\PP2019/ $A\cos(2\alpha) + F = B\sin(2\alpha)$ ∴

$\frac{A\cos(2\alpha)}{\sin(2\alpha)} + \frac{F}{\sin(2\alpha)} = B$ ∴

$Ru(R,\phi) = A\cos(2\phi) + \frac{A\cos(2\alpha)\sin(2\phi)}{\sin(2\alpha)} + \frac{F\sin(2\phi)}{\sin(2\alpha)} + F$ ∴

at $\phi = \alpha$: $u(R,\alpha) = 0$ ∴

$R(0) = A\cos(2\alpha) + \frac{A\cos(2\alpha)\sin(2\alpha)}{\sin(2\alpha)} + F\frac{\sin(2\alpha)}{\sin(2\alpha)} + F =$

$A\cos(2\alpha) + A\cos(2\alpha) + F + F = 2A\cos(2\alpha) + 2F = 0 = A\cos(2\alpha) + F$ ∴

$F = -A\cos(2\alpha)$ ∴

$Ru(R,\phi) = A\cos(2\phi) + \frac{A\cos(2\alpha)\sin(2\phi)}{\sin(2\alpha)} + \frac{\cancel{F}\sin(2\phi)(-A\cos(2\alpha))}{\sin(2\alpha)} + -A\cos(2\alpha)$

$= A(\cos(2\phi) - \cos(2\alpha)) + \frac{A\cos(2\alpha)\sin(2\phi)}{\sin(2\alpha)} - \frac{A\cos(2\alpha)\sin(2\phi)}{\sin(2\alpha)} =$

$A(\cos(2\phi) - \cos(2\alpha)) = C(\cos 2\phi - \cos 2\alpha) = Ru$ ∴

$CR^{-1}(\cos 2\phi - \cos 2\alpha) = u$

\2S/ radially inwards flow has $u<0$ ∴ $\underline{u} = u\hat{\underline{R}} = (-ve)\hat{\underline{R}}$ ∴

$u = C(\cos 2\phi - \cos 2\alpha)\frac{1}{R}$ ∴ $C<0$ ∴ for $u<0$:

$\cos 2\phi - \cos 2\alpha \geq 0$ ∴ $\cos 2\phi > \cos 2\alpha$

$\arccos(\cos 2\phi) > 2\alpha$ ∴ $2\phi > 2\alpha$ ∴ $\alpha < \phi$

for $\phi \in [0, 2\pi)$

\2S/ radially inwards flow has $u<0$ ∴ $\underline{u} = u\hat{\underline{R}} = (-ve)\hat{\underline{R}}$ ∴

$u = C(\cos 2\phi - \cos 2\alpha)\frac{1}{R}$ ∴

$C<0$ ∴ for $u<0$:

$C(\cos(2\phi) - \cos 2\alpha)\frac{1}{R} < 0$ ∴ $R \geq 0$ ∴ $\frac{1}{R} \geq 0$ ∴ $C(\cos 2\phi - \cos 2\alpha) < 0$

∴ $C<0$ ∴ $\cos 2\phi - \cos 2\alpha > 0$ ∴

$\cos 2\phi > \cos 2\alpha$ ∴ $\arccos(\cos 2\phi) > 2\alpha$ ∴

$2\phi > 2\alpha$ ∴ $\phi > \alpha$ the flow is radially inwards

with $\phi \in [0, 2\pi)$

3ai/ NS: $\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\nabla(\frac{1}{\rho_0}P) + g + \nu\nabla^2\underline{u} = -\nabla(\frac{1}{\rho_0}P) - \nabla\Pi + \nu\nabla^2\underline{u}$

∴ taking the curl of the NS equation: $\nabla\times\underline{u} = \underline{\omega}$ ∴

$\nabla\times(\frac{\partial\underline{u}}{\partial t}) = \frac{\partial}{\partial t}\nabla\times\underline{u} = \frac{\partial}{\partial t}\underline{\omega}$, cdj

$\nabla\times(-\nabla(\frac{1}{\rho_0}P)) = -\nabla\times[\nabla(\frac{1}{\rho_0}P)] = 0$,

$\nabla\times(-\nabla\Pi) = -\nabla\times(\nabla\Pi) = 0$,

$\nabla\times[\nu\nabla^2\underline{u}] = \nu\nabla\times[\nabla^2\underline{u}] = \nu\nabla^2(\nabla\times\underline{u}) = \nu\nabla^2\underline{\omega}$

$\therefore \frac{\partial\underline{\omega}}{\partial t} + \nabla\times(\underline{u}\cdot\nabla\underline{u}) = \nu\nabla^2\underline{\omega}$ ↯

3aii/ let $\frac{\partial\underline{\omega}}{\partial t} = \nabla\times(\underline{u}\times\underline{\omega}) + \nu\nabla^2\underline{\omega}$, $\nabla\cdot\underline{u} = 0$

∴ $\nabla^2\underline{u} = \nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{u}) = \nabla(\nabla\cdot\underline{u}) - \nabla\times\underline{\omega}$ ∴

$\nabla\times\nabla^2\underline{u} = \nabla\times(\nabla(\nabla\cdot\underline{u}) - \nabla\times\underline{\omega}) = \nabla\times\nabla(\nabla\cdot\underline{u}) + \nabla\times(-\nabla\times\underline{\omega}) =$ cdj

$\nabla\times\nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{\omega}) = -\nabla\times(\nabla\times\underline{\omega})$ ∵ $\nabla\times\nabla\phi \equiv 0$

$= -\nabla(\nabla\cdot\underline{\omega}) + \nabla^2\underline{\omega} = -\nabla(\nabla\cdot\underline{\omega}) + \nabla$

$-\nabla(\nabla\cdot(\nabla\times\underline{u})) + \nabla^2\underline{\omega} = -\nabla($

$-\nabla(0) + \nabla^2\underline{\omega}$ ∵ $\nabla\cdot(\nabla\times\underline{u}) = 0$ ∵ $\nabla\cdot\nabla\times \equiv 0$

$= \nabla^2\underline{\omega}$ ↯

3ai/ Let $\frac{\partial\underline{\omega}}{\partial t} = \nabla\times(\underline{u}\times\underline{\omega}) + \nu\nabla^2\underline{\omega}$, $\nabla\cdot\underline{u} = 0$

$\nabla\times(\underline{u}\times\underline{\omega}) = \underline{u}(\nabla\cdot\underline{\omega}) - \underline{\omega}(\nabla\cdot\underline{u}) + (\underline{\omega}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)\underline{\omega}$

$\therefore \nabla\cdot\underline{u} = 0$ $\therefore \nabla\times(\underline{u}\times\underline{\omega}) = \underline{u}(\nabla\cdot\underline{\omega}) - \underline{\omega}(0) + (\underline{\omega}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)\underline{\omega}$

$\therefore \nabla\times(\underline{u}\times\underline{\omega}) = \underline{u}(\nabla\cdot\underline{\omega}) + (\underline{\omega}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)\underline{\omega}$

cdj $\therefore \nabla\cdot\nabla\times \equiv 0$ $\therefore \nabla\cdot\underline{\omega} = \nabla\cdot\nabla\times\underline{u} = 0$ $\therefore \underline{u}(\nabla\cdot\underline{\omega}) = \underline{u}(0) = 0$ ∴

$\nabla\times(\underline{u}\times\underline{\omega}) = (\underline{\omega}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)\underline{\omega}$ ∴

$\frac{\partial\underline{\omega}}{\partial t} = (\underline{\omega}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)\underline{\omega} + \nu\nabla^2\underline{\omega} = -(\underline{u}\cdot\nabla)\underline{\omega} + (\underline{\omega}\cdot\nabla)\underline{u} + \nu\nabla^2\underline{\omega} = \frac{\partial\underline{\omega}}{\partial t}$

NS: $\rho(\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}) = -\nabla P + \rho g + \mu\nabla^2\underline{u}$ ∴ $\frac{\mu}{\rho} = \nu$ ∴

$\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla P + g + \nu\nabla^2\underline{u}$ ∴

$\nabla\times\frac{\partial\underline{u}}{\partial t} = \frac{\partial}{\partial t}\nabla\times\underline{u} = \frac{\partial\underline{\omega}}{\partial t}$ cdj , $g = -\nabla\Pi$ $\therefore \nabla\times g = \nabla\times(-\nabla\Pi) = -\nabla\times(\nabla\Pi) = 0$,

$\nabla\times(-\frac{1}{\rho}\nabla P) = -\frac{1}{\rho}\nabla\times\nabla P = -\frac{1}{\rho}(0) = 0$, $\nu\nabla^2\underline{u} = \nabla\times(\nu\nabla^2\underline{u}) = \nu\nabla\times\nabla^2\underline{u}$

$\frac{\partial\underline{\omega}}{\partial t} + \nabla\times(\underline{u}\cdot\nabla\underline{u}) = \nu\nabla\times\nabla^2\underline{u}$

\PP2019/ \3biA/ $\underline{u}\cdot(\underline{\omega}\cdot\nabla)\underline{u} \rightarrow [\underline{u}\cdot(\underline{\omega}\cdot\nabla)\underline{u}]_i = u_i\,((\omega\cdot\nabla)\underline{u})_i =$

$u_i(\omega\cdot\nabla)u_i = u_i(\omega_j\partial_j)u_i = \omega_j\,\partial_j\frac{(u_iu_i)}{2} = \underline{\omega}\cdot\nabla\left(\frac{|\underline{u}|^2}{2}\right)$

\3biB/ $\underline{u}\cdot\nabla(\underline{u}\cdot\underline{\omega}) \longrightarrow [\underline{u}\cdot\nabla(\underline{u}\cdot\underline{\omega})]_i = u_i(\nabla(\underline{u}\cdot\underline{\omega}))_i =$

$u_i\nabla_i(\underline{u}\cdot\underline{\omega}) = u_i\partial_i(u_j\omega_j) =$

$u_i\,\partial_i(u_j)\omega_j + u_i u_j\,\partial_i(\omega_j) =$

$\cancel{u_j u_i \partial_i \omega_j}\; u_j u_i\partial_i\omega_j + \omega_j u_i\partial_i u_j =$

$\cancel{\underline{u}\cdot(\underline{u}\cdot\nabla)\underline{\omega} + \underline{\omega}\cdot(\underline{u}\cdot\nabla}\; \cancel{u_j(\underline{u}\cdot\nabla)}$

$u_j(u_i\nabla_i)\omega_j + \omega_j(u_i\nabla_i)u_j = [\underline{u}\cdot(\underline{u}\cdot\nabla)\underline{\omega}]_i + [\underline{\omega}\cdot(\underline{u}\cdot\nabla)\underline{u}]_i =$

$[\underline{u}\cdot(\underline{u}\cdot\nabla)\underline{\omega} + \underline{\omega}\cdot(\underline{u}\cdot\nabla)\underline{u}]_i \rightarrow \underline{u}\cdot(\underline{u}\cdot\nabla)\underline{\omega} + \underline{\omega}\cdot(\underline{u}\cdot\nabla)\underline{u}$

\3biBSol/ $\underline{u}\cdot\nabla(\underline{u}\cdot\underline{\omega}) \rightarrow [\underline{u}\cdot\nabla(\underline{u}\cdot\underline{\omega})]_i = u_i\nabla_i(u_j\omega_j) =$

$u_i\partial_i(u_j\omega_j) = u_i\,\partial_i(u_j)\omega_j + u_i\,\partial_i(\omega_j)u_j =$

$u_i\,\partial_i(\omega_j)u_j + u_i\partial_i(u_j)\omega_j =$

$u_j u_i\partial_i(\omega_j) + \omega_j u_i\partial_i(u_j) =$

$u_j u_i\nabla_i(\omega_j) + \omega_j u_i\nabla_i(u_j) = [\underline{u}\cdot(\underline{u}\cdot\nabla)\underline{\omega} + \underline{\omega}\cdot(\underline{u}\cdot\nabla)\underline{u}]_i$

\3bii/ $\underline{u}\cdot(\underline{\omega}\cdot\nabla)\underline{u} = \underline{\omega}\cdot\nabla\left(\frac{u^2}{2}\right),$

$\underline{u}\cdot\nabla(\underline{u}\cdot\underline{\omega}) = \underline{u}\cdot(\underline{u}\cdot\nabla)\underline{\omega} + \underline{\omega}\cdot(\underline{u}\cdot\nabla)\underline{u}$

$\frac{\partial}{\partial t}(\underline{u}\cdot\underline{\omega}) = \underline{u}\cdot[(\underline{\omega}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)\underline{\omega}] - \underline{\omega}\cdot[(\underline{u}\cdot\nabla)\underline{u} + \nabla P]\;\therefore$

$\therefore\; \underline{u}\cdot[(\underline{\omega}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)\underline{\omega}] = \underline{u}\cdot[(\underline{\omega}\cdot\nabla)\underline{u}] - \underline{u}\cdot[(\underline{u}\cdot\nabla)\underline{\omega}]$

$= \underline{u}\cdot(\underline{\omega}\cdot\nabla)\underline{u} - \underline{u}\cdot(\underline{u}\cdot\nabla)\underline{\omega} = \underline{\omega}\cdot\nabla\left(\frac{u^2}{2}\right) - \underline{u}\cdot(\underline{u}\cdot\nabla)\underline{\omega} =$

$\underline{\omega}\cdot\nabla\left(\frac{u^2}{2}\right) - \underline{u}\cdot\nabla(\underline{u}\cdot\underline{\omega}) + \underline{\omega}\cdot(\underline{u}\cdot\nabla)\underline{u}\;,$

$\underline{\omega}\cdot[(\underline{u}\cdot\nabla)\underline{u} + \nabla P] = \underline{\omega}\cdot((\underline{u}\cdot\nabla)\underline{u}) + \underline{\omega}\cdot\nabla P$

$\therefore$ \3biiSol/ $\frac{\partial}{\partial t}(\underline{u}\cdot\underline{\omega}) = \underline{u}\cdot[\underbrace{(\underline{\omega}\cdot\nabla)\underline{u}}_{\text{use A}} - \underbrace{(\underline{u}\cdot\nabla)\underline{\omega}}_{\text{rearrang B}}] - \underline{\omega}\cdot[\underbrace{(\underline{u}\cdot\nabla)\underline{u} + \nabla P}_{\text{leave alone}}]$

$= -\underline{u}\cdot\nabla(\underline{u}\cdot\underline{\omega}) - \underline{\omega}\cdot\nabla\left(P - \frac{u^2}{2}\right) =$

$-\nabla\cdot\left[(\underline{u}\cdot\underline{\omega})\underline{u} + P\underline{\omega} - \frac{u^2}{2}\underline{\omega}\right]\;\therefore$

$\underline{u}\cdot\nabla\phi = \nabla\cdot(\phi\underline{u}) - \phi\nabla\cdot\underline{u}\;;\; \phi = \underline{u}\cdot\underline{\omega}\;\therefore$

$\underline{u}\cdot\nabla(\underline{u}\cdot\underline{\omega}) = \underline{u}\cdot\nabla(\phi) = \nabla\cdot(\phi\underline{u}) - \phi\nabla\cdot\underline{u} = \nabla\cdot((\underline{u}\cdot\underline{\omega})\underline{u} - (\underline{u}\cdot\underline{\omega})\nabla\cdot\underline{u}$

$\underline{\omega}\cdot\nabla\left(P - \frac{u^2}{2}\right) = \underline{\omega}\cdot\nabla(\theta) = \nabla\cdot(\theta\underline{\omega}) - \theta\nabla\cdot\underline{\omega} = \nabla\cdot\left(\left(P - \frac{u^2}{2}\right)\underline{\omega}\right) - \left(P - \frac{u^2}{2}\right)\nabla\cdot\underline{\omega}\;\therefore$

$\frac{\partial}{\partial t}(\underline{u}\cdot\underline{\omega}) = -\nabla\cdot\left[(\underline{u}\cdot\underline{\omega})\underline{u} + P\underline{\omega} - \frac{u^2}{2}\underline{\omega}\right]\;\therefore$

\3biii sol/ $\therefore H=\int_V \underline{u}\cdot\underline{\omega}\, dV \therefore \frac{dH}{dt} = \frac{\partial}{\partial t}\int_V (\underline{u}\cdot\underline{\omega})\, dV =$

$\int_V \frac{\partial}{\partial t}(\underline{u}\cdot\underline{\omega})\, dV = \int_V -\nabla\cdot\left[(\underline{u}\cdot\underline{\omega})\underline{u} + P\underline{\omega} - \frac{u^2}{2}\underline{\omega}\right] dV = 0 \therefore$

$H = \text{Constant}$

\4a/ incompressible $\therefore \nabla\cdot\underline{u}=0$

$\therefore$ of boundary conditions of solid no slip boundary $\therefore$

fluid at same y value of boundary has equal velocity to boundary in x axis $\therefore$

solid boundary means $\underline{u} = u\hat{\underline{i}} + 0\hat{\underline{j}} + 0\hat{\underline{k}}$ and

~~Stuff~~ no slip condition means $u=u(y,t)$ $\therefore$

$u = (u(y,t))\hat{\underline{i}} + 0\hat{\underline{j}} + 0\hat{\underline{k}}$

\4b/ N-S: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla P + \rho\underline{g} + \mu\nabla^2\underline{u}$ $\therefore \underline{g}=0 \therefore$

$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla P + \mu\nabla^2\underline{u}$ $\therefore$

$\frac{\partial \underline{u}}{\partial t} + (\underline{u}\cdot\nabla)\underline{u} = -\frac{1}{\rho}\nabla P + \frac{\mu}{\rho}\nabla^2\underline{u} = -\nabla\frac{P}{\rho} + \nu\nabla^2\underline{u}$ $\therefore$

$(\underline{u}\cdot\nabla) = u(y,t)\frac{\partial}{\partial x} \therefore (\underline{u}\cdot\nabla)\underline{u} = u(y,t)\frac{\partial}{\partial x}(u(y,t)) = 0$ $\therefore$

$\frac{\partial \underline{u}}{\partial t} = -\nabla\frac{P}{\rho} + \nu\nabla^2\underline{u}$ $\therefore$

$\frac{\partial \underline{u}}{\partial t} = \frac{\partial u}{\partial t}\hat{\underline{i}}, \quad -\nabla\frac{P}{\rho} = 0 \quad \therefore \nu\nabla^2\underline{u} = \nu\nabla^2(u(y,t),0,0) =$

$\nu\nabla^2 u(y,t)\hat{\underline{i}} = \nu u_{yy}\hat{\underline{i}}$ $\therefore$ $\therefore$ y-component is $0\hat{\underline{j}} = 0\hat{\underline{j}}$,

x-component is $\frac{\partial u}{\partial t}\hat{\underline{i}} = \nu u_{yy}\hat{\underline{i}} \therefore u_t = \nu u_{yy}$

\4a sol/ the motion of the fluid is caused by the moving boundary which moves in the x-direction, so $\underline{u} = u\hat{\underline{i}}$

$\therefore$ fluid velocity does not depend of x coordinate $\therefore$ is independent of x. the fluid at the y coordinate of the plate obeyes the no slip $\therefore$ it is dependent of y.

the velocity of the plate is different at $t=0$ and $t>0$ $\therefore$

$\therefore$ of no slip, the fluid is also dependent on time

\4b/ $(\underline{u}\cdot\nabla) = u(y,t)\frac{\partial}{\partial x} \therefore (\underline{u}\cdot\nabla\underline{u}) = u(y,t)\frac{\partial}{\partial x}u(y,t) = 0$

$\therefore \frac{\partial P}{\partial x}$ is independent of x $\therefore P = \text{constant}$.

$u_t = \nu u_{yy}$ $\therefore \nu = \mu/\rho$

\PP2019/\3aiiB/ inviscid $\therefore D=0 \therefore \frac{\partial \omega}{\partial t} - aR\frac{\partial \omega}{\partial R} = 2a\omega \therefore$

$\omega = e^{2at}F(\xi) = e^{2at}F(Re^{at}) \therefore$

$\frac{\partial \omega}{\partial t} = 2ae^{2at}F(Re^{at}) + e^{2at}F'(Re^{at})Rae^{at} \therefore$

$\frac{\partial \omega}{\partial R} = e^{2at}F'(Re^{at})e^{at} \therefore$

$\frac{\partial \omega}{\partial t} - aR\frac{\partial \omega}{\partial R} - 2a\omega =$

$2ae^{2at}F(Re^{at}) + e^{2at}F'(Re^{at})Rae^{at} - aRe^{2at}F'(Re^{at})e^{at} - 2ae^{2at}F(Re^{at}) =$

$F(Re^{at})(2ae^{2at} - 2ae^{2at}) + F'(Re^{at})(e^{2at}Rae^{at} - aRe^{2at}e^{at}) =$

$F(Re^{at})(0) + F'(Re^{at})(0) = 0 + 0 = 0 \therefore$

$\omega = e^{2at}F(Re^{at})$ is a sol in the inviscid case

$\underline{\Omega} = \omega\hat{\underline{z}} = \omega(R,t) = e^{2at}F(\xi) \therefore \underline{\omega}$ exponentially increases as t increases for $a>0$ and vice versa for $a<0$

\3biA/ $\underline{u}\cdot(\underline{\omega}\cdot\nabla)\underline{u} \to [\underline{u}\cdot(\underline{\omega}\cdot\nabla)\underline{u}]_i = u_i((\underline{\omega}\cdot\nabla)\underline{u})_i =$

$u_i(\omega\cdot\nabla)u_i = u_i(\omega_j\partial_j)u_i = \omega_j\partial_j\left(\frac{u_iu_i}{2}\right) = \underline{\omega}\cdot\nabla\left(\frac{|\underline{u}|^2}{2}\right)$

\3biB/ $\underline{u}\cdot\nabla(\underline{u}\cdot\underline{\omega}) \to [\underline{u}\cdot\nabla(\underline{u}\cdot\underline{\omega})]_i = u_i(\nabla(\underline{u}\cdot\underline{\omega}))_i = u_i\partial_i(u_j\omega_j) =$

$u_i\partial_i(u_j)\omega_j + u_i\partial_i(\omega_j)u_j = u_i\partial_i(\omega_j)u_j + u_i\partial_i(u_j)\omega_j =$

$u_ju_i\partial_i(\omega_j) + \omega_ju_i\partial_i(u_j) =$

$u_ju_i\nabla_i(\omega_j) + \omega_ju_i\nabla_i(u_j) = [\underline{u}\cdot(\underline{u}\cdot\nabla)\underline{\omega}]_i + [\omega\cdot(\underline{u}\cdot\nabla)\underline{u}]_i =$

$[\underline{u}\cdot(\underline{u}\cdot\nabla)\underline{\omega} + \underline{\omega}\cdot(\underline{u}\cdot\nabla)\underline{u}]$

\3bii/ $\frac{\partial}{\partial t}(\underline{u}\cdot\underline{\omega}) = \underline{u}\cdot[(\underline{\omega}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)\underline{\omega}] - \underline{\omega}\cdot[(\underline{u}\cdot\nabla)\underline{u} + \nabla p] =$

$\underline{u}\cdot(\underline{\omega}\cdot\nabla)\underline{u} - \underline{u}\cdot(\underline{u}\cdot\nabla)\underline{\omega} - \underline{\omega}\cdot[(\underline{u}\cdot\nabla)\underline{u} + \nabla p] \therefore$

$-\underline{u}\cdot(\underline{u}\cdot\nabla)\underline{\omega} = -\underline{u}\cdot\nabla(\underline{u}\cdot\underline{\omega}) + \underline{\omega}\cdot(\underline{u}\cdot\nabla)\underline{u}$,

$\underline{u}\cdot(\underline{\omega}\cdot\nabla)\underline{u} = \underline{\omega}\cdot\nabla\left(\frac{u^2}{2}\right) \therefore$

$\frac{\partial}{\partial t}(\underline{u}\cdot\underline{\omega}) = \underline{\omega}\cdot\nabla\left(\frac{u^2}{2}\right) - \underline{u}\cdot\nabla(\underline{u}\cdot\underline{\omega}) + \underline{\omega}\cdot(\underline{u}\cdot\nabla)\underline{u} - \underline{\omega}\cdot[(\underline{u}\cdot\nabla)\underline{u} + \nabla p] =$

$-\underline{u}\cdot\nabla(\underline{u}\cdot\underline{\omega}) + \underline{\omega}\cdot(\underline{u}\cdot\nabla)\underline{u} - \underline{\omega}\cdot\left[(\underline{u}\cdot\nabla)\underline{u} + \nabla p - \nabla\left(\frac{u^2}{2}\right)\right] =$

$-\underline{u}\cdot\nabla(\underline{u}\cdot\underline{\omega}) + \underline{\omega}\cdot(\underline{u}\cdot\nabla)\underline{u} - \underline{\omega}\cdot(\underline{u}\cdot\nabla)\underline{u} - \underline{\omega}\cdot\left[\nabla\left(p - \frac{u^2}{2}\right)\right] =$

$-\underline{u}\cdot\nabla(\underline{u}\cdot\underline{\omega}) - \underline{\omega}\cdot\nabla\left(p - \frac{u^2}{2}\right) = -\nabla(\underline{u}\cdot\underline{\omega})\cdot\underline{u} - \nabla\left(p - \frac{u^2}{2}\right)\cdot\underline{\omega} =$

$-\nabla\cdot[(\underline{u}\cdot\underline{\omega})\underline{u}] - \nabla\cdot\left[\left(p - \frac{u^2}{2}\right)\underline{\omega}\right] = -\nabla\cdot[(\underline{u}\cdot\underline{\omega})\underline{u}] - \nabla\cdot\left[p\underline{\omega} - \frac{u^2}{2}\underline{\omega}\right] =$

$-\nabla\cdot\left[(\underline{u}\cdot\underline{\omega})\underline{u} + p\underline{\omega} - \frac{u^2}{2}\underline{\omega}\right]$

3 b iii $\therefore$ let $H = \int_V \underline{u} \cdot \underline{\omega} \, dV \therefore$

$$\frac{dH}{dt} = \frac{\partial}{\partial t}\int_V \underline{u}\cdot\underline{\omega}\, dV = \int_V \frac{\partial}{\partial t}(\underline{u}\cdot\underline{\omega})\, dV = \int_V -\nabla\cdot\left[(\underline{u}\cdot\underline{\omega})\underline{u} + p\underline{\omega} - \frac{u^2}{2}\underline{\omega}\right] dV =$$

$$\int_V -\nabla\cdot(\underline{u}\cdot\underline{\omega})\underline{u} - \nabla\cdot p\underline{\omega} - \nabla\cdot\left(-\frac{u^2}{2}\underline{\omega}\right) dV =$$

$$-\int_V \nabla\cdot(\underline{u}\cdot\underline{\omega})\underline{u}\, dV - \int_V \nabla\cdot p\underline{\omega}\, dV - \int_V \nabla\cdot\left(-\frac{u^2}{2}\underline{\omega}\right) dV = 0 \therefore$$

$\frac{dH}{dt} = 0 \therefore \quad H = \text{constant}$

4 a The motion of the fluid is caused by the moving boundary which moves in the x-direction $\therefore \underline{u} = u\hat{i} \therefore$
fluid velocity does not depend on x coord $\therefore$ is independent of x
The fluid at the y coord of the plate obeyes the no slip $\therefore$ it is dependent on y.
The ~~ve~~ velocity of the plate is different at $t=0$ and $t>0 \therefore$
$\therefore$ of no slip, the fluid is also dependent on time
$\therefore$ The velocity of plate is different at $t=0$ and $t>0 \therefore \underline{u} = u(y,t)\hat{x}$

4 b N-S: $\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \cancel{\rho g} + \mu\nabla^2\underline{u} \quad \therefore \quad g = 0 \therefore$

$\rho(\partial_t \underline{u} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \mu\nabla^2\underline{u} \therefore$

$\underline{u}\cdot\nabla = u(y,t)\frac{\partial}{\partial x} \therefore \quad (\underline{u}\cdot\nabla)\underline{u} = u(y,t)\frac{\partial}{\partial x}(u(y,t)) = u(y,t)(0) = 0 \therefore$

$\rho\partial_t\underline{u} = -\nabla p + \mu\nabla^2\underline{u} \therefore$

$\nabla^2\underline{u} = \nabla^2((u(y,t),0,0)) = \nabla^2 u(y,t)\hat{i} = u_{yy}\hat{i} \therefore$

x component: $\rho\frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} + \mu u_{yy} = u_{yy} \therefore \quad \frac{\partial u}{\partial t} = \nu u_{yy}$

y component: $\rho\frac{\partial 0}{\partial t} = 0 = -\frac{\partial p}{\partial y} + 0 = 0 = \frac{\partial p}{\partial y} \therefore$

NS: $\partial_t u = \nu u_{yy}$

4 c fluid moves with the boundary on the boundary and that far away from the boundary, the fluid does know about the boundary motion at speed U

4 d let $u' = \frac{u}{U} \therefore \frac{\partial u'}{\partial t} = \frac{1}{U}\frac{\partial u}{\partial t}, \quad \frac{\partial u'}{\partial y} = \frac{1}{U}\frac{\partial u}{\partial y} \therefore$

$u'_{yy} = \frac{1}{U}u_{yy} \therefore \cancel{\frac{1}{U}u_t = \nu\frac{1}{U}u_{yy}} \therefore U u'_{yy} = u_{yy}, \quad U u'_t = u_t \therefore$

$u_t = \nu u_{yy} = U u'_t \Rightarrow \nu U u'_{yy} \quad \therefore u'_t = \nu u'_{yy} \therefore$

$u(y, t=0) = 0 = U u'(y, t=0) = 0 = u'(y, t=0), \quad u(y=0, t>0) = U = U u'(y=0, t>0) \therefore$

$u'(y=0, t>0) = 1 \quad u(y\to\infty, t>0) = U u'(y\to\infty, t>0) \to 0 \therefore \quad u'(y\to\infty, t>0) \to 0$

\PP2019/ \4e/ $[y]=L$, $[t]=T$, $\therefore \left[\frac{\partial u}{\partial t}\right]=\left[\frac{\partial}{\partial t}\right][\underline{u}]=T^{-1}[\underline{u}]=T^{-1}\times LT^{-1}=LT^{-2}$ $\therefore$

$[\nu\nabla^2\underline{u}]=\left[\frac{\partial u}{\partial t}\right]=LT^{-2}=[\nu][\nabla^2\underline{u}]=[\nu]L^{-2}[\underline{u}]=[\nu]L^{-2}LT^{-1}=[\nu]L^{-1}T^{-1}=LT^{-2}$ $\therefore$

$[\nu]=L^2T^{-1}$ $\therefore$

y, t indep, ν dependent $\therefore$

The quantity of interest is u' $\therefore$

3 params $[y]=L$, $[t]=T$, $[\nu]=L^2T^{-1}$ $\therefore$

$n=3$, $k=2$, $m=1$ $\therefore$ $a_1=y$, $a_2=t$ $\therefore$

$\Pi_1=\frac{\nu}{a_1^\alpha a_2^\beta}=\frac{\nu}{y^\alpha t^\beta}$ $\therefore$ $[\nu]=L^2T^{-1}=[y]^\alpha[t]^\beta=L^\alpha T^\beta$ $\therefore$

$\alpha=2$, $\beta=-1$ $\therefore$ $\Pi_1=\frac{\nu}{y^2t^{-1}}$ $\therefore$

$\Pi=\Phi(\Pi_1)=\Phi\left(\frac{\nu}{y^2t^{-1}}\right)$

where Π is the quantity of interest made dimensionless $\therefore$

Π_1 is the dependent params made dimensionless using the params with independent $\therefore$

$\Pi=u'$, $\Pi_1=\frac{\nu}{y^2t^{-1}}$ $\therefore$

$u'=\Phi(\eta)=\Phi\left(\frac{y}{\sqrt{4\nu t}}\right)$ $\therefore$

let $\Pi=\hat{\Phi}_1(\Pi_1)=\hat{\Phi}_1\left(\frac{\nu}{y^2t^{-1}}\right)=\hat{\Phi}_2\left(\left(\frac{\nu}{y^2t^{-1}}\right)^{-1/2}\right)=\hat{\Phi}_2\left(\left(\frac{y^2t^{-1}}{\nu}\right)^{1/2}\right)=\hat{\Phi}_2\left(\left(\frac{y^2}{\nu t}\right)^{1/2}\right)=\hat{\Phi}_2\left(\sqrt{\frac{y^2}{\nu t}}\right)=\hat{\Phi}_2\left(\frac{y}{\sqrt{\nu t}}\right)=\Phi\left(\frac{1}{16}\frac{y}{\sqrt{\nu t}}\right)=\Phi\left(\frac{1}{\sqrt{4}}\frac{y}{\sqrt{\nu t}}\right)=\Phi\left(\frac{y}{\sqrt{4\nu t}}\right)=u'$

\4f/ $u'=\Phi\left(\frac{y}{\sqrt{4\nu t}}\right)$ $\therefore$ $u'_t=\nu u'_{yy}$ $\therefore$ $u'=\Phi\left(\frac{y}{\sqrt{4\nu}}t^{-1/2}\right)$ $\therefore$

$u'_t=\Phi'\left(\frac{y}{\sqrt{4\nu}}t^{-1/2}\right)\left(-\frac{1}{2}t^{-3/2}\right)=-\frac{1}{2}t^{-3/2}\Phi'(\eta)$

$u'_y=\frac{1}{\sqrt{4\nu}}t^{-1/2}\Phi'\left(\frac{y}{\sqrt{4\nu}}t^{-1/2}\right)$ $\therefore$ $u'_{yy}=\frac{1}{4\nu}t^{-1}\Phi''(\eta)$ $\therefore$

$u'_t-\nu u'_{yy}=0=-\frac{1}{2}t^{-3/2}\Phi'-\frac{\nu}{4\nu}t^{-1}\Phi''(\eta)=0=-\frac{1}{2}t^{-3/2}\Phi'-\frac{1}{4t}\Phi''(\eta)$ $\therefore$

$\Phi''(\eta)+2t^{-3/2}t\Phi'(\eta)=\Phi''(\eta)+2\eta\Phi'(\eta)=0=\Phi''(\eta)+2\eta\Phi'$ $\therefore$

BC: $u'(y, t=0)=0$, $u'(y=0, t>0)=1$, $u'(y\to\infty, t>0)\to 0$ $\therefore$

$u'=\Phi(\eta)=\Phi\left(\frac{y}{\sqrt{4\nu t}}\right)$ $\therefore$ ~~$u'(y, t=0)$~~

$u'(y=0, t>0)=\Phi(0)=1$, $u'(y\to\infty, t>0)=\Phi(\infty)=0$ $\therefore$

$\Phi(\eta\to\infty)\to 0$

\4g/ ~~let~~ $\cancel{g}$ let $\gamma = \Phi'$ $\therefore$ $\gamma' + 2\eta\gamma = 0 = P(\eta)\gamma' + 2\eta P(\eta)\gamma = 0 =$

$\frac{\partial}{\partial \eta}(P(\eta)\Phi)$ $\therefore$ $\frac{d}{d\eta}P(\eta) = 2\eta P(\eta)$ $\therefore$ $\int \frac{P'(\eta)}{P(\eta)} d\eta = \int 2\eta \, d\eta = \eta^2 = \ln P(\eta)$ $\therefore$

$P(\eta) = e^{\eta^2}$ $\therefore$ $\frac{\partial}{\partial \eta}(e^{\eta^2}\gamma) = 0$ $\therefore$

$e^{\eta^2}\gamma = e^{\eta^2}\Phi' = A$ $\therefore$ $\cancel{\Phi = A e^{\eta}}$ $\Phi' = A e^{-\eta^2}$ $\therefore$

$\therefore$ $\cancel{g}$ $\Phi = A \int e^{-\eta^2} d\eta = \Phi(\eta)$ $\therefore$ $\cancel{\Phi(\eta) = A\int_0^{\eta} e^{-s^2} ds}$ $\therefore$

$\cancel{\Phi(0) = 1 = A\int_0^0 e^{-s^2} ds}$

$\Phi(\eta) = A \int_0^{\eta} e^{-s^2} ds + B$ $\therefore$

$\Phi(0) = 1 = A\int_0^0 e^{-s^2} ds + B = A(0) + B = B = 1$ $\therefore$

$\Phi(\eta) = A\int_0^{\eta} e^{-s^2} ds + 1$ $\therefore$

$\Phi(\eta \to \infty) = A\int_0^{\infty} e^{-s^2} ds + 1 = 0 = A(1) + 1$ $\therefore$ $A = -1$ $\therefore$

$\cancel{\Phi(\eta) = 2\int}$ $\Phi(\eta) = -\int_0^{\eta} e^{-s^2} ds + 1$

\4h/ force on area element ds with unit normal $\underline{n}$ is

$dF_i = \sigma_{ij} n_j \, dS = \Sigma_i \, dS$ $\therefore$ σ_{ij} is the stress tensor ~~of~~ $\therefore$

$\underline{n}$ normal to surface $\therefore$ $\Sigma_i^{viscous}\big|_{y=0} = -U\left(\frac{\rho\mu}{t\pi}\right)^{1/2}$

\pp2022/ \1a/ $\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u}$ is continuity equation

$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{u}) = \frac{\partial \rho}{\partial t} + \rho \nabla \cdot u + u \cdot \nabla \rho = 0$ $\therefore$

incompressible: $\nabla \cdot \underline{u} = 0 \therefore$ $[\nabla \cdot \underline{u}]_i = \partial_i u_i = 0$

$\left[\frac{D\rho}{Dt} + \rho \nabla \cdot \underline{u}\right]_i = \left[\frac{\partial \rho}{\partial t} + \rho \nabla \cdot \underline{u} + \underline{u} \cdot \nabla \rho\right]_i = \left[\frac{\partial \rho}{\partial t}\right]_i + [\rho \nabla \cdot \underline{u}]_i + [\underline{u} \cdot \nabla \rho]_i =$

$\left(\frac{\partial \rho}{\partial t}\right)_i + \rho \partial_i \cdot u_i + u_i \partial_i \rho = \left(\frac{\partial \rho}{\partial t}\right)_i + \rho (\partial_i u_i) + u_i \partial_i \rho = 0$

continuity equation $\therefore$ incompressible $\therefore \frac{D\rho}{Dt} = 0 \therefore$

$\left[\frac{D\rho}{Dt}\right]_i = \left[\frac{\partial \rho}{\partial t} + u \cdot \nabla \rho\right]_i = \left(\frac{\partial \rho}{\partial t}\right)_i + u_i \cdot \partial_i \rho = 0 \therefore$

$\left(\frac{\partial \rho}{\partial t}\right)_i + \rho (\partial_i u_i) + u_i \partial_i \rho = 0 = \rho (\partial_i u_i) = 0 = \partial_i u_i = \frac{\partial u_i}{\partial x_i} = 0$

\1bi/ incompressible $\therefore \nabla \cdot \underline{u} = 0 \therefore$

$\left[\frac{\partial \underline{u}}{\partial t}\right]_i = \frac{\partial u_i}{\partial t}$, $[\underline{u} \cdot \nabla \underline{u}]_i = [(\underline{u} \cdot \nabla) \underline{u}]_i = u_j \partial_j u_i = u_j \frac{\partial u_i}{\partial x_j}$

$[\nabla p]_i = \frac{\partial p}{\partial x_i}$ $[g]_i = g_i$ $\cancel{[\nabla \times \underline{u}]_i} = \cancel{[\nabla \cdot (\nabla \cdot \underline{u})} \cancel{[\nabla (\nabla \cdot \underline{u})]_i}$

$[\nabla^2 \underline{u}]_i = \frac{\partial^2}{\partial x_j \partial x_j} u_i \therefore$

$\rho \left(\frac{\partial u_i}{\partial t} + u_j \frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + \rho g_i + \mu \frac{\partial^2 u_i}{\partial x_j \partial x_j}$ is ith component of N-S

\1bii/ $\underline{\omega} = \nabla \times \underline{u} \therefore \omega_i = [\nabla \times \underline{u}]_i = \varepsilon_{ijk} \partial_j u_k$

\1ci/ streamline equations $\frac{dx}{u} = \frac{dy}{v} = \frac{dz}{w}$

$\therefore \underline{u} = \left(\frac{x}{1+t}, \frac{y}{1+t}, \frac{z}{1+t}\right) \therefore$

$\frac{dx}{\left(\frac{x}{1+t}\right)} = \frac{dy}{\left(\frac{y}{1+t}\right)} + \frac{dz}{\left(\frac{z}{1+t}\right)} \therefore \frac{1}{x} dx = \frac{1}{y} dy = \frac{1}{z} dz \therefore$

$\int \frac{1}{x} dx = \int \frac{1}{y} dy = \int \frac{1}{z} dz = \ln|x| = \ln|y| + c_1 = \ln|z| + c_2 \therefore$

$|x| = e^{\ln|y| + c_1} = A_1 |y| = e^{\ln|z| + c_2} = A_2 |z| \therefore$

$x = \pm A_1 y = \pm A_2 z = B_1 y = B_2 z = x$ $\therefore (1, B_1, B_2)$ B_1, B_2 are consts

~~is the ror~~ streamlines $\frac{dx_1}{u_1} = \frac{dx_2}{u_2}$, $\frac{dx_1}{u_1} = \frac{dx_3}{u_3} \therefore$

$\frac{dx}{\left(\frac{x}{1+t}\right)} = \frac{dy}{\left(\frac{y}{1+t}\right)} \therefore \frac{1}{x} dx = \frac{1}{y} dy \therefore \ln x = \ln y + c_1 \therefore x = Ay$

$\frac{dx}{\left(\frac{x}{1+t}\right)} = \frac{dz}{\left(\frac{z}{1+t}\right)} \therefore \frac{1}{x} dx = \frac{1}{z} dz \therefore \ln x = \ln z + c_2 \therefore x = Bz$

$\frac{dy}{\left(\frac{y}{1+t}\right)} = \frac{dz}{\left(\frac{z}{1+t}\right)} \therefore \frac{dy}{y} = \frac{dz}{z} \therefore \ln y = \ln z + c_3 \therefore y = Cz \therefore$

$\frac{x}{A} = Cz \therefore x = ACz = Bz \therefore AC = B \therefore C = \frac{B}{A} \therefore y = \frac{B}{A} z \therefore (x, Ay, Bz)$

\1cii/ $\therefore \frac{dx}{dt} = u$, $\frac{dy}{dt} = v$, $\frac{dz}{dt} = w$ $\therefore$

$\underline{u} = \left(\frac{x}{1+t}, \frac{y}{1+t}, \frac{z}{1+t}\right) \therefore$

$\frac{dx}{dt} = \frac{x}{1+t} \therefore \int \frac{1}{x}dx = \int \frac{1}{1+t}dt = \ln x = \ln(1+t) + C_1 \therefore$

$x = e^{\ln(1+t)+C_1} = A_1(1+t) = x$,

$\frac{dy}{dt} = \frac{y}{1+t} \therefore \int \frac{1}{y}dy = \int \frac{1}{1+t}dt = \ln y = \ln(1+t) + C_2 \therefore$

$y = e^{\ln(1+t)+C_2} = A_2(1+t) = y \therefore$

$\frac{dz}{dt} = \frac{z}{1+t} \therefore \int \frac{1}{z}dz = \int \frac{1}{1+t}dt = \ln(z) = \ln(1+t) + C_3 \therefore$

$z = e^{\ln(1+t)+C_3} = A_3(1+t) = z$ are the particle paths

$(x, y, z) = (A_1(1+t), A_2(1+t), A_3(1+t))$

\1ciii/ $\therefore (1, 1, 0) = (A_1(1+0), A_2(1+0), A_3(1+0)) = (A_1, A_2, A_3) \therefore$

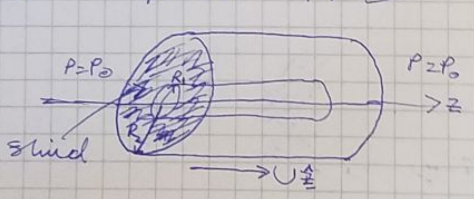
$A_1 = 1$, $A_2 = 1$, $A_3 = 0$ $\therefore \therefore (x, y, z) = (1+t, 1+t, 0) \therefore$

at $t = 2$: $(x, y, z)|_{t=2} = (1+2, 1+2, 0) = (3, 3, 0)$

\1di/ incompressible $\therefore \nabla \cdot \underline{u} = 0$

neglect gravity $\therefore g = 0$

open to athmosphere $\therefore \nabla P = 0$



\1dii/ no slip boundardy conditions $\therefore$

no special z or θ coordinates $\therefore$ flow independent of z and θ.

~~speed~~ no slip BCs $\therefore$ flow speed dissecant at ~~$R=$~~ $R = R_1$ and $R = R_2$ $\therefore$ flow dependent on R.

~~flow~~ no pressure gradient $\therefore$ flow only dependent on R directed in the $\hat{z}$ direction due to boundary movement

$\therefore \underline{u} = w\hat{z} \therefore w = w(R) \therefore \underline{u} = w(R)\hat{z}$

\1diii/ condition is $\nabla \cdot \underline{u} = 0$ $\therefore \nabla \cdot \underline{u} = \nabla \cdot w(R)\hat{z} = \nabla \cdot (0, 0, w(R)) =$

$\frac{1}{R}\frac{\partial}{\partial R}(R \times 0) + \frac{1}{R}\frac{\partial (0)}{\partial \theta} + \frac{\partial w(R)}{\partial z} = \frac{1}{R}\frac{\partial}{\partial R}(0) + \frac{1}{R}(0) + 0 =$

$0 + 0 = 0$

\PP2022/ \1div/ NS: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = \mu\nabla^2\underline{u} - \frac{1}{\rho}\nabla P \quad \frac{\mu}{\rho} = \nu$

$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = \nu\nabla^2\underline{u} - \frac{1}{\rho}\nabla P$ ∴

∴ z-component: $\frac{\partial \underline{u}}{\partial t} = 0$ ∴

$\underline{u}\cdot\nabla\underline{u} = \nu\nabla^2\underline{u}$ ∴ $\left[\nabla^2\underline{u}\right]_z = \nabla^2 w(R) = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial w(R)}{\partial R}\right)$ ∴

$\underline{u}\cdot\nabla\underline{u} = \underline{u}\cdot\nabla(0,0,w(R)) = (0,0,w(R))\cdot\nabla(0,0,w(R)) =$

$w(R)\frac{\partial}{\partial z}w(R) = w(R)\times 0 = 0$ ∴

$\frac{\partial \underline{u}}{\partial t} = \frac{\partial}{\partial t}(0,0,w(R)) = (0,0,0)$ ∴

$\nu\nabla^2\underline{u} - \frac{1}{\rho}\nabla P = 0$ ∴ $\nabla P = \nabla^2\underline{u}$ ∴

$\nabla^2\underline{u} = \nabla^2(0,0,w(R)) = \left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial w(R)}{\partial R}\right) + \frac{1}{R^2}\frac{\partial^2 w(R)}{\partial \theta^2} + \frac{\partial^2 w(R)}{\partial z^2}\right)\underline{\hat{z}} =$

$\left(\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial w(R)}{\partial R}\right)\right)\underline{\hat{z}}$ ∴

$\mu\frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{\partial w}{\partial R}\right)\underline{\hat{z}} = \nabla P$ ∴

R component: $\frac{\partial P}{\partial x} = 0$, component: $\frac{\partial P}{\partial y} = 0$

z component: $\frac{\partial P}{\partial z} = \frac{\mu}{R}\frac{d}{dR}\left(R\frac{dw}{dR}\right)$

\1dv/ cylinders are open at both sides and at both ends $P = P_0$ ∴

$\lim_{z\to-\infty} P = P_0$, $\lim_{z\to\infty} P = P_0$ ∴ $\nabla P = 0 = \frac{\partial P}{\partial R}\underline{\hat{R}} + \frac{\partial P}{\partial \theta}\underline{\hat{\theta}} + \frac{\partial P}{\partial z}\underline{\hat{z}}$ ∴

z component: $\frac{\partial P}{\partial z} = 0$ ∴ $\frac{P_0 - P_0}{L} = \frac{0}{L} = 0$

∴ There is no preassure disserential in the cylinders along the z axis

\1dvi/ $\frac{dP}{dz} = 0$ ∴ $0 = \frac{\mu}{R}\frac{d}{dR}\left(R\frac{dw}{dR}\right)$ ∴ $\frac{d}{dR}\left(R\frac{dw}{dR}\right) = 0$ ∴

$R\frac{dw}{dR} = A$ ∴ $\frac{dw}{dR} = A\frac{1}{R}$ ∴ $w = A\ln|R| + B$

but $R \geq 0$ ∴ $\ln|R| = \ln R$ ∴ $w = A\ln R + B$

\1dvii/ BCs: at $R = R_1$: $\underline{u}(R=R_1) = 0$

at $R = R_2$: $\underline{u}(R=R_2) = U\underline{\hat{z}}$ ∴

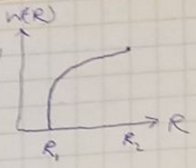
$\underline{u} = w(R)\underline{\hat{z}}$ ∴ $\underline{u} = (A\ln R + B)\underline{\hat{z}}$ ∴

$\underline{u}(R=R_1) = 0 = (A\ln R_1 + B)\underline{\hat{z}} = 0 = A\ln R_1 + B$ ∴ $-A\ln R_1 = B$ ∴

$\underline{u} = (A\ln R - A\ln R_1)\underline{\hat{z}}$ ∴ $\underline{u}(R=R_2) = U = A\ln R_2 - A\ln R_1 = A\ln\left(\frac{R_2}{R_1}\right)$ ∴ $A = \frac{U}{\ln\left(\frac{R_2}{R_1}\right)}$

$\underline{u} = \left(\frac{U}{\ln(R_2/R_1)}\ln\left(\frac{R}{R_1}\right)\right)\underline{\hat{z}}$

\1dviii/ $\frac{R_2}{R_1}$ $\therefore$ $w = \frac{U}{\ln\left(\frac{R_2}{R_1}\right)} \ln\left(\frac{R}{R_1}\right)$ $\therefore$



$\therefore w(R_2) = U$, $w(R_1) = 0$ $\therefore$

The profile agrees with no fluid movement at $R = R_1$ and fluid speed U at $R = R_2$

$\therefore$ ~~of~~ no slip boundaries.

\2a/ NS: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho g + \mu\nabla^2\underline{u}$ $\therefore$ $\frac{\mu}{\rho} = \nu$ $\therefore$

$\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\nabla\frac{p}{\rho} + g + \nu\nabla^2\underline{u}$

$\therefore \underline{\omega} = \nabla\times\underline{u}$ $\therefore$ $g = 0$ $\therefore$

$\frac{\partial \underline{u}}{\partial t} = -\underline{u}\cdot\nabla\underline{u} - \nabla\frac{p}{\rho} + \nu\nabla^2\underline{u}$ $\therefore$

$\therefore$ $u^2 = \underline{u}\cdot\underline{u}$ $\therefore$

$-\underline{u}\cdot\nabla\underline{u} = \underline{u}\times\underline{\omega} - \nabla\left(\frac{u^2}{2}\right) = \underline{u}\times(\nabla\times\underline{u}) - \nabla\left(\frac{\underline{u}\cdot\underline{u}}{2}\right) = -\underline{u}\cdot\nabla\underline{u}$ $\therefore$

$2\underline{u}\times(\nabla\times\underline{u}) - \nabla(\underline{u}\cdot\underline{u}) = -2\underline{u}\cdot\nabla\underline{u}$

~~As~~ $g = 0$ $\therefore$ $\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\nabla\frac{p}{\rho} + \nu\nabla^2\underline{u}$

$\therefore$ $\underline{\omega} = \nabla\times\underline{u}$ $\therefore$ $\frac{\partial \underline{u}}{\partial t} = -\underline{u}\cdot\nabla\underline{u} - \nabla\frac{p}{\rho} + \nu\nabla^2\underline{u}$ $\therefore$

$-2(\underline{u}\cdot\nabla)\underline{u} = -2\underline{u}\cdot\nabla\underline{u} = \underline{u}\times(\nabla\times\underline{u}) + \underline{u}\times(\nabla\times\underline{u}) - \nabla(\underline{u}\cdot\underline{u}) = 2\underline{u}\times(\nabla\times\underline{u}) - \nabla(u^2)$

$\therefore -(\underline{u}\cdot\nabla)\underline{u} = \underline{u}\times(\nabla\times\underline{u}) - \nabla\left(\frac{u^2}{2}\right) = \cancel{-(\underline{u}\cdot\nabla)}\ \underline{u}\cdot\nabla\underline{u} = \underline{u}\times\underline{\omega} - \nabla\left(\frac{u^2}{2}\right)$ $\therefore$

$\frac{\partial \underline{u}}{\partial t} = \underline{u}\times\underline{\omega} - \nabla\left(\frac{u^2}{2}\right) - \nabla\left(\frac{p}{\rho}\right) + \nu\nabla^2\underline{u} = \underline{u}\times\underline{\omega} - \nabla\left(\frac{p}{\rho} + \frac{u^2}{2}\right) + \nu\nabla^2\underline{u}$

\2b/ steady $\therefore$ $\frac{\partial \underline{u}}{\partial t} = 0$ invisid $\therefore$ $\mu = 0$ $\therefore$ $\nu = 0$ $\therefore$

$0 = \underline{u}\times\underline{\omega} - \nabla\left(\frac{p}{\rho} + \frac{u^2}{2}\right)$ $\therefore$

$\underline{u}\times\underline{\omega} = \nabla\left(\frac{p}{\rho} + \frac{u^2}{2}\right)$ $\therefore$

$\underline{u} = (u_1, u_2, u_3)$ $\therefore$ along streamlines $\underline{u}\times\underline{\omega} = 0$ $\therefore$

$\nabla\left(\frac{p}{\rho} + \frac{u^2}{2}\right) = 0$ $\therefore$ $\frac{p}{\rho} + \frac{u^2}{2} = \text{constant}$ $\therefore$ $H = \frac{p}{\rho} + \frac{u^2}{2} = \text{constant}$

\2c/ $\frac{\partial \underline{u}}{\partial t} = \underline{u}\times\underline{\omega} - \nabla\left(\frac{p}{\rho} + \frac{u^2}{2}\right) + \nu\nabla^2\underline{u}$ $\therefore$

$\nabla\times\left(\frac{\partial \underline{u}}{\partial t}\right) = \frac{\partial}{\partial t}(\nabla\times\underline{u}) = \frac{\partial \underline{\omega}}{\partial t}$

$\nabla\times\nabla^2\underline{u} = \nabla\times(\nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{u})) = \nabla\times(\nabla(\nabla\cdot\underline{u})) - \nabla\times(\nabla\times(\nabla\times\underline{u})) = -\nabla\times(\nabla\times(\underline{\omega}))$

$= -\nabla\times(\nabla\times\underline{\omega}) = -\nabla(\nabla\cdot\underline{\omega}) - \nabla^2\underline{\omega} = -\nabla(\nabla\cdot(\nabla\times\underline{\omega})) - \nabla^2\underline{\omega} = -\nabla(0) - \nabla^2\underline{\omega} = -\nabla^2\underline{\omega}$

$-\nabla\times\nabla\left(\frac{p}{\rho} + \frac{u^2}{2}\right) = -0 = 0$

$\nabla\times(\underline{u}\times\underline{\omega}) = \underline{u}(\nabla\cdot\underline{\omega}) - \underline{\omega}(\nabla\cdot\underline{u}) + (\underline{\omega}\cdot\nabla)\underline{\omega} - (\underline{u}\cdot\nabla)\underline{\omega} =$

\PP2022/ $\nabla \times (\underline{u} \times \underline{\omega}) = -\underline{\omega}(\nabla \cdot \underline{u}) + ((\underline{\omega} \cdot \nabla)\underline{u}) - (\underline{u} \cdot \nabla)\underline{\omega}$ $\therefore$

$\frac{D\underline{\omega}}{Dt} = \frac{\partial \underline{\omega}}{\partial t} + \underline{u} \cdot \nabla \underline{\omega}$ $\therefore$

$\frac{\partial \underline{\omega}}{\partial t} = -\underline{\omega}(\nabla \cdot \underline{u}) + (\underline{\omega} \cdot \nabla)\underline{u} - (\underline{u} \cdot \nabla)\underline{\omega} + \nu \nabla^2 \underline{\omega}$

$\therefore \nabla \cdot \underline{u} = 0 \therefore -\underline{\omega}(\nabla \cdot \underline{u}) = -\underline{\omega}(0) = 0 \therefore$

$(\underline{\omega} \cdot \nabla)\underline{u} = \underline{\omega} \cdot \nabla \underline{u}$, $-(\underline{u} \cdot \nabla)\underline{\omega} = -\underline{u} \cdot \nabla \underline{\omega} \therefore$

$\frac{\partial \underline{\omega}}{\partial t} + \underline{u} \cdot \nabla \underline{\omega} = \underline{\omega} \cdot \nabla \underline{u} + \nu \nabla^2 \underline{\omega} = \frac{D\underline{\omega}}{Dt} = \left(\frac{\partial}{\partial t} + \underline{u} \cdot \nabla\right)\underline{\omega}$

$\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u} \cdot \nabla$

\2d/ $\frac{D}{Dt}$ is the material derivative

$\therefore \frac{D\underline{\omega}}{Dt}$ is the material derivative of $\underline{\omega}$ $\therefore$ is the rate of change of vorticity following the fluid motion

$\nu \nabla^2 \underline{\omega}$ is the kinematic viscosity multiplied by the gradient squared vector of the vorticity

$\underline{\omega} \cdot \nabla \underline{u}$ is the vector of the vorticity dot product with the gradient operator operated on the velocity

\2ei/ $\underline{\omega} = \nabla \times \underline{u} = \nabla \times (U(y,t), -ay, az) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{z} \\ \partial x & \partial y & \partial z \\ U(y,t) & -ay & az \end{vmatrix} =$

$\hat{i}\left(\frac{\partial}{\partial y}(az) - \frac{\partial}{\partial z}(-ay)\right) - \hat{j}\left(\frac{\partial}{\partial x}(az) - \frac{\partial}{\partial z}(U(y,t))\right) + \hat{z}\left(\frac{\partial}{\partial x}(-ay) - \frac{\partial}{\partial y}U(y,t)\right) =$

$\hat{i}(0-0) - \hat{j}(0-0) + \hat{z}\left(0 - \frac{\partial U(y,t)}{\partial y}\right) = \cancel{\hat{z}} - \frac{\partial U(y,t)}{\partial y}\hat{z} = (0, 0, \omega(y,t))$

for $\omega(y,t) = -\frac{\partial U(y,t)}{\partial y}$

\2eii/ $\frac{\partial \underline{\omega}}{\partial t} + \underline{u} \cdot \nabla \underline{\omega} = \underline{\omega} \cdot \nabla \underline{u} + \nu \nabla^2 \underline{\omega}$ $\underline{\omega} = \omega \hat{z}$ $\therefore$

z component: $\frac{\partial \omega}{\partial t} + (az)\frac{\partial}{\partial z}\omega(y,t) = (\omega)\frac{\partial}{\partial z}(az) + \nu \frac{\partial^2}{\partial y^2}\omega(y,t) = \frac{\partial \omega(y,t)}{\partial t}$

$\therefore$ steady $\therefore$ independent of t $\therefore$ $\omega = \omega(y)$ $\therefore$

$\frac{\partial \omega}{\partial t} = 0 \therefore$ $\omega \frac{\partial}{\partial z}(az) = a\omega + \nu \frac{\partial^2}{\partial y^2}\omega(y) = 0$ $\therefore$

$\frac{d^2}{dy^2}\omega(y) + \frac{a}{\nu}\omega(y) = 0$ $\therefore$ let $\omega = e^{qy} \therefore \omega' = qe^{qy} \therefore \omega'' = q^2 e^{qy}$

$\therefore q^2 e^{qy} + \frac{a}{\nu}e^{qy} = 0 = q^2 + \frac{a}{\nu} \therefore q^2 = -\frac{a}{\nu} \therefore q = \pm\sqrt{\frac{a}{\nu}}\, i \therefore$

$\omega(y) = A_1 \cos\left(\sqrt{\frac{a}{\nu}}y\right) + B_1 \sin\left(\sqrt{\frac{a}{\nu}}y\right)$

\2cii/ $\omega = \omega(y) \therefore \frac{\partial \omega}{\partial t} = 0 \therefore \underline{u} \cdot \nabla \underline{\omega} = \underline{\omega} \cdot \nabla \underline{u} + \nu \nabla^2 \underline{\omega} \therefore$

$\underline{u} \cdot \nabla = \left(U \frac{\partial}{\partial x} + -ay \frac{\partial}{\partial y} + az \frac{\partial}{\partial z} \right) \therefore \quad \nu \nabla^2 \underline{\omega} = \nu \nabla^2 \omega(y) = \nu \frac{\partial^2 \omega(y)}{\partial y^2} = \nu \frac{d^2 \omega(y)}{dy^2}$

$\underline{u} \cdot \nabla \underline{\omega} = \left(U \frac{\partial}{\partial x} - ay \frac{\partial}{\partial y} + az \frac{\partial}{\partial z} \right) \omega(y) \hat{\underline{z}} =$

$-ay \frac{\partial \omega}{\partial y} \hat{\underline{z}}$

$\underline{\omega} \cdot \nabla = \left(0 \hat{\underline{x}} + 0 \hat{\underline{y}} + \omega(y) \hat{\underline{z}} \right) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right) = \omega(y) \frac{\partial}{\partial z} \therefore$

$\underline{\omega} \cdot \nabla \underline{u} = \omega(y) \frac{\partial}{\partial z} \left(U(y,t), -ay, az \right) = \left(0, 0, \omega(y) \frac{\partial (az)}{\partial z} \right) = \omega(y) a \hat{\underline{z}}$

$\therefore \hat{\underline{z}}$ component: $\omega(y) = a\omega(y) + \nu \frac{d^2 \omega}{dy^2} \therefore$

$0 = (a-1)\omega(y) + \nu \frac{\partial^2 \omega}{\partial y^2} \therefore$

let $\omega(y) = Ae^{-y^2/\delta^2} \therefore \omega' = -2yAe^{-y^2/\delta^2} \therefore$

$\omega'' = 4y^2 A e^{-y^2/\delta^2} \therefore \therefore$

$(a-1) A e^{-y^2/\delta^2} + \nu 4 y^2 A e^{-y^2/\delta}$

\2eii/ ~~$\omega(y) = Ae$~~ $\omega(y) = Ae^{-y^2/\delta^2} \therefore$

$\underline{\omega} = (0, 0, \omega(y)) = \omega(y) \hat{\underline{z}} = Ae^{-y^2/\delta^2} \therefore$

$\underline{\omega}|_{y=0} = 0 = \omega(0) \hat{\underline{z}} = Ae^{-0/\delta^2} = Ae^{-0} = Ae^{0} = A = 0$ X

$\omega(y) = Ae^{-y^2/\delta^2}$

\2eiii/ $\omega(y) = Ae^{-y^2/\delta^2} \therefore$

$-\frac{\partial U(y)}{\partial y} = \omega(y) = Ae^{-y^2/\delta^2} \therefore$

$+U(y) = -\int_0^y Ae^{-s^2/\delta^2} ds + C_1 \therefore$

$U(0) = U(y=0) = -\int_0^0 Ae^{-s^2/\delta^2} + C_1 = 0 = C_1 \therefore$

$U(y) = -\int_0^y Ae^{-s^2/\delta^2} ds = -A \int_0^y e^{-\left(\frac{s}{\delta}\right)^2} d\left(\frac{s}{\delta}\right) = -A \operatorname{erf}(y) \frac{1}{4} \pi^{1/2} =$

$\frac{\pi^{1/2} A}{4} \operatorname{erf}(y)$

\3a/ $\frac{\partial \underline{\omega}}{\partial t} = \nu \nabla^2 \underline{\omega} + \nabla \times (\underline{u} \times \underline{\omega})$

$\underline{u} = \nabla \times (\psi \underline{k}) \therefore$ ~~$\nabla \times \underline{\omega} = \nabla \times \nabla$~~ $\underline{\omega} = \nabla \times \underline{u} = \nabla \times \nabla \times (\psi \underline{k})$

\3b/ $\psi = Ex\delta(y) \therefore \nabla^2 \psi = \frac{\partial^2}{\partial x^2}(Ex\delta(y)) + \frac{\partial^2}{\partial y^2}(Ex\delta(y)) =$

$0 + Ex\delta'' = Ex\delta'' \quad \frac{\partial}{\partial t}(\nabla^2 \psi) = \frac{\partial}{\partial t}(Ex\delta''(y)) = 0$.

$\frac{\partial \psi}{\partial x} = E\delta(y), \frac{\partial}{\partial y}(\nabla^2 \psi) = \frac{\partial}{\partial y}(Ex\delta''(y)) = Ex\delta'''$.

$\frac{\partial \psi}{\partial x} \frac{\partial}{\partial y}(\nabla^2 \psi) = E\delta Ex\delta''' = E^2 x \delta\delta'''$

$\nu \nabla^2 (\nabla^2 \psi) = \nu \nabla^2 (Ex\delta'') = \nu \frac{\partial^2}{\partial y^2}(Ex\delta'') = \nu Ex\delta''''$

\9P2022/ $\frac{\partial \psi}{\partial y} = \frac{\partial}{\partial y}(Exf(y)) = Exf'$, $\frac{\partial}{\partial x}(\nabla^2\psi) = \frac{\partial}{\partial x}[Exf''] = Ef''$ ∴

$0 - EfExf''' + Exf'Ef'' = \nu Exf'''' = -E^2 x f f''' + E^2 x f' f'' =$

$\nu Exf'''' = E^2 x f' f'' - E^2 f f''' x$ ∴

$\nu E f'''' = E^2 f' f'' - E^2 f f'''$ ∴

$\nu f'''' = E f' f'' - f f'''$ ∴

$\frac{\nu}{E} f'''' = f' f'' - f f'''$ ∴

$(f'^2 - f f'')' = 2f' f'' - f' f'' - f f'''$ ∴

$+\frac{\nu}{E} f'''' = + f' f'' - f f''' = 2f' f'' - f' f'' - f f''' = (2f' f'') - (f' f'' + f f''') =$

$((f')^2)' - (f f'')' = ((f')^2 - f f'')' = (f'^2 - f f'')' = \frac{\nu}{E} f'''' = c f''''$ ∴

$c = \frac{\nu}{E}$

\3ci/ no slip BC ∴ $\underline{u}(y=0) = 0$ ∴

$\underline{u} = \nabla \times (\psi \underline{k})$, $\psi = Exf(y)$ ∴

$\underline{u} = \nabla \times (Exf(y)\underline{k}) = \begin{vmatrix} \hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & Exf(y) \end{vmatrix} = \hat{\underline{i}}\left(\frac{\partial}{\partial y}(Exf(y)) - 0\right) - \hat{\underline{j}}\left(\frac{\partial}{\partial x}(Exf(y)) - 0\right) + \hat{\underline{k}}(0-0)$

$= \hat{\underline{i}}(Exf'(y)) - \hat{\underline{j}}(Ef(y))$ ∴ $\underline{u} = (Exf'(y), -Ef(y), 0)$ ∴

∴ at $y=0$: $\underline{u} = \underline{u}(y=0) = (0, 0, 0)$ ∴

$Exf'(y=0) = 0 = f'(y=0)$, $-Ef(y=0) = 0 = f(y=0)$

\3cii/

$\underline{u}(y \to \infty) \to (Ex, -Ey, 0)$ ∴

$u(y \to \infty) = (Ex, -Ey, 0) = (Exf'(y \to \infty), -Ef(y \to \infty), 0)$ ∴

$Exf'(y \to \infty) \to Ex$ ∴ $f'(y \to \infty) \to 1$,

$-Ef(y \to \infty) \to -Ey$ ∴ $f(y \to \infty) \to y$

\3d/ $(f'^2 - f f'')' = c f''''$ ∴

$\int \frac{d}{dy}\left(\left(\frac{df(y)}{dy}\right)^2 - f(y)\frac{d^2 f(y)}{dy^2}\right) dy = \int c \frac{d^4 f(y)}{dy^4} dy =$

$c\frac{d^3 f(y)}{dy^3} + d = \left(\frac{df(y)}{dy}\right)^2 - f(y)\frac{d^2 f(y)}{dy^2} =$

$f'^2 - f f'' = c f''' + d = f'(y)^2 - f(y) f''(y) = \frac{\nu}{E} f'''(y) + d$ ∴

at $y=0$: $f(y=0) = 0$, $f'(y=0) = 0$ ∴ $f'(0)^2 - f(0) f''(0) = \frac{\nu}{E} f'''(0) + d =$

$0^2 - 0 f''(0) = 0 = \frac{\nu}{E} f'''(0) + d$ ∴ $d = -\frac{\nu}{E} f'''(0) = -c f'''(0)$

\3e/ $[y] = L$, $c = \frac{\nu}{E}$, $E = \text{constant}$ $\therefore$ $[E] = 1$ $\therefore$

$[c] = [\frac{\nu}{E}] = [\nu]$ $\therefore$

$[\frac{\partial u}{\partial t}] = [\nu \nabla^2 \underline{u}] = T^{-1}[\underline{u}] = [\nu][\nabla^2 \underline{u}] = T^{-1} L T^{-1} = [\nu] L^{-2} [\underline{u}] =$

$[\nu] L^{-2} L T^{-1} = [\nu] L^{-1} T^{-1} = L T^{-2}$ $\therefore$

$[\nu] = L^2 T^{-1}$ $\therefore$

$[y]$ and $[\nu]$ are independent

\3e/ $[y] = L$, 2 parameters $\therefore$

$[c] = L = [\frac{\nu}{E}] = [\nu][E]^{-1} = L^2 T^{-1} [E]^{-1} = L$ $\therefore$

$L T^{-1} = [E]$ $\therefore$

$[c] = L$, $[y] = L$ $\therefore$

y independent, c dependent $\therefore$ $n = 2$, $k = 1$, $m = 1$ $\therefore$

$\Pi_1 = \frac{c}{y^\alpha}$ $\therefore$ $\cancel{[y]}$ $[c] = L = [y]^\alpha = L^\alpha$ $\therefore$ $\alpha = 1$ $\therefore$

$\Pi_1 = \frac{c}{y}$ $\therefore$

$\Pi = \Phi(\Pi_1) = \Phi(\frac{c}{y})$ $\therefore$ $\Pi = \delta$ is quantity of interest $\therefore$

$\delta = \Phi(\frac{c}{y}) = c^{1/2} g(\eta) = \Phi(c y^{-1}) = \sqrt{c y^{-1}} = c^{1/2} y^{-1/2} = c^{1/2} g(\eta) = c^{1/2} \Phi(y^{-1/2})$

$\cancel{g(\eta) = y^{-1/2}}$ $\therefore$ $g(\eta) = \Phi(y^{-1/2})$ $\therefore$ $\eta = y^{-1/2}$

\3a/ NS: $\rho(\frac{\partial u}{\partial t}) + \rho(u \cdot \nabla)\underline{u} = -\nabla p + \rho \underline{f} + \mu \nabla^2 \underline{u}$ $\therefore$

$\nabla \cdot \underline{u} = 0$, $\underline{f} = 0$ $\therefore$ $\rho \frac{\partial u}{\partial t} + \rho \underline{u} \cdot \nabla \underline{u} = -\nabla p + \mu \nabla^2 \underline{u}$ $\therefore$

$\frac{\partial u}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\nabla p + \nu \nabla^2 \underline{u}$, $\frac{\mu}{\rho} = \nu$ $\therefore$ $\underline{\omega} = \nabla \times \underline{u}$ $\therefore$

$-\nabla \times \nabla p = -0 = 0$,

$\nabla \times \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \nabla \times (\underline{u}) = \frac{\partial}{\partial t} \nabla \times (\nabla \times (\psi \underline{k}))$ $\therefore$

$\nabla \times (\nabla \times (\psi \underline{k})) = \nabla(\nabla \cdot \psi \underline{k}) - \nabla^2 \psi \underline{k}$ $\therefore$

$\nu \nabla \times (\nabla^2 \underline{u}) = \nu \nabla \times (\nabla(\nabla \cdot \underline{u}) - \nabla \times (\nabla \times \underline{u})) = -\nu \nabla \times (\nabla \times (\nabla \times \underline{u}))$ $\therefore$

$\nu \nabla \times (\nabla^2 \underline{u}) = \nu \nabla \times (\nabla^2 \nabla \times (\psi \underline{k})) = -\nu \nabla \times (\nabla \times (\nabla \times (\nabla \times \psi \underline{k})))$

$= \nabla \times \nabla \times \nabla^2 (\psi \underline{k})$

$\cancel{\nabla}$ $\therefore$ $\frac{\partial u}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\nabla p + \nu \nabla^2 \underline{u}$, $\nabla \cdot \underline{u} = 0$, $-\nabla \times \nabla p = -0 = 0$ $\therefore$

$\nabla \times \frac{\partial u}{\partial t} = \frac{\partial}{\partial t} \nabla \times \underline{u}$ $\therefore$ $\nabla \times \underline{u} = \nabla \times \nabla \times (\psi \underline{k})$,

$\nu \nabla \times (\nabla^2 \underline{u}) = \nu \nabla \times \nabla(\nabla \cdot \underline{u}) - \nu \nabla \times \nabla \times (\nabla \times \underline{u}) = -\nu \nabla \times \nabla \times (\nabla \times \underline{u})$

PP2022 $\nabla \times (\psi \underline{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \psi(x,y,t) \end{vmatrix} = \hat{i}\left(\frac{\partial \psi}{\partial y}\right) - \hat{j}\frac{\partial \psi}{\partial x} = \underline{u} \therefore$

$\nabla \times \nabla \times (\psi \underline{k}) = \nabla \times \left(\hat{i}\frac{\partial \psi}{\partial y} - \hat{j}\frac{\partial \psi}{\partial x}\right) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \frac{\partial \psi}{\partial y} & -\frac{\partial \psi}{\partial x} & 0 \end{vmatrix} = \hat{\underline{k}}\left(-\frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi}{\partial y^2}\right) = -\nabla^2(\psi)\hat{\underline{k}}$

$\cancel{\underline{u} = (0, 0, \psi)} \therefore \quad \underline{u} = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0\right) \therefore$

$\underline{u} \cdot \nabla = \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y} \therefore$

$\underline{u} \cdot \nabla \underline{u} = \left(\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0\right) =$

$\left(\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial y^2}\right)\hat{\underline{i}} + \left(-\frac{\partial \psi}{\partial y}\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial \psi}{\partial x}\frac{\partial^2 \psi}{\partial x \partial y}\right)\hat{\underline{j}}$

$\nabla^2 \underline{u} = \nabla^2\left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}\right) = \cancel{\nabla^2}\left(\frac{\partial^3 \psi}{\partial x^2 \partial y} + \frac{\partial^3 \psi}{\partial y^3}, -\frac{\partial^3 \psi}{\partial x^3} - \frac{\partial^3 \psi}{\partial x \partial y^2}, 0\right) \therefore$

coefficients component:

$\frac{\partial}{\partial t}(\nabla^2 \psi) - \frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}(\nabla^2 \psi) + \frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2 \psi) = \nu \nabla^2(\nabla^2 \psi)$

$\therefore \nabla \times \nabla^2 \underline{u} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ \partial_{xx}\partial_y \psi + \partial_{yyy}\psi & -\partial_{xxx}\psi - \partial_x \partial_{yy}\psi & 0 \end{vmatrix} =$

$\hat{\underline{k}}\left(-\frac{\partial^4 \psi}{\partial x^4} - \frac{\partial^4 \psi}{\partial x^2 \partial y^2} - \frac{\partial^4 \psi}{\partial x^2 \partial y^2} - \frac{\partial^4 \psi}{\partial y^4}\right) = \hat{\underline{k}}(-\nabla^2 \nabla^2 \psi) = \cancel{\hat{k}} \; -\nabla^2(\nabla^2 \psi)\hat{\underline{k}} \therefore$

$\hat{\underline{k}}$ componente: $\partial_t \nabla^2 \psi - \partial_x \psi \, \partial_y \nabla^2 \psi + \partial_y \psi \, \partial_x \nabla^2 \psi = \nu \nabla^2(\nabla^2 \psi)$

4a incompressible $\therefore \nabla \cdot \underline{u} = 0 \qquad \nu = \frac{\mu}{\rho} \therefore$

NS: $\frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\nabla \frac{p}{\rho} + \cancel{g} + \nu \nabla^2 \underline{u}$, $\cancel{g}$ neglect gravity $\therefore g = 0 \therefore$

$\frac{dp}{dx} = 0 \qquad u(y \to \infty) \to 0 \;\therefore u(y \to \infty, t) \to 0$

$\therefore \underline{u}(u(y,t), -V, 0)$, $-V$ is constant $\therefore$

$\frac{\partial \underline{u}}{\partial t} = \frac{\partial u(y,t)}{\partial t}\hat{\underline{i}} - \frac{\partial V}{\partial t}\hat{\underline{j}} = \frac{\partial u(y,t)}{\partial t}\hat{\underline{i}} = \frac{\partial u}{\partial t}\hat{\underline{i}} \therefore$

$\hat{\underline{i}}$ component of NS: $\frac{\partial p}{\partial x} = 0 = \frac{\partial}{\partial x}\left(\frac{p}{\rho}\right) \therefore$

$\underline{u} \cdot \nabla \underline{u} = (u(y,t)\partial_x - V\partial_y)(u(y,t), -V, 0) =$

$\left(u(y,t)\frac{\partial u(y,t)}{\partial x} - V\frac{\partial u(y,t)}{\partial y}\right)\hat{\underline{i}} + \hat{j}\left(u\partial_x(-V) - V\partial_y(-V)\right) =$

$\cancel{u\frac{\partial u}{\partial x}} - V\frac{\partial u}{\partial y}\hat{\underline{i}}$

$\nu \nabla^2 \underline{u} = \nu(\partial_{xx} + \partial_{yy} + \partial_{zz})u(y,t)\hat{\underline{i}} = \nu\partial_{yy}u(y,t)\hat{\underline{i}} = \nu\frac{\partial^2 u}{\partial y^2}\hat{\underline{i}} \therefore$

x component: $\frac{\partial u}{\partial t} - V\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$

4b) Let $u(y,t) = Re\left[U\exp\left(-\frac{\lambda V y}{\nu} + int\right)\right] = Re\left[Ue^{-\frac{\lambda V y}{\nu} + int}\right] =$

$Re\left[Ue^{-\frac{\lambda V y}{\nu}} e^{int}\right] = \cancel{Re} \; \cancel{Ue^{-\frac{\lambda V y}{\nu}}}$

$\therefore \lambda^2 - \lambda - i\varepsilon = \lambda^2 - \lambda - i\frac{\nu n}{V^2} \therefore$

$u_t - V u_y = \nu u_{yy} \therefore$

$u_t = Re\left[U i n e^{-\frac{\lambda V y}{\nu} + int}\right],$

$u_y = Re\left[-U\frac{\lambda V}{\nu} e^{-\frac{\lambda V y}{\nu} + int}\right] \therefore u_{yy} = Re\left[+U\frac{\lambda^2 V^2}{\nu^2} e^{-\frac{\lambda V y}{\nu} + int}\right]$

$\therefore u_t - V u_y - \nu u_{yy} = 0 \therefore$

$Re\left[U i n e^{-\frac{\lambda V y}{\nu} + int}\right] + Re\left[+U\frac{\lambda V^2}{\nu} e^{-\frac{\lambda V y}{\nu} + int}\right] + Re\left[-U\frac{\lambda^2 V^2}{\nu} e^{-\frac{\lambda V y}{\nu} + int}\right]$

$= Re\left[U i n e^{-\frac{\lambda V y}{\nu} + int} + U\frac{\lambda V^2}{\nu} e^{-\frac{\lambda V y}{\nu} + int} - U\frac{\lambda^2 V^2}{\nu} e^{-\frac{\lambda V y}{\nu} + int}\right] =$

$Re\left[U\left(in + \frac{\lambda V^2}{\nu} - \frac{\lambda^2 V^2}{\nu}\right) e^{-\frac{\lambda V y}{\nu} + int}\right] \therefore$

$\lambda^2 - \lambda - i\varepsilon = 0 = \lambda^2 - \lambda - i\frac{\nu n}{V^2} \Rightarrow 0 = V^2\lambda^2 - V^2\lambda - i\nu n = 0 = -V^2\lambda^2 + V^2\lambda + i\nu n \therefore$

$in + \frac{\lambda V^2}{\nu} - \frac{\lambda^2 V^2}{\nu} = \frac{i\nu n + \lambda V^2 - \lambda^2 V^2}{\nu} = \frac{-V^2\lambda^2 + V^2\lambda + i\nu n}{\nu} = \frac{0}{\nu} = 0 \therefore$

$Re\left[U(0)e^{-\frac{\lambda V y}{\nu} + int}\right] = Re(0) = 0 = 0$ as required $\therefore$

$u(y,t) = Re\left[U\exp\left(-\frac{\lambda V y}{\nu} + int\right)\right]$ is a sol of $u_t - V u_y - \nu u_{yy} = 0$

4e) Mass flux $= M(t) = \int_S \rho \underline{u}\cdot d\underline{S} = \rho\int_S \underline{u}\cdot d\underline{S} = \rho\int_S (\hat{n}\cdot\underline{u})\, dS \therefore$

$d\underline{S} = \hat{n}\, dS \therefore$

$\therefore$ Surface is yz-plane $\therefore \hat{n} = \hat{i} \therefore$

$\underline{u}\cdot\hat{n} = \underline{u}\cdot\hat{i} = (u(y,t), -V, 0)\cdot(1, 0, 0) = u(y,t) = Re\left[Ue^{-\frac{\lambda V y}{\nu} + int}\right]$

$\therefore M(t) = \rho\int_S Re\left[Ue^{-\frac{\lambda V y}{\nu} + int}\right] dS = Re\left[\rho\int_S Ue^{-\frac{\lambda V y}{\nu} + int}\, dS\right] =$ $\quad y > 0$

$Re\left[\rho\int_0^\infty\int_0 Ue^{-\frac{\lambda V y}{\nu} + int}\, dz\, dy\right] = Re\left[\rho\int_0^\infty Ue^{-\frac{\lambda V y}{\nu} + int}\right] dy =$

$\rho\int_0^\infty Re\left[Ue^{-\frac{\lambda V y}{\nu} + int}\right] dy = \rho\int_0^\infty u\, dy = Re\left[\rho\int_0^\infty Ue^{-\frac{\lambda V y}{\nu}} e^{int}\, dy\right]$

$= Re\left[\rho U e^{int}\int_0^\infty e^{-\frac{\lambda V}{\nu} y}\, dy\right] = Re\left[\rho U e^{int}\left[-\frac{\nu}{\lambda V} e^{-\frac{\lambda V y}{\nu}}\right]_{y=0}^{\infty}\right] =$

$Re\left[\rho U e^{int}\left[0 + \frac{\nu}{\lambda V} e^0\right]\right] = Re\left[\rho U e^{int}\frac{\mu/\rho}{\lambda V}\right] = Re\left[\frac{\mu U}{\lambda V} e^{int}\right] \because \nu = \frac{\mu}{\rho}$

pp2022

4d $\lambda^2-\lambda-i\varepsilon=0 \therefore \lambda \approx 1,\ \lambda \approx -i\varepsilon-\varepsilon^2,\ |\varepsilon|<<1 \therefore$

$u(y,t)=Re\left[Ue^{-\frac{\lambda V y}{\nu}+int}\right] = Re\left[Ue^{-\frac{\lambda V y}{\nu}}e^{int}\right]$

$\therefore$ for small ε:

$u(y,t)=$ ~~$Re[$~~ $U Re\left[\left(e^{-\frac{Vy}{\nu}}+e^{-\frac{Vy}{\nu}(-i\varepsilon-\varepsilon^2)}\right)e^{int}\right]=$

$\varepsilon=\frac{\nu n}{V^2} \qquad \lambda^2-\lambda-i\varepsilon=0 = \lambda^2-\lambda-i\frac{\nu n}{V^2}$

$|\varepsilon|<<1 \therefore \varepsilon$ is small

$\lambda \approx 1,\ \lambda \approx -i\varepsilon-\varepsilon^2=-i\frac{\nu n}{V^2}-\left(\frac{\nu n}{V^2}\right)^2=-i\frac{\nu n}{V^2}-\frac{\nu^2 n^2}{V^4}$

$\lambda^2-\lambda-i\varepsilon=0 \therefore$

$\lambda=\frac{1\pm\sqrt{1-4(-i\varepsilon)}}{2(1)}=\frac{1}{2}\pm\frac{\sqrt{1+4\varepsilon i}}{2}=\frac{1}{2}\pm\frac{\sqrt{1+4\nu n i/V^2}}{2}$

$\therefore$ when $V>0$: $u(y,t)=Re\left[Ue^{-\frac{\lambda V y}{\nu}+int}\right]=Re\left[Ue^{int}e^{-\frac{\lambda V y}{\nu}}\right]=$

$Re\left[Ue^{int}e^{-\frac{Vy}{\nu}\cdot 1}\right]=Re\left[Ue^{-\frac{Vy}{\nu}}(\cos nt+i\sin(nt))\right]=$

$Ue^{-Vy/\nu}\cos nt$

when $V<0$: $\lambda \approx -i\varepsilon-\varepsilon^2 \therefore$

$u(y,t)=Re\left[Ue^{-\frac{\lambda V y}{\nu}}e^{int}\right]=Re\left[Ue^{-\frac{Vy}{\nu}(-i\varepsilon-\varepsilon^2)}e^{int}\right]=$

$Re\left[Ue^{+\frac{Vy}{\nu}\varepsilon^2}e^{i(nt+\frac{Vy}{\nu}\varepsilon)}\right]=Re\left[Ue^{\frac{Vy}{\nu}\frac{\nu^2n^2}{V^4}}e^{i(nt+\frac{Vy}{\nu}\frac{\nu n}{V^2})}\right]=$

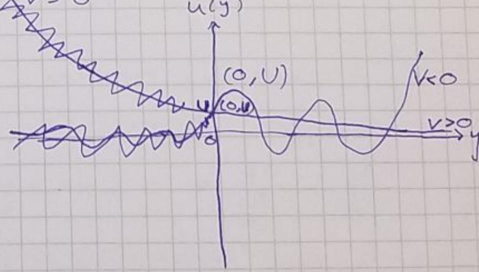
$Re\left[Ue^{\frac{\nu n^2 y}{V^3}}e^{i(nt+\frac{ny}{V})}\right]=Re\left[Ue^{\frac{\nu n^2 y}{V^3}}\left(\cos(nt+\frac{ny}{V})+i\sin(nt+\frac{ny}{V})\right)\right]=$

$Ue^{\nu n^2 y/V^3}\cos(nt+\frac{ny}{V}) \therefore$

$$u(y,t)=\begin{cases}Ue^{-\frac{Vy}{\nu}}\cos(nt), & \text{when } V>0,\\ Ue^{\nu n^2 y/V^3}\cos(nt+\frac{ny}{V}), & \text{when } V<0.\end{cases}$$

4e

$$u(y,t=0)=\begin{cases}Ue^{-Vy/\nu}\\ Ue^{\frac{\nu n^2 y}{V^3}}\cos(\frac{ny}{V})\end{cases}$$



$Ue^{\frac{\nu n^2 y}{V^3}}\cos(\frac{ny}{V})=0$ for $\cos(\frac{ny}{V})=0$

$\therefore \frac{ny}{V}=$ ~~cos~~ $\frac{(2n+1)\pi}{2} \therefore$

$y=\frac{(2n+1)\pi}{2n}$ ~~and~~ $\because y>0$

\1ai/ equation is $\frac{\partial \rho}{\partial t} + \nabla\cdot(\underline{u}\rho) = \frac{\partial \rho}{\partial t}\, \rho \nabla\cdot\underline{u} + \underline{u}\cdot\nabla\rho = 0$ ∴

$\nabla\cdot\underline{u} = 0$ ∴

$\cancel{\partial} \; 0 = \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\underline{u}\rho)\right]_i = \frac{\partial \rho}{\partial t} + \nabla_i(u\rho)_i =$

$\cancel{\frac{\partial \rho}{\partial t} + \partial_i(u_i}$ $\frac{\partial \rho}{\partial t} + \partial_i(u_i\rho) = \frac{\partial \rho}{\partial t} + \rho\partial_i u_i + u_i\partial_i\rho =$

$\frac{\partial \rho}{\partial t} + \rho(\nabla\cdot\underline{u})_i + (\underline{u}\cdot\nabla\rho)_i$ ∴

incompressible flows ∴ $0 = \frac{D\rho}{Dt}$ ∴

$0 = \left[\frac{D\rho}{Dt}\right]_i = \left[\frac{\partial \rho}{\partial t} + \underline{u}\cdot\nabla\rho\right]_i = \frac{\partial \rho}{\partial t} + u_i\partial_i\rho = 0$ ∴

$0 = \left[\frac{\partial \rho}{\partial t} + \nabla\cdot(\underline{u}\rho)\right]_i = \frac{\partial \rho}{\partial t} + \rho(\nabla\cdot\underline{u})_i + (\underline{u}\cdot\nabla)_i\rho =$

$\left[\frac{\partial \rho}{\partial t} + (\underline{u}\cdot\nabla)_i\rho\right]_i + \rho(\nabla\cdot\underline{u})_i = \rho(\nabla\cdot\underline{u})_i = 0 = (\nabla\cdot\underline{u})_i = \nabla_i u_i =$

$\partial_i u_i = 0 = \frac{\partial u_i}{\partial x_i} = 0$

\1bi/ $\rho\frac{\partial \underline{u}}{\partial t} + \rho\underline{u}\cdot\nabla\underline{u} = -\nabla P + \rho\underline{g} + \mu\nabla^2\underline{u}$ ∴

∴ $\left[\rho\frac{\partial \underline{u}}{\partial t}\right]_i = \rho\frac{\partial u_i}{\partial t}$

$[\rho\underline{u}\cdot\nabla\underline{u}]_i = [\rho(\underline{u}\cdot\nabla)\underline{u}]_i = \rho(\underline{u}\cdot\nabla)u_i = \rho(u_j\nabla_j)u_i = \rho u_j\partial_j u_i$

$[-\nabla P]_i = -\cancel{\nabla}_i P = -\partial_i P$.

$[\rho g]_i\ \rho g_i$, $[\mu\nabla^2\underline{u}]_i = \mu\nabla^2 u_i = \cancel{\mu[\nabla_j\nabla_j]u_i}$

$\mu[\nabla_j\nabla_j]u_i = \mu\partial_j\partial_j u_i$

ith component: $\rho\frac{\partial u_i}{\partial t} + \rho u_j\partial_j u_i = -\partial_i P + \rho g_i + \mu\partial_j\partial_j u_i$

\1bii/ $\nabla\times\underline{u} = \underline{\omega}$ ∴

$\omega_i = [\nabla\times\underline{u}]_i = \varepsilon_{ijk}\nabla_j u_k = \varepsilon_{ijk}\partial_j u_k$

\1ci/ stream line equations: $\frac{\partial x}{u} = \frac{\partial y}{v} = \frac{\partial z}{w}$ ∴

$\underline{u} = \left(\frac{x}{1+t}, \frac{y}{1+t}, \frac{z}{1+t}\right)$ ∴

$\frac{\partial x}{\frac{x}{1+t}} = \frac{\partial y}{\frac{y}{1+t}} = \frac{\partial z}{\frac{z}{1+t}}$ ∴ $\frac{1}{x}dx = \frac{1}{y}dy = \frac{1}{z}dz$ ∴

$\int\frac{1}{x}dx = \int\frac{1}{y}dy = \int\frac{1}{z}dz = \ln|x| = \ln|y| + c_1 = \ln|z| + c_2$ ∴

$|x| = e^{\ln|x|} = e^{\ln|y|+c_1} = e^{\ln|z|+c_2} = c_3|y| = c_4|z|$ ∴

$x = c_5 y = c_6 z$

\pp2022/ \1cii/ $\frac{dx}{dt} = u$, $\frac{dy}{dt} = v$, $\frac{dz}{dt} = w$ ∴

$\frac{dx}{dt} = \frac{x}{1+t}$ ∴ $\int \frac{1}{x} dx = \int \frac{1}{1+t} dt = \ln|x| = \ln|1+t| + c_1$ ∴

$|x| = e^{\ln|x|} = e^{\ln|1+t| + c_1} = c_2 e^{\ln|1+t|} = c_3 |1+t|$ ∴

$x = c_4(1+t)$ ∴

by symmetry $y = c_5(1+t)$, $z = c_6(1+t)$ ∴

particle paths are $(x,y,z) = (c_4(1+t),\ c_5(1+t),\ c_6(1+t))$

\1ciii/ at $t=0$: $(x,y,z) = (c_4, c_5, c_6) = (1,1,0)$ ∴

$(x,y,z) = (1+t,\ 1+t,\ 0)$ ∴

at $t=2$: $(x,y,z) = (3, 3, 0)$

\1di/

\1dii/ ∵ of no slip boundary condition

fluid flow has no special points in ϕ or z ∴

flow is independent of ϕ and z.

no slip BC ∴ flow is parallel to movement of boundary ∴

flow is only in $\hat{z}$ direction ∴ ~~fb~~

∴ ~~u=Wt~~ $\underline{u} = w\hat{z}$ ∴

~~flow~~ no slip BC ∴ flow at $R = R_2$

does not equal flow at $R = R_1$ ∴ flow has special points

in R ∴ flow depends on R ∴ ~~w=R~~ $w = w(R)$ ∴

$\underline{u} = w\hat{z} = w(R)\hat{z}$

\1diii/ incompressible is $\nabla \cdot \underline{u} = 0$ ∴

$\nabla \cdot \underline{u} = \nabla \cdot w(R)\hat{z} = \frac{\partial}{\partial z} w(R) = 0$

\1div/ ∴ $\frac{1}{\mu} P_z R = \frac{d}{dR}\left(R \frac{dw}{dR}\right)$ ∴ $A + \frac{1}{2\mu} P_z R^2 = R \frac{dw}{dR}$ ∴

$\frac{dw}{dR} = \frac{1}{2\mu} P_z R + \frac{A}{R}$ ∴

$w = w(R) = \frac{1}{4\mu} P_z R^2 + \ln|R| + B = \frac{1}{4\mu} P_z R^2 + \ln R + B$ ∵ $R \geq 0$ ∴

\1dv/ both ends of cylinder are open to atmosphere with equal pressure $P = P_0$ ∴ there is no pressure gradient ∴ along any coordinate ∴ along z there is equal pressure everywhere in the cylinder ∴ $\frac{\partial p}{\partial z} = 0$

\1dvi/ $\frac{1}{\mu}\frac{dp}{dz}R = \frac{d}{dR}\left(R\frac{dw}{dR}\right) \therefore$

$A + \frac{1}{2\mu}\frac{dp}{dz}R^2 = R\frac{dw}{dR} \therefore$

$\frac{1}{2\mu}\frac{dp}{dz}R + A\frac{1}{R} = \frac{dw}{dR} \therefore$

$w = w(R) = \frac{1}{4\mu}\frac{dp}{dz}R^2 + A\ln|R| + B = \frac{1}{4\mu}\frac{dp}{dz}R^2 + A\ln R + B \quad \because R \geq 0$

~~\1dvii/~~ $\therefore \frac{dp}{dz} = 0 \therefore w = w(R) = A\ln R + B$

\1dvii/ $\therefore w(R=R_2) = U, \quad w(R=R_1) = 0 \quad \because$ no slip $\therefore$

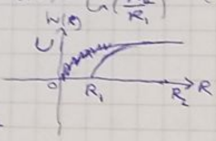
$\underline{u}(R=R_2) = U\hat{\underline{z}}, \ \underline{u}(R=R_1) = 0 \therefore$

~~where~~ $w(R=R_1) = 0 = A\ln(R_1) + B \therefore \ -A\ln(R_1) = B \therefore$

$w(R) = A\ln R - A\ln R_1 = A\ln\left(\frac{R}{R_1}\right) \therefore$

$w(R=R_2) = U = A\ln\left(\frac{R_2}{R_1}\right) \therefore \quad A = \frac{U}{\ln\left(\frac{R_2}{R_1}\right)} \therefore \quad w(R) = \frac{U}{\ln\left(\frac{R_2}{R_1}\right)}\ln\left(\frac{R}{R_1}\right) \therefore$

$\underline{u} = \frac{U}{\ln\left(\frac{R_2}{R_1}\right)}\ln\left(\frac{R}{R_1}\right)\hat{\underline{z}}$



\1dviii/ it agrees with my expectations since the theres no flow at R_1 then it slow increases to U as $R \to R_2$

\2a/ NS: $\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\nabla\frac{p}{\rho} + \underline{g} + \nu\nabla^2\underline{u}$, ~~$\nu = \frac{\rho}{\mu}$~~ $\nu = \frac{\mu}{\rho}$

constant density in gravity $\therefore$ neglect gravity $\therefore \underline{g} = 0 \therefore$

$2\underline{u}\times(\nabla\times\underline{u}) = \nabla(\underline{u}\cdot\underline{u}) - 2\underline{u}\cdot\nabla\underline{u} \therefore$

$\underline{u}\cdot\nabla\underline{u} = \frac{1}{2}\nabla|\underline{u}|^2 - \underline{u}\times(\nabla\times\underline{u}) = \nabla\frac{u^2}{2} - \underline{u}\times\nabla\times\underline{u} \quad \because \underline{\omega} = \nabla\times\underline{u} \therefore$

$\frac{\partial \underline{u}}{\partial t} + \nabla\frac{u^2}{2} - \underline{u}\times\nabla\times\underline{u} = \frac{\partial \underline{u}}{\partial t} + \nabla\frac{u^2}{2} - \underline{u}\times\underline{\omega} = -\nabla\frac{p}{\rho} + \nu\nabla^2\underline{u} \therefore$

$\frac{\partial \underline{u}}{\partial t} = \underline{u}\times\underline{\omega} - \nabla\left(\frac{p}{\rho}\right) - \nabla\frac{u^2}{2} + \nu\nabla^2\underline{u} = \underline{u}\times\underline{\omega} - \nabla\left(\frac{p}{\rho} + \frac{u^2}{2}\right) + \nu\nabla^2\underline{u}$

\2b/ $\therefore$ steady fluid $\therefore \frac{\partial \underline{u}}{\partial t} = 0$, inviscid $\therefore \nu = 0 \therefore$

$0 = \underline{u}\times\underline{\omega} - \nabla\left(\frac{p}{\rho} + \frac{u^2}{2}\right) \therefore$

$\nabla\left(\frac{p}{\rho} + \frac{u^2}{2}\right) = \nabla(H) = \underline{u}\times\underline{\omega} \therefore$

$\underline{u} = (u_1, u_2, u_3) \therefore$ along streamlines $\frac{dx}{u_1} = \frac{dy}{u_2} = \frac{dz}{u_3} \therefore$

along stream lines $\underline{u}\times\underline{\omega} = 0 \therefore$

$\nabla(H) = 0 \therefore \quad H = \text{constant} = \frac{p}{\rho} + \frac{u^2}{2}$

\2c/ $\frac{\partial \underline{u}}{\partial t} = \underline{u}\times\underline{\omega} - \nabla\left(\frac{p}{\rho} + \frac{u^2}{2}\right) + \nu\nabla^2\underline{u} \quad \because \nabla\times\nabla\phi = 0 \therefore$

taking curls: $\nabla\times\frac{\partial \underline{u}}{\partial t} = \frac{\partial}{\partial t}\nabla\times\underline{u} = \frac{\partial \underline{\omega}}{\partial t}, \ -\nabla\times\nabla\left(\frac{p}{\rho} + \frac{u^2}{2}\right) = 0,$

~~$\nu\nabla^2$~~ $\nu\nabla\times\nabla^2\underline{u} = \nu\nabla\times(\nabla(\nabla\cdot\underline{u})) - \nu\nabla\times(\nabla\times(\nabla\times\underline{u})) \therefore$

\pp2022/ $\therefore \rho = \text{constant} \therefore \frac{D\rho}{Dt} = 0 \therefore$

$\frac{D\rho}{Dt} + \rho\nabla\cdot\underline{u} = 0 \therefore \nabla\cdot\underline{u} = 0$;

$\nu\nabla\times\nabla^2\underline{u} = \nu\nabla\times(\nabla(0)) - \nu\nabla\times(\nabla\times\underline{\omega}) = -\nu\nabla\times(\nabla\times\underline{\omega}) =$

$-\nu\nabla(\nabla\cdot\underline{\omega}) + \nu\nabla^2\underline{\omega} = -\nu\nabla(0) + \nu\nabla^2\underline{\omega} = \nu\nabla^2\underline{\omega}$;

$\nabla\cdot\underline{\omega} = \nabla\cdot(\nabla\times\underline{u}) = 0$

$\therefore \nabla\times(\underline{u}\times\underline{\omega}) = \underline{u}(\nabla\cdot\underline{\omega}) - \underline{\omega}(\nabla\cdot\underline{u}) + (\underline{\omega}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)\underline{\omega} =$

$\underline{u}(0) - \underline{\omega}(0) + (\underline{\omega}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)\underline{\omega} = \underline{\omega}\cdot\nabla\underline{u} - \underline{u}\cdot\nabla\underline{\omega} \therefore$

$\frac{\partial\underline{\omega}}{\partial t} = \underline{\omega}\cdot\nabla\underline{u} - \underline{u}\cdot\nabla\underline{\omega} + \nu\nabla^2\underline{\omega} \therefore$

$\frac{\partial\underline{\omega}}{\partial t} + \underline{u}\cdot\nabla\underline{\omega} = \frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\nabla\underline{u} + \nu\nabla^2\underline{\omega} \quad \because \frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\nabla$

\2d/ $\frac{D\underline{\omega}}{Dt}$ is the material derivative of vorticity $\underline{\omega}$ $\therefore$

if $\frac{D\underline{\omega}}{Dt} \neq 0$ then vorticity changes with time and/or space.

$\underline{\omega}\cdot\nabla\underline{u}$ is the dot product of vorticity and gradient operator operated on the velocity $\underline{u}$

$\nu\nabla^2\underline{\omega}$ is the viscosity times the gradient squared of the vorticity $\therefore$

if $\nu\nabla^2\underline{\omega} \neq 0$ then $\underline{\omega}$ changes with spatial with its rate of change of also changing with space

\2e i/ $\therefore \underline{\omega} = \nabla\times\underline{u} = \nabla\times(U(y,t), -ay, az) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ U(y,t) & -ay & az \end{vmatrix} =$

$\hat{\underline{i}}\left(\frac{\partial(az)}{\partial y} - \frac{\partial(-az)}{\partial y}\right) - \hat{\underline{j}}\left(\frac{\partial(az)}{\partial x} - \frac{\partial(U(y,t))}{\partial z}\right) + \hat{\underline{k}}\left(\partial_x(-ay) - \partial_y(U(y,t))\right) =$

$\hat{\underline{i}}(0-0) - \hat{\underline{j}}(0-0) + \hat{\underline{k}}\left(0 - \frac{\partial U(y,t)}{\partial y}\right) =$

$-\frac{\partial U(y,t)}{\partial y}\hat{\underline{k}} = \left(0, 0, -\frac{\partial U(y,t)}{\partial y}\right) = (0, 0, \omega) \therefore$

$\omega = -\frac{\partial U(y,t)}{\partial y}$

\2e ii/ $\frac{D\underline{\omega}}{Dt} = \underline{\omega}\cdot\nabla\underline{u} + \nu\nabla^2\underline{\omega} = \frac{\partial\underline{\omega}}{\partial t} + \underline{u}\cdot\nabla\underline{\omega} \therefore$

$\underline{\omega} = (0, 0, \omega(y,t)) \therefore$

z component only: $\frac{\partial\omega}{\partial t}$, $\nabla^2\omega = \frac{\partial^2\omega}{\partial y^2}$,

$(\underline{u}\cdot\nabla)\omega = ((U, -ay, az)\cdot\nabla)\omega = \left(U\frac{\partial}{\partial x} - ay\frac{\partial}{\partial y} + az\frac{\partial}{\partial z}\right)\omega(y,t) =$

$-ay\frac{\partial\omega}{\partial y}$

$\underline{\omega}\cdot\nabla\underline{u} = \left(\omega(y,t)\frac{\partial}{\partial z}\right)\underline{u} = \omega\frac{\partial}{\partial z}(U(y,t), -ay, az) = \omega\frac{\partial}{\partial z}(az)\hat{\underline{k}} = \omega a\hat{\underline{k}}$

$\therefore$ z component: $\frac{\partial \omega}{\partial t} - ay\frac{\partial \omega}{\partial y} = \omega a + \nu\frac{\partial^2 \omega}{\partial y^2}$ $\therefore$

steady solution $\therefore$ $\frac{\partial \omega}{\partial t} = 0$ $\therefore$

$-ay\frac{\partial \omega}{\partial y} = a\omega + \nu\frac{\partial^2 \omega}{\partial y^2}$ $\therefore$

$\nu\frac{\partial^2 \omega}{\partial y^2} + ay\frac{\partial \omega}{\partial y} + a\omega = 0$ $\therefore$

let $\omega(y) = Ae^{-y^2/\delta^2}$ $\therefore$

$\frac{\partial \omega}{\partial y} = \frac{-2y}{\delta^2}Ae^{-y^2/\delta^2}$ $\therefore$

$\frac{\partial^2 \omega}{\partial y^2} = \frac{-2}{\delta^2}Ae^{-y^2/\delta^2} + \frac{4y^2}{\delta^4}Ae^{-y^2/\delta^2}$ $\therefore$

$\frac{-2\nu}{\delta^2}Ae^{-y^2/\delta^2} + \frac{4\nu y^2}{\delta^4}Ae^{-y^2/\delta^2} + \frac{-2y^2}{\delta^2}Ae^{-y^2/\delta^2} + a\omega =$

$y^2e^{-y^2/\delta^2}\left(\frac{4\nu}{\delta^4}A - \frac{2}{\delta^2}A\right) - \frac{2\nu}{\delta^2}Ae^{-y^2/\delta^2} + a\omega = 0$ $\therefore$

$\omega(y) = Ae^{-y^2\frac{1}{\delta^2}}$ $\therefore$

\2eiii/ $\therefore$ $\omega = -\frac{\partial U}{\partial y} = Ae^{-y^2\frac{1}{\delta^2}}$ $\therefore$

$\frac{\partial U}{\partial y} = -Ae^{-y^2\frac{1}{\delta^2}}$ $\therefore$

$U(y) = \int_0^y -Ae^{-t^2\frac{1}{\delta^2}}\,dt + C_1$ $\therefore$

$U(0) = 0 = \int_0^0 -Ae^{-t^2\frac{1}{\delta^2}}\,dt + C_1 = C_1$ $\therefore$

$U(y) = \int_0^y -Ae^{-t^2\frac{1}{\delta^2}}\,dt = -A\int_0^y e^{-\left(\frac{t}{\delta}\right)^2}dt = -\frac{A\pi^{1/2}}{2}2\pi^{-1/2}\int_0^y e^{-\left(\frac{t}{\delta}\right)^2}dt$

$\therefore$ let $\frac{t}{\delta} = s$ $\therefore$ $\frac{ds}{dt} = \frac{1}{\delta}$ $\therefore$ $\delta\,ds = dt$ $\therefore$

$t=0 : \frac{0}{\delta} = s = 0$, $t = y : \frac{y}{\delta} = s$ $\therefore$

$U(y) = -\frac{A\pi^{1/2}}{2}2\pi^{-1/2}\int_0^{y/\delta} e^{-s^2}\delta\,ds = -\frac{A\pi^{1/2}\delta}{2}2\pi^{-1/2}\int_0^{y/\delta} e^{-s^2}ds =$

$\frac{-A\pi^{1/2}\delta}{2}\operatorname{erf}\left(\frac{y}{\delta}\right)$

\3a/ let $\psi = Exf(y) = Exf$; $\nabla^2\psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)\psi$ $\therefore$

$\partial_x\psi = \partial_x(Exf) = Ef$ $\therefore$ $\partial_{xx}\psi = \partial_x(Ef) = 0$, $\partial_y\psi = \partial_y(Exf) = Exf'$ $\therefore$

$\partial_y(Exf') = Exf'' = \partial_{yy}\psi$ $\therefore$ $\nabla^2\psi = Exf''$ $\therefore$

$\nabla^2\nabla^2\psi = \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)Exf'' = Exf''''$ $\therefore$

$\frac{\partial}{\partial t}\nabla^2\psi = \frac{\partial}{\partial t}Exf'' = 0$, $\frac{\partial}{\partial y}\nabla^2\psi = \frac{\partial}{\partial y}Exf'' = Exf'''$ $\therefore$

$\frac{\partial \psi}{\partial x}\frac{\partial}{\partial y}\nabla^2\psi = EfExf''' = E^2xff'''$, $\frac{\partial}{\partial x}\nabla^2\psi = \frac{\partial}{\partial x}Exf'' = Ef''$ $\therefore$

$\frac{\partial \psi}{\partial y}\frac{\partial}{\partial x}(\nabla^2\psi) = Exf'Ef'' = E^2xf'f''$ $\therefore$

\PP2022/ $\therefore 0 - E^2 x f f''' + E^2 x f' f'' = \nu E x f'''' \therefore$

$\Rightarrow \frac{1}{\nu}\left[-E f f''' + E f' f''\right] = \frac{E}{\nu}\left[f' f'' - f f'''\right] = f'''' \therefore$

$f' f'' - f f''' = \frac{\nu}{E} f''''$

$\therefore ((f')^2 - f f'')' = 2 f' f'' - f' f'' - f f''' = f' f'' - f f''' \therefore$

$(f'^2 - f f'')' = \frac{\nu}{E} f'''' = c f'''' \quad \therefore \quad c = \frac{\nu}{E}$

\3ci/ $\underline{u} = \nabla \times (\psi \underline{k}) = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \psi \end{vmatrix} =$

$\hat{i}\left(\frac{\partial \psi}{\partial y} - 0\right) - \hat{j}\left(\frac{\partial \psi}{\partial x} - 0\right) + \hat{k}(0-0) = \left(\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0\right) = \cancel{(Ex, -Ey, 0)} =$

$\left(\frac{\partial}{\partial y}(E x f), -\frac{\partial}{\partial x}(E x f), 0\right) = (E x f', -E f, 0) \therefore$

at $y = 0$: ~~$E x f' \to E x$, $-E f \to -E y$~~ $\therefore$ solid boundary and no slip condition on stationary $\therefore$ no fluid motion $\therefore \underline{u} = 0 \therefore$

$\therefore$ at $y = 0$: $\underline{u} = (E x f'(0), -E f(0), 0) = (0, 0, 0)$

$\therefore E x f'(0) = 0, \quad -E f(0) = 0 \therefore$

$f'(0) = 0, \quad f(0) = 0$

\3cii/ as $y \to \infty$: $\underline{u} = (E x f', -E f, 0) \to (E x, -E y, 0) \therefore$

$E x f' \to E x, \quad -E f \to -E y \therefore$

$f' \to 1, \quad f \to y$ as $y \to \infty$

\3d/ $(f'^2 - f f'')' = c f'''' = \frac{\partial}{\partial y}(f'^2 - f f'') = c \frac{\partial}{\partial y} f''' \therefore$

$f'^2 - f f'' = c f''' + d \therefore$

at $y = 0$: $f(0) = 0, \quad f'(0) = 0 \therefore$

solid stationary boundary $\therefore (f'(0))^2 - f(0) f''(0) = c f'''(0) + d =$

$0^2 - 0 f''(0) = c f'''(0) + d = 0 \therefore d = c f'''(0) = \frac{\nu}{E} f'''(0) \therefore$

\3e/ $\therefore [y] = L$, ~~$c = \frac{\nu}{E}$, E constant~~

$[y]$ and $[c]$ are dependent $\therefore$

$\left[\frac{\partial u}{\partial t}\right] = L T^{-1} T^{-1} = L T^{-2} = [\nu \nabla^2 \underline{u}] = [\nu][\nabla^2][\underline{u}] = L^{-2} L T^{-1} = L^{-1} T^{-1} \therefore$

$L^2 T^{-1} = [\nu] \quad , \quad \therefore [c] = \left[\frac{\nu}{E}\right] = [\nu][E]^{-1} = L = L^2 T^{-1} [E]^{-1} \therefore$

$[E]^{-1} = L L^{-2} T^{1} = L^{-1} T^{1} \therefore [E] = L T^{-1} \therefore$

$[c] = \left[\frac{\nu}{E}\right] = L \therefore$

$n=2$, y is independent, c is dependent $\therefore$

$$\Pi_1 = \frac{c}{y^\alpha} \therefore$$

\3e/ $[\text{force}] = MLT^{-2} \therefore [E] = MLT^{-2}L = ML^2T^{-2} \therefore$

$\left[\frac{\partial u}{\partial t}\right] = LT^{-1}T^{-1} = LT^{-2} = [\nu \nabla^2 \underline{u}] = [\nu] L^{-2} LT^{-1} = L^{-1}T^{-1} \therefore$

$L^2 T^{-1} \therefore \quad c = \frac{\nu}{E} \therefore \quad [c] = [\nu][E]^{-1} = L^2 T^{-1} M^{-1} L^{-2} T^2 = M^{-1} T^1$

\3e/ $n=2 \therefore y$ is independent, c is dependent $\therefore$

$\Pi_1 = \frac{c}{y^\alpha} \therefore [c] = [y]^\alpha = L^\alpha \therefore$

$\Pi_2 = \frac{c}{y^\alpha} \therefore$

$\Pi = \Phi\left(\frac{c}{y^\alpha}\right) = c^{1/2} g(\eta) = \Phi(c y^{-\alpha}) \therefore \quad \Phi(x) = \sqrt{x} = x^{1/2} \therefore$

$\varsigma = \Pi = \Phi(c y^{-\alpha}) = c^{1/2}(y^{-\alpha})^{1/2} = c^{1/2} y^{-\frac{1}{2}\alpha} \doteq c^{1/2} g(\eta) \therefore$

$\cancel{g(\eta) = y^{-\frac{1}{2}\alpha}}$

$\varsigma = c^{1/2} g(\eta) \propto \Pi = \Phi(c y^{-\alpha}) = c^{1/2} y^{-\frac{1}{2}\alpha} \therefore$

$\eta = y^{-1/2\,\alpha}$

\4a/ incompressible $\therefore \nabla \cdot \underline{u} = 0$ neglect gravity $\therefore g = 0$

$\frac{\partial P}{\partial x} = 0 \therefore \quad NS: \frac{\partial \underline{u}}{\partial t} + \underline{u} \cdot \nabla \underline{u} = -\nabla P + \nu \nabla^2 \underline{u}, \quad \nu = \frac{\mu}{\rho} \therefore$

x component: $\frac{\partial u}{\partial t}$ $\quad -\frac{\partial P}{\partial x} = 0, \quad \nu \nabla^2 u = \nu \frac{\partial^2 u}{\partial y^2} = \nu \nabla^2 u(t, y)$

$= \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right) u(y,t)$

$(\underline{u} \cdot \nabla) u = (u, -V, 0) \cdot \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, 0\right) = \left(u\frac{\partial}{\partial x} - V\frac{\partial}{\partial y}\right) u(y,t) = -V\frac{\partial u}{\partial y} \therefore$

x component of NS: $\frac{\partial u}{\partial t} - V\frac{\partial u}{\partial y} = 0 + \nu \frac{\partial^2 u}{\partial y^2} = \nu \frac{\partial^2 u}{\partial y^2}$

\4b/ $\therefore$ ~~u_t Z R~~ $\quad u = R\left[U e^{-\frac{\lambda V y}{\nu} + int}\right] \therefore$

$u_t = R\left[U i n e^{-\frac{\lambda V y}{\nu} + int}\right] \therefore$

$u_y = R\left[-\frac{\lambda V}{\nu} U e^{-\frac{\lambda V y}{\nu} + int}\right] \therefore \quad u_{yy} = R\left[+\frac{\lambda^2 V^2}{\nu^2} U e^{-\frac{\lambda V y}{\nu} + int}\right] \therefore$

$u_t - V u_y - \nu u_{yy} = 0 \doteq R\left[U i n e^{-\frac{\lambda V y}{\nu} + int}\right] + R\left[-\frac{\lambda V^2 U}{\nu} e^{-\frac{\lambda V y}{\nu} + int}\right] + R\left[-\frac{\lambda^2 V^2 U}{\nu} e^{-\frac{\lambda V y}{\nu} + int}\right]$

$= R\left[\left(in + -\frac{\lambda V^2}{\nu} - \frac{\lambda^2 V^2}{\nu}\right) U e^{-\frac{\lambda V y}{\nu} + int}\right] \therefore$

$\cancel{in - \frac{\lambda V^2}{\nu} - \frac{\lambda^2 V^2}{\nu} = 0} \therefore \lambda^2 - \lambda - i\varepsilon = \lambda^2 - \lambda - i\frac{\nu n}{V^2} = 0 = +in + \frac{\lambda V^2}{\nu} - \frac{\lambda^2 V^2}{\nu} = 0 \therefore$

\PP 2022/ $\therefore\ u_t - Vu_y - \nu u_{yy} = \mathbb{R}\left[(0)Ue^{-\frac{\lambda V y}{\nu}+int}\right] = \mathbb{R}[0] = 0$

$\therefore\ u = \mathbb{R}\left[Ue^{-\frac{\lambda V y}{\nu}+int}\right]$

\1c/ mass flux $= M(t) = \rho\int_S \underline{u}\cdot \underline{dS} = \rho\int_S u\cdot \hat{\underline{n}}\,dS\ \therefore$

yz-plane $\therefore \hat{\underline{n}} = \hat{\underline{x}} = \hat{\underline{i}}\ \therefore$

$M(t) = \rho\iint_S \underline{u}\cdot\hat{\underline{i}}\,dz\,dy = \rho\iint_S (u,-V,0)\cdot(1,0,0)\,dz\,dy =$

$\rho\iint_S u\,dz\,dy = \rho\iint_S \mathbb{R}\left[Ue^{-\frac{\lambda V y}{\nu}+int}\right]dz\,dy = \rho\int_0^\infty\int_0^1 \mathbb{R}\left[Ue^{-\frac{\lambda V y}{\nu}+int}\right]dz\,dy$

$= \rho\int_0^\infty \mathbb{R}\left[Ue^{-\frac{\lambda V y}{\nu}+int}\right]dy = \mathbb{R}\left[\int_0^\infty \rho U e^{-\frac{\lambda V y}{\nu}+int}\right]dy = \mathbb{R}\left[\rho U\int_0^\infty e^{-\frac{\lambda V y}{\nu}}\left[e^{int}\right]dy\right]$

$= \mathbb{R}\left[\rho U e^{int}\int_0^\infty e^{-\frac{\lambda V y}{\nu}}dy\right] = \mathbb{R}\left[\rho U e^{int}\left[-\frac{\nu}{\lambda V}e^{-\frac{\lambda V y}{\nu}}\right]_0^\infty\right] =$

$\mathbb{R}\left[\rho U e^{int}\left[-\frac{\nu}{\lambda V}(0-1)\right]\right] = \mathbb{R}\left[\rho U e^{int}\left(\frac{\left(\frac{\mu}{\rho}\right)}{\lambda V}\right)\right] = \mathbb{R}\left[\frac{\mu U}{\lambda V}e^{int}\right]$

\3a/ $\nabla\cdot\underline{u} = 0 = \nabla\cdot\nabla\times(\psi\underline{k})$ neglect gravity $\therefore g = 0$

$\therefore$ NS: $\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\nabla P + \nu\nabla^2\underline{u}$, $\nu = \frac{\mu}{\rho}\ \therefore$

taking curls: $\nabla\times\frac{\partial \underline{u}}{\partial t} = \frac{\partial}{\partial t}\nabla\times\underline{u} = \frac{\partial}{\partial t}\nabla\times\nabla\times(\psi\underline{k})\ \therefore$

$\nabla\times\nabla\times(\psi\underline{k}) = \nabla(\nabla\cdot\psi\underline{k}) - \nabla^2\psi\underline{k} = \nabla\left(\frac{\partial}{\partial z}(\psi(x,y,t))\right) - \nabla^2\psi\underline{k} = \nabla(0) - \nabla^2\psi\underline{k}$

$= -\nabla^2\psi\underline{k}\ \therefore$

$\nabla\times\frac{\partial \underline{u}}{\partial t} = \frac{\partial}{\partial t}(-\nabla^2\psi\underline{k}) = -\frac{\partial}{\partial t}\nabla^2\psi\underline{k}$

$\therefore$ taking components only: $\nabla\times\nabla^2\underline{u} = \nabla^2\nabla\times\underline{u} = \nabla^2\nabla\times\nabla\times(\psi\underline{k}) =$

$\nabla^2[\nabla(\nabla\cdot\psi\underline{k})] - \nabla^2[\nabla^2\psi\underline{k}] = \nabla^2\left[\nabla\left(\frac{\partial}{\partial z}\psi(x,y,t)\right)\right] - \nabla^2\nabla^2\psi\underline{k} =$

$\nabla^2[\nabla(0)] - \nabla^2\nabla^2\psi\underline{k} = -\nabla^2\nabla^2\psi\underline{k}$

$-\nabla\times\nabla P \equiv 0$.

$\underline{u} = \nabla\times(\psi\underline{k}) = \begin{vmatrix}\hat{\underline{i}} & \hat{\underline{j}} & \hat{\underline{k}} \\ \partial_x & \partial_y & \partial_z \\ 0 & 0 & \psi\end{vmatrix} = \left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0\right)\ \therefore$

$\cancel{(\underline{u}\cdot\nabla)\psi\underline{k} = \left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right.}$

$\nabla\times[\underline{u}\cdot\nabla\underline{u}] = \nabla\times[(\underline{u}\cdot\nabla)\underline{u}] = (\underline{u}\cdot\nabla)\nabla\times\underline{u} + \nabla(\underline{u}\cdot\nabla)\times\underline{u}$

$\therefore (\underline{u}\cdot\nabla) = \left(\frac{\partial\psi}{\partial y}\frac{\partial}{\partial x} - \frac{\partial\psi}{\partial x}\frac{\partial}{\partial y}\right)\left(\frac{\partial\psi}{\partial y}, -\frac{\partial\psi}{\partial x}, 0\right)$

$\therefore\ \nabla\times((\underline{u}\cdot\nabla)\underline{u}) =$

$\nabla \times [\underline{u} \cdot \nabla \underline{u}] = \nabla \times [\nabla \times (\psi \underline{k}) \cdot \nabla (\underline{u})] = \nabla \times [\nabla \times (\psi \underline{k}) \cdot \nabla (\nabla \times (\psi \underline{k}))]$

$\underline{u} \cdot \nabla \underline{u} = \nabla \times (\psi \underline{k}) \cdot \nabla \nabla \times (\psi \underline{k}) = (\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0) \cdot \nabla \nabla \times (\psi \underline{k}) =$

$(\frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y}) \nabla \times (\psi \underline{k}) =$

$(\frac{\partial \psi}{\partial y} \frac{\partial}{\partial x} - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y}) (\frac{\partial \psi}{\partial y}, -\frac{\partial \psi}{\partial x}, 0) =$

$(\frac{\partial \psi}{\partial y} \psi_{xy} - \psi_x \psi_{yy}, -\psi_y \psi_{xx} + \psi_x \psi_{xy}, 0) \therefore$

$[\nabla \times (\underline{u} \cdot \nabla \underline{u})]_{z\,\text{component}} =$

$\frac{\partial}{\partial x}(-\psi_y \psi_{xx} + \psi_x \psi_{xy}) - \frac{\partial}{\partial y}(\psi_y \psi_{xy} - \psi_x \psi_{yy}) =$

~~$-\psi_{xy}\psi_{xxy}$~~

$-\psi_{xy}\psi_{xx} - \psi_y \psi_{xxx} + \psi_{xx}\psi_{xy} + \psi_x \psi_{xxy} + \psi_{yy}\psi_{xy} + \psi_y \psi_{xyy} - \psi_{xy}\psi_{yy} - \psi_x \psi_{yyy}$

$= -\psi_y \psi_{xxx} + \psi_x \psi_{xxy} + \psi_y \psi_{xyy} - \psi_x \psi_{yyy} = \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y}(\nabla^2 \psi) - \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x}(\nabla^2 \psi) \therefore$

z component:

$-\frac{\partial}{\partial t}(\nabla^2 \psi) + \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y}(\nabla^2 \psi) - \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x}(\nabla^2 \psi) = -\nu \nabla^2(\nabla^2 \psi) \therefore$

$\frac{\partial}{\partial t}(\nabla^2 \psi) - \frac{\partial \psi}{\partial x} \frac{\partial}{\partial y}(\nabla^2 \psi) + \frac{\partial \psi}{\partial y} \frac{\partial}{\partial x}(\nabla^2 \psi) = \nu \nabla^2(\nabla^2 \psi)$

\1 d. for $\lambda \approx 1$: $u = \mathbb{R}[U e^{-\frac{Vy}{\nu}} e^{int}] = \mathbb{R}[U e^{-\frac{Vy}{\nu}} [\cos(nt) + i \sin nt]] =$
$U e^{-\frac{Vy}{\nu}} \cos nt$

for $\lambda \approx -i\varepsilon - \varepsilon^2$: $u = \mathbb{R}[U e^{-\frac{(-i\varepsilon - \varepsilon^2)Vy}{\nu} + int}] =$

$\mathbb{R}[U e^{\frac{\varepsilon^2 Vy}{\nu} + \frac{\varepsilon Vy}{\nu} i} e^{int}] = \mathbb{R}[U e^{\frac{\varepsilon^2 Vy}{\nu}} e^{i(nt + \frac{\varepsilon Vy}{\nu})}] =$

$\mathbb{R}[U e^{\frac{(\frac{\nu n}{V^2})^2 Vy}{\nu}} (\cos(nt + \frac{\varepsilon Vy}{\nu}) + i \sin(nt + \frac{\varepsilon Vy}{\nu}))] =$

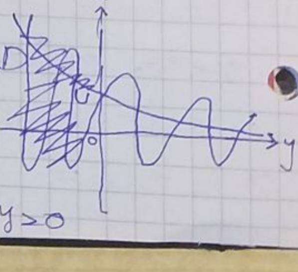
$U e^{\frac{(\frac{\nu n}{V^2})^2 Vy}{\nu}} \cos(nt + \frac{\varepsilon Vy}{\nu}) = U e^{\frac{\frac{\nu^2 n^2}{V^4} Vy}{\nu}} \cos(nt + \frac{\frac{\nu n}{V^2} Vy}{\nu}) =$
$U e^{\frac{\nu n^2 y}{V^3}} \cos(nt + \frac{ny}{V})$

$\therefore \lambda^2 - \lambda - i\varepsilon = \lambda^2 - \lambda - i(\frac{\nu n}{V^2}) = \lambda^2 - \lambda - (\frac{\nu n}{V^2}) i$

has discriminant: $(-1)^2 - 4(1)(-\frac{\nu n}{V^2} i) = 1 + 4\frac{\nu n}{V^2} i =: D$

$D \to 1$ for large $V > 0$, $\therefore u(y,t)$

$$u(y,t) = \begin{cases} U \exp(-\frac{Vy}{\nu}) \cos(nt), & V > 0, \\ U \exp(\frac{\nu n^2 y}{V^3}) \cos(nt + \frac{ny}{V}), & V < 0. \end{cases}$$



at $t = 0$ $U e^{-\frac{Vy}{\nu}}$

$U e^{\frac{\nu n^2 y}{V^3}} \cos(\frac{ny}{V})$ $\because y > 0$

\PP2020/ \1ai/ N-S: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = \mu^2\nabla^2\underline{u} + \cancel{\nabla p} - g =$

$\nabla p + \nabla\Gamma + \mu^2\nabla^2\underline{u}$

\1aii/ $\cancel{d\phi}$ $\cancel{\nabla\phi + \underline{u}\cdot\nabla\underline{\omega} = 0}$ $\therefore$ $\frac{d\rho}{dt} + \cancel{\Delta\rho}\ \underline{u}\cdot\nabla\underline{\omega} = 0 \therefore$

incompressible $\therefore$ $\frac{d\rho}{dt} = 0 \therefore \underline{u}\cdot\nabla\underline{\omega} = 0$, $\underline{\omega} = \nabla\times\underline{u}$ $\therefore$

$\underline{u}\cdot\nabla(\nabla\times\underline{u}) = 0$ $\therefore$ $\cancel{u\,\phi}$ $\nabla\cdot(\phi\underline{u}) = \phi\nabla\cdot\underline{u} + \underline{u}\cdot\nabla\phi$

$\therefore \nabla\cdot\underline{u} = 0$

\1bi/ $\therefore (R, \phi, z)$ $\therefore$ incompressible is $\nabla\cdot\underline{u} = 0 \therefore$

$\nabla\cdot\underline{u} = \nabla\cdot\left(\frac{1}{R}\hat{\underline{R}} + R\hat{\underline{\phi}}\right) = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{1}{R}\right) + \frac{1}{R}\frac{\partial}{\partial\phi}(R) = \frac{1}{R}\frac{\partial}{\partial R}(1) + \frac{1}{R}(0) =$

$\frac{1}{R}(0) + 0 = 0$ $\therefore$ flow is incompressible

\1bii/ irrotational is $\underline{\omega} = 0$ $\therefore \nabla\times\underline{u} = 0 \therefore$

$$\nabla\times\left(\frac{1}{R}, R, 0\right) = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ \frac{1}{R} & R(R) & 0 \end{vmatrix} = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ \frac{1}{R} & R^2 & 0 \end{vmatrix} =$$

$$\cancel{\hat{\phi}}\ \frac{1}{R}\left[\hat{\underline{R}}\left(\partial_\phi(0) - \partial_z(R^2)\right) - R\hat{\underline{\phi}}\left(\partial_R(0) - \partial_z\left(\tfrac{1}{R}\right)\right) + \hat{\underline{z}}\left(\partial_R(R^2) - \partial_\phi\left(\tfrac{1}{R}\right)\right)\right] =$$

$$\frac{1}{R}\left[\hat{\underline{R}}(0-0) - R\hat{\underline{\phi}}(0-0) + \hat{\underline{z}}(2R - 0)\right] = \frac{1}{R}2R\hat{\underline{z}} = 2\hat{\underline{z}} \neq 0 \therefore$$

flow is not irrotational.

\1biii/ $(\underline{u}\cdot\nabla)\phi$, $\nabla = \hat{\underline{R}}\frac{\partial}{\partial R} + \hat{\underline{\phi}}\frac{1}{R}\frac{\partial}{\partial\phi} + \hat{\underline{z}}\frac{\partial}{\partial z} \therefore$

$(\underline{u}\cdot\nabla) = \left(\frac{1}{R}, R, 0\right)\cdot\left(\frac{\partial}{\partial R}, \frac{1}{R}\frac{\partial}{\partial\phi}, \frac{\partial}{\partial z}\right)\cancel{\phi}$

$= \frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial}{\partial\phi}$ $\therefore$

$(\underline{u}\cdot\nabla)\underline{u} = \left(\frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial}{\partial\phi}\right)\left(\frac{1}{R}\hat{\underline{R}} + R\hat{\underline{\phi}}\right) =$

$\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\hat{\underline{R}} + R\hat{\underline{\phi}}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{R}\hat{\underline{R}} + R\hat{\underline{\phi}}\right) =$

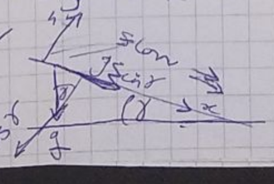
$\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\hat{\underline{R}}\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(R\hat{\underline{\phi}}\right) + \frac{\partial}{\partial\phi}\left(\frac{1}{R}\hat{\underline{R}}\right) + \frac{\partial}{\partial\phi}\left(R\hat{\underline{\phi}}\right) =$

$\hat{\underline{R}}\frac{1}{R}\frac{\partial}{\partial R}(R^{-1}) + \hat{\underline{\phi}}\frac{1}{R}\frac{\partial}{\partial R}(R) + \hat{\underline{R}}\frac{\partial}{\partial\phi}\left(\frac{1}{R}\right) + \frac{1}{R}\frac{\partial}{\partial\phi}(\hat{\underline{R}}) + \hat{\underline{\phi}}\frac{\partial}{\partial\phi}(R) + R\frac{\partial}{\partial\phi}(\hat{\underline{\phi}}) =$

$\hat{\underline{R}}(-1)\frac{1}{R}R^{-2} + \hat{\underline{\phi}}\frac{1}{R} + 0 + \frac{1}{R}\hat{\underline{\phi}} + 0 + R(-1)\hat{\underline{R}} =$

$-R^{-3}\hat{\underline{R}} - R\hat{\underline{R}} + \left(\frac{1}{R} + \frac{1}{R}\right)\hat{\underline{\phi}} = \left(-\frac{1}{R^3} - R\right)\hat{\underline{R}} + \frac{2}{R}\hat{\underline{\phi}}$

\1ci/



$y = 0$ is a solid boundary $\therefore$ at $y = 0$:

$\underline{u}(u, 0)$, no slip means $\underline{u}(u(y=0), 0) = \underline{u}(0, 0)$

$\therefore \underline{u} = (u(y), 0)$

\1c ii/ N-S: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = \nabla P - \underline{g} + \mu\nabla^2\underline{u}$ ∴

$\frac{\partial \underline{u}}{\partial t} = 0$, is $\underline{u}\cdot\nabla\underline{u}$, $\underline{u}\cdot\nabla = u(y)\frac{\partial}{\partial x}$,

$u(y)\frac{\partial}{\partial x}\cdot(u(y), 0) = \underline{\hat{x}}\, u(y)\frac{\partial}{\partial x}u(y) = 0$ ∴

$0 = \nabla P - \underline{g} + \mu\nabla^2\underline{u}$, is $\nabla^2\underline{u} = \frac{\partial^2}{\partial x^2}(u(y)) = 0$ ∴

∴ $0 = \frac{\nabla P}{\rho} - \frac{\underline{g}}{\rho}$ ∴ $\underline{g} = \begin{pmatrix} g\sin\gamma \\ -g\cos\gamma \end{pmatrix} = \begin{pmatrix} dP/dx \\ dP/dy \end{pmatrix}$,

\1c iii/ let $-g\cos\gamma = \frac{dP}{dy} = \frac{d}{dy}(P)$ ∴

$\int -g\cos\gamma\, dy = P = -g\cos(\gamma)y + C$ ∴ at $y = h$:

$P_0 = -g\cos(\gamma)h + C$ ∴ $P_0 + g\cos(\gamma)h = P_0 + h\,g\cos\gamma = C$ ∴

$P = P_0 + hg\cos\gamma - g\cos(\gamma)y = P_0 + (h-y)\rho g\cos\gamma$

~~\1c iv/~~ let $\frac{d}{dy}P = -\rho g\cos\gamma$ ∴

$P = -\rho g\cos(\gamma)y + C$ ∴ at $y = h$: $P = P_0 = -\rho g\cos(\gamma)h + C$ ∴

$P_0 + \rho g\cos(\gamma)h = C$ ∴ $P = -\rho g\cos(\gamma)y + P_0 + \rho g\cos(\gamma)h = P_0 + (h-y)\rho g\cos\gamma$

\1c iv/ $\frac{d}{dy}P = -\rho g\cos\gamma$ ∴ $\frac{d}{dx}P = \rho g\sin\gamma$ ∴

$P = \int \rho g\sin\gamma\, dx = \rho g\sin(\gamma)x + C$.

\1d i/ $\frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\nabla$ ∴ N-S: $\rho\frac{D\underline{u}}{Dt} = \nabla P - \underline{g} + \mu\nabla^2\underline{u}$ ∴

$\nabla\times\left(\rho\frac{D}{Dt}\underline{u}\right) = \rho\frac{D}{Dt}(\nabla\times\underline{u}) = \rho\frac{D}{Dt}\underline{\omega} = \nabla\times\nabla P - \nabla\times\underline{g} + \mu\nabla\times\nabla^2\underline{u}$

\1d ii A/ vorticity: $\nabla\times\underline{u} = \underline{\omega}$ ∴ $\underline{\omega} = \nabla\times\underline{u} = \nabla\times(0, aRz, 0)$

$\frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & aR^2z & 0 \end{vmatrix} = \frac{1}{R}\left[\hat{\underline{R}}\left(\partial_\phi(0) - \partial_z(aR^2z)\right) - R\hat{\underline{\phi}}\left(\partial_R(0) - \partial_z(0)\right) + \hat{\underline{z}}\left(\partial_R(aR^2z) - \partial_\phi(0)\right)\right]$

$= \frac{1}{R}\left[\hat{\underline{R}}(-aR^2) + \hat{\underline{z}}(2aRz)\right] = -aR\hat{\underline{R}} + 2az\hat{\underline{z}}$ ∴

$\nabla\cdot\underline{\omega} = \nabla\cdot(-aR\hat{\underline{R}} + 2az\hat{\underline{z}}) = \frac{1}{R}\frac{\partial}{\partial R}(R(-1)aR) + \frac{\partial}{\partial z}(2az) = \frac{1}{R}\frac{\partial}{\partial R}(-aR^2) + \frac{\partial(2az)}{\partial z}$

$= \frac{-2aR}{R} + 2a = -2a + 2a = 0$

\1d ii B/ $\frac{dR}{R} = \frac{d\phi}{d\phi} + \frac{dz}{z}$

\PP2020/ \1 dii C/ $\underline{u} = \nabla \times (\psi \hat{\underline{z}})$ ∴

$\underline{\omega} = \nabla \times \underline{u} = \nabla \times \nabla \times (\psi \hat{\underline{z}}) = \nabla(\nabla \cdot \psi \hat{\underline{z}}) - \nabla^2 \psi \hat{\underline{z}}$

\2 a/ $x' = \frac{x}{L}$, $t' = \frac{t}{T}$, $u' = \frac{u}{U}$ ∴

$\frac{dx'}{dt} =$

\2ci/ $\nabla \cdot \underline{u} = 0$ is incompressible easily seen ∴

$\nabla \cdot \underline{u} = \nabla \cdot \nabla \times \left(\frac{\psi}{r \sin\theta} \hat{\underline{\phi}}\right)$ and $\nabla \cdot (\nabla \times \underline{v}) \equiv 0$ ∴ easily see

$\underline{u} = (u, v, 0)$ ∴ $\underline{u} = \nabla \times \left(\frac{\psi}{r \sin\theta} \hat{\underline{\phi}}\right) = \frac{1}{r^2 \sin\theta} \begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta \hat{\underline{\phi}} \\ \partial_r & \partial_\theta & \partial_\phi \\ 0 & 0 & \psi \end{vmatrix} =$

$\frac{1}{r^2 \sin\theta}\left[\hat{\underline{r}}(\partial_\theta(\psi) - \partial_\phi(0)) - r\hat{\underline{\theta}}(\partial_r(\psi) - \partial_\phi(0)) + r\sin\theta \hat{\underline{\phi}}(\partial_r(0) - \partial_\theta(0))\right] =$

$\frac{1}{r^2 \sin\theta}\left[\hat{\underline{r}} \frac{\partial \psi}{\partial \theta} - r\hat{\underline{\theta}} \frac{\partial \psi}{\partial r}\right] = \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta} \hat{\underline{r}} - \frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r} \hat{\underline{\theta}}$ ∴

$\Rightarrow u = \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta}$, $v = -\frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r}$

\2ciii/ $E^2 = \frac{\partial^2}{\partial r^2} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta}\right)$ ∴ $\psi = f(r)\sin^2\theta$.

$E^2\psi = E^2(f(r)\sin^2\theta) = \frac{\partial^2}{\partial r^2}(f(r)\sin^2\theta) + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta} \frac{\partial}{\partial \theta}(f(r)\sin^2\theta)\right) =$

$\sin^2\theta f''(r) + \frac{\sin\theta}{r^2} \frac{\partial}{\partial \theta}\left(\frac{1}{\sin\theta} \cancel{\frac{\partial}{\partial\theta}} f(r) 2\sin\theta\cos\theta\right) =$

$\sin^2\theta f''(r) + \frac{\sin\theta}{r^2} f(r) 2 \frac{\partial}{\partial \theta}(\cos\theta) =$

$\sin^2\theta f''(r) + \frac{2\sin\theta}{r^2} f(r)(-\sin\theta) = \sin^2\theta f''(r) - \frac{2\sin^2\theta}{r^2} f(r)$ ∴

$f(r) = r^\alpha$ ∴ $f''(r) = \alpha(\alpha-1) r^{\alpha-2} = \alpha^2 r^{\alpha-2} - \alpha r^{\alpha-2}$ ∴

$E^4\psi = E^2(E^2\psi) = E^2\left(\sin^2\theta f''(r) - 2\sin^2\theta f(r) \frac{1}{r^2}\right) =$

$\frac{\partial^2}{\partial r^2}\left(\sin^2\theta f''(r) - 2\sin^2\theta f(r)\frac{1}{r^2}\right) + \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta} \frac{\partial}{\partial\theta}\left(\sin^2\theta f''(r) - 2\sin^2\theta f(r)\frac{1}{r^2}\right)\right) =$

$\sin^2\theta f''''(r) - 2\sin^2\theta f''(r) + 6(-2)\sin^2\theta f(r)\frac{1}{r^4} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}(f''(r) - 2f(r)\frac{1}{r^2}) 2\sin\theta\cos\theta\right) =$

$\sin^2\theta f^{(4)}(r) - 2\sin^2\theta f''(r) - 12\sin^2\theta f(r)\frac{1}{r^4} + \frac{\sin\theta}{r^2} \frac{\partial}{\partial\theta}\left(2f''(r)\cos\theta - 4f(r)\frac{1}{r^2}\cos\theta\right) =$

$\sin^2\theta f^{(4)}(r) - 2\sin^2\theta f''(r) - 12\sin^2\theta f(r)\frac{1}{r^4} + \frac{\sin\theta}{r^2} 2f''(r)(-\sin\theta) - \frac{4\sin\theta f(r)}{r^4}(-\sin\theta) =$

$\sin^2\theta f^{(4)}(r) - 2\sin^2\theta f''(r) - 12\frac{1}{r^4} f(r)\sin^2\theta - \frac{2\sin^2\theta f''(r)}{r^2} + \frac{4\sin^2\theta f(r)}{r^4} = 0$ ∴

$f^{(4)}(r) - 2f''(r) - 12\frac{1}{r^4} f(r) - 2\frac{f''(r)}{r^2} + \frac{4f(r)}{r^4} = 0 =$

~~$r^4 f^{(4)}$ $f^{(4)}(r) - 2f''(r)(1 + \frac{1}{r^2})$~~ $f^{(4)}(r) - 2f''(r) - 2f''(r)\frac{1}{r^2} - 8f(r)\frac{1}{r^4} = 0$

$f^{(3)}(r) = \alpha^2(\alpha-2)r^{\alpha-3} - \alpha(\alpha-2)r^{\alpha-3} = \alpha^3 r^{\alpha-3} - 2\alpha^2 r^{\alpha-3} - \alpha^2 r^{\alpha-3} + 2\alpha r^{\alpha-3}$

\2civ/ $\underline{\omega} = \nabla \times \underline{u} = \nabla \times \left(\frac{\psi}{r\sin\theta} \hat{\underline{\phi}} \right) = \nabla \times \left(\frac{f(r)\sin^2\theta}{r\sin\theta} \hat{\underline{\phi}} \right) =$

$\nabla \times \left(\frac{1}{r} \sin\theta f(r) \hat{\underline{\phi}} \right) = \frac{1}{r^2\sin\theta} \begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\hat{\underline{\phi}} \\ \partial_r & \partial_\theta & \partial_\phi \\ 0 & 0 & \sin^2\theta f(r) \end{vmatrix} =$

$\frac{1}{r^2\sin\theta}\left[\hat{\underline{r}} \left(\partial_\theta (\sin^2\theta f(r)) - \partial_\phi(0) \right) - r\hat{\underline{\theta}} \left(\partial_r (\sin^2\theta f(r) - 0) \right) + r\sin\theta \hat{\underline{\phi}} (0-0) \right] =$

$\frac{1}{r^2\sin\theta}\left[\hat{\underline{r}}\, 2\sin\theta\cos\theta f(r) - r\hat{\underline{\theta}} \sin^2\theta f'(r) \right] =$

$2\cos\theta \frac{1}{r^2} f(r) \hat{\underline{r}} - \sin\theta \frac{1}{r} f'(r) \hat{\underline{\theta}}$

\1ai/ NS: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} \right) = -\nabla P + \rho \underline{g} + \mu\nabla^2\underline{u} \therefore \underline{g} = -\nabla M \therefore$

$\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} \right) = -\nabla P - \rho\nabla M + \mu\nabla^2\underline{u}$

\1aii/ equation is : $\frac{D\rho}{Dt} + \rho\nabla\cdot\underline{u} = 0 = \frac{\partial\rho}{\partial t} + \underline{u}\cdot\nabla\rho + \rho\nabla\cdot\underline{u}$

$\therefore$ for incompressible fluids the material derivative of ρ is $0 \therefore \frac{D\rho}{Dt} = 0 \therefore 0 + \rho\nabla\cdot\underline{u} = 0 = \rho\nabla\cdot\underline{u} = 0 = \nabla\cdot\underline{u}$

\1bi/ $\nabla\cdot\underline{u} = \nabla\cdot\left(\frac{1}{R}\hat{\underline{R}} + R\hat{\underline{\phi}} + 0\hat{\underline{z}} \right) = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{1}{R}\right) + \frac{1}{R}\frac{\partial(R)}{\partial\phi} + \frac{\partial(0)}{\partial z} =$

$\frac{1}{R}\frac{\partial}{\partial R}(1) + \frac{1}{R}(0) + 0 = \frac{1}{R}(0) + 0 = 0 + 0 = 0 = \nabla\cdot\underline{u} \therefore$

$\underline{u}$ is incompressible

\1bii/ $\underline{\omega} = \nabla\times\underline{u} = \nabla\times\left(\frac{1}{R}\hat{\underline{R}} + R\hat{\underline{\phi}} + 0\hat{\underline{z}} \right) = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ \frac{1}{R} & RR & 0 \end{vmatrix} =$

$\frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ \frac{1}{R} & R^2 & 0 \end{vmatrix} = \frac{1}{R}\left[\hat{\underline{R}}(0-0) - R\hat{\underline{\phi}}(0-0) + \hat{\underline{z}}\left(\frac{\partial}{\partial R}(R^2) - \frac{\partial}{\partial\phi}\left(\frac{1}{R}\right) \right) \right] =$

$\frac{1}{R}\left[\hat{\underline{z}}(2R - 0) \right] = \frac{2R}{R}\hat{\underline{z}} = 2\hat{\underline{z}} \neq 0$ $\therefore$ vorticity $\underline{\omega} \neq 0 \therefore$

the flow is not irrotational

\1biii/ $(\underline{u}\cdot\nabla)\underline{u} = \left(\left(\frac{1}{R}\hat{\underline{R}} + R\hat{\underline{\phi}} + 0\hat{\underline{z}} \right) \cdot \left(\hat{\underline{R}}\frac{\partial}{\partial R} + \hat{\underline{\phi}}\frac{\partial}{\partial\phi} + \hat{\underline{z}}\frac{\partial}{\partial z} \right) \right)\underline{u} =$

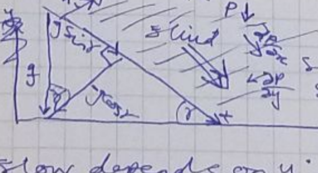
$\left(\frac{1}{R}\frac{\partial}{\partial R} + R\frac{\partial}{\partial\phi} \right)\underline{u} = \left(\frac{1}{R}\frac{\partial}{\partial R} + R\frac{\partial}{\partial\phi} \right)\left(\frac{1}{R}\hat{\underline{R}} + R\hat{\underline{\phi}} \right) =$

$\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\hat{\underline{R}}\right) + \frac{1}{R}\frac{\partial}{\partial R}(R\hat{\underline{\phi}}) + R\frac{\partial}{\partial\phi}\left(\frac{1}{R}\hat{\underline{R}}\right) + R\frac{\partial}{\partial\phi}(R\hat{\underline{\phi}}) =$

$\frac{1}{R}(-1)\frac{1}{R^2}\hat{\underline{R}} + \frac{1}{R}(1)\hat{\underline{\phi}} + R\frac{1}{R}\hat{\underline{\phi}} + RR(-\hat{\underline{R}}) =$

$\frac{-1}{R^3}\hat{\underline{R}} + \frac{1}{R}\hat{\underline{\phi}} + \hat{\underline{\phi}} - R^2\hat{\underline{R}} = \left(-R^2 - \frac{1}{R^3}\right)\hat{\underline{R}} + \left(1 - \frac{1}{R}\right)\hat{\underline{\phi}}$

\1ci/ $\therefore \nabla\cdot\underline{u} = 0$



no slip BC $\therefore$ $y=0$ is solid Boundary $\therefore$ no special points in x, flow is only in x-direction

$\therefore \underline{u} = (u, 0)$, flow depends on y $\because$ of BCs $\therefore$ $\underline{u} = (u(y), 0) \therefore$

$u = u(y)$ only

\RP2020/ $\frac{\partial \omega}{\partial t} - (\omega\cdot\nabla)u + (u\cdot\nabla)\omega = \nu\nabla^2\omega$ ∴

$\frac{\partial \omega}{\partial t} + u\cdot\nabla\omega = (\omega\cdot\nabla)u + \nu\nabla^2\omega = \frac{D\omega}{Dt}$

\1dii/A: $\omega = \nabla\times u = \nabla\times(0\hat{R} + aRz\hat{\phi} + 0\hat{z}) = \frac{1}{R}\begin{vmatrix}\hat{R} & R\hat{\phi} & \hat{z}\\ \partial_R & \partial_\phi & \partial_z \\ 0 & aR^2z & 0\end{vmatrix}$

$= \frac{1}{R}\left[\hat{R}(0-aR^2) - R\hat{\phi}(0-0) + \hat{z}(2aRz - 0)\right] =$

$-aR\hat{R} + 2a\cancel{R}z\hat{z}$ ∴

$\nabla\cdot\omega = \nabla\cdot(-R\hat{R} + 2a\cancel{R}z\hat{z}) = \frac{1}{R}\frac{\partial}{\partial R}(-R^2a) + \frac{\partial}{\partial z}(2a\cancel{R}z) = 0$

\1dii/A/:

$\omega = \nabla\times u = \nabla\times(0\hat{R} + aRz\hat{\phi} + 0\hat{z}) = \frac{1}{R}\begin{vmatrix}\hat{R} & R\hat{\phi} & \hat{z}\\ \partial_R & \partial_\phi & \partial_z \\ 0 & aR^2z & 0\end{vmatrix} =$

$\frac{1}{R}\left[\hat{R}(0-aR^2) - R\hat{\phi}(0-0) + \hat{z}(2aRz)\right] =$

$-aR\hat{R} + 2a\cancel{R}z\hat{z}$ ∴

$\nabla\cdot\omega = \cancel{\frac{1}{R}}\nabla\cdot(-aR\hat{R} + 2aRz\hat{z}) = \frac{1}{R}\frac{\partial}{\partial R}(-aR^2) + \frac{\partial}{\partial z}(2a\cancel{R}z) =$

$-\frac{1}{R}2Ra + 2a\cancel{R} = 2a - 2a = 0$

~~\1dii/B~~ \1diiB/ $\cancel{\frac{dr}{ds}} = \omega(r,t)$ $\quad c = \text{Constant}$

$\cancel{z}\,\delta(R) = \text{constant}$ ∴

$\omega = -aR\hat{R} + 2az\hat{z}$ ∴

$\cancel{dz}$ $\cancel{dz}$ ∴ $\frac{d}{dz}(z\delta(R)) = \frac{d}{dz}(c) = 0 = \delta(R)$

\1diic/ ∴ $u = \nabla\times(\psi\hat{z}) = \frac{1}{R}\begin{vmatrix}\hat{R} & R\hat{\phi} & \hat{z}\\ \partial_R & \partial_\phi & \partial_z \\ 0 & 0 & \psi\end{vmatrix} =$

$\frac{1}{R}\left[\hat{R}\left(\frac{\partial\psi}{\partial\phi}\right) - R\hat{\phi}\left(\frac{\partial\psi}{\partial R}\right) - \hat{z}(0)\right] = \frac{1}{R}\left[\hat{R}\frac{\partial\psi}{\partial\phi} - R\hat{\phi}\frac{\partial\psi}{\partial R}\right] =$

$\hat{R}\frac{1}{R}\frac{\partial\psi}{\partial\phi} - \hat{\phi}\frac{\partial\psi}{\partial R} = 0\hat{R} + aRz\hat{\phi}$ ∴

$\frac{1}{R}\frac{\partial\psi}{\partial\phi} = 0$ ∴ $\frac{\partial\psi}{\partial\phi} = 0$ ∴ $\cancel{\frac{\partial\psi}{\partial}}$ $\psi = \psi(R,z)$ ∴

$+\frac{\partial\psi}{\partial R} = -aRz$ ∴ $\psi = -\frac{a}{2}R^2z = \psi(R,z)$

\1diiD/ $\Gamma = \int_C u\cdot dl_\phi$

∴ $\cancel{\hat{n} = \hat{z}\,\hat{R}}$ ∴ $\cancel{u\cdot dl = u\cdot\hat{\phi}dR} =$ ∴ $\hat{n} = \hat{\phi}$ ∴ $u\cdot\hat{\phi} = aRz$ ∴

$dl = \hat{\phi}\,dR$ ∴ $\Gamma = \int_0^R aRz\,dR = \left[\frac{1}{2}aR^2z\right]_{R=0}^{R} = \frac{1}{2}aR^2z$

\2a/ $\therefore\ u \sim U,\quad \frac{\partial}{\partial t} \sim \frac{1}{T} = \frac{U}{L},\quad \nabla \sim \frac{1}{L}$

$\therefore\ \frac{\partial u}{\partial t} \sim \frac{U}{L} U = \frac{U^2}{L}$

$\underline{u} \cdot \nabla \underline{u} \sim U \frac{1}{L} U = \frac{U^2}{L}\quad \therefore \frac{\partial u}{\partial t} \sim \underline{u} \cdot \nabla \underline{u}\ \therefore$

$\therefore\ \mu \nabla^2 \underline{u} \sim \mu \frac{1}{L^2} U = \mu \frac{U}{L^2}\quad \therefore$

$\frac{\rho(\partial_t \underline{u})}{\mu \nabla^2 \underline{u}} \sim \frac{1}{\nu} \frac{(U^2/L)}{(U/L^2)} = \frac{UL}{\nu} = Re\ \therefore$

for $Re << \cancel{0}\ 1\ \therefore$

$\rho(\partial_t \underline{u}) \sim \rho(\underline{u} \cdot \nabla \underline{u}) << \mu \nabla^2 \underline{u}\ \therefore$

May neglect $\partial_t \underline{u}$, $\underline{u} \cdot \nabla \underline{u}\ \therefore$

$\rho(0+0) = 0 = -\nabla p + \mu \nabla^2 \underline{u}$

\2bi/ (i) toothpaste is very viscous $\therefore \nu >> 1$ is very large very viscous fluid flow

(ii) ~~toothpaste~~ ~~paste~~ ~~at~~ honey is a small distance L $\therefore$ very high Re $\therefore$ very viscous

\2c/ $\nabla \times \nabla \equiv 0\ \therefore\ \underline{\omega} = \nabla \times \underline{u} = \nabla \times \nabla \times \left(\frac{\psi}{r\sin\theta} \hat{\underline{\phi}}\right)$

$\therefore\ \nabla \cdot \underline{u} = \nabla \cdot \nabla \times \left(\frac{\psi}{r\sin\theta} \hat{\underline{\phi}}\right) \equiv 0\ \therefore$ flow is always incompressible

$$\underline{u} = \nabla \times \left(\frac{\psi}{r\sin\theta} \hat{\underline{\phi}}\right) = \frac{1}{r^2\sin\theta} \begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\hat{\underline{\phi}} \\ \partial r & \partial\theta & \partial\phi \\ 0 & 0 & \psi \end{vmatrix} =$$

$$\frac{1}{r^2\sin\theta}\left[\hat{\underline{r}}\left(\frac{\partial\psi}{\partial\theta}\right) - r\hat{\underline{\theta}}\left(\frac{\partial\psi}{\partial r}\right) + r\sin\theta\hat{\underline{\phi}}(0)\right] =$$

$$\frac{1}{r^2\sin\theta}\left[\hat{\underline{r}}\frac{\partial\psi}{\partial\theta} - r\frac{\partial\psi}{\partial r}\hat{\underline{\theta}}\right] = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\hat{\underline{r}} - \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\hat{\underline{\theta}}$$

\2cii/ $\therefore\ \underline{u}(R \to \infty) = U\underline{k}\ \therefore$

$\underline{u} = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\hat{\underline{r}} - \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\hat{\underline{\theta}} = u\hat{\underline{r}} + v\hat{\underline{\theta}}\ \therefore$

$\underline{r} = x\hat{\underline{i}} + y\hat{\underline{j}} + z\hat{\underline{k}} = r\sin\theta\cos\phi\,\hat{\underline{i}} + r\sin\theta\sin\phi\,\hat{\underline{j}} + r\cos\theta\,\hat{\underline{k}} = r\hat{\underline{r}}\ \therefore$

$\therefore\ \hat{\underline{r}} = \sin\theta\cos\phi\,\hat{\underline{i}} + \sin\theta\sin\phi\,\hat{\underline{j}} + \cos\theta\,\hat{\underline{k}}$ &

$\hat{\underline{\theta}} = \cos\theta\cos\phi\,\hat{\underline{i}} + \cos\theta\sin\phi\,\hat{\underline{j}} - \sin\theta\,\hat{\underline{k}}\ \therefore$

$\cos\theta\,\hat{\underline{r}} - \sin\theta\,\hat{\underline{k}} = \cancel{\sin\theta\cos\theta}$

$\sin\theta\cos\theta\cos\phi\,\hat{\underline{i}} + \sin\theta\cos\theta\sin\phi\,\hat{\underline{j}} + \cos^2\theta\,\hat{\underline{k}} - \sin\theta\cos\theta\cos\phi\,\hat{\underline{i}} - \sin\theta\cos\theta\sin\phi\,\hat{\underline{j}} + \sin^2\theta\,\hat{\underline{k}}$

$= \cos^2\theta\,\hat{\underline{k}} + \sin^2\theta\,\hat{\underline{k}} = \hat{\underline{k}}\ \therefore\ \cos\theta\,\hat{\underline{r}} - \sin\theta\,\hat{\underline{\theta}} = \hat{\underline{z}}\ \therefore$ as $r \to \infty$:

$U(\hat{\underline{z}}) = U(\cos\theta\,\hat{\underline{r}} - \sin\theta\,\hat{\underline{\theta}}) = U\cos\theta\,\hat{\underline{r}} - U\sin\theta\,\hat{\underline{\theta}} = u\hat{\underline{r}} + v\hat{\underline{r}}\ \therefore$

PP2020 / Vc ii / NS: $\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\nabla\frac{p}{\rho} + \underline{g} + \nu\nabla^2\underline{u}$, $\nu = \frac{\mu}{\rho}$

$\nabla\cdot\underline{u} = 0$ ∴

$\underline{u} = (u(y), 0)$ ∴ $\frac{\partial \underline{u}}{\partial t} = \left(\frac{\partial}{\partial t}u(y), \frac{\partial}{\partial t}(0)\right) = (0,0) = 0$

$\underline{u}\cdot\nabla = (u(y), 0)\cdot(\partial_x, \partial_y) = u(y)\frac{\partial}{\partial x}$ ∴

$\underline{u}\cdot\nabla\underline{u} = u(y)\frac{\partial}{\partial x}(u(y), 0) = u(y)\frac{\partial}{\partial x}(u(y)) = u(y)(0) = 0$ ∴

$\nu\nabla^2\underline{u} = \nu\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right)(u(y), 0) = \nu\frac{\partial^2 u(y)}{\partial y^2}\hat{\underline{i}}$ ∴

$-\nabla\frac{p}{\rho} = \cancel{\rho\partial p}\ -\frac{1}{\rho}\frac{\partial p}{\partial x}\hat{\underline{i}} + -\frac{1}{\rho}\frac{\partial p}{\partial y}\hat{\underline{j}}$ ∴

x component: $0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + g\sin\gamma + \nu\frac{\partial^2 u(y)}{\partial y^2}$

y component: $0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g\cos\gamma$

Vc iii / ∴ $0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g\cos\gamma$ ∴

$\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\cos\gamma$ ∴ $\frac{\partial p}{\partial y} = -\rho g\cos\gamma$ ∴

$p(y) = -\rho g(\cos(\gamma))y + c_1$ ∴

$p(y=h) = p_0 = -\rho g\cos(\gamma)h + c_1$ ∴ $c_1 = p_0 + \rho g\cos(\gamma)h$ ∴

$p(y) = -\rho g\cos(\gamma)y + p_0 + \rho g\cos(\gamma)h = p = p_0 + (h-y)\rho g\cos\gamma$

Vc iv / ∴ $\underline{u}(y=h) = (\text{constant}, .V)$ ∴

$p = p_0 + (h-y)\rho g\cos\gamma$ ∴ $\frac{\partial p}{\partial x} = 0$ ∴ $-\frac{1}{\rho}\frac{\partial p}{\partial x} = 0$ ∴

$0 = g\sin\gamma + \nu\frac{\partial^2 u(y)}{\partial y^2}$ ∴

$-\frac{g}{\nu}\sin\gamma = \frac{\partial^2 u(y)}{\partial y^2}$ ∴ $-\frac{g}{\nu}\sin(\gamma)y + c_2 = \frac{\partial u(y)}{\partial y}$ ∴

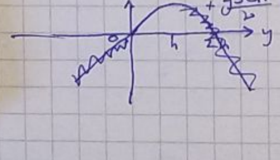
$-\frac{g}{\nu}\sin(\gamma)h + c_2 = \frac{\partial u(y=h)}{\partial y} = 0$ ∴ $c_2 = \frac{g}{\nu}\sin(\gamma)h = \frac{gh}{\nu}\sin\gamma$ ∴

$\frac{\partial u(y)}{\partial y} = -\frac{g}{\nu}\sin(\gamma)y + \frac{gh}{\nu}\sin\gamma$ ∴

$u(y) = \frac{-g}{2\nu}\sin(\gamma)y^2 + \frac{gh}{\nu}\sin(\gamma)y + c_3$ ∴

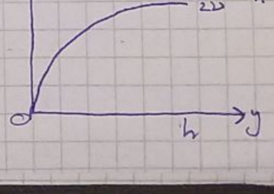
no slip BC ∴ $u(y=0) = 0 = \frac{-g}{2\nu}\sin(\gamma)(0)^2 + \frac{gh}{\nu}\sin(\gamma)(0) + c_3 = 0 = c_3$

∴ $u(y) = \frac{-g}{2\nu}\sin(\gamma)y^2 + \frac{gh}{\nu}\sin(\gamma)y = \frac{g\sin\gamma}{\nu}\left[-\frac{1}{2}y^2 + hy\right] = \frac{g\sin\gamma}{\nu}\left[-\frac{1}{2}y + h\right]y$



$+\frac{g\sin\gamma}{\nu}\left[-\frac{1}{2}h^2 + h^2\right] = \frac{g\sin\gamma}{\nu}\left[\frac{1}{2}h^2\right] = u(h)$

$y > 0$



\1c ii/ $Q = \int_0^h u\,dy = \int_0^h -\frac{g}{2\nu}\sin(\gamma)y^2 + \frac{gh}{\nu}\sin(\gamma)y\,dy =$

$\left[-\frac{g}{6\nu}\sin(\gamma)y^3 + \frac{gh}{2\nu}\sin(\gamma)y^2\right]_{y=0}^{h} =$

$\frac{-g}{6\nu}\sin(\gamma)\left[h^3-0^3\right] + \frac{gh}{2\nu}\sin(\gamma)\left[h^2-0^2\right] =$

$\frac{-gh^3\sin\gamma}{6\nu} + \frac{gh^3\sin\gamma}{2\nu} = \frac{gh^3\sin\gamma}{3\nu}$

\1c iii/ mass flux: ~~rate~~ $M = \rho\int_S \mathbf{u}\cdot d\mathbf{S} = \rho\int_S \mathbf{u}\cdot\hat{\mathbf{n}}\,dS$ ∴

as $\hat{\mathbf{n}} = \hat{\mathbf{i}}$ ∴ $\mathbf{u}\cdot\hat{\mathbf{n}} = \mathbf{u}\cdot(1,0) = (u(y),0)\cdot(1,0) = u(y)$ ∴

$M = \rho\int_0^h u(y)\,dy = \rho\int_0^h u\,dy = \frac{\rho g h^3\sin\gamma}{3\nu}$ ∴

$M_x = \rho\int_0^H u(y)|_x\,dy = \rho\int_0^{2h} u(y)|_x\,dy = \rho\int_0^{2h} \frac{-g}{2\nu}\sin(x)y^2 + \frac{gh}{\nu}\sin(x)y\,dy$

$= -\rho\frac{g\sin x}{6\nu}\left[(2h)^3\right] + \frac{\rho gh\sin x}{2\nu}\left[(2h)^2-0\right] =$

$\frac{-\rho g\sin(x)8h^3}{6\nu} + \frac{\rho gh\sin(x)8h^3}{2\nu} = \frac{8\rho gh^3\sin(x)}{3\nu} = \frac{\rho gh^3\sin\gamma}{3\nu}$ ∴

$8\sin x = \sin\gamma$ ∴ $\sin\theta$ is an increasing function ∴

$x < \gamma$ ∴ ∴ $0 \le x < \gamma < \frac{\pi}{2}$

which agrees with my expectations ∴ a steeper angle of γ would require a higher $y=h$ value for the fluid to not notice there is a slope. ~~[illegible]~~

\1d i/ $\frac{D\boldsymbol{\omega}}{Dt} = \frac{\partial\boldsymbol{\omega}}{\partial t} + \mathbf{u}\cdot\nabla\boldsymbol{\omega}$ ∴ $\frac{\partial\mathbf{u}}{\partial t} + \mathbf{u}\cdot\nabla\mathbf{u} = -\frac{1}{\rho}\nabla P - g\nabla\Pi + \nu\nabla^2\mathbf{u}$

∴ $\nu = \frac{\mu}{\rho}$ ∴ ~~$\nabla\times\frac{\partial\mathbf{u}}{\partial t}$~~ $\boldsymbol{\omega} = \nabla\times\mathbf{u}$ ∴ $\nabla\times\frac{\partial\mathbf{u}}{\partial t} = \frac{\partial}{\partial t}\nabla\times\mathbf{u} = \frac{\partial\boldsymbol{\omega}}{\partial t}$

~~$\nu\nabla\times\nabla^2\mathbf{u} =$~~ $\nabla\cdot\mathbf{u} = 0$ ∴ $\nu\nabla\times\nabla^2\mathbf{u} = \nu\nabla\times(\nabla(\nabla\cdot\mathbf{u})) - \nu\nabla\times(\nabla\times(\nabla\times\mathbf{u})) =$

$\nu\nabla\times(\nabla(0)) - \nu\nabla\times(\nabla\times(\boldsymbol{\omega})) = -\nu\nabla\times(\nabla\times(\boldsymbol{\omega})) = -\nu\nabla(\nabla\cdot\boldsymbol{\omega}) + \nu\nabla^2\boldsymbol{\omega} =$

$-\nu\nabla(\nabla\cdot\nabla\times\mathbf{u}) + \nu\nabla^2\boldsymbol{\omega} = -\nu\nabla(0) + \nu\nabla^2\boldsymbol{\omega} = \nu\nabla^2\boldsymbol{\omega}$

$-\nabla\times\nabla\frac{P}{\rho} = -0 = 0$, $-g\nabla\times\nabla\Pi = -0 = 0$ ∴

~~$2\mathbf{u}\cdot\nabla\mathbf{u} =$~~ ∴ $\frac{\partial\boldsymbol{\omega}}{\partial t} + \nabla\times(\mathbf{u}\cdot\nabla\mathbf{u}) = \nu\nabla^2\boldsymbol{\omega}$ ∴

$2(\mathbf{u}\cdot\nabla)\mathbf{u} = \nabla(\mathbf{u}\cdot\mathbf{u}) - 2\mathbf{u}\times(\nabla\times\mathbf{u})$ ∴ $\mathbf{u}\cdot\nabla\mathbf{u} = \nabla\left(\frac{\mathbf{u}\cdot\mathbf{u}}{2}\right) - \mathbf{u}\times(\nabla\times\mathbf{u})$

∴ $\nabla\times(\mathbf{u}\cdot\nabla\mathbf{u}) = \nabla\times\nabla\left(\frac{\mathbf{u}\cdot\mathbf{u}}{2}\right) - \nabla\times(\mathbf{u}\times(\nabla\times\mathbf{u})) = 0 - \nabla\times(\mathbf{u}\times\boldsymbol{\omega}) =$

$-\mathbf{u}(\nabla\cdot\boldsymbol{\omega}) + \boldsymbol{\omega}(\nabla\cdot\mathbf{u}) - (\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega} = -(\boldsymbol{\omega}\cdot\nabla)\mathbf{u} + (\mathbf{u}\cdot\nabla)\boldsymbol{\omega}$ ∴

\PP2020/ $u = U\cos\theta$, $v = -U\sin\theta = \frac{-1}{r\sin\theta}\frac{\partial\psi}{\partial r}$,

$U\cos\theta = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$ ∴

$(U\sin^2\theta) r = \frac{\partial\psi}{\partial r}$ ∴ $\psi = \frac{1}{2}Ur^2\sin^2\theta + f(\theta)$ ∴

$\frac{\partial\psi}{\partial\theta} = Ur^2\cos\theta\sin\theta = \frac{1}{2}U2(r^2)\sin\theta\cos\theta + f'(\theta) = Ur^2\cos\theta\sin\theta + f'(\theta)$

∴ $f'(\theta) =$ ~~constant~~ 0 ∴

$f(\theta)$ indep of θ ∴ $f(\theta) = 0$ ∴

$\cancel{\partial\psi}$ $\psi = \frac{1}{2}Ur^2\sin^2\theta$ as $r\to\infty$ with $\underline{u}\to U\hat{\underline{z}}$ as $r\to\infty$

\2ciii/ $E^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right)$ ∴

∴ $E^2\psi = E^2 f(r)\sin^2\theta$ ∴

$E^2\psi = \sin^2\theta f'' + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{\sin\theta}{\sin\theta}2\cos\theta f\right) = \sin^2\theta f'' + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}(2\cos\theta f)$

$= \sin^2\theta f'' + \frac{\sin\theta}{r^2}(-2)\sin\theta f = \sin^2\theta f'' - \frac{2\sin^2\theta}{r^2}f = \left(f'' - \frac{2f}{r^2}\right)\sin^2\theta$

∴ $E^4\psi = E^2E^2\psi = E^2\left(\left(f'' - \frac{2f}{r^2}\right)\sin^2\theta\right)$

$\cancel{\sin^2\theta\left(f'''' - \frac{2f''}{r^2}\right.}$

$\frac{\partial}{\partial r}(f'' - 2r^{-2}f) = f''' - 2r^{-2}f' + 4r^{-3}f$

∴ $\frac{\partial^2}{\partial r^2}\left(f'' - \frac{2f}{r^2}\right) = \frac{\partial}{\partial r}(f''' - 2r^{-2}f' + 4r^{-3}f) = f'''' - 2r^{-2}f'' + 4r^{-3}f' + 4r^{-3}f' - 12r^{-4}f$

$= f'''' - 2r^{-2}f'' + 8r^{-3}f' - 12r^{-4}f$

$\cancel{\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\left(f'' - \frac{2f}{r^2}\right)\sin^2\theta\right) = \frac{\partial}{\partial\theta}\left(\sin\theta\left(f'' - \frac{2f}{r^2}\right)\right) = \cos\left(f'' - \frac{2f}{r^2}\right)}$

$\cancel{E^2\psi}$ $E^4\psi = E^2E^2\psi = E^2(E^2\psi) = E^2\left(\left(f'' - \frac{2f}{r^2}\right)\sin^2\theta\right)$ ∴

$\frac{\partial}{\partial\theta}\left(\left(f'' - \frac{2f}{r^2}\right)\sin^2\theta\right) = \left(f'' - \frac{2f}{r^2}\right)2\sin\theta\cos\theta$ ∴

$\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\left(f'' - \frac{2f}{r^2}\right)\sin^2\theta\right)\right) = \frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\left(f'' - \frac{2f}{r^2}\right)2\sin\theta\cos\theta\right) = \frac{\partial}{\partial\theta}\left(2\left(f'' - \frac{2f}{r^2}\right)\cos\theta\right) =$

$-2\left(f'' - \frac{2f}{r^2}\right)\sin\theta = \left(\frac{4f}{r^2} - 2f''\right)\sin\theta = \cancel{\sin\theta}\, 4r^{-2}f\sin\theta - 2f''\sin\theta$ ∴

$E^4\psi = f'''' - 2r^{-2}f'' + 8r^{-3}f' + \cancel{4r^{-3}f} - 12r^{-4}f + 4r^{-2}f\sin\theta - 2f''\sin\theta = 0$

∴ $f = r^\alpha$ ∴ $f' = \alpha r^{\alpha-1}$ ∴ $f'' = \alpha(\alpha-1)r^{\alpha-2}$ ∴ $f''' = \alpha(\alpha-1)(\alpha-2)r^{\alpha-3}$ ∴ $f'''' = \alpha(\alpha-1)(\alpha-2)(\alpha-3)r^{\alpha-4}$

∴ $E^4\psi = f'''' + f''(-2r^{-2} - 2\sin\theta) + f'(8r^{-3}) + f(-12r^{-4} + 4r^{-2}\sin\theta) =$

$\alpha(\alpha-1)(\alpha-2)(\alpha-3)r^{\alpha-4} + \alpha(\alpha-1)r^{\alpha-2}(-2r^{-2} - 2\sin\theta) + \alpha r^{\alpha-1}8r^{-3} + r^\alpha(-12r^{-4} + 4r^{-2}\sin\theta)$

\2 ciii/ $E^2\psi = \frac{\partial^2\psi}{\partial r^2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial\psi}{\partial\theta}\right) =$

$\frac{\partial}{\partial r^2}(f\sin^2\theta) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(f\sin^2\theta)\right) =$

$f''\sin^2\theta + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}2f\sin\theta\cos\theta\right) = f''\sin^2\theta + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}(2f\cos\theta) =$

$f''\sin^2\theta + \frac{\sin\theta}{r^2}2f(-\sin\theta) = f''\sin^2\theta - 2r^{-2}\sin^2\theta f =$

$\sin^2\theta(f'' - 2r^{-2}f) = \sin^2\theta\left(\frac{d^2}{dr^2}f - 2r^{-2}f\right) =$

$\sin^2\theta\left(\frac{d^2}{dr^2}(r^\alpha) - 2r^{-2}r^\alpha\right) = \sin^2\theta\left(\frac{d}{dr}(\alpha r^{\alpha-1}) - 2r^{\alpha-2}\right) =$

$\sin^2\theta(\alpha(\alpha-1)r^{\alpha-2} - 2r^{\alpha-2}) = \sin^2\theta(\alpha(\alpha-1)-2)r^{\alpha-2}$ ∴

$E^4\psi = E^2(E^2\psi) = E^2(\sin^2\theta(\alpha(\alpha-1)-2)r^{\alpha-2}) =$

$\frac{\partial^2}{\partial r^2}(\sin^2\theta(\alpha(\alpha-1)-2)r^{\alpha-2}) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin^2\theta(\alpha(\alpha-1)-2)r^{\alpha-2})\right) =$

$\sin^2\theta(\alpha(\alpha-1)-2)(\alpha-2)\frac{\partial}{\partial r}(r^{\alpha-3}) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}2\sin\theta\cos\theta(\alpha(\alpha-1)-2)r^{\alpha-2}\right) =$

$\sin^2\theta(\alpha(\alpha-1)-2)(\alpha-2)(\alpha-3)r^{\alpha-4} + \frac{\sin\theta}{r^2}(\alpha(\alpha-1)-2)2\frac{\partial}{\partial\theta}(\cos\theta)r^{\alpha-2}$

$\sin^2\theta(\alpha(\alpha-1)-2)(\alpha-2)(\alpha-3)r^{\alpha-4} - r^{\alpha-4}(\alpha(\alpha-1)-2)2\sin^2\theta =$

$r^{\alpha-4}\sin^2\theta\left[(\alpha(\alpha-1)-2)(\alpha-2)(\alpha-3) - 2(\alpha(\alpha-1)-2)\right] =$

$r^{\alpha-4}(\alpha(\alpha-1)-2)\sin^2\theta\left[(\alpha-2)(\alpha-3)-2\right] = 0$ ∴

$\alpha(\alpha-1)-2 = 0$ or $(\alpha-2)(\alpha-3)-2 = 0$ ∴

$\alpha^2-\alpha-2 = (\alpha+1)(\alpha-2) = 0$, $\alpha^2+6-5\alpha-2 = \alpha^2-5\alpha+4 = (\alpha-1)(\alpha-4)$ ∴

$\alpha=-1$, $\alpha=2$, $\alpha=1$, $\alpha=4$ ∴

$f = r^\alpha$ ∴

General Solution: $f = Ar^{-1} + Br^1 + Cr^2 + Dr^4$

\2civ/ $\omega = \nabla\times\underline{u} = \nabla\times\left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\hat{\underline{r}} - \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\hat{\underline{\theta}} + 0\hat{\underline{\phi}}\right) =$

$$\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\hat{\underline{\phi}}\\ \partial_r & \partial_\theta & \partial_\phi\\ \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} & -\frac{1}{\sin\theta}\frac{\partial\psi}{\partial r} & 0\end{vmatrix} =$$

$\cancel{\frac{1}{r^2\sin\theta}\left[\hat{\underline{r}}\left(\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial r^2} - \frac{\partial}{\partial\theta}\left(r^{-2}(\sin\theta)^{-1}\frac{\partial\psi}{\partial\theta}\right)\right)\right.}$

$\frac{1}{r^2\sin\theta}\left[\hat{\underline{r}}(0-0) - r\hat{\underline{\theta}}(0-0) + r\sin\theta\hat{\underline{\phi}}\left(-\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial r^2} - \frac{\partial}{\partial\theta}\left(r^{-2}(\sin\theta)^{-1}\frac{\partial\psi}{\partial\theta}\right)\right)\right] =$

$\frac{1}{r^2\sin\theta}\left[r\sin\theta\hat{\underline{\phi}}\left(-\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial r^2} - r^{-2}(\sin\theta)^{-1}\frac{\partial^2\psi}{\partial\theta^2} - r^{-2}(-1)(\sin\theta)^{-2}(\cos\theta)\frac{\partial\psi}{\partial\theta}\right)\right] =$

$\frac{1}{r}\hat{\underline{\phi}}\left(-\frac{1}{\sin\theta}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^2\sin\theta}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\cos\theta}{r^2\sin^2\theta}\frac{\partial\psi}{\partial\theta}\right)$ ∴

$\psi = f\sin^2\theta$ ∴ $\frac{\partial\psi}{\partial\theta} = f2\sin\theta\cos\theta$, $\frac{\partial^2\psi}{\partial\theta^2} = 2f\cos\theta\cos\theta + 2f\sin\theta(-1)\sin\theta =$

$2f\cos^2\theta - 2f\sin^2\theta$, $\frac{\partial\psi}{\partial\theta} = f\sin(2\theta)$ ∴ $\frac{\partial^2\psi}{\partial\theta^2} = 2f\cos(2\theta)$, PP2020 $\frac{\partial \psi}{\partial r} = f' \sin^2\theta \therefore \frac{\partial^2 \psi}{\partial r^2} = f'' \sin^2\theta \therefore$

$f' = \frac{\partial f}{\partial r} = -Ar^{-2} + B + 2Cr + 4Dr^3$

$\therefore f'' = 2Ar^{-3} + 2C + 12Dr^2 \therefore$

$\frac{\partial^2 \psi}{\partial r^2} = (2Ar^{-3} + 2C + 12Dr^2)\sin^2\theta \therefore$

$\underline{\omega} = \left(-\frac{1}{r\sin\theta}\frac{\partial^2\psi}{\partial r^2} - \frac{1}{r^3\sin\theta}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\cos\theta}{r^3\sin^2\theta}\frac{\partial\psi}{\partial\theta}\right)\hat{\underline{\phi}} =$

$\cancel{\left(\frac{-(2Ar^{-3}+2C+12Dr^2)\sin^2\theta}{r\sin\theta}\right.}$

$\left(-\frac{1}{r\sin\theta} f''\sin^2\theta - \frac{1}{r^3\sin\theta}(2f\cos^2\theta - 2f\sin^2\theta) + \frac{\cos\theta}{r^3\sin^2\theta} 2f\sin\theta\cos\theta\right)\hat{\underline{\phi}} =$

$\left(-\frac{\sin\theta}{r} f'' - \frac{2\cos^2\theta}{r^3\sin\theta} f + \frac{2\sin\theta}{r^3} f + \frac{2\cos^2\theta}{r^3\sin\theta} f\right)\hat{\underline{\phi}} =$

$\left(-\frac{\sin\theta}{r} f'' + \frac{2\sin\theta}{r^3} f\right)\hat{\underline{\phi}}$

The strength of a vortex tube through C is equal to the flux of vorticity the open surface S that is bounded by the closed curve C

flux of vorticity $= \int_S \underline{\omega}\cdot d\underline{S} = \int_S \nabla\times\underline{u}\cdot dS = \int_S (\nabla\times\underline{u})\,\hat{z}\, dS =$

$\int_C \underline{u}\cdot d\underline{l} = \int_a^b \underline{u}\cdot\hat{\underline{n}}\, dr \quad \int_a^b \underline{u}\cdot\hat{\underline{r}}\, dr = \int_a^b \left(\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\hat{\underline{r}} - \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\hat{\underline{\theta}}\right)\cdot(1\hat{\underline{r}} + 0\hat{\underline{\theta}})\, dr =$

$\int_a^b \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\, dr = \int_a^b \frac{1}{r^2\sin\theta} 2f\sin\theta\cos\theta\, dr =$

$\cancel{\int_a^b \frac{2\cos\theta}{r^2}} \quad \int_a^b 2r^{-2}\cos\theta\,(f)\,dr = \int_a^b 2r^{-2}\cos\theta\,(Ar^{-1} + Br + Cr^2 + Dr^4\, dr =$

$2\cos\theta\int_a^b r^{-3} + Br^{-1} + C + Dr^2\, dr =$

$2\cos\theta\left[\frac{1}{-2}r^{-2} + B\ln(r) + Cr + \frac{D}{3}r^3\right]_a^b =$

$2\cos\theta\left[-\frac{1}{2}\left(\frac{1}{b^2} - \frac{1}{a^2}\right) + B(\ln b - \ln a) + C(b-a) + \frac{D}{3}(b^3 - a^3)\right] =$

$2\cos\theta\left[\frac{1}{2}\left(\frac{1}{a^2} - \frac{1}{b^2}\right) + B\ln\left(\frac{b}{a}\right) + C(b-a) + \frac{D}{3}(b^3 - a^3)\right]$

3a $\therefore \underline{u}(|x|=\infty, t) = (0, 0, \omega(|x|=\infty, t)) = (0,0,0)$

$\frac{\partial\underline{u}}{\partial x} = \left(0, 0, \frac{\partial\omega}{\partial x}\right) \therefore \frac{\partial\underline{u}(|x|=\infty, t)}{\partial x} = \left(0, 0\, \frac{\partial\omega(|x|=\infty, t)}{\partial x}\right) = (0,0,0)$

NS: $\rho\left(\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho g + \mu\nabla^2\underline{u} \therefore \frac{\mu}{\rho} = \nu \therefore$

$\frac{\partial\underline{u}}{\partial t} = \frac{\partial}{\partial t}(0,0,\omega) = \left(0, 0, \frac{\partial\omega}{\partial t}\right) = \frac{\partial\omega}{\partial t}\hat{\underline{k}} \therefore$

$+\nabla p = -\nabla p = 0, \quad g = 0 = \rho g \therefore \quad \cancel{\rho(\underline{u}\cdot\nabla} \quad \left(\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = \nu\nabla^2\underline{u} \therefore$

$\nabla^2\underline{u} = \nabla^2(0,0,\omega(x,t)) = \frac{\partial^2}{\partial x^2}\omega(x,t) = \frac{\partial^2\omega}{\partial t^2}\hat{\underline{k}}, \therefore$

$\underline{u}\cdot\nabla\underline{u} = (\underline{u}\cdot\nabla)(0,0,\omega) = ((0,0,\omega)\cdot(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}))(0,0,\omega) =$

$\omega\frac{\partial}{\partial z}(0,0,\omega(x,t)) = (0,0,\omega\frac{\partial\omega(x,t)}{\partial z}) = (0,0,0)$ ∴

$\frac{\partial\omega}{\partial t}\hat{\underline{k}} = \nu\frac{\partial^2\omega}{\partial x^2}\hat{\underline{k}}$

∴ taking $\hat{\underline{k}}$ component of NS: $\frac{\partial\omega}{\partial t} = \nu\frac{\partial^2\omega}{\partial x^2}$

\3b/ $\frac{dQ}{dt} = \frac{\partial Q}{\partial t} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty}\omega\,dx = \int_{-\infty}^{\infty}\frac{\partial\omega}{\partial t}dx = \int_{-\infty}^{\infty}\nu\frac{\partial^2\omega}{\partial x^2}dx =$

$\nu\left[\frac{\partial\omega}{\partial x}\right]_{x=-\infty}^{\infty} = \nu\left[\lim_{x\to\infty}\frac{\partial\omega}{\partial x} - \lim_{x\to-\infty}\frac{\partial\omega}{\partial x}\right] = \nu[0-0] = \nu(0) = 0$

\3c/ $[x]=L$, $[t]=T$

$\left[\frac{\partial\omega}{\partial t}\right] = [\nu\nabla^2\underline{u}] = T^{-1}LT^{-1} = LT^{-2} = [\nu]L^{-2}LT^{-1} = L^{-1}T^{-1}$ ∴

$[\nu] = L\,L\,T^{-2}T = L^2T^{-1}$

∴ $n=3$, x,t independent, ν dependent ∴

$\Pi_1 = \frac{\nu}{x^\alpha t^\beta}$ ∴ $[\nu] = [x]^\alpha[t]^\beta = L^\alpha T^\beta = L^2T^{-1}$ ∴ $\alpha=2$, $\beta=-1$ ∴

$\Pi_1 = \frac{\nu}{x^2t^{-1}} = \frac{\nu t}{x^2}$ ∴

$\Pi = \omega' = \phi(\frac{\nu t}{x^2}) = G((\frac{\nu t}{x^2})^{-1/2}) = G((\frac{x^2}{\nu t})^{1/2}) = G(\frac{x}{\sqrt{\nu t}}) = G(\psi)$

$\psi = \frac{x}{\sqrt{\nu t}}$

\3d/ $Q = \int_{-\infty}^{\infty}w\,dx$, $\frac{dQ}{dt}=0$ ∴ Q = constant ∴

$\omega = \frac{Q}{\sqrt{4\nu t}}F(\frac{x}{\sqrt{4\nu t}}) = \frac{1}{x}Q\frac{x}{\sqrt{4\nu t}}F(\frac{x}{\sqrt{4\nu t}}) = \omega = \frac{Q}{\sqrt{4\nu t}}F(\xi)$

$\omega\to 0$ as $|x|\to\infty$, $\frac{\partial\omega}{\partial x}\to 0$ as $|x|\to\infty$ ∴

$\frac{\partial\omega}{\partial x} = \frac{Q}{\sqrt{4\nu t}}F'(\frac{x}{\sqrt{4\nu t}})\frac{1}{\sqrt{4\nu t}} = \frac{Q}{4\nu t}F'(\frac{x}{\sqrt{4\nu t}}) = \frac{Q}{4\nu t}F'(\xi)$

∴ $\omega = \frac{Q}{\sqrt{4\nu t}}F(\xi)\to 0$ as $|x|\to\infty$ ∴

∴ $\frac{Q}{\sqrt{4\nu t}}F(\frac{x}{\sqrt{4\nu t}})\to 0$ as $|x|\to\infty$ ∴

$F(\frac{x}{\sqrt{4\nu t}})\to 0$ as $|x|\to\infty$ ∴ $F(\xi)\to 0$ as $|x|\to\infty$

$\frac{\partial\omega}{\partial x} = \frac{Q}{4\nu t}F'(\xi)\to 0$ as $|x|\to\infty$ ∴

$F'(\xi)\to 0$ as $|x|\to\infty$

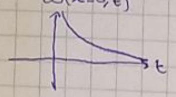
\3e/ (a) $F(\xi) = \pi^{-1/2}e^{-\xi^2} = \pi^{-1/2}e^{-(\frac{x}{\sqrt{4\nu t}})^2} = \cancel{\pi^{-1/2}e^{-\frac{x^2}{(4\nu t)}}}$

$\pi^{-1/2}e^{-\frac{x^2}{4\nu t}}$ ∴

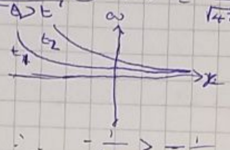
$\omega = \frac{Q}{\sqrt{4\nu t}}F(\xi) = \frac{Q}{\sqrt{4\nu t}}\pi^{-1/2}e^{-\frac{x^2}{4\nu t}} = \frac{Q}{\sqrt{4\pi\nu t}}e^{-\frac{x^2}{4\nu t}}$.

\PP2020/ e^{-x^2} is a decreasing function $\therefore$

$$\max_x \omega = \omega|_{x=0} = \frac{Q}{\sqrt{4\pi\nu t}} e^{-\frac{0^2}{4\nu t}} = \frac{Q}{\sqrt{4\pi\nu t}} e^0 = \frac{Q}{\sqrt{4\pi\nu}\sqrt{t}} = \frac{Q}{\sqrt{4\pi\nu}} t^{-1/2}$$



$\therefore$



$$\omega = \frac{Q}{\sqrt{4\pi\nu t}} e^{-\frac{x^2}{4\nu t}}$$

$\therefore t_2 > t_1 \therefore \frac{1}{t_1} > \frac{1}{t_2} \therefore -\frac{1}{t_2} > -\frac{1}{t_1}, \frac{1}{\sqrt{t_1}} > \frac{1}{\sqrt{t_2}}$

$\therefore e^{-\frac{1}{t_2}} > e^{-\frac{1}{t_1}} \therefore \omega|_{t=t_2} > \omega|_{t=t_1}$

$$\omega|_{x=0} = \frac{Q}{\sqrt{4\pi\nu t}} e^{-\frac{0^2}{4\nu t}} = \frac{Q}{\sqrt{4\pi\nu t}} \therefore$$

$$\omega|_{x=0} \times e^{-1} = \omega|_{x=x_*} = \frac{Q}{\sqrt{4\pi\nu t}} e^{-1} = \frac{Q}{\sqrt{4\pi\nu t}} e^{-\frac{x_*^2}{4\nu t}} \therefore$$

$$e^{-\frac{x_*^2}{4\nu t}} = e^{-1} \therefore -1 = -\frac{x_*^2}{4\nu t} \therefore 4\nu t = x_*^2 \therefore$$

$x_* = \sqrt{4\nu t}$ is the distance x must travel for $e^{-1}\omega|_{x=0} = \omega|_{x=x_*}$

\38/ $E(t) = \int_{-\infty}^{\infty} |\underline{u}|^2 \, dx =$

$$\therefore \frac{dE(t)}{dt} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} |\underline{u}|^2 \, dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} |\underline{u}|^2 \, dx = \int_{-\infty}^{\infty} 2 |\underline{u}| \frac{\partial \underline{u}}{\partial t} \, dx =$$

$$2\int_{-\infty}^{\infty} \frac{\partial}{\partial t} |(0,0,\omega)| \frac{\partial (0,0,\omega)}{\partial t} \, dx = 2\int_{-\infty}^{\infty} |(0,0,\omega)| \frac{\partial}{\partial t} \omega \underline{\hat{k}} \, dx =$$

$$2\int_{-\infty}^{\infty} |\omega| \frac{\partial}{\partial t} \omega \, dx = 2\int_{-\infty}^{\infty} \left|\frac{Q}{\sqrt{4\pi\nu t}} e^{-\frac{x^2}{4\nu t}}\right| \frac{\partial}{\partial t} \omega \, dx =$$

$$2\int_{-\infty}^{\infty} \frac{Q}{\sqrt{4\pi\nu t}} e^{-\frac{x^2}{4\nu t}} \frac{\partial \omega}{\partial t} \, dx = 2\int_{-\infty}^{\infty} \omega \frac{\partial \omega}{\partial t} \, dx = 2\int_{-\infty}^{\infty} \omega \nu \frac{\partial^2 \omega}{\partial x^2} \, dx =$$

$$2\nu \int_{-\infty}^{\infty} \omega \frac{\partial^2 \omega}{\partial x^2} \, dx = 2\nu\left(\left[\omega \frac{\partial \omega}{\partial x}\right]_{x=-\infty}^{\infty} - \int_{-\infty}^{\infty} \frac{\partial \omega}{\partial x} \frac{\partial \omega}{\partial x} \, dx\right) =$$

$$2\nu\left(\left[\lim_{x\to\infty} \omega \frac{\partial \omega}{\partial x} - \lim_{x\to-\infty} \omega \frac{\partial \omega}{\partial x}\right] - \left[\frac{\partial}{\partial x}(\omega)\,\omega\right]_{x=-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{\partial^2 \omega}{\partial x^2} \omega \, dx =\right.$$

$$2\nu\left(\left[\omega \frac{\partial \omega}{\partial x}\right]_{x=-\infty}^{\infty} - \left[\omega \frac{\partial \omega}{\partial x}\right]_{-\infty}^{\infty}\right) + 2\nu\int_{-\infty}^{\infty} \omega \frac{\partial^2 \omega}{\partial x^2} \, dx = 2\nu \int_{-\infty}^{\infty} \omega \frac{\partial^2 \omega}{\partial x^2} \, dx$$

$$\frac{dE(t)}{dt} = \frac{\partial}{\partial t}\int_{-\infty}^{\infty} |(0,0,\omega)|^2 \, dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} |(0,0,\omega)|^2 \, dx = \int_{-\infty}^{\infty} \frac{\partial}{\partial t} \omega^2 \, dx =$$

$$2\int_{-\infty}^{\infty} \omega \frac{\partial \omega}{\partial t} \, dx = 2\int_{-\infty}^{\infty} \omega \nu \frac{\partial^2 \omega}{\partial x^2} \, dx = 2\nu \int_{-\infty}^{\infty} \omega \frac{\partial^2 \omega}{\partial x^2} \, dx \therefore$$

$\frac{dE}{dt} \propto t^p \therefore \frac{dE}{dt} = h(x) t^p$ where h is a function independent of t $\therefore$

$\frac{dE}{dt} = h(x) t^{-1/2} = h(x) t^p \therefore p = -1/2$

4a) $\underline{u}\times(\nabla\times\underline{u})+\underline{u}\times(\nabla\times\underline{u})=\nabla(\underline{u}\cdot\underline{u})-(\underline{u}\cdot\nabla)\underline{u}-(\underline{u}\cdot\nabla)\underline{u}=$

$2\underline{u}\times(\nabla\times\underline{u})=\nabla(\underline{u}\cdot\underline{u})-2(\underline{u}\cdot\nabla)\underline{u}$ $\therefore$

$\underline{u}\times(\nabla\times\underline{u})=\frac{1}{2}\nabla(\underline{u}\cdot\underline{u})-(\underline{u}\cdot\nabla)\underline{u}$

$\therefore [\underline{u}\times(\nabla\times\underline{u})]\rightarrow[\underline{u}\times(\nabla\times\underline{u})]_i=\varepsilon_{ijk}u_j(\nabla\times\underline{u})_k=\varepsilon_{ijk}u_j\varepsilon_{klm}\partial_l u_m=$

$\cancel{\varepsilon_{ijk}u_j\varepsilon}$ $\varepsilon_{kij}u_j\varepsilon_{klm}\partial_l u_m$

$[\underline{u}\cdot\underline{u}]_i=u_iu_i$ $\therefore$

$[\frac{1}{2}\nabla(\underline{u}\cdot\underline{u})]_i=\frac{1}{2}\partial_j u_ju_j$

$[\underline{u}\cdot\nabla]_i=u_i\partial_i$

$[(\underline{u}\cdot\nabla)\underline{u}]_i=u_j\partial_j u_i$

$[\underline{u}\times(\nabla\times\underline{u})]_i=\varepsilon_{ijk}u_j(\nabla\times\underline{u})_k=\varepsilon_{ijk}u_j\varepsilon_{kmn}\partial_m u_n=$

$\varepsilon_{kji}\varepsilon_{kmn}u_j\partial_m u_n=(\delta_{jm}\delta_{in}-\delta_{jn}\delta_{im})u_j\partial_m u_n=$

$\delta_{jm}\delta_{in}u_j\partial_m u_n-\delta_{jn}\delta_{im}u_j\partial_m u_n=$

$u_j\partial_j u_i-u_j\partial_i u_j=u_j\frac{\partial u_i}{\partial u_j}-u_j\frac{\partial u_j}{\partial_i}$

$[\frac{1}{2}\nabla(\underline{u}\cdot\underline{u})]_i-[(\underline{u}\cdot\nabla)\underline{u}]_i=\frac{1}{2}\partial_i(\underline{u}\cdot\underline{u})-(\underline{u}\cdot\nabla)\underline{u}_i=\frac{1}{2}\partial_i(u_ju_j)-(u_j\cdot\partial_j)u_i=$

$\frac{1}{2}\partial_i(u_j)u_j+\frac{1}{2}\partial_i(u_j)u_j-u_j\frac{\partial u_i}{\partial_j}=\cancel{\partial_i}\, u_j\frac{\partial u_j}{\partial x_i}-u_j\frac{\partial u_i}{\partial x_j}$

~~4a~~ $\therefore [\underline{u}\times(\nabla\times\underline{u})]_i=\varepsilon_{ijk}u_j(\nabla\times\underline{u})_k=\varepsilon_{ijk}u_j\varepsilon_{kmn}\partial_m u_n=$

$\varepsilon_{kij}u_j\varepsilon_{kmn}\partial_m u_n=\varepsilon_{kij}\varepsilon_{kmn}u_j\partial_m u_n=$

$(\delta_{im}\delta_{jn}-\delta_{in}\delta_{jm})u_j\partial_m u_n=\delta_{im}\delta_{jn}u_j\partial_m u_n-\delta_{jn}\delta_{im}u_j\partial_m u_n=$

$u_j\partial_i u_j-u_j\partial_j u_i=\frac{1}{2}u_j\partial_i u_j+\frac{u_j}{2}\partial_i u_j-u_j\partial_i u_j=$

$\frac{1}{2}(u_j\partial_i u_j+u_j\partial_i u_j)-u_j\nabla_j u_i=$

$\frac{1}{2}[\nabla(\underline{u}\cdot\underline{u})]_i-[\underline{u}\cdot\nabla\underline{u}]_i=[\frac{1}{2}\nabla(\underline{u}\cdot\underline{u})-\underline{u}\cdot\nabla\underline{u}]_i$ $\therefore$

$u\times(\nabla\times\underline{u})=\frac{1}{2}\nabla(\underline{u}\cdot\underline{u})-(\underline{u}\cdot\nabla)\underline{u}$

4b) inviscid $\therefore \nu=\mu=0$, irrotational $\therefore$ $\cancel{\nabla}$ $\underline{\omega}=\nabla\times\underline{u}=0$

$\rho=$ constant $\therefore \frac{D\rho}{Dt}=\frac{\partial\rho}{\partial t}+u\cdot\nabla\rho=0$ $\therefore$

incompressible $\therefore \nabla\cdot\underline{u}=0$

$\therefore \nabla\times\underline{u}=0$ $\therefore$ $\underline{u}=\nabla\phi$

fluid flow is irrotational $\therefore$ $\underline{\omega}=\nabla\times\underline{u}=0$ $\therefore$

$\nabla\times\nabla\phi\equiv0$ $\therefore$ if $\cancel{\nabla\phi}$ $u=\nabla\phi$ then $\cancel{\nabla}$ $\underline{\omega}=\nabla\times\nabla\phi=0$ is always true

$\therefore$ incompressible $\therefore \nabla\cdot\underline{u}=0$ $\therefore$ $\nabla^2\phi=\cancel{\nabla^2}$ $\cancel{\nabla\cdot\nabla\phi}$

$\nabla\cdot\underline{u}=\nabla\cdot\nabla\phi=\nabla^2\phi=0$

\PP2020/

\4b ii/ $\nabla^2\phi = 0 \Rightarrow \frac{\partial^2}{\partial x^2}\phi + \frac{\partial^2}{\partial y^2}\phi = 0$

$\therefore |\underline{u}| < \infty$ as $y \to \infty \therefore$

$\underline{u} = \nabla\phi \Rightarrow \underline{u} = \nabla f(y)\sin(kx-\omega t) \Rightarrow$

$\frac{\partial}{\partial x} f(y)\sin(kx-\omega t)\hat{i} + \frac{\partial}{\partial y} f(y)\sin(kx-\omega t)\hat{j} =$

$f(y)k\cos(kx-\omega t)\hat{i} + f'(y)\sin(kx-\omega t)\hat{j} \therefore$

$u(y=\infty) < \infty \therefore$

$\cancel{f(y)}$ $f(y=-\infty) < \infty$, $f'(y=\infty) < \infty \therefore$

$|f(y)| < \infty$ as $y \to \infty$, $|f'(y)| < \infty$ as $y \to -\infty$

$\therefore$ let $f(y) = Ae^{ky} \therefore f'(y) = Ake^{ky} \therefore$

$\nabla \cdot \underline{u} = 0 = \frac{\partial}{\partial x}(f(y)k\cos(\cancel{k}-\omega t)) + \frac{\partial}{\partial y} f(y)\sin(kx-\omega t) \Rightarrow$

$-f(y)k^2\sin(kx-\omega t) + f''(y)\sin(kx-\omega t) = 0 \therefore$

$-f(y)k^2 + f''(y) = 0 = f''(y) - k^2 f(y) = 0 \therefore$

let $f(y) = e^{qy} \therefore f''(y) = q^2 e^{qy} \therefore$

$q^2e^{qy} - k^2e^{qy} = 0 = q^2 - k^2 = 0 \therefore q^2 = k^2 \therefore q = \pm k \therefore k > 0 \therefore$

$f(y) = Be^{ky} + Ce^{-ky} \therefore f(y=-\infty) = Be^{-\infty} + Ce^{\infty}$ ~~$< \infty \therefore C=0$~~

~~$f(y) = Ae^{ky}$~~ $\therefore = B(0) + Ce^{\infty} = Ce^{\infty} < \infty \therefore C = 0 \therefore$

~~f~~ ~~f~~ let $B = A \therefore f(y) = Ae^{ky}$

\4biii/ NS: $\rho\left(\frac{\partial u}{\partial t} + u\cdot\nabla\underline{u}\right) = -\nabla P + \rho g + \mu\nabla^2\underline{u} \therefore$

$\frac{\partial}{\partial t}\underline{u} = \frac{\partial}{\partial t}\nabla\phi = \nabla\frac{\partial\phi}{\partial t} \cancel{z} = \frac{\partial}{\partial x}\frac{\partial\phi}{\partial t}\hat{i} + \frac{\partial}{\partial y}\frac{\partial\phi}{\partial t}\hat{j} \therefore$

$\nabla^2\underline{u} = \nabla^2(\nabla\phi) = \nabla\cdot\nabla(\nabla\phi)$ $\quad g = -g\hat{j}$, $\mu = 0 \therefore$

inviscid $\therefore \rho\left(\frac{\partial u}{\partial t} + \underline{u}\cdot\nabla u\right) = -\nabla P + \rho g \therefore$

~~the~~ $\hat{j}$ component of NS:

~~$\left(\frac{\partial u}{\partial t}\right)_{\hat{j}}$~~ $\frac{\partial u}{\partial t} = \nabla\frac{\partial\phi}{\partial t} = \frac{\partial}{\partial x}\frac{\partial\phi}{\partial t}\hat{i} + \frac{\partial}{\partial y}\frac{\partial\phi}{\partial t}\hat{j} + \frac{\partial}{\partial z}\frac{\partial\phi}{\partial t}\hat{k}$,

$\underline{u} = \nabla\phi = \frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j} + \frac{\partial\phi}{\partial z}\hat{z} \therefore$

$\underline{u}\cdot\nabla = \frac{\partial\phi}{\partial x}\frac{\partial}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial y} + \frac{\partial\phi}{\partial z}\frac{\partial}{\partial z} \therefore$

two dimensional $\therefore \underline{u}\cdot\nabla\underline{u} = \left(\frac{\partial\phi}{\partial x}\frac{\partial}{\partial x} + \frac{\partial\phi}{\partial y}\frac{\partial}{\partial y}\right)\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j}\right) =$

$\left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x^2} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y\partial x}\right)\hat{i} + \left(\frac{\partial\phi}{\partial x}\frac{\partial^2\phi}{\partial x\partial y} + \frac{\partial\phi}{\partial y}\frac{\partial^2\phi}{\partial y^2}\right)\hat{j}$

$\nabla\cdot\underline{u} = 0 = \nabla\cdot\left(\frac{\partial\phi}{\partial x}\hat{i} + \frac{\partial\phi}{\partial y}\hat{j}\right) = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = 0$

true

$\nabla p = \frac{\partial p}{\partial x}\hat{i} + \frac{\partial p}{\partial y}\hat{j}$,

$\nabla^2 \underline{u} = \nabla^2\left(\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j}\right) = \left(\frac{\partial}{\partial x^2}\frac{\partial \phi}{\partial x} + \frac{\partial^2}{\partial y^2}\frac{\partial \phi}{\partial x}\right)\hat{i} + \left(\frac{\partial}{\partial x^2}\frac{\partial \phi}{\partial y} + \frac{\partial}{\partial y^2}\frac{\partial \phi}{\partial y}\right)\hat{j}$

$\therefore$ $\hat{j}$ component of NS:

$\rho\frac{\partial}{\partial y}\frac{\partial \phi}{\partial t} + \rho\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial x \partial y} + \rho\frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial y^2} \mp \frac{\partial p}{\partial y} + \rho g = 0$

$\therefore$ integrating w.r.t y:

$\rho\frac{\partial \phi}{\partial t} + \rho\int \frac{\partial \phi}{\partial x}\frac{\partial^2}{\partial y \partial x}\phi \, dy + \rho\int\frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial y^2}dy + p + \rho g y = C_1(t)$

$\therefore \frac{\partial \phi}{\partial t} + \int\left(\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial y^2}\right)dy + \frac{p}{\rho} + g \cdot y = C_2(t)$

$\therefore \int\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial y \partial x} + \frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial y^2}dy = \frac{1}{2}\int 2\frac{\partial \phi}{\partial x}\frac{\partial^2 \phi}{\partial y \partial x}dy + \frac{1}{2}\int 2\frac{\partial \phi}{\partial y}\frac{\partial^2 \phi}{\partial y^2}dy =$

$\frac{1}{2}\int\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial x}\right)^2 dy + \frac{1}{2}\int\frac{\partial}{\partial y}\left(\frac{\partial \phi}{\partial y}\right)^2 dy = \frac{1}{2}\left[\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2\right] = \frac{1}{2}\left(\sqrt{\left(\frac{\partial \phi}{\partial x}\right)^2 + \left(\frac{\partial \phi}{\partial y}\right)^2}\right)^2 =$

$\frac{1}{2}\left|\frac{\partial \phi}{\partial x}\hat{i} + \frac{\partial \phi}{\partial y}\hat{j}\right|^2 = \frac{1}{2}|\nabla\phi|^2 \therefore$

$\frac{\partial \phi}{\partial t} + \frac{1}{2}|\nabla\phi|^2 + \frac{p}{\rho} + gy = C_2(t) = C(t)$

$C(t)$ is independent of x $\because$ there are no special points in x

$\therefore$ IC and BC independent of x

\4.biv✓ on free surface $y = \eta(x,t)$

$\frac{\partial \phi}{\partial t} + g\eta = 0$, $\frac{\partial \phi}{\partial y} = \frac{\partial \eta}{\partial t}$ on $y = 0$ is free surface

group speed is C_g , phase speed is C_p

$\therefore C_g = \frac{d\omega}{dk}$ $\therefore C_p = \frac{\omega}{k}$ $\therefore$

$g\eta = -\frac{\partial \phi}{\partial t}$ $\therefore$ $\eta = -\frac{1}{g}\frac{\partial \phi}{\partial t}$ $\therefore$ $\frac{\partial \eta}{\partial t} = -\frac{1}{g}\frac{\partial^2 \phi}{\partial t^2}$ $\therefore$

$\frac{\partial \phi}{\partial y} = -\frac{1}{g}\frac{\partial^2 \phi}{\partial t^2}$ $\therefore$ $\frac{\partial^2 \phi}{\partial t^2} = -g\frac{\partial \phi}{\partial y}$ $\therefore$ $\frac{\partial^2 \phi}{\partial t^2} \mp g\frac{\partial \phi}{\partial y} = 0$

$\therefore \frac{\partial^2 \phi}{\partial t \partial y} = g\frac{\partial \eta}{\partial y}$, $\frac{\partial^2 \phi}{\partial t \partial y} = \frac{\partial^2 \eta}{\partial t^2}$

$\phi = Ae^{ky}\sin(kx - \omega t)$ $\therefore$

$\frac{\partial \phi}{\partial y} = Ake^{ky}\sin(kx - \omega t)$, $\frac{\partial \phi}{\partial t} = Ae^{ky}(-\omega)\cos(kx - \omega t)$ $\therefore$

$\frac{\partial^2 \phi}{\partial t^2} = -Ae^{ky}(-\omega)^2\sin(kx - \omega t) = -Ae^{ky}\omega^2\sin(kx - \omega t)$ $\therefore$

$-Ake^{ky}\sin(kx - \omega t)g = -Ae^{ky}\omega^2\sin(kx - \omega t)$ $\therefore$ $kg = \omega^2$ $\therefore$ $\omega = \pm\sqrt{kg}$

\pr2020/ is dispersion relation ∴

$\omega = \pm\sqrt{k}\, g^{1/2} = \pm\sqrt{g}\, k^{1/2} = \omega(k)$ ∴

∴ ~~sp~~ group speed $C_g = \frac{d\omega}{dk} = \frac{d}{dk}(\pm\sqrt{g}\,k^{1/2}) = \pm\sqrt{g}\,(\frac{1}{2})k^{-1/2} =$

$\pm\sqrt{g}\,\frac{1}{2}\,\frac{1}{\sqrt{k}} = \pm\frac{1}{2}\sqrt{\frac{g}{k}}$

~~sp~~hase speed $C_p = \frac{\omega}{k} = \pm\sqrt{g}\,k^{1/2}/k = \pm\sqrt{g}\,k^{-1/2} = \pm\sqrt{\frac{g}{k}}$

\4v/ width $= 3000 \times 10^3$ m $= 3000000$ m

depth $= 3\times10^3 = 3000$ m

∴ ~~p~~hase speed $= C_p = \pm\sqrt{\frac{g}{k}}$ ∴

~~as~~ if waves have different wavelength λ then they have different wave number k ∴ phase speed are different for different wavelengths ∴ waves are dispersive ∴

∴ very long waves means very small wave number

∴ phase speed $= C_p = \pm\sqrt{\frac{g}{k}}$, group speed $C_g = \pm\frac{1}{2}\sqrt{\frac{g}{k}}$ ∴

both phase and group speed are very big for these waves which is why they arrive at the coast first then waves with small speeds

∴ $T = 30$ s ∴ frequence $= \frac{1}{30}$ per second $= k$ ∴

$\lambda = \frac{2\pi}{k} = \frac{2\pi}{(1/30)} = 60\pi = 188.5$ is wavelength

phase speed $= C_p = \frac{\omega}{k} = \pm\sqrt{\frac{g}{k}}$ ∴

$T\,c = \lambda = 30\times(\pm\sqrt{\frac{g}{k}}) = 30\times\sqrt{10}\,\frac{1}{\sqrt{k}}$ ∴ $T = 30$ s ∴ $\frac{1}{T} = \frac{1}{30} = k$ ∴

$30\times\sqrt{10}\times\frac{1}{\sqrt{1/30}} = \lambda = 519.6$ m ∴

∴ 519.6 m in 30 seconds ∴ 17.32 ms⁻¹ ∴

Middle to coast $= \frac{1}{2}(3000\times10^3) = 1500000$ m ∴

$1500000/17.32 = 86602.5$ seconds $= 1443.4$ minutes $= 24.1$ hours from middle of Atlantic to shores.

\2cii/ $\hat{r}\cos\theta - \hat{\theta}\sin\theta =$

$\sin\theta\cos\theta\cos\phi\,\hat{i} + \sin\theta\cos\theta\sin\phi\,\hat{j} + \cos^2\theta\,\hat{k} - \sin\theta\cos\theta\cos\phi\,\hat{i} - \sin\theta\cos\theta\sin\phi\,\hat{j} + \sin^2\theta\,\hat{k} =$

$\cos^2\theta\,\hat{k} + \sin^2\theta\,\hat{k} = \hat{k}$ ∴

for large r: $\underline{u} \approx U\underline{k} = U(\cos\theta\,\hat{\underline{r}} - \sin\theta\,\hat{\underline{\theta}}) = U\cos\theta\,\hat{r} - U\sin\theta\,\hat{\underline{\theta}} =$

$\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\hat{r} - \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\hat{\underline{\theta}}$ ∴

$\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = U\cos\theta \qquad -\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = -U\sin\theta$ ∴ $\frac{\partial\psi}{\partial\theta} = Ur^2\sin\theta\cos\theta$ ∴

$\frac{\partial\psi}{\partial r} = Ur\sin^2\theta$ ∴ $\psi = \frac{1}{2}Ur^2\sin^2\theta + f(\theta)$ ∴

$\frac{\partial\psi}{\partial\theta} = \frac{1}{2}Ur^2\,2\sin\theta\cos\theta + f'(\theta) = Ur^2\sin\theta\cos\theta + f'(\theta) = Ur^2\sin\theta\cos\theta$

∴ $f'(\theta) = 0$ ∴ $f(\theta)$ is constant ∴ $f(\theta)$ is independent of θ

∴ $f(\theta) = 0$ ∴

$\psi = \frac{1}{2}Ur^2\sin^2\theta$ as $r \to \infty$

$E^2\psi = E^2(r^\alpha\sin^2\theta) = \frac{\partial^2}{\partial r^2}(r^\alpha\sin^2\theta) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(r^\alpha\sin^2\theta)\right) =$

$\sin^2\theta\frac{\partial}{\partial r}(\alpha r^{\alpha-1}) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}r^\alpha 2\sin\theta\cos\theta\right) =$

$\sin^2(\theta)\alpha(\alpha-1)r^{\alpha-2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}(\cos\theta)r^\alpha 2 =$

$\sin^2(\theta)\alpha(\alpha-1)r^{\alpha-2} + \sin\theta(-\sin\theta)2r^{\alpha-2} =$

$\sin^2(\theta)\alpha(\alpha-1)r^{\alpha-2} - \sin^2(\theta)2r^{\alpha-2} = \sin^2(\theta)r^{\alpha-2}[\alpha(\alpha-1)-2]$ ∴

$E^4\psi = E^2(E^2\psi) = E^2(\sin^2(\theta)r^{\alpha-2}[\alpha(\alpha-1)-2])$

\NS to vorticity equation:/ NS: $\rho(\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}) = -\nabla p + \rho\underline{g} + \mu\nabla^2\underline{u}$,

$\frac{\mu}{\rho} = \nu$, $\nabla\times\underline{u} = \underline{\omega}$, $\nabla\cdot\underline{u} = 0$ ∴ $g = -\nabla\Pi$ ∴

$\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p - \nabla\Pi + \nu\nabla^2\underline{u}$ ∴ $\qquad \underline{u}\cdot\nabla\underline{u} = \nabla\frac{\underline{u}^2}{2} - \underline{u}\times\underline{\omega}$ ∴

$\nabla\times(\frac{\partial\underline{u}}{\partial t}) = \frac{\partial}{\partial t}\nabla\times\underline{u} = \frac{\partial\underline{\omega}}{\partial t}$ $\qquad \nabla\times(\underline{u}\cdot\nabla\underline{u}) = \nabla\times\nabla\frac{\underline{u}^2}{2} - \nabla(\underline{u}\times\underline{\omega}) \equiv -\nabla(\underline{u}\times\underline{\omega})$

$= -\underline{u}(\nabla\cdot\underline{\omega}) + \underline{\omega}(\nabla\cdot\underline{u}) + (\underline{\omega}\cdot\nabla)\underline{u} + (\underline{u}\cdot\nabla)\underline{\omega} \equiv$

$-\nabla\times\nabla\Pi \equiv 0$ $\qquad \underline{\omega}(\nabla\cdot\underline{u}) - (\underline{\omega}\cdot\nabla)\underline{u} + (\underline{u}\cdot\nabla)\underline{\omega} = -(\underline{\omega}\cdot\nabla)\underline{u} + (\underline{u}\cdot\nabla)\underline{\omega}$ ∴

$-\frac{1}{\rho}\nabla\times\nabla p \equiv 0$ $\qquad \frac{\partial\underline{\omega}}{\partial t} - \nabla(\underline{u}\times\underline{\omega}) = \nu\nabla^2\underline{\omega} = \frac{\partial\underline{\omega}}{\partial t} - (\underline{\omega}\cdot\nabla)\underline{u} + (\underline{u}\cdot\nabla)\underline{\omega}$ ∴

$\frac{\partial\underline{\omega}}{\partial t} = \nabla(\underline{u}\times\underline{\omega}) + \nu\nabla^2\underline{\omega}$, $\quad \frac{\partial\underline{\omega}}{\partial t} + (\underline{u}\cdot\nabla\underline{\omega}) = (\underline{\omega}\cdot\nabla)\underline{u} + \nu\nabla^2\underline{\omega}$

∴ $\frac{\partial\underline{\omega}}{\partial t} + \nabla\times(\underline{u}\cdot\nabla\underline{u}) = \nu\nabla\times\nabla^2\underline{u}$ ∴

$\nabla^2\underline{u} = \nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{u}) = \nabla(\nabla\cdot\underline{u}) - \nabla\times\underline{\omega}$ ∴

$\nabla\times\nabla^2\underline{u} = \nabla\times\nabla(\nabla\cdot\underline{u}) - \nabla\times(\nabla\times\underline{\omega}) \equiv 0 - \nabla\times(\nabla\times\omega) = -\nabla\times(\nabla\times\underline{\omega}) =$

$-\nabla(\nabla\cdot\underline{\omega}) + \nabla^2\underline{\omega} = -\nabla(\nabla\cdot\nabla\times\underline{u}) + \nabla^2\underline{\omega} \equiv -\nabla(0) + \nabla^2\underline{\omega} = \nabla^2\underline{\omega}$

∴ $\frac{\partial\underline{\omega}}{\partial t} + \nabla\times(\underline{u}\cdot\nabla\underline{u}) = \nu\nabla^2\underline{\omega}$

∴ $\underline{u}\times(\nabla\times\underline{u}) + \underline{u}\times(\nabla\times\underline{u}) = \nabla(\underline{u}\cdot\underline{u}) - (\underline{u}\cdot\nabla)\underline{u} - (\underline{u}\cdot\nabla)\underline{u} =$

$2\underline{u}\times(\nabla\times\underline{u}) = \nabla u^2 - 2\underline{u}\cdot\nabla\underline{u}$ ∴

$\underline{u}\cdot\nabla\underline{u} = \frac{1}{2}\nabla u^2 - \underline{u}\times(\nabla\times\underline{u}) = \nabla\frac{u^2}{2} - \underline{u}\times(\nabla\times\underline{u}) = \nabla\frac{\underline{u}^2}{2} - \underline{u}\times\underline{\omega}$

\PP2020/ \1ai/ NS: $\rho\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla p + \rho \underline{g} + \mu\nabla^2\underline{u}$ ∴

∴ $\underline{g} = -\nabla \Pi = -\nabla\Pi$ ∴

$\left(\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u}\right) = -\nabla \frac{p}{\rho} - \nabla\Pi + \nu\nabla^2\underline{u}$, $\nu = \frac{\mu}{\rho}$ ∴

\1aii/ equation $\frac{D\rho}{Dt} + \rho\nabla\cdot\underline{u} = \frac{\partial\rho}{\partial t} + \underline{u}\cdot\nabla\underline{u} + \rho\nabla\cdot\underline{u} = 0$ ∴

for incompressable: $\frac{D\rho}{Dt} = 0$ ∴

$0 + \rho\nabla\cdot\underline{u} = 0 = \rho\nabla\cdot\underline{u} = 0 = \nabla\cdot\underline{u}$

\1bi/ incompressible is $\nabla\cdot\underline{u} = 0$ ∴

$\nabla\cdot\underline{u} = \nabla\cdot\left(\frac{1}{R}\hat{\underline{R}} + R\hat{\underline{\phi}} + 0\hat{\underline{z}}\right) = \frac{1}{R}\frac{\partial}{\partial R}\left(R\frac{1}{R}\right) + \frac{1}{R}\frac{\partial}{\partial\phi}(R) + \frac{\partial}{\partial z}(0) =$

$\frac{1}{R}\left(\frac{\partial}{\partial R}(1)\right) + \frac{1}{R}(0) + 0 = \frac{1}{R}(0) + 0 = 0$

\1bii/ irrotational is $\underline{\omega} = \nabla\times\underline{u} = 0$ ∴

$\underline{\omega} = \nabla\times\underline{u} = \nabla\times\left(\frac{1}{R}\hat{\underline{R}} + R\hat{\underline{\phi}} + 0\hat{\underline{z}}\right) = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ \frac{1}{R} & RR & 0 \end{vmatrix} = \frac{1}{R}\begin{vmatrix} \hat{\underline{R}} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ \frac{1}{R} & R^2 & 0 \end{vmatrix} =$

$\frac{1}{R}\left[\hat{\underline{R}}(0-0) - R\hat{\underline{\phi}}(0-0) + \hat{\underline{z}}\left(\frac{\partial}{\partial R}(R^2) - 0\right)\right] = \frac{1}{R}\hat{\underline{z}}\, 2R = 2\hat{\underline{z}} \neq 0$

∴ flow is not irrotational

\1biii/ $\underline{u}\cdot\nabla = \left(\frac{1}{R}\hat{\underline{R}} + R\hat{\underline{\phi}} + 0\hat{\underline{z}}\right)\cdot\left(\hat{\underline{R}}\frac{\partial}{\partial R} + \hat{\underline{\phi}}\frac{1}{R}\frac{\partial}{\partial\phi} + \hat{\underline{z}}\frac{\partial}{\partial z}\right) =$

$\frac{1}{R}\frac{\partial}{\partial R} + \frac{\partial}{\partial\phi} + 0 = \frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R}\frac{\partial}{\partial\phi}$ ∴

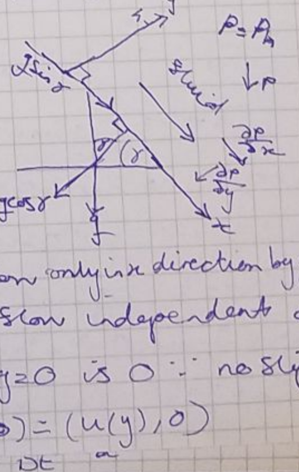
$\underline{u}\cdot\nabla\underline{u} = (\underline{u}\cdot\nabla)\underline{u} = \left(\frac{1}{R}\frac{\partial}{\partial R} + \frac{1}{R}\frac{\partial}{\partial\phi}\right)\left(\frac{1}{R}\hat{\underline{R}} + R\hat{\underline{\phi}}\right) =$

$\frac{1}{R}\frac{\partial}{\partial R}\left(\frac{1}{R}\hat{\underline{R}}\right) + \frac{1}{R}\frac{\partial}{\partial R}\left(R\hat{\underline{\phi}}\right) + \frac{1}{R}\frac{\partial}{\partial\phi}\left(\frac{1}{R}\hat{\underline{R}}\right) + \frac{1}{R}\frac{\partial}{\partial\phi}\left(R\hat{\underline{\phi}}\right) =$

$\frac{1}{R}\frac{\partial}{\partial R}(R^{-1})\hat{\underline{R}} + \frac{1}{R}\frac{\partial}{\partial R}(R)\hat{\underline{\phi}} + \frac{1}{R}\frac{\partial}{\partial\phi}(R^{-1})\hat{\underline{R}} + \frac{1}{R}\frac{\partial}{\partial\phi}(\hat{\underline{R}})\frac{1}{R} + \frac{1}{R}\frac{\partial}{\partial\phi}(\hat{\underline{\phi}})R =$

$\frac{1}{R}(-1)\frac{1}{R^2}\hat{\underline{R}} + \frac{1}{R}\hat{\underline{\phi}} + 0 + \frac{1}{R^2}\hat{\underline{\phi}} + (-\hat{\underline{R}}) =$

$\left(-\frac{1}{R^3} - 1\right)\hat{\underline{R}} + \left(\frac{1}{R^2} + \frac{1}{R}\right)\hat{\underline{\phi}}$



\1ci/ $y = 0$ is a solid stationary boundary

∴ the flow being incompressible has no $y\,\hat{\underline{y}}$ component ∴ $\underline{u} = (u, 0)$ ∵ flow only in x direction by gravity

∴ there are no special points in x ∴ flow independent of x.

there are special points in y ∵ flow at $y=0$ is 0 ∵ no slip BC

∴ flow dependent of y ∴ $\underline{u} = (u, 0) = (u(y), 0)$

Dt ∴ Dt

\1c ii/ NS: $\frac{\partial \underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\nabla\frac{p}{\rho} + \underline{g} + \nu\nabla^2\underline{u}$, $\nu = \frac{\mu}{\rho}$ ∴

$\nabla\cdot \underline{u} = 0$ ∴ $\frac{\partial \underline{u}}{\partial t} = \frac{\partial}{\partial t}(u(y),0)) = (0,0)$

∴ $\nu\nabla^2 \underline{u} = \left(\frac{\partial^2}{\partial x^2}+\frac{\partial^2}{\partial y^2}\right)(u(y),0) = \left(\frac{\partial^2 u(y)}{\partial y^2}, 0\right)$

$-\nabla\frac{p}{\rho} = \left(-\frac{1}{\rho}\frac{\partial p}{\partial x}, -\frac{1}{\rho}\frac{\partial p}{\partial y}\right)$

$\underline{g} = (g\sin\gamma, -g\cos\gamma)$

$\underline{u}\cdot\nabla = (u(y),0)\cdot\left(\frac{\partial}{\partial x},\frac{\partial}{\partial y}\right) = \left(u(y)\frac{\partial}{\partial x} + 0\right) = u(y)\frac{\partial}{\partial x}$ ∴

$\underline{u}\cdot\nabla\underline{u} = u(y)\frac{\partial}{\partial x}(u(y),0) = \left(u(y)\frac{\partial}{\partial x}u(y), 0\right) = (0,0)$ ∴

y component: $0+0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g\cos\gamma + 0 = 0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g\cos\gamma$

x component: $0+0 = -\frac{1}{\rho}\frac{\partial p}{\partial x} + g\sin\gamma + \nu\frac{\partial^2 u(y)}{\partial y^2} = 0$

no special points in x

\1c iii/ y component: $0 = -\frac{1}{\rho}\frac{\partial p}{\partial y} - g\cos\gamma$ ∴

$\frac{1}{\rho}\frac{\partial p}{\partial y} = -g\cos\gamma$ ∴ $\frac{\partial p}{\partial y} = -\rho g\cos\gamma$ ∴

$p = \int -\rho g\cos\gamma\, dy = -\rho g\cos(\gamma)y + C$ ∴

∴ at $y=h$: $p_0 = -\rho g\cos(\gamma)h + c$ ∴ $p_0 + \rho g h\cos\gamma = c$ ∴

$p = -\rho g\cos(\gamma)y + p_0 + \rho g h\cos\gamma = \rho g\cos(\gamma)(h-y) + p_0 = p_0 + (h-y)\rho g\cos\gamma$

\1c iv/ $p = p_0 + (h-y)\rho g\cos\gamma$ ∴

$\frac{\partial p}{\partial x} = 0$ ∴ $-\frac{1}{\rho}(0) + g\sin\gamma + \nu\frac{\partial^2 u(y)}{\partial y^2} = 0 = g\sin\gamma + \nu\frac{\partial^2 u(y)}{\partial y^2}$ ∴

$\frac{\partial^2 u(y)}{\partial y^2} = -\frac{g}{\nu}\sin\gamma$ ∴ $\frac{\partial u(y)}{\partial y} = -\frac{g}{\nu}\sin(\gamma)y + C_2$ ∴

at $h = y$: $\frac{\partial u(y=h)}{\partial y} = -\frac{g}{\nu}\sin(\gamma)h + C_2 = 0$ ∴ $C_2 = \frac{g}{\nu}h\sin\gamma$ ∴

$\frac{\partial u(y)}{\partial y} = -\frac{g}{\nu}\sin(\gamma)y + \frac{g}{\nu}h\sin\gamma = (h-y)\frac{g}{\nu}\sin\gamma$ ∴

$u(y) = (hy - \frac{1}{2}y^2)\frac{g}{\nu}\sin\gamma + C_3$ ∴

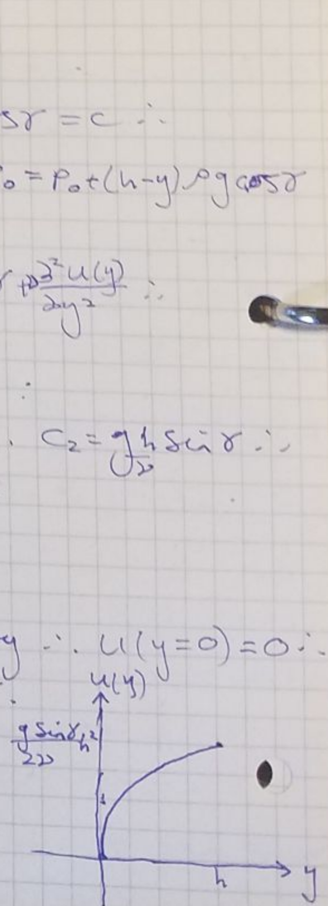
at $y=0$ is no slip stationary solid boundary ∴ $u(y=0)=0$ ∴

$u(0) = 0 = (h(0) - \frac{1}{2}(0))\frac{g}{\nu}\sin\gamma + C_3 = 0 + C_3 = C_3 = 0$ ∴

$u(y) = (hy - \frac{1}{2}y^2)\frac{g}{\nu}\sin\gamma = y(h-\frac{1}{2}y)\frac{g}{\nu}\sin\gamma$ ∴

$u(y) = -\frac{1}{2}\frac{g}{\nu}\sin(\gamma)y^2 + \frac{gh\sin\gamma}{\nu}y$

$y\geq 0$ at $y=h$: $u(h) = h(h-\frac{1}{2}h)\frac{g\sin\gamma}{\nu} = \frac{g\sin\gamma}{2\nu}h^2$ ∴



\pp2020/ \1ci/ $Q = \int_0^h u\,dy = \int_0^h -\frac{1}{2}\frac{g}{\nu}\sin(\gamma)y^2 + \frac{gh\sin\gamma}{\nu}y\,dy$

$= \left[-\frac{1}{6}\frac{g}{\nu}\sin(\gamma)y^3 + \frac{gh\sin\gamma}{2\nu}y^2\right]_0^h = -\frac{1}{6}\frac{g}{\nu}\sin(\gamma)h^3 - \frac{gh\sin\gamma}{2\nu}h^2 =$

$-\frac{1}{6}\frac{g\sin(\gamma)h^3}{\nu} + \frac{1}{2}\frac{g\sin(\gamma)h^3}{\nu} = \frac{gh^3\sin\gamma}{3\nu}$

\1cii/ mass flux $= M = \rho\int_S \underline{u}\cdot d\underline{S} = \rho\int_S \underline{u}\cdot\hat{\underline{n}}\,dS$

$\therefore\ \hat{\underline{n}} = \hat{\underline{i}}\ \therefore\ \underline{u}\cdot\hat{\underline{n}} = \underline{u}\cdot\hat{\underline{i}} = \underline{u}\cdot(1,0) = (u(y),0)\cdot(1,0) = u(y)\ \therefore$

$M = \rho\int_0^h u(y)\,dy = \frac{\rho g h^3\sin\gamma}{3\nu}\ \therefore$

$\cancel{M_x = \rho\int_0^H u(y)|_x\,dy} = \quad \cancel{M_x = \rho\int_0^H}\quad M_x = \rho\int_S \underline{u}|_x\cdot d\underline{S} = \rho\int_0^H \underline{u}|_x\cdot\hat{\underline{n}}\,dS$

$= \rho\int_0^H u(y)|_x\,dy = \frac{\rho g H^3\sin x}{3\nu} = \frac{\rho g(2h)^3\sin x}{3\nu} = \frac{8\rho g h^3\sin x}{3\nu}$

$= M = \frac{\rho g h^3\sin\gamma}{3\nu}\ \therefore\quad 8\sin x = \sin\gamma\ \therefore$

$\therefore\ \sin x = \frac{1}{8}\sin\gamma\ \therefore\ x = \sin^{-1}\left(\frac{1}{8}\sin\gamma\right)\ \therefore$

$x, \gamma > 0,\quad x, \gamma << 1\ \therefore$

$\sin$ is increasing for very small angles $\therefore\ \sin x = \frac{1}{8}\sin\gamma\ \therefore$

$\therefore\ x$ is smaller than γ which agrees with my expectations since a steeper angle would require a higher ~~h for the~~ $y=h$ value for the fluid not to notice there is a slope

\1di/ NS: $\frac{\partial\underline{u}}{\partial t} + \underline{u}\cdot\nabla\underline{u} = -\frac{1}{\rho}\nabla p - \nabla\Pi + \nu\nabla^2\underline{u}\ \therefore$

curls: $\nabla\times\frac{\partial\underline{u}}{\partial t} = \frac{\partial}{\partial t}\nabla\times\underline{u} = \frac{\partial\underline{\omega}}{\partial t}\ \therefore\quad \nabla\cdot\underline{u} = 0\ \therefore$

$\nabla\times\nabla^2\underline{u} = \nabla\times(\nabla(\nabla\cdot\underline{u})) - \nabla\times\left[\nabla\times(\nabla\times\underline{u})\right] - \nabla\times(\nabla\times(\nabla\times\underline{u})) =$

$-\nabla\times(\nabla\times\underline{\omega}) = -\nabla(\nabla\cdot\underline{\omega}) + \nabla^2\underline{\omega} = -\nabla(\nabla\cdot\nabla\times\underline{u}) + \nabla^2\underline{\omega} \equiv \nabla^2\underline{\omega}$

$\nabla\times\nabla p \equiv 0,\quad \nabla\times\nabla\Pi \equiv 0\ \therefore$

$2\underline{u}\times(\nabla\times\underline{u}) + \cancel{\underline{u}\times(\nabla\times\underline{u})} = \nabla(\underline{u}\cdot\underline{u}) - 2(\underline{u}\cdot\nabla)\underline{u}\ \therefore$

$\underline{u}\cdot\nabla\underline{u} = \nabla\left(\frac{u^2}{2}\right) - \underline{u}\times(\nabla\times\underline{u}) = \nabla\left(\frac{u^2}{2}\right) - \underline{u}\times(\underline{\omega})$

$\nabla\times(\underline{u}\cdot\nabla\underline{u}) = \nabla\times\nabla\left(\frac{u^2}{2}\right) - \nabla\times(\underline{u}\times\underline{\omega}) \equiv -\nabla(\underline{u}\times\underline{\omega})\ \therefore$

$-\nabla(\underline{u}\times\underline{\omega}) = -u(\nabla\cdot\underline{\omega})^{\to 0} + \underline{\omega}(\nabla\cdot\underline{u})^{\to 0} - (\underline{\omega}\cdot\nabla)\underline{u} + (\underline{u}\cdot\nabla)\underline{\omega} \equiv$

$-(\underline{\omega}\cdot\nabla)\underline{u} + (\underline{u}\cdot\nabla)\underline{\omega}\ \therefore$

$\frac{\partial\underline{\omega}}{\partial t} - (\underline{\omega}\cdot\nabla)\underline{u} + (\underline{u}\cdot\nabla)\underline{\omega} = \nu\nabla^2\underline{\omega}\ \therefore$

$\frac{\partial\underline{\omega}}{\partial t} + (\underline{u}\cdot\nabla)\underline{\omega} = (\underline{\omega}\cdot\nabla)\underline{u} + \nu\nabla^2\underline{\omega} = \frac{D\underline{\omega}}{Dt}\ \therefore\ \frac{D}{Dt} = \frac{\partial}{\partial t} + \underline{u}\cdot\nabla$

\1d ii/ A) $\underline{u} = aRz\hat{\underline{\phi}}$ ∴

$\underline{\omega} = \nabla \times \underline{u} = \frac{1}{R}\left|\cancel{\begin{matrix}\hat{R} \\ \partial_R\end{matrix}}\right.$ $\nabla \times (0\hat{\underline{R}} + aRz\hat{\underline{\phi}} + 0\hat{\underline{z}}) = \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & aR^2z & 0 \end{vmatrix} =$

$\frac{1}{R}\left[\hat{\underline{R}}(0 - aR^2) - R\hat{\underline{\phi}}(0-0) + \hat{\underline{z}}(2aRz - 0)\right] =$

$\frac{1}{R}\left[-aR^2\hat{\underline{R}} + 2aRz\hat{\underline{z}}\right] = -aR\hat{\underline{R}} + 2az\hat{\underline{z}}$ ∴

$\nabla \cdot \underline{\omega} = \nabla \cdot (-aR\hat{\underline{R}} + 2az\hat{\underline{z}}) = \frac{1}{R}\frac{\partial}{\partial R}(-aR^2) + \frac{\partial}{\partial z}(2az) =$

$\frac{1}{R}(-2aR) + 2a = -2a + 2a = 0$

~~\1\1ii~~ 1dii B) ~~v~~ vortex lines: $\frac{dr}{ds} = \omega(r,t)$ $c = \text{constant}$ ∴

$\underline{\omega} = -aR\hat{\underline{R}} + 2az\hat{\underline{z}}$ ∴

∴ let $z\,f(R) = c$ ∴ $\cancel{\frac{\partial}{\partial z}S}$ ∴ $f(R) = 0$,

$z\,f'(R) = c$ ∴

$\frac{dr}{dR} = -aR$, $\frac{dR}{\omega_R} = \frac{dz}{\omega_z}$ ∴ $\frac{dR}{-aR} = \frac{dz}{2az}$

∴ $\int \frac{1}{-aR}\,dR = \int \frac{1}{2az}\,dz = \cancel{2a}$

$-\frac{1}{a}\ln|R| = \cancel{2a}\ \frac{1}{2a}\ln|z| + C_1 = -\frac{1}{a}\ln R$ ∵ $R \geq 0$ ∴

∴ $-\frac{1}{2}\ln|z| + C_2 = \ln R = \ln|z^{-1/2}| + C_2$ ∴

$R = e^{\ln|z^{-1/2}| + C_2} = C_3|z^{-1/2}| = C_4 z^{-1/2}$ ∴

$z^{1/2}R = C_5$ ∴ $zR^2 = z\,f(R) = C$ ∴

$f(R) = R^2$

\1dii C/ $\underline{u} = (0, aRz, 0) = \nabla \times (\psi\hat{\underline{z}}) = \cancel{\frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & R\psi & 0 \end{vmatrix}} =$

$\cancel{\frac{1}{R}\left|\hat{\underline{R}}\left(0 - R\frac{\partial\psi}{\partial z}\right) - R\hat{\underline{\phi}}(0-0) + \hat{\underline{z}}(\right.}$ $= \frac{1}{R}\begin{vmatrix} \hat{R} & R\hat{\underline{\phi}} & \hat{\underline{z}} \\ \partial_R & \partial_\phi & \partial_z \\ 0 & 0 & \psi \end{vmatrix} =$

$\frac{1}{R}\left[\hat{\underline{R}}\left(\frac{\partial\psi}{\partial\phi}\right) - R\hat{\underline{\phi}}\left(\frac{\partial\psi}{\partial R}\right) - 0\hat{\underline{z}}\right] = \left(\frac{1}{R}\frac{\partial\psi}{\partial\phi}, -\frac{\partial\psi}{\partial R}, 0\right)$ ∴

$\frac{1}{R}\frac{\partial\psi}{\partial\phi} = 0$ ∴ $\frac{\partial\psi}{\partial\phi} = 0$, $aRz = -\frac{\partial\psi}{\partial R}$ ∴ $\frac{\partial\psi}{\partial R} = -aRz$ ∴

$\psi = -\frac{1}{2}aR^2z + f(\phi)$ ∴

$\frac{\partial\psi}{\partial\phi} = \frac{\partial}{\partial\phi}\left(-\frac{1}{2}aR^2z + f(\phi)\right) = -\frac{1}{2}aR^2z + f'(\phi) = 0$ ∴ $f'(\phi) = 0$

∴ $f(\phi) = 0$ ∴ $\cancel{\frac{\partial}{\partial z}}$ $\psi = -\frac{1}{2}aR^2z = \psi(R,z)$

\PP2020/ \1diiD/ $\Gamma = \int_C \underline{u} \cdot d\underline{L} = \int_S (\nabla \times \underline{u}) \cdot \hat{\underline{n}}\, dS = \int_S \underline{\omega} \cdot \hat{\underline{n}}\, dS$

$= \int_C aRz\hat{\underline{\phi}} \cdot \hat{\underline{n}}\, dL = \int_0^R aRz\hat{\underline{\phi}} \cdot \hat{\underline{\phi}}\, dL = \int_0^R aRz\, dR =$

$\left[\frac{a}{2}R^2 z\right]_0^R = \frac{a}{2}R^2 z$

\2a/ $T = \frac{L}{U}$, $\frac{\partial}{\partial t} \sim \frac{1}{T}$, $\nabla \sim \frac{1}{L}$, $\underline{u} \sim U$

$\therefore \frac{\partial}{\partial t} \sim \frac{1}{T} = \frac{U}{L}$ $\nabla^2 \sim \frac{1}{L^2} \therefore$

$\frac{\partial \underline{u}}{\partial t} \sim \frac{1}{T} U = \frac{U}{L} U = \frac{U^2}{L}$, $\underline{u} \cdot \nabla \underline{u} \sim U \frac{1}{L} U = \frac{U^2}{L} \sim \frac{\partial \underline{u}}{\partial t}$

$\nu \nabla^2 \underline{u} \sim \nu \frac{1}{L^2} U = \nu \frac{U}{L^2} \therefore$

$$\frac{\partial \underline{u}/\partial t}{\nu \nabla^2 \underline{u}} \sim \frac{\underline{u} \cdot \nabla \underline{u}}{\nu \nabla^2 \underline{u}} \sim \frac{U^2/L}{\nu U/L^2} = \frac{UL}{\nu} = R_e << 1$$

for small Reynolds number $\therefore$

neglect $\frac{\partial \underline{u}}{\partial t}$, $\underline{u} \cdot \nabla \underline{u}$ $\therefore$ neglect gravity $\therefore g = 0 \therefore$

$\therefore \rho(0) = -\nabla p + \mu \nabla^2 \underline{u} = 0$

\2bi/ $R_e = \frac{UL}{\nu}$. honey has large kinematic viscosity.

$\therefore$ very viscous

toothpaste has very small L $\therefore$ small distance

\2ci/ $\therefore \nabla \cdot \underline{u} = 0$ $\nabla \cdot \nabla \times \equiv 0 \therefore$

$\nabla \cdot \underline{u} = \nabla \cdot \nabla \times \left(\frac{\psi}{r\sin\theta}\hat{\underline{\phi}}\right) \equiv 0$ always shows $\nabla \cdot \underline{u} = 0$ incompressible

~~\2cii/~~ $\underline{u} = (u, v, 0) = \nabla \times \left(\frac{\psi}{r\sin\theta}\hat{\underline{\phi}}\right) = \frac{1}{r^2\sin\theta}\begin{vmatrix} \hat{\underline{r}} & r\hat{\underline{\theta}} & r\sin\theta\hat{\underline{\phi}} \\ \partial_r & \partial_\theta & \partial_\phi \\ 0 & 0 & \psi \end{vmatrix}$

$= \frac{1}{r^2\sin\theta}\left[\hat{\underline{r}}\left(\frac{\partial\psi}{\partial\theta}\right) - r\hat{\underline{\theta}}\left(\frac{\partial\psi}{\partial r}\right)\right] = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\hat{\underline{r}} - \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\hat{\underline{\theta}} \equiv u\hat{\underline{r}} + v\hat{\underline{\theta}}$

$\therefore u = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$, $-\frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = v \therefore$

as $r \to \infty$: $\underline{u} \approx U\hat{\underline{k}} = U(\cos\theta\hat{\underline{r}} - \sin\theta\hat{\underline{\theta}}) = U\cos\theta\,\hat{\underline{r}} - U\sin\theta\,\hat{\underline{\theta}}$

$= \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\hat{\underline{r}} - \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\hat{\underline{\theta}} \therefore$

$U\cos\theta = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}$, $+\frac{\partial\psi/\partial r}{r\sin\theta} = +U\sin\theta \therefore \frac{\partial\psi}{\partial\theta} = Ur^2\sin\theta\cos\theta \therefore$

$\frac{\partial\psi}{\partial r} = Ur\sin^2\theta \therefore \psi = \frac{1}{2}Ur^2\sin^2\theta + f(\theta) \therefore$

$\frac{\partial\psi}{\partial\theta} = Ur^2\sin\theta\cos\theta + f'(\theta) = Ur^2\sin\theta\cos\theta \therefore f'(\theta) = 0 \therefore f(\theta) = 0 \therefore$

$\psi = Ur\sin^2\theta$

\2iii/ let $f = r^\alpha$, $\therefore \psi = f(r)\sin^2\theta = r^\alpha \sin^2\theta \therefore$

$E^2\psi = E^2 r^\alpha \sin^2\theta = \frac{\partial}{\partial r^2}(r^\alpha \sin^2\theta) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(r^\alpha \sin^2\theta)\right) =$

$\sin^2\theta \frac{\partial}{\partial r}(\alpha r^{\alpha-1}) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta} r^\alpha 2\sin\theta\cos\theta\right) =$

$\alpha \sin^2(\theta)(\alpha-1)r^{\alpha-2} + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}(2\cos\theta)r^\alpha =$

$\alpha(\alpha-1)r^{\alpha-2}\sin^2\theta + -2r^{\alpha-2}\sin^2\theta =$

$[\alpha(\alpha-1)-2]r^{\alpha-2}\sin^2\theta \therefore$

$E^4\psi = E^2(E^2\psi) = E^2([\alpha(\alpha-1)-2]r^{\alpha-2}\sin^2\theta) =$

$\frac{\partial}{\partial r^2}([\alpha(\alpha-1)-2]r^{\alpha-2}\sin^2\theta) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}([\alpha(\alpha-1)-2]r^{\alpha-2}\sin^2\theta)\right) =$

$[\alpha(\alpha-1)-2]\sin^2(\theta)(\alpha-2)\frac{\partial}{\partial r}(r^{\alpha-3}) + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}\left(\frac{[\alpha(\alpha-1)-2]r^{\alpha-2}}{\sin\theta}2\sin\theta\cos\theta\right) =$

$[\alpha(\alpha-1)-2](\alpha-2)(\alpha-3)r^{\alpha-4}\sin^2\theta + \frac{\sin\theta}{r^2}\frac{\partial}{\partial\theta}(\cos\theta)2[\alpha(\alpha-1)-2]r^{\alpha-2}$

$= [\alpha(\alpha-1)-2]\left[(\alpha-2)(\alpha-3)r^{\alpha-4}\sin^2\theta - 2r^{\alpha-4}\sin^2\theta\right] =$

$[\alpha(\alpha-1)-2]r^{\alpha-4}\sin^2\theta\,[(\alpha-2)(\alpha-3)-2] = 0 \therefore$

$\alpha(\alpha-1)-2 = 0$ or $(\alpha-2)(\alpha-3)-2 = 0 \therefore$

$\alpha^2-\alpha-2 = 0$, $\alpha^2+6-5\alpha-2 = \alpha^2-5\alpha+4 = 0 \therefore$

$(\alpha-2)(\alpha+1) = 0$, $(\alpha-4)(\alpha-1) = 0 \therefore$

$\alpha = 2$, $\alpha = -1$, $\alpha = 4$, $\alpha = 1 \therefore$

$f(r) = \sum_\alpha r^\alpha = Ar^{-1} + Br^1 + Cr^2 + Dr^4 = A\frac{1}{r} + Br + Cr^2 + Dr^4 \therefore$

\2iv/ $\underline{u} = u\hat{r} + v\hat{\theta} \therefore$ $\underline{\omega} = \nabla\times\underline{u} = \nabla\times(u\hat{r} + v\hat{\theta}) =$

$\frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \partial r & \partial\theta & \partial\phi \\ u & rv & 0\end{vmatrix} = \frac{1}{r^2\sin\theta}\begin{vmatrix}\hat{r} & r\hat{\theta} & r\sin\theta\hat{\phi} \\ \partial r & \partial\theta & \partial\phi \\ \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} & -\frac{1}{\sin\theta}\frac{\partial\psi}{\partial r} & 0\end{vmatrix}$

$\cancel{\underline{u} = u\hat{r} + v\hat{\theta}} =$ $\underline{u} = \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}\hat{r} - \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r}\hat{\theta}$

$\psi = f(r)\sin^2\theta \therefore \psi_\theta = 2f\sin\theta\cos\theta$, $\psi_r = f'\sin^2\theta \therefore$

$\therefore \underline{u} = \frac{1}{r^2\sin\theta}2f\sin\theta\cos\theta\,\hat{r} - \frac{1}{r\sin\theta}f'\sin^2\theta\,\hat{\theta}$

$2\frac{1}{r^2}f\cos\theta\,\hat{r} - \frac{1}{r}f'\sin\theta\,\hat{\theta} \therefore$

$\underline{\omega} = \nabla\times\underline{u} = \frac{1}{r^2}$ $\therefore$ $\underline{\omega} = \left(-\frac{\sin\theta}{r}f'' + \frac{2\sin\theta}{r^3}f\right)\hat{\phi} \therefore$

$\oint_C \underline{u}\cdot d\underline{l} = \int_a^b \underline{u}\cdot\hat{n}\,dr = \int_a^b \underline{u}\cdot\hat{r}\,dr = \int_S \underline{\omega}\cdot d\underline{S} = \int_S \nabla\times\underline{u}\cdot d\underline{S} =$

$\int_a^b \frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta}dr = \int_a^b \frac{1}{r^2\sin\theta}2f\sin\theta\cos\theta\,dr = 2\cos\theta\left[\frac{1}{2}\left(\frac{1}{a^2}-\frac{1}{b^2}\right) + B\ln\left(\frac{b}{a}\right) + C(b-a) + \frac{D}{3}(b^3-a^3)\right]$