MTH3024 Stochastic Processes

CW 1 10% Due 4/3/2022   CW2 10% Due 1/4/2022

office hours: (~~[illegible]~~ 11, week 6-10, [illegible] 12:30 to on Friday

\Menti/ \1/

Sigma algebra described as collection of sets

Σ closed under unions and intersections

\2/ $P(A_w) = 0.8P(A_w|dry) + 0.2P(A_w|wet) = 0.4 + 0.02 = 0.42$

$P(D) = 0.8P(D|dry) + P(D|wet) 0.2 = 0.08 + 0.1 = 0.18$

$P(I_w) = 0.8P(I_w|dry) + 0.2P(I_w|wet) = 0.8 \cdot 0.4 + 0.2 \cdot 0.4 = 0.4$ $\therefore$

$P(A_w) > P(I_w)$

$X \equiv$ who wins? $x=0$ Aust $x=\frac{1}{2}$ Draw $x=1$ Ind

$Y \equiv$ weather $Y=0$ dry, $Y=1$ rain

$\cancel{E[X|Y]}$ $E[X|Y=0] = 0 \times 0.5 + 0.5 \times 0.1 + 1 \times 0.4 = 0.45$

$E[X|Y=1] = 0 \times 0.1 + 0.5 \times 0.5 + 1 \times 0.4 = 0.65$

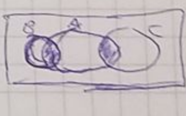
$E[X] = \sum_{j=0}^{1} E[X|Y=j]P(Y=j)$ LoTP   law of total expectation

$= E[X|Y=0]P(Y=0) + E[X|Y=1]P(Y=1) = 0.45 \times 0.8 + 0.65 \times 0.2 = 0.36 + 0.13 =$

0.49

\Ex/ $P(A|B \cup C) = P(A|B) + P(A|C)$ $P(B \cap C) = 0$

$P(A \cap (B \cup C)) = P(A \cap B) + P(A \cap C)$



$P(B \cup C) = P(B) + P(C)$

\Ex/ $\sum_{n=1}^{\infty} \frac{x^{2n}}{n}$ $f(x) = \frac{x^{2n}}{n}$ $f'(x) = 2x^{2n-1}$ $\sum_{n=1}^{\infty} y^n = \frac{y}{1-y}$

$S'(x) = \sum_{n=1}^{\infty} f'(x) = \frac{2}{x} \sum_{n=1}^{\infty} x^{2n} = \frac{2}{x} \sum_{n=1}^{\infty} (x^2)^n$ $\therefore y = x^2$

$= \frac{2}{x} \sum_{n=1}^{\infty} y^n = \frac{2}{x} \frac{y}{1-y} = \frac{2x}{(1-x)^2}$

$S(x) = \sum_{n=1}^{\infty} S(x) = \sum_{n=1}^{\infty} \frac{x^{2n}}{n} = \int \frac{2x}{1-x^2} dx = -\int \frac{-2x}{1-x^2} dx = -\log(1-x^2) + C$

$\therefore S(0) = 0 \Rightarrow -\log(1-0^2) = -\log(1) = 0 = C \Rightarrow S(x) = -\log(1-x^2)$

Weeks 7-8-9

we have a $P: \Sigma \to \mathbb{R}$, $B(\mathbb{R}) = \sigma(\{(-\infty, x) \mid x \in \mathbb{R}\})$

$\{X = x\} = \{\omega \in \Omega \mid X(\omega) = x\} \in \Sigma$

\Ex/ toss a single coin: $\Omega = \{H, T\}$, $X(T) = 0$, $X(H) = 1$

toss a coin twice: let $X \equiv \# H_s$ (num heads)

$\omega = (\omega_1, \omega_2)$ what is $P(X = 1)$?

$\therefore \omega(H, T)$ or $\omega = (T, H)$ $\therefore$ let X_n be Z r.v. for Z n'th toss

$X_n(\omega) = 1$ if Z n'th toss is H

$\{\omega \in \Omega \mid X_1(\omega) + X_2(\omega) = 1\}$, $P(X = 1) = \frac{1}{2}$

\Def/ a r.v. X is discrete if it takes values over a countable set $\Omega_X = \{x_i \mid i \in \mathbb{N}\}$. write $p_i = P(X = x_i)$ & call this Z probab mass func. must have (this is normalised) $\sum_{i \in \mathbb{N}} p_i = 1$

\Def/ a r.v. X is continuous if its range Ω_X is an interval (uncountable) & $\exists$ a density func $f_X(x)$ st $\int_{\Omega_X} f_X(x)\,dx = 1$

write $P(A) = \int_A f_X(x)\,dx$, $A \subset B(\Omega_X)$

write $F_X(x) = P(X \leq x)$ & call this Z cumulative density func

\Expectation/ let $g: \mathbb{R} \to \mathbb{R}$ be a func. Z expectation of $g(x)$ is given by discrete: $E[g(X)] = \sum_{i \in \mathbb{N}} g(x_i) p_i$ $\quad p = P(X = x_i)$

continuous $E[g(x)] = \int_{\Omega_X} g(x) f_X(x)\,dx$ $\quad f_X(x) = P(X = x)$

Z expectation of a r.v. X is given by

Discrete $E[X] = \sum_{i \in \mathbb{N}} x_i p_i$

Continuous $E[X] = \int_{\Omega_X} x f_X(x)\,dx$

if X, Y are two r.vs, then $E[\alpha X + \beta Y] = \alpha E[X] + \beta E[Y]$.
$\alpha, \beta \in \mathbb{R}$

\Variance/ Z variance of a r.v. X is given by $V(X) = E[(X - E(X))^2]$

$\therefore Var(X) = E[(X - E[X])^2] = E[X^2 - 2E[X]X + E[X]^2] = E[X^2] - 2E[X]E[X] + E[X]^2$

$= E[X^2] - 2E[X]^2 + E[X]^2 = E[X^2] - E[X]^2$

if X, Y are two r.v.s then $Var(\alpha X) = \alpha^2 Var(X)$, $\alpha \in \mathbb{R}$

$Var(X + Y) = Var(X) + Var(Y) + 2\,Cov(X, Y)$

Week 1 / Probability theory /

Z set Ω is Z sample space for a probabilistic experiment

- $\omega \in \Omega \equiv$ outcome $\quad F \subset \Omega \equiv$ events

toss a coin once $\Omega = \{H, T\}$

toss a coin twice $\Omega = \{HH, HT, TH, TT\}$, $A \subset \Omega$,

$A = \{HT, TH, TT\}$

Def / a sigma altegbra σ-algebra, $A \subset P(\Omega)$ {P is power set}

is a ~~Subs~~ set of subsets of Ω (i) $\Omega \in A$

(ii) for all $F \in A$, we have $F^c \in A$, $F^c = \Omega / F \in A$

(iii) if $F_j \in A$, $j \in \mathbb{N}$ then $\bigcup_{j=1}^{\infty} F_j \in A$

Ex / is $\emptyset \in A$? $\Omega = \emptyset^c \in A$ from (i)

$\therefore$ from (ii) $\emptyset = \Omega^c \in A$

Coin tossing : $A = P(\Omega)$, $\Omega = \{H, T\}$ $\therefore P(\Omega) = \{\emptyset, \Omega, \{H\}, \{T\}\}$

Experiments on $\mathbb{R}$, Z so-called Borel σ-algebra is formed by taking unions, intersections & complements of Z intervals of Z form $(-\infty, x]$, $x \in \mathbb{R}$. write $B(\mathbb{R}) = \sigma(\{(-\infty, x) \mid x \in \mathbb{R}\}$

Def / given a σ-algebra, Σ , a func $P : \Sigma \to \mathbb{R}$ is called a probability measure if : (Kolmogorov's Axioms)

(i) $P(\Omega) = 1$, $P(\emptyset) = 0$

(ii) if $F \in \Sigma$, $P(F) \in [0, 1]$

(iii) if $F_i, F_j \in \Sigma$ disjoint, $F_i \cap F_j = \emptyset$ for $i \neq j$

$P\left(\bigcup_{i=1}^{n} F_i\right) = \sum_{i=1}^{n} P(F_i)$ true as $n \to \infty$

Def / the tuple (Ω, Σ, P) is a probability space

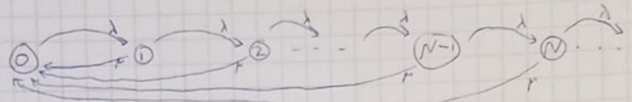
Ex / coin tossing $\Omega = \{H, T\}$ $\therefore \Sigma = \{\emptyset, \Omega, \{H\}, \{T\}\}$

$P(\{H\}) = \frac{1}{2} = P(\{T\})$ $\quad P(\Omega) = P(\{H, T\}) = P(\{H\}) = P(\{T\}) = \frac{1}{2} + \frac{1}{2} = 1$

Random variables /

Def / a random variable (r.v.) is a map $X : \Omega \to \mathbb{R}$ st, $\forall$ $\Sigma \in B(\mathbb{R})$, we have $\{X \in \Sigma\} = \{\omega \in \Omega \mid X(\omega) \in \Sigma\} \in \Sigma$

arrive rate 1 per 10 min   all leave rate 1 per 30 min



$\frac{dP_0}{dt} = -\lambda_0 P_0 + \mu P_1 + \mu P_2 + \mu P_3 + \dots$ {eg how many ways to enter & leave state 0?}

$= -\lambda_0 + \mu \sum_{n=1}^{\infty} P_n = -\lambda P_0 + \mu(1-P_0) = \left\{-\lambda_0 + \mu_n\left[-P_0 + \sum_{n=0}^{\infty} P_n\right]\right\}$

steady state: $\frac{P_0}{dt} = 0 \Rightarrow \lambda P_0 = \mu(1-P_0) \Rightarrow P_0 = \frac{\mu}{\mu+\lambda} = \frac{2}{8} = \frac{1}{4}$

for 6 per hour & 2 per hour $\therefore \frac{1}{4} = P_0$ is steady state probab case is empty

$i > 0$: $\frac{dP_i}{dt} = \lambda P_{i-1} - \lambda P_i - \mu P_i$

steady state $\frac{dP_i}{dt} = 0 \Rightarrow (\lambda+\mu) P_i = \lambda P_{i-1} \Rightarrow P_i = \frac{\lambda P_{i-1}}{\lambda+\mu} = \frac{3}{4} P_{i-1}$

in general: $P_i = \left(\frac{3}{4}\right)^i P_0 = \left(\frac{3}{4}\right)^i \left(\frac{1}{4}\right) \Rightarrow$

$P_5 = \left(\frac{3}{4}\right)^5 \left(\frac{1}{4}\right) \approx 0.0593$

is st st probab there are 5 in case

\Ex / Apples 5 per day   Bananas 3 per day

probab more apples in 2 days than bananas in 3 days given no more than 2 bananas over 3 days?

$A_t \sim$ num apples eaten over time t

$B_t \sim$ no. bananas eaten over time t

$A_t \sim Poiss(5t)$ & $B_t \sim Poiss(3t)$

$\therefore A_2 \sim Poiss(10)$ $B_3 \sim Poiss(9)$

$P(A_2 > B_3 \mid B_3 \leq 2) = \sum_{k=0}^{2} P(A_2 > k \mid B_3 = k) P(B_3 = k) =$

$\sum_{k=0}^{2} P(A_2 > k) P(B_3 = k)$ (since A_t, B_t independent)

$= \sum_{k=0}^{2} (1 - P(A_2 \leq k)) P(B_3 = k) =$

$P(B_3 = 0)(1 - P(A_2 = 0)) + P(B_3 = 1)(1 - P(A_2 = 0) - P(A_2 = 1)) +$

$P(B_3 = 2)(1 - P(A_2 = 0) - P(A_2 = 1) - P(A_2 = 2)) =$

\2/ $p_{yellow} = \frac{2}{3}$ probab that time 2 3rd yellow bulb is greater than 2 years $\quad 0.2381$

$\lambda = \frac{2}{3}$ $\qquad P(Y(t=2) < 2) =$

$\lambda t = \frac{4}{3} \quad \therefore \quad \left(\frac{4}{3}\right)$

\Sol/ $N(t) \sim Poiss(\lambda t) = Poiss(3t)$

$P(N(2) = 0 \quad 0.025$

\Sol/ $B(t) \sim Poiss(\frac{1}{3} \times 3t) = Poiss(t)$

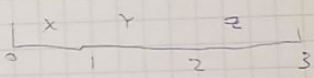
$Y(t) \sim Poiss(\frac{2}{3} \times 3t) = Poiss(2t)$ $\qquad P(S_3^Y > 2) =$

$P(Y(2) < 3) = P(Y(2) = 0) + P(Y(2) = 1) + P(Y(2) = 2) =$

$e^{-4}(\underset{Y=0}{1} + \underset{Y=1}{4} + \underset{Y=2}{4^2/2!}) \approx 0.2381$

Probab 2 bulbs in first 2 years & bulbs in years 1 to 3

$N_{(a,b)}(t)$ = no. of events between $t=a$ & $t=b$



$X \sim N_{[0,1)}$, $Y = N_{[1,2)}$, $Z = N_{[2,3)}$

$X, Y, Z \sim Poisson(3)$

$P(N_{[0,2)} = 2 \;\&\; N_{[1,3)} = 2) = P(X+Y=2 \;\&\; Y+Z=2) =$

$\sum_{k=0}^{2} P(X+Y=2 \;\&\; Y+Z \mid Y=k) P(Y=k) =$

$P(X=2, Z=2) P(Y=0) + P(X=1, Z=1) P(Y=1) + P(X=0, Z=0) P(Y=2) =$

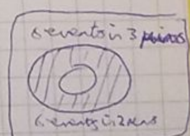
$P(X=2) P(Z=2) P(Y=0) + P(X=1) P(Y=1) P(Z=1) + P(X=0) P(Z=0) P(Y=2) =$

$\left(\frac{e^{-3} 3^2}{2!}\right)^2 e^{-3} + (3e^{-3})^3 + (e^{-3})^2 \frac{e^{-3} 3^2}{2!} \approx 0.0064$

\Sheet 2/ \1c/ X = poisson process rate 3 $\qquad P(2 \le S_6 \le 3)$

$\{N(t) \ge r+1\} \subset \{N(t) \ge r\}$

$\{N(t) \ge r\} = \bigcup_{k=r}^{\infty} \{N(t) = k\}$ $\qquad \{N(t) \ge r+1\} = \bigcup_{k=r+1}^{\infty} \{N(t) = k\}$



$\{N(t) = r\}$

$P(N(3) \ge 6) - P(N(2) \ge 6) = (1 - P(N(3) < 5)) - (1 - P(N(2) \le 5)) =$

$P(N(2) \le 5) - P(N(3) \le 5)$

$G_x(\theta) = e^{\lambda(\theta-1)}$, $\lambda > 0$ **\Ex/**

$G_x(\theta) = e^{\lambda(\theta-1)}$ what is $P(S_1=2, S_2=2)$?

$P(S_1=2, S_2=2) = P(S_2=2 \mid S_1=2)P(S_1=2)$

$P(S_1=2) = P(X=2) = \frac{\lambda^2 e^{-\lambda}}{2!}$

$P(S_2=2 \mid S_1=2)$ has three ways of happening $\therefore$

$P(X_1=1, X_2=1) \overset{\text{indep}}{=} P(X_1=1)P(X_2=1) = (\lambda e^{-\lambda})^2$

$P(X_1=2, X_2=0) = P(X_1=2)P(X_2=0) = \frac{\lambda^2 e^{-\lambda}}{2!} \cdot e^{-\lambda}$

$P(X_1=0, X_2=2) = \cancel{\frac{\lambda^2 e^{-\lambda}}{2!}}\, e^{-\lambda} \cdot \frac{\lambda^2 e^{-\lambda}}{2!}$

$P(S_2=2 \mid S_1=2)P(S_1=2) =$

$\frac{\lambda^2 e^{-\lambda}}{2!}\left[(\lambda e^{-\lambda})^2 + \frac{\lambda^2 e^{-2\lambda}}{2!} + \frac{\lambda^2 e^{-2\lambda}}{2!}\right] =$

$\frac{\lambda^2 e^{-\lambda}}{2!}\left[\lambda^2 e^{-2\lambda} + \lambda^2 e^{-2\lambda}\right] = \frac{\lambda^2 e^{-\lambda}}{2!}(2\lambda^2 e^{-2\lambda}) = \lambda^4 e^{-3\lambda}$

\Ex/ $var(S_n) = \sigma_n^2$ $var(S_n)$ in terms of σ_{n-1}^2 & Z mean μ_x & var σ_x^2 of X

\Sol/ let $\sigma_n^2 = var(S_n)$, $\sigma_x^2 = var(X)$, $\mu_x = E[X]$, $\mu_n = E[S_n]$

$G_x''(\theta)\big|_{\theta=1} = E[X(X-1)] = E[X^2] - \underbrace{E[X]}_{\mu_x}$

$var(X) = \sigma_x^2 = E[X^2] - E[X]^2 = E[X^2] - \mu_x^2 = G_x''(\theta)\big|_{\theta=1} + \mu_x - \mu_x^2$

$G_n(\theta) = G_x(G_{n-1}(\theta)) \Rightarrow G_n'(\theta) = G_x'(G_{n-1}(\theta))G_{n-1}'(\theta) \Rightarrow$

$G_n''(\theta) = G_x''(G_{n-1}(\theta))(G_{n-1}'(\theta))^2 + G_x'(G_{n-1}(\theta))G_{n-1}''(\theta) \Rightarrow$

$G_n''(1) = G_x''(1)(G_{n-1}'(1))^2 + G_x'(1)G_{n-1}''(1) =$

$G_x''(1)(\mu_x^{n-1})^2 + \mu_x G_{n-1}''(1) = \mu_x^{2n-2}[\sigma_x^2 - \mu_x + \mu_x^2] + \mu[\sigma_{n-1}^2 - \mu_x^{n-1} + \mu_x^{2n-2}] =$

$\sigma_n^2 - \mu_x^n + \mu_x^{2n} \Rightarrow \sigma_n^2 = \mu^{2n-2}\sigma_x^2 + \mu\sigma_{n-1}^2$

\Menti 1/ $\lambda = 3/\text{year}$ $P((\mathbb{I}(t=3)=0) \cap (\mathbb{I}(t=4)=0) \cap (\mathbb{I}(t=5)=0)) =$

0.025 Memoryless $3 \cdot 2 = 6 = \lambda t$ $\therefore$ same as any 2 year period $P((t=2)=0) = \frac{(6)^0(e^{-6})}{\cancel{12}\ 0!} = e^{-6}$

$W = V_1 + V_2 + \dots + V_Z \qquad G_W(\theta) = G_Z(G_V(\theta))$

$E[W] = G_W'(\theta)\big|_{\theta=1} = (G_Z(G_V(\theta)))'_{\theta=1} = G_Z'(G_V(\theta))\,G_V'(\theta)\big|_{\theta=1} = G_Z'(1)G_V'(1) =$

$E[Z]E[V] = 10.5 \times 3.5 \times 0.5 = 18.375$ try to go to 3 significant figures

\problem sheet 1/ \Q4 recap/ $N \sim Poiss(6)$ $Y = X_1 + X_2 + \dots X_N$

$P(X=0)=0.2 \quad P(X=1)=0.5, \quad P(X=2)=0.2$

$G_X(\theta) = 0.2 + 0.5\theta + 0.3\theta^2$

$G_N(\theta) = e^{6(\theta-1)} = \exp\{6(\theta-1)\}$

$G_Y(\theta) = G_N(G_X(\theta)) = \exp\{6(0.2+0.5\theta+0.3(\theta^2)-1)\} =$

$\exp\{1.2+3\theta+1.8\theta^2-6\} = \exp\{-4.8+3\theta+1.8\theta^2\} = \exp\{-4.8\}\exp\{3\theta+1.8\theta^2\} =$ $\left\{e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!}\right.$

$\exp\{-4.8\}\exp\{\theta(3+1.8\theta)\}$

$= e^{-4.8}\left(1+\theta(3+1.8\theta)+\theta^2\frac{(3+1.8\theta)^2}{2!}+\dots\right)$

$= e^{-4.8}\left(1.8\theta^2 + \frac{9}{2}\theta^2 + \text{other terms not of } O(\theta^2)\right)$

$\Rightarrow P(Y=2) = e^{-4.8}\left(1.8+\frac{9}{2}\right)$

$P(X=0)=0.3, \quad P(X=1)=0.5 \quad P(X=2)=0.2$?

$E[X] = 0\times0.3 + 1\times0.5 + 2\times0.2 = 0.9 < 1$ $\therefore$ population dies out

\Menti 1/ $G_X(\theta) = 0.4 + 0.3\theta + 0.3\theta^2$ $\therefore$

$G_X^{10}(\theta) \qquad E[X] = 0.4\cdot0 + 0.3\cdot1 + 0.3\cdot2 = 0.3+0.6 = 0.9$ $\therefore$

$100 \times 0.9^{10} = 34.87 \approx 35$

$P(X=0)=0.4$, ~~P=k~~ $P(X=1)=P(X=2)=0.3$

$X \sim$ no. of spring per individual

$G_X(\theta) = 0.4 + 0.3\theta + 0.3\theta^2$ $\therefore$

$E[X] = G_X'(\theta)\big|_{\theta=1} = (0.3+0.6)_{\theta=1} = 0.9$

$E[S_n] = E[X]^n \Rightarrow E[S_{10}] = 0.9^{10} \approx 0.349$ for $S_0 = 1$

$\therefore$ if $\tilde{S}_0 = 100$: $Y \sim$ no individuals at gen 0

$E[\tilde{S}_{10}] = E[Y]E[S_{10}] = 100 \times 0.349 \sim 35$

$E[X] < 1 \Rightarrow e = 1 \qquad E[X] > 1 \quad e = G_X(e)$

$$\sum_{n=1}^{\infty} \frac{x^n}{n!} \quad \therefore f(x) = f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \frac{f'''(0)}{3!}x^3 + \dots + \frac{f^{(n)}(0)x^n}{n!} + \dots$$

$$= \sum_{n=0}^{\infty} f^{(n)}(0)\, x^n \frac{1}{n!}$$

$x = a$, a $\quad (x-a)^n$

$$e^x = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \cancel{\frac{x^4}{4}} \dots + \frac{x^n}{n!} + \dots$$

X is dice number $\therefore x = 1, 2, 3, 4, 5, 6$

Y is coin heads $\therefore Y = 0, 1, 3, 4, 5, 6$

$E[Y \mid X = 1] = E(Y = 0) =$

$E\left[\frac{1+2+3+4+5+6}{6}\right] = \frac{21}{6} \qquad E\left[\frac{0+1}{2}\right] = \frac{1}{2} \quad \therefore \frac{21}{6} \times \frac{1}{2}$ $\because$ not grouping

1.75

Probability generating func $G_X(\theta)$

for $G_X(\theta) = 0.4 + 0.1\theta + 0.7\theta^2$ does coeffs total to 1? $0.4 \cancel{+} 0.1 + 0.7 = \cancel{1.2} \neq 1$

$G_X(\theta)$ is $\theta = 1$ subbed in does it equal 1?

eg $e^{\theta^4 - 1} \Rightarrow e^{1^4 - 1} = e^{1-1} = e^0 = 1$

$\frac{1}{3-\theta} \Rightarrow \frac{1}{3-1} = \frac{1}{2} \neq 1 \qquad \frac{1}{(2-\theta)^3} \Rightarrow \frac{1}{(2-1)^3} = \frac{1}{1^3} = 1$

$0.4 \cancel{+} 0.1\theta + 0.7\theta^2 \Rightarrow 0.4 \cancel{+} 0.1(1) + 0.7(1)^2 = 1$ but can't have negitive probab eg -0.1 $\therefore$ not a valid probab Generating func

$\sum_{i=1}^{20} i = \frac{1}{2}(20)(20+1) = 210 \quad \therefore \frac{210}{20} = 10.5$

$(1+2+3+4+5+6)/6 = 3.5 \qquad 0.5 \quad \therefore 10.5 \times 3.5 \times 0.5 = 18.375$

$X \sim$ Coin toss outcome $\quad Y \sim$ D6 outcome $\quad Z \sim$ D20 outcome

state finite state space: $\Omega_X = \{0, 1\}$ 1 is head, 0 = tail

$\Omega_Y = \{1, 2, 3, 4, 5, 6\}$

$\Omega_Z = \{1, 2, \dots, 20\}$

$V = X_1 + X_2 + \dots + X_Y \quad \therefore G_V(\theta) = G_Y(G_X(\theta))$

$E[V] = G_V'(\theta)\big|_{\theta=1} = G_V'(1) = (G_Y(G_X(\theta)))'_{\theta=1} = G_Y'(\underbrace{G_X(\theta)}_{1 \text{ for } \theta=1})G_X'(\theta)\big|_{\theta=1}$

$= G_Y'(1)G_X'(1) = E[Y]E[X] = 3.5 \times 0.5$

Moment Generating Funcs/ Let X be an r.v. its k^{th} moment is $E[X^k]$, $k \geq 1$

Def/ the MGF of X, $M_X(t) = E[e^{tX}]$, $t \in \mathbb{R}$

discrete: $E[e^{tX}] = \sum_i e^{tx_i} P(X=x_i)$

Continuous: $E[e^{tX}] = \int_{\Omega_X} f_X(x) e^{tx} dx$

properties 1) $M_X(0) = E[X^0] = E[1] = 1$

2) $E[X^k] = M_X^{(k)}(0)$, $M_X^{(k)} = \frac{d^k M_X}{dt^k}$

3) if X, Y are r.v.s & $M_X(t)$, $M_Y(t)$ are continuous (cts) in a neighbood around $t=0$, if $M_X(t) = M_Y(t)$ then X, Y are equal in distri

4) if X, Y are indep, & $Z = X+Y$ then $M_Z(t) = M_X(t) M_Y(t)$

Ex/ $X \sim$ Binomial(p, n) $M_X(t) = E[e^{tX}] = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} e^{tk} = \sum_{k=0}^{n} \binom{n}{k} (e^t p)^k (1-p)^{n-k}$

{Binomial thm: $\sum_{k=0}^{n} \binom{n}{k} x^k y^{n-k} = (x+y)^n$ $\therefore x = e^t p$, $y = 1-p$ }

$= M_X(t) = (1-p+pe^t)^n$

Ex/ $X \sim$ Poisson(λ) $M_X(t) = E[e^{tX}] = \sum_{k=0}^{\infty} e^{tk} P(X=k) = \sum_{k=0}^{\infty} e^{tk} \frac{\lambda^k e^{-\lambda}}{k!} =$

$e^{-\lambda} \sum_{k=0}^{\infty} (e^t \lambda)^k / k!$ $\quad \left\{ e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} = \sum_{k=0}^{\infty} \frac{x^k}{k!} \quad x = \lambda e^t \therefore \right\}$

$\Rightarrow M_X(t) = e^{-\lambda} e^{\lambda e^t} = \cancel{e^{\lambda(e^t - 1)}}\ e^{\lambda(e^t-1)}$

Application: suppose $X_n \sim$ Bernoulli(p), $P(X_n=1) = p$ $P(X_n=0) = 1-p$

what is the distri of $X = X_1 + X_2 + \dots + X_n$ where $p = \frac{\lambda}{n}$ in the limit as $n \to \infty$? $M_{X_n}(t) = 1 - p + pe^t$

assume X_n i.i.d.

Property 4 $\Rightarrow M_X(t) = M_{X_1}(t) * M_{X_2}(t) \cdot \dots \cdot M_{X_n}(t) = (1-p+pe^t)^n \equiv$ MGF Binomial

{Set $p = \lambda/n$} $= (1 - \frac{\lambda}{n} + \frac{\lambda}{n} e^t)^n = (1 - \frac{\lambda(1-e^t)}{n})^n$

$\left\{ \lim_{n\to\infty} (1 + \frac{x}{n})^n = e^x, x \in \mathbb{R} \right\}$ $= (1 + \frac{\lambda(e^t-1)}{n})^n \to e^{\lambda(e^t-1)}$ as $n \to \infty$

property 3 $\Rightarrow X \sim$ Poisson(λ), $\lambda > 0$ as $n \to \infty$

what is $E_Y[E_X[X|Y=y]]$?

\proposition: (thm)/ \Law of Total Expectation (the tower law)/

$E_Y[E_X[X|Y=y]] = E_X[X]$

\proof:/ $E_Y[E_X[X|Y|y]] = \sum_j E_X[X|Y=y_j]\, f_Y(y_j) =$

$\sum_j \left(\sum_i x_i \frac{f_{XY}(x_i, y_j)}{f_Y(y_j)} \right) f_Y(y_j) = \sum_j \left(\sum_i x_i f_{XY}(x_i, y_j) \right) =$

$\sum_i x_i \sum_j f_{XY}(x_i, y_j) = \sum_i x_i f_X(x_i) = E_X[X]$ $\square$

\Week 2/

\Limit theorems/ a statement $S(\omega)$ holds <u>almost</u> <u>surely</u> w.r.t. P if $P(\{\omega \in \Omega \mid S(\omega) \text{ true}\}) = 1$

convergence of r.v.s suppose we have a sequence of r.v.s $\{X_n\}_{n \in \mathbb{N}}$ & an r.v. X

1) $X_n \to X$ almost surely (a.s.) as $n \to \infty$ if ~~$P(\{\omega \in \Omega \mid \lim_{n\to\infty} X_n(\omega)$~~

$P(\{\omega \in \Omega \mid \lim_{n\to\infty} X_n(\omega) = X\}) = P(X_n \to X) = 1$

2) $X_n \to X$ in probab as $n \to \infty$ if $\lim_{n\to\infty} P(\{\omega \in \Omega \mid |X_n(\omega) - X(\omega)| < \varepsilon\}) = 1$, $\forall \varepsilon > 0$ if $X_n \to X$ a.s., then $X_n \to X$ in probab.

<u>Law of Large numbers</u> $\{X_n\}_{n \in \mathbb{N}}$, i.i.d. $E[X] = \mu$, $Var(X) = \sigma^2$

- strong Law (of large numbers) $S_N = \frac{1}{N} \sum_{n=1}^{N} X_n \to \mu$ a.s. $N \to \infty$
- weak Law (of large numbers) $S_N = \frac{1}{N} \sum_{n=1}^{N} X_n \to \mu$ in probab as $N \to \infty$

\Ex: coin tossing/ $X_n \equiv$ result of the n^{th} toss

$P(X_n = 1) = P(X_n = 0) = \frac{1}{2}$

$S_N \to \frac{1}{2} = E[X_i]$ as $N \to \infty$

eg $\omega = HHHH\cdots$ $S_N = \frac{1}{N} \sum_{n=1}^{N} X_n(\omega) = 1 \neq \frac{1}{2}$

what is the probab of this sequence occuring (ω)?

$P(\{X_n = 1 \text{ for } n = 1, \dots, N\}) = (\frac{1}{2})^N \to 0$ as $N \to \infty$

\Central limit thm:/ $\{X_n\}_{n \in \mathbb{N}}$ i.i.d. $E[X] = \mu$, $Var(X) = \sigma^2 < \infty$

let $S_n = \frac{1}{N} \sum_{n=1}^{N} X_n$, $Z_N = \frac{\sqrt{N}}{\sigma}(S_N - \mu) \sim N(0,1)$

toss a coin 10^6 $\therefore$ CLT $\Rightarrow Z_n = (S_n - \frac{1}{2}) \cdot \frac{1000}{1/2} = 2000(S_n - \frac{1}{2}) =$

$2000 S_n - 1000 \sim N(0,1)$

$Cov(X,Y) = E[XY] - E[X]E[Y]$

Independence two r.v.s X, Y are indep if $P(X=x, Y=y) = P(X=x)P(Y=y)$

$p_{ij}^{XY} = p_i^X p_j^Y$ $\quad p_{ij}^{XY} = p(X=x_i, Y=y_j)$, $p_i^X = P(X=x_i)$, $p_j^Y = P(Y=y_j)$

$E[XY] = \sum_{i,j} p_{ij}^{XY} x_i y_j = \sum_i \sum_j (p_i^X x_i)(p_j^Y y_j) = \sum_i (p_i^X x_i) \sum_j (p_j^Y y_j) = E[X]E[Y] \Rightarrow$

$Var(X+Y) = Var(X) + Var(Y)$ if $Cov(X,Y) = 0$

Say that Z r.v.s X_i, $i = 1, \dots, N$ are i.i.d if they are indep & identitcally distried

$E[X_1 + X_2 + \dots + X_N] = N\mu$, $\mu = E[X_i]$

$Var(X_1 + X_2 + \dots + X_N) = N\sigma^2$, $\sigma^2 = Var(X_i)$

Conditional probabilities / suppose probab space (Ω, Σ, P) & are intrested in two events $A, B \in \Sigma$

assume that $P(B) \neq 0$. Z conditional probab of A given B is

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ Say that $\{B_i\}_{i=1}^N$ is a partition of Ω if $B_i \cap B_j \neq \emptyset$ $\forall i \neq j$ & if B_i cover Ω ie $\bigcup_{i=1}^N B_i = \Omega$

if $\{B_i\}_{i=1}^N$ form a partition of Ω, have $P(A) = \sum_i P(A|B_i)P(B_i)$ is the Law of Total probab

Bayes thm / $P(B_i|A) = \frac{P(A|B_i)P(B_i)}{\sum_j P(A|B_j)P(B_j)}$

Suppose that X, Y are two r.v.s $X \in \Omega_X = \{x_i \mid i \in \mathbb{N}\}$
$Y \in \Omega_Y = \{y_j \mid j \in \mathbb{N}\}$

Joint probab distri of X & Y is given by $f_{XY}(x_i, y_j) = P(X=x_i, Y=y_j)$

Normalisation $\Rightarrow \sum_{ij} f_{XY}(x_i, y_j) = 1$

Z marginal distri of X given $f_{XY}(x_i, y_j)$ is given by

$f_X(x_i) = P(X=x_i) = \sum_{j=1}^N f_{XY}(x_i, y_j)$

Z conditional expectation of X given Y is $E[X|Y=y_j] = \sum_i x_i P(X_i|Y=y_j) = \sum_i x_i \frac{f_{XY}(x_i, y_j)}{f_Y(y_j)}$

\Splitting poisson process/ let $Z(t)$ be a combined poiss process with rate λ, st $Z(t) = A(t) + B(t)$ (indep) suppose Z probab a given event comes from $A(t)$ is $p \in (0,1)$ & Z probab it comes from $B(t)$ is $1-p$ whats Z distri of $B(t)$?

$P(B(t)=0) = P(\text{no events at all over interval } t) + P(k \text{ events, but all from } A(t)) =$

$P(Z(t)=0) + \sum_{k\geq 1} P(Z(t)=k \cap \text{all } k \text{ events from } A(t)) =$

$e^{-\lambda t} + \sum_{k \geq 1} \frac{(\lambda t)^k e^{-\lambda t}}{k!} p^k = e^{-\lambda t}\left(1 + \sum_{k\geq 1} \frac{(p\lambda t)^k}{k!}\right) = e^{-\lambda t} e^{p\lambda t} = e^{-\lambda t(1-p)}$ ∴

Z time to Z first event from $B(t)$ is distrib according to a exponential distri with rate $\lambda(1-p)$ ∴ $B(t)$ is poiss process with rate $\lambda(1-p)$

a similar argument shows $A(t)$ is a poisson process with rate $p\lambda$

\Ex/ Suppose three types type_1 type_2 type_3 with a combined replacement rate of 10/year probab 10% 40% 50% Q: whats expected time betw replacements of type_1? type_1 is a poiss proc with rate $0.1 \times 10 = 1$/year

$E[T^{\text{type}_1}] = \frac{1}{1} = 1$ year

Q: whats probab need to replace 2 $\Uparrow$ in one year? $P(\Uparrow(t=1)=2) =$

$\frac{(\lambda t)^2 e^{-\lambda t}}{2!} = \frac{5^2 e^{-5}}{2!}$

\Conditioning on Arrivals/ let $Z(t)$ be a poiss process with rate λ Suppose know n events occured in $[0,T]$ how many events occured in $[0,t]$, $t \leq T$?

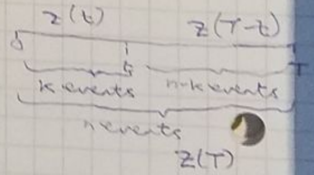
\proposition/ let X be Z number of events in $[0,t]$ then $X \sim \text{Binomial}(n, t/T)$

\proof:/ $P(Z(t)=k \mid Z(T)=n) = \frac{P(\{Z(t)=k\} \cap \{Z(T)=n\})}{P(Z(t)=n)}$

$= P(\{Z(t)=k\} \cap \{Z(T-t)=n-k\}) / P(Z(t)=n)$

$= P(Z(t)=k) P(Z(T-t)=n-k) / P(Z(T)=n) =$

$\frac{((\lambda t)^k e^{-\lambda t}/k!) \times (\lambda(T-t))^{n-k} e^{-\lambda(T-t)}/(n-k)!}{[(\lambda T)^n e^{-\lambda T}/n!]} =$



$P(\text{Time to } k^{th} \text{ event} \le t) - P(\text{Time to } (k+1)^{th} \text{ event} \le t) =$

$P(S_k \le t) - P(S_{k+1} \le t) = \int_0^t g_k(x)\,dx - \int_0^t g_{k+1}(x)\,dx$

$\int_0^t g_{k+1}(x)\,dx = \frac{\lambda^{k+1}}{\Gamma(k+1)}\int_0^t \underset{u}{x^k} \overset{dv}{e^{-\lambda x}} = \frac{\lambda^{k+1}}{\Gamma(k+1)}\left[-\frac{x^k e^{-\lambda x}}{\lambda}\right]_0^t - \frac{\lambda^{k+1}}{\Gamma(k+1)}\int_0^t -\frac{1}{\lambda}kx^{k-1}e^{-\lambda x} =$

$-\frac{\lambda^k}{\Gamma(k+1)}(t^k e^{-\lambda t}) + \frac{\lambda^k k}{k\Gamma(k)}\int_0^t x^{k-1}e^{-\lambda x}\,dx = \frac{-\lambda^k}{\Gamma(k+1)}(t^k e^{-\lambda t}) + \underbrace{\frac{\lambda^k}{\Gamma(k)}\int_0^t x^{k-1}e^{-\lambda x}\,dx}_{\int_0^t g_k(x)dx}$ $\therefore$

$P(S_k \le t) - P(S_{k+1} \le t) = \frac{(\lambda t)^k (e^{-\lambda t})}{\Gamma(k+1)} = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$ $\square$

Ex/ suppose replacements of a light bulb $\aleph(t)$ follow a poisson process with rate 1/year Q: what is the probab that no replacements occure in 1 year?

$P(\aleph(t=1)=0) = e^{-1}$

Q: what is probab that one replacement occures in 2 years?

$P(\aleph(t=2)=1) = \frac{e^{-2}2^1}{1!} = 2e^{-2}$

Q: what is probab that atleast 3 years pass before third replacement?

$P(S_3 \ge 3) = P(\aleph(t=3) < 3) = e^{-3}\left(3^0 + \frac{3^1}{1!} + \frac{3^2}{2!}\right) = e^{-3}(\overset{k=0}{1} + \overset{k=1}{3} + \overset{k=2}{\tfrac{9}{2}})$

Combining poisson processes/ let $A(t)$, $B(t)$ be two indep poisson processes with rates λ_A, λ_B respectively

let $Z(t) = A(t) + B(t)$, what is distri of $Z(t)$?

thm:/ $Z(t)$ is a poisson process with rate $\lambda = \lambda_A + \lambda_B$

proof:/ let $G_{A,B,Z}(\theta)$ be the PGF of A, B, Z

have $G_A(\theta) = \exp(\lambda_A(\theta-1))$, $G_B(\theta) = \exp(\lambda_B(\theta-1))$

by indep $G_Z(\theta) = G_A(\theta)G_B(\theta) = e^{\lambda_A(\theta-1)}e^{\lambda_B(\theta-1)} = e^{(\lambda_A+\lambda_B)(\theta-1)}$ $\square$

Ex/ Suppose there are two types of lightbulbs: $\aleph$ Υ replacement rate 1/year, 4/year Q: expected time to next light bulb replacement?

$Z(t) = \aleph(t) + \Upsilon(t)$ $E[T^Z] = \frac{1}{\lambda_A + \lambda_B} = \frac{1}{5}$ years

Q: whats probab no bulb needs replacing in 1 year? $P(Z(1)=0) =$

$G_Z(0) = e^{(\lambda_A+\lambda_B)(\theta-1)}\big|_{\theta=0} = e^{-5}$

\Ex/ let $G_X(\theta) = 0.3 + 0.5\theta + 0.2\theta^2$ suppose $S_0 = 6$ Z PGF for Y is $G_Y(\theta) = \theta^6 \Rightarrow G_1(\theta) = (0.3 + 0.5\theta + 0.2\theta^2)^6$ ∴

Q: $P(S_1 = k)$ (use PGF II ideas)

$E[S_n] = E[Y]\mu_X^n = 6 \cdot (0.9)^n$ ∴ $\lim_{n\to\infty} E[S_n] = \lim_{n\to\infty}(6(0.9)^n) = 0$

have $\tilde{e}_0 = e_0 = 0$, $\tilde{e}_1 = 0.3 \Rightarrow e_1 = (0.3)^6$

$\tilde{e}_2 = 0.468 \Rightarrow e_2 = (0.468)^6$ ...

∵ $\tilde{e} = 1$, $e = 1$

\Ex/ suppose $Y \sim Binomial(3, 0.4)$ have $G_Y(\theta) = (0.6 + 0.4\theta)^3$

then $G_n(\theta) = G_Y(G_X^n(\theta))$ & $E[Y] = np = 0.4 \times 3 = 1.2 \Rightarrow$

$E[S_n] = 1.2 \times (0.9)^n$

in general $e_n = (0.6 + 0.4\tilde{e}_n)^3$

\Poisson Processes/ rv $T \sim Exp(\lambda)$ follows an exponential distri with rate λ if: $f_T = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0 \\ 0 & , x < 0 \end{cases}$, $\lambda > 0$

Compute MGF $M_T(t) = E[e^{tx}] = \int_0^\infty e^{tx}\lambda e^{-\lambda x}\,dx = \lambda\int_0^\infty e^{-(\lambda - t)x}\,dx =$

$\lambda\left[\frac{e^{-(\lambda - x)x}}{t - \lambda}\right]_0^\infty = \frac{\lambda}{\lambda - t}$ ∴

$E[T] = M_T'(0) = \left(\frac{-\lambda}{(\lambda - t)^2}(-1)\right)\Big|_{t=0} = \frac{\lambda}{\lambda^2} = \frac{1}{\lambda}$

Z exponential distri is memoryless: $P(T > s+t \mid T > s) = P(T > t)$

to show this, note $P(T > s+t \mid T > s) = P(T > s+t \cap T > s)/P(T > s) = \frac{P(T > s+t)}{P(T > s)}$

$\left\{ \text{aside: } P(T > s) = \int_s^\infty \lambda e^{-\lambda x}\,dx = -\left[e^{-\lambda x}\right]_s^\infty = e^{-\lambda s} \right\}$

$P(T > s+t \mid T > s) = P(T > s+t)/P(T > s) =$

$e^{-\lambda(s+t)}/e^{-\lambda s} = e^{-\lambda t} = P(T > t)$ □

\Def/ we say events occur as a Poisson Process, $N(t)$, if the time between successive events (inter-event times) are indep & all follow an exponential distri with rate $\lambda > 0$ (inter-event times are i.i.d)

\K'th time distri/ let T_k be Z time between Z $(k-1)^{th}$ & k^{th} event. assume that Z zero'th event occured at $t = 0$

Poisson process $\Rightarrow T_n \sim Exp(\lambda)$ whats probab distri of Z n'th event time? let $S_n = T_1 + T_2 + \dots + T_n$

what is Z distri of S_n?

recall of Z Gamma distri $g_n(x) = \frac{\lambda(\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!} = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma(n)}$

aside: $\Gamma(n) = (n-1)! = \int_0^\infty e^{n-1} e^{-x} dx$ $\therefore g_n(x) = \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma(n)}$

assume that an r.v $Y \sim Gamma(n, \lambda)$, then its MGF is:

$M_Y(t) = E[e^{tY}] = \int_0^\infty e^{tx} g_n(x) dx = \frac{\lambda^n}{\Gamma(n)} \int_0^\infty x^{n-1} e^{-\lambda x} e^{tx} dx = \frac{\lambda^n}{\Gamma(n)} \int_0^\infty x^{n-1} e^{-(\lambda - t)x} dx$

how to evaluate Z integral? have that $\int_0^\infty g_n(x) dx = 1 =$

$\int_0^\infty \frac{\lambda^n x^{n-1} e^{-\lambda x}}{\Gamma(n)} dx = \frac{\lambda^n}{\Gamma(n)} \int_0^\infty x^{n-1} e^{-\lambda x} dx \Rightarrow \int_0^\infty x^{n-1} e^{-\lambda x} dx = \frac{\Gamma(n)}{\lambda^n}$ $\therefore \tilde{\lambda} = \lambda - t$

$\int_0^\infty x^{n-1} e^{-\tilde{\lambda} x} dx = \frac{\Gamma(n)}{\tilde{\lambda}^n}$ $\therefore$ have $M_Y(t) = \frac{\lambda^n}{\Gamma(n)} \int_0^\infty x^{n-1} e^{-\tilde{\lambda} x} dx = \frac{\lambda^n}{\Gamma(n)} \frac{\Gamma(n)}{\tilde{\lambda}^n}$

$= \frac{\lambda^n}{(\lambda - t)^n}$

\thm:/ if $S_n = T_1 + T_2 + \dots + T_n$, then $S_n \sim Gamma(n, \lambda)$,

$T_k \sim Exp(\lambda)$

\proof:/ what is Z MGF of S_n? have $M_{S_n}(t) = E[e^{tS_n}] =$

$E[e^{t(T_1 + T_2 + \dots + T_n)}] = E[e^{tT_1 + tT_2 + \dots + tT_n}] = E[e^{tT_1} e^{tT_2} \dots e^{tT_n}] =$

$E[e^{tT_1}] E[e^{tT_2}] \dots E[e^{tT_n}] = (E[e^{tT}])^n = (M_T(t))^n = \left(\frac{\lambda}{\lambda - t}\right)^n = \frac{\lambda^n}{(\lambda - t)^n}$

uniqueness of MGFs $\Rightarrow S_n \sim Gamma(n, \lambda)$ $\square$

\counting processes/ suppose $N(t)$ is a poisson process with rate λ

what is Z distri of $N(t)$? what is Z probab that we observe k events over a time interval t?

\thm:/ $P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!}$, $k = 0, 1, 2, \dots$

$N(t) \sim Poiss(\lambda t)$, $E[N(t)] = \lambda t$

\proof:/ P(at least k events over $[0,t]$) $= P$(Time to k^{th} event $\leq t$)

$T_k \equiv$ time interval between $(k-1)^{th}$ & k^{th} event

$S_n \equiv T_1 + T_2 + T_3 + \dots + T_n$

have $\{N(t) \geq k+1\} \subset \{N(t) \geq k\}$ & $\therefore P(N(t) = k) =$

P(at least k events over $[0,t]$) $- P$(at least k+1 events over $[0,t]$) $=$

\Ex/ Suppose $G_X(\theta) = 0.3 + 0.5\theta + 0.2\theta^2$

have $E[X] = G_X'(1) = 0.5 + 0.4 = 0.9 < 1 \Rightarrow$ ① system $\Rightarrow e = 1$

$E[S_n] = \mu^n = (0.9)^n \Rightarrow \lim_{n\to\infty} E[S_n] = \lim_{n\to\infty} 0.9^n = 0$

\Ex/ Suppose $G_X(\theta) = 0.2 + 0.4\theta + 0.3\theta^2 + 0.1\theta^3$

$E[X] = G_X'(1) = 0.4 + 0.6 + 0.3 = 1.3 > 0 \quad \therefore$ ② $\quad \therefore$

$\lim_{n\to\infty} E[S_n] = \lim_{n\to\infty} 1.3^n = \infty \quad \therefore$

solve $G_X(e) - e = 0 = 0.2 - 0.6e + 0.3e^2 + 0.1e^3 \quad \therefore$ to make things easier:

$10(G_X(e) - e) = 0 = 2 - 6e + 3e^2 + e^3 =$

$(e-1)(e^2 - 4e - 2) \qquad \therefore e = 1, \quad e^2 - 4e - 2 = 0 \therefore$

$e_\pm = -2 \pm \sqrt{6} \quad \Rightarrow e - 2 + \sqrt{6} \approx 0.45$

\initial population/ change of initial pop size. let $S_n \equiv$ pop size at gen n. each individual evolves according to iid r.v. X

before. $S_0 = 1$ $\qquad$ now: $S_0 = Y$ where Y has PGF $G_Y(\theta)$

$G_0(\theta) = \theta \qquad G_0(\theta) = G_Y(\theta) \qquad S_1 = X_1 + X_2 + \dots + X_Y \Rightarrow$

$G_1(\theta) = G_Y(G_X(\theta))$

\proposition/ if $S_0 = Y$ where Y has PGF $G_Y(\theta)$, then

$G_n(\theta) = G_Y(\underbrace{G_X^n(\theta)}_{\text{PGF for } S_n \text{ when } S_0=1})$

$S_2 = X_1 + X_2 + \dots + X_{S_1} \Longrightarrow G_{S_2}(\theta) = G_1(G_X(\theta)) = G_Y(G_X^2(\theta)) \quad \therefore$ by induction

① let $\tilde{\mu}_n \equiv$ expected pop size at gen n when $S_0 = 1$

$\tilde{\mu}_n = (G_X^n(\theta))'|_{\theta=1}$ $\quad$ & $\mu_n \equiv$ expected pop size at gen n when $S_0 = Y$

then $\mu_n = E[Y]\tilde{\mu}_n = E[Y]\mu_X^n$, where $\mu_X = E[X]$

{ $\mu_n = E[S_n] = G_n'(\theta)|_{\theta=1} = (G_Y(G_n(\theta)))'|_{\theta=1} = G_Y'(G_n(\theta))G_n'(\theta)|_{\theta=1} =$

$G_Y'(1)G_n'(1) = E[Y]E[S_n | S_0 = 1] = E[Y]\tilde{\mu}_n$

② let $\tilde{e}_n, e_n \equiv$ probab of extinction at gen n $\quad (S_0 = 1, S_0 = Y)$

then $e_n = P(S_n = 0) = G_Y(\tilde{e}_n)$ note $\tilde{e}_n = G_X^n$

③ let $\tilde{e} = \lim_{n\to\infty} \tilde{e}_n$, $e = \lim_{n\to\infty} e_n$ then $e = \lim_{n\to\infty} P(S_n = 0) = \lim_{n\to\infty} G_n(0)$

$= \lim_{n\to\infty} G_Y(\tilde{e}_n) = G_Y(\lim_{n\to\infty} \tilde{e}_n) = G_Y(\tilde{e})$

\Corollary/ if $\tilde{e} = 1$, $\therefore e = 1$

variance: $var(S_n) = E[S_n^2] - E[S_n]^2 = E[S_n^2] - \mu_x^{2n}$

$G_{S_n}''(1) = E[S_n(S_n - 1)] = E[S_n^2] - E[S_n] = E[S_n^2] - \mu_x^n \Rightarrow$

$var(S_n) = G_{S_n}''(1) + \mu_x^n - \mu_x^{2n} = (G_x^n(\theta))''|_{\theta=1} + \mu_x^n - \mu_x^{2n}$

Extinction probabs / $S_n \equiv$ num of individuals at gen n

individuals "evolve" obeying iid r.v. X

probab that pop is extinct at gen n is $P(S_n = 0) = e_n$

ultimate probab of extinction is $e = \lim_{n\to\infty} e_n$

$\because S_0 = 1$, $e_0 = P(S_0 = 0) = 0$

$e_1 = P(S_1 = 0)$ in general, $P(S_n = 0) = G_{S_n}(0) \Rightarrow e_1 = G_{S_1}(0) = G_x(0)$

$e_n = P(S_n = 0) = G_{S_n}(0) = (G_x \circ G_x \circ \dots \circ G_x)(0) =$ $\{G_{S_n}(\theta) = G_x^n(\theta)\}$

$G_x(G_{S_{n-1}}(0)) = G_x(P(S_{n-1} = 0)) = G_x(e_{n-1})$

$= G_x(e_{n-1})$

Ex/ suppose $G_x(\theta) = 0.3 + 0.5\theta + 0.2\theta^2$ whats probab that pop goes extinct at gen 2? $\because e_0 = 0$ $e_1 = G_x(0) = 0.3$ $\therefore$

$e_2 = G_x(e_1) = G_x(0.3) = 0.3 + 0.5 \cdot 0.3 + 0.2 \cdot 0.3^2 = 0.468$

Ultimate extinction probabs / S_n is number of individuals at Z n'th gen. evolve with iid r.v. X, $S_0 = 1$

what is Z probab that Z pop ultimately goes extinct?

$e = \lim_{n\to\infty} e_n$ ① $P(X=0) = 0 \Rightarrow e_n = 0 \Rightarrow e = 0$

② $P(X=0) \neq 0$ have $G_x(0) \neq 0$. Over $\theta \in [0,1]$, G_x, G_x' are non-decreasing funcs of θ $e_{n+1} = G_x(e_n) > e_n$

③ $G_x(1) = 1$, $G_x'(1) = E[X] = \mu$

④ Z num e is a "fixed pt" of $G_x(e)$, ie $e = G_x(e)$

if $e = \lim_{n\to\infty} e_n$, then $e = \lim_{n\to\infty} e_{n+1}$ then $e = \lim_{n\to\infty} e_{n+1} = \lim_{n\to\infty} G_x(e_n) =$

$G_x(\lim_{n\to\infty} e_n) = G_x(e)$

⑤ for case ① we have $e=1$. also have $\mu \leq 1$

⑥ $\because G(1) = 1$, $e=1$ is always a root of $G_x(e) - e$

we have the PGF for $Z = X_1 + X_2 + \cdots + X_N$ is

$G_Z(\theta) = G_N(G_X(\theta))$ $\therefore$

$E[Z] = G_Z'(1) = (G_N(G_X(\theta)))'|_{\theta=1} = G_N'(G_X(\theta))G_X'(\theta)|_{\theta=1} =$

$G_N'(G_X(1))G_X'(1) = G_N'(1)G_X'(1) =$

$E_N[N]E_X[X] = 3.5 \times 0.5 = \left\{ \frac{1+2+3+4+5+6}{6} * \frac{0+1}{2} = 3.5 \times 0.5 \right\} = 1.75$

\Week 3/ \Branching Process/

Let S_n be the number of individuals in a population at generation n assume that each individual evolves according to a r.v. X and that individuals evolve indep of one another

Suppose that $G_X(\theta)$ is the PGF for X and that $S_0 = 1$

\then:/ the PGF for S_n, $G_n(\theta) \equiv G_{S_n}(\theta)$ is given by

$G_n(\theta) = \underbrace{G_X \circ G_X \circ \cdots \circ G_X(\theta)}_{n \text{ times}} = G_X^n(\theta)$

\proof:/ have $P(S_0 = 1) = 1 \Rightarrow G_0(\theta) = \theta$ & that

$G_1(\theta) = G_X(\theta)$ at gen 2, $S_n = X_1 + X_2 + \cdots + X_{S_1}$

{& is $Z = X_1 + \cdots + X_N$ then $G_Z(\theta) = G_N(G_X(\theta))$}

in our case $G_N = G_X \Longrightarrow G_Z(\theta) = G_X(G_X(\theta))$

continue using induction. Assume that this is true up to gen n

we have $S_{n+1} = X_1 + X_n + \cdots + X_{S_n}$ & $G_n = G_X^n(\theta)$

then $G_{n+1}(\theta) = G_n(G_X(\theta)) = G_X^n(G_X(\theta)) = G_X^{n+1}(\theta)$

$\therefore$ induction implies hypothesis true for $n \geq 1$ $\square$

\population mean/ let S_n denote the # of individuals at gen n & assume iid evolution of individuals according to r.v. X

\proposition/ let $\mu_n \equiv$ pop mean at gen n & let μ_X = mean of X

then $\mu_n = E[S_n] = \mu_X^n$

\proof:/ $E[S_n] = G_n'(1) = (G_X \circ G_X \circ \cdots \circ G_X)'(\theta)|_{\theta=1} =$

$(G_X \circ G_{n-1})'(\theta)|_{\theta=1} = (G_X(G_{n-1}(\theta)))'|_{\theta=1} = G_X'(G_{n-1}(\theta))G_{n-1}'(\theta)|_{\theta=1} =$

$G_X'(G_{n-1}(1))G_{n-1}'(1) = G_X'(1)G_{n-1}'(1) = \mu_X G_{n-1}'(1) = \mu_X(\mu_X G_{n-2}'(1)) =$

$\mu_X^2 G_{n-2}'(1) = \mu_X^2(\mu_X G_{n-3}'(1)) = \mu_X^3 G_{n-3}'(1) = \mu_X^n$ $\square$

\probab Generating funcs/ let X be a r.v. taking vals over Z non-negative integers, its probab Generating func (PGF) is

$G_X(\theta) = E[\theta^X] = \sum_{k=0}^{\infty} \theta^k P(X=k)$, $\theta \in \mathbb{R}$

$\theta \sim e^t \Rightarrow G_X(\theta) = M_X(\log(\theta))$

Properties: 1) $G_X(1) = E[1] = \sum_{k=0}^{\infty} P(X=k) = 1$

2) $G_X^{(k)}(1) = E[X(X-1)(X-2)\cdots(X-r+1)]$ $\quad G_X^{(k)}(\theta) = \frac{d^k G_X}{d\theta^k}$

3) if X, Y are r.v.s & $G_X(\theta)$, $G_Y(\theta)$ are cts & equal in a neighbood of $\theta=1$, then X, Y are equal in distri

4) if X, Y are indep, $Z = X+Y$ $\quad G_Z(\theta) = G_X(\theta) G_Y(\theta)$

5) if $G_X(\theta)$ is given as a power series, then Z coeffs in front of θ^k is equiv to $P(X=k)$ $\quad$ in particular $G_X(0) = P(X=0)$

\Ex/ PGF of an r.v. $X \sim \text{Poisson}(\lambda)$ $\quad M_X(t) = e^{\lambda(e^t - 1)} \Rightarrow$

$G_X(\theta) = e^{\lambda(e^{\log\theta} - 1)} = e^{\lambda(\theta-1)}$

\Ex/
Let X be an r.v. with PGF $G_X(\theta) = 0.5 + 0.2\theta + 0.2\theta^2 + 0.1\theta^3$

i) whats $P(X=1)$? whats $E[X]$?

$P(X=1) = 0.2$ $\quad E[X] = G_X'(1) = 0.2 + 2\cdot(0.2)\theta + 3(0.1)\theta^2 \Big|_{\theta=1} =$

$0.2 + 0.4 + 0.3 = 0.9$

\Ex/ let X be an r.v. with PGF $G_X(\theta) = \frac{2}{2-\theta} - 1$

$\{G_X(1) = \frac{2}{2-1} - 1 = 1$ $\therefore$ is satisfied as a PGF$\}$

ii) what is $P(X=0)$? $P(X=k)$?

$G_X(\theta) = \frac{1}{1-\frac{\theta}{2}} - 1$ $\qquad \{(1-x)^{-1} = 1 + x + x^2 + x^3 + \cdots, \ |x|<1\}$ $\therefore x = \frac{\theta}{2} \Rightarrow$

$\frac{\theta}{2} < 1$, $|\theta| < 2 \therefore \}$ $\quad G_X(\theta) = 1 + (\frac{\theta}{2}) + (\frac{\theta}{2})^2 + (\frac{\theta}{2})^3 + \cdots - 1 =$

$\frac{\theta}{2} + (\frac{\theta}{2})^2 + (\frac{\theta}{2})^3 + \cdots$ $\quad \therefore P(X=0) = 0$, $P(X=k) = \frac{1}{2^k}$

\Ex/ let X_1, X_2, X_3, X_4 be iid r.v. with PGF

$G_X(\theta) = 0.5 + 0.2\theta + 0.2\theta^2 + 0.1\theta^3$ $\quad$ let $Y = X_1 + X_2 + X_3 + X_4$

whats $P(Y=12)$, $P(Y=13)$, $P(Y=2)$? $\quad \therefore$ from property 4:

$G_Y(\theta) = G_{X_1}(\theta) G_{X_2}(\theta) G_{X_3}(\theta) G_{X_4}(\theta) = (G_X(\theta))^4$ $\quad P(X_i = 3) = 0.1$

$\therefore P(Y=12) = 0.1^4 = P(Y = 3\times 4)$ $\qquad \therefore P(Y=13) = 0$

$P(Y=2)$? $\quad (x_1 + x_2 + x_3 + \cdots + x_k)^n$ multinomial expansion tells us

that Z coeffs in front of $x_1^{l_1}x_2^{l_2}\cdots x_k^{l_k}$ is given by
$\frac{n!}{l_1!l_2!\cdots l_k!}$ where $l_1+l_2+\cdots+l_k=n$

() ∴ let $x_1=0.5$, $x_2=0.2\theta$, $x_3=0.2\theta^2$, $x_4=0.1\theta^3$, $n=4$

$x_1^{l_1}x_2^{l_2}x_3^{l_3}x_4^{l_4}=(0.5)^{l_1}(0.2\theta)^{l_2}(0.2\theta^2)^{l_3}(0.1\theta^3)^{l_4}$ ∴ $l_4=0$ ∴

① $l_4=0$, $l_3=1$, $l_2=0$, $l_1=3$

② $l_4=0$, $l_3=0$, $l_2=2$, $l_1=2$ are combinations of θ^2 ∴

$P(Y=2)=\frac{4!}{1!3!}(0.5)^3(0.2)+\frac{4!}{2!2!}(0.2)^2(0.5)^2$ {give 3.s.f.}

\Random sums/ let $\{X_n\}_{n\in\mathbb{N}}$ be iid r.v. taking vals over Z non-negative integers

let $Z=X_1+X_2+\cdots+X_N$ if $N=0$, then $Z=0$

assume N is an r.v., indep of X_n

whats distri of Z?

suppose that PGF of X_n is $G_X(\theta)$ & that Z PGF of N is given by $G_N(\theta)$

\proposition:/ Z PGF of Z is given by $G_N\circ G_X(\theta)=G_N(G_X(\theta))$

\proof:/ $G_Z(\theta)=E_Z[\theta^Z]=E_N[E_Z[\theta^Z|N]]$ (law of total expectation)

$=\sum_{k=0}^{\infty}E_Z[\theta^Z|N]P(N=k)=\sum_{k=0}^{\infty}E_Z[\theta^{X_1}\theta^{X_2}\cdots\theta^{X_k}]P(N=k)=$

$\sum_{k=0}^{\infty}E_X[\theta^{X_1}]E[\theta^{X_2}]\cdots E[\theta^{X_k}]P(N=k)=$

$\sum_{k=0}^{\infty}G_{X_1}(\theta)\cdot G_{X_2}(\theta)\cdots G_{X_k}(\theta)P(N=k)=$

$\sum_{k=0}^{\infty}(G_X(\theta))^kP(N=k)=G_N(G_X(\theta))$ □

\Ex/ ①: roll a fair 6 sided die ② toss a coin Z number of times given by Z outcome of Z die roll how many heads do we expect to see?

let $N\equiv$ outcome of Z die roll $\equiv$ number of coin tosses

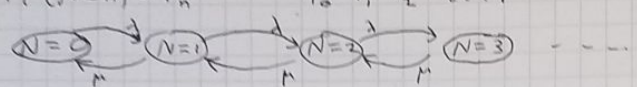
) $X_n\equiv$ outcome of Z n'th coin toss

PGF for N: $G_N(\theta)=\frac{1}{6}(\theta+\theta^2+\theta^3+\theta^4\theta^5+\theta^6)$

assume $X_n(H)=1$ then $G_X(\theta)=\frac{1}{2}(1+\theta)$ ∴ from proposition

∴ $N_q = N-1$ unless $N=0$ then $N_q=0$

$P_r(N=n) = P_n$ ∴ $P_0 \quad P_1 \quad P_2 \quad \dots P$



$\frac{dP_0}{dt} = \mu P_1 - \lambda P_0$ $\qquad \frac{dP_1}{dt} = \lambda P_0 - \lambda P_1 + \mu P_2 - \mu P_1$

steadystates: $\frac{dP_0}{dt} = 0$ $\qquad \frac{dP_1}{dt} = 0$ $\quad --- \quad \frac{dP_\infty}{dt} = 0$ steady state

∴ $\frac{dP_0}{dt} = \mu P_1 - \lambda P_0 = 0$, $\frac{dP_1}{dt} = \lambda P_0 - \lambda P_1 + \mu P_2 - \mu P_1 = 0$ --- steady state

$\lambda P_0 = \mu P_1 \qquad \lambda P_1 = \mu P_2 \qquad \lambda P_2 = \mu P_3 \dots$ ∴ $\frac{\lambda}{\mu} = \rho < 1$

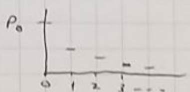
∴ $P_1 = \rho P_0$, $P_2 = \rho P_1 \qquad \rho_3 = \rho P_2 \dots$ ∴

$\cancel{\sum_{n=0}^{\infty}}$ $\sum_{n=0}^{P} P_n = 1 \Rightarrow P_0 + \rho P_0 + \rho^2 P_0 + \dots = 1$ ∴

$P_0(1+\rho+\rho^2+\dots) = 1$ ∴ $\cancel{P_0\sum}$ $\cancel{P_0\sum}$ $P_0 \sum_{n=0}^{\infty} \rho^n = P_0 \frac{1}{1-\rho} = 1$ ∴

$P_0 = 1-\rho$ ∴

$P_n = \rho^n P_0$ geometric distri



$P_n = \rho^n (1-\rho)$

{∴ Mean queue size} {∴ length of the queue}

Length of Z syst: $L_s = E(N) = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n\rho^n P_0 = \frac{\rho}{1-\rho}$

$\left\{ \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n\rho^n P_0 = P_0 \sum_{n=0}^{\infty} n\rho^n = (1-\rho)\sum_{n=0}^{\infty} n\rho^n = \frac{\rho}{1-\rho} \right.$ ∴

$\left. \cancel{\sum_{n=0}^{\infty} \frac{\rho}{1}} \qquad \sum_{n=0}^{\infty} n\rho^n = \frac{\rho}{(1-\rho)(1-\rho)} = \frac{\rho}{(1-\rho)^2} \right\}$

$L_s = E[N] = \sum_{n\geq 0} nP_n = \sum_{n=0}^{\infty} n\rho^n P_0 = \sum_{n=0}^{\infty} n\rho^n(1-\rho)$

$\left\{ \sum_{n\geq 1} nx^{n-1} = (1-x)^{-2} = \sum_{n\geq 1} nx^n \frac{1}{x} = \frac{1}{x}\sum_{n\geq 1} nx^n \right\}$ ∴ $L_s = \sum_{n=0}^{\infty} n\rho^n(1-\rho) = (1-\rho)\sum_{n=0}^{\infty} n\rho^{n-1}\rho$

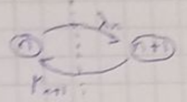
$= \rho(1-\rho)\sum_{n=0}^{\infty} n\rho^{n-1} = \rho(1-\rho) = \rho(1-\rho)(1-\rho)^{-2} \cdot \rho(1-\rho) = \frac{1}{1-\rho}$

m/m/1 queue $\quad P_r(N=n) = P_n = \rho^n(1-\rho)$ geometric distri

$E(N) = \sum_{n=0}^{\infty} nP_n = \sum_{n=0}^{\infty} n\rho^n(1-\rho)$

apply normalisation: $\sum_{n=0}^{N} P_n = 1 \Rightarrow$

$$\left(\sum_{n=1}^{N} \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_1} + 1\right) P_0 = 1$$



equating probab flows across green line:

$\rightarrow: P_n\lambda_n$ $\quad \leftarrow: P_{n+1}\mu_{n+1}$ $\quad$ equating $\Rightarrow P_{n+1}\mu_{n+1} = P_n\lambda_n$

General approach: • 1) Draw state diagram with appropriate rates

• 2) write steady state eqns by evaluating probab flows

• 3) solve eqns, typically iteratively, to express P_n in terms of P_0

• 4) apply normalisation to find P_0. Sub this into expression for P_n

Stochastic processes / r.v. $X(t)$ ← time — $\mathbb{R}$ continuous time, $\mathbb{Z}$ discrete time (discrete, continuous)

$X(t, s)$ $\quad$ s = space

Branching process, poisson process

Queuing processes, Markov chains, random walk, gambler's ruin

Queuing process / arrive at rate λ ∴ average arrival time is $\frac{1}{\lambda}$

than servers S have r of them so r servers ∴ S_r

N_q is number in queue

N in the system ∴ $\{N_q + S_r = N?\}$

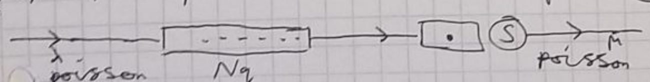
rate of service μ ∴ each till has different rate

∴ at S_1 is μ_1 ... S_2 is μ_2 ... S_r is μ_r

$\left\{\sum_{i=1}^{r} \mu_i > \lambda \text{ must be true?}\right\}$

queue can contain 0 to ∞ or 0 to finite number

M/M/1 queue with ∞ capacity:



∴ M/M/1 ∴ 1 ∵ only 1 server, Markovian process

N	N_q
0	0
1	0
2	1
3	2
4	3

N = number of individuals in Z syst

N_q = number of individuals in Z queue ∴

$P_0(t)(1-\lambda_0\delta t) + P_1(t)\mu_1\delta t + o(\delta t)$

$P_N(t+\delta t) = P_N(t)P(\text{no departures over } \delta t) + P_{N-1}(t)P(\text{one arrival over } \delta t) +$
$\sum_{k=0}^{N-2} P_k(t)P(N-k \text{ arrivals over } \delta t) =$
$P_N(t)(1-\mu_N\delta t) + P_{N-1}(t)\lambda_{N-1}\delta t + o(\delta t)$

$P_n(t+\delta t) = P_{n-1}(t)\lambda_{n-1}\delta t + P_{n+1}(t)\mu_{n+1}\delta t + P_n(t)(1-\lambda_n\delta t - \mu_n\delta t) + o(\delta t)$

for $1 \le n \le N-1$

$P_n(t+\delta t) = P_n(t) - (\lambda_n+\mu_n)\delta t P_n(t) + P_{n-1}(t)\lambda_{n-1}\delta t + P_{n+1}\mu_{n+1}\delta t$

$\frac{P_n(t+\delta t) - P_n(t)}{\delta t} = -(\lambda_n+\mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t)$

let $\delta t \to 0$ then: $\frac{dP_n(t)}{dt} = -(\lambda_n+\mu_n)P_n(t) + \lambda_{n-1}P_{n-1}(t) + \mu_{n+1}P_{n+1}(t)$

similarly $\frac{dP_0(t)}{dt} = \mu_1P_1(t) - \lambda_0P_0(t)$

$\frac{dP_N(t)}{dt} = \lambda_{N-1}P_{N-1}(t) - \mu_N P_N(t)$ Z above are Z <u>Poisson rate eqns</u>

\Steady state analysis/ have poisson rate eqns

$\frac{dP_0}{dt} = \mu_1P_1(t) - \lambda_0P_0(t)$

$\frac{dP_n}{dt} = -(\lambda_n+\mu_n)P_n(t) + \mu_{n+1}P_{n+1}(t) + \lambda_{n-1}P_{n-1}(t)$, $1 \le n \le N-1$

$\frac{dP_N}{dt} = \lambda_{N-1}P_{N-1}(t) - \mu_N P_N(t)$

what happens as $t\to\infty$? what is Z steady state sol?

steady state described by $\frac{dP_n}{dt} = 0 \ \forall n \in \{0,1,\dots,N-1\}$

when does a steady state exist? · Guaranteed if N finite $\Rightarrow$
$\lambda_n = 0 \ \forall n \ge N$

- if for $n \ge n_0$ Z arrival rate is smaller than Z departure rate

in particular, $\lim_{n\to\infty} \frac{\lambda_n}{\mu_n} < 1$

Set $\frac{dP_n}{dt} = 0 \ \forall n$: $0 = \mu_1P_1 - \lambda_0P_0$

$0 = -(\lambda_n+\mu_n)P_n + \lambda_{n-1}P_{n-1} + \mu_{n+1}P_{n+1}$

$0 = \lambda_{N-1}P_{N-1} - \mu_N P_N$

for $n=0$: $P_1 = \frac{\lambda_0}{\mu_1}P_0$ $\quad n=1$: $0 = -(\lambda_1+\mu_1)P_1 + \lambda_0P_0 + \mu_2P_2 =$

$-\lambda_1P_1 - \mu_1P_1 + \mu_1P_1 + \mu_2P_2 = -\lambda_1P_1 + \mu_2P_2 \Rightarrow P_2 = \frac{\lambda_1}{\mu_2}P_1 = \frac{\lambda_1\lambda_0}{\mu_2\mu_1}P_0$

in general, $P_n = \frac{\lambda_{n-1}}{\mu_n}P_{n-1} = \frac{\lambda_{n-1}\lambda_{n-2}\lambda_{n-3}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_n}P_0$, $1 \le n \le N$

$$\frac{n!\,(\lambda^k \lambda^{n-k}\lambda^{-n})}{k!(n-k)!}\left(e^{-\lambda t}e^{-\lambda T}e^{\lambda t}e^{\lambda T}\right)\frac{t^k(T-t)^{n-k}}{T^n} =$$

$$\frac{n!}{k!(n-k)!}\times\frac{t^k(T-t)^{n-k}}{T^n} = \frac{n!}{k!(n-k)!}\left(\frac{t}{T}\right)^k\left(1-\frac{t}{T}\right)^{n-k} \sim \text{Binomial}\left(n, \frac{t}{T}\right)$$ $\square$

Week 5

Small interval limit

Def: (Little 'o' notation) suppose have a func $f(x)$, say $f(x) = o(x)$ is $\lim_{x\to 0}\frac{f(x)}{x} = 0$

eg i) $f(x) = x^2$ $\quad \lim_{x\to x}\frac{f(x)}{x} = \lim_{x\to 0}\frac{x^2}{x} = \lim_{x\to 0} x = 0 \Rightarrow f(x) = x^2 = o(x)$

ii) $f(x) = 1+x^2$ $\quad \lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{1}{x} + \lim_{x\to 0} x = \infty \Rightarrow f(x) = 1+x^2 \neq o(x)$

iii) $f(x) = \sqrt{x}$ $\quad \lim_{x\to 0}\frac{f(x)}{x} = \lim_{x\to 0}\frac{x^{1/2}}{x} = \lim_{x\to 0} x^{-1/2} = \infty \Rightarrow f(x) = \sqrt{x} \neq o(x)$

let $N(t) \sim \text{Poiss}(\lambda t) \equiv$ number of people arriving to a case over an interval t $\quad$ what happens as $t \to 0$?

let $t = \delta t \to 0$ $\quad$ have: $P(N(\delta t) = 0) = e^{-\lambda\delta t} = 1 - \lambda\delta t + \frac{(\lambda\delta t)^2}{2} + \dots = 1 - \lambda\delta t + o(\delta t)$

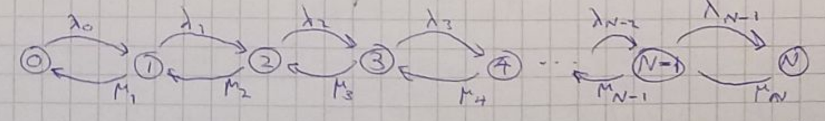
$P(N(\delta t) = 1) = (\lambda\delta t)e^{-\lambda\delta t} = \lambda\delta t(1 - \lambda\delta t + \dots) = \lambda\delta t - (\lambda\delta t)^2 + \dots = \lambda\delta t + o(\delta t)$

$\underset{k\geq 2}{P(N(\delta t = k)} = \frac{(\lambda\delta t)^k e^{-\lambda\delta t}}{k!} = \frac{\lambda^k}{k!}\delta t^k(1 - \lambda\delta t + \dots) = o(\delta t)$

$\Rightarrow P(N(\delta t) \geq 2) = o(\delta t)$ $\quad$ for δt small, treat probabs $o(\delta t)$ as insignificant

Poisson rate eqns: suppose system which arrivals follow a poiss process with rate λ_n & departures follow a poiss process with rate μ_n where n is the number of individuals in the system. Assume that arrivals + departures are indep of one another. Key pt: as n changes so do arrival/departure rates.

State diagram:



whats probab that n customers are in the case at time t? $= P_n(t)$

$P_0(t+\delta t) = P_0(t)\,P(\text{no arrivals over } \delta t) + P_1(t)\,P(\text{one departure over } \delta t) + \sum_{k=2}^{N} P_k(t)\,P(k \text{ departures over } \delta t) =$

(of time length $n-k$) $\therefore$ returning again at time n $\therefore$
$P(E_n) = t_i^{(n)}$ it follows $t_i^{(n)} = f_i^{(n)} + \sum_{k=1}^{n-1} f_i^{(k)} t_i^{(n-k)}$ for $n>1$ &
$t_i^{(1)} = f_i^{(1)}$
define now $G_t(z)$ & $G_f(z)$ for $z \in \mathbb{C}$ by: $G_t(z) = \sum_{n\geq 1} t_i^{(n)} z^n$,
$G_f(z) = \sum_{n\geq 1} f_i^{(n)} z^n$ $\therefore$
$G_t(z) = G_f(z) + G_f(z) G_t(z)$ $\therefore G_t(z) = G_f(z)(1-G_f(z))^{-1}$ $\therefore$
at $z=1$ $G_f(1) = f_i$ & $G_t(1) = \sum_n t_i^{(n)}$ $\therefore$
$\sum_n t_i^{(n)} = f_i (1-f_i)^{-1}$ if state i is transient $\therefore f_i < 1$ $\therefore$
$\sum_n t_i^{(n)} < \infty$ but if state i is recurrent $\therefore f_i = 1$ & $\therefore \sum_n t_i^{(n)} = \infty$ $\square$

the alternative criteria for determining recurrence fails in the case of time dependent

\Ex/ $T = T(n) = \begin{pmatrix} 1 & 0 \\ 1/(n+1) & n/(n+1) \end{pmatrix}$ see $t_2^{(n)} = P(X_n = 2 \mid X_0 = 2) = 1/(n+1)$ $\therefore$
divergent series, from the transition ~~matrix~~ diag see once we escape 2 we never return - the probab of eventual escape is 1
$\therefore$ 2 is transient ^state^

the summability cond on $t_i^{(n)}$ leading to transience is equiv to the divergent sum cond on $t_i^{(n)}$ but Markov chains not indep processes $\therefore$ BC2 not applied to reach conclusion of recurrence

using $t_i^{(n)}$ not useful trying to determine posi or null recurrence, need $f_i^{(n)}$

\time indep (non homog) Markov chains/ if T depend on time n
$\therefore T = T(n)$ $\therefore$ still use transition diags but the probab labels on the arrows will now depend on times the transitions are being make. if $p^{(n)}$ denotes the probab state vec at time n $\therefore p^{(1)} = p^{(0)} T(1)$, $p^{(2)} = p^{(1)} T(2)$, ..., $p^{(n)} = p^{(n-1)} T^{(n)}$ $\therefore$
$[T^{(n)}]_{ij} = P(X_n = j \mid X_0 = i)$, $T^{(n)} = T(1)T(2)\cdots T(n-1)T(n)$

\Ex 7.5/ $\Omega = \{1,2\}$ $\therefore$ $T = T(n) = \begin{pmatrix} 1/2 & 1/2 \\ 1/(n+1) & n/(n+1) \end{pmatrix}$ $\therefore$
$T^{(n)} = T(1)T(2)\cdots T(n-1)T(n)$ $[T^{(n)}]_{ij} = P(X_n = j \mid X_0 = i)$, $n \geq 1$ $\therefore$
as time evolves see ~~state~~ state 2 is visited it more

\Ex 7.2/ $T\begin{pmatrix} 1/2 & 1/2 \\ 1/3 & 2/3 \end{pmatrix}$ ∴ is an irreducible chain with 2 recurrent aperiodic states ∴ chain is ergodic ∴ chain has a limiting distri $\lim_{n\to\infty} p^{(n)} = \tilde{p}$ where $\tilde{p} = (2/5, 3/5)$

$\{ f_1^{(1)} = 1/2$, $f_1^{(2)} = 1/2$ ∴ $\mu_1 = 1\times\frac{1}{2} + 2\times\frac{1}{2} = \frac{3}{2}$

$f_2^{(1)} = \frac{1}{3}$ $f_2^{(2)} = \frac{2}{3}$ ∴ $\mu_2 = 1\times\frac{1}{3} + 2\times\frac{2}{3} = \frac{5}{3}$

$\tilde{p} = (\mu_1^{-1}, \mu_2^{-1})$ }

if $T = \begin{pmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 0 & 1 & 0 \end{pmatrix}$ ∴ $\Omega = \{1, 2, 3\}$ ∴ $\lim_{n\to\infty} p^{(n)}$ doesn't converge but oscillates between $(4/9, 1/3, 2/9)$ & $(2/9, 2/3, 1/9)$ on alternative time steps ∴ Z stationary distri is $(1/3, 1/2, 1/6)$ is Z mean of Z two oscillating states

$\{ ((\frac{4}{9} + \frac{2}{9})/2 , (1/3 + \frac{2}{3})/2 , (\frac{2}{9} + \frac{1}{9})/2) = (\frac{1}{3}, \frac{1}{2}, \frac{1}{6}) \}$

\alternative view of recurrence/ using Z current def of recurrence: calc Z probabs $f_i^{(n)}$ & f_i to decide if state i is recurrent or transient, when T is a const (time indep) transition Mat

alternatively: let $t_i^{(n)} = (T^n)_{ii}$ ∴ $t^{(n)} = P(X_n = i \mid X_0 = i)$ ∴ $t_i^{(n)} \geq f_i^{(n)}$ ∵ former probabs accounts for intermediate returns before time n while Z latter probabs account for first returns only

\proposition 7.3/ T is time indep trasition Mat ∴

① if $\sum_n t_i^{(n)} = \infty$ i is recurrent

② if $\sum_n t_i^{(n)} < \infty$ i is transient

\proof event R_n defined for $n \geq 1$ ∴ $P(R_0) = 0$ Set $t_i^{(0)} = 1$ ∵ initially in state i Let E_n denote Z event of being in state i at time n ∴ $P(E_n) = P(E_n \cap (\bigcup_{k=1}^{n} R_k)) = P(\bigcup_{k=1}^{n} (E_n \cap R_k)) =$

$\sum_{k=1}^{n} P(E_n \cap R_k) = P(R_n) + \sum_{k=1}^{n-1} f_i^{(k)} t_i^{(n-k)}$ ∵ $\{R_k\}_{k\leq n}$ form a joint partition of E_n, & $E_n \cap R_k$ denotes Z event of making a first return to state i at time k ∴ another excursion from ~~time~~ i

stationary distri $\tilde{P}_{1\times m} = \tilde{P}_{1\times m} = \tilde{P}_{1\times m} T_{m\times m}$

~~$T^T = \lambda \tilde{P}^T$~~ Ex $T^T = \tilde{P}^T = \lambda \tilde{P}$ $\lambda = 1$

if have irreducible periodic chain limit doesnot exist

but $\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} p^{(i)} \to \tilde{P}$

Markov chain Monte Carlo

\Ex 7.1/ $U = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \end{pmatrix}$ $V = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} \\ 1 & 0 \end{pmatrix}$ ∴

mat U: states $\{1, 2, 3\}$ form a periodic (period 3)

{1 to 2 is consumed 2 to 3 is 100 , 3 to 1 is 100 ∴ period 3}

irreducible subchain, Z positively recurrent

∴ state 4 is transient

for V: both states accessible from one another ∴ chain is irreducible & both states positively recurrent & aperiodic ∴ chain is ergodic

\limiting stationary distris / state probabs in limit as $n\to\infty$

for ergodic chains: limiting distri $\lim_{n\to\infty} p^{(n)}$ exists & is equal to Z stationary distri $\tilde{P}$ Z sol of Z steady state eqn $\tilde{P} = \tilde{P} T$

Z stationary distri represents Z fraction of time spent in each state ie Z probabs are Z reciprocals of Z mean recurrence times $\tilde{P} = (\mu_1^{-1}, \mu_1^{-2}, \ldots, \mu_m^{-1})$

for chains with periodic states, Z probabs dont converge to a limiting distri but instead oscillate between states but any posi-recurrent irreducible chain, long-term time mean of Z probabs converge to Z stationary distri $\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} p^{(0)} T^k = \tilde{P}$

$\tilde{P}$ represents Z fraction of time spent in each state.

Z stationary distri ~~found~~ $\tilde{P}$ found by solving Z eigvec eqn $T' \tilde{P}' = \tilde{P}'$ $(T^T \tilde{P}^T = \tilde{P}^T)$ analytically

$f = \alpha + (1-\alpha)(1-\beta) + (1-\alpha)\beta(1-\beta) + (1-\alpha)\beta^2(1-\beta) + \dots =$ $\{ |\beta| < 1 \}$

$\alpha + (1-\alpha)(1-\beta)(1+\beta+\beta^2+\beta^3+\dots) = \alpha + (1-\alpha)(1-\beta)\sum_{n=0}^{\infty}\beta^n =$

$\alpha + (1-\alpha)(1-\beta)\frac{1}{1-\beta} = \alpha + (1-\alpha)(1-\beta)(1-\beta)^{-1} =$

$\alpha + (1-\alpha) = 1$ ∴ definite prob of returning to state 1 ∴ state 1 is recurrent

∴ recurrent if $f = 1$ transient if $f < 1$

null recurrent $\mu = \infty$ positively recurrent $\mu < \infty$

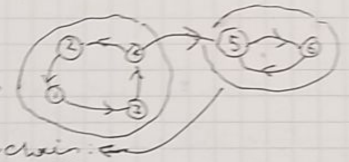
∴ use mean recurrence time $\mu_i = \sum_{n=1}^{\infty} n f_i^{(n)}$

\ Classification of Chains / state j is "accessible" from i if $\exists n > 0$ $(T)^n_{ij} > 0$

i is "accessible" from j if $\exists m > 0$ $(T)^m_{ij} > 0$

∴ if have both then : states i & j "communicate"

Communication class := all states that communicate with eachother



Subchains

∴ irreducible Subchain

Class properties: periodicity ∴ if one is periodic, they all are

recurrence, if one is recurrent, they all are

$f=1$ ~~transience~~

null $\mu < \infty$

positively $\mu < \infty$

transience: $f < 1$ ∴ if one is transient in the irreducible Sub chain, they all are

irreducible subchain, aperiodic, positively recurrent = ergodic Subchain

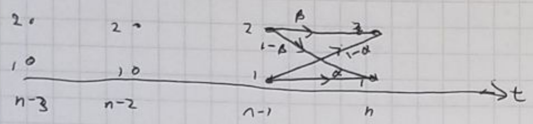
\ Limiting distri / $\lim_{n\to\infty} p^{(n)} = ?$

$p^{(n)} = p^{(n-1)}T = p^{(0)}T^n$ ∴ does Z limit exist? if so what is it?

ergodic chain: irreducible, aperiodic, positively recurrent

$\Longrightarrow$ the limit does exist.

2 state Markov chain / r.v $X_n = 1, 2$ $\quad n = 0, 1, 2, 3, \dots$



$$Pr(X_n=1) = Pr(X_{n-1}=1)\underbrace{Pr(X_n=1 \mid X_{n-1}=1)}_{\alpha} + Pr(X_{n-1}=2)\underbrace{Pr(X_n=1 \mid X_{n-1}=2)}_{1-\beta}$$

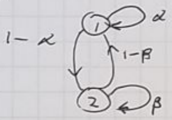
$$Pr(X_n=2) = Pr(X_{n-1}=1)\underbrace{Pr(X_n=2 \mid X_{n-1}=1)}_{1-\alpha} + Pr(X_{n-1}=2)\underbrace{Pr(X_n=2 \mid X_{n-1}=2)}_{\beta}$$

$p^{(n)} = (Pr(X_n=1), Pr(X_n=2))$

$p^{(n)} = p^{(n-1)}T \qquad T = \begin{array}{c|cc} & 1 & 2 \\ \hline 1 & \alpha & 1-\alpha \\ 2 & 1-\beta & \beta \end{array}$

$\therefore p^{(n)} = p^{(n-1)}T = p^{(n-2)}T^2 = p^{(n+3)}T^3 = p^{(0)}\cancel{T}^n = p^{(n)}$

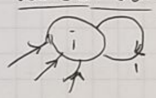
directed graph



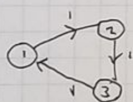
transition diag

M states
has M rows & columns in transition matrix

Classification of states / absorbing state: $\exists$ state i : $T_{ii}=1$ & ~~T_{ij} $\forall j$~~ $T_{ij} = 0 \ \forall j \neq i$



periodic state



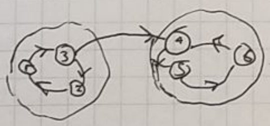
$(T_{ii})^n > 0$ for $n = k, 2k, 3k, 4k$

for $k > 1$ period $\quad (T_{ii}) = 0$ otherwise

or aperiodic if $\nexists k$

recurrent state (even if it takes an infinite amount of time)

transient state:

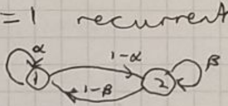


~~probab of "first" return to state i~~

recurrent state: probab of "first" return to state i

$f_i^{(n)}$ $\quad n = 1, 2, \dots, \infty \qquad f = \sum_{n=1}^{\infty} f_i^{(n)} = 1$ recurrent

if < 1 then transient

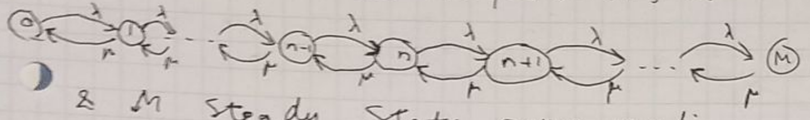


$\therefore$

first return probabs: $f_1^{(1)} = \alpha$, $\quad f_1^{(2)} = (1-\alpha)(1-\beta)$, $\quad f_1^{(3)} = (1-\alpha)\beta(1-\beta)$,

$f_1^{(4)} = (1-\alpha)\beta\beta(1-\beta)$ $\quad \dots \quad \therefore$

$N-1$ possible states $n \in \{0, 1, \cdots, N\}$ $\therefore$



& M steady state eqns read:

$\lambda P_0 = \mu P_1$, $\lambda P_1 = \mu P_2$, $---$, $\lambda P_n = \mu P_{n+1}$, $---$, $\lambda P_{M-1} = \mu P_M$ $\therefore$

$\rho = \frac{\lambda}{\mu}$ $\therefore$ $P_{n+1} = \rho P_n$ $\therefore$ $P_n = \rho^n P_0$ $\forall n \in \{0, \cdots, N\}$ $\therefore$

P_n describes a probab distri:

$1 = \sum_{n=0}^{M} P_n = P_0 \sum_{n=0}^{M} \rho^n$ $\therefore$ $P_0 = \frac{1-\rho}{1-\rho^{M+1}}$ $\therefore$ $P_n = \frac{\rho^n(1-\rho)}{1-\rho^{M+1}}$

for $0 \leq n \leq M$ & $\rho \neq 1$, if $\rho = 1$ $\therefore$

$P_n = P_0 \; \forall \; 0 \leq n \leq M$ $\therefore$ $\cancel{P_n = P_0}$ $P_n = P_0 = \frac{1}{M+1}$ $\therefore$

Z mean syst size L_s: $L_s = \sum_{n=0}^{M} n P_n = \frac{\rho(1-(M+1)\rho^M + M\rho^{M+1})}{(1-\rho)(1-\rho^{M+1})}$

by PGF: $G_X(\theta) = \frac{P_0(1-(\theta\rho)^{N+1})}{1-\theta\rho}$ $\therefore$ deriv & $\theta = 1$ to get L_s

L_q is similar

if $\rho = 1$: $L_s = \sum_{n\geq 0} n P_n = \sum_{n \geq 0} \frac{n}{M+1} = \frac{M}{2}$

\Little's thm & Z effective arrival rate/

λ_{eff} is effective arrival rate $= \lambda_{eff} =$

λ_{eff} = rate at which individuals arrive & actually join queue = arrival rate for individuals who get served eventually

$\therefore$ steady state dia grate consts λ_n $\therefore$

$\lambda_{eff} = \sum_{n\geq 0} \lambda_n P_n$ $\therefore$ if steady st $\therefore$ Littles thm

\thm 6.1 Littles thm/ L_s, L_q denote Z expected syst & queue sizes respectively. Let W_s, W_q denote Z expected waiting times in Z syst & queue respectively

assuming Z syst is st st: $L_s = \lambda_{eff} W_s$, $L_q = \lambda_{eff} W_q$

λ_{eff} is Z effec arrival rate

$\therefore$ for M/M/1 & M/M/2 (both infinite capacity) $\therefore$

$\lambda_{eff} = \lambda$ but for ~~$\lambda_{eff} = \lambda$~~ M/M/1 with finite capacity

M: $\lambda_{eff} = \sum_{n \geq 0}^{M} \lambda_n P_n = \sum_{n\geq 0}^{M-1} \lambda P_n = \lambda(1 - P_M)$ $\therefore$ $\lambda_M = 0$

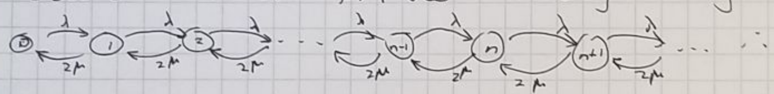
mean waiting time for syst: W_s is mean waiting time for Z queue to be empty plus Z mean time to be served ∴

$W_s = W_q + \frac{1}{\mu}$ ∴ $L_s = \rho(1-\rho)^{-1}$ & $\mu = \frac{\lambda}{\rho}$ ∴

$W_q = \frac{\rho^2}{1-\rho}\frac{1}{\lambda}$ ∴

$W_s = \frac{\rho}{1-\rho}\frac{1}{\lambda}$ ∴ $L_q = \lambda W_q$, $L_s = \lambda W_s$ Little's Law

\Z M/M/2 syst with infinite capacity ∴ $\mu_i = 2\mu$ $\forall i \in N$
two servers at an individual service rate μ ∴
~~$N_q Z N$~~ if ∃ one in the syst ∴ one of Z servers will be idel ($N_q = N-1$)
both servers will be active if ∃ two or more individuals in Z syst ($N_q = N-2$)
assume all arrive at rate λ ∴ steady state diag:



steady state eqns: $\lambda P_0 = \mu P_1$, $\lambda P_1 = 2\mu P_2$, ... , $\lambda P_2 = 2\mu P_3$, ...
∴ $\lambda P_n = 2\mu P_{n+1}$ $\forall n \geq 1$ but not for $n = 0$ ∵ single is served at rate μ ∴
if set $\rho = \lambda/2\mu$ ∴ $P_n = 2\rho^n P_0$ $\forall n \geq 1$ ∵ $\sum_{n\geq 0} P_n = 1$ ∴
$P_0\left(1 + 2\sum_{n\geq 1}\rho^n\right) = 1$ ∴ $P_0 = \frac{1-\rho}{1+\rho}$ ∴ $P_n = 2\frac{\rho^n(1-\rho)}{1+\rho}$ $\forall n \geq 1$
only valid for $\rho < 1$ ∴
mean syst size L_s : $L_s = \sum_{n\geq 0} n P_n = \frac{2\rho}{1-\rho^2}$
mean queue size L_q : $L_q = \sum_{n\geq 2}(n-2)P_n = \frac{2\rho^3}{1-\rho^2}$
again if ∃ $N > 2$ in syst: $N-2$ in Z queue & ∴ 2 getting served
∴ to get Z expected waiting times in Z syst & queue:
$L_s = \lambda W_s$ & $L_q = \lambda W_q$ by littles law

\Z M/M/1 syst with finite capacity M ∴ rate μ , λ
capacity M of syst ∴ if full no extra to queue
underlying process assume poisson ∴

$(1-\rho)^{-2} = 1 + 2\rho + 3\rho^2 + \dots + \frac{(-2)(-2-1)\cdots(-2-r+1)}{r!}\rho^r$

$\rho(1-\rho)^{-2} = \rho + 2\rho^2 + 3\rho^3 + \dots$ $\therefore$

$E(N) = (1-\rho)\rho(1-\rho)^{-2} = \frac{\rho}{1-\rho}$ $\qquad \rho = \frac{\lambda}{\mu} < 1$

or $M(t) = E(e^{tN}) = \sum_{n=0}^{\infty} \rho^n (1-\rho) e^{tn} = \frac{1-\rho}{1-\rho e^t} = (1-\rho)\sum_{n=0}^{\infty} (\rho e^t)^n =$

$(1-\rho)\frac{1}{1-\rho e^t} = \frac{1-\rho}{1-\rho e^t}$ $\therefore$

$E(N) = \frac{dM}{dt}\Big|_{t=0} = \frac{\rho}{1-\rho}$

Waiting time: wait: $W = T_1 + T_2 + \dots + T_N + T$

T is time for you to be serviced

T_i for $i = 1, \dots, N$ is time for people infront of you in Z queue to be serviced

thm: Wald's thm:

$E(W) = E(T)E(N) + E(T) = (E(N)+1)E(T)$ $\qquad \therefore E(T) = \frac{1}{\mu}$

$= \left(\frac{\rho}{1-\rho} + \frac{1-\rho}{1-\rho}\right)\frac{1}{\mu} = \frac{1}{1-\rho}\frac{1}{\mu}$, $\quad \frac{\lambda}{\mu} = \rho \quad \therefore \frac{1}{\mu} = \frac{\rho}{\lambda}$ $\therefore$

$E(W) = \frac{1}{1-\rho}\frac{1}{\mu} = \left(\frac{\rho}{1-\rho}\right)\frac{1}{\lambda}$ $\therefore$

$E(W) = E(N)\frac{1}{\lambda}$ $\quad \therefore$

$E(N) = \lambda E(W)$ is Little's Law

Ex/ 4 mins average to get served, 3/min on average to arrive $\therefore$

3/min × 4 min = 12 people expected on average to be stood in a bar at any one time

Mean queue size: $L_q = E(N_q)$ $\quad \therefore L_q = E(N_q) = \sum_{n \geq 1} (n-1) P_n = \frac{\rho^2}{1-\rho}$

$\therefore L_s =$ (Mean number in queue) + (Mean number being served) $=$

$L_q + \sum_{n \geq 0} n P$(n customers being served)

$= L_q + 0 \times P$(system empty) $+ 1 \times P$(server busy)

$= L_q + 1 \times (1 - P_0) = L_q + \rho = L_q + (1 - P_0) = L_q + \rho$

$\therefore$ Mean waiting times: $T_q = T_1 + T_2 + \dots + T_N$ $\quad T_i$ indep expon distried

$\therefore E[T] = 1/\mu$ $\therefore$ Mean waiting time for queue to clear $W_q = E[T_q] = E[T]E[N] = L_s/\mu$

\Sheet 1/ \6b Sol/ $P(S_2 > 1) = 1 - P(S_2 \leq 1) = 1 - (P(S_2=0) + P(S_2=1)) =$

$1 - (G_X(G_X(0)) + \text{coeff of } \theta \text{ in } G_{S_2}) \approx 1 - (0.2736 + 0.1332) = 0.598$

$P(S_2=0) = G_X(G_X(0)) = e_2 = G_X(G_X(0)) = 0.2 + 0.3(0.2) + 0.3(0.2)^2 + 0.2(0.2)^3$

$P(S_2=1) = \text{coeff of } \theta \text{ in } G_{S_2} = (0.3)^2 + (0.3)\binom{2}{1}(0.2)0.3 + 0.2\binom{3}{1}(0.2)^2 0.3$

\6c/ $e = \lim_{n\to\infty} e_n$ $\quad P(S_0=0)=0$ $\quad P(S_1=0)=0.2$

$P(S_2=0)=0.2736$

$\mu_X = E(X) = 1.5 \therefore e \neq 1$

\6c Sol/ $E(X) = 1.5 > 1 \therefore e < 1 \therefore$ solve $G_X(\theta) = \theta \therefore$

$\theta = 1, \frac{1}{2}(-2.5 \pm \sqrt{10.25}) \therefore e \approx 0.35$ $\{\because e = 1$ is always a root of $G_X(\theta) = \theta\}$

\6c/ $G_X(\theta) = 0.2 + 0.3\theta + 0.3\theta^2 + 0.2\theta^3$

$G_X'(1) = E(X) = 0.3 + 0.3(2)1 + 0.2(3)1 = 1.5 > 1 \therefore$

$e < 1 \therefore$

$G_X(\theta) - \theta = 0.2 + 0.3\theta + 0.3\theta^2 + 0.2\theta^3 - \theta = 0$

~~0.2+0.3~~ $0.2 - 0.7\theta + 0.3\theta^2 + 0.2\theta^3 = 0$

$G_X(\theta) - \theta$ has root $\theta = 1 \therefore \theta - 1 = 0 \therefore$

$(\theta - 1)(a\theta^2 + b\theta + c) = 0.2\theta^3 + 0.3\theta^2 - 0.7\theta + 0.2 =$

$a\theta^3 + \dots \therefore a = 0.2$

$-1c = 0.2 \therefore c = -0.2 \therefore$

$-1b - 0.2 = -0.7 \therefore b = 0.5 \therefore$

$G_X(\theta) - \theta = (\theta - 1)(0.2\theta^2 + 0.5\theta - 0.2) = 0 \therefore$

is $0.2\theta^2 + 0.5\theta - 0.2 = 0$

$\theta_{1,2} = (-0.5 \pm \sqrt{0.5^2 - 4(0.2)(-0.2)})/(2(0.2))$

$\therefore \theta_1 = 0.3508$

$+0.3\theta+0.3\theta^2+0.2\theta^3)^3$ $\quad \theta_2 = -2.85 \therefore$

Ultimate extinction probability is $e = 0.351$ (3sf)

\6d/ $S_0 = 6 \therefore G_Y(\theta) = E(\theta^Y) = E(\theta^{x_1+x_2+x_3+x_4+x_5+x_6}) = E(\theta^{6X})$ iid $= E(\theta^X)^6$

$= 0.2 + 0.3(\theta^6) + 0.3(\theta^6)^2 + 0.2(\theta^6)^3 =$

$0.2 + 0.3\theta^6 + 0.3\theta^{12} + 0.2\theta^{18}$

$E(S_6) = E(Y)\mu_X^6 =$

\3 sol/ $\therefore$ X ~ in

\5b/ have $E(X) = G_X'(1) = \mu_X = 0.9$ $\therefore E(S_n) = \mu_n = \mu_X^n = 0.9^n$

\5c/ $e_0 = P(S_0 = 0) = 0$

$e_1 = G_X(0) = P(X=0) = 0.4$

$e_2 = G_X(e_1) = 0.4 + 0.3(0.4) + 0.3(0.4)^2 = \cancel{0.78}\ 0.568$

$e_3 = G_X(e_2) = 0.4 + 0.3(0.568) + 0.3(0.568)^2 = 0.6671872$

$e_4 = G_X(e_3) = 0.4 + 0.3(0.667) + 0.3(0.667)^2 = 0.734$ (3 s.f.)

$e_0 = P(S_0 = 0) = 0$ $\quad e_1 = P(S_1 = 0) = G_X(e_0) = 0.4$

$e_2 = G_X(e_1) = 0.568$

$e_3 = G_X(e_2) = 0.667$ $\quad e_4 = G_X(e_3) = 0.7337$

\5d/ $P(X=0) \neq 0$

$e = \lim_{n\to\infty} e_{n+1} = \lim_{n\to\infty} G_X(e_n) = G_X(\lim_{n\to\infty} e_n) = G_X(e)$

$G_X(\theta) = 0.4 + 0.3\theta + 0.3\theta^2$;

$G_X'(\theta) = 0.3 + 0.6\theta$

$E(X) = 0.9 < 1$ $\therefore$ pop will ultimately go extinct

$\therefore$ ~~prob~~ prob of ultimately extinct is 1

\6a/ $G_X(\theta) = 0.2 + 0.3\theta + 0.3\theta^2 + 0.2\theta^3$

$\mu_X = E(X) = 0.3 + 0.6\theta + 0.6\theta^2 \big|_{\theta=1} = 1.5$ $\therefore$

$E(S_6) = \mu_X^6 = 1.5^6 = 11.4$ (3 s.f.)

\6b/ E? $G_2(\theta) = G_X \circ G_X(\theta)$

\5d sol/ $\because E(X) < 1$, ultimate extinction prob $e = 1$

\6a sol/ $G_X(\theta) = 0.2 + 0.3\theta + 0.3\theta^2 + 0.2\theta^3$ & $G_{S_n} = G_X^n(\theta)$

$E(X) = G_X'(1) = \mu_X = 1.5$ & $\therefore E(S_6) = \mu_X^6 = 1.5^6 = 11.4$

\6b/ $G_2(\theta) = G_X \circ G_X(\theta) = G_X(0.2 + 0.3\theta + 0.3\theta^2 + 0.2\theta^3)$

$= 0.2 + 0.3(0.2 + 0.3\theta + 0.3\theta^2 + 0.2\theta^3) + 0.3(0.2 + 0.3\theta + 0.3\theta^2 + 0.2\theta^3)^2 + 0.2(0.2 + 0.3\theta + 0.3\theta^2 + 0.2\theta^3)^3$

$P(S_2 > 1) = \cancel{P(S_2 \geq 1)} = 1 - P(S_2 \leq 1) = 1 - P(S_2 = 0) - P(S_2 = 1)$

$P(S_2 = 0) = G_2(0) = 0.2 + (0.3)0.2 + (0.3)0.2^2 + (0.2)0.2^3 = 0.2736$

$P(S_2 = 1) =$ coefficient of θ $\therefore$ {coeff of θ in G_{S_2}}

$P(S_2 = 1) = 0.3 \times 0.3 + 0.3\binom{2}{1}(0.2)(0.3) + 0.2\binom{3}{1}(0.2)(0.2)(0.3) =$

$0.09 + 0.036 + 7.2\times10^{-3} = 0.1332$ $\therefore$

$P(S_2 > 1) = 1 - 0.2736 - 0.1332 = 0.5932$

Sheet 1 $\cancel{2}$ $N \sim Poiss(6)$ $\therefore \cancel{\lambda}$ and $P(N=k) = \frac{\lambda^k e^{-\lambda}}{k!}$

$G_{X_i}(\theta) = 0.2 + 0.5\theta + 0.3\theta^2$

- $G_N(\theta) = E(\theta^N) = \sum_{k=0}^{\infty} \theta^k P(N=k) = \sum_{k=0}^{\infty} \theta^k \frac{\lambda^k e^{-\lambda}}{k!} = e^{\lambda(\theta-1)} = e^{6(\theta-1)}$ $\therefore$

$Y = X_1 + X_2 + \dots + X_N$ $\therefore$ $\cancel{G_N}$ $G_N \circ G_{X_i}(\theta) = G_N(G_X(\theta)) =$

$e^{6(0.2+0.5\theta+0.3\theta^2-1)} = G_Y(\theta) = e^{-4.8+3\theta+1.8\theta^2}$ $\therefore$

$E(Y) = G_Y^{(1)}(1) = G_Y^{(1)}(\theta)|_{\theta=1} = \frac{d}{d\theta} G_Y(\theta)|_{\theta=1} =$

$e^{-4.8+3\theta+1.8\theta^2}(3+3.6\theta)|_{\theta=1} = 1(3+3.6) = 6.6$

$G_Y^{(2)}(1) = \frac{d}{dt} G_Y^{(1)}(\theta)|_{\theta=1} =$

$e^{-4.8+3\theta+1.8\theta^2}(3+3.6\theta)^2 + e^{-4.8+3\theta+1.8\theta^2}(3.6)|_{\theta=1} =$

$(3+3.6)^2 + 3.6 = 47.16$ $\therefore$

$E($ $var(Y) = G_Y^{(2)}(1) + E(Y) - E(Y)^2 = 47.16 + 6.6 - 6.6^2 = 10.2$

$P(Y=2) =$ coeff of θ^2 $\therefore$

$G_Y(\theta) = e^{-4.8} e^{3\theta+1.8\theta^2} = e^{\theta(3+1.8\theta)} =$

$e^{-4.8}\left[\frac{(\theta(3+1.8\theta))^0}{0!} + \frac{(\theta(3+1.8\theta))^1}{1!} + \frac{(\theta(3+1.8\theta))^2}{2!} + \dots\right] =$

$e^{-4.8}\left[1 + 3\theta + 1.8\theta^2 + \theta^2(9 + 1.8^2\theta^2 + 3 \cdot 1.8\theta)\frac{1}{2} + \dots\right] =$

$e^{-4.8}\left[1 + 3\theta + 1.8\theta^2 + \theta^2\frac{9}{2} + 1.62\theta^3 + 2.7\theta + \dots\right] =$

$P(Y=2) = e^{-4.8}(1.8 + \frac{9}{2}) = 6.3e^{-4.8}$

5a S_n = number of individuals at gen n

$\cancel{G_2}$ $n=2$ $\therefore G_{S_2}(\theta) = G_X \circ G_X(\theta)$ $\cancel{G_X}$

$G_X(\theta) = 0.4 + 0.3\theta + 0.3\theta^2$ $\therefore$

$G_{S_2} = G_X(G_X(\theta)) = 0.4 + 0.3(G_X(\theta)) + 0.3(G_X(\theta))^2 =$

~~0.4 + 0.3(G~~

$0.4 + 0.3(0.4 + 0.3\theta + 0.3\theta^2) + 0.3(0.4 + 0.3\theta + 0.3\theta^2)^2 =$

- $0.4 + 0.12 + 0.09\theta + 0.09\theta^2 + 0.3(0.16 \dots)$

5b $E(S_n) = G_{S_n}'(1)$ $\mu_x = E(X) = G_X^{(1)}(1) = G_X^{(1)}(\theta)|_{\theta=1} =$

$\frac{d}{d\theta}(0.4 + 0.3\theta + 0.3\theta^2)|_{\theta=1} = 0.3 + 0.6\theta|_{\theta=1} = 0.3 + 0.6 = 0.9$ $\therefore$

$E(S_n) = \mu_x^n = 0.9^n$

\3 Sol/ observe $X \sim \text{Binomial}(n,p)$ with PGF $G_X(\theta) = (1-p+p\theta)^n$, $n=10$, $p=0.3$. For X have $E[X]$, var(2.1) & for $Y=6X$

$E(Y) = 6E(X) = 18$ & $var(Y) = 36 var(X) = 75.6$

using binomial expansion: $P(Y=24) = P(X=4) = \binom{10}{4}(0.7)^6(0.3)^4$

ie 4 successes & 6 failures

for 2 PGF: $G_Y(\theta) = E(\theta^Y) = E(\theta^{6X}) = E((\theta^6)^X) = (0.7+0.3(\theta^6))^{10}$ ∴

$G_X(\theta) = G_X(\theta^1) = (0.7+0.3\theta)^{10} = (0.7+0.3(\theta^1))^{10}$ ∴

$G_X(\theta^6) = (0.7+0.3(\theta^6))^{10}$

\3 retry/ $G_X(\theta) = G_X(\theta^1) = (0.7+0.3(\theta^1))^{10}$ ∴

$G_X(\theta^6) = (0.7+0.3\theta^6)^{10}$

$\cancel{E}$ $G_Y(\theta) = E[\theta^Y] = E(\theta^{6X}] = E((\theta^6)^X) = G_X(\theta^6) = (0.7+0.3\theta^6)^{10}$

∴ $\cancel{G_Y}$ $E(Y) = G_Y^{(1)}(1) = G_Y^{(1)}(\theta)\big|_{\theta=1} = \frac{d}{d\theta}(0.7+0.3\theta^6)^{10}\big|_{\theta=1} =$

$10(0.7+0.3\theta^6)^9(6 \cdot 0.3\theta^5)\big|_{\theta=1} = 10(0.7+0.3(1))^9(6 \cdot 0.3(1)) =$

$10(1)(1.8) = 18$ ✓

$G_Y^{(2)}(1) = G_Y^{(2)}(\theta)\big|_{\theta=1} = \frac{d}{dt}\left[10(0.7+0.3\theta^6)^9(1.8\theta^5)\right]\big|_{\theta=1} =$

$10 \cdot 9(0.7+0.3\theta^6)^8(1.8\theta^5) + 10(0.7+0.3\theta^6)^9(5 \cdot 1.8\theta^4)\big|_{\theta=1} =$

$90(0.7+0.3)^8 \cdot (1.8) + 10(1)^9(9) =$

$90 \cdot 1.8 + 90 = 252$ $= E(X(X-1)) = E(X^2) - E(X)$ ∴

$var(X) = E(X^2) - E(X)^2 = E(X^2) - E(X) + E(X) - E(X)^2 =$

$G_Y^{(2)}(1) + G_Y^{(1)}(1) - (G_Y^{(1)}(1))^2 = 252 + 18 - 18^2 = -54$ ✗ 75.6

$var(Y) = \cancel{E(6X)} = var(6X) = 36 var(X) =$ ∴

$G_X^{(1)}(\theta) = \frac{d}{d\theta}(0.7+0.3\theta)^{10} = 10(0.7+0.3\theta)^9(0.3)$ ∴

$G_X^{(2)}(\theta) = \frac{d}{d\theta}\left[10(0.7+0.3\theta)^9(0.3)\right] = 10 \cdot 9(0.7+0.3\theta)^8 0.3^2$ ∴

$E(X(X-1)) = E(X^2) - E(X) = G_X^{(2)}(1) = G_X^{(2)}(\theta)\big|_{\theta=1} = 10 \cdot 9(0.7+0.3)^8\, 0.9 = 81$ ∴

$var(X) = E(X^2) - (E(X))^2 = E(X^2) - E(X) + E(X) - (E(X))^2 =$ ∴

$E(X) = G_X^{(1)}(1) = G_X^{(1)}(\theta)\big|_{\theta=1} = 10(0.7+0.3)^9(0.3) = 3$ ∴

$var(X) = 81 + 3 - 3^2 = 75$

$P(Y=24) = P(6X=24) = P(X=4) = \binom{10}{4}(0.7)^6(0.3)^4 \approx 0.200\ (3sf)$ ✓

\Sheet 1/

\1b Sol/ $q = 1-p$: $G_X(\theta) = \frac{p}{q}\sum_{i=1}^{\infty}(q\theta)^i = \frac{p\theta}{1-q\theta}$ $\therefore E[X] = 1/p$ when $\theta = e^{t}$ & $var[X] = 2pq/(1-q)^2 = \frac{1}{p} - \frac{1}{p^2} = \frac{1-p}{p^2}$

\1c Sol/ $G_X(\theta) = 0.1 + 0.2\theta + 0.3\theta^2 + 0.4\theta^3$ $E[X] = G_X'(1) = 2$ &
$var[X] = G_X''(1) + G_X'(x) - (G_X'(1))^2 = 1$

\2 Sol/ X_1, X_2 have PGF $G_X(\theta) = 0.1 + 0.2\theta + 0.3\theta^2 + 0.4\theta^3$ by independ
$G_Y(\theta) = G_{X_1+X_2}(\theta) = (G_X(\theta))^2$ $\therefore E(Y) = E(X_1) + E(X_2) = 4$ $\therefore$ Sor

$P(Y=3)$ use multinomial expansion:

$P(Y=3)$ = Coeff. of θ^3 =

coeff of $(1\times\theta^3)$ + coeff of $(\theta\times\theta^2)$ =

$\binom{2}{1}(0.1)(0.4) + \binom{2}{1}(0.2)(0.3) = 0.3$

\3/ $Y = 6X = X+X+X+X+X+X$ $\therefore$ $G_Y(\theta) = G_{X+X+X+X+X+X} = ((G_X(\theta))^4 =$
$((0.7+0.3\theta)^{10})^4 = (0.7+0.3\theta)^{40}$ $\therefore$

$P(X=0) = 0.7$ $P(X=1) = 0.3$ $\therefore$

$P(Y=24)$ = Coeff of θ^{24}

Multinomial ~~noted~~ expansion: Coeff of $x_1^{l_1}x_2^{l_2}x_3^{l_3}\cdots x_k^{l_k}$ is
$\frac{n!}{l_1!\,l_2!\,l_3!\cdots l_k!}$ $l_1+l_2+l_3+\cdots+l_k = n$ $\therefore$

$x_1 = 0.7$ $x_2 = 0.3\theta$, $n = 40$ $\therefore$

$x_1^{l_1}x_2^{l_2} = 0.7^{l_1}(0.3\theta)^{l_2}$ $\therefore$

$Y = 6X$ $\therefore$ $P(Y=24) = P(6X=24) = P(X=4)$ $\therefore$

$P(X=4)$ use multinomial thm:

$P(X=4)$ = Coeff θ^4 = Coeff of $(1\times1\times1\times1\times1\times1\times\theta\times\theta\times\theta\times\theta)$

~~coeff of $(1\times\theta^4)+(1\times\theta\times\theta^3)+(1\times\theta^2\times\theta^2)$~~ =

$\binom{10}{1}(0.7)^6(0.3)^4$ ~~$+\binom{10}{1}$~~ $= 10\times(0.7)^6\times0.3^4 = 9.53\times10^{-3}$ (3S.F.) =
0.00953 X $\binom{10}{4}(0.7)^6(0.3)^4$

~~\3 Sol/~~ $E[Y] = G_Y^{(1)}(1) = G_Y^{(1)}(\theta)|_{\theta=1} = \frac{d}{d\theta}G_Y(\theta)|_{\theta=1} =$
$\frac{d}{d\theta}[(0.7+0.3\theta)^{40}]|_{\theta=1} = 40(0.7+0.3\theta)^{39}0.3|_{\theta=1} = 40(0.7+0.3(1))^{39}0.3 =$
$40(0.7+0.3)^{39}0.3 = 40(1)^{39}0.3 = 40(1)0.3 = 40\times0.3 = 12$ X

$\frac{(p)^2[-2p-p^2]-(-p)2(p)(-1+p)}{p^4} = \frac{-2p^3-p^4+2p^2(-1+p)}{p^4} = \frac{-2p^3-p^4-2p^2+2p^3}{p^4} =$

$\cancel{=}$ $\frac{-p^4-2p^2}{p^4}$ $\therefore$

$\cancel{E[}$ $var[\cancel{X^2}] = E[X^2]-E[X]^2 = \left[E[X^2]-E[X]\right] + E[X] - E[X]^2 =$

$\frac{-p^4-2p^2}{p^4} + \frac{1}{p} - \frac{1}{p^2} = -\frac{p^4}{p^4} - \frac{2p^2}{p^4} + \frac{1}{p} - \frac{1}{p^2} =$

$-1 - \frac{2}{p^2} + \frac{1}{p} - \frac{1}{p^2} = -1 - \frac{3}{p^2} + \frac{1}{p} = -\frac{p^2}{p^2} - \frac{3}{p^2} + \frac{p}{p^2} = \frac{-3-p-p^2}{p^2}$ X

$q = 1-p \therefore var[X] = (1-p)/p^2$

\1c/ $G_X(\theta) = \cancel{\theta\theta^2+P}\ 0.1\theta^0 + 0.2\theta^1 + 0.3\theta^2 + 0.4\theta^3 = 0.1 + 0.2\theta + 0.3\theta^2 + 0.4\theta^3$

$G_X^{(1)}(\theta)\big|_{\theta=1} = G_X^{(1)}(1) = E[X] = \frac{d}{d\theta}\left[0.1+0.2\theta+0.3\theta^2+0.4\theta^3\right]\Big|_{\theta=1} =$

$0.2 + 0.6\theta + 1.2\theta^2 \big|_{\theta=1} = 0.2+0.6+1.2 = 2$ ✓

$E[X(X-1)] = E[X^2] - E[X] = G_X^{(2)}(1) = G_X^{(2)}(\theta)\big|_{\theta=1} = \frac{d}{d\theta}\left[0.2+0.6\theta+1.2\theta^2\right]\Big|_{\theta=1} =$

$0.6 + 2.4\theta\big|_{\theta=1} = 0.6+2.4 = 3$ $\therefore$

$E[X^2] - E[X]^2 = var[X] = G_X^{(2)}(1) + G_X^{(1)}(1) - \left(G_X^{(1)}(1)\right)^2 =$

$\cancel{3+2}$ $3+2-2^2 = 5-4 = 1$ ✓

\2/ $Y = X_1 + X_2$ $\therefore G_Y(\theta) = G_{X_1}(\theta)\,G_{X_2}(\theta) =$

$(0.1+0.2\theta+0.3\theta^2+0.4\theta^3)^2$ $\therefore$

$E[Y] = G_Y^{(1)}(1) = G_Y^{(1)}(\theta)\big|_{\theta=1} = \frac{d}{d\theta}\left[(0.1+0.2\theta+0.3\theta^2+0.4\theta^3)^2\right]\Big|_{\theta=1} =$

$2(0.1+0.2\theta+0.3\theta^2+0.4\theta^3)(0.2+0.6\theta+1.2\theta^2)\big|_{\theta=1} =$

$2(0.1+0.2+0.3+0.4)(0.2+0.6+1.2) =$

$2(1)(2) = 4$ ✓

$P(Y=3)$ is coeff of θ^3 $\therefore$

$G_Y(\theta) = (0.1+0.2\theta+0.3\theta^2+0.4\theta^3)(0.1+0.2\theta+0.3\theta^2+0.4\theta^3)$ $\therefore$

$P(Y=3) = 0.1(0.4) + 0.2(0.3) + 0.3(0.2) + 0.4(0.1) =$

$0.04 + 0.06 + 0.06 + 0.04 = 0.1 + 0.1 = 0.2$ ✓

\1a sol/ $X \sim Poisson(\lambda)$ $\therefore G_X(\theta) = \sum_{n=0}^{\infty} \frac{e^{-\lambda}\lambda^n\theta^n}{n!} = e^{-\lambda}\sum_{n=0}^{\infty}\frac{(\lambda\theta)^n}{n!} =$

$e^{-\lambda}e^{\lambda\theta} = e^{\lambda(\theta-1)}$ $\therefore E[X] = var[X] = \lambda$

\Sheet 1/ $\frac{(1-1(1-p))p - 1p(-(1-p))}{(1-1(1-p))^2} = (1-$

\Sheet 2/ $\lambda = 3$

\1a/ $\lambda = 3$ $\quad P(X=x) = \frac{\lambda^x e^{-\lambda}}{x!}$

\1b/ $\lambda = 3$ $\quad P(N(t) =$ $\quad$ 7th event $n = 7$

$\frac{3(3x)^{7-1} e^{-\lambda x}}{6!} = \frac{3(9)^6 e^{-9}}{6!} = 0.2736 \qquad 1 - 0.732 = 0.7267$

$g_n(x) = \frac{\lambda(\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!}$

$P(S(3) < 7) \qquad 1 - P(6 \leq S(3))$

$g_n(x) = \frac{\lambda(\lambda x)^{n-1} e^{-\lambda x}}{(n-1)!} \qquad P(S(3) > 7) = 1 - P(S_n(7) > 6)$.

$g_n(3) = \frac{3(3 \cdot 3)^{6-1} e^{-3 \cdot 3}}{(6-1)!} = 0.1822 \qquad 1 - 0.1822 = 0.818 \ (358)$

$P(X \leq x) \qquad F \qquad \int_0^x f\,dt \qquad P(X = 3$

\1b/ Z number $N(t)$ of events in any time period t follows

$P(N(t) = k) = \frac{(\lambda t)^k e^{-\lambda t}}{k!} \quad \therefore \quad P(N(0) =$

\Sheet 1/ \1b/ $r \in \mathbb{N} \therefore G_X(\theta) = E(\theta^X) = \sum_{r=1}^{\infty} \theta^r P(X=r) = \sum_{r=1}^{\infty} \theta^r p(1-p)^{r-1} =$

$p \sum_{r=1}^{\infty} \theta^r (1-p)^r (1-p)^{-1} = \frac{p}{1-p} \sum_{r=1}^{\infty} (\theta(1-p))^r = \frac{p}{1-p} \frac{\theta(1-p)}{1-\theta(1-p)} = \frac{\theta p}{1-\theta(1-p)} \therefore$

$E[X] = G_X^{(1)}(1) = G_X^{(1)}(\theta)\big|_{\theta=1} = \frac{d}{d\theta}\left(\frac{\theta p}{1-\theta(1-p)}\right)\Big|_{\theta=1} = \frac{(1-\theta(1-p))p - \theta p(-(1-p))}{(1-\theta(1-p))^2}\Big|_{\theta=1} =$

$\frac{(1-(1-p))p - p(-(1-p))}{(1-(1-p))^2} = \frac{p(1-(1-p)+(1-p))}{\cancel{(1-p)}(1-1+p)^2} = \frac{p(1)}{p^2} = \frac{1}{p}$

$G_X^{(2)} = G_X^{(2)}(\theta)\big|_{\theta=1} = E[X(X-1)] = E[X^2] - E[X] = \frac{d}{d\theta}\left[\frac{(1-\theta(1-p))p - \theta p(-(1-p))}{(1-\theta(1-p))^2}\right]\Big|_{\theta=1}$

$\frac{d}{d\theta}\left[\frac{p(1-\theta+p)+\theta p(1-p)}{(1-\theta+\cancel{p\theta})^2}\right]\Big|_{\theta=1} = \frac{d}{d\theta}\left[\frac{p-\theta p+p^2-\theta p-\theta p^2}{(1-\theta+\theta p)^2}\right]\Big|_{\theta=1} =$

$\frac{d}{d\theta}\left[\frac{p-2\theta p+p^2-\theta p^2}{(1-\theta+\theta p)^2}\right]\Big|_{\theta=1} = \frac{(1-\theta+\theta p)^2[-2p-p^2] - (p-2\theta p+p^2-\theta p^2)\,2(1-\theta+\theta p)(-1+p)}{(1-\theta+\theta p)^4}\Big|_{\theta=1}$

$= \frac{(1-1+1p)^2[-2p-p^2] - (p-2p+p^2-p^2)\,2(1-1+p)(-1+p)}{(1-1+1p)^4} =$

$$G_X(\theta) = E[\theta^X] = \sum_{k=0}^{\infty} P(X=k) = \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} e^{-\lambda} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^k}{k!} = e^{-\lambda} e^{\lambda} =$$

$$e^{-\lambda+\lambda} = e^0 = 1 \quad X$$

$$G_X(\theta) = E[\theta^X] = \sum_{k=0}^{\infty} \theta^k P(X=k) = \sum_{k=0}^{\infty} \theta^k \frac{\lambda^k}{k!} e^{-\lambda} = \cancel{e^{-\lambda} \sum_{k=0}^{\infty} \theta^k e}$$

$$e^{-\lambda} \sum_{k=0}^{\infty} \theta^k \frac{\lambda^k}{k!} = e^{-\lambda} \sum_{k=0}^{\infty} \frac{(\theta\lambda)^k}{k!} = e^{-\lambda} e^{\theta\lambda} = e^{\theta\lambda - \lambda} = e^{\lambda(\theta-1)}$$

$$\mathrm{var}(X) = \lambda \qquad E(X) = \lambda$$

$$\cancel{\text{Var}} \quad G_X^{(1)}(\theta)\Big|_{\theta=1} = \frac{d' G_X}{d\theta'}\Big|_{\theta=1} = E[(X-1+1)]\Big|_{\theta=1} = E[(X)]\Big|_{\theta=1} = E[X]\Big|_{\theta=1} = \frac{d}{d\theta}\left[e^{\lambda(\theta-1)}\right]\Big|_{\theta=1} =$$

$$e^{\lambda(\theta-1)} \frac{d}{d\theta}[\lambda(\theta-1)]\Big|_{\theta=1} = e^{\lambda(\theta-1)} \cdot \lambda\Big|_{\theta=1} = \lambda e^{\lambda(\theta-1)}\Big|_{\theta=1} = \lambda e^{\lambda(1-1)} = \lambda e^0 = \lambda$$

$$\cancel{E[X^2] = E \frac{d^r G_X}{}} \quad \cancel{G} \quad G_X^{(2)}(\theta) = \frac{d^2 G_X(\theta)}{d\theta^2} = \frac{d}{d\theta}\left[\lambda e^{\lambda(\theta-1)}\right] = \left[\lambda^2 e^{\lambda(\theta-1)}\right].$$

$$\therefore G_X^{(2)}(1) = E[(X-(2-1)+1)(X-2+1)] = E[(X+1)(X-1)]$$

$$E[X^2] = g'' + g' \qquad G_X^{(2)}(1) = E[(X-2+1)(X-1+1)] = E[(X-1)(X)]$$

$$G_X^{(2)}(1) = E[X(X-1)] = E(X^2 - X) = E(X^2) - E(X)$$

$$\therefore E(X^2) - E(X) + E(X) = G_X^{(2)}(1) + G_X^{(1)}(1)$$

$$E(X^2) - E(X)^2 = E(X^2) - (G_X^{(1)}(1))^2$$

$$G_X(\theta) = e^{\lambda(\theta-1)} \quad \therefore G_X^{(2)}(\theta) = \frac{d^2 G_X(\theta)}{d\theta^2} = \frac{d}{d\theta}\left[\frac{d}{d\theta}\left(e^{\lambda(\theta-1)}\right)\right] = \frac{d}{d\theta}\left(\lambda e^{\lambda(\theta-1)}\right) =$$

$$\lambda^2 e^{\lambda(\theta-1)} \quad \therefore G_X^{(2)}(1) = G_X^{(2)}(\theta)\Big|_{\theta=1} = \lambda^2 e^{\lambda(1-1)} = \lambda^2 e^{\lambda(0)} = \lambda^2 e^0 = \lambda^2$$

$$\cancel{\therefore E(X^2)} \quad \therefore G_X^{(1)}(1) = G_X^{(1)}(\theta)\Big|_{\theta=1} = \lambda e^{\lambda(1-1)} = \lambda \therefore$$

$$\cancel{E(X^2)} = E(X) = G_X^{(1)}(1) = \lambda$$

$$E(X^2) = E(X^2 - X) + E(X) = G_X^{(2)}(1) + G_X^{(1)}(1) = \lambda^2 + \lambda \therefore$$

$$\mathrm{var}(X) = E(X^2) - E(X)^2 = \lambda^2 + \lambda - \lambda^2 = \lambda$$

\1b/ $G_X(\theta) = E(\theta^X] = \sum_{k=0}^{\infty} \theta^k P(X=k) = \sum_{k=0}^{\infty} \theta^k p(1-p)^{k-1} = p \sum_{k=0}^{\infty} \theta^k (1-p)^{k-1} =$

$$p \sum_{k=0}^{\infty} \theta^k (1-p)^k (1-p)^{-1} = \frac{p}{1-p} \sum_{k=0}^{\infty} \theta^k (1-p)^k = \frac{p}{1-p} \sum_{k=0}^{\infty} (\theta(1-p))^k = \qquad \left\{ \sum_{k=0}^{\infty} x^k = \frac{1}{1-x} \right\}$$

$$\frac{p}{1-p} \frac{1}{1-\theta(1-p)} = \cancel{p} \quad X$$

\1b redo/ $r \in \mathbb{N} \therefore G_X(\theta) = E(\theta^X) = \sum_{r=1}^{\infty} \theta^r P(X=r) = \sum_{r=1}^{\infty} \theta^r p(1-p)^{r-1} =$

$$p \sum_{r=1}^{\infty} \theta^r (1-p)^r (1-p)^{-1} = \frac{p}{1-p} \sum_{r=1}^{\infty} (\theta(1-p))^r = \frac{p}{1-p} \frac{\theta(1-p)}{1-\theta(1-p)} = \frac{\theta p}{1-\theta(1-p)}$$

$$E(X) = G_X^{(1)}(1) = G_X^{(1)}(\theta)\Big|_{\theta=1} = \frac{d}{d\theta}\left(\frac{\theta p}{1-\theta(1-p)}\right)\Big|_{\theta=1} = \frac{(1-\theta(1-p))p - \theta p(-(1-p))}{(1-\theta(1-p))^2}\Big|_{\theta=1} =$$

\Sheet 0/ coeffs of x^2: $e(1+\frac{1}{2}) = \frac{3}{2}e$

of x^3: $e(1+\frac{1}{2}\times 2+\frac{1}{6}) = \frac{13}{6}e$

\5/ use comparison or ratio tests as appropriate & conclude (i) converges, (ii) diverges, (iii) converges (can be seen by subbing $x=e$ that the limit is $e^e - 1$) (iv) converges

\6a/ for μ, σ^2 : $\exp(tx)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right) = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\exp\left(-\frac{(x-\mu')^2}{2\sigma^2}\right)$ for some μ' (Ex: calc μ') Now for $Y \sim N(\mu, \sigma^2)$ $\therefore$

$M_Y(t) = E[e^{tY}] = \int_{-\infty}^{\infty} \exp(tx)\exp\left(-\frac{(x-\mu)^2}{2\sigma^2}\right)dx =$

$\exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)\int_{-\infty}^{\infty}\exp\left(-\frac{(x-\mu')^2}{2\sigma^2}\right)dx = \exp\left(\mu t + \frac{\sigma^2 t^2}{2}\right)$ $\because X_x$ are indep,

$M_X(t) = M_{X_1}(t)M_{X_2}(t) = \exp\left(\frac{t^2}{2}\right)\exp(t^2) = \exp\left(\frac{3t^2}{2}\right)$ & $E[X] = M_X'(0) = 0$

$\left\{\int_{-\infty}^{\infty}\exp\left(-\frac{(x-\mu')^2}{2\sigma^2}\right)dx = 1 \because\right.$ its a normal distri $\left.N(\mu', \sigma^2)\right\}$

\6b/ Set $q = 1-p$ $\therefore M_X(t) = E[e^{tX}] = \sum_{r=1}^{\infty} e^{rt}pq^{r-1} =$

$\frac{p}{q}\sum_{r=1}^{\infty}(qe^t)^r = pe^t(1-qe^t)^{-1}$ $\therefore$ $M_X(0) = p/(1-q) = p(1-1+p) = 1$

$\therefore E[X] = M_X'(0) = (pe^t/(1-qe^t)^2\big|_{t=0} = 1/p$

\6c/ for $Y \sim \text{Uniform}[0, a]$ $\therefore M_Y(t) = \frac{e^{at}-1}{at}$ (does this converge at $t=0$?) $\therefore$ using independence: $M_X(t) = M_{X_1}(t)M_{X_2}(t) =$

$\left(\frac{e^t-1}{t}\right)\left(\frac{e^{2t}-1}{2t}\right) = \frac{1}{2t^2}\left(t+\frac{t^2}{2}+\frac{t^3}{3!}+\dots\right)\left(2t+\frac{(2t)^2}{2}+\frac{(2t)^3}{3!}+\dots\right) =$

$\frac{1}{2}\left(1+\frac{t}{2}+\frac{t^2}{3!}+\dots\right)\left(2+2t+\frac{2(2t)^2}{3!}+\dots\right)$ $\therefore E[X] = \frac{3}{2}$ as Σ coeffs of t & $E[X^2] = \frac{8}{3}$ as Σ coeffs of $\frac{t^2}{2}$

\Sheet 1/ \1a/ $X \sim \text{Poi}(\lambda)$ $M_X(t) = E[e^{tX}] =$

~~$\sum_i e^{tx_i} p(X=x_i$~~ ~~$\int_{\Omega_X} e^{tx} f_X(x)\,dx = \int_{\Omega_X} e^{tx}\frac{\lambda^x}{x!}e^{-\lambda}\,dx$~~ $\sum_i e^{tx_i}p(X=x_i) =$

$\sum_i e^{tx_i}\frac{\lambda^x}{x!}e^{-\lambda} = \sum_{x=0}^{\infty} e^{tx}\frac{\lambda^x e^{-\lambda}}{x!} = \sum_{x=0}^{\infty}\frac{(e^t\lambda)^x e^{-\lambda}}{x!} = e^{-\lambda}\sum_{x=0}^{\infty}\frac{(e^t\lambda)^x}{x!}$

$\cancel{= e^{-\lambda}}$ $\{e^t\lambda = y\}$ $= e^{-\lambda}\sum_{x=0}^{\infty}\frac{y^x}{x!} = e^{-\lambda}e^y = e^{-\lambda}e^{e^t\lambda} = e^{e^t\lambda-\lambda} = e^{\lambda(e^t-1)}$

Sheet 0/ 1a/ if A is an event & A & A^c are indep if $P(A) \in \{0,1\}$ this can be seen noting $P(A \cap A^c) = 0$, whereas $P(A)P(C) = 0$ iff $P(C)$ or $P(A) = 1$

1b/ for indep A, B: $P(A^c \cap B^c) = P((A \cup B)^c) = 1 - P(A \cup B) =$
$1 - (P(A) + P(B) - P(A \cap B)) = 1 - P(A) - P(B) + P(A \cap B) =$
$1 - P(A) - P(B) + P(A)P(B) = (1 - P(A))(1 - P(B)) = P(A^c)P(B^c)$ $\therefore A^c$ & B^c are indep

2a/ in general eqn does not hold: $P(A|B \cup C) = \frac{P(A \cap (B \cup C))}{P(B \cup C)} =$
$\frac{P(A \cap B) + P(A \cap C)}{P(B) + P(C)} \neq \frac{P(A \cap B)}{P(B)} + \frac{P(A \cap C)}{P(C)} = P(A|B) + P(A|C)$

2b/ for A, B disjoint $P(A \cup B | C) = \frac{P((A \cup B) \cap C)}{P(C)} = \frac{P(A \cap C) + P(B \cap C)}{P(C)} =$
$\frac{P(A \cap C)}{P(C)} + \frac{P(B \cap C)}{P(C)} = P(A|C) + P(B|C)$

3i/ $\sum_{n=1}^{\infty} (2x)^n = \frac{2x}{1-2x}$ converges for $|x| < \frac{1}{2}$

3ii/ $\sum_{n=1}^{\infty} nx^n = \frac{x}{(1-x)^2}$ converges for $|x| < 1$

3iii/ $\sum_{n=1}^{\infty} n^2 x^n = \frac{x(x+1)}{(1-x)^3}$ converges for $|x| < 1$

3iv/ $\sum_{n=1}^{\infty} \frac{x^n}{n} = -\log(1-x)$ converges for $|x| < 1$

3v/ $\sum_{n=1}^{\infty} \frac{x^n}{n!} = e^x - 1$ converges for all $x \in \mathbb{R}$

4i/ $(1+2x)^4$ Binomial thm $\Rightarrow$ coeff of x^2: 24, x^3: 32

4ii/ $x(2-x)^{-1}$ taylor series expansion $\Rightarrow$ coeff of x^2: $1/4$, x^3: $1/8$

4iii/ e^{x^2} taylor series expansion $\Rightarrow$ coeff of x^2: 1, x^3: 0

4iv/ $(1+x+x^2)^4$ multinomial thm $\Rightarrow$ coeff of x^2: 10, coeff of x^3: 16

4v/ $(1+x+x^3+x^4)^5$: coeff of x^2: $\underbrace{\binom{5}{3}}_{\text{terms } 1 \times 1 \times 1 \times x \times x} = \frac{5!}{3!2!}$

coeff of x^3: $\underbrace{\binom{5}{3}}_{\text{terms } 1 \cdot 1 \cdot x \cdot x \cdot x} + \underbrace{\binom{5}{1}}_{\text{terms } 1 \cdot 1 \cdot 1 \cdot 1 \cdot x^3} = \frac{5!}{3!2!} + \frac{5!}{4!1!}$

4vi/ $\exp((1-x)^{-1})$ expand & simplify: $\exp((1-x)^{-1}) = \exp(1+x+x^2+\dots) =$
$(e)\exp(x+x^2+\dots) = (e)\exp(x(1+x+x^2+\dots)) =$
$(e)(1 + x\underbrace{(1+x+x^2+\dots)}_{(1-x)^4} + \frac{x^2}{2!}(1+x+\dots)^2 + \frac{x^3}{3!}(1+\dots)^3 + \dots)$ $\therefore$

\Sheet 0 lecture/ \3d/ $\frac{d}{dx}\frac{x^n}{n} = \frac{nx^{n-1}}{n} = x^{n-1}$ $\therefore$

$$\frac{d}{dx}\sum_{n=1}^{\infty}\frac{x^n}{n} = \sum_{n=1}^{\infty} x^{n-1} = \sum_{n=1}^{\infty}\frac{1}{x}x^n = \frac{1}{x}\sum_{n=1}^{\infty}x^n = \frac{1}{x}\frac{x}{1-x} = \frac{1}{1-x}$$

$$\frac{d}{dx}\sum_{n=1}^{\infty}\frac{x^n}{n} = \sum_{n=1}^{\infty}\frac{nx^{n-1}}{n} = \sum_{n=1}^{\infty}x^{n-1} = \sum_{n=1}^{\infty}x^{-1}x^n = x^{-1}\sum_{n=1}^{\infty}x^n = x^{-1}\frac{x}{1-x} = \frac{1}{1-x}$$

\3e/ $\frac{d}{dx}\frac{x^n}{n!} = \frac{nx^{n-1}}{n!} = \frac{nx^{n-1}}{n(n-1)!} = \frac{x^{n-1}}{(n-1)!}$ $\therefore$

$$\frac{d}{dx}\sum_{n=1}^{\infty}\frac{x^n}{n!} = \sum_{n=1}^{\infty}\frac{x^{n-1}}{(n-1)!} = \frac{x^0}{1} + \sum_{n=2}^{\infty}\frac{x^{n-1}}{(n-1)!} = \cancel{1+\sum_{n=1}^{\infty}\frac{x^n}{(n-1)!}}\ 1 + \sum_{n=1}^{\infty}\frac{x}{n!}$$ $\therefore$

$$\frac{d}{dx}f(x) = 1 + f(x) = 1 + y = \frac{dy}{dx}$$ $\therefore$

$$y - \frac{dy}{dx} = -1 \therefore h(x)y - h(x)\frac{dy}{dx} = -1\,h(x)$$ $\therefore$

$$\cancel{\frac{d}{dx}h(x) = -h(x)}\quad h(x) = h(x)\frac{dy}{dx} - h(x)y$$ $\therefore$

$$\frac{d}{dx}(+h(x)) = -h(x) \quad \therefore \frac{h'(x)}{h(x)} = -1 \quad \therefore \ln(h(x)) = -x \therefore h(x) = e^{-x}$$

$$\therefore h(x) = \frac{d}{dx}(h(x)y) = \cancel{h} \; e^{-x} = \frac{d}{dx}(e^{-x}y) \therefore -e^{-x} = e^{-x}y \therefore y = -1$$

$$\therefore \sum_{n=1}^{\infty}\frac{x}{n!} = -1$$

\4a/ $1 + 4\cdot1^3\cdot2x + 6\cdot1^2\cdot(2x)^2 + 4\cdot1\cdot(2x)^3 + 1\cdot(2x)^4$ $\therefore$

$6\cdot4x^2 = 24x^2 \therefore 24$ & $x^3(4\cdot8) = 32x^3 \therefore 32$

\4b/ $1\cdot x^2 = x^2$ $0x^3$

\4c/ $(1+x+x^2)^4 = ((1+x)+x^2)^4 = 1\cdot(1+x)^4 + \cdots$

\5a/ $\lim_{n\to\infty}\frac{1}{n^3} = 0$ $\sum_{n=1}^{\infty}\frac{1}{n^2} \in \mathbb{R}$ & $\cancel{\Sigma}\ \frac{1}{n^2} > \frac{1}{n^3} \therefore \sum_{n=1}^{\infty}\frac{1}{n^3}$ Converges

~~5a~~ 5b/ $$\lim_{n\to\infty}\frac{n^2+1}{n^3+6} = \frac{n^2(1+\frac{1}{n^2})}{n^3(1+\frac{6}{n^3})} = \frac{(1+\frac{1}{n^2})}{n+\frac{6}{n^2}} \therefore \lim_{n\to\infty}\frac{(1+\frac{1}{n^2})}{n+\frac{6}{n^2}} = \frac{\lim_{n\to\infty}1+0}{\cancel{\infty}+0} = 0$$

$$\sum_{n=1}^{\infty}\frac{n^2+1}{n^3+6} = \sum_{n=1}^{\infty}\frac{n^2}{n^3+6} + \sum_{n=1}^{\infty}\frac{1}{n^3+6} \qquad \sum_{n=1}^{\infty}\frac{n^2}{n^2(n+\frac{6}{n^2})} = \sum_{n=1}^{\infty}\frac{1}{n+\frac{6}{n^2}}$$

$$\sum_{n=1}^{\infty}\frac{1}{n^3+6} < \sum_{n=1}^{\infty}\frac{1}{n^3}$$

\5c/ $$\lim_{n\to\infty}\frac{e^n}{n!} = \lim_{n\to\infty}\frac{\frac{d}{dn}e^n}{\frac{d}{dn}n!} = \frac{e^n}{1} = e^n \cancel{\neq 0} \therefore \lim_{n\to\infty}e^n = 0$$

\5d/ $$\lim_{n\to\infty}\frac{n!}{n^n} = \lim_{n\to\infty}\frac{n!}{e^{\ln(n^n)}} = \lim_{n\to\infty}\frac{n!}{e^{n\ln n}}$$

\6a/ $X_1 \sim N(0,1)$ $X_2 \sim N(0,2)$ $M_X(t) = E[e^{tX}] = E[e^{t(X_1+X_2)}] =$

$E[X] = 0$ ~~E[X]~~ $\mathrm{var}[X] = 1+2 = 3 \therefore E[e^{tX}] = \therefore E[tx] = tE[x] = t(0) = 0$

$\therefore E[e^{tx}] = \exp(E[tx]) = \exp(0) = 1$

\1a/

Independence: $P(A \cap A^c) = P(A)P(A^c)$ is independent

is independent $P(A \cap A^c) = 1$ but $P(A)P(A^c) \neq 1$ $\therefore$ not independent

is $P(A) = 1$ $\therefore$ $P(A^c) = 0$ $\therefore$ $P(A)P(A^c) = 0 \neq 1$ is $P(A) < 1$ $\therefore$

$P(A)P(A^c) < 1 \neq 1$

\1b/ $P(A^c \cap B^c) = 1 - P(A \cup B) = 1 - (P(A) + P(B) - P(A \cap B)) =$

$1 - P(A) - P(B) + P(A \cap B) = 1 - P(A) - P(B) + P(A)P(B) = (P(A) - 1)(P(B) - 1) =$

$(-1)(-1)(1 - P(A))(1 - P(B)) = (1 - P(A))(1 - P(B)) = P(A^c)P(B^c)$ $\therefore$ independent

B, C disjoint $\therefore P(B \cap C) = 0$

\2a/ $P(A \mid B \cup C) = \dfrac{P(B \cup C \mid A)P(A)}{P(B \cup C)} = \dfrac{P(A \cap (B \cup C))}{P(B \cup C)}$

$= [P(A) + P(B \cup C) - P(A \cup (B \cup C))] / P(B \cup C) =$

$[P(A) + P(B \cup C) - P(A \cup B \cup C)] / P(B \cup C) = \dfrac{[P(A) + P(B) + P(C) - P(B \cap C) - P(A \cup B \cup C)]}{P(B \cup C)} =$

$[P(A) + P(B) + P(C) - P(\cancel{A \cup B \cup C})] / P(B \cup C) =$

$[P(A) + P(B) + P(C) - [P(A) + P(B) + P(C) - P(A \cap C) - P(B \cap C) - P(A \cap B) + P(A \cap B \cap C)]] / P(B \cup C)$

$= [P(A \cap C) + P(B \cap C) + P(A \cap B) - P(A \cap B \cap C)] / P(B \cup C) =$ $\{P(B \cap C) = 0 = P(A \cap B \cap C)\}$

$[P(A \cap C) + P(A \cap B)] / P(B \cup C)$

$= \dfrac{P(A \cap C) + P(A \cap B)}{P(B) + P(C) - P(B \cap C)} = \dfrac{P(A \cap C) + P(A \cap B)}{P(B) + P(C)} = \dfrac{P(A \cap C)}{P(B) + P(C)} + \dfrac{P(A \cap B)}{P(B) + P(C)}$

\2b/ $P(A \cup B \mid C) = \dfrac{P(C \mid A \cup B)P(A \cup B)}{P(C)} =$ $P(A \cap B) = 0$

$P(C \mid A \cup B)[P(A) + P(B) - P(A \cap B)] / P(C) = P(C \mid A \cup B)[P(A) + P(B)] / P(C) =$

$\dfrac{P(C \mid A \cup B)P(A)}{P(C)} + \dfrac{P(C \mid A \cup B)P(B)}{P(C)} = \dfrac{P(A \cup B \mid C)P(C)P(A)}{P(A \cup B)P(C)} + \dfrac{P(A \cup B \mid C)P(C)P(B)}{P(A \cup B)P(C)}$

\3b/ $\displaystyle\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ $(2x)^n = 2^n x^n$ $\therefore$ $\dfrac{d}{dx}(2^n x^n) = 2^n n x^{n-1}$

$\displaystyle\sum_{n=1}^{\infty} a^n = \frac{1}{1-a} - 1 = \frac{1}{1-a} - \frac{1-a}{1-a} = \frac{a}{1-a}$ $\displaystyle\sum_{n=1}^{\infty} (2x)^n = \frac{2x}{1-2x}$

$\dfrac{d}{dx}((2x)^n) = n(2x)^{n-1} 2 = 2n(2x)^{n-1} = 2n(2^{n-1})x^{n-1} = 2^n n x^{n-1}$ $\therefore$

$\displaystyle\sum_{n=1}^{\infty} x^n = \frac{x}{1-x}$ $\therefore$ $\displaystyle\frac{d}{dx}\sum_{n=1}^{\infty} x^n = \sum_{n=1}^{\infty} n x^{n-1} = \frac{d}{dx}\frac{x}{1-x} = \frac{(1-x)1 - x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$ $\therefore$

$\displaystyle\sum_{n=1}^{\infty} n x^n = x\sum_{n=1}^{\infty} n x^{n-1} = x\frac{1}{(1-x)^2} = \frac{x}{(1-x)^2}$

$\displaystyle\frac{d}{dx}\sum_{n=1}^{\infty} n x^n = \sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{d}{dx}\frac{x}{(1-x)^2} = \frac{(1-x)^2 1 - x(2)(1-x)(-1)}{(1-x)^4} = \frac{(1-x)^2 + 2x - 2x^2}{(1-x)^4} =$

$\displaystyle\frac{1 + x^2 - 2x + 2x - 2x^2}{(1-x)^4} = \frac{1-x^2}{(1-x)^4} = \frac{(1-x)(1+x)}{(1-x)^4} = \frac{1+x}{(1-x)^3}$ $\therefore$ $\displaystyle\sum_{n=1}^{\infty} n^2 x^n = x\sum_{n=1}^{\infty} n^2 x^{n-1} = x\frac{1+x}{(1-x)^3} = \frac{x+x^2}{(1-x)^3}$

6b $E[X] = \sum_{r=1}^{\infty} r P(X=r) = \sum_{r=1}^{\infty} r p(1-p)^{r-1} = \frac{1}{p}$ $M_X(t) = E[e^{tX}]$ [illegible]

$var(X) = \frac{1-p}{p^2}$ $M_X(0) = M_X(t=0) = E[e^{0X}] = E[e^0] = E[1] = 1$

6c $E[X_1] = \frac{0+1}{2} = \frac{1}{2}$ $E[X_2] = \frac{0+2}{2} = 1$ $var(X_2) = \frac{(2-0)^2}{12}$

$\frac{(1-0)^2}{12} = \frac{1}{12} = var[X_1]$ $var[X_2] = \frac{(2-0)^2}{12} = \frac{4}{12}$ [illegible]

$E[X] = E[X_1] + E[X_2] = \frac{1}{2} + 1 = \frac{3}{2}$ $M_X(t) = E[e^{tX}]$ [illegible]

MTH3024 Stochastic Processes

CW1 10% due 4/3/2022 CW2 10% due 1/4/2022

~~office~~ hours: (LSIT03.13, week 1-5, Laver 611, week 6-11) · Fri 13:35–14:35

\Sheet 1 Lecture/ \1a/ As $x \in A \Rightarrow x \notin A^c$ ∴ A & A^c are indep \1b/ $P(x \in A^c) = 1 - P(x \in A)$ ∴ $P(x \in B^c) = 1 - P(x \in B)$ ∴ and $P(x \in A)$ is indep of $P(x \in B)$ ∴ A^c & B^c are indep also

\2a/ ~~P(A)~~ $P(x \in (B \cap C)) = 0$ ∴ $P(x \mid (B \cap C)) = 0$ ∴ $P(A \mid (B \cap C)) = 0$ ∴ $P(A \mid B \cup C) = P((A \mid B) \cup (A \mid C)) = P(A \mid B) + P(A \mid C)$

$P(A \cap B \mid C) = P(A \cap B) = 0$ ∴ $P(A \cup B \mid C) = P((A \mid C) \cup (B \mid C)) = P(A \mid C) + P(B \mid C)$

\3/ $\underbrace{\sum_{n=1}^{\infty} (2x)^n}_{a = 2x \quad \sum_{n=1}^{\infty} a^n} = \sum_{n=1}^{\infty} 2^n x^n$ ∴ if $a = 2x$ ∴ $\sum_{n=1}^{\infty} (2x)^n = \sum_{n=1}^{\infty} a^n = \left(\sum_{n=0}^{\infty} a^n\right) - a^0$

$= \left(\sum_{n=0}^{\infty} a^n\right) - 1 = \frac{1}{1-a} - 1 = \frac{1}{1-a} - \frac{1-a}{1-a} =$

$\frac{1 - 1 + a}{1-a} = \frac{a}{1-a} = \frac{2x}{1-2x}$

$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a}$ ∴ $\sum_{n=1}^{\infty} n x^n = \sum_{n=1}^{\infty} \left((n^{\frac{1}{n}})^n x^n\right) = \sum_{n=1}^{\infty} (n^{\frac{1}{n}} x)^n$

∴ if $a = n^{\frac{1}{n}} x$: $= \sum_{n=1}^{\infty} a^n = \frac{a}{1-a} = \frac{n^{\frac{1}{n}} x}{1 - n^{\frac{1}{n}} x}$

$\sum_{n=1}^{\infty} n^2 x^n = \sum_{n=1}^{\infty} ((n^2)^{\frac{1}{n}})^n x^n = \sum_{n=1}^{\infty} (n^{\frac{2}{n}})^n x^n = \sum_{n=1}^{\infty} (n^{\frac{2}{n}} x)^n$

∴ if $a = n^{\frac{2}{n}} x$: $= \sum_{n=1}^{\infty} a^n = \frac{a}{1-a} = \frac{n^{\frac{2}{n}} x}{1 - n^{\frac{2}{n}} x}$

$\sum_{n=1}^{\infty} \frac{x^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n} x^n = \sum_{n=1}^{\infty} \left(\left(\frac{1}{n}\right)^{\frac{1}{n}}\right)^n x^n = \sum_{n=1}^{\infty} \left((n^{-1})^{\frac{1}{n}}\right)^n x^n = \sum_{n=1}^{\infty} (n^{-\frac{1}{n}} x)^n$

∴ if $a = n^{-\frac{1}{n}} x$: $= \sum_{n=1}^{\infty} a^n = \frac{a}{1-a} = \frac{n^{-\frac{1}{n}} x}{1 - n^{-\frac{1}{n}} x}$

$\frac{d}{dx} \sum_{n=1}^{\infty} (2x)^n = \frac{d}{dx} \sum_{n=1}^{\infty} 2^n x^n = \sum_{n=1}^{\infty} 2^n x^{n-1}$

$\frac{d}{dx} x^n = n x^{n-1}$ ∴ $\sum_{n=1}^{\infty} (1x)^n = \frac{x}{1-x}$ ∴ ~~$\sum_{n=1}^{\infty} x^{n-1}$~~

$\frac{d}{dx}\left[\sum_{n=1}^{\infty} x^n\right] = \sum_{n=1}^{\infty} n x^{n-1} = \frac{d}{dx}\left[\frac{x}{1-x}\right] = \frac{(1-x)1 - x(-1)}{(1-x)^2} = \frac{1}{(1-x)^2}$ ∴

$\sum_{n=1}^{\infty} n x^n = x \sum_{n=1}^{\infty} n x^{n-1} = x \frac{1}{(1-x)^2} = \frac{x}{(1-x)^2}$

$\sum_{n=1}^{\infty} n^2 x^n$ $\quad \frac{d}{dx} n^2 x^n = n^3 x^{n-1}$

\3b/ P(next plain is Ryanair) $= \frac{12}{10+12} = 0.54 = \frac{6}{11}$

\3b sol/ Let T_E, T_R be Z respective times of arrival; these are exponentially ~~with~~ distributed with $\lambda_E = 10$ & $\lambda_R = 12$ respectively. to calc Z probab $P(T_R < T_E)$ we can integrate Z joint density $f_{T_E, T_R}(x, y) = (10e^{-10x})(12e^{-12y})$ for $x, y \geq 0$ now

$$P(T_R < T_E) = \int_0^\infty \int_0^x f_{T_E, T_R}(x,y)\,dx\,dy = \int_0^\infty \int_0^x (10e^{-10x})(12e^{-12y})\,dy\,dx = \frac{6}{11}$$

More generally, have for $X \sim Exp(\lambda)$ $Y \sim Exp(\mu)$ that

$P(T_X < T_Y) = \frac{\lambda}{\lambda+\mu}$; $P(T_Y < T_X) = \frac{\mu}{\lambda+\mu}$

\4/ $\lambda = 6$

$i = 1, 2, \ldots, 10$ $\quad \lambda_i = \frac{6}{10}$

$E[T^1] = \sum \frac{1}{(6/10)} = \frac{10}{6}$ Mins $= 100$ seconds

$E[T^1]E[T^2] \cdots E[T^{10}] = \left(\frac{10}{6}\right)^{10}$ Mins $= 100^{10} = 1000$ seconds

$\therefore \lambda_i = \frac{\lambda}{N}$ $\therefore E[T^i] = \frac{1}{\lambda_i} = \frac{1}{(\lambda/N)} = \frac{N}{\lambda}$ $\therefore$

$E[T^1]E[T^2] \cdots E(T^N) = \left(\frac{N}{\lambda}\right)^N = \frac{N^N}{\lambda^N}$

\4 sol/ known as Z matching problem: Z overall rate of arrivals is $\lambda = 6$ per min. Z expected arrival of Z first bird (irrespective of species) is $E[T^{(10)}) = \frac{1}{6}$. once Z first bird has passed, there are 9 species left. Splitting yields that Z remaining bird species arrive at a rate of $\lambda^{(9)} = \frac{9}{10} \times 6$ mins $\therefore$ Z mean time to see Z next bird of any of Z remaining 9 species is $E[T^{(9)}) = \frac{1}{\lambda^{(9)}} = \frac{10}{9 \times 6}$ iterate this process to obtain Z ~~total~~ expected time $\frac{10}{6}\left(\frac{1}{10} + \frac{1}{9} + \frac{1}{8} + \cdots + \frac{1}{2} + 1\right) \approx 4.88$

in general Z answer - obtained in Z same way is total expected time $= \frac{N}{\lambda} \sum_{k=1}^{N} \frac{1}{k} \sim \frac{N\log(N)}{\lambda}$ for large N

\4/ $\lambda = 6$ $\quad E(T^{(10)}) = \frac{10}{10} \times \frac{1}{6} = \frac{1}{6}$

~~$E(T^{(9)})$~~ $\quad \lambda^{(9)} = \frac{9}{10} \times 6 = \therefore E(T^{(9)}) = \frac{1}{\lambda^{(9)}} = \frac{1}{\left(\frac{9}{10} \times 6\right)} = \frac{10}{9 \times 6}$ $\therefore$

$\sum_{n=1}^{10} E(T^{(n)}) = \frac{10}{6}\left(\frac{1}{10} + \frac{1}{9} + \frac{1}{8} + \cdots + \frac{1}{2} + 1\right) \approx 4.88$ $\therefore$

$P(X = 5) = (3)(3)(3) \cdots$

$-(\lambda_n + \mu_n)t$

\Sheet 2/ \2a/ $\lambda = 5$

1 $\lambda_b = 2.5$, $\lambda_r = 1.5$ $\lambda_w = 1$

Probab white car is 1 or more ∴

~~P(~~ $P(W(t=0)=1) = \frac{e^{-1\cdot 0}(1\cdot 0)^1}{1!} = \frac{e^0 \, 0^1}{1!} = 0$

$P(W(t=1)=1) = \frac{e^{-1\cdot 1}(1\cdot 1)^1}{1!} = e^{-1}$

$P(W(t=1)=0) = \frac{e^{-1\cdot 1}(1\cdot 1)^0}{0!} = e^{-1}$

\2 ~~a~~/ have independence of events in the future from events in the past. ∴ $P(\text{next car white}) = 0.2$

\2b/ $\lambda_r = 1.5$ ~~pt~~ expected time between two red cars ∴

$5 \times 0.3 = 1.5$ mins ✗ 40 seconds

\2b Sol/ red cars arrive at a rate of $\lambda = 5 \times 0.3 = 1.5$ per min

∴ the expected time between red cars $E[T^{red}] = \frac{1}{\lambda} = 40$ seconds

\2b/ 1.5 mins $= \lambda_r$ ∴ $\frac{1}{\lambda_r} = 0.\dot{6}\dot{6}$ mins $= 40$ seconds

\3a/ $\lambda_E = 10$ $\lambda_R = 12$ ∴ $\frac{5}{3} = \lambda_E = 1.\dot{6}$ per 10 mins $\lambda_R = 2$ per 10 mins

$P(E(t=1)=3)\,P(R(t=1)=5) = \frac{(\frac{5}{3}\cdot 1)^3 e^{-\frac{5}{3}\cdot 1}}{3!} \; \frac{(2\cdot 1)^5 e^{-2\cdot 1}}{5!} = 0.00526$

$\lambda_E = \frac{1}{6}$ per min $\lambda_R = 0.2$ per min ∴

$P(E(t=10)=3)\,P(R(t=10)=5) = \frac{(\frac{1}{6}\cdot 10)^3 e^{-\frac{1}{6}\cdot 10}}{3!} \; \frac{(0.2\times 10)^5 e^{-0.2\times 10}}{5!} =$

0.00526 ✓

\3a Sol/ 10 mins are $t = \frac{1}{6}$ hours

let $E(t)$ denote the number of easyJet planes & $R(t)$ the number of Ryanair planes, we have $E(t) \sim \text{Poisson}(10t)$, $R(t) \sim \text{Poisson}(12t)$ ∴ $E(t) + R(t) \sim \text{Poisson}(22t)$ ∴

have a poisson process with rate $\lambda = 3$ per min & ∴

$N(t) \sim \text{Poisson}(\lambda t)$

10 mins are $t = \frac{1}{6}$ hours ∴ by indep of the processes:

$$P(E(t)=3, R(t)=5) = P(E(t)=3)P(R(t=5) = \frac{e^{-\frac{10}{6}}\left(\frac{10}{6}\right)^3}{3!} \times \frac{e^{-2}\,2^5}{5!} \approx 0.005$$

P(6 events between 2 and 3 mins) =

~~P(5 events~~

P(less then 6 events at 2 mins) - P(less then 6 events at 3 mins) =

$P(N(t=2)<6) - P(N(t=3)<6) =$

P(5 or less events at 2 mins) - P(5 or less events at 3 mins) =

$P(N(t=2)\leq 5) - P(N(t=3)\leq 5) = P(N(2)\leq 5) - P(N(3)\leq 5)$ ∴

$P(N(2)\leq 5) =$

$$e^{-3\cdot 2}\left[\frac{(3\cdot 2)^0}{0!} + \frac{(3\cdot 2)^1}{1!} + \frac{6^2}{2!} + \frac{6^3}{3!} + \frac{6^4}{4!} + \frac{6^5}{5!}\right] =$$

$$e^{-6}\left[1 + 6 + \frac{6^2}{2!} + \frac{6^3}{3!} + \frac{6^4}{4!} + \frac{6^5}{5!}\right] = 0.446$$

$P(N(3)\leq 5) =$

$$e^{-9}\left[1 + 9 + \frac{(3\cdot 3)^2}{2!} + \frac{(3\cdot 3)^3}{3!} + \frac{9^4}{4!} + \frac{9^5}{5!}\right] = e^{-9}\left[1 + 9 + \frac{9^2}{2!} + \frac{9^3}{3!} + \frac{9^4}{4!} + \frac{9^5}{5!}\right] = 0.116$$

$\therefore P(N(2)\leq 5) - P(N(3)\leq 5) = 0.446 - 0.116 = 0.33$

\1d/ P(Z time between every 4th event is less then 3 mins) =

$P(S_4 < 3) = P(N(t=3) > 4) = 1 - P(N(t=3)\leq 4) = 1 - P(N(3)\leq 4)$ ∴

$P(N(3)\leq 4) = (\lambda t)^k e^{-\lambda t}/k! =$

$$e^{-(3\cdot 3)}\left[\frac{(3\cdot 3)^0}{0!} + \frac{(3\cdot 3)^1}{1!} + \frac{9^2}{2!} + \frac{9^3}{3!} + \frac{9^4}{4!}\right] = e^{-9}\left[1 + 9 + \frac{9^2}{2!} + \frac{9^3}{3!} + \frac{9^4}{4!}\right] = 0.0550$$

$\therefore 1 - P(N(3)\leq 4) = 1 - 0.0550 = 0.945$ (3SS) ✗ 0.979

\1d sn/ ∵ events in Z future are indep from events in Z ~~past~~: $P(S_4\leq 3) = P(N(3)\geq 4) = 1 - P(N(3)<4) = 1 - P(N(3)\leq 3)$ ∴

$$\left\{e^{-(3\cdot 3)}\left[\frac{(3\cdot 3)^0}{0!} + \frac{(3\cdot 3)^1}{1!} + \frac{(3\cdot 3)^2}{2!} + \frac{9^3}{3!}\right] = e^{-9}\left[1 + 9 + \frac{9^2}{2!} + \frac{9^3}{3!}\right] = 0.0212 \therefore\right\}$$

$1 - P(N(3)\leq 3) = 0.979$ (3SS)

\1e/ 10 events in 12 mins ∴ $n = 10$, $12 = T$ ∴ $t = 4$

~~X =~~ X be events in $[0, 4]$ ∴ $X \sim Bin(10, 4/12) = Bin(10, 1/3)$

$\therefore P(X=3) = \binom{10}{3}\left(\frac{1}{3}\right)^3\left(1-\frac{1}{3}\right)^7 = 0.260$ (3SS)

\1e sn/ let X be Z R.V. describes Z events in Z interval $[0, t]$ (given N events in $[0, T]$) then $X \sim Bin(N, \frac{t}{T})$ ∴ $N = 10$, $T = 12$, $t = 4$ ∴

$P(X=3) = \binom{10}{3}\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^7 \approx 0.26$

Sheet 2 / 1b sol / $S_n = T_1 + \cdots + T_n$ where T_n are Z

exponentially distrib. inter-event times

$P(S_7 > 3) = P(6 \text{ or less events in } 3 \text{ min}) =$

$P(N(3) \le 6) = e^{-9}\left(1 + 9 + \cdots + \frac{9^6}{6!}\right) \simeq 0.207$

1b redo / T_k is time interval between $(k-1)^{th}$ & k^{th} event

$\therefore S_n = T_1 + T_2 + \cdots + T_n \therefore$

$P(S_7 > 3) = P(N(t=3) \le 6) = P(N(t=3) \le 6) = P(N(3) \le 6) = P(N(3) = 0) +$

$P(N(3) = 1) + P(N(3) = 2) + P(N(3) = 3) + P(N(3) = 4) + P(N(3) = 5) + P(N(3) = 6) =$

$(e^{-3 \cdot 3})\left[\frac{(3 \cdot 3)^0}{0!} + \frac{9^1}{1!} + \frac{9^2}{2!} + \frac{9^3}{3!} + \frac{9^4}{4!} + \frac{9^5}{5!} + \frac{9^6}{6!}\right] =$

$e^{-9}\left[1 + 9 + \frac{9^2}{2!} + \frac{9^3}{3!} + \frac{9^4}{4!} + \frac{9^5}{5!} + \frac{9^6}{6!}\right] \simeq 0.207\ (358)$

1c / $P(2 \le S_6 \le 3) = P(S_6 \le 3) - P(S_6 < 2) =$

$P(N(t=3) \le 6) - P(N(t=2) \le 6)$

$P(N(t=3) \le 6) \simeq 0.207$

$P(N(t=2) \le 6) = P(N(2) \le 6) =$

$e^{-3 \cdot 2}\left[\frac{(3 \cdot 2)^0}{0!} + \frac{6^1}{1!} + \frac{6^2}{2!} + \frac{6^3}{3!} + \frac{6^4}{4!} + \frac{6^5}{5!} + \frac{6^6}{6!}\right] = 0.606$

$P(2 \le S_6 \le 3) = P(S_6 \ge 2) - P(S_6 > 3)$

$0.606 - 0.207 = 0.399$ X 0.33

1c sol / $P(2 \le S_6 \le 3) = P(S_6 \le 3) - P(S_6 \le 2) =$

$P(N(t=3) \ge 6) - P(N(t=2) \ge 6) =$

$P(N(3) \ge 6) - P(N(2) \ge 6) =$

$P(N(2) < 6) - P(N(3) < 6) =$

$P(N(2) \le 5) - P(N(3) \le 5) = 0.33$

1c retry / $P(2 \le S_6 \le 3) = P(S_6 \le 3) - P(S_6 < 2) = \therefore$

$P(S_6 < 2) = P(N(t=2) \ge 7)$, $P(S_6 \le 3) = P(N(t=3) \ge 7)$ X

1c retry / $P(2 \le S_6 \le 3) = P(S_6 \le 3) - P(S_6 \le 2) =$

$P(N(t=3) \ge 6) - P(N(t=2) \ge 6) = 1 - P(N(t=3) < 6) - (1 - P(N(t=2) < 6)) =$

$P(N(t=2) < 6) - P(N(t=3) < 6) = P(N(t=2) \le 5) - P(N(t=3) \le 5) =$

\6d redo/ $G_Y(\theta) = (0.2+0.3\theta+0.3\theta^2+0.2\theta^3)^6$

$E(S_n) = E(Y)\mu_x^6 = 6\cdot(1.5)^6 = 6\cdot 1.5^6 = 68.3$ (358)

$G_{Y_2}(\theta) = G_Y \circ G_Y(\theta) = G_Y((0.2+0.3\theta+0.3\theta^2+0.2\theta^3)^6)$

$(0.2+0.3(0.2+0.3\theta+0.3\theta^2+0.2\theta^3)^6+0.3((0.3\theta+0.3\theta^2+0.2\theta^3+0.2)^6)^2+0.2((0.2+0.3\theta+0.3\theta^2+0.2\theta^3)^6)^3$

$G_Y(\theta) = 0.2^6 + \binom{6}{1}0.2^5(0.3\theta) + \binom{6}{1}0.2^5(0.3\theta) + \binom{6}{1}0.2^4(0.3\theta)^2 + \dots =$

$\frac{1}{15625} + \frac{9}{15625}\theta$ ✗

\6d Sol/ Now for $Y = S_0 = 6$ $G_Y = \theta^6$ & $\mu_Y = E(Y) = 6$ ∴

$E(S_6) = \mu_Y \mu_x^6 = 6(1.5)^6 \approx 68.3$

$e \approx G_Y(0.35) = 0.35^6 \approx 0.0019$

\6e/ $Y \sim \text{Binomial}(6, 0.6)$ ∴ $G_Y(\theta) = (0.4+0.6\theta)^6$ ∴

$G_n(\theta) = G_Y(G_x^n(\theta))$ $E(Y) = 6\cdot 0.6 = 3.6$

$E(S_n) = 3.6\,\mu_x^6 = 3.6\cdot(1.5)^6 \approx 41.0$ ✓ (358)

$e_n = (0.4+0.6\tilde{e}_n)^6$ $\tilde{e}_n = 0.3508$ ∴

$(0.4+0.6(0.3508))^6 = 0.0518$ ✓ (358)

\6e Sol/ Similarly for $S_0 = Y$ with $G_Y(\theta) = (0.4+0.6\theta)^6$:

$E(Y) = 3.6$ ∴ $E(S_6) = \mu_Y \mu_x^6 = 3.6(1.5)^6 \approx 41$ &

$e = G_Y(0.35) \approx 0.052$

\Sheet 2/

\1a/ $\lambda = 3$

\1a Sol/ Z probab no events happen in two mins is $P(N(2)=0) =$ e^{-6}

\1a retry/ P(observe 0 events over time interval $t=0$) =

$P(N(t=2)=0) = \frac{(\lambda 2)^0 e^{-\lambda 2}}{0!} = (3\cdot 2)^0 e^{-3\cdot 2}/1 = 6^0 e^{-6} = e^{-6}$ ✓

\1b/ $g_n(x) = \frac{\lambda(\lambda x)^{n-1}}{(n-1)!} e^{-\lambda x}$ $P(S(3)>7) = 1 - P(S_n(7) \geq 6)$

$P(S(3)<7) = 1 - P(6 \leq S(3))$

P(7th event more than 3 mins means)

P(6th event less than 3 mins) is

Probab 6 events in 3 mins is ~~P(N(t)=6)~~ $P(N(t=3)=6) =$

$P(N(3)=6) = \frac{(\lambda 3)^6 e^{-\lambda 3}}{6!} = \frac{(3\cdot 3)^6 e^{-3\cdot 3}}{6!} = \frac{9^6 e^{-9}}{720} = 0.0911$ (358) ✗ 0.207