MTH2009 term 2
COMPLEX ANALYSIS

\week 1/ \Pages 13-16 LN/

complex numbers are the set $\mathbb{C} = \{x+iy : x, y \in \mathbb{R}\}$ and recall $i^2 = -1$

let $z = a+ib$, $w = c+id$ $\therefore z+w = \underbrace{(a+ib)}_{z} + \underbrace{(c+id)}_{w} = a+c+i(b+d)$

$zw = (a+ib)(c+id) = ac+aid+ibc+ibid = ac+aid+ibc-bd$
$= (ac-bd)+i(ad+bc)$

$z = x+iy$ $\therefore$ inverse: $\frac{1}{z} = \frac{1}{x+iy}$

complex conjugate of z is: $\bar{z} = x-iy$ (sometimes denote $\bar{z} = z^*$)

$\therefore \frac{1}{z} = \frac{1}{x+iy} = \frac{x-iy}{(x+iy)(x-iy)} = \frac{x-iy}{x^2+y^2} = \frac{x}{x^2+y^2} - \frac{y}{x^2+y^2} i$

$z = x+iy$ $\therefore$ $Re(z) = x$ $Im(z) = y$ $\therefore \bar{z} = x-iy$

$\therefore z\bar{z} = (x+iy)(x-iy) = x^2 - xiy + xiy - i^2y^2 = x^2+y^2$

absolute value (modulus) of $z = x+iy$ is: $|z| = \sqrt{x^2+y^2}$

argand diagram:

\lemma 2.4:/ if $z, w \in \mathbb{C}$ then $(z \pm w)^* = z^* \pm w^*$ $\{w^* = \bar{w}\}$

$(zw)^* = z^* w^*$ $\left(\frac{z}{w}\right)^* = \frac{z^*}{w^*}$ if $w^* \neq 0$

\corollary P15/ if ~~z,w~~ $z, w \in \mathbb{C}$ then $|zw| = |z||w|$

\proof:/ $|zw|^2 = (zw)(\bar{z}\bar{w}) = (z\bar{z})(w\bar{w}) = |z|^2 \cdot |w|^2$ $\therefore$

$\sqrt{|zw|^2} = \sqrt{|z|^2 \cdot |w|^2} = |zw| = |z||w|$ $\square$

\corollary: triangle inequality/ if $z, w \in \mathbb{C}$ $\therefore |z+w| \leq |z|+|w|$

polar form of complex number: $z = Re^{i\theta}$

θ is argument of z where $e^{i\theta} = \cos\theta + i\sin\theta$

& $R = |z|$

\lemma/ for $\theta, \phi \in \mathbb{R}$ have: $e^{i(\theta+\phi)} = e^{i\theta}e^{i\phi}$

\corollary $(Re^{i\theta})(Se^{i\phi}) = RSe^{i(\theta+\phi)}$ □

\roots of complex numbers/

Every complex number has exactly n distinct n-th roots

let $z = Re^{i\theta} = R(\cos\theta + i\sin\theta)$, $\mu = \rho(\cos\alpha + i\sin\alpha)$

then $R(\cos\theta + i\sin\theta) = \rho^n(\cos\alpha + i\sin\alpha)^n$

{by de Moivres formula} $= \rho^n(\cos(n\alpha) + i\sin(n\alpha))$

this implies:

$\rho^n = R$, $n\alpha = \theta + 2\pi k$ (k integer) $\therefore$

$\rho = R^{1/n}$, $\alpha = \frac{\theta}{n} + \frac{2\pi k}{n}$

\Ex1/ the n-th roots of $z=1$

take $z^n = 1$ $\therefore z = 1^{1/n}$ $\therefore 1 = e^{2m\pi i}$

$\therefore 1^{1/n} = e^{2m\pi i/n} = \cos\left(\frac{2m}{n}\pi\right) + i\sin\left(\frac{2m}{n}\pi\right)$

\E2/ find the cubic roots of $z = -1+i$

take $u = (-1+i)^{1/3}$ $\therefore$ ~~$u = (\sqrt{2})^{1/3}[$~~

$u = (\sqrt{2})^{1/3}\left[\cos\left(\frac{3\pi}{4}\cdot\frac{1}{3} + \frac{2\pi}{3}k\right) + i\sin\left(\frac{3\pi}{4}\cdot\frac{1}{3} + \frac{2\pi k}{3}\right)\right]$ $k = 0,1,2$

□

\Ex3/ let $z = 3+4i$ find the 10-th roots of z express your answer in polar form

let ~~m~~ $z = 3+4i = Re^{i\theta}$ $R = \sqrt{3^2+4^2}$

and $\theta = \tan^{-1}\left(\frac{4}{3}\right)$ the ~~10-th~~ 10-th roots of z are:

$5^{1/10}\cdot e^{i\left(\frac{\theta}{10} + \frac{\pi}{5}n\right)}$ for $n = 0, 1, \ldots, 9$

\Powers of complex numbers/

let $z = Re^{i\theta}$ if n is an integer then have:

- $z^n = (Re^{i\theta})^n = R^n \cdot e^{in\theta}$ {de Moivre's formula}

$= R^n \cdot (\cos\theta + i\sin\theta)^n = R^n [\cos(n\theta) + i\sin(n\theta)$

$n = 0, \pm 1, \pm 2, \dots$ $\square$

\the exponential func & complex logarithms/

\Exponential func/

\lemma/ for any $z \in \mathbb{C}$ have:

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

\properties of the exponential func/

$e^{z+w} = e^z \cdot e^w$ # $e^{z+2\pi i} = e^z$ (the exponential func is a periodic func of period $2\pi i$

$|e^z|$ can write $z = x+iy$ $\therefore$ $|e^z| = |e^{x+iy}|$

$= |e^x| \cdot |e^{iy}| = |e^x| \cdot |\cos y + i\sin y|$

$= |e^x| \cdot (\cos^2 y + \sin^2 y)^{1/2} = |e^x| \cdot (1)^{1/2} = |e^x|$

$= e^x = e^{Re(z)}$

the trigonometric funcs $\cos z = \dfrac{e^{iz} + e^{-iz}}{2}$

and ~~sin~~ $\sin z = \dfrac{e^{iz} - e^{-iz}}{2i}$

\complex logarithm / $e^z = Re^{i\theta}$ has sols

$z = \log R + i(\theta + 2n\pi)$, $n \in \mathbb{Z}$

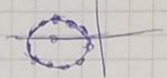
\Ex / find $\log(-3-4i)$:

- $\log(-3-4i) = \log(|-3-4i|) - \theta i$

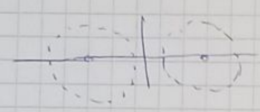
$= \log 5 - i(\frac{\pi}{2} + \arctan\frac{3}{4})$ $\square$

\topology of complex plane/

• an open disc is a set, $D(a,R)=\{z\in\mathbb{C} : |z-a|<R\}$ with $a\in\mathbb{C}$ and $R\in\mathbb{R}^+$

~~ex~~ eg domain:  point $(-2,0)$ radius 1

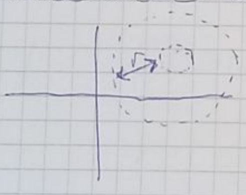
$\therefore D(-2,1)$ staggered line to show open disc

$\therefore$  is $D(-2,1)\cup D(2,1)$

centered at -2 radius 1

• a closed disc $\bar{D}(a,r)=\{z\in\mathbb{C} : |z-a|\leq r\}$

• a punctured disc: $D'(a,r)=\{z\in\mathbb{C} : 0<|z-a|<r\}$



{LN page 21}

\topology of complex plane/

~~ex~~ let $A\subseteq\mathbb{C}$ each point $a\in\mathbb{C}$ is either:

• an interior point of A is $\exists r>0$ with $D(a,r)\subseteq A$

• an exterior point of A is $\exists r>0$ with $D(a,r)\cap A\neq\emptyset$

• an boundary point of A is a is neither an interior or an exterior point of A

\open and closed sets/ an subset $A\subseteq\mathbb{C}$ is open is all $a\in A$ are interior points of A

an Subset $A \subseteq \mathbb{C}$ is closed is (its complement) $\mathbb{C} \setminus A$ (the complement of A in $\mathbb{C}$) is open

\lemma/ $D(a,r)$ is open and $\bar{D}(a,r)$ is closed

an set can be both open and closed and also neither open or closed

\limit points/ let $S \subseteq \mathbb{C}$ $z \in \mathbb{C}$ is an limit pt of S is for every $\varepsilon > 0$, the punctured disc $D'(z, \varepsilon)$ contains a pt of S

\Ex/ take an open set then every pt of this set is an limit pt of it $\square$

let $S \subseteq \mathbb{C}$ the closure of S: $(\bar{S})$ is the union of S and its limit pts (it is indeed the interior and the boundary of the set)

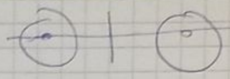
\lma/ is S is an subset of $\mathbb{C}$, then $\bar{S}$ is a closed set

an subset $A \subseteq \mathbb{C}$ is bounded is $\exists r > 0$ st $A \subseteq D(0,r)$

an Subset $A \subseteq \mathbb{C}$ is compact is its closed and bounded

~~the~~ the $D(a,r)$, $\bar{D}(a,r)$, $D'(a,r)$ are all examples of bounded sets (LN P21 for intuition) $\therefore$ $\bar{D}(a,r)$ is compact

an open set ~~$U \subseteq \mathbb{C}$~~ $V \subseteq \mathbb{C}$ is connected is any 2 pts $x, y \in V$ can be joined by a finite sequence of straight lines segments contained within V

\Ex/  Not connected

∴ Not a domain

$A \subseteq \mathbb{C}$ is an <u>domain</u> if A is non-empty, open and connected

\Recall: def of Continuity/

$f \underset{z\to a}{\to} l$ if $\forall \varepsilon > 0$ there exists $\delta > 0$ st

if $z \in D(a,\delta) \cap A$ <u>then</u> $f(z) \in D(l,\varepsilon)$

\vid 1 (EN p27-31)

\Holomorphic funcs/ let $A \subseteq \mathbb{C}$ (open subset of $\mathbb{C}$), $f: A \to \mathbb{C}$ we say f is <u>differentiable</u> at $a \in A$ if $\lim_{z\to a} \frac{f(z)-f(a)}{z-a}$ exists when this limit exists we call it $f'(a)$ ∴

an func on an open set U which is <u>differentiable</u> at every pt of U is called an <u>holomorphic</u> func on U

\thm/ if f and g diffable then $f \pm g$, fg, f/g, $f\circ g$, $g \circ f$ are diffable

\Cauchy-Riemann eqns/ $U \subseteq \mathbb{C}$ suppose $f: U \to \mathbb{C}$ given by $f(x+iy) = u(x,y) + iv(x,y)$ with $x, y, u(x,y), v(x,y) \in \mathbb{R}$

if $z_0 \in U$ and if f is diffable at z_0, then $\frac{\partial u}{\partial x}(z_0) = \frac{\partial v}{\partial y}(z_0)$ and $\frac{\partial u}{\partial y}(z_0) = -\frac{\partial v}{\partial x}(z_0)$

~~as~~ are the Cauchy-Riemann eqns

\Lma/ Suppose $f(x+iy)=u(x,y)+iv(x,y)$ is an func on an open set U and suppose that $z_0 \in U$

if f satisfies the Cauchy-Riemann eqns at z_0 and if $\frac{\partial u}{\partial x}$ and $\frac{\partial v}{\partial y}$ are continuous at z_0 then f is diffable at z_0 $\square$

\Ex/ $f(z)=e^z$ $\quad z=x+iy$ $\quad f(x+iy)=e^{x+iy}=e^x \cdot e^{iy}$

$=e^x \cdot (\cos y + i\sin y) = \underbrace{e^x \cos y}_{u(x,y)} + i\underbrace{e^x \sin y}_{v(x,y)}$ $\quad u,v$ cont

$$\left.\begin{aligned} \frac{\partial u}{\partial x} &= e^x \cos y = \frac{\partial v}{\partial y} \\ \frac{\partial v}{\partial x} &= e^x \sin y = -\frac{\partial u}{\partial y} \end{aligned}\right\} \text{ f satisfies the Cauchy-Riemann eqns}$$

$\therefore$ f is Holomorphic $\square$

\LNP33/ a path is a continuous map

$\gamma:[a,b]\to\mathbb{C}$ moreover it is called smooth if γ is diffable and γ' is cont

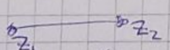
write: $\gamma(t)=x(t)+iy(t)$; $x,y:[a,b]\to\mathbb{R}$

γ is cont if $x(t)$ and $y(t)$ are cont

γ is diffable if $x(t)$ and $y(t)$ are diffable

\Ex/ let $z_1, z_2 \in \mathbb{C}$ the line segment from z_1 to z_2 is the path $\gamma:[0,1]\to\mathbb{C}$

$\gamma(t)=z_1+t(z_2-z_1)$ (sometimes denoted $[z_1,z_2]$)

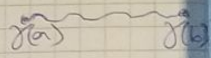


\Ex/ let $z_0\in\mathbb{C}$, $r>0$, $\alpha,\beta\in\mathbb{R}$, $\alpha<\beta$

define $\gamma:[\alpha,\beta]\to\mathbb{C}$ $\quad \gamma(t)=z_0+re^{it}$

γ is an arc of the circle with centre z and radius r

let us look at the path $\gamma : [a,b] \to \mathbb{C}$

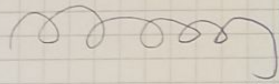


- $\gamma(a)$ is the start pt
- $\gamma(b)$ is the end pt

if $\gamma(a) = \gamma(b) \Rightarrow \gamma$ is closed we say

we say the path is simple if $t, s \in (a,b)$ and $\gamma(t) = \gamma(s) \Rightarrow t = s$

$\therefore$ it doesn't do anything like



$\gamma^- = \gamma(-t)$ is a map

$\gamma^- : [-b,-a] \to \mathbb{C}$ "the reversal of γ" $\square$

\P34-35 LN/ \path integral/ let f be cont on an open set U let $\gamma : [a,b] \to \mathbb{C}$ be a smooth path contained in U

the path integral $\int_\gamma f(z)\,dz$ is defined by

$$\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt$$

\Ex/ let us compute the integral of $f(z) = z$ over γ, the line segment from 1 to $2+2i$

let us first define the curve γ

we know (P33) that the line segment from z_1 to z_2 is the path $\gamma : [0,1] \to \mathbb{C}$ defined by

$\gamma(t) = z_1 + t(z_2 - z_1)$ $\quad z_1 = 1,\ z_2 = 2+2i$ $\therefore$ we get the path $\gamma(t) = 1 + t(2+2i-1)$ $\quad \therefore \gamma(t) = 1 + t(1+2i)$

$\gamma(t) = 1 + t(1+2i)$; $\gamma'(t) = 1+2i$ $\quad \therefore$ we may write

$$\int_\gamma f(z)\,dz = \int_0^1 \underbrace{(1+(1+2i)t)}_{f(\gamma(t))} \cdot \underbrace{(1+2i)}_{\gamma'(t)}\,dt = -\frac{1}{2} + 4i$$

\ Properties of path integrals /

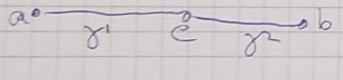
① $\int_\gamma (f(z)+g(z))dz = \int_\gamma f(z)dz + \int_\gamma g(z)dz$

② $\int_\gamma a f(z)dz = a\int_\gamma f(z)dz \quad , \forall a \in \mathbb{C}$

③ if γ^- is the reversal of γ, then

$$\int_{\gamma^-} f(z)dz = -\int_\gamma f(z)dz$$

④ $\int_\gamma f(z)dz = \int_{\gamma^1} f(z)dz + \int_{\gamma^2} f(z)dz \qquad \gamma:[a,b]\to\mathbb{C}$



$\gamma^1:[a,c]\to\mathbb{C} \qquad \gamma^2:[c,b]\to\mathbb{C}$

\ fundamental thm of calculus / assume $f: U\to\mathbb{C}$ is holomorphic and U is open assume also that f' is cont then

$$\int_\gamma f'(z)dz = f(\gamma(b)) - f(\gamma(a))$$

if γ is a closed path

then $\int_\gamma f'(z)dz = 0$

\ Ex / compute $\int_1^{2+2i} z\,dz$ using the fundamental thm of calculus

$$\int_1^{2+2i} z\,dz = \int_1^{2+2i} \frac{d}{dz}\left(\frac{z^2}{2}\right)dz = \underbrace{\frac{(2+2i)^2}{2}}_{f(\gamma(b))} - \underbrace{\frac{1^2}{2}}_{f(\gamma(a))}$$

$= -\frac{1}{2} + 4i$

Sequences and series of complex numbers

let (a_n) be an sequence of complex numbers
let $a \in \mathbb{C}$ we say: a is the limit of (a_n)
as $n \to \infty$ if $\forall \varepsilon > 0 \; \exists N \in \mathbb{R} \; \forall n > N, |a_n - a| < \varepsilon$
{ $|a_n - a| < \varepsilon$ means the elements of our sequence become closer and closer to an ε-close }

Ex $z_n = \frac{1+i}{n} \xrightarrow[n\to\infty]{} 0$ let us use the ε-δ def

let $z = 0$ $\quad |z_n - z| = \left|\frac{1+i}{n} - 0\right| = \frac{|1+i|}{n} = \frac{\sqrt{2}}{n}$

$\forall \varepsilon > 0$, choose $N = \frac{\sqrt{2}}{\varepsilon}$ if $n > N$ then

$|z_n - z| = \frac{\sqrt{2}}{n} < \frac{\sqrt{2}}{N} < \varepsilon$ because $N = \sqrt{2}/\varepsilon$ $\square$

thm let (z_n) be an sequence of complex numbers
let $z \in \mathbb{C}$ then the following are equivalent

① $z_n \to z$ as $n \to \infty$

② $|z_n - z| \to 0$ as $n \to \infty$

③ $Re(z_n) \to Re(z)$ and $Im(z_n) \to Im(z)$ as $n \to \infty$

Cauchy Sequences an sequence (a_n) of complex numbers is an Cauchy sequence if
$\forall \varepsilon > 0, \exists N \in \mathbb{R}$ such that if $n, m > N$ then
$|a_n - a_m| < \varepsilon$

Ex $\frac{1}{n}$ is a cauchy sequence
fix $\varepsilon > 0$, $\forall N > 0$ if $n, m > N$ then

$\left|\frac{1}{n} - \frac{1}{m}\right| \leq \frac{1}{n} - \frac{1}{m} < \frac{2}{N}$

since $m > N$ we have $\frac{1}{m} < \frac{1}{N}$ $\square$

\Lemma/ if (an) (a_n) is an convergent sequence of complex numbers then (a_n) is cauchy

\Proof/ since we are assuming our series is convergent then $a_k \to l$ as $n \to \infty$
now take $\varepsilon > 0$ by definition there is N such that if $n > N$, then $|a_n - l| < \varepsilon/2$
suppose $n, m > N$, $|a_n - a_m| = |a_n - l + l - a_m|$
$< \underbrace{|a_n - l|}_{<\varepsilon/2} + \underbrace{|a_m - l|}_{<\varepsilon/2} < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\square$

\Convergence of series/ ① let $\sum_{n=0}^{\infty} z_n$ be an infinite series of complex nums
we say the series converges if $S_N = \sum_{n=0}^{N} z_n$ converges {the sum of the first $N+1$ elements

② $\sum_{n=0}^{\infty} z_n$ is absolutely convergent if $\sum_{n=0}^{\infty} |z_n|$ converges.

\Lemma/ if $\sum_{n=0}^{\infty} |z_n|$ converges then so does $\sum_{n=0}^{\infty} z_n$

\Corollary/ let $(z_n)_{n=0}^{\infty}$ be an sequence of complex nums if $\forall \varepsilon > 0 \ \exists N$ st $\left| \sum_{m+1}^{n} z_k < \varepsilon \right|$ $\forall n, m > N$ then $\sum_{k=0}^{\infty} z_k$ is convergent

\Ex/ $\sum_{n=1}^{\infty} \frac{1}{n^2}$ is convergent

$k^2 > k^2 - k \therefore \frac{1}{k^2} < \frac{1}{k^2 - k}$

let $z_k = \frac{1}{k^2}$ $\sum_{k=m+1}^{n} \frac{1}{k^2} < \sum_{k=m+1}^{n} \frac{1}{k(k-1)}$ now using partial fractions can write: $\frac{1}{k(k-1)} = \frac{1}{k-1} - \frac{1}{k}$ (recall)

$\sum_{k=m+1}^{n} \frac{1}{k(k-1)} = \sum_{k=m+1}^{n} \frac{1}{k-1} - \frac{1}{k} =$

$\underbrace{\left(\frac{1}{m+1-1} - \frac{1}{m+1}\right)}_{k=m+1} + \underbrace{\left(\frac{1}{m+2-1} - \frac{1}{m+2}\right)}_{k=m+2} + \cdots + \underbrace{\left(\frac{1}{n-1} - \frac{1}{n}\right)}_{k=n}$

$= \frac{1}{m} + \left(-\frac{1}{m+1} + \frac{1}{m+1}\right) + \left(-\frac{1}{m+2} + \frac{1}{m+2}\right) + \dots + \left(-\frac{1}{n-1} + \frac{1}{n-1}\right) + \left(-\frac{1}{n}\right)$

$= \frac{1}{m} - \frac{1}{n} < \frac{1}{m}$ $\therefore$ let $\varepsilon > 0$, $N = \frac{1}{\varepsilon}$ if $n, m > N$

then $\left|\sum_{m+1}^{k} \frac{1}{k^2}\right| < \varepsilon$ thus $\sum_{k=1}^{\infty} \frac{1}{n^2}$ is convergent $\square$

Sequences of functions ① let (f_n) be an sequence of funcs on U let f be a func on U f is the pointwise limit of the sequence (f_n) if $\forall x \in U$, we have $f_n(x) \to f(x)$ as $n \to \infty$

② let (f_n) be an sequence of funcs in U we say: f is the uniform limit of (f_n) if $\sup_{x \in U} |f_n(x) - f(x)| \xrightarrow[n\to\infty]{} 0$ we say (f_n) is uniformly convergent to f

Ex $f_n : \mathbb{R} \to \mathbb{R}$ st $f_n(x) = x^2 + \frac{\sin(x)}{n}$; $f(x) = x^2$

let us see that $f_n \to f$ as $n \to \infty$ want to show $\sup_{x \in \mathbb{R}} |f_n(x) - f(x)| \to 0$ as $n \to \infty$ $|f_n(x) - f(x)|$

$= \left|x^2 + \frac{\sin x}{n} - x^2\right| = \left|\frac{\sin x}{n}\right| \leq \frac{1}{n}|\sin x| \leq \frac{1}{n}(1) = \frac{1}{n} \xrightarrow[n\to\infty]{} 0$

thm let (f_n) be an sequence of continuous funcs that converge uniformly to f on an subset ~~$U \in \mathbb{R}$~~ $U \subseteq \mathbb{C}$ suchthat every point of U is an limit point of U {limit pts p22 LN} then f is continuous $\square$

thm let γ be an contour and (f_n) an sequence of funcs integrable on γ assume that $f_n \to f$ uniformly on γ then $\int_\gamma f_n(z)\,dz \to \int_\gamma f(z)\,dz$

note: the proof of this thm (p44) uses the "tools" from last lectures $\square$

let $(f_n)_{n=1}^{\infty}$ defined on U is uniformly Cauchy on U if $\forall \varepsilon>0 \; \exists N$ st $\forall n,m>N, \forall z \in U$, $|f_n(z)-f_m(z)|<\varepsilon$

\lemma/ an sequence of funcs defined on U is uniformly convergent on U if and only if it is uniformly Cauchy on U

let $(f_n)_{n=1}^{\infty}$ be an sequence of funcs on U the series $\sum_{n=1}^{\infty} f_n(z)$ converges unifly on U if $S_N(z)=\sum_{n=1}^{N} f_n(z)$ converges unifly on U

\weierstrass M-test thm/ let $(f_n)_{n=1}^{\infty}$ be an sequence of funcs defined on U the series $\sum_{n=1}^{\infty} f_n(z)$ converges unifly and absolutely on U if there exists an sequence $(M_n)_{n=1}^{\infty}$ of non-negative real numbers st $\forall n \in \mathbb{N}, \forall z \in U$ have $|f_n(z)| \le M_n$ and $\sum_{n=1}^{\infty} M_n$ converges

\try Ex 6.13 P47LN/ hint: for any real num $0<r<1$, the geometric series $\sum_{n=0}^{\infty} z^n$ converges uniformly on $\bar{D}(0,r)$

note: $\forall z \in \bar{D}(0,r)$: $|z^n| \le r^n$ but $\sum_{n=0}^{\infty} r^n$ converges so by the weierstrass M-test:

$\sum_{n=0}^{\infty} z^n$ converges unifly on $\bar{D}(0,r)$ $\square$

\Cauchy's integral formula / let U be a domain ($A \subseteq \mathbb{C}$ is a domain if A is a nonempty, open, connected set)

let γ be a positively oriented, simple contour with its image and interior lying entirely within U

suppose that a is an point in the interior of γ

by Cauchy's integral formula: if f is holomorphic on U (holomorphic: it is differentiable at every point of U) then $f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz$

\ex / let γ be the circle with centre 0 and radius 2 then $\int_\gamma \frac{e^{z^2}}{z+1}\,dz = 2\pi i e$

let us compare our integral with Cauchy's integral formula: $f(a) = \frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}\,dz \rightarrow \int \frac{f(z)}{z-a}\,dz = 2\pi i f(a)$

$a=-1$, $f(z) = e^{z^2}$

$\int_\gamma \frac{e^{z^2}}{z+1}\,dz = 2\pi i f(-1) = 2\pi i \cdot e^{(-1)^2} = 2\pi i e$ $\square$

Cauchy's integral formula for the n-th derivative

\thm/ let U be a domain, γ a positively oriented simple contour with its image and interior lying entirely within U

suppose a is an path in the interior of γ

if f is holomorphic on U, then $f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz$

\Ex/ let γ be the unit circle compute

$\int_\gamma \frac{\sin z}{z^4}\,dz$ $\quad f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(z)}{(z-a)^{n+1}}\,dz$ take $a=0$, $n=3$

$\int_\gamma \frac{f(z)}{z^4}\,dz = \frac{2\pi i}{3!}\cdot f'''(0)$ $\quad f(z) = \sin z \therefore f'''(0) = -1 \therefore$

$\int_\gamma \frac{\sin z}{z^4}\, dz = -\frac{2\pi i}{3!}$ $\square$

week 5 tutorial $\left|\int_\gamma \frac{3z^3\sin(z)}{2z^5-1}\,dz\right| \le 6\pi e$

γ is unit circle $\{z\in\mathbb{C} \mid |z|=1\}$ transversed anticlockwise

$\left|\int_\gamma f(z)\,dz\right| \le ML$ L is length 2π

$\left|\int_\gamma \frac{3z^3\sin(z)}{2z^5-1}\,dz\right| \le ML$ use ML bound

$\left|\int_\gamma f(z)\,dz\right| \le ML$ L is length of ~~γ~~ path (curve) travelled

L is length of γ M is the "maximum" of $f(z)$ ie that

$|f(z)| \le M$ for $z\in\gamma$

$f(z) = \frac{3z^3\sin z}{2z^5-1}$ $L = 2\pi$ {circumference of circle}

$|f(z)| = \left|\frac{3z^3\sin z}{2z^5-1}\right|$ reverse triangle inequality

in complex numbers for z so $\sin(z)$ is not bounded

$\frac{d}{dz}\left(\frac{3z^3\sin z}{2z^5-1}\right) = \frac{d}{dz}\left(3z^3\sin z(2z^5-1)^{-1}\right)$

$= (9z^2\sin z + 3z^3\cos z)(2z^5-1)^{-1} - 3z^3\sin z(2z^5-1)^{-2}(10z^4)$

~~as the~~ looking for the max on the circle, nothing to say max will be on the circle

$|z-w| \ge |z|-|w|$

$|f(z)| = \left|\frac{3z^3\sin z}{2z^5-1}\right| \le \frac{|3z^3\sin z|}{|2z^5-1|} \le \frac{|3z^3\sin z|}{|2z^5|-|1|}$

$= \frac{|3z^3|\left|\frac{e^{iz}-e^{-iz}}{2i}\right|}{|2z^5|-|1|} = \frac{|3||z^3|\left|\frac{1}{2i}\right||e^{iz}-e^{-iz}|}{|2||z^5|-|1|} =$ $\{|z|=1$ given$\}$

$= \frac{3(1)\left|\frac{1}{2i}\right||e^{iz}-e^{iz}|}{|2|(1)-1} = \frac{3\frac{1}{2i}|e^{iz}-e^{iz}|}{1} = \frac{3}{2i}|e^{iz}-e^{iz}|$

$= \frac{3}{2}\left|e^{iz} + (-e^{-iz})\right| \leq \frac{3}{2}\left(|e^{-y}| + |e^{y}|\right)$ {triangle inequality}

{as $-1 \leq y \leq 1$} $\leq \frac{3}{2}(e+e) = 3e = M$

$\therefore ML = (3e)(2\pi) = 6\pi e$

\5a/ Cauchy integral formula: if $f(z)$ is holomorphic on U and γ is a simple closed path in U and $a \in U$ is contained in γ then $f^{(n)}(a) = \frac{n!}{2\pi i}\oint_\gamma \frac{f(z)}{(z-a)^{n+1}}dz$

$\int_C \frac{\sin(\pi z)}{(z-(-1))^4}dz$ $\quad a = -1 \quad n = 3$

$\frac{2\pi i}{3!}f^{(3)}(-1) = \oint_\gamma \frac{f(z)}{(z-(-1))^4}dz = \frac{1}{3}\pi i f^{(3)}(-1)$

~~$=\cos(\pi z) =$~~ clearly $f(z) = \sin(\pi z)$

$-\cos(-\pi) = -(-1) = 1 \quad \therefore \frac{1}{3}\pi^4 i \qquad f'''(z) = -\pi^3\cos(\pi z)$

\Morera's theorem/ \thm/ let U be a domain let f be a func continuous on U such that $\int_\gamma f(z)dz = 0$ for every positively oriented simple contour γ such that γ^* and its interior is contained in U then $\exists$ a func g defined on U st $g'(z) = f(z)$ $\forall z$ in U

\Morera's thm/ let U be a domain let f be continuous on U if $\int_\gamma f(z)dz = 0$ for every positively oriented, simple, closed contour γ st γ^* and its interior is contained in U then f is holomorphic on U

\Cauchy's estimate/ let f be holomorphic on a domain containing the closed disc $\bar{D}(a,r)$

if M is an upper bound for $|f(z)|$ on the boundary of the disc, so that $|f(z)| \le M$ $\forall z$ with $|z-a| = r$ then $|f^{(n)}(a)| \le \frac{n!M}{r^n}$ $\forall$ non-negative integer n

(recall the notion of closed, bounded, compact sets on p22 & 23 of LN)

proposition let U be a compact subset of $\mathbb{C}$ and let $f: U \to \mathbb{C}$ be a continuous func then f is bounded (recall. f is bounded if $\exists M \ge 0$ st $|f(z)| \le M$, $\forall x \in U$)

an entire func f is an holomorphic func on $\mathbb{C}$

\Liouville's thm/ let f be an entire func if f is bounded then f is constant $\square$

\generalised Liouville's thm/ let f be an entire func if $\exists n, C, R$ st $|f(z)| \le C \cdot |z|^n$ whenever $|z| > R$, then f is a polynomial of degree at most n

\ex/ suppose f is an entire func satisfying $|f(z)| \le |z| + 1$ $\forall z \in \mathbb{C}$ prove $f(z) = Az + B$ where $|A| \le 1$ & $|B| \le 1$ want to: show that there are real consts C & R st $|f(z)| \le C|z|$ whenever $|z| > R$

\generalised Liouville's thm/: $|f(z)| \le C \cdot |z|^n$, $|z| > R$ with $n=1$ let us find C & R we know: if $\underbrace{|z| \ge 1}_{R}$ $\quad |z| + 1 \le \underbrace{2}_{C}|z|$ so we may take $C = 2$ & $R = 1$

find the bounds for the coeffs of f $\quad f(z) = Az + B$

$f(0) = B$ but in our case $|B| = |f(0)| \leq |0| + 1 = 1$ $\quad |B| \leq 1$

$A = f'(0)$ use Cauchy's estimate $|f^{(n)}(a)| \leq \frac{n!\,M}{r^n}$

$n = 1 \quad \therefore \quad |f'(a)| \leq \frac{M}{r} \qquad |A| = |f'(0)| \leq 1 \qquad \square$

\ power series / let $a \in \mathbb{C}$, (a_n) a sequence of complex numbers for each $n \geq 0$ defined $f_n : \mathbb{C} \to \mathbb{C}$ by $f_n(z) = a_n(z-a)^n$

$\sum_{n=0}^{\infty} f_n$: power series about a $\quad \sum_{n=0}^{\infty} a_n(z-a)^n$

any differentiable complex func has a local power series expansion (local means something around a)

\ taylors thm / let f be holomorphic on a domain U & suppose that $D(a,R) \subseteq U$ where $a \in \mathbb{C}$, $R > 0$ then $\exists$ an sequence $(a_n)_{n=0}^{\infty}$ of complex numbers st for any point ~~in~~ $z \in D(a,R)$ $\quad f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$

where $a_n = \frac{f^{(n)}(a)}{n!} = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}\,dw$ where γ is any circular contour with centre a and radius r where ~~$<R$~~ $r < R$

\ proof of taylors thm / assume $a = 0$ let f be holomorphic on an domain U suppose $D(0,R) \subseteq U$ let $z \in D(0,R)$ so that $|z| < R$ let Γ be an circular contour with centre 0 and radius S with $|z| < S < R$ by Cauchy's integral formula have

$f^{(n)}(a) = \frac{n!}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}\,dw$ take $n = 0$ in Cauchy's integral formula: $f(a) = \frac{1}{2\pi i}\int \frac{f(w)}{w-a}\,dw$

$$f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w-z}\,dw = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w(1-\frac{z}{w})}\,dw =$$

$$\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w}\cdot\frac{1}{1-\frac{z}{w}}\,dw = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{w}\cdot\underbrace{\frac{1}{1-\frac{z}{w}}}_{\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n}\,dw$$

$= \frac{1}{2\pi i}\int_{\mu} \frac{f(w)}{w}\cdot\sum_{n=0}^{\infty}\left(\frac{z}{w}\right)^n dw = \frac{1}{2\pi i}\int_{\mu} \sum_{n=0}^{\infty} \left(\frac{f(w)}{w^{n+1}}\cdot z^n\right)dw$

$= \frac{1}{2\pi i}\sum_{n=0}^{\infty} \int_{\mu}\left(\frac{f(w)}{w^{n+1}}\cdot z^n\right)dw$ {can swich Σ and $\int_\gamma$ because have uniform convergence}

$= \sum_{n=0}^{\infty} \underbrace{\left(\frac{1}{2\pi i}\int_{\mu} \frac{f(w)}{w^{n+1}}\,dw\right)}_{a_n} z^n$ $\square$

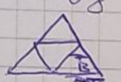
Week 4 live lecture / P45-54LN /

recall/ an func is holomorphic on an set U if it is diffable at every point of U

"vague" Cauchy's thm/ if f is holomorphic at every point inside an contour γ, then $\int_\gamma f(z)dz = 0$ (recall fundamental thm of calculus P35LN)

Cauchys thm for a triangle/ given any two points on distinct edges any point on the interval between these two points is an interior point

Cauchy's thm for an triangle/ if f is holomorphic on an domain U and ~~T~~ (triangle) $T \subseteq U$ is a triangle then $\int_{\partial T} f(z)dz = 0$ (where ∂T is the boundary of triangle)

lemma/ take an triangle T with vertices w_1, w_2, w_3 (P50LN) subdivide T into subtriangles T_1, T_2, T_3, T_4 (where each subtriangle has half the dimension of the original triangle) 

then

$$\int_{\partial T} f(z)dz = \sum_{j=1}^{4}\int_{\partial T_j} f(z)dz$$

\lemma/ let f be holomorphic in an open $U \subseteq \mathbb{C}$, take $\alpha \in U$ then $\exists$ an func $v(z)$ defined on U st $f(z) = f(\alpha) + (z-\alpha) f'(\alpha) + (z-\alpha) v(z)$ and st $v(z) \to 0$ as $z \to \alpha$

\nested sequence of complex sets/

\lemma/ let U be an closed subset of $\mathbb{C}$, let (a_n) be ~~an~~ convergent sequence of elements of U with limit a then $a \in U$ let $U_1 \supseteq U_2 \supseteq \dots \supseteq U_n \supseteq \cdots$ be an nested decreasing sequence of compact subsets of $\mathbb{C}$ then there is an point ~~is~~ α st $\alpha \in U_n$ for each $n \in \mathbb{N}$.

an point z_0 is an star centre of U is $\forall z \in U$ the line segment from z to z_0 lie's inside U

an star domain is an domain in $\mathbb{C}$ with an star centre

\thm/ is f is holomorphic on an star domain U then $f = g'$ for some g holomorphic on U

\cauchy thm for an star domain/ is U is an star domain with f holomorphic on U and γ is an closed contour on U then $\int_\gamma f(z)\,dz = 0$

\cauchy thm/ let U be an domain (simply connected on set) let γ be an closed contour st U contains γ^* and the interior of γ (boundary and interior) let f be holomorphic on U then $\int_\gamma f(z)\,dz = 0$ $\square$

\Jordan Curve thm/ let γ be an simple closed curve then $\mathbb{C} \setminus \gamma^*$ is the disjoint union of: an bounded region called the interior of γ and an unbounded region called the exterior of γ (the image γ^* is the common boundary of these regions)

\thm/ let f be an func holomorphic on an domain U let γ_1, γ_2 be contours with the same start and end points st U contains γ_1^*, γ_2^* and the region between the contours then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$ (deformation thm)

\def/ an simple closed curve is said to be positively oriented if the interior of the curve is to the left of the curve when travelling in the direction of the contour

\thm/ γ_1, γ_2 positively oriented simple contours with γ_2^* lying in the interior of γ_1 if f is holomorphic in some domain that contains γ_1^*, γ_2^* and the region between ~~them~~ contours ~~the~~ then $\int_{\gamma_1} f(z)dz = \int_{\gamma_2} f(z)dz$

\ex P45LN/ $f_n(z) = z^3 + \frac{2\sin z}{n}$

$\therefore$ as $n \to \infty$; $\frac{2\sin z}{n} \to 0$, $f_n(z) \to z^3 = f(z)$

\claim/ $f_n \to f$ uniformly on the disc $\overline{D}(0,1)$

\claim/ f_n does not converge uniformly on $\mathbb{C}$

$|f_n(z) - f(z)| = \left|z^3 + \frac{2\sin z}{n} - z^3\right| = \left|\frac{2\sin z}{n}\right| = \frac{2}{n}|\sin z|$

{as $n > 0$} $= \frac{2}{n}\left|\frac{e^{iz} - e^{-iz}}{2i}\right|$ { note: $|i| = 1$ }

triangle inequality

$= \frac{2}{n} \cdot \frac{1}{2} |e^{iz} - e^{-iz}| = \frac{1}{n}|e^{iz} - e^{-iz}| \leq \frac{1}{n}\left(|e^{iz}| + |e^{-iz}|\right)$

$\therefore \quad |f_n(z) - f(z)| \leq \frac{1}{n}\left(|e^{iz}| + |e^{-iz}|\right)$

{ $|e^{iz}| = |e^{i(x+iy)}| = |e^{ix-y}| = |e^{-y}e^{ix}| = |e^{-y}||e^{ix}|$

$= |e^{-y}||\cos x + i\sin x| = |e^{-y}|\sqrt{\cos^2 x + \sin^2 x} = |e^{-y}|(1)$

$= |e^{-y}| = e^{-y} \leq e$ } this is true for every ~~closed~~ $\overline{D}(0,1)$ so

$\sup_{z \in \overline{D}(0,1)} |f_n(z) - f(z)| \leq \frac{2}{n} e$ goes to 0 as $n \to \infty$

$\therefore$ uniformly convergence on $\overline{D}(0,1)$ //

ex P45LN / $f_n(z) = z^3 + \frac{2\sin(z)}{n}$: $f(z) = z^3$

claim f_n does not converge uniformly on $\mathbb{C}$ /

Suppose by contradiction that f_n converges uniformly on $\mathbb{C}$ $\therefore$ ~~for~~ $|f_n(z) - f(z)| \geq \frac{1}{n}\left(|e^{iz}| - |e^{-iz}|\right)$

~~$\geq \frac{1}{n}$~~ {by reverse triangle inequality} $= \frac{1}{n}(e^{-y} - e^{y})$

{ for convergence: $f_n \xrightarrow[\overline{D}(0,1)]{} f$ uniformly $|f_n - f| \leq \ldots$

doesn't converge: $f_n \not\to f$ uniformly $|f_n - f| \geq \ldots$ }

$|f_n(z) - f(z)| \geq \frac{1}{n}(e^{-y} - e^{y})$ fix some k with $k = -y$

$|f_n(z) - f(z)| \geq \frac{1}{n}(e^{k} - e^{-k}) \geq \frac{1}{n}(e^{k} - 1)$ { $e^{-k} = \frac{1}{e^{k}} \leq 1$

$\therefore -\frac{1}{e^{k}} \geq -1 \quad \therefore -e^{-k} \geq -1$ }

$\frac{1}{n}(e^{k} - 1) \geq \frac{k}{n}$ {by taylor series for e^{k} (P19LN)} //

\Complex integration along an smooth path/

$\gamma:[a,b]\to D$ (D domain)

$\int_\gamma f = \int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt$

\ex/ $f(z)=z^2$, $\gamma(t)=t^2+it$, $0\le t\le 1$

$\int_\gamma f = \int_0^1 f(\gamma(t))\cdot \gamma'(t)\,dt = \int_0^1 (t^2+it)^2\cdot(2t+i)\,dt$

$= \int_0^1 (t^4+2t^3 i - t^2)\cdot(2t+i)\,dt = -\frac{2}{3}+\frac{2}{3}i$ □

\M-L lemma:/ if there is an constant $M\ge 0$ such that $|f(z)|\le M$ for all points z on γ. then $\left|\int_\gamma f\right|\le M\cdot L(\gamma)$ where $L(\gamma)$ is the length of the path (contour)

\ex/ let γ be the upper half of the unit circle traversed anticlockwise show that (P36)

$\left|\int_\gamma \frac{e^z}{z}\,dz\right|\le \pi\cdot e$

$L(\gamma)=\int_0^\pi|\gamma'(t)|\,dt=\pi$

take $\gamma(t)=e^{it}$, $0\le t\le\pi$ $\quad\gamma'(t)=ie^{it}$ $\therefore |\gamma'(t)|=1$

$\therefore \left|\frac{e^z}{z}\right|=\left|\frac{e^{\cos t}}{1}\right|\le e$ $\quad\left\{ \left|\frac{e^{x+iy}}{x+iy}\right| = \left|\frac{1}{x+iy}\right||e^x e^{iy}|\right.$

$\left.= \left|\frac{1}{x+iy}\right||e^x||e^{iy}| = \left|\frac{1}{x+iy}\right||e^x| = \frac{1}{\sqrt{x^2+y^2}}e^x \le e \text{ unit circle}\right\}$

$\frac{3}{2}\cdot\frac{1}{2}\left|e^{2zi}+(-e^{-2zi})\right|\le \frac{3}{4}\left(|e^{2zi}|+|-e^{-2zi}|\right)$

$=\frac{3}{4}\left(|\cos 2z + i\sin 2z|+|e^{-2zi}|\right)=$

$\frac{3}{4}\left(|e^{2(x+iy)i}|+|e^{-2(x+iy)i}|\right)=\frac{3}{4}\left(|e^{2xi-2y}|+|e^{-2xi+2y}|\right)=$

$\frac{3}{4}\left(|e^{-2y}||e^{2xi}|+|e^{2y}e^{-2xi}|\right)=\frac{3}{4}\left(|e^{-2y}|\cdot 1+|e^{2y}||e^{-2xi}|\right)$

$= \frac{3}{4}\left(|e^{-2y}| + |e^{2y}|\right) = \frac{3}{4}\left(e^{-2y} + e^{2y}\right) \leq \frac{3}{4}\left(e^{2\cdot 1} + e^{2\cdot 1}\right)$

$= \frac{3}{4}\left(e^2 + e^2\right) = \frac{3}{4}\left(2e^2\right) = \frac{3}{2}e^2 = M = \text{constant}$

$\frac{3}{2}e^2 \cdot 2\pi = 3\pi e^2 \quad \therefore \ \forall_{\gamma} \ |f(z)| \leq M$

\week 6 vid 1/ \taylors thm:/ let f be holomorphic on a domain U and suppose $D(a,R) \subseteq U$ where $a \in \mathbb{C}$ and $R > 0$ then, $\exists$ a sequence $(a_n)_{n=0}^{\infty}$ of complex numbers st $\forall z \in D(a,R)$ $f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ where

$a_n = \frac{f^{(n)}(a)}{n!}$

{find $f^{(n)}(a)$ using Cauchy's integral formula}

γ is any circular contour with centre a and radius r $(r<R)$

\radius of convergence/ the sum $\sum_{n=0}^{\infty} z^n$ converges if $|z|<1$ and diverges if $|z|>1$ this is, the series converges inside a disc of radius 1, centred at 0 and diverges outside the disc

$\sum_{n=0}^{\infty} a_n (z-a)^n$ for which values of z does the series converge? three possibilities:

① the series converges only when $z=a$

② the series converges everywhere $\forall z \in \mathbb{C}$

③ $\exists$ some radius $R > 0$ st the series converges if $z \in D(a,R)$ and diverges if $|z|>R$

\lemma/ let $\sum_{n=0}^{\infty} a_n (z-a)^n$ be a power series if the series converges for some ~~$z = z_0 \in \mathbb{C}$~~ $z = z_0 \in \mathbb{C}$ with $z_0 \neq a$, then $\forall r$ with $0<r<|z_0 - a|$ the series converges uniformly (and absolutely) on the disc $\overline{D}(a,r)$

\thm (radius of convergence)/ let $\sum_{n=0}^{\infty} a_n(z-a)^n$ be an power series suppose that $\exists$ some $z_0 \neq a$ st the power series converges when $z = z_0$ if the series doesn't converge $\forall z \in \mathbb{C}$, then $\exists R>0$, $R \in \mathbb{R}$ st

- the series converges absolutely when $|z-a|<R$ and
- diverges when $|z-a|>R$ $\square$ (proof 68-69 LN)

the number R is called the radius of convergence of the power series

- if an power series converges for every $z \in \mathbb{C}$, we say it has *infinite* radius of convergence
- if the series converges only at a point $\underline{a}$ we define its radius of convergence to be *zero*

let f be holomorphic on an domain U and suppose $D(a,R) \subseteq U$ where $a \in \mathbb{C}$ and $R>0$ then $\exists$ an sequence $(a_n)_{n=0}^{\infty}$ of complex numbers st $\forall z \in D(a,R)$
$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ where $a_n = \frac{f^{(n)}(a)}{n!}$ and

find $f^{(n)}(a)$ using Cauchy's integral formula here in what we are doing γ is any circular contour with centre $\underline{a}$ and radius r $(r<R)$

\thm/ let f be an function of $z \in \mathbb{C}$ defined by $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ with radius of convergence R then f is holomorphic $D(a,R)$ and $f'(z) = \sum_{n=1}^{\infty} n a_n (z-a)^{n-1}$ $\forall z \in D(a,R)$

\thm (uniqueness of power series)/ if $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ is an power series that converges in an domain containing an disc $D(a,R)$ where $a \in \mathbb{C}$

and $R>0$ ($R \in \mathbb{R}$) then f is holomorphic on $D(a,R)$
and $a_n = \frac{f^{(n)}(a)}{n!}$ $\forall$ integer $n \geq 0$

\ power series of $f(z) = z\sin(z)$ about π / we know that $\forall w \in \mathbb{C}$ $\sin(w) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot w^{2n+1}$ {need to know}

$w = z - \pi$ $\quad \sin(z-\pi) = -\sin z$

$\sin(z-\pi) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} \cdot (z-\pi)^{2n+1}$

$z\sin(z) = (w+\pi)\sin(w+\pi) = w\sin(w+\pi) + \pi\sin(w+\pi) =$
$-w\sin w - \pi\sin w = -w \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} w^{2n+1} - \pi \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} w^{2n+1}$

\ Ex previous Mayexam / find the taylor series around ~~z_0~~ $z=0$ and the radius of convergence of $f(z) = \cos(3z^3)$ / let $w = 3z^3$ let us use the taylor series for cosine ($\cos w$)

~~cos~~ $\cos w$ $\quad f(z) = \underbrace{\sum_{n=0}^{\infty} (-1)^n \cdot \frac{w^{2n}}{(2n)!}}_{\text{series for } \cos(w)} = \sum_{n=0}^{\infty} (-1)^n \cdot \frac{(3z^3)^{2n}}{(2n)!}$

$= \sum_{n=0}^{\infty} (-1)^n \cdot \frac{9^n z^{6n}}{(2n)!}$ the series converges $\forall w \longrightarrow$ converges $\forall z$ $\square$

\ zeros of holomorphic functions (p73-77 LN) /
let f be an holomorphic func of an complex variable an zero of f is an complex number z_0 st $f(z_0) = 0$
suppose f is holomorphic in an domain (an domain is an nonempty, open, connected set) containing an pt $a \in \mathbb{C}$ then $\exists$ an real number ~~$R>0$~~ $r>0$ st f has an power series expansion
$f(z) = \sum_{n=0}^{\infty} a_n (z-a)^n$ in $D(a,r)$

$f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ Suppose a is an zero of f then:

- $a_n = 0$, $\forall n > 0 \Rightarrow$ f is identically zero on $D(a,r)$
- if $\exists$ some $N \in \mathbb{N}$ st $a_0 = a_1 = \dots = a_{N-1} = 0$; $a_N \neq 0$

we say f has an zero of order N at a

by taylors thm, f has an zero of order N at $a \in \mathbb{C}$ if $f(a) = f'(a) = \dots = f^{(n-1)}(a)$; $f^{(n)}(a) \neq 0$

$\left\{ f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n \quad a_n = \frac{f^{(n)}(a)}{n!} \quad a_1 = a_2 = \dots = a_{n-1} = 0,\ \frac{a_n \neq 0}{f^{(n)}(a) \neq 0} \right\}$

Ex/ an zero of order 1 is said to be an simple zero

$f'(a) \neq 0$

an zero of order 2 is said to be an double zero . . .

Ex/ $\cdot f(z) = z$; $f'(z) = 1 \neq 0$ $\therefore$ simple zero/ zero of order 1

~~if~~ $\cdot f(z) = z^2$; $f'(z) = 2z$, $f(0) = 0$, $f'(0) = 0$, $f''(z) = 2 \neq 0$

$f''(0) \neq 0$ $\therefore$ zero of order 2

Lemma/ if f and g have zeros of order n & m respectively at $a \in \mathbb{C}$, then $f \cdot g$ has an zero of order ~~m+~~ $n+m$ at a

~~f has order n~~ f has zeros order n g has zeros order m

$\therefore f \cdot g$ has zeros order $n+m$

Lemma (Isolated zero)/ let f be holomorphic on an domain U containing an pt a if $\exists$ $m \in \mathbb{N}$ st f has an zero of order m at a then the zero is isolated (an zero is isolated if $\exists$ some $r > 0$ st $f(z) \neq 0$ if $z \in D'(a,r)$

\thm/ let f be holomorphic on an domain U if $\exists$ $a\in U$ and $r>0$ st $D(a,r)\subseteq U$ and st $f(z)=0$ $\forall$ point $z\in D(a,r)$ then $f(z)=0$ $\forall z\in U$

this is, if an function is locally zero then it is globally zero

$f(z)=0$ in $D(a,r) \implies f(z)=0$ in U

let S be an open subset of $\mathbb{C}$ consider all $A\subseteq S$ st ① A is open ② $S\setminus A$ is open (complement) if the only sets A that satisfy ① and ② are ~~the~~ $\emptyset$ and S itself $\Rightarrow$ S is topologically connected

~~if S is~~

S is topologically connected $\iff$ {if S} S is connected

\identity thm/ let U be an domain and let $f: U\to\mathbb{C}$ be an holomorphic func

the following are equivalent: ① $f(z)=0$ $\forall z\in U$

② $\exists a\in U, r>0$ st $f(z)=0$ $\forall z\in D(a,r)$

③ the set ~~of~~ S of zeros of f has an limit pt ~~z_0 &~~ $z_0\in U$

\Laurents Series (P 79-81 LN) / $f(z)=\sum_{n=0}^{\infty} a_n(z-a)^n = a_0+a_1z+a_2z^2+\cdots$

\Laurents thm/ if f is holomorphic on a annulus $A=\{z\in\mathbb{C} : R<|z-a|<S\}$ for $0\le R\le S\le\infty$ then $\exists$ an sequence $(b_n)_{n\in\mathbb{Z}}\in\mathbb{C}$ st $f(z)=\sum_{n=-\infty}^{\infty} f_n(z-a)^n$ the Lauren Series $\forall z\in A$

$f(z)=\sum_{n=-\infty}^{\infty} b_n(z-a)^n$ Moreover $\forall r$ st $R<r<S$ and $\forall n\in\mathbb{Z}$ if γ is the circular contour with centre a

and radius R, then $b_n = \frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}dw$

- suppose $\sum_{n=-\infty}^{\infty} a_n(z-a)^n$ is an laurent series convergent in an annulus $A=\{z\in\mathbb{C} : R<|z-a|<S\}$, $R<S$ then $\sum_{n=-\infty}^{-1} a_n(z-a)^n$ is the principle part of the laurent series.

thm (uniqueness of laurent series) let $A=\{z\in\mathbb{C} : R<|z-a|<S\}$, $0<R<S\leq\infty$ if $\sum_{n=-\infty}^{\infty} b_n(z-a)^n$ converges $\forall z\in A$, then $f(z)=\sum_{n=-\infty}^{\infty} b_n(z-a)^n$ is holomorphic on A, and $\forall n\in\mathbb{Z}$ $b_n=\frac{1}{2\pi i}\int_\gamma \frac{f(w)}{(w-a)^{n+1}}dw$, where γ is any circular contour with centre a and radius r with $R<r<S$

ex/ $A=\{z\in\mathbb{C} : 1<|z|<2\}$ $f:\mathbb{C}\setminus\{1,2\}\to\mathbb{C}$ $f(z)=\frac{1}{(z-1)(z-2)}$

want: the laurent series about the pt 0

$f(z)=\frac{1}{(z-1)(z-2)}=\frac{1}{z-2}-\frac{1}{z-1}$ $1<|z|<2$ $|z|<2$ so must

consider $|\frac{z}{2}|<1$ $\therefore \frac{1}{z-2}=\frac{1}{2(1-\frac{z}{2})}=\frac{1}{2}\cdot\frac{1}{1-\frac{z}{2}}=\frac{1}{2}\sum_{n=0}^{\infty}\left(\frac{z}{2}\right)^n$

- we have $|z|>1$ $\therefore$ need $|\frac{1}{z}|<1$ $\therefore \frac{1}{z-1}=\frac{1}{z(1-\frac{1}{z})}=\frac{1}{z}\sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^n$

$=\frac{1}{z}\sum_{n=0}^{\infty}\frac{1}{z^n}=\sum_{n=0}^{\infty}\frac{1}{z^{n+1}}=\sum_{n=1}^{\infty}\frac{1}{z^n}=\sum_{n=1}^{\infty}z^{-n}=\sum_{n=-\infty}^{-1}z^n$ $\therefore$

$f(z)=\frac{1}{(z-1)(z-2)}=\frac{1}{z-2}-\frac{1}{z-1}=-\frac{1}{2}\sum_{n=0}^{\infty}\frac{z^n}{2^n}-\sum_{n=-\infty}^{-1}z^n$ the laurent expansion of f about 0

\ex PS1 LN/ (using laurents thm to compute integrals)
let μ be the circular contour with centre 0 and radius 1.5

$\therefore$ compute $I = \int_\mu f(z)\,dz$; $f(z) = \frac{1}{(z-1)(z-2)}$

$f(z) = \sum_{n=-\infty}^{\infty} a_n z^n$; $a_n = \frac{1}{2\pi i}\int_\mu \frac{f(w)}{w^{n+1}}\,dw$ if take $n=-1$

$a_{-1} = \frac{1}{2\pi i}\int_\mu \frac{f(w)}{w^0}\,dw$ $\therefore 2\pi i a_{-1} = \int_\mu f(w)\,dw$

and $f(z) = -\sum_{n=-\infty}^{-1} z^n - \sum_{n=0}^{\infty} \frac{z^n}{2^{n+1}}$ ← coeffs of our series

corres to $\frac{1}{z}$ is $-z^{-1}$ $\therefore a_{-1} = -1$ this is

$\int_\mu f(w)\,dw = -2\pi i$ $\square$

\singularities/ let U be a domain on which f is holomorphic if a is an point not in U but st Z punctured disc $D'(a,r)$ is an subset of U for some $r>0$ then f has an isolated singularity at a

suppose $f(z) = \frac{\sin z}{z}$ $\therefore$ ~~f: ℂ{~~ $f: \mathbb{C}\setminus\{0\} \to \mathbb{C}$ $\therefore z=0$ is an singularity

if $f: \mathbb{C}\setminus\{0\} \circlearrowleft$ $f(z) = e^{1/z}$ $\therefore$ st $f: \mathbb{C}\setminus\{0\} \to \mathbb{C}\setminus\{0\}$

$\therefore z=0$ is an singularity

if $f(z) = \frac{1}{(z+1)(z+2)}$ ~~f: ℂ{~~ $f: \mathbb{C}\setminus\{-1,-2\} \to \mathbb{C}$ $\therefore z=-1$ and $z=-2$ are singularities

if f has an isolated singularity at a, by laurents thm $f(z) = \sum_{n=-\infty}^{\infty} b_n (z-a)^n$ about $D'(a,r)$ for some $r>0$ we can use Z laurents expansion to study (classify) such singularities ex 1/ $f(z) = \frac{\sin z}{z} = \frac{1}{z}\sin z$ f has Laurent Series:

$f(z) = \frac{1}{z}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots\right) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$ this has zero principle parts so no $z^{-n} = \frac{1}{z}$

$\therefore$ f has an removable singularity at 0 (singularity at $z=0$)

ex/ $f(z) = e^{1/z}$ $z=0$ is an singularity f has Laurent Series $f(z) = \sum_{n=0}^{\infty} \left(\frac{1}{z}\right)^n / n! = \sum_{n=0}^{\infty} \frac{1}{z^n \cdot n!} = \sum_{n=-\infty}^{0} \frac{1}{(-n)!} z^n =$

$\frac{1}{z^0 n!} + \sum_{n=-\infty}^{-1} \frac{1}{(-n)!} z^n$ {there are infinitely many terms in the principle part} $\therefore$ essential singularity

$f(z) = $ eg $\frac{1}{z-3}$ singularity at $z=3$ Laurent Series:

$\dots + \frac{1}{z^3} + \frac{1}{z^2} + \frac{1}{z} + \dots$ ① all zero $\therefore$ f has an removable ~~singul~~ singularity or

② not all zero $\therefore$ (infinitely many different from 0) f has an essential singularity

thm (Picard's great thm) if f is defined on $D'(a,r)$ and has an essential singularity at a, then f takes all complex vals with at most one exception on $D'(a,r)$ eg $f(z) = \frac{1}{z-5}$ $\therefore f: \mathbb{C}\setminus\{5\} \to \mathbb{C}$ $\therefore$ singularity at $z=5$

Cauchy's residue thm/ if f is an function holomorphic in an punctured disc $D'(a,r)$ for some $a \in \mathbb{C}$ and $r > 0$, with Laurent series $\sum_{n=-\infty}^{\infty} b_n (z-a)^n$ for $z \in D'(a,r)$ then the residue of f at a is

$\text{Res}(f,a) = b_{-1}$ $\therefore$ $\dots + b_{-1} \cdot \frac{1}{z} + \dots$ look at coeffs

the residue of f at a is: $Res(f,a)=b_{-1}$ if f has an removeable singularity at $a \implies Res(f,a)=0$
if f has an simple pole at a then $\implies Res(f,a)\neq 0$

thm Cauchy's residue thm/ if γ is an closed, simple contour, traversed anticlockwise, if f is an holomorphic function on an domain containing the image and the interior of γ except for an finite number of isolated singularities in Z interior $(a_1,\dots,a_n)$ then $\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{k} Res(f,a_j)$

Singularities/ let U be an domain on which f is holomorphic if a is an pt not in U but st Z punctured disc $D'(a,r)$ is an subset of U for some $r>0$ then say f has an isolated singularity at a

important:/ if f is holomorphic in an domain U & if f has an zero of order m at $a\in U$ then Z func $\frac{1}{f}$ has an isolated singularity at a

to classify singularities:/ if f has an isolate singularity at a then $f(z)=\sum_{n=-\infty}^{\infty} b_n(z-a)^n$ by laurents thm now look at Z coeffs b_n

- f has an removeable singularity at a if $b_{-m}=0$ $\forall m$, this is $f(z)=\sum_{n=0}^{\infty} b_n(z-a)^n$
- f has an essential singularity at a if $b_{-m}\neq 0$ for infinitely many m
- f has an pole of order N at a if Z principal pt part of Z laurent series is given by $\frac{b_{-N}}{(z-a)^N}+\dots+\frac{b_{-1}}{z-a}$

Let func $f(z) = \sin(\frac{1}{z})$ for $z \neq 0$ $\therefore z=0$ is an isolated singularity $\therefore$ know Z expansion of $\sin z$ around 0 is given by $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$ $\therefore$ can write $\sin(\frac{1}{z}) = \frac{1}{z} - \frac{1}{3!z^3} + \frac{1}{5!z^5} - \dots$ $\therefore$ have that f has an essential singularity at 0

Consider func $f(z) = \frac{\sin z}{z^4}$ for $z \neq 0$ $\therefore z=0$ is an isolated singularity & know Z expansion of $\sin z$ around 0 is given by $\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \dots$ $\therefore$ can write $\frac{\sin z}{z^4} = \frac{z}{z^4} - \frac{z^3}{3!z^4} + \frac{z^5}{5!z^4} - \dots = \frac{1}{z^3} - \frac{1}{3!z} + \frac{z}{5!} - \dots$

$\therefore z=0$ is an pole of order 3

let γ be Z circle with centre 0 & radius 2 Compute $\int_\gamma \frac{e^z}{z^2(z+1)} dz$ $\therefore f$ is diffable except at Z pts where $z^2(z+1)=0$ $\therefore z=0$ & $z=-1$

recall partial fracs $\frac{1}{z^2(z+1)} = \frac{A}{z} + \frac{B}{z^2} + \frac{C}{z+1}$ find A, B, C $\therefore$

$\therefore \frac{1}{z^2(z+1)} = \frac{Az(z+1) + B(z+1) + Cz^2}{z^2(z+1)} = \frac{Az^2 + Az + Bz + B + Cz^2}{z^2(z+1)} =$

$\frac{(A+B)z + (A+C)z^2 + B}{z^2(z+1)}$ $\therefore A+B=0,\ A+C=0,\ B=1$ $\therefore$

$A=-1,\ C=1,\ B=1$ $\therefore \frac{1}{z^2(z+1)} = \frac{-1}{z} + \frac{1}{z^2} + \frac{1}{z+1}$ $\therefore$

$\int_\gamma \frac{e^z}{z^2(z+1)} dz = \int_\gamma \frac{-e^z}{z} dz + \int_\gamma \frac{e^z}{z^2} dz + \int_\gamma \frac{e^z}{z+1} dz$ $\therefore$ Compute each using Cauchy's integral formula

Cauchy's integral formula $f^{(n)}(a) = \frac{n!}{2\pi i} \int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz$

$\therefore \int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$

$\therefore$ for $\int_\gamma \frac{-e^z}{z} dz$ $a=0,\ n=0,\ f(z) = -e^z$

$\therefore f(z) = -e^z$, $a = 0$, $n = 0$ $\therefore \int_\gamma \frac{-e^z}{z} dz = \frac{2\pi i}{0!} f(0) =$

$\frac{2\pi i}{1}(-e^0) = -2\pi i = \int_\gamma \frac{-e^z}{z} dz$

for $\int_\gamma \frac{e^z}{z^2} dz$ $\therefore f(z) = e^z$, $a=0$, $n=1$ for $\int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$

$\therefore \int_\gamma \frac{e^z}{e^z} dz = \frac{2\pi i}{1!} f'(0) = 2\pi i e^0 = 2\pi i$

for $\int_\gamma \frac{e^z}{z+1} dz$ $\therefore f(z) = e^z$, $a=-1$, $n=0$ for $\int_\gamma \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$

$\therefore \int_\gamma \frac{e^z}{z+1} dz = \int_\gamma \frac{e^z}{(z-(-1))} dz = \frac{2\pi i}{0!} f(-1) = \frac{2\pi i}{1} e^{-1} = 2\pi i e^{-1}$

$\therefore$ adding 3 three integrals:

$\int_\gamma \frac{e^z}{z^2(z+1)} dz = \int_\gamma \frac{-e^z}{z} dz + \int_\gamma \frac{e^z}{z^2} dz + \int_\gamma \frac{e^z}{z+1} dz = -2\pi i + 2\pi i + \frac{2\pi i}{e}$

$= \frac{2\pi i}{e} = 2\pi i e^{-1}$ $\square$

\ex/ $\int_C \frac{\sin \pi z}{(z+1)^4} dz$ C: circle with centre 0 & radius 2

Cauchy's integral formula $\int_C \frac{f(z)}{(z-a)^{n+1}} dz = \frac{2\pi i}{n!} f^{(n)}(a)$

$\therefore f(z) = \sin \pi z$, $a = -1$, $n = 3$ $\{(n+1) = 4 \therefore n = 3\}$ $\therefore$

$\int_C \frac{\sin \pi z}{(z+1)^4} dz = \frac{2\pi i}{3!} f'''(-1)$ $f(z) = \sin \pi z \therefore f'(z) = \pi \cos \pi z$

$\therefore f''(z) = -\pi^2 \sin \pi z \therefore f'''(z) = -\pi^3 \cos \pi z \therefore$

$f'''(-1) = -\pi^3 \cos(\pi(-1)) = -\pi^3 \cos(-\pi) = -\pi^3 \cos \pi = -\pi^3(-1) = \pi^3$

$\therefore \int_C \frac{\sin \pi z}{(z+1)^4} = \frac{2\pi i}{3!} \pi^3 = \frac{\pi^4 i}{3}$ $\square$ didn't use the contour but used assumption $\therefore z = -1$ must be inside the contour ex/ $\int_C \frac{2(z^3-z^2-z-1)}{(z^2+1)(z+1)^2}dz$ $\therefore$ $\frac{2(z^3-z^2-z-1)}{(z^2+1)(z+1)^2} =$

$\frac{A}{z-i} + \frac{B}{z+i} + \frac{C}{z+1} + \frac{D}{(z+1)^2}$

Computing residues (P85-92 LN)/ if f has an laurent series $\sum_{n=-\infty}^{\infty} b_n(z-a)^n$, for $z \in D'(a,r)$ then $Res(f,a) = b_{-1}$

$\therefore$ the residue of f at a is the coeff of $\frac{1}{z}$ in our laurent's series

Cauchys residue thm/ suppose: γ is an closed, simple contour, traversed anticlockwise f is holomorphic except at an finite number of isolated singularities, say, $a_1, a_2, \dots, a_k$, then $\int_\gamma f(z)dz = 2\pi i \sum_{j=1}^{k} Res(f, a_j)$

recall P73 LN/ $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ f has an zero of order N at a if $f(a) = f'(a) = \dots = f^{(n-1)}(a) = 0$ & $f^{(n)}(a) \neq 0$

ex/ $f(z) = \sin z$: $f(0) = 0$, $f'(0) = \cos 0 = 1 \neq 0$ $\therefore$ 0 is an pole of order 1 as the first derivative is different to 0 ($\neq 0$) since $f(z) = \sin z$ has order 1 at 0 then ~~$\sin z$~~ $\sin^3 z = \underset{1}{\sin z} + \underset{1}{\sin z} + \underset{1}{\sin z} = 3$ $\therefore \sin^3 z$ has order $1+1+1 = 3$ at 0 $\square$

Computing residues II/ $f(z) = \sum_{n=0}^{\infty} a_n(z-a)^n$ f has an zero of order N at a if $f(a) = f'(a) = \dots = f^{(n-1)}(a) = 0$ & $f^{(n)}(a) \neq 0$

if f has a simple pole at a, then f has laurent series $f(z) = \frac{b_{-1}}{z-a} + b_0 + b_1(z-a) + b_2(z-a)^2 + \dots$ & have

$Res(f,a) = b_{-1} = \lim_{z \to a}(z-a)f(z)$ (*)

\ex/ let γ be a circular contour with centre 0 & radius 3 $\quad f(z)=\frac{1}{(z-1)(z-2)}$ f has two singularities

$\therefore z=1$ & $z=2$ lie inside γ $\therefore$

$Res(f,1)=\lim_{z\to1}(z-1)\cdot f(z)=\lim_{z\to1}(z-1)\cdot\frac{1}{(z-1)(z-2)}=\lim_{z\to1}\frac{1}{z-2}=\frac{1}{1-2}=-1$

$Res(f,2)=\lim_{z\to2}(z-2)\cdot f(z)=\lim_{z\to2}(z-2)\cdot\frac{1}{(z-1)(z-2)}=\lim_{z\to2}\frac{1}{z-1}=\frac{1}{2-1}=1$ $\therefore$

apply Cauchy's residue thm: $\int_\gamma\frac{1}{(z-1)(z-2)}dz=$

$2\pi i\sum_{j=1}^{k}Res(f,a_j)=2\pi i\left[\underbrace{Res(f,1)}_{-1}+\underbrace{Res(f,2)}_{1}\right]=$

$2\pi i(-1+1)=0$ $\square$

\Computing residues III/ \lemma/ let $f(z)=\frac{h(z)}{k(z)}$ has an isolated singularity at a, h & k are holomorphic (differentiable) in a disc centred at a, if $h(a)\neq0$ & k has a simple zero at a, then $Res(f,a)=\frac{h(a)}{k'(a)}$

\ex/ $f(z)=\tan(z)=\frac{\sin(z)}{\cos(z)}$ $\therefore$ $Res(f,\frac{3\pi}{2})=\frac{\sin(\frac{3\pi}{2})}{(\cos(\frac{3\pi}{2}))'}=$

$\frac{\sin(\frac{3\pi}{2})}{-\sin(\frac{3\pi}{2})}=-1$

for poles of higher order, we need to look at the Laurent series

\ex/ compute a residue of $f(z)=\frac{\sin z}{(z-\pi)^6}$ at $z=\pi$

$\therefore$ let $w=z-\pi$ $\therefore$ $\sin z=\sin(w+\pi)=-\sin w$ $\therefore$

$f(w)=-\frac{\sin w}{w^6}=-\frac{1}{w^6}\cdot\sin w=-\frac{1}{w^6}\left(w-\frac{w^3}{3!}+\frac{w^5}{5!}-\dots\right)=$

$-\frac{1}{w^5}+\frac{1}{3!w^3}-\frac{1}{5!w}$ $\therefore$ the coeffs of $\frac{1}{\cancel{z}}$ is $-\frac{1}{5!}$ since

$-\frac{1}{5!w}=\frac{1}{5!(z-\cancel{\pi})}$ $\therefore$ $Res(f,\pi)=-\frac{1}{5!}$ $\square$

Computing residues IV/ notation: write:
$1+w+w^2+w^3+\underbrace{\ldots}_{h.o.t}$ higher-order-terms as
$1+w+w^2+w^3+O(w^4)$

proposition/ Suppose f has a pole of order n at $a \in \mathbb{C}$ then $\text{Res}(f,a)=\lim_{z\to a}\frac{g^{(n-1)}(z)}{(n-1)!}$ where
$g(z)=(z-a)^n \cdot f(z)$

ex/ let $f(z)=\frac{\sin z}{(z-1)^3}$ ∴ f has a triple pole at $z=1$ ∴
write $g(z)=(z-1)^3 \cdot f(z)$ with $n=3$ ∴
$\text{Res}(f,1)=\lim_{z\to 1}\frac{g''(z)}{2!}=-\frac{1}{2}\sin 1$ ∴
$\text{Res}(f,a)=\lim_{z\to a}\frac{g^{(n-1)}(z)}{(n-1)!}$

residues at essential singularities/ have to compute the Laurent series
ex/ find the residue at 0 of $f(z)=e^{2/z}$ ∴ let $w=\frac{2}{z}$
then $f(w)=e^w=1+w+w^2+\ldots=1+\frac{2}{z}+\frac{4}{z^2}+\ldots$ ∴ $\text{Res}(f,0)=2$
since coeff of $\frac{1}{z}$ is 2 □

integrating trigonometric funcs/ if $z=e^{it}$, then may write $\cos t=\frac{1}{2}(z+z^{-1})$; $\sin t=\frac{1}{2i}(z-z^{-1})$ ∴ want to write
$\int_0^{2\pi}F(\cos t,\sin t)\,dt$ as an func of z

ex 14.3 p94 LN/ compute $I=\int_0^{2\pi}\frac{\cos 2t}{5-3\cos t}dt$ write $z=e^{it}$,
$\cos t=\frac{1}{2}(z+z^{-1})$ ∴ $\cos 2t=\frac{1}{2}(\cos 2t+i\sin 2t+\cos 2t-i\sin 2t)=$
$\frac{1}{2}(\cos 2t+i\sin 2t+\cos(-2t)+i\sin(-2t))=\frac{1}{2}(e^{i2t}+e^{-i2t})=$
$\frac{1}{2}((e^{it})^2+(e^{-it})^2)=\frac{1}{2}(z^2+z^{-2})=\cos 2t$ ∴

$I=\int_0^{2\pi}\frac{\frac{1}{2}(z^2+z^{-2})}{5-3\cdot\frac{1}{2}(z+z^{-1})}dt$ since $z=e^{it}$ ∴ $dz=ie^{it}dt$ ∴
$dt=\frac{dz}{ie^{it}}=\frac{dz}{iz}$ ∴ $dt=\frac{dz}{iz}$ ∴ $I=\int_0^{2\pi}\frac{\frac{1}{2}(z^2+z^{-2})}{5-3\cdot\frac{1}{2}(z+z^{-1})}\cdot\frac{dz}{iz}$

take $f(z) = \frac{1}{iz} \cdot \frac{\frac{1}{2}(z^2+\frac{1}{z^2})}{5-\frac{3}{2}(z+\frac{1}{z})} = \frac{\frac{1}{2}(z^2+\frac{1}{z^2})}{\frac{1}{2}(10-3(z+\frac{1}{z}))} \cdot \frac{1}{iz} =$

$\frac{z^2+\frac{1}{z^2}}{iz(10-3(z+\frac{1}{z}))} = \frac{z^2+\frac{1}{z^2}}{iz(10-3z-\frac{3}{z})} = \frac{\frac{1}{z^2}(z^4+1)}{iz \cdot \frac{1}{z}(10z-3z^2-3)} =$

$\frac{z^4+1}{iz^2(10z-3z^2-3)} = f(z) = \frac{i(z^4+1)}{z^2(3z^2-10z+3)}$ {want to find its poles} ∴

$3z^2-10z+3=0 \Longrightarrow z=3$ & $z=\frac{1}{3}$ f has an double pole at 0 since z and z^2 since z^2 also means z

since $h(z)=z=z^2$ ∴ $h(0)=0$, $h'(z)=2z$ ∴ $h'(0)=0$ $h''(z)=2$ ∴ $h''(0)=2\neq 0$ ∴ $h(z)$ has order 2 at 0 ∴ has an double pole at 0

∴ f has an double pole at 0 and has simple poles at $z=3$ & $z=\frac{1}{3}$

∴ Contour $z=e^{it}$ (circle with radius 1) ∴ $z=0$ & $z=\frac{1}{3}$ are inside the unit circle but $z=3$ is not inside Z contour

only the poles $z=0$ & $z=\frac{1}{3}$ are inside Z contour (∴ by proposition 13.13 P90 LN) $Res(f,a)=\lim_{z\to a}\frac{g^{(n-1)}(z)}{(n-1)!}$ ~~with~~ (if f has an pole of order n) with:

$g(z)=(z-a)^{(n)}f(z)$ ∴ if f has an ~~order~~ pole of order 2 at 0 ∴ Sub $n=2$, $a=0$ ∴ f has an double pole (order 2) at 0 ∴ $Res(f,0)=\lim_{z\to 0}\frac{g'(z)}{1!}$; $g(z)=(z-0)^2 \cdot f(z)$ ∴

$Res(f,0)=\lim_{z\to 0} g'(z)$ ∴ $g(z)=z^2 f(z) = \frac{i(z^4+1)}{z^2(3z^2-10z+3)} \cdot z^2 =$

$\frac{i(z^4+1)}{(3z^2-10z+3)} = g(z)$ ∴ $g'(z) = \frac{i\left[4z^3(3z^2-10z+3)-(z^4+1)(6z-10)\right]}{(3z^2-10z+3)^2}$ ∴

$Res(f,0) = \lim_{z\to 0} g'(z) = \frac{i\cdot 10}{9} = \frac{10i}{9} = Res(f,0)$

∴ want to now compute $Res(f,\frac{1}{3})$ where $\frac{1}{3}$ is an simple pole {residue at an simple pole given by formula 13.1 P87 LN} ∴

$\text{Res}(f,\frac{1}{3}) = \lim_{z\to\frac{1}{3}} (z-\frac{1}{3})\cdot f(z) = \cancel{(z-\frac{1}{3})\cdot\frac{i(z^4+1)}{z^2(3z^2-10z+3)}} =$

$\lim_{z\to\frac{1}{3}} (z-\frac{1}{3})\frac{i(z^4+1)}{z^2(3z^2-10z+3)} = \lim_{z\to\frac{1}{3}} (z-\frac{1}{3})\cdot\frac{i(z^4+1)}{z^2(z-3)(3z-1)} =$

$\lim_{z\to\frac{1}{3}} (z-\frac{1}{3})\cdot\frac{i(z^4+1)}{z^2(z-3)(z-\frac{1}{3})} = -i\cdot\frac{41}{36} = -\frac{41i}{36}$ ∴

$\text{Res}(f,0) = \frac{10i}{9}$; $\text{Res}(f,\frac{1}{3}) = -\frac{41}{36}i$ ∴ by Cauchy's residue thm (P85LN) $\int_\gamma f(z)\,dz = 2\pi i\left[\text{Res}(f,0)+\text{Res}(f,\frac{1}{3})\right]$

{where 0 and $\frac{1}{3}$ are the poles inside γ} ∴

$\int_\gamma f(z)\,dz = 2\pi i\left[\frac{10i}{9} - \frac{41i}{36}\right] = \frac{\pi}{18} = \int_0^{2\pi}\frac{\cos 2t}{5-3\cos t}\,dt$ □

\Semicircle ~~Method~~/ want to compute $\int_{-\infty}^{\infty} f(x)\,dx$ using semicircle contour with radius R and letting $R\to\infty$

\Ex 1 P96LN/ Compute $I = \int_{-\infty}^{\infty}\frac{1}{1+x^2}\,dx$ ∴

$f(z) = \frac{1}{1+z^2}$ and consider the contour semi-circle upper half plane ~~from~~ between $-R$ and R and can divide into line segment $-R$ to R on the real axis with a straight line γ_1 and the semi-circle line γ_2 anti-clockwise ∴

γ_1 is the interval $[-R,R]$

γ_2 is the semicircle in the upper half plane of radius R going from R to $-R$

$f(z) = \frac{1}{1+z^2}$ has singularities $z^2+1=0 \Rightarrow z=\pm i$

γ is the ~~joint~~ join of γ_1 & γ_2 ∴ $-i$ is not inside γ so $+i$ is the only singularity in the upper half plane assuming $i<R$

$z=i$ is a simple inside the contour γ to compute the residue at a simple pole we use formula [B.] in LN $\therefore$ $\text{Res}(f,a) = \lim_{z\to a}(z-a)f(z)$ $\therefore$

$$\text{Res}(f, i) = \lim_{z\to i} f(z)(z-i) = \lim_{z\to i}(z-i)\cdot\frac{1}{(z+i)(z-i)} =$$

$$\lim_{z\to i}\frac{1}{z+i} = \frac{1}{2i} = \frac{i}{2i^2} = -\frac{i}{2}$$ $\therefore$ have that

$$\int_\gamma f(z)\,dz = 2\pi i\left(-\frac{i}{2}\right) = \pi$$

$$\int_\gamma f(z)\,dz = \int_{\gamma_1} f(z)\,dz + \int_{\gamma_2} f(z)\,dz = \int_{-R}^{R} f(z)\,dz + \int_{\gamma_2} f(z)\,dz$$

want to estimate the integral on γ_2 will use the M-L inequality (P37) suppose $|f(z)| \le M$, $\forall z \in \gamma$

$$\left|\int_\gamma f(z)\,dz\right| \le M\cdot L(\gamma)$$

suppose $z \in \gamma_2^*$ so that $|z| = R$ $\therefore$ upper half plane semicircle has radius R $\therefore$

$|1+z^2| \ge |z^2| - 1 = R^2 - 1$ if $R>1$ then $|f(z)| = \left|\frac{1}{1+z^2}\right| \le \frac{1}{R^2-1}$ $= M$ $\quad L(\gamma_2) = \pi\cdot R$ $\therefore$ apply the M-L inequality: $\left|\int_\gamma f(z)\,dz\right| \le M\cdot L(\gamma_2) \le \frac{1}{R^2-1}\cdot\pi R \xrightarrow[R\to\infty]{} 0$

had $\int_{-R}^{R} f(z)\,dz = \pi - \int_{\gamma_2} f(z)\,dz = \pi - 0 = \pi \implies \int_{-R}^{R} f(z)\,dz = \pi$ ∎

{Jordan's inequality/lemma/ if $0 < t \le \frac{\pi}{2}$ then $\sin t \ge \frac{2t}{\pi}$

compute/ $I = \int_0^\infty \frac{x\sin x}{x^2+1}\,dx$ take $f(z) = \frac{ze^{iz}}{z^2+1} =$

$$\frac{z(\cos z + i\sin z)}{z^2+1} \quad \therefore \quad I = \text{Im}\left(\int_0^\infty \frac{ze^{iz}}{z^2+1}\right) \quad \therefore$$

~~$\gamma_2 = (t) = Re^{it}$~~ $\gamma_2(t) = Re^{it}$ for $0 \le t \le \pi$

$I_2 = \int_\gamma f(z)\,dz$ $\qquad I_2 = \left|\int_0^\pi \underbrace{\frac{Re^{it}e^{iRe^{it}}}{(Re^{it})^2+1}}_{f(\gamma_2(t))} \cdot \underbrace{iRe^{it}}_{\gamma_2'(t)}\,dt\right| \le$

$\int_0^\pi \left|\frac{Re^{it}e^{iRe^{it}}}{(Re^{it})^2+1}\cdot iRe^{it}\right| dt \le \left(\frac{R^2}{R^2-1}\right)\int_0^\pi \left|e^{iRe^{it}}\right| dt$ ∴

$\left|e^{iRe^{it}}\right| = \left|e^{iR(\cos t + i\sin t)}\right| = \left|e^{iR\cos t}e^{iRi\sin t}\right| =$

$\left|e^{iR\cos t}\right|\left|e^{-R\sin t}\right| = \left|\cos(R\cos t)+i\sin(R\cos t)\right|\left|e^{-R\sin t}\right| =$

$= |1|\left|e^{-R\sin t}\right| = e^{-R\sin t}$ ∴

$|I_2| \le \left(\frac{R^2}{R^2-1}\right)\int_0^\pi e^{-R\sin t}\,dt = 2\left(\frac{R^2}{R^2-1}\right)\int_0^{\pi/2} e^{-R\sin t}\,dt$

{Since sin t is symmetrical about $\frac{\pi}{2}$} ∴

by Jordan's inequality $e^{-R\sin t}\le e^{-\frac{2Rt}{\pi}}$ ∴

$|I_2| \le 2\left(\frac{R^2}{R^2-1}\right)\int_0^{\pi/2} e^{-\frac{2Rt}{\pi}}\,dt = 2\left(\frac{R^2}{R^2-1}\right)\left[-\frac{\pi}{2R}\cdot e^{-\frac{2Rt}{\pi}}\right]_0^{\pi/2} =$

$2\left(\frac{R^2}{R^2-1}\right)\frac{\pi}{2R}\left(1-e^{-R}\right)$ ∴

$f(z)=\frac{ze^{iz}}{z^2+1}$ ∴ f has one singularity in the interior of γ since i is in the interior but $-i$ is not ∴ $z=i$ is a simple pole ∴ $Res(f,i) =$

$\lim_{z\to i}(z-i)\cdot\frac{ze^{iz}}{(z-i)(z+i)} = \lim_{z\to i}\frac{ze^{iz}}{z+i} = \frac{ie^{i\cdot i}}{i+i} = \frac{ie^{-1}}{2i} = \frac{1}{2e}$

∴ by the residue thm $\int_\gamma f(z)\,dz = 2\pi i\, Res(f,i) =$

$2\pi i\frac{1}{2e} = \frac{\pi i}{e}$ ∴ $\int_{-\infty}^{\infty} \frac{x\sin x}{x^2+1}\,dx = \frac{\pi}{e}$ since

taking the imaginary part of $\frac{\pi}{e}i$ $\blacksquare$

~~the Semi-Circle Method Ex p97 LN~~

$I = \int_0^\infty \frac{dx}{(1+x^2)^2}$ is an even func $\therefore$

$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2}$ $\therefore$ let $f(z) = \frac{1}{(1+z^2)^2}$ $\therefore$ apply semi-circle meth with γ_1 being line segment from $+R$ to $-R$ and γ_2 is semi-circle from $-R$ to $+R$ in upper half of ~~plane~~ complex plane

if γ is the ~~joint~~ join of γ_1 and γ_2 let $f(z) = \frac{1}{(1+z^2)^2}$

the only ~~of~~ singularity of f inside γ is i assuming $(R > i)$ i is a double pole for f

$\therefore$ Recall proposition 13.13 in our LN let

$g(z) = (z-i)^2 f(z) = \frac{1}{(z+i)^2}$ $\therefore$ $g'(z) = \frac{-2(z+i)}{(z+i)^4} = \frac{-2}{(z+i)^3}$ $\therefore$

$Res(f, i) = \lim_{z \to i} g'(z) = \lim_{z \to i} \frac{-2}{(z+i)^3} = \frac{1}{4i}$ $\therefore$

$Res(f, i) = \frac{1}{4i}$ $\therefore$ by Cauchy's Residue thm:

$\int_\gamma f(z)dz = 2\pi i Res(f, i) = 2\pi i \cdot \frac{1}{4i} = \frac{\pi}{2} = \int_\gamma f(z)dz$

if $z \in \gamma_2^*$, then $|f(z)| = \left|\frac{1}{(z+i)^2}\right| \leq \frac{1}{(R^2-1)^2}$ via M-L bound,

~~γ~~ $\left|\int_{\gamma_2} f(z)dz\right| \leq \frac{1}{(R^2-1)^2} \cdot \underbrace{\pi R}_{L(\gamma)} \xrightarrow[R\to\infty]{} 0$ $\therefore$

γ is the join of γ_1 & γ_2 $\therefore$ $\int_\gamma f(z)dz = \int_{\gamma_1} f(z)dz + \int_{\gamma_2} f(z)dz$

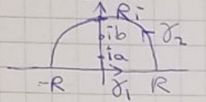
and $\int_{\gamma_2} f(z)dz \xrightarrow[R\to\infty]{} 0$ $\therefore$

$I = \frac{1}{2}\int_{-\infty}^{\infty} \frac{dx}{(1+x^2)^2} = \frac{1}{2} \cdot \frac{\pi}{2} = \frac{\pi}{4}$ $\boxed{}$

Semi-circle Method Ex3 p99 LN / Suppose $0<a<b$

Compute $I = \int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)}$ ∴ let

$f(z) = \frac{1}{(z^2+a^2)(z^2+b^2)}$ $z^2=-a^2$ ∴ $z=\pm ia$, $z^2=-b^2$ ∴ $z=\pm ib$



assume $R>b>a>0$ then for $z \in \gamma_2^*$

have the estimate $|f(z)| \geq \frac{1}{(R^2-a^2)(R^2-b^2)}$ by Z M-L inequality $\left|\int_{\gamma_2} f(z)\,dz\right| \leq \frac{1}{(R^2-a^2)(R^2-b^2)} \cdot \pi R \xrightarrow[R\to\infty]{} 0$

by Cauchy's Residue thm ~~$\int_\gamma f(z)dz = Res(f, ia)+$~~

$\int_\gamma f(z)\,dz = Res(f,ia) + Res(f,ib)$ so must compute Residues

$Res(f,ia) = \lim_{z\to ia}(z-ia) f(z) = \lim_{z\to ia}(z-ia) \cdot \frac{1}{(z+ia)(z-ia)(z^2+b^2)}$

$= \frac{1}{2ia(-a^2+b^2)}$ $Res(f,ib) = \frac{-1}{2ib(b^2-a^2)}$ ∴

$\int_\gamma f(z)\,dz = 2\pi i\left[\frac{1}{2ia(-a^2+b^2)} - \frac{1}{2ib(b^2-a^2)}\right] = -\frac{\pi}{(b-a)(b^2-a^2)}$

∴ $\int_{-\infty}^{\infty} \frac{dx}{(x^2+a^2)(x^2+b^2)} = \frac{\pi}{ab(a+b)}$ ☐

The slice of pie Method / want to compute $I = \int_0^\infty \frac{dx}{1+x^{100}}$ the zeros of $f(z) = \frac{1}{1+z^{100}}$ are the 100th roots of $1+z^{100}=0$ these are $\exp\left(\frac{i\pi(1+2n)}{100}\right)$, $n \in \mathbb{Z}$ take $\alpha = \exp\left(\frac{i\pi}{100}\right)$ integrate over a contour γ: the boundary of a secor of a circle with centre 0, radius R and angle $\frac{2\pi}{100}$

only ~~α~~ $= \exp\left(\frac{i\pi}{100}\right)$ is enclosed in γ the pole at α is a simple pole let $k(z) = 1 + z^{100}$ $\quad k(\alpha) = 0 \quad k'(z) = 1 + 100 z^{99}$

by lemma 13.9 $\operatorname{Res}(f, \alpha) = \frac{h(\alpha)}{k'(\alpha)} = \frac{1}{100\alpha^{99}} = \frac{\alpha}{100\alpha^{100}}$

& $\alpha^{100} = -1$ $\therefore$ $\operatorname{Res}(f, \alpha) = \frac{-\alpha}{100}$ $\therefore$ by the residue

thm: $\int_\gamma f(z)\,dz = 2\pi i \cdot \operatorname{Res}(f,\alpha) = 2\pi i \cdot \left(\frac{-\alpha}{100}\right)$ is the

integral along γ

looking at integral of γ_2: $|f(z)| \leq \frac{1}{R^{100}-1}$ the arc

length of γ_2 is $R \cdot \frac{2\pi}{100}$ {since only segment} $\therefore$

by Z M-L inequality,

$\left|\int_\gamma f(z)\,dz\right| \leq \frac{1}{R^{100}-1} \cdot R \cdot \frac{2\pi}{100} \xrightarrow[R\to\infty]{} 0$

let us look now at the integral on γ_3

take $\gamma_3^-(t) = \alpha^2 t$ for $0 \leq t \leq R$ {γ_3^- is γ_3 but in

opposite direction}

$\int_{\gamma_3} f(z)\,dz = -\int_{\gamma_3^-} f(z)\,dz = -\int_0^R \frac{\alpha^2}{1+(\alpha^2 t)^{100}}\,dt$ $\quad \{(\alpha^2)^{100} = \alpha^{200} = (-1)^2 = 1\}$

$= -\alpha^2 \int_0^R \frac{1}{1+t^{100}}\,dt = -\alpha^2 \int_{\gamma_1} f(z)\,dz$

$\underbrace{\int_\gamma f(z)\,dz}_{\frac{-2\pi i\alpha}{100}} = \int_{\gamma_1} f(z)\,dz + \underbrace{\int_{\gamma_2} f(z)\,dz}_{\xrightarrow[R\to\infty]{} 0} + \underbrace{\int_{\gamma_3} f(z)\,dz}_{-\alpha^2\int_{\gamma_1} f(z)\,dz}$ $\therefore$

$\frac{-2\pi i\alpha}{100} = \int_{\gamma_1} f(z)\,dz - \alpha^2 \int_{\gamma_1} f(z)\,dz = (1-\alpha^2)\int_{\gamma_1} f(z)\,dz =$

$(1-\alpha^2)\int_0^R \frac{1}{1+x^{100}}\,dx = \frac{-2\pi i\alpha}{100}$ $\therefore$

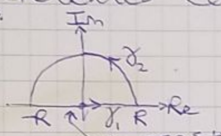
$\int_0^R \frac{1}{1+x^{100}}\,dx = \frac{-2\pi i\alpha}{100} \cdot \frac{1}{(1-\alpha^2)}$ $\therefore$ verify this is

a real number:

$+\frac{2\pi i}{100}\cdot\frac{\alpha}{\alpha^2-1} = \frac{2\pi i}{100}\cdot\frac{1}{\alpha-\alpha^{-1}} = \frac{\pi}{100}\cdot\frac{2i}{\alpha-\alpha^{-1}} =$

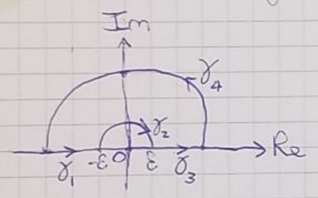
$\frac{\pi}{100}\cdot\frac{2i}{e^{\frac{i\pi}{100}}-e^{-\frac{i\pi}{100}}} = \frac{\pi}{100}\operatorname{cosec}\frac{\pi}{100}$ which is a real number ☑

\indented contours/ \Ex/ evaluate $I=\int_{-\infty}^{\infty}\frac{\sin x}{x}\,dx$



passes through 0

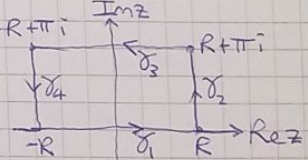
So the semicircle γ passes through the singularity $x=0$



\Rectangle Method/ \Ex P107 LN/ Compute

$I=\int_{-\infty}^{\infty}\frac{dx}{\cosh x}$ define $f(z)=\frac{1}{\cosh z}$ $\left\{\cosh z=\frac{e^{z}+e^{-z}}{2}\right.$ $\therefore$

$\left.\cosh z=\cos(iz)\right\}$ $\therefore f(z)=\frac{2}{e^{z}+e^{-z}}$



$\therefore$ take a rectangular contour γ which is the join of $\gamma_1,\gamma_2,\gamma_3,\gamma_4$ $\therefore$ $\int_\gamma f(z)\,dz=\int_{\gamma_1}f(z)\,dz+\int_{\gamma_2}f(z)\,dz+\int_{\gamma_3}f(z)\,dz+\int_{\gamma_4}f(z)\,dz$

$\therefore$ Look at γ_3 first: $\int_{\gamma_1}f(z)\,dz=+\int_{\gamma_3^-}f(z)\,dz$ $\therefore$

$\gamma_3^-=t+\pi i$, $-R\le t\le R$

$\therefore \int_{\gamma_3}f(z)\,dz=-\int_{-R}^{R}\frac{2}{e^{t+\pi i}+e^{-t-\pi i}}\,dt=-\int_{-R}^{R}\frac{2}{e^{t}\cdot e^{\pi i}+e^{-t}\cdot e^{-\pi i}}\,dt$

$=-\int\frac{2}{e^{t}+e^{-t}}=\int_{\gamma_1}f(z)\,dz$

if $z \in \gamma_2$, then $z = R + it$ for $0 \le t \le \pi$

$|\cosh z| = \left|\frac{e^{R+it} + e^{-R-it}}{2}\right|$ {by reverse triangle inequality}

$\ge \frac{e^R - e^{-R}}{2}$ since $|\cosh z| \ge \frac{e^R - e^{-R}}{2}$

$|f(z)| = \frac{1}{|\cosh z|} \le \frac{2}{e^R - e^{-R}}$ {the length of γ_2 is π}

$\therefore \left|\int_{\gamma_2} f(z)\,dz\right| \le \frac{2}{e^R - e^{-R}} \cdot \pi \xrightarrow[R\to\infty]{} 0$ {M-L inequality} $\therefore$

$\int_{\gamma_2} f(z)\,dz \xrightarrow[R\to\infty]{} 0$ and $\int_{\gamma_4} f(z)\,dz \xrightarrow[R\to\infty]{} 0$ since

$\int_{-\infty}^{\infty} f(z)\,dz = \int_{-R}^{R} f(z)\,dz$ when $R \to \infty$

now compute: $\int_{\gamma} f(z)\,dz$ $f(z) = \frac{1}{\cosh z}$ singularities of f are the zeros of $\cosh z = 0$

$\cos(iz) = 0 \implies iz = \pi \cdot (n + \frac{1}{2}),\ n \in \mathbb{Z} \implies$

$z = -i\pi(n + \frac{1}{2}),\ n \in \mathbb{Z}$ $\therefore$ the only singularity that lies in Z interior of γ is $z = \frac{\pi i}{2}$

$f(z) = \frac{1}{\cosh z}$ $z = \frac{\pi i}{2}$ but this is a simple zero

since $\cosh'(\frac{\pi i}{2}) = \sinh(\frac{\pi i}{2}) = i\sin(\frac{\pi}{2}) = i \ne 0$ $\therefore$

$\mathrm{Res}(f, \frac{\pi i}{2}) = \frac{1}{\cosh'(\frac{\pi i}{2})} = -i$ Residue thm $\implies$

$\int_{\gamma} f(z)\,dz = 2\pi i \cdot \mathrm{Res}(f, \frac{\pi i}{2}) = 2\pi i \cdot (-i) = 2\pi$ $\therefore$

$\underbrace{\int_{\gamma} f(z)\,dz}_{2\pi} = \int_{\gamma_1} f(z)\,dz + \underbrace{\int_{\gamma_2} f(z)\,dz}_{\xrightarrow[R\to\infty]{} 0} + \int_{\gamma_3} f(z)\,dz + \underbrace{\int_{\gamma_4} f(z)\,dz}_{\xrightarrow[R\to\infty]{} 0}$ $\therefore$

$2\pi \to 2\int_{-\infty}^{\infty} \frac{1}{\cosh x}\,dx$ as $R \to \infty$ $\therefore$

$\int_{-\infty}^{\infty} \frac{1}{\cosh x}\,dx = \pi$ ▨

\Sheet 1 Changed/

\8 changed/ $\bigcup_{n=1}^{\infty} \bar{D}(0, 3-\frac{1}{n})$ an open set? ∴

Let $z \in \bar{D}(0, 3-\frac{1}{n})$ ∴ $|z| \leq 3 - \frac{1}{n} < 3$ ∴ $z \in D(0,3)$ $\forall n \in \mathbb{N}$

∴ $\bigcup_{n=1}^{\infty} \bar{D}(0, 3-\frac{1}{n}) \subset D(0,3)$

Let $z \in D(0,3)$ ∴ ~~3-|z|>0~~ $|z| < 3$ ∴ $3-|z| > 0$ ∴

$\frac{1}{3-|z|} > 0$ ∴ is pos & finite ∴ by archimedean property of reals

pos integer m st $m \in \mathbb{N}$ ∴ $\frac{1}{3-|z|} < m$ ∴

$\frac{1}{m} < 3-|z|$ ∴ $-\frac{1}{m} > -(3-|z|)$ ∴

$3-(3-|z|) < 3-\frac{1}{m}$ ∴ $|z| < 3 - \frac{1}{m}$ ∴

$z \in \bar{D}(0, 3-\frac{1}{m}) \subset \bigcup_{n=1}^{\infty} \bar{D}(0, 3-\frac{1}{n})$ ∴ z is arbit element of $D(0,2)$

∴ $D(0,2) \subset \bigcup_{n=1}^{\infty} \bar{D}(0, 2-\frac{1}{n})$ ∴

∴ $\bigcup_{n=1}^{\infty} \bar{D}(0, 2-\frac{1}{n}) = D(0,2)$ ∴ it is an open disk ∴

its an open set

$\cancel{P(B|A)P(A)}$ $\quad P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$

$$P(A|B) = \frac{P(B|A)P(A)}{P(B|A)P(A) + P(B|\neg A)P(\neg A)}$$

$P(A|B) = \frac{P(A \cap B)}{P(B)}$ $\quad P(x_1 = 1 | z = 1) = 0.8$ ∴ $P(x_1 = 0 | z=1) = 0.2$

$P(z=0 | x_1 = 0) = (P(x_1 = 0 | z = 0)\, P(z=0)) / P(x_1 = 0)$

$P(x_1 = 1 | z = 0) = 0.1$ ∴ $P(x_1 = 0 | z = 0) = 0.9$

$$P(z=0|x_1=0) = \frac{P(x_1=0|z=0)P(z=0)}{P(x_1=0|z=0)P(z=0) + P(x_1=0|z=1)P(z=1)}$$

∴ $P(z=0) = 0.8$ ∴ $\frac{0.9 \times 0.8}{0.9 \times 0.8 + 0.2 \times 0.2} = \frac{9}{14}$

$$P(z=0 \mid (x_1=0 \cap x_3=1)) = \frac{P(z=0 \cap (x_1=0 \cap x_3=1))}{P(x_1=0 \cap x_3=1)}$$

$$P(z=0 \mid (x_1=0 \cap x_3=1)) = \frac{P((x_1=0 \cap x_3=1) \mid z=0)P(z=0)}{P((x_1=0 \cap x_3=1) \mid z=0)P(z=0) + P((x_1=0 \cap x_3=1 \mid z=1)P(z=1)}$$

~~proto of~~ for $z=0$ x_1 and x_3 are independent

$$\therefore \quad \frac{P(x_1=0 \mid z=0)P(x_3=1 \mid z=0)P(z=0)}{P(x_1=0 \mid z=0)P(x_3=1 \mid z=0)P(z=0) + P(x_1=0 \mid z=1)P(x_3=1 \mid z=1)P(z=1)}$$

$\therefore$ $P(x_3=1 \mid z=0)=0.9$

$P(x_3=1 \mid z=1)=0.2$ $\therefore$

$$\frac{0.9 \times 0.9 \times 0.8}{0.9 \times 0.9 \times 0.8 + 0.2 \times 0.2 \times 0.2} = \frac{81}{82}$$

$$P(z=0 \mid x_1=0 \cap x_2=1 \cap x_3=1) =$$

$$\frac{P(x_1=0 \cap x_2=1 \cap x_3=1 \mid z=0)P(z=0)}{P(x_1=0 \cap x_2=1 \cap x_3=1 \mid z=0)P(z=0) + P(x_1=0 \cap x_2 \cap 1 \cap x_3=1 \mid z=1)P(z=1)}$$

$$= \frac{P(x_1=0 \mid z=0)P(x_2=1 \mid z=0)P(x_3=1 \mid z=0)P(z=0)}{P(x_1=0 \mid z=0)P(x_2=1 \mid z=0)P(x_3=1 \mid z=0)P(z=0) + P(x_1=0 \mid z=1)P(x_2=1 \mid z=1)P(x_3=1 \mid z=1)P(z=1)}$$

$P(x_2=1 \mid z=0)=0.5$ $\qquad$ $P(x_2=1 \mid z=1)=0.5$ $\therefore$

$$\frac{0.9 \times 0.5 \times 0.9 \times 0.8}{0.9 \times 0.5 \times 0.9 \times 0.8 + 0.2 \times 0.5 \times 0.2 \times 0.2} = \frac{81}{82}$$

2009 Sheet 1 Complex analysis

1a/ $z = x + iy \therefore \bar{z} = x - iy$ $\quad w = a + ib \therefore \bar{w} = a - ib$

$\therefore \bar{z}\bar{w} = (x - iy)(a - ib) = xa - ixb - iay - by = (xa - yb) + i(-xb - ya)$

$zw = (x + iy)(a + ib) = xa - yb + ixb + iya = (xa - yb) + i(xb + ya)$

$\therefore \overline{zw} = (xa - yb) + i(-xb - ya) \quad \therefore \overline{zw} = \bar{z}\bar{w}$

3/ by observation: $2(2i) + i(2i)^2 = 4i + (4)(-1)i = 4i - 4i = 0$

$\cancel{z \to} \; z = x + iy \quad z \to 2i \therefore \cancel{x \to 0} \quad (x + iy) \to 2i \therefore x \to 0,\ y \to 2$

$\therefore \cancel{2(0) + i(\sqrt{2}} \quad 2(0) + i(2)^2 = 4i$

$\therefore$ for $\varepsilon > 0$: $\cancel{f(x,y) = 2x + iy^2}$ $\therefore$ for $0 < |2i| < \delta$

$\cancel{|f(x,y) - f(0,2}$ $\quad f(x + iy) = 2x + iy^2$

$|f(x + iy) - f(4i)| = |2x + iy^2 - 4i| = |2x + i(y^2 - 4)|$

$= \sqrt{(2x)^2 + (y^2 - 4)^2}$

$\forall \varepsilon > 0, \exists \delta > 0$ st $|2x + iy^2 - 4i| < \varepsilon$ whenever $0 < |z - 2i| < \delta$

by norm: $|2x + iy^2 - 4i| \leq 2|x| + |y^2 - 4| = 2|x| + |(y-2)(y+2)|$

$= 2|x| + |y - 2||y + 2|$ $\quad$ let $2|x| < \varepsilon/2$

& $|y - 2||y + 2| < \varepsilon/2$

let $|x| < \varepsilon/4$ $\quad$ is $|y - 2| < 1 \therefore |y + 2| = |y - 2 + 4| \leq |y - 2| + |4|$

$= |y - 2| + 4 < 1 + 4 = 5$ $\therefore$ require $|y - 2| < \mathrm{Min}(1, \varepsilon/10)$

$\therefore |y - 2||y + 2| < \frac{\varepsilon}{10} \times 5 = \varepsilon/2$ $\qquad z = x + iy \therefore$ need $|x| < \varepsilon/4$

& $|y - 2| < \mathrm{Min}(1, \varepsilon/10)$ is a rectangle with sides parallel to the x & y axes centred at pt $(0,2)$ or equiv pt $2i$ in z plane the height (length y direction) is $\mathrm{Min}(1, \varepsilon/10)$ & width (length in x direction) is $\varepsilon/4$ Can fit disk centred at pt $(0,2)$ of radius $\mathrm{Min}(1, \varepsilon/10)$ inside rectangle by def of open disk everypoint inside disk is at most $\mathrm{Min}(1, \varepsilon/10)$ away from $2i$ $\therefore \delta = \mathrm{Min}(1, \varepsilon/10)$