\Week 4 ~~\4d/~~ $\int_0 g(x)f(x)\,dx = \int_0 \frac{g(x)f(x)}{h(x)} h(x)\,dx = E_h\left[\frac{g(x)f(x)}{h(x)}\right]$

\4b redo/ $g(x)f(x) = x^a(1-x)^b e^{-cx} =$

$[(1-x)^b][x^{(a+1)-1}e^{-cx}] = \frac{\Gamma(a+1)}{c^{a+1}}(1-x)^b \frac{c^{a+1}}{\Gamma(a+1)} x^{(a+1)-1} e^{-cx}$ for $x>0$ ∴

$g(x) = \frac{\Gamma(a)}{c^a}(1-x)^b$, $f(x) = \frac{c^{a+1}}{\Gamma(a+1)} x^{(a+1)-1} e^{-cx} \sim \text{Gamma}(a+1, c)$

\4a redo/ $g(x)f(x) = x^a(1-x)^b e^{-cx} = [e^{-cx}][x^{(a+1)-1}(1-x)^{(b+1)-1}]$

$= \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} e^{-cx} \frac{\Gamma(a+1+b+1)}{\Gamma(a+1)\Gamma(b+1)} x^{(a+1)-1}(1-x)^{(b+1)-1}$

∴ $f(x) = \frac{\Gamma(a+1+b+1)}{\Gamma(a+1)\Gamma(b+1)} x^{(a+1)-1}(1-x)^{(b+1)-1} \sim \text{Beta}(a+1, b+1)$

$g(x) = \frac{\Gamma(a+1)\Gamma(b+1)}{\Gamma(a+b+2)} e^{-cx}$

\4d/ Clearly Z Beta version is much better here. Z reason is that g is usually 0, particularly with Z gamma formulation. For part d., we want to get most of our samples in [0, 0.5] in our importance sample. I will use a Beta(5, 20) distri for this as it's mass is in Z right area, but any choice that targets [0, 0.5] should be fine ∴

$h(x) = \frac{\Gamma(25)}{\Gamma(5)\Gamma(20)} x^4(1-x)^{19}$ & $\frac{g(x)f(x)}{h(x)} = \frac{\Gamma(5)\Gamma(20)\,\mathbb{1}(0<x<0.5)x^2}{\Gamma(25)(1-x)^{15}e^{x/2}}$

$\frac{1}{h(x)} = \frac{\Gamma(5)\Gamma(20)}{\Gamma(25)x^4(1-x)^{19}}$ & $a=6, b=4, c=0.5$ & ∴

$g(x)f(x) = x^6(1-x)^4 e^{-0.5x}$ ∴ $\frac{g(x)f(x)}{h(x)} = \frac{\Gamma(5)\Gamma(20)\,x^6(1-x)^4 e^{-\frac{1}{2}x}}{\Gamma(25)x^4(1-x)^{19}} =$

$\frac{\Gamma(5)\Gamma(20)\,x^2}{\Gamma(25)(1-x)^{15}e^{x/2}}$ but keepers $= \mathbb{1}(0<x<0.5)$ ∴

$f(x) = \frac{\Gamma(a+1+b+1)}{\Gamma(a+1)\Gamma(b+1)} x^{(a+1)-1}(1-x)^{(b+1)-1}\,\mathbb{1}(0<x<0.5)$ ∴

$g(x)f(x) = x^6(1-x)^4 e^{-0.5x}\,\mathbb{1}(0<x<0.5)$ ∴ $\frac{g(x)f(x)}{h(x)} = \frac{\Gamma(5)\Gamma(20)\,\mathbb{1}(0<x<0.5)x^2}{\Gamma(25)(1-x)^{15}e^{x/2}}$

∴ code it as function} so running Z sampler

∴ Z importance sampler of Beta(5, 20) clearly works best ∵ g ~~is~~ has Z least number of zeros using this method (Z importance sampler method)

$X \sim unif(a,b) \quad \therefore f(x) = \frac{1}{b-a}$

$X \sim N(\mu, \sigma^2) \therefore \quad f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$X \sim Gamma(\alpha, \beta) \quad f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$

$X \sim Beta(a,b) \quad f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$

$X \sim Exp(\lambda) \quad f(x;\lambda) = \lambda e^{-\lambda x} \qquad X \sim Exp(\lambda) \equiv Gamma(1,\lambda)$

$X \sim Poi(\lambda) \quad f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$

$X \sim unif(a,b) \quad f(x) = \frac{1}{b-a} \quad X \sim N(\mu,\sigma^2) \therefore f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$X \sim Gamma(\alpha,\beta): f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad X \sim Beta(a,b) \quad f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$

$X \sim Exp(\lambda) \quad f(x) = \lambda e^{-\lambda x} \quad X \sim Exp(\lambda) \equiv Gamma(1,\lambda) \quad X \sim Poi(\lambda) \quad f(x) = \frac{\lambda^x e^{-\lambda}}{x!}$

$f(x) = \frac{1}{b-a} \quad f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \quad f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$

$f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \quad f(x) = \lambda e^{-\lambda x} \quad f(x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad f(x) = \frac{1}{b-a}$

$f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \quad \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad f(x) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$

$f(x) = \lambda e^{-\lambda x} \quad f(x) = \frac{\lambda^x e^{-\lambda}}{x!} \quad unif \sim \frac{1}{b-a} \quad N \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$Gamma \sim \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad Beta \sim \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \quad Exp \sim \lambda e^{-\lambda x}$

$Poi \sim \frac{\lambda^x e^{-\lambda}}{x!} \quad unif \sim \frac{1}{b-a} \quad N \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \quad Gamma \sim \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$

$Beta \sim \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \quad Exp \sim \lambda e^{-\lambda x} \quad Poi \sim \frac{\lambda^x e^{-\lambda}}{x!} \quad unif \sim \frac{1}{b-a}$

$N \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \quad Gamma \sim \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad Beta \sim \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$

$Exp \sim \lambda e^{-\lambda x} \quad Poi \sim \frac{\lambda^x e^{-\lambda}}{x!} \quad unif \sim \frac{1}{b-a} \quad N \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$Gamma \sim \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad Beta \sim \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}\cancel{(1-x)}^{b-1} \quad Exp \sim \lambda e^{-\lambda x} \quad Poi \sim \frac{\lambda^x e^{-\lambda}}{x!}$

$unif \sim \frac{1}{b-a} \quad N \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \quad Gamma \sim \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad Beta \sim \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$

$Exp \sim \lambda e^{-\lambda x} \quad Poi \sim \frac{\lambda^x e^{-\lambda}}{x!} \quad unif \sim \frac{1}{b-a} \quad \cancel{A.R} \; N \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$Gamma \sim \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad Beta \sim \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \quad Exp \sim \lambda e^{-\lambda x} \quad Poi \sim \frac{\lambda^x e^{-\lambda}}{x!}$

$unif \sim \frac{1}{b-a} \quad N \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \quad Gamma \sim \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad Beta \sim \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$

$Exp \sim \lambda e^{-\lambda x} \quad Poi \sim \frac{\lambda^x e^{-\lambda}}{x!} \quad unif \sim \frac{1}{b-a} \quad N \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$

$Gamma \sim \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad Beta \sim \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \quad Exp \sim \lambda e^{-\lambda x} \quad Poi \sim \frac{\lambda^x e^{-\lambda}}{x!}$

$unif \sim \frac{1}{b-a} \quad N \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \quad Gamma \sim \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad Beta \sim \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1}$

$Exp \sim \lambda e^{-\lambda x} \quad Poi \sim \frac{\lambda^x e^{-\lambda}}{x!} \quad unif \sim \frac{1}{b-a} \quad N \sim \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} \quad Gamma \sim \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$

$Beta \sim \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \quad Exp \sim \lambda e^{-\lambda x} \quad Poi \sim \frac{\lambda^x e^{-\lambda}}{x!} \quad unif \sim \frac{1}{b-a} \quad N \sim \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2\sigma}(x-\mu)^2}$

$Gamma \sim \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x} \quad Beta \sim \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} x^{a-1}(1-x)^{b-1} \quad Exp \sim \lambda e^{-\lambda x} \quad Poi \sim \frac{\lambda^x e^{-\lambda}}{x!}$

\Week 5 Sheet / \1a/ if we can find Z CDF, $F(x)$, & its inverse exists, then set $X_* = F^{-1}(U)$ U is a sample from Unif $(0,1)$ to obtain X_* from Z distn with pdf $F(x) = \int_0^x 2t\,dt = x^2$ $\therefore$ setting $U = x^2$ $\therefore x = \sqrt{U}$ $\therefore F^{-1}(U) = \sqrt{U}$ $\{U \sim \text{unif}(0,1) \therefore F(x) = \int_0^x 2t\,dt = \left[\frac{1}{2}2t^2\right]_0^x = x^2 - 0^2 = x^2 = P(X \leq x) \therefore F(x) \sim \text{Unif}(0,1) \therefore x^2 = U \therefore \sqrt{U} = x \therefore x = F^{-1}(F(x)) \therefore F^{-1}(x^2) = x = F^{-1}(U) = x = \sqrt{U}\}$

$\{U \sim \text{unif}(0,1) \therefore F(x) = P(X \leq x) = \int_0^x 2t\,dt = x^2 \therefore F(x) \sim \text{Unif}(0,1) \therefore F(x) = x^2 = U \therefore x = \sqrt{U} \therefore F^{-1}(F(x)) = x = F^{-1}(U) = \sqrt{U}\}$

\2a/ Z envelope func has to be bigger than $f(x)\ \forall x$. $f(x)$ takes its max at $x=1$ when $f(x)=2$ $\therefore g_u(x) = 2f_u(x)$ is Z envelope with Z largest acceptance rate (any larger value of k would have more area under $g_u(x)$ that is not under $f(x)$). Z acceptance rate is c/k where, in this probl $c=1$ $\therefore$ Z acceptance rate is $1/2$

\2b/ Z rejection sampling alg uses Z first unif to propose an X_u from $f_u(x)$. another is then proposed, U, & accept X_u if $Ug_u(X_u) \leq cf(X_u)$ $\therefore$ 5 steps: $X_u = 0.467$, $U * g_u(X_u) = 0.748 \times 2 = 1.496 > f(X_u) = 0.934$, reject

$X_u = 0.228$, $U * g_u(X_u) = 0.164 \times 2 = 0.328 < f(X_u) = 0.456$, accept

\3a/ must first find Z cdf & $\therefore$ must integrate Z actual $f(x)$ $\therefore$ need to find Z normalising const. on $[0,3]$: $f(x) \propto x^3$ $\therefore$ $f(x) = cx^3$ $\therefore f(x) = \frac{1}{\int_0^3 x^3\,dx} x^3 = \frac{4}{3^4}x^3 = \frac{4}{81}x^3$ $\{\int_0^3 cx^3\,dx = 1 = c\frac{1}{4}[x^4]_0^3 = \frac{1}{4}c(3^4 - 0^4) = \frac{81}{4}c = 1 \therefore c = \frac{4}{81} \therefore f(x) = \frac{4}{81}x^3$ $\therefore$

$F(x) = \int_0^x \frac{4}{81}t^3\,dt = \left[\frac{t^4}{81}\right]_0^x = \left(\frac{x}{3}\right)^4$ & $F(x) = U = \left(\frac{x}{3}\right)^4$ $\therefore F^{-1}(U) = 3U^{1/4}$

$\{f(x) \propto x^3 \therefore f(x) = cx^3 \therefore \int_0^3 cx^3\,dx = 1 = \frac{81}{4}c \therefore c = \frac{4}{81} \therefore f(x) = \frac{4}{81}x^3 \therefore$

$F(x) = P(X \leq x) = \int_0^x \frac{4}{81}t^3\,dt = \left(\frac{x}{3}\right)^4 \therefore U \sim \text{unif}(0,1) \therefore F(x) \sim \text{Unif}(0,1) \therefore$

$F(x) = \left(\frac{x}{3}\right)^4 = U \therefore U^{1/4} = \frac{x}{3} \therefore 3U^{1/4} = x \therefore F^{-1}(F(x)) = x = F^{-1}(U) = 3U^{1/4}$

\3b/ $\therefore$ taking samples $\underline{x}$ $\therefore$ $3\underline{x}^{1/4}$ gives samples from $f(x)$

\3d/ $f_u(x)$ is Unif$(0, M)$ where M is $\max g(x) = cf(x)$. here it makes sense to set $g(x) = x^3$ $\therefore M = 27$ & $f_u(x) = \frac{1}{3}$ on $[0,3]$. we require $g_u(x) = kf_u(x) \geq 27$ $\therefore k = 81$. Z acceptance rate is then $\frac{c}{k} = \frac{81/4}{81} = \frac{1}{4}$

$\{ f(x) = \frac{4}{81}x^3 \therefore C_1 = \frac{4}{81}$ $\quad 0 \le x \le 3 \therefore f_u(x) \sim \text{Unif}(0, M)$

$M = \max g(x) = c f(x) = \frac{4}{81} f(x) = \cancel{\frac{4}{81}x^3}$ X

M is $\max g(x) = c f(x) \therefore$ ~~set~~ $f(x) = c_1 x^3$ X $\}$

$\{ f(x) = \frac{4}{81}x^3$, $f_u(x) \sim \text{Unif}(0, M)$ $\quad M$ is $\max g(x) = c f(x)$

$f(x) = \frac{4}{81}x^3$ & $\max g(x) = c f(x) \therefore$ set $g(x) = x^3 \therefore$

$0 \le x \le 3 \therefore \max_{x \in [0,3]} x^3 = 3^3 = 27 = M \therefore M = 27 \therefore f_u(x) \sim \text{Unif}(0, 27)$

$\therefore \cancel{\frac{1}{27-0} = f_u(x) =}$ $x \in [0, 3] \therefore f_u(x) = \frac{1}{3-0} = \frac{1}{3} \therefore$ require

$g_u(x) = k f_u(x) \ge M = 27 \therefore k \frac{1}{3} \ge 27 \therefore k \ge 81 \therefore k = 81$

$g(x) = x^3$, $f(x) = \frac{4}{81}x^3 \therefore g(x) = x^3 = c f(x) = c \frac{4}{81} x^3 = x^3 \therefore \frac{4c}{81} = 1 \therefore$

$c = \frac{81}{4} \therefore$ Z acceptance rate: $\frac{c}{k} = \frac{81/4}{81} = \frac{1}{4}$ $\}$

\3e/ $X_u = 3 \times 0.252 = 0.756$, $U g_u(X_u) = 27 \times 0.367 = 9.909 > g(X_u) =$

$0.756^3 = 0.432 \therefore$ reject

$X_u = 3 \times 0.399 = 1.197$, $U g_u(X_u) = 27 \times 0.060 = 1.620 < g(X_u) = 1.197^3 = 1.715 \therefore$ accept

\4a/ $y_i | \theta \sim N(\theta, \sigma^2) \therefore f(y_i | \theta) = \frac{1}{\sqrt{2\pi}\sigma} e^{\frac{-1}{2\sigma^2}(y_i - \theta)^2}$

$\text{Gamma}(a, b) \therefore f(\theta | a, b) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}$

\5a/ $\lambda e^{-\lambda x} = f(x)$ Let R be Z event it rains on any given day $\therefore$ by Z law of total prob: $P(X < x | \lambda) = P(X < x | R, \lambda) P(R) + P(X < x | \tilde{R}, \lambda) P(\tilde{R}) =$

$q \int_0^x \lambda e^{-\lambda t} dt + (1 - q)$ $\quad \{ P(X < | \tilde{R}, \lambda) = 1$, $P(R) = q \therefore P(\tilde{R}) = 1 - q$,

$P(X < | R, \lambda) = \int_0^x \lambda e^{-\lambda t} dt \}$ $\quad = q \left[-e^{-\lambda t} \right]_0^x + (1 - q) =$

$q(1 - e^{-\lambda x}) + (1 - q) = q - q e^{-\lambda x} + 1 - q = 1 - q e^{-\lambda x} \therefore$

$F(x) = P(X < x | \lambda) = \begin{cases} 0, & x < 0 \\ 1 - q e^{-\lambda x}, & x \ge 0 \end{cases}$

note, at $x = 0$: $F(x) = 1 - q \therefore \exists$ there is discontinuity in $F(x)$ st X is both discrete & continuous (it is continuous for $X > 0$ & there is prob $1 - q$ that $X = 0$)

\5b/ Z general inverse CDF method for sampling a value X_* from Z distri for X is to construct $F_*^{-1}(p) = \min_{F(x) \ge p}(x)$ then sample $U \sim \text{Unif}(0,1)$ & set $X_* = F_*^{-1}(U)$

\5c/ to construct $F_*^{-1}(p) = \cancel{\min_{F(x) \ge}}$ consider range $p \in [0, 1]$ sequentially $\therefore$ firstly: $F(x) = 0 \; \forall x < 0$. So $F_*^{-1}(0) = \min_{F(x) \ge 0}(x) = -\infty$. next:

\Week 5 Sheet/ $F(0)=1-q$, so for $p\in(0,1-q]$, $F_*^{-1}(p)=$ $\min_{F(x)\geq p}(x)=0$. $F(x)$ is continuous for $x\geq 0$ & gives values $p\in[1-q,1]$, so we can compute its inverse directly &

$F_*^{-1}(p)=F^{-1}(p)$ for $p\in[1-q,1]$. to compute $F^{-1}(x)$, for $x>0$, $p=1-qe^{-\lambda x}$

$\therefore x=\frac{1}{\lambda}\ln\left(\frac{q}{1-p}\right)$ $\Big\{ \because qe^{-\lambda x}=1-p \therefore e^{-\lambda x}=\frac{1-p}{q} \therefore -\lambda x=\ln\left(\frac{1-p}{q}\right)\therefore$

$x=\frac{1}{\lambda}\left(-\ln\left(\frac{1-p}{q}\right)\right)=\frac{1}{\lambda}\ln\left(\left(\frac{1-p}{q}\right)^{-1}\right)=\cancel{\frac{1}{\lambda}\left(\frac{1}{1-}\right)}\, x=\frac{1}{\lambda}\ln\left(\frac{q}{1-p}\right)\Big\}\therefore$

$$F_*^{-1}(p)=\begin{cases}-\infty & p=0\\ 0 & p\in(0,1-q)\\ \frac{1}{\lambda}\ln\left(\frac{q}{1-p}\right) & p\in[1-q,1]\end{cases}\quad\therefore$$

$F_*^{-1}(0.0836)=0$ $\quad F_*^{-1}(0.6042)=\frac{1}{0.05}\ln\left(\frac{0.8}{1-0.6042}\right)=14.07$

\6a/ $c=\int_{-2}^{2}(2+x)(2-x)\,dx=\frac{32}{3}$ $\Big\{ f(x)\propto(2+x)(2-x)=4+-x^2 \therefore x\in[-2,2]\therefore$

$\cancel{\int_{-2}^{2}(2+x}$ $\therefore f(x)=c_1(2+x)(2-x)=c_1(4-x^2)=4c_1-c_1x^2\therefore$

$\int_{-2}^{2}c_1(2+x)(2-x)\,dx=\int_{-2}^{2}4c_1-c_1x^2\,dx=1=\left[4c_1x-\frac{1}{3}c_1x^3\right]_{-2}^{2}=4c_1[2+2]-\frac{1}{3}c_1(2^3-(-2)^3)=$

$16c_1-\frac{1}{3}c_1(8--8)=16c_1-\frac{16}{3}c_1=1=\frac{32}{3}c_1\therefore c_1=\frac{3}{32}\therefore f(x)=\frac{3}{32}(2+x)(2-x)$

$\therefore g(x)=(2+x)(2-x)=cf(x)=c\frac{3}{32}(2+x)(2-x)\therefore 1=c\frac{3}{32}\therefore\frac{32}{3}=c\Big\}$

\6b/ Z max value of $g(x)$ is 4 at $x=0$ & $f_u(x)=\frac{1}{4}$ so $g_u(x)=kf_u(x)\geq g(x)=cf(x)$ holds if $k=16$. So Z acceptance rate for this proposal is $c/k=(32/3)/16=2/3$

$\Big\{ g(x)=cf(x)=(2+x)(2-x)=4-x^2$, $c=\frac{32}{3}$

$\therefore\max_{x\in[-2,2]}g(x)=\max_{x\in[-2,2]}4-x^2=4=4-0=4-0^2=g(0)\therefore M=4\therefore$

$x\in[-2,2]\therefore$ if $f_u(x)\sim$ Unif: $f_u(x)=\frac{1}{2--2}=\frac{1}{4}\therefore g_u(x)=kf_u(x)\geq g(x)=cf(x)$

$\therefore g_u(x)=k\frac{1}{4}\geq 4\therefore k=16$, $c=\frac{32}{3}\therefore\frac{c}{k}=\frac{(32/3)}{16}=\frac{2}{3}$ is Z acceptance rate of Z proposal is $\frac{2}{3}$

\6d/ Let $h(x)=\ln(g(x))=\ln(4-x^2)\therefore h'(x)=\frac{-2x}{4-x^2}\therefore$

$h''(x)=-\frac{2}{4-x^2}-\frac{4x^2}{(4-x^2)^2}<0$ for $x\in[-2,2]\therefore g(x)$ is log-concave.

\6e/ Let $\hat{x}=\operatorname{argmax}(g(x))$ & $H=\sup h''(x)<0$, $\therefore$ Z alg is to generate sampling using $f_u(x)$ Z pdf of a normal distri $N(\hat{x},\frac{1}{-H})$, with Z envelope $g_u(x)=e^{h(\hat{x})}e^{\frac{H}{2}(x-\hat{x})^2}$ st Z k st $g_u(x)=kf_u(x)\leq g(x)$ is

$k = e^{h(\hat{x})}\sqrt{\frac{2\pi}{-H}}$ in full, we generate X_u from $f_u(x)$, then generate $U \sim \text{Unif}(0,1)$, & accept X_u if $U g_u(X_u) < g(X_u)$

\6f/ $\therefore h'''(x) = -\frac{12x}{(4-x^2)^2} - \frac{4x^2}{(4-x^2)^2}$ which is 0 if $x=0$ or if $12 - 3x^2 + x = 0$ ie if $x = \frac{1 \pm \sqrt{145}}{6}$ & is $-\frac{1}{2} < 0$ there in Z interval $[-2,2]$ $h''(x)$ is maximised at $x=0$ & so $H = -\frac{1}{2}$, & Z conditions of Z alg have been met. now $\hat{x} = 0$ & $g(\hat{x}) = 4 \therefore R = \frac{c}{k} = \frac{(32/3)}{4\sqrt{\frac{2\pi}{(1/2)}}} = \frac{(32/3)}{4\sqrt{4\pi}} = \frac{4}{3\sqrt{3}}$

\7a/ $f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$. Z gamma distri has pdf $f(x) \propto x^{a-1} e^{-bx}$ $\therefore$ Let $g(x) = c\,f(x) = x^{a-1} e^{-bx}$

$\{ f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$ $\therefore \frac{b^a}{\Gamma(a)} \in \mathbb{R} \therefore f(x) \propto x^{a-1} e^{-bx}$, $\therefore$ let $g(x) = c\,f(x) = x^{a-1} e^{-bx} \}$ & let $h(x) = \ln(g(x))$ st $h(x) = (a-1)\ln(x) - bx$ now, $h'(x) = \frac{a-1}{x} - b \therefore h''(x) = -\frac{a-1}{x^2} \therefore$ $h''(x) < 0 \ \forall x$ if $a > 1$ $\therefore$ Z Gamma pdf is log-concave.

\7b/ though we have log-concavity, $H = \sup h''(x) = 0$ violating Z clause required for our rejection sampling

\7a/ $\{ f(x) = \frac{b^a}{\Gamma(a)} x^{a-1} e^{-bx}$ $\therefore \frac{b^a}{\Gamma(a)} \in \mathbb{R} \therefore f(x) \propto x^{a-1} e^{-bx}$, $\therefore$ let $g(x) = c\,f(x) = x^{a-1} e^{-bx}$ $\therefore$ let $h(x) = \ln(g(x)) = \ln(x^{a-1} e^{-bx}) = \ln(x^{a-1}) + \ln(e^{-bx}) = (a-1)\ln(x) - bx \therefore h'(x) = \frac{a-1}{x} - b \therefore h''(x) = -\frac{a-1}{x^2} \therefore$ $x^2 \geq 0 \therefore -\frac{1}{x^2} \leq 0 \therefore -\frac{a-1}{x^2} < 0$ for ~~all~~ $a - 1 > 0 \therefore$ $h''(x) < 0 \ \forall x$ if $a > 1 \therefore$ Z Gamma pdf is log concave & $\sup(h''(x)) = 0 \}$

\Week 7 Sheet/ \1a/ have $\pi(\sigma) \propto 1 \therefore$ must use Z change of variable formula to find $\pi(\sigma^2)$. Let $Y = g(\sigma) = \sigma^2$.

$\therefore \pi(y) = \left|\frac{d}{dy} g^{-1}(y)\right| \pi(g^{-1}(y))$ $\quad g^{-1}(y) = \sqrt{y}$ $\{\because g(y) = y^2\}$ $\therefore \frac{d}{dy} g^{-1}(y) = \frac{d}{dy}\sqrt{y} = \frac{d}{dy}(y^{1/2}) = \frac{1}{2} y^{-1/2} = \frac{1}{2\sqrt{y}}$ & $\pi(\sqrt{y}) = 1$ $\{ \pi(\sqrt{y}) = \left|\frac{d}{dy} g^{-1}(\sqrt{y})\right| \pi(g^{-1}(\sqrt{y})) = \left|\frac{d}{dy}[(y^{1/2})^{1/2}]\right| \pi((y^{1/2})^{1/2}) = \left|\frac{d}{dy}(y^{1/4})\right| \pi(y^{1/4}) \}$ $\therefore \pi(y) = \frac{1}{2\sqrt{y}} \therefore \pi(\sigma^2) \propto \frac{1}{\sigma}$

\week 7 Sheet ~~3~~/ $Y = g(\sigma) = \sigma^2$ $\therefore$ $\pi(y) = \left|\frac{d}{dy} g^{-1}(y)\right| \pi(g^{-1}(y))$

$\therefore g^{-1}(y) = \sqrt{y}$ $\{\because g(y) = y^2 \therefore$ let $\hat{y} = y^2 \therefore \sqrt{\hat{y}} = y$ $\therefore$ let $\sqrt{y} = \hat{y}$ $\therefore$

$g^{-1}(y) = \sqrt{y}\}$ & $\frac{d}{dy} g^{-1}(y) = \frac{1}{2\sqrt{y}}$ $\{\because g^{-1}(y) = \sqrt{y}$ $\therefore \frac{d}{dy} g^{-1}(y) =$

$\frac{d}{dy}\sqrt{y} = \frac{d}{dy} y^{1/2} = \frac{1}{2} y^{-1/2} = \frac{1}{2\sqrt{y}}\}$ & $\pi(\sqrt{y}) = 1$, $\therefore$

$\pi(y) = \frac{1}{2\sqrt{y}}$ $\{\because \pi(y) = \left|\frac{d}{dy} g^{-1}(y)\right| \pi(g^{-1}(y)) = \left|\frac{d}{dy}\sqrt{y}\right| \pi(\sqrt{y}) =$

$\left|\frac{1}{2\sqrt{y}}\right|(1) = \left|\frac{1}{2\sqrt{y}}\right| = \frac{1}{2\sqrt{y}}\}$ $\therefore \pi(\sigma^2) \propto \frac{1}{\sigma}$ $\{\because \pi(y) = \frac{1}{2\sqrt{y}}$ $\therefore \pi(y^2) = \frac{1}{2\sqrt{y^2}} =$

$\frac{1}{2y}$ $\therefore \pi(\sigma^2) = \frac{1}{2\sigma} = \frac{1}{2}\frac{1}{\sigma}$ $\therefore \pi(\sigma^2) \propto \frac{1}{\sigma}\}$

\1b/ first $\pi(\log(\sigma)) \propto 1$, & $x = \log(\sigma)$ with $y = e^{2\log(\sigma)} = g(x)$ $\therefore$

$g^{-1}(y) = \frac{\log(y)}{2}$ $\{g(x) = g(\log(\sigma)) = e^{2\log(\sigma)} = e^{\log(\sigma^2)} = \sigma^2$ $\therefore$

$g(\sigma) = g(e^{\ln\sigma}) = e^{\sigma^2}$ $\therefore$ let $\hat{g} = e^{\sigma^2}$ $\therefore \ln(\hat{g}) = \ln(e^{\sigma^2}) = \sigma^2$ $\therefore$

$(\ln(\hat{g}))^{1/2} = \sigma$ $\therefore$ let $\hat{g} = \sqrt{\ln(\sigma)}$ $\therefore g^{-1}(\sigma) = \sqrt{\ln(\sigma)}$ $\therefore$

$g^{-1}(y) = \cancel{g^{-1}(\sigma^2)} = \sqrt{\ln(y)}$ X $\}$ $g^{-1}(y) = \frac{\log(y)}{2}$ & $\frac{d}{dy} g^{-1}(y) = \frac{1}{2y}$ $\{\because$

$g^{-1}(y) = \frac{\log(y)}{2}$ $\therefore \frac{d}{dy} g^{-1}(y) = \frac{1}{2}\frac{1}{y} = \frac{1}{2y} = \frac{d}{dy}\left(\frac{1}{2}\log(y)\right)\}$ $\therefore$

$\pi(y) \propto \frac{1}{y}$ $\{\because$ is $\pi(y) = \frac{1}{2y} = \frac{1}{2}\frac{1}{y}$ $\therefore \pi(y) \propto \frac{1}{y}\}$ $\therefore \pi(\sigma^2) \propto \frac{1}{\sigma^2}$

\2a/ start at values $X_1^{(0)}$ & $X_2^{(0)}$. For $t = 1, 2, \ldots$, seq, sample $X_1^{(t)}, X_2^{(t)}$ by first sampling $X_1^{(t)}$ from $p(X_1 | X_2 = X_2^{(t-1)})$, & then sampling $X_2^{(t)}$ from $p(X_2 | X_1 = X_1^{(t)})$.

\2b/ Suppose X is discrete taking values $x_1, \ldots, x_m$ with probs $p_1, \ldots, p_m$ respectively. to sample from X, first define F_j $F_j = p_1 + \ldots + p_j$ then $F^{-1}(p) = x_j$ for $p \in (F_{j-1}, F_j]$. So, generate unif values on $[0,1]$ & treate these as p in Z above to get sample values X_U.

\2c/ Z joint probs are proportional to those in Z table, & $\because$ each row & column sums to 0.4, have, e.g. $p(X_1 = 1 | X_2 = 1) = \frac{P(X_1 = 1, X_2 = 1)}{P(X_2 = 1)} = \frac{0.2k}{0.4k} = 0.5$

& $P(X_1 = 2 | X_2 = 1) = \frac{P(X_1 = 2, X_2 = 1)}{P(X_2 = 1)} = \frac{0.1k}{0.4k} = 0.25$ due to symmetry of X_1 & X_2

here, can see $P(X_1 = i | X_2 = j) = \begin{cases} \frac{1}{2} & i = j \\ \frac{1}{4} & i \neq j \end{cases}$

& Z same is true for $X_2|X_1$. our gibbs sample now runs as:

$F(X_1^{(1)}|X_2^{(0)}=3)=(F_1,F_2,F_3)=(\frac{1}{4},\frac{1}{2},1)\cdot U_1=0.672$ $\therefore X_1^{(1)}=3$

$F(X_2^{(1)}|X_1^{(1)}=3)=(\frac{1}{4},\frac{1}{2},1)\cdot U_2=0.796$ $\therefore X_2^{(1)}=3$

$F(X_1^{(2)}|X_2^{(1)}=3)=(\frac{1}{4},\frac{1}{2},1)\cdot U_3=0.202$ $\therefore X_1^{(2)}=1$

4 a) $\{f(x)=\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}=\frac{2^3}{\Gamma(3)}x^{3-1}e^{-2x}=\frac{8}{(3-1)!}x^2e^{-2x}=4x^2e^{-2x}\}$

b) Some: first: $X\sim Gamma(\alpha,\beta)$, its pdf is $f(x)=\frac{\beta^\alpha}{\Gamma(\alpha)}x^{\alpha-1}e^{-\beta x}$, $x>0$ as Z random walk metropolis has symmetric proposal $(q(x^*|x^{t-1})=q(x^{t-1}|x^*))$ Z metropolis ratio is $r=f(x^*)/f(x^{t-1})$ working in log space (as usual with MCMC) $\log r=\log f(x^*)-\log f(x^{t-1})=(\alpha-1)\log\frac{x^*}{x^{t-1}}-\beta(x^*-x^{t-1})$, & we accept each x^* with prob $\min(1,r)$. obtaining $N(0,1)$ numbers from $Unif(0,1)$ numbers requires Z Box-Muller method. recall, $U_1,U_2\sim Unif(0,1)$, then $W=-2\log(1-U_1)$, $\Theta=2\pi U_2$, $Z_1=\sqrt{W}\cos\Theta$, $Z_2=\sqrt{W}\sin\Theta$, & $Z_1,Z_2\sim N(0,1)$. so a key question is how to order Z use of Z random numbers

$\therefore$ ratio is 1.69 is accepted as again $r>1$ $\therefore X^{(1)}=0.998$

~~[illegible]~~ d is takes measures cost 100 $\tilde{d}$ dont take measures θ is severe weather $\tilde{\theta}$ is not severe weather

	θ	$\tilde{\theta}$
d	-300	-100
$\tilde{d}$	-1000	0

$P(\theta)=P\in[0,1]$ $\therefore$ EMV: $EMV(d)=\sum_\theta M(d,\theta)P(\Theta=\theta)=$

$M(d,\theta)P(\Theta=\theta)+M(d,\tilde{\theta})P(\Theta=\tilde{\theta})=M(d,\theta)P+M(d,\tilde{\theta})(1-P)=$

$-300P-100(1-P)=-300P-100+100P=-200P-100=-100(1+2P)$

$EMV(\tilde{d})=\sum_\theta M(\tilde{d},\theta)P(\Theta=\theta)=M(\tilde{d},\theta)P(\Theta=\theta)+M(\tilde{d},\tilde{\theta})P(\Theta=\tilde{\theta})=$

$M(\tilde{d},\theta)P+M(\tilde{d},\tilde{\theta})(1-P)=-1000P-0(1-P)=-1000P$ ~~if EMV(d) > EMV(d̃)~~

if $EMV(d)>EMV(\tilde{d})$ choose d $\therefore$ let $P(\Theta=\theta)=0.2$ $\therefore P=0.2$ $\therefore$

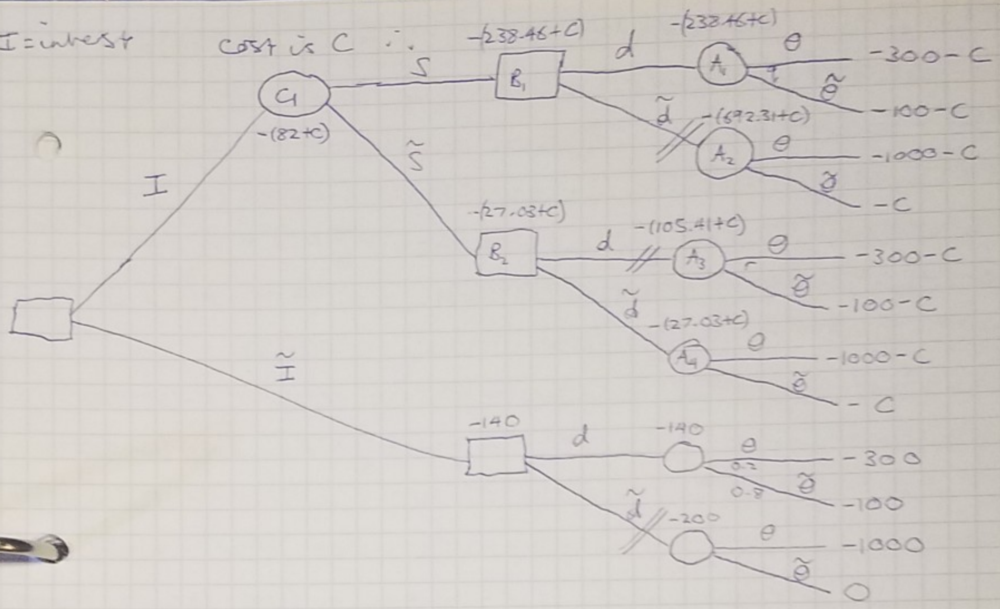
$EMV(d)=-100(1+2\times0.2)=-140$, $EMV(\tilde{d})=-1000(0.2)=-200$ $\therefore$

$-140>-200$ $\therefore EMV(d)>EMV(\tilde{d})$ $\therefore$ choose d.

let S be event system gives severe forecast then 90% accuracy means: $P(S|\theta)=0.9=P(\tilde{S}|\tilde{\theta})$ $\therefore P(S|\tilde{\theta})=1-P(S|\theta)=1-0.9=0.1$ $\therefore$

I = invest

cost is C ∴



$q = P(\theta \mid S)$, $r = P(\theta \mid \hat{S})$

$P(\theta|S)$, $P(\theta|\tilde{S})$, $P(S)$, $P(\theta)=0.2$, $P(\tilde{\theta})=0.8$ ∴

$P(\theta|S) = \frac{P(S|\theta)P(\theta)}{P(S)}$ (by Bayes)

$P(S) \overset{LOTP}{=} P(S|\theta)P(\theta) + P(S|\tilde{\theta})P(\tilde{\theta}) = 0.9P(\theta) + 0.1P(\tilde{\theta}) = 0.9(0.2) + 0.1(0.8) = \frac{13}{50}$

$\therefore P(\tilde{S}) = 1 - P(S) = 1 - \frac{13}{50} = \frac{37}{50}$ ∴

$P(\theta|S) = \frac{0.9 \times 0.2}{(13/50)} = \frac{9}{13}$ ∴

$P(\tilde{\theta}|S) = \frac{P(S|\tilde{\theta})P(\tilde{\theta})}{P(S)} = \frac{0.1 \times 0.8}{(13/50)} = \frac{4}{13}$

$\cancel{P(\theta|\tilde{S})} = P(\tilde{S}|\theta) = \frac{P(\theta|\tilde{S})P(\tilde{S})}{P(\theta)}$, $\cancel{P(\tilde{S}}$

$\{ P(\tilde{S}|\theta) = P((\tilde{S}|\theta)) \overset{LOTP}{=} \cancel{P((\tilde{S}|\theta)|\theta) + P((\tilde{S}|\theta)} P((\tilde{S}|\theta)|\theta)P(\theta) + P((\tilde{S}|\theta)|\tilde{\theta})P(\tilde{\theta}) =$

$P((\tilde{S}|\theta)|\theta)0.2 + P((\tilde{S}|\theta)|\tilde{\theta})0.8 = P(\tilde{S}|\theta)0.2 + 0 \times 0.8 = P(\tilde{S}|\theta)0.2 = P(\tilde{S}|\theta)$ X }

$\left\{ P(\tilde{S}|\theta) = \frac{P(\tilde{S} \cap \theta)}{P(\theta)} \right\}$

$P(\tilde{S}|\theta)$ = P(not ~~severe~~ severe forecast given ~~severe~~ severe weather) = P(system is wrong ~~correct~~) =

1 − P(system is correct ~~wrong~~) = 1 − P(not severe forecast given s̃e weather = $1 - P(\tilde{S}|\tilde{\theta}) = 1 - 0.9 = 0.1$ }

$\therefore P(\theta|\tilde{S}) = \frac{P(\tilde{S}|\theta)P(\theta)}{P(\tilde{S})} = \frac{0.1 \times 0.2}{(37/50)} = \frac{1}{37}$ ∴

$P(\tilde{\theta}|\tilde{S}) = \frac{P(\tilde{S}|\tilde{\theta})P(\tilde{\theta})}{P(\tilde{S})} = \frac{0.9 \times 0.8}{(37/50)} = \frac{36}{37}$

$$EMV(A_1)=\sum_{\theta} M(S,d,\theta)P(\Theta=\theta)=M(S,d,\theta)P(\theta|S)+M(S,d,\tilde{\theta})P(\tilde{\theta}|S)=$$
$$(-300-c)P(\theta|S)+(-100-c)P(\tilde{\theta}|S)=(-300-c)\frac{9}{13}+(-100-c)\frac{4}{13}=$$
$$-\frac{2700}{13}-\frac{9}{13}c-\frac{400}{13}-\frac{4}{13}c=-\frac{3100}{13}-c\approx-238-c=EMV(d|I,S)\ \therefore$$
$$EMV(d|I,S)=-(300+c)P(\theta|S)-(100+c)P(\tilde{\theta}|S)=-(238.46+c)$$
$$EMV(A_2)=EMV(\tilde{d}|I,S)=\sum_{\theta} M(\tilde{d},\theta|S)P(\Theta=\theta)=\sum_{\theta} M(\tilde{d},\theta,S)P(\Theta=\theta)=$$
$$\sum_{\theta} M(S,\tilde{d},\theta)P(\Theta=\theta)=M(S,\tilde{d},\theta)P(\Theta=\theta)+M(S,\tilde{d},\tilde{\theta})P(\Theta=\tilde{\theta})=$$
$$(-1000-c)P(\Theta=\theta)+(-c)P(\Theta=\tilde{\theta})=(-1000-c)P(\theta|S)+(-c)P(\tilde{\theta}|S)=$$
$$(-1000-c)\frac{9}{13}+(-c)\frac{4}{13}=-\frac{9000}{13}-\frac{9}{13}c-\frac{4}{13}c=-\frac{9000}{13}-c=-692.3-c=-(692.3+c)$$
$$EMV(A_3)=EMV(d|I,\tilde{S})=\sum_{\theta} M(d,\theta|I,\tilde{S})P(\Theta=\theta)=\sum_{\theta} M(I,\tilde{S},d,\theta)P(\Theta=\theta)=$$
$$M(I,\tilde{S},d,\theta)P(\Theta=\theta)+M(I,\tilde{S},d,\tilde{\theta})P(\Theta=\tilde{\theta})=M(I,\tilde{S},d,\theta)P(\theta|\tilde{S})+M(I,\tilde{S},d,\tilde{\theta})P(\tilde{\theta}|\tilde{S})$$
$$=(-300-c)P(\theta|\tilde{S})+(-100-c)P(\tilde{\theta}|\tilde{S})=(-300-c)\frac{1}{37}+(-100-c)\frac{36}{37}=$$
$$-\frac{300}{37}-\frac{1}{37}c-\frac{3600}{37}-\frac{36}{37}c=-\frac{3900}{37}-c=-105.4-c=-(105.4+c)$$
$$EMV(A_4)=EMV(\tilde{d}|I,\tilde{S})=\sum_{\theta} M(\tilde{d},\theta|I,\tilde{S})P(\Theta=\theta)=\sum_{\theta} M(I,\tilde{S},\tilde{d},\theta)P(\Theta=\theta)=$$
$$M(I,\tilde{S},\tilde{d},\theta)P(\Theta=\theta)+M(I,\tilde{S},\tilde{d},\tilde{\theta})P(\Theta=\tilde{\theta})=M(I,\tilde{S},\tilde{d},\theta)P(\theta|\tilde{S})+M(I,\tilde{S},\tilde{d},\tilde{\theta})P(\tilde{\theta}|\tilde{S})$$
$$=(-1000-c)P(\theta|\tilde{S})+(-c)P(\tilde{\theta}|\tilde{S})=(-1000-c)\frac{1}{37}+(-c)\frac{36}{37}=-\frac{1000}{37}-\frac{1}{37}c-\frac{36}{37}c=$$
$$-\frac{1000}{37}-c=-27.03-c=-(27.03+c)\ \therefore$$
$$EMV(A_1)=-(238.46+c)>-(692.3+c)=EMV(A_2)\quad\therefore EMV(A_1)>EMV(A_2)$$
$$EMV(A_4)=-(27.03+c)>-(105.4+c)=EMV(A_3)\quad\therefore\quad EMV(A_4)>EMV(A_3)$$
$$EMV(B_1)=EMV(A_1)=-(238.46+c)$$
$$EMV(B_2)=EM(A_4)=-(27.03+c)\ \therefore\qquad EMV(B_2)>EMV(B_1)\ \therefore$$
$$EMV(C_1)=EMV(B_2)=-(27.03+c)\ \therefore$$

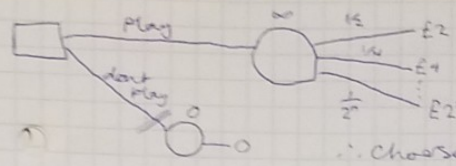
choose I is $EMV(C_1)>-140\ \therefore\quad -(27.03+c)>-140\ \therefore$

$-27.03-c>-140\ \therefore$

is $112.97>c$: choose I, choose invest $\cancel{}$

$$EMV(I)=EMV(C_1)=EMV(B_1)P(S)+EMV(B_2)P(\tilde{S})=-(238.46+c)P(S)+-(27.03+c)P(\tilde{S})$$
$$=-(238.46+c)\frac{13}{50}+-(27.03+c)\frac{37}{50}=-61.9996-\frac{13}{50}c-20.0022-\frac{37}{50}c=$$
$$-82.0018-c=-(82+c)\ \therefore$$

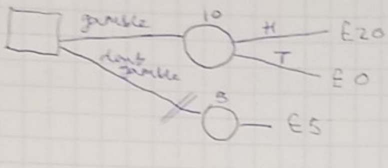
choose I, invest is: $-(82+c)=-82-c>-140\ \therefore\ c<58$
is a severe forecast, take the measures, else don't take the measures.



EMV(dont play) = 0

$EMV(play) = \sum_{i=1}^{\infty} p_i x_i = \sum_{i=1}^{\infty} \frac{1}{2^i} 2^i = \sum_{i=1}^{\infty} 1 = \infty$

∴ choose play whatever the cost



EMV(dont gamble) = 0

$EMV(gamble) = \cancel{\sum_{\Theta}} \sum_{\Theta} M(\Theta) P(\Theta = \theta) =$

$M(H)P(H) + M(T)P(T) = 20P(H) + 0P(T) =$

$20 \times \frac{1}{2} + 0(\frac{1}{2}) = 10$ ∴ always choose gamble

\Week 12 Sheet/

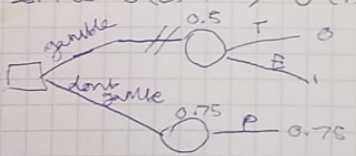
\1a/ Edinburgh, Paris, Rome, Tenerife

less travel time is worth more ∴ $U(S_1) > U(S_2)$ if travel time of S_1 is less than S_2

\1a sol/ for me, Rome $^*\geq$ Paris $^*\geq$ Edinburgh $^*\geq$ Tenerife

label these R, P, E, T respectively. using Z method of defining a utility func: $U(R) = 1$, $U(T) = 0$. set $U(E)$ as Z val of p st I am indifferent betw E for sure & a gamble with probab p of R & (1−p) of T ie: $E \sim^* pR +_g (1-p)T$, by doing this for E & similarly for T, $\{P \sim^* qR +_g (1-q)T\}$, my utilities are $U(E) = 0.7$, $U(P) = 0.75$, utilities are coherent with preferences,

\1b/ for me: $U(E) = 1$, $U(T) = 0$, $U(P) = 0.75$ ∴ $P(T) = 0.5$, $P(E) = 0.5$ ∴



∴ EMV(dont gamble) = 0.75

$EMV(gamble) = \sum_{\Theta} M(\Theta) P(\Theta = \theta) =$

$M(T)P(T) + M(E)P(E) = 0(\frac{1}{2}) + 1(\frac{1}{2}) = \frac{1}{2}$ ∴

dont gamble unless $U(P) < 0.5$

\1b sol/ for me, no, as $P \,^*\!> E$. so $P \,^*\!> 0.5E +_g 0.5T$

\2a sol/ having a linear utility for money means that we can work EMV throughout. Let θ_1 be event high demand for 10 years.

Let θ_2 be event low demand for 10 years. Let θ_3 be event high demand for 2 years then low demand for remaining 8 years.

Let θ_4 be event high demand for 2 years.

Let OP denote decision to open cake shop let L denote decision to open large shop

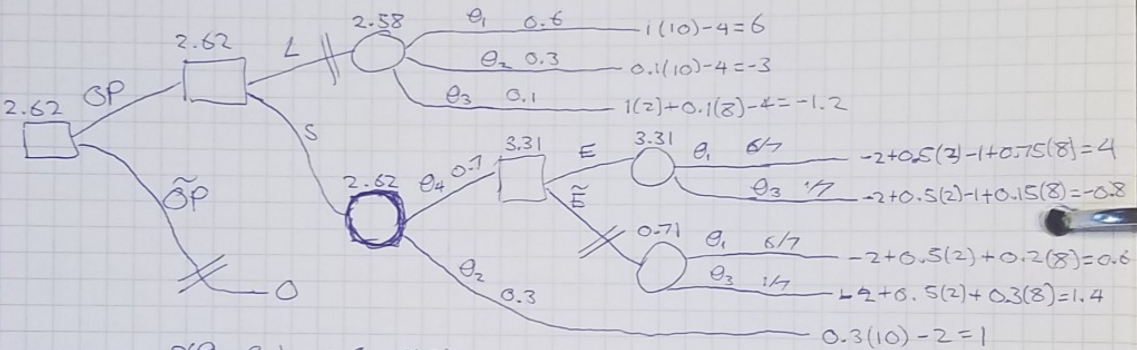
let S denote decision to open small shop

let E denote decision to expand small shop after 2 years.

$\therefore P(\theta_1) = 0.6$, $P(\theta_2) = 0.3$, $P(\theta_3) = 0.1$

$P(\theta_4) = P(\text{high demand for first 2 years}) = P(\theta_1) + P(\theta_3) = 0.6 + 0.1 = 0.7$ $\therefore$

$P(\theta_3 \mid \theta_4) = \dfrac{P(\theta_3, \theta_4)}{P(\theta_4)} = \dfrac{P(\theta_3)}{P(\theta_4)} = \dfrac{0.1}{0.7} = \dfrac{1}{7}$ all gains in £M $\therefore$:



$P(\theta_1 \mid \theta_4) = \dfrac{P(\theta_1, \theta_4)}{P(\theta_4)} = \dfrac{P(\theta_1)}{P(\theta_4)} = \dfrac{0.6}{0.7} = \dfrac{6}{7}$

$EMV(OP, L) = \sum_{\theta_i} M(\theta_i) P(\theta_i = \theta) = \sum_{\theta_i} M(OP, L, \theta_i) P(\theta_i) =$

$M(OP, L, \theta_1) P(\theta_1) + M(OP, L, \theta_2) P(\theta_2) + M(OP, L, \theta_3) P(\theta_3) = 6(0.6) - 3(0.3) - 1.2(0.1) = 2.58$

$EMV(OP, S, E) = \sum_{\theta} M(OP, S, E, \theta) P(\theta) = M(OP, S, E, \theta_1) P(\theta_1) + M(OP, S, E, \theta_3) P(\theta_3) =$

$4(\frac{6}{7}) - 0.8(\frac{1}{7}) = 3.31$

$EMV(OP, S, \tilde{E}) = \sum_{\theta} M(OP, S, \tilde{E}, \theta) P(\theta) = M(OP, S, \tilde{E}, \theta_1) P(\theta_1) + M(OP, S, \tilde{E}, \theta_3) P(\theta_3)$

$= 0.6(\frac{6}{7}) + 1.4(\frac{1}{7}) = 0.71$

$EMV(OP, S) = \sum_{\theta} M(OP, S, \theta) P(\theta) = M(OP, S, \theta_4) P(\theta_4) + M(OP, S, \theta_2) P(\theta_2) =$

$3.31(0.7) + 1(0.3) = 2.62$

2b/

$\therefore$ Mary should listen to Mel & Sue & open small shop then expand after 2 years if demand is high

3/ suppose R, r, Z best & worst rewards. then: $U(R) - U(r) > 0$;

$V(R) - V(r) > 0$ as U & V are utility funcs. this $\therefore \exists\, a > 0, b$ st

$aU(R) + b = V(R)$ $aU(r) + b = V(r)$ $\therefore$

for any reward s with $r <^* s <^* R$ $\exists$ a gamble: $pR +_g (1-p) r \sim^* s$

& $V(s) = pV(R) + (1-p) V(r)$ as V is a utility func $\therefore$

Week 12 Sheet

$V(S) = p(aU(R)+b) + (1-p)(aU(r)+b) =$

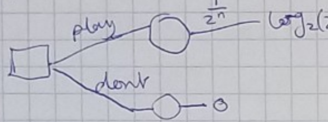
1- $apU(R) + pb + a(1-p)U(r) + (1-p)b = a(pU(R) + (1-p)U(r)) + pb + (1-p)b =$

$a(pU(R) + (1-p)U(r)) + b = aU(pR +_g (1-p)r) + b = aU(S) + b$

4/ by induction. know Z property holds for $k=2$. suppose Z property holds for some integer k. we now show it must then hold for $k+1$. write: $p_1 r_1 +_g \dots +_g p_k r_k +_g p_{k+1} r_{k+1} = (1-p_{k+1})S +_g p_{k+1} r_{k+1}$ where S is Z gamble $p_1^* r_1 +_g \dots +_g p_k^* r_k$, & $p_i^* = p_i(1-p_{k+1})$ (each Z prob of getting reward i given that we dont get reward $k+1$). ∴ we know that Z property holds for $k=2$, $U(p_1 r_1 +_g \dots +_g p_k r_k +_g p_{k+1} r_{k+1}) =$
$(1-p_{k+1})U(S) + p_{k+1}U(r_{k+1}) = (1-p_{k+1})(p_1^* U(r_1) + \dots + p_k^* U(r_k)) + p_{k+1}U(r_{k+1}) =$
$p_1 U(r_1) + \dots + p_k U(r_k) + p_{k+1} U(r_{k+1})$ by inductive assumption.

5/ without loss of generality, we can set Z utility of Z best & worst rewards at 1 & 0 respectively ∴ let $U(£2500) = 1$, $U(£=0) = 0$. ∴ set $U(£500) = x$ for $x \in [0,1]$. for these preferences to be compatible with utility theorey, must have $U(G_1) > U(G_2)$ & $U(G_4) > U(G_3)$. but,
$U(G_1) > U(G_2)$ ∴ $x > p + qx$ $\{U(G_1) > U(G_2)$ ∴ $U(500) > U(2500p +_g 500q)$ ∴ $x > U(2500p +_g 500q)$ ∴ $x > p + qx\}$ ∴ $x > \frac{p}{1-q}$ &
$U(G_4) > U(G_3)$ ∴ $p > x(p+s)$ $\{U(G_4) = U(2500p) = U(2500)p = 1p = p$
$U(G_3) = U(500(p+s)) = U(500)(p+s) = x(p+s)$,
$U(G_2) = U(2500p +_g 500q) = U(2500p) + U(500q) = U(2500)p + U(500)q =$
$1p + xq = p + qx$ ∴ $U(G_4) > U(G_3)$ ∴ $p > x(p+s)\}$ ∴ $1-q = p+s$ ∴
$p > x(1-q)$ ∴ $x < \frac{p}{1-q}$, these cannot be true simultaneously. Z preferences break Z assump that gambles canbe coherently compared. in particular, substitutionality, which requires, for rewards $S_1 \succ^* S_2$, probabs p^*, $q^* = 1-p^*$, & for anyreward r that we have $pS_1 +_g qr \succ^* pS_2 +_g qr$. let r be Z reward that we recieve £500 for sure, & similarly for S_1. Let t be Z reward that we receive £0 for sure. then let: $S_2 = \frac{p}{p+s}2500 +_g \frac{s}{p+s}0$, we have that $G_1 \succ^* G_2$ ∴ $(p+s)S_1 +_g qr \succ^* (p+s)S_2 +_g qr$, but $G_4 \succ^* G_3$: $(p+s)S_1 +_g qr \succ^* (p+s)S_2 +_g qr$ which violates coherence.

\Week 12 Sheet extra Ex/ \1a/ $U(x) = \log_2(x)$



$EMV(dont) = 0$

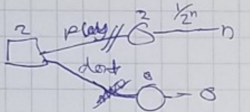
$EMV(play) = \sum_i x M(i)\, p_i = \sum_{i=1}^{\infty} i \frac{1}{2^i} < \infty$

\1 or Sol/ $U(x) = \log_2(x)$ ∴ gamble: $G = \sum_{r=1}^{\infty} , \frac{1}{2^r}$ £2^r ∴

$E(U(G)) = \sum_{r=1}^{\infty} \frac{1}{2^r} U(2^r) = \sum_{r=1}^{\infty} \frac{1}{2^r} \log_2(2^r) = \sum_{r=1}^{\infty} \frac{r}{2^r} = 2$ ∵ ($\sum_{k=1}^{\infty} k z^k = \frac{z}{(1-z)^2}$ for $|z| < 1$)

{ ∴ $\sum_{r=1}^{\infty} k 2^k = \sum_{r=1}^{\infty} k z^k = \frac{z}{(1-z)^2} = \frac{2}{(1-2)^2} = \frac{2}{(-1)^2} = \frac{2}{1} = 2$

\1b/



∴ $2 < 5$ ∴ shouldnt play

\1b/ $U(5) = \log_2(5) = 2.32$ ∴ $U(£5) > U(G)$ & you should keep £5.

Z Max you should pay is £z with $U(z) = 2$ ∴ $\log_2(z) = 2$ ∴ $z = 2^2 = 4$

\2/ desire decisions & outcomes:
let S be Z event Z new restaurant is profitable
let T_1 be Z event Telesurvey 1 predicts S
let T_2 be Z event Telesurvey 2 predicts S
let d_R be decision to open a restaurant
let d_{T_1} be decision to pay for telesurvey 1
let d_{T_2} be decision to pay for telesurvey 2
∴ Z utilities are:
$U(S, d_{T_1}, d_{T_2}) = 8 - 0.7 - 0.8 = 6.5$
$U(S, d_{T_1}, \tilde{d}_{T_2}) = 8 - 0.7 = 7.3$

$U(\tilde{S}, d_{T_1}, d_{T_2}) = -5 - 0.7 - 0.8 = -6.5$
$U(\tilde{S}, d_{T_1}, \tilde{d}_{T_2}) = -5 - 0.7 = -5.7$
we already know that, in Z absence of data from Z survey, he wont open Z resurant ∵ he hasnt already opened Z resurant ∴ he's pessimistic without Z surveys ∴
$E(U(S, \tilde{d}_{T_1})] = 0.3 \times 8 - 0.7 \times 5 = -1.1 < U(\tilde{S}, \tilde{d}_{T_1}) = 0$ ∴
if Z expected utility ~~at~~ this chance node is > 0, should take Z survey, else, be satisfied with not opening Z restaurant.
Probabs:
$P(S) = 0.3$, $P(\tilde{S}) = 0.7$, $P(T_1|S) = 0.8 = P(\tilde{T}_1|\tilde{S})$, $P(T_1|\tilde{S}) = 0.2 = P(\tilde{T}_1|S)$,
$P(T_2|S) = 0.9 = P(\tilde{T}_2|\tilde{S})$, $P(T_2|\tilde{S}) = 0.1 = P(\tilde{T}_2|S)$ independence of surveys given outcome

~~T_1, T_2 indep ∴ $P(T_1, T_2) = P(T_1)P(T_2)$~~ ∴ $P(T_1, T_2|S) = P(T_1|S)P(T_2|S)$ ∴

$P(T_1) = P(T_1|S)P(S) + P(T_1|\tilde{S})P(\tilde{S})$ (LoTP) $= 0.8 \times 0.3 + 0.2 \times 0.7 = 0.38$ ∴
$P(\tilde{T}_1) = 1 - P(T_1) = 0.62$ ∴
$P(T_1, T_2) = P(T_1, T_2|S)P(S) + P(T_1, T_2|\tilde{S})P(\tilde{S})$ (LoTP) $=$
$P(T_1|S)P(T_2|S)P(S) + P(T_1|\tilde{S})P(T_2|\tilde{S})P(\tilde{S}) = 0.8 \times 0.9 \times 0.3 + 0.2 \times 0.1 \times 0.7 = 0.23$ ∴

$P(T_2|T_1) = P(T_1, T_2)/P(T_1) = {0.23}/{0.38} = \frac{23}{38} = 0.61$ ∴
$P(\tilde{T}_2|T_1) = 1 - P(T_2|T_1) = 1 - 0.61 = 1 - \frac{23}{38} = 15/38 = 0.39$

$P(T_2, \tilde{T}_1) = P(T_2, \tilde{T}_1|S)P(S) + P(T_2, \tilde{T}_1|\tilde{S})P(\tilde{S})$ (LoTP) $= P(T_2|S)P(\tilde{T}_1|S)P(S) + P(T_2|\tilde{S})P(\tilde{T}_1|\tilde{S})P(\tilde{S}) =$
$0.9 \times 0.2 \times 0.3 + 0.1 \times 0.8 \times 0.7 = 0.11$ ∴

Week 12 Sheet etra Ex / $P(T_2|\tilde{T}_1) = \frac{P(T_2, \tilde{T}_1)}{P(\tilde{T}_1)} = \frac{0.11}{0.62} = \frac{11}{62} = 0.18$ ∴

$P(\tilde{T}_2|\tilde{T}_1) = 1 - P(T_2|\tilde{T}_1) = \frac{51}{62} = 0.82$

$P(S|T_1) = \frac{P(T_1|S)P(S)}{P(T_1)}$ (by Bayes) $= \frac{0.8 \times 0.3}{0.38} = \frac{12}{19} = 0.63$

$P(\tilde{S}|T_1) = \frac{P(T_1|\tilde{S})P(\tilde{S})}{P(T_1)} = \frac{0.2 \times 0.7}{0.38} = \frac{7}{19} = 1 - P(S|T_1) = 1 - \frac{12}{19} = \frac{7}{19} = 0.37$

$P(S|\tilde{T}_1) = \frac{P(\tilde{T}_1|S)P(S)}{P(\tilde{T}_1)} = \frac{0.2 \times 0.3}{0.62} = \frac{3}{31} = 0.10$ (by bayes) ∴

$P(\tilde{S}|\tilde{T}_1) = 1 - P(S|\tilde{T}_1) = 1 - 0.10 = \frac{28}{31} = 0.90$

$P(S|T_1, T_2) = P(T_1, T_2|S)P(S) / P(T_1, T_2)$ (by Bayes) $= P(T_1|S)P(T_2|S)P(S) / P(T_1, T_2) = \frac{0.8 \times 0.9 \times 0.3}{0.23} = \frac{108}{115} = 0.94$

$\therefore P(\tilde{S}|T_1, T_2) = 1 - P(S|T_1, T_2) = 1 - 0.94 = \frac{7}{115} = 0.06$

$P(S|T_1, \tilde{T}_2) = P(T_1, \tilde{T}_2|S)P(S) / P(T_1, \tilde{T}_2)$ (by bayes) $= P(T_1|S)P(\tilde{T}_2|S)P(S) / P(T_1, \tilde{T}_2) = \frac{0.8 \times 0.1 \times 0.3}{P(T_1, \tilde{T}_2)}$

$\therefore P(T_1, \tilde{T}_2) = P(T_1, \tilde{T}_2|S)\cancel{P(S)} + P(T_1, \tilde{T}_2|\tilde{S})P(\tilde{S}) = P(T_1|S)P(\tilde{T}_2|S)P(S) + P(T_1|\tilde{S})P(\tilde{T}_2|\tilde{S})P(\tilde{S}) =$

$0.8 \times 0.1 \times 0.3 + 0.2 \times 0.9 \times 0.7 = 0.15$ ∴

$P(S|T_1, \tilde{T}_2) = 0.8 \times 0.1 \times 0.3 / P(T_1, \tilde{T}) = 0.024 / 0.15 = 0.16$

$P(\tilde{S}|T_1, \tilde{T}_2) = 1 - 0.16 = 0.84$

$P(S|\tilde{T}_1, T_2) = P(\tilde{T}_1, T_2|S)P(S) / P(\tilde{T}_1, T_2) = P(\tilde{T}_1|S)P(T_2|S)P(S) / P(\tilde{T}_1, T_2) = \frac{0.2 \times 0.9 \times 0.3}{0.11} = \frac{27}{55} = 0.49$

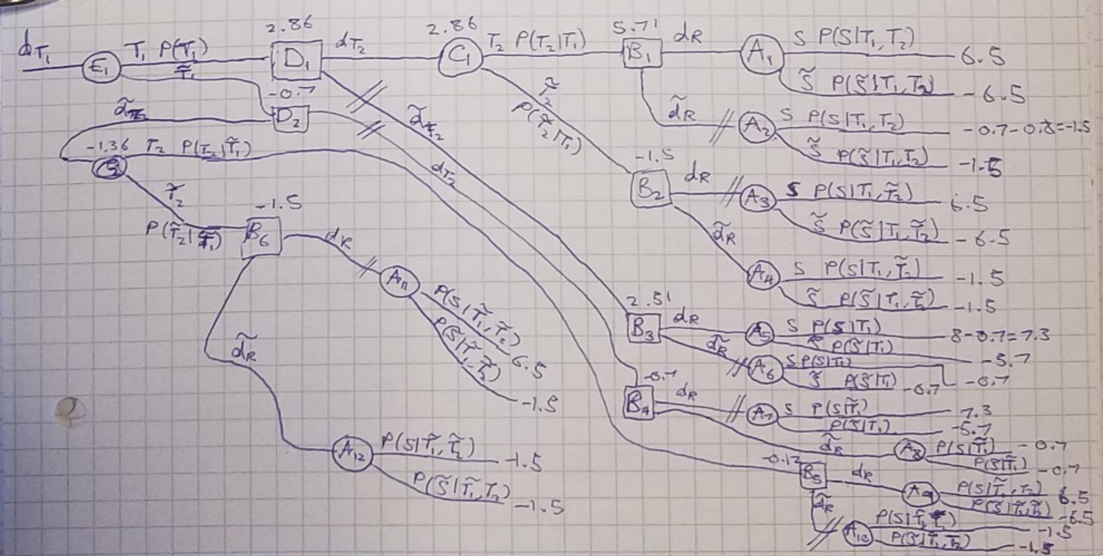
$\therefore P(\tilde{S}|\tilde{T}_1, T_2) = 1 - P(S|\tilde{T}_1, T_2) = 1 - \frac{27}{55} = \frac{28}{55} = 0.51$

$P(S|\tilde{T}_1, \tilde{T}_2) = \frac{P(\tilde{T}_1, \tilde{T}_2|S)P(S)}{P(\tilde{T}_1, \tilde{T}_2)} = \frac{P(\tilde{T}_1|S)P(\tilde{T}_2|S)P(S)}{P(\tilde{T}_1, \tilde{T}_2)} = \frac{0.2 \times 0.1 \times 0.3}{P(\tilde{T}_1, \tilde{T}_2)} = \frac{0.006}{P(\tilde{T}_1, \tilde{T}_2)}$ ∴

$P(\tilde{T}_1, \tilde{T}_2) = P(\tilde{T}_1, \tilde{T}_2|S)P(S) + P(\tilde{T}_1, \tilde{T}_2|\tilde{S})P(\tilde{S}) \overset{(LOTP)}{=} P(\tilde{T}_1|S)P(\tilde{T}_2|S)P(S) + P(\tilde{T}_1|\tilde{S})P(\tilde{T}_2|\tilde{S})P(\tilde{S}) =$
$0.2 \times 0.1 \times 0.3 + 0.8 \times 0.9 \times 0.7 = 0.51$ ∴

$P(S|\tilde{T}_1, \tilde{T}_2) = 0.006 / 0.51 = 0.01$ ∴

$P(\tilde{S}|\tilde{T}_1, \tilde{T}_2) = 1 - P(S|\tilde{T}_1, \tilde{T}_2) = 1 - 0.01 = 0.99$ ∴



$\therefore A_1 = 6.5\left(\frac{108}{115}\right) - 6.5\left(\frac{7}{115}\right) = 6.5\left(\frac{108}{115} - \frac{7}{115}\right) = 5.71$, $A_2 = -1.5p - 1.5(1-p) = -1.5(1) = -1.5$

$A_3 = 6.5\left(\frac{12-63}{75}\right) = -4.42$ $\quad A_4 = -1.5p_2 - 1.5(1-p_2) = -1.5(1) = -1.5$

$A_5 = 7.3 \times \frac{12}{19} - 5.7 \times \frac{7}{19} = 2.51$ $\quad A_6 = -0.7$, $A_7 = 7.3 \times \frac{3}{31} - 5.7 \times \frac{28}{31} = -4.44$

$A_8 = -0.7$ $\quad A_9 = 6.5\left(\frac{27-28}{55}\right) = -0.12$ $\quad A_{10} = -1.5$ $\quad A_{11} = -6.35$ $\quad A_{12} = -1.5$ $\therefore$

$B_1 = \text{Max}(A_1, A_2) = 5.71 \quad (d_R)$ $\qquad B_4 = \text{Max}(A7, A8) = -0.7 \quad (\tilde{d}_R)$

$B_2 = \text{Max}(A_3, A_4) = -1.5 \quad (\tilde{d}_R)$ $\qquad B_5 = \text{Max}(A9, A10) = -0.12 \quad (d_R)$

$B_3 = \text{Max}(A_5, A_6) = 2.51 \quad (d_R)$ $\qquad B_6 = \text{Max}(A11, A12) = -1.5 \quad (\tilde{d}_R)$

$C_1 = (5.71) \times \left(\frac{23}{38}\right) - 1.5 \times \frac{15}{38} = 2.86$

$C_2 = (-0.7) \times \frac{11}{62} - 1.5 \times \frac{51}{62} = -1.36$

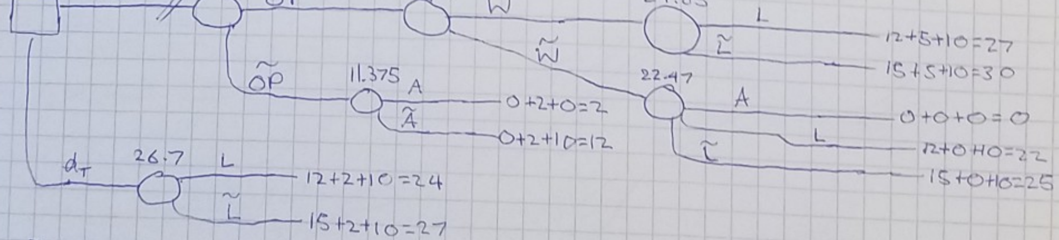
$D_1 = \text{Max}(C_1, B_3) = 2.86 \quad (d_{T_2})$

$D_2 = \text{Max}(C_2, B_4) = -0.7 \quad (\tilde{d}_{T_2})$ $\therefore$

$E_1 = (2.86) \times \frac{19}{50} - 0.7 \times \frac{31}{50} = 0.65 > 0$ $\therefore$ he should buy Z first survey (d_{T_1}) &, if T_1, buy Z 2nd survey (d_{T_2}), otherwise he shouldnt open a restaurant. if after buying Z 2nd survey, T_2, he should open a restaurant. otherwise he shouldnt open restaurant.

3 abs/describe these decisions & events:
d_d: decision to drive over Z mountains
d_T: decision to take Z train
OP: event Z pass is open
W: event Z weather on Z mountain is good
A: event Z man has a serious accident
L: event he arrives late

note: only accident or pass closure events mean he misses appointment & will always mean it $\therefore$ dont need extra notation



$\therefore E[U(d_d, OP, W)] = P(L \mid OP, W, d_d)U(L, W, OP, d_d) + P(\tilde{L} \mid OP, W, d_d)U(\tilde{L}, W, OP, d_d)$

$P(L \mid OP, W, d_d)(12+5+10) + P(\tilde{L} \mid OP, W, d_d)(15+5+10) = \frac{1}{20}(27) + \left(\frac{19}{20}\right)30 = 29.85$

$\because Pr(\text{ontime} \mid \text{good weather}) = 1 - Pr(\text{late} \mid \text{good weather}) = 1 - \frac{1}{20} = \frac{19}{20}$

\Week 12 Sheet extra Ex/ Pr(~~not~~ late | bad conditions) =

1 − Pr(accident | bad conditions) − Pr(late | bad conditions) $= 1 - \frac{1}{4} - \frac{1}{16} = \frac{11}{16}$ ∴

$E[U(d_d, OP, \tilde{W})] = P(A|OP, \tilde{W}, d_d)U(A, \tilde{W}, OP, d_d) + P(L|OP, \tilde{W}, d_d)U(L, \tilde{W}, OP, d_d) +$

$P(\tilde{L}|OP, \tilde{W}, d_d)U(\tilde{L}, \tilde{W}, OP, d_d) = \frac{1}{16}\times 0 + \frac{1}{4}\times 22 + \frac{11}{16}\times 25 = 22.47$

$P(\tilde{A}|\tilde{OP}) = 1 - P(A|\tilde{OP}) = 1 - \frac{1}{16} = \frac{15}{16}$ ∴

$E[U(d_d, \tilde{OP})] = P(A|\tilde{OP}, d_d)U(A, \tilde{OP}, d_d) + P(\tilde{A}|\tilde{OP}, d_d)U(\tilde{A}, \tilde{OP}, d_d) =$

$\frac{1}{16}\times 2 + \frac{15}{16}\times 12 = 11.375$

P(train on time) = 1 − P(train late) $= 1 - \frac{1}{10} = \frac{9}{10}$ ∴

$E(U(d_T)] = P(L|d_T)U(L, d_T) + P(\tilde{L}|d_T)U(\tilde{L}, d_T) = \frac{1}{10}\times 24 + \frac{9}{10}\times 27 = 26.7$

P(not good weather | OP) $= P(\tilde{W}|OP) = 1 - P(W|OP) = 1 - \frac{2}{3} = \frac{1}{3}$ ∴

$E[U(d_d, OP)] = P(W|OP, d_d)U(W, OP, d_d) + P(\tilde{W}|OP, d_d)U(\tilde{W}, OP, d_d) =$

$\frac{2}{3}\times 29.85 + \frac{1}{3}\times 22.47 = 27.39$

$P(\tilde{OP}|d_d) = 1 - P(OP|d_d) = 1 - \frac{4}{5} = \frac{1}{5}$ ∴

$E[U(d_d)] = P(OP|d_d)U(OP, d_d) + P(\tilde{OP}|d_d)U(\tilde{OP}, d_d) = \frac{4}{5}\times 27.39 + \frac{1}{5}\times 11.375 = 24.19$

$E[U(d_T)] > E[U(d_d)]$ ∴ Z man should take Z train

\Week 1 Sheet/ \1a/ By Bayes thm: Z posterior: $\pi(\theta|y) \propto \pi(\theta)P(y|\theta)$

$\propto \theta^{a-1}(1-\theta)^{b-1}p(y|\theta) \propto \theta^{a-1}(1-\theta)^{b-1}\prod_{i=1}^{n}\theta(1-\theta)^{y_i-1} \propto \theta^{a-1}(1-\theta)^{b-1}\theta^{n}(1-\theta)^{\sum_{i=1}^{n}(y_i-1)} \propto$

$\theta^{a+n-1}(1-\theta)^{b-1}(1-\theta)^{-n+n\bar{y}} \propto \theta^{(a+n)-1}(1-\theta)^{b+n\bar{y}-n-1} \propto \theta^{(a+n)-1}(1-\theta)^{(b+n\bar{y}-n)-1}$ ∴

$\pi(\theta|y) \propto \theta^{(a+n)-1}(1-\theta)^{(b+n\bar{y}-n)-1}$ ∴

~~θ|y~~ $\theta|y \sim Beta(a+n, b+n\bar{y}-n)$ ∵

$\pi(\theta|y)$ is proportional to a Beta density ∵ $\theta \in (0,1)$ ∴

$$\pi(\theta|y) = \frac{\Gamma(a+n+b+n\bar{y}-n)}{\Gamma(a+n)\Gamma(b+n\bar{y}-n)}\theta^{a+n-1}(1-\theta)^{b+n\bar{y}-n-1} = \frac{\Gamma(a+b+n\bar{y})}{\Gamma(a+n)\Gamma(b+n\bar{y}-n)}\theta^{a+n-1}(1-\theta)^{b+n\bar{y}-n-1}$$

\1b/ ∴ $a_1 = a+n$, $b_1 = b+n\bar{y}-n$ ∴ $\pi(\theta|y) = \frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\theta^{a_1-1}(1-\theta)^{b_1-1}$ for $0<\theta<1$

∴ Z posterior distri is also a beta distri $Beta(a_1, b_1)$

\1c/ Z posterior expectation: $E(\theta|y) = \int_0^1 \theta\pi(\theta|y)\,d\theta =$

$$\int_0^1 \theta\frac{\Gamma(a+b+n\bar{y})}{\Gamma(a+n)\Gamma(b+n\bar{y}-n)}\theta^{a+n-1}(1-\theta)^{b+n\bar{y}-n-1}\,d\theta =$$

$$\frac{\Gamma(a+b+n\bar{y})}{\Gamma(a+n)\Gamma(b+n\bar{y}-n)}\int_0^1 \theta\theta^{a+n-1}(1-\theta)^{b+n\bar{y}-n-1}\,d\theta = \frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\int_0^1\theta\theta^{a_1-1}(1-\theta)^{b_1-1}\,d\theta =$$

$$\frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\int_0^1\theta^{(a_1+1)-1}(1-\theta)^{b_1-1}\,d\theta = \frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\frac{\Gamma(a_1+1)\Gamma(b_1)}{\Gamma(a_1+1+b_1)}\int_0^1\frac{\Gamma(a_1+1+b_1)}{\Gamma(a_1+1)\Gamma(b_1)}\theta^{(a_1+1)-1}(1-\theta)^{b_1-1}\,d\theta$$

$$= \frac{\Gamma(a_1+b_1)\Gamma(a_1+1)\Gamma(b_1)}{\Gamma(a_1)\Gamma(b_1)\Gamma(a_1+1+b_1)} = \frac{(a_1+b_1-1)!\,(a_1)!}{(a_1-1)!\,(a_1+b_1)!} = \frac{a_1}{a_1+b_1}$$

\2a/ by Bayes thm: Z posterior density for θ given by y is

$\therefore \pi(\theta|y) \propto \pi(\theta) f(y|\theta) \propto \pi(\theta)\prod_{i=1}^{n} f(y_i|\theta) = \pi(\theta)\prod_{i=1}^{n}\theta e^{-\theta y_i} \propto \pi(\theta)\theta^n e^{-\theta\sum_{i=1}^{n} y_i} \propto$

$\pi(\theta)\theta^n e^{-\theta n\bar{y}} \propto \theta^{a-1}e^{-b\theta}\theta^n e^{-\theta n\bar{y}} \propto \theta^{a+n-1}e^{-b\theta-\theta n\bar{y}} \propto \theta^{(a+n)-1}e^{-(b+n\bar{y})\theta}$

$\therefore$ comparing this to Z density of Z Gamma(a,b) prior, shows it is proportional to Z density of a Gamma(a_1,b_1) distri, $a_1=a+n$, $b_1=b+n\bar{y}$ & $\therefore$ Z posterior is a Gamma distri with

$\pi(\theta|y) = \frac{b_1^{a_1}}{\Gamma(a_1)}\theta^{a_1-1}e^{-b_1\theta}$ for $\theta>0$ eg $\pi(\theta|y) = \frac{(b+n\bar{y})^{a+n}}{\Gamma(a+n)}\theta^{a+n-1}e^{-(b+n\bar{y})\theta}$

\2b/ taking logs, differentiating & setting to zero, Z MAP esti, $\hat{\theta}$, satisfies Gamma density Gamma$(a,b)=\pi(\theta)=\frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}$ $\theta>0$ $\therefore$

Gamma$(a_1,b_1)=\pi(\theta|y)=\frac{b_1^{a}}{\Gamma(a_1)}\theta^{a_1-1}e^{-b_1\theta}$ for $\theta>0$ $\therefore$

$\ln(\pi(\theta|y)) = \ln\left[\frac{b_1^{a_1}}{\Gamma(a_1)}\theta^{a_1-1}e^{-b_1\theta}\right] = \ln(b_1^{a_1}) - \ln(\Gamma(a_1)) - \ln(\theta^{a_1-1}) + \ln(e^{-b_1\theta}) =$

$a_1\ln(b_1) - \ln(\Gamma(a_1)) + (a_1-1)\ln(\theta) - b_1\theta$ $\therefore$

$\frac{\partial[\ln(\pi(\theta|y))]}{\partial\theta} = 0 - 0 + \frac{a_1-1}{\theta} - b_1 = \frac{a_1-1}{\theta} - b_1$ $\therefore$ $\frac{a_1-1}{\hat{\theta}} - b_1 = 0$ $\therefore$

$\frac{a_1-1}{\hat{\theta}} = b_1$ $\therefore$ $\frac{a_1-1}{b_1} = \hat{\theta}$ is Z MAP esti of $\hat{\theta}$

\3/ prior distri Gamma $\therefore$ Gamma$(a,b) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}$ for $\theta>0$ $\therefore$

$\pi(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}$ for $\theta>0$ $\therefore$ by Bayes thm: posterior $\pi(\theta|y) \propto \pi(\theta)f(y|\theta) \propto$

$\pi(\theta)\prod_{i=1}^{n} f(y_i|\theta) \propto \pi(\theta)\prod_{i=1}^{n}\theta y_i^{-\theta-1} \propto \theta^{a-1}e^{-b\theta}\prod_{i=1}^{n}\theta y_i^{-\theta-1} \propto \theta^{a-1}e^{-b\theta}\theta^n\prod_{i=1}^{n} y_i^{-\theta-1} \propto$

$\theta^{a+n-1}e^{-b\theta}\prod_{i=1}^{n}e^{\ln(y_i^{-\theta-1})} \propto \theta^{a+n-1}e^{-b\theta}\prod_{i=1}^{n}e^{(-\theta-1)\ln y_i} \propto \theta^{a+n-1}e^{-b\theta}e^{\sum_{i=1}^{n}(-\theta-1)\ln y_i} \propto$

$\theta^{(a+n)-1}e^{-b\theta}e^{(-\theta-1)\sum_{i=1}^{n}\ln y_i} \propto \theta^{(a+n)-1}e^{-b\theta}e^{-\theta(\sum_{i=1}^{n}\ln(y_i)) - \sum_{i=1}^{n}\ln y_i} \propto$

$\theta^{(a+n)-1}e^{-b\theta}e^{-\theta\sum_{i=1}^{n}\ln y_i}e^{-\sum_{i=1}^{n}\ln y_i} \propto \theta^{(a+n)-1}e^{-b\theta}e^{-\theta\sum_{i=1}^{n}\ln y_i} \propto \theta^{(a+n)-1}e^{-b\theta-\theta\sum_{i=1}^{n}\ln y_i} \propto$

$\theta^{(a+n)-1}e^{-(b+\sum_{i=1}^{n}\ln y_i)\theta} \propto \theta^{a_1-1}e^{-b_1\theta}$ $\therefore a_1 = a+n$, $b_1 = b+\sum_{i=1}^{n}\ln y_i$ $\therefore$

$\theta|y \sim$ Gamma(a_1,b_1) & $\pi(\theta|y) = \frac{b_1^{a_1}}{\Gamma(a_1)}\theta^{a_1-1}e^{-b_1\theta}$ for $\theta>0$

\1b/ Z posterior density $\pi(\theta|y)$ is proportional to Z Beta density:

$\pi(\theta|y) = \frac{\Gamma(a+b+n\bar{y})}{\Gamma(a+n)\Gamma(b+n\bar{y}-n)}\theta^{a+n-1}(1-\theta)^{b+n\bar{y}-n-1}$ $\therefore$ $\theta|y \sim$ Beta$(a+n, b+n\bar{y}-n)$ $\therefore$

Z posterior distri is also a beta distri Beta(a_1,b_1)

\1d/ $a=1$, $b=9$, $n=30$ $\therefore$ 95% $\therefore$ $1-0.95=0.05$ $\therefore$ $\frac{0.05}{2}=0.025$ $\therefore$ $1-0.025=0.975$

$1-\frac{1-\alpha}{2} = \frac{2}{2}-\frac{1-\alpha}{2} = \frac{1+\alpha}{2}$ $\quad 1-\frac{1+\alpha}{2} = 1-0.975=0.025$ $\therefore$ $q(0.025, 0.955) =$

$(q_{0.025}, q_{0.975}) = (0.00281, 0.336)$ 1d))

\2c/ $a=5$, $b=2$, $n=20$, $\alpha=0.95$ $\therefore$ $(0.025, 0.975)$ $\therefore$ $(q_{0.025}, q_{0.975}) = (0.359, 0.957)$ for Gamma$(5,2)$

Monte carlo $\therefore$ $(1.02, 4.6)$ for param Gamma$(5,2)$

$\therefore$ [illegible]

\Week 1 Sheet\ \4a/ By Bayes thm $P(A|b) = \frac{P(b|A)P(A)}{P(b)}$, Z LoTP, ∴

$P(b) = P(b|A)P(A) + P(b|B)P(B) + P(b|C)P(C)$ as A, B, C for a partition ∴

$P(b|B) = 0$ as Z jailer is truthful, & $P(b|C) = 1$ ∵ Z jailer is truthful & cannot inform Alan he is to be executed ∴ $P(b) =$

$p \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = p \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} = P(b|A)P(A) + P(b|C)P(C) = P(b|A)\frac{1}{3} + P(b|C)\frac{1}{3} = \frac{1}{3}(p+1)$ &

∴ $P(A|b) = x = \frac{P(b|A)P(A)}{P(b)} = \frac{p \cdot \frac{1}{3}}{\frac{1}{3}(p+1)} = \frac{p}{p+1}$

\4b/ Alan argues that telling him Z identity of another that is to be executed gives no info ∴ $P(A) = P(A|b) = x = \frac{1}{3}$ & $x = \frac{p}{p+1}$ ∴ $p = \frac{x}{1-x}$ ∴

$x = \frac{1}{3}$ ∴ $p = \frac{1}{2}$ ∴ if Alan is to go free, Z jailer is equally likely to nominate Bernard or Charles in his conversation with Alan. this is perhaps not unreasonable for Alan to think ie Alan doesn't believe Z jailer is more/less likely to nominate Bernard over Charles

\4c/ Alan actually thinks $P(A|b) = x = \frac{1}{2}$ ∴ $p = 1$ ie even if Alan is to go free, Z jailer would always say Bernard rather than Charles. this seem unreasonable. how could alan be certain that Z jailer would behave like this?

\4d/ as long as coherence isn't violated: $P(b|A) = P(c|A) = \frac{1}{2}$ seems reasonable but another might be $p > \frac{1}{2}$ as Alan feels $p > \frac{1}{2}$ as Alan feels it is more likely that Z jailer gives Z first name he can.

\5a/

	Jacket	Coat
right hand	0.125	0.25
left hand	0.125	0.5

$1 - \frac{3}{4} = \frac{1}{4}$ ∴ $\frac{1}{4} \times \frac{1}{2} = 0.125$ $\quad \frac{3}{4} \times \frac{1}{3} = \frac{1}{4}$ $\quad$ ∴ $\frac{1}{4} \times 2 = \frac{1}{2}$ ∴

prob of phone in Jacket is $\frac{1}{4}$

\5b/ with events J, C, L, R desired eg Z joint distr in table form:

	L	R
J	1/8	1/8
C	1/2	1/4

\5c/ no ∵ it could be elsewhere eg stolen

\6a/ let θ denote events he dies & y event sayer says so as e. ∴ $\pi(\theta|y) = 0.92$ ∴ he should be concerned

\6b/ shouldn't believe sayer unless evidence they were accurate that updates a prior on sayer ability beyond random chance. in absence of evidence view sayer as guessing ∴ $P(y|\theta=1) = P(y|\theta=0) = 0.5$ ∴ $\pi(\theta|y) = 0.94 = \pi(\theta)$ makes sense as under Z view that saying is guessing, here "data" shouldn't change your prior. ∴ have a prior on sayers skill & then view 2 brochure numbers as datas before hearing 2 prediction will require hierarchical modelling

\week2 sheet/\1a/ for normal prior distri for $\mu=m, \sigma^2=v^2$:

$\pi(\mu=m, \sigma^2=v^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2} = \frac{1}{v\sqrt{2\pi}} e^{-\frac{1}{2v^2}(x-m)^2}$ ∴

by bayes thm $\pi(\theta|y) = \pi(\mu=m, \sigma^2=v^2|y) \propto \pi(\theta) f(y|\theta) \propto \pi(\mu=m, \sigma^2=v^2) f(y|\theta) \propto$
$\pi(\mu=m, \sigma^2=v^2) f(y|\mu, \sigma^2)$ ∴

normal prior dist. i for μ, σ^2 with density $\pi(\mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(x-\mu)^2}$ ∴

need posterior distri $\mu, \sigma^2|y$ ∴ by bayes thm $\pi(\theta|y) \propto \pi(\theta) f(y|\theta)$ ∴

$\pi(\mu, \sigma^2|y) \propto \pi(\mu, \sigma^2) f(y|\mu, \sigma^2)$ & $\mu=m, \sigma^2=v^2$ ∴ for $\theta=(\mu, \sigma^2)$ ∴

$\pi(\theta) = \pi(\mu, \sigma^2) = \frac{1}{v\sqrt{2\pi}} e^{-\frac{1}{2v^2}(\theta-m)^2}$, $f(y|\mu, \sigma^2) = \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2} =$

$\left(\frac{1}{\sigma\sqrt{2\pi}}\right)^n e^{\sum_{i=1}^{n}(-\frac{1}{2\sigma^2}(y_i-\mu)^2} = \prod_{i=1}^{n} f(y_i|\theta) = \prod_{i=1}^{n} f(y_i|\mu, \sigma^2)$ ∴ by bayes thm:

$\pi(\theta|y) \propto \pi(\theta) f(y|\theta) \propto \frac{1}{v\sqrt{2\pi}} e^{-\frac{1}{2v^2}(\theta-m)^2} f(y|\theta) \propto e^{-\frac{1}{2v^2}(\theta-m)^2} f(y|\theta) \propto$

$e^{-\frac{1}{2v^2}(\theta-m)^2} f(y|\theta) \propto \cancel{\frac{1}{v\sqrt{2\pi}}} e^{-\frac{1}{2v^2}(\theta-m)^2} f(y|\mu, \sigma^2) \propto e^{-\frac{1}{2v^2}(\theta-m)^2} \prod_{i=1}^{n} f(y_i|\mu, \sigma^2) \propto$

$e^{-\frac{1}{2v^2}(\theta-m)^2} \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2} \propto e^{-\frac{1}{2v^2}(\theta-m)^2} \prod_{i=1}^{n} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2} \propto$

$e^{-\frac{1}{2v^2}(\theta-m)^2} e^{\sum_{i=1}^{n}[-\frac{1}{2\sigma^2}(y_i-\mu)^2]} \propto e^{-\frac{1}{2v^2}(\theta-m)^2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2} \propto$

$\exp\left[-\frac{1}{2}\left(\frac{1}{v^2}(\theta-m)^2 + \frac{1}{\sigma^2}\sum_{i=1}^{n}(y_i^2 - 2\mu y_i + \mu^2)\right)\right] \propto \exp\left[-\frac{1}{2}\left(\frac{1}{v^2}(\theta^2 - 2m\theta + m^2) + \frac{1}{\sigma^2}\left(\sum_{i=1}^{n} y_i - 2\mu\sum_{i=1}^{n} y_i + \sum_{i=1}^{n}\mu^2\right)\right)\right]$

$\propto \exp\left[-\frac{1}{2}\left(\frac{1}{v^2}(\theta - 2m\theta) + \frac{1}{\sigma^2}(-2n\bar{y}\mu + n\mu^2)\right)\right]$ ∴ $\pi(\mu|y) \sim N(\mu_n, \sigma_n^2)$ with

$\sigma_n^2 = \frac{1}{\frac{1}{v^2}+\frac{n}{\sigma^2}}$, $\mu_n = \frac{\frac{m}{v^2}+\frac{n\bar{y}}{\sigma^2}}{\frac{1}{v^2}+\frac{n}{\sigma^2}}$ ∴ $\pi(\mu, y) \sim N(\mu_n, \sigma_n^2) = N\left(\frac{\frac{m}{v^2}+\frac{n\bar{y}}{\sigma^2}}{\frac{1}{v^2}+\frac{n}{\sigma^2}}, \frac{1}{\frac{1}{v^2}+\frac{n}{\sigma^2}}\right)$

posterior distri for μ: $\pi(\mu|y)$ ∴ by bayes thm $\pi(\mu|y) \propto \pi(\mu) f(y|\mu) \propto$
$\frac{1}{v\sqrt{2\pi}} e^{-\frac{1}{2v^2}(\mu-m)^2} f(y|\mu) \propto e^{-\frac{1}{2v^2}(\mu-m)^2} f(y|\mu) \propto e^{-\frac{1}{2v^2}(\mu-m)^2} \prod_{i=1}^{n} f(y_i|\mu) \propto$

$e^{-\frac{1}{2v^2}(\mu-m)^2} \prod_{i=1}^{n} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2} \propto e^{-\frac{1}{2v^2}(\mu-m)^2} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2} \propto$

$\exp\left[-\frac{1}{2v^2}(\mu^2 - 2m\mu + m^2) + \frac{1}{2\sigma^2}\left(\sum_{i=1}^{n}(y_i^2 - 2y_i\mu + \mu^2)\right)\right] \propto \exp\left[-\frac{1}{2}\left(\frac{1}{v^2}(\mu^2 - 2m\mu) + \frac{1}{\sigma^2}(-2n\bar{y} + n\mu^2)\right)\right] \propto$

$\exp\left[-\frac{1}{2}\left(\frac{1}{v^2}\mu^2 - \frac{2m}{v^2}\mu - \frac{2n\bar{y}}{\sigma^2}\mu + \frac{n}{\sigma^2}\mu^2\right)\right] \propto \exp\left[-\frac{1}{2}\left(\left[\frac{1}{v^2}+\frac{n}{\sigma^2}\right]\mu^2 - 2\left[\frac{m}{v^2}+\frac{n\bar{y}}{\sigma^2}\right]\mu\right)\right] =$

$\exp\left[-\frac{1}{2}(a\mu^2 - 2b\mu)\right] \propto \exp\left[-\frac{a}{2}(\mu^2 - 2\frac{b}{a}\mu)\right] \propto \exp\left[-\frac{a}{2}\left((\mu-\frac{b}{a})^2 - \frac{b^2}{a^2}\right)\right] \propto \exp\left[-\frac{a}{2}(\mu-\frac{b}{a})^2\right] \propto$

$\exp\left[-\frac{1}{2a^{-1}}(\mu-\frac{b}{a})^2\right]$ ∴ $\mu|y \sim N(\frac{b}{a}, a^{-1})$ ∴ $\pi(\mu|y)$ is proportional to a normal density

∴ $\pi(\mu|y) = \frac{1}{a^{-\frac{1}{2}}\sqrt{2\pi}} e^{-\frac{1}{2a^{-1}}(\mu-\frac{b}{a})^2} = \frac{1}{\sigma_n\sqrt{2\pi}} \exp\left(-\frac{1}{2\sigma_n^2}(\mu-\mu_n)^2\right)$ ∴

$\pi(\mu|y) \sim N(\mu_n, \sigma_n^2) = N\left(\frac{\frac{m}{v^2}+\frac{n\bar{y}}{\sigma^2}}{\frac{1}{v^2}+\frac{n}{\sigma^2}}, \frac{1}{\frac{1}{v^2}+\frac{n}{\sigma^2}}\right)$ ∵ $\mu_n = \frac{\frac{m}{v^2}+\frac{n\bar{y}}{\sigma^2}}{\frac{1}{v^2}+\frac{n}{\sigma^2}}$, $\sigma_n^2 = \frac{1}{\frac{1}{v^2}+\frac{n}{\sigma^2}}$ ∴

$a = \frac{1}{v^2} + \frac{n}{\sigma^2}$, $b = \frac{m}{v^2} + \frac{n\bar{y}}{\sigma^2}$

posterior: $\pi(\theta|y) \propto \pi(\theta)p(y|\theta)$ $\pi(\theta|y) \propto \pi(\theta)p(y|\theta)$
$\pi(\theta|y) \propto \pi(\theta)p(y|\theta)$ $\pi(\theta|y) \propto \pi(\theta)p(y|\theta)$ $\pi(\theta|y) \propto \pi(\theta)p(y|\theta)$

\\Week 2/ \1b/ $N(\mu_n, \sigma_n^2)$ $\sigma^2=2$, $m=1.75$, $v^2=0.5$ ∴ $N(\mu_n, \sigma_n^2)$ has 95% credible interval $(1.592775, 3.074368)$ where $z_{0.975}=1.96$ is Z 97.5th quantile of Z standard normal distri. for Z numbers in Z table: $n=10$, $\bar{y}=2.567$ ∴ $\sigma_n^2 = \frac{1}{(2+5)} = \frac{1}{7}$, & $\mu_n = (1.75/0.5 + 25.67/2)/7 = 2.33$ ∴ 95% credible interval is $2.33 \pm \frac{1.96}{\sqrt{7}} = [1.59, 3.07]$

\1c/ $N(\mu_n, \sigma_n^2)$ ∴ $P(\mu>2 \mid y) = 0.811$ ∴ $1-P(\mu<2 \mid y)$ ∴ $P(\mu>2 \mid y) = 0.809$

\1d/ for $m=2$ ∴ $m_2=2$ ∴ cred interval $(1.66, 3.15)$ & ~~P(~~ $P(\mu>2 \mid y) = 0.858$ under Z alternative prior for μ, σ_n is unchanged & $\mu_n = 2.41$ ∴ Z 95% confidence interval is $[1.67, 3.15]$ & $P(\mu>2 \mid y) = 0.861$

\2a/ $\theta \sim Beta(a,b)$ ∴ Beta prior distri for θ with density ∴ $\pi(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1}(1-\theta)^{b-1}$, $y_i \mid \theta \sim Geo(\theta)$ ∴ $P(y_i \mid \theta) = \theta(1-\theta)^{y_i-1}$ ∴ By Bayes thm: $\pi(\theta \mid y) \propto \pi(\theta) P(y \mid \theta) \propto \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1}(1-\theta)^{b-1} P(y \mid \theta) \propto \theta^{a-1}(1-\theta)^{b-1} P(y \mid \theta) \propto \theta^{a-1}(1-\theta)^{b-1} \prod_{i=1}^{n} P(y_i \mid \theta) \propto \theta^{a-1}(1-\theta)^{b-1} \prod_{i=1}^{n} \theta(1-\theta)^{y_i-1}$
$\propto \theta^{a-1}(1-\theta)^{b-1} \theta^n (1-\theta)^{\sum_{i=1}^{n}(y_i-1)} \propto \theta^{a+n-1}(1-\theta)^{b-1}(1-\theta)^{n\bar{y}-n} \propto \theta^{(a+n)-1}(1-\theta)^{(b+n\bar{y}-n)-1}$

$g(\theta) = \phi$: $\pi_\phi(\phi) = \left| \det\left(\frac{\partial g^{-1}(\phi)}{\partial \phi}\right) \right| \pi_\theta(g^{-1}(\phi))$

$\phi = \frac{\theta}{1-\theta}$ ∴ $\theta = \frac{\phi}{1+\phi} = g^{-1}(\phi) = h(\phi)$ ∵ $\phi(1-\theta) = \theta = \phi - \theta\phi$ ∴ $\theta + \theta\phi = \phi = \theta(1+\phi)$ ∴ $\frac{\phi}{1+\phi} = \theta$ ∴

$\theta = \frac{\phi}{1+\phi} = g^{-1}(\phi) = h(\phi)$ ∴ $\frac{\partial \theta}{\partial \phi} = \frac{(1+\phi)1 - \phi(1)}{(1+\phi)^2} = \frac{1+\phi-\phi}{(1+\phi)^2} = \frac{1}{(1+\phi)^2}$ ∵ change of variabs formula for $\phi = g(\theta)$: $\pi_\phi(\phi) = \left| \det\left(\frac{\partial g^{-1}(\phi)}{\partial \phi}\right) \right| \pi_\theta(g^{-1}(\phi))$ ∴ using Z change of variabs formula have prior: $\pi(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \theta^{a-1}(1-\theta)^{b-1} = \pi_\theta(g^{-1}(\phi)) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} (g^{-1}(\phi))^{a-1}(1-g^{-1}(\phi))^{b-1}$
$= \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \left(\frac{\phi}{1+\phi}\right)^{a-1} \left(1-\frac{\phi}{1+\phi}\right)^{b-1}$ ∴ $\pi_\phi(\phi) = \pi_\theta(g^{-1}(\phi)) \left| \det\left(\frac{\partial g^{-1}(\phi)}{\partial \phi}\right) \right| = \pi_\theta(g^{-1}(\phi)) \left| \det\left(\frac{\partial \theta}{\partial \phi}\right) \right|$

$= \pi_\theta(\theta) \left| \det\left(\frac{1}{(1+\phi)^2}\right) \right| = \pi_\theta(\theta) \left| \frac{1}{(1+\phi)^2} \right| = \pi_\theta(\theta) \frac{1}{(1+\phi)^2} = \frac{1}{(1+\phi)^2} \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \left(\frac{\phi}{1+\phi}\right)^{a-1} \left(1-\frac{\phi}{1+\phi}\right)^{b-1}$
$= \pi_\phi(\phi) = \pi(h(\phi)) \left| \frac{\partial \theta}{\partial \phi} \right| = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \left(\frac{\phi}{1+\phi}\right)^{a-1} \left(1-\frac{\phi}{1+\phi}\right)^{b-1} \frac{1}{(1+\phi)^2} =$
$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \phi^{a-1} \left(\frac{1}{1+\phi}\right)^{a-1} \left(\frac{1}{1+\phi}\right)^{b-1} \left(\frac{1}{1+\phi}\right)^2 = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \phi^{a-1} \left(\frac{1}{1+\phi}\right)^{a-1+b-1+2} =$
$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \phi^{a-1} \left((1+\phi)^{-1}\right)^{a+b} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} \phi^{a-1}(1+\phi)^{-a-b}$ for $\phi > 0$ ∴

By Bayes thm: $\pi(\phi \mid y) \propto \pi(\phi) P(y \mid \phi) \propto \frac{\Gamma(a+b)\phi^{a-1}}{\Gamma(a)\Gamma(b)} (1+\phi)^{-a-b} P(y \mid \phi) \propto$
$\phi^{a-1}(1+\phi)^{-a-b} P(y \mid \phi) \propto \phi^{a-1}(1+\phi)^{-a-b} \prod_{i=1}^{n} P(y_i \mid \phi) \propto$
$\phi^{a-1}(1+\phi)^{-a-b} \prod_{i=1}^{n} \phi(1-\phi)^{y_i-1} \propto \phi^{a-1}(1+\phi)^{-a-b} \phi^n (1-\phi)^{\sum_{i=1}^{n}(y_i-1)} \propto$
$\phi^{a-1}(1+\phi)^{-a-b} \phi^n (1-\phi)^{n\bar{y}-n} \propto \phi^{(a+n)-1}(1+\phi)^{(-a-b+n\bar{y}-n+1)-1}$ ∴ a similar argument yields Z posterior $\pi_\phi(\phi \mid x) = \frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)} \phi^{a_1-1}(1+\phi)^{-a_1-b_1}$ for $\phi > 0$ ∴ $\pi(\phi \mid y) \propto$
$\phi^{(a+n)-1}(1+\phi)^{(-a-b+n\bar{y}-n+1)-1}$ ∴ posterior distri is beta distri :

$\phi | y \sim Beta(a+n, -a-b+n\bar{y}-n+1)$ & posterior density $\pi(\phi|y) =$

$\frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\phi^{(a+n)-1}(1+\phi)^{-(a+n)-(b-n\bar{y})} = \frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\phi^{a_1-1}(1+\phi)^{-a_1-b_1}$ for $\phi > 0$

\3a/ $\phi = \frac{1}{\theta} \therefore \theta = \frac{1}{\phi} = g^{-1}(\phi) = h(\phi) = \phi^{-1} \therefore \frac{\partial\theta}{\partial\phi} = -\phi^{-2} = -\frac{1}{\phi^2} \therefore$

$\pi_\phi(\phi) = |\det(\frac{\partial g^{-1}(\phi)}{\partial\phi})|\pi_\theta(g^{-1}(\phi)) \therefore \pi_\theta(\theta) = \pi_\theta(g^{-1}(\phi)) = \cancel{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}(g^{-1}(\phi))^{a-1}(1-g^{-1}(\phi))^{b-1}} =$

$\cancel{\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}}\ \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta} = \frac{b^a}{\Gamma(a)}(g^{-1}(\phi))^{a-1}e^{-b(g^{-1}(\phi))} = \frac{b^a}{\Gamma(a)}(\phi^{-1})^{a-1}e^{-b(\phi^{-1})} \therefore$

$\pi_\phi(\phi) = \frac{b^a}{\Gamma(a)}\phi^{-a+1}e^{-b\phi^{-1}}(-\phi^{-2}) = \frac{b^a}{\Gamma(a)}\phi^{-a-1}e^{-b\phi^{-1}} \therefore$

$f(y_i|\phi) = \phi e^{-\phi y_i}$ $y_i > 0 \therefore$ by Bayes thm:

$\pi(\phi|y) \propto \pi(\phi)f(y|\phi) \propto \frac{b^a}{\Gamma(a)}\phi^{-a-1}e^{-b\phi^{-1}}f(y|\phi) \propto \phi^{-a-1}e^{-b\phi^{-1}}\phi^n e^{-n\bar{y}\phi} \propto$

$\phi^{-a+n-1}e^{-b\frac{1}{\phi}-n\bar{y}\phi} \therefore$

$\phi = \frac{1}{\theta} \therefore \theta = \frac{1}{\phi} = g^{-1}(\phi) = h(\phi)$ & $\therefore \frac{\partial\theta}{\partial\phi} = -\frac{1}{\phi^2} \therefore$ Z prior:

$\pi_\phi(\phi) = \pi(h(\phi))|\frac{\partial\theta}{\partial\phi}| = \frac{b^a}{\Gamma(a)}\phi^{1-a}e^{-b\phi}\phi^{-2} = \frac{b^a}{\Gamma(a)}\phi^{-a-1}e^{-b/\phi}$ for $\phi > 0 \therefore$

a similar argument : Z posterior $\pi_\phi(\phi) = \frac{b_1^{a_1}}{\Gamma(a_1)}\phi^{-a_1-1}e^{-b_1/\phi}$ for $\phi > 0 \therefore$

$f(y_i|\phi) = \phi e^{-\phi y_i} \therefore$ By Bayes thm:

$\pi(\phi|y) \propto \pi(\phi)f(y|\phi) \propto \pi_\phi(\phi)f(y|\phi) \propto \frac{b^a}{\Gamma(a)}\phi^{-a-1}e^{-b/\phi}f(y|\phi) \propto$

$\phi^{-a-1}e^{-b/\phi}f(y|\phi) \propto \phi^{-a-1}e^{-b/\phi}\prod_{i=1}^n f(y_i|\phi) \propto \phi^{-a-1}e^{-b/\phi}\prod_{i=1}^n \phi e^{-\phi y_i} \propto$

$\phi^{-a-1}e^{-b/\phi}\phi^n\prod_{i=1}^n e^{-\phi y_i} \propto \phi^{-a+n-1}e^{-b/\phi}e^{\sum_{i=1}^n(-\phi y_i)} \propto \phi^{-a+n-1}e^{-b/\phi}\frac{n}{n}\sum_{i=1}^n y_i \propto$

$\phi^{-(a-n)-1}e^{-b/\phi}e^{-n\bar{y}\phi} \propto \phi^{-(a-n)-1}e^{-b/\phi-n\bar{y}\phi} \propto \phi^{-(a-n)-1}e^{-(b/\phi+n\bar{y}\phi^2/\phi)} \propto$

$\phi^{-(a-n)-1}e^{-(b+n\bar{y}\phi^2)/\phi}$ for $\phi > 0 \therefore$

$\pi(\phi|y) \propto \phi^{-(a-n)-1}e^{-b/\phi} \propto \phi^{-a_1-1}e^{-b_1/\phi}$ for $\phi > 0$ with $a_1 = a-n \therefore$

posterior $\pi_\phi(\phi) = \frac{b_1^{a_1}}{\Gamma(a_1)}\phi^{a_1-1}e^{b_1/\phi}$ for $\phi > 0$

\3b/ Expected info $J(\theta) = E[-\frac{\partial^2 l}{\partial\theta^2}] \therefore$ Jeffreys prior is $J(\theta)^{1/2} \therefore$

$y_i|\theta \sim Exp(\theta) = \theta e^{-\theta y_i} \therefore L(\theta) = \prod_{i=1}^n \theta e^{-\theta y_i} = \theta^n e^{-n\bar{y}\theta} \therefore l(\theta) = \ln(L(\theta)) =$

$\ln(\theta^n e^{-n\bar{y}\theta}) = \ln(\theta^n) + \ln(e^{-n\bar{y}\theta}) = n\ln\theta - n\bar{y}\theta \therefore \frac{\partial l}{\partial\theta} = \frac{\partial}{\partial\theta}(n\ln\theta - n\bar{y}\theta) = n\frac{1}{\theta} - n\bar{y} \therefore$

$\frac{\partial^2 l}{\partial\theta^2} = \frac{\partial}{\partial\theta}(n\frac{1}{\theta} - n\bar{y}) = -n\frac{1}{\theta^2} \therefore J(\theta) = -E(-n\frac{1}{\theta^2}) = n\frac{1}{\theta^2}E(1) = \frac{n}{\theta^2} \therefore$

Jeffreys prior is $J(\theta)^{1/2} = (\frac{n}{\theta^2})^{1/2} = \sqrt{\frac{n}{\theta^2}} = \frac{\sqrt{n}}{\theta}$ is Jeffreys prior for θ

~~3b~~ let $l(\theta)$ denote loglihood $\therefore l(\theta) = c + n\ln(\theta) - \theta n\bar{y} \therefore l'(\theta) = \frac{n}{\theta} - n\bar{y} \therefore$

$l''(\theta) = -\frac{n}{\theta^2}$ $\therefore$ as y is not involved : $J(\theta) = \frac{n}{\theta^2}$ & $\therefore$ Z Jeffreys prior is

$\pi_J(\theta) \propto \frac{1}{\theta}$ $\therefore$ Jeffrey's prior is Z distri with density proportional to $J(\theta)^{1/2}$

& $\cancel{J(\theta)^{1/2} = \frac{\sqrt{n}}{\theta}}$ $J(\theta)^{1/2} = \frac{\sqrt{n}}{\theta} \therefore \pi_J(\theta) \propto J(\theta)^{1/2} = \frac{\sqrt{n}}{\theta} \propto \frac{1}{\theta} \therefore \pi_J(\theta) \propto \frac{1}{\theta}$

\Week 2 Sheet/ \3 c/ $y_i|\phi \sim Exp(\phi) = \phi e^{-\phi y_i}$ $\therefore L(\phi) = \phi^n e^{-n\bar{y}\phi}$ $\therefore$

$l(\phi) = n\ln\phi - n\bar{y}\phi$ $\therefore \frac{\partial l}{\partial\phi} = \frac{n}{\phi} - n\bar{y}$ $\therefore \frac{\partial^2}{\partial\phi^2} = -n/\phi^2$ $\therefore$ $J(\phi) = -E(-\frac{n}{\phi^2}) =$

$\frac{n}{\phi^2}E(1) = \frac{n}{\phi^2}$ $\therefore J(\phi)^{1/2} = \frac{\sqrt{n}}{\phi}$ $\therefore$ $\pi_J(\phi) \propto \frac{1}{\phi}$ $\therefore$ Similar argument $\therefore$

$\pi_J(\theta) \propto \frac{1}{\theta}$

\4a/ $p(y) = \int_{-\infty}^{\infty} p(y|\theta)\pi(\theta)d\theta$ $y_i|\theta \sim Bin(n,\theta)$ $\therefore$ $p(y_i|\theta) = \binom{n}{y_i}\theta^{y_i}(1-\theta)^{n-y_i}$ $\therefore$

$p(y|\theta) = \prod_{i=1}^{n} p(y_i|\theta) = \prod_{i=1}^{n}\binom{n}{y_i}\theta^{y_i}(1-\theta)^{n-y_i} = \prod_{i=1}^{n}\binom{n}{y_i}\theta^{\sum_{i=1}^{n} y_i}(1-\theta)^{\sum_{i=1}^{n}(n-y_i)} =$

$\prod_{i=1}^{n}\binom{n}{y_i}\theta^{n\bar{y}}(1-\theta)^{n^2-n\bar{y}}$ $\therefore$ By Bayes thm:

$\pi(\theta|y) \propto \pi(\theta)p(y|\theta)$ $\therefore$ prior predictive distri: $p(y) = \int_{-\infty}^{\infty} p(y|\theta)\pi(\theta)d\theta$ $\therefore$

$p(\tilde{y}) = \int_0^1 p(y|\theta)p(\theta)d\theta$ $\therefore$ $p(y|\theta) = \prod_{i=1}^{n}\left(\frac{n!}{(n-y_i)!\,y_i!}\right)\theta^{n\bar{y}}(1-\theta)^{n^2-n\bar{y}} =$

$\cancel{(n!)}$ $(n!)^n\left(\prod_{i=1}^{n}\frac{1}{(n-y_i)!\,y_i!}\right)(\theta^{(n\bar{y})})^n((1-\theta)^{n^2-n\bar{y}})^n$

$\theta|y \sim Beta(\alpha+N\bar{y}, \beta+N(n-\bar{y}))$ $\therefore$ $y_i \sim Bin(n,\theta)$ $\therefore$ $y_i|\theta \sim Bin(n,\theta)$ $\therefore$

$p(y_i|\theta) = \binom{N}{y_i}\theta^{y_i}(1-\theta)^{N-y_i}$ for $i = 0, 1, \ldots, N$ $\therefore$ $p(y|\theta) = \prod_{i=1}^{n} p(y_i|\theta) = \prod_{i=1}^{n}\binom{n}{y_i}\theta^{y_i}(1-\theta)^{n-y_i}$

$\propto \theta^{N\bar{y}}(1-\theta)^{N(n-\bar{y})}$ $\therefore$ $\pi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$ $\therefore$

$\pi(\theta|y) \propto \pi(\theta)p(y|\theta) \propto \pi(\theta)\theta^{N\bar{y}}(1-\theta)^{N(n-\bar{y})} \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}\theta^{N\bar{y}}(1-\theta)^{N(n-\bar{y})} \propto$

$\theta^{(\alpha+N\bar{y})-1}(1-\theta)^{(\beta+N(n-\bar{y})-1}$ $\therefore$ $\pi(\theta|y) = \frac{\Gamma(\alpha+N\bar{y}+\beta+N(n-\bar{y}))}{\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))}\theta^{\alpha+N\bar{y}-1}(1-\theta)^{\beta+N(n-\bar{y})-1}$

$\theta|y \sim Beta(\alpha+N\bar{y}, \beta+N(n-\bar{y}))$ is Z posterior distri $\therefore$

$p(\tilde{y}) = \int_0^1 p(y|\theta)\pi(\theta)d\theta$ $\therefore$ $\tilde{y}|y$ is Beta Binomial & $p(\tilde{y}|y) =$

$(\Gamma(n+1)\Gamma(\alpha+\beta+Nn)\Gamma(\alpha+N\bar{y}+\tilde{y})\Gamma(n+\beta+N(n-\bar{y})-\tilde{y}))/(\Gamma(\tilde{y}+1)\Gamma(n-\tilde{y}+1)\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))\Gamma(\alpha+\beta+n+Nn))$

for $\tilde{y} = 0, 1, \ldots, n$ $\therefore$ $\pi(\theta|y) = \frac{\Gamma(\alpha+\beta+Nn)}{\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))}\theta^{\alpha+N\bar{y}-1}(1-\theta)^{\beta+N(n-\bar{y})-1}$ $\therefore$

$p(y) = \int_0^1 p(y|\theta)\pi(\theta)d\theta$ $\therefore$ posterior predictive $p(\tilde{y}|y) = \int_0^1 p((\tilde{y}|y)|\theta)p(\theta)$ $\therefore$ Bayes:

$\pi(\theta|y) \propto \pi(\theta)p(y|\theta)$ $\therefore$ $\theta|y \sim Beta(\alpha+N\bar{y}, \beta+N(n-\bar{y}))$ $\therefore$ $\tilde{y}|y \sim Beta(\alpha+N\bar{y}, \beta+N(n-\bar{y}))$ $\therefore$

$\pi(\tilde{y}|y) = \frac{\Gamma(\alpha+\beta+Nn)}{\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))}\tilde{y}^{\alpha+N\bar{y}-1}(1-\tilde{y})^{\beta+N(n-\bar{y})-1}$ $\therefore$

\4a/ By Bayes thm $\pi(\theta|y) \propto \pi(\theta)p(y|\theta)$ θ has a Beta prior distri $Beta(\alpha,\beta)$

$\therefore$ $\pi(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}$ for $0<\theta<1$ y is binomially distri'd $\therefore$ $p(y_i|\theta) =$

$\binom{n}{y_i}\theta^{y_i}(1-\theta)^{n-y_i} = \frac{n!}{(n-y_i)!\,y_i!}\theta^{y_i}(1-\theta)^{n-y_i}$ $\therefore$ $\pi(\theta|y) \propto \pi(\theta)p(y|\theta) \propto$

$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}p(y|\theta) \propto \theta^{a-1}(1-\theta)^{b-1}p(y|\theta) \propto \theta^{a-1}(1-\theta)^{b-1}\prod_{i=1}^{n}p(y_i|\theta) \propto$

$\theta^{a-1}(1-\theta)^{b-1}\prod_{i=1}^{n}\frac{n!}{(n-y_i)!\,y_i!}\theta^{y_i}(1-\theta)^{n-y_i} \propto \theta^{a-1}(1-\theta)^{b-1}\prod_{i=1}^{n}\theta^{y_i}(1-\theta)^{n-y_i} \propto$

$\theta^{a-1}(1-\theta)^{b-1}\theta^{\sum_{i=1}^{n}y_i}(1-\theta)^{\sum_{i=1}^{n}(n-y_i)} \propto \theta^{a-1}(1-\theta)^{b-1}\theta^{n\bar{y}}(1-\theta)^{n^2-n\bar{y}} \propto \theta^{(a+n\bar{y})-1}(1-\theta)^{(b+n^2-n\bar{y})-1} \propto$

$\theta^{a_1-1}(1-\theta)^{b_1-1}$ with $a_1 = a+n\bar{y}$, $b_1 = b+n^2-n\bar{y}$ $\therefore$ posterior distri is Beta distri

Beta (a_1, b_1) & posterior density $\pi(\theta|\mathbf{y}) = \frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\theta^{a_1-1}(1-\theta)^{b_1-1}$ $\therefore$

$P(y) = \int_0^1 \binom{n}{y}\theta^y(1-\theta)^{n-y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}d\theta = \int_0^1 \frac{n!}{(n-y)!\,y!}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}d\theta =$

$\int_0^1 \frac{\Gamma(n+1)}{\Gamma(n-y+1)\Gamma(y+1)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}d\theta = \frac{\Gamma(n+1)}{\Gamma(y+1)\Gamma(n-y+1)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}d\theta$

$\therefore P(y) = \frac{\Gamma(n+1)}{\Gamma(y+1)\Gamma(n-y+1)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+y)\Gamma(n+\beta-y)}{\Gamma(\alpha+\beta+n)}$ $\quad\theta \sim \text{Beta}(\alpha, \beta)$ $\therefore$

$P(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$ for $0 \le \theta \le 1$ ~~$y|\theta$~~ $y|\theta$ is binomially distried $\therefore$

$y|\theta \sim \text{Bin}(n, \theta)$; $P(y|\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$ for $y = 0, 1, ---, n$ $\therefore$ prior predictive

distri $P(y) = \int_0^1 P(y|\theta)P(\theta)d\theta$ $\because$ By Bayes thm $\pi(\theta|\mathbf{y}) \propto P(\mathbf{y}|\theta)\pi(\theta)$ $\therefore$

$P(y) = \int_{-\infty}^{\infty}P(y|\theta)\pi(\theta)d\theta = \int_0^1 P(y|\theta)P(\theta)d\theta = \int_0^1\binom{n}{y}\theta^y(1-\theta)^{n-y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}d\theta =$

$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\binom{n}{y}\int_0^1\theta^y(1-\theta)^{n-y}\theta^{\alpha-1}(1-\theta)^{\beta-1}d\theta = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{n!}{(n-y)!\,y!}\int_0^1\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}d\theta =$

$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(n+1)}{\Gamma(n-y+1)\Gamma(y+1)}\frac{\Gamma(y+\alpha)\Gamma(n-y+\beta)}{\Gamma(y+\alpha+n-y+\beta)}\int_0^1\frac{\Gamma(y+\alpha+n-y+\beta)}{\Gamma(y+\alpha)\Gamma(n-y+\beta)}\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}d\theta =$

$\frac{\Gamma(n+1)}{\Gamma(y+1)\Gamma(n-y+1)}\frac{\Gamma(\alpha+y)\Gamma(n+\beta-y)}{\Gamma(\alpha+\beta+n)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$ $\because F(y=\infty) = F(\infty) = F(1) = P(y \le \infty) = P(y \le 1) = 1$

4b $y_i|\theta \sim \text{Bin}(n, \theta)$ $\therefore P(y_i|\theta) = \binom{n}{y_i}\theta^{y_i}(1-\theta)^{n-y_i}$ $\therefore P(\mathbf{y}|\theta) = \prod_{i=1}^n P(y_i|\theta) =$

$\prod_{i=1}^n\binom{n}{y_i}\theta^{y_i}(1-\theta)^{n-y_i} = \prod_{i=1}^n\binom{n}{y_i}\theta^{\sum_{i=1}^n y_i}(1-\theta)^{\sum_{i=1}^n(n-y_i)} = \prod_{i=1}^n\binom{n}{y_i}\theta^{n\bar{y}}(1-\theta)^{n^2-n\bar{y}}$ $\therefore$ by Bayes thm

$\pi(\theta|\mathbf{y}) \propto \pi(\theta)P(\mathbf{y}|\theta)$ prior predictive distri $P(y) = \int_{-\infty}^{\infty}P(y|\theta)\pi(\theta)d\theta$ $\therefore$

$P(\tilde{y}) = \int_0^1 P(\mathbf{y}|\theta)P(\theta)d\theta$ $\therefore P(\mathbf{y}|\theta) = \prod_{i=1}^n\left(\frac{n!}{(n-y_i)!\,y_i!}\right)\theta^{n\bar{y}}(1-\theta)^{n^2-n\bar{y}} = \frac{(n!)^n}{((n-y_i)!)^n(y_i!)^n}(\theta^{n\bar{y}})^n((1-\theta)^{n^2-n\bar{y}})^n$

$\theta|\mathbf{y} \sim \text{Beta}(\alpha+N\bar{y}, \beta+N(n-\bar{y}))$ $\therefore y_i \sim \text{Bin}(n, \theta)$ $\therefore y_i|\theta \sim \text{Bin}(n, \theta)$ $\therefore$

$P(y_i|\theta) = \binom{N}{y_i}\theta^{y_i}(1-\theta)^{N-y_i}$ for $i = 0, 1, ---, N$ $\therefore P(\mathbf{y}|\theta) = \prod_{i=1}^N\binom{n}{y_i}\theta^{y_i}(1-\theta)^{n-y_i} \propto$

$\theta^{N\bar{y}}(1-\theta)^{N(n-\bar{y})}$ $\therefore \pi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$ $\therefore$ by Bayes thm:

$\pi(\theta|\mathbf{y}) \propto \pi(\theta)P(\mathbf{y}|\theta) \propto \pi(\theta)\theta^{N\bar{y}}(1-\theta)^{N(n-\bar{y})} \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}\theta^{N\bar{y}}(1-\theta)^{N(n-\bar{y})} \propto$

$\theta^{(\alpha+N\bar{y})-1}(1-\theta)^{(\beta+N(n-\bar{y}))-1}$ $\therefore \pi(\theta|\mathbf{y}) = \frac{\Gamma(\alpha+N\bar{y}+\beta+N(n-\bar{y}))}{\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))}\theta^{\alpha+N\bar{y}-1}(1-\theta)^{\beta+N(n-\bar{y})-1}$ $\therefore$

$\theta|\mathbf{y} \sim \text{Beta}(\alpha+N\bar{y}, \beta+N(n-\bar{y}))$ is Z posterior distri $\therefore P(\tilde{y}) = \int_0^1 P(\mathbf{y}|\theta)\pi(\theta)d\theta$ $\therefore$

$\tilde{y}|\mathbf{y}$ is Betabinomial & $P(\tilde{y}|\mathbf{y}) = \frac{\Gamma(n+1)\Gamma(\alpha+\beta+Nn)\Gamma(\alpha+N\bar{y}+\tilde{y})\Gamma(n+\beta+N(n-\bar{y})-\tilde{y})}{\Gamma(\tilde{y}+1)\Gamma(n-\tilde{y}+1)\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))\Gamma(\alpha+\beta+n+Nn)}$

for $\tilde{y} = 0, 1, \ldots, n$ $\therefore \pi(\theta|\mathbf{y}) = \frac{\Gamma(\alpha+\beta+Nn)}{\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))}\theta^{\alpha+N\bar{y}-1}(1-\theta)^{\beta+N(n-\bar{y})-1}$

$P(y) = \int_0^1 P(y|\theta)\pi(\theta)d\theta$ $\therefore$ $\frac{\Gamma(n+1)}{\Gamma(\tilde{y}+1)}\frac{\Gamma(\alpha+N\bar{y}+\tilde{y})\Gamma(n+\beta+N(n-\bar{y})-\tilde{y})}{\Gamma(n-\tilde{y}+1)\Gamma(\alpha+\beta+n+Nn)}$ $\therefore$ posterior predictive

$P(\tilde{y}|\mathbf{y}) = \int_0^1 P((\tilde{y}|\mathbf{y})|\theta)P(\theta)$ $\because$ ~~$\pi(\theta|\mathbf{y}) \propto \pi$~~ by bayes $\pi(\theta|\mathbf{y}) \propto \pi(\theta)P(\mathbf{y}|\theta)$ $\therefore$

$\theta|\mathbf{y} \sim \text{Beta}(\alpha+N\bar{y}, \beta+N(n-\bar{y}))$ $\therefore \tilde{y}|\mathbf{y}) = \frac{\Gamma(\alpha+\beta+Nn)}{\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))}\tilde{y}^{\alpha+N\bar{y}-1}(1-\tilde{y})^{\beta+N(n-\bar{y})-1}$

$\theta|\mathbf{y} \sim \text{Beta}(\alpha+N\bar{y}, \beta+N(n-\bar{y}))$ $\therefore y_i \sim \text{Bin}(n, \theta)$ $\therefore y_i|\theta \sim \text{Bin}(n, \theta)$ $\therefore$

$P(y_i|\theta) = \binom{n}{y_i}\theta^{y_i}(1-\theta)^{n-y_i}$

monte carlo [illegible] $\therefore \tilde{y}|\mathbf{y}$ is Beta-Binomial $P(y) = \int_{-\infty}^{\infty}P(y|\theta)\pi(\theta)d\theta$ $\therefore$

\Week 2 Sheet/ $\tilde{y}|\theta \sim Bin(n,\theta)$ $\therefore P(\tilde{y}|\theta)=\binom{n}{\tilde{y}}\theta^{\tilde{y}}(1-\theta)^{n-\tilde{y}}$;∴

$\int_0^1 P(\tilde{y}|\theta)\pi(\theta|y)\,d\theta = \int_0^1 \left(\binom{n}{\tilde{y}}\theta^{\tilde{y}}(1-\theta)^{n-\tilde{y}}\right)\pi(\theta|y)\,d\theta = \binom{n}{\tilde{y}}\int_0^1 \theta^{\tilde{y}}(1-\theta)^{n-\tilde{y}}\pi(\theta|y)\,d\theta =$

$\frac{n!}{(n-\tilde{y})!\tilde{y}!}\int_0^1 \theta^{\tilde{y}}(1-\theta)^{n-\tilde{y}}\pi(\theta|y)\,d\theta = \cancel{\frac{\Gamma(n+1)}{\Gamma(n-\tilde{y}+1)\Gamma(\tilde{y}+1)}\int_0^1 \theta^{\tilde{y}}(1-\theta)^{n-\tilde{y}}\frac{\Gamma(\alpha+\beta+Nn)}{\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))}}$

$\frac{\Gamma(n+1)}{\Gamma(n-\tilde{y}+1)\Gamma(\tilde{y}+1)}\int_0^1 \theta^{\tilde{y}}(1-\theta)^{n-\tilde{y}}\frac{\Gamma(\alpha+\beta+Nn)}{\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))}\theta^{\alpha+N\bar{y}-1}(1-\theta)^{\beta+N(n-\bar{y})-1}\,d\theta =$

Beta$(\alpha+N\bar{y}+\tilde{y}, n+\beta+N(n-\bar{y})-\tilde{y})$

$\frac{\Gamma(n+1)\Gamma(\alpha+\beta+Nn)}{\Gamma(n-\tilde{y}+1)\Gamma(\tilde{y}+1)\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))}\frac{\Gamma(\alpha+N\bar{y}+\tilde{y})\Gamma(n+\beta+N(n-\bar{y})-\tilde{y})}{\Gamma(\alpha+N\bar{y}+\tilde{y}+n+\beta+Nn-N\bar{y}-\tilde{y})}\int_0^1 \frac{\Gamma(\alpha+N\bar{y}+\tilde{y}+n+\beta+Nn-N\bar{y}-\tilde{y})}{\Gamma(\alpha+N\bar{y}+\tilde{y})\Gamma(n+\beta+N(n-\bar{y})-\tilde{y})}\theta^{\alpha+N\bar{y}+\tilde{y}-1}(1-\theta)^{\dots}\,d\theta = P(\tilde{y}|y)$

$= \Gamma(n+1)\Gamma(\alpha+\beta+Nn)\Gamma(\alpha+N\bar{y}+\tilde{y})\Gamma(n+\beta+N(n-\bar{y})-\tilde{y}) / (\Gamma(\tilde{y}+1)\Gamma(n-\tilde{y}+1)\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))\Gamma(\alpha+\beta+n+Nn))$

$\theta|y \sim Beta(\alpha+N\bar{y}, \beta+N(n-\bar{y}))$ $\therefore \pi(\theta|y)\sim Beta$, $\tilde{y}|\theta \sim Bin(n,\theta)$ $\therefore P(\tilde{y}|\theta)\sim Bin$

$\therefore P(\tilde{y}|y) = \int_0^1 P(\tilde{y}|\theta)\pi(\theta|y)\,d\theta$

\4 c/ $\theta \sim Beta(\alpha,\beta)$ ∴ prior of θ: $\pi(\theta) = \frac{\Gamma(1+1)}{\Gamma(1)\Gamma(1)}\theta^{1-1}(1-\theta)^{1-1} = \frac{1!}{0!0!}\theta^0(1-\theta)^0 = \frac{1}{1\cdot 1}1\cdot(1)=1$

∴ $\theta \sim Unif[0,1]$ ∴ Model the number of patients responding, y with in any group 5 tested via $y\sim Bin(5,\theta)$ $\theta\sim Unif[0,1]$ $P(y)=\int_0^1 P(y|\theta)\pi(\theta)\,d\theta$ ∴

$\pi(\theta)=\frac{1}{\theta}$ $\therefore P(y|\theta)=\binom{5}{y}\theta^y(1-\theta)^{n-y} = \frac{5!}{(5-y)!5!}\theta^y(1-\theta)^{n-y}$ ∴

$P(y)=\int_0^1 \frac{5!}{(5-y)!y!}\theta^y(1-\theta)^{n-y}\frac{1}{\theta}\,d\theta = \frac{5!}{(5-y)!y!}\int_0^1 \theta^y(1-\theta)^{n-y}\theta^{-1}\,d\theta =$

$\frac{120}{\Gamma(5-y+1)\Gamma(y)}\int_0^1 \theta^{y-1}(1-\theta)^{(n-y+1)-1}\,d\theta = \frac{120\,\Gamma(y)\,\Gamma(n-y+1)}{\Gamma(5-y+1)\Gamma(y)\Gamma(n+1)}\int_0^1 \frac{\Gamma(y+n-y+1)}{\Gamma(y)\Gamma(n-y+1)}\theta^{y-1}(1-\theta)^{(n-y+1)-1}\,d\theta$

$\frac{120\,\Gamma(y)\Gamma(n-y+1)}{\Gamma(5-y+1)\Gamma(y)\Gamma(n+1)} = P(y)$ ∴ $y\sim Bin(5,\theta)$ with ~~Bet~~ $\theta\sim Beta(1,1)$, the prob for the event none of the patients in your group respond is $P(y=0)$ ∴

$P(y) = \frac{\Gamma(6)\Gamma(2)\Gamma(1+y)\Gamma(5+1-y)}{\Gamma(y+1)\Gamma(5-y+1)\Gamma(1)\Gamma(1)\Gamma(1+1+5)} = \frac{\Gamma(6)}{\Gamma(7)} = \frac{5!}{6!} = \frac{1}{6} = P(y=0)$ is obtained from

the prior predictive distri from $P(y=0) = \frac{\Gamma(6)\Gamma(2)\Gamma(1)\Gamma(6)}{\Gamma(1)\Gamma(6)\Gamma(1)\Gamma(1)\Gamma(7)} = \frac{1}{6}$

\4 d/ $\bar{y} = \frac{0+0+1}{3} = \frac{1}{3}$ ∴ from part (b) ∴ $P(\tilde{y}|y)$ ∴ $P(\tilde{y}=0|y)$ ∴ $N=3$ ∴ $P(\tilde{y}=0|y)=$

$\Gamma(5+1)\Gamma(1+1+3\cdot 5)\Gamma(1+3\tfrac{1}{3}+0)\Gamma(5+1+3(5-\tfrac{1}{3})-0) / (\Gamma(0+1)\Gamma(5-0+1)\Gamma(1+3\cdot\tfrac{1}{3})\Gamma(1+3(5-\tfrac{1}{3}))\Gamma(1+1+5+3\cdot 5)) =$

$\Gamma(6)\Gamma(17)\Gamma(2)\Gamma(22) / (\Gamma(1)\Gamma(6)\Gamma(2)\Gamma(15)\Gamma(22)) = \frac{\Gamma(17)}{\Gamma(15)} = \frac{16!}{14!} = 16\times 15 = 240$ our prob bearing seen the results of the other experiments has change through learning about θ

2 is given by the posterior predictive prob $P(\tilde{y}=0|y)$ with $y=(0,0,1)$, $N=3$ ∴

$N\bar{y}=1$ & $N(n-\bar{y})=1$ ∴ into posterior predictive distri: $P(\tilde{y}=0|y) = \frac{\Gamma(6)\Gamma(17)\Gamma(2)\Gamma(20)}{\Gamma(1)\Gamma(6)\Gamma(2)\Gamma(15)\Gamma(22)} = \frac{4}{7}$

\5/ posterior predictive $P(\tilde{y}|y) = \int_\theta P(\tilde{y}|\theta)P(\theta|y)\,d\theta$

prior predictive $P(y) = \int_{-\infty}^{\infty} P(y|\theta)\pi(\theta)\,d\theta$ ∴ $P(y|\theta)=\theta(1-\theta)^{y-1}$, $\pi(\theta)=\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}$

$0<\theta<1$ ∴ $P(y)=\int_0^1 P(y|\theta)\pi(\theta)\,d\theta = \int_0^1 \theta(1-\theta)^{y-1}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}\,d\theta =$

Beta(a_1, b_1) & posterior density $\pi(\theta|y) = \frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\theta^{a_1-1}(1-\theta)^{b_1-1}$ ∴

$p(y) = \int_0^1 \binom{n}{y}\theta^y(1-\theta)^{n-y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}d\theta = \int_0^1 \frac{n!}{(n-y)!\,y!}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}d\theta =$

$\int_0^1 \frac{\Gamma(n+1)}{\Gamma(n-y+1)\Gamma(y+1)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}d\theta = \frac{\Gamma(n+1)}{\Gamma(y+1)\Gamma(n-y+1)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}d\theta$

∴ $p(y) = \frac{\Gamma(n+1)}{\Gamma(y+1)\Gamma(n-y+1)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(\alpha+y)\Gamma(n+\beta-y)}{\Gamma(\alpha+\beta+n)}$ $\theta \sim$ Beta(α, β) ∴

$p(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}$ for $0 \le \theta \le 1$ ~~$y|\theta$~~ $y|\theta$ is binomially distried ∴

$y|\theta \sim$ Bin(n, θ) ∴ $p(y|\theta) = \binom{n}{y}\theta^y(1-\theta)^{n-y}$ for $y = 0, 1, ---, n$ ∴ prior predictive

distri $p(y) = \int_0^1 p(y|\theta)p(\theta)d\theta$ ∵ By Bayes thm $\pi(\theta|y) \propto p(y|\theta)\pi(\theta)$ ∴

$p(y) = \int_{-\infty}^{\infty} p(y|\theta)\pi(\theta)d\theta = \int_0^1 p(y|\theta)p(\theta)d\theta = \int_0^1 \binom{n}{y}\theta^y(1-\theta)^{n-y}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}d\theta =$

$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\binom{n}{y}\int_0^1 \theta^y(1-\theta)^{n-y}\theta^{\alpha-1}(1-\theta)^{\beta-1}d\theta = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{n!}{(n-y)!\,y!}\int_0^1 \theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}d\theta =$

$\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\frac{\Gamma(n+1)}{\Gamma(n-y+1)\Gamma(y+1)}\frac{\Gamma(y+\alpha)\Gamma(n-y+\beta)}{\Gamma(y+\alpha+n-y+\beta)}\int_0^1 \frac{\Gamma(y+\alpha+n-y+\beta)}{\Gamma(y+\alpha)\Gamma(n-y+\beta)}\theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}d\theta =$

$\frac{\Gamma(n+1)}{\Gamma(y+1)\Gamma(n-y+1)}\frac{\Gamma(\alpha+y)\Gamma(n+\beta-y)}{\Gamma(\alpha+\beta+n)}\frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}$ ∵ $F(y=\infty) = F(\infty) = F(1) = P(y \le \infty) = P(y \le 1) = 1$

2b $y_i|\theta \sim$ Bin(n, θ) ∴ $p(y_i|\theta) = \binom{n}{y_i}\theta^{y_i}(1-\theta)^{n-y_i}$ ∴ $p(y|\theta) = \prod_{i=1}^{n} p(y_i|\theta) =$

$\prod_{i=1}^{n}\binom{n}{y_i}\theta^{y_i}(1-\theta)^{n-y_i} = \prod_{i=1}^{n}\binom{n}{y_i}\theta^{\sum_{i=1}^{n} y_i}(1-\theta)^{\sum_{i=1}^{n}(n-y_i)} = \prod_{i=1}^{n}\binom{n}{y_i}\theta^{n\bar{y}}(1-\theta)^{n^2-n\bar{y}}$ ∴ by Bayes thm

$\pi(\theta|y) \propto \pi(\theta)p(y|\theta)$ prior predictive distri $p(y) = \int_{-\infty}^{\infty} p(y|\theta)\pi(\theta)d\theta$ ∴

$p(\tilde{y}) = \int_0^1 p(y|\theta)p(\theta)d\theta$ ∵ $p(y|\theta) = \prod_{i=1}^{n}\left(\frac{n!}{(n-y_i)!\,y_i!}\right)\theta^{n\bar{y}}(1-\theta)^{n^2-n\bar{y}} = \frac{(n!)^n}{((n-y_i)!)^n(y_i!)^n}(\theta^{(n\bar{y})})^n((1-\theta^{n^2-n\bar{y}})^n$

$\theta|y \sim$ Beta$(\alpha+N\bar{y}, \beta+N(n-\bar{y}))$ ∴ $y_i \sim$ Bin(n, θ) ∴ $y_i|\theta \sim$ Bin(n, θ) ∴

$p(y_i|\theta) = \binom{N}{y_i}\theta^{y_i}(1-\theta)^{N-y_i}$ for $i = 0, 1, ---, N$ ∴ $p(y|\theta) = \prod_{i=1}^{N}\binom{n}{y_i}\theta^{y_i}(1-\theta)^{n-y_i} \propto$

$\theta^{N\bar{y}}(1-\theta)^{N(n-\bar{y})}$ ∴ $\pi(\theta) = \frac{\Gamma(\alpha+\beta)}{\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1} \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}$ ∴ by Bayes thm:

$\pi(\theta|y) \propto \pi(\theta)p(y|\theta) \propto \pi(\theta)\theta^{N\bar{y}}(1-\theta)^{N(n-\bar{y})} \propto \theta^{\alpha-1}(1-\theta)^{\beta-1}\theta^{N\bar{y}}(1-\theta)^{N(n-\bar{y})} \propto$

$\theta^{(\alpha+N\bar{y})-1}(1-\theta)^{(\beta+N(n-\bar{y}))-1}$ ∴ $\pi(\theta|y) = \frac{\Gamma(\alpha+N\bar{y}+\beta+N(n-\bar{y}))}{\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))}\theta^{\alpha+N\bar{y}-1}(1-\theta)^{\beta+N(n-\bar{y})-1}$ ∴

$\theta|y \sim$ Beta$(\alpha+N\bar{y}, \beta+N(n-\bar{y}))$ is Z posterior distri ∴ $p(\tilde{y}) = \int_0^1 p(y|\theta)\pi(\theta)d\theta$ ∴

$\tilde{y}|y$ is BetaBinomial & $p(\tilde{y}|y) = \frac{\Gamma(n+1)\Gamma(\alpha+\beta+Nn)\Gamma(\alpha+N\bar{y}+\tilde{y})\Gamma(n+\beta+N(n-\bar{y})-\tilde{y})}{\Gamma(\tilde{y}+1)\Gamma(n-\tilde{y}+1)\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))\Gamma(\alpha+\beta+n+Nn)}$

for $\tilde{y} = 0, 1, \dots, n$ ∴ $\pi(\theta|y) = \frac{\Gamma(\alpha+\beta+Nn)}{\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))}\theta^{\alpha+N\bar{y}-1}(1-\theta)^{\beta+N(n-\bar{y})-1}$ ∴

$p(y) = \int_0^1 p(y|\theta)\pi(\theta)d\theta$ ∴ $\frac{\Gamma(n+1)}{\Gamma(\tilde{y}+1)}\frac{\Gamma(\alpha+N\bar{y}+\tilde{y})\Gamma(n+\beta+N(n-\bar{y})-\tilde{y})}{\Gamma(n-\tilde{y}+1)\Gamma(\alpha+\beta+n+Nn)}$ ∴ posterior predictive

$p(\tilde{y}|y) = \int_0^1 p((\tilde{y}|y)|\theta)p(\theta)$ ∵ ~~$\pi(\theta|y) \propto \pi$~~ by bayes $\pi(\theta|y) \propto \pi(\theta)p(y|\theta)$ ∴

$\theta|y \sim$ Beta$(\alpha+N\bar{y}, \beta+N(n-\bar{y}))$ ∴ $\tilde{y}|y) = \frac{\Gamma(\alpha+\beta+Nn)}{\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))}\tilde{y}^{\alpha+N\bar{y}-1}(1-\tilde{y})^{\beta+N(n-\bar{y})-1}$

$\theta|y \sim$ Beta$(\alpha+N\bar{y}, \beta+N(n-\bar{y}))$ ∴ $y_i \sim$ Bin(n, θ) ∴ $y_i|\theta \sim$ Bin(n, θ) ∴

$p(y_i|\theta) = \binom{n}{y_i}\theta^{y_i}(1-\theta)^{n-y_i}$ ∴ $\tilde{y}|y$ is Beta-Binomial $p(y) = \int_{-\infty}^{\infty} p(y|\theta)\pi(\theta)d\theta$ ∴

\Week 2 Sheet/ $\tilde{y}|\theta \sim Bin(n,\theta)$ $\therefore p(\tilde{y}|\theta) = \binom{n}{\tilde{y}}\theta^{\tilde{y}}(1-\theta)^{n-\tilde{y}}$, $\therefore$

$\int_0^1 p(\tilde{y}|\theta)\pi(\theta|y)\,d\theta = \int_0^1 \left(\binom{n}{\tilde{y}}\theta^{\tilde{y}}(1-\theta)^{n-\tilde{y}}\right)\pi(\theta|y)\,d\theta = \binom{n}{\tilde{y}}\int_0^1 \theta^{\tilde{y}}(1-\theta)^{n-\tilde{y}}\pi(\theta|y)\,d\theta =$

$\frac{n!}{(n-\tilde{y})!\tilde{y}!}\int_0^1 \theta^{\tilde{y}}(1-\theta)^{n-\tilde{y}}\pi(\theta|y)\,d\theta = \frac{\Gamma(n+1)}{\Gamma(n-\tilde{y}+1)\Gamma(\tilde{y}+1)}\int_0^1 \theta^{\tilde{y}}(1-\theta)^{n-\tilde{y}}\frac{\Gamma(\alpha+\beta+Nn)}{\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))}$

$\frac{\Gamma(n+1)}{\Gamma(n-\tilde{y}+1)\Gamma(\tilde{y}+1)}\int_0^1 \theta^{\tilde{y}}(1-\theta)^{n-\tilde{y}}\frac{\Gamma(\alpha+\beta+Nn)}{\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))}\theta^{\alpha+N\bar{y}-1}(1-\theta)^{\beta+N(n-\bar{y})-1}\,d\theta =$

Beta$(\alpha+N\bar{y}+\tilde{y},\ n+\beta+N(n-\bar{y})-\tilde{y})$

$\frac{\Gamma(n+1)\Gamma(\alpha+\beta+Nn)}{\Gamma(n-\tilde{y}+1)\Gamma(\tilde{y}+1)\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))}\frac{\Gamma(\alpha+N\bar{y}+\tilde{y})\Gamma(n+\beta+N(n-\bar{y})-\tilde{y})}{\Gamma(\alpha+N\bar{y}+\tilde{y}+n+\beta+Nn-N\bar{y}-\tilde{y})}\int_0^1 \frac{\Gamma(\alpha+N\bar{y}+\tilde{y}+n+\beta+Nn-N\bar{y}-\tilde{y})}{\Gamma(\alpha+N\bar{y}+\tilde{y})\Gamma(n+\beta+N(n-\bar{y})-\tilde{y})}\theta^{\alpha+N\bar{y}+\tilde{y}}(1-\theta)\,d\theta = p(\tilde{y}|y)$

$= \Gamma(n+1)\Gamma(\alpha+\beta+Nn)\Gamma(\alpha+N\bar{y}+\tilde{y})\Gamma(n+\beta+N(n-\bar{y})-\tilde{y}) / \left(\Gamma(\tilde{y}+1)\Gamma(n-\tilde{y}+1)\Gamma(\alpha+N\bar{y})\Gamma(\beta+N(n-\bar{y}))\Gamma(\alpha+\beta+n+Nn)\right)$

$\theta|y \sim Beta(\alpha+N\bar{y}, \beta+N(n-\bar{y}))$ $\therefore \pi(\theta|y) \sim Beta$, $\tilde{y}|\theta \sim Bin(n,\theta)$ $\therefore p(\tilde{y}|\theta) \sim Bin$

$\therefore p(\tilde{y}|y) = \int_0^1 p(\tilde{y}|\theta)\pi(\theta|y)\,d\theta$

\4e/ $\theta \sim Beta(\alpha,\beta)$ $\therefore$ prior of θ: $\pi(\theta) = \frac{\Gamma(1+1)}{\Gamma(1)\Gamma(1)}\theta^{1-1}(1-\theta)^{1-1} = \frac{1!}{0!0!}\theta^0(1-\theta)^0 = \frac{1}{1\cdot 1}1\cdot(1) = 1$

$\therefore \theta \sim Unif[0,1]$ $\therefore$ Model 2 number of patients responding, y with in any group 5 tested via $y \sim Bin(5,\theta)$ $\theta \sim Unif[0,1]$ $p(y) = \int_0^1 p(y|\theta)\pi(\theta)\,d\theta$ $\therefore$

$\pi(\theta) = \frac{1}{\theta}$ $\therefore p(y|\theta) = \binom{5}{y}\theta^y(1-\theta)^{n-y} = \frac{5!}{(5-y)!5!}\theta^y(1-\theta)^{n-y}$ $\therefore$

$p(y) = \int_0^1 \frac{5!}{(5-y)!y!}\theta^y(1-\theta)^{n-y}\frac{1}{\theta}\,d\theta = \frac{5!}{(5-y)!y!}\int_0^1 \theta^y(1-\theta)^{n-y}\theta^{-1}\,d\theta =$

$\frac{120}{\Gamma(5-y+1)\Gamma(y)}\int_0^1 \theta^{y-1}(1-\theta)^{(n-y+1)-1}\,d\theta = \frac{120\,\Gamma(y)\Gamma(n-y+1)}{\Gamma(5-y+1)\Gamma(y)\Gamma(n+1)}\int_0^1 \frac{\Gamma(y+n-y+1)}{\Gamma(y)\Gamma(n-y+1)}\theta^{y-1}(1-\theta)^{(n-y+1)-1}\,d\theta$

$\frac{120\,\Gamma(y)\Gamma(n-y+1)}{\Gamma(5-y+1)\Gamma(y)\Gamma(n+1)} = p(y)$ $\therefore y \sim Bin(5,\theta)$ with ~~Bet~~ $\theta \sim Beta(1,1)$, 2 prob for 2 event none of 2 patients in your group respond is $p(y=0)$ $\therefore$

$p(y) = \frac{\Gamma(6)\Gamma(2)\Gamma(1+y)\Gamma(5+1-1)}{\Gamma(y+1)\Gamma(5-y+1)\Gamma(1)\Gamma(1)\Gamma(1+1+5)} = \frac{\Gamma(6)}{\Gamma(7)} = \frac{5!}{6!} = \frac{1}{6} = p(y=0)$ is obtained from 2 prior predictive distri from $p(y=0) = \frac{\Gamma(6)\Gamma(2)\Gamma(1)\Gamma(6)}{\Gamma(1)\Gamma(6)\Gamma(1)\Gamma(1)\Gamma(7)} = \frac{1}{6}$

\4d/ $\bar{y} = \frac{0+0+1}{3} = \frac{1}{3}$ $\therefore$ from part (b) $\therefore p(\tilde{y}|y)$ $\therefore p(\tilde{y}=0|y)$ $\therefore$ $N=3$ $\therefore$ $p(\tilde{y}=0|y) =$

$\Gamma(5+1)\Gamma(1+1+3-5)\Gamma(1+3\tfrac{1}{3}+0)\Gamma(5+1+3(5-\tfrac{1}{3})-0) / \left(\Gamma(0+1)\Gamma(5-0+1)\Gamma(1+3\cdot\tfrac{1}{3})\Gamma(1+3(5-\tfrac{1}{3}))\Gamma(1+1+5+3\cdot 5)\right) =$

$\Gamma(6)\Gamma(17)\Gamma(2)\Gamma(22) / \left(\Gamma(1)\Gamma(6)\Gamma(2)\Gamma(15)\Gamma(22)\right) = \frac{\Gamma(17)}{\Gamma(15)} = \frac{16!}{14!} = 16\times 15 = 240$ our prob bearing seen 2 results of 2 other experiments has change through learning about θ 2 is given by 2 posterior predictive prob $p(\tilde{y}=0|y)$ with $y=(0,0,1)$, $N=3$ $\therefore$

$N\bar{y}=1$ & $N(n-\bar{y})=1$ $\therefore$ into posterior predictive distri: $p(\tilde{y}=0|y) = \frac{\Gamma(6)\Gamma(17)\Gamma(2)\Gamma(20)}{\Gamma(1)\Gamma(6)\Gamma(2)\Gamma(15)\Gamma(22)} = \frac{4}{7}$

\\5/ posterior predictive $p(\tilde{y}|y) = \int_\theta p(\tilde{y}|\theta)p(\theta|y)\,d\theta$

prior predictive $p(y) = \int_{-\infty}^{\infty} p(y|\theta)\pi(\theta)\,d\theta$ $\therefore p(y|\theta) = \theta(1-\theta)^{y-1}$, $\pi(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}$

$0<\theta<1$ $\therefore p(y) = \int_0^1 p(y|\theta)\pi(\theta)\,d\theta = \int_0^1 \theta(1-\theta)^{y-1}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}\,d\theta =$

$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 \theta^{(a+1)-1}(1-\theta)^{(b+y-1)-1}\,d\theta = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(a+1)\Gamma(b+y-1)}{\Gamma(a+b+y)}\int_0^1 \frac{\Gamma(a+1+b+y-1)}{\Gamma(a+1)\Gamma(b+y-1)}\theta^{(a+1)-1}(1-\theta)^{(b+y-1)-1}\,d\theta$

$\because$ Z integrand is a density $= \frac{\Gamma(a+b)\Gamma(a+1)\Gamma(b+y-1)}{\Gamma(a)\Gamma(b)\Gamma(a+b+y)}$ is Z prior predictive distri

By Bayes thm: $\pi(\theta|\underline{y}) \propto \pi(\theta)p(\underline{y}|\theta)$ $\therefore$ $\pi(\theta|\underline{y}) \propto \pi(\theta)p(\underline{y}|\theta)$ $\because$

$\pi(\theta) \propto \theta^{a-1}(1-\theta)^{b-1}$, $p(\underline{y}|\theta) \propto \prod_{i=1}^{n} p(y_i|\theta) = \prod_{i=1}^{n}\theta(1-\theta)^{y_i-1} \propto \theta^n(1-\theta)^{-n+n\bar{y}}$ $\therefore$

$\pi(\theta|\underline{y}) \propto \pi(\theta)p(\underline{y}|\theta) \propto \theta^{a-1}(1-\theta)^{b-1}p(\underline{y}|\theta) \propto \theta^{a-1}(1-\theta)^{b-1}\theta^n(1-\theta)^{-n+n\bar{y}} \propto$

$\theta^{(a+n)-1}(1-\theta)^{(b+n\bar{y}-n)-1} \propto \theta^{a_1-1}(1-\theta)^{b_1-1}$ with $a_1 = a+n$, $b_1 = b+n\bar{y}-n$ $\therefore$

posterior distri is Beta distri $Beta(a_1, b_1)$ & posterior density

$\pi(\theta|\underline{y}) = \frac{\Gamma(a_1+b_1)}{\Gamma(a_1)\Gamma(b_1)}\theta^{a_1-1}(1-\theta)^{b_1-1} = \frac{\Gamma(a+n+b+n\bar{y}-n)}{\Gamma(a+n)\Gamma(b+n\bar{y}-n)}\theta^{a+n-1}(1-\theta)^{b+n\bar{y}-n-1} =$

$\frac{\Gamma(a+b+n\bar{y})}{\Gamma(a+n)\Gamma(b+n\bar{y}-n)}\theta^{a+n-1}(1-\theta)^{b+n\bar{y}-n-1}$ $\therefore$ $f(y|\theta) = \theta(1-\theta)^{y-1}$ for posi integers y &

$\pi(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}$ for $0<\theta<1$ $\therefore$ Z prior predictive distri has mass

func $\pi(y) = \int_0^1 f(y|\theta)\pi(\theta)\,d\theta = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 \theta(1-\theta)^{y-1}\theta^{a-1}(1-\theta)^{b-1}\,d\theta =$

$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 \theta(1-\theta)^{y-1}\theta^{a-1}(1-\theta)^{b-1}\,d\theta = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 \theta^{(a+1)-1}(1-\theta)^{(b+y-1)-1}\,d\theta =$

$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(a+1)\Gamma(b+y-1)}{\Gamma(a+b+y)}\int_0^1 \frac{\Gamma(a+b+y)}{\Gamma(a+1)\Gamma(b+y-1)}\theta^{(a+1)-1}(1-\theta)^{(b+y-1)-1}\,d\theta = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\frac{\Gamma(a+1)\Gamma(b+y-1)}{\Gamma(a+b+y)}$ $\because$

Z integrand is a density. Similarly Z posterior predictive distri

has mass func $\pi(y|\underline{x}) = \frac{\Gamma(a+n+1)\Gamma(b+\sum_{i=1}^{n}x_i-n+y-1)}{\Gamma(a+b+\sum_{i=1}^{n}x_i+y)}\frac{\Gamma(a+b+\sum_{i=1}^{n})}{\Gamma(a+n)\Gamma(b+\sum_{i=1}^{n}x_i-n)}$ $\therefore$

$p(y|\underline{x}) = \int_0^1 p(y|\theta)\pi(\theta|\underline{x})\,d\theta$ $\therefore$ $\pi(\theta|\underline{x}) = \frac{\Gamma(a+b+n\bar{x})}{\Gamma(a+n)\Gamma(b+n\bar{x}-n)}\theta^{a+n-1}(1-\theta)^{b+n\bar{x}-n-1}$ $\therefore$

$p(y|\underline{x}) = \int_0^1 \theta(1-\theta)^{y-1}\frac{\Gamma(a+b+n\bar{x})}{\Gamma(a+n)\Gamma(b+n\bar{x}-n)}\theta^{a+n-1}(1-\theta)^{b+n\bar{x}-n-1}\,d\theta =$

$\frac{\Gamma(a+b+n\bar{x})}{\Gamma(a+n)\Gamma(b+n\bar{x}-n)}\int_0^1 \theta^{(a+n+1)-1}(1-\theta)^{(b+y+n\bar{x}-n-1)-1}\,d\theta =$

$\frac{\Gamma(a+n+1)\Gamma(b+n\bar{x}-n+y-1)}{\Gamma(a+b+n\bar{x}+y)}\frac{\Gamma(a+b+n\bar{x})}{\Gamma(a+n)\Gamma(b+n\bar{x}-n)}\int_0^1 \frac{\Gamma(a+n+1+b+y+n\bar{x}-n-1)}{\Gamma(a+n+1)\Gamma(b+n\bar{x}-n+y-1)}\theta^{(a+n+1)-1}(1-\theta)^{(b+y+n\bar{x}-n-1)-1}\,d\theta$

$= \frac{\Gamma(a+n+1)\Gamma(b+n\bar{x}-n+y-1)\Gamma(a+b+n\bar{x})}{\Gamma(a+b+n\bar{x}+y)\Gamma(a+n)\Gamma(b+n\bar{x}-n)} = p(y|\underline{x})$ is Z posterior predictive distri $\therefore$

$P(y=1) = \frac{\Gamma(a+b)\Gamma(a+1)\Gamma(b+1-1)}{\Gamma(a)\Gamma(b)\Gamma(a+b+1)} = \frac{\Gamma(a+b)\Gamma(a+1)\Gamma(b)}{\Gamma(a)\Gamma(b)\Gamma(a+b+1)} = \frac{\Gamma(a+b)\Gamma(a+1)}{\Gamma(a)\Gamma(a+b+1)}$

$P(y=1|\underline{x}) = \Gamma(a+n+1)\Gamma(b+n\bar{x}-n+1-1)\Gamma(a+b+n\bar{x})/(\Gamma(a+b+n\bar{x}+1)\Gamma(a+n)\Gamma(b+n\bar{x}-n)) =$

$\Gamma(a+n+1)\Gamma(b+n\bar{x}-n)\Gamma(a+b+n\bar{x})/(\Gamma(a+b+n\bar{x}+1)\Gamma(a+n)\Gamma(b+n\bar{x}-n))$ $\therefore$ $P(y=1|\underline{x}) > P(y=1)$ $\therefore$

$\Gamma(a+n+1)\Gamma(b+n\bar{x}-n)\Gamma(a+b+n\bar{x})/(\Gamma(a+b+n\bar{x}+1)\Gamma(a+n)\Gamma(b+n\bar{x}-n)) > \Gamma(a+b)\Gamma(a+1)/(\Gamma(a)\Gamma(a+b+1))$ $\therefore$

$\frac{(a+n)!(b+n\bar{x}-n-1)!(a+b+n\bar{x}-1)!}{(a+b+n\bar{x})!(a+n-1)!(b+n\bar{x}-n-1)!} > \frac{(a+b-1)!a!}{(a-1)!(a+b)!}$ $\therefore$

$\frac{(a+n)(a+n-1)!(b+n\bar{x}-n-1)!(a+b+n\bar{x}-1)!}{(a+b+n\bar{x})!(a+n-1)!(b+n\bar{x}-n-1)!} > \frac{(a+b-1)!a(a-1)!}{(a-1)!(a+b)(a+b-1)!}$ $\therefore$

$\frac{(a+n)(b+n\bar{x}-n-1)!(a+b+n\bar{x}-1)!}{(a+b+n\bar{x})(a+b+n\bar{x}-1)!(b+n\bar{x}-n-1)!} > \frac{a}{a+b}$ $\therefore$ $\frac{(a+n)(b+n\bar{x}-n-1)!}{(a+b+n\bar{x})} > \frac{a}{a+b}$ $\therefore$

$(a+n)(b+n\bar{x}-n-1)! > \frac{a(a+b+n\bar{x})}{a+b}$ $\therefore$ $(b+n\bar{x}-n-1)! > \frac{a(a+b+n\bar{x})}{(a+b)(a+n)}$ $\because$ $\Gamma(t+1) = t\Gamma(t)\ \forall t \in \mathbb{R}$ $\because$

\Week 2 Sheet/ subbing $y=1$ into these distri's yields:

$\pi(y=1)=\frac{\Gamma(a+b)\Gamma(a+1)\Gamma(b)}{\Gamma(a)\Gamma(b)\Gamma(a+b+1)}=\frac{a}{a+b}$ & $\pi(y=1|\underline{y})=\frac{\Gamma(a+n+1)\Gamma(b+\sum_{i=1}^{n}y_i-n)}{\Gamma(a+b+\sum_{i=1}^{n}y_i+1)}\frac{\Gamma(a+b+\sum_{i=1}^{n}y_i)}{\Gamma(a+n)\Gamma(b+\sum_{i=1}^{n}y_i-n)}$

$\frac{(a+n)\Gamma(a+n)\Gamma(b+\sum_{i=1}^{n}y_i-n)}{(a+b+\sum_{i=1}^{n}y_i)\Gamma(a+b+\sum_{i=1}^{n}y_i)}\frac{\Gamma(a+b+\sum_{i=1}^{n}y_i)}{\Gamma(a+n)\Gamma(b+\sum_{i=1}^{n}y_i-n)}=\frac{a+n}{(a+b+\sum_{i=1}^{n}y_i)}=\frac{a+n}{a+b+\sum_{i=1}^{n}y_i}$ $\therefore$

$\frac{a+n}{a+b+n\bar{y}}>\frac{a}{a+b}$ $\therefore$ $\frac{(a+n)(a+b)}{a}>a+b+n\bar{y}$ $\therefore$ $a+b+n\bar{y}<\frac{a^2+ab+an+bn}{a}=a+b+n+\frac{bn}{a}$ $\therefore$

$n\bar{y}<n+\frac{bn}{a}$ $\therefore$ $\bar{y}<1+\frac{b}{a}$ $\therefore$ $\pi(y=1|\underline{y})>\pi(y=1)$ $\therefore$ $(a+n)(a+b)>a(a+b+n\bar{y})$ $\therefore$

$\bar{y}<1+\frac{b}{a}$

\6/ Gamma prior for λ $\therefore$ $\pi(\lambda)=\frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}$ for $\lambda>0$ $\therefore$ $f(y|\lambda)=\frac{\lambda^y}{y!}e^{-\lambda}$ $\therefore$

By Bayes thm: $\pi(\lambda|\underline{y})\propto\pi(\lambda)f(\underline{y}|\lambda)\propto\frac{b^a}{\Gamma(a)}\lambda^{a-1}e^{-b\lambda}f(\underline{y}|\lambda)\propto\lambda^{a-1}e^{-b\lambda}f(\underline{y}|\lambda)\propto$

$\lambda^{a-1}e^{-b\lambda}f(\underline{y}|\lambda)\propto\lambda^{a-1}e^{-b\lambda}\prod_{i=1}^{n}f(y_i|\lambda)\propto\lambda^{a-1}e^{-b\lambda}\prod_{i=1}^{n}\frac{\lambda^{y_i}}{y_i!}e^{-\lambda}\propto$

$\lambda^{a-1}e^{-b\lambda}\prod_{i=1}^{n}\left(\frac{1}{(y_i)!}\right)\lambda^{\sum_{i=1}^{n}y_i}(e^{-\lambda})^n\propto\lambda^{a-1}e^{-b\lambda}\lambda^{n\bar{y}}e^{-n\lambda}\propto\lambda^{a_1-1}e^{-b_1\lambda}$, with

$a_1=a+n\bar{y}$, $b_1=b+n$ $\therefore$ posterior density $\pi(\lambda|\underline{y})=\frac{(b+n)^{a+n\bar{y}}}{\Gamma(a+n\bar{y})}\lambda^{a+n\bar{y}-1}e^{-(b+n)\lambda}$ $\therefore$

posterior distri is Gamma distri: Gamma(a_1,b_1) for $\lambda>0$

\Week 3 Sheet/ \1a/ unknown random quantity assigning 1 to true & 0 to false whose true vals is unknown with only two possible vals true & false, assigning 1 to true & 0 to false

\2b/ $P(A\vee(B\wedge C))$ $\therefore$ $A\vee(B\wedge C)=\sim(\tilde{A}\wedge\sim(B\wedge C))=1-((1-A)(1-BC))=1-(1-A-BC+ABC)=$

$A+BC-ABC$, $(A\vee B)\wedge(A\vee C)=(\sim(\tilde{A}\wedge\tilde{B})\wedge(\sim(\tilde{A}\wedge\tilde{C}))=(1-(1-A)(1-B))(1-(1-A)(1-C))=$

$(1-1+A+B-AB)(1-1+A+C-AC)=A^2+AC+A^2C+BA+BC-ABC-A^2B-ABC+A^2BC$

$\therefore$ A is 1 or 0 $\therefore$ A^2 is just A $\therefore$ $A^2=A$ $\therefore$ $=A+(AC-AC)+BA+BC-ABC-AB+(-ABC+ABC)=$

$A+BC-ABC$ $\therefore$ $P(A\vee(B\wedge C))=P((A\vee B)\wedge(A\vee C))$ otherwise we have violated coherence

$P(A\vee(B\wedge C))=P(A)+P(BC)-P(ABC)$ by linearity of expectation

\3/ $(A\wedge B)\vee(C\wedge D)=\sim((\widetilde{A\wedge B})\wedge(\widetilde{C\wedge D}))=1-((1-AB)(1-CD))=1-(1-AB-CD+ABCD)=$

$AB+CD-ABCD$ $\therefore$ $P((AB)\vee(C\wedge D))=E[AB+CD-ABCD]=$ by l.o.e

$E(AB)+E(CD)-E(ABCD)=0.5+0.4-0.2=0.7$

\4/ w.l.o.g: assume X takes n possible vals $x_1,\dots,x_n$ $\therefore$ $X=x_i$ $i=1,\dots,n$

form a partition $\therefore$ $X=x_1(X=x_1)\vee\dots\vee x_n(X=x_n)=\sum_{i=1}^{n}x_i(X=x_i)$ (partition)

$E(X)=$ coherence $E(\sum_{i=1}^{n}x_i(X=x_i))=$ l.o.e $\sum_{i=1}^{n}x_i E(X=x_i)=\sum_{i=1}^{n}x_i P(X=x_i)$

\5/ $E_1, \ldots, E_n$ a partion, A an r.q. $E(A) = \sum_{i=1}^{n} E(A|E_i)P(E_i) = E(E(A|E))$

$A = A(E_1 \vee E_2 \vee \ldots \vee E_n) = A(E_1 + E_2 + \ldots + E_n)$ (partion) l.o.e & Coherence ∴ $E(A) =$

$E(AE_1) + E(AE_2) + \ldots + E(AE_n)$ ∴ thm of compound probs: for E event,

A r.q. $E(AE) = E(A|E)P(E)$ (event) $P(AE) = P(A|E)P(E)$ ∴ ~~E(A)~~=

$E(A) = E(A|E_1)P(E_1) + \ldots + E(A|E_n)P(E_n) = \sum_{i=1}^{n} E(A|E_n)P(E_i) = \sum_{i=1}^{c} P(A|E_i)P(E_i)$

\1a/ $A \vee B \vee C = A \vee (B \vee C) = \sim(\tilde{A} \wedge (\sim(B \vee C))) = \sim(\tilde{A} \wedge (\sim(\sim(\tilde{B} \wedge \tilde{C}))))=$

$\sim(\tilde{A} \wedge (\tilde{B} \wedge \tilde{C})) = \sim(\tilde{A} \wedge \tilde{B} \wedge \tilde{C})$ ∴

$A \vee B \vee C = \sim(\tilde{A} \wedge \tilde{B} \wedge \tilde{C}) = 1 - (\tilde{A} \wedge \tilde{B} \wedge \tilde{C}) = 1 - (1-A)(1-B)(1-C) =$

$1 - (1-A)(1-B-C+BC) = 1 - (1-B-C+BC-A+AB+AC-ABC) =$

$1 - 1 + B + C + A - BC - AB - AC + ABC = A + B + C - AB - AC - BC + ABC$ ∴

$P(A \vee B \vee C) = P(A+B+C-AB-AC-BC+ABC) =$

$P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$

\revision lecture/ do the M508 mock paper aswell

how to implement algorithems

Jerreys prior is expected information $I(\theta) = J(\theta) = -E\left(\frac{\partial^2 l(\theta)}{\partial\theta^2}\right)$

$l(\theta) : \ln(f(\underline{y}|\theta))$ ∴

Jerreys prior: $\pi(\theta) \propto J(\theta)^{1/2}$

$y|\theta \sim N(\theta, \sigma^2)$ ∴ $L(\theta)$

$f(\underline{y}|\theta) = \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2\sigma^2}(y_i-\theta)^2} = (2\pi)^{-\frac{n}{2}} \sigma^{-n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\theta)^2}$

$l(\theta) = \ln(L(\theta)) = -\frac{n}{2}\ln(2\pi) - n\ln(\sigma) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i^2 - 2y_i\theta + \theta^2)$ ∴

$\frac{\partial l}{\partial\theta} = \frac{1}{\sigma^2} n\bar{y} - \frac{1}{\sigma^2} n\theta = \frac{n\bar{y}}{\sigma^2} - \frac{n}{\sigma^2}\theta$ ∴

$\frac{\partial^2 l}{\partial\theta^2} = -\frac{n}{\sigma^2}$, $J(\theta) = \frac{n}{\sigma^2}$ ∴

$\pi(\theta) \propto J(\theta)^{1/2} \propto \frac{\sqrt{n}}{\sigma} \propto 1$ ∵ no θ so every is constant ∴

$\theta \sim \text{Unif} \propto 1$ ∴ $\pi_J(\theta) \propto 1$ ∴ $\theta \sim \text{Unif} \propto 1$

$y|\theta \sim Bin(n, \theta)$, $\pi(\theta) \sim Beta(\alpha, \beta)$ ∴ $0 < \theta < 1$ ∴ prior predictive

$P(y) = \int_{-\infty}^{\infty} P(y|\theta)\pi(\theta)\,d\theta = \int_0^1 \frac{\Gamma(n+1)\theta^y(1-\theta)^{n-y}\Gamma(\alpha+\beta)}{\Gamma(y+1)\Gamma(n-y-1)\Gamma(\alpha)\Gamma(\beta)}\theta^{\alpha-1}(1-\theta)^{\beta-1}\,d\theta =$

$\frac{\Gamma(n+1)\Gamma(\alpha+\beta)}{\Gamma(y+1)\Gamma(n-y+1)\Gamma(\alpha)\Gamma(\beta)}\int_0^1 \theta^{y+\alpha-1}(1-\theta)^{n-y+\beta-1}\,d\theta$

$\propto Beta(y+\alpha, n-y+\beta)$ $\frac{\Gamma(y+\alpha)\Gamma(n-y+\beta)}{\Gamma(\alpha+n+\beta)}$ ∴ $P(y=1)$, $P(y=0)$, $P(y>4)$

∴ posterior predictive distribution:

$p(y|\underline{y}) = \int_{-\infty}^{\infty} p(y|\theta)\pi(\theta|\underline{y})d\theta$

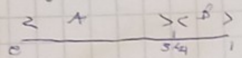
posterior; $\theta|\underline{y} \sim Beta(\alpha+n, \beta+n\bar{y}) = Beta(\alpha_1, \beta_1)$

$\pi(\theta|y) \propto \pi(\theta) f(\underline{y}|\theta) \propto \theta^{\alpha-1}(1-\theta)^{\beta-1} \prod_{i=1}^{n} \theta^{y_i}(1-\theta)^{1-y_i}$ $\quad \left\{ \prod_{i=1}^{n} \theta^{y_i}(1-\theta)^{n-y_i} \right.$

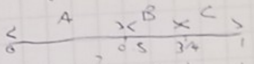
$\theta^{\alpha-1+\sum_{i=1}^{n} y_i}(1-\theta)^{\beta-1+n-\sum_{i=1}^{n} y_i}$

\ Xinverse cdf / $x_1, \ldots, x_m$ $\quad p_1, \ldots, p_m$ ∴

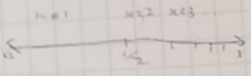
$F_j = P(X \le x_j) = p_1 + p_2 + \ldots + p_j$ $\quad$ ∴ $F_x^{-1}(p) = x_j$ when $p \in [F_{j-1}, F_j]$



$\{A, B\}$ $p = \frac{3}{4}$



$\{A, B, C\}$ $p = \frac{1}{2}, \frac{1}{4}, \frac{1}{4}$



$X \sim Geo(\theta)$, $P(X=x) = (1-\theta)^{x-1}\theta$ $\quad x = 1, 2, \ldots$

0.649 $\quad$ 0.826 $\quad$ 0.062 $\quad$ 0.963 $\quad$, 0.276

$F_1 = P(X=1) = \frac{1}{2} = 0.5$ $\quad F_2 = P(X \le 2) = \frac{1}{2} + \frac{1}{4} = \frac{3}{4} = 0.75$, $P_3 = \frac{3}{4} + (\frac{1}{2})^3 = \frac{7}{8} = 0.875$

$F_4 = \frac{15}{16} = 0.9375$, $F_5 = \frac{31}{32} = 0.96875$ ∴ no numbers bigger than F_5 ∴ F_6 not needed

∴ $x_1 = 2$, $x_2 = 3$, $x_3 = 1$, $x_4 = 5$, $x_5 = 1$

cakes C $\quad$ Bars B $\quad$, tea T ∴ $C \overset{*}{>} B \overset{*}{>} T$ ∴

$U(C) = 1$, $\quad U(T) = 0$ ∴

$U(B) = \min p$ st $\quad B \sim^* pC +_g (1-p)T$

is rewards and gambles are coherently comparable and ∴

utility respects preferences and $U(g) = E(U(g))$

∴ $U(B) = pU(c) + (1-p)U(T) = U(pC +_g (1-p)T) = p$

rewards being coherently comparable ∴ $C \overset{*}{>} B$, $B \overset{*}{>} T$ ∴ $C \overset{*}{>} T$

is $p > \frac{1}{2}$, $\quad C \overset{*}{>} B$: $pC +_g (1-p)B \overset{*}{>} pB +_g (1-p)C$

is $S \overset{*}{>} r$: $pS +_g (1-p)v \overset{*}{>} pr +_g (1-p)v$

can choose descion preference order to make decision tree easy

\PPECM3741PP2017/ \1C/ $\quad \mu|\underline{y} \sim N\left(\underbrace{\frac{\frac{m}{v^2} + \frac{n\bar{\bar{y}}}{\sigma^2}}{\frac{1}{v^2} + \frac{n}{\sigma^2}}}_{\hat{m}}, \underbrace{\frac{1}{\frac{1}{v^2} + \frac{n}{\sigma^2}}}_{\hat{v}}\right)$ ∴

$\bar{\bar{y}} = \frac{1}{n}\sum_{i=1}^{n}\log(y_i)$

$\mu|\underline{y} \sim N(\hat{m}, \hat{v})$ ∴

95% credible interval is $[a,b]$: $\int_a^b \pi(\mu \mid y)\,d\mu = 0.95$

$[\hat{m} - \hat{\sigma} Z_{0.975}, \hat{m} + \hat{\sigma} Z_{0.975}]$ $\therefore$ $Z_{0.975} = 1.96$

$P(\mu > z \mid y)$ $\therefore$ $1 - \Phi\left(\frac{z - \hat{m}}{\hat{\sigma}}\right)$

\PP2021/ T is transition time $\therefore$ $T_i \sim Exp(\lambda)$.

$T_p = T_{I_1} + T_{I_2}$ Show $T_p \sim Gamma(2, \lambda)$ $\therefore$

$f_{T_p}(z) = \frac{\lambda^2}{\Gamma(2)} z^{2-1} e^{-\lambda z} = \lambda^2 z e^{-\lambda z}$ $\therefore$

$f_{T_p}(z) = \int_{-\infty}^{\infty} f_{I_1, I_2}(t, z-t)\,dt = \int_{-\infty}^{\infty} f_{I_1}(t) f_{I_2}(z-t)\,dt =$

$\int_{-\infty}^{\infty} \lambda e^{-\lambda t} \lambda e^{-\lambda(z-t)}\,dt = \int_0^z \lambda^2 e^{-\lambda z}\,dt$ $\because t > 0, z - t > 0$ $\therefore t \in [0, z)$

$= \lambda^2 e^{-\lambda z} [t]_0^z$ $\therefore$ $\Gamma(2) = 1$ $\therefore$ $f_{T_p}(z) = \frac{\lambda^2}{\Gamma(2)} z e^{-\lambda z}$

\(b)/ $\lambda \sim Gamma(\alpha, \beta)$ $\underline{t} = t_1, \ldots, t_n$ $f(t_i \mid \lambda) \sim Gamma(2, \lambda)$ $\therefore$

By Bayes theorem: $\pi(\lambda \mid \underline{t}) \propto \pi(\lambda) \prod_{i=1}^{n} f(t_i \mid \lambda) \propto$

$\lambda^{\alpha-1} e^{-\beta\lambda} \prod_{i=1}^{n} \lambda^2 t_i e^{-\lambda t_i} \propto \lambda^{\alpha+2n-1} e^{-\lambda n \bar{t}} \propto \lambda^{\alpha+2n-1}$

$\lambda^{\alpha+2n-1} e^{-\beta\lambda} e^{-\lambda n \bar{t}} \propto \lambda^{\alpha+2n-1} e^{-(\beta + n\bar{t})\lambda}$ propor to a gamma density

~~$\lambda \mid \underline{t} \sim Gam$~~ $\lambda \mid \underline{t} \sim G(\alpha + 2n, \beta + n\bar{t}$

(c) $\therefore t = 7$ $\therefore t_1 = 7$ $\therefore \bar{t} = 7$, $n = 1$, $n\bar{t} = 7$

$T_{I_1} + T_{I_2} \mid \lambda \sim Gamma(2, \lambda)$

$\therefore$ ~~$T = I$~~ $T = T_{I_1} + T_{I_2} \mid \lambda$ $\therefore$ $P(T \leq 7 \mid t = 7)$

$T_i \mid \lambda \sim Exp(\lambda)$ $\therefore T_p \mid \lambda \sim Gamma(2, \lambda)$ $\therefore T_p = 7$

$\therefore P(T \leq 7 \mid t = 7)$ = posterior predictive $= \int_0^7 \int_0^\infty P(T \mid \lambda)\, \pi(\lambda \mid T_p = 7)\, d\lambda\, dT$

$= \int_0 \lambda^2 T e^{-\lambda T} \frac{\beta_n^{\alpha_n}}{\Gamma(\alpha_n)} \lambda^{\alpha_n - 1} e^{-\lambda \beta_n}\, d\lambda\, dT$ $\because$ β_n, α_n from posterior

\revision videos/ if ever stuck in Bayes, you can always move to the next part without issue

\chapter 2: 1) classical vs Bayes. 2) Axioms of probability from Expectation

3): 2 definitions of expectation, coherence

4): proved theorems 5) random quantity-event duality $\wedge$ 'and' $\vee$ 'or' $\sim$ 'not' $A \vee B$ true is A or B both happen

$\therefore x, y$ numbers $x \wedge y = \min(x, y)$ $x \vee y = \max(x, y)$ $\tilde{x} = \sim x = 1 - x$

\revision videos/ $A \vee B = \sim(\tilde{A} \wedge \tilde{B}) = 1-(1-A)(1-B) = A+B-AB$

for event A: $P(A) = E(A)$ $\therefore P(A \vee B) = P(A+B-AB) = P(A)+P(B)-P(AB)$

by linearity of expectation , $A^2 = A$

\chapter 3/ Applying Bayes thm: $\pi(\theta|y) \propto \pi(\theta)P(y|\theta)$

By Bayes theorem $\therefore \pi(\theta|y) \propto \pi(\theta)P(y|\theta)$

conjugacy: is the form of posterior and prior are the same the have a conjugate prior for this model

The posterior distribution $\neq$ posterior density

$\therefore$ ~~of~~ $\theta|y \sim N(\mu, \sigma^2)$ is a distribution and is not a density

\Exchangeability/

\prediction/: prior predictive: $P(y) = \int_{-\infty}^{\infty} P(y,\theta)\,d\theta = \int_{-\infty}^{\infty} P(y|\theta)\pi(\theta)\,d\theta$

posterior predictive: $P(y|y) = \int_{-\infty}^{\infty} P(y,\theta|y)\,d\theta = \int_{-\infty}^{\infty} P(y|\theta)\pi(\theta|y)\,d\theta$

\inference/ Estimation $E(\theta|y)$, MAP, Credible intervals (these are all integrals or involve integrals) - they all have frequentist analogues:

MAP is like MLE   credible intervals have a parallel with confidence intervals

• Priors : subjective vs objective debate

proper vs improper priors $\therefore$ its proper is it integrates to 1 $\therefore$, is $\theta \in (-\infty, \infty)$, $\pi(\theta) \propto 1$ $\therefore$ doesnt integrate to 1 $\therefore$ improper prior

• Normal approximations, asymptotics

$\therefore$ Normal approximation to posterior: $\theta|y \,\dot\sim\, N(\hat{\theta}, I(\hat{\theta})^{-1})$,

$\hat{\theta}$ = MAP (posterior mode)

$$I(\hat{\theta}) = \left[-\frac{d^2}{d\theta^2}\log\pi(\theta|y)\right]\Big|_{\theta=\hat{\theta}}$$

$\theta \sim Beta(a,b)$ $y|\theta \sim Ber(\theta)$, $\theta|y_1,\ldots,y_n) \sim Beta(a+n, b+\sum_{i=1}^{n} y_i - n)$

$\theta|y \to N(\theta_0, (nJ(\theta_0))^{-1})$ $\theta_0 = \underset{\theta}{\text{True}}$ (minKL)

$$J(\theta) = E\left[-\frac{\partial^2 l(x|\theta)}{\partial\theta^2}\right]$$

\Chapter 4/1)• B.H.M.s NHM

3)• Monte Carlo: $E(g(\theta)) = \int_\theta g(\theta)p(\theta)d\theta \approx \frac{1}{N}\sum_{i=1}^{n} g(\theta_i)$,

$\theta_1,\dots,\theta_N$ samples from $p(\theta)$

3)• sampling simple distributions: inverse CDF (continuous/Discrete/mixtures)

• Box-muller

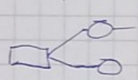
• Rejection Sampling

• Rejection Sampling for log-concave distributions, $H<0$

4)• MCMC, Gibbs sampler, MH, HMC, AMH

MCMC, Gibbs sample, Metropolis hastings (MH) are most important

\chapter 5/1) Decision trees, solving them with roll back



Utility

2) Utility theorey: Def's, Notation: $\overset{*}{>}$, $\overset{*}{\sim}$, $+_g$

G gamble: $G_1 = pR_1 +_g (1-p)R_2$ recieve rewards R_1, probability p and R_2 otherwise ∴

$U(r) \geq U(R) \iff r \overset{*}{\geq} R$

(ii) $U(\text{gamble}) = E[U(\text{gamble})]$ ∴

$U(G_1) = U(pR_1 +_g (1-p)R_2) = pU(R_1) + (1-p)U(R_2)$

• Method for constructing Utility

• Assumptions: rewards and gambles being coherently compared

\Danny Exam/\PP2020Q1C:i/ θ outcomes, d decision,

$\pi(\theta)$, $U(\theta,d)$ ∴ d^* is optimal decision ∴

$$d^* = \arg\max_d E[U(\theta,d)] = \arg\max_d \int_{-\infty}^{\infty} U(\theta,d)\pi(\theta)d\theta$$

\1cii/ $U(\theta,d) - (\theta-d)^2$, $E(U(\theta,d)|y) = -(E[\theta^2|y] - 2dE[\theta|y] + d^2)$

$E(U(\theta)) = -[E[\theta^2|y] - 2dE[\theta|y] + d^2)$ ∴

$$E[U(\theta)] = \int_{-\infty}^{\infty} U(\theta)\pi(\theta|y)d\theta = \int_{-\infty}^{\infty} -(\theta-d)^2\pi(\theta|y)d\theta = \int_{-\infty}^{\infty} -(\theta^2 - 2d\theta + d^2)\pi(\theta|y)d\theta$$

$$= -\int_{-\infty}^{\infty} \theta^2\pi(\theta|y)d\theta + \int_{-\infty}^{\infty} 2d\theta\,\pi(\theta|y) - \int_{-\infty}^{\infty} d^2\pi(\theta|y)d\theta =$$

\Danny Exam:/ $-E[\theta^2|y] + E[2d\theta|y] - E[d^2|y] =$ by L.o.E

$-E[\theta^2|y] + 2dE[\theta|y] - d^2E[1|y] =$

$-E[\theta^2|y] + 2dE[\theta|y] - d^2 = -(E[\theta^2|y] - 2dE[\theta|y] + d^2)$

\1c iii/ show $d^* = E(\theta|y)$ $\therefore$ $d^* = \arg\max_d E[U(\theta,d)]$ $\therefore$

$E[U(\theta,d)|y] = -(E[\theta^2|y] - 2dE[\theta|y] + d^2)$ is negative $\therefore$

minimise $E[\theta^2|y] - 2dE[\theta|y] + d^2$ $\therefore$

$\frac{dU}{dd^*} = \frac{dU}{d(d)}\Big|_{d=d^*} = -2E[\theta|y] + 2d\Big|_{d=d^*} = -2E[\theta|y] + 2d^* = 0$ $\therefore$

$d^* = E[\theta|y]$

\PP2020 Q2/ \2a/ $X \sim f(x)$ $g(x) = cf(x)$

X_* a random point under g

$\therefore$ pdf of X_*: $f_{X_*}(x) = \lim_{\Delta x \to 0} \frac{F_{X_*}(x+\Delta x) - F_{X^*}(x)}{\Delta(x)}$

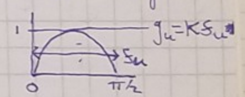
$= \lim_{\Delta x \to 0} \frac{P(X^* \in [x, x+\Delta x])}{\Delta x}$ $\therefore$

Area under $g(x) = c$ $\therefore$ $P(X^* \in [x, x+\Delta x]) \approx \frac{cf(x)\Delta x}{c} = f(x)\Delta x$

$\therefore f_{X^*}(x) = \lim_{\Delta x \to 0} \frac{P(X^* \in [x, x+\Delta x])}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x)\Delta x}{\Delta x} = \lim_{\Delta x \to 0} f(x) = f(x)$ $\therefore$

$f_{X^*}(x) = f(x)$

\2b/ θ pdf: $f(\theta) = \sin\theta$ $\theta \in [0, \frac{\pi}{2}]$ "optimal uniform envelope rejection method for sampling from $f(\theta)$"



$\therefore$ let $f_u(\theta)$ be pdf $\text{Unif}[0, \frac{\pi}{2}]$

$g_u(\theta) = Kf_u(\theta)$, $K = \max f(\theta) = 1$ $\therefore$

Sample θ_u from f_u and U from $\text{Unif}(0,1)$

$\therefore$ Accept θ_u if $Ug_u(\theta_u) \le \frac{2}{\pi}$

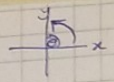
\2c/ Acceptance rate: $\frac{\text{Area under } f(x)}{\text{Area under } g_u(x)} = \frac{1}{(\pi/2)} = \frac{2}{\pi}$

\2d/ $f(x,y) = \frac{y}{x^2+y^2}$; $x,y \ge 0$ $x^2+y^2 \le 2$

$\therefore$ let $y = r\sin\theta$, $x = r\cos\theta$ $\therefore x^2+y^2 = r^2$ $\therefore$

Jacobian is: $J(r,\theta) = \begin{vmatrix} \frac{dx}{dr} & \frac{dx}{d\theta} \\ \frac{dy}{dr} & \frac{dy}{d\theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix} = r\cos^2\theta + r\sin^2\theta = r$ $\therefore$

$\cancel{f(r,\theta) \propto r\sin\theta}$ $f(r,\theta) \propto \frac{J(r,\theta)\, r\sin\theta}{r^2} \propto \frac{r\, r\sin\theta}{r^2} \propto \sin\theta$ $\therefore$

$x \ge 0$, $y \ge 0$ means $\theta \in [0, \frac{\pi}{2}]$ ∴ 

$x^2+y^2 \le 2$ ∴ $r^2 \le 2$ ∴ $r \le \sqrt{2}$ ∴

$\iint f(r,\theta)\, d\theta\, dr = 1 = \int_0^{\sqrt{2}}\int_0^{\pi/2} k\sin\theta\, d\theta\, dr = \sqrt{2}\int_0^{\pi/2} k\sin\theta\, d\theta = \sqrt{2}\, k\left[-\cos\theta\right]_0^{\pi/2} =$

$\sqrt{2}\,k\left[-0+1\right] = \sqrt{2}\,k = 1$ ∴ $k = \frac{1}{\sqrt{2}}$ ∴

$f(r,\theta) = \frac{1}{\sqrt{2}}\sin\theta$ ∴

$f(\theta) = \int_0^{\sqrt{2}} \frac{1}{\sqrt{2}}\sin\theta\, dr = \sin\theta$

∴ $f(r,\theta) = f(r|\theta)f(\theta)$ ∴ $f(r|\theta) = \frac{1}{\sqrt{2}} = f(r)$ ∴

$r \sim \text{Unif}(0,\sqrt{2})$ ∴

algorithm is: i: Sample R^* from $\text{Unif}(0,\sqrt{2})$

(ii) Sample θ^* as in part (b)

(iii) if θ^* accepted $X^* = R^*\cos\theta^*$, $Y^* = R^*\sin\theta^*$

PP2020 Q2a $X \sim f(x)$ X^* a random point under g

∴ pdf of X^*: $f_{X^*}(x) = \lim_{\Delta x \to 0} \frac{F_{X^*}(x+\Delta x) - F_{X^*}(x)}{\Delta x} = \lim_{\Delta x\to 0} \frac{P(X^* \in [x, x+\Delta x])}{\Delta x}$

∴ area under $g(x) = c$ ∴ $P(X^* \in [x, x+\Delta x]) = f(x)\Delta x$ ∴

$f_{X^*}(x) = \lim_{\Delta x\to 0} \frac{f(x)\Delta x}{\Delta x} = \lim_{\Delta x \to 0} f(x) = f(x)$

2b let $f_u(\theta)$ be pdf of $\text{Unif}[0,\frac{\pi}{2}]$

∴ $g_u(\theta) = k f_u(\theta)$, $k = \text{Max} f(\theta)$ ∴ $\text{Max} f(\theta) = f(\frac{\pi}{4}) = \sin(\frac{\pi}{4}) = 1$ ∴

$k = 1$ ∴

Sample θ_u from f_u and U from $\text{Unif}(0,1)$

∴ Accept θ_u if $U g_u(\theta_u) \le \frac{2}{\pi}$

2c Acceptance rate $= \frac{\text{Area under } f(x)}{\text{Area under } g_u(x)}$

2d let $y = r\sin\theta$, $x = r\cos\theta$ ∴ $x^2+y^2 = r^2$ ∴ Jacobian: $J = r$ ∴

$f(r,\theta) \propto \frac{J\, r\sin\theta}{r^2} = \frac{r\, r\sin\theta}{r^2} = \sin\theta$ ∴ $f(r,\theta) = k\sin\theta$ ∴

$x \ge 0$, $y \ge 0$: $\theta \in [0, \frac{\pi}{2}]$

$x^2+y^2 \le 2$ ∴ $r^2 \le 2$ ∴ $r \le \sqrt{2}$ ∴

$1 = \int_0^{\sqrt{2}}\int_0^{\pi/2} k\sin\theta\, d\theta\, dr = \sqrt{2}\int_0^{\pi/2} k\sin\theta\, d\theta = \sqrt{2}\,k\left[-\cos\theta\right]_0^{\pi/2} = \sqrt{2}\,k\left[-0+1\right] = \sqrt{2}\,k = 1$

∴ $k = \frac{1}{\sqrt{2}}$ ∴

\Dany Exam:/ ~~PP2022~~ $f(r,\theta)=\frac{1}{\sqrt{2}}\sin\theta$ ∴

$f(\theta)=\int_0^{\sqrt{2}}\frac{1}{\sqrt{2}}\sin\theta\, dr=\sin\theta$ ∴

$f(r,\theta)=f(r|\theta)f(\theta)$ ∴ $f(r|\theta)=\frac{1}{\sqrt{2}}=f(r)$ ∴

$r\sim \text{unif}(0,\sqrt{2})$ ∴

algorithem is : i: sample r^* from $\text{unif}(0,\sqrt{2})$

ii: Sample θ^* from f_u, $f_u\sim\text{Unif}[0,\frac{\pi}{2}]$ as in part (b)

∴ Accept θ^* if $Ug_u(\theta^*)\leq\frac{2}{\pi}$, U sampled from $\text{Unif}(0,1)$,

$g_u=\frac{1}{\sqrt{2}}f_u$

iii: if θ^* accepted : $X^*=r^*\cos\theta^*$, $Y^*=r^*\sin\theta^*$

\PP2020Q1C:/ $d^*=\arg\max_d E[U(\theta,d)]=\arg\max_d\int_{-\infty}^{\infty}U(\theta,d)\pi(\theta)\,d\theta$

\Cii/ $E[U(\theta,d)|y]=\int_{-\infty}^{\infty}U(\theta,d)\pi(\theta|y)\,d\theta=\int_{-\infty}^{\infty}U(\theta,d)\pi(\theta|y)\,d\theta=$

$\int_{-\infty}^{\infty}-(\theta-d)^2\pi(\theta|y)\,d\theta=-\int_{-\infty}^{\infty}(\theta^2-2d\theta+d^2)\pi(\theta|y)\,d\theta=$

$-\left(\int_{-\infty}^{\infty}\theta^2\pi(\theta|y)\,d\theta-2d\int_{-\infty}^{\infty}\theta\pi(\theta|y)+d^2\int_{-\infty}^{\infty}1\,\pi(\theta|y)\,d\theta\right)=$

$-(E[\theta^2|y]-2dE[\theta|y]+d^2E[1])=-(E[\theta^2|y]-2dE[\theta|y]+d^2)$

\1Ciii/ ∴ $E[U(\theta,d)|y]$ is negative ∴ minimise it ∴ ∴

$d^*=\arg\max_d E[U(\theta,d)]$ ~~argmax~~ ∴

$\frac{dU}{dd^*}=\frac{dU}{d(d)}\Big|_{d=d^*}=+2E[\theta|y]-2d\Big|_{d=d^*}=0$ ∴

$2E[\theta|y]-2d^*=0$ ∴ $d^*=E[\theta|y]$

\1Ciii/ $d^*=\arg\max_d E[U(\theta,d)]$, $E[U(\theta,d)|y]$ is negative ∴

minimise $-E[\theta^2|y]-2dE[\theta|y]+d^2$ ∴

$\frac{dU}{dd^*}=\frac{dU}{dd}\Big|_{d=d^*}=\frac{d}{dd}\left(-E[\theta^2|y]-2dE[\theta|y]+d^2\right)\Big|_{d=d^*}=-2E[\theta|y]+2d\Big|_{d=d^*}=$

$-2E[\theta|y]+2d^*=0$ ∴ $d^*=E[\theta|y]$

\pp2017-8 Q3/ \3a/ $f(y|\sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}$ will be likelihood func with y.

prior: $\pi(\sigma^2) = \frac{(S^2 \nu/2)^{\nu/2}}{\Gamma(\nu/2)} \sigma^{-(2+\nu)} e^{-\frac{\nu S^2}{2\sigma^2}}$, $\sigma^2 > 0$

$\therefore$ By Bayes theorem: $\pi(\theta|y) \propto \pi(\theta) p(y|\theta)$ $\therefore$

$\pi(\sigma^2|y) \propto \pi(\sigma^2) f(y|\sigma^2) \propto \cancel{\pi(\sigma^2) f(y) \propto}\ \pi(\sigma^2) \prod_{i=1}^{n} f(y_i|\sigma^2) \propto$

$\frac{(S^2\nu/2)^{\nu/2}}{\Gamma(\nu/2)} \sigma^{-(2+\nu)} e^{-\frac{\nu S^2}{2\sigma^2}} \prod_{i=1}^{n} f(y_i|\sigma^2) \propto \sigma^{-(2+\nu)} e^{-\frac{\nu S^2}{2\sigma^2}} \prod_{i=1}^{n} f(y_i|\sigma^2) \propto$

$\sigma^{-(2+\nu)} e^{-\frac{\nu S^2}{2\sigma^2}} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2} \propto \sigma^{-(2+\nu)} e^{-\frac{\nu S^2}{2\sigma^2}} \prod_{i=1}^{n} \frac{1}{\sigma} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2} \propto$

$\sigma^{-(2+\nu)} e^{-\frac{\nu S^2}{2\sigma^2}} \sigma^{-n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2} \propto \sigma^{-(2+\nu+n)} e^{-\frac{1}{2\sigma^2}\left[\nu S^2 + \sum_{i=1}^{n}(y_i-\mu)^2\right]} \propto$

$\sigma^{-(2+\nu_n)} e^{-\frac{1}{2\sigma^2}\nu_n S_n^2}$; $\nu_n = \nu + n$, $\nu_n S_n^2 = \nu S^2 + \sum_{i=1}^{n}(y_i-\mu)^2$ $\therefore$

$S_n^2 = \frac{1}{\nu_n}\left[\nu S^2 + \sum_{i=1}^{n}(y_i-\mu)^2\right] = \frac{1}{\nu+n}\left[\nu S^2 + \sum_{i=1}^{n}(y_i-\mu)^2\right]$ $\therefore$

$\sigma^2|y \sim \mathrm{Inv}\text{-}\chi^2(\nu_n, S_n^2)$;

$\nu_n = \nu + n$, $\frac{\nu_n S_n^2}{2\sigma^2} = -\frac{1}{2\sigma^2}(\nu S^2 + \sum_{i=1}^{n}(y_i-\mu)^2)$ $\therefore$

$S_n^2 = (\nu S^2 + \sum_{i=1}^{n}(y_i-\mu))/(\nu+n)$

\3b/ Posterior predictive $p(y|y) = \int_{-\infty}^{\infty} p(y|\sigma^2)\pi(\sigma^2|y)\,d\sigma^2$

$\because p(y|y) = \int_{-\infty}^{\infty} p(y|\theta)\pi(\theta|y)\,d\theta$ $\because$ $\cancel{p(y,\theta) = p(y}$ $p(y,\theta|y) = p(y|\theta)\pi(\theta|y)$

$\therefore p(y|y) = \int_{-\infty}^{\infty} p(y,\theta|y)\,d\theta = \int_{-\infty}^{\infty} p(y|\theta)\pi(\theta|y)\,d\theta$

$\because p(y,\theta|y) = p(y|\theta)\pi(\theta|y)$ $\therefore p(y|y) = \int_{-\infty}^{\infty} p(y,\theta|y)\,d\theta = \int_{-\infty}^{\infty} p(y|\theta)\pi(\theta|y)\,d\theta$

\3b/ posterior predictive: $p(y|y) = \int_{-\infty}^{\infty} p(y|\theta)\pi(\theta|y)\,d\theta =$
$\int_{-\infty}^{\infty} p(y|\sigma^2)\pi(\sigma^2|y)\,d\sigma^2 =$

$\int_{0}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(y-\mu)^2} \frac{(S_n^2\nu_n/2)^{\nu_n/2}}{\Gamma(\nu_n/2)} \sigma^{-(2+\nu_n)} e^{-\frac{\nu_n S_n^2}{2\sigma^2}}\,d\sigma^2 =$ $\because \sigma^2 \geq 0$

$\frac{(S_n^2\nu_n/2)^{\nu_n/2}}{\Gamma(\nu_n/2)\sqrt{2\pi}} \int_{0}^{\infty} \sigma^{-(3+\nu_n)} e^{-\frac{1}{2\sigma^2}(y-\mu)^2 - \frac{1}{2\sigma^2}\nu_n S_n^2}\,d\sigma^2 =$

$\frac{(S_n^2\nu_n/2)^{\nu_n/2}}{\Gamma(\nu_n/2)\sqrt{2\pi}} \int_{0}^{\infty} \sigma^{-(3+\nu_n)} e^{-\frac{1}{2\sigma^2}((y-\mu)^2 + \nu_n S_n^2)}\,d\sigma^2$

$\frac{(S_n^2\nu_n)^{\nu_n/2} 2^{-1/2} 2^{-\nu_n/2}}{\Gamma(\nu_n/2)\sqrt{\pi}} \int_{0}^{\infty} \sigma^{-(3+\nu_n)} e^{-\frac{1}{2\sigma^2}((y-\mu)^2 + \nu_n S_n^2)}\,d\sigma^2 =$

$\frac{(S_n^2\nu_n)^{\nu_n/2} 2^{-\frac{1}{2}(1+\nu_n)}}{\Gamma(\nu_n/2)\sqrt{\pi}} \int_{0}^{\infty} \sigma^{-(3+\nu_n)} e^{-\frac{1}{2\sigma^2}((y-\mu)^2 - \nu_n S_n^2)}\,d\sigma^2$

$\therefore K = \sqrt{2}$

\Danny Exam/\PP2018 Q3/3a/ $f(y|\sigma^2) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(y-\mu)^2}$

Prior: $\pi(\sigma^2) = \frac{(S^2\nu/2)^{\nu/2}}{\Gamma(\nu/2)} \sigma^{-(2+\nu)} e^{-\frac{\nu S^2}{2\sigma^2}}, \sigma^2 > 0$

$\therefore$ By Bayes theorem: $\pi(\theta|\underline{y}) \propto \pi(\theta) f(\underline{y}|\theta) \therefore$

$$\pi(\sigma^2|\underline{y}) \propto \pi(\sigma^2) f(\underline{y}|\sigma^2) \propto \frac{(S^2\nu/2)^{\nu/2}}{\Gamma(\nu/2)} \sigma^{-(2+\nu)} e^{-\frac{\nu S^2}{2\sigma^2}} f(\underline{y}|\sigma^2) \propto$$

$$\sigma^{-(2+\nu)} e^{-\frac{\nu S^2}{2\sigma^2}} f(\underline{y}|\sigma^2) \propto \sigma^{-(2+\nu)} e^{-\frac{\nu S^2}{2\sigma^2}} \prod_{i=1}^{n} f(y_i|\sigma^2) \propto$$

$$\sigma^{-(2+\nu)} e^{-\frac{\nu S^2}{2\sigma^2}} \prod_{i=1}^{n} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2} \propto$$

$$\sigma^{-(2+\nu)} e^{-\frac{\nu S^2}{2\sigma^2}} \prod_{i=1}^{n} \sigma^{-1} e^{-\frac{1}{2\sigma^2}(y_i-\mu)^2} \propto$$

$$\sigma^{-(2+\nu)} e^{-\frac{\nu S^2}{2\sigma^2}} \sigma^{-n} e^{-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(y_i-\mu)^2} \propto \sigma^{-(2+\nu+n)} e^{-\frac{1}{2\sigma^2}(\nu S^2 + \sum_{i=1}^{n}(y_i-\mu)^2)} \propto$$

$$\sigma^{-(2+\nu_n)} e^{-\frac{1}{2\sigma^2}\nu_n S_n^2} \quad ; \nu_n = \nu + n, \quad \nu_n S_n^2 = \nu S^2 + \sum_{i=1}^{n}(y_i-\mu)^2 = (\nu+n) S_n^2 \therefore$$

$$S_n^2 = \frac{1}{\nu+n}\left(\nu S^2 + \sum_{i=1}^{n}(y_i-\mu)^2\right) \therefore$$

$$\sigma^2|\underline{y} \sim Inv-\chi^2(\nu_n, S_n^2)$$

\3b/ Posterior predictive: $p(y|\underline{y}) = \int_{-\infty}^{\infty} p(y|\theta)\pi(\theta|\underline{y})\,d\theta \therefore \sigma^2 > 0 \therefore$

$$p(y|\underline{y}) = \int_{-\infty}^{\infty} p(y|\sigma^2)\pi(\sigma^2|\underline{y})\,d\sigma^2 = \int_{0}^{\infty} \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{1}{2\sigma^2}(y-\mu)^2} \pi(\sigma^2|\underline{y})\,d\sigma^2 =$$

$$\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \sigma^{-1} e^{-\frac{1}{2\sigma^2}(y-\mu)^2} \pi(\sigma^2|\underline{y})\,d\sigma^2 =$$

$$\frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} \sigma^{-1} e^{-\frac{1}{2\sigma^2}(y-\mu)^2} \frac{(S_n^2\nu_n/2)^{\nu_n/2}}{\Gamma(\nu_n/2)} \sigma^{-(2+\nu_n)} e^{-\frac{\nu_n S_n^2}{2\sigma^2}}\,d\sigma^2 =$$

$$\frac{(S_n^2\nu_n/2)^{\nu_n/2}}{\Gamma(\nu_n/2)\sqrt{2}\sqrt{\pi}} \int_{0}^{\infty} \sigma^{-1-(2+\nu_n)} e^{-\frac{1}{2\sigma^2}(y-\mu)^2 - \frac{1}{2\sigma^2}\nu_n S_n^2}\,d\sigma^2 =$$

$$\frac{(S_n^2\nu_n)^{\nu_n/2}(\frac{1}{2})^{\nu_n/2} 2^{-1/2}}{\Gamma(\nu_n/2)\sqrt{\pi}} \int_{0}^{\infty} \sigma^{-(3+\nu_n)} e^{-\frac{1}{2\sigma^2}((y-\mu)^2 + \nu_n S_n^2)}\,d\sigma^2 =$$

$$\frac{(S_n^2\nu_n)^{\nu_n/2} 2^{-\nu_n(\frac{1}{2})} 2^{-(\frac{1}{2})}}{\Gamma(\nu_n/2)\sqrt{\pi}} \int_{0}^{\infty} \sigma^{-(3+\nu_n)} e^{-\frac{1}{2\sigma^2}((y-\mu)^2 + \nu_n S_n^2)}\,d\sigma^2 =$$

$$\frac{(S_n^2\nu_n)^{\nu_n/2} 2^{-\frac{1}{2}(1+\nu_n)}}{\Gamma(\nu_n/2)\sqrt{\pi}} \int_{0}^{\infty} \sigma^{-(3+\nu_n)} e^{-\frac{1}{2\sigma^2}((y-\mu)^2 - \nu_n S_n^2)}\,d\sigma^2$$

\3C/ $p(y|\underline{y}) = \frac{(S_n^2\nu_n)^{\nu_n/2}\,2^{-\frac{1}{2}(1+\nu_n)}}{\Gamma(\nu_n/2)\sqrt{\pi}} \int_0^\infty \sigma^{-(3+\nu_n)} e^{-\frac{1}{2\sigma^2}((y-\mu)^2+\nu_n S_n^2)}\, d\sigma^2$

$\Gamma(\alpha) = \int_0^\infty z^{\alpha-1} e^{-z}\, dz$ ∴

let $z = +\frac{1}{2\sigma^2}((y-\mu)^2+\nu_n S_n^2) = \frac{1}{2\sigma^2}(B)$ for $B = (y-\mu)^2+\nu_n S_n^2$ ∴

$(y-\mu)^2+\nu_n S_n^2$ is constant wrt σ^2 ∴

∴ $\int_0^\infty d\sigma^2 \rightarrow \int_\infty^0$

∵ $\frac{dz}{d(\sigma^2)} = \frac{d}{d\sigma^2}\left(\frac{B}{2\sigma^2}\right) = \frac{d}{d\sigma^2}\left[\frac{B}{2}(\sigma^2)^{-1}\right] = (-1)\frac{B}{2}(\sigma^2)^{-2} = \cancel{(-1)}\ \frac{-B}{2(\sigma^2)^2}$ ∴

$\sigma^2 = 0 \rightarrow \cancel{\frac{B}{2\sigma}}\ z = \frac{1}{2\sigma^2}(B)\Big|_{\sigma^2=0} = \infty$, $\sigma^2 = \infty \rightarrow z = \frac{B}{2\sigma^2}\Big|_{\sigma^2=\infty} = 0$ ∴

$-\frac{2(\sigma^2)^2}{B}\, dz = d\sigma^2$ ∴ $\int_0^\infty d\sigma^2 = \int_\infty^0 -\frac{2(\sigma^2)^2}{B}\, dz$

∵ $\frac{dz}{d\sigma^2} = \frac{dz}{d(\sigma^2)} = \frac{d}{d(\sigma^2)}\left(\frac{B}{2\sigma^2}\right) = \frac{d}{d(\sigma^2)}\left[\frac{B}{2}(\sigma^2)^{-1}\right] = \frac{B}{2}(-1)(\sigma^2)^{-2} = \frac{-B}{2\sigma^4}$

∴ $dz = \frac{-B}{2\sigma^4}\, d\sigma^2$, $\sigma^2 = \frac{B}{2z}$ ∴ $\sigma^4 = \frac{B^2}{4z^2}$ ∴

$\cancel{dz = \frac{-B}{2(B}}$ $dz = \frac{-B}{2\sigma^4}\, d\sigma^2 = -\frac{B}{2}(\sigma^4)^{-1}\, d\sigma^2 = -\frac{B}{2}\left(\frac{B^2}{4z^2}\right)^{-1} d\sigma^2 = \frac{-B}{2}\,\frac{4z^2}{B^2}\, d\sigma^2 =$

$\frac{-2z^2}{B}\, d\sigma^2$ ∴ $d\sigma^2 = \frac{-B}{2z^2}\, dz$ ∴ $\sigma^2 = \frac{B}{2z}$ ∴ $\sigma = \left(\frac{B}{2z}\right)^{1/2}$

$p(y|\underline{y}) = \frac{(S_n^2\nu_n)^{\nu_n/2}\,2^{-\frac{1}{2}(1+\nu_n)}}{\Gamma(\nu_n/2)\sqrt{\pi}} \int_\infty^0 \left(\left(\frac{B}{2z}\right)^{1/2}\right)^{-(3+\nu_n)} e^{-z}\, \frac{-B}{2z^2}\, dz =$

$\frac{(S_n^2\nu_n)^{\nu_n/2}\,2^{-\frac{1}{2}(1+\nu_n)}}{\Gamma(\nu_n/2)\sqrt{\pi}} \int_\infty^0 \left(\frac{B}{2z}\right)^{-\frac{1}{2}(3+\nu_n)} e^{-z}\, \frac{-B}{2z^2}\, dz =$

$\frac{(S_n^2\nu_n)^{\nu_n/2}\,2^{-\frac{1}{2}(1+\nu_n)}}{\Gamma(\nu_n/2)\sqrt{\pi}} \int_0^\infty B^{-\frac{1}{2}(3+\nu_n)}\, 2^{\frac{1}{2}(3+\nu_n)}\, z^{\frac{1}{2}(3+\nu_n)} e^{-z} B^1 2^{-1} z^{-2}\, dz =$

$\frac{(S_n^2\nu_n)^{\nu_n/2}\,2^{-\frac{1}{2}(1+\nu_n)}}{\Gamma(\nu_n/2)\sqrt{\pi}} \int_0^\infty B^{-\frac{1}{2}(3+\nu_n)+1}\, 2^{\frac{1}{2}(3+\nu_n)-1}\, z^{\frac{1}{2}(3+\nu_n)-2}\, e^{-z}\, dz$

$\frac{(S_n^2\nu_n)^{\nu_n/2}\,2^{-\frac{1}{2}(1+\nu_n)}\,2^{\frac{1}{2}(3+\nu_n)-\frac{2}{2}}\,B^{-\frac{1}{2}(3+\nu_n)+\frac{2}{2}}}{\Gamma(\nu_n/2)\sqrt{\pi}} \int_0^\infty z^{\frac{1}{2}(3+\nu_n)-\frac{4}{2}} e^{-z}\, dz =$

$\frac{(S_n^2\nu_n)^{\nu_n/2}\,2^{-\frac{1}{2}(1+\nu_n)}\,2^{\frac{1}{2}(1+\nu_n)}\,B^{-\frac{1}{2}(1+\nu_n)}}{\Gamma(\nu_n/2)\sqrt{\pi}} \int_0^\infty \cancel{z^{\frac{1}{2}(-2+\nu_n)}}\ z^{\frac{1}{2}(3+\nu_n)-\frac{2}{2}-1} e^{-z}\, dz =$

$\frac{(S_n^2\nu_n)^{\nu_n/2}\,2^{-\frac{1}{2}(1+\nu_n)+\frac{1}{2}(1+\nu_n)}\,B^{-\frac{1}{2}(1+\nu_n)}}{\Gamma(\nu_n/2)\sqrt{\pi}} \int_0^\infty z^{\frac{1}{2}(1+\nu_n)-1} e^{-z}\, dz =$

$\frac{(S_n^2\nu_n)^{\nu_n/2}\,2^0\,B^{-\frac{1}{2}(1+\nu_n)}}{\Gamma(\nu_n/2)\sqrt{\pi}} \int_0^\infty z^{\frac{\nu_n+1}{2}-1} e^{-z}\, dz = \frac{(S_n^2\nu_n)^{\nu_n/2}\,B^{-\frac{1}{2}(1+\nu_n)}}{\Gamma(\nu_n/2)\sqrt{\pi}} \int_0^\infty z^{\frac{\nu_n+1}{2}-1} e^{-z}\, dz$

$= \frac{(S_n^2\nu_n)^{\nu_n/2}\,B^{-\frac{1}{2}(1+\nu_n)}}{\Gamma(\nu_n/2)\sqrt{\pi}}\, \Gamma\left(\frac{\nu_n+1}{2}\right) =$

\DannyExam/ $\frac{(S_n^2\nu_n)^{\nu_n/2}\Gamma\left(\frac{\nu_n+1}{2}\right)}{\Gamma(\nu_n/2)\sqrt{\pi}}\left((y-\mu)^2+\nu_n S_n^2\right)^{-\frac{1}{2}(1+\nu_n)} =$

$$\frac{\Gamma\left(\frac{\nu_n+1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu_n/2)}\frac{(S_n^2\nu_n)^{\frac{\nu}{2}+\frac{1}{2}}}{(S_n^2\nu_n)^{\frac{1}{2}}}\left((y-\mu)^2+\nu_n S_n^2\right)^{-\frac{1}{2}(1+\nu_n)} =$$

$$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu_n/2)\,S\sqrt{\nu}}\left(\frac{(y-\mu)^2+S_n^2\nu_n}{S_n^2\nu_n}\right)^{-\frac{1}{2}(1+\nu_n)} =$$

$$\frac{\Gamma\left(\frac{\nu+1}{2}\right)}{\sqrt{\pi}\,\Gamma(\nu_n/2)\,S\sqrt{\nu}}\left(1+\frac{y-\mu^2}{S_n^2\nu}\right)^{-\frac{1+\nu_n}{2}} =$$

$$\frac{\Gamma\left(\frac{\nu_n+1}{2}\right)}{\Gamma\left(\frac{\nu_n}{2}\right)\sqrt{\pi\nu}\,S_n}\left(1+\frac{y-\mu^2}{S_n^2\nu}\right)^{-\frac{1+\nu_n}{2}} = \frac{\Gamma\left(\frac{\nu_n+1}{2}\right)}{\Gamma\left(\frac{\nu_n}{2}\right)\sqrt{\pi}\,S_n}\left(1+\frac{1}{\nu_n}\left(\frac{y-\mu}{S_n}\right)^2\right)^{-\frac{\nu_n+1}{2}}$$

\Week 7/ \2a/ Start at values $X_1^{(0)}$ and $X_2^{(0)}$. For $t=1,2,\dots$ set $X_1^{(t)}, X_2^{(t)}$ by first sampling $X_1^{(t)}$ from $P(X_1 \mid X_2 = X_2^{(t-1)})$, then $X_2^{(t)}$ from $P(X_2 \mid X_1 = X_1^{(t)})$ $\therefore$ $X_1^{(1)}$ from $P(X_1 \mid X_2 = X_2^{(0)})$, then $X_2^{(1)}$ from $P(X_2 \mid X_1 = X_1^{(1)})$ ...

\2b/ Suppose X is discrete taking values $x_1,\dots,x_m$ with probabilities $p_1,\dots,p_m$ respectively. To sample from X, first define $F_j = p_1+\dots+p_j$ then $F^{-1}(p) = x_j$ for $p \in (F_{j-1}, F_j]$

$\therefore$ generate uniform values on $[0,1]$ and treat these as p in the above to get sample values X_U

\2c/ The joint probabilities are proportional to those in the table and $\because$ each row and column sums to 0.4:

$$P(X_1=1 \mid X_2=1) = \frac{P(X_1=1, X_2=1)}{P(X_2=1)} = \frac{0.2k}{0.4k} = 0.5$$

$$P(X_1=2 \mid X_2=1) = P(X_1=3 \mid X_2=1) = \frac{P(X_1=2, X_2=1)}{P(X_2=1)} = \frac{0.1k}{0.4k} = 0.25$$

due to symmetry of X_1 and X_2 here: $P(X_1=i \mid X_2=j) = \begin{cases}\frac{1}{2} & i=j\\ \frac{1}{4} & i\neq j\end{cases}$

and $P(X_2=i \mid X_1=j) = \begin{cases}1/2 & i=j\\ \frac{1}{4} & i\neq j\end{cases}$ $\therefore$ Gibbs sampler:

$F(X_1^{(1)} \mid X_2^{(0)}=3) = (F_1, F_2, F_3) = (\frac{1}{4}, \frac{1}{2}, 1)$ $U_1 = 0.872 > \frac{1}{2} = 0.5 \therefore X_1^{(1)} = 3$ $\therefore$

$F(X_2^{(1)} \mid X_1^{(1)}=3) = (F_1, F_2, F_3) = (\frac{1}{4}, \frac{1}{2}, 1)$ $\therefore U_2 = 0.796 > 0.5 = \frac{1}{2} \therefore$ $X_2^{(1)} = 3$ $\therefore (3,3)$

$F(X_1^{(2)} \mid X_2^{(1)}=3) = (F_1, F_2, F_3) = (\frac{1}{4}, \frac{1}{2}, 1)$ $\therefore U_3 = 0.202 < 0.25 = \frac{1}{4} \therefore X_1^{(2)} = 1$ $\therefore$

$F(X_2^{(2)} \mid X_1^{(2)}=1) = (F_1, F_2, F_3) = (\frac{1}{2}, \frac{3}{4}, 1)$ $\therefore U_4 = 0.136 < 0.5 = \frac{1}{2} \therefore X_2^{(2)} = 1$ $\therefore (1,1)$

$F(X_1^{(3)} \mid X_2^{(2)}=1) = (F_1, F_2, F_3) = (\frac{1}{2}, \frac{3}{4}, 1)$ $\therefore U_5 = 0.983 > 0.75 = 3/4 \therefore X_1^{(3)} = 3$ $\therefore$

$F(X_2^{(3)} \mid X_1^{(3)}=3) = (F_1, F_2, F_3) = (\frac{1}{4}, \frac{1}{2}, 1)$ $\therefore U_6 = 0.525 > 0.5 = \frac{1}{2} \therefore X_2^{(3)} = 3$ $\therefore (3,3)$ are samples

\week2/Q2./ change of variables: $\pi_\phi(\phi) = \left|\frac{\partial g^{-1}(\phi)}{\partial \phi}\right| \pi_\theta(g^{-1}(\phi))$

By Bayes theorem: $\pi(\phi|y) \propto \pi(\phi) f(y|\phi)$

$\therefore \theta \sim \text{Beta}(a,b)$, $\phi = \frac{\theta}{1-\theta} \therefore$

$\phi(1-\theta) = \phi - \phi\theta = \theta \therefore \quad \phi = \theta + \phi\theta = \theta(1+\phi) \therefore$

$\frac{\phi}{1+\phi} = \theta \therefore \quad \pi_\theta(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1} \therefore$

$\pi_\theta(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\left(\frac{\phi}{1+\phi}\right)^{a-1}\left(1-\frac{\phi}{1+\phi}\right)^{b-1} \therefore$

$1-\frac{\phi}{1+\phi} = \frac{1+\phi}{1+\phi} - \frac{\phi}{1+\phi} = \frac{1}{1+\phi} = (1+\phi)^{-1} \therefore \left(1-\frac{\phi}{1+\phi}\right)^{b-1} = \left[(1+\phi)^{-1}\right]^{b-1} = (1+\phi)^{-b+1} \therefore$

$\frac{\phi}{1+\phi} = \phi(1+\phi)^{-1} \therefore \left(\frac{\phi}{1+\phi}\right)^{a-1} = \left[\phi(1+\phi)^{-1}\right]^{a-1} = \phi^{a-1}(1+\phi)^{-a+1} \therefore$

$\pi_\theta(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\phi^{a-1}(1+\phi)^{-a+1}(1+\phi)^{-b+1} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\phi^{a-1}(\phi+1)^{-a-b+2}$

$= \pi_\theta(g^{-1}(\phi))$

$\therefore g^{-1}(\phi) = \theta \therefore g(\theta) = \phi \therefore \quad g(\theta) = \frac{\theta}{1-\theta}$

\2/ change of variables: $\pi_\phi(\phi) = \left|\frac{\partial g^{-1}(\phi)}{\partial \phi}\right| \pi_\theta(g^{-1}(\phi)) \therefore$

$\phi = \frac{\theta}{1-\theta} = g(\theta) \quad \therefore$

$\phi(1-\theta) = \phi - \theta\phi = \theta \therefore \quad \phi = \theta\phi + \theta = \theta(1+\phi) \therefore \quad \theta = \frac{\phi}{\phi+1} \therefore$

$\cancel{g^{-1}}$ let $y = \frac{\theta}{1-\theta} \therefore \quad \frac{y}{y+1} = \theta \therefore \quad g^{-1}(\theta) = \frac{\theta}{\theta+1} \therefore$

$g^{-1}(\phi) = \frac{\phi}{\phi+1} \therefore$

$\frac{\partial g^{-1}(\phi)}{\partial \phi} = \frac{\partial}{\partial \phi}(\phi(\phi+1)^{-1}) = (\phi+1)^{-1} - \phi(\phi+1)^{-2} = \frac{\phi+1}{(\phi+1)^2} - \frac{\phi}{(\phi+1)^2} = \frac{1}{(\phi+1)^2} = (\phi+1)^{-2} \therefore$

$\pi_\theta(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1} \therefore$

$\pi_\theta(g^{-1}(\phi)) = \pi_\theta\left(\frac{\phi}{\phi+1}\right) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\left(\frac{\phi}{\phi+1}\right)^{a-1}\left(1-\frac{\phi}{\phi+1}\right)^{b-1} =$

$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\phi^{a-1}(\phi+1)^{-a+1}(\phi+1)^{-b+1} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\phi^{a-1}(\phi+1)^{-a-b+2} \therefore$

$\pi_\phi(\phi) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\phi^{a-1}(\phi+1)^{-a-b+2}(\phi+1)^{-2} = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\phi^{a-1}(\phi+1)^{-a-b}$

$y_i|\phi \sim \text{Geo}(\theta) \therefore f(y_i|\phi) = \phi(1-\phi)^{y_i-1} \therefore \cancel{f(y_i|\phi) = \frac{\phi}{\phi+1}\left(1-\frac{\phi}{\phi+1}\right)^{y_i-1} = \phi(\phi+1)^{-y_i} \ldots}$

$\therefore$ By Bayes: $\pi_\phi(\phi|y) \propto \pi_\phi(\phi) f(y|\phi) \propto \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\phi^{a-1}(\phi+1)^{-a-b} f(y|\phi) \propto$

$\phi^{a-1}(\phi+1)^{-a-1}\prod_{i=1}^{n}\phi(1-\phi)^{y_i-1} \propto \phi^{a-1}(\phi+1)^{-a-b}\phi^n(1-\phi)^{-n+n\bar{y}}$ X

Week 7 $\sigma \sim Unif(0, \infty)$ $\therefore P(\sigma) = const, P(\sigma) \propto 1$

$g(\theta) = \theta^2$ $\therefore$ given σ $\therefore$ trying $\phi = \sigma^2$

$g^{-1}(\theta) = \theta^{1/2}$ $\therefore$

$\therefore \pi(\theta) = \text{constant}$ $\therefore \pi(g^{-1}(\phi)) = \text{constant}$

$$\frac{d g^{-1}(\theta)}{d\theta} = \frac{1}{2}\theta^{-1/2}$$

$$\therefore \frac{d g^{-1}(\phi)}{d\phi} = \frac{1}{2}\phi^{-1/2}$$

$\frac{1}{2}\phi^{-1/2} \times \text{constant} = \phi^{-1/2} \times \text{constant}$

$\therefore \pi_\phi(\phi) \propto \phi^{-1/2}$ $\therefore$ $\phi = \sigma^2$ $\therefore$ $\pi_{\sigma^2}(\sigma^2) \propto (\sigma^2)^{-1/2} = \sigma^{-1}$

$\pi(\log\sigma) \propto 1$ $\pi(\log\sigma) = \text{constant}$

$\log\sigma$, $g(\theta) = \sigma^2$

$\theta = \log\sigma$ $\therefore$ $g(\theta) = g(\log\sigma) = \sigma^2$

$\therefore$ $g(\theta) = (e^\theta)^2 = e^{2\theta}$

$\therefore \pi_\theta(\theta) = \text{constant} = \pi_\theta(g^{-1}(\sigma^2))$

$y = e^{2\theta}$ $\therefore$ $\ln y = 2\theta$ $\therefore$ $\frac{1}{2}\ln y = \theta = \ln y^{1/2}$ $\therefore$

$y = \frac{1}{2}\ln\theta$ $\frac{d}{d\theta}g^{-1}(\theta) = \frac{1}{2\theta}$

~~$\pi_\phi(\phi) = \pi_{\sigma^2}(\sigma^2) = \frac{1}{2\log\sigma} \times \text{constant} =$~~

$\pi_\phi(\phi) = \pi_{\sigma^2}(\sigma^2) = \frac{1}{2\sigma^2} \times \text{constant} = \frac{1}{\sigma^2}\,\text{constant} \propto \frac{1}{\sigma^2}$

~~what~~

when given uniform distri of $Unif(0,1)$ values

do Box Muller $\therefore$

with pairs of $Unif(0,1)$ values eg U_1, U_2

do: $W = -2\ln(1-U_1)$, $\Theta = 2\pi(U_2) = 2\pi U_2$

$X_1 = \sqrt{W}\cos\Theta = \sqrt{-2\ln(1-U_1)}\cos(2\pi U_2)$

$X_2 = \sqrt{W}\sin\Theta = \sqrt{-2\ln(1-U_1)}\sin(2\pi U_2)$

where X_1, X_2 will be from a $N(0,1)$ distribution

$\therefore$ and is $Y \sim N(0, \frac{1}{2})$ $\therefore$ $\frac{y-0}{\sqrt{1/2}} = Z \sim N(0,1)$ $\therefore$

$Y = \sqrt{\frac{1}{2}}Z \sim N(0, \frac{1}{2})$

\week 7 4a/ Metropolis hastings/ let $x \sim Gamma(\alpha, \beta)$ ∴

$f(x) = \frac{\beta^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\beta x}$ ∴ $X^{(0)} = 1.5$ ∴

$r = f(x^*)/f(x^{t-1})$ ∴

x^* is after a random walk

∴ let random walk $= 0.5$ ∴ $x^* = x^{(0)} + 0.5 = \cancel{1}.5 + 0.5 = 2$ ∴

$$r = f(x^*)/f(x^{t-1}) = \frac{\frac{\beta^\alpha}{\Gamma(\alpha)} x^{*\alpha-1} e^{-\beta x^*}}{\frac{\beta^\alpha}{\Gamma(\alpha)} x^{t-1\alpha-1} e^{-\beta x^{t-1}}} = \frac{x^{*\alpha-1} e^{-\beta x^*}}{x^{t-1\alpha-1} e^{-\beta x^{t-1}}}$$ ∴

∴ walking through log space ∴

$$\ln r = \ln(f(x^*)/f(x^{t-1})) = \ln(f(x^*)) - \ln(f(x^{t-1})) =$$

$$\ln\left(\frac{x^{*\alpha-1}}{x^{t-1\alpha-1}}\right) + \ln(e^{-\beta x^* + \beta x^{t-1}}) = \ln(x^{*\alpha-1}) - \ln(x^{t-1\alpha-1}) + -\beta x^* + \beta x^{t-1} =$$

$$(\alpha-1)\ln\left(\frac{x^*}{x^{t-1}}\right) + \beta(x^{t-1} - x^*) = (\alpha-1)\ln\frac{x^*}{x^{t-1}} - \beta(x^* - x^{t-1})$$

∴ find $\min(1, r)$ ∴

if $1 \leq r$ then accept x^* with probability 1 ∴

definitely accept x^*

∴ 10 unif(0,1) values given ∴ use $W = -2\ln(1-U_1)$, $\Theta = 2\pi U_2$,

$Z_1 = \sqrt{W}\cos\Theta$, $Z_2 = \sqrt{W}\sin\Theta$ to convert to normal $Z_1, Z_2 \sim N(0,1)$

pairs of the unif(0,1) values ∴

\week 7/ \1a/ uniform improper prior σ ∴

$\pi(\sigma) = constant$ ∴ $\pi(\sigma) \propto 1$ ∴ $\pi_\sigma(\sigma) = constant \propto 1$ ∴

let $g(\sigma) = \sigma^2$ ∴ change of variable formula $\pi_\phi(\phi) = \left|\det\left(\frac{\partial g^{-1}(\phi)}{\partial \phi}\right)\right| \pi_\sigma(g^{-1}(\phi))$

∴ $\cancel{g^{-1}}$ $g^{-1}(\sigma) = \sqrt{\sigma} = \sigma^{1/2}$ ∴

$\pi_\sigma(g^{-1}(\sigma)) = constant$

∴ $\frac{dg^{-1}(\sigma)}{d\phi} = \frac{1}{2}\sigma^{-1/2}$ ∴ $\frac{dg^{-1}(\sigma^2)}{d\phi} = \frac{1}{2}(\sigma^2)^{-1/2} = \frac{1}{2}\frac{1}{\sigma}$ ∴

$\pi_{\sigma^2}(\sigma^2) = \left|\frac{1}{2}\frac{1}{\sigma}\right| constant = \frac{1}{2}\frac{1}{\sigma} constant = \frac{1}{\sigma} constant \propto \frac{1}{\sigma}$

\1b/ uniform improper prior on $\log(\sigma)$ ∴

$\pi_{\log(\sigma)}(\log(\sigma)) = constant \propto 1$ ∴

~~$g(\sigma) = \sigma^2$~~ ∴ ~~$g(\ln(\sigma)) = (\ln(\sigma))^2$~~ ∴ $\theta = \log(\sigma)$ ∴

$g(\theta) = \sigma^2 = g(\ln\sigma)$ ∴ $(e^{\ln\sigma})^2 = e^{2\ln\sigma} = \sigma^2$ ∴

$g(\theta) = e^{2\theta}$ ∴ $y = e^{2\theta}$ ∴ $\ln y = 2\theta$ ∴ $\frac{1}{2}\ln y = \theta$ ∴

$g^{-1}(\theta) = \frac{1}{2}\ln\theta$ ∴

\week 7/ $\cancel{\pi(\log\sigma)}$ $\pi_\phi(\phi) = \left|\det\left(\frac{\partial g^{-1}(\phi)}{\partial \phi}\right)\right| \pi_\theta(g^{-1}(\phi))$ ∴

$\pi_\theta(g^{-1}(\phi)) = \pi_{\ln\sigma}(g^{-1}(\phi)) = \pi_{\ln\sigma}(g^{-1}(\sigma^2)) = \pi_{\ln\sigma}(\frac{1}{2}\ln(\sigma^2)) = \pi_{\ln\sigma} =$ constant

∴ $g^{-1}(\theta) = \frac{1}{2}\ln\theta$ ∴ $\frac{\partial g^{-1}(\theta)}{\partial\theta} = \frac{1}{2}\frac{1}{\theta}$ ∴

$\partial\sigma$/ $\frac{\partial g^{-1}(\phi)}{\partial\phi} = \frac{\partial g^{-1}(\sigma^2)}{\partial\sigma^2} = \frac{1}{2}\frac{1}{\cancel{\phi}} = \frac{1}{2}\frac{1}{\sigma^2}$ ∴

$\left|\frac{\partial g^{-1}(\phi)}{\partial\phi}\right| = \left|\frac{1}{2}\frac{1}{\sigma^2}\right| = \frac{1}{2}\frac{1}{\sigma^2}$ ∴

$\pi_\phi(\phi) = \pi_{\sigma^2}(\sigma^2) = \frac{1}{2}\frac{1}{\sigma^2}$ constant $= \frac{1}{2}\frac{1}{\sigma^2}\pi_{\ln\sigma} = \frac{1}{2}\frac{1}{\sigma^2}$ const $= \frac{1}{\sigma^2}$ constant $\propto \frac{1}{\sigma^2}$

∴ $\pi_\phi(\phi) \propto \frac{1}{\sigma^2}$

\week 2/ \2/ $\pi_\phi(\phi) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\phi^{a-1}(1+\phi)^{-a-b}$

$\pi_\phi(\phi) = \left|\frac{\partial g^{-1}(\phi)}{\partial\phi}\right|\pi_\theta(g^{-1}(\phi))$

$\cancel{\phi = \frac{\theta}{1-\theta}}$ $\phi = \frac{\theta}{\theta-1} = g(\theta)$

$\cancel{y = \frac{\theta}{\theta-1}}$ ∴ let $\cancel{t = \frac{\theta}{\theta-1}}$ ∴ $g(\theta)(\theta-1) = \theta = g(\theta)\theta - g(\theta)$ ∴

$\theta - g(\theta)\theta = -g(\theta) = \theta(1-g(\theta))$

$\phi = \frac{\theta}{\theta-1} = g(\theta)$ ∴ $g = \frac{\theta}{\theta-1}$ ∴ $\cancel{g}$ $(\theta-1)g = \theta = \theta g - g$ ∴

$g = \theta g - \theta = \theta(g-1)$ ∴ $\frac{g}{g-1} = \theta$ ∴ $g^{-1}(\theta) = \frac{\theta}{\theta-1}$

\2/ $\cancel{\phi = \frac{\theta}{\theta-1}}$ $\phi = \frac{\theta}{1-\theta} = g(\theta)$ ∴ $\frac{\theta}{1-\theta} = g$ ∴ $\theta = g - \theta g$ ∴ $\theta + \theta g = g = \theta(1+g)$

∴ $\frac{g}{1+g} = \theta$ ∴ $\frac{\theta}{1+\theta} = g^{-1}(\theta)$ ∴ $g^{-1}(\phi) = \frac{\phi}{1+\phi}$

∴ $\pi_\phi(\phi) = \left|\frac{\partial g^{-1}(\phi)}{\partial\phi}\right|\pi_\theta(g^{-1}(\theta))$ $\pi_\theta(\theta) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}$ ∴

$\pi_\phi(\phi) = \frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\phi^{a-1}(1+\phi)^{-a-b}$, $f(y_i|\theta) = (1-\theta)^{y_i-1}\theta$

∴ $\pi_\phi(\phi|y) = \left|\frac{\partial g^{-1}(\phi)}{\partial\phi}\right|\pi_\theta(g^{-1}(\phi)|y)$ ∴ $\frac{\partial g^{-1}(\phi)}{\partial\phi} = \frac{\partial}{\partial\phi}\left(\frac{\phi}{1+\phi}\right) = (1+\phi)^{-2}$

$\pi_\theta(\theta|y) \propto \pi_\theta(\theta)f(y|\theta) \propto \cancel{(1-\theta)}\ \theta^{a-1}(1-\theta)^{b-1}\prod_{i=1}^{n}(1-\theta)^{y_i-1}\theta \propto$

$\theta^{a+n-1}(1-\theta)^{b+n\bar{y}-n-1}$ ∴ let $a_n = a+n$, $b_n = b+n\bar{y}-n$ ∴

$\theta|y \sim \mathrm{Beta}(a_n, b_n)$ ∴ $\pi_\theta(\theta|y) = \frac{\Gamma(a_n+b_n)}{\Gamma(a_n)\Gamma(b_n)}\theta^{a_n-1}(1-\theta)^{b_n-1}$ ∴

$\cancel{\pi_\phi(\phi|y) = (1+\phi)^{-2}}$ $\pi_\theta(g^{-1}(\phi)|y) = \frac{\Gamma(a_n+b_n)}{\Gamma(a_n)\Gamma(b_n)}\left(\frac{\phi}{1+\phi}\right)^{a_n-1}\left(1-\frac{\phi}{1+\phi}\right)^{b_n-1} =$

$\frac{\Gamma(a_n+b_n)}{\Gamma(a_n)\Gamma(b_n)}(\phi(\phi+1)^{-1})^{a_n-1}((1+\phi)^{-1})^{b_n-1} = \frac{\Gamma(a_n+b_n)}{\Gamma(a_n)\Gamma(b_n)}\phi^{a_n-1}(\phi+1)^{-a_n+1}(1+\phi)^{-b_n+1} =$

$\cancel{\frac{\Gamma(a_n+b_n)}{\Gamma(a_n)\Gamma(b_n)}}$ $\frac{\Gamma(a_n+b_n)}{\Gamma(a_n)\Gamma(b)}\phi^{a_n-1}(1+\phi)^{-a_n-b_n+2}$ ∴

$\cancel{\pi_\phi(g)}$ $\pi_\phi(\phi|y) = \frac{\Gamma(a_n+b_n)}{\Gamma(a_n)\Gamma(b_n)}\phi^{a_n-1}(1+\phi)^{-a_n-b_n}$

\3a/ $\phi = \frac{1}{\theta} = g(\theta)$, $\pi_\theta(\theta) = \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}$ $f(y_i|\theta)=\theta e^{-\theta y_i}$

Change of variables: $\pi_\phi(\phi) = \left|\frac{\partial g^{-1}(\phi)}{\partial\phi}\right|\pi_\theta(g^{-1}(\phi))$

∴ $\frac{1}{\theta}=g$ ∴ $\frac{1}{g}=\theta$ ∴ $\frac{1}{\theta}=g$ ∴ $g^{-1}(\theta)=\frac{1}{\theta}$ ∴

$g^{-1}(\phi)=\frac{1}{\phi}$ ∴

$\pi_\theta(g^{-1}(\phi)) = \pi_\theta(\frac{1}{\phi}) = \frac{b^a}{\Gamma(a)}(\frac{1}{\phi})^{a-1}e^{-b(\frac{1}{\phi})} = \cancel{\frac{b^a}{\Gamma(a)}(\frac{1}{\phi})} \frac{b^a}{\Gamma(a)}\phi^{-a+1}e^{-b\phi^{-1}}$ ∴

$\left|\frac{\partial g^{-1}(\phi)}{\partial\phi}\right| = \left|\frac{\partial}{\partial\phi}(\phi^{-1})\right| = |-\phi^{-2}| = \phi^{-2}$ ∴

$\pi_\phi(\phi) = \frac{b^a}{\Gamma(a)}\phi^{-a+1}e^{-b\frac{1}{\phi}}(+\phi^{-2}) = \frac{b^a}{\Gamma(a)}\phi^{-a-1}e^{-b\frac{1}{\phi}}$

By Bayes theorem: $\pi_\theta(\theta|y) \propto \pi_\theta(\theta)f(y|\theta) \propto \pi_\theta(\theta)\prod_{i=1}^n f(y_i|\theta) \propto$
$\pi_\theta(\theta)\prod_{i=1}^n\theta e^{-\theta y_i} \propto \pi_\theta(\theta)\theta^n e^{-\theta n\bar{y}} \propto \frac{b^a}{\Gamma(a)}\theta^{a-1}e^{-b\theta}\theta^n e^{-n\bar{y}\theta} \propto$
$\theta^{a-1}e^{-b\theta}\theta^n e^{-n\bar{y}\theta} \propto \theta^{(a+n)-1}e^{-(b+n\bar{y})\theta}$

∴ $\pi_\theta(\theta|y)$ is proportional to a Gamma distribution ∴

Let $a_n=a+n$, $b_n=b+n\bar{y}$ ∴ $\theta|y \sim \text{Gamma}(a_n,b_n)$ ∴

$\pi_\theta(\theta|y)=\frac{b_n^{a_n}}{\Gamma(a_n)}\theta^{a_n-1}e^{-b_n\theta}$ ∴

$\pi_\phi(\phi|y) = \left|\frac{\partial g^{-1}(\phi)}{\partial\phi}\right|\pi_\theta(g^{-1}(\phi)|y)$ ∴

$\pi_\theta(g^{-1}(\phi)|y)=\pi_\theta(\frac{1}{\phi}|y)=\frac{b_n^{a_n}}{\Gamma(a_n)}(\frac{1}{\phi})^{a_n-1}e^{-b_n\frac{1}{\phi}}=\frac{b_n^{a_n}}{\Gamma(a_n)}\phi^{-a_n+1}e^{-b_n\frac{1}{\phi}}$

∴ $\pi_\phi(\phi|y)=\phi^{-2}\frac{b_n^{a_n}}{\Gamma(a_n)}\phi^{-a_n+1}e^{-b_n\frac{1}{\phi}}=\frac{b_n^{a_n}}{\Gamma(a_n)}\phi^{-a_n-1}e^{-b_n\frac{1}{\phi}}$

\3b/ Jeffreys prior is = $\pi(\theta)\propto$ (expected information)$^{1/2}$ =
$(E(-\frac{\partial^2 l}{\partial\theta^2}))^{1/2} = (E(-I(\theta)))^{1/2}$ ∴

$\prod_{i=1}^n f(y_i|\theta)=\prod_{i=1}^n\theta e^{-\theta y_i}=\theta^n e^{-\theta n\bar{y}}=\theta^n e^{-n\theta\bar{y}}=L(y|\theta)$ ∴

$l(y|\theta)=\ln L(y|\theta)=\ln(\theta^n)+\ln(e^{-n\theta\bar{y}})=n\ln\theta-n\theta\bar{y}$ ∴

$\frac{\partial l(y|\theta)}{\partial\theta}=n\frac{1}{\theta}-n\bar{y} = n\theta^{-1}-n\bar{y}$

$\frac{\partial^2 l(y|\theta)}{\partial\theta^2} = -n\theta^{-2}$ ∴ $I(\theta)= -E(\frac{\partial^2 l}{\partial\theta^2}) = -E(-n\theta^{-2}) = nE(\frac{1}{\theta^2})=\frac{n}{\theta^2}$

∴ Jeffreys prior of θ is: $\pi_{J(\theta)}(\theta)\propto(\frac{n}{\theta^2})^{1/2}=(\frac{\sqrt{n}}{\theta})$ ∴
$\pi_{J(\theta)}(\theta)\propto\frac{1}{\theta}$

\3c/ $\pi_{J(\phi)}(\phi)=\left|\frac{\partial g^{-1}(\phi)}{\partial\phi}\right|\pi_{J(\theta)}(g^{-1}(\phi))$ ∴ $\left|\frac{\partial g^{-1}(\phi)}{\partial\phi}\right|=\phi^{-2}$, $\pi_{J(\theta)}(g^{-1}(\phi))=$
$\pi_{J(\theta)}(\frac{1}{\phi})\propto\frac{1}{(\frac{1}{\phi})}=\phi$ ∴ $\pi_{J(\theta)}(\phi)\propto\phi^{-2}\phi=\frac{1}{\phi}$ ∴ $\pi_{J(\phi)}(\phi)\propto\frac{1}{\phi}$ is
Jeffreys prior of ϕ

Week 2 / prior predictive: $p(y)=P(Y=y)=\int_{-\infty}^{\infty} p(y|\theta)\pi(\theta)\,d\theta$

posterior predictive $p(y|\mathbf{y})=\int_{-\infty}^{\infty} p(y|\theta)\pi(\theta|\mathbf{y})\,d\theta$

5 / prior predictive $=p(y)=P(Y=y)=\int_{-\infty}^{\infty} p(y|\theta)\pi(\theta)\,d\theta=$

$=\int_0^1 \theta(1-\theta)^{y-1}\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\theta^{a-1}(1-\theta)^{b-1}\,d\theta=$

$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)}\int_0^1 \theta^{(a+1)-1}(1-\theta)^{(b-y-1)-1}\,d\theta$ $\therefore$

$a_n=a+1$, $b_n=b-y-1$ $\therefore \int_0^1 \frac{\Gamma(a_n+b_n)}{\Gamma(a_n)\Gamma(b_n)}\theta^{a_n-1}(1-\theta)^{b_n-1}\,d\theta=1$

$\because$ integrating Beta(a_n,b_n) over domain $\therefore$

$p(y)=\frac{\Gamma(a+b)\Gamma(a_n)\Gamma(b_n)}{\Gamma(a)\Gamma(b)\Gamma(a_n+b_n)}\int_0^1 \frac{\Gamma(a_n+b_n)}{\Gamma(a_n)\Gamma(b_n)}\theta^{a_n-1}(1-\theta)^{b_n-1}\,d\theta=$

$\frac{\Gamma(a+b)\Gamma(a_n)\Gamma(b_n)}{\Gamma(a)\Gamma(b)\Gamma(a_n+b_n)}$.

By Bayes theorem: $\pi(\theta|\mathbf{y})\propto\pi(\theta)p(\mathbf{y}|\theta)\propto\pi(\theta)\prod_{i=1}^{n}p(y_i|\theta)\propto$

$\pi(\theta)\prod_{i=1}^{n}\theta(1-\theta)^{y_i-1}\propto\pi(\theta)\theta^n(1-\theta)^{-n+n\bar{y}}\propto\theta^{a-1}(1-\theta)^{b-1}\theta^n(1-\theta)^{-n+n\bar{y}}\propto$

$\theta^{(a+n)-1}(1-\theta)^{(b+n\bar{y}-n)-1}$

$\therefore$ ~~$\pi(\theta|\mathbf{y})$~~ $\therefore$ $a_2=a+n$, $b_2=b+n\bar{y}-n$ $\therefore$

$\pi(\theta|\mathbf{y})\propto\theta^{a_2-1}(1-\theta)^{b_2-1}$ $\therefore$ $\pi(\theta|\mathbf{y})$ is proportional to a beta distribution

$\therefore \theta|\mathbf{y}\sim$ Beta(a_2,b_2) $\therefore \pi(\theta|\mathbf{y})=\frac{\Gamma(a_2+b_2)}{\Gamma(a_2)\Gamma(b_2)}\theta^{a_2-1}(1-\theta)^{b_2-1}$

$\therefore$ ~~predictive~~ posterior predictive: $p(y|\mathbf{y})=\int_0^1 p(y|\theta)\pi(\theta|\mathbf{y})\,d\theta=$

$\int_0^1\theta(1-\theta)^{y-1}\frac{\Gamma(a_2+b_2)}{\Gamma(a_2)\Gamma(b_2)}\theta^{a_2-1}(1-\theta)^{b_2-1}\,d\theta=\frac{\Gamma(a_2+b_2)}{\Gamma(a_2)\Gamma(b_2)}\int_0^1\theta^{(a_2-1)-1}(1-\theta)^{(b_2+y-1)-1}\,d\theta$

$=\frac{\Gamma(a_2+b_2)\Gamma(a_3)\Gamma(b_3)}{\Gamma(a_2)\Gamma(b_2)\Gamma(a_3+b_3)}\int_0^1\frac{\Gamma(a_3+b_3)}{\Gamma(a_3)\Gamma(b_3)}\theta^{a_3-1}(1-\theta)^{b_3-1}\,d\theta=$

$\frac{\Gamma(a_2+b_2)\Gamma(a_3)\Gamma(b_3)}{\Gamma(a_2)\Gamma(b_2)\Gamma(a_3+b_3)}$, $a_3=a_2-1$, $b_3=b_2+y-1$

\PP2021/

\1ai/ By Bayes theorem: $\pi(\theta|\underline{y}) \propto \pi(\theta)\, p(\underline{y}|\theta) \propto$

$\pi(\theta) \prod_{i=1}^{n} p(y_i|\theta) \propto \pi(\theta) \prod_{i=1}^{n} p(y_i|\theta) \propto \pi(\theta) \prod_{i=1}^{n} e^{\theta - y_i} \propto$

$\frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} \prod_{i=1}^{n} e^{\theta - y_i} \propto$

$\theta^{a-1} e^{-b\theta} \prod_{i=1}^{n} e^{\theta - y_i} \propto \theta^{a-1} e^{-b\theta} \prod_{i=1}^{n} e^{\theta} e^{-y_i} \propto \theta^{a-1} e^{-b\theta} \prod_{i=1}^{n} e^{\theta} \propto$

$\theta^{a-1} e^{-b\theta} (e^{\theta})^n = \theta^{a-1} e^{-b\theta} e^{n\theta} = \theta^{a-1} e^{-b\theta + n\theta} \propto \theta^{a-1} e^{-(b-n)\theta}$

which is proportional to a gamma desity with $a_n = a$, $b_n = b - n$

\1a/ $\pi(\theta) = \frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta}$ $\therefore$

By Bayes theorem: $\pi(\theta|\underline{y}) \propto \pi(\theta)\, p(\underline{y}|\theta) \propto$

$\frac{b^a}{\Gamma(a)} \theta^{a-1} e^{-b\theta} p(\underline{y}|\theta) \propto \theta^{a-1} e^{-b\theta} p(\underline{y}|\theta) \propto \theta^{a-1} e^{-b\theta} \prod_{i=1}^{n} p(y_i|\theta) \propto$

$\theta^{a-1} e^{-b\theta} \prod_{i=1}^{n} e^{\theta - y_i} \propto \theta^{a-1} e^{-b\theta} \prod_{i=1}^{n} e^{\theta} e^{-y_i} \propto \theta^{a-1} e^{-b\theta} \prod_{i=1}^{n} e^{\theta} \propto$

$\theta^{a-1} e^{-b\theta} (e^{\theta})^n \propto \theta^{a-1} e^{-b\theta} (e^{n\theta}) \propto \theta^{a-1} e^{-b\theta} e^{-b\theta + n\theta} \propto$

$\theta^{a-1} e^{-(b-n)\theta}$ which is proportional to a gamma density with $a_n = a$, $b_n = b - n$

\1aii/ The cdf is $F(y) = \int_{\theta}^{y} e^{\theta - x}\, dx = \left[-e^{\theta - x}\right]_{x=\theta}^{y} =$

$-e^{\theta - y} - (-e^{\theta - \theta}) = 1 - e^{\theta - y}$ $\therefore$

Letting $U \sim \text{unif}(0,1)$: $U = 1 - e^{\theta - y}$ $\therefore$

$e^{\theta - y} = 1 - U$ $\therefore$ $\theta - y = \ln(1-U)$ $\therefore$

$y = \theta - \ln(1-U)$ $\therefore$

for $\theta = 1$: $y = 1 - \ln(1-U)$ $\therefore$

The 3 radom draws are: $y = 1 - \ln(1 - 0.502) = 1.70$ (3 s.f.), $y = 1 - \ln(1 - 0.221) = 1.25$ (3 s.f.), $y = 1 - \ln(1 - 0.893) = 3.23$ (3 s.f.) $\therefore$

~~If log(U) is used, the alternative draws are~~

\1aii/ The cdf is $F(y) = P_r(Y \leq y) = \int_{\theta}^{y} p(x|\theta)\, dx = \int_{\theta}^{y} e^{\theta - x}\, dx$

$= \int_{\theta}^{y} e^{\theta} e^{-x}\, dx = e^{\theta} \int_{\theta}^{y} e^{-x}\, dx = e^{\theta} \left[-e^{-x}\right]_{\theta}^{y} = e^{\theta}\left[-e^{-y} + e^{-\theta}\right] =$

$-e^{\theta - y} + e^{\theta - \theta} = 1 - e^{\theta - y}$ $\therefore$

let $U \sim \text{Unif}(0,1)$ ∴ $U = 1 - e^{\theta - y}$ ∴

$e^{\theta - y} = 1 - U$ ∴ $\theta - y = \ln(e^{\theta - y}) = \ln(1-U)$ ∴

$y = \theta - \ln(1-U)$ ∴ ~~for~~ $\theta = 1$ ∴

$y = 1 - \ln(1-U)$ ∴

$U = 0.502, 0.221, 0.893$ ∴

$y = 1 - \ln(1 - 0.502) = 1.70,\ 1 - \ln(1-0.221) = 1.25,\ 1 - \ln(1-0.893) = 3.23)$

are 3 samples from $P(y|\theta)$

\1b/ Bayesian inference and prediction involves integration wrt the posterior that mostly can only be done numerically, and Monte Carlo integration are derivatives like MCMC enable such integrals to be estimated and their error controlled whatever the dimension of the posterior; and will give an integral written out.

\1b/ Bayesian inference and prediction involves integration wrt the posterior that mostly can only be done numerically; and Monte Carlo integration and derivatives like MCMC enable such integrals to be estimated and their error controlled whatever the dimensions of the posterior. ~~and with ga~~

\1ci/ $P(y>3) = E[\mathbb{1}(y>3)] = \int_0^\infty \mathbb{1}(y>3) \frac{3^2}{\Gamma(2)} y^{2-1} e^{-3y}\, dy =$
$\int_0^\infty \mathbb{1}(y>3)\, 9\, y e^{-3y}\, dy$ is $\int g(x) f(x)\, dx$; $g(x) = \mathbb{1}(y>3)$,
$f(x)$ is gamma$(2,3)$ pdf ∴

drawing N samples y_i from $f(y)$, the Monte Carlo estimate is $P(y>3) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}(y_i > 3)$

\1ci/ $g(y) = \mathbb{1}(y>3)$, $f(y) \sim \text{Gamma}(2,3)$ ∴

$f(y) = \frac{b^a}{\Gamma(a)} y^{a-1} e^{-by} = \frac{3^2}{\Gamma(2)} y^{2-1} e^{-3y} = \frac{9}{(2-1)!} y^1 e^{-3y} = 9 y e^{-3y}$ ∴

$P(y>3) = E[\mathbb{1}(y>3)] = \int_0^\infty \cancel{\mathbb{1}(y>3)} \int_0^\infty g(y) f(y)\, dy = \int_0^\infty \mathbb{1}(y>3)\, 9 y e^{-3y}\, dy$

∴ drawing N samples y_i from $f(y)$, the Monte Carlo estimate is $P(y>3) = \frac{1}{N} \sum_{i=1}^{N} \mathbb{1}(y_i > 3)$

\PP2021/

\1cii/ $P(y>3)=1-P(y<3)=1-\int_0^3 f(y) = 1-\int_0^3 \frac{3^2}{\Gamma(2)} y^{2-1}e^{-3y}dy =$

$1-\int_0^3 \frac{9}{\cancel{\Gamma}(2-1)!} y^1 e^{-3y}dy = 1-\int_0^3 9ye^{-3y}dy$

The pdf of a Unif(0,3) random quantity is $\frac{1}{3}$, so the integral can be written as $\int_0^3 g(y)f(y)dy$ with $f(y)=\frac{1}{3}$ the pdf of the uniform on $[0,3]$ and $g(y)=27ye^{-3y}$. This integral can be estimated by Monte carlo as follows: let $u_1,\dots,u_N$ Unif(0,1) and $y_i=3u_i$ then $P(y>3)\approx 1-\frac{1}{N}\sum_{i=1}^{N}27y_ie^{-3y_i}$

\1cii/ $P(y>3)=1-P(y<3)=1-\int_0^3 f(y)= 1-\int_0^3 \frac{3^2}{\Gamma(2)} y^{2-1}e^{-3y}dy=$

$1-\int_0^3 \frac{9}{(2-1)!} y^1 e^{-3y}dy = 1-\int_0^3 9ye^{-3y}dy$

The pdf of a Unif(0,3) random quantity is $\frac{1}{3}$ $\therefore$ the integral can be written as $\int_0^3 g(y)f(y)dy$ with $f(y)=\frac{1}{3}$ the pdf of the uniform on $[0,3]$ and $g(y)=27ye^{-3y}$. This integral can be estimated by Monte Carlo $\therefore$ let $u_1,\dots,u_N$ Unif(0,1) and $y_i=3u_i$ then $P(y>3)\approx 1-\frac{1}{N}\sum_{i=1}^{N}27y_ie^{-3y_i}$

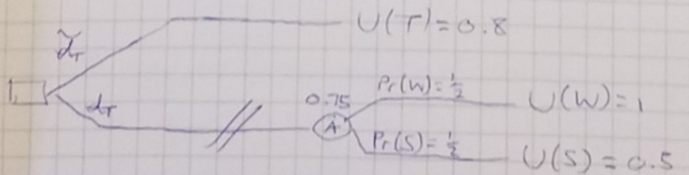
\1di/ Preference ordering $W^* > T^* > S^* > A$

$\therefore$ set $U(W)=1$, $U(A)=0$ and set $U(T)=0.8$ and $U(S)=0.5$

\1dii/ Preference $W^* > T^* > S^* > A$ $\therefore$

let $U(W)=1$, $U(A)=0$, $\therefore$ let $U(T)=0.8$, $U(S)=0.5$

\1dii/ let d_T be the decision to trade.



$\cancel{U(A)=\frac{1}{2}U(W)}$ $U(A)=Pr(W)U(W)+Pr(S)U(S)=\frac{1}{2}(1)+\frac{1}{2}(0.5)=0.75$

$\therefore U(T)=0.8>0.75=U(A)$ $\therefore$ choose $\tilde{d}_T$

$\therefore$ I should not trade

[illegible] event random quantity duality

[illegible] $= P(\sim(\sim(A \wedge B) \wedge \sim(A \wedge \tilde{B}))) =$

[illegible] $= P(AB + A - A^2 B - AB + A^2 B^2) =$

[illegible] $= P(A)$

[illegible] event random quantity duality.

[illegible] $= P(\sim(\sim(A \wedge B) \wedge \sim(A \wedge \tilde{B}))) =$

[illegible] $P(1-(1-AB)(1-A(1-B))) =$

[illegible] $) =$

[illegible] $) = P(1-(1-A+A^2 B - A^2 B^2)) =$

[illegible] $) = P(A - A^2 B + A^2 B^2) = P(A - AB + AB) = P(A)$

[illegible] $(A \wedge B)$ and $(A \wedge \tilde{B})$ are incompatible

[illegible]

[illegible] $= P(A \wedge B) + P(A \wedge \tilde{B})$

[illegible] $(A \wedge B)$ and $(A \wedge \tilde{B})$ are incompatible $\therefore$

[illegible] $= P((A \wedge B) \vee (A \wedge \tilde{B})) = P(A)$

[illegible] is committed to all bets with gain

[illegible] $(A\tilde{B} - P_{A\tilde{B}}) + c_3(A - P_A)$

[illegible] happens: $G = c_1(A - P_{AB}) - c_2 P_{A\tilde{B}} + c_3(A - P_A)$

[illegible] can remove the random quantities by setting

[illegible] $G = c_1(P_A - P_{AB}) - c_2 P_{A\tilde{B}}$

[illegible] $A - c_2 P_{A\tilde{B}} - c_1 A + c_1 P_A$

[illegible] random quantity by setting $c_3 = -c_1$

[illegible] The given inequality implies that the term in [illegible] is always positive and $\therefore$ our opponent can [illegible] to ensure we certainly lose violating coherence

[illegible] is committed to all bets with gain

[illegible] $+ c_2(A\tilde{B} - P_{A\tilde{B}}) + c_3(A - P_A)$

[illegible] $+ c_2(0 - P_{A\tilde{B}}) + c_3(A - P_A) = c_1(A - P_{AB}) - c_2 P_{A\tilde{B}} + c_3(A - P_A)$

$\therefore G = c_1(A - P_{AB}) - c_2 P_{A\tilde{B}} - c_1(A - P_A) = c_1(P_A - P_{AB}) - c_2 P_{A\tilde{B}}$

\PP2 ~~of~~ 1/ if $\tilde{B}$: ~~B~~ $AB=0$ $\therefore$ $G = +C_1(0-P_{AB}) + C_2(A\tilde{B} - P_{A\tilde{B}}) + C_3(A-P_A)$

$= -C_1P_{AB} + C_2(A\tilde{B} - P_{A\tilde{B}}) + C_3(A-P_A)$ $\therefore$

~~let~~ $\tilde{B}=1$ $\therefore$ $A\tilde{B} = A(1) = A$ $\therefore$

$G = -C_1P_{AB} + C_2(A - P_{A\tilde{B}}) + C_3(A-P_A)$ $\therefore$

let $C_3 = -C_2$ $\therefore$ $G = -C_1P_{AB} + C_2(A) - C_2P_{A\tilde{B}} - C_2(A-P_A) =$

$-C_1P_{AB} + C_2(P_A - P_{A\tilde{B}})$

\2c/ The collaborator is committed to all bets with gain

$G = C_1(AB - P_{AB}) + C_2(A\tilde{B} - P_{A\tilde{B}}) + C_3(A-P_A)$

if B happens: $B=1$ $\therefore$ $AB = A(1) = A$ $\therefore$ $\tilde{B}=0$ $\therefore$ $A\tilde{B}=0$ $\therefore$

$G = C_1(A-P_{AB}) + C_2(0-P_{A\tilde{B}}) + C_3(A-P_A) =$

$C_1(A-P_{AB}) - C_2P_{A\tilde{B}} + C_3(A-P_A)$ $\therefore$

let $C_3 = -C_1$ $\therefore$

$G = C_1(A-P_{AB}) - C_2P_{A\tilde{B}} - C_1(A-P_A) = C_1(P_A - P_{AB}) - C_2P_{A\tilde{B}}$

$\therefore$ $G = C_1(P_A - P_{AB}) - C_2P_{A\tilde{B}}$

if $\tilde{B}$ ~~G~~ $G = C_1(AB - P_{AB}) + C_2(A\tilde{B} - P_{A\tilde{B}}) + C_3(A-P_A)$ $\therefore$

$\tilde{B}=1$ $\therefore$ $A\tilde{B} = A(1) = A$, $B=0$ $\therefore$ $AB = A(0) = 0$ $\therefore$

$G = C_1(0-P_{AB}) + C_2(A-P_{A\tilde{B}}) + C_3(A-P_A) =$

$-C_1(P_{AB}) + C_2A - C_2P_{A\tilde{B}} + C_3(A) - C_3P_A$ and $C_3 = -C_1$ $\therefore$

$G = -C_1P_{AB} + C_2A - C_2P_{A\tilde{B}} - C_1A + C_1P_A$

$\therefore$ let $C_2 = C_1$ $\therefore$

$G = -C_1P_{AB} + C_1A - C_1P_{A\tilde{B}} - C_1A + C_1P_A = C_1(P_A - P_{AB} - P_{A\tilde{B}}) = G$ $\therefore$

$P_A - P_{AB} - P_{A\tilde{B}} > 0$ always $\therefore$ our opponent can set C_1 negative to ensure we certainly lose, violating coherence $\therefore$

$P_A > P_{AB} + P_{A\tilde{B}}$

\2c/ The collaborator is committed to all bets with gain

$G = C_1(AB - P_{AB}) + C_2(A\tilde{B} - P_{A\tilde{B}}) + C_3(A-P_A)$ $\therefore$

if B: $B=1$ $\therefore$ $AB = A(1) = A$, $\tilde{B}=0$ $\therefore$ $A\tilde{B} = A(0) = 0$ $\therefore$

$G = C_1(A-P_{AB}) + C_2(0-P_{A\tilde{B}}) + C_3(A-P_A) = C_1(A-P_{AB}) - C_2P_{A\tilde{B}} + C_3(A-P_A)$

$\therefore$ let $C_3 = -C_1$ $\therefore$

$G = C_1(A - P_{AB}) - C_2 P_{A\tilde{B}} - C_1(A - P_A) = C_1(P_A - P_{AB}) - C_2 P_{A\tilde{B}}$ $\therefore$

$G = C_1(AB - P_{AB}) + C_2(A\tilde{B} - P_{A\tilde{B}}) - C_1(A - P_A)$ $\therefore$

$\because \tilde{B} = 1 \quad A\tilde{B} = A(1) = A, \quad B = 0 \therefore AB = A(0) = 0$

$G = C_1(0 - P_{AB}) + C_2(A - P_{A\tilde{B}}) - C_1(A - P_A) =$

$-C_1 P_{AB} + C_2(A - P_{A\tilde{B}}) - C_1(A - P_A)$ $\therefore$

let $C_2 = C_1$ $\therefore$

$G = -C_1 P_{AB} + C_1(A - P_{A\tilde{B}}) - C_1(A - P_A) = -C_1 P_{AB} - C_1 P_{A\tilde{B}} + C_1 P_A =$

$C_1(P_A - P_{AB} - P_{A\tilde{B}})$ $\therefore$

The given inequality: $P_A - P_{AB} - P_{A\tilde{B}} > 0 \therefore P_A > P_{AB} + P_{A\tilde{B}}$

$\therefore$ our opponent can set C_1 negative ~~to be~~ $\therefore$

$C_1(P_A - P_{AB} - P_{A\tilde{B}}) < 0$ to ensure we certainly lose 100% of the time, violating coherence

\3a/ The y_{ij} are the data layer, the θ form the process layer, and the prior layer contains σ^2, μ, τ.
The rest of the parameters are hyperparameters and not treated as randomquantities.

\3a/ The y_{ij} are the data layer, the θ form the process layer, and the prior layer contains σ^2, μ, τ.
The rest of the parameters are hyper-parameters and not treated as random quantites.

\3b/ By Bayes: $\pi(\mu \mid \theta, \sigma^2, \tau^2, \underset{\sim}{y}) \propto \pi(\mu \mid \tau^2, \sigma^2, \theta) P(\underset{\sim}{y} \mid \ldots) \propto$
$\pi(\mu \mid \tau^2, \sigma^2)\, \pi(\theta \mid \mu, \tau^2) \propto \exp\left\{-\frac{1}{2\tau_0^2}(\mu - \mu_0)^2\right\} \prod_{j=1}^{J} \exp\left\{-\frac{1}{2\tau^2}(\theta_j - \mu)^2\right\} \propto$
$\exp\left\{-\frac{1}{2}\left(\frac{1}{\tau_0^2}(\mu - \mu_0)^2 + \frac{1}{\tau^2}\sum_{j=1}^{J}(\theta_j - \mu)^2\right)\right\} \propto \exp\left\{-\frac{1}{2}\left(\left(\frac{1}{\tau_0^2} + \frac{J}{\tau^2}\right)\mu^2 - 2\mu\left(\frac{\mu_0}{\tau_0^2} + \frac{J\bar{\theta}}{\tau^2}\right)\right)\right\}$
$\propto \exp\left\{-\frac{1}{2}\left(\left(\frac{1}{\tau_0^2} + \frac{J}{\tau^2}\right)\left(\mu - \frac{\frac{\mu_0}{\tau_0^2} + \frac{J\bar{\theta}}{\tau^2}}{\frac{1}{\tau_0^2} + \frac{J}{\tau^2}}\right)^2\right)\right\}$
which is proportional to a normal density $\therefore$
$\mu \mid \theta, \sigma^2, \tau^2, \underset{\sim}{y} \sim N\left(\frac{\frac{\mu_0}{\tau_0^2} + \frac{J\bar{\theta}}{\tau^2}}{\frac{1}{\tau_0^2} + \frac{J}{\tau^2}}, \frac{1}{\tau_0^2} + \frac{J}{\tau^2}\right)$

\3c/ By Bayes: $\pi(\tau^2 \mid \mu, \theta, \sigma^2, \underset{\sim}{y}) \propto \pi(\tau^2)\, \pi(\theta \mid \mu, \tau^2) \propto$

\PP2021/ $\tau^{-2(\frac{\eta}{2}+1)} \exp\left\{-\frac{\eta\xi^2}{2\tau^2}\right\} \prod_{j=1}^{J} \frac{1}{\tau} \exp\left\{-\frac{1}{2\tau^2}(\theta_j-\mu)^2\right\} \propto$

$$\tau^{-2(\frac{\eta}{2}+1)-J} \exp\left\{-\frac{1}{2\tau^2}\left(\eta\xi^2+\sum_{j=1}^{J}(\theta_j-\mu)^2\right)\right\} \propto$$

$$\tau^{-2(\frac{\eta+J}{2}+1)} \exp\left\{-\frac{1}{2\tau^2}\left(\eta\xi^2+\sum_{j=1}^{J}(\theta_j-\mu)^2\right)\right\}$$

which is proportional to the inverse chi-squared distribution.

$\therefore \tau^2 \mid \mu, \theta, \sigma^2, y \sim \text{Inv-}\chi^2\left(\eta-J, \frac{\eta\xi^2+\sum_{j=1}^{J}(\theta_j-\mu)^2}{\eta+J}\right)$

$\therefore$ the scale of the parameter comes from

$(\eta+J)\xi_n^2 = \eta\xi^2+\sum_{j=1}^{J}(\theta_j-\mu)^2)$

\3b/ By Bayes: $\pi(\mu \mid \theta, \sigma^2, \tau^2, y) \propto \pi(\mu \mid \tau^2, \sigma^2, \theta) p(y \mid \dots) \propto$

$\pi(\mu \mid \tau^2, \sigma^2) \pi(\theta \mid \mu, \tau^2) \propto$

$\exp\left\{-\frac{1}{2\tau_0^2}(\mu-\mu_0)^2\right\} \prod_{j=1}^{J} \exp\left\{-\frac{1}{2\tau^2}(\theta_j-\mu)^2\right\} \propto$

$\exp\left\{-\frac{1}{2}\left(\frac{1}{\tau_0^2}(\mu-\mu_0)^2+\frac{1}{\tau^2}\sum_{j=1}^{J}(\theta_j-\mu)^2\right)\right\} \propto$

$\exp\left\{-\frac{1}{2}\left(\left(\frac{1}{\tau_0^2}+\frac{J}{\tau^2}\right)\mu^2-2\mu\left(\frac{\mu_0}{\tau_0^2}+\frac{J\bar{\theta}}{\tau^2}\right)\right)\right\} \propto$

$\exp\left\{-\frac{1}{2}\left(\left(\frac{1}{\tau_0^2}+\frac{J}{\tau^2}\right)\left(\mu-\frac{\frac{\mu_0}{\tau_0^2}+\frac{J\bar{\theta}}{\tau^2}}{\frac{1}{\tau_0^2}+\frac{J}{\tau^2}}\right)^2\right)\right\}$

$p(\theta \mid a) \propto p(\theta) p(a \mid \theta)$

$p(\theta \mid a, y) \propto p(\theta \mid a) p(y \mid \theta, a)$

$\propto p(\theta) p(y, a \mid \theta)$

$\propto p(\theta) p(y \mid \theta, a) p(a \mid \theta)$

$\propto p(\theta \mid a) p(y \mid \theta, a)$

$\propto p(\theta \mid y) p(a \mid \theta, y)$

which is proportional to a Normal density $\therefore$

$$\mu \mid \theta, \sigma^2, \tau^2, y \sim N\left(\frac{\frac{\mu_0}{\tau_0^2}+\frac{J\bar{\theta}}{\tau^2}}{\frac{1}{\tau_0^2}+\frac{J}{\tau^2}}, \frac{1}{\tau_0^2}+\frac{J}{\tau^2}\right)$$

\3c/ By Bayes $\pi(\tau^2 \mid \mu, \theta, \sigma^2, y) \propto \pi(\tau^2) \pi(\theta \mid \mu, \tau^2) \propto$

$\tau^{-2(\frac{\eta}{2}+1)} \exp\left\{-\frac{\eta\xi^2}{2\tau^2}\right\} \prod_{j=1}^{J} \frac{1}{\tau} \exp\left\{-\frac{1}{2\tau^2}(\theta_j-\mu)^2\right\} \propto$

$\tau^{-2(\frac{\eta}{2}+1)-J} \exp\left\{-\frac{1}{2\tau^2}\left(\eta\xi^2+\sum_{j=1}^{J}(\theta_j-\mu)^2\right)\right\} \propto$

$\tau^{-2(\frac{\eta+J}{2}+1)} \exp\left\{-\frac{1}{2\tau^2}\left(\eta\xi^2+\sum_{j=1}^{J}(\theta_j-\mu)^2\right)\right\}$

$\pi(\theta \mid y) \propto p(\theta) p(y \mid \theta)$

$p(\theta \mid y, a) \propto p(\theta) p(y, a \mid \theta)$

$\propto p(\theta \mid a) p(y \mid \theta, a)$

which is proportional to the density of an inverse chi squared distribution $\therefore$

$$\tau^2 \mid \mu, \theta, \sigma^2, y \sim \text{Inv-}\chi^2\left(\eta+J, \frac{\eta\xi^2+\sum_{j=1}^{J}(\theta_j-\mu)^2}{\eta+J}\right)$$

$\therefore$ the scale of the parameter comes from $(\eta+J)\xi_n^2 = \eta\xi^2+\sum_{j=1}^{J}(\theta_j-\mu)^2)$

\4a/ Let T_E be the time spent in the E class and T_1 be the time in I_1 $\therefore$ $T_P = T_E + T_1$ and

$T_1 \mid \lambda \sim Exp(\lambda)$, $T_E \mid \lambda \sim Exp(\lambda)$ $\therefore$

$f(T_P)(z) = \int_0^\infty f_{T_E, T_1}(x, z-x)\,dx = \int_0^z f_{T_E}(x) f_{T_1}(z-x)\,dx$ (independence)

$= \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)}\,dx = \int_0^z \lambda^2 e^{-\lambda z} e^{-\lambda x + \lambda x}\,dx = \int_0^z \lambda^2 e^{-\lambda z}\,dx =$

$\left[\lambda^2 e^{-\lambda z} x\right]_{x=0}^{z} = \lambda^2 z e^{-\lambda z} = \frac{\lambda^2}{1!} z^{2-1} e^{-\lambda z} = \frac{\lambda^2}{\Gamma(2)} z^{2-1} e^{-\lambda z}$

which is the pdf of a Gamma$(2, \lambda)$ distribution

$\because$ exponential is only defined for positive argument $\therefore$

$z - x$ only works for $x \in [0, z]$

\4a/ Let T_E be the time spent in the E class and T_1 be the time in I_1 $\therefore$ $T_P = T_E + T_1$ and

$T_1 \mid \lambda \sim Exp(\lambda)$, $T_E \mid \lambda \sim Exp(\lambda)$ $\therefore$

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$\because$ exponential distribution is only defined for positive argument $\therefore$ $z - x$ only works as long as $x \in [0, z]$ $\therefore$

$f(T_P)(z) = \int_0^z f_{T_E}(x) f_{T_1}(z-x)\,dx = \int_0^z \lambda e^{-\lambda x} \lambda e^{-\lambda(z-x)}\,dx =$

$\int_0^z \lambda^2 e^{-\lambda z} e^{-\lambda x + \lambda x}\,dx = \int_0^z \lambda^2 e^{-\lambda z}\,dx = \left[\lambda^2 e^{-\lambda z}\right]_{x=0}^{z} =$

$\lambda^2 z e^{-\lambda z} = \frac{\lambda^2}{1} z^1 e^{-\lambda z} = \frac{\lambda^2}{1!} z^{2-1} e^{-\lambda z} = \frac{\lambda^2}{\Gamma(2)} z^{2-1} e^{-\lambda z} = \lambda^2 z e^{-\lambda z}$

which is the pdf of a Gamma$(2, \lambda)$ distribution ~

\4b/ Suppose t_i for $i = 1, \ldots, n$ are exchangeable observations of the relevant time, by Bayes theorem:

$\pi(\lambda \mid \underline{t}) \propto \pi(\lambda) f(\underline{t} \mid \theta) \propto \lambda^{\alpha-1} e^{-\beta\lambda} \prod_{i=1}^{n} \lambda^2 t_i e^{-\lambda t_i} \propto$

$\lambda^{\alpha+2n-1} e^{-\lambda(\beta)} e^{-\lambda \sum_{i=1}^{n} t_i} \propto \lambda^{\alpha+2n-1} e^{-\lambda(\beta + \sum_{i=1}^{n} t_i)}$

which is proportional to a Gamma density $\therefore$

$\lambda \mid \underline{t} \sim Gamma(\alpha + 2n, \beta + \sum_{i=1}^{n} t_i)$ $\therefore$

$\lambda \sim Gamma(\alpha, \beta)$ and $\lambda \mid \underline{t} \sim Gamma(\alpha + 2n, \beta + \sum_{i=1}^{n} t_i)$

$\therefore$ the prior is conjugate

\PP2021/ \4b/ suppose t_i for $i=1,\dots,n$ are exchangeable observations of the relevant time, by Bayes theorem:

$$\pi(\lambda|\underline{t}) \propto \pi(\lambda) f(\underline{t}|\theta) \propto \lambda^{\alpha-1} e^{-\beta\lambda} \prod_{i=1}^{n} \lambda^2 t_i e^{-\lambda t_i} \propto$$
$$\lambda^{\alpha+2n-1} e^{-\lambda\beta} e^{-\lambda \sum_{i=1}^{n} t_i} \propto \lambda^{\alpha+2n-1} e^{-\lambda(\beta + \sum_{i=1}^{n} t_i)}$$

which is proportional to a Gamma density $\therefore$

$\lambda|\underline{t} \sim \text{Gamma}(\alpha+2n, \beta+\sum_{i=1}^{n} t_i)$ $\therefore$

and $\lambda \sim \text{Gamma}(\alpha, \beta)$ $\therefore$

The prior is conjugate

\4e/ Let $T_R = T_2 + T_3$, where T_2 is the time spent in I_2, T_3 in I_3 $\therefore$ T_R is time spent ~~from~~ showing symptoms to recovery $\therefore$

by part (a) $T_R|\lambda \sim \text{Gamma}(2,\lambda)$. The required probability is a posterior predictive probability $\therefore$

$$p(t_p \le 7 | T_p = 7) = \int_0^7 \int_0^\infty p(t_R|\lambda)\, \pi(\lambda|T_p = 7)\, d\lambda\, dt_R =$$
$$\int_0^7 \int_0^\infty \lambda^2 t_R e^{-\lambda t_R} \frac{(\beta+7)^{\alpha+2}}{\Gamma(\alpha+2)} \lambda^{\alpha+1} e^{-\lambda(\beta+7)}\, d\lambda\, dT_R =$$
$$\int_0^7 \frac{t_R(\beta+7)^{\alpha+2}\Gamma(\alpha+4)}{\Gamma(\alpha+2)(t_R+\beta+7)^{\alpha+4}} \int_0^\infty \frac{(t_R+\beta+7)^{\alpha+4}}{\Gamma(\alpha+4)} \lambda^{\alpha+3} e^{-\lambda(t_R+\beta+7)}\, d\lambda\, dt_R =$$
$$\int_0^7 \frac{t_R(\beta+7)^{\alpha+2}\Gamma(\alpha+4)}{\Gamma(\alpha+2)(t_R+\beta+7)^{\alpha+4}}\, dt_R$$

and $\alpha = \beta = 1$ $\therefore$

$$p(t_p \le 7 | T_p = 7) = \int_0^7 \frac{t_R(1+7)^{1+2}\Gamma(1+4)}{\Gamma(1+2)(t_R+1+7)^{1+4}}\, dt_R = \int_0^7 \frac{t_R(8)^3\Gamma(5)}{\Gamma(3)(t_R+8)^5}\, dt_R =$$
$$\frac{8^3(5-1)!}{(3-1)!} \int_0^1 \frac{t_R}{(t_R+8)^5}\, dt_R = 6144 \int_0^7 \frac{t_R}{(t_R+8)^5}\, dt_R = \quad \text{(by IBP)}$$
$$6144 \int_0^7 t_R (t_R+8)^{-5}\, dt_R = 6144\left(\left[t_R \frac{1}{-4}(t_R+8)^{-4}\right]_0^7 - \int_0^7 1\left(\frac{1}{-4}\right)(t_R+8)^{-4}\, dt_R\right)$$
$$= 6144\left(7\left(\frac{1}{-4}\right)(7+8)^{-4} + \frac{1}{4}\left[\frac{1}{-3}(t_R+8)^{-3}\, dt_R\right]_0^7\right) =$$
$$6144\left(-\frac{7}{4(15^4)} + \frac{1}{4}\left[\frac{-1}{3(7+8)^3}\right]\right) = 0.636$$

\4c/ let $T_R = T_2 + T_3$, where T_R is the time spent from showing symptoms to recovery, T_2 is time spent in I_2, T_3 in I_3