MTH3042 Integral equations Term 2

~~[illegible]~~ Mondays ~~[illegible]~~

Fridays ~~[illegible]~~ tuesdays ~~[illegible]~~

thursdays ~~[illegible]~~ Queens Building ~~[illegible]~~

tutorial like sessions on Mondays

Coursework 20% (12 pm Friday 1st april)

given table of Laplace transforms in Exams formula table

Laplace transforms

Def / given a function $F:[0,\infty)\to\mathbb{R}$ $F(t) = \boxed{}$

the laplace transform of F is given by

$$f(s) = \int_0^\infty e^{-st} F(t)\,dt$$

if $F(t) = \underbrace{\text{constant}}_{k}$ $\boxed{}$

the laplace transform of $kF(t)$ is

k·laplace transform of $F(t)$.

Notation: $\mathcal{L}(F(t)) = f(s)$ ← laplace transform

Ex / $I = \int_0^1 \frac{1}{x^p}\,dx$ for which p does this integral exist?

$$\int_\varepsilon^1 \frac{1}{x^p}\,dx = \left[\frac{x^{1-p}}{1-p}\right]_\varepsilon^1 = \frac{1}{1-p} - \lim_{\varepsilon\to 0}\frac{\varepsilon^{1-p}}{1-p}$$

∴ when $p<1$ limit $\to 0$ & if $p>1$ tends to ∞

integral exists if $p<1$ & diverges otherwise

when $p=1$: $\int_\varepsilon^1 \frac{1}{x}\,dx = \lim_{\varepsilon\to 0}\left[\log(x)\right]_\varepsilon^1 = \log(1) - \underbrace{\lim_{\varepsilon\to 0}\log(\varepsilon)}_{\to -\infty}$

∴ the integral does not exist for $p=1$

Proposition / let $\mathcal{L}(F(t)) = f(s)$ denote the laplace transform of $F(t)$, then

(i) $F(t)=1$ $\mathcal{L}(1)(s) = \frac{1}{s}$ $s>0$ (LT1)

(ii) $F(t)=t^n$ $\mathcal{L}(t^n)(s) = \frac{n!}{s^{n+1}}$ (LT2)

(iii) $F(t)=e^{at}$ $\mathcal{L}(e^{at})(s) = \frac{1}{s-a}$ $s>a$ (LT4)

(iv) $F(t)=\cos(\omega t)$ $\quad \mathcal{L}(F(t))(s) = \frac{s}{s^2+\omega^2}$ $\quad s>0$

(v) $F(t)=\sin(\omega t)$ $\quad \mathcal{L}(F(t))(s) = \frac{\omega}{s^2+\omega^2}$ $\quad s>0$

\Proofs/ (i) $F(t)=1$ $\quad s>0$ $\quad \mathcal{L}(F)(s) = \int_0^\infty e^{-st}\cdot 1\, dt =$

$$\lim_{R\to\infty}\int_0^R e^{-st}\,dt = \lim_{R\to\infty}\left[-\frac{1}{s}e^{-st}\right]_0^R = -\frac{1}{s}\Big(\underbrace{\lim_{R\to\infty} e^{-sR}}_{\to 0\ (s>0)} - 1\Big) = \frac{1}{s}$$

(iii) $F(t)=e^{at}$ $\quad \mathcal{L}(F)(s) = \int_0^\infty e^{at}e^{-st}\,dt = \int_0^\infty e^{at-st}\,dt = \int_0^\infty e^{t(a-s)}\,dt$

$$= \lim_{R\to\infty}\left[\frac{1}{a-s}e^{t(a-s)}\right]_0^R = \lim_{R\to\infty}\Big(\underbrace{\frac{e^{aR-sR}}{a-s}}_{\to 0\ \because s>a,\ s>a} - \frac{1}{a-s}\Big) = \frac{1}{s-a}$$

for (iv) & (v) Reminder: $\cos(x) = \frac{e^{ix}+e^{-ix}}{2}$ $\quad \sin(x) = \frac{e^{ix}-e^{-ix}}{2i}$

so $e^{ix} = \cos x + i\sin(x) = \frac{e^{ix}+e^{-ix}}{2} + i\left(\frac{e^{ix}-e^{-ix}}{2i}\right) = e^{ix}$

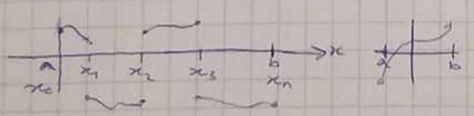
let $x=\omega t$ $\quad$ lets find $\mathcal{L}(\overbrace{e^{i\omega t}}^{\cos(\omega t)+i\sin(\omega t)})$

$$\mathcal{L}(e^{i\omega t})(s) = \int_0^\infty e^{-st}e^{i\omega t}\,dt = \int_0^\infty e^{t(i\omega-s)}\,dt = \lim_{R\to\infty}\int_0^R e^{t(i\omega-s)}\,dt =$$

$$\lim_{R\to\infty}\Big(\underbrace{\frac{1}{i\omega-s}e^{R(i\omega-s)}}_{t=R} - \underbrace{\frac{1}{i\omega-s}}_{t=0}\Big) \qquad \left\{ e^{R(i\omega-s)} = \underbrace{e^{Ri\omega}}_{\text{bounded}}\,\underbrace{e^{-Rs}}_{\to 0 \text{ as } R\to\infty} \right\}$$

$$= \frac{1}{s-i\omega} = \frac{s+i\omega}{\underbrace{s^2+\omega^2}_{(s-i\omega)(s+i\omega)}} \qquad \text{so } \mathcal{L}(e^{i\omega t}) = \underbrace{\frac{s}{s^2+\omega^2}}_{\mathcal{L}(\cos(\omega t))} + i\cdot\underbrace{\frac{\omega}{s^2+\omega^2}}_{\mathcal{L}(\sin(\omega t))}$$

\Def/ let (a,b) be a finite interval, $F(a,b)\to\mathbb{R}$ is piecewise continuous if $\exists$ a finite number of points x_i $\quad a=x_0<x_1<x_2<\dots<x_n=b$ where F is continuous on each subinterval with a finite limit at each end point



\Def/ a function $F:(0,\infty)\to\mathbb{R}$ is of <u>exponential order</u> $\gamma\in\mathbb{R}$ if $\exists M>0$ st $|F(t)|\le Me^{\gamma t}$ $\forall t>0$ $\{\gamma \text{ is order}\}$

\Ex/ every polynomial is of exponential order. take $\underbrace{e^{at}}_{t^n \le \frac{e^{at}n!}{a^n}} = \sum_{n=0}^\infty \frac{t^n a^n}{n!} \ge \underbrace{\frac{t^n a^n}{n!}}_{\{a \text{ is } \gamma\}}$

for the $\mathcal{L}(F(t))(s)$ to exist, we need: the integral $\int_0^R e^{-st}F(t)\,dt$ to exist for $R\to\infty$

\thm/ let $F:(0,\infty)\to\mathbb{R}$ be a piecewise continuous func of exponential order γ, then the $\mathcal{L}(F(t))$

exists $s > \gamma$

proof: $\mathcal{L}(F(t))$: ($|s| \leq 5$ 11)

$$\left|\int_0^R e^{-st} F(t)\,dt\right| \leq \int_0^R |F(t)| e^{-st}\,dt$$

$$\leq \int_0^R e^{-st} M e^{\gamma t}\,dt$$ using (*) since F is of exp order

$$= M\int_0^R e^{-t(s-\gamma)}\,dt$$

properties of Laplace transforms

① Linearity: let $F(t)$, $G(t)$ be two functs with Laplace transforms $f(s)$, $g(s)$, then for $s > \gamma$, for $\alpha, \beta \in \mathbb{R}$ $\quad \mathcal{L}(\alpha F(t) + \beta G(t)) = \alpha\,\mathcal{L}(F)(s) + \beta\,\mathcal{L}(G)(s) =$

$\alpha f(s) + \beta g(s)$ (LT7)

② $a > 0$ $\quad \mathcal{L}(F(at))(s) = \frac{1}{a} f\left(\frac{s}{a}\right)$ (LT8)

proof ①: $\mathcal{L}(\alpha F + \beta G)(s) = \lim_{R\to\infty}\int_0^R (\alpha F + \beta G) e^{-st}\,dt =$

$\lim_{R\to\infty}\left(\int_0^R \alpha F e^{-st}\,dt + \int_0^R \beta G e^{-st}\,dt\right) =$

$$\alpha \underbrace{\lim_{R\to\infty}\int_0^R F e^{-st}\,dt}_{\mathcal{L}(F)} + \beta \underbrace{\lim_{R\to\infty}\int_0^R G e^{-st}\,dt}_{\mathcal{L}(G)} = \alpha f(s) + \beta g(s) \quad \text{(LT8)}$$

proof ②: $\mathcal{L}(F(at))(s) = \lim_{R\to\infty}\int_0^R e^{-st} F(at)\,dt$

let $u = at$ $\quad u' = a$ $\quad \frac{du}{dt} = a$

$$= \lim_{R\to\infty}\int_0^R e^{-\frac{u}{a}s} F(u)\frac{du}{a} = \frac{1}{a}\lim_{R\to\infty}\int_0^R e^{-\frac{su}{a}} F(u)\,du$$

$$= \frac{1}{a}\mathcal{L}(F)\left(\frac{s}{a}\right) \quad \therefore \mathcal{L}(F(at))(s) = \frac{1}{a}\mathcal{L}(F)\left(\frac{s}{a}\right)$$

Ex: apply LT8 (ie property ②) to $F(t) = \cos(t)$

we have $\mathcal{L}(F) = \mathcal{L}(\cos(t)) = \frac{s}{s^2+1}$

$$\mathcal{L}(\cos(\omega t)) = \frac{1}{\omega}\left(\frac{\frac{s}{\omega}}{\left(\frac{s}{\omega}\right)^2+1}\right) = \frac{1}{\omega^2}\left(\frac{s}{\frac{s^2}{\omega^2}+1}\right) = \frac{s}{s^2+\omega^2}$$

Ex: Apply LT7 (property ①) to $\mathcal{L}(P(t))$,

$$P(t) = \sum_{i=1}^{n} a_i t^i = a_1 t + a_1 t + a_2 t^2 + \dots + a_n t^n$$

$\mathcal{L}(P(t)) = \mathcal{L}\left(\sum_{i=1}^{n} a_i t^i\right)$ $\quad$ {in LT2 : n is natural number}

$= \mathcal{L}(a_0) = a_1\mathcal{L}(t) + a_2\mathcal{L}(t^2) + \dots$ arealdl

$= \sum_{i=0}^{n} a_i \mathcal{L}(t^i)(s)$ $\therefore$ using LT2:

$\mathcal{L}(p(t)) = \sum_{i=0}^{n} a_i \frac{i!}{s^{i+1}}$

First shift thm thm: let F be a func with Laplace transform $f(s)$, $s > \gamma$ then $\mathcal{L}(e^{at}F(t))(s) = f(s-a)$ (LT13)

proof: $\mathcal{L}(e^{at}F(t)) = \int_0^\infty e^{-st} e^{at} F(t)\,dt = \int_0^\infty e^{at-st} F(t)\,dt$

$= \int_0^\infty \underbrace{e^{t(a-s)}}_{e^{-t(s-a)}} F(t)\,dt$

$= f(s-a)$ by z def of z Laplace transform

Ex Find $\mathcal{L}(G(t))$, $G(t) = e^{at}$ here

$F(t) = 1$, $\mathcal{L}(1 \cdot e^{at})$ using 1st shift thm, $\mathcal{L}(1)$ with s replaced with $s-a$ $\mathcal{L}(e^{at}) = \frac{1}{s-a}$ (agrees with LT4)

Ex $G(t) = e^{-2t}\cos(3t)$ $\therefore a = -2$, $F(t) = \cos(3t)$

find Laplace transform of $\cos(3t)$ & replace s with $s+2$ $\{(s-2)\}$

$\therefore \mathcal{L}(\cos(3t)) = \frac{s}{s^2+9} = \mathcal{L}(e^{-2t}\cos(3t)) = \frac{s+2}{(s+2)^2+9} = \frac{s+2}{s^2+4s+13}$

Ex $F(t) = (2t-3)e^{\frac{t-2}{3}}$ $\therefore$ find $\mathcal{L}(F(t))$.

$F(t) = 2te^{\frac{t-2}{3}} - 3e^{\frac{t-2}{3}}$

$\mathcal{L}(F(t)) = \mathcal{L}\left(2te^{\frac{t-2}{3}} - 3e^{\frac{t-2}{3}}\right) = \int_0^\infty e^{-st}\left(2te^{\frac{t-2}{3}} - 3e^{\frac{t-2}{3}}\right)dt =$

$\int_0^\infty 2te^{-st+\frac{t-2}{3}} - 3e^{-st+\frac{t-2}{3}}\,dt = \lim_{R\to\infty}\int_0^R 2te^{-st+\frac{t-2}{3}} - 3e^{-st+\frac{t-2}{3}}\,dt =$

$\lim_{R\to\infty}\int_0^R 2te^{-st+\frac{t-2}{3}}\,dt + \lim_{R\to\infty}\int_0^R -3e^{-st+\frac{t-2}{3}}\,dt =$

$2\lim_{R\to\infty}\int_0^R te^{-st+\frac{t-2}{3}}\,dt - 3\lim_{R\to\infty}\int_0^R e^{-st+\frac{t-2}{3}}\,dt$

$F(t) = 2te^{\frac{t+2}{3}} - 3e^{\frac{t+2}{3}} = 2te^{\frac{2}{3}}e^{\frac{t}{3}} - 3e^{\frac{2}{3}}e^{\frac{t}{3}} =$

$2e^{\frac{2}{3}}te^{t/3} - 3e^{2/3}e^{t/3} = \underbrace{2e^{2/3}\left(\frac{1}{s^2}\right)}_{LT13,\ LT2\ \text{replace } s \text{ with } s-\frac{1}{3}} - \underbrace{3e^{2/3}\frac{1}{s-\frac{1}{3}}}_{\text{by LT4}} =$

func of exponential ... , ...

$$2e^{2/3}\left(\frac{1}{(s-\frac{1}{3})^2}\right) - \frac{3e^{2/3}}{s-\frac{1}{3}} = \frac{27e^{2/3}(1-s)}{(3s-1)^2}$$ {(my) seems to always combine into a single denominator}

Make common denominator take $\frac{1}{3}$ out of bracket

Let F' denote the derivative of F

$\mathcal{L}(F'(t))(s) = \int_0^\infty e^{-st} F'(t)\,dt$ $\therefore$ using integration by parts (IBP)

$u = e^{-st}$, $v' = F'(t)$ $\quad u' = -se^{-st}$, $v = F(t)$

$= \lim_{R\to\infty}\left[e^{-st}F(t)\right]_0^R + s\int_0^\infty e^{-st}F(t)\,dt =$

$\underbrace{\lim_{R\to\infty} e^{-sR}F(R)}_{0} - F(0) + s\int_0^\infty e^{-st}F(t)\,dt$

$= -F(0) + sf(s)$ $\therefore$ $\mathcal{L}(F'(t))(s) = sf(s) - F(0)$

Thm/ Let $F(t)$, $F:(0,\infty)\to\mathbb{R}$ be a piecewise continuous ~~function~~ function with piecewise continuous derivative $F':(0,\infty)\to\mathbb{R}$ and of exponential order.

$\mathcal{L}(F'(t))(s) = s\mathcal{L}(F(t))(s) - F(0)$

(assuming $\lim_{t\to 0^+} F(t)$ is $F(0)$)

Lets generalise this result:

Thm/ for the nth derivative: $\mathcal{L}(F^{(n)}(t))$ $n\geq 2$, F is continuous, as well as its derivatives up to order $n-1$, ~~at~~ nth derivative piecewise continuous of finite exponential order

$\mathcal{L}(F^{(n)}(t))(s) = s^n\mathcal{L}(F(t))(s) - \sum_{k=0}^{n-1} s^k F^{(n-1-k)}(0)$ (LT11) & (LT15 - $\mathcal{L}(F')$)

when $n=2$, $k=0,1$

$\mathcal{L}(F''(t))(s) = s^2\mathcal{L}(F(t)) - \underbrace{sF(0)}_{k=1} - \underbrace{F'(0)}_{k=0}$

Ex/ find the laplace transform of F' where $F(t) = \cos(t)$

$\mathcal{L}(F'(t)) = s\underbrace{f(s)}_{=\frac{s}{s^2+1}} - \underbrace{F(0)}_{=1}$ $\therefore$ $\mathcal{L}(F') = s\left(\frac{s}{s^2+1}\right) - 1 = -\frac{1}{s^2+1}$

Ex. Find laplace transform of $F(t)=(t+a)^2 \quad a\in\mathbb{R}$

$=t^2+2ta+a^2 \therefore \mathcal{L}(F(t)) = \underbrace{\mathcal{L}(t^2)}_{LT2} + 2a\underbrace{\mathcal{L}(t)}_{LT2} + a^2\underbrace{\mathcal{L}(1)}_{LT1} =$

$\frac{2!}{s^3} + \frac{2a}{s^2} + \frac{a^2}{s}$

Ex find the laplace transform of $F(t) = t\sin(2t)$

$\left\{ \frac{1}{s}(e^{2s}-1) - \frac{1}{s^2}e^{2s} \quad -\frac{1}{s}(e^{-2s}-1) + \frac{1}{s^2}e^{-2s} \quad -\frac{1}{s}(e^{2s}-1) - \frac{1}{s^2}e^{2s} \right\}$

want to find $\mathcal{L}(F(t)) = f(s)$ $\quad F'(t) = \sin(2t) + 2t\cos(2t)$

$F''(t) = 2\cos(2t) + 2\cos(2t) - 4t\sin(2t$

$F''(t) = 4\cos(2t) - 4t\sin(2t) \therefore$

$\underbrace{\mathcal{L}(F''(t))}_{LT10} = \underbrace{\mathcal{L}(4\cos(2t))}_{LT6} - 4\underbrace{\mathcal{L}(t\sin(2t))}_{f(2)}$

$= s^2f(s) - sF(0) - F'(0)$

$= s^2f(s) - s(0) - 0 \quad \therefore \quad s^2f(s) = \frac{4s}{s^2+4} - 4f(s)$

make $f(s)$ subject

$\therefore$ so $f(s) = \frac{4s}{(s^2+4)^2}$

thm: F is piecewise continuous of finite exponential order $\gamma>0$. then $\mathcal{L}\left(\int_0^t F(u)\,du\right) = \frac{f(s)}{s}$ (i) (LT20)

for $a>0 \quad \mathcal{L}\left(\int_a^t F(u)\,du\right) = \frac{1}{s}\left(f(s) - \int_0^a F(t)\,dt\right)$ (LT21)

Ex/ Find the laplace transform of $\int_0^t \underbrace{\sin(au)\cos(au)}_{F(u)}\,du$

$\therefore$ using (i) $\mathcal{L}\left(\int_0^t \sin(au)\cos(au)\,du\right) = \frac{\mathcal{L}(\sin(au)\cos(au))}{s} \therefore$

$\frac{1}{2}\sin(2au) = \sin(au)\cos(au) \therefore$

$\mathcal{L}\left(\int_0^t \sin(au)\cos(au)\,du\right) = \frac{\frac{1}{2}\left(\frac{2a}{s^2+(2a)^2}\right)}{s} = \frac{f(s)}{s}$

$= \frac{1}{2s}\mathcal{L}(\sin(2au)) = \frac{1}{2s}\left(\frac{2a}{s^2+4a^2}\right) = \frac{a}{s(s^2+4a^2)}$

Ex/ find the LT of $\int_0^t \underbrace{e^{au}\cos(bu)}_{F(u)}\,du = \frac{f(s)}{s}$

$\therefore f(s) = \mathcal{L}(\cos(bu))$ with s replaced with $s-a$ $\because$ (using 1st shift thm)

$= \frac{(s-a)}{(s-a)^2+b^2} \quad \therefore \mathcal{L}\left(\int_0^t e^{au}\cos(bu)\,du\right) = \frac{(s-a)}{s((s-a)^2+b^2)}$

$F(t) = \begin{cases} 1 & 0 \le t < 2 \\ t-2 & t \ge 2 \end{cases}$ $\quad \mathcal{L}(F(t))$ using def :

$= -\frac{1}{s}(e^{-2s}-1) + \frac{1}{s^2}e^{-2s}$

$$f(t) = \int_0^\infty e^{-st}F(t)\,dt = \int_0^2 e^{-st}\,1\,dt + \int_2^\infty e^{-st}(t-2)\,dt =$$

$$\left[\frac{1}{-s}e^{-st}\right]_0^2 + \lim_{R\to\infty}\int_2^R te^{-st}\,dt - \lim_{R\to\infty}\int_2^R 2e^{-st}\,dt =$$

$$\frac{1}{-s}\left[e^{-2s}-e^0\right] + \lim_{R\to\infty}\left(\left[t\frac{1}{-s}e^{-st}\right]_2^R - \int_2^R \frac{1}{-s}e^{-st}\,dt\right) - \lim_{R\to\infty}\left[\frac{2}{-s}e^{-st}\right]_0^R =$$

$$\frac{1}{-s}\left[e^{-2s}-1\right] + \lim_{R\to\infty}\left(\left[R\frac{1}{-s}e^{-sR} - 2\frac{1}{-s}e^{-2s}\right] + \frac{1}{s}\left[\frac{1}{-s}e^{-st}\right]_2^R\right) - \lim_{R\to\infty}\left[\frac{2}{-s}e^{-sR} - \frac{2}{-s}\right]$$

Ex: $F(t) = \begin{cases} 1 & 0 \le t < 2 \\ t-2 & t \ge 2 \end{cases}$ $\quad$ recall: $\mathcal{L}(F(t)) = \int_0^\infty e^{-st}F(t)\,dt$

$$= \int_0^2 e^{-st}\cdot 1\,dt + \underbrace{\int_2^\infty e^{-st}(t-2)\,dt}_{IBP} \qquad \left\{u = t-2 \quad u' = 1 \quad v' = e^{-st} \quad v = -\frac{1}{s}e^{-st}\right\}$$

$$= -\frac{1}{s}e^{-2s} + \frac{1}{s} + \underbrace{\left[(t-2)\cdot\frac{-1}{s}e^{-st}\right]_2^R}_{\text{as } R\to\infty} + \frac{1}{s}\int_2^\infty e^{-st}\,dt$$

$$= -\frac{1}{s}(e^{-2s}-1) + (0-0) + \frac{1}{s}\cdot\underbrace{\frac{-1}{s}\left[e^{-st}\right]_2^\infty}_{\text{as } R\to\infty}$$

$$= -\frac{1}{s}(e^{-2s}-1) + \frac{1}{s^2}e^{-2s}$$

the inverse Laplace transform here we are given $f(s)$ & we want to find $F(t)$

Def: Let $f(s) = \mathcal{L}(F(t))(s)$, then the inverse Laplace transform of $f(s)$ is $\mathcal{L}^{-1}(f(s)) = F(t)$ we have linearity: $\mathcal{L}^{-1}(\alpha f(s) + \beta g(s)) = \mathcal{L}^{-1}(\alpha f(s)) + \mathcal{L}^{-1}(\beta g(s)) = \alpha\mathcal{L}^{-1}(f(s)) + \beta\mathcal{L}^{-1}(g(s)) = \alpha F(t) + \beta G(t)$

Ex: find the inverse Laplace transform of

$$f(s) = \underbrace{\frac{1}{s^3}}_{\substack{LT2 \\ n=2}} + \frac{6}{s^2+4}$$

$$\mathcal{L}^{-1}(f(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^3}\right) + \mathcal{L}^{-1}\Big(\underbrace{\frac{6}{s^2+4}}_{\omega^2\ LT5}\Big) = \frac{1}{2!}t^2 + \frac{6}{2}\sin(2t) = \frac{t^2}{2} + 3\sin(2t)$$

\Ex/ find the inverse Laplace transform of $f(s)=\frac{2s+3}{s^2-4s+20}$

$$=\frac{2s+3}{(s-2)^2-4+20}=\frac{2s+3}{(s-2)^2+16}=\frac{2s}{(s-2)^2+16}+\frac{3}{(s-2)^2+16}=$$

$$\frac{2(s-2)+4}{(s-2)^2+16}+\frac{3}{(s-2)^2+16}=\frac{2(s-2)}{(s-2)^2+16}+\frac{3+4}{(s-2)^2+16}=\frac{2(s-2)}{(s-2)^2+16}+\frac{7}{(s-2)^2+16}$$

LT6+LT13 ω^2 LT5+LT13 $\to \omega$

$$=2\cos(4t)e^{2t}+\frac{7}{4}\sin(4t)e^{2t}$$ {used 1st shift then adjusted consts}

\Initial value thm/ let $F:(0,\infty)\to\mathbb{R}$ be continuous with piecewise continuous derivatives F' of finite exponential order

$f(s)=\mathcal{L}(F(t))$ $\therefore$ $\lim_{t\to 0^+}F(t)=\lim_{s\to\infty} s f(s)$

\Final value thm/ $\lim_{t\to\infty}F(t)=\lim_{s\to 0^+} s f(s)$

(which hold) when the limits exist

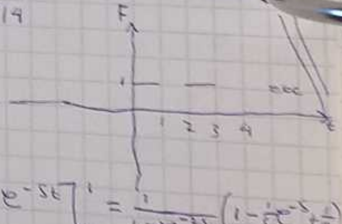
\Ex/ $F(t)=\cos(\omega t)$ $\therefore$ $\lim_{t\to 0^+}F(t)\overset{IVT}{=}\lim_{s\to\infty} s f(s)=\lim_{s\to\infty} s\left(\frac{s}{s^2+\omega^2}\right)=$

$\lim_{s\to\infty}\frac{s^2}{s^2+\omega^2}=1$

we cannot find $\lim_{t\to\infty}\cos(\omega t)$ using FVT $\because$ the limit doesn't exist

\Periodic funcs/ if F is a piecewise continuous func of period $T>0$ $\quad F(t+T)=F(t)\ \forall t\in\mathbb{R}$

then $\mathcal{L}(F(t))(s)=\frac{1}{1-e^{-sT}}\int_0^T e^{-st}F(t)\,dt$ LT14



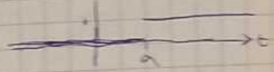
\Ex/ $T=2$ $\therefore$ $\mathcal{L}(F(t))=\frac{1}{1-e^{-2s}}\int_0^2 e^{-st}F(t)\,dt=$

$$\frac{1}{1-e^{-2s}}\left(\int_0^1 e^{-st}\,dt+\underbrace{\int_1^2 e^{-st}\cdot 0\,dt}_{=0}\right)=\frac{1}{1-e^{-2s}}\left[-\frac{1}{s}e^{-st}\right]_0^1=\frac{1}{1-e^{-2s}}\left(1-\frac{1}{s}e^{-s}+\frac{1}{s}\right)$$

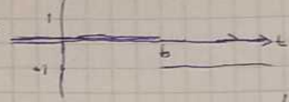
$$=\frac{e^s}{s(e^s+1)}$$

\Def/ the Heaviside func $H(t)$ is given by $H(t)=\begin{cases}1 & t\geq 0\\ 0 & t<0\end{cases}$

Ex / Sketch $F(t) = H(t-a) - H(t-b)$ $H(t-a)$:

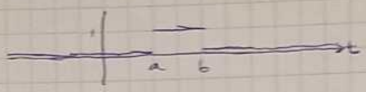


$-H(t-b)$:



$\therefore$ adding 2 graphs:

$F(t) = H(t-a) - H(t-b)$



$$F(t) = \begin{cases} 0 & t < a \\ 1 & a \le t \le b \\ 0 & t > b \end{cases}$$

$\mathcal{L}(H(t-a)) = \frac{e^{-as}}{s}$ LT15 $F(t) = 3H(t-7)$ $\mathcal{L}(F(t)) = \frac{3e^{-7s}}{s}$

thm: Second shift thm / $\mathcal{L}(F(t-a)H(t-a))(s) = e^{-as} \underbrace{\mathcal{L}(F(t))(s)}_{f(s)}$

LT17

Ex / find the inverse Laplace transformation of $f(s) = \frac{e^{-5s}}{(s-2)^4}$

$\therefore$ in the form $e^{-as} f(s) = \frac{e^{-5s}}{(s-2)^4}$ $a = 5$, $f(s) = \frac{1}{(s-2)^4}$

$\mathcal{L}^{-1}\left(\frac{e^{-5s}}{(s-2)^4}\right) = F(t-a)H(t-a) = F(t-5)H(t-5)$ we need $F(t)$ then

replace t with $t-5$ $F(t) \rightarrow \mathcal{L}^{-1}(f(s)) = \mathcal{L}^{-1}\left(\frac{1}{(s-2)^4}\right)$

LT2 with $n=3$, & LT13, $a=12$

$\mathcal{L}^{-1}(f(s)) = \frac{e^{2t}t^3}{3!} = \frac{e^{2t}t^3}{6}$ $\left(\mathcal{L}(t^3) = \frac{n!}{s^{n+1}} = \frac{3!}{s^4}\right)$

$F(t-5) = \frac{e^{2(t-5)}(t-5)^3}{6}$ $\therefore$ $\mathcal{L}^{-1}\left(\frac{e^{-5s}}{(s-2)^4}\right) = \frac{H(t-5)e^{2(t-5)}(t-5)^3}{6}$

Ex / inverse Laplace transform of $f(s) = \frac{se^{-4s}}{(3s+2)(s-2)}$

$(3s+2)(s-2) = 3s^2 - 6s + 2s - 4 = 3s^2 - 4s - 4 = 3(s^2 - \frac{4}{3}s) - 4 =$

$3((s-\frac{2}{3})^2) - \frac{4}{3} - 4 = 3((s-\frac{2}{3})^2) - \frac{16}{3} = 3((s-\frac{2}{3})^2 - \frac{16}{9})$

$f(s) = \frac{se^{-4s}}{3(\underbrace{(s-\frac{2}{3})^2}_{a} - \underbrace{\frac{16}{9}}_{\omega^2})}$ $H(t-4)\left(\frac{1}{12}e^{-2/3(t-4)} + \frac{1}{4}e^{2(t-4)}\right)$

find $\mathcal{L}^{-1}\left(\frac{se^{-4s}}{(3s+2)(s-2)}\right)$ use LT17 $a = 4$, $f(s) = \frac{s}{(3s+2)(s-2)}$

$\therefore \mathcal{L}^{-1}(e^{-4s}f(s)) = F(t-4)H(t-4)$ (*). find $F(t) = \mathcal{L}^{-1}\left(\frac{s}{(3s+2)(s-2)}\right)$

• replace t with $t-4$ • Put into (*)

$f(s) = \frac{s}{(3s+2)(s-2)} = \frac{A}{3s+2} + \frac{B}{s-2}$ $\therefore$ $s = A(s-2) + B(3s+2)$ $\therefore A = \frac{1}{4}$, $B = \frac{1}{4}$

$f(s) = \dfrac{\frac{1}{4}}{3(s+\frac{2}{3})} + \dfrac{\frac{1}{4}}{(s-2)} \xrightarrow[\text{LT1+LT3)}]{\text{LT4 (or}} F(t) = \frac{1}{12}e^{-\frac{2}{3}t} + \frac{1}{4}e^{2t}$

$\therefore F(t-4) = \frac{1}{12}e^{-\frac{2}{3}(t-4)} + \frac{1}{4}e^{2(t-4)}$

$\therefore \mathcal{L}^{-1}(e^{-4s}f(s)) = H(t-4)\left(\frac{1}{12}e^{-\frac{2}{3}(t-4)} + \frac{1}{4}e^{2(t-4)}\right)$

Differential eqns/ use Laplace transforms to transform an ODE (or a system of ODEs into an algebraic eqn to solve Z ODE

Ex/ $\ddot{x}(t) - 3\dot{x}(t) + 2x(t) = 4e^{2t}$ ① I.C. $x(0) = -3$, $\dot{x}(0) = 5$

$\ddot{x}$ 2nd deriv $\dot{x}$ 1st deriv

Step 1: find Z laplace transform of each term

$\mathcal{L}(4e^{2t}) = 4 \cdot \frac{1}{s-2}$ use LT4 $a=2$ $\mathcal{L}(e^{at}) = \frac{1}{s-a}$

$\mathcal{L}(2x(t)) = 2x(s)$

$\mathcal{L}(3\dot{x}(t)) = 3\mathcal{L}(\dot{x}(t)) = 3(sx(s) - X(0)) = 3sx(s) + 9$ LT9

$\mathcal{L}(\ddot{x}) = s^2x(s) - sX(0) - \dot{X}(0)$ LT10

$= s^2x(s) + 3s - 5$

Step 2: put these Laplace transforms back into ① (ODE)

$s^2x(s) + 3s - 5 - 3sx(s) - 9 + 2x(s) = \frac{4}{s-2}$

Make $x(s)$ subject: $x(s)(s^2 - 3s + 2) = \frac{4}{s-2} + 9 + 5 - 3s$

$x(s) = \dfrac{-3s^2 + 20s - 24}{(s-2)(s^2 - 3s + 2)}$

Step 3: find Z inverse Laplace transform of $x(s)$

$X(t) = \mathcal{L}^{-1}(x(s))$ $x(s) = \dfrac{-3s^2+20s-24}{(s-2)(s-1)(s-2)} = \dfrac{-3s^2+20s-24}{(s-1)(s-2)^2} =$

$\dfrac{A}{(s-2)^2} + \dfrac{B}{(s-2)} + \dfrac{C}{s-1}$

finding A, B, C : $-3s^2 + 20s - 24 = A(s-1) + B(s-2)(s-1) + C(s-2)^2$

compare coeffs: s^0: $-24 = -A + 2B + 4C$

s^1: $20 = A - 2B - 2C$ s^2: $-3 = B + C$ Solve: $A = 4$, $B = 4$, $C = -7$

$\frac{4}{(s-2)^2} + \frac{4}{s-2} - \frac{7}{s-1} = x(s)$ $\therefore$

$X(t) = \underbrace{4te^{2t}}_{\substack{\text{LT2 } n=1 \\ \text{LT13, } a=2}} + \underbrace{4e^{2t}}_{\substack{\text{LT4} \\ a=2}} - \underbrace{7e^{t}}_{\substack{\text{LT4} \\ a=1}}$ $\therefore X(t) = 4te^{2t} + 4e^{2t} - 7e^{t}$

Ex/ Solve: $X'(t) = X(t) - Y(t)$ ①

$Y'(t) = 1 + Y(t) - X(t)$ ② I.C. $X(0) = 2$, $Y(0) = 1$

find Z Laplace transform of each term.

$\mathcal{L}(Y(t)) = y(s)$ $\mathcal{L}(X(t)) = x(s)$

$\mathcal{L}(X'(t)) = \underset{LT9}{sx(s) - X(0)} = sx(s) - 2$

$\mathcal{L}(Y'(t)) = sy(s) - Y(0) = sy(s) - 1$

$\mathcal{L}(1) = \frac{1}{s}$ LT1 sub these into Z ODEs ①, ②:

① : $sx(s) - 2 = x(s) - y(s)$

② : $sy(s) - 1 = \frac{1}{s} + y(s) - x(s)$

① $x(s)(s-1) + y(s) = 2$

② $x(s) + y(s)(s-1) = \frac{1}{s} + 1$ now solve for $x(s)$ Z $y(s)$

$$\begin{bmatrix} s-1 & 1 \\ 1 & s-1 \end{bmatrix}\begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} 2 \\ \frac{1}{s}+1 \end{bmatrix} \quad \begin{bmatrix} x(s) \\ y(s) \end{bmatrix} = \begin{bmatrix} s-1 & 1 \\ 1 & s-1 \end{bmatrix}^{-1}\begin{bmatrix} 2 \\ \frac{1}{s}+1 \end{bmatrix} \quad \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1} = \frac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \right.$$

$$\begin{pmatrix} x(s) \\ y(s) \end{pmatrix} = \frac{1}{(s-1)^2 - 1}\begin{pmatrix} s-1 & -1 \\ -1 & s-1 \end{pmatrix}\begin{bmatrix} 2 \\ \frac{1}{s}+1 \end{bmatrix} = \frac{1}{s^2 - 2s}\begin{bmatrix} 2s - 2 - \frac{1}{s} - 1 \\ -2 + 1 + s - \frac{1}{s} - 1 \end{bmatrix} = \frac{1}{s^2 - 2s}\begin{bmatrix} -\frac{1}{s} + 2s - 3 \\ -2 - \frac{1}{s} + s \end{bmatrix}$$

$\therefore\ x(s) = \dfrac{-\frac{1}{s} + 2s - 3}{s^2 - 2s}$ $y(s) = \dfrac{-2 - \frac{1}{s} + s}{s^2 - 2s}$ simplify $\left(\frac{\text{poly}}{\text{poly}}\right)$ then split

using partial fracs for $x(s), y(s)$:

$x(s) = \dfrac{1}{4(s-2)} + \dfrac{7}{4s} + \dfrac{1}{2s^2}$ $y(s) = \dfrac{-1}{4(s-2)} + \dfrac{5}{4s} + \dfrac{1}{2s^2}$

now: $X(t) = \mathcal{L}^{-1}(x(s))$, $Y(t) = \mathcal{L}^{-1}(y(s))$

$X(t) = \mathcal{L}^{-1}\left(\frac{1}{4(s-2)} + \frac{7}{4s} + \frac{1}{2s^2}\right) = \underbrace{\frac{1}{4}e^{2t}}_{LT4,\ a=2} + \underbrace{\frac{7}{4}}_{LT1} + \underbrace{\frac{1}{2}t}_{LT2,\ n=1}$

$Y(t) = \mathcal{L}^{-1}\left(-\frac{1}{4(s-2)} + \frac{5}{4s} + \frac{1}{2s^2}\right) = \underbrace{-\frac{1}{4}e^{2t}}_{LT4\ a=2} + \underbrace{\frac{5}{4}}_{LT1} + \underbrace{\frac{1}{2}t}_{LT2,\ n=1}$

now look at Z Convolution thm:

Def/ let $F, G : \mathbb{R} \to \mathbb{R}$. Z Convolution of F Z G is denoted by $F * G$ is $(F*G)(t) = \int_{-\infty}^{\infty} F(u)G(t-u)\,du$

in our case, F & G have domain $[0,\infty)$ ∴

$(F*G)(t)=\int_0^\infty F(u)G(t-u)\,du$

note: the convolution is <u>not</u> a product, it's an integral operator which acts on the product of F, G

\thm Convolution thm:/ let $f(s)$ & $g(s)$ be $\mathcal{L}(F(t))$ & $\mathcal{L}(G(t))$ respectively, then $\mathcal{L}(F*G)(s)=f(s)g(s)$

\Convolution thm / \Def/ let $F,G:\mathbb{R}\to\mathbb{R}$. The Convolution $F*G$ is $(F*G)(t)=\int_{-\infty}^{\infty}F(u)G(t-u)\,du$ in our case F & G have domain ~~(F*G)~~ $[0,\infty)$ ∴

$\underbrace{(F*G)(t)}_{\text{not a product}}=\int_0^\infty F(u)G(t-u)\,du$

\Convolution thm/ let $f(s)$ & $g(s)$ be $\mathcal{L}(F(t))$ & $\mathcal{L}(G(t))$ respectively then $\mathcal{L}(F*G)(s)=f(s)g(s)$

applying the inverse laplace transform: $\mathcal{L}^{-1}\mathcal{L}(F*G)=\mathcal{L}^{-1}(f(s)g(s))$

∴ $F*G=\mathcal{L}^{-1}(f(s)g(s))$

usefulness: ① have $F(t)=\cdots$ $G(t)=\cdots$

want $\mathcal{L}(F*G)$ • find $\mathcal{L}(F)=f(s)$ $\mathcal{L}(G)=g(s)$

• multiply them: $f(s)g(s)$

② have $\hat{f}(s)=\underbrace{\Delta}_{f(s)}\cdot\underbrace{\square}_{g(s)}$ would like $\mathcal{L}^{-1}(\hat{f}(s))$

• find $\mathcal{L}^{-1}(\Delta)$, $\mathcal{L}^{-1}(\square)$

• $\mathcal{L}^{-1}(\hat{f}(s))=F*G$

• find $F*G=\int_0^\infty F(u)G(t-u)\,du$

③ have $F(t)=\cdots$ $G(t)=\cdots$ want $F*G$

• find $f(s)$, $G(s)$ • inverse laplace of $f(s)g(s)$

\Ex/ given $F(t)=1$ $G(t)=\sin(t)$ $t\in[0,\infty)$ find $\mathcal{L}(F*G)(s)$ (a) by using def of $F*G$

(b) using Convolution thm

(a): $F * G = \underbrace{(1)}_{F(t)} * \underbrace{(\sin(t))}_{G(t)} = \int_0^\infty 1 \cdot \sin(t-u)\,du = [\cos(t-u)]_0^\infty \overset{\leftarrow t>u}{=}$ $\nwarrow u=0$

$\uparrow = \int_0^t 1 \cdot \sin(t-u)\,du$

$1 - \cos(t)$

$\mathcal{L}(F*G) = \mathcal{L}(1-\cos(t)) = \mathcal{L}(1) - \mathcal{L}(\cos(t)) = \underbrace{\frac{1}{s}}_{LT1} - \underbrace{\frac{s}{s^2+1}}_{LT6} = \frac{1}{s(s^2+1)}$

(b): $\mathcal{L}(F*G)(s) = f(s)g(s)$ $\qquad \mathcal{L}(F) = f(s) = f(1) = \frac{1}{s}$

$\mathcal{L}(G) = g(s) = \mathcal{L}(\sin(t)) = \frac{1}{s^2+1}$ $\therefore$ by the convolution thm $\mathcal{L}(F*G) =$

$\frac{1}{s} \cdot \frac{1}{s^2+1} = \frac{1}{s(s^2+1)}$

\Ex/

problem: given: $\hat{f}(s) = \frac{1}{s(s^2+1)}$ find $\mathcal{L}^{-1}(\hat{f}(s))$

<u>technique 1</u>: using partial fractions: $\frac{1}{s(s^2+1)} = \frac{1}{s} - \frac{s}{s^2+1}$

$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) - \mathcal{L}^{-1}\left(\frac{s}{s^2+1}\right) = 1 - \cos(t)$

<u>technique 2</u>: $\frac{1}{s(s^2+1)} = \underbrace{\frac{1}{s}}_{\Delta} \cdot \underbrace{\frac{1}{s^2+1}}_{\square}$ $\qquad \mathcal{L}^{-1}(\Delta) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) = 1 = F(t)$

$\mathcal{L}^{-1}(\square) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin(t) = G(t)$ $\therefore$

using ⊛ we have: $\mathcal{L}^{-1}(f(s)g(s)) = F*G$

$\mathcal{L}^{-1}\left(\frac{1}{s(s^2+1)}\right) = 1 * \sin(t)$ $\qquad 1 * \sin(t) = \int_0^t 1 \cdot \sin(t-u)\,du = 1 - \cos(t)$

\Ex/ $f(s) = \frac{6}{s^3 - 10s^2 + 33s - 36}$ find $\mathcal{L}^{-1}(f(s))$

here 4 is a root of the denominator so $(s-4)$ is a factor

$f(s) = \frac{6}{(s-4)(s^2-6s+9)} = \frac{6}{(s-4)(s-3)^2} = \underbrace{\frac{6}{s-4}}_{\Delta} \cdot \underbrace{\frac{1}{(s-3)^2}}_{\square}$

$\mathcal{L}^{-1}(\Delta) = \mathcal{L}^{-1}\left(\frac{6}{s-4}\right) = \cancel{4}\, \overset{\swarrow \hat{F}}{6e^{4t}}$ $\{LT4, a=4\}$

$\mathcal{L}^{-1}(\square) = \mathcal{L}^{-1}\left(\frac{1}{(s-3)^2}\right) = \underbrace{te^{3t}}_{G}$ $\{LT2, n=1 \quad LT13, a=3\}$

using convolution thm: from ⊛ $\mathcal{L}^{-1}(f(s)) = (6e^{4t}) * (te^{3t})$

$= \int_0^t 6e^{4u} \cdot (t-u)e^{3(t-u)}\,du = 6\int_0^t (t-u)\overset{\swarrow e^{3t}e^{u}}{e^{3t+u}}\,du$

$\{IBP: U = t-u \quad U' = -1 \quad V' = e^{3t+u} \quad V = e^{3t+u}\}$

$= 6\left[(t-u)e^{3t+u}\right]_0^t + 6\int_0^t e^{3t+u}\,du = -6te^{3t} + 6\left[e^{3t+u}\right]_0^t =$

$-6te^{3t} + 6\left[e^{3t+u}\right]_0^t = -6te^{3t} + 6e^{4t} - 6e^{3t} = 6e^{3t}(-1-t) + 6e^{4t}$ is our $\mathcal{L}^{-1}(f(s))$

Ex/ find $\mathcal{L}^{-1}(\hat{f}(s))$, $\hat{f}(s) = \frac{1}{(s^2+1)^2}$

$\hat{f}(s) = \underbrace{\frac{1}{(s^2+1)}}_{\Delta = f(s)} \cdot \underbrace{\frac{1}{(s^2+1)}}_{\square = g(s)}$ use the convolution thm:

$F(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin(t)$

$G(t) = \mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin(t)$

$\mathcal{L}^{-1}(\hat{f}(s)) = \sin(t) * \sin(t) = \int \cancel{\sin(t)\sin(t)} \int_0^t \sin(u)\sin(t-u)\,du$

{use $\sin(A)\sin(B) = \frac{1}{2}(\cos(A-B) - \cos(A+B))$}

integrate each term then put in integration limits

$= \int_0^t \frac{1}{2}(\cos(2u-t) - \cos(t))\,du \stackrel{\ell}{=} -\frac{1}{2}(\sin(t) - t\cos(t))$

Ex/ Solve $Y''(t) + Y'(t) - 6Y(t) = R(t)$ $\quad Y(0) = 0,\ Y'(0) = 0$

$R(t)$ is an unknown func but is continuous & of exponential order (ie it has a laplace transform)

$\mathcal{L}(R(t)) = r(s)$ $\qquad \mathcal{L}(6Y(t)) = 6y(s)$

$\mathcal{L}(Y'(t)) = sy(s) - \underbrace{Y(0)}_{=0} = sy(s)$

$\mathcal{L}(Y''(t)) = s^2y(s) - (s)(0) - 0 = s^2y(s)$ $\qquad$ into ODE:

$s^2y(s) + sy(s) - 6y(s) = r(s)$ $\quad \therefore$

$y(s) = \frac{r(s)}{s^2+s-6 \leftarrow (s-2)(s+3)}$ $\quad \therefore \quad y(s) = \underbrace{r(s)}_{\Delta} \cdot \underbrace{\frac{1}{(s-2)(s+3)}}_{\square}$

$\therefore \mathcal{L}^{-1}(\Delta) = \mathcal{L}^{-1}(r(s)) = R(t)$ $\qquad \square = \frac{1}{5(s-2)} - \frac{1}{5(s+3)}$ by partial fractions.

$\mathcal{L}^{-1}(\square) = \mathcal{L}^{-1}\left(\frac{1}{(s-2)(s+3)}\right) = \mathcal{L}^{-1}\left(\frac{1}{5(s-2)} - \frac{1}{5(s+3)}\right) = \underbrace{\frac{1}{5}e^{2t}}_{LT4,\ a=2} - \underbrace{\frac{1}{5}e^{-3t}}_{LT4,\ a=-3}$

$\therefore$ using the convolution thm $Y(t) = R(t) * \left(\frac{1}{5}e^{2t} - \frac{1}{5}e^{-3t}\right) =$

$\frac{1}{5}\int_0^t R(u)\left(e^{2(t-u)} - e^{-3(t-u)}\right)du = Y(t)$

Chapter 2: Integral eqns

Def/ an integral eqn is an eqn where $y(x)$, the unknown func lies under the integral, & the derivative of $y(x)$ does not appear in the eqn $\quad y(x) - \int_0^a \underbrace{k(x,t)}_{\text{kernel}}\underbrace{y(t)}_{\text{unknown}}\,dt = f(x)$

$y(x) = \ldots \leftarrow$ only depends on x

this is a 1-Dim integral eqn.

A 2-Dim integral eqn looks like $Z(x,y) - \int_a^b \int_a^b Z(s,t) \cdot \frac{ds\,dt}{((x-s)^2+(y-t)^2)^{1/2}} = 0$

we will only focus on 1-Dim IEqns (IE)

if have: $y'(x) - \int_0^a k(x,t)y(t)\,dt = f(x)$ this is an integro-differential eqn, ∵ $y'(x)$ appears in Z eqn, will not be looking into IDE

Ex/ $y(x) - \int_0^1 e^{x-t}\, y(t)\,dt = e^x$

$y(x) - \int_0^x (x^2+t)\,y(t)\,dt = \sin(x)$

$y^2(x) + \int_0^{2\pi} (x-t)\,y(t)\,dt = x^3+1$

Aim: solve Z integral eqns for $y(x)$

Classifying integral eqns (IE)

Def/ A fredholm integral eqn is an IE where both of Z integration limits are const $y(x) - \int_a^b k(x,t)\,y(t)\,dt = f(x)$ (a, b are const)

Def/ A volterra integral eqn is an IE where one of Z limits is variable $y(x) - \int_0^x k(x,t)\,y(t) = f(x)$ eg x is variable

Def/ an integral eqn of Z first kind is one where Z unknown does not appear outside Z integral $\int_0^x k(x,t)\,y(t)\,dt = f(x)$

Def/ an IE of Z second kind is one where Z unknown does appear outside Z integral $y(x) - \int_a^x k(x,t)\,y(t)\,dt = f(x)$

Def/ a homogeneous integral eqn is one where Z $f(x)$ is zero
$y(x) - \int_0^x e^{x-t}\,y(t)\,dt = 0$

Def/ a non homogeneous IE is one where Z $f(x)$ is non zero
$y(x) - \int_0^x e^{x-t}\,y(t)\,dt + \sin(x) = 0$ ∴ $y(x) - \int_0^x e^{x-t}\,y(t)\,dt = \underbrace{-\sin(x)}_{f(x)}$

Def/ a linear IE is one where $y(x)$, Z unknown, only appears linearly everywhere
↳ (in the form $y(x)$)

Def/ a non linear IE is one where $y(x)$ appears non linearly at least once whether thats inside Z integral or outside (eg: $y^2(x)$, $y^{1/3}(x)$, $e^{y(x)}$, $\sin(3y(x))$)

an example of a non linear IE:

$\lambda y(x) - \int_a^b U(x,t,y(t))\,dt = f(x)$ eg $U(x,t,y(t)) = \frac{xt}{t+y(t)}$

this is called a Urysohn type IE $\quad U(x,t,y(t)) = (xt)\cdot\frac{1}{t+y(t)} =$

$k(x,t)\cdot H(y(t),t)$ then our IE looks like

$\lambda y(x) - \int_a^b k(x,t)\,H(y(t),t)\,dt = f(x)$ is called a Hammerstein IE

eg if $U(x,t,y(t)) = \cancel{e^{x}}\; e^{x^2 t\, y(t)}\cdot x$

if $\lambda y(x) - \int_a^x U(x,t,y(t))\,dt = f(x)$ this is a nonlinear ~~Volterra~~

~~IE~~ Urysohn Volterra IE of 2 second kind when $\lambda \neq 0$ &

nonhomogeneous when $f(x) \neq 0$

\Ex on Convolution/ let $F(t) = e^{-t}$ $\quad G(t) = \sin(t)$ would like to find

$F*G$ $\quad \therefore F*G = \int_0^\infty F(u)\,G(t-u)\,du$

$F*G = e^{-t} * \sin(t) = \int_0^\infty e^{-t}\sin(t-u)\,du =$

$\mathcal{L}(F*G)(s) = f(s)\,g(s) \quad \therefore$

$F(t) = e^{-t} \;\therefore\; \mathcal{L}(F(t))(s) = f(s) = \mathcal{L}(e^{-t}) = \frac{1}{s--1} = \frac{1}{s+1}$

$\mathcal{L}(G(t))(s) = g(s) = \mathcal{L}(\sin(t)) = \frac{1}{s^2+1} \;\therefore$

$\mathcal{L}(F*G)(s) = f(s)\,g(s) = \frac{1}{s+1}\,\frac{1}{s^2+1} = \frac{1}{(s+1)(s^2+1)} = \frac{A}{s+1} + \frac{Bs+C}{s^2+1} \;\therefore$

$(s+1)(s^2+1) = A(s^2+1) + (Bs+C)(s+1) \;\therefore$

if $s=-1$ $\quad 0 = 2A \;\therefore\; A = 0$ $\quad$ if

$s = 0: \quad 1\cdot 1 = 1 = C(1) = C = 1$

$(s+1)(s^2+1) = (Bs+1)(s+1)$

\Sol/ $F*G = \int_0^t F(u)\,G(t-u)\,du = \int_0^t e^{-u}\sin(t-u)\,du$

$U = e^{-u}$, $U' = -e^{-u}$, $V' = \sin(t-u)$, $V = \cos(t-u)$

$F*G = \left[e^{-u}\cos(t-u)\right]_0^t + \int_0^t e^{-u}\cos(t-u)\,du$

$U = e^{-u}$ $\quad U' = -e^{-u}$ $\quad V' = \cos(t-u)$, $V = -\sin(t-u)$

$F*G = \left[e^{-u}\cos(t-u)\right]_0^t + \left[-e^{-u}\sin(t-u)\right]_0^t - \underbrace{\int_0^t e^{-u}\sin(t-u)\,du}_{F*G}$

$2(F*G) = e^{-t} - \cos(t) + \sin(t)$ $\quad F*G = \frac{1}{2}\left(e^{-t} - \cos(t) + \sin(t)\right)$

\Biological Models/

n satisfies $\quad n(t) = f(t)\,n_0 + \int_0^t f(t-s)\,r(n(s))\,ds$

a simple model of population Dynamics $n(t)$ be population at t $t \ge 0$ f be Z survival func

value $f(t)$ is Z fraction of Z initial population

assume population growth rate is birth minus death rate

equals some nonlinear func r of $n(t)$ n satisfies

$n(t) = f(t)n_0 + \int_0^t f(t-s)\,r(n(s))\,ds$

{2nd kind non homogeneous nonlinear volterra

F: Fredholm V: volterra]2

1 1st kind 2: 2nd kind]2

Homogeneous nonhomogeneous]2

linear nonlinear]2 $2^4 = 16$

func $K(t)$ buying stock S $K(t)S + \int_0^t K(t-s)\,\phi(s)\,ds$

$\therefore S = K(t)S + \int_0^t K(t-s)\,\phi(s)\,ds$ $\therefore$ equates $S(1-K(t))$

is 2nd kind nonlinear volterra nonhomogeneous

Sol

volterra 1st kind nonhomogeneous linear

Menti 3 $u(r,\theta) = \frac{1}{2\pi}\int_0^{2\pi} \frac{1-r^2}{1-2r\cos(\theta-\phi)+r^2}\,g(\phi)\,d\phi$

{Fredholm 1st kind linear non homogeneous

$\int_0^t \frac{(t-\tau)^{\beta/b-1}\,y^\beta(\tau)}{(y^\beta(t)-y^\beta(\tau))^{\beta/b}}\,d\tau = \frac{b}{\beta}$ volterra nonhomogeneous nonlinear

1st kind eg nonlinear nonhomogeneous volterra IE of Z 1st kind

analytical sols to IE / technique for Fredholm IE:

"p-method"

Ex: classify & solve Z IE $y(x) = \int_0^1 e^{x+t}\,y(t)\,dt = e^x$

is a linear non homog Fredholm IE of Z 2nd kind

$k(x,t) = e^{x+t} = e^x \cdot e^t$ $\therefore$

$y(x) - e^x \int_0^1 e^t\,y(t)\,dt = e^x$ ⊛ $\therefore$ –

Pmethod Step 1: write our Sol $y(x) = \dots$

$y(x) = e^x + e^x \underbrace{\int_0^1 e^t y(t)\,dt}$ → when found is a constant

\Step 2:/ Let $P = \int \dots dt$ ↑only t ∴ $P = \int_0^1 e^t y(t)\,dt$

∴ Sub this p into ⊛

∴ $y(x) = e^x + e^x p$ is Sol of Z IE ∴

\Step 3:/ Sub $y(x)$ in step 2 into Z expression for P in Step 1

$P = \int_0^1 e^t(e^t + e^t p)\,dt$

$P = \int_0^1 e^{2t} + pe^{2t}\,dt = P = (1-p)\left[\frac{1}{2}e^{2t}\right]_0^1 = P = (1-p)\left(\frac{1}{2}e^2 - \frac{1}{2}\right)$ ∴ Make P Z

Subject: $P\left(1 - \frac{1}{2}e^2 + \frac{1}{2}\right) = \cancel{\frac{1}{2}e^2 - \frac{1}{2}}\ \frac{1}{2}e^2 - \frac{1}{2}$ ∴ Solve for P:

$\cancel{P = \frac{e^2-1}{3e^2}}$ $P = \frac{e^2-1}{3-e^2}$

\Step 4:/ put P back into Z Sol in Step 2 $y(x) = e^x(1+p)$ ∴

$y(x) = e^x\left(1 + \frac{e^2-1}{3-e^2}\right) = \frac{2}{3-e^2}e^x$

\Ex/ Classify & Solve Z IE $y(x) - \int_0^1 e^{x-\alpha t} y(t)\,dt = 0$

is a linear, homog, Fredholm IE of Z 2nd kind

$k(x,t) = e^{x-\alpha t} = e^x \cdot e^{-\alpha t}$ ∴ $y(x) = \int_0^1 e^x e^{-\alpha t} y(t)\,dt = e^x \int e^{-\alpha t} y(t)\,dt$

∴ $P = \int_0^1 e^{-\alpha t} y(t)\,dt$ ∴ $y(x) = e^x p$ ∴ $y(t) = e^t p$ ∴

$P = \int_0^1 e^{-\alpha t} e^t p\,dt$ ⊛ ∴ $P = P\left[\frac{e^{t-\alpha t}}{1-\alpha}\right]_0^1 = P\left[\frac{e^{t-\alpha t}}{1-\alpha}\right]_0^1 =$

$P = P\left(\frac{1}{1-\alpha}e^{1-\alpha} - \frac{1}{1-\alpha}\right)$ ∴

* when $\alpha = 1$: ⊛ gives $P = P\int_0^1 1\,dt$ ∴ $P = P$ ∴ $y(x) = p e^x$

* when $\alpha \neq 1$: $\cancel{P\left(1 - \frac{e^{1-\alpha}}{1-\alpha}\right) =}$ $P = \frac{P}{1-\alpha}(e^{1-\alpha} - 1)$ ∴

$P(1-\alpha) = P(e^{1-\alpha} - 1)$ ∴ $P(1 - \alpha - e^{1-\alpha} - 1) = 0 = P(2 - \alpha - e^{1-\alpha})$

is 0 when $P = 0$ so $y(x) = 0$

and when $2 - \alpha - e^{1-\alpha} = 0$ ∴ $2 - \alpha = e^{1-\alpha}$ ∴

$\alpha = 1$ Satisfies this ∴ when $\alpha \neq 1$ $y(x) = 0$

$\therefore$ sol to the IE is $y(x) = \begin{cases} \beta e^x & \alpha = 1, \beta \in \mathbb{R} \\ 0 & \alpha \neq 1 \end{cases}$

$\Rightarrow$ now look at solving Volterra type IEs

Ⓐ technique 1: using Laplace transforms

Ex:/ classify $y(x) - \int_0^x (x-t) y(t) dt = \sin(x)$

this is a linear, Volterra, nonhomog IE of the 2nd type

kernel $K(x,t) = x-t$ convolution kernel $k(x,t) = k(x-t)$

Step 1:/ Laplace transform of each term in the IE

(let $\hat{y}$ denote $\mathcal{L}(y)$) $\hat{y}(s) - \boxed{} = \frac{1}{s^2+1} = \hat{y}(s) - \frac{1}{s^2}\hat{y}(s)$

$\underset{LT22}{\mathcal{L}}\left(\int_0^x \underbrace{(x-t)}_{G(x-t)} \underbrace{y(t)}_{F(t)} dt\right) = f \cdot g = \hat{y}(s) \cdot \frac{1}{s^2}$

$G(x) = x \to \frac{1}{s^2}$ $\quad F(t) \to \hat{y}(s)$

$\left\{ \text{if } \int_0^x \underbrace{e^{x-t}(x-t)^2 \sin(x-t)}_{G(x-t)} \cdot \underbrace{y(t)}_{F(t)} dt \quad F(t) \to f = \hat{y}(s) \right.$

$\left. G(x) = e^x x^2 \sin(x) \to g = \mathcal{L}(e^x x^2 \sin(x)) \right\}$

• Step 2: Solve for $\hat{y}(s)$ then find inverse Laplace transform to find $y(x)$ $\quad \hat{y}(s)\left(1 - \frac{1}{s^2}\right) = \frac{1}{s^2+1} \quad \hat{y}(s) = \frac{s^2}{s^4 - 1} = \frac{A}{s^2-1} + \frac{B}{s^2+1} \quad \therefore$

$\hat{y}(s) = \frac{1}{2}\left(\frac{1}{s^2+1} + \frac{1}{s^2-1}\right) \quad \therefore y(x) = \mathcal{L}$

$y(x) = \mathcal{L}^{-1}(\hat{y}(s)) = \frac{1}{2}(\sin(x) + \sinh(x))$

Ⓑ technique 2: using differentiation look at:

~~Ex~~ $y(x) - \int_0^x (x-t) y(t) dt = \sin(x)$ will transform our IE into an ODE

• Step 1: diff each term in the IE $\quad y'(x) - \frac{d}{dx}\left(\int_0^x (x-t) y(t) dt\right) = \cos(x)$

$\left\{ \text{recall: } \frac{d}{dx}\left(\int_{g(x)}^{f(x)} h(t) dt\right) = h(f(x)) f'(x) - h(g(x)) \cdot g'(x) \right.$

$\left. \Rightarrow \frac{d}{dx}\left(\int_0^x y(t) dt\right) = y(x) \cdot 1 - y(0)(0) = y(x) \right\}$

$\therefore \frac{d}{dx}\left(\int_0^x (x-t) y(t) dt\right) = \underbrace{\frac{d}{dx}\int_0^x x y(t) dt}_{①} - \underbrace{\frac{d}{dx}\int_0^x t y(t) dt}_{②}$

Sor ①: $\frac{d}{dx}\left(x\int_0^x y(t)\,dt\right)$

$= xy(x) + \int_0^x y(t)\,dt$

Sor ②: $\frac{d}{dx}\int_0^x t\,y(t)\,dt = x(y(x))\cdot 1 - 0\cdot 0 = xy(x)$ ∴

Say our DE looks like:

$y'(x) - \Big(\underbrace{xy(x) + \int_0^x y(t)\,dt}_{①} \underbrace{- xy(x)}_{②}\Big) = \cos(x) = y'(x) - \left(\int_0^x y(t)\,dt\right)$ ∴

$y'(x) - \int_0^x y(t)\,dt = \cos(x)$ ☺ Step 2: now dissing again. $y''(x) - y(x) = -\sin(x)$ Step 3: ∴ $x=0$ into IE:

$y(0) = 0$ ∴ putting $x=0$ into $y'(0) = 1$ } IC

$y'(x) - \left(xy(x) + \int_0^x y(t)\,dt - xy(x)\right) = \cos(x)$

$y'(x) - \int_0^x y(t)\,dt = \cos(x)$ ☺

- Step 2: ∴ $y''(x) - y(x) = -\sin(x)$ by dissing again

- Step 3: putting $x=0$ into IE $y(0)=0$ } IC
  putting $x=0$ into ☺ $y'(0) = 1$ }

- Step 4: solve Z IVP $y''(x) - y(x) = 0$ (homog case)

Sol: $y = Ae^{x} + Be^{-x}$ ∴

particular sol: $y_p = C\sin(x)$ ∴

$y_p' = C\cos x$, $y_p'' = -C\sin(x)$ ∴ into $y_p'' - y_p = -\sin(x)$

giving $-C\sin(x) - C\sin(x) = -\sin(x)$ ∴ $C = \frac{1}{2}$ ∴

$y(x) = Ae^{x} + Be^{-x} + \frac{1}{2}\sin(x)$ ∴

find A, B: $y(0) = 0$: $0 = A + B$

$y'(0) = 0$: $y' = Ae^{x} - Be^{-x} + \frac{1}{2}\cos(x)$ ∴ $0 = A - B + \frac{1}{2}$ ∴

$A = \frac{1}{4}$, $B = -\frac{1}{4}$ ∴ $y(x) = \frac{1}{4}e^{x} - \frac{1}{4}e^{-x} + \frac{1}{2}\sin(x)$

$= \frac{1}{2}\sinh(x) + \frac{1}{2}\sin(x) = \frac{1}{2}(\sinh x + \sin x)$

\Structive of kernel:/ $\lambda(y) - \int_a^x \underbrace{k(x,t)}_{\text{kernel}} y(t)\,dt = f(x)$

\Def/ a kernel is <u>symmetric</u> if $k(x,t) = k(t,x)$

eg $k(x,t) = \sin(x+t)e^{tx} = k(t,x)$

but $k(x,t) = (x+t)e^{t} \neq (x+t)e^{x} = k(t,x)$ $\therefore$ $k(x,t) \neq k(t,x)$

\Def/ a kernel is a convolution kernel if $k(x,t) = g(x-t)$

\Ex/ $k(x,t) = \cos(x-t)e^{x-t} = k(x-t)$

$k(x,t) = \cos(x-t)e^{x}t^{3} \neq k(x-t)$

\Def/: a kernel is degenerate (or seperable) if $k(x,t) = k_1(x)\cdot k_2(t)$

eg $k(x,t) = \sin(x)e^{x+t} = e^{x}\sin(x)\cdot e^{t}$

$k(x,t) = e^{x}+3t \neq k_1(x)k_2(t)$ not seperable

~~\Def~~ kernels can continuous or discontinuous

strongly singular weakly singular

\Def/ a discontinuous kernel is <u>weakly</u> singular (at $x=t$) if $k(x,t)$ is continuous at $x \neq t$ & if $\exists\, \alpha \in (0,1)$ & $c>0$ st

$|k(x,t)| \leq C|x-t|^{-\alpha}$ for $x \neq t$ on its set ~~def~~ definition

\Ex/: ① $\int_{-1}^{1} \underbrace{(x-t)^2}_{\text{continuous}} y(t)\,dt = \tan(x)$

② $y(x) - \int_{-1}^{1} \frac{y(t)}{x-t}\,dt = 0$ $\therefore$ not continuous strongly singular

$k(x,t) = \frac{1}{x-t}$ strong singular

if $k(x,t) = \frac{1}{(x-t)^{1/2}}$ weak $\frac{1}{(x-t)^{3/2}}$ strong $\frac{1}{(x-t)^{2}}$ strong

$\frac{1}{(x-t)^{-1/2}}$ continuous

\Def/ a piecewise continuous kernel has the form

$$k(x,t) = \begin{cases} k_1(x,t) & a \leq t \leq x \\ k_2(x,t) & x \leq t \leq b \end{cases}$$

\Def/ generally the kernel $k(x,t) = \sum_{i=1}^{n} k_1^{(i)}(x)\, k_2^{(i)}(t)$ (**)

Fredholm IE with degenerate kernels

Rank 1 : $n=1$ in the sum (**) $\therefore$

let $\lambda \neq 0$ $\therefore$ $\lambda y(x) - \int_a^b k(x,t)y(t)\,dt = g(x)$ $\therefore$

$\lambda y(x) - k_1(x) \underbrace{\int_a^b k_2(t) y(t)\, dt}_{p} = g(x)$ $\therefore p = \int_a^b k_2(t) y(t)\, dt$ $\therefore$

$\lambda y(x) = g(x) + k_1(x) p$ $\therefore$ $y(x) = \dfrac{g(x) + p k_1(x)}{\lambda}$ $\therefore$

$p = \int_a^b k_2(t) \left(\dfrac{g(t) + p k_1(t)}{\lambda} \right) dt$ $\therefore$

$\lambda p - p \int_a^b k(t,t)\, dt = \int_a^b g(t) k_2(t)\, dt = \int_a^b g(t)\, k_2(t)\, dt$ $\therefore$

$p\left(\lambda - \int_a^b k(t,t)\, dt\right) = \int_a^b g(t) k_2(t)\, dt$ $\therefore$

$p = \dfrac{\int_a^b g(t) k_2(t)\, dt}{\lambda - \int_a^b k(t,t)\, dt}$ Θ

two cases: ① if $\lambda \neq \int_a^b k(t,t)\, dt$ then we accept p Θ

& say $y(x) = \frac{1}{\lambda} \left(g(x) + \dfrac{\int_a^b g(t) k_2(t)\, dt}{\lambda - \int_a^b k(t,t)\, dt} k_1(x) \right)$

case ②: if $\lambda = \int_a^b k(t,t)\, dt$ then (a) $\int_a^b g(t) k_2(t)\, dt \neq 0$ then Z

IE has no Sols

(b) $\int_a^b g(t) k_2(t)\, dt = 0$ & Z IE has infinitely many Sols:

$y(x) = \frac{1}{\lambda} (g(x) + p k_1(x))$

Fredholm IE of degenerate kernel Rank n

eg ~~EX~~ / eg $k(x,t) = e^{x+t} + 3x t^2 + t e^x = k_1^{(1)} k_2^{(1)}(t) + k_1^{(2)}(x) k_2^{(2)}(t) + k_1^{(3)}(x) k_2^{(3)}(t)$

$= e^x e^t + 3x \cdot t^2 + e^x \cdot t$ in general:

IE: $\lambda y(x) - \int_a^b k(x,t) y(t)\, dt = f(x)$

$\lambda y(x) - \int_a^b \sum_{j=1}^{n} k_1^{(j)}(x)\, k_2^{(j)}(t) \Big) y(t)\, dt = f(x)$

{denote $p_j = \int_a^b k_2^{(j)}(t) y(t)\, dt$ $j = 1, \dots, n$}

in terms of p: $\lambda y(x) - \sum_{j=1}^{n} k_1^{(j)}(x) p_j = f(x)$ $\int_a^b \lambda y(t) k_2(t)\, dt$

Multiply each term by k_2 & integrate $\lambda p_i - \sum_{j=1}^{n} \underbrace{c_{ij}}\, p_j = \underbrace{d_i}$ $\to \int f(t) k_2^{(i)}(t)\, dt$

$\therefore$ a System $\overset{\vec{p}}{A\vec{x}} = \vec{b}$ of eqn (linear) $\to \int_a^b k_2^{(i)}(x) k_1^{(j)}(x)\, dt$

know that this System has a Unique Sol if Z determinant is

nonzero: $\det(\lambda \delta_{ij} - a_{ij})$ is $\neq 0$ {δ_{ij} is kronecker delta} if $\det(\lambda\delta_{ij} - a_{ij}) = 0$ then have no or infinitely many sols

$\left\{ \delta_{ij} = \begin{cases} 1 & i=j \\ 0 & i \neq j \end{cases} \right.$

λ sols to $\det(A - \lambda I) = 0$ eigvals of matrix

continuing: for our Fredholm IE with degenerate kernel of rank n

if $\lambda \neq 0$ we have 2 cases

① if the homog system has only the zero sol, then the non-homog case has a unique sol $\{p_j\}_{j=1}^{n}$ & $\therefore$ our IE has the sol

$y(x) = \frac{1}{\lambda}\left(\sum_{j=1}^{n} k_i^{(j)}(x) p_j + f(x)\right)$

② if the homog system has non-zero sols, then the non homog system has either no sols or infinitely many sols

Ex/ $\lambda y(x) - \int_0^1 \underbrace{e^{x-t}}_{e^x e^{-t}} y(t)\,dt = e^x$ $\qquad$ $\lambda y(x) - e^x \underbrace{p}_{\int_0^1 e^{-t} y(t)\,dt} = e^x$ ⊛

$y(x) = \frac{1}{\lambda}(e^x + e^x p)$

① if $\lambda \neq 0$: $p = \int_0^1 e^{-t} \lambda(e^t + pe^t)\,dt$ $\therefore$ $\lambda p = \int_0^1 e^0 + pe^0\,dt$ $\therefore$

$\lambda p = 1 + p$ $\therefore$ $p = \frac{1}{\lambda - 1}$

two subcases: Ⓐ $\lambda = 1$ not possible, no sols.

& Ⓑ $\lambda \neq 1$: $p = \frac{1}{\lambda-1}$ $\xrightarrow[\text{put p into sol } y(x)]{}$ $\therefore y(x) = \frac{e^x}{\lambda - 1}$

② if $\lambda = 0$: $0 - e^x p = e^x$ from ⊛ $\quad \therefore p = -1$ $\therefore$

$-1 = \int_0^1 e^{-t} y(t)\,dt$

$\therefore$ we can have $y(x) = -e^x$ as a sol

we can also have: $\int_0^1 e^{-t}(y_1 - y_2)(t)\,dt = 0$

where each of y_1, y_2 are orthogonal to e^{-t}

$\{\langle y_1, f_1 \rangle = \int y_1 \cdot f_1\,dx = 0$ orthogonal$\}$

$\therefore$ we have infinitely many sols

Chapter 3/ linear operators:/

- integral operator $(\underset{\swarrow}{ky})(x) = \int_a^b k(x,t)\,y(t)\,dt$

integral operator acting on y

$\therefore$ $(ky)(x)$, $[ky](x)$, $k(y)(x)$, $ky(x)$

\Ex/ apply operator k to $k(x,t)=e^{-xt}$ on $a=0$, $b=2$

① $y(x)=2$ ∴ ~~$k+y$~~ $(ky)(x)=\int_0^2 2e^{-xt}\,dt=\left[-\frac{2}{x}e^{-xt}\right]_0^2=\frac{2}{x}(1-e^{-2x})$

when $x=0$, $(ky)(x)=\int_0^2 2\,dt=[2t]_0^2=4$

$$(ky)(x)=\begin{cases}\frac{2}{x}(1-e^{-2x}) & 0<x\le 2\\ 4 & x=0\end{cases}$$

② $y(x)=e^{cx}$, $c\in\mathbb{R}$ $(ky)(x)=\int_0^2 e^{-xt}e^{ct}\,dt=$

$\int_0^2 e^{t(c-x)}\,dt=\left[\frac{1}{c-x}e^{t(c-x)}\right]_0^2=\frac{1}{c-x}\left(e^{2(c-x)}-1\right)$ $x\neq c$

when $x=c$: $(ky)(x)=\int_0^2 1\,dt=[t]_0^2=2$

$$(ky)(x)=\begin{cases}\frac{1}{c-x}\left(e^{2(c-x)}-1\right) & x\neq c\\ 2 & x=c\end{cases}$$

now we can rewrite our IE

$\lambda y(x)-\int_a^b k(x,t)y(t)\,dt=f(x)$ $a\le x\le b$

$\lambda y(x)-(ky)(x)=f(x)$ ∴

$(\lambda I-k)y=f$

<u>problem</u>: given f, find y st $(\lambda I-k)y=f$

use $C([a,b])$ to denote the set of funcs defined & continuous on $[a,b]$

\Def/ consider $(\lambda I-k)y=f$

① if $f\neq 0$ (nonhomog IE) if sol y is unique then λ is a <u>regular val</u>

② if $f\neq 0$ (nonhomog IE) if we have no sols or infinitely many sol then λ is a <u>spectrum pt</u>

③ if $f=0$ (homog IE) & a non-zero sol then λ is an <u>eigval</u>

\Ex/ for our previous example Ex/: $\lambda y(x)-\int_0^1 e^{x-t}y(t)\,dt=e^x$

$\lambda=1$: no sols then λ is a spectrum pt

linear operators in $C([a,b])$ when we dont have (doesnt exist or too difficult to find) an <u>exact</u> (analytical sol) we use numerical approxs. for that we fix the func class there we use

$\S$ $C([a,b])$

Z Sup norm:

\Def/ for a func y continuous on $[a,b]$ we call Z Maximum of $|y|$ on $[a,b]$ in $C([a,b])$ Z norm (sup-norm of y)

$\|y\|_{C([a,b])} = \max\{|y(x)| : x \in [a,b]\}$

notation : $\|y\|_\infty = \sup\{|y(x)| : x \in [a,b]\}$

Properties :

- $\|x\| \geq 0 \quad \forall x \in \mathbb{R}^n$, $\|x\|$ iff $x=0$
- ~~$\forall \alpha$~~ $\|\alpha x\| = |\alpha|\|x\| \quad \forall \alpha \in \mathbb{R}$, $x \in \mathbb{R}^n$
- $\|x+y\| \leq \|x\| + \|y\| \quad \forall x,y \in \mathbb{R}^n$
- $|y(x)| \leq \|y\|_\infty \quad x \in [a,b]$
- $\left|\int_a^b y(x)dx\right| \leq \int_a^b |y(x)|dx \leq (b-a)\|y\|_\infty$ {taking 1 rectangle for our Riemann integral}

\Ex/ find Z sup-norm of y

(a) $y:[3,5) \to \mathbb{R}$, $y(x) = x^2 - x$

$y' = 2x - 1 \qquad y' = 0 \quad \therefore x = \frac{1}{2} \quad , y(\frac{1}{2}) = -\frac{1}{4}$

$y(3) = 6 = y(a) \qquad y(5) = 20 = y(b)$

$\|y\|_\infty = \max\{|6|, |20|, |-\frac{1}{4}|\} = 20$

(b) $y:[-1,1] \to \mathbb{R}$, $y(x) = x^2 - 1 \quad \therefore y' = 2x \quad \therefore x = 0$

$y(0) = -1$, $y(-1) = 0$, $y(1) = 0$

$\|y\|_\infty = \max\{|-1|, |0|, |0|\} = 1$

(c) $y:[a,b] \to \mathbb{R}$, $y(x) = \cos(x) \qquad \|y\|_\infty \leq 1$ if

$x = k\pi$, $k \in \mathbb{Z}$, $x[a,b] \quad \therefore \|y\|_\infty = 1 \qquad |\cos(k\pi)| = |\pm 1| = 1$

otherwise Max/min of cos is on Z boundary $\|y\| = \max\{|\cos(a)|, |\cos(b)|\}$

\reminder:/ linear map $\cdot f(x+y) = f(x) + f(y) \quad \cdot f(\alpha x) = \alpha f(x)$

\Def/ a mapping L is a linear operator if

$L(\alpha y_1 + y_2) = \alpha L(y_1) + L(y_2) \quad \forall y_1, y_2 \in C([a,b]) \quad \forall \alpha \in \mathbb{R}$

Def/ A Mapping L is a linear operator if
$L(\alpha y_1 + y_2) = \alpha L(y_1) + L(y_2) \quad \forall y_1, y_2 \in C([a,b]) \; \forall \alpha \in \mathbb{R}$

Lemma/ if $k(x,t)$ is cont or piecewise cont or weakly singular the K given by $(K_y)(x) = \int_a^b k(x,t)y(t)\,dt$ is a linear operator

Ex/ deduce whether K with kernel $k(x,t) = \frac{1}{\sqrt[3]{|x-t|}}$ is a linear operator

$\because$ $k(x,t)$ is a weakly singular kernel K is a linear operator
$\int_0^1 |k(x,t)|\,dt = \int_0^1 \frac{1}{\sqrt[3]{|x-t|}}\,dt = \int_0^x \frac{1}{\sqrt[3]{x+t}}\,dt + \int_x^1 \frac{1}{\sqrt[3]{t-x}}\,dt =$
$\frac{3}{3}\left(\sqrt[3]{x^2} + \sqrt[3]{(x-1)^2}\right)$

Proposition/ consider $Ly=0$ homog $Ly=f$ nonhomog
① any linear combination of sols to the hom case is also a sol to the homg case
eg $y''+p(x)y=0$ $\therefore$ if y_1, y_2 both sols then
$\square y_1 + \square y_2$ (is a sol to the nonhomgcase) is also a sol to $y''+p(x)y=0$
② the difference of two sols of the nonhomg case is also a sol of the homg case
eg $y''+p(x)y=f(x)$ y_1, y_2 both sols then $\square y_1 - \square y_2$ is a sol to $y''+p(x)y=0$
③ sols of the homgcase can be added to the sols of the nonhomg case to give sols of the nonhomog case
eg y_1 sol of $y''+p(x)y=0$ y_2 sol of $y''+p(x)y=f(x)$ then
$\square y_1 + \square y_2$ sol of $y''+p(x)y=f(x)$

The Fredholm Alternative/

Def/ a linear operator L satisfies the Fredholm Alternative if
- either $\exists$ a sol of the nonhomog case $Ly=f$
- or the homog case $Ly=0$ has a non-zero sol

\proposition/ if Z homog eqn $Ly=0$ has only Z trivial sol
($y=0$) then there is at most 1 sol to Z non homog eqn $Ly=f$

\proposition/ if L satisfies Z Fredholm Alternative
then $Ly=f$ has a unique sol

for IE $\lambda I-K$ is our linear operator $(\lambda I-K)y=f$ (L)
$\lambda y(x)-\int_a^b k(x,t)y(t)dt=f(x)$

\thm/ for IE of Z 2nd kind ($\lambda\neq0$), Z linear operator
$(\lambda I-K):C([a,b])\to C([a,b])$ with a kernel of finite rank,
Z integral ~~op~~ operator satisfies Z FA (Fredholm Alternative)

\Def/ let $k:[a,b]\times[a,b]\to\mathbb{R}$ be a kernel. denote k^*
to be Z adjoint kernel given by $k^*(x,t)=k(t,x)$
similarly if K is an integral operator then K^* is Z adjoint
integral operator with kernel k^*

\lemma/: ① when k is of finite rank, Z M^* (matrix) corresponding to K^*
is Z transpose of M (matrix) where $M_{ij}=\int_a^b k_2^{(i)}(t)k_1^{(j)}(t)dt$

eg $\lambda y-\int_0^1(\underbrace{\cos(x)e^t}_{k_1^{(1)}(x)k_2^{(1)}(t)}+\underbrace{\sin(x)t^2}_{k_1^{(2)}(x)k_2^{(2)}(t)}+\underbrace{tx}_{k_1^{(3)}(x)k_2^{(3)}(t)})y(t)dt=f(x)$

$M_{11}=\int_0^1 k_2^{(1)}(t)k_1^{(1)}(t)\,dt=\int_0^1 e^t\cos(t)\,dt$

$\vdots$

$M_{32}=\int_0^1 k_2^{(3)}(t)k_1^{(2)}(t)=\int_0^1 t\cdot\sin(t)dt$
$(i=3,j=2)$

$M=\begin{pmatrix}(\cdot)&(\cdot)&(\cdot)\\(\cdot)&(\cdot)&(\cdot)\\(\cdot)&(\cdot)&(\cdot)\end{pmatrix}$ $M^*=\begin{pmatrix}(\cdot)&(\cdot)&(\cdot)\\(\cdot)&(\cdot)&(\cdot)\\(\cdot)&(\cdot)&(\cdot)\end{pmatrix}$ taking Z adjoint kernel $M^*=M^T$

\lemma/ ② $\int_a^b(ky)(x)\cdot z(x)dx=\int_a^b(k^*z)(x)\cdot y(x)\,dx$ $\forall y,z\in C([a,b])$

now look at Z Fredholm Thm/

\Z Fredholm thm/ let $(\lambda I-K)y=f$ be a non homog linear
Fredholm IE of Z 2nd kind ($\lambda\neq0$) with finite rank kernel in $C([a,b])$

~~$\lambda\in\mathbb{R}/\{0\}$~~ $\lambda\in\mathbb{R}/\{0\}$

① each λ is either a regular val of K or an eval of K

② for any pos num Λ there are only finite many evals λ st $|\lambda| > \Lambda$

continuing ③ if λ is an eval of K then

ⓐ λ is also an eval of K^*

ⓑ the spaces of sols of the homog eqn $(\lambda I - K)y = 0$ & its adjoint $(\lambda I - K^*)z = 0$ have the same dimension

④ $(\lambda I - K)y = f$ has a sol iff $\int_a^b f(x)z(x)\,dx = 0$ where $z(x)$ satisfies the adjoint case $(\lambda I - K^*)z = 0$

so if a homog eqn has non-zero sol $z(x)$, then we know that sols of the nonhomog case exists iff f is orthogonal to the sols of the adjoint homog eqn

⑤ if λ is a regular val of K then

ⓐ the sol of $(\lambda I - K)y = f(x)$ has the form

$$y(x) = \frac{f(x)}{\lambda} + \frac{1}{\lambda}\int_a^b \underbrace{r_\lambda(x,t)}_{\text{resolvent kernel}} f(t)\,dt$$

or write as $y = \frac{1}{\lambda}(f + \underbrace{R_\lambda f}_{\text{resolvent operator}})$

ⓑ that $r_\lambda(x,t)$ is smooth (infinitely many times continuously diffable).

⑥ if λ is a regular val of K then $\|y\|_\infty \leq \overset{\text{const}}{C}\|f\|_\infty$ where

$C \leq \frac{1}{|\lambda|}\left(1 + (b-a)\|r_\lambda\|_\infty\right)$

Ex/ Consider the IE $\lambda y(x) - \int_0^1 \overbrace{xt^2}^{k(x,t)} y(t)\,dt = f(x)$

Solve this & discuss sols for different vals of λ. State the resolvent kernel

$\therefore$ using the p-method: $\lambda y(x) - x\underbrace{\int_0^1 t^2 y(t)\,dt}_{p} = f(x)$ $\quad y(x) = \frac{f(x)}{\lambda} + \frac{1}{\lambda}p$

$\int_0^1 t^2 y(t)\,dt = p = \int_0^1 t^2\left(\frac{f(x)}{\lambda} + \frac{t}{\lambda}p\right)dt = p = \frac{1}{\lambda}\int_0^1 t^2 f(t)\,dt + \frac{1}{\lambda}p\int_0^1 t^3\,dt =$

$p = \frac{1}{\lambda}\int_0^1 t^2 f(t)\,dt + \frac{p}{\lambda}\left[\frac{t^4}{4}\right]_0^1 = p = \frac{1}{\lambda}\int_0^1 t^2 f(t)\,dt + \frac{p}{4\lambda}\ \therefore$

$p\left(1 - \frac{1}{4\lambda}\right) = \frac{1}{\lambda}\int_0^1 t^2 f(t)\,dt \quad \therefore \quad p = \frac{1}{\lambda\left(1 - \frac{1}{4\lambda}\right)}\int_0^1 t^2 f(t)\,dt$

taking $\lambda \neq 0$

• when $\lambda \neq \frac{1}{4}$ have $p = \frac{4}{4\lambda - 1}\int_0^1 t^2 f(t)\,dt$ so our sol is (into *)

$y(x) = \frac{f(x)}{\lambda} + \frac{x}{\lambda}\frac{4}{4\lambda-1}\int_0^1 t^2 f(t)\,dt$ ∴

So $y(x) = \frac{f(x)}{\lambda} + \frac{1}{\lambda}\int_0^1 \underbrace{\frac{4x}{4\lambda - 1}}_{\to\ \Gamma_\lambda(x,t)} t^2 f(t)\,dt$ {see ⑤ in Fredholm thm} Z resolvent kernel

• when $\lambda = \frac{1}{4}$ need $\int_0^1 t^2 f(t)\,dt = 0$ ∴ ie t^2 & f are orthogonal

{see ④ in Fredholm thm}

∴ $\int_0^1 \underbrace{x^2}_{\to\ Z(x)\text{ sol of adjoint homog case}} f(x)\,dx = 0$

$\lambda Z(x) - \int_0^1 x^2 t Z(t)\,dt = 0$

solving : $Z(x) = \frac{x^2 p}{\lambda}$ $p = \int_0^1 t Z(t)\,dt$ ∴ $p = \int_0^1 t\left(\frac{t^2 p}{\lambda}\right)dt$ ∴

$p = \frac{p}{\lambda}\left[\frac{t^4}{4}\right]_0^1 = \frac{p}{\lambda}\cdot\frac{1}{4}$ ∴ $p = \frac{p}{4\lambda}$ ∵ $\lambda = \frac{1}{4}$ ∴ $p = p$

∴ $Z(x) = \frac{x^2 p}{\lambda} = 4x^2 p$ {is $p = 1$?}

\Ex/ verify Z result in ② of lemma for $k(x,t) = \sin(x)t$

$Z(x) = x$ $y(x) = x^2$ $[a,b] = [0,\pi]$

$\int_a^b (Ky)(x)\, Z(x)\,dx = \int_a^b (K^* Z)(x)\, y(x)\,dx$

$(Ky)(x) = \int_a^b k(x,t) y(t)\,dt =$

$a = 0$ $b = \pi$ $\int_0^\pi \sin(x) t x^2\,dt = \sin(x) x^2 \int_0^\pi t\,dt =$

$x^2\sin(x)\left[\frac{1}{2}t^2\right]_0^\pi = x\sin(x)\frac{1}{2}\left[\pi^2 - 0\right] = \frac{x}{2}\sin(x)\pi^2 = \frac{1}{2}\pi^2 x\sin(x)$ ∴

$\int_0^\pi \frac{1}{2}\pi^2 x\sin(x)\, x\,dx = \int_0^\pi \frac{1}{2}\pi^2\, x^2\sin(x)\,dx = I =$

$\frac{1}{2}\pi^2\int_0^\pi x^2\sin x\,dx = \frac{1}{2}\pi^2\left[-\cos(x)x^2\right]_0^\pi - \int_0^\pi 2x(-\cos(x))\,dx =$

$-\frac{1}{2}\pi\left[x^2\cos x\right]_0^\pi + 2\int_0^\pi x\cos(x)\,dx =$

$-\frac{1}{2}\pi\left[\pi^2\cos\pi - 0\right] + \cancel{2}\int_0^\pi 2x\cos(x)\,dx =$

$\frac{1}{2}\pi^3 + \left[2x\sin(x)\right]_0^\pi - \int_0^\pi 2(-\sin x)\,dx =$

$\frac{1}{2}\pi^3 + \left[2\pi\sin\pi - 0\right] + \int_0^\pi 2\sin x\,dx = \frac{1}{2}\pi^3 + 2\left[-\cos x\right]_0^\pi =$

$\frac{1}{2}\pi^3 + 2\left[+1 + \cos 0\right] = \frac{1}{2}\pi^3 \cancel{+} 2[1+1] = \frac{1}{2}\pi^3 + 4$ ✗ $\pi^{5/4}$

Sol / LHS: $(ky)(x) = \int_0^\pi \underbrace{\sin(x)t}_{k(x,t)} \cdot \underbrace{t^2}_{y(t)} dt = \sin(x)\left(\left[\frac{t^4}{4}\right]_0^\pi\right) = \frac{\pi^4}{4}\sin x$ ..

$\int_0^\pi \frac{\pi^4}{4}\sin(x)x\,dx = \frac{\pi^4}{4}\left[\sin x - x\cos x\right]_0^\pi = \frac{\pi^5}{4}$

RHS: $k^*(x,t) = \sin(t)x$ $(k^*z)(x) = \int_0^\pi \sin(t)x\,t\,dt =$
$x\int_0^\pi t\sin(t)\,dt = x\pi$ $\therefore$

$\int_0^\pi (k^*z)(x)y(x)\,dx = \int_0^\pi x\pi x^2\,dx = \left[\frac{x^4}{4}\pi\right]_0^\pi = \frac{\pi^5}{4}$

partial fractions: / $\frac{P(x)}{Q(x)}$ · what is Z order of $Q(x)$, $P(x)$

- Can we factorise $Q(x)$?

is have $\frac{x^n + ax^{n-1} + \cdots}{x^m + bx^{m-1} + \cdots}$ $n > m$

$= \boxed{} + \frac{A}{\cdots} + \frac{B}{\cdots}$
poly of order $n-m$

Ex / $\frac{6x^3 - 5x^2 - 7}{3x^2 - 2x - 1 \to (3x+1)(x-1)}$

$= Ax + B + \frac{C}{3x+1} + \frac{D}{x-1}$ - to find consts: · compare coeffs

· by $P(x) = Ax(\cdots) + B(\cdots)$

› or set specific x values

$\therefore 6x^3 - 5x^2 - 7 = Ax(3x+1)(x-1) + B(3x+1)(x-1) + C(x-1) + D(3x+1)$

$x = 0: -7 = -B - C + D$ ①

$x = 1: -6 = 4D \to D = -3/2$

in ① now that $D = -\frac{3}{2}$: $-\frac{11}{2} = -B - C$ ①

$x = -\frac{1}{3}: -\frac{70}{9} = C\left(-\frac{4}{3}\right)$ $\therefore$ $C = \frac{35}{6}$

$\therefore$ in ① : $B + \frac{35}{6} = \frac{11}{2}$, $B = -\frac{1}{3}$

Compare coeffs of x^3: $6 = 3A$ $\therefore$ $A = 2$

$\therefore \frac{6x^3 - 5x^2 - 7}{3x^2 - 2x - 1} = 2x - \frac{1}{3} + \frac{35}{6}\left(\frac{1}{3x+1}\right) + -\frac{3}{2}\left(\frac{1}{x-1}\right)$

Ex / $\frac{1}{x^4(x+1)} = \frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{D}{x^4} + \frac{E}{x+1}$

$1 = Ax^3(x+1) + Bx^2(x+1) + Cx(x+1) + D(x+1) + Ex^4 =$

$-\frac{1}{x} + \frac{1}{x^2} + -\frac{1}{x^3} + \frac{1}{x^4} + \frac{1}{x+1}$

\Ex/ $\dfrac{3}{x^2\underbrace{(x^2+2x+1)}_{\to (x+1)^2}}$

$= \dfrac{A}{x} + \dfrac{B}{x^2} + \dfrac{C}{x+1} + \dfrac{D}{(x+1)^2}$ ∴

$3 = Ax(x^2+2x+1) + B(x+1)^2 + Cx^2(x+1) + Dx^2$

then $A=-6$, $B=3$, $C=6$, $D=3$

\integration by parts/ $\displaystyle\int_a^b u(x)\,v'(x)\,dx = \Big[u(x)v(x)\Big]_a^b - \int_a^b u'(x)\,v(x)\,dx$

gets difficult if have (poly)·(trig), (poly)(exp), (trig)(exp), (trig)(trig)

Sometimes integration by parts (IBP) is required multimple times

$I = \displaystyle\int_0^{\pi} \underbrace{e^{3x}}_{u} \underbrace{\cos(2x)}_{v'(x)}\,dx$ ∴ $u' = 3e^{3x}$, $v = \frac{1}{2}\sin(2x)$ ∴

$I = \Big[\frac{1}{2}e^{3x}\sin(2x)\Big]_0^{\pi} - \frac{3}{2}\displaystyle\int_0^{\pi} e^{3x}\sin(2x)\,dx$ ∴

$u = e^{3x}$, $u' = 3e^{3x}$ $v' = \sin(2x)$, $v = -\frac{1}{2}\cos(2x)$

$I = \Big[\frac{1}{2}e^{3x}\sin(2x)\Big]_0^{\pi} - \frac{3}{2}\Big[-\frac{1}{2}e^{3x}\cos(2x)\Big]_0^{\pi} - \frac{3\cdot 3}{2\cdot 2}\underbrace{\displaystyle\int_0^{\pi} e^{3x}\cos(2x)\,dx}_{I}$ ∴ $(I + \frac{9}{4}I) =$

$\frac{13}{4}I = \frac{3}{4}(e^{3\pi}(+1) - 1)$ $I = \frac{3}{13}(+e^{3\pi} - 1)$

\ODEs/ 1st order DE - seperation of variables $dy/dx = f(x)g(y)$

$\displaystyle\int \frac{1}{g(y)}\,dy = \int f(x)\,dx$

integrating factor: $\frac{dy}{dx} + P(x)y = R(x)$ $Q(x)\frac{dy}{dx} + P(x)y = R(x)$ ∴

$\frac{dy}{dx} + \frac{P(x)}{Q(x)}y = \frac{R(x)}{Q(x)}$ ∴ I.F. $e^{\int P(x)dx} = I.F$

ODE: $\frac{d}{dx}(y \cdot IF) = R(x)\,I.F.$

$y(x)\cdot IF = \displaystyle\int R(x)\,IF\,dx$ $y(x) = \frac{1}{IF}\displaystyle\int R(x)\,IF\,dx$

· 2nd ODES $ay''(x) + by' + cy = f(x)$ ∴ Sol $y = y_h + y_p$

take: $am^2 + bm + c = 0$ charac eqn $\Delta = b^2 - 4ac$ $\Delta > 0$, $y_h = C_1e^{\lambda_1 x} + C_2e^{\lambda_2 x}$

2 distrinct real roots λ_1, λ_2 of charac eqn

$\Delta = 0$: $y_h = (C_1 + C_2x)e^{\lambda x}$ repeated root

$\Delta < 0$: complex roots $y_h = (C_1\cos(\beta x) + C_2\sin(\beta x))e^{\alpha x}$, $\lambda_1 = \alpha + i\beta$ $\lambda_2 = \alpha - i\beta$

for y_p use: $y_p = -y_1\displaystyle\int \frac{y_2 f}{W}\,dx + y_2\int \frac{y_1 f}{W}\,dx$ W wronskian $W = y_1y_2' - y_2y_1'$

homog case can also use trial method

$f(x)$	y_p
$e^{\alpha x}$	$Pe^{\alpha x}$
Poly of order n	$Px^n + Qx^{n-1} \cdots Z$
$\cos(\alpha x)$ or $\sin(\alpha x)$	$P\cos(\alpha x)+Q\sin(\alpha x)$

take y_p, y_p', y_p'' into our ODE to find consts

eg $y'' + 2y' + 5y = 5t^2 + 12$ find y_h? y_p? $y=$?

reminder: From Z Fredholm thm: $(\lambda I - k)y = f$ ~~iff~~ has a sol iff $\int_a^b f(x)\underbrace{z(x)}\,dx = 0$ sol of Z adjoint homog IE

\Numericals Sols to IE/

\Def/ a linear operator is bounded if $\exists$ C_0 st

$\underbrace{\|Ly\|_\infty}_{\text{supnorm of linear operator acting on } y} \le \underbrace{C_0}_{\text{const}}\,\underbrace{\|y\|_\infty}_{\text{sup norm of } y}$ $\forall$

For integral operators:

\proposition/ let k be an integral operator with kernel k which is cont, piecewise cont, or weakly singular, then $\|k\|_\infty = \max_{x\in[a,b]} \underbrace{\int_a^b |k(x,t)|\,dt}_{g(x)}$

\Ex/ - find $g(x) = \int_a^b |k(x,t)|\,dt$

- find $\max_{x\in[a,b]} g(x) =$ • $g(a)$, • $g(b)$ • $g'(x)=0$

$k(x,t) = \sin(t)x^2 \qquad \|k\|_\infty = \max_{x\in[a,b]} \int_0^{2\pi} |\sin(t)x^2|\,dt$

$$k(x,t) = \begin{cases} x^3 t & 0 \le t \le x \\ t^2(x^2+1) & x \le t \le 2 \end{cases}$$

$$g(x) = \int_a^b |k(x,t)|\,dt = \int_0^x x^3 t\,dt + \int_x^2 t^2(x^2+1)\,dt$$

integrating both parts, sub limits of integration

$g(x) = \frac{x^5}{2} + (x^2+1)\left(\frac{8}{3} - \frac{x^3}{3}\right)$ so

$\|k\|_\infty = \max\left\{ |g(0)|, |g(2)|, |g'(x)=0| \right\}$

solve $g'=0$ for stationary pts

\Iteration techniques/ we would like to use <u>Neumann series</u> to approx Z sol of our IE (Note: "Analytical" → exact sol "numerical" → approx sol)

Consider : $\lambda y - My = f$ we can rewrite this as

$(I - \lambda^{-1} M) y = \lambda^{-1} f$ $\quad y = (I - \lambda^{-1} M)^{-1} \lambda^{-1} f$ ← soln is it exists

Inverse operators:

\Def/ an operator L^{-1} is inverse to operator L is $L^{-1}L = LL^{-1} = I$

\lemma/ let L_1 & L_2 be two operators which are bounded then $L_1 L_2$ is also bounded & $\|L_1 L_2\|_\infty \leq \|L_1\|_\infty \|L_2\|_\infty$ ⊛

\proposition/ Let L be a bounded operator with bounded inverse L^{-1} then $\|L^{-1}\|_\infty \geq \frac{1}{\|L\|_\infty}$

(use ⊛ with $L_2 = L$ & $L_1 = L^{-1}$ & rearrange inequality)

Note: similarly with funcs & matrices, not all linear operators have an inverse.

\Def/ elements of Z following $y_n = \frac{1}{\lambda}(f + My_{n-1}) = \frac{f}{\lambda} + \sum_{k=1}^{n} \frac{1}{\lambda^{k+1}} M^k f$ $\quad n \in \mathbb{N}$

are Z Neuman iterations & $y = \sum_{k=0}^{\infty} \lambda^{-(k+1)} \overbrace{M^k f}^{\text{integral operator applied k times}}$ is Z infinite Neuman Series

$M^0 f = If = f$

$M^1 f = Mf$ $\quad M^2 = M(Mf)$ $\quad M(M(Mf)) \cdots$

$M^k f = M(M^{k-1} f)$

\Ex/ consider $y(x) - \int_0^1 x t^3 y(t)\,dt = x^2$ state Z zeroth iteration, y_0, & ∴ compute Z neumann iteration y_1, y_2, y_3

∴ $y(x) = x^2 + \int_0^1 x t^3 y(t)\,dt$ $\quad$ ∴ $\left\{ y_n = \frac{f}{\lambda} + \frac{1}{\lambda} M y_{n-1} \right\}$ ⊛ $\lambda = 1$, $f(x) = x^2$

$y_0 : \frac{f}{\lambda} = x^2$ using $n=0$

$y_1 = x^2 + \frac{1}{1} x \int_0^1 t^3 \overbrace{y_0(t)}^{x_2}\,dt$ ← for $n=1$ $\quad \left\{ \because \int_0^1 x t^3 y(t)\,dt \text{ is integral operator acting on } y \right\}$

$y_1 = x^2 + x \int_0^1 t^3 (t^2)\,dt = x^2 + \frac{x}{6} = x(x + \frac{1}{6})$ using $n=1$ integrate & simplify $y_1(x)$

now $n=2$ ⊛ becomes: $y_2 = \frac{f}{\lambda} + \frac{1}{\lambda} M y_1$ $\quad y_2 = x^2 + \int_0^1 x t^3 y_1(t)\,dt$

$= x^2 + \int_0^1 x t^3 \left[x(x + \frac{1}{6}) \right] dt$ ∴ integrating & simplifying :

$y_2(x) = x^2 + x(\frac{1}{6} + \frac{1}{36}) = x(x + \frac{1}{5})$

$y_3(x)$: ⊛ with $n=3$ $\quad y_3(x) = x^2 + \int_0^1 x t^3 y_2(t)\,dt$

$y_3(x) = x^2 + x\int_0^1 t^3 (t(t+\frac{1}{5}))\,dt$

$y_3(x) = x^2 + x\left(\frac{1}{6} + \frac{1}{25}\right)$

\thm/ ☺ Let M be a bounded operator & $|\lambda| > \|M\|_\infty$ then:

① $\lambda y - My = f$ has a unique sol $y(x)$ & Z Neumann iterations forming Z neuman series converge to Z exact sol $y(x)$

② Z operator $L = \lambda I - M$ has an inverse & $\|L^{-1}\|_\infty \leq \frac{1}{|\lambda| - \|M\|_\infty}$ ⊛

Notes: Ⓐ For const C (see item 6 in Z Fredholm thm) we have ~~Hr~~ $\|r_\lambda\|_\infty \leq \|(\lambda I - k)^{-1}\|_\infty \sum_{i,j=1}^{n} \|k_1^{(i)}\|_\infty \|k_2^{(j)}\|_\infty$ if $|\lambda| > \|k\|_\infty$ then $\|(\lambda I - k)^{-1}\|_\infty \leq \frac{1}{|\lambda| - \|k\|_\infty}$ from ⊛

so $\|r_\lambda\|_\infty \leq \frac{1}{(|\lambda| - \|k\|_\infty)} \sum_{i,j=1}^{n} \|k_1^{(i)}\|_\infty \|k_2^{(j)}\|_\infty$

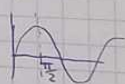
③ this thm ☺ tells us that if $|\lambda| > \|k\|_\infty$ then we have Neumann iterations that converge to $y(x)$ Z unique sol as $n \to \infty$ then: $\|y - y_n\|_\infty \leq \cancel{\frac{1}{|\lambda|}} \frac{1}{|\lambda| - \|M\|_\infty}\left(\frac{\|M\|_\infty}{|\lambda|}\right)^{n-1} \|f\|_\infty$

$|\lambda|$; sup norm of operator ; sup norm of RHS

\Ex/ $k: C[0,1] \to C[0,1]$ $\quad k(x,t) = \sin(x^2+t)$ is there a unique sol $y - Ky = f$? $\quad \lambda = 1 \quad \therefore |\lambda| = 1$ we want $|\lambda| > \|k\|_\infty$

$\|k\|_\infty = \max_{x\in[a,b]} \int_a^b |k(x,t)|\,dt = \max_{x\in[0,1]} \int_0^1 |\sin(x^2+t)|\,dt =$ (integrate & sub limits)

~~know~~ $\|-\cos(\underbrace{x^2+1}_{\alpha}) + \cos(\underbrace{x^2}_{\beta})\|_\infty$ {we know: $\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ $\quad \therefore 2\sin(\frac{1}{2}) \cdot \|\sin(x^2 + \frac{1}{2})\|_\infty$



$\|k\|_\infty = 2\sin(\frac{1}{2})\sin(\frac{3}{2}) \approx \underbrace{0.95}_{\|k\|_\infty} < \underbrace{1}_{|\lambda|}$ $\therefore$ our IE $y - ky = f$ has a unique sol by thm ☺

\Ex/ Find $\|k\|_\infty$ where $k(x,t) = \begin{cases} t(1-x) & 0 \leq t \leq x \\ x(1-t) & x \leq t \leq 1 \end{cases}$ $\Rightarrow \frac{1}{8}$

$\therefore$ for $0 \leq t \leq x$: $\quad \|k\|_\infty \cancel{B} = \|t(1-x)\| = \ldots \quad 0(1-x) = 0$

$x(1-x) = x - x^2$ $\qquad \ldots t = 0 \quad \therefore$ ✗

$\|k\|_\infty = \max_{x\in[a,b]} \int_a^b |k(x,t)|\,dt = \cancel{\max} \max\left(\int_0^x |t(1-x)|\,dt, \int_x^1 x(1-t)\,dt\right.$

\Ex/ $\lambda y(x) - \int_0^1 k(x,t)\, y(t)\, dt = f(x) \quad 0 \le x \le 1$

$k(x,t) = \begin{cases} t(1-x) & 0 \le t \le x \\ x(1-t) & x \le t \le 1 \end{cases}$

$\therefore \|k\|_\infty = \int_0^x t(1-x)\,dt + \overbrace{\int_x^1 x(1-t)\,dt}^{\int x - xt} =$

$\left[(1-x)\frac{t^2}{2}\right]_0^x + \left[xt - \frac{xt^2}{2}\right]_x^1 = \underbrace{\frac{x}{2}(1-x)}_{g(x)} = \frac{x}{2} - \frac{x^2}{2}$

~~Max~~ $\max_{[0,1]} \left\{ \frac{x}{2}(1-x)\Big|_{x=0},\ \frac{x}{2}(1-x)\Big|_{x=1},\ \text{at } g'(x)=0 \text{ sols} \right\}$

$g'(x) = \frac{1}{2}x$ $\therefore$ max occurs at $x = \frac{1}{2}$ $\therefore$

$\|K\|_\infty = \frac{1}{8}$

\Ex/ For this kernel, under which condition can we find a sol using Z Neumann iteration method for $\lambda y - Ky = f$ $\therefore |\lambda| > \frac{1}{8}$

require $|\lambda| > \|K\|_\infty$ $\therefore |\lambda| > \frac{1}{8}$

\Ex-c/ let $f = 1$, $\lambda = \frac{1}{2}$

\ci/ give Z starting pt (y_0) & Z next two iterations (y_1, y_2) for Z neumann method $\quad |\lambda| = \frac{1}{2} > \frac{1}{8}$

$y_0 = \frac{f}{\lambda} = \frac{1}{1/2} = 2$

$y_1 = \frac{1}{\lambda}(f + Ky_0) = 2(1 + Ky_0) = 2 + 2\left(\int_0^x \underset{y_0}{2}\, t(1-x)\,dt + \int_x^1 \underset{y_0}{2}\, x(1-t)\,dt\right) =$

$2 + 4\left(\int_0^x t(1-x)\,dt + \int_x^1 x(1-t)\,dt\right) \therefore$

$y_1 = 2 + 2x(1-x) \quad \therefore$

$y_2 = \frac{1}{\lambda}(f + Ky_1) = 2 + 2\left(\int_0^x t(1-x)\,y_1(t)\,dt + \int_x^1 x(1-t)\,y_1(t)\,dt\right) =$

$2 + 2\left(\int_y^1 t(1-x)\left\{2 + 2t(1-t)\right\}dt + \int_x^1 x(1-t)\left\{2 + 2t(1-t)\right\}dt\right) \therefore$

$y_2 = 2 + \frac{1}{3}(x-1)(x^2 - x - 7)$

\Cii/ how many iterates are needed to get an approx of Z IE with an error not exceeding 10^{-6} $\therefore$

$\therefore$

use $\|y - y_n\|_\infty \le \frac{1}{|\lambda| - \|k\|_\infty}\left(\frac{\|k\|_\infty^{n+1}}{|\lambda|^{n+1}}\right)\|f\|_\infty$ $\therefore$

exact Sol, approx, $|\lambda|$, sup norm of k, sup norm of f

would like $\|y - y_n\|_\infty \le 10^{-6}$ $\therefore$

$\frac{1}{|\lambda| - \|k\|_\infty}\left(\frac{\|k\|_\infty^{n+1}}{|\lambda|^{n+1}}\right)\|f\|_\infty \le 10^{-6}$ $\therefore$

$\frac{1}{\frac{1}{2} - \frac{1}{8}}\left(\frac{\frac{1}{8}}{1/2}\right)^{n-1} \times 1 \le 10^{-6}$ $\quad\therefore \frac{8}{3}\left(\frac{1}{4}\right)^{n+1} \le 10^{-6}$

$\frac{2}{3}\left(\frac{1}{4}\right)^n\left(\frac{1}{4}\right) \le 10^{-6}$ $\quad\therefore \left(\frac{2}{3}\right)\left(\frac{1}{4}\right)^n \le 10^{-6}$ $\therefore$

$\log\left(\left(\frac{2}{3}\right)\left(\frac{1}{4}\right)\right) \le \log(10^{-6})$ $\quad\therefore \log\left(\frac{2}{3}\right) + \underbrace{\log\left(\left(\frac{1}{4}\right)^n\right)}_{n\log(1/4)} \le \log(10^{-6})$

$\log\left(\frac{2}{3}\right) + n\log(1/4) \le \log(10^{-6})$ $\therefore$

$-\log\left(\frac{2}{3}\right) - n\log(1/4) \ge -\log(10^{-6})$ $\quad\{4 < 5 \Rightarrow -4 > -5\}$ $\quad\{\log(1/4) = \log(4^{-1})$

$-\log(2/3) + n\log(4) \ge -\log(10^{-6})$ so $n\log(4) \ge \log\left(\frac{2}{3}\right) + \log(10^{6})$

$\therefore n \ge (\log(2) - \log(3) + 6)/\log(4)$ $\quad\therefore n \ge 9.67$ $\quad\therefore$ need 10 iterations

to get an error of less than 10^{-6}

\Neumann Series/ \Lemma:/ Let K & L be two integral operators on $C([a,b])$ with k, l being Z kernels respectively. then KL is an operator given by $\tilde{k}(x,t) = \int_a^b k(x,s)\,l(s,t)\,ds$ (*)

\Ex/ $k(x,t) = e^{3x}(3 + xt)$ $\quad l(x,t) = \sin(x+t)$

$\tilde{k}(x,t) = \int_a^b e^{3x}(3 + xs)\sin(s+t)\,ds$

\proposition:/ K is an integral operator with kernel $k(x,t)$ then $\forall n \in \mathbb{N}$ $\quad K^n = \underbrace{K(K(K\cdots))}_{n\text{ times}}$

- this is also an operator with K_n given by $k_n(x,t) = \int_a^b k(x,s)\,k_{n-1}(s,t)\,ds$

$n \in \mathbb{N}$ $\quad k_1 = k$ $\quad$ using (*) with $k = k$, $k = l$

Z neumann Series is $\quad y = \frac{f}{\lambda} + \sum_{n=1}^{\infty} \frac{1}{\lambda^{n+1}}\int_a^b k_n(x,t)\,f(t)\,dt$ (**)

$\therefore$ lets see how we can use iterated kernels to find Z Sol using Neumann series

Ex / Let $\lambda y(x) - \int_0^1 x t^2 y(t)\,dt = f(x)$

ⓐ Compute $\|K\|_\infty$ $\therefore \|K\|_\infty = \max_{x\in[0,1]} \int_0^1 x t^2\,dt =$

$\max_{x\in[0,1]} \left(x \left[\frac{1}{3} t^3\right]_0^1 \right) = \max_{x\in[0,1]} \left(\frac{x}{3}\right) = \frac{1}{3}$

ⓑ under which condition can we use Neumann series : $|\lambda| > 1/3$

ⓒ use Z Neumann iteration technique to solve IE

$k(x,t) = xt^2$ $\quad k_{\underset{n=2}{n}}(x,t) = \int_0^1 \underbrace{xs^2}_{k(x,s)} \cdot \underbrace{st^2}_{k_1(s,t)}\,ds = \frac{1}{4} x t^2 = k_2(x,t)$ $\left\{ \therefore k_2(s,t) = \frac{1}{4} s t^2 \right\}$

$k_3(x,t) = \int_0^1 xs^2\, k_2(s,t)\,ds = \int_0^1 xs^2 \frac{1}{4} s t^2\,ds \quad \therefore$

$k_3(x,t) = \left(\frac{1}{4}\right)^2 x t^2$.

$k_4(x,t) = \int_0^1 xs^2 \cdot \left(\frac{1}{4}\right)^2 s t^2\,ds = \left(\frac{1}{4}\right)^3 x t^2$

we notice a pattern: $k_n(x,t) = \left(\frac{1}{4}\right)^{n-1} x t^2$

$\therefore$ into ⊗⊗ we can write our sol y in terms of Z neumann series

$$y(x) = \frac{f(x)}{\lambda} + \sum_{n=1}^{\infty} \frac{1}{\lambda^{n+1}} \int_0^1 \underbrace{\left(\frac{1}{4}\right)^{n-1} x t^2}_{k_n(x,t)} f(t)\,dt$$

$$y(x) = \frac{f(x)}{\lambda} + \frac{x}{\lambda^2}\left(\sum_{n=0}^{\infty} \frac{1}{(4\lambda)^n} \right) \int_0^1 t^2 f(t)\,dt$$

we used: $\underbrace{\sum_{n=1}^{\infty} \frac{1}{\lambda^{n+1}} \left(\frac{1}{4}\right)^{n-1}}_{\text{index shift}} = \sum_{n=0}^{\infty} \frac{1}{\lambda^{(n+1)+1}} \left(\frac{1}{4}\right)^{n+1-1}$

$$= \sum_{n=0}^{\infty} \frac{1}{\lambda^2} \cdot \frac{1}{\lambda^n} \left(\frac{1}{4}\right)^n = \frac{1}{\lambda^2} \sum_{n=0}^{\infty} \left(\frac{1}{4\lambda}\right)^n$$

we can find $\sum_{n=0}^{\infty} \left(\frac{1}{4\lambda}\right)^n$ using $\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$ if $|r| < 1$

and $|\lambda| > \frac{1}{3}$ $\therefore$ we need $\left|\frac{1}{4\lambda}\right| < 1$ ie $|\lambda| > \frac{1}{4}$ yes since $|\lambda| > \frac{1}{3}$

so $y(x) = \frac{f(x)}{\lambda} + \frac{x}{\lambda^2} \left(\frac{1}{1 - \left(\frac{1}{4\lambda}\right)} \right) \int_0^1 t^2 f(t)\,dt$

$$y(x) = \frac{f(x)}{\lambda} + \frac{4x}{\lambda(4\lambda - 1)} \int_0^1 t^2 f(t)\,dt \quad ④$$

ⓓ / Solve Z IE exactly $\lambda y(x) - x \overbrace{\int_0^1 t^2 y(t)\,dt}^{P} = f(x)$

$y(x) = \frac{f(x)}{\lambda} + \frac{xP}{\lambda}$ find P: $P = \int_0^1 t^2 y(t)\,dt$ $\therefore$

$P = \int_0^1 t^2 \left(\frac{f(t)}{\lambda} + \frac{tP}{\lambda} \right) dt \quad P = \int_0^1 \frac{t^2}{\lambda} f(t)\,dt + \frac{P}{\lambda} \int_0^1 t^3\,dt \quad \therefore$

$p = \int_0^1 \frac{t^2}{\lambda} f(t)\,dt + \frac{p}{\lambda}\left(\frac{1}{4}\right)$ $\qquad p\left(1-\frac{1}{4\lambda}\right) = \int_0^1 \frac{t^2}{\lambda} f(t)\,dt$

$p = \frac{4}{4\lambda-1}\int_0^1 t^2 f(t)\,dt$ $\qquad y(x) = \frac{f(x)}{\lambda} + \frac{x}{\lambda}\left(\frac{4}{4\lambda}-1\right)\int_0^1 t^2 f(t)\,dt$

Same as $\triangle$

we will now look at first kind integral eqns ~~the~~

\thm:/ let k be an integral operator with continuous kernel, $\lambda \in \mathbb{R}\setminus\{0\}$ then $(\lambda I - k)$ satisfies Z FA (Fredholm Alternative)

\thm:/ let k be an integral operator with piecewise continuous or weakly singular kernel, $\lambda \in \mathbb{R}\setminus\{0\}$ then $(\lambda I - k)$ satisfies Z FA

{reminder: first kind IE: $Ky = f$ $\left\{ \int_a^b k(x,t)\,y(t)\,dt = f(x) \right\}$

this first kind IE has a sol if Z range of $f(x)$ belongs to Z range of K

\proposition:/ ⓐ let K be an integral operator, with continuous kernel k & Z first j partial derivs of $k(x,t)$ wrt x exist & are continuous, weakly singular, or piecewise-continuous. Then Z range of K consists of j-times continuously differentiable funcs

ⓑ if Z integral operator K has a finite rank kernel, then Z range of K is finite dimension

\Ex/ Does $\int_{-1}^1 \frac{y(t)}{1+(x-t)^2}\,dt = |x|$ $\quad -1 < x < 1$ has a sol?

$\therefore$ Although $k(x,t) = \frac{1}{1+(x-t)^2}$ is smooth $\in C^\infty$ infinitely many times differentiable & Z derivs are cont

our f is problematic

$\because$ $|x|$ is not differentiable at $x=0$ & 0 is in our x range

\Ex/ 1st kind IE $k(x,t) = e^{-xt}$ $\quad y \equiv 2$ on $[0,2]$

$(Ky)(x) = \int_0^2 e^{-xt}\,y(t)\,dt = 2\int_0^2 e^{-xt}\,dt = \frac{2}{x}(1-e^{-2x})$ when $x \neq 0$

when $x=0$ $k(x,t) = e^0 = 1$ $\quad (Ky)(x) = \int_0^2 y(t)\,dt = \int_0^2 2\,dt = 4$

So $(Ky)(x) = \begin{cases} 4 & x=0 \\ \frac{2}{x}(1-e^{-2x}) & 0<x\le 2 \end{cases}$

this is infinitely many times continuously diffable

$Ky = f \quad \int_0^2 e^{-xt} y(t)\,dt = f(t)$

a well known problem arising with first kind integral eqns is ill-posedness

a well-posed probl is one where we have a sol (<u>existence</u>) & it does not have more than one sol (<u>uniqueness</u>), & that sol depends continuously on the params (data) of the probl (<u>stability</u>)

in the case of an ill-posed IE we don't have a C st $\|y\|_\infty \le C\|f\|_\infty$ {(thm 3.17)}

Ex $k(x,t) = xt$, $y(x) = \cos(2x)$ $\|y\|_\infty = 1$ $\{x \in [0,\pi]\}$

$(Ky)(x) = \int_0^\pi xt\cos(2t)\,dt = x\int_0^\pi t\cos(2t)\,dt$

$(Ky)(x) = f$ ∴ $u=t$, $v' = \cos(2t)$ ∴ IBP

$(Ky)(x) = x\left[\frac{\cos(2t)}{4}\right]_0^\pi = 0$

$\|K\|_\infty = 0$ $\{\|K\|_\infty = \max|(Ky)(x)|\,|_{x\in[0,\pi]}\}$

1st kind IE: $\|f\|_\infty = 0$ so $\exists\, C>0$ st $\|y\|_\infty = 1 \le C\underbrace{\|Ky\|_\infty}_{=0}$

$1 \le C(0)$? No! ∴ so $\|y\|_\infty \le C\|f\|_\infty$

FA: $\exists$ a unique sol to $\lambda y - Ky = f$ <u>or</u> the homogen $\lambda y - Ky = 0$ has a <u>nontrivial</u> sol ~~volterra~~

Chapter 4 Volterra IE

a linear volterra IE

$\lambda y(x) - \int_a^x \underbrace{k(x,t)}_{\text{kernel}} y(t) = f(x) \quad a\le x\le b$

Def $(Ky)(x) = \int_a^x k(x,t)\,y(t)\,dt$ is a volterra integral operator

$K: C([a,b]) \to C[(a,b)]$

would like to introduce a thm that tells us Z if Fredholm alternative is always true for Volterra IE eqns of Z 2nd kind

\Lemma/ let K be a Volterra integral operator & $a \neq -\infty$

then $|(K^n y)(x)| \leq \frac{1}{n!}(x-a)^n \|K\|_\infty^n \|y\|_\infty$ $\forall n \in \mathbb{N}$ ①

\proof/ using induction: • $n=1$

$|(Ky)(x)| = \left|\int_a^x k(x,t)y(t)\,dt\right| \leq \int_a^x |k(x,t)|\,|y(t)|\,dt \leq \int_a^x \|K\|_\infty \|y\|_\infty\,dt$ ②

~~$\leq (x-a)\|K\|_\infty \|y\|_\infty$~~ $\leq (x-a)\|K\|_\infty \|y\|_\infty$

$|(Ky)(x)| \leq (x-a)\|K\|_\infty \|y\|_\infty$ (= RHS in ① for $n=1$)

• $n=2$: $|\underbrace{K^2(y)}_{K(K(y))}(x)| \leq \int_a^x \underbrace{|k(x,t)|}_{\leq \|K\|_\infty} \underbrace{|(Ky)(t)|}_{\text{operator } K \text{ on } y}\,dt$

$\leq \int_a^x (t-a)\|K\|_\infty^2 \|y\|_\infty$ ← used ②

~~used ②:~~ ~~$\leq \int_a$~~ $\leq \frac{(x-a)^2}{2}\|K\|_\infty^2 \|y\|_\infty$

(same as RHS in ① for $n=2$)

assume statement ① is true for n

$|K^n(y)(x)| \leq \frac{1}{n!}(x-a)^n \|K\|_\infty^n \|y\|_\infty$

lets take $n+1$: $\left\{ \text{goal: } |(K^{n+1}y)(x)| \leq \frac{1}{(n+1)!}(x-a)^{n+1}\|K\|_\infty^{n+1}\|y\|_\infty \right\}$

$|(K^{n+1}y)(x)| \leq \int_a^x |k(x,t)|\,|(K^n y)(t)|\,dt$ (using assumption:)

$\leq \int_a^x \|K\|_\infty \frac{1}{n!}(t-a)^n \|K\|_\infty^n \|y\|_\infty\,dt$

$\leq \frac{(x-a)^{n+1}}{n!(n+1)}\|K\|_\infty^{n+1}\|y\|_\infty = \frac{(x-a)^{n+1}}{(n+1)!}\|K\|_\infty^{n+1}\|y\|_\infty$ (same as our goal)

by induction: Z statement ① is true $\forall n \in \mathbb{N}$

\Lemma/ $-\infty < a < b \leq \infty$ & $\|K\|_\infty$ on $[a,b]$ is finite. Z homog Volterra IE $\lambda y - Ky = 0$ on $[a,b]$ has <u>only</u> Z trivial sol for $\lambda \neq 0$

\proof/ if y is a sol of Z homog IE then $y = \frac{1}{\lambda}Ky$

$= \frac{1}{\lambda}K\left(\frac{1}{\lambda}Ky\right) = \frac{1}{\lambda^2}K^2 y = \frac{1}{\lambda}K^2\left(\frac{1}{\lambda}Ky\right) = \frac{1}{\lambda^3}K^3(y)$ etc ...

so $y = \frac{1}{\lambda^n}K^n y$ so $\|y\|_\infty \leq \frac{1}{n!}\left(\frac{b-a}{\lambda}\|K\|_\infty\right)^n \|y\|_\infty$ ②

($\because \|y\|_\infty = \|\frac{1}{\lambda^n}K^n y\|_\infty \leq \frac{1}{\lambda^n}\cdot\frac{1}{n!}(x-a)^n \|K\|_\infty^n \|y\|_\infty$)

For (*) to be true we must have $\|y\|_\infty = 0$ so $y = 0$ is Z sol of Z homog Volterra IE

\thm:/ Let $-\infty \le a \le b \le \infty$ on $[a,b]$ $\|K\|_a$ is finite. then $\exists$ a unique cont sol for Z volterra IE for $\lambda \neq 0$

\proof from Z Fredholm alternative, $\exists$ a unique sol of $\lambda y(x) - \int_a^x k(x,t) y(t)\,dt = f(x)$ $\because$ Z previous ~~lemma~~ lemma showed a zero sol of Z homog case (ie 2nd FA doesn't hold)

we will not look at IE with a convolution kernel. $\{k(x,t) = k(x-t)\}$

\thm:/ Let k be a volterra integral operator with $k(x,t) = k(x-t)$ when it exists, Z Laplace transform of Z sol of

$$\lambda y(x) - \int_a^x \underbrace{k(x,t)}_{k(x-t)} y(t)\,dt = f(x) \quad \text{is} \quad \hat{y}(s) = \cancel{\frac{\hat{f}(s)}{\lambda - \hat{k}(s)}} \; \frac{\hat{f}(s)}{\lambda - \hat{k}(s)} \quad \lambda \neq 0$$

where $\hat{y} = \mathcal{L}(y(x))$, $\hat{f}(s) = \mathcal{L}(f(x))$

$\hat{k}(s) = \mathcal{L}(\underline{k(x)})$ $k(x-t) = k(x,t)$

\proof/ take $\lambda y(x) - \int_a^x k(x-t)\, y(t)\,dt = f(x)$

$\lambda \hat{y}(s) - \hat{k}(s)\hat{y}(s) = \hat{f}(s)$

$$\hat{y}(s)(\lambda - \hat{k}(s)) = \hat{f}(s) \qquad \hat{y}(s) = \frac{\hat{f}(s)}{\lambda - \hat{k}(s)} \quad \square$$

\Def/ Z degree of a rational func $f = \frac{f_1}{f_2}$ is degree(f_1) - degree(f_2)

\proposition:/ suppose degree of $\hat{k}$ is negn then

- when $\lambda \neq 0$ Z degree of $\hat{y}$ = degree of $\hat{f}$
- when $\lambda = 0$ Z degree of $\hat{y}$ = degree of $\hat{f}$ - degree of $\hat{k}$

so whatever choice of $\lambda \neq 0$ & $f \in C[0,\infty)$ we still have a sol. if $\lambda = 0$ & $f \in C[0,\infty)$ then sol depends on Z degree of $\hat{f}$ & $\hat{k}$

Ex/ find the sol of the Volterra IE using Laplace transforms

a) ① $y(x) - \int_0^x \underbrace{(x-t)}_{k(x,t) = k(x-t)} y(t)\,dt = x^2$

$\lambda = 1 \quad k(x) = x \quad f(x) = x^2 \quad \hat{y}(s) = \dfrac{\hat{f}}{1 - \hat{k}(s)} \overset{\text{LT} = \text{top \& bottom}}{=}$

$\dfrac{2s^{-3}}{1 - s^{-2}} = \dfrac{2/s^3}{1 - 1/s^2} \therefore$

$g(s) = \dfrac{2}{s(s^2-1)} = \text{(partial fractions)} \quad -\dfrac{2}{s} + \dfrac{2s}{s^2-1} \therefore$

(inverse Laplace transforms) $y(x) = -2 + 2\cosh(x)$

$\int_0^x (x-t)y(t)\,dt = x$

$\lambda = 0$ 1st kind using LT22 $\mathcal{L}\left(\int_0^x F(u)G(t-u)\,du\right) = f(s)g(s)$

$g(s) \cdot \frac{1}{s^2} = s^2$

$\hat{g}(s) = 1 \rightarrow$ right hand column

$\mathcal{L}^{-1}(\hat{g}(s)) = \mathcal{L}^{-1}(1) = \delta(x)$ dirac delta func

using differentiation : $\int_0^x (x-t)y(t)\,dt = x$

$x = x\int_0^x y(t)\,dt - \int_0^x t y(t)\,d + \underbrace{x(y(t))}_{} + 1 + \cancel{\int_0^x y(t)\,dt - xy(t)} = 1$

$\cancel{\int_0^x y(t)\,dt = }$ $\quad \underbrace{x(y(t))}_{} + 1 + \int_0^x y(t)\,dt - \underbrace{xy(t)}_{} = 1$

$\int_0^x y(t)\,dt = 1 \quad x = 0,\ 0 = 1$ not true

Ex/ solve the Volterra IE $y(x) - \int_0^x \underbrace{\cos(x-t)}_{\mathcal{K}(x-t)} y(t)\,dt = f(x)$

$\hat{g}(s) = \dfrac{\hat{f}(s)}{\underset{1}{\lambda} - \hat{\mathcal{K}}(s)}$ $\quad \hat{\mathcal{K}}$ = Laplace transform of $\mathcal{K}(x)$ ie of $\cos(x)$

so $\hat{g}(s) = \dfrac{\hat{f}(s)}{1 - \underbrace{\frac{s}{s^2+1}}_{\cos(x)}}$ $\quad$ make common dinominator & simplify

$\hat{g}(s) = \hat{f}(s) \cdot \underbrace{\left(\dfrac{s^2+1}{s^2-s+1}\right)}_{\hat{g}(s)} \quad \therefore \quad$ LT22:

$y(x) = \int_0^x f(t) \cdot$ (the inverse Laplace transform of $\hat{g}(s)$ but with x replaced with $x-t$) dt

{using LT22:} $\mathcal{L}^{-1}(f \cdot g) = \int_0^t F(u)G(t-u)\,du$

now we need $\mathcal{L}^{-1}(\hat{g}(s)) = \mathcal{L}^{-1}\left(\dfrac{s^2+1}{s^2-s+1}\right)$ {can't use partial fracs ∵ already not factorised}

$\hat{g}(s) = \dfrac{s^2+1-s+s}{s^2-s+1} = \dfrac{(s^2+1-s)}{s^2-s+1} + \dfrac{s}{s^2-s+1} = 1 + \underbrace{\dfrac{s}{s^2-s+1}}_{\hat{c}(s)} \therefore$

$$\hat{C}(s) = \frac{s}{(s-\frac{1}{2})^2+\frac{3}{4}}$$ {s must be shifted everywhere its already shifted}

$$= \frac{s-\frac{1}{2}+\frac{1}{2}}{(s-\frac{1}{2})^2+\frac{3}{4}} = \underbrace{\frac{s-\frac{1}{2}}{(s-\frac{1}{2})^2+\frac{3}{4}}}_{LT13,\,LT6} + \underbrace{\frac{\frac{1}{2}}{(s-\frac{1}{2})^2+\frac{3}{4}}}_{LT13,\,LT5}$$ ∴ using LT13 & LT6 or LT5

$$C(x) = \cos\left(\tfrac{\sqrt{3}}{2}x\right)e^{\frac{1}{2}x} + \tfrac{1}{2}\cdot\tfrac{2}{\sqrt{3}}\sin\left(\tfrac{\sqrt{3}}{2}x\right)e^{\frac{1}{2}x}$$

$$\hat{y}(s) = \hat{f}(s)\hat{g}(s) = \hat{f}(s)(1+\hat{C}(s)) = \hat{f}(s) + \hat{f}(s)\hat{C}(s)$$

$$y(x) = f(x) + \int_0^x f(t)C(x-t)\,dt$$

$$y(x) = f(x) - \int_0^x f(t)\left\{\cos\left(\tfrac{\sqrt{3}}{2}(x-t)\right)e^{\frac{1}{2}(x-t)} + \tfrac{1}{\sqrt{3}}\sin\left(\tfrac{\sqrt{3}}{2}(x-t)\right)e^{\frac{1}{2}(x-t)}\right\}dt$$

\Ex/ $$y(x) - \mu\int_0^x \overbrace{e^{x-t}}^{\mathcal{U}=e^{x-t}}\,y(t)\,dt = f(t)$$

$$\hat{y}(s) = \frac{\hat{f}(s)}{1-\underbrace{\frac{\mu}{s-1}}_{\to\,\mu\mathcal{L}(e^x)}}$$ simplifying gives:

$$\hat{y}(s) = \hat{f}(s)\cdot\underbrace{\left(\frac{s-1}{s-1-\mu}\right)}_{\hat{g}(s)}$$

$$\hat{g}(s) = \frac{s-1-\mu+\mu}{s-1-\mu} = 1 + \overbrace{\frac{\mu}{s-1-\mu}}^{C(s)} \to \left\{\hat{y}(s) = \hat{f}(s)\left[1+\frac{\mu}{s-1-\mu}\right]\right\}$$

$$y(x) = f(x) + \mathcal{L}^{-1}\left(\hat{f}(s)\cdot\hat{C}(s)\right)$$ (using LT22)

$$C(x) = \mathcal{L}^{-1}\left(\frac{\mu}{s-1-\mu}\right) = \mu\,\mathcal{L}^{-1}\left(\frac{1}{s-(\mu+1)}\right)$$ (∴ using LT4 with $a=\mu+1$:)

$$C(x) = \mu\cdot e^{(\mu+1)x} \qquad y(x) = f(x) + \mu\int_0^x f(t)e^{(\mu+1)(x-t)}\,dt$$

we will now revisit volterra IE using differentiation to find 2 sol

recall: we can transform an IE into an IVP

\proposition/ let $k(x,t)$, $f(x)$ be continuously diffable in x, then

$\lambda y(x) - \int_a^x k(x,t)y(t)\,dt = f(x)$ for continuous y is equivalent to

$$\lambda y'(x) - k(x,x)y(x) - \int_a^x \frac{\partial k(x,t)}{\partial t}y(t)\,dt = f'(x)$$

$\lambda y(a) = f(a)$

we can use $\frac{\partial}{\partial x}\left(\int_a^b k(x,t)\,y(t)\,dt\right) = k(x,x)y(x) + \int_a^x \frac{\partial k(x,t)}{\partial x}y(t)\,dt$

(remark 2.21 in notes)

$y_{n+1} = \frac{f}{\lambda} + \frac{1}{\lambda}\int_a^x k(x,t)\, y_n(t)\,dt$ verifying the sol

$y(x) = \dots \sum_{n=0}^{\infty}$ (using induction)

④ Solve the IE exactly (using Laplace transforms or differentiation)

⑤ compare our exact sol to numerical one (Maclaurin series)

\Ex/ $y(x) - \int_0^x (x-t)\,y(t)\,dt = x^2$ Show by using Neumann iteration

Method, the sol is given by $y_n(x) = \sum_{k=0}^{n} \frac{2}{(2k-2)!} x^{2k+2}$

$y_0 = \frac{f}{\lambda} = x^2$

$y_1 = \frac{f}{\lambda} + \frac{1}{\lambda} K y_0$ ④

induction : y_1 $n=1$

goal: $y_1 = \sum_{k=0}^{1} \frac{2}{(2k+2)!} x^{2k+2}$

$y_1 = \underbrace{\left(\frac{2}{2!}x^2\right)}_{k=0} + \underbrace{\left(\frac{2}{4!}x^4\right)}_{k=1} = \underbrace{x^2 + \frac{1}{12}x^4}_{\text{goal}}$

using ④ $y_1 = x^2 + \int_0^x (x-t)t^2\,dt \overset{\text{wrt } t}{=} x^2 + \frac{1}{12}x^4$ = goal

$\therefore$ true for base clause $\therefore$

Let $y_n(x) = \sum_{k=0}^{n} \frac{2}{(2k+2)!} x^{2k+2}$ $\forall n \in \mathbb{N}$

now for the induction step : $n+1$

goal: $y_{n+1}(x) = \sum_{k=0}^{n+1} \frac{2}{(2k+2)!} x^{2k+2}$

$y_{n+1} = x^2 + \int_0^x (x-t) \sum_{k=0}^{n} \frac{2}{(2k+2)!} t^{2k+2}\,dt =$

$x^2 + \sum_{k=0}^{n} \frac{2}{(2k+2)!}\left(\int_0^x xt^{2k+2} - t^{2k+3}\,dt\right) =$

$x^2 + \sum_{k=0}^{n} \frac{2}{(2k+2)!}\left[\frac{xt^{2k+3}}{2k+3} - \frac{1}{2k+4}t^{2k+4}\right]_0^x =$

$x^2 + \sum_{k=0}^{n} \frac{2}{(2k+2)!}\left(\frac{x^{2k+4}}{2k+3} - \frac{x^{2k+4}}{2k+4}\right) \longrightarrow \frac{1}{2k+3} - \frac{1}{2k+4} = \frac{1}{(2k+3)(2k+4)}$

$= x^2 + \sum_{k=0}^{n} \frac{2x^{2k+4}}{(2k+2)!(2k+3)(2k+4)} = x^2 + \sum_{k=0}^{n} \frac{2x^{2k+4}}{(2k+4)!}$

$= x^2 + \sum_{k=1}^{n+1} \frac{2x^{2(k-1)+4}}{(2(k-1)+4)!} = x^2 + \sum_{k=1}^{n+1} \frac{2x^{2k+2}}{(2k+2)!} = \sum_{k=0}^{n+1} \frac{2x^{2k+2}}{(2k+2)!}$ $\left\{\text{when } k=0 \;\; \frac{2x^2}{2!} = x^2\right\}$

we reached our goal $\therefore$ Z truth for n implies its truth for n+1 $\therefore$ true $\forall n \in \mathbb{N}$

Volterra integral eqns of Z 1st kind: $\int_a^x k(x,t)\,y(t)\,dt = f(x)$ (**) $a \le x \le b$

For a finite a, $f(a)=0$ is a necessary condition for this IE (**) to have a sol. Can use diff or IBP to convert (**) into an IE of Z 2nd kind

proposition/ suppose $-\infty < a < b < \infty$, $k(x,x) \neq 0$ $\forall x \in [a,b]$ & $\frac{\partial k(x,t)}{\partial x}$ & $f'(x)$ are cont. then (**) has a unique sol iff $f(a)=0$

proof/ (iff) $\Longrightarrow$ sub $x=a$ into our IE (**) gives $f(a)=0$ $\Longleftarrow$ Take IE (**) $\int_a^x k(x,t)\,y(t)\,dt = f(x)$ & diff wrt x ∴ $k(x,x)\,y(x) + \int_a^x \frac{\partial k(x,t)}{\partial x}\,y(t)\,dt = f'(x)$ ∴ $\div k(x,x)$:

$$y(x) + \frac{1}{k(x,x)}\int_a^x \frac{\partial k(x,t)}{\partial x}\,y(t)\,dt = \frac{f'(x)}{k(x,x)}$$ know from a previous thm that this 2nd kind IE has a sol □

now apply IBP on IE (**) $\int_a^x \underbrace{k(x,t)}_{u}\underbrace{y(t)}_{v'}\,dt = f(x)$

$u' = \frac{\partial k(x,t)}{\partial t}$, $v = \int_a^x y(t)\,dt = \tilde{y}$

so our integral becomes:

$$k(x,x)\underbrace{\int_a^x y(t)\,dt}_{\tilde{y}(x)} - \int_a^x \frac{\partial k(x,t)}{\partial t}\,\tilde{y}(t)\,dt = f(x) \quad ∴ \div k(x,x):$$

$$\tilde{y}(x) - \int_a^x \underbrace{\frac{\frac{\partial k(x,t)}{\partial t}}{k(x,x)}}_{\tilde{k}(x,t)}\,\tilde{y}(t)\,dt = \underbrace{\frac{f(x)}{k(x,x)}}_{\tilde{f}(x)} \quad ∴$$

$\tilde{y}(x) - \int_a^x \tilde{k}(x,t)\,\tilde{y}(t)\,dt = \tilde{f}(x)$ is a 2nd kind IE with unknown $\tilde{y}(x)$, kernel $\tilde{k}(x,t) = \frac{1}{k(x,x)}\frac{\partial k(x,x)}{\partial t}$, RHS $\tilde{f}(x) = \frac{f(x)}{k(x,x)}$

Ex/ in $\int_0^x \cos(x-t)\,y(t)\,dt = C - \cos(x)$ what condition must have on C st have a cont sol to our IE ∴

Diff: $\underbrace{\cos(x-x)\,y(x)}_{k(x,x)y(x)} + \int_0^x \underbrace{-\sin(x-t)}_{\partial k/\partial x}\,y(t)\,dt = \sin(x)$ ∴

$$y(x) - \int_0^x \sin(x-t)\,y(t)\,dt = \sin x \quad \text{IE of Z 2nd kind}$$

at $x=0$ $(a=0)$ $0 = C-1$ ∴ $C=1$ must have $C=1$ (from our 1st kind IE)

\Ex/ Find Z sol in terms of f of Z following IE

$\int_0^1 k(x,t)\,y(t)\,dt = f \qquad k(x,t) = \begin{cases} 2x & 0<t<x \\ t & x<t<1 \end{cases}$

- write our integral by splitting it in two

$\int_0^x 2x\,y(t)\,dt + \int_x^1 t\,y(t)\,dt = f(x)$

{reminder: $\frac{\partial}{\partial x}\int_{g(x)}^{f(x)} h(t)\,dt = f'h(f(x)) - g'h(g(x))$}

- diff $2x\,y(x) + 2\int_0^x y(t)\,dt + 0 - x\,y(x) = f'(x)$ ∴

$x\,y(x) + 2\int_0^x y(t)\,dt = f'(x)$

diff again: $x\,y'(x) + y(x) + 2y(x) = f''(x)$

we need $f \in C^2[0,1]$ ∴

$x\,y'(x) + 3y(x) = f''(x)$

- solve Z ODE $\quad y'(x) + \frac{3}{x}y(x) = \frac{f''(x)}{x}$ ∴

IF $e^{\int \frac{3}{x}dx} = e^{3\ln x} = x^3$ ∴

$(yx^3) = \int x^2 f''(x)\,dx$ ∴ $\quad y = \frac{1}{x^3}\int \cancel{x} f''(x)\,dx + C$

$f'(0) = 0$, & $3y(0) = f''(0)$ ie $\frac{1}{3}f''(0) = y(0)$

\Ex/ $y(x) = 1 + \int_0^x e^t y(t)\,dt$ solve Z IE will solve analytically & numerically

analytically: diff $\quad y'(x) = e^x y(x) \qquad y(0) = 1 \qquad y' = e^x y$

$\frac{dy}{dx} = e^x y$ seperation of variabls

$\int \frac{1}{y}\,dy = \int e^x\,dx \qquad \ln(y) = e^x + C$ at

$x=0 \quad \ln(1) = 1 + c$ ∴ $C = -1$ ∴

$y = e^{(e^x - 1)}$

numerically: $y_0 = \frac{f}{\lambda} = 1$

$y_1 = 1 + \int_0^x e^t y_0\,dt = 1 + \int_0^x e^t\,dt = 1 + (e^x - 1)$

$y_2 = 1 + \int_0^x e^t y_1\,dt = \int_0^x e^t(1 + (e^t - 1))\,dt = 1 + e^x - 1 + \frac{(e^x-1)^2}{2}$

by induction lets first assume

$y_n = 1 + \sum_{j=1}^{n} \frac{(e^x-1)^j}{j!} \qquad \cancel{y_n} = \quad y_n = \sum_{j=0}^{n} \frac{(e^x-1)^j}{j!}$

$y_{n+1} = 1 + \int_0^x e^t \sum_{j=0}^{n} \frac{(e^t-1)^j}{j!}\,dt \qquad y_{n+1} = 1 + \sum_{j=0}^{n} \int_0^x \frac{e^t(e^t-1)^j}{j!}\,dt$ ∴

integrate wrt t: $y_{n+1} = \sum_{j=0}^{n+1} \frac{(e^x-1)^j}{j!}$

{ for y_2: $y_2 = 1 + (e^x-1) + \int_0^x e^{2t} - e^t \, dt = \left[\frac{1}{2}e^{2t} - e^t\right]_0^x =$
$\frac{1}{2}e^{2x} - e^x = \frac{1}{2}e^{2x} - e^x + \frac{1}{2} = \frac{1}{2}(e^x-1)^2$ }

$\therefore e^Y = \sum_{j=0}^{n} \frac{Y^j}{j!}$ $\therefore$ $y_n = e^{e^x-1}$ same exact sol

CW only need up to week 8 max

1 page too small $\therefore$ maybe 3-4-5 pages?

5 to 10 references

week 9 thursday lec 2 really useful CW info

no coverpage needed

for a take laplace?

Non linear IE / the following IE $y(x) - \int_a^b k(x,t) H(t, y(t)) dt = f(x)$ is called a Hammerstein IE eg

$$y(x) - \int_a^b \sin(x+t) \underbrace{\frac{y(t)}{y^2(t)+t}}_{H(t,y(t))} dt = 0$$

Def / $(Ky)(x) = \sum_{j=1}^{n} \int_a^b k^{(j)}(x) H^{(j)}(y(t), t) \, dt$ defines a finite n rank Hammerstein integral operator

$K : C[a,b] \to C[a,b]$

so a finite rank Hammerstein IE has the form

$$y(x) - \sum_{j=1}^{n} \int_a^b k^{(j)}(x) H^{(j)}(y(t), t) \, dt = f(x)$$

eg $$y(x) - \int_a^2 \overbrace{e^x}^{k^{(1)}} \underbrace{(t + y(t))^2}_{H^{(1)}} dt + \int_1^2 \overbrace{e^{2t}}^{k^{(2)}} \underbrace{t\, y^2(t)}_{H^{(2)}} dt = f(x)$$

we solve such IE using the same method as that used for linear Fredholm IE (φ-method)

Lemma: the IE $y(x) - \int_a^b U(x, t, y(t)) \, dt = h(x)$ * is equiv to
(nonhomog nonlinear IE)

$$z(x) - \int_a^b g(x, t, z(t)) \, dt = 0$$

(homog nonlinear IE)

proof/ Let $z(x) = y(x) - h(x)$ & sub into (*)

$$\underbrace{y(x) - h(x)}_{z(x)} = \int_a^b U(x, t, z(t) + h(t))\,dt$$

$$\therefore \int_a^b g(x, t, z(t))\,dt = z(x) \qquad \square$$

thm:/ consider $y(x) - \sum_{j=1}^{n} \int_a^b k^{(j)} H^{(j)}(y(t), t)\,dt = 0$

define $p_j = \int_a^b H^{(j)}(y(s), s)\,ds \quad 1 \le j \le n$ (i) ∴

Z IE is equiv to $\vec{p} - \vec{F}(\vec{p}) = 0$

where $\vec{F}(\vec{p}) = (F_1(p), \underbrace{F_2(p)}_{j=2}, \dots)$

$\vec{p} = (p_1, p_2, \dots, p_n)$

$F_j(p) = \int_a^b H^{(j)}\left(\sum_{i=1}^{n} p_i k^{(i)}(x), x\right) dx \quad 1 \le j \le n$

proof/ $y = \sum_{j=1}^{n} p_j k^{(j)}(x)$ into (i)

$$p_j = \int_a^b H^{(j)}\left(\sum_{j=1}^{n} p_j k^{(j)}(s), s\right) ds \quad 1 \le j \le n \qquad \square$$

Ex1:/ Solve Z following nonlinear IE for different λ:

$$y(x) - \lambda \int_0^1 \overbrace{x^2}^{k(x)} \underbrace{t^2(1 + y^2(t))}_{H(t, y(t))}\,dt = 0 \quad \therefore \text{HZ}$$

$n = 1 \therefore \; y(x) - \lambda x^2 p = 0 \quad \therefore y(x) = \lambda x^2 p \quad \therefore y(t) = \lambda t^2 p \;\therefore$

$$p = \int_0^1 t^2 (1 + y^2(t))\,dt$$

$$p = \int_0^1 t^2 (1 + (\lambda x^2 p)^2)\,dt \quad \therefore \quad p = \int_0^1 t^2\,dt + \lambda^2 \int_0^1 p^2 t^6\,dt \;\therefore$$

$$p = \left[\frac{1}{3}t^3 + \frac{\lambda^2}{7} t^7 p^2\right]_0^1 \quad \therefore \quad p = \frac{1}{3} + \frac{\lambda^2}{7} p^2 \;\therefore$$

when $\lambda = 0$: $y(0) = 0$ from IE

when $\lambda \ne 0$: $\overbrace{3\lambda p^2}^{a} \overbrace{- 21 p}^{b} + \overbrace{7}^{c} = 0 \quad \therefore \quad p_{1,2} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

$p_{1,2} = \frac{7}{21\lambda^2} \pm \frac{7}{2\lambda^2} \sqrt{1 - \frac{4}{21}\lambda^2}$ (**) ∴ different cases for λ :

cases for λ :

(a) : when $\frac{4}{21}\lambda^2 > 1$ we have no real sols ie $|\lambda| > \sqrt{\frac{21}{4}}$

(b): $|\lambda| = \sqrt{\frac{21}{4}}$ ie $\sqrt{1-1} = 0$

So $\rho = \frac{7}{2\lambda^2}$ put into y to get

$y(x) = \frac{7}{\sqrt{21}} x^2$ (1 unique sol)

(c): $0 < |\lambda| < \sqrt{\frac{21}{4}}$ 2 sols:

$y(x) = \lambda x^2 \rho_{1,2}$ $\rho_{1,2} \to (**)$

Ex2 / given $a(x) \in C[0,1]$

posi on $(0,1)$ consider Z Hammerstein IE

$$y(x) - \int_0^1 \overbrace{a(x)}^{k(x,t)} \overbrace{\{y(t) + y^2(t)\}}^{H(t,y(t))} dt = 0$$

$y(x) = a(x) \cdot \rho$

$\rho = \int_0^1 y(t) + y^2(t)\,dt$ ∴ $\rho = \int_0^1 a(t) + a^2(t)\rho^2\,dt$ ∴

$\rho = \rho \underbrace{\int_0^1 a(t)\,dt}_{I_1} + \rho^2 \underbrace{\int_0^1 a^2(t)\,dt}_{I_2}$ $I_k = \int_0^1 a^k(t)\,dt$

$\rho = \rho I_1 + \rho^2 I_2$

$\rho = 0$ is a sol $y = 0$

$\rho^2 I_2 - (1 - I_1) = 0$

$\rho_{1,2} = \pm\sqrt{\frac{1 - I_1}{I_2}}$ ∴ $y_1(x) = \sqrt{\frac{1-I_1}{I_2}}\, a(x)$ $y_2(x) = -\sqrt{\frac{1-I_1}{I_2}}\, a(x)$

now let $a(x) = \lambda(1+x)$ ∴

$I_1 = \int_0^1 a(x)\,dx = \int_0^1 \lambda(1+x)\,dx = \frac{3\lambda}{2}$

$I_2 = \int_0^1 \lambda^2(1+x)^2\,dx = \frac{15}{4}\lambda^2$

Sol $y_{1,2}(x) = \pm\sqrt{\frac{2}{\lambda} - 6}\,(1+x) = \lambda(1+x)\sqrt{\frac{1-\frac{3}{2}\lambda}{(\frac{15}{4}\lambda)}}$

for real sols need $\frac{2}{\lambda} - 6 \geq 0$ ie $\lambda \leq \frac{2}{3}$

when $\lambda = \frac{2}{3}$, $y(x) = 0$

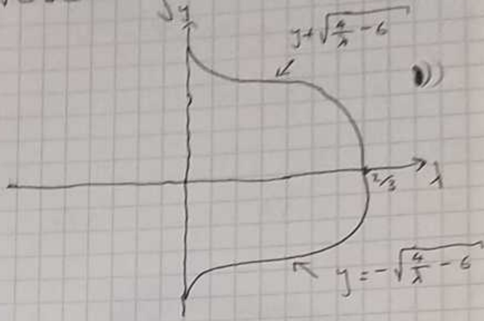
when $\lambda < \frac{2}{3}$ have two real sols:

$y_{1,2} = \pm\sqrt{\frac{2}{\lambda} - 6}\,(1+x)$

we can sketch Z bifurcation diagram to sketch

2 sol of 2 since (y) representing different cases for 2 param (λ)

Set $x=0$



EX: IE on an infinite interval:

$$y(x)-\lambda\int_0^\infty \big(\underbrace{e^{-x}}_{h^{(1)}}\underbrace{y(t)}_{H^{(1)}}+\underbrace{e^{-2x}}_{h^{(2)}}\underbrace{y^3(t)}_{H^{(2)}}\big)dt=0 \qquad 0\le x<\infty$$

here $n=2$ ∴ $y(x)=\lambda(P_1e^{-x}+P_2e^{-2x})$

$P_1 = \int_0^\infty y(t)\,dt \qquad P_2=\int_0^\infty y^3(t)\,dt$

$$P_1=\int_0^\infty \lambda P_1e^{-t}+\lambda P_2e^{-2t}\,dt=\left[-\lambda P_1e^{-t}-\frac{\lambda P_2}{2}e^{-2t}\right]_0^\infty=$$

$$\Big[\underbrace{-\frac{\lambda P_1}{e^t}}_{\to 0 \text{ as } t\to\infty}-\underbrace{\frac{\lambda P_2}{2e^{2t}}}_{\to 0 \text{ as } t\to\infty}\Big]_0^\infty$$

$P_1=0-\left(-\frac{\lambda P_1}{1}-\frac{1}{2}\lambda\frac{P_2}{1}\right) \qquad P_1=\lambda P_1+\frac{1}{2}\lambda P_2$ ①

For $P_2=\int_0^\infty(\lambda P_1e^{-t}+\lambda P_2e^{-2t})^3$

$(a+b)^3=a^3+3a^2b+3ab^2+b^3$ ∴

$P_2=\lambda^3\left(\frac{1}{3}P_1^3+\frac{3}{4}P_1^2P_2+\frac{3}{5}P_1P_2^2+\frac{1}{6}P_2^3\right)$ ②

(as $t\to\infty$ all terms $\to 0$)

From ① make P_1 subject $P_1=\frac{\lambda P_2}{2(1-\lambda)}$

- when $\lambda=0$, $y=0$
- when $\lambda\neq 0$: when $\lambda=1$: eqn ① gives $P_2=0$ ∴

from ② $P_1=1$ ∴ $y(x)=0$

when $\lambda\neq 1$ sub: $P_1=\frac{\lambda P_1}{2(1-\lambda)}$ into ② to find P_2

∴ $P_2=P_2^3\left(\frac{\lambda(20\lambda^3+45\lambda^2+36\lambda+10)}{480(1-\lambda)^3}\right)$

either $P_2=0$ or $1=P_2^2\left(\frac{\lambda^3(20\lambda^3+45\lambda^2+3\lambda+10)}{480(1-\lambda)^3}\right)$ ∴

Solving: $P_2 = \pm\sqrt{\dfrac{480(1-\lambda)^3}{\lambda^3(20\lambda^3+45\lambda^2+36\lambda+10)}}$

$\therefore P_2 = \pm 4 \underbrace{\left(\dfrac{30(1-\lambda)^3}{\lambda^3(20\lambda^3+45\lambda^2+3\lambda+10)}\right)}_{*}^{1/2}$

need $*$ to be posi $\therefore$ need λ to satisfy

$\lambda \in (-\infty, \lambda^*) \cup (0,1)$

$\hookrightarrow$ root of $20\lambda^3+45\lambda^2+3\lambda+10=0$ $\therefore$

$y_\pm(x) \pm 4\left(\dfrac{30(1-\lambda)^3}{\lambda^3(20\lambda^3+45\lambda^2+36\lambda+10)}\right)^{1/2}(e^{-x}+(1-\lambda)e^{-2x})$

$\{y(x) = \lambda P_1 e^{-x} + \lambda P_2 e^{-2x}\}$

Non linear Volterra IE / $y(x) - \int_a^x U(x,t,y(t))\,dt = 0$

cont on $[a,b]\times[a,b]\times[-c,c]$

$a<b$ $c \in \mathbb{R}$

Def / Z formula $(Ky)(x) = \int_a^x U(x,t,y(t))\,dt$ is a nonlinear Volterra integral operator $K: C[a,b] \to C[a,b]$

we can transform our ODEs into IE

if we have $y'(x) = f(x,y(x))$ $y(x_0) = y_0$

integrating Z DE: $y(x) - y_0 = \int_{x_0}^x f(t,y(t))\,dt$

Lemma: / Z non homog eqn: $z(x) - \int_a^x \hat{U}(x,t,z(t))\,dt = g(x)$ ☺

$g \in C[a,b]$ is equiv to

$y(x) - \int_a^x U(x,t,y(t))\,dt = 0$

proof / Set $z(x) = y(x) + g(x)$ put into ☺

$y(x) + \cancel{g(x)} - \int_a^x \hat{U}(x,t,y(t)+g(t))\,dt = \cancel{g(x)}$

$y(x) - \int_a^x U(x,t,y(t))\,dt = 0$

note: $\hat{U}(x,t,z(t)) = \hat{U}(x,t,y+g) = U(x,t,y)$

Ex / consider: $y(x) - \int_0^x \overbrace{\dfrac{y^2(t)}{1+t^2}}^{h(t)}\,dt = \arctan(x)$

deriv: (recall: $\frac{d}{dx}\int_0^x h(t)\,dt = h(x)$)

$y'(x) - \dfrac{y^2(x)}{1+x^2} = \dfrac{1}{1+x^2}$ $\therefore$ $y'(x) = \dfrac{1+y^2}{1+x^2}$ $\therefore$ Separation of variables

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2} \qquad \int \frac{dy}{1+y^2} = \int \frac{dx}{1+x^2}$$

$\therefore \arctan(y) = \arctan(x) + C$

$\therefore$ from our IE $y(0)=0$ $\quad 0 = 0 + C \quad \therefore C = 0$

$y(x) = x$

Ex/ Solve $\phi(x) - \int_0^x \frac{1+t^2}{1+\phi^2(t)}\,dt = 0 \quad \phi(0) = 0$

deriv: $\phi'(x) - \frac{1+x^2}{1+\phi^2(x)} = 0$

$\phi'(x)(1+\phi^2(x)) = 1 + x^2 \quad \therefore$ integrate both sides:

$\int \phi'(x)(1+\phi^2(x))\,dx = \int 1 + x^2\,dx \quad \therefore$

$\int \phi'(x)\,dx + \int \phi'(x)\phi^2(x)\,dx = x + \frac{1}{3}x^3 + C$

$\phi(x) + \frac{1}{3}\phi^3(x) = x + \frac{1}{3}x^3 + C$

{note for $\int \phi'(x)\phi^2(x)\,dx = \frac{1}{3}\phi^3(x)$ use IBP $(u = \phi^2, v' = \phi')$ }

$\phi(x) + \frac{1}{3}\phi^3(x) = x + \frac{1}{3}x^3 + C$ at $x = 0$ $\phi(0) = 0 \quad \therefore C = 0 \therefore$

$\phi - x = \frac{1}{3}(x^3 - \phi^3) = \frac{1}{3}(x - \phi)(\phi^2 + \phi x + x^2) \quad \therefore$

$\phi(x) = x$ is a sol $\therefore$

$1 = -\frac{1}{3}(x^2 + \phi x + \phi^2) \quad \therefore \phi^2 + \phi x + (3 + x^2) = 0 = A\phi^2 + B\phi + C = 0 \therefore$

Solve for $\phi \therefore \phi = \frac{-x \pm \sqrt{x^2 - 4(1)(3+x^2)}}{2} \quad \therefore$

$\phi = -\frac{x}{2} \pm \frac{1}{2}\sqrt{-3x^2 - 12} \therefore \rightarrow$ always nega $\therefore$ no real sol except

$\therefore$ only sol $\phi(x) = x$

we can use our Neumann iteration method to solve non linear IE:

$y_n = \frac{f}{\lambda} + K y_{n-1}$

Ex:/ find Z first & 2nd iterates of Z Neumann iteration for Z IE $\quad \phi(x) - \int_0^x (\sqrt{\phi(t)} + t)\,dt = 1 \quad \therefore$

$\phi_n = \frac{f}{\lambda} + K\phi_{n-1} \quad \therefore$

Want ϕ_1, ϕ_2 $\therefore$

$\phi_0 = 1$ $\therefore$

$\phi_1 = 1 + K\phi_0 = \cancel{1}\ 1 + \int_0^x (\sqrt{\phi_0(t)} + t)\,dt =$
$1 + \int_0^x 1 + t\,dt = 1 + x + \frac{1}{2}x^2$ $\therefore$

$\phi_2(x) = 1 + \int_0^x (\sqrt{\phi_1(t)} + t)\,dt = 1 + \int_0^x \sqrt{1 + t + \frac{1}{2}t^2} + t\,dt$

use Sub: $\tau = 1 + t$ $\therefore$ $\frac{d\tau}{dt} = 1$ $\therefore$ $\tau = 1 + t$ $\therefore$

$\sqrt{\tau + \frac{1}{2}(\tau - 1)^2} = \sqrt{\tau + \frac{1}{2}\tau^2 - \tau + \frac{1}{2}}$

$\phi_2(x) = 1 + \frac{1}{2}x^2 + \frac{1}{\sqrt{2}}\underbrace{\int_1^{1+x}(1 + \tau^2)^{1/2}\,d\tau}_{I}$

$\therefore$ want $I = \int_1^{1+x}(1 + \tau^2)^{1/2}\,d\tau$ $\therefore$

Let $\tau = \tan(u)$ $\frac{d\tau}{du} = \sec^2(u)$ $\therefore$ $u = \arctan(\tau)$ $\therefore$

$I = \int_1^{1+x}\sqrt{1 + \tau^2}\,d\tau = \int_{\pi/4}^{\arctan(1+x)}\sqrt{\underbrace{1 + \tan^2(u)}_{\sec^2(u)}}\cdot\sec^2(u)\,du$

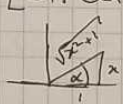
$= \int_{\pi/4}^{\arctan(1+x)}\sec^3(u)\,du$ $\therefore$ IBP: $u = \sec(u)$ & $u' = \sec(u)\tan(u)$ $\therefore$

$v' = \sec^2(u)$ $v = \tan(u)$

$I = \left[\sec(u)\tan(u)\right]_{\pi/2}^{\arctan(1+x)} - \int_{\pi/4}^{\arctan(1+x)}\sec(u)\underbrace{\tan^2(u)}_{\sec^2(u)-1}\,du$ $\therefore$

$I = \left[\sec(u)\tan(u)\right]_{\pi/4}^{\arctan(1+x)} - I + \int_{\pi/4}^{\arctan(x+1)}\sec(u)\,du$ $\therefore$

$2I = \left[\sec(u)\tan(u)\right]_{\pi/4}^{\arctan(1+x)} + \left[\ln|\tan(u) + \sec(u)|\right]_{\pi/4}^{\arctan(1+x)}$

{side note $\sec(\arctan(x)) = \sqrt{x^2+1}$  $\therefore$ $\sec(\arctan(x))$:

$\tan(\alpha) = \frac{x}{1}$ $\arctan(x) = \alpha$ }

$\therefore$ $\sec(\arctan(x))$: $\sec(\alpha) = \frac{1}{\cos(\alpha)}$ $\therefore$ $\cos(\alpha) = \frac{1}{\sqrt{x^2+1}}$ $\therefore$

$\sec(\alpha) = \sqrt{x^2+1}$ $\therefore$

$I = \frac{1}{2}\Big(\sec[\arctan(x+1)](x+1) - \sec(\frac{\pi}{4})\tan(\frac{\pi}{4})\Big) +$
$\ln|(x+1) + \sec[\arctan(x+1)]| - \ln|\tan(\pi/4) + \sec(\pi/4)|\Big)$ $\therefore$

$I = \frac{1}{2}\Big\{\Big((x+1)\sqrt{(x+1)^2+1} - \frac{2}{\sqrt{2}}\Big) + \ln|(x+1) + \sqrt{(x+1)^2+1}| - \ln|1 + \frac{2}{\sqrt{2}}|\Big\}$ $\therefore$

$\phi_2(x) = \frac{1+x^2}{2} + \frac{x+1}{2\sqrt{2}}\sqrt{1 + x + \frac{1}{2}x^2} + \frac{1}{2\sqrt{2}}\ln\left(\frac{\sqrt{x^2+2x+2} + (x+1)}{1+\sqrt{2}}\right)$

Springbreak 25 days

exercises done in notes

- 10 sheets : 2hrs on each day 20 days $\frac{1}{2}$ sheet
- ex done in mentions : 2hrs on day 21

Ex: (in purple font in notes) 2hrs on day 22 - 23

- 2 past papers 2hrs on day 24, 25
- put stars or make note of any difficulties
- last symbol question not examinable

\Sheet week 10/

\3/ Solve $y(x) - \left(\int_0^1 \frac{1}{2} y(t) + \lambda x y^2(t) dt\right) = 0$

$P_1 = \int_0^1 y(t) dt$, $P_2 = \int_0^1 y^2(t) dt$

$y(x) = \frac{1}{2} P_1 + \lambda x P_2$ (*) $\therefore$ $y(t) = \frac{1}{2} P_1 + \lambda t P_2$

$P_1 = \int_0^1 \frac{1}{2} P_1 - \lambda t P_2 dt = \frac{1}{2} P_1 [t]_0^1 - \lambda P_2 [\frac{1}{2} t^2]_0^1 =$

$\frac{1}{2} P_1 + \frac{1}{2} \lambda P_2 = P_1$ $\therefore$ $P_1 = \lambda P_2$ (1)

$P_2 = \int_0^1 (\frac{1}{2} P_1 + \lambda t P_2)^2 dt$ $\therefore$ expand & underin (integr)

$P_2 = \int_0^1 \frac{1}{4} P_1^2 t + P_1 \lambda t P_2 + \lambda^2 t^2 P_2^2 \, dt =$

$\frac{1}{4} P_1^2 + \frac{1}{2} \lambda P_1 P_2 + \frac{1}{3} \lambda P_2^2 = P_2$ (2) $\therefore$

$P_1 = 0$ & $P_0 = 0$ is a sol of (1) & (2) $\therefore$ $y(x) = 0$ (into (*))

now sub $P_1 = \lambda P_2$ into (2)

$P_2 = \frac{1}{4} (\lambda P_2)^2 + \frac{1}{2} \lambda (\lambda P_2) P_2 + \frac{1}{3} \lambda^2 P_2^2$

$P_2 = 0$

$1 = \frac{\lambda^2 P_2}{4} + \frac{\lambda^2 P_2}{2} + \frac{\lambda^2 P_2}{3}$ $\therefore$ $1 = \frac{P_2}{12} (\lambda^3 \cdot 3 + 6 \lambda^2 + 4 \lambda^2)$ $\therefore$

$1 = \frac{13 \lambda^2}{12} P_2$ $\therefore$ $P_2 = \frac{12}{13 \lambda^2}$ $\therefore$ into (1) :

$P_1 = \lambda \left(\frac{12}{13 \lambda^2}\right) = \frac{12}{13 \lambda}$ into (*) our sol:

$y(x) = \frac{1}{2} \left(\frac{12}{13 \lambda}\right) + \lambda \left(\frac{12}{13 \lambda^2}\right)$ $y(x) = \frac{6}{13 \lambda} + \frac{12 x}{13 \lambda}$

note: when $\lambda = 0$ our IE is : $y(x) - \frac{1}{2} \int_0^1 y(t) dt = 0$ $\therefore$ $y(x) = \frac{1}{2} P$ $\therefore$

$y(t) = \frac{1}{2} P$ $\therefore$ $P = \int_0^1 \frac{1}{2} P dt$, $\therefore$ $P = \frac{1}{2} P [t]_0^1$ $\therefore$ $P = \frac{1}{2} P$ $\therefore$ $P = 0$ $\therefore$ $y(x) = 0$

Sheet week 10 / Q5 / $\phi(x) - \int_{-1}^{1} (x+t)^2 \phi^2(t)\,dt = 0 \quad \therefore$

$$\phi(x) - \int_{-1}^{1} x^2 \phi^2(t)\,dt - 2x\int t\phi^2(t)\,dt - \int_{-1}^{1} t^2 \phi^2(t)\,dt = 0 \quad \therefore$$

$$\phi(x) - x^2 \underbrace{\int_{-1}^{1} \phi^2(t)\,dt}_{P_1} - 2x \underbrace{\int_{-1}^{1} t\phi^2(t)\,dt}_{P_2} - \underbrace{\int_{-1}^{1} t^2 \phi^2(t)\,dt}_{P_2} \cancel{\phi} = 0$$

$\therefore \phi(x) = x^2 P_1 + 2x P_2 + P_3 \quad \therefore \phi(t) = t^2 P_1 + 2t P_2 + P_3 \quad \therefore$

$$P_1 : \int_{-1}^{1} \phi^2(t)\,dt = \int_{-1}^{1} (\underbrace{t^2 P_1}_{A} + \underbrace{2tP_2 + P_3}_{B})^2\,dt =$$

$$P_1 = \int_{-1}^{1} t^4 P_1^2\,dt + 2\int_{-1}^{1} (t^2 P_1)(2t P_2 + P_3)\,dt + \int_{-1}^{1} 4t^2 P_2^2 + 2(2tP_2 + P_3) + P_3\,dt$$

$\therefore P_1 = \frac{2}{5} P_1^2 + \frac{2}{3} P_1 P_3 + \frac{8}{3} P_2^2 + 2P_3^2$ ① {can see $P_1 = P_2 = P_3 = 0$ is sol}

$$P_2 = \int_{-1}^{1} t\,\phi^2(t)\,dt = \int_{-1}^{1} t(t^2 P_1 + 2tP_2 + P_3)^2\,dt =$$

$$P_2 = \int_{-1}^{1} t^5 P_1^2\,dt + 2\int_{-1}^{1} t^3 P_1 (2tP_2 + P_3)\,dt + \int_{-1}^{1} (4t^3 P_2^2 + \cancel{2}t(2tP_2 + P_3) + P_3 t)\,dt$$

$P_2 = \frac{8}{5} P_1 P_2 + \frac{8}{3} P_2 P_3$ ② {again $P_1 = P_2 = P_3 = 0$ is sol}

now: $P_3 = \int_{-1}^{1} t^2 \phi^2(t)\,dt = \cancel{\int_{-1}^{1} t^6 P_1^2}$

$$\int_{-1}^{1} t^6 P_1^2\,dt + 2\int_{-1}^{1} t^4 P_1 (2tP_2 + P_3)\,dt + \int_{-1}^{1} \left[4t^4 P_2^2 + 2t^2(2tP_2 + P_3) + P_3 t^2\right] dt \quad \therefore$$

$P_3 = \frac{2}{7} P_1^2 + \frac{8}{5} P_2^2 + \frac{2}{3} P_3^2 + \frac{4}{5} P_1 P_3$ ③ $\therefore$ Solve by sub then into ϕ:

$\phi(x) = x^2 P_1 + 2x P_2 + P_3 \quad \therefore$ Solving: take $P_2 = 0$ from ② $\therefore$

$\to P_1 = P_3 = 0$

$P_2 = \frac{8}{5} P_1 P_2 + \frac{8}{3} P_2 P_3$

$P_2\left(1 - \frac{8}{5} P_1 - \frac{8}{3} P_3\right) = 0 \quad \therefore 1 - \frac{8}{5} P_1 - \frac{8}{3} P_3 = 0$

$\frac{8}{5} P_1 = 1 - \frac{8}{3} P_3 \quad \therefore$

$P_1 = \frac{5}{8}\left(1 - \frac{8}{3} P_3\right)$ into every P_1 of ① & ③ have two eqns: P_2, P_3

$\therefore$ solve: 2 approx sols: 5 sets of sols: $(0, 0, 0)$,

$(P_1, P_2, P_3) = (0.305727, -0.241772, 0.191564)$,

$(0.305727, 0.241772, 0.191564)$,

$(3.8104, -1.29149i, -1.91124)$, $(3.8104, 1.29149i, -1.91124)$

recap: Chap 1: Laplace transfers

- find $\mathcal{L}(F(t))$ (and piecewise)
- find $\mathcal{L}^{-1}(f(s))$ • Convolution thm (LT22)
- heaviside func • Solving ODEs using Laplace.

Chap 2: - how to classify IE

- classify kernels (must know if weakly singular
- Methods of solve IE fredhom (by P method),

volterra IE (by differentiation or Laplace)

if differentiation: $\frac{\partial}{\partial x}\int_{g(x)}^{f(x)} h(t)\,dt$ or $\frac{\partial}{\partial x}\int_{g(x)}^{f(x)} h(x,t)\,dt$

$\int h(x)\cdot f(t)\,dt$

$A(x)\int_{g(x)}^{f(x)} f(t)\,dt \therefore$ deriv: $A(x)\cdot\frac{\partial}{\partial x}\int_{f(x)} f(t)\,dt + A'(x)\int_{g(x)}^{f(x)} f(t)\,dt$ $= A(x)\int_{g(x)}^{f(x)} f(t)\,dt$

But $\cancel{\frac{\partial}{\partial x}\int_{g(x)}^{f(x)} k(x,t)\,dt}$ $\frac{\partial}{\partial x}\int_{g(x)}^{f(x)} k(x,t)\,y(t) = k(x,x)\,y(x) + \int \frac{\partial k(x,t)}{\partial x}\,y(t)\,dt$

Question. . . . .
Write on both sides of the paper

Do not write in either margin

(Q1) $\frac{d^2}{dt^2}x(t) - 2\frac{d}{dt}x(t) + 2x(t) = 2e^t \quad \therefore$

taking Laplacian: $\therefore$ Let $\mathcal{L}(x(t)) = \hat{x}(s) \therefore$

$\mathcal{L}\left(\frac{d^2}{dt^2}x(t) - 2\frac{d}{dt}x(t) + 2x(t)\right) = \mathcal{L}(2e^t) = 2\mathcal{L}(e^t) =$

$2\mathcal{L}(e^{(1)t}) = 2\frac{1}{s-1} = \mathcal{L}\left(\frac{d^2}{dt^2}x(t)\right) - 2\mathcal{L}\left(\frac{d}{dt}x(t)\right) + 2\mathcal{L}(x(t))$

$= s^2\hat{x}(s) - sx(0) - \dot{x}(0) - 2s\hat{x}(s) - 2x(0) + 2\hat{x}(s) =$

$s^2\hat{x}(s) - s(0) - 1 - 2s\hat{x}(s) - 2(0) + 2\hat{x}(s) =$

$s^2\hat{x}(s) - 1 - 2s\hat{x}(s) + 2\hat{x}(s) = \hat{x}(s)\left[s^2 - 2s + 2\right] - 1 =$

$2\frac{1}{s-1} \therefore \quad \hat{x}(s)\left[s^2 - 2s + 2\right] = 2\frac{1}{s-1} + 1 \quad \therefore$

~~$\hat{x}(s)$~~ $\hat{x}(s) = 2\frac{1}{(s^2-2s+2)(s-1)} + \frac{1}{(s^2-2s+2)}$

(Q1 SA) $\ddot{x}(t) - 2\dot{x}(t) + 2x(t) = 2e^t \qquad x(0) = 0,\ \dot{x}(0) = 1$

using LT10, LT9, take Laplace transform of each term in our ODE: $\quad \mathcal{L}(x) = x(s)$

$s^2x(s) - 1 - 2sx(s) + 2\underbrace{\mathcal{L}(x)}_{x(s)} = \frac{2}{s-1}$ ← LT4

$x(s) = \frac{s-1+2}{s-1} \cdot \frac{1}{s^2-2s+2} \qquad x(s) = \frac{s+1}{(s-1)(s^2-2s+2)} \quad \therefore$

partial fractions: $\frac{s+1}{(s-1)(s^2-2s+2)} = \frac{A}{s-1} + \frac{Bs+C}{s^2-2s+2}$

$(s+1) = A(s^2-2s+2) + (Bs+C)(s-1) \quad \therefore$

$s + 1 = As^2 - 2As + 2A + Bs^2 - Bs + Cs + C \quad \therefore 0 = A + B,\ A = -B \therefore$

$1 = -2A - B + C \quad \therefore 1 = 2A - C \therefore C = 2A - 1 \therefore 1 = 2A - 1 - A \therefore A = 2$

$\therefore B = -2,\ C = 3 \quad \therefore$

$\frac{2}{s-1} + \frac{3-2s}{s^2-2s+2}$ (3 mark) $\quad \therefore \frac{2}{s-1} + \frac{3-2s}{(s-1)^2+1} =$

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Question. . . . .
Write on both sides of the paper

2 Marks

$$\frac{2}{s-1} + \frac{3-2(s-1)-2}{(s-1)^2+1} = \frac{2}{s-1} + \frac{1}{(s-1)^2+1} - \frac{2(s-1)}{(s-1)^2+1}$$

$x(t) = 2e^t + e^t \sin(t) - 2e^t \cos(t)$ 2 Marks

Question. . . . .
Write on both sides of the paper

Do not write in either margin

(Q2)

$\mathcal{L}(t) = \mathcal{L}(t^1) = \frac{1!}{s^{1+1}} = \frac{1}{s^2}$

$\mathcal{L}(8-3t) = \cancel{\mathcal{L}} 8\mathcal{L}(1) - 3\mathcal{L}(t) = 8\frac{1}{s} - 3\frac{1}{s^2}$

$\mathcal{L}(t-4) = \mathcal{L}(t) - 4\mathcal{L}(1) = \frac{1}{s^2} - 4\frac{1}{s}$

$\mathcal{L}(0) = 0$ ∴

$\mathcal{L}(F(t)) = f(s) = \begin{cases} 1/s^2 & t<2 \\ 8\frac{1}{s} - 3\frac{1}{s^2} & 2 \le t < 3 \\ \frac{1}{s^2} - 4\frac{1}{s} & 3 \le t < 4 \\ 0 & 4 \le t \end{cases}$

(Q2 Sol:) $F(t) = \begin{cases} t & t<2 \\ 8-3t & 2 \le t < 3 \\ t-4 & 3 \le t < 4 \\ 0 & 4 \le t \end{cases}$

{is forget Laplace definition, look at LT|4 ∴

$\mathcal{L}(F(t)) = \int_0^T e^{-st} F(t)\,dt$ } ∴

$\mathcal{L}(F(t)) = \int_0^2 t e^{-st}\,dt + \int_2^3 (8-3t)e^{-st}\,dt + \int_3^4 (t-4)e^{-st}\,dt$

$= \frac{1}{s^2} + e^{-2s}\left(-\frac{2}{3} - \frac{1}{s^2}\right) + \frac{1}{s}e^{-3s} + \frac{2}{s}e^{-2s} + \frac{3}{s^2}e^{-3s} -$

$\frac{3}{s^2}e^{-2s} - \frac{1}{s}e^{-3s} - \frac{1}{s^2}e^{-4s} + \frac{1}{s^2}e^{-3s}$

$= \frac{1}{s^2}\left(1 + 4e^{-3s} - 4e^{-2s} - e^{-4s}\right)$

but using haviside function:

$F(t) = t(H(t) - H(t-2)) + (8-3t)(H(t-2) - H(t-3)) +$

$(t-4)(H(t-3) - H(t-4)) =$

$t\,H(t) - 4(t-2)H(t-2) + 4(t-3)H(t-3) - (t-4)H(t-4)$

∴ LT17 with $a = 0, 2, 3, 4$ ∴

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Question. . . . .
Write on both sides of the paper

$\mathcal{L}(F(t)) = \frac{1}{s^2}\left(1 - 4e^{-2s} + 4e^{-3s} - e^{-4s}\right)$

eg $F = \begin{cases} 0 & 0 \le t < 3 \\ 3t^2 & t \ge 3 \end{cases}$:

$F = 3t^2(H(\cancel{3}t - 3))$

$\cdot \int_3^\infty 3te^{-st}\,dt$

(Q3a) $(\lambda I - K)y = f(x)$ $\quad f \in C([0, 2\pi])$ ∴
$(Ky)(x) = \int_0^{2\pi} \cos(x-t)y(t)\,dt$ obtain values of $\lambda \neq 0$ for which the IE has a sol

$\lambda y(x) - \int_0^{2\pi} \cos(x-t)y(t)\,dt = f(x)$

{is Fredholm: P-Method.
Volterra: Laplace or differentiation}
{differentiation method also used for solving 1st kind IE}

$\lambda y(x) - \cos(x)\underbrace{\int_0^{2\pi}\cos(t)y(t)\,dt}_{P_1} - \sin(x)\underbrace{\int_0^{2\pi}\sin(t)y(t)\,dt}_{P_2} = f(x)$ ∴

$\lambda y(x) = f(x) + \cos(x)P_1 + \sin(x)P_2$ (2marks)

$P_1 = \frac{1}{\lambda}\int_0^{2\pi} \cos(t)(f(t) + \cos(t)P_1 + \sin(t)P_2)\,dt$

$P_1 = \frac{1}{\lambda}\int_0^{2\pi}\cos(t)f(t)\,dt + P_1\frac{1}{\lambda}\int_0^{2\pi}\cos^2(t)\,dt + P_2\frac{1}{\lambda}\int_0^{2\pi}\underbrace{\cos(t)\sin(t)}_{\frac{1}{2}\sin(2t)}\,dt$ $\quad \cos(2\theta) = 2\cos^2\theta - 1$

$P_1 = \frac{1}{\lambda}\int_0^{2\pi}\cos(t)f(t)\,dt + \frac{1}{\lambda}P_1\int_0^{2\pi}\frac{1}{2} + \frac{1}{2}\cos(2t)\,dt + P_2\frac{1}{\lambda}\left[-\frac{1}{4}\cos(2t)\right]_0^{2\pi}$

$P_1 = \frac{1}{\lambda}\int_0^{2\pi}\cos(t)f(t)\,dt + P_1\frac{1}{\lambda}\left[\frac{1}{2}t + \frac{1}{4}\sin(2t)\right]_0^{2\pi} + P_2\underbrace{\left(-\frac{1}{4} + \frac{1}{4}\right)}_{=0}\frac{1}{\lambda}$

$P_1\left(1 - \frac{\pi}{\lambda}\right) = \frac{1}{\lambda}\int_0^{2\pi}\cos(t)f(t)\,dt$ $\qquad P_1 = \frac{1}{\lambda}\left(\pi P_1 + \int_0^{2\pi}\cos(t)f(t)\,dt\right)$

$P_1 = \frac{1}{\lambda - \pi}\int_0^{2\pi}\cos(t)f(t)\,dt$

Question. . . . .
Write on both sides of the paper

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For P_2:

$$P_2 = \frac{1}{\lambda}\left(\int_0^{2\pi} f(t)\sin(t)\,dt + P_1\int_0^{2\pi}\frac{1}{2}\sin(2t)\,dt + P_2\int_0^{2\pi}\underbrace{\sin^2(t)}_{\frac{1}{2}-\frac{1}{2}\cos(2t)}\,dt\right) \therefore$$

$$P_2 = \frac{1}{\lambda}\left(\int_0^{2\pi} f(t)\,dt + P_1\overbrace{\left(-\frac{1}{4}+\frac{1}{4}\right)}^{=0} + P_2\left[\frac{t}{2} - \frac{1}{4}\sin(2t)\right]_0^{2\pi}\right)$$

$$P_2 = \left(1 - \frac{\pi}{\lambda}\right) = \frac{1}{\lambda}\int_0^{2\pi} f(t)\sin(t)\,dt$$

$$P_2 = \frac{1}{\lambda - \pi}\int_0^{2\pi} f(t)\sin(t)\,dt$$ (2 marks)

$\therefore \lambda \neq \pi$ for a sol to exist (1 mark)

(Q3b) Sol is $y(x) = \frac{1}{\lambda}\left(f(x) + P_1\cos(x) + P_2\sin(x)\right)$

$$= \frac{1}{\lambda}\left(f(x) + \frac{1}{\lambda - \pi}\int_0^{2\pi}\cos(x-t)f(t)\,dt\right) \quad \lambda \neq \pi$$

(Q3c) thing that sits in front of f is resolvent kernel

$$r(x,t) = \frac{1}{\lambda - \pi}\cos(x-t)$$

Question. . . . .
Write on both sides of the paper

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(Q 3a)

$(\lambda I - K)y = \delta(x) = \lambda y(x) - \int_0^{2\pi} \cos(x-t)y(t)dt = \delta(x)$

$\therefore$ for $x=t$: $\lambda y(t) - \int_0^{2\pi} \cos(t-t)y(t)dt = \delta(t) =$

$\lambda y(t) - \int_0^{2\pi} \cos(0)y(t)dt = \lambda y(t) - \int_0^{2\pi} y(t)dt = \delta(t)$

taking derivative: $\frac{d}{dx}\left[\lambda y(x) - \int_0^{2\pi} \cos(x-t)y(t)dt\right] = \frac{d}{dx}(\delta(x)) =$

$\frac{d}{dx}(\lambda y(x)) + \frac{d}{dx}\left[-\int_0^{2\pi}(\cos x\cos t + \sin x \sin t)\, y(t)dt\right] = \delta'(x) =$

$\lambda y'(x) + \frac{d}{dx}\left[-\cos x \int_0^{2\pi} \cos(t)y(t)dt\right] + \frac{d}{dx}\left[-\sin x \int_0^{2\pi} \sin(t)y(t)dt\right]$

$\lambda y'(x) + \sin x \int_0^{2\pi} \cos(t)y(t)dt - \cos x \int_0^{2\pi} \sin(t)y(t)dt = \delta'(x) \therefore$

Let $P_1 = \int_0^{2\pi} \cos(t)y(t)dt$, $P_2 = \int_0^{2\pi} \sin(t)y(t)dt \therefore$

$\lambda y'(x) + \sin(x)P_1 - \cos(x)P_2 = \delta'(x) \therefore$

$\lambda y'(t) + \sin(t)P_1 - \cos(t)P_2 = \delta'(t) \therefore$

$y'(t) = \frac{1}{\lambda}\delta'(t) - \frac{1}{\lambda}\sin(t)P_1 + \frac{1}{\lambda}\cos(t)P_2$,

$\lambda y(x) - \cos x \int_0^{2\pi} \cos(t)y(t)dt - \sin x \int_0^{2\pi} \sin(t)y(t)dt = \delta(x) =$

$\lambda y(x) - \cos(x)P_1 - \sin(x)P_2 = \delta(x) \therefore$

$\lambda y(t) = \delta(t) + \cos(t)P_1 + \sin(t)P_2 \therefore$

$y(t) = \frac{1}{\lambda}\delta(t) + \frac{1}{\lambda}\cos(t)P_1 + \frac{1}{\lambda}\sin(t)P_2$

taking derivative:

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Question. . . . .
Write on both sides of the paper

(Q 3b)

$f''(x) = \lambda y''(x) + \cos x \int_0^{2\pi} \cos(t) y(t)\,dt + \sin x \int_0^{2\pi} \sin(t) y(t)\,dt =$

$\lambda y''(x) + \int_0^{2\pi} (\cos x \cos t + \sin x \sin t) y(t)\,dt =$

$\lambda y''(x) + \int_0^{2\pi} \cos(x-t)\, y(t)\,dt =$

$\lambda y''(x) + \lambda y(x) - f(x) = f''(x) \quad \therefore$

$y''(x) + y(x) = \frac{1}{\lambda} f''(x) + \frac{1}{\lambda} f(x)$

(Q3c) resolvent kernel is k^*

MTH3042 Mini Mock 2021-2022, duration: 40 minutes

Table of Laplace Transform:

Name	$F(t)$ $n \in \mathbb{N}, a, \omega \in \mathbb{R}$	$f(s) = \mathscr{L}(F(t))$
LT1	1	$\frac{1}{s}$
LT2	t^n	$\frac{n!}{s^{n+1}}$
LT3	$t^a \ (a \geq -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$
LT4	e^{at}	$\frac{1}{s-a}$
LT5	$\sin(\omega t)$	$\frac{\omega}{s^2+\omega^2}$
LT6	$\cos(\omega t)$	$\frac{s}{s^2+\omega^2}$
LT7	$\alpha F + \beta G$	$\alpha f(s) + \beta g(s)$
LT8	$F(at)$, $a > 0$	$\frac{1}{a} f\left(\frac{s}{a}\right)$
LT9	$F'(t)$	$sf(s) - F(0^+)$
LT10	$F''(t)$	$s^2 f(s) - sF(0^+) - F'(0^+)$
LT11	$F^{(n)}(t)$	$s^n f(s) - s^{(n-1)} F(0^+) - \ldots - F^{(n-1)}(0^+)$
LT12	$t^n F(t)$	$(-1)^n f^{(n)}(s)$
LT13	$e^{at} F(t)$	$f(s-a)$
LT14	$F(t+T) = F(t)$	$\frac{1}{1-e^{-sT}} \int_0^T e^{-st} F(t) dt$
LT15	$H(t-a)$	$\frac{e^{-as}}{s}$
LT16	$F(t)H(t-a)$	$e^{-as} \mathscr{L}(F(t+a))$
LT17	$F(t-a)H(t-a)$	$e^{-as} f(s)$
LT18	$\delta(t-a)$	e^{-sa}, $a > 0$, $s > 0$
LT19	$\delta(t-a)F(t)$	$F(a)e^{-sa}$, $a > 0$, $s > 0$
LT20	$\int_0^t F(u) du$	$\frac{f(s)}{s}$
LT21	$\int_0^t F(u) du$, $a > 0$	$\frac{1}{s}\left(f(s) - \int_0^a F(t) dt\right)$
LT22	$\int_0^t F(u) G(t-u) du$	$f(s) g(s)$

/continued ...

Question. . . . .
Write on both sides of the paper

Do not write in either margin

$$\frac{d}{dx}\left[\lambda y'(x)+\sin x\int_0^{2\pi}\cos(t)\,y(t)\,dt-\cos x\int_0^{2\pi}\sin(t)\,y(t)\,dt\right]=\frac{d}{dx}\left(\delta'(x)\right)=$$

$$=\lambda y''(x)+\frac{d}{dx}\left[\sin x\int_0^{2\pi}\cos(t)\,y(t)\,dt\right]+\frac{d}{dx}\left[-\cos x\int_0^{2\pi}\sin(t)\,y(t)\,dt\right]$$

$$=\delta''(x)=\lambda y''(x)+\cos x\int_0^{2\pi}\cos(t)\,y(t)\,dt+\sin(x)\int_0^{2\pi}\sin(t)\;y(t)\,dt\ \therefore$$

Sheet 1 / 1 / $f(s) = \mathcal{L}(F(t)) = \int_0^\infty e^{-st} F(t)\,dt = \mathcal{L}(F)(s)$ ∴

for $F(t) = t^2$: $\mathcal{L}(F(t)) = \int_0^\infty e^{-st} t^2\,dt = \left[-t^2 \frac{1}{s} e^{-st}\right]_0^\infty - \int_0^\infty -\frac{2t}{s} e^{-st}\,dt$

$= \left[-t^2 \frac{1}{s} e^{-st}\right]_0^\infty + \int_0^\infty \frac{2t}{s} e^{-st}\,dt = \left[-t^2 \frac{1}{s} e^{-st}\right]_0^\infty + \left[-2t \frac{1}{s^2} e^{-st}\right]_0^\infty - \int_0^\infty -2\frac{1}{s^2} e^{-st}\,dt$

$= \left[\left(-t^2 - \frac{2t}{s}\right) \frac{1}{s} e^{-st}\right]_0^\infty + \left[-\frac{1}{s^3} e^{-st}\right]_0^\infty$

2 i / $\mathcal{L}(F(at)) = \frac{1}{a} f\left(\frac{s}{a}\right)$ ∴ $\mathcal{L}(\sinh(at)) = \frac{1}{a} f\left(\frac{s}{a}\right)$

$f(s) = \mathcal{L}(\sinh(t))$. $\sinh(t) = \frac{e^t}{2} - \frac{e^{-t}}{2}$ ∴

$\mathcal{L}(e^{at}) = \frac{1}{s-a}$ ∴ $\mathcal{L}(e^t) = \frac{1}{s-1}$ $\mathcal{L}(e^{-t}) = \frac{1}{s+1}$ ∴

$\mathcal{L}(\sinh(t)) = \mathcal{L}\left(\frac{e^t}{2} - \frac{e^{-t}}{2}\right) = \mathcal{L}\left(\frac{e^t}{2}\right) + \mathcal{L}\left(-\frac{1}{2}e^{-t}\right) = \frac{1}{2}\mathcal{L}(e^t) - \frac{1}{2}\mathcal{L}(e^{-t}) =$

$\frac{1}{2}\frac{1}{s-1} - \frac{1}{2}\frac{1}{s+1} = f(s)$ ∴ $\mathcal{L}(at) = \frac{1}{2}\frac{1}{\frac{s}{a}-1} - \frac{1}{2}\frac{1}{\frac{s}{a}+1} =$

$\frac{1}{2}\frac{a}{s-a} - \frac{1}{2}\frac{a}{s+a} = \frac{1}{2}\frac{a(s+a)}{s^2-a^2} - \frac{1}{2}\frac{a(s-a)}{s^2-a^2} = \frac{1}{2}\frac{as+a^2-as+a^2}{s^2-a^2} = \frac{a^2}{s^2-a^2}$ ✗ $\frac{a}{s^2-a^2}$

2 ii / $\cosh(at) = \frac{e^{at}+e^{-at}}{2}$ $\mathcal{L}(\cosh(at)) = \frac{1}{2}\mathcal{L}(e^{at}) + \frac{1}{2}\mathcal{L}(e^{-at}) =$

$\frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a} = \frac{1}{2}\frac{s+a}{s^2-a^2} + \frac{1}{2}\frac{s-a}{s^2-a^2} = \frac{1}{2}\frac{s+a+s-a}{s^2-a^2} = \frac{s}{s^2-a^2}$

3 c ii / $F(t) = t^2\cos(t)$ $F'(t) = 2t\cos(t) - t^2\sin(t)$

$F''(t) = 2\cos(t) - 2t\sin(t) - t^2\cos(t) - 2t\sin(t) =$

$2\cos(t) - t^2\cos(t) - 4t\sin(t)$

$\mathcal{L}(RHS) = \mathcal{L}(LHS)$ {we use LT10: $\mathcal{L}(F''(t)) = s^2 f(s) - sF(0) - F'(0)$

$s^2 f(s) - 0 - 0 = \mathcal{L}(2\cos(t)) - \underbrace{\mathcal{L}(t^2\cos(t))}_{f(s)} - 4\underbrace{\mathcal{L}(t\sin(t))}_{*}$

{* similarly to $\mathcal{L}(t\sin(2t))$ using $\mathcal{L}(F'')$}

$= \frac{2\mathcal{L}(\cos(t))}{s^2+1}$ $s^2 f(s) = \overbrace{\frac{2s}{s^2+1}}^{LT6} - f(s) - \frac{8\mathcal{L}(\cos(t))}{s^2+1}$ LT6

$s^2 f(s) + f(s) = \frac{2s}{s^2+1} - \frac{8s}{(s^2+1)(s^2+1)}$ $\cancel{f(s)(s^2+1) = \frac{-6s}{s^2+1}}$ $\cancel{f(t)}$

$f(s)(s^2+1) = \frac{2s(s^2+1) - 8s}{(s^2+1)^2}$ $f(s) = \frac{2s^3-6s}{(s^2+1)^3}$

alternative method /

$F(t) = \underbrace{t^2}_{t^n}\underbrace{\cos(t)}_{G(t)}$ {LT12: $\mathcal{L}(t^n G(t)) = (-1)^n g^{(n)}(s)$} ∴ $n=2$,

$G(t) = \cos(t)$ $\overset{\therefore}{LT6}$ $g(s) = \frac{s}{s^2+1}$ ∴ $\mathcal{L}(F(t)) = \mathcal{L}(t^2\cos(t)) = (-1)^2\left(\frac{s}{s^2+1}\right)'' =$

$$\left(\frac{(s^2+1)1-s(2s)}{(s^2+1)^2}\right)' = \left(\frac{1-s^2}{(s^2+1)^2}\right)' = \frac{-2s(s^2+1)^2-(1-s^2)2(s^2+1)(2s)}{(s^2+1)^3}$$

$$= \frac{-2s^3-2s-4s+4s^3}{(s^2+1)^3} = \frac{2s^3-6s}{(s^2+1)^3}$$

\3iii/ $F(t) = \cos^3(t)$ $\therefore$ $F'(t) = -3\cos^2(t)\sin(t) =$

$-3\sin(t)(1-\sin^2(t)) = -3\sin(t) + 3\sin^3(t)$ $\therefore$

{LT9: $\mathcal{L}(F'(t)) = sf(s) - F(0)$ $f(s) = \mathcal{L}(\cos^3(t))$}

$sf(s) - 1 = -3\mathcal{L}(\sin(t)) + 3\mathcal{L}(\sin^3(t))$ ☺

{ $\mathcal{L}(\sin^3(t)) = \sin^3(t) = \sin(t)(1-\cos^2(t)) = \sin(t) - \sin(t)\cos^2(t)$ }

Let $G(t) = \sin^3(t)$ $\therefore$ $\underbrace{G'(t)}_{\text{use LT9}} = 3\sin^2(t)\cos(t) = 3\cos(t)(1-\cos^2(t)) =$

$3\cos(t) - 3\cos^3(t)$

$\therefore$ ~~stg~~ $sg(s) - G(0) = \frac{3s}{s^2+1} - 3f(s)$ {by LT of both sides}

$\therefore$ $g(s) = \frac{3s}{s(s^2+1)} - \frac{3f(s)}{3} = \frac{3}{s^2+1}$ ~~+~~ $- 3\frac{f(s)}{s}$

Backinto ☺ $sf(s) - 1 = -\frac{3}{s^2+1} + \frac{9}{s^2+1} - \frac{9f(s)}{s}$ $\therefore$

$f(s)\left(s+\frac{9}{s}\right) = \frac{6}{s^2+1} + 1$ $\therefore$ $f(s) = \frac{s^2+1+6}{s^2+1}$ $\therefore$

$$f(s) = \frac{(s^2+7)s}{(s^2+1)(s^2+9)}$$

\4/ $\frac{d}{dt}\cosh(3t) = 3\sinh(3t)$

$\mathcal{L}(\sinh(at)) = \frac{a}{s^2-a^2}$ $\mathcal{L}(\cosh(at)) = \frac{s}{s^2-a^2}$ using $\sinh(x) = \frac{e^x-e^{-x}}{2}$ $\cosh(x) = \frac{e^x+e^{-x}}{2}$

$\therefore \mathcal{L}(\underbrace{\tfrac{d}{dt}\cosh(3t)}_{F'(t)}) = \mathcal{L}(3\sinh(3t))$ $\therefore$ using LT9: $s\mathcal{L}(\cosh(3t)) - \cosh(3\cdot 0) =$

$3\left(\frac{3}{s^2-9}\right) \leftarrow$ {by $\mathcal{L}(\sinh(at)) = \frac{a}{s^2-a^2}$}

$= s\cdot\frac{s}{s^2-9} - 1 = \frac{9}{s^2-9}$ $\therefore$ $\frac{s^2-s^2+9}{s^2-9} = \frac{9}{s^2-9}$

\Ex/ $F(t) = te^{2t}\sin(3t)$ $\mathcal{L}(F(t)) = ?$

$\mathcal{L}(F(t)) = \mathcal{L}(te^{2t}\sin(3t)) = f(s)$ $\therefore$ $\frac{d}{dt}F(t) = \frac{d}{dt}(te^{2t}\sin(3t)) =$

$e^{2t}\sin(3t) + 2te^{2t}\sin(3t) + 3te^{2t}\cos(3t)$

is $G(t) = e^{2t}\sin(3t)$ is $H(t) = \sin(3t)$ $\therefore$ $e^{2t}\sin(3t) = e^{2t}H(t)$ $\therefore$

$\mathcal{L}(\sin(3t)) = \frac{3}{s^2+3^2} = \frac{3}{s^2+9}$ $\therefore$ $\mathcal{L}(e^{2t}\sin(3t)) = \frac{3}{(s-2)^2-9}$ $\therefore$

$\frac{6(s-2)}{((s-2)^2+9)^2}$ $\therefore \mathcal{L}(te^{2t}\sin(3t)) = (-1)^1 g^{(1)}(s)$

$\therefore \frac{d}{ds}\left(\frac{3}{(s-2)^2+9}\right)(-1) = \mathcal{L}(te^{2t}\sin(3t))$

\Sol/ $F(t) = te^{2t}\sin(3t)$ $\therefore t^nG(t)$

LT12 $\mathcal{L}(t^nG(t)) = (-1)^n g^{(n)}(s)$ $\quad n=1 \quad \mathcal{L}(F(t)) = (-1)^1 g'(s)$

$t^nG(t) = t^1e^{2t}\sin(3t)$ · find $\mathcal{L}(e^{2t}\sin(3t))$ · dissertiate it

· $\mathcal{L}(e^{2t}\sin(3t))$ {LT13 a=2 (first shift thm)

$= \frac{3}{s^2+9}\Big|$ replace s with (s-2 $\quad = \frac{3}{(s-2)^2+9}$ $\quad$ dissertiate : $\frac{0-3(2(s-2)^1)}{((s-2)^2+9)^2}$

$\mathcal{L}(F(t)) = \frac{6(s-2)}{((s-2)^2+9)^2}$

\in 3v/ find $\sin^2(t)$ can use the methods

\3ii/ $G(t) = e^{2t}$ $\therefore \mathcal{L}(G(t)) = \frac{1}{s-2} = \mathcal{L}(e^{2t})$

$\mathcal{L}(t^2e^{2t}) = (-1)^2 \mathcal{L}^{(2)}(s) = (+1)\mathcal{L}^{(2)}(e^{2t}) = \left(\frac{1}{s-2}\right)'' = \left(\frac{(s-2)(0)-1(1)}{(s-2)^2}\right)' = \left(\frac{-1}{(s-2)^2}\right)' =$

$= \left((s-2)^2(0)+1(2)(s-2)(1)\right)/(s-2)^4 = \frac{2}{(s-2)^3}$

\3ii/ $\mathcal{L}(t^2\cos(t)) = (-1)^2 \mathcal{L}^{(2)}(\cos(t)) = \left(\frac{s}{s^2+1}\right)'' = \left(\frac{(s^2+1)1-s(2s)}{(s^2+1)^2}\right)' =$

$\left(\frac{s^2+1-2s^2}{(s^2+1)^2}\right)' = \left(\frac{1-s^2}{(s^2+1)^2}\right)' = \left((s^2+1)^2(-2s)-(1-s^2)2(s^2+1)2s\right)/(s^2+1)^2$ ↯

↯ \3iii/ $\mathcal{L}(\cos^3(t)) = \mathcal{L}((\cos(t))^3) = s^2\mathcal{L}(\cos(t)) - s\cos(0) - \frac{d}{dt}\cos(t)\Big|_{t=0}$

$= s^2 \cdot \frac{s}{s^2+1} - s(1) + \sin(t)\Big|_{t=0} = \frac{s^3}{s^2+1} - s + 0 = \frac{s^3}{s^2+1} - \frac{s^3+s}{s^2+1} = \frac{-s}{s^2+1}$ ✗

\3iii/ $\cos^3(t) = \cos(t)(1-\sin^2(t)) =$

$F(t) = \cos^3(t)$ $\therefore F'(t) = \frac{d}{dt}\cos^3(t) = -3\cos^2(t)(\sin(t)) = -3\sin(t)(1-\sin^2(t)) =$

~~3sin(t)~~ $-3\sin(t)+3\sin^2(t)$ $\therefore \mathcal{L}(F'(t)) = sf(s) - F(0^+)$

$f(s) = \mathcal{L}(\cos^3(t))$ $\therefore G(t) = \sin^3(t)$ $\therefore G'(t) = 3\sin^2(t)\cos(t) =$

$3\cos(t)(1-\cos^2(t)) = 3\cos(t) - 3\cos^3(t)$

$\mathcal{L}(G'(t)) = s\,g(s) - G(0) = s\,\mathcal{L}(G(s)) - \sin^3(0) = s\mathcal{L}(\sin^3(t)) - 0 = s\mathcal{L}(\sin^3(t))$

$g(s) = \mathcal{L}(G(t))$

$F(t) = \cos^3(t)$ $\therefore F'(t) = -3\cos^2(t)\sin(t) = -3\sin(t)(1-\sin^2(t)) =$

$-3\sin(t)+3\sin^3(t)$ $\therefore \mathcal{L}(F'(t)) = sf(s) - F(0)$ $\qquad f(s) = \mathcal{L}(\cos^3(t))$

$\mathcal{L}(F'(t)) = \mathcal{L}(-3\sin(t)+3\sin^3(t)) = -3\mathcal{L}(\sin(t)) + 3\mathcal{L}(\sin^3(t)) = sf(s) - \cos^3(0) = sf(s) - 1$

$\therefore \mathcal{L}(\sin^3(t)) = \mathcal{L}(\sin(t)(1-\cos^2(t))) = \mathcal{L}(\sin(t) - \sin(t)\cos^2(t))$

$G(t) = \sin^3(t) \therefore G'(t) = 3\sin^2(t)\cos(t) = 3\cos(t)(1-\cos^2(t)) =$

$3\cos(t) - 3\cos^3(t)$

$\mathcal{L}(G'(t)) = \mathcal{L}(3\cos(t) - 3\cos^3(t)) = s\,g(s) - G(0) = s\,g(s) - \sin^3(0) = s\,g(s) - 0 = s\,g(s)$

\5a✓ $F'(t) = -3\cos^2(t)\sin(t) = -3\sin(t) + 3\sin^3(t)$ $\quad \{\mathcal{L}(F'(t)) = s\,f(s) - F(0)\}$

$\mathcal{L}(F'(t)) = \mathcal{L}((\cos^3(t))') = s\,f(s) - F(0) = s\,f(s) - 1 = s\,\mathcal{L}(\cos^3(t)) - 1 =$

$\mathcal{L}(-3\sin(t) + 3\sin^3(t)) = -3\mathcal{L}(\sin(t)) + 3\mathcal{L}(\sin^3(t))$

$\hat{F}(t) = \sin^3(t) \therefore \hat{F}'(t) = 3\sin^2(t)\cos(t) = 3\cos(t) - 3\cos^3(t) \therefore$

$\mathcal{L}(\sin^3(t)) = \mathcal{L}(\hat{F}(t)) =$

\3 iii✓ $F(t) = \cos^3(t) \therefore F'(t) = -3\cos^2(t)\sin(t) = -3\sin(t) + 3\sin^3(t)$

$\mathcal{L}(F'(t)) = s\,f(s) - F(0) = s\,\mathcal{L}(F(t)) - F(0) = s\,\mathcal{L}(\cos^3(t)) - 1 \therefore$

$s\,\mathcal{L}(\cos^3(t)) - 1 = -3\mathcal{L}(\sin t) + 3\mathcal{L}(\sin^3 t)$ ②

$\hat{F}(t) = \sin^3(t) \therefore \hat{F}'(t) = 3\cos(t) - 3\cos^3(t) \therefore$ $\quad \mathcal{L}((\sin^3(t))') = \mathcal{L}(3\sin^2 t\cos t) =$

$\mathcal{L}(3\cos t - 3\cos^3 t) = 3\mathcal{L}(\cos t) - 3\mathcal{L}(\cos^3 t)$

$\mathcal{L}(\sin^3(t)) = 3\mathcal{L}(\cos t) - 3\mathcal{L}(\cos^3(t))$ ① $\quad = s\,f(s) - \sin^3(0) = s\,\mathcal{L}(\sin^3(t)) \therefore$

$\mathcal{L}(F'(t)) = s\,f(s) \therefore f(s) = \frac{\mathcal{L}(F'(t))}{s} \therefore$ ① into ② : $\mathcal{L}(\sin^3 t) = \frac{3}{s}\mathcal{L}(\cos t) - \frac{3}{s}\mathcal{L}(\cos^3 t)$

$s\,\mathcal{L}(\cos^3(t)) - 1 = -3\mathcal{L}(\sin t) + 3\mathcal{L}(\sin^3(t)) =$

$-3\mathcal{L}(\sin t) + 3\left(\frac{3}{s}\mathcal{L}(\cos t) - \frac{3}{s}\mathcal{L}(\cos^3(t))\right) = -3\mathcal{L}(\sin t) + \frac{9}{s}\mathcal{L}(\cos t) - \frac{9}{s}\mathcal{L}(\cos^3 t)$

$\therefore (s + \frac{9}{s})\mathcal{L}(\cos^3 t) = -3\mathcal{L}(\sin t) + \frac{9}{s}\mathcal{L}(\cos t) + 1 = -\frac{3}{s^2+1} + \frac{9}{s^2+1} + 1 \therefore$

$\mathcal{L}(\cos^3(t)) = \left(\frac{s^2+7}{s^2+1}\right)\left(\frac{s}{s^2+9}\right) = \frac{s(s^2+7)}{(s^2+1)(s^2+9)}$

\3 iv✓ $F(t) = e^t\sin(t)$ $\quad \mathcal{L}(e^t\sin t) = \mathcal{L}(\sin(t-1)) = \frac{1}{(s-1)^2+1}$ ✓

\3 v✓ $F(t) = \sinh^2(t)$ $\quad F'(t) = 2\sinh(t)\cosh(t)$

$\cosh(2t) = 1 - 2\sinh^2(t) \therefore \sinh^2(t) = -\frac{1}{2} + \frac{1}{2}\cosh(2t) \therefore$

$\mathcal{L}(\sinh^2(t)) = \mathcal{L}(-\frac{1}{2} - \frac{1}{2}\cosh(2t)) = \mathcal{L}(-\frac{1}{2}) + \frac{1}{2}\mathcal{L}(\cosh(2t)) =$

$-\frac{1}{2}\mathcal{L}(1) + \frac{1}{2}\mathcal{L}(\frac{1}{2}e^{2t} + \frac{1}{2}e^{-2t}) = \frac{1}{2s} + \frac{1}{4}\mathcal{L}(e^{2t}) + \frac{1}{4}\mathcal{L}(e^{-2t}) =$

$-\frac{1}{2s} + \frac{1}{4}\frac{1}{s-2} + \frac{1}{4}\frac{1}{s+2} = \frac{s}{2(s^2-4)}$

\3 vi✓ $F(t) = e^{-t}\cos(2t) \therefore \mathcal{L}(F(t)) = \mathcal{L}(e^{-t}\cos(2t)) \therefore$

$\mathcal{L}(\cos(2t)) = \frac{s}{s^2+2^2} = \frac{s}{s^2+4} \therefore \mathcal{L}(e^{-t}\cos(2t)) = \frac{s}{(s+1)^2+4}$ ✓

Sheet 2 / 1:v / $\bar{f}(s) = \frac{6s}{s^2-16}$

remember: $\cosh(\omega t) = \frac{1}{2}(e^{\omega t} + e^{-\omega t})$

$\therefore \mathcal{L}[\cosh(\omega t)] = \mathcal{L}[\frac{1}{2}e^{\omega t}] + \mathcal{L}(\frac{1}{2}e^{-\omega t}) = \frac{1}{2}\left(\frac{1}{s-\omega}\right) + \frac{1}{2}\left(\frac{1}{s+\omega}\right) = \frac{1}{2}\left(\frac{2s}{s^2-\omega^2}\right)$

$= \frac{s}{s^2-\omega^2}$

using $\mathcal{L}[\cosh(\omega t)] = \frac{s}{s^2-\omega^2}$

$\mathcal{L}^{-1}\left[\frac{6s}{s^2-16}\right] = 6\cosh(4t)$

5 / $\bar{f}(s) = \frac{1}{s^4+1}$ $\quad s^4+1 = (s^2+\square+1)(s^2+-\square+1) = (s^2+?+1)(s^2-?+1)$

s^3 term: $0 = -\square s^3 + \square s^3$

s^2 term: $s^2 + s^2 - \square^2 s^2 = 0 = s^2 + s^2 - \square\square s^2 = 0$

$\therefore 2s^2 - \square^2 s^2 = 0 \quad \therefore \square^2 = 2 \quad \therefore \square = \sqrt{2}$

... s term: $0 = \square - \square$ $\quad \therefore$

$s^4+1 = (s^2+\sqrt{2}s+1)(s^2-\sqrt{2}s+1)$

$\bar{f}(s) = \frac{1}{s^4+1} = \frac{1}{(s^2+\sqrt{2}s+1)(s^2-\sqrt{2}s+1)} = \bar{f}(s) = \frac{As+B}{s^2+\sqrt{2}s+1} + \frac{Cs+D}{s^2-\sqrt{2}s+1}$

$1 = (As+B)(s^2-\sqrt{2}s+1) + (Cs+D)(s^2+\sqrt{2}s+1)$

Compare coeffs: s^0: $1 = B+D \quad \therefore B = 1-D$

s^1: $0 = A - \sqrt{2}B + C + \sqrt{2}D$ ②

s^2: $0 = -\sqrt{2}A + B + \sqrt{2}C + D$ ③ $\quad s^3$: $0 = A + C \quad \therefore \quad A = -C$

from ②: $0 = A - \sqrt{2}B - A + \sqrt{2}(1-B)$

$0 = -\sqrt{2}B + \sqrt{2} - \sqrt{2}B \quad -2\sqrt{2}B = -\sqrt{2} \quad \therefore B = \frac{1}{2} \quad \therefore D = 1-B = 1-\frac{1}{2} = \frac{1}{2} = D$

from ③: $0 = -\sqrt{2}A + \frac{1}{2} + \sqrt{2}(-A) + \frac{1}{2} \quad \therefore 0 = 1 - 2\sqrt{2}A \quad \therefore A = \frac{1}{2\sqrt{2}} \quad \therefore C = -\frac{1}{2\sqrt{2}}$

$\bar{f}(s) = \frac{\frac{1}{2\sqrt{2}}s + \frac{1}{2}}{s^2+\sqrt{2}s+1} - \frac{\frac{1}{2\sqrt{2}}s - \frac{1}{2}}{s^2-\sqrt{2}s+1} = \frac{s+\sqrt{2}}{2\sqrt{2}(s^2+\sqrt{2}s+1)} - \frac{s+-\sqrt{2}}{2\sqrt{2}(s^2-\sqrt{2}s+1)} =$

$\frac{s+\sqrt{2}}{2\sqrt{2}((s+\frac{\sqrt{2}}{2})^2 - \frac{1}{2} + 1)} - \frac{s+-\sqrt{2}}{2\sqrt{2}((s-\frac{\sqrt{2}}{2})^2 - \frac{1}{2} + 1)} = \frac{s+\sqrt{2}}{2\sqrt{2}((s+\frac{\sqrt{2}}{2})^2+\frac{1}{2})} - \frac{s-\sqrt{2}}{2\sqrt{2}((s-\frac{\sqrt{2}}{2})^2+\frac{1}{2})}$

in order to use sin & cos from table, need to just 2 numerator

$\frac{s+\sqrt{2} - \frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}{2\sqrt{2}((s+\frac{\sqrt{2}}{2})^2+\frac{1}{2})} - \frac{s-\sqrt{2}+\frac{\sqrt{2}}{2}-\frac{\sqrt{2}}{2}}{2\sqrt{2}((s-\frac{\sqrt{2}}{2})^2+\frac{1}{2})} = \left\{\frac{1}{2\sqrt{2}}\left(\frac{s+\frac{\sqrt{2}}{2}}{(s+\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\right) - \frac{1}{2\sqrt{2}}\left(\frac{s-\frac{\sqrt{2}}{2}}{(s-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\right)\right\}$ ~~X~~

$\frac{1}{2\sqrt{2}}\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{4}$ want $\frac{1}{2}$ $\quad \frac{1}{2\sqrt{2}}\left(\frac{\sqrt{2}}{2}\right) = \frac{1}{4}$ want $\frac{1}{2}$

$\omega = \frac{1}{\sqrt{2}}$ $\quad a = -\frac{\sqrt{2}}{2}$

$= \frac{1}{2\sqrt{2}}\left(\frac{s+\frac{\sqrt{2}}{2}}{(s+\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\right) + \frac{1}{4}\left(\frac{1}{(s+\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\right) + -\frac{1}{2\sqrt{2}}\left(\frac{s-\frac{\sqrt{2}}{2}}{(s-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\right) + \frac{1}{4}\left(\frac{1}{(s-\frac{\sqrt{2}}{2})^2+\frac{1}{2}}\right)$

find $\mathcal{L}^{-1}$ of each of the four terms: $\quad a = \frac{\sqrt{2}}{2}$

$\mathcal{L}^{-1}(f(s)) = \frac{1}{2\sqrt{2}} e^{-\frac{t}{\sqrt{2}}} \cos(\sqrt{\tfrac{1}{2}}\, t) - \frac{1}{2\sqrt{2}} e^{+\frac{t}{\sqrt{2}}} \cos(\sqrt{\tfrac{1}{2}}\, t) + \frac{1}{4}\sqrt{2} e^{-\frac{t}{\sqrt{2}}} \sin(\sqrt{\tfrac{1}{2}}\, t) +$

$\frac{1}{4}\sqrt{2} e^{\frac{t}{\sqrt{2}}} \sin(\sqrt{\tfrac{1}{2}}\, t)$

\Sheet 3/ \1 iii/ $f(s) = \frac{1}{(s+1)^2(s^2+4)} = \frac{1}{(s+1)^2} \cdot \frac{1}{s^2+4}$

$\frac{1}{(s+1)^2} \xrightarrow{\mathcal{L}^{-1}} te^{-t}$ LT2, n=1, LT13; $\frac{1}{s^2+4} \xrightarrow{\mathcal{L}^{-1}} \frac{1}{2}\sin(2t)$

now : convolute: $(te^{-t}) * (\frac{1}{2}\sin(2t)) = \frac{1}{2}\int_0^t ue^{-u} \sin(2(t-u))\,du = I$

IBP: $U = ue^{-u}$ $U' = e^{-u} - ue^{-u}$ $V' = \sin(2(t-u))$, $V = \frac{1}{2}\cos(2t-2u)$

$I = \frac{1}{2}[UV]_0^t - \frac{1}{4}\int_0^t (e^{-u} - ue^{-u})\cos(2t-2u)\,du$

$I = \frac{1}{2}[UV]_0^t - \frac{1}{4}\int_0^t e^{-u}\cos(2t-2u)\,du + \frac{1}{4}\int_0^t ue^{-u}\cos(2t-2u)\,du$

$J = \int_0^t e^{-u}\cos(2t-2u)\,du$

lets find J : $U = e^{-u}$, $U' = -e^{-u}$ $V' = \cos(2t-2u)$, $V = -\frac{1}{2}\sin(2t-2u)$

$J = [-e^{-u}\frac{1}{2}\sin(2t-2u)]_0^t - \frac{1}{2}\int_0^t e^{-u}\sin(2t-2u)\,du$ $\downarrow$ IBP

$U = e^{-u}$ $U' = -e^{-u}$ $V' = \sin(2t-2u)$ $V = \frac{1}{2}\cos(2t-2u)$

$J = [-e^{-u}\frac{1}{2}\sin(2t-2u)]_0^t - \frac{1}{2}[\frac{1}{2}e^{-u}\cos(2t-2u)]_0^t - \frac{1}{2}(\frac{1}{2}\underbrace{\int_0^t e^{-u}\cos(2t-2u)\,du}_{J})$

$J = +e^0\frac{1}{2}\sin(2t) - \frac{1}{2}(-\frac{1}{2}e^{-t} - \frac{1}{2}e^0\cos(2t)) - \frac{1}{4}J$

$\frac{5}{4}J = \frac{1}{2}\sin(2t) + \frac{1}{4}e^{-t} + \frac{1}{4}\cos(2t)$ $\therefore$

so $J = \frac{2}{5}\sin(2t) + \frac{1}{5}e^{-t} + \frac{1}{5}\cos(2t)$

$I = \frac{1}{2}[UV]_0^t - \frac{1}{4}J + \frac{1}{4}\int_0^t ue^{-u}\cos(2t-2u)\,du$

\Sheet 5/ from Z Fredholdum thm: $(\lambda I - k)y = f$ has a sol iff

$\int_a^b f(x)\, Z(x)\, dx = 0$ — sol of Z adjoint homog IE

\Q3/ $k(x,t) = \begin{cases} 1-t & 0 \le x \le t \le 1 \\ 1-x & 0 \le t \le x \le 2\pi \end{cases}$

\a/ show $\int_0^1 |k(x,t)|\,dt < 1$ $x \in [0,1]$

$\int_0^1 |k(x,t)|\,dt = \int_0^x (1-x)\,dt + \int_x^1 (1-t)\,dt = \frac{1}{2}(1-x^2)$ max in $[0,1]$

$\|k\|_\infty = \frac{1}{2} < 1$ occurs at $x = 0$

\b/ solve $y - ky = 1$

\c/ solve $y - ky = x$

So $y(x) = \frac{1}{\cos(1)}\cos(x)$ for when $f(x)=1$

\c / $f(x) = x$ ODE $y''(x)+y(x)=0$ $y(x)=A\cos(x)+B\sin(x)$

IC. ① $y'(0)=1$ ② $y(0)=\int_0^1 (1-t)y(t)\,dt$

$y'(x) = -A\sin(x)+B\cos(x)$ ① $1=B$

② $A=\int_0^1 (1-t)(A\cos(t)+\sin(t))\,dt$

$A=\int_0^1 A\cos(t)+\sin(t)\,dt + \int_0^1 t\sin(t)-At\cos(t)\,dt$ ∴

$A=[A\sin(t)-\cos(t)]_0^1 + \left(-\int_0^1 t(\sin(t)+A\cos(t))\,dt\right)$ $u=t$, $u'=1$ $v'=\sin(t)+A\cos(t)$

$v=-\cos(t)+A\sin(t)$

$-\int_0^1 t(\sin(t)+A\cos(t))\,dt = -[-t(\cos(t)+A\sin(t))]_0^1 + \int_0^1 A\sin(t)-\cos(t)\,dt =$

$- - (\cos(1)+A\sin(1)) + [-A\cos(t)-\sin(t)]_0^1 =$

$\cos(1) - A\sin(1) - A\cos(1) - \sin(1) + A$ ∴ So

$A = A\sin(1)-\cos(1)+1+\cos(1)-A\sin(1)-A\sin(1)-A\cos(1)-\sin(1)+A$

$A\cos(1)=1-\sin(1)$ $A=\frac{1-\sin(1)}{\cos(1)}$ ∴

$y(x)=\frac{1-\sin(1)}{\cos(1)}\cos(x)+\sin(x)$

4 / $y-ky=f$ $k(x,t)=\frac{1}{2}+x(3t^2-1)$ $x\in[-1,1]$

consider Z adjoint homog case ~~$z(x)-kz(x)$~~ Z $z(x)-k^*z(x)=0$

$k^*(x,t)=\frac{1}{2}+t(3x^2-1)$

$z(x)-\int\left(\frac{1}{2}+t(3x^2-1)\right)z(t)\,dt=0$

$z(x)-\int_{-1}^1 \frac{1}{2}z(t)\,dt-(3x^2-1)\int_{-1}^1 t\,z(t)\,dt=0$

$z(x)=\frac{1}{2}P_1+(3x^2-1)P_2$ $P_1=\int_{-1}^1 z(t)\,dt$, $P_2=\int_{-1}^1 t\,z(t)\,dt$ ∴

$P_1=\int_{-1}^1 \frac{1}{2}P_1+(3t^2-1)P_2\,dt$ ①, $P_2=\int_{-1}^1 t\left(\frac{1}{2}P_1+(3t^2-1)P_2\right)dt$ ②

can solve Z eqns for P_1 & P_2: $P_1=P_1$ $P_1\in\mathbb{R}$ & $P_2=0$ ∴

$z(x)=\frac{1}{2}P_1$

we know we have a sol to Z nonhomog case if $\int f(x)\cdot\frac{1}{2}P_1\,d=0$ ie

ie $\int_{-1}^1 f(x)\,dx=0$ for Z adjoint original IE with $f(x)\neq 0$:

$y(x)-\int_{-1}^1\left(\frac{1}{2}+x(3t^2-1)\right)y(t)\,dt=f(x)$

Sheet 5/ $y - ky = f(x)$ $\quad y(x) - \int_0^1 k(x,t)y(t)\,dt = f(x)$

~~$y(x) - \{$~~ $y(x) - \Big\{ \underbrace{\int_0^x (1-t)y(t)\,dt + \int_x^1 (1-t)y(t)\,dt}_{x \le t} + \underbrace{\int_0^x (1-x)y(t)\,dt + \int_x^{2\pi} (1-x)y(t)\,dt}_{x \ge t} \Big\} = f(x)$

$y(x) - \Big\{ \int_0^x y(t)\,dt - \int_0^x t\,y(t)\,dt + \int_x^1 y(t)\,dt - \int_x^1 t y(t)\,dt + \int_0^x y(t)\,dt - \int_0^x x y(t)\,dt + \int_x^{2\pi} y(t)\,dt - \int_x^{2\pi} x y(t)\,dt \Big\}$

$= f(x)$

$y(x) - \Big\{ \int_0^1 y(t)\,dt - \int_0^x x y(t)\,dt - \int_x^1 t y(t)\,dt \Big\} = f(x)$

$y(x) - \int_0^1 y(t)\,dt + \overbrace{\int_0^x x y(t)\,dt}^{x\int_0^x y(t)dt} + \int_x^1 t y(t)\,dt = f(x)$ $\quad (*)$

we need to solve this then put $f(x) = 1$ ((a)) and $f(x) = x$ (b)

~~Di~~fferentiate: use $\frac{d}{dx} \int_{g(x)}^{f(x)} h(x)\,dt = h(f(x))f'(x) - h(g(x))g'(x)$

$y'(x) - 0 + xy(x) + \int_0^x y(t)\,dt + 1\,y(1)\cdot(0) - xy(x)\cdot 1 = f'(x)$

$y'(x) + \int_0^x y(t)\,dt = f'(x)$ $\quad (**)$

differentiate again: $y''(x) + y(x) = f''(x)$

from $(**)$ ~~$y'(0)$~~ $x = 0$ $\quad y'(0) = f'(0)$

from $(*)$ $\quad y(0) + \int_0^1 (t-1)\,y(t)\,dt = f(0)$

$y(0) = f(0) + \int_0^1 (1-t)y(t)\,dt$

(b) for $f(x) = 1$ $\quad y''(x) + y(x) = 0$

$y(x) = A\cos(x) + B\sin(x)$ $\quad \therefore m^2 + 1 = 0 \quad m = \pm i \quad \alpha \pm i\beta \quad \alpha = 0,\ \beta = 1$

$y_h = A\cos(x) + B\sin(x)$

$y'(0) = 0$ $\qquad y'(x) = -A\sin(x) + B\cos(x)$ $\qquad \therefore 0 = B$

$y(x) = A\cos(x)$

IC: $y(0) = 1 + \int_0^1 (1-t)\,y(t)\,dt$ $\qquad \underset{A\cos(0)}{A} = \underbrace{1 + \int_0^1 (1-t)A\cos(t)\,dt}_{1 + \int_0^1 (1-t)y(t)dt}$

$A = 1 + A\int_0^1 \cos(t)\,dt - \overbrace{A\int_0^1 t\cos(t)\,dt}^{IBP}$ $\quad u = t \quad u' = 1 \quad v' = \cos(t) \quad v = \sin(t)$

$A = 1 + [A\sin(t)]_0^1 - A[t\sin(t)]_0^1 + A\int_0^1 \sin(t)\,dt$

$A = 1 + A\sin(1) - 0 - A\sin(1) + A(0) + A[-\cos(t)]_0^1$

$\cancel{A} = 1 + \cancel{A\sin(1)} - \cancel{A\sin(1)} - A\cos(1) + \cancel{A}$ $\qquad 0 = 1 - A\cos(1)$ $\quad \therefore A = \frac{1}{\cos(1)}$

Same "P-method" $y(x) - \underbrace{\int_{-1}^{1} \frac{1}{2} y(t)\,dt}_{P_1} - x\underbrace{\int_{-1}^{1}(3t^2-1)y(t)\,dt}_{P_2} = f(x)$ ∴

$y(x) = f(x) + \frac{1}{2}P_1 + xP_2$

) $P_1 = \int_{-1}^{1} f(t) + \frac{1}{2}P_1 - tP_2\,dt$ $\qquad P_1 = \int_{-1}^{1} f(t)\,dt + \left[\frac{P_1 t}{2}\right]_{-1}^{1} - \left[\frac{t^2 P_2}{2}\right]_{-1}^{1}$ ∴

$P_1 = \int_{-1}^{1} f(t)\,dt + P_1$ ∴ $\int_{-1} f(t)\,dt$

for P_2: $P_2 = \int_{-1}^{1}(3t^2-1)\left(f(t) + \frac{1}{2}P_1 - tP_2\right)dt$ ∴ So

$P_2 = \int_{-1}^{1}(3t^2-1)f(t)\,dt + 0 \leftarrow \int_{-1}^{1}\frac{1}{2}P_1(3t^2-1)\,dt - \int_{-1}^{1} P_2 t(3t^2-1)\,dt = 0$

$y(x) = f(x) + \frac{1}{2}P_1 + x\underbrace{\int_{-1}^{1}(3t^2-1)f(t)\,dt}_{\text{but } \int_{-1}^{1} f(t) = 0}$ ∴ So

$y(x) = f(x) + \frac{1}{2}P_1 + x\int_{-1}^{1} 3t^2 f(t)\,dt$

\Sheet 74/ §1/ $y(x) - \int_0^1 \ln\left(\frac{x}{t}\right) y(t)\,dt = 1$ ∴

$y(x) - \int_0^1 (\ln(x) - \ln(t))\,y(t)\,dt = 1$

$y(x) - \ln(x)\underbrace{\int_0^1 y(t)\,dt}_{P_1} + \underbrace{\int_0^1 \ln(t)\,y(t)\,dt}_{P_2} = 1$

$y(x) = 1 + P_1\ln(x) - P_2$ ∴

for P_1: $P_1 = \int_0^1 1 + P_1\ln(t) - P_2\,dt$ ∴ So

$P_1 = \int_0^1 (1-P_2)\,dt + P_1\int_0^1 \ln(t)\,dt$

$P_1 = \left[(1-P_2)t\right]_0^1 - P_1$

$2P_1 = (1-P_2)$ ∴ $P_2 = 1 - 2P_1$

$P_2 = \int_0^1 \ln(t)\left(1 + P_1\ln(t) - P_2\right)dt$

$P_2 = \int_0^1 \ln(t)\,dt\,(1-P_2) + P_1\int_0^1 \ln^2(t)\,dt$

$P_2 = (1-P_2)(-1) + P_1(2)$ have:

$P_1 = \frac{1}{2}$, $P_2 = 0$ ∴ So

$y(x) = 1 + \frac{1}{2}\ln(x) = 1 + \ln\sqrt{x}$

$\int_0^1 \ln(t)\,dt$ IBP $u = \ln(t)$ $u' = \frac{1}{t}$ $v' = 1$

$v = t$ $= \left[t\ln(t)\right]_0^1 - \int_0^1 1\,dt = 0 - 0 - \left[t\right]_0^1 = -1$

$\int_0^1 \ln^2(t)\,dt$ IBP $u = \ln^2(t)$ $u' = 2\ln(t)\cdot\frac{1}{t}$

$v' = 1$, $v = t$ $= \left[t\ln^2(t)\right]_0^1 - \int_0^1 2\ln(t)\,dt =$

$0 - 2\int_0^1 \ln(t)\,dt = 2$

2a/ $y(x) - 2\int_0^x \overbrace{\cos(x-t)}^{G(t-u)}\,\overbrace{y(t)}^{F(u)}\,dt = 1$ in LT 22

Z Laplace transform of each term $\hat{y}(s) - 2\,\overbrace{\hat{y}(s)\cdot \mathcal{L}(\cos(x))}^{f(s)\cdot g(s) \text{ in LT 22}} = \frac{1}{s}$ $\therefore$

$\hat{y}(s) - 2\hat{y}(s)\cdot\frac{s}{s^2+1} = \frac{1}{s}$

$\hat{y}\left(1 - \frac{2s}{s^2+1}\right) = \frac{1}{s}$ $\qquad \hat{y}(s) = \frac{s^2+1}{s(s-1)^2} = \frac{A}{s} + \frac{B}{(s-1)} + \frac{C}{(s-1)^2}$ $\therefore$

$\hat{y}(s) = \frac{1}{s} + \frac{2}{(s-1)^2}$ $\quad \therefore\ y(x) = 1 + \underline{2xe^x}$, $\rightarrow$ LT2 n=1 LT13

using differentiation: $y(x) - 2\int_0^x \cos(x-t)y(t)\,dt = 1$ $\therefore$

$\frac{d}{dx}\left(\int_{g(x)}^{f(x)} h(t)\,dt = h(f(x))f'(x) - h(g(x))g'(x)\right)$ $\therefore$

Diff: $y'(x) - 2\{\quad\} = 0$ $\therefore$

integral term: $\int_0^x \cos(x)\cos(t)y(t)\,dt + \int_0^x \sin(x)\sin(t)y(t)\,dt =$

$\cos(x)\int_0^x \cos(t)y(t)\,dt + \sin(x)\int_0^x \sin(t)y(t)\,dt$ $\therefore$ So

$y'(x) - 2\left\{\cos(x)\cos(x)y(x) - \sin(x)\int_0^x \cos(t)y(t)\,dt + \sin(x)\sin(x)y(x) + \cos(x)\int_0^x \sin(t)y(t)\,dt\right\} =$
0

$y'(x) - 2\left(y(x) - \sin(x)\int_0^x \cos(t)y(t)\,dt + \cos(x)\int_0^x \sin(t)y(t)\,dt\right) = 0$

diff again: $y''(x) - 2\Big(y'(x) - \underline{\sin(x)}\cos(x)y(x) - \cos(x)\int_0^x \cos(t)y(t)\,dt +$

$\underline{\cos(x)}\sin(x)y(x) - \sin(x)\int_0^x \sin(t)y(t)\,dt\Big) = 0$ $\therefore$

$y''(x) - 2\left(y'(x) - \cos(x)\int_0^x \cos(t)y(t)\,dt - \sin(x)\int_0^x \sin(t)y(t)\,dt\right) = 0$ $\therefore$

$y''(x) - 2\left(y'(x) - \int_0^x \cos(x-t)y(t)\,dt\right) = 0$

$\left\{ \circledast\ y(x) - 2\int_0^x \cos(x-t)y(t)\,dt = 1 \right\}$ $\therefore$ $\left\{\frac{1}{2}y(x) - \frac{1}{2} = \int_0^x \cos(x-t)y(t)\,dt\right\}$ $\therefore$

from Z original IE $y''(x) - 2y'(x) + y(x) = 1$ $\therefore$

from $\circledast$ putting $x=0$: $y'(0) - 2y(0) = 0$ from original IE $y(0) = 1$

$y'(0) = 2$ $\therefore$

$y(x) = 1 + 2xe^x$

\Sheet 7/ \Q2/ given $k(x,t) = \cos(x+t) \rightarrow K$

$h(x,t) = \sin(x+t) \rightarrow H$ on $[0,1] \times [0,1]$ kernels of KH & HK

$\therefore$ we have: KH: kernel $a(x,t) = \int_0^1 k(x,t) h(s,t)\, ds$

similarly: HK: kernel $b(x,t) = \int_0^1 h(x,s) k(s,t)\, ds$

$a(x,t) = \int_0^1 \cos(x+s) \sin(s+t)\, ds$

$\{$use identity: $\sin(\alpha)\cos(\beta) = \frac{1}{2}(\sin(\alpha+\beta) + \sin(\alpha-\beta))$ $\}$

$a(x,t) = \frac{1}{2}\int_0^1 \sin(2s+x+t) + \sin(t-x)\, ds =$

$\frac{1}{2}\left[-\frac{\cos(2s+x+t)}{2} + s\sin(t-x)\right]_0^1 = \frac{1}{2}\left(-\frac{1}{2}\cos(2+x+t) + \sin(t-x) + \frac{1}{2}\cos(x+t)\right)$

$\frac{1}{4}\left(2\sin(t-x) + \cos(x+t) - \cos(2+x+t)\right) = \frac{1}{4}\left(2\sin(t-x) - 2\sin\left(\frac{2+2x+2t}{2}\right)\sin\left(-\frac{2}{2}\right)\right)$

$\{$use identity: $\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ $\}$

$= \frac{1}{4}\left(2\sin(t-x) + 2\sin(1+x+t)\sin(1)\right)$

$= \frac{1}{2}\sin(t-x) + \frac{1}{2}\sin(1)\sin(1+x+t)$

$b(x,t) = \int_0^1 h(x,s) k(s,t)\, ds = \int_0^1 \sin(x+s)\cos(s+t)\, ds =$

$\frac{1}{2}\int_0^1 \sin(x+t+2s) + \sin(x+t)\, ds =$

$\frac{1}{2}\left(-\frac{\cos(x+t+2)}{2} + \sin(x+t) + \frac{\cos(x+t)}{2}\right) =$

$\frac{1}{2}\sin(1)\sin(x+t+1) - \frac{1}{2}\sin(x-t)$ $\{$identity $\cos(\alpha) - \cos(\beta) = -2\sin\left(\frac{\alpha+\beta}{2}\right)\sin\left(\frac{\alpha-\beta}{2}\right)$ $\}$

\3/ $y(x) - \int_0^1 x t^3 y(t)\, dt = x^2 \quad 0 \le x \le 1$

\3a/ solve the IE $\therefore$ p-method $\quad y(x) = x^2 + xp$

$p = \int_0^1 t^3 y(t)\, dt \quad \therefore y(t) = t^2 + tp \therefore \quad p = \int_0^1 t^3(t^2 + pt)\, dt = \int_0^1 t^5 + pt^4\, dt =$

$p = \left[\frac{1}{6}t^6 + \frac{1}{5}pt^5\right]_0^1 = \frac{1}{6} + \frac{1}{5}p \quad \therefore \quad \frac{4}{5}p = \frac{1}{6} \therefore \quad p = \frac{5}{24} \therefore$

$y(x) = x^2 + \frac{5}{24}x$

\3b/ y_0, y_1, y_2, y_3

$y_0 = \frac{f}{\lambda} = \frac{x^2}{1} \quad \therefore y_1 = \frac{f}{\lambda} + x\int_0^1 t^3 y_0(t)\, dt = x^2 + x\int_0^1 t^3 \cdot t^2\, dt = x^2 + \frac{x}{6}$

$y_2 = \frac{f}{\lambda} + x\int_0^1 t^3 y_1(t)\,dt = x^2 + x\int_0^1 t^3\left(t^2 + \frac{t}{6}\right)dt$

$= x^2 + \frac{x}{5}$

$y_3 = x^2 + x\int_0^1 t^3\left(t^2 + \frac{t}{5}\right)dt = x^2 + \frac{31}{150}x$

\3c/ give an esti on how much this y_3 differs from Z exact

Sol

$$\|y - y_3\|_\infty \le \frac{1}{|\lambda| - \|k\|_\infty}\left(\frac{\|k\|_\infty}{|\lambda|}\right)^{n+1}\|f\|_\infty \quad \therefore$$

$\|k\|_\infty = \max_{x\in[0,1]}\int_0^1 x t^3\,dt = \max_{x\in[0,1]}\left(\frac{x}{4}\right) = \frac{1}{4}$, $\quad n+1 \to 4$, $\quad \|f\|_\infty \to 1$

So $\|y - y_3\|_\infty \le \frac{1}{1-\frac{1}{4}}\left(\frac{1/4}{1}\right)^4 \cdot 1 = \frac{1}{192}$

\3d/ $y(x) - \int_0^1 x t^3 y(t)\,dt = f(x)$

$\therefore$ p-method : $y(x) = f(x) + xp \qquad p = \int_0^1 t^3 y(t)\,dt = \int_0^1 t^3(f(t) + pt)\,dt \quad \therefore$

$p = \int_0^1 t^3 f(t)\,dt + p\left[\frac{1}{5}t^5\right]_0^1 \quad \therefore$

$\frac{4}{5}p = \int_0^1 t^3 f(t)\,dt \quad \therefore$

$p = \frac{5}{4}\int_0^1 t^3 f(t)\,dt \quad \therefore$

So $y(x) = f(x) + \frac{5}{4}x\int_0^1 t^3 f(t)\,dt$

$\Gamma_\lambda(x,t) = \frac{5}{4}xt^3$

now $f(x) = x^2 \quad \therefore$

$y(x) = x^2 + \frac{5}{4}x\int_0^1 t^3 t^2\,dt \qquad y(x) = x^2 + \frac{5}{24}x$

\4/ $k(x,t) = \begin{cases} x-t & 0 \le t \le x \\ 0 & x \le t \le 1 \end{cases} \qquad K \subset [0,1]$

\4a/ $\int_0^1 |k(x,t)|\,dt = \int_0^x x-t\,dt + \underbrace{\int_0}_{=0}$

$= -\frac{1}{2}\left[(x-t)^2\right]_0^x = \frac{1}{2}x^2 \le \frac{1}{2}x^2 \le \frac{1}{2} \quad \forall x \in [0,1]$

\4b/ $y - Ky = x \qquad y(x) - \int_0^x (x-t)y(t)\,dt = x$

$y_0 = \frac{f}{\lambda} = x$

$y_1 = x + \int_0^x (x-t)y_0(t)\,dt = x + \int_0^x (x-t)t\,dt \to \int_0^x xt\,dt - \int_0^x t^2\,dt$

$y_1 = x + \frac{1}{3!}x^3$

$y_2 = x + \int_0^x (x-t)(t+\frac{1}{3!}t^3)\,dt = x + \left[\frac{x}{4\cdot 3!}t^4 + \frac{xt^2}{2} - \frac{t^5}{5\cdot 3!} - \frac{t^3}{3}\right]_0^x =$

$x + \left(\frac{x^5}{4} + \frac{x^3}{2} - \frac{x^5}{5\cdot 3!} - \frac{x^3}{3}\right) = x + \frac{x^3}{3!} + \frac{x^5}{5!}$

Claim $y_n(x) = x\sum_{k=0}^{n} \frac{x^{2k}}{(2k+1)!}$

use induction ~~for~~ $n+1$:

goal: $y_{n+1}(x) = x\sum_{k=0}^{n+1} \frac{x^{2k}}{(2k+1)!}$

$y_{n+1} = x + \int_0^x (x+t)\, y_n(t)\,dt =$ $\quad t\sum \frac{t^{2k}}{(2k+1)!}$

$x + x\int_0^x t\left(\sum_{k=0}^{n} \frac{t^{2k}}{(2k+1)!}\right)dt - \int_0^x t\left(t\sum_{k=0}^{n} \frac{t^{2k}}{(2k+1)!}\right)dt$

$+x\sum_{k=0}^{n}\left[\frac{t^{2k+2}}{(2k+1)!(2k+2)}\right]_0^x - \sum_{k=0}^{n}\left[\frac{t^{2k+3}}{(2k+1)!(2k+3)}\right]_0^x$

\Sheet1/ \Ex1/ $\mathcal{L}(t^2)(s) = \int_0^\infty e^{-st}t^2\,dt =$

$\lim_{R\to\infty}\left[-\frac{1}{s}t^2e^{-st}\right]_0^R + \frac{1}{s}\int_0^\infty 2te^{-st}\,dt = \lim_{R\to\infty}\left[-\frac{1}{s}t^2e^{-st} - \frac{2}{s^2}te^{-st} + \frac{2}{s^3}e^{-st}\right]_0^R$

$= \frac{2}{s^3} \quad \because \lim_{R\to\infty} e^{-st} = \lim_{R\to\infty}\frac{1}{e^{st}} = 0$

\Ex2:/ $\sinh(x) = \frac{e^x - e^{-x}}{2} \qquad \mathcal{L}(\sinh(at)) = \mathcal{L}\left(\frac{e^{at}-e^{-at}}{2}\right) =$

$\frac{1}{2}\mathcal{L}(e^{at}) - \frac{1}{2}\mathcal{L}(e^{-at}) = \frac{1}{2}\frac{1}{s-a} - \frac{1}{2}\frac{1}{s+a} = \frac{1}{2}\frac{s+a-(s-a)}{s^2-a^2} = \frac{a}{s^2-a^2}$

\2:ii/ $\cosh(x) = \frac{e^x+e^{-x}}{2} \quad \therefore \quad \mathcal{L}(\cosh(at)) = \mathcal{L}\left(\frac{e^{at}+e^{-at}}{2}\right) =$

$\frac{1}{2}\mathcal{L}(e^{at}) + \frac{1}{2}\mathcal{L}(e^{-at}) = \frac{1}{2}\frac{1}{s-a} + \frac{1}{2}\frac{1}{s+a} =$

$\frac{1}{2}\frac{s+a+(s-a)}{s^2-a^2} = \frac{s}{s^2-a^2}$

\3:/ $\mathcal{L}(e^{-at}F(t)) = f(s-a),\ \mathcal{L}(F) = f(s) \therefore$

$\mathcal{L}(t^2) = \frac{2!}{s^3} = \frac{2}{s^3} \quad \therefore \mathcal{L}(e^{2t}t^2) = \mathcal{L}(e^{-(-2)t}F(t)) = f(s-(-2)) = f(s+2)$

$\mathcal{L}(e^{2t}t^2) = \frac{2}{(s-2)^3}$

\3:/ $\mathcal{L}(t^2\cos t) = (-1)^2 f''(s) = f''(s) \qquad f(s) = \mathcal{L}(\cos(t)) \therefore$

$\left(\frac{s}{s^2+1}\right)'' = \frac{2s^3-6s}{(s^2+1)^3}$

or: $F'(t) = 2t\cos(t) - t^2\sin(t)$ & $F''(t) = 2\cos(t) - 4t\sin(t) - t^2\cos(t)$

& $\mathcal{L}(F''(t)) = s^2f(s) - sF(0) - F'(0) = s^2\mathcal{L}(t^2\cos(t))$ &

$\mathcal{L}(F''(t)) = \mathcal{L}(2\cos(t)) - \mathcal{L}(4t\sin(t)) - \mathcal{L}(t^2\cos(t))$ ∴

$\mathcal{L}(2\cos(t)) - \mathcal{L}(4t\sin(t)) = (s^2+1)\,\mathcal{L}(t^2\cos t)$

$F'(t) = 4\sin t + 4t\cos t$

$F''(t) = 8\cos t - 4t\sin t$ ∴

$\mathcal{L}(F''(t)) = 8\,\mathcal{L}(\cos t) - 4\mathcal{L}(t\sin t) = s^2\mathcal{L}(4t\sin t)$ ∴

$4\mathcal{L}(t\sin t) = \frac{8\mathcal{L}(\cos t)}{s^2+1}$ ∴

$\frac{2s}{s^2+1} - \frac{8s}{(s^2+1)^2} = (s^2+1)\mathcal{L}(t^2\cos t)$ ∴ $\quad \mathcal{L}(t^2\cos t) = \frac{2s^3-6s}{(s^2+1)^3}$

\3 iii/ $F'(t) = -3\cos^2 t\sin t = -3\sin t + 3\sin^3 t$

$\cos^2 t = 1-\sin^2 t$ ∴ $\mathcal{L}(F'(t)) = s\mathcal{L}(\cos^3(t))(s) - 1$ by LT9 ∴

$s\mathcal{L}(\cos^3 t)(s) - 1 = -3\mathcal{L}(\sin t) + 3\mathcal{L}(\sin^3 t)$

Let $\hat{F}(t) = \sin^3 t$ ∴ $\hat{F}'(t) = 3\sin^2 t\cos t = 3\cos t - 3\cos^3 t$ ∴

$\mathcal{L}(\sin^3 t)(s) = \frac{3}{s}\mathcal{L}(\cos t) - \frac{3}{s}\mathcal{L}(\cos^3(t))$

∵ LT9: is $\mathcal{L}(F'(t)) = sf(s)$ ∴ $f(s) = \frac{\mathcal{L}(F'(t))}{s}$ ∴

$s\mathcal{L}(\cos^3 t) - 1 = -3\mathcal{L}(\sin t) + \frac{9}{s}\mathcal{L}(\cos t) - \frac{9}{s}\mathcal{L}(\cos^3 t)$ ∴

$(s+\frac{9}{s})\mathcal{L}(\cos^3 t) = -3\mathcal{L}(\sin t) + \frac{9}{s}\mathcal{L}(\cos t) + 1 = -\frac{3}{s^2+1} + \frac{9}{s^2+1} = 1$ ∴

$\mathcal{L}(\cos^3 t) = \left(\frac{s^2+7}{s^2+1}\right)\left(\frac{s}{s^2+9}\right) = \frac{s(s^2+7)}{(s^2+1)(s^2+9)}$

\3 iv/ $\mathcal{L}(F(t)) = \mathcal{L}(e^t\sin t)$ ∴ $F'(t) = e^t\sin t + e^t\cos t$ ∴

$\mathcal{L}(F'(t)) = \mathcal{L}(e^t\sin t + e^t\cos t) = \mathcal{L}(e^t\sin(t)) + \mathcal{L}(e^t\cos t)$

∴ LT9: $\mathcal{L}(F'(t)) = s\mathcal{L}(F(t))$ $\quad$ ∵ $F(0)=0$ ∴

Let $\hat{F}(t) = e^t\cos t$ ∴ $\hat{F}'(t) = e^t\cos t - e^t\sin t$ ∴

LT9: $\mathcal{L}(\hat{F}(t)) = s(\mathcal{L}(e^t\cos(t))) - 1$

∵ $\hat{F}'(t) = e^t\cos t - e^t\sin t$ ∴

$\mathcal{L}(\hat{F}'(t)) = s(\mathcal{L}(e^t\cos(t))) - 1 = \mathcal{L}(e^t\cos t) - \mathcal{L}(e^t\sin t)$ ∴

$s(\mathcal{L}(e^t\cos t)) - 1 = \mathcal{L}(e^t\cos t) - \mathcal{L}(e^t\sin t)$ ∴

~~L(cost~~ $\mathcal{L}(e^t\cos t) = \frac{1}{s-1}(1-\mathcal{L}(e^t\sin t))$ ∴

$\mathcal{L}(F'(t)) = \mathcal{L}(e^t\sin(t)) + \mathcal{L}(e^t\cos t) = \mathcal{L}(e^t\sin t) + \frac{1}{s-1}(1-\mathcal{L}(e^t\sin t)) =$

$s\mathcal{L}(e^t\sin t)$ ∴

$\mathcal{L}(\sin(t)e^t) = \frac{1}{s^2-2s+2} = \frac{1}{(s-1)^2+1}$

\Sheet 1/ \3v/ $F(t)=\sinh^2(t)$ ∴

$\cosh(2\theta)=1+2\sinh^2(\theta)$ ∴ $\mathcal{L}(\sinh^2(t)) = \mathcal{L}(\frac{1}{2}\cosh(2t)-\frac{1}{2}) =$

$\frac{s}{2(s^2-4)} - \frac{1}{2s} = \frac{2}{s(s^2-4)}$

\3vii/ $F(t)=t^2e^{3t}$ ∴ $\mathcal{L}(e^{3t}t^2)=\frac{2}{(s-3)^3}$ by LT2 & FST

\3vi/ $F(t)=e^{-t}\cos(2t)$ ∴ FTS:

$\mathcal{L}(e^{-t}\cos(2t))=\frac{s+1}{(s+1)^2+4} = \frac{s+1}{s^2+2s+5}$

\4/ $\mathcal{L}(\sinh at)=\frac{a}{s^2-a^2}$ $\quad \mathcal{L}(\cosh at)=\frac{s}{s^2-a^2}$

& $\cosh 0=1$ ∴ $\mathcal{L}(\frac{d}{dt}\cosh 3t)= s\mathcal{L}(\cosh 3t)-1 = s\frac{s}{s^2-9}-1=$

$s\frac{s}{s^2-9} - \frac{s^2-9}{s^2-9} = \frac{9}{s^2-9}$

$\mathcal{L}(3\sinh 3t)=3\frac{3}{s^2-9}=\frac{9}{s^2-9}$

\Sheet 2/

\1i/ $f(s)=\frac{1}{s^4}$ $\quad F(t)=\frac{1}{6}t^3$ by LT2

\1ii/ $\frac{4}{s-2}=f(s)$ ∴ $F(t)=4e^{2t}$ by LT1, LT13

\1iii/ $f(s)=\frac{1}{s^2+9}$ ∴ $F(t)=\frac{1}{3}\sin(3t)$ by LT5

\1iv/ $f(s)=\frac{6s}{s^2-16}$ ∴ $F(t)=6\cosh 4t$

\1v/ $f(s)=\frac{4s+12}{s^2+8s+16}$ ∴ $\mathcal{L}^{-1}(\frac{4s+12}{s^2+8s+16})=\mathcal{L}^{-1}(\frac{4(s+4)-4}{(s+4)^2})=$

$\mathcal{L}^{-1}(\frac{4}{s+4})-\mathcal{L}^{-1}(\frac{4}{(s+4)^2}) = 4e^{-4t} - 4te^{-4t} = 4e^{-4t}(1-t)$

\1vi/ $f(s)=\frac{3s+7}{s^2-2s-3}$ $\quad \mathcal{L}(\frac{3s+7}{s^2-2s-3}) = \mathcal{L}^{-1}(\frac{3(s-1)+10}{(s-1)^2-4})=$

$\mathcal{L}^{-1}(\frac{3(s-1)}{(s-1)^2-4})+\mathcal{L}^{-1}(\frac{10}{(s-1)^2-4}) = 3\cosh(2t)e^t + 5\sinh(2t)e^t =$

$e^t(3\cosh(2t)+5\sinh(2t))$ $\quad$ ∴ $\cosh t=\frac{1}{2}(e^t+e^{-t})$ & $\sinh t=\frac{1}{2}(e^t-e^{-t})$

∴ $F(t)=\frac{3}{2}e^{3t}+\frac{3}{2}e^{-t}+\frac{5}{2}e^{3t}-\frac{5}{2}e^{-t} = 4e^{3t}-e^{-t}$

\1vii/ $f(s)=\frac{3s^2+s-1}{(s+2)(s-1)^2} = \frac{A}{s+2}+\frac{B}{s-1}+\frac{C}{(s-1)^2}$ ∴ $A=1, B=2, C=1$ ∴

$f(s)=\frac{1}{s+1}+\frac{2}{s-1}+\frac{1}{(s-1)^2}$ ∴

$F(t)=e^{-2t}+2e^t+te^t$

\1viii/ FST: $\mathcal{L}^{-1}(\frac{1}{(s-a)^n}) = e^{at}\mathcal{L}^{-1}(\frac{1}{s^n}) = e^{at}(\frac{t^{n-1}}{(n-1)!})$

\1ix/ $f(s)=\frac{s^2}{s^3+a^3}$ $\quad a\in\mathbb{R}$ ∴ $f(s)=\frac{s^2}{(s+a)(s^2-as+a^2)}=\frac{A}{s+a}+\frac{Bs+C}{s^2-as+a^2}$

∴ $A=\frac{1}{3}$, $B=\frac{2}{3}$, $C=-\frac{a}{3}$ ∴

$f(s) = \frac{1/3}{s+a} + \frac{(2/3)s - a/3}{s^2 - as + a^2}$ $\therefore$ LT4 $\therefore$

$f(s) = \frac{1/3}{s+a} + \frac{2}{3}\frac{s - a/2}{(s-a/2)^2 + 3/4\, a^2}$ $\therefore$ FST & LT6 $\therefore$

$F(t) = \frac{1}{3} e^{-at} + \frac{2}{3} e^{(a/2)t} \cos(\frac{\sqrt{3}}{2} at)$

\1 x/ $f(s) = \frac{(s+1)e^{-\pi s}}{s^2+2s+1}$ $\therefore$ LT17 $a = \pi$ $\therefore \frac{s+1}{s^2+2s+1} = \frac{s+1}{(s+1)^2} = \frac{1}{s+1}$ $\therefore$

$\mathcal{L}^{-1}(\frac{1}{s+1}) = e^{-t}$ $\therefore$ LT17: $\mathcal{L}(F(t-a)H(t-a)) = e^{-as} f(s)$

$F(t) = H(t-\pi)e^{\pi - t}$ $\therefore F(t-a)H(t-a) = \mathcal{L}^{-1}(e^{-as} f(s))$ $\therefore$

$\hat{f}(s) = \frac{s+1}{s^2+2s+1} = \frac{1}{s+1}$ $\therefore \mathcal{L}^{-1}(e^{-\pi s} \frac{s+1}{s^2+2s+1}] = \mathcal{L}^{-1}(e^{-as} \frac{1}{s+1}) = \mathcal{L}^{-1}(e^{-as} \hat{f}(s))$

$\therefore \mathcal{L}^{-1}(\hat{f}(s)) = \mathcal{L}^{-1}(\frac{1}{s+1}) = e^{-t}$ $\therefore \mathcal{L}^{-1}(f(s)) = \mathcal{L}^{-1}(\frac{1}{s+1} e^{-as}) =$

$\mathcal{L}(F(t)) = \mathcal{L}(F(t-a)H(t-a)) = e^{-as} f(s) = e^{-\pi s} \frac{s}{s+1} = e^{-\pi s} \frac{s+1}{s^2+2s+1}$ $\therefore$

$F(t) = \mathcal{L}^{-1}(\frac{1}{s+1}) = e^{-t}$ $\therefore F(t-a) = F(t-\pi) = e^{-(t-\pi)} = e^{-t+\pi}$ $\therefore$

$\mathcal{L}^{-1}(e^{-\pi s} \frac{s+1}{s^2+2s+1}) = F(t-a)H(t-a) = e^{-t+\pi} H(t-a) = F(t) = e^{-t+\pi} H(t-\pi)$

\1 x i/ $f(s) = \ln(\frac{s^2+1}{s(s+3)})$ LT112 $n = 1$: $\mathcal{L}(t^n F(t)) = (-1)^n f^{(n)}(s)$ $\therefore$

$\mathcal{L}(t^1 F(t)) = -1 f^{(1)}(s)$ $\therefore \mathcal{L}(tF(t)) = -f'(s)$ $\therefore F(t) = -\frac{\mathcal{L}^{-1}(f'(s))}{t}$ $\therefore$

$\therefore f'(s) = \frac{3s^2 - 2s - 3}{s(s^2+1)(s+3)} = \frac{A}{s} + \frac{Bs + C}{s^2+1} + \frac{D}{s+3}$ $\therefore A = -1, B = 2, C = 0, D = -1$ $\therefore$

$f'(s) = -\frac{1}{s} + \frac{2s}{s^2+1} - \frac{1}{s+3}$ $\therefore$

$\mathcal{L}^{-1}(f'(s)) = -1 - e^{-3t} + 2\cos t$ $\therefore$

$F(t) = -\frac{\mathcal{L}^{-1}(f'(s))}{t} = \frac{1}{t}(1 + e^{-3t} - 2\cos t)$

\2 sol/ This func is periodic with $T = 1$ $\therefore$ Sub $F(t) = t$ & $T = 1$ into LT14 & $\therefore \mathcal{L}(F(t+T)) = \mathcal{L}(F(t)) = \mathcal{L}[F(t+T) = F(t)] = \frac{1}{1-e^{-sT}}\int_0^T e^{-st} F(t)\,dt$

$\therefore \mathcal{L}(F(t)) = \mathcal{L}(F(t+1)) = \frac{1}{1-e^{-s}} \int_0^1 e^{-st} t\, dt =$

$\frac{1}{1-e^{-s}}\left(\left[-\frac{t}{s}e^{-st}\right]_0^1 + \frac{1}{s}\int_0^1 e^{-st}\,dt\right) = \frac{1}{s(1-e^{-s})}\left(\frac{1}{s} - e^{-s} - \frac{1}{s}e^{-s}\right) = \frac{1 - e^{-s}(s+1)}{s^2(1-e^{-s})}$

\3 i/ Heaviside func $H(t)$: $H(t) = \begin{cases} 1 & t \geq 0 \\ 0 & t < 0 \end{cases}$

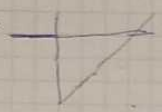
$\therefore G_1(t) = (t-2)H(t)$ $\therefore$ $t < 0$: $G_1(t) = 0$ $\quad 0 \leq t \leq 2$: $t - 2$

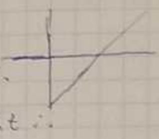
$t \geq 2$: $\cancel{t-2}$ $t - 2$

Sheet 2

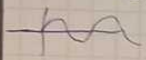
3ii $H(t-1) = \begin{cases} 1 & t \geq 1 \\ 0 & t \leq 1 \end{cases}$ $\therefore G_2(t) = (t-2)H(t-1) \therefore$

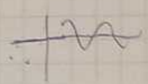
$\therefore t < 1 : G_2 = 0$, $t \geq 1 : G_2 = t-2 \therefore$ 

3iii 

3iv $G_4(t) = (t-3)H(t-1)$ $\therefore$ at $t=1$: $1-3 = -2 = G_3(t) \therefore$ 

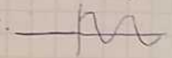
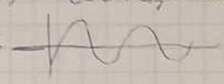
3v $G_5(t) = \sin(t)H(t)$ $\therefore$ $t \leq 0$ $G_5 = 0$, $t \geq 0$ $G_5 = \sin t \therefore$



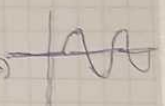
3vi $G_6(t) = \sin(t)H(t-0.7)$ $\therefore$ $H(t-0.7) = \begin{cases} 1 & t \geq 0.7 \\ 0 & t < 0.7 \end{cases}$ $\therefore$ 

$\sin(0.7) > 0 \therefore$ $0.7 < \frac{\pi}{2} \therefore$

3vii $G_7 = \sin(t-0.7)H(t)$ $\therefore$ $t<0 : G_7 = 0$ ~~$\sin(t-0.7)$~~ $t \geq 0 : G_7 = \sin(t-0.7)$

$\therefore$  X $t=0 : \sin(0-0.7) = \sin(-0.7) = G_7$: 

3viii $G_8(t) = \sin(t-0.7)H(t-0.7) \therefore t < 0.7 :$ $G_8 = 0$

$t \geq 0.7$ $G_8 = \sin(t-0.7)$ $\therefore$ $t = 0.7 :$ $G_8 = \sin(0.7-0.7) = \sin(0)$ 

4i $F'' - 4F' + 3F = 1$ $F(0) = F'(0) = 0$ $\therefore$ is $\mathcal{L}(F) = f(s)$ $\therefore$

$\mathcal{L}(F') = s f(s)$ & $\mathcal{L}(F'') = s^2 f(s) \therefore$

$\mathcal{L}(F'' - 4F' + 3F) = \mathcal{L}(1) = \mathcal{L}(F'') - 4\mathcal{L}(F') + 3\mathcal{L}(F) =$

$s^2 f(s) - 4s f(s) + 3 f(s) = \frac{1}{s} = (s^2 - 4s + 3) f(s)$ $\therefore f(s) = \frac{1}{s} \frac{1}{s^2 - 4s + 3}$

$= \frac{1}{s(s-3)(s-1)} = f(s) = \frac{A}{s} + \frac{B}{s-3} + \frac{C}{s-1}$ $\therefore$

$1 = A(s-3)(s-1) + Bs(s-1) + Cs(s-3)$ $\therefore$ $s=0 :$ $A = 1/3$,

$s = 3 = B = 1/6$ $s = 1 : C = -1/2 \therefore$ $f(s) = \frac{1}{3s} + \frac{1}{6(s-3)} - \frac{1}{2(s-1)} \therefore$

$F(t) = \mathcal{L}^{-1}(f(s)) = \mathcal{L}^{-1}\left(\frac{1}{3s} + \frac{1}{6(s-3)} - \frac{1}{2(s-1)}\right) = \frac{1}{3}\mathcal{L}^{-1}\left(\frac{1}{s}\right) + \frac{1}{6}\mathcal{L}^{-1}\left(\frac{1}{s-3}\right) - \frac{1}{2}\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) =$

$\frac{1}{3} + \frac{1}{6}e^{3t} - \frac{1}{2}e^{t} = F(t)$

4ii $F'' - 2aF' + a^2 F = 0$ $F'(0) = 1$, $F(0) = 0$ $\therefore$

$\mathcal{L}(F'') = s^2 f(s) - sF(0) - F'(0) = s^2 f(s) - s(0) - 1 = s^2 f(s) - 1$

$\mathcal{L}(F') = s f(s) - F(0) = s f(s) - 0 = s f(s)$ $\therefore \mathcal{L}(-2aF') = -2a\mathcal{L}(F') = -2as f(s)$

$\mathcal{L}(F) = f(s) \therefore$ $\mathcal{L}(a^2 F) = a^2 \mathcal{L}(F) = a^2 f(s)$ $\therefore$

$\mathcal{L}(F'' - 2aF' + a^2F) = \mathcal{L}(F'') - 2a\mathcal{L}(F') + a^2\mathcal{L}(F) =$

$s^2y(s) - 1 - 2asy(s) + a^2y(s) = \mathcal{L}(0) = 0 = y(s)(s^2 - 2as - a^2) - 1 \therefore$

$1 = (s^2 - 2as - a^2)y(s) = (s-a)^2y(s) \quad \therefore \quad y(s) = \frac{1}{(s-a)^2} \quad \therefore LT2, LT13$

$\mathcal{L}^{-1}\left(\frac{1}{(s-a)^2}\right) = \mathcal{L}^{-1}(\hat{f}(s-a)) = e^{at}\hat{F}(t), \quad \hat{F}(t) = \mathcal{L}^{-1}(\hat{f}(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = \mathcal{L}^{-1}\left(\frac{1!}{s^{1+1}}\right) =$

$t^1 = t \quad \therefore \quad \mathcal{L}^{-1}\left(\frac{1}{(s-a)^2}\right) = e^{at}t \quad \therefore$

$\mathcal{L}^{-1}(y(s)) = F(t) = \mathcal{L}^{-1}\left(\frac{1}{(s-a)^2}\right) = e^{at}t = F(t) = te^{at}$

4 iii $\quad F'' - 4F' + 3F = 0 \quad F(0) = 1, \; F'(0) = 1 \therefore$

$\mathcal{L}(F') = sf - F(0), \quad \mathcal{L}(F'') = s^2f - sF(0) - F'(0) \therefore$

$s^2f(s) - s - 1 - 4sf(s) + 4 + 3f(s) = 0 \therefore$

$(s^2 - 4s + 3)f(s) = s - 3 \therefore \quad f(s) = \frac{1}{s-1} \quad \therefore F(t) = e^t$

5 $\quad f(s) = \frac{1}{s^4+1} = \frac{1}{(s^2+\sqrt{2}s+1)(s^2-\sqrt{2}s+1)} = \frac{As+B}{s^2+\sqrt{2}s+1} + \frac{Cs+D}{s^2-\sqrt{2}s+1} =$

$\frac{\frac{1}{2\sqrt{2}}s + \frac{1}{2}}{s^2+\sqrt{2}s+1} + \frac{\frac{1}{2} - \frac{1}{2\sqrt{2}}s}{s^2-\sqrt{2}s+1} = \frac{s+\sqrt{2}}{2\sqrt{2}(s^2+\sqrt{2}s+1)} - \frac{s-\sqrt{2}}{2\sqrt{2}(s^2-\sqrt{2}s+1)} =$

$\frac{1}{2\sqrt{2}} \cdot \frac{s+\sqrt{2}/2}{(s+\sqrt{2}/2)^2+\frac{1}{2}} + \frac{1}{4} \cdot \frac{1}{(s+\frac{\sqrt{2}}{2})^2+\frac{1}{2}} - \frac{1}{2\sqrt{2}} \cdot \frac{s-\frac{\sqrt{2}}{2}}{(s-\frac{\sqrt{2}}{2})^2+\frac{1}{2}} + \frac{1}{4} \cdot \frac{1}{(s-\frac{\sqrt{2}}{2})^2+\frac{1}{2}} \therefore$

by $\mathcal{L}^{-1}(\;)$:

$F(t) = \frac{1}{2\sqrt{2}}e^{-\frac{t}{\sqrt{2}}}\cos\left(\sqrt{\tfrac{1}{2}}t\right) + \frac{1}{4}\sqrt{2}e^{-\frac{t}{\sqrt{2}}}\sin\left(\sqrt{\tfrac{1}{2}}t\right) - \frac{1}{2\sqrt{2}}e^{\frac{t}{\sqrt{2}}}\cos\left(\sqrt{\tfrac{1}{2}}t\right) + \frac{1}{4}\sqrt{2}e^{t/\sqrt{2}}\sin\left(\sqrt{\tfrac{1}{2}}t\right)$

Sheet 3 $\quad \mathcal{L}\left(\int_0^t F(u)G(t-u)\,du\right) = f(s)g(s) \quad \therefore \mathcal{L}^{-1}(f(s)g(s)) = \int_0^t F(u)G(t-u)\,du$

i

$f(s) = \frac{s}{(s^2+\omega^2)^2} \quad f(s) = g(s)k(s) \therefore \quad \frac{s}{(s^2+\omega^2)^2} = \frac{s}{s^2+\omega^2} \cdot \frac{1}{s^2+\omega^2} \quad \therefore$

$g(s) = \frac{s}{s^2+\omega^2} \quad \& \quad k(s) = \frac{1}{s^2+\omega^2} \quad \therefore$

$G(t) = \cos(\omega t) \quad \& \quad K(t) = \frac{1}{\omega}\sin(\omega t) \quad \Big\{ \mathcal{L}^{-1}(k(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^2+\omega^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{\omega}\frac{\omega}{s^2+\omega^2}\right) =$

$\frac{1}{\omega}\mathcal{L}^{-1}\left(\frac{\omega}{s^2+\omega^2}\right) = \frac{1}{\omega}\sin(\omega t) = K(t) \Big\}$

$g(s)k(s) = \mathcal{L}(G(t) * K(t)) \quad \therefore \mathcal{L}^{-1}(g(s)k(s)) = G(t) * K(t) \therefore$

$\mathcal{L}^{-1}(g(s)k(s)) = G(t) * K(t) = \cos(\omega t) * \frac{1}{\omega}\sin(\omega t) = \int_0^t \frac{1}{\omega}\cos(\omega u)\sin(\omega(t-u))\,du$

$\therefore \{\cos(A)\sin(B) = \frac{1}{2}(\sin(A+B) - \sin(A-B))\}$

$\therefore \; = \frac{1}{2\omega}\int_0^t \sin(\omega t) - \sin(2\omega u - \omega t)\,du = \frac{1}{2\omega}t\sin(\omega t)$

1 ii $\quad f(s) = \frac{1}{s^2(s+1)^2} = \frac{1}{s^2}\frac{1}{(s+1)^2} \quad \therefore$

$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t, \quad \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) = te^{-t} \quad \therefore$

$\mathcal{L}^{-1}\left(\frac{1}{s^2(s+1)^2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\frac{1}{(s+1)^2}\right) = \mathcal{L}^{-1}(g(s)k(s)) = G(t) * K(t) = t * te^{-t} =$

\Sheet week 3/ $t * te^{-t} = \int_0^t u(t-u)e^{-(t-u)}\,du =$

$\int_0^t ute^{u-t} - u^2e^{u-t}\,du \quad \therefore$ by IBP :

$= e^{-t}(t+2) + t - 2$

$\int_0^t ute^{u-t} - u^2e^{u-t}\,du = \int_0^t ute^{u-t}\,du + \int_0^t -u^2e^{u-t}\,du \quad \therefore$

$\int_0^t ute^{u-t}\,du = \left[ute^{u-t}\right]_0^t - \int_0^t te^{u-t}\,du =$

$tte^{t-t} - \left[te^{u-t}\right]_0^t = t^2 - te^{t-t} + te^{0-t} = t^2 - t + te^{-t}$

$\int_0^t -u^2 e^{u-t}\,du = \left[-u^2e^{u-t}\right]_0^t - \int_0^t -2ue^{u-t}\,du =$

$-t^2e^{t-t} - \left[-2ue^{u-t}\right]_0^t + \int_0^t -2e^{u-t}\,du =$

$-t^2 + 2te^{t-t} + \left[-2e^{u-t}\right]_0^t = -t^2 + 2t - 2e^{t-t} + 2e^{0-t} = -t^2 + 2t - 2 + 2e^{-t} \quad \therefore$

$\therefore \int_0^t ute^{u-t} - u^2e^{u-t}\,du = t^2 - t + te^{-t} - t^2 + 2t - 2 + 2e^{-t} = t + te^{-t} - 2 + 2e^{-t} =$

$e^{-t}(t+2) + t - 2$ ✓

\iii/ $S(s) = \frac{1}{(s+1)^2(s^2+4)} = \frac{1}{(s+1)^2} \cdot \frac{1}{(s^2+4)} \quad \therefore \quad \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) = te^{-t}$,

$\mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = \mathcal{L}^{-1}\left(\frac{1}{2}\frac{2}{s^2+4}\right) = \mathcal{L}^{-1}\left(\frac{1}{2}\frac{2}{s^2+2^2}\right) = \frac{1}{2}\mathcal{L}^{-1}\left(\frac{2}{s^2+2^2}\right) = \frac{1}{2}\sin(2t) \quad \therefore$

$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2(s^2+4)}\right) = \mathcal{L}^{-1}\left(\frac{1}{(s+1)^2}\right) * \mathcal{L}^{-1}\left(\frac{1}{s^2+4}\right) = te^{-t} * \frac{1}{2}\sin(2t) =$

$\frac{1}{2}\int_0^t ue^{-u}\sin(2t-2u)\,du = 2I \quad \therefore \quad I = \int_0^t ue^{-u}\sin(2t-2u)\,du$ &

$J = \int_0^t e^{-u}\sin(2t-2u)\,du = \left[\frac{1}{2}e^{-u}\cos(2t-2u)\right]_0^t + \left[-\frac{1}{2}e^{-u}\sin(2t-2u)\right]_0^t - \frac{1}{2}J$

$\therefore J = \frac{2}{5}e^{-t} - \frac{2}{5}\cos(2t) + \frac{1}{5}\sin(2t) \quad \therefore \quad U = ue^{-u}$, $V' = \sin(2t-2u) \quad \therefore$

$I = \left[\frac{1}{2}ue^{-u}\cos(2t-2u)\right]_0^t + \frac{1}{4}\left[(e^{-u} - ue^{-u})\sin(2t-2u)\right]_0^t + \frac{1}{4}\int_0^t (2e^{-u} - ue^{-u})\sin(2t-2u)\,du$

$= \frac{te^{-t}}{2} - \frac{1}{4}\sin(2t) + \frac{1}{2}J - \frac{1}{4}I \quad \therefore$

$I = \frac{2}{5}te^{-t} + \frac{4e^{-t}}{25} - \frac{4}{25}\cos(2t) - \frac{3}{25}\sin(2t)$

\iv/ $S(s) = \frac{e^{-3s}}{(s-1)(s-2)} = e^{-3s} \cdot \frac{1}{s-1} \cdot \frac{1}{s-2} \quad \therefore$

$\mathcal{L}^{-1}\left(\frac{1}{s-1}\right) = e^t$, $\mathcal{L}^{-1}\left(\frac{1}{s-2}\right) = e^{2t} \quad \therefore \quad \mathcal{L}^{-1}\left(\frac{1}{(s-1)(s-2)}\right) = e^t * e^{2t} = \int_0^t e^u e^{2(t-u)}\,du =$

$\int_0^t e^{2t-u}\,du = -e^{t} + e^{2t} \quad \therefore \quad$ LT17 $a = 3$

$\mathcal{L}^{-1}\left(\frac{e^{-3s}}{(s-1)(s-2)}\right) = \mathcal{L}^{-1}\left(e^{-3s}\frac{1}{(s-1)(s-2)}\right) = \mathcal{L}^{-1}\left(e^{-as}\hat{S}(s)\right) = \hat{F}(t-a)H(t-a) =$

$\hat{F}(t-3)H(t-3) = H(t-3)\left(e^{2(t-3)} - e^{t-3}\right)$

\2a/ {nonlin nonvolterra ∴ Fredholm, 2nd kind nonhomog}
nonline, nonhomog, Fredholm IE of Z 2nd kind
\2b/ {homog 2nd kind linear volterra}
linear, homog, volterra IE of Z 2nd kind
\2c/ {1st kind linear Fredholm non homog}
linear, nonhomog, Fredholm IE of Z 1st kind
\2d/ {Fredholm nonhomog 2nd kind linear}
linear, nonhomog, Fredholm IE of Z 2nd kind

\Sheet 4/

\1a/ $y(x) - \int_0^1 x^2 t^2 y(t)\,dt = 1$

Let $P = \int_0^1 t^2 y(t)\,dt$ ∴ $y(x) = 1 + \int_0^1 x^2 t^2 y(t)\,dt = 1 + Px^2$ ∴ $y(t) = 1 + Pt^2$ ∴

$P = \int_0^1 t^2(1+Pt^2)\,dt = \frac{1}{3} + \frac{P}{5}$ ∴ $P = \frac{5}{12}$ ∴

$y(x) = 1 + \frac{5}{12}x^2$

\1b/ $y(x) - \int_0^1 xt\,y(t)\,dt = \sin x$

Let $P = \int_0^1 t\,y(t)\,dt$ ∴ $y(x) = \sin x + x\int_0^1 t\,y(t)\,dt = \sin x + Px$ $y(t) = \sin t + Pt$ ∴

$P = \int_0^1 t(\sin t + Pt)\,dt = \int_0^1 t\sin t\,dt + P\int_0^1 t^2\,dt = \sin 1 - \cos 1 + \frac{P}{3}$ ∴

$P = \frac{3}{2}(\sin 1 - \cos 1)$ ∴ $y(x) = \sin x + \frac{3}{2}(\sin 1 - \cos 1)x$

\1c/ $y(x) - \int_0^1 \ln\left(\frac{x}{t}\right) y(t)\,dt = 1 = y(x) - \int_0^1 (\ln x - \ln t)\,y(t)\,dt$ ∴

$y(x) - \ln x\int_0^1 y(t)\,dt + \int_0^1 \ln t\,y(t)\,dt = 1$ ∴

Let $P_1 = \int_0^1 y(t)\,dt$, $P_2 = \int_0^1 \ln t\,y(t)\,dt$ ∴ $y(x) = 1 + P_1\ln x - P_2$ ∴ $y(t) = 1 + P_1\ln t - P_2$

∴ $y(x) - \ln x\,P_1 + P_2 = 1$ ∴

{$y(x) - \ln x\int_0^1 1 + P_1\ln t - P_2\,dt + \int_0^1 \ln t\,(1 + P_1\ln t - P_2)\,dt = 1$}

$P_1 = 1 + P_1\int_0^1 \ln t\,dt - P_2$

$P_2 = \int_0^1 \ln t\,dt + P_1\int_0^1 (\ln t)^2\,dt - P_2\int_0^1 \ln t\,dt$

{ $P_1 = \int_0^1 y(t) = \int_0^1 1 + P_1\ln t - P_2\,dt = \int_0^1 1\,dt + \int_0^1 P_1\ln t - \int_0^1 P_2\,dt =$

$[t]_0^1 + P_1\int_0^1 \ln t - P_2\int_0^1 1\,dt = 1 + P_1\int_0^1 \ln t - P_2$

$P_2 = \int_0^1 \ln t\,y(t)\,dt = \int_0^1 \ln t\,(1 + P_1\ln t - P_2)\,dt = \int_0^1 \ln t\,dt + P_1\int_0^1 (\ln t)^2\,dt - P_2\int_0^1 \ln t\,dt$ }

Sheet 4 / 2b / $y(x) \mp \int_0^x \sin(x-t)\,y(t)\,dt = \sin x$ ∴

$y(x) - \varepsilon\int_0^x \sin(x-t)\,y(t)\,dt = \sin x,\ \varepsilon = \pm 1$ ∴ diffs ∴ $y(x) - \sin x = \varepsilon\int_0^x \sin(x-t)\,y(t)\,dt$

∴ $\frac{d}{dx}\left(y(x) - \varepsilon\int_0^x \sin(x-t)\,y(t)\,dt\right) = \frac{d}{dx}(\sin x) = \cos x =$

$\frac{d}{dx}(y(x)) - \varepsilon\frac{d}{dx}\left(\int_0^x (\sin x\cos t - \cos x\sin t)\,y(t)\,dt\right) =$

$y'(x) - \varepsilon\frac{d}{dx}\left(\sin x\int_0^x \cos t\,y(t)\,dt\right) - \varepsilon\frac{d}{dx}\left(-\cos x\int_0^x \sin t\,y(t)\,dt\right) =$

$y'(x) - \varepsilon\cos x\int_0^x \cos t\,y(t)\,dt - \varepsilon\sin x\cos x\,y(x) - \varepsilon\sin x\int_0^x \sin t\,y(t)\,dt + \varepsilon\cos x\sin x\,y(x) =$

$y'(x) - \varepsilon\int_0^x (\sin x\sin t + \cos x\cos t)\,y(t)\,dt = y'(x) - \varepsilon\int_0^x \cos(x-t)\,y(t)\,dt = \cos(x)$ ∴

$\frac{d}{dx}\left(y'(x) - \varepsilon\int_0^x \cos(x-t)\,y(t)\,dt\right) = \frac{d}{dx}(\cos x) = -\sin x =$

$y''(x) - \varepsilon\frac{d}{dx}\left(\sin x\int_0^x \sin t\,y(t)\,dt\right) - \varepsilon\frac{d}{dx}\left(\cos x\int_0^x \cos t\,y(t)\,dt\right) =$

$y''(x) - \varepsilon\cos x\int_0^x \sin t\,y(t) - \varepsilon\sin x\sin x\,y(x) + \varepsilon\sin x\int_0^x \cos t\,y(t)\,dt - \varepsilon\cos x\cos x\,y(x) =$

$y''(x) + \varepsilon\int_0^x (\sin x\cos t - \cos x\sin t)\,y(t)\,dt - \varepsilon y(x)$

$y''(x) + \varepsilon\int_0^x \sin(x-t)\,y(t)\,dt - \sin x = y''(x) + y(x) - \sin x - \varepsilon y(x) = -\sin x$ ∴

$y'' + (1-\varepsilon)\,y(x) = 0$ IC : $y(0) = 0$, $y'(0) = 1$ when $\varepsilon = 1$;

$y(x) = x$, when $\varepsilon = -1$: $y(x) = \frac{1}{\sqrt{2}}\sin\sqrt{2}\,x$

by Laplace : $\mathcal{L}\left(y(x) - \varepsilon\int_0^x \sin(x-t)\,y(t)\,dt\right) = \mathcal{L}(\sin x) = \frac{1}{s^2+1} =$

$\hat{y}(s) - \varepsilon\mathcal{L}\left(\int_0^x y(t)\sin(x-t)\,dt\right) = \hat{y}(s) - \varepsilon\mathcal{L}\left(\int_0^x y(u)\sin(x-u)\,du\right) =$

$\hat{y}(s) - \varepsilon\mathcal{L}\left(\int_0^x F(u)\,G(x-u)\,du\right) = \hat{y}(s) - \varepsilon f(s)\,g(s) = \hat{y}(s) - \varepsilon\hat{y}(s)\,g(s) =$

$\hat{y}(s) - \varepsilon\hat{y}(s)\,\mathcal{L}(\sin x) = \hat{y}(s) - \varepsilon\hat{y}(s)\frac{1}{s^2+1} = \frac{1}{s^2+1}$ ∴ $\hat{y}(s) = \frac{1}{s^2+1} + \varepsilon\frac{1}{s^2+1}\hat{y}(s)$ ∴

$\hat{y}(s) = \frac{1}{s^2+(1-\varepsilon)}$ ∴ when $\varepsilon = 1$: $\hat{y}(s) = \frac{1}{s^2}$, when $\varepsilon = -1$, $\hat{y}(s) = \frac{1}{sp^2+2}$

∴ $\mathcal{L}^{-1}(\hat{y}(s)) = \mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = y(x) = x$ for $\varepsilon = 1$

when $\varepsilon = -1$: $\mathcal{L}^{-1}(\hat{y}(s)) = \mathcal{L}^{-1}\left(\frac{1}{sp^2+2}\right) = \mathcal{L}^{-1}\left(\frac{1}{s^2+2}\right) = \mathcal{L}^{-1}\left(\frac{1}{\sqrt{2}}\frac{\sqrt{2}}{s^2+2}\right) = \frac{1}{\sqrt{2}}\mathcal{L}^{-1}\left(\frac{\sqrt{2}}{s^2+(\sqrt{2})^2}\right) =$

$\frac{1}{\sqrt{2}}\sin(\sqrt{2}\,x) = y(x)$

Sheet 27/ $\therefore \int_0^1 \ln t\,dt = \int_0^1 1\ln t\,dt = [t\ln t]_0^1 - \int_0^1 \frac{1}{t} t\,dt = [t\ln t - t]_0^1$

$= -1 \qquad \left\{ \int_0^1 (\ln t)^2 dt = \int_0^1 \ln t \ln t\,dt = [(\ln t)(t\ln t - t)]_0^1 - \int_0^1 \frac{1}{t}(t\ln t - t)dt \right\}$

$\int_0^1 (\ln t)^2 dt = \int_0^1 1\cdot(\ln t)^2 dt = [t(\ln t)^2]_0^1 - \int_0^1 2(\ln t)\frac{1}{t} t = [\cancel{t}(\ln t)^2]_0^1 - 2\int_0^1 \ln t\,dt = 2$

$\therefore P_1 = 1 + P_1 \int_0^1 \ln t\,dt - P_2 = 1 - P_1 - P_2 = P_1 \quad \therefore$

$P_2 = \int_0^1 \ln t\,dt + P_1 \int_0^1 (\ln t)^2 dt - P_2 \int_0^1 \ln t\,dt = P_2 = -1 + 2P_1 + P_2 \quad \therefore$

$P_1 = \frac{1}{2}, \; P_2 = 0 \quad \therefore \; y(x) = 1 + P_1 \ln x - P_2 = y(x) = 1 + \ln(\sqrt{x}) = \ln(e\sqrt{x})$

12a/ $y(x) - 2\int_0^x \cos(x-t)\,y(t)\,dt = 1$

by Laplace: let $\hat{y}(s) = \mathcal{L}(y(x)) \therefore \mathcal{L}\left(y(x) - 2\int_0^x \cos(x-t)\,y(t)\,dt\right) = \mathcal{L}(1) =$

$\frac{1}{s} = \mathcal{L}(y(x)) - 2\mathcal{L}\left(\int_0^x \cos(x-t)y(t)dt\right) = \mathcal{L}(y(x)) - 2\mathcal{L}\left(\int_0^x \cos(x-t)y(t)dt\right)$

$\mathcal{L}\left(\int_0^x \cos(x-t)\,y(t)\,dt\right) = \mathcal{L}\left(\int_0^x \cos(x-u)\,y(u)\,du\right) =$

$\mathcal{L}\left(\int_0^x y(u)\cos(x-u)\,du\right) = \mathcal{L}\left(\int_0^x F(u)G(t-u)\,du\right) \quad \{LT22\}$

$= f(s)g(s) = \hat{y}(s)g(s) = \hat{y}(s)\,\mathcal{L}[\cos(x)] = \hat{y}(s)\frac{1}{s^2+1}$

$\mathcal{L}\left(y(x) - 2\int_0^x \cos(x-t)\,y(t)\,dt\right) = \mathcal{L}(1) = \frac{1}{s} = \mathcal{L}(y(x)) - 2\mathcal{L}\left(\int_0^x y(t)\cos(x-t)\,dt\right) =$

$\frac{1}{s} = \hat{y}(s) - 2\frac{s}{s^2+1}\hat{y}(s) = \hat{y}(s)\left(1 - 2\frac{s}{s^2+1}\right) \therefore$

$\hat{y}(s) = \frac{s^2+1}{s(s-1)^2} = \frac{1}{s} + \frac{2}{(s-1)^2} \quad \therefore$

$\mathcal{L}^{-1}(\hat{y}(s)) = \mathcal{L}^{-1}\left(\frac{1}{s} + \frac{2}{(s-1)^2}\right) = y(s) = \mathcal{L}^{-1}\left(\frac{1}{s}\right) + 2\mathcal{L}\left(\frac{1}{(s-1)^2}\right) = 1 + 2xe^x$

by deriv: diffing: $y'(x) - 2\int_0^x \sin(x-t)\,y(t)\,dt = 2y(x)$

$\left\{ y(x) - 2\int_0^x \cos(x-t)\,y(t)\,dt = 1 \right. \therefore \frac{d}{dx}\left[y(x) - 2\int_0^x \cos(x-t)\,y(t)\,dt\right] = \frac{d}{dx}(1) = 0 =$

$\frac{d}{dx}(y(x)) - 2\frac{d}{dx}\left(\int_0^x \cos(x-t)\,y(t)\,dt\right) = y'(x) - 2\frac{d}{dx}\left(\int_{g(x)}^{h(x)} h(t)\,dt\right) =$

$y'(x) - 2\left[h(f(x))f'(x) - h(g(x))\cdot g'(x)\right]$

$\frac{d}{dx}\left[\int_0^x \cos(x-t)\,y(t)\,dt\right] = \cancel{h(x)} \quad \therefore \cos(x-t) = \cos(x)\cos t + \sin x \sin t \therefore$

$\int_0^x \cos(x-t)\,y(t)\,dt = \int_0^x \cos x \cos t\,y(t)\,dt + \int_0^x \sin x \sin t\,y(t)\,dt =$

$\cos(x)\int_0^x \cos t\,y(t)\,dt + \sin x \int_0^x \sin t\,y(t)\,dt \quad \therefore$

$\frac{d}{dx}\left(\int_0^x \cos(x-t)\,y(t)\,dt\right) = \frac{d}{dx}\left(\cos x\int_0^x \cos t\,y(t)\,dt\right) + \frac{d}{dx}\left(\sin x \int_0^x \sin t\,y(t)\,dt\right) =$

Sheet 4 2a continued

$-\sin x\int_0^x \cos t\, y(t)\,dt + \cos x \cos x\, y(x) + \cos x\int_0^x \sin t\, y(t)\,dt + \sin x \sin x\, y(x) =$

$-\sin x\int_0^x \cos t\, y(t)\,dt + \cos x\int_0^x \sin t\, y(t)\,dt + y(x) =$

$y(x) + \int_0^x (\cos x \sin t - \sin x \cos t)\, y(t)\,dt =$

$y(x) - \int_0^x (\sin x \cos t - \cos x \sin t)\, y(t)\,dt$ $\cancel{= y(x) + \int_0^x}$

$= y(x) - \int_0^x \sin(x-t)\, y(t)\,dt \Rightarrow \frac{d}{dx}\left(\int_0^x \cos(x-t)\, y(t)\right) \therefore$

$\frac{d}{dx}\left[y(x) - 2\int_0^x \cos(x-t)\, y(t)\,dt\right] = \frac{d}{dx}(1) = 0 =$

$y'(x) - 2\frac{d}{dx}\left[\int_0^x \cos(x-t)\, y(t)\,dt\right] = y'(x) - 2\left[y(x) - \int_0^x \sin(x-t)\, y(t)\,dt\right] =$

$y'(x) - 2y(x) + 2\int_0^x \sin(x-t)\, y(t)\,dt = 0 \therefore$

$y'(x) + 2\int_0^x \sin(x-t)\, y(t)\,dt = 2y(x) \therefore$

$\frac{d}{dx}\left(y'(x) + 2\int_0^x \sin(x-t)\, y(t)\,dt\right) = y'' + 2\frac{d}{dx}\left(\int_0^x \sin(x-t)\, y(t)\,dt\right) = 2y'(x)$

$\frac{d}{dx}\left(\int_0^x \sin(x-t)\, y(t)\,dt\right) = \frac{d}{dx}\left(\int_0^x (\sin x \cos t - \cos x \sin t)\, y(t)\,dt\right) =$

$\frac{d}{dx}\left(\sin x\int_0^x \cos t\, y(t)\,dt\right) - \frac{d}{dx}\left(\cos x\int_0^x \sin t\, y(t)\,dt\right) =$

$\cos x\int_0^x \cos t\, y(t)\,dt + \sin x \cos x\, y(x) + \sin(x)\int_0^x \sin t\, y(t)\,dt - \cos x \sin x\, y(x) =$

$\int_0^x (\cos x \cos t + \sin x \sin t)\, y(t)\,dt = \int_0^x \cos(x-t)\, y(t) \therefore$

$y''(x) + 2\int_0^x \cos(x-t)\, y(t)\,dt = 2y'(x) = y''(x) + (y(x) - 1) = 2y'(x) \therefore$

$y''(x) - 2y'(x) + y(x) = 1$, $y(0) = 1$, $y'(0) = 2 \therefore$

$y(x) = Ae^x + Bx e^x + 1 \quad \therefore A = 0,\ B = 2 \therefore \quad y(x) = 1 + 2x e^x$

2c $y(x) + \int_0^x \cos(x-t)\, y(t)\,dt = \cos x \therefore$

$y'(x) + \frac{d}{dx}\left(\int_0^x (\cos x \cos t + \sin x \sin t)\, y(t)\,dt\right) = -\sin x \Rightarrow$

$y'(x) + \frac{d}{dx}\left(\cos x\int_0^x \cos t\, y(t)\,dt\right) + \frac{d}{dx}\left(\sin x\int_0^x \sin t\, y(t)\,dt\right) =$

$y'(x) - \sin x\int_0^x \cos t\, y(t)\,dt + \cos x \cos x\, y(x) + \cos x\int_0^x \sin t\, y(t)\,dt + \sin x \sin x\, y(x) =$

$y'(x) + y(x) - \int_0^x (\sin x \cos t - \cos x \sin t)\, y(t)\,dt =$

$y'(x)+y(x)-\int_0^x \sin(x-t)y(t)\,dt=-\sin x$ $\therefore$ $+y(x)\cancel{+}\cos x=-\int_0^x \cos(x-t)y(t)\,dt$

$\therefore$ $y''(x)+y'(x)+\frac{d}{dx}\left(-\sin x\int_0^x \cos t\, y(t)\,dt\right)+\frac{d}{dx}\left(\cos x\int_0^x \sin t\, y(t)\,dt\right)=-\cos x=$

$y''(x)+y'(x)-\cos x\int_0^x \cos t\, y(t)\,dt-\sin x\cos x\, y(x)-\sin x\int_0^x \sin t\, y(t)\,dt+\cos x\sin x\, y(x)=$

$y''(x)+y'(x)-\int_0^x (\cos x\cos t+\sin x\sin t)y(t)\,dt=$

$y''(x)+y'(x)-\int_0^x \cos(x-t)y(t)\,dt=y''(x)+y'(x)+y(x)-\cos x=-\cos x$ $\therefore$

$y''(x)+y'(x)+y(x)=0$, $y(0)=1$, $y'(0)=-1$ $\therefore$

$y(x)=e^{-x/2}\left(\cos\frac{\sqrt{3}}{2}x-\frac{1}{\sqrt{3}}\sin\frac{\sqrt{3}}{2}x\right)$

Laplace: $\mathcal{L}\{y(x)+\int_0^x \cos(x-t)y(t)\,dt\}=\mathcal{L}\{\cos x\}=\frac{s}{s^2+1}=$

$\hat{y}(s)+\mathcal{L}\left(\int_0^x y(t)\,dt\cos(x-t)\,dt\right)=\hat{y}(s)+\mathcal{L}\{y(x)\}\mathcal{L}(\cos(x))=$

$\hat{y}(s)+\hat{y}(s)\frac{s}{s^2+1}=\hat{y}(s)\left(1+\frac{s}{s^2+1}\right)=\frac{s}{s^2+1}=\hat{y}(s)\left(\frac{s^2+1+s}{s^2+1}\right)$ $\therefore$

$\hat{y}(s)=\frac{s}{s^2+s+1}=\frac{s+\frac{1}{2}-\frac{1}{2}}{(s+\frac{1}{2})^2+1-\frac{1}{4}}=\frac{s+\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}-\frac{\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}=\hat{y}(s)$ $\therefore$

$\mathcal{L}^{-1}(\hat{y}(s))=\mathcal{L}^{-1}\left(\frac{s-\frac{1}{2}}{(s+\frac{1}{2})^2+\frac{3}{4}}\right)-\mathcal{L}^{-1}\left(\frac{1}{\sqrt{3}}\frac{(\sqrt{3}/2)}{(s+\frac{1}{2})^2+(\frac{\sqrt{3}}{2})^2}\right)=$

$e^{-\frac{1}{2}x}\cos\left(\frac{\sqrt{3}}{2}x\right)-\frac{1}{\sqrt{3}}e^{-\frac{1}{2}x}\sin\left(\frac{\sqrt{3}}{2}x\right)=e^{-\frac{x}{2}}\left(\cos\left(\frac{\sqrt{3}}{2}x\right)-\frac{1}{\sqrt{3}}\sin\left(\frac{\sqrt{3}}{2}x\right)\right)=y(x)$

\3 sol/ $kx f(x)=\int_0^x f(t)\,dt$ $\therefore$ diff:

$\frac{d}{dx}(kx f(x))=\frac{d}{dx}\left(\int_0^x f(t)\,dt\right)=kf(x)+kxf'(x)=f(x)$ $\therefore$

$kxf'(x)=f(x)-kf(x)=f(x)(1-k)$ $\therefore$ $\frac{f'(x)}{f(x)}=\frac{1-k}{kx}=\frac{f'}{f}$ $\therefore$

$\int\frac{f'}{f}dx=\int\frac{1-k}{k}\frac{1}{x}dx=\ln(f)+(\ln x)\left(\frac{1-k}{k}\right)+C=\ln\left(x^{(1-k)/k}\right)$ $\therefore$

$f(x)=e^{\ln(x^{(1-k)/k})+c_2}=e^{c_2}e^{\ln(x^{(1-k)/k})}=cx^{(1-k)/k}=f(x)=Cx^{\frac{1-k}{k}}$

\4 sol/ $\lambda y(x)-\int_0^1 (x-t)y(t)\,dt=1=\lambda y(x)-x\int_0^1 y(t)\,dt+\int_0^1 t y(t)\,dt=1$ $\therefore$

$P_1=\int_0^1 y(t)\,dt$, $P_2=\int_0^1 t y(t)\,dt$ $\therefore$

$\lambda y(x)=1+P_1x\cancel{+}P_2$ $\therefore$ $\lambda y(t)=1+P_1t+P_2$ $\therefore$ $y(t)=\frac{1}{\lambda}+\frac{1}{\lambda}P_1t+\frac{1}{\lambda}P_2$ $\therefore$

$\therefore$ $P_1=\int_0^1 \frac{1}{\lambda}+\frac{1}{\lambda}P_1t\cancel{+}\frac{1}{\lambda}P_2\,dt$ $\therefore$ $\lambda P_1=\int_0^1 1+P_1t\cancel{+}P_2\,dt=\left[t+\frac{1}{2}P_1t^2\cancel{+}P_2t\right]_0^1=$

$\lambda P_1=1+\frac{1}{2}P_1-P_2$ $\quad P_2=\int_0^1 t\frac{1}{\lambda}(1+P_1t+P_2)\,dt$ $\therefore$ $\lambda P_2=\int_0^1 t+P_1t^2+P_2t\,dt=$

$\left[\frac{1}{2}t^2+\frac{1}{3}P_1t^3+\frac{1}{2}P_2t^2\right]_0^1=\lambda P_2=\frac{1}{2}+\frac{1}{3}P_1-\frac{1}{2}P_2$ $\therefore$

Sheet 4 $\therefore$ $\lambda P_1 = 1 + \frac{1}{2}P_1 - P_2$, $\lambda P_2 = \frac{1}{2} + \frac{1}{3}P_1 - \frac{1}{2}P_2$ $\therefore \lambda \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \lambda P_1 \\ \lambda P_2 \end{bmatrix} = \lambda P = \begin{bmatrix} 1 + \frac{1}{2}P_1 - P_2 \\ \frac{1}{2} + \frac{1}{3}P_1 - \frac{1}{2}P_2 \end{bmatrix} \therefore$

$1 = (\lambda - \frac{1}{2})P_1 + P_2$, $\frac{1}{2} = (\lambda + \frac{1}{2})P_2 - \frac{1}{3}P_1$ $\therefore$

$P = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \therefore \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} (\lambda - \frac{1}{2})P_1 + P_2 \\ -\frac{1}{3}P_1 + (\lambda + \frac{1}{2})P_2 \end{bmatrix} = \begin{bmatrix} (\lambda - \frac{1}{2}) & 1 \\ -\frac{1}{3} & \lambda + \frac{1}{2} \end{bmatrix} \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \begin{bmatrix} \lambda - \frac{1}{2} & 1 \\ -\frac{1}{3} & \lambda + \frac{1}{2} \end{bmatrix} P = A(\lambda)P$

Syst has a unique sol $\forall \lambda \in \mathbb{R} \because \det[A(\lambda)] = \lambda^2 + \frac{1}{12} \neq 0 \therefore$

Z sol is : $(A(\lambda))^{-1} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} = \frac{1}{\lambda^2 + (\frac{1}{12})} \begin{bmatrix} \lambda + \frac{1}{2} & -1 \\ \frac{1}{3} & \lambda - \frac{1}{2} \end{bmatrix} \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix} = \begin{bmatrix} P_1 \\ P_2 \end{bmatrix} \therefore$

$P_1 = \frac{12\lambda}{12\lambda^2 + 1}$, $P_2 = \frac{6\lambda + 1}{12\lambda^2 + 1}$

and $\lambda y(x) = 1 + P_1 x - P_2$ $\therefore$ when $\lambda \neq 0$:

$$y(x) = 1 + P_1 x + P_2 = 1 + \frac{12\lambda}{12\lambda^2 + 1}x + \frac{6\lambda + 1}{12\lambda^2 + 1} = \frac{12x + 12\lambda - 6}{1 + 12\lambda^2}$$

When $\lambda = 0$ $\frac{12x + 12(0) - 6}{1 + 12(0)^2} = y_p(x) = 12x - 6$ is particular sol $\therefore$

$\lambda = 0 : 1 = -\int_0^1 (x-t)y(t)dt$ $\therefore$ add any sol to homog eqn to (PS) y_p $\therefore$

$\int_0^1 (x-t)y(t)dt = x\int_0^1 y(t)dt - \int_0^1 ty(t)dt = 0 = x\hat{P}_1 + \hat{P}_2$ $\therefore$

compare coeffs : $\int_0^1 y(t)dt = 0$, $\int_0^1 ty(t)dt = 0$ $\therefore$

Let $y(x) = a + bx + x^2 f(x)$ cont $f \in C[0,1]$ $\therefore y(t) = a + bt + t^2 f(t)$ $\therefore$

$x\int_0^1 a + bt + t^2 f(t)dt - \int_0^1 t(a + bt + t^2 f(t))dt =$

$x\int_0^1 a + bt + t^2 f(t)dt - \int_0^1 at + bt^2 + t^3 f(t)dt = 0$

$y_p(x) = 12x - 6$ $\quad y(x) = a + bx + x^2 f(x)$ $\therefore$

$x\int_0^1 y(t)dt - \int_0^1 ty(t)dt = 0$ $\therefore$ $a = 2\int_0^1 (3t^3 - 2t^2)f(t)dt$, $b = 6\int_0^1 (t^2 - 2t^3)f(t)dt$

$\therefore \int_0^1 a + bt\, dt = [at + \frac{1}{2}bt^2]_0^1 = a + \frac{1}{2}b$ $\therefore$

$\int_0^1 at + bt^2 dt = [\frac{1}{2}at^2 + \frac{1}{3}bt^3]_0^1 = \frac{1}{2}a + \frac{1}{3}b$ $\therefore$

$ax + \frac{1}{2}bx + x\int_0^1 t^2 f(t)dt - \frac{1}{2}a - \frac{1}{3}b - \int_0^1 t^3 f(t)dt = 0$

$a = \int_0^1 (6t^3 - 4t^2)f(t)dt$, $b = \int_0^1 (6t^2 - 12t^3)f(t)dt$

$y_p(t) = 12t - 6$ $\therefore$ $-\int_0^1 ty(t)dt = -\int_0^1 t(12t - 6)dt = \int_0^1 6t - 12t^2 dt$

$x\int_0^1 y(t)\,dt - \int_0^1 t y(t)\,dt = 0 \quad \therefore \int_0^1 y(t)\,dt = 0, \ \int_0^1 t y(t)\,dt = 0 \ \therefore$

$y(x) = a + bx + x^2 f(x) \ \therefore \quad y(t) = a + bt + t^2 f(t) \ \therefore$

$\int_0^1 a + bt + t^2 f(t)\,dt = 0 = \int_0^1 a + bt\,dt + \int_0^1 t^2 f(t)\,dt =$

$\left[at + \frac{1}{2}bt^2\right]_0^1 + \int_0^1 t^2 f(t)\,dt = a + \frac{1}{2}b + \int_0^1 t^2 f(t)\,dt = 0$

$\int_0^1 t(a + bt + t^2 f(t))\,dt = 0 = \int_0^1 at + bt^2\,dt + \int_0^1 t^3 f(t)\,dt =$

$\left[\frac{1}{2}at^2 + \frac{1}{3}bt^3\right]_0^1 + \int_0^1 t^3 f(t)\,dt = \frac{1}{2}a + \frac{1}{3}b + \int_0^1 t^3 f(t)\,dt = 0 \ \therefore$

$a = -\frac{1}{2}b - \int_0^1 t^2 f(t)\,dt \ \therefore$

$\frac{1}{2}\left(-\frac{1}{2}b - \int_0^1 t^2 f(t)\,dt\right) + \frac{1}{3}b + \int_0^1 t^3 f(t)\,dt = 0 =$

$-\frac{1}{4}b - \frac{1}{2}\int_0^1 t^2 f(t)\,dt + \frac{1}{3}b + \int_0^1 t^3 f(t)\,dt = 0 \ \therefore$

$\frac{1}{12}b + \int (t^3 - \frac{1}{2}t^2) f(t)\,dt \quad \therefore \quad b = 6\int_0^1 (t^2 - 2t^3) f(t)\,dt$

$\therefore a = -\frac{1}{2} \times 6\int_0^1 (t^2 - 2t^3) f(t)\,dt - \int_0^1 t^2 f(t)\,dt = a = 2\int_0^1 (3t^2 - 2t^2) f(t)\,dt$

$\therefore$ GS for $\lambda = 0$: ~~$y(x) = 12x - 6 + x^2 f(x)$~~

$y(x) = 12x - 6 + x^2 f(x) + 2\int_0^1 (3t^2 - 2t^2) f(t)\,dt + 6x\int_0^1 (t^2 - 3t^2) f(t)\,dt$

for any choice of $f \in C[0,1]$

\3 Sol/ $k(x,t) = g(x,t)\ln|x-t|$

$\because g$ is bounded: $|k(x,t)| = |g(x,t)|\,|\ln|x-t|\,| \le C|\ln|x-t|\,| =$

$C|\ln|x-t|\,|\frac{|x-t|^\beta}{|x-t|^\beta} = C\frac{|\ln|x-t|\,|\,|x-t|^\beta}{|x-t|^\beta}$ $\therefore$ L'Hospital's rule

shows numerator is bounded as $t \to x$ be $\lim_{y\to 0} y^\beta \ln(y) = 0$ for

$y > 0,\ 0 < \beta < 1 \ \because \lim_{y\to 0} y^\beta \ln(y) = \lim_{y\to 0}\frac{\ln y}{y^{-\beta}} = \lim_{y\to 0}\frac{1/y}{-y^{-\beta-1}} = -\lim_{y\to 0} y^\beta = 0$

$\therefore \lim_{t\to x} C\frac{|\ln|x-t|\,|\,|x-t|^\beta}{|x-t|^\beta} \ \therefore \lim_{t\to x} C|\ln|x-t|\,|\,|x-t|^\beta = C\lim_{t\to x}|\ln|x-t|\,|\,|x-t|^\beta$

$= C\lim_{y\to 0}\ln(y)\,y^\beta = 0 \ \therefore \lim_{t\to x}|\ln|x-t|\,|\,|x-t|^\beta = 0 \ \therefore$

$|x-t|^\beta \ln|x-t|$ is bounded near $x = t$ $\therefore$

$g(x,t)\ln|x-t|$ is weakly singular

Sheet 4/ $y''(x) - y(x) - \int_0^1 e^{x-t} y(t)\,dt = 0 \quad y(0) = y_0, \; y(1) = y_1$

$\therefore y''(x) - y(x) = e^x \int_0^1 e^{-t} y(t)\,dt \quad \therefore \int_0^1 e^{-t} y(t)\,dt = p \;\therefore$

$y''(x) - y(x) = e^x p \quad \therefore$

$y(x) = Ae^x + Be^{-x} + \frac{px}{2} e^x \quad \therefore \quad y(0) = y_0, \; y(1) = y_1 \;\therefore$

$3p - 4A - 2B(1 - e^{-2}) = 0, \quad A + B = y_0, \quad ep + 2eA + 2e^{-1}B = 2y_1 \;\therefore$

$p = \frac{1}{e(2e^2-1)}(ey_0(e^2-3) + y_1(e^2+1)) \quad A = \frac{1}{2(2e^2-1)}(-y_0(e^2+2) + 3ey_1],$

$B = \frac{e}{2(2e^2-1)}(5ey_0 - 3y_1) \;\therefore$

$y(x) = \frac{1}{2(2e^2-1)}[e^x(-y_0(e^2+2) + 3ey_1) + e^{1-x}(5ey_0 - 3y_1) + xe^{x-1}(ey_0(e^2-3) + y_1(e^2+1))]$

Sheet 5/ 1 a/ $y(x) - \int_{-1}^1 (\frac{1}{2} + xt) y(t)\,dt = f(x) \;\therefore$

Let $a = \int_{-1}^1 [y(t)]\,dt$, $b = \int_{-1}^1 t\,y(t)\,dt \;\therefore y(x) = \frac{1}{2}a + bx \;\therefore$

the integral eqn is rank 2 $\quad y(x) + \int_{-1}^1 -\frac{1}{2} y(t)\,dt + \int_{-1}^1 -xt\,y(t)\,dt = f(x)$

$\therefore y(x) - \frac{1}{2}\int_{-1}^1 y(t)\,dt - x\int_{-1}^1 t\,y(t)\,dt = f(x) \;\therefore$

$\therefore y(x) - \frac{1}{2}a - xb = f(x) \;\therefore \quad y(x) = \frac{1}{2}a - xb \;\therefore\; y(t) = \frac{1}{2}a - tb$

$a = \int_{-1}^{1} y(t)\,dt = \int_{-1}^1 \frac{1}{2}a - tb\,dt = \left[\frac{1}{2}at - \frac{1}{2}t^2 b\right]_{-1}^1 =$

$\frac{1}{2}a(1 - -1) - \frac{1}{2}(1^2 - (-1)^2)b = \frac{1}{2}a(2) - \frac{1}{2}(1-1)b = a \quad \therefore$ a is arbit

$b = \int_{-1}^1 t\,y(t)\,dt = \int_{-1}^1 t(\frac{1}{2}a - tb)\,dt = \int_{-1}^1 \frac{1}{2}at - bt^2\,dt = \left[\frac{1}{4}at^2 - \frac{1}{3}bt^3\right]_{-1}^1 =$

$\frac{1}{4}a(1^2 - (-1)^2) - \frac{1}{3}b(1^3 - (-1)^3) = \frac{1}{4}a(1-1) - \frac{1}{3}b(1 - -1) = -\frac{1}{3}b(1+1) = -\frac{2}{3}b \;\therefore \frac{5}{3}b = 0 \;\therefore b = 0$

the const funcs are sols of the homog eqn $\therefore$ we are in the 2nd alternative

1 b/ the adjoint eqn is the same, $\therefore$ the const funcs are the sols of the homog adjoint eqn

1 c/ from (b) we know the IE has a sol iff $\int_{-1}^1 f(t)z(t)\,dt = c\int_{-1}^1 f(t)\,dt = 0$

for $f_1(x) = 3x - 1$: $\int_{-1}^1 (3x-1)\,dx = -2 \neq \;\therefore \nexists$ no sols to the IE

when $f_2(x) = 3x^2 - 1$, $\therefore \int_{-1}^1 f_2(x)\,dx = 0 \;\therefore \exists$ infinite number of sols, we know f_2 is a particular sol, we only need to add the sols of the homog eqn: $y(x) = f_2(x) + C = 3x^2 - 1 + C$ for $C \in \mathbb{R}$

\2a/ $y = Ky$ Z IE is rank 2: Let $a = \int_0^1 y(t)\,dt$, $b = \int_0^1 y(t)\cos(\pi t)\,dt$

$k(x,t) = 1 + \cos(\pi x)\cos(\pi t)$ ∴ Z sol of Z homog eqn:

$y(x) = a + b\cos(\pi x)$ ∴ $a = \int_0^1 (a + b\cos(\pi t))\,dt = a$, ∴

$b = \int_0^1 (a + b\cos(\pi t))\cos(\pi t)\,dt = \frac{b}{2}$ ∴ Z const funcs $y \equiv a$ are

sols of Z homog IE

\2a/ $k(x,t) = 1 + \cos(\pi x)\cos(\pi t)$ ∴ $\int_0^1 k(x,t)y(t)\,dt = \int_0^1 (1 + \cos(\pi x)\cos(\pi t))y(t)\,dt$

$= \int_0^1 y(t)\,\cancel{dt} + \int_0^1 \cos(\pi x)\cos(\pi t)y(t)\,dt = \int_0^1 y(t)\,dt + \cos(\pi x)\int_0^1 \cos(\pi t)y(t)\,dt$

∴ Let $a = \int_0^1 y(t)\,dt$, $b = \int_0^1 y(t)\cos(\pi t)\,dt$ ∴ Z sol of Z homog eqn is

$y(x) = a + b\cos(\pi x)$ ∴ $y(t) = a + b\cos(\pi t)$ ∴

$a = \int_0^1 y(t)\,dt = \int_0^1 a + b\cos(\pi t)\,dt = \int_0^1 a\,dt + b\int_0^1 \cos(\pi t)\,dt =$

$[at]_0^1 + b\left[\frac{1}{\pi}\sin(\pi t)\right]_0^1 = a(1-0) + \frac{1}{\pi}\sin(\pi) - \frac{1}{\pi}\sin 0 = a$ ∴ a is arbit

$b = \int_0^1 y(t)\cos(\pi t)\,dt = \int_0^1 (a + b\cos(\pi t))\cos(\pi t)\,dt = a\int_0^1 \cos(\pi t)\,dt + b\int_0^1 \cos^2(\pi t)\,dt$

$= a\left[\frac{1}{\pi}\sin(\pi t)\right]_0^1 + b\int_0^1 \frac{1}{2}\cos(2\pi t) + \frac{1}{2}\,dt = 0 + b\left[\frac{1}{2\pi}\sin(2\pi t) + \frac{1}{2}t\right]_0^1 =$

$b\left[\cancel{0} + \frac{1}{2}(1)\right] = \frac{1}{2}b$ ∴ $\frac{1}{2}b = 0$ ∴ $b = 0$ ∴ Z const funcs

$y \equiv a$ are sols of Z homog IE

~~2a~~ / $k(x,t) = 1 + \cos(\pi x)\cos(\pi t)$ $C([0,1])$; $y = Ky$ ∴ Let

$y(x) = \int_0^1 k(x,t)y(t)\,dt = \int_0^1 y(t)\,dt + \cos(\pi x)\int_0^1 \cos(\pi t)\,y(t)\,dt = y(x)$ ∴

$a = \int_0^1 y(t)\,dt$, $b = \int_0^1 y(t)\cos(\pi t)\,dt$ ∴ $y(x) = a + \cos(\pi x)\,b$ ∴

$y(t) = a + \cos(\pi t)\,b$ ∴ $a = \int_0^1 y(t)\,dt = \int_0^1 a + b\cos(\pi t)\,dt = a$ ∴ a is arbit

$b = \int_0^1 y(t)\cos(\pi t)\,dt = \int_0^1 (a + \cos(\pi t)b)\cos(\pi t)\,dt = \frac{b}{2}$ ∴ $b = 0$ ∴

Z const funcs $y \equiv a$ are sols of Z homog IE

\2b/ $k(x,t) = 1 + \cos(\pi x)\cos(\pi t)$ ∴ $k(t,x) = 1 + \cos(\pi t)\cos(\pi x) =$

$1 + \cos(\pi x)\cos(\pi t) = k(x,t)$ ∴ Z IE is self adjoint ∵ $k^* = k$ ∴ Z

const funcs are also Z sols of Z homog adjoint eqn ∴

Sheet 5 / ~~extra~~ $\int_0^1 f(t) = 0$ ∴ for $f(x) = x^2$; $f(t) = t^2$ ∴

$\int_0^1 f(t)\,dt = \int_0^1 t^2\,dt = \left[\frac{1}{3}t^3\right]_0^1 = \frac{1}{3}[1^3 - 0^3] = \frac{1}{3} \neq 0$ ∴ $f(x) = x^2$ doesn't meet this condition

2d/ $y = f + Ky$ $k(x,t) = 1 + \cos(\pi x)\cos(\pi t)$?

$y(x) = f(x) + \int_0^1 (1 + \cos(\pi x)\cos(\pi t))y(t)\,dt$ ✗

$a = \int_0^1 y(t)\,dt$ $b = \int_0^1 y(t)\cos(\pi t)\,dt$ ∴

$y(x) = \int_0^1 (f(t) + a + b\cos\pi t)\,dt = \int_0^1 f(t)\,dt + a = a$

$b = \int_0^1 (f(t) + a + b\cos\pi t)\cos\pi t\,dt = \int_0^1 f(t)\cos\pi t\,dt = \frac{b}{2}$

$y(x) = f(x) + \int_0^1 (1 + \cos\pi x\cos\pi t)\,y(t)\,dt =$

$f(x) + \int_0^1 y(t)\,dt + \cos\pi x\int_0^1 \cos\pi t\,y(t)\,dt = f(x) + a + \cos(\pi x)b = y(x)$ ∴

$y(t) = f(t) + a + \cos(\pi t)b$ ∴

$a = \int_0^1 y(t)\,dt = \int_0^1 (f(t) + a + b\cos\pi t)\,dt = \int_0^1 f(t)\,dt + a = a,$

$b = \int_0^1 \cos(\pi t)y(t)\,dt = \int_0^1 (f(t) + a + b\cos\pi t)\cos\pi t\,dt = \int_0^1 f(t)\cos\pi t\,dt = \frac{b}{2}$ ∴

$b = 2\int_0^1 f(t)\cos(\pi t)\,dt$ ∴ 2 final form for y is

$y(x) = f(x) + 2\left(\int_0^1 f(t)\cos(\pi t)\,dt\right)\cos(\pi x) + C$, C is arb const

3a/ $x \in [0,1]$ ∴ $\int_0^1 |k(x,t)|\,dt = \int_0^x |k(x,t)|\,dt + \int_x^1 |k(x,t)|\,dt =$

$\int_0^x (1-x)\,dt + \int_x^1 (1-t)\,dt = \frac{1}{2}(1-x^2)$ ∴ $\max\left|\int_0^1 |k(x,t)|\,dt\right| = \max\left|\frac{1}{2}(1-x^2)\right| = \frac{1}{2}\max|1-x^2| =$

$\frac{1}{2}\times 1 = \frac{1}{2} < 1$ ∴ $\|K\|_\infty = \frac{1}{2} < 1$

3b/ $y - Ky = 1$ ∴ 2 IE is: $y(x) - \int_0^1 y(t)\,dt + x\int_0^x y(t)\,dt + \int_x^1 t\,y(t)\,dt = f(x)$

$y(x) - \int_0^1 k(x,t)\,y(t)\,dt =$

~~$y(x) - \int_0^x 1 - x\,dt + \int_x^1 1 - t\,dt = \int_0^x 1\,dt - x \quad y(x) - \int_0^x 1\,dt + x\int_0^x$~~

$y(x) - \int_0^x (1-x)y(t)\,dt - \int_x^1 (1-t)y(t)\,dt =$

$y(x) - \int_0^x y(t)\,dt + x\int_0^x y(t)\,dt - \int_x^1 y(t)\,dt + \int_x^1 t\,y(t)\,dt =$

$y(x) - \int_0^1 y(t)\,dt + x\int_0^x y(t)\,dt + t\int_x^1 t\,y(t)\,dt = f(x)$ ∴

$\left\{ \frac{d}{dx}\left(y(x) - \int_0^1 y(t)dt + x\int_0^x y(t)dt + \int_x^1 t\,y(t)dt\right) = \frac{d}{dx}(f(x)) = f'(x) = \right.$

$y'(x) - 0 + \int_0^x y(t)dt + xy(x) + 0 = y' + \int_0^x y(t)dt + xy(x) = f'(x) \Big\}$ X

$y(x) - \int_0^1 y(t)dt + x\int_0^x y(t)dt + \int_x^1 t\,y(t)dt = f(x)$ ∴ deriv:

$y'(x) + \int_0^x y(t)dt = f'(x)$ ∴

$\left\{ y'(x) + \int_0^x y(t)dt + xy(x) - xy(x) = f'(x) = y'(x) + \int_0^x y(t)dt = f'(x) \right\}$

∴ deriv: $y''(x) + y(x) = f''(x)$; $y(0) = f(0) + \int_0^1 (1-t)y(t)dt$ & $y'(0) = f'(0)$ ∴

for $f(x) = 1$: $y''(x) + y(x) = 0$ ∴ IC : $y(0) = 1 + \int_0^1 (1-t)y(t)dt$ &

$y'(0) = 0$ ∴ $y(x) = A\cos x$ ∴

$A = 1 + A\sin 1 - A\sin 1 + A\int_0^1 \sin x\,dx = 1 - A\cos 1 + A$ ∴ $A = \frac{1}{\cos 1}$ ∴

$y(x) = \frac{\cos x}{\cos 1}$

\3c/ $y - Ky = x$ ∴ $f(x) = x$ ∴ $y''(x) + y(x) = 0$, IC : $y(0) = \int_0^1 (1-t)y(t)dt$

& $y'(0) = 1$ ∴ $y(x) = A\cos x + B\sin x$ ∴ $y'(0) = B = 1$ ∴

$A = y(0) = 1 - A\cos 1 + A - \sin 1$ ∴ $A = \frac{1-\sin 1}{\cos 1}$ ∴

$y(x) = \left(\frac{1-\sin 1}{\cos 1}\right)\cos x + \sin x$

\4/ $y - Ky = f$, $k(x,t) = \frac{1}{2} + x(3t^2-1)$, $x \in [-1, 1]$

first we interested in Z sols of Z corres homog eqns, if a homog eqn has non trivial sols, we know that sols of Z non homog exist iff f is orthogonal to Z sols of Z adjoint homog eqn

homog eqn : $y(x) - \int_{-1}^1 \left[\frac{1}{2} + x(3t^2-1)\right] y(t)dt = 0$

this is eqn of finite rank 2 ∴ Let $P_1 = \int_{-1}^1 y(t)dt$,

$P_2 = \int_{-1}^1 (3t^2-1)y(t)dt$ ∴ $y(x) = \frac{1}{2}P_1 + P_2 x$ ∴ $y(t) = \frac{1}{2}P_1 + P_2 t$ ∴

$P_1 = \int_{-1}^1 \frac{1}{2}P_1 + P_2 t\,dt = P_1$, $P_2 = \int_{-1}^1 (3t^2-1)(\frac{1}{2}P_1 + P_2 x)dt = 0$ ∴ ∃ non-trivial sols of Z eqn which are $y(x) = \frac{1}{2}P$ P_1 is arb const

Z adjoint eqn has kernel $k^*(x,t) = \frac{1}{2} t(3x^2-1)$ ∴ calc $y^* = K^* y^*$ has Z sols $y^*(x) = 1$ ∴ Z origin non homog eqn has a