y(t)e^{2(x-t)}\,dt = y(x)$ $\therefore$ taking laplacian $\therefore$ ~~g at~~ if $x=0$. $y(0) = 2 + \int_0^0 y(t)e^{2(x-t)}\,dt = 2 + 0 = 2 = y(0)$ $\therefore$

$\mathcal{L}(y(x)) = \mathcal{L}(2) + \mathcal{L}\left(\int_0^x y(t)\,e^{2(x-t)}\,dt\right) = \hat{y}(s) = \frac{2}{s} + \mathcal{L}(g(x))\mathcal{L}(e^{2x}) =$

$\hat{y}(s) = \frac{2}{s} + \hat{g}(s)\frac{1}{s-2}$ $\therefore$ $\hat{g}(s) - \frac{1}{s-2}\hat{g}(s) = \frac{2}{s} = \left(1 - \frac{1}{s-2}\right)\hat{g}(s) = \left(\frac{s-3}{s-2}\right)\hat{g}(s)$

$\therefore \hat{y}(s) = \frac{2(s-2)}{s(s-3)} = \frac{A}{s} + \frac{B}{s-3}$ $\therefore$ $2(s-2) = 2s-4 = A(s-3) + Bs$ $\therefore$

\3042PP2020/ let $s=0$: $2(0)-4=-4=A(0-3)+0B=-3A=-4 \therefore A=\frac{4}{3}$

let $s=3$: $2(3)-4=2=A(0)+3B=3B=2 \therefore B=\frac{2}{3} \therefore$

$\hat{y}(s)=\frac{4}{3}\frac{1}{s}+\frac{2}{3}\frac{1}{s-3} \therefore$

$\mathcal{L}^{-1}(\hat{y}(s))=y(x)=\frac{4}{3}(1)+\frac{2}{3}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right)=\frac{4}{3}+\frac{2}{3}e^{3x}$

\1e/ Equation is Fredholm, non homogeneous, nonlinear, of the second kind. $\therefore$

$y(x)=2-x\int_0^1\sqrt{t}\,(y(t))^2dt \therefore$ let $p=\int_0^1\sqrt{t}\,(y(t))^2\,dt \therefore$

$y(x)=2-px \therefore y(t)=2-pt \therefore$

$p=\int_0^1\sqrt{t}\,(y(t))^2\,dt=\int_0^1\sqrt{t}\,(2-pt)^2dt=\int_0^1 t^{1/2}(4+p^2t^2-4pt)\,dt=$

$\int_0^1 4t^{1/2}-4pt^{3/2}+p^2t^{5/2}\,dt=\left[4\left(\frac{2}{3}\right)t^{3/2}-4p\left(\frac{2}{5}\right)t^{5/2}+p^2\left(\frac{2}{7}\right)t^{7/2}\right]_0^1=$

$\frac{8}{3}\left[1^{3/2}-0^{3/2}\right]-\frac{8}{5}p\left[1^{5/2}-0^{5/2}\right]+\frac{2}{7}p^2\left[1^{7/2}-0^{7/2}\right]=$

$\frac{8}{3}-\frac{8}{5}p+\frac{2}{7}p^2=p \therefore \frac{2}{7}p^2-\frac{13}{5}p+\frac{8}{3} \therefore$

$p=\dfrac{\frac{13}{5}\pm\sqrt{\left(-\frac{13}{5}\right)^2-4\left(\frac{2}{7}\right)\left(\frac{8}{3}\right)}}{2\left(\frac{2}{7}\right)}=\frac{91}{20}\pm\dfrac{\sqrt{\frac{1949}{525}}}{(4/7)}\approx\frac{91}{20}\pm\left(\frac{7}{4}\right)1.93$ (3 S.F.)

$\therefore$ ~~p = 7.92~~ $p=7.92,\ 1.18$ (3 S.F.) $\therefore$

$y(x)=2-px$, $p=7.92, 1.18 \therefore$

$y(x)=2-7.92x$, $y(x)=2-1.18x$

\2i/ $\lambda=1 \therefore y_0(x)=f=2=y_0(t) \therefore$

~~$y_{n+1}(x)$~~ $y_n(x)=2+\int_0^x t^2y_{n-1}(t)\,dt \therefore$

$y_1(x)=2+\int_0^x t^2(2)\,dt=2+2\int_0^x t^2\,dt=2+2\left[\frac{1}{3}t^3\right]_0^x=2+2\left[\frac{1}{3}x^3-\frac{1}{3}(0)^3\right]=$

$2+2\left[\frac{1}{3}x^3\right]=$ ~~2~~ $2\times\frac{1}{0!}\left(\frac{x^3}{3}\right)^0+2\times\frac{1}{1!}\left(\frac{x^3}{3}\right)^1=2\sum_{j=0}^{1}\frac{1}{j!}\left(\frac{x^3}{3}\right)^j \therefore$

$\therefore$ sequence is true for $n=1 \therefore$

assuming true for $n=k$: $y_k(x)=2\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{x^3}{3}\right)^j \therefore$

for $n=k+1$: $y_{k+1}(x)=2+\int_0^x t^2y_k(t)\,dt=2+\int_0^x t^2\,2\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{t^3}{3}\right)^j dt=$

$2+\int_0^x 2\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{1}{3}\right)^j t^{3j}t^2\,dt = 2+\int_0^x 2\sum_{j=0}^{n}\frac{1}{j!}\left(\frac{1}{3}\right)^j t^{3j+2}\,dt =$

$2+\left[2\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{1}{3}\right)^j\frac{1}{3j+3}t^{3j+3}\right]_0^x =$

$2+2\sum_{j=0}^{k}\frac{1}{j!}\left(\frac{1}{3}\right)^j\frac{1}{3}\frac{1}{j+1}\left(t^3\right)^{j+1} = 2+\sum_{j=0}^{k}\frac{1}{(j+1)!}\left(\frac{x^3}{3}\right)^{j+1} =$

$2\sum_{j=0}^{k+1}\frac{1}{j!}\left(\frac{x^3}{3}\right)^j$ $\therefore$ state true also for $n=k+1$ $\therefore$

by induction: statement is true $\forall n\in\mathbb{N}$

\2aii/ $\frac{d}{dx}\left(\int_0^x k(x,t)y(t)\,dt\right) = k(x,x)y(x)+\int_0^x\frac{\partial k(x,t)}{\partial t}y(t)\,dt$ $\therefore$

$\frac{\partial}{\partial x}(t^2)=0$ $\therefore$ taking the derivative of the equation:

$y'(x)=0+x^2y(x)+\int_0^x(0)y(t)\,dt = x^2y(x)=y'(x)$ $\therefore$

$y'(x)-x^2y(x)=0$ is the equation as an ODE $\therefore$

~~$y'(x)$~~ $x^2y(x)=\frac{dy}{dx}$ $\therefore$ $x^2=\frac{1}{y(x)}\frac{dy}{dx}$ $\therefore$

$\int x^2\,dx=\int\frac{1}{y(x)}\frac{dy}{dx}\,dx=\int\frac{1}{y(x)}\,dy=\ln|y(x)|=\frac{1}{3}x^3+c_1$ $\therefore$

$|y(x)|=e^{\frac{1}{3}x^3+c_1}=e^{c_1}e^{\frac{1}{3}x^3}$ $\therefore y(x)=c_2e^{\frac{1}{3}x^3}$ $\therefore$

~~$y(0)$~~ at $x=0$: $y(0)=2+\int_0^0 t^2y(t)\,dt=2+0=2=y(0)$ $\therefore$

$y(0)=2=c_2e^{\frac{1}{3}(0)^3}=c_2e^0=c_2=2$ $\therefore$

$y(x)=2e^{\frac{1}{3}x^3}$ $\therefore$

The previous result is the taylor series of $2e^{\frac{1}{3}x^3}$

and are both equal.

\2b/ equation is nonlinear, non homogeneous, Fredholm, of the second kind

$\therefore$ for $\lambda=0$: $4=\int_{-1}^{1}xt^2(y(t))^2\,dt=x\int_{-1}^{1}t^2(y(t))^2\,dt$ $\therefore$

let ~~P~~ $P=\int_{-1}^{1}t^2(y(t))^2\,dt$ $\therefore$ $4=Px=(0)x+4$ $\therefore$ by comparing coefficients: $4=Px$ is impossible $\therefore\lambda\neq0$ $\therefore$

$4-x\int_{-1}^{1}t^2(y(t))^2\,dt=\lambda y(x)=4-xP=4-Px=\lambda y(x)$ $\therefore$

$y(x)=\frac{4}{\lambda}-\frac{1}{\lambda}Px$ $\therefore$ ~~$y(t)=\frac{1}{\lambda}4$~~ $y(t)=4\frac{1}{\lambda}-\frac{1}{\lambda}Px$ $\therefore$

\3042PP2020/ $P=\int_{-1}^{1} t^2\left(4\frac{1}{\lambda}-\frac{1}{\lambda}Pt\right)^2 dt=\int_{-1}^{1} t^2\left(16\frac{1}{\lambda^2}+\frac{1}{\lambda^2}P^2t^2-8\frac{1}{\lambda^2}Pt\right)dt$

$=\int_{-1}^{1} 16\frac{1}{\lambda^2}t^2-8\frac{1}{\lambda^2}Pt^3+\frac{1}{\lambda^2}P^2t^4\,dt=\left[\frac{16}{3}\frac{1}{\lambda^2}t^3-2\frac{1}{\lambda^2}Pt^4+\frac{1}{5}\frac{1}{\lambda^2}P^2t^5\right]_{-1}^{1}$

$\frac{16}{3}\frac{1}{\lambda^2}\left[(1)^3-(-1)^3\right]-2\frac{1}{\lambda^2}P\left[(1)^4-(-1)^4\right]+\frac{1}{5}\frac{1}{\lambda^2}P^2\left[1^5-(-1)^5\right]=$

$\frac{32}{3}\frac{1}{\lambda^2}+0+\frac{2}{5}\frac{1}{\lambda^2}P^2=\frac{32}{3}\frac{1}{\lambda^2}+\frac{2}{5}\frac{1}{\lambda^2}P^2=P$ $\therefore$

$\frac{2}{5\lambda^2}P^2-P+\frac{32}{\lambda^2}=0$ $\therefore$

$P=\frac{1\pm\sqrt{1^2-4\left(\frac{2}{5\lambda^2}\right)\left(\frac{32}{3\lambda^2}\right)}}{2\left(\frac{2}{5\lambda^2}\right)}=\frac{5\lambda^2}{4}\pm\left(\sqrt{1-\frac{256}{15\lambda^4}}\right)\frac{5\lambda^2}{4}$ $\therefore$

$y(x)=\frac{4}{\lambda}-\frac{1}{\lambda}Px=\frac{4}{\lambda}-\frac{1}{\lambda}\left(\frac{5\lambda^2}{4}\pm\left(\sqrt{1-\frac{256}{15\lambda^4}}\right)\frac{5\lambda^2}{4}\right)=$

$\frac{4}{\lambda}-\frac{5}{4}\lambda\mp\left(\sqrt{1-\frac{256}{15\lambda^4}}\right)\frac{5\lambda}{4}$ has solutions for

$1-\frac{256}{15\lambda^4}\geq 0$ $\therefore$ $\lambda^4-\frac{256}{15}\geq 0$ $\therefore$ $\lambda^4\geq\frac{256}{15}$ $\therefore$

$\lambda\geq 0$ $\therefore$ $\lambda^2\geq\sqrt{\frac{256}{15}}$ $\therefore$ $\lambda\geq\left(\frac{256}{15}\right)^{1/4}$

\4a/ let $\mathcal{L}(x(t))=x(s)$ $\therefore$ $\mathcal{L}(x'(t))=sx(s)-x(0)=sx(s)+1$

$\therefore$ $\mathcal{L}(x''(t))=s^2x(s)-s\,x(0)-x'(0)=s^2x(s)-s(-1)-6=s^2x(s)+s-6$ $\therefore$

$\mathcal{L}(3t^2+t-1)=3\frac{2!}{s^{2+1}}+\frac{1!}{s^{1+1}}-\frac{1}{s}=\frac{6}{s^3}+\frac{1}{s^2}-\frac{1}{s}=$

$s^2x(s)+s-6-(sx(s)+1)-6x(s)=$

$s^2x(s)+s-6-sx(s)-1-6x(s)=(s^2-s-6)x(s)+(s-7)=\frac{6}{s^3}+\frac{1}{s^2}-\frac{1}{s}=$

$(s-3)(s+2)x(s)+(s-7)$ $\therefore$

$(s-3)(s+2)x(s)=\frac{6}{s^3}+\frac{1}{s^2}-\frac{1}{s}-s+7=\frac{6}{s^3}+\frac{s}{s^3}-\frac{s^2}{s^3}-\frac{s^4}{s^3}+\frac{7s^3}{s^3}=$

$\frac{-s^4+7s^3-s^2+s+6}{s^3}$ $\therefore$

$x(s)=\frac{-s^4+7s^3-s^2+s+6}{s^3(s-3)(s+2)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D}{s-3}+\frac{E}{s+2}$ $\therefore$

$-s^4+7s^3-s^2+s+6=As^2(s-3)(s+2)+Bs(s-3)(s+2)+C(s-3)(s+2)+Ds^3(s+2)+Es^3(s-3)$ $\therefore$

$s=0$: $6=C(-3)(2)=-6C$ $\therefore$ $C=-1$ $\therefore$

$s=3$: $-3^4-7(3)^3-3^2+3+6=D3^3(3+2)=135D=-270$ $\therefore$ $\cancel{D=\frac{-42}{5/2}}$ $D=-\frac{1}{2}$

$s=-2$: $-2^4-7(2)^3-2^2-2+6=E(-2)^3(-2-3)=-72=40E$ $\therefore$ $E=-1.8$ $\therefore$

coefficient of s^4: $-1 = \cancel{A} + D + E = A - \frac{1}{2} - 1.8 = A - 2.3 \therefore A = 1.3$

at $s=1$: $-1+7-1+1+8 = 1.3(1)^2(1-3)(1+2) + B(1)(1-3)(1+2) - 1(1-3)(1+2)t - \frac{1}{2}(1)^4(1+2) - 1.8(1)^4(1-3)$

$= -7.8 + 12B + 6 - \frac{3}{2} + 3.6 = 0.3 + 12B = 12 \therefore 12B = 11.7 \therefore B = \frac{39}{40}$

$\therefore$ ~~x(s) = 2/3~~

$x(s) = 1.3\frac{1}{s} + \frac{39}{40}\frac{1}{s^2} - \frac{1}{s^3} - \frac{1}{2}\frac{1}{s-3} - 1.8\frac{1}{s+2} \therefore$

$\mathcal{L}^{-1}(x(s)) = x(t) = 1.3 + \frac{39}{40}t - \frac{1}{2}t^2 - \frac{1}{2}e^{3t} - 1.8e^{-2t}$

\4bi/ $u = 2t-1 \therefore \frac{du}{dt} = 2 \therefore \frac{1}{2}du = dt$

$t=1 \rightarrow 2(1)-1 = 1 = u \quad t=0 \rightarrow 2(0)-1 = -1 = u \therefore$

$y(x) = 1 + \int_0^1 y(t)\,dt - 4x\int_0^1 \frac{1}{1+(y(t))^2}dt \therefore \int_0^1 y(t)\,dt = q_1,\ \int_0^1 \frac{1}{1+(y(t))^2}dt = q_2$

$\therefore y(x) = 1 + q_1 - 4q_2x \therefore \quad y(t) = 1 + q_1 - 4q_2t = (1+q_1) - 4q_2t$

$q_1 = \int_0^1 y(t)\,dt = \int_0^1 1 + q_1 - 4q_2t\,dt = [t + q_1t - 2q_2t^2]_0^1 =$

$(1-0) + q_1(1-0) - 2q_2(1^2 - 0^2) = 1 + q_1 - 2q_2 = q_1 \therefore$

$1 - 2q_2 = 0 \therefore \quad 1 = 2q_2 \therefore \quad q_2 = \frac{1}{2} \therefore$

$q_2 = \frac{1}{2} = \int_0^1 \frac{1}{1+((1+q_1)-4q_2t)^2}dt \therefore$

$\frac{d}{dt}\left[((1+q_1)-4q_2t)^{-1}\right] = -1(-4q_2)\left[(1+q_1)-4q_2t\right]^{-2} \therefore$

$\frac{1}{2} = \frac{1}{-4q_2}\int_0^1 \frac{-4q_2}{1+((1+q_1)-4q_2t)^{-2}}dt = -\frac{1}{2}\int_0^1 \frac{-2}{1+((1+q_1)-2t)^{-2}}dt$

$= -\frac{1}{2}\int_{-1}^1 \frac{-2}{1+((1+q_1)-(2t-1)-1)^2}\frac{1}{2}du = \frac{1}{2}\int_{-1}^1 \frac{1}{1+(q_1-u)^2}du = \frac{1}{2} \therefore$

~~$\int_{-1}^1 \frac{1}{1+}$~~ let $p = q_1 \therefore \int_{-1}^1 \frac{1}{1+(p-u)^2}du = 1 \therefore$

$y(x) = 1 + p - 4\left(\frac{1}{2}\right)x = 1 + p - 2x$

\4bii/ $\therefore$ let $v = p - u \therefore \frac{dv}{du} = -1 \therefore -dv = du \therefore$

$u=1: p-1 = v \quad u=-1: p+1 = v \therefore$

$1 = \int_{p+1}^{p-1} \frac{1}{1+v^2}(-1)\,dv = \int_{p-1}^{p+1}\frac{1}{1+v^2}dv$ and $\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) =$

$\frac{\cos x(\cos x) - \sin x(-\sin x)}{\cos x^2} = \frac{\cos^2 + \sin^2 x}{\cos x^2} = \frac{1}{\cos^2 x}$

\3a+2PP2020/ let $z = \arctan x \therefore \tan z = x \therefore$

$$\frac{dx}{dz} = \frac{1}{\cos^2 x} \therefore \frac{dz}{dx} = \cos^2 x = \cos^2(\tan z) = \cos^2\left(\frac{\sin z}{\cos z}\right)$$

let $\int \frac{1}{1+v^2}\,dv = \arctan v \therefore$

$$1 = \left[\arctan v\right]_{p-1}^{p+1} = \arctan(p+1) - \arctan(p-1)$$

$\therefore \arctan(p+1)$

let ~~v=~~ ~~v = sinh θ~~ ~~∴ 1 sinh~~ v

let $v = \sin\theta \therefore v^2 = \sin^2\theta = 1-\cos^2\theta \therefore$

$1+v^2 = 1+1-\cos^2\theta = 2-\cos^2\theta \therefore \frac{dv}{d\theta} = \cos\theta$

$v = \sinh\theta \therefore v^2 = \sinh^2\theta \quad \therefore \cosh^2\theta - \sinh^2\theta = 1 \therefore$

$\frac{dv}{d\theta} = \cosh\theta \therefore \frac{1}{\cosh\theta}\,dv = d\theta \quad \therefore 1+v^2 = 1+\sinh^2\theta = \cosh^2\theta \therefore$

~~let~~ $v = p+1$

\3ai/ $\|k\|_\infty = \max_{x\in[0,1]} \int_0^1 |k(x,t)|\,dt =$

$$\max_{x\in[0,1]}\left[\int_0^x |xt|\,dt + \int_x^1 |(1+x)t|\,dt\right] =$$

$$\max_{x\in[0,1]}\int_0^x |xt|\,dt + \max_{x\in[0,1]}\int_x^1 |(1+x)t|\,dt =$$

$$\max_{x\in[0,1]}\int_0^x xt\,dt + \max_{x\in[0,1]}\int_x^1 (1+x)t\,dt = \int_0^1 t\,dt + \int_0^1 (1+0)t\,dt = \left[\tfrac{1}{2}t^2\right]_0^1 + \left[\tfrac{1}{2}t^2\right]_0^1 =$$

$$\left[t^2\right]_0^1 = 1^2 - 0^2 = 1$$

\3aii/ $y_0(x) = \frac{f(x)}{\lambda}$ $\therefore$ neuman series exists for $\lambda \neq 0$

$\therefore y_n(x) = \frac{f(x)}{\lambda} + \frac{1}{\lambda}K y$

\3aiii/ $\|y - y_n\|_\infty \leq \frac{1}{|\lambda| - \|k\|_\infty}\left(\frac{\|k\|_\infty}{|\lambda|}\right)^{n+1}\|f\|_\infty \therefore |\lambda| = 10 \therefore$

$$10y - \int_0^1 [k(x,t)]\,y(t)\,dt = x \therefore$$

$y_0(x) = f(x) = x \quad , \therefore y = \frac{1}{10}x + \frac{1}{10}\int_0^1 k(x,t)\,y(t)\,dt \therefore y_0(t) = t \therefore$

$$y_1(x) = \frac{1}{10}x + \frac{1}{10}\int_0^1 k(x,t)\,y_0(t)\,dt = \frac{1}{10}x + \frac{1}{10}\int_0^1 k(x,t)\,t\,dt =$$

$$\frac{1}{10}x + \frac{1}{10}\int_0^x xt\,t\,dt + \int_x^1 (1+x)t\,t\,dt = \frac{1}{10}x + \frac{1}{10}x\int_0^x t^2\,dt + \frac{(1+x)}{10}\int_x^1 t^2\,dt =$$

$\frac{1}{10}x + \frac{1}{10}x\left[\frac{1}{2}t^2\right]_0^x + \frac{(1+x)}{10}\left[\frac{1}{2}t^2\right]_x^1 =$

$\frac{1}{10}x + \frac{1}{10}x\,\frac{1}{2}x^2 + \frac{1}{10}(1+x)\frac{1}{2}(1-x^2) =$

$\cancel{\frac{1}{10}x}\ \frac{1}{10}x + \frac{1}{20}x^3 + \frac{1}{20}(1+x)(1-x^2) =$

$\frac{1}{10}x + \frac{1}{20}x^3 + \frac{1}{20}\left[1 - x^2 - x^2 + x\right] = \frac{3}{20}x - \frac{1}{20}x^2 + \frac{1}{20} = y_1(x)$ ∴

$g(x) = x$ ∴ $\|g(x)\|_\infty = \max_{x\in[0,1]} x \,dx = 1 = \|g\|_\infty$ ∴

$\|y - y_n\|_\infty = \frac{1}{10-1}\left(\frac{1}{10}\right)^{n+1},\ = \frac{1}{9}(0.1)^{n+1}$ ∴

$\|y - y_0\|_\infty = \frac{1}{9}(0.1)^{0+1} = \frac{1}{90}$

$\|y - y_1\|_\infty = \frac{1}{9}(0.1)^{1+1} = \frac{1}{9}(0.1)^2 = \frac{1}{900}$

\3b i/ $g(s) = \frac{e^s}{se^s} + \frac{1}{se^s} = \frac{1}{s} + \frac{1}{se^s}$ ∴

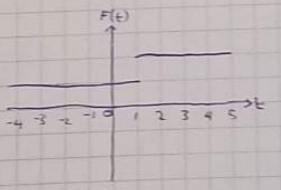
$\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$, $\frac{1}{se^s} = \cancel{\frac{1}{s}\frac{1}{e^s}}\ \frac{1}{se^{1s}} = \frac{e^{-1s}}{s}$ ∴

$\mathcal{L}^{-1}\left(\frac{1}{se^s}\right) = \mathcal{L}^{-1}\left(\frac{e^{-1s}}{s}\right) = H(t-1)$ ∴

$\mathcal{L}^{-1}(g(s)) = F(t) = \mathcal{L}^{-1}\left(\frac{e^s+1}{se^s}\right) = \mathcal{L}^{-1}\left(\frac{e^s}{se^s}\right) + \mathcal{L}^{-1}\left(\frac{1}{se^s}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{e^{-1s}}{s}\right) =$

$1 + H(t-1)$

\3b ii/



\3042 Mock/

laplace $x(t)$ — laplace $\dot{x}(t)$

\1/ let $\mathcal{L}(x(t)) = x(s)$ ∴ $\mathcal{L}(\dot{x}(t)) = s\,x(s) - x(0) = s\,x(s)$ ∴

laplace $\ddot{x}(t)$

$\mathcal{L}(\ddot{x}(t)) = s^2 x(s) - s\,x(0) - \dot{x}(0) = s^2 x(s) - 1$ ∴

laplace $2e^t$ — add laplaces

$\mathcal{L}(2e^t) = 2\frac{1}{s-1} = s^2 x(s) - 1 - 2s\,x(s) + 2x(s) = x(s)(s^2-2s+2) - 1$ ∴

enclose & combine fracs

$(s^2-2s+2)x(s) = \frac{2}{s-1} + 1 = \frac{2+s-1}{s-1} = \frac{s+1}{s-1}$ ∴

write $x(s)$ as partial frac

$x(s) = \frac{s+1}{(s-1)(s^2-2s+2)} = \frac{A}{s-1} + \frac{Bs+C}{s^2-2s+2}$ ∴

rearrange partial frac

$s+1 = A(s^2-2s+2) + (Bs+C)(s-1)$ ∴

$s=1:$ $1+1=2 = A(1^2-2(1)+2) + (Bs+C)(0) = A = 2$ ∴

solve partially frac

$s^0:$ $1 = 2(2) - C = 4 - C$ ∴ $C=3$ ∴

$s^2:$ $0 = 2 + B$ ∴ $B = -2$ ∴

write $x(s)$

$x(s) = 2\frac{1}{s-1} + \frac{-2s+3}{s^2-2s+2}$ ∴

since inverse laplaces — notice its form

$\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^t$, $s^2-2s+2 = (s-1)^2+2-1 = (s-1)^2+1$ ∴

so $\frac{-2s+3}{s^2-2s+2} = \frac{-2s+3}{(s-1)^2+1} = \frac{-2s}{(s-1)^2+1} + \frac{3}{(s-1)^2+1} = \frac{-2(s-1)+3-2}{(s-1)^2+1} =$

rearrange to nice form

$-2\frac{s-1}{(s-1)^2+1} + \frac{1}{(s-1)^2+1}$ ∴

find inverse laplaces

$\mathcal{L}^{-1}\left(\frac{-2s+3}{s^2-2s+2}\right) = -2\mathcal{L}^{-1}\left(\frac{s-1}{(s-1)^2+1}\right) + \mathcal{L}^{-1}\left(\frac{1}{(s-1)^2+1}\right) = -2e^t\cos(t) + e^t\sin(t)$

add to get final answer

∴ $\mathcal{L}^{-1}(x(s)) = x(t) = 2e^t - 2e^t\cos(t) + e^t\sin(t) =$

$e^t[2 - 2\cos(t) + \sin(t)]$

write in terms of heavyside func

\2/ $F(t) = (t-4)[H(t-3) - H(t-4)] + 0[H(t-4)] + (8-3t)[H(t-2) - H(t-3)]$

$+ t[1 - H(t-2)] =$

expand everything

$t - tH(t-2) + 8H(t-2) - 3tH(t-2) - 8H(t-3) + 3tH(t-3) + tH(t-3) - 4H(t-3)$

$- tH(t-4) + 4H(t-4) =$

group by heavyside func

$t + (-4t+8)H(t-2) + (4t-12)H(t-3) + (-t+4)H(t-4) =$

notice its on table

$t - 4(t-2)H(t-2) + 4(t-3)H(t-3) - (t-4)H(t-4)$ ∴

take laplace

$\mathcal{L}(F(t)) = f(s) =$

$\frac{1}{s^2} - 4e^{-2s}\frac{1}{s^2} + 4e^{-3s}\frac{1}{s^2} - e^{-4s}\frac{1}{s^2}$

$= \frac{1}{s^2}(1 - 4e^{-2s} + 4e^{-3s} - e^{-4s})$

\3a ✓ $\therefore \lambda y(x) - \int_0^{2\pi} \cos(x-t)y(t)\,dt = f(x) \therefore \lambda y(x) = \int_0^{2\pi} \cos(x-t)y(t)\,dt + f(x)$

$\therefore y(x) = \frac{1}{\lambda}\int_0^{2\pi} \cos(x-t)\,y(t)\,dt + \frac{1}{\lambda} f(x) \therefore$

$\cos(x-t) = \cos(x)\cos(t) + \sin(x)\sin(t) \therefore$

$y(x) = \frac{1}{\lambda}\cos(x)\int_0^{2\pi}\cos(t)y(t)\,dt + \frac{1}{\lambda}\sin(x)\int_0^{2\pi}\sin(t)y(t)\,dt + \frac{1}{\lambda}f(x) \therefore$

let $p = \int_0^{2\pi}\cos(t)y(t)\,dt$, $q = \int_0^{2\pi}\sin(t)y(t)\,dt \therefore$

$y(x) = \frac{1}{\lambda}\cos(x)\,p + \frac{1}{\lambda}\sin(x)\,q + \frac{1}{\lambda}f(x) \therefore$

$y(t) = \frac{1}{\lambda}p\cos(t) + \frac{1}{\lambda}q\sin(t) + \frac{1}{\lambda}f(t) \therefore$

$p = \int_0^{2\pi}\cos(t)y(t)\,dt = \int_0^{2\pi}\cos(t)\left(\frac{1}{\lambda}p\cos(t) + \frac{1}{\lambda}q\sin(t) + \frac{1}{\lambda}f(t)\right)dt =$

$\frac{1}{\lambda}p\int_0^{2\pi}\cos^2(t)\,dt + \frac{q}{\lambda}\int_0^{2\pi}\cos(t)\sin(t)\,dt + \frac{1}{\lambda}\int_0^{2\pi}f(t)\,dt \therefore$

$\int_0^{2\pi}\cos^2(t)\,dt = \int_0^{2\pi}\frac{1}{2} - \frac{1}{2}\cos(2t)\,dt = \left[\frac{1}{2}t - \frac{1}{4}\sin(2\pi)\right]_0^{2\pi} =$

$\frac{1}{2}[2\pi - 0] - \frac{1}{4}[\sin(4\pi) - \sin(0)] = \pi - \frac{1}{4}(0-0) = \pi$,

$\int_0^{2\pi}\cos(t)\sin(t)\,dt = \frac{1}{2}\int_0^{2\pi}\sin(2t)\,dt \quad \frac{1}{4}\left[-\cos(2t)\right]_0^{2\pi} =$

$-\frac{1}{4}[\cos(4\pi) - \cos(0)] - \frac{1}{4}[1-1] = 0 \therefore$

$p = \frac{1}{\lambda}p\pi + \frac{1}{\lambda}q(0) + \frac{1}{\lambda}\int_0^{2\pi}f(t)\,dt \therefore$

$p - \frac{\pi}{\lambda}p = \left(1 - \frac{\pi}{\lambda}\right)p = \frac{1}{\lambda}\int_0^{2\pi}f(t)\,dt \therefore p = \frac{1}{\lambda\left(1-\frac{\pi}{\lambda}\right)}\int_0^{2\pi}f(t)\,dt$

for $1 - \frac{\pi}{\lambda} \neq 0 \therefore 1 \neq \frac{\pi}{\lambda} \therefore \lambda \neq \pi \therefore p = \frac{1}{\lambda - \pi}\int_0^{2\pi}f(t)\,dt$,

$q = \int_0^{2\pi}\sin(t)y(t)\,dt = \int_0^{2\pi}\sin(t)\left[\frac{1}{\lambda}p\cos(t) + \frac{1}{\lambda}q\sin(t) + \frac{1}{\lambda}f(t)\right]dt =$

$\frac{p}{\lambda}\int_0^{2\pi}\cos(t)\sin(t)\,dt + \frac{q}{\lambda}\int_0^{2\pi}\sin^2(t)\,dt + \frac{1}{\lambda}\int_0^{2\pi}f(t)\,dt \therefore$

$\int_0^{2\pi}\cos(t)\sin(t) = 0$, $\int_0^{2\pi}\sin^2(t)\,dt = \int_0^{2\pi}\frac{1}{2} - \frac{1}{2}\cos(2t)\,dt = \left[\frac{1}{2}t - \frac{1}{4}\sin(2t)\right]_0^{2\pi}$

$= \frac{1}{2}[2\pi - 0] - \frac{1}{4}[\sin(4\pi) - \sin(0)] = \pi \therefore$

$q = \frac{p}{\lambda}(0) + \frac{q}{\lambda}\pi + \frac{1}{\lambda}\int_0^{2\pi}f(t)\,dt \therefore q - \frac{q\pi}{\lambda} = q\left(1 - \frac{\pi}{\lambda}\right) = \frac{1}{\lambda}\int_0^{2\pi}f(t)\,dt \therefore$

$q = \frac{1}{\lambda\left(1-\frac{\pi}{\lambda}\right)}\int_0^{2\pi}f(t)\,dt = \frac{1}{\lambda-\pi}\int_0^{2\pi}f(t)\,dt \therefore$

Solutions for $\lambda \neq \pi$

\3042 mock/ $\therefore\ y(x) = \frac{1}{\lambda}\frac{1}{\lambda-\pi}\int_0^{2\pi} f(t)\,dt\,\cos(x) + \frac{1}{\lambda}\frac{1}{\lambda-\pi}\int_0^{2\pi} f(t)\,dt\,\sin(x) + \frac{1}{\lambda}f(x)$

$= \left[\frac{\cos(x)+\sin(x)}{\lambda^2-\pi\lambda}\right]\int_0^{2\pi} f(t)\,dt + \frac{f(x)}{\lambda} = y(x)\ \therefore$

resolvent kernel $R_\lambda(x,t) = \frac{\cos(x)+\sin(t)}{\lambda^2-\pi\lambda}$

\3a redo/ $\lambda y - Ky = f(x)\ \therefore\ \lambda y = f(x) + Ky\ \therefore\ y(x) = \frac{f(x)}{\lambda} + \frac{1}{\lambda}Ky\ \therefore$

$\cancel{y(x)}\ y(x) = \frac{f(x)}{\lambda} + \frac{1}{\lambda}\int_0^{2\pi}\cos(x-t)y(t)\,dt = \frac{f(x)}{\lambda} + \frac{1}{\lambda}\int_0^{2\pi}(\cos x\cos t + \sin x\sin t)y(t)\,dt$

$= \frac{f(x)}{\lambda} + \frac{1}{\lambda}\cos x\int_0^{2\pi}\cos t\,y(t)\,dt + \frac{1}{\lambda}\sin x\int_0^{2\pi}\sin t\,y(t)\,dt\ \therefore$

let $\int_0^{2\pi}\cos(t)y(t)\,dt = p$, $\int_0^{2\pi}\sin(t)y(t)\,dt = q\ \therefore$

$y(x) = \frac{f(x)}{\lambda} + \frac{1}{\lambda}p\cos(x) + \frac{1}{\lambda}q\sin(x)\ \therefore\ y(t) = \frac{f(t)}{\lambda} + \frac{1}{\lambda}p\cos(t) + \frac{1}{\lambda}q\sin(t)$

$\therefore\ p = \int_0^{2\pi}\cos(t)y(t)\,dt = \int_0^{2\pi}\cos(t)\left[\frac{f(t)}{\lambda} + \frac{1}{\lambda}p\cos(t) + \frac{1}{\lambda}q\sin(t)\right]dt =$

$\frac{1}{\lambda}\int_0^{2\pi}\cos(t)f(t)\,dt + \frac{1}{\lambda}p\int_0^{2\pi}\cos^2(t)\,dt + \frac{1}{\lambda}q\int_0^{2\pi}\cos(t)\sin(t)\,dt\ \therefore$

$\cos(t)$ is even, $\sin(t)$ is odd $\therefore\ \cos(t)\sin(t)$ is odd function

$\therefore\ \int_0^{2\pi}\cos(t)\sin(t)\,dt = \frac{1}{2}\int_0^{2\pi}2\sin(t)\cos(t)\,dt = \frac{1}{2}\int_0^{2\pi}\sin(2t)\,dt = \frac{1}{4}\left[\cos(2t)\right]_0^{2\pi}$

$\frac{1}{4}\left[\cos(4\pi) - \cos(0)\right] = \frac{1}{4}(1-1) = 0$,

$\int_0^{2\pi}\cos^2(t)\,dt = \int_0^{2\pi}\frac{1}{2} + \frac{1}{2}\cos(2t)\,dt = \left[\frac{1}{2}t + \frac{1}{4}\sin(2t)\right]_0^{2\pi} = \frac{1}{2}(2\pi) + 0 = \pi\ \therefore$

$p = \frac{1}{\lambda}\int_0^{2\pi}\cos(t)f(t)\,dt + \pi\frac{1}{\lambda}p\ \therefore\ p - \pi\frac{1}{\lambda}p = \left(1-\pi\frac{1}{\lambda}\right)p = \frac{1}{\lambda}\int_0^{2\pi}\cos(t)f(t)\,dt\ \therefore$

$p = \frac{1}{\lambda\left(1-\frac{\pi}{\lambda}\right)}\int_0^{2\pi}\cos(t)f(t)\,dt = \frac{1}{\lambda-\pi}\int_0^{2\pi}\cos(t)f(t)\,dt$

$q = \int_0^{2\pi}\sin(t)y(t)\,dt = \int_0^{2\pi}\sin(t)\left[\frac{f(t)}{\lambda} + \frac{1}{\lambda}p\cos(t) + \frac{1}{\lambda}q\sin(t)\right]dt =$

$\frac{1}{\lambda}\int_0^{2\pi}\sin(t)f(t)\,dt + \frac{1}{\lambda}p\int_0^{2\pi}\sin(t)\cos(t)\,dt + \frac{q}{\lambda}\int_0^{2\pi}\sin^2(t)\,dt\ \therefore\ \int_0^{2\pi}\sin(t)\cos(t)\,dt = 0$,

$\int_0^{2\pi}\sin^2(t)\,dt = \int_0^{2\pi}\frac{1}{2} - \frac{1}{2}\cos(2t)\,dt = \left[\frac{1}{2}t - \frac{1}{4}\sin(2t)\right]_0^{2\pi} = \frac{1}{2}(2\pi-0) - \frac{1}{4}(0-0) = \pi\ \therefore$

$q = \frac{1}{\lambda}\int_0^{2\pi}\sin(t)f(t)\,dt + \pi\frac{1}{\lambda}q\ \therefore\ q - \pi\frac{1}{\lambda}q = \left(1-\pi\frac{1}{\lambda}\right)q = \frac{1}{\lambda}\int_0^{2\pi}\sin(t)f(t)\,dt\ \therefore$

$q = \frac{1}{\lambda\left(1-\frac{\pi}{\lambda}\right)}\int_0^{2\pi}\sin(t)f(t)\,dt = \frac{1}{\lambda-\pi}\int_0^{2\pi}\sin(t)f(t)\,dt\ \therefore$

$y(x) = \frac{f(x)}{\lambda} + \frac{1}{\lambda}\cos(x)\frac{1}{\lambda-\pi}\int_0^{2\pi}\cos(t)f(t)\,dt + \frac{1}{\lambda}\frac{\sin(x)}{\lambda-\pi}\int_0^{2\pi}\sin(t)f(t)\,dt =$

$$\frac{g}{\lambda} + \frac{1}{\lambda}\frac{1}{\lambda - \pi}\int_0^{2\pi}\left[\cos(x)\cos(t) + \sin(x)\sin(t)\right] g(t)\,dt = \frac{g}{\lambda} + \int_0^{2\pi}\frac{1}{\lambda(\lambda-\pi)}\cos(x-t)\, g(t)\,dt$$

$\therefore$ Solutions for $\lambda \neq \pi$

\3b/ $\therefore\ y(x) = \frac{g(x)}{\lambda} + \frac{1}{\lambda^2 - \pi\lambda}\int_0^{2\pi}\cos(x-t)\, g(t)\,dt = \frac{g}{\lambda} + \int_0^{2\pi}\frac{1}{\lambda(\lambda-\pi)}\cos(x-t)\, g(t)\,dt$

\3c/ $r_\lambda(x,t)$ is resolvent kernel $\therefore\ r_\lambda(x,t) = \frac{1}{\lambda(\lambda-\pi)}\cos(x-t) =$

$$\frac{\cos(x-t)}{\lambda^2 - \pi\lambda}$$

\3042 ~~P~~ 2021/

\1ai/ ~~nm~~ linear, volterra, nonhomogeneous equation of the second kind ~~which~~ with a kernel of finite rank 1, and seperable

\1aii/ $\int_1^x \frac{1}{t} y(t)\,dt = y(x) - 2 \therefore$

$\frac{1}{x} y(x) + \int_1^x (0) y(t)\,dt = y'(x) = \frac{1}{x} y(x)$,

at $x = 1$: $\int_1^1 \frac{y(t)}{t}\,dt = y(1) - 2 = 0 \therefore \quad y(1) = 2 \therefore$

$\frac{y'(x)}{y(x)} = \frac{1}{x} \therefore \quad \int \frac{y'(x)}{y(x)}\,dx = \int \frac{1}{x}\,dx = \ln|x| + C_1 = \ln|y(x)| \therefore$

$|y(x)| = e^{\ln|x| + C_1} = e^{C_1} e^{\ln|x|} = C_2 |x| \therefore y(x) = C_3 x \therefore$

$y(1) = 2 = C_3(1) = C_3 = 2 \therefore \quad y(x) = 2x$

\1bi/ nonhomogenous, Fredholm, linear equation of the 2nd kind, with seperable kernel of finite rank 2.

\1bii/ $y(x) = x^2 + \sin(x) \int_{-\pi}^{\pi} \cos(t) y(t)\,dt + x^2 \int_{-\pi}^{\pi} \sin(t) y(t)\,dt \therefore$

let $p = \int_{-\pi}^{\pi} \cos(t) y(t)\,dt$, $\quad q = \int_{-\pi}^{\pi} \sin(t) y(t)\,dt \therefore$

$y(x) = x^2 + \sin(x) p + x^2 q \therefore \quad y(t) = (1+q)t^2 + p\sin(t) \therefore$

$p = \int_{-\pi}^{\pi} \cos(t) y(t)\,dt = \int_{-\pi}^{\pi} \cos(t)\left((1+q)t^2 + p\sin(t)\right) dt =$

$(1+q) \int_{-\pi}^{\pi} t^2 \cos(t)\,dt + p \int_{-\pi}^{\pi} \cos(t)\sin(t)\,dt \therefore$

$(-1)\pi = -\pi \therefore \cos(t)$ is even, $\sin(t)$ is odd $\therefore$

$\cos(t)\sin(t)$ is odd, $\therefore \int_{-\pi}^{\pi} \cos(t)\sin(t)\,dt = 0 \therefore$

$p = (1+q) \int_{-\pi}^{\pi} t^2 \cos(t)\,dt \therefore$

$\int_{-\pi}^{\pi} t^2 \cos(t)\,dt = \left[t^2 \sin(t)\right]_{-\pi}^{\pi} - \int_{-\pi}^{\pi} 2t\sin(t)\,dt =$

$\left[\pi^2 \sin(\pi) - (-\pi^2)\sin(-\pi)\right] - \left[2t(-1)\cos(t)\right]_{-\pi}^{\pi} + \int_{-\pi}^{\pi} 2(-1)\cos(t)\,dt =$

$0 + 0 + \left[2\pi\cos(\pi) - 2(-\pi)\cos(-\pi)\right] - \left[2\sin(t)\right]_{-\pi}^{\pi} =$

$2\pi(-1) + 2\pi(-1) - 2\left[\sin(\pi) - \sin(-\pi)\right] = -4\pi - 2[0 - 0] = -4\pi \therefore$

$p = (1+q)(-4\pi)$,

$q = \int_{-\pi}^{\pi} \sin(t) y(t)\,dt = \int_{-\pi}^{\pi} \sin(t)\left((1+q)t^2 + p\sin(t)\right) dt =$

$(1+q)\int_{-\pi}^{\pi} t^2 \sin(t)\,dt + p\int_{-\pi}^{\pi} \sin^2(t)\,dt$ $\therefore$

t^2 is even, $\sin(t)$ is odd function, $\therefore$ $t^2\sin(t)$ is odd $\therefore$

$\int_{-\pi}^{\pi} t^2 \sin(t)\,dt = 0$ $\therefore$

$q = p\int_{-\pi}^{\pi} \sin^2(t)\,dt$ $\therefore$ $\quad \cos(2t) = \cos^2(t) - \sin^2(t) = 1-\sin^2(t) - \sin^2(t) =$

$1 - 2\sin^2(t)$ $\therefore$ $2\sin^2(t) = 1-\cos(2t)$ $\therefore$ $\sin^2(t) = \frac{1}{2} - \frac{1}{2}\cos(2t)$ $\therefore$

$\int_{-\pi}^{\pi} \sin^2(t)\,dt = \int_{-\pi}^{\pi} \frac{1}{2} - \frac{1}{2}\cos(2t)\,dt = \left[\frac{1}{2}t - \frac{1}{4}\sin(2t)\right]_{-\pi}^{\pi} =$

$\frac{1}{2}[\pi - (-\pi)] - \frac{1}{4}[\sin(2\pi) - \sin(-2\pi)] = \frac{1}{2}[\pi+\pi] - \frac{1}{4}[0-0] = \pi$ $\therefore$

$q = \pi p$ $\therefore$ $\quad p = (1+q)(-4\pi)$ $\therefore$ $\quad \frac{q}{\pi} = p$ $\therefore$

$\frac{q}{\pi} = (1+q)(-4\pi)$ $\therefore$ $\quad -4\pi^2(1+q) = q = -4\pi^2 - 4\pi^2 q = q$ $\therefore$

$-4\pi^2 = q + 4\pi^2 q = q(1+4\pi^2) = -4\pi^2$ $\therefore$ $\quad q = \frac{-4\pi^2}{1+4\pi^2}$ $\therefore$

$\frac{q}{\pi} = \frac{-4\pi}{1+4\pi^2} = p$ $\therefore$

$y(x) = \left(1 + \frac{-4\pi^2}{1+4\pi^2}\right)x^2 + \frac{-4\pi}{1+4\pi^2}\sin(x)$

$\therefore$ I defined the IE & its kernel then seperated the IE into an eqn with two integrals & said those integrals were unknown consts & wrote the IE ~~with the~~ integrals replaced with these consts then wrote it in terms of t then plugged the new found y(t) into the new definition of p & q 's integrals and expanded 2 integrated to find a simultaneous eqn of p & q & solved for p & q then plugged p & q into the eqn of y(x)

1c / $\mathcal{L}(F(t)) = f(s) = \mathcal{L}(e^t\cos(t)) + \mathcal{L}(t\sin(2t))$ $\therefore$

$\mathcal{L}(\cos(t)) = \frac{s}{s^2+1}$ $\therefore$ $\mathcal{L}(e^t\cos(t)) = \frac{(s-1)}{(s-1)^2+1}$,

$\mathcal{L}(\sin(2t)) = \frac{2}{s^2+4}$ $\therefore$ $\quad t = t^1$ $\therefore$ $\frac{d}{ds}\mathcal{L}(\sin(2t)) = \frac{d}{ds}\frac{2}{s^2+4} =$

$\frac{(s^2+4)(0) - 2(2s)}{(s^2+4)^2} = \frac{-4s}{(s^2+4)^2}$ $\therefore$ $\quad \mathcal{L}(t\sin(2t)) = (-1)^1\frac{-4s}{(s^2+4)^2} = \frac{4s}{(s^2+4)^2}$ $\therefore$

$\mathcal{L}(F(t)) = \frac{(s-1)}{(s-1)^2+1} + \frac{4s}{(s^2+4)^2}$

$\therefore$ I took the Laplacian of F(t) & seperated into 2 laplacians & solved the 1st , then the 2nd , then added them up ~~for~~ $\mathcal{L}(F(t))$

\30+2 PP2021/

\1di/ $F(t) = \mathcal{L}^{-1}(f(s))$ $\quad f(s) = \frac{1}{(s+3)^2(s-2)} = \frac{A}{s+3} + \frac{B}{(s+3)^2} + \frac{C}{s-2} \therefore$

$1 = A(s+3)(s-2) + B(s-2) + C(s+3)^2 \therefore$

$s=2: \; 1 = 0A + 0B + C(2-3)^2 = 1C = C = 1$

$s=-3: \; 1 = 0A + B(-3-2) + 0C = -5B = 1 \quad \therefore B = -\frac{1}{5} \therefore$

$s^2: \; 0 = A + C = A + 1 = 0 \therefore \quad A = -1 \therefore$

$f(s) = -\frac{1}{s+3} - \frac{1}{5}\frac{1}{(s+3)^2} + \frac{1}{s-2} \therefore$

$\mathcal{L}^{-1}\left(\frac{1}{s+3}\right) = \cancel{4} e^{-3t}$ $\quad , \mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t},$

$\mathcal{L}^{-1}\left(\frac{1}{(s+3)^2}\right) = e^{-3t}t^1 = e^{-3t}t \therefore$

$\mathcal{L}^{-1}(f(s)) = -e^{-3t} - \frac{1}{5}te^{-3t} + e^{2t}$

$\therefore$ I seperated $f(s)$ into partially fractions & solved the partial fraction. then solved ~~for~~ the individually inverse laplace of each component of the 3 fractions then added them for $\mathcal{L}^{-1}(f(s))$

\1dii/ let $\mathcal{L}(F) = \mathcal{L}(F(t)) = f(s) \therefore \mathcal{L}(F') = sf(s) - F(0) = sf(s)$

$\therefore \mathcal{L}(F'') = s^2 f(s) - sF(0) - F'(0) = s^2 f(s) - s(0) - 0 = s^2 f(s) \therefore$

$s^2 f(s) + sf(s) - 6f(s) = \mathcal{L}(e^{-3t}) = \frac{1}{s+3} = (s^2+s-6)f(s) = (s+3)(s-2)f(s) \therefore$

$f(s) = \frac{1}{(s+3)(s+3)(s-2)} = \frac{1}{(s+3)^2(s-2)} \therefore$

$F = F(t) = \mathcal{L}^{-1}(f(s)) = \mathcal{L}^{-1}\left(\frac{1}{(s+3)^2(s-2)}\right) = -e^{-3t} - \frac{1}{5}te^{-3t} + e^{2t}$

$\therefore$ I took the laplace of individual LHS & added them & took laplace of RHS & factored out $f(s)$ & rearranged equation for $f(s)$ then inverse laplace of $f(s)$ for $F(t)$ already found

\2a/ nonlinear, nonhomogeneous, Fredholm equation of the 2nd kind with seperatable kernel of finite rank $\therefore$

$y(x) = 2 + \int_1^2 y(t)\,dt - 2x\int_1^2 \frac{1}{(y(t))^2}\,dt \therefore$ let $\int_1^2 y(t) = p$, $\int_1^2 \frac{1}{(y(t))^2}\,dt = q \therefore$

$y(x) = 2 + p - 2xq \therefore \quad y(t) = (2+p) - 2qt \therefore$

$p = \int_1^2 y(t)\,dt = \int_1^2 (2+p) - 2qt\,dt = \left[(2+p)t - 2q\frac{1}{2}t^2\right]_1^2 = (2+p)[2-1] - q[2^2-1^2] =$

$(2+p)(1) - q(3) = 2 + p - 3q = p \quad \therefore 2 - 3q = 0 \quad \therefore 2 = 3q \quad \therefore \frac{2}{3} = q \therefore$

$$y(t) = (2+p) - 2\left(\tfrac{2}{3}\right)t = (2+p) - \tfrac{4}{3}t \therefore$$

$$q = \int_1^2 \frac{1}{(y(t))^2}\,dt = \int_1^2 \frac{1}{\left((2+p)-\frac{4}{3}t\right)^2}\,dt = -\frac{3}{4}\int_1^2 \frac{-\left(\frac{4}{3}\right)}{\left((2+p)-\frac{4}{3}t\right)^2}\,dt = \cancel{-\frac{3}{4}}$$

$$-\frac{3}{4}\left[\frac{1}{-1}\left((2+p)-\frac{4}{3}t\right)^{-1}\right]_1^2 = \frac{3}{4}\left[\left((2+p)-\frac{4}{3}(2)\right)^{-1} - \left((2+p)-\frac{4}{3}(1)\right)^{-1}\right] =$$

$$\frac{3}{4}\left[\left(2+p-\frac{8}{3}\right)^{-1} - \left(\frac{2}{3}+p\right)^{-1}\right] = \frac{3}{4}\left[\frac{1}{\left(-\frac{2}{3}+p\right)} - \frac{1}{\left(\frac{2}{3}+p\right)}\right] = q = \frac{2}{3} \therefore$$

$$\frac{1}{p-\frac{2}{3}} - \frac{1}{p+\frac{2}{3}} = \frac{8}{9} = \frac{p+\frac{2}{3}}{\left(p-\frac{2}{3}\right)\left(p+\frac{2}{3}\right)} - \frac{p-\frac{2}{3}}{\left(p-\frac{2}{3}\right)\left(p+\frac{2}{3}\right)} = \frac{\left(\frac{4}{3}\right)}{p^2-\frac{4}{9}} = \frac{8}{9} \therefore$$

$$\frac{8}{9}p^2 - \frac{32}{81} = \frac{4}{3} \therefore \frac{8}{9}p^2 = \frac{140}{81} \therefore p^2 = \frac{35}{18} \therefore p = \pm\sqrt{\frac{35}{18}} = \pm\frac{\sqrt{70}}{6} \therefore$$

$$y(x) = 2 \pm \frac{\sqrt{70}}{6} - \frac{4}{3}x$$

$\therefore$ I classifies the equation & its kernel then seperated the integrals & took out anything indep of t then set the two integrals as consts and so now has the equation with consts & x then plugged eqn into definition of consts to find a value for q then plug in def of q to find value for p then wrote equation of $y(x)$

\2b

$$F(s) = \frac{1}{(s^2+a^2)} \times \frac{1}{(s^2+b^2)} \therefore$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+a^2}\right) = \frac{1}{a}\mathcal{L}^{-1}\left(\frac{a}{s^2+a^2}\right) = \frac{1}{a}\sin(at) \therefore$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+b^2}\right) = \frac{1}{b}\sin(bt) \therefore$$

$$\mathcal{L}^{-1}(F(s)) = \int_0^t \frac{1}{a}\sin(au)\frac{1}{b}\sin(b(t-u))\,du = \frac{1}{ab}\int_0^t \sin(au)\sin(bt-bu)\,du =$$

$$\frac{1}{ab}\int_0^t \frac{1}{2}\left[\cos(au-bt+bu) - \cos(au+bt-bu)\right]du =$$

$$\frac{1}{2ab}\left[\frac{1}{a+b}\sin(au-bt+bu) - \frac{1}{a-b}\sin(au+bt-bu)\right]_{u=0}^{t} =$$

$$\frac{1}{2ab}\left[\frac{1}{a+b}\left[\sin(at-bt+bt) - \sin(-bt)\right] - \frac{1}{a-b}\left[\sin(at+bt-bt) - \sin(bt)\right]\right] =$$

$$\frac{1}{2ab}\left[\frac{1}{a+b}\left[\sin(at)+\sin(bt)\right] - \frac{1}{a-b}\left[\sin(at)-\sin(bt)\right]\right] =$$

$$\frac{1}{2ab}\left(\frac{a-b}{(a+b)(a-b)}\left[\sin(at)+\sin(bt)\right] - \frac{a+b}{(a-b)(a+b)}\left[\sin(at)-\sin(bt)\right]\right) =$$

$$\frac{1}{2ab}\left(\frac{1}{a^2-b^2}\right)\left(a\sin(at) + a\sin(bt) - b\sin(at) - b\sin(bt) - a\sin(at) + a\sin(bt) + b\sin(at) - b\sin(bt)\right)$$

\3o+2PP2021/

$$= \frac{1}{2ab(a^2-b^2)}\left[2a\sin(bt) - 2b\sin(bt)\right] =$$

$$\frac{1}{ab(a+b)(a-b)}\sin(bt)\left[a-b\right] = \frac{1}{ab(a+b)}\sin(bt)$$

$\therefore$ I seperated $f(s)$ into a product & found their individual inverse laplace then pluged them into convolution integral then integrated then simplified

\2ci/ $\therefore z(t)=1 \therefore (Kz)(x) = \int_0^1 e^{x+t} z(t)\,dt = \int_0^1 e^{x+t}(1)\,dt =$

$$\int_0^1 e^{x+t}\,dt = e^x\int_0^1 e^t\,dt = e^x\left[e^t\right]_0^1 = e^x\left[e^1-e^0\right] = e^x\left[e-1\right] \therefore$$

$(Kz)(t) = e^t[e-1) \therefore$

$$K^2z = K(Kz) = K(Kz)(x) = \int_0^1 e^{x+t}(Kz)(t)\,dt = \int_0^1 e^{x+t}e^t(e-1)\,dt$$

$$= e^x\int_0^1 e^t e^t(e-1)\,dt = e^x(e-1)\int_0^1 e^{2t}\,dt = (e-1)e^x\left[\tfrac{1}{2}e^{2t}\right]_0^1 =$$

$$(e-1)e^x\tfrac{1}{2}\left[e^{2(1)}-e^0\right] = (e-1)\frac{e^2-1}{2}e^x$$

\2ci/ plugged $z=1$ into Ky definition & integrated & simplified then plugged this new into Ky def $\therefore$ integrated & simplified

\2cii/ for $n=1 \therefore K^1z = Kz = (e-1)\left(\frac{e^2-1}{2}\right)^0 e^x = (e-1)\left(\frac{e^2-1}{2}\right)^{1-1}e^x \therefore$

true for $n=1 \therefore$

assuming true for $n=k \therefore K^r z = (e-1)\left(\frac{e^2-1}{2}\right)^{r-1}e^x \therefore$

for $n=r+1$: $K^{r+1}z = K((K^r)(t))(x) = \int_0^1 e^{x+t}(K^r)(t)\,dt =$

$$e^x\int_0^1 e^t(e-1)\left(\frac{e^2-1}{2}\right)^{r-1}e^t\,dt = (e-1)\left(\frac{e^2-1}{2}\right)^{r-1}e^x\int_0^1 e^{2t}\,dt =$$

$$(e-1)\left(\frac{e^2-1}{2}\right)^{r-1}e^x\left[\tfrac{1}{2}e^{2t}\right]_0^1 = (e-1)\left(\frac{e^2-1}{2}\right)^{r-1}e^x\left(\frac{(e^2-1)}{2}\right)^1 =$$

$(e-1)\left(\frac{e^2-1}{2}\right)^r e^x \therefore$ true for $n=k+1 \therefore$

true by induction $\forall n \in \mathbb{N}$ that $K^n z = (e-1)\left(\frac{e^2-1}{2}\right)^{n-1}e^x$

$\therefore$ showed plugging in for $n=1$ holds it true then assumed true for $n=k$ then proved it for $n=r+1$ plug into formula and replace with $n=r$ assumption then ~~replace~~ integrate & simplify to given form so statement true.

\3ai/ $y_n(x) = 1 + \int_0^x t\, y_{n-1}(t)\,dt$ $\therefore$ $f(x) = y_0(x) = 1$ $\therefore$

$y_1(x) = 1 + \int_0^x t\, y_0(t)\,dt = 1 + \int_0^x t(1)\,dt = 1 + \int_0^x t\,dt = 1 + \left[\frac{1}{2}t^2\right]_0^x = 1 + \frac{1}{2}x^2 =$

$\frac{1}{0!}\left(\frac{x^2}{2}\right)^0 + \frac{1}{1!}\left(\frac{x^2}{2}\right)^1 = \sum_{j=0}^{1} \frac{1}{j!}\left(\frac{x^2}{2}\right)^j$ $\therefore$ true for $n=1$ $\therefore$

assuming true for $n = r$ $\therefore$ $y_r(x) = \sum_{j=0}^{r} \frac{1}{j!}\left(\frac{x^2}{2}\right)^j$ $\therefore$

for $n = r+1$ $\therefore$ $y_{r+1} = 1 + \int_0^x t\, y_r(t)\,dt = 1 + \int_0^x t \sum_{j=0}^{r} \frac{1}{j!}\left(\frac{t^2}{2}\right)^j dt$

$1 + \int_0^x \sum_{j=0}^{r} \frac{1}{j!}\left(\frac{1}{2}\right)^j t^{2j+1}\,dt = 1 + \left[\sum_{j=0}^{r} \frac{1}{j!}\left(\frac{1}{2}\right)^j \frac{1}{2j+2} t^{2j+2}\right]_0^x =$

$1 + \sum_{j=0}^{r} \frac{1}{j!}\left(\frac{1}{2}\right)^j \frac{1}{j+1}\frac{1}{2}(x^2)^{j+1} = 1 + \sum_{j=0}^{r} \frac{1}{(j+1)!}\left(\frac{x^2}{2}\right)^{j+1} = \sum_{j=0}^{r+1} \frac{1}{j!}\left(\frac{x^2}{2}\right)^j$ $\therefore$

true ~~for~~ $\therefore$ true $\forall n \in \mathbb{N}$ by induction

\3aii/ $\therefore$ $y'(x) = 0 + x y(x) + \int_0^x (0) y(t)\,dt = x y(x) + 0 = x y(x) = y'(x)$

$\therefore$ $x = \frac{y'(x)}{y(x)}$ $\therefore$ at $x=0$: $y(0) = 1 + \int_0^0 t\, y(t)\,dt = 1 + 0 = 1 = y(0)$ $\therefore$

$\int x\,dx = \int \frac{y'(x)}{y(x)}\,dx = \ln|y(x)| = \frac{1}{2}x^2 + C_1$ $\therefore$

$|y(x)| = e^{\frac{1}{2}x^2 + C_1} = e^{C_1} e^{\frac{1}{2}x^2} = C_2 e^{\frac{1}{2}x^2}$ $\therefore$

$y(x) = C_3 e^{\frac{1}{2}x^2}$ $\therefore$ $y(0) = 1 = C_3 e^{\frac{1}{2}(0)^2} = C_3 e^0 = C_3 = 1$ $\therefore$

$y(x) = 1e^{\frac{1}{2}x^2} = e^{\frac{1}{2}x^2}$ $\therefore$

the result from (i) is equivelent $\because$ it is the taylor series of $e^{\frac{1}{2}x^2}$

$\therefore$ I ~~stat~~ the formula for $y_n(x)$ then pluged y_0 the solved for y_1 then assume it held for for y_n then found y_{n+1} by pugging into formula & subing in y_n then solve to form $\therefore$ true by induction then found deriv of original equat & found BC at $x=0$ then seperate for x then solve and plug in IC to solve for C then wrote $y(x)$ & said it has a tayler expansion

\3bi/ for $\lambda = 0$: $0 = f(x) + \int_0^x t\, y(t)\,dt$ $\therefore$ $f(x)$ must be continuous and differentiable $\therefore$

$f'(x) + x y(x) + \int_0^x (0) y(t)\,dt = 0 = f'(x) + x y(x)$ $\therefore$ $-f'(x) = x y(x)$ $\therefore$ $-\frac{f'(x)}{x} = y(x)$

$\therefore$ I said $f(x)$ must be cont & diffable then deriv eqn & rearrange for $y(x)$

\30428PP2021/

\3bii/ $\therefore\ y(x) = \frac{1}{\lambda} f(x) + \frac{1}{\lambda}\int_0^x t\,y(t)\,dt \ \therefore$

$y'(x) = \frac{1}{\lambda} f'(x) + \frac{1}{\lambda} x y(x) + \frac{1}{\lambda}\int_0^x (0) y(t)\,dt = y'(x) = \frac{1}{\lambda} f'(x) + \frac{1}{\lambda} x y(x) \ \therefore$

$y'(x) - \frac{1}{\lambda} x y(x) = \frac{1}{\lambda} f'(x)$,

at $x=0$: $\lambda y(0) = f(0) + 0 = f(0) \ \therefore\ y(0) = \frac{1}{\lambda} f(0) \ \therefore$

$IF = e^{\int -\frac{1}{\lambda} x\,dx} = e^{-\frac{1}{2\lambda}x^2} \ \therefore$

$\frac{d}{dx}\left(e^{-\frac{1}{2\lambda}x^2} y(x)\right) = \frac{1}{\lambda} e^{-\frac{x^2}{2\lambda}} f'(x) \ \therefore\ y(x) = e^{\frac{1}{2\lambda}x^2}\int_0^x \frac{1}{\lambda} e^{-\frac{t^2}{2\lambda}} f'(t)\,dt$

$e^{-\frac{1}{2\lambda}x^2} y(x) = \int \frac{1}{\lambda} e^{-\frac{1}{2\lambda}x^2} f'(x)\,dx = \left[\frac{1}{\lambda} e^{-\frac{1}{2\lambda}x^2} f(x)\right] - \int -\frac{1}{\lambda^2} x e^{-\frac{1}{2\lambda}x^2} f(x)\,dx + C$

$\therefore$ at $x=0$: $e^0 y(0) = \frac{1}{\lambda} f(0) = \cancel{\left[\frac{1}{\lambda} e^0 f(0)\right] + \int} \ \therefore$

$e^{-\frac{1}{2\lambda}x^2} y(x) = \int_0^x \frac{1}{\lambda} e^{-\frac{t^2}{2\lambda}} f'(t)\,dt =$

$\left[\frac{1}{\lambda} e^{-\frac{t^2}{2\lambda}} f(t)\right]_0^x - \int_0^x -\frac{1}{\lambda^2} t e^{-\frac{1}{2\lambda}t^2} f(t)\,dt \ \therefore$ at $x=0$:

$\cancel{e^0 y(0) = \frac{1}{\lambda} f(0) =}$

$e^{-\frac{1}{2\lambda}x^2} y(x) = \frac{1}{\lambda} e^{-\frac{1}{2\lambda}x^2} f(x) - \frac{1}{\lambda} e^0 f(0) - \frac{1}{\lambda}\int_0^x -\frac{1}{\lambda} t e^{-\frac{1}{2\lambda}t^2} f(t)\,dt \ \therefore$

$y(x) = \frac{1}{\lambda} f(x) - \frac{1}{\lambda} f(0) e^{\frac{1}{2\lambda}x^2} + \frac{1}{\lambda^2}\int_0^x t e^{\frac{1}{2\lambda}x^2} e^{-\frac{1}{2\lambda}t^2} f(t)\,dt =$

$\frac{1}{\lambda} f(x) - \frac{1}{\lambda} f(0) e^{\frac{1}{2\lambda}x^2} + \frac{1}{\lambda^2}\int_0^x t e^{\frac{1}{2\lambda}(x^2-t^2)} f(t)\,dt$

$\therefore$ I rearrange Z eqn for $y(x)$ then took Z deriv & found BCs at $x=0$ then rearranged for IF form then found IF then integrated but switched vars for t from x to 0 then IBP then plug in values then rearrange for $y(x)$

\3c/ $\therefore\ \cos(x) = e^{-xx} y(x) + \int_0^x \frac{\partial e^{-xt}}{\partial x} y(t)\,dt = e^{-x^2} y(x) + \int_0^x -t e^{-xt} y(t)\,dt$

$\therefore\ y(x) e^{-x^2} = \cos(x) + \int_0^x t e^{-xt} y(t)\,dt \ \therefore$

$y(x) = e^{x^2}\cos(x) + e^{x^2}\int_0^x t e^{-xt} y(t)\,dt = e^{x^2}\cos(x)\int_0^x t e^{x^2-xt} y(t)\,dt =$

$y(x) = e^{x^2}\cos(x)\int_0^x e^{x(x-t)} t\,y(t)\,dt$ $\therefore$ this equation has a unique solution $\therefore$ the original equation only has a unique solution

$\therefore$ I took deriv then rearrange for $y(x)$ & has unique sol $\therefore$ original does also

\4a.i/ $k(x,t) = \frac{s(x-t)}{6+5\sin(x-t)}$ ∴

$\int_0^{2\pi} |k(x,t)|\,dt = \int_0^{2\pi} \left|\frac{\cos(x-t)}{6+5\sin(x-t)}\right| dt = \int_0^{2\pi} \frac{|\cos(x-t)|}{|6+5\sin(x-t)|}\,dt$

\4a.ii/ $\|K\|_\infty = \max_{x\in[0,2\pi]} \int_0^{2\pi} \left|\frac{\cos(x-t)}{6+5\sin(x-t)}\right| dt =$

$\max_{x\in[0,2\pi]} \int_0^{2\pi} \frac{|\cos(x-t)|}{|6+5\sin(x-t)|}\,dt$ ∴ let $x-t=u$ ∴ $\frac{du}{dt}=-1$ ∴ $-du=dt$

$t=2\pi \to x-2\pi=u \quad t=0 \to x-0=x=u$ ∴

$\|K\|_\infty = \max_{x\in[0,2\pi]} \int_x^{x-2\pi} \frac{|\cos(u)|}{|6+5\sin(u)|}(-1)\,du = \max_{x\in[0,2\pi]} \int_{x-2\pi}^{x} \frac{|\cos(u)|}{|6+5\sin(u)|}\,du =$

$\int_0^{2\pi} \frac{|\cos(u)|}{|6+5\sin(u)|}\,du$

\4b/ $\lambda=3$ ∴ $|\lambda|=3$ ∴ $\|K\|_\infty = \ln(11) \simeq 2.398 < 3$ ∴

$\|K\|_\infty < |\lambda|$ ∴ ~~K~~ and $k(x,t)$ is continuous of finite rank, by the Fredholm alternative, the equation has only one unique continuous solution.

\4c/ $\|y-y_n\| \le \frac{1}{|\lambda|-\|K\|_\infty}\left(\frac{\|K\|_\infty}{|\lambda|}\right)^{n-1}\|g\|_\infty$ ∴

$|\lambda|=3, \quad \|K\|_\infty=\ln(11), \quad \|g\|_\infty=4$ ∴ $\|y-y_n\|_\infty \le 10^{-2}=0.01$ ∴

$\frac{1}{3-\ln(11)}\left(\frac{\ln(11)}{3}\right)^{n-1}(4) \le 0.01$ ∴

$\left(\frac{\ln(11)}{3}\right)^{n-1} \le 1.505\times10^{-3}$ ∴

$(n-1)\ln\left(\frac{\ln(11)}{3}\right) \le \ln(0.001505) \doteq -6.499$

~~n-1 ≥ 29.009~~ ∴ $n-1 \ge 29.009$ ∴ $n \ge 30.009$ ∴

$n > 31$ ∴ $n=31$

\4a.i/ $\|K\|_\infty = \max_x \int_0^{2\pi} |k(x,t)| = \max_{x\in[0,2\pi]} \int_0^{2\pi} \left|\frac{\cos(x-t)}{6+5\sin(x-t)}\right| dt = \max_{x\in[0,2\pi]} \int_0^{2\pi} \frac{|\cos(x-t)|}{|6+5\sin(x-t)|}\,dt$

∴ $u=t-x$ ∴ $x-t=-u$ ∴ $du=dx$, $t=2\pi: 2\pi-x=u$, $t=0: 0-x=u=-x$ ∴

$\|K\|_\infty = \max_{x\in[0,2\pi]} \int_{-x}^{2\pi-x} \frac{|\cos(-u)|}{|6+5\sin(-u)|}\,du = \int_0^{2\pi} \frac{|\cos(-u)|}{|6+5\sin(-u)|}\,du = \int_0^{2\pi} \frac{|\cos(u)|}{|6-5\sin(u)|}\,du$

∴ $6-5\sin(u)>0$ ∴ $\|K\|_\infty = \int_0^{2\pi} \frac{|\cos(u)|}{6-5\sin(u)}\,du = 2\int_{-\pi/2}^{\pi/2} \frac{\cos u}{6-5\sin(u)}\,du = \ln(11)$

\4a.i/ $\|k\|_\infty = \max_{x\in[0,2\pi]} \left|\frac{\cos(x-t)}{6+5\sin(x-t)}\right| = \frac{1}{6+(-5)} = \frac{1}{1} = 1$, $|(Ky)(x)| \le \int_0^{2\pi} \|k\|_\infty \|y\|_\infty\,dt \le \int_0^{2\pi} 1\,|y(t)|\,dt = \|y\| 2\pi$

\3042PP2020\

\1a\ $\mathcal{L}(t^2) = \frac{2!}{s^{2+1}} = \frac{2}{s^3}$ $\quad \mathcal{L}((t+1)e^{-t}) = \mathcal{L}(e^{-t}t) + \mathcal{L}(e^{-t}) =$

$\frac{1}{(s+1)^2} + \frac{1}{s+1}$, $\sinh^2(t) = \left(\frac{1}{2}(e^t - e^{-t})\right)^2 = \frac{1}{4}(e^{2t} + e^{-2t} - 2)$ ∴

$\mathcal{L}(\sinh^2(t)) = \frac{1}{4}\frac{1}{s-2} + \frac{1}{4}\frac{1}{s+2} - \frac{1}{2}\frac{1}{s}$ ∴

$\mathcal{L}(F(t)) = \frac{4}{s^3} - \frac{1}{(s+1)^2} - 4\frac{1}{s+1} + \frac{1}{4(s-2)} + \frac{1}{4(s+2)} - \frac{1}{2s}$

\1b\ $g(s) = \frac{5}{s^2} \cdot \frac{1}{(2-s)^2} = 5 \cdot \frac{1!}{s^2} \cdot \frac{1!}{(s-2)^2}$ ∴

$\mathcal{L}^{-1}\left(\frac{1!}{s^2}\right) = t' = t$, $\mathcal{L}^{-1}\left(\frac{1!}{(s-2)^2}\right) = e^{2t}t$ ∴

$\mathcal{L}^{-1}(g(s)) = 5\int_0^t (u)(e^{2(t-u)}(t-u))du = 5e^{2t}\int_0^t tue^{-2u} - u^2e^{-2u}\,du =$

$5e^{2t}\,t\int_0^t ue^{-2u}\,du - 5e^{2t}\int_0^t u^2e^{-2u}\,du$ ∴

$\int_0^t ue^{-2u}du = \left[u\frac{1}{-2}e^{-2u}\right]_{u=0}^{t} - \int_0^t -\frac{1}{2}e^{-2u}\,du =$

$-\frac{1}{2}\left[te^{-2t} - 0\right] + \frac{1}{2}\left[\frac{1}{-2}e^{-2u}\right]_{u=0}^{t} = -\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + \frac{1}{4}$,

$\int_0^t u^2e^{-2u}du = \left[u^2\frac{1}{-2}e^{-2u}\right]_{u=0}^{t} - \int_0^t 2u\frac{1}{-2}e^{-2u}\,du =$

$\left[\frac{-t^2}{2}e^{-2t}\right] + \int_0^t ue^{-2u}du = -\frac{t^2}{2}e^{-2t} - \frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + \frac{1}{4}$ ∴

$\mathcal{L}^{-1}(g(s)) = 5te^{2t}\left[-\frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + \frac{1}{4}\right] - 5e^{2t}\left[-\frac{1}{2}t^2e^{-2t} - \frac{1}{2}te^{-2t} - \frac{1}{4}e^{-2t} + \frac{1}{4}\right]$

$= -\frac{5}{2}t^2 - \frac{5}{4}t + \frac{5}{4}te^{2t} + \frac{5}{2}t^2 + \frac{5}{2}t + \frac{5}{4} - \frac{5}{4}e^{2t} = \frac{5}{4}te^{2t} - \frac{5}{4}e^{2t} + \frac{5}{4}$ ✗

\1ci\ A: $k(x,t) = \cos(x-t)$ ∴ continuous ∴ not weakly singular

B: $k(x,t) = \frac{1}{(x-t)^{1/2}}$ and $(x-t)^{1/2} < x-t$, $k(x,t)$ not continuous and not ~~weakly~~ strongly singular ∴ weakly singular,

C: $k(x,t) = \cos(x-t)\frac{1}{|x-t|^{1/2}}$ ∴ not continuous at $x=t$,

$|x-t|^{1/2} < x-t$ ∴ not strongly singular ∴ weakly singular

\1cii\ a kerel is weakly singular is it is not contineous and $\frac{1}{k(x,t)} < x-t$

\1d/ volterra, of the 2nd kind, nonhomogeneous, linear
separable finite rank

$\therefore$ let $\mathcal{L}(y(x)) = \hat{y}(s)$ $\therefore$ $\hat{y}(s) = 2\frac{1}{s} + \mathcal{L}[e^{2x}]\,\mathcal{L}[y(x)] =$
$2\frac{1}{s} + \frac{1}{s-2}\hat{y}(s)$ $\therefore$ $\hat{y}(s) - \frac{1}{s-2}\hat{y}(s) = \frac{2}{s} = \hat{y}(s)\left[\frac{s-2}{s-2} + \frac{-1}{s-2}\right] =$

$\frac{s-3}{s-2}\hat{y}(s) = \frac{2}{s}$ $\therefore$ $\hat{y}(s) = \frac{2(s-2)}{s(s-3)} = \frac{A}{s} + \frac{B}{s-3}$ $\therefore$

$2(s-2) = 2s-4 = A(s-3) + Bs$ $\therefore$

$s=0$: $-4 = -3A$ $\therefore$ $A = \frac{4}{3}$,

$s=3$: $2(3)-4 = 2 = 0A + 3B = 2$ $\therefore$ $B = \frac{2}{3}$ $\therefore$

$\hat{y}(s) = \frac{4}{3}\frac{1}{s} + \frac{2}{3}\frac{1}{s-3}$ $\therefore$ $y(x) = \frac{4}{3}\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \frac{2}{3}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) = \frac{4}{3}e^{3t} + \frac{2}{3}$

\1e/ Fredholm, nonlinear, nonhomogeneous, second kind
separable, finite rank. $\therefore$ continuous kernel.

$y(x) = 2 - x\int_0^1 \sqrt{t}\,(y(t))^2\,dt$ $\therefore$ $\int_0^1 \sqrt{t}\,(y(t))^2\,dt = p$ $\therefore$

$y(x) = 2 - px$ $\therefore$ $y(t) = 2 - pt$ $\therefore$

$p = \int_0^1 \sqrt{t}\,(2-pt)^2\,dt = \int_0^1 t^{1/2}(4 + p^2t^2 - 4pt)\,dt = \int_0^1 4t^{1/2} - 4pt^{3/2} + p^2t^{5/2}\,dt =$

$\left[4\left(\frac{2}{3}\right)t^{3/2} - 4p\left(\frac{2}{5}\right)t^{5/2} + p^2\left(\frac{2}{7}\right)t^{7/2}\right]_0^1 =$

$\frac{8}{3}(1) - \frac{8}{5}(1)p + \frac{2}{7}p^2(1) - 0 = \frac{8}{3} - \frac{8}{5}p + \frac{2}{7}p^2 = p$ $\therefore$

$\frac{2}{7}p^2 - \frac{13}{5}p + \frac{8}{3}$ $\therefore$ $p = \dfrac{\frac{13}{5} \pm \sqrt{\left(\frac{13}{5}\right)^2 - 4\left(\frac{2}{7}\right)\left(\frac{8}{3}\right)}}{2\left(\frac{2}{7}\right)}$

$\frac{91}{20} \pm \sqrt{\frac{1949}{525}} = p$ $\therefore$

$y(x) = 2 - px = 2 - \left(\frac{91}{20} \pm \sqrt{\frac{1949}{525}}\right)x$

\2ai/ $y_n(x) = 2 + \int_0^x t^2 y_{n-1}(t)\,dt$ $\therefore$ $y_0 = 2$ $\therefore$

$y_1(x) = 2 + \int_0^x t^2 2\,dt = 2 + 2\left[\frac{1}{3}t^3\right]_0^x = 2\left(\frac{1}{0!}\right)\left(\frac{x^3}{3}\right)^0 + 2\left(\frac{1}{1!}\right)\left(\frac{x^3}{3}\right)^1 = 2\sum_{j=0}^{1}\frac{1}{j!}\left(\frac{x^3}{3}\right)^j$

$\therefore$ true for $n=1$ $\therefore$

assuming true for $n=r$: $y_r(x) = 2\sum_{j=0}^{r}\frac{1}{j!}\left(\frac{x^3}{3}\right)^j$ $\therefore$

for $n=r+1$: $y_{r+1}(x) = 2 + \int_0^x t^2 y_r(t)\,dt = 2 + \int_0^x t^2\, 2\sum_{j=0}^{r}\frac{1}{j!}\left(\frac{t^3}{3}\right)^j dt =$

$2 + \int_0^x 2\sum_{j=0}^{r}\frac{1}{j!}\left(\frac{1}{3}\right)^j t^{3j+2}\,dt = 2 + \left[2\sum_{j=0}^{r}\frac{1}{j!}\left(\frac{1}{3}\right)^j\frac{1}{3j+3}t^{3j+3}\right]_0^x =$

\3042PP2020/ $2\left(\frac{1}{0!}\right)\left(\frac{x^3}{3}\right)^0 + 2\sum_{j=0}^{r} \frac{1}{j!}\left(\frac{1}{3}\right)^j \frac{1}{3}\frac{1}{j+1}(x^3)^{j+1} =$

$2\left(\frac{1}{0!}\right)\left(\frac{x^3}{3}\right)^0 + 2\sum_{j=0}^{r} \frac{1}{(j+1)!}\left(\frac{x^3}{3}\right)^{j+1} = 2\sum_{j=0}^{r+1}\frac{1}{j!}\left(\frac{x^3}{3}\right)^{j+1}$ $\therefore$ true for $n=r+1$

$\therefore$ true $\forall n \in \mathbb{N}$ by induction

\2aii/ $\therefore y'(x) + 0 + x^2 y(x) + \int_0^x (0) y(t)\,dt = x^2 y(x) = y'(x)$,

at $x=0$: $y(0) = 2 + \int_0^0 t^2 y(t)\,dt = 2 + 0 = 2 = y(0)$ $\therefore$

$x^2 = \frac{y'(x)}{y(x)}$ $\therefore$ $\int x^2\,dx = \frac{1}{3}x^3 + C_1 = \int \frac{y'(x)}{y(x)}\,dx = \ln|y(x)|$ $\therefore$

$e^{\ln|y(x)|} = |y(x)| = e^{\frac{1}{3}x^3 + C_1} = C_2 e^{\frac{1}{3}x^3}$ $\therefore$

~~$|y(x)|$~~ $y(x) = C_3 e^{\frac{1}{3}x^3}$ $\therefore$ $y(0) = 2 = C_3 e^{\frac{1}{3}(0)^3} = 1C_3 = C_3 = 2$ $\therefore$

$y(x) = 2e^{\frac{1}{3}x^3}$ $\therefore$

the previous result from (2.a.i) is the taylor series of $2e^{\frac{1}{3}x^3}$

\2b/ nonlinear nonhomogeneous Fredholm of the second kind equation with seperable kernel of finite rank.

$\therefore \lambda y(x) = 4 - x\int_{-1}^{1} t^2 (y(t))^2\,dt$ $\therefore$ $y(x) = \frac{4}{\lambda} - \frac{x}{\lambda}\int_{-1}^{1} t^2 (y(t))^2\,dt$ $\therefore$

$p = \int_{-1}^{1} t^2 (y(t))^2\,dt$ $\therefore$ $y(x) = \frac{4}{\lambda} - \frac{x}{\lambda}p$ $\therefore$ $y(t) = \frac{4}{\lambda} - \frac{p}{\lambda}t$ $\therefore$

$p = \int_{-1}^{1} t^2 (y(t))^2 = \int_{-1}^{1} t^2\left(\frac{4}{\lambda} - \frac{p}{\lambda}t\right)^2 dt = \int_{-1}^{1} t^2\left(\frac{16}{\lambda^2} + \frac{p^2}{\lambda^2}t^2 - 8p\frac{1}{\lambda}t\right) dt =$

$\int_{-1}^{1} \frac{16}{\lambda^2}t^2 + \frac{p^2}{\lambda^2}t^4 - 8p\frac{1}{\lambda}t^3\,dt = \left[\frac{16}{3}\frac{1}{\lambda^2}t^3 + \frac{p^2}{5}\frac{1}{\lambda^2}t^5 - 2p\frac{1}{\lambda}t^4\right]_{-1}^{1} =$

$\frac{16}{3}\frac{1}{\lambda^2}\left[1^3 - (-1)^3\right] + \frac{p^2}{5}\frac{1}{\lambda^2}\left(1^5 - (-1)^5\right) - 2p\frac{1}{\lambda}\left[1^4 - (-1)^4\right] =$

$\frac{16}{3}\frac{1}{\lambda^2}[2] + \frac{p^2}{5}\frac{1}{\lambda^2}[2] - 2p\frac{1}{\lambda}(0) = \frac{32}{3}\frac{1}{\lambda^2} + \frac{2}{5}\frac{1}{\lambda^2}p^2 = p$ $\therefore$

$\frac{2}{5\lambda^2}p^2 - p + \frac{32}{3\lambda^2} = 0$ $\therefore$ $p = \frac{1 \pm \sqrt{1 - 4\left(\frac{2}{5\lambda^2}\right)\left(\frac{32}{3\lambda^2}\right)}}{2\left(\frac{2}{5\lambda^2}\right)} = \frac{5\lambda^2}{4} \pm \frac{\sqrt{1 - \frac{256}{15}\frac{1}{\lambda^4}}}{\left(\frac{4}{5\lambda^2}\right)}$

$= \frac{5\lambda^2}{4} \pm \frac{5\lambda^2}{4}\sqrt{1 - \frac{256}{15}\frac{1}{\lambda^4}} = p$ $\therefore$

$y(x) = \frac{4}{\lambda} - \frac{1}{\lambda}\left(\frac{5\lambda^2}{4} \pm \frac{5\lambda^2}{4}\sqrt{1 - \frac{256}{15}\lambda^4}\right)x = \frac{4}{\lambda} - \left(\frac{5\lambda}{4} \pm \frac{5\lambda}{4}\sqrt{1 - \frac{256}{15}\lambda^4}\right)x$ $\therefore$

has solutions for $1 - \frac{256}{15}\lambda^4 \geq 0$ $\therefore$ $1 \geq \frac{256}{15}\lambda^4$ $\therefore$ $\frac{15}{256} \geq \lambda^4$ $\therefore$

$\lambda^2 \geq 0$ $\therefore$ $+\sqrt{\frac{15}{256}} = \frac{\sqrt{15}}{16} \geq \lambda^2 \geq 0$ $\therefore$ $-\left(\frac{\sqrt{15}}{16}\right)^{1/2} \leq \lambda \leq +\left(\frac{\sqrt{15}}{16}\right)^{1/2}$ and ~~$\lambda \neq 0$~~

if $\lambda = 0$: $4 = \int_{-1}^{1} x t^2 (y(t))^2 dt = 4 = px$ ∴ by comparing coefficients is impossible ∴ $\lambda \neq 0$

\3ai/ $\|k\|_\infty = \max_{x\in[0,1]} \int_0^1 |k(x,t)|\,dt = \max_{x\in[0,1]} \left[\int_0^x |xt|\,dt + \int_x^1 |(1+x)t|\,dt \right] =$

~~Max~~ ∴ for $x \in [0,1]$: $|xt| \leq |(1-x)t|$ ∴

$\|k\|_\infty = \max_{x\in[0,1]} \int_x^1 |(1+x)t|\,dt = \int_0^1 |(1+0)t|\,dt = \int_0^1 |t|\,dt = \int_0^1 t\,dt = \left[\frac{1}{2}t^2\right]_0^1 = \frac{1}{2}(1^2 - 0^2) = \frac{1}{2}$

\3aii/ $\lambda y(x) - \int_0^1 k(x,t) y(t) = f(x)$ exists for $|\lambda| > \|k\|_\infty = \frac{1}{2}$ ∴

$\frac{1}{2} < |\lambda|$

\3aiii/ $\|y - y_n\|_\infty \leq \frac{1}{|\lambda| - \|k\|_\infty} \left(\frac{\|k\|_\infty}{|\lambda|}\right)^{n+1} \|f\|_\infty$ ∴

$y_0(x) = \frac{f}{\lambda} = \frac{x}{10}$ ∴ $y_0(t) = \frac{t}{10}$ ∴ $y_n(x) = x + \int_0^1 k(x,t) y_{n-1}(t)\,dt$ ∴

$y_1(x) = x + \int_0^1 k(x,t) y_0(t)\,dt = x + \int_0^1 k(x,t) \frac{t}{10}\,dt = x + \int_0^x k(x,t)\frac{t}{10}\,dt + \int_x^1 k(x,t)\frac{t}{10}\,dt$

$= x + \int_0^x xt \frac{t}{10}\,dt + \int_x^1 (1+x)t\frac{t}{10}\,dt = x + \frac{x}{10}\int_0^x t^2\,dt + \frac{(1+x)}{10}\int_x^1 t^2\,dt =$

$x + \frac{x}{10}\left[\frac{1}{3}t^3\right]_0^x + \frac{(1+x)}{10}\left[\frac{1}{3}t^3\right]_x^1 =$

$x + \frac{x}{30}\left[x^3\right] + \frac{(1+x)}{30}\left[1 - x^3\right] = x + \frac{1}{30}x^4 + \frac{1}{30} - \frac{1}{30}x^3 + \frac{1}{30}x - \frac{1}{30}x^4 =$

$-\frac{1}{30}x^3 + \frac{1}{30}x + x + \frac{1}{30}$

∴ ~~$\|y - y_0\|$~~ $\|f\|_\infty = \max_{x\in[0,1]} |f(x)| = \max_{x\in[0,1]} |x| = |1| = 1$ ∴ $\|k\|_\infty = \frac{1}{2}$, $|\lambda| = 10$ ∴

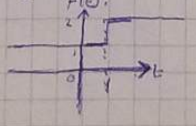
$\|y - y_0\|_\infty \leq \frac{1}{10 - \frac{1}{2}} \left(\frac{\frac{1}{2}}{10}\right)^{0+1} (1) = \frac{2}{19}\left(\frac{1}{20}\right)^1 = \frac{1}{190} \approx 0.00526$ (3 s.f.)

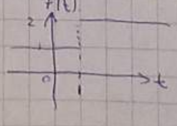
$\|y - y_1\| \leq \frac{1}{10 - \frac{1}{2}} \left(\frac{\frac{1}{2}}{10}\right)^{1+1} = \frac{2}{19}\left(\frac{1}{20}\right)^2 = \frac{1}{3800} = 0.000263$ (3 s.f.)

\3bi/ $F(t) = \mathcal{L}^{-1}(f(s)) = \mathcal{L}^{-1}\left(\frac{e^s + 1}{se^s}\right) = \mathcal{L}^{-1}\left(\frac{e^s}{se^s} + \frac{1}{se^s}\right) = \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{e^{-s}}{s}\right) =$

$\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \mathcal{L}^{-1}\left(\frac{e^{-s}}{s}\right) = 1 + H(t-1)$ ∴ where $H(t)$ is heaviside func. ∴ $F(t) = \begin{cases} 1, & t \leq 1 \\ 2, & t \geq 1 \end{cases}$

\3bii/





\3042FP2020/

\4a/ let $\mathcal{L}(x(t)) = x(s)$ $\therefore$ $\mathcal{L}(x'(t)) = sx(s) - x(0) = sx(s) + 1$ $\therefore$

$\mathcal{L}(x''(t)) = s^2x(s) - sx(0) - x'(0) = s^2x(s) + s - 6$ $\therefore$

$\mathcal{L}(3t^2+t-1) = 3\frac{2!}{s^{2+1}} + \frac{1!}{s^{1+1}} - \frac{1}{s} = 6\frac{1}{s^3} + \frac{1}{s^2} - \frac{1}{s} = s^2x(s) + s - 6 - sx(s) - 1 - 6x(s)$

$= x(s)(s^2 - s - 6) + s - 7 = \frac{6}{s^3} + \frac{s}{s^3} - \frac{s^2}{s^3} = \frac{-s^2+s+6}{s^3} = x(s)(s-3)(s+2) + s - 7$

$\therefore x(s)(s-3)(s+2) = \frac{-s^2+s+6}{s^3} - s + 7 = \frac{-s^2+s+6}{s^3} - \frac{s^4}{s^3} + \frac{7s^3}{s^3} = \frac{-s^4+7s^3-s^2+s+6}{s^3}$

$\therefore x(s) = \frac{-s^4+7s^3-s^2+s+6}{s^3(s-3)(s+2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s^3} + \frac{D}{s-3} + \frac{E}{s+2}$ $\therefore$

$-s^4+7s^3-s^2+s+6 = As^2(s-3)(s+2) + Bs(s-3)(s+2) + C(s-3)(s+2) + Ds^3(s+2) + Es^3(s-3)$ $\therefore$

$0: 0+0-0-0+6 = 6 = 0A + 0B + C(-3)(2) + 0D + 0E = -5C = 6$ $C = -\frac{6}{5}$

$s=3: -3^4+7\cdot 3^3-3^2+3+6 = 0A + 0B + 0C + D(3^3)(3+2) + 0E = 108 = 135D$ $\therefore$ $D = \frac{4}{5}$

$s=-2: -2^4-7(2^3)-2^2-2+6 = 0A + 0B + 0C + 0D + E(-2^3)(-2-3) = 40E = -72$ $\therefore$ $E = -\frac{9}{5}$

$s^4: -1 = A + D + E = A + \frac{4}{5} - \frac{9}{5} = -1 + A = -1$ $\therefore A = 0$ $\therefore$

$s=1: -1+7-1+1+6 = 0 + B(1)(1-3)(1+2) - \frac{6}{5}(1-3)(1+2) + \frac{4}{5}(1)^3(1+2) - \frac{9}{5}(1)^3(1-3) =$

$-6B + \frac{36}{5} + \frac{12}{5} + \frac{18}{5} = -6B + \frac{66}{5} = 12$ $\therefore -\frac{6}{5} = 6B$ $\therefore$ $B = \frac{1}{5}$ $\therefore$

$x(s) = \frac{1}{5}\frac{1}{s^2} - \frac{6}{5}\frac{1}{s^3} + \frac{4}{5}\frac{1}{s-3} - \frac{9}{5}\frac{1}{s+2}$ $\therefore$

$\mathcal{L}^{-1}(x(s)) = x(t) = \frac{1}{5}\mathcal{L}^{-1}\left(\frac{1!}{s^{1+1}}\right) - \frac{6}{5\cdot 2}\mathcal{L}^{-1}\left(\frac{2!}{s^{2+1}}\right) + \frac{4}{5}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) - \frac{9}{5}\mathcal{L}^{-1}\left(\frac{1}{s+2}\right) =$

$\frac{1}{5}t^1 + \frac{6}{10}t^2 + \frac{4}{5}e^{3t} - \frac{9}{5}e^{-2t} = \frac{1}{5}t + \frac{6}{10}t^2 + \frac{4}{5}e^{3t} - \frac{9}{5}e^{-2t}$

\4bi/ $y(x) = 1 + \int_0^1 y(t)\,dt - 4x\int_0^1 \frac{1}{1+(y(t))^2}\,dt$ $\therefore$

let $p = \int_0^1 y(t)\,dt$, $q = \int_0^1 \frac{1}{1+(y(t))^2}\,dt$ $\therefore$

$y(x) = (1+p) - 4qx$ $\therefore$ $y(t) = (1+p) - 4qt$ $\therefore$

$p = \int_0^1 y(t)\,dt = \int_0^1 (1+p) - 4qt\,dt = \left[(1+p)t - 2qt^2\right]_0^1 =$

$(1+p)[1] - 2q[1^2] = 1 + p - 2q = p$ $\therefore$ $1 - 2q = 0$ $\therefore$ $1 = 2q$ $\therefore$ $q = \frac{1}{2}$ $\therefore$

$q = \frac{1}{2} = \int_0^1 \frac{1}{1+(y(t))^2}\,dt = \int_0^1 \frac{1}{1+(1+p-2t)^2}\,dt$ $\therefore$ let $u = 2t-1$ $\therefore$

$\frac{du}{dt} = 2$ $\therefore$ $\frac{1}{2}du = dt$, $t=1: 2(1)-1 = 2-1 = 1 = u$, $t=0: 2(0)-1 = -1 = u$ $\therefore$

$\cancel{\frac{1}{2} = \int_{-1}^{1} \frac{1}{1+(1+p-2(2t-1))^2}}$

$\frac{1}{2} = \int_0^1 \frac{1}{1+(1+p-(2t))^2}\,dt = \int_0^1 \frac{1}{1+(1+p-(2t-1)-1)^2}\,dt = \int_0^1 \frac{1}{1+(p-(2t-1))^2}\,dt =$

$\frac{1}{2} = \int_{-1}^1 \frac{1}{1+(p-u)^2}\,\frac{du}{2} \therefore 1 = \int_{-1}^1 \frac{1}{1+(p-u)^2}\,du$ as required

\4bii ∴ let $v = p-u$ ∴ $\frac{dv}{du} = -1$ ∴ $-dv = du$,

$u=1$: $p-(1) = p-1 = v$, $u=-1$: $p-(-1) = p+1 = v$ ∴

$1 = \int_{p+1}^{p-1} \frac{1}{1+v^2}(-1)\,dv = \int_{p-1}^{p+1} \frac{1}{1+v^2}\,dv = 1 = \left[\arctan(v)\right]_{p-1}^{p+1} =$

$\arctan(p+1) - \arctan(p-1) = 1$ ∴

$\arctan(p+1) = 1 + \arctan(p-1)$ ∴ $\tan(\arctan(p+1)) = p+1 = \tan(1+\arctan(p-1))$

$\tan(1-(-\arctan(p-1))) = \frac{\tan(1) - \tan(-\arctan(p-1))}{1+\tan(1)\tan(-\arctan(p-1))} =$

$\tan(-x) = -\tan(x)$ ∵ tan is odd ∴

$p+1 = \frac{\tan(1) + \tan(\arctan(p-1))}{1-\tan(1)\tan(\arctan(p-1))} = \frac{\tan(1)+p-1}{1-\tan(1)(p-1)}$

$(p+1)(1-(p-1)\tan(1)) = \tan(1)+p-1 = (p+1)(1-p\tan(1)+\tan(1)) =$

$(p+1)(-p\tan(1) + (1+\tan(1))) = -p^2\tan(1) + p(1+\tan(1)) - p\tan(1) + (1+\tan(1)) =$

$-p^2\tan(1) + p + (1+\tan(1)) = \tan(1) + p - 1$

$-p^2\tan(1) + 1 + \tan(1) = \tan(1) - 1$ ∴

$2 = p^2\tan(1)$ ∴ $\frac{2}{\tan(1)} = p^2$ ∴

$p = \pm\sqrt{\frac{2}{\tan(1)}} \approx \pm 1.13$ (3 s.f.) ∴

$y(x) = (1+p) - 2x = (1 \pm 1.13) - 2x$ ∴

$y(x) = 2.13 - 2x,\ -0.133 - 2x$

\1aii so a kernel is weakly singular if it is discontinuous, and $k(x,t)$ is continuous when $x \neq t$ and if $\exists$ constants $\alpha \in (0,1)$ & $C>0$ st $|k(x,t)| \leq C|x-t|^{-\alpha}$ ∴ $k(x,t) = \frac{1}{(x-t)^\alpha}$, $\alpha < 1$ ∴ $|k(x,t)| \leq C\frac{1}{|x-t|^\alpha}$ for $x \neq t$ as this set is definition

\1cii A: $k(x,t) = \cos(x-t)$ ∴ not weakly singular ∵ continuous

\3042PP2020/ B: $k(x,t) = (x-t)^{-1/2} = \frac{1}{(x-t)^{\alpha}} = \frac{1}{(x-t)^{1/2}}$, $\alpha = \frac{1}{2} < 1$ ∴ weakly singular

C: $k(x,t) = \cos(x-t)|x-t|^{-1/2} = \cos(x-t)\frac{1}{|x-t|^{1/2}}$ ∴

$|k(x,t)| = \left|\cos(x-t)\frac{1}{|x-t|^{1/2}}\right| \leq |\cos(x-t)|\frac{1}{|x-t|^{1/2}} \leq 1\frac{1}{|x-t|^{1/2}} = \frac{1}{|x-t|^{1/2}}$

$\leq 1\frac{1}{|x-t|^{1/2}} = 1\frac{1}{|x-t|^{\alpha}}$ ∴ $\alpha = \frac{1}{2}$ ∴ $\alpha < 1$ ∴ $|k(x,t)| \leq C\frac{1}{|x-t|^{\alpha}}$

∴ $|k(x,t)| \leq |x-t|^{-1/2}$ ∴ weakly singular

\2ai/ $y(x) = 2 + \int_0^x t^2 y(t)dt = 2 + Ky(x)$ ∴ $y_n = \sum_{j=1}^{n} K^j f$, $f(x) = 2$

~~if~~ $K^0 f = f = 2$, $K^1 f = \int_0^x t^2 \cdot 2dt = \frac{2}{3}x^3$ ∴ if

$K^j f = 2 \cdot \frac{1}{j!}\left(\frac{x^3}{3}\right)^j$ ∴ $K^{j+1} f = 2\frac{1}{j!}\int_0^x \frac{t^2 t^{3j}}{3^j}dt = \frac{2}{j!}\left(\frac{x^{3(j+1)}}{(j+1)3^{j+1}}\right) =$

$\frac{2}{(j+1)!}\left(\frac{x^3}{3}\right)^{j+1}$ ∴ by induction, $y_n = 2\sum_{j=0}^{n}\frac{1}{j!}\left(\frac{x^3}{3}\right)^j$

\2aii/ $y'(x) = x^2 y(x)$, $y(0) = 2$ ∴ $\frac{dy}{y} = x^2 dx$ ∴ $\ln|y| = \frac{1}{3}x^3 + C$ ∴

$y = Ae^{x^3/3}$ ∴ $y(0) = 2$ ∴ $A = 2$ ∴ $y(x) = 2e^{x^3/3}$ ∴

Z series $2\sum_{j=0}^{n}\frac{1}{j!}\left(\frac{x^3}{3}\right)^j$ indeed converges to $2e^{x^3/3}$

\2b/ equation is non-linear Fredholm (Hammerstein) of the second kind when $\lambda \neq 0$ and first kind when $\lambda = 0$, non homogeneous

when $\lambda = 0$, setting $x = 0$: $4 = 0$ ∴ no solutions ~~wh~~

when $\lambda \neq 0$: let $P = \int_{-1}^{1} t^2 y^2(t)dt$ ∴ $y(x) = \frac{1}{\lambda}(4 - Px)$ ∴

$P = \int_{-1}^{1} t^2\left(\frac{4-Px}{\lambda}\right)^2 dt = \frac{1}{\lambda}\int_{-1}^{1} t^2(4-Pt)^2 dt = \frac{1}{\lambda^2}\int_{-1}^{1}(16t^2 - 8Pt^3 + P^2t^4)dt =$

$\frac{1}{\lambda^2}\left(\frac{32}{3} + \frac{2}{5}P^2\right)$ ∴ $P^2 + \frac{5\lambda^2}{2}P + \frac{80}{3} = 0$ ∴ $P = \frac{1}{2}\left(+\frac{5}{2}\lambda^2 \pm \sqrt{\frac{25}{4}\lambda^2 - \frac{320}{3}}\right)$

∴ if D: $\frac{25}{4}\lambda^4 - \frac{320}{3} < 0$ ∴ $\frac{25}{4}\lambda^4 < \frac{320}{3}$ ∴ $\lambda^4 < \frac{256}{15}$ ∴

$\lambda^2 < \sqrt{\frac{256}{15}} = \frac{16}{\sqrt{15}}$ ∴ $-\sqrt{\frac{16}{\sqrt{15}}} < \lambda < \sqrt{\frac{16}{\sqrt{15}}}$ then no real sols

if $D = 0$: $\lambda = \pm\sqrt{\frac{16}{\sqrt{15}}}$ ∴ a unique solution, $y(x) = \frac{4}{\lambda} + \frac{5}{4}\lambda x$

otherwise if $D > 0$, ∃ two sols:

$y(x) = \frac{4}{\lambda} - \frac{1}{2}\left(\frac{5}{2}\lambda^2 \pm \sqrt{\frac{25}{4}\lambda^4 - \frac{320}{3}}\right)x$

\3ai/ let $g(x) = \int_0^1 k(x,t)\,dt = \int_0^1 k(x,t)\,dt = \int_0^x k(x,t)\,dt + \int_x^1 k(x,t)\,dt =$

$\int_0^x xt\,dt + \int_x^1 (1+x)t\,dt = x\int_0^x t\,dt + (1+x)\int_x^1 t\,dt =$

$x\left[\frac{1}{2}t^2\right]_0^x + (1+x)\left[\frac{1}{2}t^2\right]_x^1 = \frac{1}{2}x\left[x^2 - 0^2\right] + (1+x)\frac{1}{2}\left[1^2 - x^2\right] =$

$\frac{1}{2}x^3 + \frac{1}{2}(1+x) - \frac{1}{2}x^2 - \frac{1}{2}x^3 = -\frac{1}{2}x^2 + \frac{1}{2}x + \frac{1}{2}$ ∴

$g(0) = g(1) = \frac{1}{2}$, $g'(x) = -x + \frac{1}{2} = 0$ ∴ $\frac{1}{2} = x$, $g''(x) = -1 < 0$ ∴

$\text{Max}\, g(x) = g\left(\frac{1}{2}\right) = -\frac{1}{2}\left(\frac{1}{2}\right)^2 + \frac{1}{2}\left(\frac{1}{2}\right) + \frac{1}{2} = \frac{5}{8}$ ∴

$\|K\|_\infty = \max_{x\in[0,1]} g(x) = g\left(\frac{1}{2}\right) = \frac{5}{8}$

∴ $\int_0^1 |k(x,t)|\,dt = \int_0^x |k(x,t)|\,dt + \int_x^1 |k(x,t)|\,dt =$

$\int_0^x |xt|\,dt + \int_x^1 |(1+x)t|\,dt = \int_0^x xt\,dt + \int_x^1 (1+x)t\,dt = \int_0^1 k(x,t)\,dt = g(x)$ ∴

$\|K\|_\infty = \max_{x\in[0,1]} \int_0^1 |k(x,t)|\,dt = \max_{x\in[0,1]} g(x) = g\left(\frac{1}{2}\right) = \frac{5}{8}$

\3aii/ $\|K\|_\infty = \frac{5}{8}$ ∴ is $\|K\|_\infty < |\lambda|$ ∴ $\frac{5}{8} < |\lambda|$ is the condition for λ

\3aiii/ $y_0 = \frac{x}{10}$ ∴ $y_1 = \frac{x}{10} + \frac{x}{100}\int_0^x t^2\,dt + \frac{(1+x)}{100}\int_x^1 t^2\,dt =$

$\frac{x}{10} + \frac{x^4}{300} + (1+x)\frac{1}{300}(1-x^3) = \frac{1}{300}(1+x-x^3) + \frac{x}{10}$

$\Big\{ y_n = \frac{f}{\lambda} + \frac{1}{\lambda}\int_0^1 k(x,t)\,y_{n-1}(t)\,dt = \frac{x}{10} + \frac{1}{10}\int_0^1 k(x,t)\,y_{n-1}(t)\,dt$ ∴

$y_0(x) = \frac{x}{10}$ ∴ $y_0(t) = \frac{t}{10}$ ∴ $y_1(x) = \frac{x}{10} + \frac{1}{10}\int_0^1 k(x,t)\frac{t}{10}\,dt =$

$\frac{x}{10} + \frac{1}{100}\int_0^x xt\,t\,dt + \frac{1}{100}\int_x^1 (1+x)t\,t\,dt = \frac{x}{10} + \frac{x}{100}\int_0^x t^2\,dt + \frac{(1+x)}{100}\int_x^1 t^2\,dt =$

$\frac{x}{10} + \frac{x}{100}\left[\frac{1}{3}t^3\right]_0^x + \frac{(1+x)}{100}\left[\frac{1}{3}t^3\right]_x^1 = \frac{x}{10} + \frac{x}{300}\left[x^3\right] + \frac{(1+x)}{300}\left[1 - x^3\right] =$

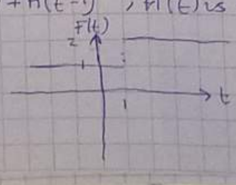
$\frac{x}{10} + \frac{x^4}{300} + \frac{1}{300} - \frac{x^3}{300} + \frac{x}{300} - \frac{x^4}{300} = -\frac{1}{300}x^3 + \frac{31}{300}x + \frac{1}{300} \Big\}$ ∴

$\|y - y_1\|_\infty \le \frac{1}{|\lambda| - \|K\|_\infty}\left(\frac{\|K\|_\infty}{|\lambda|}\right)^2 \|f\|_\infty$ ∴ $\max_{x\in[0,1]} |f(x)| = \|f\|_\infty = \max_{x\in[0,1]} x = 1$ ∴

$\|y - y_1\|_\infty = \frac{1}{10 - \frac{5}{8}}\left(\frac{5/8}{10}\right)^2 \cdot 1 = 0.000417$

\3bi/ $F(t) = \mathcal{L}^{-1}\left(\frac{1}{s} - \frac{1}{s e^{s}}\right) = 1 + H(t-1)$, $H(t)$ is Heaviside function

∴ $F(t) = \begin{cases} 1 & t \le 1 \\ 2 & t > 1 \end{cases}$



\3bii/

$\mathcal{L}(F(t)) = \int_{-\infty}^{\infty} F(t)e^{-st}\,dt$ but not $-\infty$ to ∞ based on range for t of $F(t)$

eg $\mathcal{L}(e^{2t}\cos(3t)) = \frac{s-2}{(s-2)^2-9}$ $\therefore$ $\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$ for t

useful to use definition of laplace transform for piecewise funcs

\ex/ IVP: $4\ddot{y}(t)+y(t) = g(t)$ $\quad y(0)=3,\ \dot{y}(0)=-7$ $\therefore$ laplacian:

$\mathcal{L}(\ddot{y}(t)) = s^2Y(s)-sy(0)-\dot{y}(0)$ $\therefore$ $\mathcal{L}(y(t)) = Y(s)$ $\therefore$

$\mathcal{L}(y(t)) = Y(s)$ $\therefore$ $4(s^2Y(s)-sy(0)-\dot{y}(0))+Y(s) = G(s)$ $\quad \therefore \mathcal{L}(g(t)) = G(s)$ $\therefore$

$4(s^2Y(s)-3s+7)+Y(s) = G(s)$ $\therefore$ $Y(s)(4s^2+1) = G(s)+12s-28$ $\therefore$

$Y(s) = \frac{G(s)}{4s^2+1} + \frac{12s-28}{4s^2+1}$ $\therefore$ $Y(s) = \frac{12s}{4s^2+1} - \frac{28}{4s^2+1} + \frac{G(s)}{4s^2+1} =$

$\frac{3s}{s^2+\frac{1}{4}} - \frac{7}{s^2+\frac{1}{4}} + \frac{G(s)}{4(s^2+\frac{1}{4})}$ $\therefore$

$y(t) = 3\cos\left(\frac{t}{2}\right) \frac{-7}{(1/2)}\sin\left(\frac{t}{2}\right) + \frac{1}{4}\frac{1}{(1/2)}\int_0^t \sin\left(\frac{u}{2}\right)g(t-u)\,du$ $\therefore$

$y(t) = 3\cos\left(\frac{t}{2}\right) - 14\sin\left(\frac{t}{2}\right) + \frac{1}{2}\int_0^t \sin\left(\frac{u}{2}\right)g(t-u)\,du$

\Ex/ $\int_0^x \sin(x-t)y(t)\,dt = x^2$

any problems as t approaches x in the kernel

is kernel weakly singular, strongly singular, continuous

weakly singular means only a problem at $x=t$

For a seperable (degenerate) kernel 2 rank is 2 number of k where

$k(x,t) = k_1^{(1)}(x)k_1^{(2)}(t) + k_2^{(1)}(x)k_2^{(2)}(t) + k_3^{(1)}(x)k_3^{(2)}(t)$ $\quad \frac{1}{\sqrt{x-t}}$ compared to $\frac{1}{x-t}$
weakly singular chaps 3, 4, 5

$\therefore$ ~~laplace~~ eqn = $\sin(x) * y(x) = x^2$ $\therefore$ Laplace:

$\frac{Y(s)}{s^2+1} = \frac{2}{s^3}$ $\therefore$ $Y(s) = \frac{2(s^2+1)}{s^3} = \frac{2}{s} + \frac{2}{s^3}$ $\therefore$ $y(x) = 2 + x^2$

\Ex/ $y(x) - \int_0^x e^{x-t}y(x)\,dx = x$ $\therefore$

$\hat{y}(s) - \mathcal{L}(e^{x} * y(x)) = \frac{1}{s}$ $= -\frac{1}{s-1}\hat{y}(s) + \hat{y}(s) = \frac{1}{s} = \hat{y}(s)$ ~~$\left(\frac{1}{s-1}+1\right) - \hat{y}(s)\left(\frac{s}{s-1}\right)$~~

$\therefore$ ~~$\frac{s-1}{s^2} = \hat{y}(s)$~~

$\frac{Y(s)}{s-1} = Y(s) - \frac{1}{s^2}$ $\therefore$ $Y(s) = \frac{s-1}{s^2(s-2)} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} =$

$-\frac{1/4}{s} + \frac{1/2}{s^2} + \frac{1/4}{s-2}$ $\therefore$ $y(x) = -\frac{1}{4} + \frac{x}{2} + \frac{1}{4}e^{2x}$

\Sheet 7 Q3/

\Sheet 7:/ \5/ $y(x) - \frac{1}{80}\int_0^{\pi}(\sin(x) + e^t)y(t)\,dt = 1$

find y_2 & esti error: $\|y - y_2\|_\infty$

$y_n = \frac{f}{\lambda} + \frac{1}{\lambda}Ky_{n-1} \qquad y_0 = \frac{f}{\lambda} = \frac{1}{1} = 1 \therefore$

$y_1 = \frac{f}{\lambda} + \frac{1}{\lambda}Ky_0 = \frac{f}{\lambda} + \frac{1}{\lambda}K(1) = 1 + 1\left(\frac{1}{80}\int_0^{\pi}(\sin(x) + e^t)\,1\,dt\right) =$

$1 + \frac{1}{80}\int_0^{\pi}\sin(x) + e^t\,dt = \frac{1}{80}\left[t\sin(x) + e^t\right]_0^{\pi} = \frac{79 + e^{\pi}}{80} + \frac{\pi}{80}\sin(x) \therefore$

$y_2 = \frac{f}{\lambda} + \frac{1}{\lambda}Ky_1 = 1 + 1\int_0^{\pi}\frac{1}{80}(\sin(\cancel{x}) + e^t)\left(\frac{79 + e^{\pi}}{80} + \frac{\pi}{80}\sin(t)\right)dt =$

$1 + \int_0^{\pi}\frac{1}{80}(\sin(x) + e^t)\left(\frac{79 + e^{\pi}}{80}\right)dt + \int_0^{\pi}\frac{1}{80}(\sin(x) + e^t)\left(\frac{\pi}{80}\sin(t)\right)dt$

$\therefore$ for $\frac{\pi}{80^2}\overbrace{\int_0^{\pi}e^t\sin(t)\,dt}^{I}$ need IBP $u = e^t$, $u' = e^t$, $v' = \sin(t)$, $v = -\cos(t)$

$I = \left[-e^t\cos t\right]_0^{\pi} + \int_0^{\pi}e^t\cos(t)\,dt \qquad u = e^t, u' = e^t, v' = \cos(t), v = \sin(t)$

$= e^{\pi} + 1 + \left[e^t\sin(t)\right]_0^{\pi} - \underbrace{\int_0^{\pi}e^t\sin(t)\,dt}_{I} \qquad \therefore I = \frac{e^{\pi} + 1}{2}$

$\therefore y_2 = 1 + \frac{1}{80^2}\left((79 + e^{\pi})(e^{\pi} - 1) + \frac{\pi}{2}(e^{\pi} - 1)\right) + \frac{81\pi + e^{\pi}}{80^2}\sin(x)$

$\therefore \|y - y_n\|_\infty \leq \frac{1}{|\lambda| - \|K\|_\infty}\left(\frac{\|\ \|_\infty}{|\lambda|}\right)^{n+1}\|f\|_\infty$ ($n+1 \to 2$, $\|f\|_\infty \to 1$, $|\lambda| \to 1$)

$\therefore \|K\|_\infty = \max_{x\in[0,\pi]}\frac{1}{80}\int_0^{\pi}|\sin(x) + e^t|\,dt = \max_{x\in[0,\pi]}\left(\frac{1}{80}\left[t\sin(x) + e^t\right]_0^{\pi}\right) =$

$\max_{x\in[0,\pi]}\left(\left|\frac{\pi\sin(x) + e^{\pi} - 1}{80}\right|\right) = \frac{\pi + e^{\pi} - 1}{80} \quad \therefore \|y - y_2\|_\infty \leq \frac{(\pi + e^{\pi} - 1)^3}{80^2(81 - \pi - e^{\pi})} = 0.0462$

\6/ $p, q \in \mathbb{N}$ $\quad k_p(x,t) = xt^p$, $L_q(x,t) = x^q t$ $\quad K, L$ for $x \in [0,1]$

K_pL_q has kernel: $\int_0^1 k(x,s)l(s,t)\,ds = \int_0^1 xs^p s^q t\,ds = \int_0^1 xts^{p+q}\,ds = \frac{1}{p+q+1}xt$

$L_qK_p = \int_0^1 x^q s \cdot st^p\,ds = \frac{1}{3}x^q t^p$ $\therefore$ Two kernel are equal when $p = q = 1$

6b/ for sup norms: $\|K_p\|_\infty = \int_0^1 t^p\,dt = \frac{1}{p+1}$, $\|L_q\|_\infty = \int_0^1 t\,dt = \frac{1}{2}$ $\because p, q \in \mathbb{N}$

$\therefore$ ~~RH~~ $\|K_pL_q\|_\infty = \frac{1}{2(p+q+1)} \leq \frac{1}{2}\cdot\frac{1}{p+1} = \|L_q\|_\infty\|K_p\|_\infty$

$\|L_qK_p\|_\infty = \frac{1}{3(p+1)} \leq \frac{1}{2}\cdot\frac{1}{p+1}$

\remider $\lambda y - Ky = f$ chap 3 def 3.2.1 $(I - \lambda^{-1}Ky) = \lambda^{-1}f \therefore$

$y = (I - \lambda^{-1}K)^{-1}(\lambda^{-1}f)$

$y - x\int_0^1 t^p y(t)\,dt = f(x) \therefore y = f + x\alpha \xrightarrow{\text{st}} \int_0^1 t^p y(t)\,dt \therefore$

$\alpha = \int_0^1 t^p(f(t) + t\alpha)\,dt \therefore \alpha = \frac{\alpha}{p+2} + \int_0^1 t^p f(t)\,dt \quad \therefore y(x) = f(x) + \frac{p+2}{p+1}x\int_0^1 t^p f(t)\,dt$

$\therefore (I-K_p)^{-1} 1 = 1 + \frac{p+2}{p+1} \int_0^1 t^p \cdot 1 \, dt \therefore$

$\| (I-K_p)^{-1} \|_\infty = 1 + \frac{p+2}{(p+1)^2} \leq 1 + \frac{1}{p}$

$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$

$\cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$

$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

\2020 Q2b/ $4 - \lambda y(x) = \int_{-1}^1 x t^2 y(t) \, dt$

if $\lambda = 0$ its 1st kind if $\lambda \neq 0$ its 2nd kind

$p = \int_{-1}^1 t^2 y^2(t) \, dt \quad \therefore y = \frac{1}{\lambda}(4 - px)$

$\therefore p = \int_{-1}^1 t^2 \left(\frac{1}{\lambda}(4 - pt)\right)^2 dt \quad \therefore p = \frac{1}{\lambda^2}\int_{-1}^1 t^2 (4-pt)^2 \, dt = \frac{1}{\lambda^2}\left(\frac{32}{3} + \frac{2}{5}p^2\right) \therefore$

$p^2 - \frac{5\lambda^2}{2} p + \frac{80}{3} = 0 \quad \therefore$ solve for p

$p = \frac{1}{2}\left(\frac{5}{2}\lambda^2 \pm \sqrt{\frac{25}{4}\lambda^4 - \frac{320}{3}}\right) \quad \therefore y(x) = \frac{4}{\lambda} - \frac{1}{2}\Big(\frac{5}{2}\lambda^2 \pm \underbrace{\sqrt{\frac{25}{4} - \frac{320}{3}}}_{D}\Big)x$

if $D = 0$ have a unique sol,

if $D < 0$. no real sols $-\sqrt{\frac{16}{\sqrt{15}}} < \lambda < \sqrt{\frac{16}{\sqrt{15}}}$

$D >$ we have two sols

\2021 paper/\3b/ $\lambda y(x) = f(x) + \int_0^x t y(t) \, dt \quad x \geq 0$

when $\lambda = 0$ $0 = f(x) + \int_0^x t y(t) \, dt$ need $f(x)$ to be cont & differentiable

need f to be continuous and differentiable and $f(0) = 0$ $\therefore$

$-f'(x) = x y(x) \quad \therefore y(x) = -\frac{f'(x)}{x}$

\3b/ $y' - \frac{x}{\lambda} y = \frac{f'}{\lambda} \quad \therefore \quad IF = e^{\int -\frac{x}{\lambda} dx} = e^{-\frac{1}{2\lambda}x^2}$

$\frac{d}{dx}\left(y \cdot e^{-\frac{x^2}{2\lambda}}\right) = \int_0^x \frac{f'}{\lambda} e^{-t^2/2\lambda} \, dt + C$

$a^x = e^{\ln a^x} = e^{x \ln a} \quad \therefore \int a^x dx = \int e^{\ln(a) x} dx = \frac{1}{\ln(a)} e^{x \ln(a)} = \frac{1}{\ln(a)} a^x$

$\int \tan x \, dx = \int \frac{\sin x}{\cos x} dx = -\int \frac{-\sin x}{\cos x} dx = -\ln|\cos x| = \ln(|\cos x|^{-1}) = \ln\left|\frac{1}{\cos x}\right|$

$\int \ln x \, dx = \int (\ln x) 1 \, dx = [\ln(x) x] - \int \frac{1}{x} x \, dx = x\ln(x) - x$

$\int \sec x \, dx = \int \frac{1}{\cos x} dx = \ln|\sec x + \tan x| = \ln\left|\frac{1}{\cos x} + \frac{\sin x}{\cos x}\right|$

$\int \sec^2 x = \int \frac{1}{\cos^2 x} dx = \tan x \qquad \int \sec x \tan x \, dx = \int \frac{1}{\cos x} \tan x \, dx = \sec x = \frac{1}{\cos x}$

$\int \frac{1}{1+x^2} dx = \arctan x \quad \therefore$ let ~~$u = \tan x$~~ let $u = \arctan x \therefore$

$\tan u = x \quad \therefore \quad \frac{dx}{du} = \frac{d}{du}\frac{\sin u}{\cos u} = \frac{1}{\cos^2 u}$

$\int \frac{1}{1+x^2} dx = \arctan x, \quad \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x \quad \int \frac{-1}{\sqrt{1-x^2}} dx = \arccos x$

$\int \frac{-1}{1+x^2} dx = \frac{1}{\tan x}, \quad \int \frac{1}{x\sqrt{x^2-1}} dx = \frac{1}{\cos x}, \quad \int \frac{-1}{x\sqrt{x^2-1}} dx = \frac{1}{\sin x} \quad \int \frac{1}{\sqrt{1-x^2}} dx = \arcsin x$

$\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$ $a \neq b$ $\frac{px+q}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}$ $\frac{px+q}{(x-a)^2} = \frac{A}{x-a} + \frac{B}{(x-a)^2}$

$\frac{px^2+qx+r}{(x-a)(x-b)(x-c)} = \frac{A}{x-a} + \frac{B}{x-b} + \frac{C}{x-c}$ $\frac{px^2+qx+r}{(x-a)^2(x-b)} = \frac{A}{x-a} + \frac{B}{(x-a)^2} + \frac{C}{x-b}$

$\frac{px^2+qx+r}{(x-a)(x^2+bx+c)} = \frac{A}{x-a} + \frac{Bx+C}{x^2+bx+c}$ $\mathcal{L}(1) = \frac{1}{s}$ $\mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1$ $\mathcal{L}(t^n) = \frac{n!}{s^{n+1}}$ $\mathcal{L}^{-1}\left(\frac{n!}{s^{n+1}}\right) = t^n$

$\mathcal{L}(t^n) = \frac{\Gamma(n+1)}{s^{n+1}}$ $\mathcal{L}^{-1}\left(\frac{\Gamma(n+1)}{s^{n+1}}\right) = t^n$ $\mathcal{L}(e^{at}) = \frac{1}{s-a}$ $\mathcal{L}^{-1}\left(\frac{1}{s-a}\right) = e^{at}$ $\mathcal{L}(\sin(\omega t)) = \frac{\omega}{s^2+\omega^2}$

$\mathcal{L}^{-1}\left(\frac{\omega}{s^2+\omega^2}\right) = \sin(\omega t)$ $\mathcal{L}(\sinh(\omega t)) = \frac{\omega}{s^2-\omega^2}$ $\mathcal{L}^{-1}\left(\frac{\omega}{s^2-\omega^2}\right) = \sinh(\omega t)$

$\mathcal{L}(\cos(\omega t)) = \frac{s}{s^2+\omega^2}$ $\mathcal{L}^{-1}\left(\frac{s}{s^2+\omega^2}\right) = \cos(\omega t)$ $\mathcal{L}(\cosh(\omega t)) = \frac{s}{s^2-\omega^2}$ $\mathcal{L}^{-1}\left(\frac{s}{s^2-\omega^2}\right) = \cosh(\omega t)$

$\mathcal{L}(F(at)) = \frac{1}{a} f\left(\frac{s}{a}\right)$ $\mathcal{L}^{-1}\left(\frac{1}{a} f\left(\frac{s}{a}\right)\right) = F(at)$ $\mathcal{L}(F'(t)) = sf(s) - F(0)$

$\mathcal{L}^{-1}(sf(s) - F(0)) = F'(t)$ $\mathcal{L}(F''(t)) = s^2 f(s) - sF(0) - F'(0)$ $\mathcal{L}^{-1}(s^2 f(s) - sF(0) - F'(0)) = F''(t)$

$\mathcal{L}(F^{(n)}(t)) = s^n f(s) - s^{(n-1)}F(0) - \dots - F^{(n-1)}(0)$ $\mathcal{L}^{-1}(s^n f(s) - s^{(n-1)}F(0) - \dots - F^{(n-1)}(0)) = F^{(n)}(t)$

$\mathcal{L}(t^n F(t)) = ((-1)^n f^{(n)}(s))$ $\therefore \mathcal{L}^{-1}((-1)^n f^{(n)}(s)) = t^n F(t)$ $\mathcal{L}(e^{at}F(t)) = f(s-a)$

$\mathcal{L}^{-1}(f(s-a)) = e^{at}F(t)$ $\mathcal{L}(F(t+T) = F(t)) = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t)\,dt$ $\therefore \mathcal{L}(F(t)) = \int_0^t e^{-st} F(t)\,dt$

$\mathcal{L}(H(t-a)) = \frac{e^{-as}}{s}$ $\mathcal{L}^{-1}\left(\frac{e^{-as}}{s}\right) = H(t-a)$ $\mathcal{L}(F(t-a)H(t-a)) = e^{-as} f(s)$

$\mathcal{L}(F(t)H(t-a)) = e^{-as}\mathcal{L}(F(t-a))$ $\therefore \mathcal{L}^{-1}(e^{-as} f(s)) = F(t-a)H(t-a)$ $\mathcal{L}^{-1}(e^{-as}\mathcal{L}(F(t-a))) = F(t)H(t-a)$

$\mathcal{L}(\delta(t-a)) = e^{-sa}$ $\mathcal{L}^{-1}(e^{-as}) = \delta(t-a)$ $\mathcal{L}(\delta(t-a)F(t)) = F(a)e^{-sa}$ $\mathcal{L}^{-1}(F(a)e^{-sa}) = \delta(t-a)F(t)$

$\mathcal{L}\left(\int_0^t F(u)\,du\right) = \frac{f(s)}{s}$ $\mathcal{L}^{-1}\left(\frac{f(s)}{s}\right) = \int_0^t F(u)\,du$ $\mathcal{L}\left(\int_a^t F(u)\,du\right) = \frac{1}{s}\left(f(s) - \int_a^0 F(t)\,dt\right)$

$\mathcal{L}\left(\int_0^t F(u)G(t-u)\,du\right) = f(s)g(s)$ $\mathcal{L}^{-1}(f(s)g(s)) = \int_0^t F(u)G(t-u)\,du$

\Sheet 5/ \1/ $y(x) - \frac{1}{2}\underbrace{\int_{-1}^{1} y(t)\,dt}_{a} - x\underbrace{\int_{-1}^{1} t\,y(t)\,dt}_{b} = f(x)$ $y(x) = \frac{1}{2}a + bx$

$y(t) = \frac{1}{2}a + bt$

$y(x) - \frac{1}{2}a - bx = f(x)$ $y(x) - \frac{1}{2}a - bx = 0$ $\therefore y(x) = \frac{1}{2}a + bx$

$y(t) = \frac{1}{2}a + bt$ $\therefore a = \int_{-1}^{1} y(t)\,dt = \int_{-1}^{1} \frac{1}{2}a + bt\,dt$ t^1 is odd

t^0 is even $\therefore a = \left[\frac{1}{2}at\right]_{-1}^{1} = \frac{1}{2}a[1 - -1] = a$

$b = \int_{-1}^{1} (\frac{1}{2}a + bt)t\,dt = \int_{-1}^{1} \frac{1}{2}at + bt^2\,dt = \left[\frac{1}{3}bt^3\right]_{-1}^{1} = \frac{2}{3}b = b$ $\therefore 0 = \frac{1}{3}b$ $\therefore b = 0$

$y(x) = \frac{1}{2}a$ $\therefore$ sol to homogeneous equation is not 0 $\therefore$ and has $\frac{1}{2}a$ as a sol. $\therefore$ we are in the 2nd alternative

adjoint kernel is $k^*(x,t) = k(x,t)$ eg is $k(x,t) = x+t$ $\therefore k(t,x) = t+x = x+t = k(x,t)$

the $k(x,t)$ is symmetric

in the 2nd alternative $\therefore$ the adjoint eqn is the same $\therefore$ a is also a sol to the homog adjoint eqn $\therefore$ IE has a sol is $\int_{-1}^{1} f(t)\,dt = 0$

[illegible] 2020 / [illegible] / $F(t) = 2t^2 - [illegible] e^{-t} + [illegible]$

$\sinh(t) = \frac{1}{2}(e^{t} - e^{-t})$ ∴ $F(t) = [illegible]$

$F(s) = \mathcal{L}[F(t)] = 2\left(\frac{2}{s^3}\right) - [illegible]$

$F(s) = \frac{4}{s^3} - [illegible]$

$F(s) = \frac{5}{s^2(s-2)^2} = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s-2} + \frac{D}{(s-2)^2}$

$5 = As(s-2)^2 + B(s-2)^2 + C s^2 (s-2) + D s^2$

$A = \frac{5}{4}$, $B = \frac{5}{4}$, $C = -\frac{5}{4}$, $D = \frac{5}{4}$

$F(t) = \mathcal{L}^{-1}[F(s)] = \mathcal{L}^{-1}\left[\frac{5}{4s} + \frac{5}{4s^2} - \frac{5}{4(s-2)} + \frac{5}{4(s-2)^2}\right] =$

$\frac{5}{4}\left(1 + t(1 + e^{2t}) - e^{2t}\right)$

c i / a ~~[illegible]~~ kernel is weakly singular if it is discontinuous and $k(x,t)$ is continuous when $x \neq t$ [illegible] constants $\alpha \in (0,1)$ & $C > 0$ st $|k(x,t)| \le \frac{C}{|x-t|^{\alpha}}$ for $x \neq t$ [illegible] def of definition

c ii / A: $k(x,t) = \cos(x-t)$ continuous ∴ not weakly singular

B: $k(x,t) = (x-t)^{-1/2} = \frac{1}{(x-t)^{1/2}} = \frac{1}{(x-t)^{\alpha}}$ ∴ $\alpha = \frac{1}{2} < 1$ ∴ weakly singular

C: $k(x,t) = \cos(x-t)|x-t|^{-1/2}$ ∴ $|k(x,t)| = |\cos(x-t)| \frac{1}{|x-t|^{1/2}}$

$\le \frac{1}{|x-t|^{1/2}} = \frac{1}{|x-t|^{\alpha}}$ ∴ $\alpha = \frac{1}{2} < 1$ ∴ $|k(x,t)| \le C\frac{1}{|x-t|^{\alpha}}$ ∴ weakly singular

d /
Z eqn is linear, non homog, Volterra, of the second kind

∴ $Y(s) = \frac{2}{s} + \frac{1}{s-2}Y(s)$ ∴ [illegible] ∴ have convolution kernel ∴

$Y(s) = \frac{2(s-2)}{s(s-3)}$ ∴ $Y(s) = \frac{4}{3s} + \frac{2}{3(s-3)}$ ∴ $y(x) = \frac{4}{3} + \frac{2}{3}e^{3x}$

1e / non linear, second kind, Hammerstein, continuous and separable kernel, non homog ∴ $a = \int_0^1 \sqrt{t}\, y^2(t)\,dt$ ∴

$y(x) = 2 - ax$ ∴ $a = \int_0^1 \sqrt{t}\,(2-at)^2\,dt = \int_0^1 4t^{1/2} - 4at^{3/2} + a^2 t^{5/2}\,dt =$

$\frac{8}{3} - \frac{8}{5}a + \frac{2}{7}a^2 = a$ ∴ $\frac{2}{7}a^2 - \frac{13}{5}a + \frac{8}{3} = 0$ ∴ $a_{\pm} = \frac{7}{4}\left(\frac{13}{5} \pm \sqrt{\frac{169}{25} - \frac{64}{21}}\right)$

$= \frac{91}{20} \pm \frac{7}{20}\sqrt{\frac{1949}{21}}$ ∴ $y(x) = 2 - a_{\pm} x$

2 a ii / this series converges to $2e^{x^3/3}$ as $n \to \infty$

$\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ $\quad \sum_{k=1}^{n} k^2 = \frac{1}{6}n(n+1)(2n+1)$ $\quad e^x = \sum_{k\geq 0} \frac{x^k}{k!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} = \lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$ $\quad \cos(2x) = 2\cos^2 x - 1$

$\sin(x\pm y)\ \sin x \cos y \pm \cos x \sin y$ $\quad \sin 2x = 2\sin x \cos x$

$2\sin x \cos y = \sin(x-y) + \sin(x+y)$ $\quad 2\sin x \sin y = \cos(x-y) - \cos(x+y)$

$2\cos x \cos y = \cos(x-y) + \cos(x+y)$ $\quad \sum_{k=0}^{\infty} r^k = \frac{1}{1-r}$ $\quad \sum_{k=1}^{\infty} r^k = \frac{r}{1-r}$

$\cosh^2 x = 1 + \sinh^2 x$

$\sum_{k=0}^{n} r^k = \frac{1-r^{n+1}}{1-r}$ $\quad \therefore \sum_{k=1}^{\infty} k r^{k-1} = \frac{1}{(1-r)^2}$ $\quad \sum_{k=1}^{n} k r^{k-1} = \frac{1-(n+1)r^n + nr^{n+1}}{(1-r)^2}$

$\sum_{k=0}^{n} a r^k = a\left(\frac{1-r^{n+1}}{1-r}\right)$ $\quad \sin x + \sin y = 2\sin\left(\frac{x+y}{2}\right)\cos\left(\frac{x-y}{2}\right)$

$\sin(x) - \sin y = 2\cos\left(\frac{x+y}{2}\right)\sin\left(\frac{x-y}{2}\right)$

$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \dots = \sum_{k\geq 1} (-1)^{k+1}\frac{x^k}{k}$ for $|x|<1$

$\sum_{n=1}^{\infty} \frac{1}{n^\alpha} < \infty$ if $\alpha > 1$ $\quad \sum_{n=1}^{\infty} \frac{1}{n^\alpha} = \infty$ if $\alpha \leq 1$ $\quad \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right) = \gamma = \text{const}$

$\cos(a)\sin(b) = \frac{1}{2}(\sin(a+b) - \sin(a-b))$ $\quad \sin\left(x+\frac{\pi}{2}\right) = \cos(x)$

$\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$ $\quad$ is the taylor series as $n\to\infty$

$\cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$ $\quad \sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$

$\cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$

$\cos(a)\sin(b) = \frac{1}{2}[\sin(a+b) - \sin(a-b)] = -\frac{1}{2}[\sin(a-b) - \sin(a+b)]$

$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$ $\quad \cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$

$\cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$ $\quad \sin a \cos b = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$

$\sin a \cos b = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$ $\quad \sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$

$\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$ $\quad \sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$\sin a - \sin b = \sin a + \sin(-b) = 2\sin\left(\frac{a+(-b)}{2}\right)\cos\left(\frac{a-(-b)}{2}\right) = \sin\left(\frac{a-b}{2}\right)\cos\left(\frac{a+b}{2}\right)$

$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$ $\quad \sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$ $\quad \sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$ $\quad \cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$

$\sin a \cos b = \frac{1}{2}[\sin(a-b) + \sin(a+b)]$ $\quad \sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$ $\quad \sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$ $\quad \cos(a) + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$\sin a \sin b = \frac{1}{2}(\cos(a-b) - \cos(a+b))$ $\quad \cos a \cos b = \frac{1}{2}(\cos(a-b) + \cos(a+b))$

$\sin a \cos b = \frac{1}{2}(\sin(a-b) + \sin(a+b))$

$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$ $\quad \cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

$\cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$ $\quad \cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$\cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$ $\quad \cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$ $\quad \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$

$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$ $\quad \sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$ $\quad \sin(a) + \sin(b) = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$

$\sin(a) + \sin(b) = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$ $\quad \sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$ $\quad \sin(a) + \sin(b) = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$

$\cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$ $\quad \cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$ $\quad \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$ $\quad 1 + \cot^2 x = \operatorname{cosec}^2 x$

$\cos^2\theta = \frac{1}{2} + \frac{1}{2}\cos(2\theta)$ $\quad \sin^2\theta = \frac{1}{2} - \frac{1}{2}\cos(2\theta)$

$\sin a \sin b = \frac{1}{2}[\cos(a-b) - \cos(a+b)]$ $\quad \cos a \cos b = \frac{1}{2}[\cos(a-b) + \cos(a+b)]$

$\sin(a) - \sin(b) = 2\cos\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$ $\quad \cosh^2 x - \sinh^2 x = 1$

$\sin(a) + \sin(b) = 2\cos\left(\frac{a-b}{2}\right)\sin\left(\frac{a+b}{2}\right)$ $\quad \cos(a) - \cos(b) = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

$\cos(a) + \cos(b) = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$ $\quad \sin(2a) = 2\sin a\cos a$

$\cos(2\theta) = \cos^2\theta - \sin^2(\theta)$ $\quad \tan(a \pm b) = \frac{\tan(a) \pm \tan(b)}{1 \mp \tan(a)\tan(b)}$ $\therefore \tan(2a) = \frac{2\tan(a)}{1-\tan^2(a)}$

$1 + \tan^2 x = \sec^2 x$ $\quad 1 - \tanh^2 x = \operatorname{sech}^2 x$ $\quad \cosh(-x) = \cosh x$, $\sinh(-1) = -\sinh(x)$

$k^*(x,t) = k(t,x)$ $\therefore$ if $k(t,x) = k(x,t)$ then kernel is symmetric $\therefore$

$k^*(x,t) = k(x,t)$ $\therefore$ kernel is self adjoint $\therefore$ the sols to the homog eqn is also 2 sols of homog ajoint eqn

need $\int_{-1}^{1} f(t) = 0$ for a sol

4/ $y - Ky = f$, $\quad k(x,t) = \frac{1}{2} + x(3t^2-1)$ $\therefore$ $y(x) = f(x) + (Ky)(x)$ $\therefore$ $y(x) = f(x) + \int_{-1}^{1}\left(\frac{1}{2} + x(3t^2-1)\right)y(t)\,dt$

$\therefore$ homog is $f(x) = 0$ $\therefore$ $y(x) = \int_{-1}^{1}\left(\frac{1}{2} + x(3t^2-1)\right)y(t)\,dt = \frac{1}{2}\underbrace{\int_{-1}^{1} y(t)\,dt}_{p} - x\int_{-1}^{1} y(t)\,dt - 3x\underbrace{\int_{-1}^{1} t^2 y(t)\,dt}_{q}$

$\therefore p = p$, $q = 0$ 2 $\quad y(x) = \int_{-1}^{1}\left(\frac{1}{2} + 3xt^2 - x\right)y(t)\,dt$

$y(x) = \int_{-1}^{1} \frac{1}{2}y(t) + x(3t^2-1)y(t)\,dt = \frac{1}{2}\int_{-1}^{1} y(t)\,dt + x\int_{-1}^{1}(3t^2-1)\,y(t)\,dt = \frac{1}{2}p + qx$ $\therefore$

$y(t) = \frac{1}{2}p + qt$ $\therefore p = p$, $\quad q = 0$ $\therefore$ $\quad y(x) = \frac{1}{2}p$ $\quad \therefore$

$k^*(x,t) = k(t,x) = \frac{1}{2} + t(3x^2-1) = \frac{1}{2} + 3x^2t - t$ $\therefore \frac{1}{2} + (3x^2-1)t$ $\therefore$

$y = K^*y$ $\therefore$ $y(x) = \int_{-1}^{1}\left(\frac{1}{2} + (3x^2-1)t\right)y(t)\,dt = \frac{1}{2}\int_{-1}^{1} y(t)\,dt + (3x^2-1)\int_{-1}^{1} t\,y(t)\,dt = \frac{1}{2}p_2 + (3x^2-1)q_2$

$\therefore p_2 = p_2$, $\quad q_2 = 0$ $\therefore$ $y^*(x) = \frac{p_2}{2}$ $\therefore$ p_2 arbit $\therefore$ let $p_2 = 2$ $\therefore$ $y^*(x) = 1$

$\therefore$ $\int_{-1}^{1} f(t) \times y^*(t)\,dt = 0 = \int_{-1}^{1} f(t) \times 1\,dt = \int_{-1}^{1} f(t)\,dt$

$\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$ $\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$ $\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

$\sin a + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$ $\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$ $\sin(a) + \sin b = 2\sin\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$

$\cos a + \cos b = 2\cos\left(\frac{a+b}{2}\right)\cos\left(\frac{a-b}{2}\right)$ $\cos a - \cos b = -2\sin\left(\frac{a+b}{2}\right)\sin\left(\frac{a-b}{2}\right)$

$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$ $\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$

$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$ $= \tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$ $1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \operatorname{cosec}^2 x$ $\int \tan x = \ln|\sec x|$ $\int \sec^2 x \, dx = \tan x$

$y = \arctan x$ $\tan y = x$ $\sec^2 y \frac{dy}{dx} = 1$ $\therefore \frac{dy}{dx} = \frac{1}{\sec^2 y} = \cos^2 y$

$\int \frac{1}{1+x^2} dx = \arctan x$ $\int \sec x \tan x = \sec x$ $\int \sec x = \ln|\sec x + \tan x|$

$\int \sec x \tan x = \sec x$ $\int \sec x = \ln|\sec x + \tan x|$ $\int \sec x \tan x = \sec x$

$\int \sec x = \ln|\sec x + \tan x|$ $\int \sec x \tan x = \sec x$

$k(x,t) = k(x-t)$ convolution kernel $\|K\|_\infty = \max_{x\in[0,2\pi]} \int_0^{2\pi} |k(x,t)| dt$

$\|K\|_\infty = \max_{x\in[0,2\pi]} \int_0^{2\pi} |k(x,t)| dt$ $\|y - y_n\|_\infty = \frac{1}{|\lambda| - \|K\|_\infty}\left(\frac{\|K\|_\infty}{|\lambda|}\right)^{n+1} \|f\|_\infty$

$\|K\|_\infty < |\lambda|$ $y_n(x) = \frac{f(x)}{\lambda} + \int_0^{2\pi} k(x,t) y_{n-1}(t) dt$

FA : $|\lambda| > \|K\|_\infty$ $\therefore$ unique cont sol $\|y - y_n\|_\infty < 10^{-2}$

~~be~~ ~~sere~~ result is taylor series as $n \to \infty$

no sols for $\lambda = 0$ two sols, no real sols, real sols

volterra homogeneous non linear, Fredholm finite rank, continuous, second kind, first kind

adjoint

(1a) $F(t)=t,\ G(t)=e^{at}$

$(F*G)(t)=t*e^{at}=$

$e^{at}*t = Z$

$Z=\int_0^t e^{au}(t-u)\,du=$

$t\int_0^t e^{au}\,du-\int_0^t ue^{au}\,du$

$\int_0^t e^{au}\,du=\left[\frac{1}{a}e^{au}\right]_{u=0}^{t}=\frac{1}{a}\left[e^{at}-e^{0}\right]=\frac{1}{a}e^{at}-\frac{1}{a}$,

$\int_0^t ue^{au}=\left[u\frac{1}{a}e^{au}\right]_{u=0}^{t}-\int_0^t\frac{1}{a}e^{au}\,du=$

$=\left[\frac{t}{a}e^{at}-(0)e^{0}\right]-\frac{1}{a}\left[\frac{1}{a}e^{au}\right]_{u=0}^{t}=$

$\frac{1}{a}te^{at}-\frac{1}{a^2}\left[e^{at}-e^{0}\right]=$

$\frac{1}{a}te^{at}-\frac{1}{a^2}e^{at}+\frac{1}{a^2}$

$(F*G)(t)=t\left[\frac{1}{a}e^{at}-\frac{1}{a}\right]-\left[\frac{1}{a}te^{at}-\frac{1}{a^2}e^{at}+\frac{1}{a^2}\right]=$

$\frac{1}{a}te^{at}-\frac{1}{a}t-\frac{1}{a}te^{at}+\frac{1}{a^2}e^{at}-\frac{1}{a^2}\ Z$

$=-\frac{1}{a}t+\frac{1}{a^2}e^{at}-\frac{1}{a^2}$

(2a) Let $\mathcal{L}[f(x)]=\hat{f}(s)$ $\quad \mathcal{L}\left(\int_0^x k(x-t)y(t)\,dt\right)=\mathcal{L}(k(x))\mathcal{L}(y(x))=$

$\hat{f}(s)=\mathcal{L}(k(x))\hat{y}(s)$

$\frac{\hat{f}(s)}{\mathcal{L}(k(x))}=\hat{y}(s)$

$y(x)=\int_0^x\mathcal{L}^{-1}\left(\frac{1}{\mathcal{L}(k(x))}\right)\mathcal{L}^{-1}(\hat{f}(s))\,du=$

(1b) $s^2-4s+20 = (s-2)^2+20-4 = (s-2)^2+16$,

$6(s)-4 = 6(s-2)-4+12 = 6(s-2)+8$ ∴

$F(s) = \frac{6(s-2)+8}{(s-2)^2+16} = 6\frac{s-2}{(s-2)^2+\cancel{16}} + \frac{8}{(s-2)^2+16}$ A

$= 6\frac{1}{s-2} + 2\frac{4}{(s-2)^2+4^2}$ ∴

$\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t}$, $\mathcal{L}^{-1}\left(\frac{4}{(s-2)^2+4^2}\right) = e^{2t}\sin(4t)$ ∴

$\mathcal{L}^{-1}(F(s)) = 6e^{2t} + 2e^{2t}\sin(4t)$

(1c) volterra, nonhomogeneous, linear, 2nd kind

kernel is continuous finite rank 1, Separable.

∴ $y'(x) = 3\cdot 2x + 2xy(x) + \int_0^x (0)y(t)\,dt$ at $x=0$: $y(0) = 0+2(0) = 0$

$= 6x + 2xy(x) = y'(x)$ ∴

$y'(x) - 2xy(x) = 6x$ ∴

$IF = e^{\int -2x\,dx} = e^{-x^2}$ ∴

$\frac{d}{dx}\left(e^{-x^2}y(x)\right) = 6xe^{-x^2}$ ∴

$e^{-x^2}y(x) = \int_0^x 6te^{-t^2}\,dt$

$= -3\int_0^x -2te^{-t^2}\,dt = -3\left[e^{-t^2}\right]_0^x$ A

$= -3\left[e^{-x^2} - e^0\right] = -3e^{-x^2} + 3$ ∴

$y(x) = -3e^{x^2}e^{-x^2} + 3e^{x^2} = -3 + 3e^{x^2}$

$x-t<0 \quad x \not> t \qquad |x-t| = -(x-t)$

$x-t>0 \quad x > t \quad \therefore |x-t| = (x-t)$

$$k(x,t) = \begin{cases} -(x-t), & x<t<2\pi \\ x-t, & 0<t<2\pi \end{cases}$$

$$f(x) = \int_0^{2\pi} k(x,t)\cancel{y(t)}dt = \int_0^x k(x,t)y(t)dt + \int_x^1 k(x,t)y(t)dt =$$

$$\int_0^x (x-t)y(t)dt + \int_x^1 -(x-t)y(t)dt$$

$$\int_0^{2\pi} f(x) = 0$$

non homogeneous: $0 = \int_0^{2\pi} k(x,t)y(t)dt$

$y(x)$ nontrivial:

$|t-x| = |-(x-t)| = |x-t|$

$k^*(x,t) = k(x,t) = |t-x| = |x-t| = k(x,t) \therefore$

self adjoint kernel

$\therefore$ adjoint non homogeous also has nontrivial solutions

$\cancel{\int_0^{2\pi} f(x)\, y^*(x)dx = 0} \qquad \int_0^{2\pi} f(x)\, k^*(x)dx = 0$

$$y(x) = \frac{-4\pi^2}{\pi^2-4}\sin(x) + \frac{-4\pi}{\pi^2-4}\cos(x) + \sin(x)$$

$\sin x \cos t - \cos x \sin t \qquad p = -\frac{4}{\pi^2} q$

$q = \frac{1}{\pi} + \frac{-4}{\pi^2} p$

$$\int_0^{2\pi} \delta(x) = 0$$

$$\int_0^{2\pi} \delta(x)\, k^{*}(x,t) = 0$$

$$y'(x) - \int_0^x -\sin(x-t)\, y(t)\, dt - \cos(x-x)\, y(x) =$$

$$y'(x) - y(x) + \int_0^x \sin(x-t)\, y(t) = \cos x$$

$$y''(x) - y'(x) + \sin(x-x)\, y(x) + \int_0^x \cos(x-t)\, y(t) =$$

$$y''(x) - y'(x) + \left(y(x) - \sin x\right) = -\sin x \;\therefore$$

$$y''(x) - y'(x) + y(x) = 0 \;\therefore \qquad y(x) = e^{2x}$$

$$q^2 - q + 1 = 0 \;\therefore$$

$$q = \frac{1 \pm \sqrt{1^2 - 4(1)(1)}}{2(1)} = \frac{1}{2} \pm \frac{\sqrt{-3}}{2} = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i \;\therefore$$

$$y(x) = A e^{\left(\frac{1}{2} + \frac{\sqrt{3}}{2} i\right)x} + B e^{\left(\frac{1}{2} - \frac{\sqrt{3}}{2} i\right)x}$$

$$= e^{\frac{1}{2}x}\left(A e^{\frac{\sqrt{3}}{2} i x} + B e^{-\frac{\sqrt{3}}{2} i x}\right) =$$

$$y(x) = e^{\frac{1}{2}x}\left(C \cos\left(\frac{\sqrt{3}}{2}x\right) + D \sin\left(\frac{\sqrt{3}}{2}x\right)\right)$$

$$y(0) - \int_0^0 \cos(0-t)\, y(t)\, dt = \sin(0) = 0 = y(0) - 0 \quad \therefore\; y(0) = 0,\; y'(0) = 1$$

$$y'(0) -$$

$$y(x) = \frac{2}{\sqrt{3}} e^{\frac{1}{2}x} \sin\left(\frac{\sqrt{3}}{2}x\right)$$

$y(x) = \lambda \sin(x) \int_0^{\pi} \sin(t) \, dt$

$\sin(2A) = 2\sin A \cos A$

$\cos(2A) = \cos^2 A - \sin^2 A$

$= 1 - 2\sin^2 A$

$= 2\cos^2 A - 1$

$y(x) = \lambda \sin(x) p$

$p = \int_0^{\pi} \sin(t) \left(\lambda p \sin(t)\right)^2 dt$

$= \lambda^2 p^2 \int_0^{\pi} \sin^3(t) \, dt$

$\frac{1}{2}\omega$

$\sin(t)\left(1 - \cos^2(t)\right) = \sin(t) - \sin(t)\left(\cos(t)\right)^2$

$\int_0^{\pi} -\sin(t)\left(\cos(t)\right)^2 dt = \left[\frac{1}{3}\left(\cos(t)\right)^3\right]_0^{\pi}$

$\frac{1}{3}\left[\left(\cos(\pi)\right)^3 - \left(\cos(0)\right)^3\right] = \frac{1}{3}\left\{(-1)^3 - (1)^3\right\} = \frac{1}{3}\left[-1 - 1\right] = -\frac{2}{3}$

$\frac{1}{3}\left(\cos(t)\right)^3$

$\cancel{\int \cos(t) \cos^2(t)}$

$\cos(t)$

$\int_0^{\pi} \sin(t) \, dt = \left[-\cos(t)\right]_0^{\pi} = -\left[\cos\pi - \cos 0\right] = 2$

$\frac{1}{3}\cos^3(t) = \frac{1}{3}\left(\cos(t)\cos^2(t)\right)$

$\frac{1}{3}\left(\cos(t)\left(\cos(2t) + 1\right)\right)$

$\frac{1}{6}\left(\cos(t)\left(\cos(2t) - 1\right)\right)$

$\frac{1}{6}\left(\cos(t)\cos(2t) - \cos(t)\right)$

$\frac{1}{6}\left(\cos(t)\cos(2t)\right)$

$\cos(x) = y$

$\frac{1}{3}(y)^3 \qquad \frac{d}{dx} = \frac{1}{3}(3)y^2 \frac{dy}{dx}$

$= \left(\cos(x)\right)^2(-\sin x)$

$-\cos(t) + \frac{1}{3}\left(\cos(t)\right)^3$

$\lambda^2 p^2 \frac{4}{3} = p$

p

is $\lambda = 0 \qquad y(x) = 0$

(Task 1 a:)

Bessel functions named from Freidrich Bassel are solutions of the Bessel equation:

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^2 - \nu^2)y = 0 \,, \qquad (1)$$

$\nu \in \mathbb{C}$ and is constant.

The order of the bessel function is denoted by ν. [1]

This equation (1) has general solution:

$$y(x) = A\,J_\nu(x) + B\,Y_\nu(x) \,, \qquad (2)$$

Where A and B are constants. Where $J_\nu(x)$ is the bessel function of the first kind, and $Y_\nu(x)$ is the bessel function of the second kind, which are linearly independent. [1]

ν and $-\nu$ produce the same solution for real ν.

$J_\nu(x)$ can be desined by:

$$J_\nu(x) = \sum_{m=0}^{\infty} \frac{(-1)^m}{\Gamma(m+1)\Gamma(m+\nu+1)} \left(\frac{x}{2}\right)^{2m+\nu} . \qquad (3)$$

$J_\nu(x)$ is sinite at the origin $x=0$. [4]

For non integer order, $J_\nu(x)$ and $J_{-\nu}(x)$ are linearly independent.

If ν is an integer $\nu = n$:

$$J_n(x) = \sum_{m=0}^{\infty} \frac{(-1)^m x^{2m+n}}{2^{2m+n}\, m!\,(n+m)!} . \qquad (4)$$

or:

$$J_n(x) = \frac{1}{\pi}\int_0^{\pi} \cos(x\sin\theta - n\theta)\,d\theta . \qquad (5)$$

And:

$$J_{-n}(x) = (-1)^n J_n(x) . \qquad (6)$$

Bessel functions of the second kind are known as Neumann functions, and are developed as a linear combination of bessel funclions of the first order described: [3] for non integer values ν:

$$Y_\nu(x) = \frac{J_\nu(x)\cos(\nu\pi) - J_{-\nu}(x)}{\sin(\nu\pi)} \qquad (7)$$

$$KL = \int_0^{\pi} k(x,s)\, L(s,t)\, dS = \int_0^{\pi} \sin(x+s)\, st\, dS$$

$$= t \int_0^{\pi} \sin(x+s)\, s\, dS$$

$$\sin(x+s) = \sin x \cos s + \cos x \sin s$$

$$t \sin x \int_0^{\pi} s \cos(s)\, dS + t \cos x \int_0^{\pi} s \sin(s)\, dS$$

$$\int_0^{\pi} s \cos(s)\, dS \; - \; \ldots$$

$$\int_0^{\pi} s \sin(s)\, dS \; \ldots$$

$$KL = t \sin(x) \cdots \; + t \cos(x) \cdots$$

(Task 1a Continued:)

For integral values of ν, the expression of $Y_\nu(x)$ has an indeterminate form, and $Y_\nu(x)\big|_{x=0} = \pm\infty$. And the expression for Y_ν is valid for any value of ν for the limit of this function for $x \neq 0$. [3]

In the case of integer order n, the function $Y_\nu(x)$ is defined by taking the limit as a non integer ν tends to n: [4]

$$Y_n(x) = \lim_{\nu \to n} Y_\nu(x). \qquad (8)$$

And for integer values n:

$$Y_n(x) = \frac{2}{\pi} J_n(x)\left(\ln\frac{x}{2} + C\right) - \frac{1}{\pi}\sum_{m=0}^{n-1}\frac{(n-m-1)!}{m!}\left(\frac{x}{2}\right)^{-n+2m} - \frac{1}{\pi}\sum_{m=0}^{\infty}\frac{(-1)^m (x/2)^{n+2m}}{m!(m+n)!}\left\{\sum_{k=1}^{n+m}\frac{1}{k} + \sum_{k=1}^{m}\frac{1}{k}\right\}, \qquad (9)$$

Where C is Eulers constant, approximately 0.5772157. [2]

The zeros, or roots, of the bessel functions are the values of x where the value of the bessel function goes to zero ($J_\nu(x) = 0$). [3] Frequently, the zeros are found in tabulated formats, as they must be numerically evaluated. Bessel functions of the first and second kind have an infinite number of zeroes as the value of $x \to \infty$. The zeroes of the functions can be seen in the crossing points of their graphs. [3]

(Task 1b:)

$$J_0(x) = \sum_{k=0}^{\infty}(-1)^k \frac{\left(\frac{1}{4}x^2\right)^k}{(k!)^2} \qquad (10)$$

equally:

$$J_0(x) = \frac{1}{\pi}\int_0^{\pi} e^{ix\cos\theta}\, d\theta \qquad (11)$$

converges for all x. [5]

The bessel functions of order $\pm 1/2$ are defined as:

$$J_{-1/2}(x) = \sqrt{\frac{2}{\pi x}}\cos x, \qquad (12)$$

$$J_{1/2}(x) = \sqrt{\frac{2}{\pi x}}\sin x. \qquad (13)$$

(Task 1b Continued:)

Asymptotic forms for the bessel functions are:

$$J_\nu(x) \approx \frac{1}{\Gamma(\nu+1)}\left(\frac{x}{2}\right)^\nu , \quad (14)$$

for $x<1$.

And

$$J_\nu(x) \approx \sqrt{\frac{2}{\pi x}} \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) , \quad (15)$$

for $x > |\nu^2 - \frac{1}{4}|$.

Some identities of bessel functions of the first kind are:

$$\frac{d}{dx}\left[x^\nu J_\nu(x)\right] = x^\nu J_{\nu-1}(x). \quad (16)$$

$$\int_0^u u' J_0(u')\,du' = u J_1(u). \quad (17)$$

$$\sum_{k=-\infty}^{\infty} J_k(x) = 1 . \quad (18)$$

Bessel functions are not unlike the trigonometric functions.[1] They enjoy the orthogonality property in the appropriate range of integration with the appropriate weight function. [1]

(Task 1C:)

Bessel functions can be expected to make their appearance in the solutions of axisymmetric problems. [1] And problems whose solutions are connected with the application of cylindrical and spherical coordinates. [2]

They are used when solving problems of acoustics, radio physics, hydrodynamics, atomic and nuclear physics. Applications of bessel functions can be applied to heat conduction theory, including dynamical and linked problems. [2]

In elasticity theory the solutions in bessel functions are effective for all spatial problems, which are solved in spherical or cylindrical coordinates; also for different problems concerning the oscillations of

(Task 1C continued:)

plates on an elastic foundation; for a series of questions of the theory of shells; for problems on the concentration of the stresses near cracks. [2]

Bessel functions are used to model electromagnetic waves in a cylindrical waveguide, pressure amplitudes of inviscid rotational flows, heat conduction in a cylindrical object, models of vibration of a thin circular (or annular) artificial membrane, diffusion problems on a lattice, dynamics of floating bodies, solutions to the radial Schrodinger equation for a free particle, solving for patterns of acoustical radiation. [5]

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[3] Jennifer Niedziela (2008). Bessel Functions and Their Applications.

[4] Elina Shishkina, Sergei Sitnik (2020). Transmutations, Singular and Fractional Differential Equations with Applications to Mathematical Physics.

[5] Shahna Haider (2020). Bessel Functions. https://www.youtube.com/watch?v=LiTpgo-NjSo.

(Task 2a:)

$0=(xy'(x))'+xy(x)=\frac{d}{dx}\left(x\frac{d}{dx}y\right)+xy=0=x\frac{d^2}{dx^2}y+\frac{d}{dx}y+xy=0$

$=xy''+y'+xy=0$ This ODE has the solution $J_0(x)$ ∴

Let $y(x)=J_0(x)$ ∴

$x\frac{d^2}{dx^2}(J_0)+\frac{d}{dx}(J_0)+xJ_0=0=x\frac{d^2}{dx^2}(J_0(x))+\frac{d}{dx}(J_0(x))+xJ_0(x)=0$ ∴

Taking the Laplacian of ~~this~~ ODE:

$\mathcal{L}\left(x\frac{d^2}{dx^2}(J_0)+\frac{d}{dx}(J_0)+xJ_0\right)=\mathcal{L}(0)=0=\mathcal{L}\left(x\frac{d^2}{dx^2}(J_0)\right)+\mathcal{L}\left(\frac{d}{dx}(J_0)\right)+\mathcal{L}(xJ_0)$

∴ $\mathcal{L}\left(x\frac{d^2}{dx^2}(J_0)\right)=\mathcal{L}\left(x^1\frac{d^2}{dx^2}(J_0)\right)=(-1)^1\frac{d}{ds}\left[\mathcal{L}\left(\frac{d^2}{dx^2}(J_0)\right)\right]$ (by LT12)

$=(-1)\frac{d}{ds}\left[s^2\mathcal{L}(J_0)-sJ_0(0)-J_0'(0)\right]$ (by LT10)

$\mathcal{L}\left(\frac{d}{dx}(J_0)\right)=s\mathcal{L}(J_0)-J_0(0)$ (by LT9) ∴

$\mathcal{L}(xJ_0)=\mathcal{L}(x^1J_0)=(-1)^1\frac{d}{ds}(\mathcal{L}(J_0))=(-1)\frac{d}{ds}(\mathcal{L}(J_0))$ (by LT12) ∴

Subbing into ODE:

$-\frac{d}{ds}\left[s^2\mathcal{L}(J_0)-sJ_0(0)-J_0'(0)\right]+s\mathcal{L}(J_0)-J_0(0)-\frac{d}{ds}(\mathcal{L}(J_0))=0$ ∴

Let $\mathcal{L}(J_0)=\hat{J}$ ∴

$-\frac{d}{ds}\left[s^2\hat{J}-sJ_0(0)-J_0'(0)\right]+s\hat{J}-J_0(0)-\frac{d}{ds}\hat{J}=0=$

$-\frac{d}{ds}\left[s^2\hat{J}\right]+\frac{d}{ds}\left[sJ_0(0)\right]+\frac{d}{ds}\left[J_0'(0)\right]+s\hat{J}-J_0(0)-\frac{d}{ds}\hat{J}=0=$

$-2s\hat{J}-s^2\frac{d}{ds}\hat{J}+J_0(0)+0+s\hat{J}-J_0(0)-\frac{d}{ds}\hat{J}=0=$

$-s\hat{J}-s^2\frac{d}{ds}\hat{J}-\frac{d}{ds}\hat{J}=0=$

$-(s^2+1)\frac{d}{ds}\hat{J}-s\hat{J}=0$ ∴

$-(s^2+1)\frac{d}{ds}\hat{J}=s\hat{J}$ ∴

$\frac{1}{\hat{J}}\frac{d}{ds}\hat{J}=-\frac{s}{s^2+1}$ ∴

$\int\frac{1}{\hat{J}}\frac{d\hat{J}}{ds}ds=\int-\frac{s}{s^2+1}ds=-\frac{1}{2}\int\frac{2s}{s^2+1}ds=\int\frac{1}{\hat{J}}d\hat{J}=-\frac{1}{2}\ln|s^2+1|+C_1=$

(Task 2a Continued:)

$-\frac{1}{2}\ln(s^2+1)+C_1 \qquad (\because s^2 \geq 0 \therefore s^2+1 \geq 1 \therefore |s^2+1| = s^2+1)$

$= \ln([s^2+1]^{-1/2}) + C_1 = \ln|\hat{J}| \therefore$

$e^{\ln|\hat{J}|} = |\hat{J}| = e^{\ln([s^2+1]^{-1/2})+C_1} = e^{C_1}e^{\ln([s^2+1]^{-1/2})} =$

$C_2 e^{\ln([s^2+1]^{-1/2})} = C_2[s^2+1]^{-1/2} = C_2\frac{1}{(s^2+1)^{1/2}} = C_2\frac{1}{\sqrt{s^2+1}} = |\hat{J}| \therefore$

$\mathcal{L}(J_0) = \hat{J} = C\frac{1}{\sqrt{s^2+1}} = \mathcal{L}(J_0(x))$, where C_1, C_2, C are ~~arbitrary~~ constants.

and $J_0(x) = y(x) \therefore 1 = y(0) = J_0(0) \therefore$

By the initial value theorem: $\lim_{x\to0} J_0(x) = \lim_{s\to\infty} s\hat{J}(s) =$

$\lim_{s\to\infty} s\frac{C}{\sqrt{s^2+1}} = \lim_{s\to\infty}\frac{sC}{\sqrt{s^2(1+\frac{1}{s^2})}} = \lim_{s\to\infty}\frac{sC}{\sqrt{s^2}\sqrt{1+\frac{1}{s^2}}} = \cancel{\lim_{s\to\infty}\frac{sC}{s\sqrt{1+s^2}}}$

$= \lim_{s\to\infty}\frac{sC}{s\sqrt{1+\frac{1}{s^2}}} = \lim_{s\to\infty}\frac{C}{\sqrt{1+\frac{1}{s^2}}} = \frac{C}{\sqrt{1+0}} = \frac{C}{\sqrt{1}} = C = J_0(0) = 1 \therefore$

$C = 1 \therefore$

$\cancel{\mathcal{L}(J_0(x)) = \mathcal{L}(J_0(x)(s)) = \mathcal{L}}$

$\mathcal{L}(J_0(x)) = \mathcal{L}(J_0(x))(s) = \mathcal{L}(J_0)(s) = \frac{1}{\sqrt{s^2+1}} = \frac{1}{\sqrt{1+s^2}}$

(Task 2b:)

Taking the Laplacian: Let $\mathcal{L}(y(x)) = \hat{y}(s) \therefore$

$\mathcal{L}\left[\int_0^x J_0(x-t)y(t)\,dt\right] = \mathcal{L}(\sin x) = \frac{1}{s^2+1^2} = \frac{1}{s^2+1} = \mathcal{L}\left[\int_0^x y(t)J_0(x-t)\,dt\right] =$

$\mathcal{L}\left[\int_0^x y(u)J_0(x-u)\,du\right] =$ (by LT22:) $\mathcal{L}(y(x))\mathcal{L}(J_0(x)) =$

$\hat{y}(s)\mathcal{L}(J_0(x)) = \hat{y}(s)\mathcal{L}(J_0)(s) = \hat{y}(s)\frac{1}{\sqrt{1+s^2}} = \frac{1}{s^2+1} = \frac{1}{\sqrt{s^2+1}\sqrt{s^2+1}} \therefore$

$\frac{\sqrt{1+s^2}}{\sqrt{1+s^2}\sqrt{1+s^2}} = \hat{y}(s) = \frac{1}{\sqrt{1+s^2}} = \mathcal{L}(J_0)(s) = \hat{y}(s) \therefore$

$\mathcal{L}^{-1}(\hat{y}(s)) = \mathcal{L}^{-1}[\mathcal{L}(J_0)(s)] = \mathcal{L}^{-1}[\mathcal{L}(J_0(x)(s)] = \mathcal{L}^{-1}(\mathcal{L}(y(x))) = J_0(x) = y(x)$

(Task 2C:)

Note: the binomial theorem: $(1+b)^{\gamma} = \sum_{n=0}^{\infty} \frac{\gamma^n}{n!} b^n = \sum_{n=0}^{\infty} \binom{\gamma}{n} b^n$

$\therefore (1+s)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} s^n \therefore$

$$\left(1+\left(\tfrac{1}{s}\right)^2\right)^{-1/2} = \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\left(\tfrac{1}{s}\right)^2\right)^n = \sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\tfrac{1}{s}\right)^{2n} \quad \therefore$$

$$\mathcal{L}(J_0)(s) = \mathcal{L}(J_0) = \frac{1}{\sqrt{1+s^2}} = \frac{1}{\sqrt{s^2\left(\frac{1}{s^2}+1\right)}} = \frac{1}{\sqrt{s^2}\sqrt{\left(\frac{1}{s^2}+1\right)}} = \frac{1}{s\sqrt{1+\left(\frac{1}{s^2}\right)}} =$$

$$\frac{1}{s\sqrt{1+\left(\frac{1}{s}\right)^2}} = \frac{1}{s}\left(\frac{1}{\sqrt{1+\left(\frac{1}{s}\right)^2}}\right) = \frac{1}{s}\left(\frac{1}{1+\left(\frac{1}{s}\right)^2}\right)^{1/2} = \frac{1}{s}\left(1+\left(\tfrac{1}{s}\right)^2\right)^{-1/2} =$$

$$\frac{1}{s}\sum_{n=0}^{\infty} \binom{-1/2}{n} \left(\tfrac{1}{s}\right)^{2n} = \mathcal{L}(J_0)(s).$$

Note: the binomial coefficient of minus half:

$$\binom{r}{n}\binom{r-1/2}{n} = \left(\binom{2r}{n}\binom{2r-n}{n}\right)/4^n \quad \therefore$$

Let $r = -\frac{1}{2}$:

$$\binom{-1/2}{n}\binom{-\frac{1}{2}-\frac{1}{2}}{n} = \binom{2(-\frac{1}{2})}{n}\binom{2(-\frac{1}{2})-n}{n}\frac{1}{4^n} = \binom{-1/2}{n}\binom{-1}{n} = \frac{1}{4^n}\binom{-1}{n}\binom{-1-n}{n} \quad \therefore$$

$$\binom{-1/2}{n} = \frac{1}{4^n}\binom{-1-n}{n}.$$

Note: the negated upper index of binomial coefficient:

$$\binom{r}{n} = (-1)^n \binom{n-r-1}{n} \quad \therefore$$

$$\binom{-1-n}{n} = \binom{(-1-n)}{n} = \binom{n-(-1-n)-1}{n}(-1)^n = (-1)^n\binom{n+1+n-1}{n} = (-1)^n\binom{2n}{n} \quad \therefore$$

$$\frac{1}{4^n}\binom{-1-n}{n} = \frac{1}{4^n}(-1)^n\binom{2n}{n} = \binom{-1/2}{n} \quad \therefore$$

$$\mathcal{L}(J_0)(s) = \frac{1}{s}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\tfrac{1}{s}\right)^{2n} = \frac{1}{s}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}\binom{2n}{n}\left(\tfrac{1}{s}\right)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}\binom{2n}{n}\left(\tfrac{1}{s}\right)^{2n}\left(\tfrac{1}{s}\right)$$

$$= \sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}\binom{2n}{n}\left(\tfrac{1}{s}\right)^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}\binom{2n}{n}\frac{1}{s^{2n+1}} = \mathcal{L}(J_0)(s)$$

and $\binom{b}{g} = C^b_g = {}_bC_g = C(b,g) = \frac{b!}{g!(b-g)!} = \binom{b}{g} \therefore$

(Task 2C continued:)

$$\binom{2n}{n} = \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{n!n!} = \frac{(2n)!}{(n!)^2} \therefore$$

$$\mathcal{L}(J_0)(s) = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}\binom{2n}{n}\frac{1}{s^{2n+1}} = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^n}\frac{(2n)!}{(n!)^2}\frac{1}{s^{2n+1}} =$$

$$\sum_{n=0}^{\infty}\left[\frac{\left(\frac{(-1)^n(2n)!}{4^n(n!)^2}\right)}{s^{2n+1}}\right] = \sum_{n=0}^{\infty}\frac{\alpha_n}{s^{2n+1}} \quad \text{for } \alpha_n = \frac{(-1)^n(2n)!}{4^n(n!)^2}$$

(Task 2d:)

$$J_0(x) = \mathcal{L}^{-1}(\mathcal{L}(J_0(x))(s)) = \mathcal{L}^{-1}\left[\sum_{n=0}^{\infty}\frac{\alpha_n}{s^{2n+1}}\right] = \mathcal{L}^{-1}\left[\sum_{n=0}^{\infty}\frac{\left(\frac{(-1)^n(2n)!}{4^n(n!)^2}\right)}{s^{2n+1}}\right] =$$

$$\mathcal{L}^{-1}\left[\frac{\left(\frac{(-1)^0(0)!}{4^0(0!)^2}\right)}{s^1} + \frac{\left(\frac{(-1)^1(2)!}{4^1(1!)^2}\right)}{s^3} + \frac{\left(\frac{(-1)^2(4)!}{4^2(2!)^2}\right)}{s^5} + \frac{\left(\frac{(-1)^3(6)!}{4^3(3!)^2}\right)}{s^7} + \cdots\right] =$$

$$\mathcal{L}^{-1}\left[\frac{1}{4^0(0!)^2}\frac{1}{s} - \frac{1}{4^1(1!)^2}\frac{2!}{s^3} + \frac{1}{4^2(2!)^2}\frac{4!}{s^5} - \frac{1}{4^3(3!)^2}\frac{6!}{s^7} + \cdots\right] =$$

$$\frac{1}{4^0(0!)^2}\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{4^1(1!)^2}\mathcal{L}^{-1}\left(\frac{2!}{s^3}\right) + \frac{1}{4^2(2!)^2}\mathcal{L}^{-1}\left(\frac{4!}{s^5}\right) - \frac{1}{4^3(3!)^2}\mathcal{L}^{-1}\left(\frac{6!}{s^7}\right) + \cdots =$$

$(\therefore$ by LT1, LT2:$)$ $\frac{+1}{4^0(0!)^2}(1) - \frac{1}{4^1(1!)^2}x^2 + \frac{1}{4^2(2!)^2}x^4 - \frac{1}{4^3(3!)^2}x^6 + \cdots =$

$$+\frac{1}{4^0(0!)^2}x^{2(0)} - \frac{1}{4^1(1!)^2}x^{2(1)} + \frac{1}{4^2(2!)^2}x^{2(2)} - \frac{1}{4^3(3!)^2}x^{2(3)} + \cdots = J_0(x)$$

$\therefore$ Let $r=0$: $\sum_{n=0}^{r}(-1)^n\frac{x^{2n}}{4^n(n!)^2} = \sum_{n=0}^{0}(-1)^n\frac{1}{4^n(n!)^2}x^{2n} = (-1)^0\frac{1}{4^0(0!)^2}x^{2(0)}$

$\therefore$ Sum is true for $r=0$ $\therefore$

Let $r=k$, where $k \in \mathbb{Z}_{\geq 0}$ $\therefore$

assume: $\sum_{n=0}^{k}(-1)^n\frac{x^{2n}}{4^n(n!)^2} =$

$$\frac{1}{4^0(0!)^2}x^{2(0)} - \frac{1}{4^1(1!)^2}x^{2(1)} + \frac{1}{4^2(2!)^2}x^{2(2)} - \frac{1}{4^3(3!)^2}x^{2(3)} + \cdots + (-1)^k\frac{1}{4^k(k!)^2}x^{2k}$$

$\therefore$ Let $r=k+1$ $\therefore$ $\sum_{n=0}^{r}(-1)^n\frac{x^{2n}}{4^n(n!)^2} = \sum_{n=0}^{k+1}(-1)^n\frac{x^{2n}}{4^n(n!)^2} =$

$$\sum_{n=0}^{k}(-1)^n\frac{x^{2n}}{4^n(n!)^2} + \sum_{n=k+1}^{k+1}(-1)^n\frac{x^{2n}}{4^n(n!)^2} =$$

(Task 2d Continued:)

$$\frac{1}{4^0(0!)^2}x^{2(0)} - \frac{1}{4^1(1!)^2}x^{2(1)} + \frac{1}{4^2(2!)^2}x^{2(2)} - \frac{1}{4^3(3!)^2}x^{2(3)} + \dots + (-1)^k \frac{x^{2k}}{4^k(k!)^2} + \sum_{n=k+1}^{k+1} (-1)^n \frac{x^{2n}}{4^n(n!)^2}$$

$$= \frac{1}{4^0(0!)^2}x^{2(0)} - \frac{1}{4^1(1!)^2}x^{2(1)} + \frac{1}{4^2(2!)^2}x^{2(2)} - \frac{1}{4^3(3!)^2} + \dots + (-1)^k \frac{x^{2k}}{4^k(k!)^2} + (-1)^{k+1} \frac{x^{2(k+1)}}{4^{k+1}((k+1)!)^2}$$

$\therefore$ because sum is true for $r=0$, and assuming it is true for $r=k$, then $\sum_{n=0}^{r} (-1)^n \frac{x^{2n}}{4^n(n!)^2} =$

$$\frac{1}{4^0(0!)^2}x^{2(0)} - \frac{1}{4^1(1!)^2}x^{2(1)} + \frac{1}{4^2(2!)^2}x^{2(2)} - \frac{1}{4^3(3!)^2}x^{2(3)} + \dots + (-1)^{k+1} \frac{x^{2(k+1)}}{4^{k+1}((k+1)!)^2}$$

for $r = k+1$ by induction $\therefore$ sum is true for $r = \lim_{k\to\infty}(k)$ by induction $\therefore$

$$J_0(x) = \frac{1}{4^0(0!)^2}x^{2(0)} - \frac{1}{4^1(1!)^2}x^{2(1)} + \frac{1}{4^2(2!)^2}x^{2(2)} - \frac{1}{4^3(3!)^2}x^{2(3)} + \dots =$$

$$\lim_{r\to\infty} \sum_{n=0}^{r} \frac{(-1)^n}{4^n(n!)^2} x^{2(n)} = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{4^n(n!)^2} = J_0(x)$$

(Q2 a) $(xy'(x))' + xy(x) = 0$ $\therefore$

$$\frac{dy}{dx} + x\frac{d^2y}{dx^2} + xy = 0 \quad \therefore \quad x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0 \quad \therefore$$

$$x\frac{d^2y}{dx^2} + \frac{dy}{dx} + (x^2 - n^2)y = 0 = x\frac{d^2y}{dx^2} + \frac{dy}{dx} + xy = 0 \quad \therefore$$

$n = 0$ $\therefore$ the sol of the ODE is $J_0(x)$ $\therefore$

to show the Laplace transform of J_0 is $\mathcal{L}(J_0)(s) = \frac{1}{\sqrt{1+s^2}}$

we sub: $y(x) = J_0(x)$ $\therefore$

$$x\frac{d^2J_0}{dx^2} + \frac{dJ_0}{dx} + xJ_0 = 0 \quad \therefore$$

take the Laplace transform of the ODE:

$$\mathcal{L}\left(x\frac{d^2J_0}{dx^2} + \frac{dJ_0}{dx} + xJ_0\right) = 0 \quad \therefore$$

$$\mathcal{L}\left(x\frac{d^2J_0}{dx^2}\right) + \mathcal{L}\left(\frac{dJ_0}{dx}\right) + \mathcal{L}(xJ_0) = 0 \quad \therefore$$

$$\mathcal{L}\left(x\frac{d^2J_0}{dx^2}\right) = \mathcal{L}\left(x^1\frac{d^2J_0}{dx^2}\right) = (-1)^1\frac{d}{ds}\left[\mathcal{L}\left(\frac{d^2J_0}{dx^2}\right)\right] \quad \{\text{by LT12}\}$$

$$= (-1)\frac{d}{ds}\left[s^2\mathcal{L}(J_0) - sJ_0(0) - J_0'(0)\right] \quad \{\text{by LT10}\}$$

$$\mathcal{L}\left(\frac{dJ_0}{dx}\right) = s\mathcal{L}(J_0) - J_0(0) \quad \{\text{by LT 9}\} \quad \therefore$$

$$\mathcal{L}(xJ_0) = \mathcal{L}(x^1J_0) = (-1)^1\frac{d}{ds}(\mathcal{L}(J_0)) = (-1)\frac{d}{ds}(\mathcal{L}(J_0)) \quad \{\text{by LT12}\} \quad \therefore$$

ODE is:

$$-\frac{d}{ds}\left[s^2\mathcal{L}(J_0) - sJ_0(0) - J_0'(0)\right] + s\mathcal{L}(J_0) - J_0(0) - \frac{d}{ds}(\mathcal{L}(J_0)) = 0$$

$\therefore$ Let $\mathcal{L}(J_0) = F$:

$$-\frac{d}{ds}\left[s^2F - sJ_0(0) - J_0'(0)\right] + sF - J_0(0) - \frac{d}{ds}F = 0 \quad \therefore$$

$$\left\{-\frac{d}{ds}[s^2F] + \frac{d}{ds}[sJ_0(0)] + \frac{d}{ds}[J_0'(0)] + sF - J_0(0) - \frac{d}{ds}F = 0 \quad \therefore\right.$$

$$\left.-2sF - s^2\frac{d}{ds}F + J_0(0) + 0 + sF - J_0(0) - \frac{d}{ds}F = 0\right\} \quad \therefore$$

$$-sF - s^2\frac{d}{ds}F - \frac{d}{ds}F = 0 \quad \therefore$$

$$-(s^2+1)\frac{d}{ds}F - sF = 0 \quad \therefore \quad -(s^2+1)\frac{d}{ds}F = sF \quad \therefore$$

$$\frac{1}{F}\frac{dF}{ds} = -\frac{s}{s^2+1} \quad \therefore \quad \int\frac{1}{F}\frac{dF}{ds}\,ds = \int -\frac{s}{s^2+1}\,ds = -\frac{1}{2}\int\frac{2s}{s^2+1}\,ds = \int\frac{1}{F}\,dF \quad \therefore$$

$$-\frac{1}{2}\ln|s^2+1| + C_1 = \ln|F| \quad \therefore$$

$$\ln(|s^2+1|^{-1/2}) + C_1 = \ln|F| \quad \therefore \quad e^{\ln(|s^2+1|^{-1/2}) + C_1} = C_2e^{\ln(|s^2+1|^{-1/2})} = \ln|F| \quad \therefore$$

$s^2 \geq 0 \therefore s^2+1 \geq 0 \therefore |s^2+1| = s^2+1 \therefore$

$C_2 e^{\ln((s^2+1)^{-1/2})} = \ln|F| = C_2 (s^2+1)^{-1/2} = C_2 \frac{1}{(s^2+1)^{1/2}} = C_2 \frac{1}{\sqrt{s^2+1}} = \ln|F|$

$\therefore \mathcal{L}(J_0) = F = C \frac{1}{\sqrt{s^2+1}} = \mathcal{L}(J_0(x))$

and $y(0) = 1 \therefore$

$\mathcal{L}^{-1}\left[\mathcal{L}(J_0(x))\right] = \mathcal{L}^{-1}\left[\mathcal{L}(J_0(x))(s)\right] = \mathcal{L}^{-1}\left[\mathcal{L}(J_0)(s)\right] = \mathcal{L}^{-1}\left[C \frac{1}{\sqrt{s^2+1}}\right] =$

$C \mathcal{L}^{-1}\left[\frac{1}{\sqrt{s^2+1}}\right] = J_0(x) = J_0$

and $J_0(0) = y(0) = 1 = J_0(0) \therefore$

$\left\{ \mathcal{L}(J_0(x)) \Big|_{x=0} = \mathcal{L}(J_0(0)) = \mathcal{L}(1) = \frac{1}{s} \quad \{\text{by LT1}\} \right\}$

$\therefore$ By the initial value theorem: $1 = y(0) = J_0(0) =$

$\lim_{x \to 0} J_0(x) = \lim_{s \to \infty} sF(s) = \lim_{s \to \infty} s \frac{C}{\sqrt{s^2+1}} = \lim_{s \to \infty} \frac{sC}{\sqrt{s^2(1+\frac{1}{s^2})}} = \lim_{s \to \infty} \frac{sC}{\sqrt{s^2}\sqrt{1+\frac{1}{s^2}}} =$

$\lim_{s \to \infty} \frac{sC}{s\sqrt{1+\frac{1}{s^2}}} = \lim_{s \to \infty} \frac{C}{\sqrt{1+\frac{1}{s^2}}} = \frac{C}{\sqrt{1+0}} = \frac{C}{\sqrt{1}} = C = 1 \therefore$

$\mathcal{L}(J_0(x)) = \mathcal{L}(J_0(x)(s)) = \mathcal{L}(J_0)(s) = \frac{1}{\sqrt{s^2+1}}$

$\frac{1}{4^n}\binom{-1-n}{n} = \frac{(-1)^n}{4^n}\binom{n-(-1-n)-1}{n} = \frac{(-1)^n}{4^n}\binom{n+1+n-1}{n} = \frac{(-1)^n}{4^n}\binom{2n}{n} = \binom{-1/2}{n}$

Note: negated upper index of binomial coefficient:

$\binom{r}{n} = (-1)^n \binom{n-r-1}{n}$

$\binom{-1-n}{n} = \binom{(-1-n)}{n} = \binom{n-(-1-n)-1}{n}(-1)^n = \binom{n+1+n-1}{n}(-1)^n = (-1)^n \binom{2n}{n} \therefore$

$\frac{1}{4^n}\binom{-1-n}{n} = \frac{1}{4^n}(-1)^n \binom{2n}{n}$

(Task 2 Q a) $(xy'(x))' + xy(x) = 0 = \frac{d}{dx}\left(x\frac{d}{dx}y(x)\right) + xy(x) =$

$\frac{d}{dx}y(x) + x\frac{d}{dx}\left(\frac{d}{dx}y(x)\right) + xy(x) = \cancel{x\frac{d^2}{dx^2}y(x) + xy(x) = x}$

$\frac{d}{dx}y(x) + x\frac{d^2}{dx^2}y(x) + xy(x) = y'(x) + xy''(x) + xy(x) = 0$ ∴

Let $\mathcal{L}(y(x)) = \hat{y}(s)$ ∴

$\cancel{\mathcal{L}((xy(x))')}$ $\mathcal{L}((xy'(x))' + xy(x)) = \mathcal{L}(0) = 0 =$

$\mathcal{L}(y'(x) + xy''(x) + xy(x)) =$

$\mathcal{L}(y'(x)) + \mathcal{L}(xy''(x)) + \mathcal{L}(xy(x))$

(Task 2 Q b) $\mathcal{L}(J_0)(s) = \mathcal{L}(J_0) = \frac{1}{\sqrt{1+s^2}}$

$\int_0^x J_0(x-t)\, y(t)\, dt = \sin x \quad x \geq 0$ ∴ let $\mathcal{L}(y(x)) = \hat{y}(s)$

$\mathcal{L}(J_0(x)) = \frac{1}{\sqrt{1+s^2}}$ ∴ by LT22; LT5:

$\mathcal{L}\left(\int_0^x J_0(x-t)\, y(t)\, dt\right) = \mathcal{L}(\sin x) = \frac{1}{s^2+1^2} = \frac{1}{s^2+1} =$

$\mathcal{L}\left(\int_0^x y(t) J_0(x-t)\, dt\right) = \mathcal{L}\left(\int_0^x y(u) J_0(x-u)\, du\right) = \mathcal{L}(y(x))\,\mathcal{L}(J_0(x)) =$

$\hat{y}(s)\,\mathcal{L}(J_0(x)) = \hat{y}(s)\frac{1}{\sqrt{1+s^2}} = \frac{1}{s^2+1}$ ∴

$\hat{y}(s) = \frac{\sqrt{1+s^2}}{s^2+1} = \frac{\sqrt{s^2+1}}{\sqrt{s^2+1}\sqrt{s^2+1}} = \frac{1}{\sqrt{s^2+1}} = \hat{y}(s)$ ∴

$\mathcal{L}^{-1}(\hat{y}(s)) = \mathcal{L}^{-1}\left(\frac{1}{\sqrt{s^2+1}}\right) = y(x) = \mathcal{L}^{-1}(\mathcal{L}(J_0)(s)) = J_0(x) = y(x)$

(Task 2 Q c) $\mathcal{L}(J_0(x))(s) = \frac{1}{\sqrt{1+s^2}}$

show $\mathcal{L}(J_0) = \sum_{n=0}^{\infty}\frac{\alpha_n}{s^{2n+1}} = \mathcal{L}(J_0(x))(s) = \sum_{n=0}^{\infty}\frac{\left(\frac{(-1)^n(2n)!}{4^n(n!)^2}\right)}{s^{2n+1}}$ ∴

$\alpha_n = \frac{(-1)^n(2n)!}{4^n(n!)^2}$ ∴

$\frac{1}{\sqrt{1+s^2}} = \sum_{n=0}^{\infty}\frac{\left(\frac{(-1)^n(2n)!}{4^n(n!)^2}\right)}{s^{2n+1}}$

(Q2 c:) $\int_0^x J_0(x-t)y(t)\,dt = \sin x$ ∴ $\frac{d}{dx}\left(\int_0^x J_0(x-t)y(t)\,dt\right) = \frac{d}{dx}(\sin x) =$

$J_0(x-x)y(x) = \cos x = J_0(0)y(x)$ and $J_0(x) = y(x)$ ∴ $J_0(0) = y(0)$

∴ $y(0) = 1$ ∴ $J_0(0) = 1$ ∴ $\cos x = y(x)$ ✗ wrong! obviously

$$\int \frac{dy}{y} = -\int \frac{s\,ds}{s^2+1} = \int \frac{1}{y}\,dy = -\frac{1}{2}\int \frac{2s}{s^2+1}\,ds =$$

$$\ln|y| = -\frac{1}{2}\ln|s^2+1| + C_3 \quad \therefore$$

$$\cancel{|y| = e^{\frac{1}{2}\ln}} \quad |y| = e^{-\frac{1}{2}\ln|s^2+1| + C_3} = e^{C_3} e^{-\frac{1}{2}\ln|s^2+1|} =$$

$$e^{C_3} e^{\ln(|s^2+1|^{-1/2})} = e^{C_3} e^{\ln((s^2+1)^{-1/2})} = Ce^{\ln((s^2+1)^{-1/2})} =$$

$$C(s^2+1)^{-1/2} = y = \frac{C}{\sqrt{s^2+1}}$$

(Q2C:) note: the binomial theorem: $\cancel{1+b}$

$$(1+b)^n = 1 + nb + \frac{n(n-1)}{2!}b^2 + \frac{n(n-1)(n-2)}{3!}b^3 + \cdots \cancel{+b^n} \quad \therefore$$

$$\frac{1}{\sqrt{1+s^2}} = \frac{1}{(1+s^2)^{1/2}} \quad \text{and} \quad (1-x)^{-2} = \frac{1}{(1-x)^2} = \sum_{n=0}^{\infty} n x^{n-1} \quad \therefore$$

$$\frac{1}{\sqrt{1+s^2}} = (1+s^2)^{-1/2} \quad \therefore \quad \cancel{1-x} \quad (1-s)^{-2} = \sum_{n=0}^{\infty} n s^{n-1} \quad \therefore$$

$$(1-s^2)^{-2} = \sum_{n=0}^{\infty} n (s^2)^{n-1}$$

$$\mathcal{L}(J_0)(s) = \mathcal{L}(J_0) = \frac{1}{\sqrt{1+s^2}} = \frac{1}{\sqrt{s^2\left(\frac{1}{s^2}+1\right)}} = \frac{1}{\sqrt{s^2}\sqrt{\left(\frac{1}{s^2}+1\right)}} = \frac{1}{\sqrt{s^2}\sqrt{1+\left(\frac{1}{s^2}\right)}} =$$

$$\frac{1}{\sqrt{s^2}\sqrt{1+\left(\frac{1}{s}\right)^2}} = \frac{1}{s\sqrt{1+\left(\frac{1}{s}\right)^2}} = \frac{1}{s}\frac{1}{\sqrt{1+\left(\frac{1}{s}\right)^2}} = \frac{1}{s}\left(\frac{1}{1+\left(\frac{1}{s}\right)^2}\right)^{1/2} = \frac{1}{s}\left(1+\left(\frac{1}{s}\right)^2\right)^{-1/2}$$

$$= \frac{1}{s}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\frac{1}{s}\right)^{2n} \quad \therefore$$

by binomial theorem: $\cancel{(a+b)^n = \sum_{n=0}}$ $(1+b)^{\gamma} = \sum_{n=0}^{\infty} \frac{\gamma^n}{n!} b^n = \sum_{n=0}^{\infty}\binom{\gamma}{n} b^n \quad \therefore$

$$(1+s)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}s^n \quad \therefore \quad \left(1+\left(\frac{1}{s}\right)^2\right)^{-1/2} = \sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\left(\frac{1}{s}\right)^2\right)^n = \sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\frac{1}{s}\right)^{2n}$$

$$\therefore \mathcal{L}(J_0)(s) = \frac{1}{s}\sum_{n=0}^{\infty}\binom{-1/2}{n}\left(\frac{1}{s}\right)^{2n}$$

note: ~~b~~ binomial coefficient minus half:

$$\binom{r}{n}\binom{r-\frac{1}{2}}{n} = \left(\binom{2r}{n}\binom{2r-n}{n}\right)/4^n \quad \therefore \quad \text{let } r = -\frac{1}{2}:$$

$$\binom{-1/2}{n}\binom{-\frac{1}{2}-\frac{1}{2}}{n} = \binom{2(-\frac{1}{2})}{n}\binom{2(-\frac{1}{2})-n}{n}\frac{1}{4^n} = \binom{-1/2}{n}\binom{-1}{n} = \frac{1}{4^n}\binom{-1}{n}\binom{-1-n}{n} \quad \therefore$$

$$\binom{-1/2}{n} = \frac{1}{4^n}\binom{-1-n}{n} = \frac{(-1)^n}{4^n}\binom{n-(-1-n)-1}{n} = \frac{(-1)^n}{4^n}\binom{2n}{n} = \binom{-1/2}{n} \quad \therefore$$

(Task 2b:)

Taking the Laplacian: Let $\mathcal{L}(y(x)) = \hat{y}(s)$ ∴

$$\mathcal{L}\left[\int_0^x J_0(x-t)\,y(t)\,dt\right] = \mathcal{L}(\sin x) = \frac{1}{s^2+1^2} = \frac{1}{s^2+1} =$$

$$\mathcal{L}\left[\int_0^x y(t)\,J_0(x-t)\,dt\right] = \mathcal{L}\left[\int_0^x y(u)\,J_0(x-u)\,du\right] = \quad \text{(by LT22:)}$$

$$\mathcal{L}(y(x))\,\mathcal{L}(J_0(x)) = \hat{y}(s)\,\mathcal{L}(J_0(x)) = \hat{y}(s)\,\mathcal{L}(J_0)(s) =$$

$$\hat{y}(s)\,\frac{1}{\sqrt{1+s^2}} = \frac{1}{s^2+1} = \frac{1}{\sqrt{s^2+1}\,\sqrt{s^2+1}} \quad ∴$$

$$\frac{\sqrt{1+s^2}}{\sqrt{1+s^2}\,\sqrt{1+s^2}} = \hat{y}(s) = \frac{1}{\sqrt{1+s^2}} = \mathcal{L}(J_0)(s) = \hat{y}(s) \quad ∴$$

$$\mathcal{L}^{-1}(\hat{y}(s)) = \mathcal{L}^{-1}[\mathcal{L}(J_0)(s)] = \mathcal{L}^{-1}[\mathcal{L}(J_0(x))(s)] =$$

$$\mathcal{L}^{-1}(\mathcal{L}(y(x))) = J_0(x) = y(x)$$

(~~RP~~ 2b) note: $\frac{d}{dx}\left[\int_{g(x)}^{f(x)} k(x,t)y(t)dt\right] = k(x,x)y(x) + \int_{g(x)}^{f(x)} \frac{d}{dx}(k(x,t))y(t)dt$

$\therefore$ ~~taking~~ the Laplacian: $\therefore$ Let $\mathcal{L}(y(x)) = \hat{y}(s)$ $\therefore$

$\mathcal{L}\left[\int_0^x J_0(x-t)y(t)dt\right] = \mathcal{L}(\sin x) = \frac{1}{s^2+1^2} = \frac{1}{s^2+1} =$

$\mathcal{L}\left[\int_0^x y(t)J_0(x-t)dt\right] = \mathcal{L}\left[\int_0^x y(t)J_0(x-u)du\right] =$ $\{$by LT22:$\}$

$\cancel{f(s)}$ $\mathcal{L}(y(x))\,\mathcal{L}(J_0(x)) = \hat{y}(s)\,\mathcal{L}(J_0(x)) = \hat{y}(s)\,\mathcal{L}(J_0)(s) =$

$\hat{y}(s)\frac{1}{\sqrt{1+s^2}} = \frac{1}{s^2+1} = \frac{1}{\sqrt{s^2+1}\sqrt{s^2+1}}$ $\therefore$

$\frac{\sqrt{1+s^2}}{\sqrt{s^2+1}\sqrt{s^2+1}} = \hat{y}(s) = \frac{1}{\sqrt{s^2+1}} = \cancel{\mathcal{L}}\ \frac{1}{\sqrt{1+s^2}} = \mathcal{L}(J_0)(s) = \hat{y}(s)$ $\therefore$

$\mathcal{L}^{-1}(\hat{y}(s)) = \cancel{\mathcal{L}^{-1}(J_0)(s)}\ \mathcal{L}^{-1}\left[\mathcal{L}(J_0)(s)\right] = \mathcal{L}^{-1}\left[\mathcal{L}(J_0(x))(s)\right] =$

$\mathcal{L}^{-1}(\mathcal{L}(y(x))) = J_0(x) = y(x)$

$\frac{d}{dx}\left[\int_0^x J_0(x-t)y(t)dt\right] = \frac{d}{dx}(\sin x) = \cos x =$

$J_0(x-x)y(x) + \left[\int_0^x \frac{d}{dx}[J_0(x-t)]y(t)dt\right]$

(Q2C contin.) $\mathcal{L}(J_0)(s) = \frac{1}{s}\sum_{n=0}^{\infty}\binom{-\frac{1}{2}}{n}\left(\frac{1}{s}\right)^{2n} =$

$$\frac{1}{s}\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}\binom{2n}{n}\left(\frac{1}{s}\right)^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}\binom{2n}{n}\left(\frac{1}{s}\right)^{2n}\left(\frac{1}{s}\right) =$$

$$\sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}\binom{2n}{n}\left(\frac{1}{s}\right)^{2n+1} = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}\binom{2n}{n}\frac{1}{s^{2n+1}} = \mathcal{L}(J_0)(s)$$

and $\binom{b}{g} = C^b_g = {}_bC_g = {}^b_gC = \frac{b!}{g!(b-g)!} = \binom{b}{g}$ ∴

$$\binom{2n}{n} = \frac{(2n)!}{n!(2n-n)!} = \frac{(2n)!}{n!\,n!} = \frac{(2n)!}{(n!)^2} \;\therefore$$

$$\mathcal{L}(J_0)(s) = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}\binom{2n}{n}\frac{1}{s^{2n+1}} = \sum_{n=0}^{\infty}\frac{(-1)^n}{4^n}\frac{(2n)!}{(n!)^2}\frac{1}{s^{2n+1}} =$$

$$\sum_{n=0}^{\infty}\left[\frac{\left(\frac{(-1)^n(2n)!}{4^n(n!)^2}\right)}{s^{2n+1}}\right] = \sum_{n=0}^{\infty}\frac{\alpha^n}{s^{2n+1}} \quad \text{for } \alpha_n = \frac{(-1)^n(2n)!}{4^n(n!)^n} \quad \square$$

(Q2d) $\mathcal{L}^{-1}\left[\sum_{n=0}^{\infty}\frac{x_n}{s^{2n+1}}\right] = \mathcal{L}^{-1}\left[\sum_{n=0}^{\infty}\frac{\left(\frac{(-1)^n(2n)!}{4^n(n!)^2}\right)}{s^{2n+1}}\right] = \mathcal{L}^{-1}\left(\mathcal{L}(J_0(x))(s)\right) = J_0(x)$

$$= \mathcal{L}^{-1}\left[\frac{\left(\frac{(-1)^0(0)!}{4^0(0!)^2}\right)}{s^1} + \frac{\left(\frac{(-1)^1(2)!}{4^1(1!)^2}\right)}{s^3} + \frac{\left(\frac{(-1)^2(4)!}{4^2(2!)^2}\right)}{s^5} + \frac{\left(\frac{(-1)^3(6)!}{4^3(3!)^2}\right)}{s^7} + \dots\right] =$$

$$\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{2s^3} + \frac{3}{8s^5} - \frac{5}{16s^7} + \dots\right] =$$

$$\mathcal{L}^{-1}\left[\frac{1}{s} - \frac{1}{4}\frac{2!}{s^3} + \frac{1}{4^2\times 2^2}\frac{4!}{s^5} - \frac{1}{4^3\times 6^2}\frac{6!}{s^7} + \dots\right] =$$

$$\mathcal{L}^{-1}\left(\frac{1}{s}\right) - \frac{1}{4^1\times(1!)^2}\mathcal{L}^{-1}\left(\frac{2!}{s^3}\right) + \frac{1}{4^2\times(2!)^2}\mathcal{L}^{-1}\left(\frac{4!}{s^5}\right) - \frac{1}{4^3\times(3!)^2}\mathcal{L}^{-1}\left(\frac{6!}{s^7}\right) + \dots \quad \{\because \text{by LT2}\}$$

$$= +\frac{1}{4^0(0!)^2}\times 1 - \frac{1}{4^1\times(1!)^2}x^2 + \frac{1}{4^2\times(2!)^2}\mathcal{L}^{-1}\left(\frac{4!}{s^5}\right) - \frac{1}{4^3\times(3!)^2}\mathcal{L}\left(\frac{6!}{s^7}\right) + \dots =$$

$$+\frac{1}{4^0(0!)^2}\times 1 - \frac{1}{4^1(1!)^2}x^2 + \frac{1}{4^2(2!)^2}x^4 - \frac{1}{4^3\times(3!)^2}x^6 + \dots =$$

$$+\frac{1}{4^0(0!)^2}\times x^{2\times 0} - \frac{1}{4^1(1!)^2}x^{2\times 1} + \frac{1}{4^2(2!)^2}x^{2\times 2} - \frac{1}{4^3\times(3!)^2}x^{2\times 3} + \dots =$$

$$\sum_{n=0}^{\infty}(-1)^n\frac{1}{4^n(n!)^2}x^{2n} = \sum_{n=0}^{\infty}(-1)^n\frac{x^{2n}}{4^n(n!)^2} = J_0(x)$$

(Q2d) $J_0(x) = \mathcal{L}^{-1}(\mathcal{L}(J_0)(s)) = \mathcal{L}^{-1}\left(\sum_{n=0}^{\infty} \frac{a_n}{s^{2n+1}}\right) =$

$\frac{1}{4^0(0!)^2}x^{2(0)} - \frac{1}{4^1(1!)^2}x^{2(1)} + \frac{1}{4^2(2!)^2}x^{2(2)} - \frac{1}{4^3(3!)^2}x^{2(3)} + \dots = J_0(x)$ ∴

Let $r = 0$. $\sum_{n=0}^{r}(-1)^n \frac{x^{2n}}{4^n(n!)^2} = \sum_{n=0}^{0}(-1)^n \frac{1}{4^n(n!)^2}x^{2n} = (-1)^0 \frac{1}{4^0(0!)^2}x^{2(0)}$

∴ true for $r = 0$ ∴

Let $r = k$, where $k \in \mathbb{Z}_{\geq 0}$ ∴

assume: $\sum_{n=0}^{k}(-1)^n \frac{x^{2n}}{4^n(n!)^2} =$

$\frac{1}{4^0(0!)^2}x^{2(0)} - \frac{1}{4^1(1!)^2}x^{2(1)} + \frac{1}{4^2(2!)^2}x^{2(2)} - \frac{1}{4^3(3!)^2}x^{2(3)} + \dots + (-1)^k \frac{x^{2k}}{4^k(k!)^2}$

∴ Let $r = k+1$ ∴

$\sum_{n=0}^{r}(-1)^n \frac{x^{2n}}{4^n(n!)^2} = \sum_{n=0}^{k+1}(-1)^n \frac{x^{2n}}{4^n(n!)^2} = \sum_{n=0}^{k}(-1)^n \frac{x^{2n}}{4^n(n!)^2} + \sum_{n=k+1}^{k+1}(-1)^n \frac{x^{2n}}{4^n(n!)^2} =$

$\frac{1}{4^0(0!)^2}x^{2(0)} - \frac{1}{4^1(1!)^2}x^{2(1)} + \frac{1}{4^2(2!)^2}x^{2(2)} - \frac{1}{4^3(3!)^2}x^{2(3)} + \dots + (-1)^k \frac{x^{2k}}{4^k(k!)^2} + \sum_{n=k+1}^{k+1}(-1)^n \frac{x^{2n}}{4^n(n!)^2}$

$= \frac{1}{4^0(0!)^2}x^{2(0)} - \frac{1}{4^1(1!)^2}x^{2(1)} + \frac{1}{4^2(2!)^2}x^{2(2)} - \frac{1}{4^3(3!)^2}x^{2(3)} + \dots + (-1)^k \frac{x^{2k}}{4^k(k!)^2} + (-1)^{k+1} \frac{x^{2(k+1)}}{4^{k+1}((k+1)!)^2}$

∴ because it's true for $r = 0$, and assuming it's true for $r = k$, then $\sum_{n=0}^{r}(-1)^n \frac{x^{2n}}{4^n(n!)^2} =$

$\frac{1}{4^0(0!)^2}x^{2(0)} - \frac{1}{4^1(1!)^2}x^{2(1)} + \frac{1}{4^2(2!)^2}x^{2(2)} - \frac{1}{4^3(3!)^2}x^{2(3)} + \dots + (-1)^{k+1} \frac{x^{2(k+1)}}{4^{k+1}((k+1)!)^2}$

for $r = k+1$ by induction ∴

$J_0(x) = \frac{1}{4^0(0!)^2}x^{2(0)} - \frac{1}{4^1(1!)^2}x^{2(1)} + \frac{1}{4^2(2!)^2}x^{2(2)} - \frac{1}{4^3(3!)^2}x^{2(3)} + \dots =$

$\lim_{r \to \infty} \sum_{n=0}^{r} \frac{(-1)^n}{4^n(n!)^2}x^{2(n)} = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{4^n(n!)^2} = J_0(x)$ as required □