MTH2008 Real Analysis Coursework 1 ①

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1.(i) $S=\left\{\frac{(-1)^n}{n} \middle| n\in\mathbb{N}\right\}=\left\{\frac{(-1)^1}{1},\frac{(-1)^2}{2},\frac{(-1)^3}{3},\frac{(-1)^4}{4},\frac{(-1)^5}{5},\dots\right\}$

$=\left\{\frac{-1}{1},\frac{1}{2},\frac{-1}{3},\frac{+1}{4},\frac{-1}{5},\dots\right\}$

$=\left\{\frac{-1}{1}\right\}\cup\left\{\frac{1}{2}\right\}\cup\left\{\frac{-1}{3}\right\}\cup\left\{\frac{1}{4}\right\}\cup\left\{\frac{-1}{5}\right\}\cup\dots$

the interior of S is S°

the point $x_0=\frac{1}{2}$ is in S but there is no ε-neighborhoods about x_0 that is contained in S

so $x_0=\frac{1}{2}$ is not an interior point of S

same is true $\forall x\in S$ so S has no interior points

so

$S^\circ=\{\ \}=\emptyset$ is the set of interior points of S

1.(ii)/ the exterior of S is $(S^c)^o$ which is the interior of the complement of S

so S^c

$$= \left(\left\{\frac{-1}{1}\right\} \cup \left\{\frac{1}{2}\right\} \cup \left\{\frac{-1}{3}\right\} \cup \left\{\frac{1}{4}\right\} \cup \left\{\frac{-1}{5}\right\} \cup \left\{\frac{1}{6}\right\} \cup \left\{\frac{-1}{7}\right\} \cup \left\{\frac{1}{8}\right\} \cup \dots\right)^c$$

$$= \left(\left\{\frac{-1}{1}\right\} \cup \left\{\frac{-1}{3}\right\} \cup \left\{\frac{-1}{5}\right\} \cup \left\{\frac{-1}{7}\right\} \cup \dots \cup \dots \cup \left\{\frac{1}{8}\right\} \cup \left\{\frac{1}{6}\right\} \cup \left\{\frac{1}{4}\right\} \cup \left\{\frac{1}{2}\right\}\right)^c$$

$$= (-\infty,-1) \cup \left(-1,\frac{-1}{3}\right) \cup \left(\frac{-1}{3},\frac{-1}{5}\right) \cup \dots \cup \dots \cup \left(\frac{1}{8},\frac{1}{4}\right) \cup \left(\frac{1}{4},\frac{1}{2}\right) \cup \left(\frac{1}{2},\infty\right)$$

the point $x_0 = \frac{-1}{8}$ is in S^c and there is an ε-neighborhood about x_0 that is contained in S^c so $x_0 = \frac{-1}{8}$ is an interior point of S^c

the point $x_0 = \frac{1}{9}$ is in S^c and there is an ε-neighborhood about x_0 that is contained in S^c so $x_0 = \frac{1}{9}$ is an interior point of S^c

so $\forall x \in (S^c)$; x is an interior point of S^c

so $(S^c)^o = S^c$ therefore this is the exterior of S so the set of exterior points of S is S^c

1.(iii) the point $x_0 = \frac{-1}{9}$ is in S but there exists a small enough ε such that a deleted ε-neighborhood about x_0 does not contain any point in S so $x_0 = \frac{-1}{9}$ is not a limit point of S, for $\varepsilon > 0$

the point $x_0 = \frac{1}{10}$ is in S but there exists a small enough $\varepsilon \in \mathbb{R}$ such that a deleted ε-neighborhood about x_0 does not contain any point in S so $x_0 = \frac{1}{10}$ is not a limit point of S, for $\varepsilon > 0$

~~so $\forall x \in S$, x is not a limit~~

$\forall x \in S$; x is not a limit point of S so S has no limit points

So the set of limit points of S is $\{\ \} = \emptyset$

④

\1.(iv)/ the point $x_0 = \frac{-1}{9}$ is in S, so every neighborhood of x_0 contains at least one point in S and at least one point not in S which will be below x_0 so $x_0 = \frac{-1}{9}$ is a boundary point of S

the point $x_0 = \frac{1}{10}$ is in S, so every neighborhood of x_0 contains at least one point in S and also at least one point not in S, which will be above x_0, so $x_0 = \frac{1}{10}$ is a boundary point of S

so $\forall x \in S$; x is a boundary point of S

so the set ~~of~~ of boundary points of S is the boundary of S, ~~so $\partial S = S$~~, which is ∂S, so

$\partial S = S$

the closure of S is $\bar{S}$

and $\bar{S} = S \cup \partial S = S \cup S = S$

so the closure of S is S

⑤

1.(v) the point $x_0 = \frac{-1}{9}$ is in S, and for a small enough ~~ε∈R there exists an ε-neighborhood about~~ $\varepsilon > 0$ there exists an ε-neighborhood about x_0 that contains only one point in S, being the point x_0, so $x_0 = \frac{-1}{9}$ is an isolated point of S

the point $x_0 = \frac{1}{10}$ is in S, and for a small enough $\varepsilon > 0$ there exists an ε-neighborhood about x_0 that contains only one point in S, being the point x_0, so $x_0 = \frac{1}{10}$ is an isolated point of S

so $\forall x \in S$; x is an isolated point of S

so the set of isolated points of S is S

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\1.(vi)/ from question \1.(i)/ it is shown $S^\circ = \emptyset$

but is S was open then ~~it would contain~~ for any point x_0 in S ~~would have~~ there would be an neighborhood about x_0 which is contained in S so ~~$\forall x$~~ all points in S are interior to S so $\forall x \in S$; $x \in S^\circ$ so $S \subseteq S^\circ$ but since all points in S° are interior to S then $\forall x \in S^\circ$; $x \in S$ so $S^\circ \subseteq S$ so $S = S^\circ$ so $S = S^\circ = \emptyset$ but $S \neq \emptyset$ which is a contradiction so S is not open.

$S^c =$

$(-\infty,-1)\cup(-1,\frac{-1}{3})\cup(\frac{-1}{3},\frac{-1}{5})\cup\dots\cup\dots\cup(\frac{1}{6},\frac{1}{4})\cup(\frac{1}{4},\frac{1}{2})\cup(\frac{1}{2},\infty)$

$\cancel{\forall x \in S; x \in S^\circ \text{ so } S^\circ}$ $\forall x \in S^c; x \in (S^c)^\circ$

so $(S^c)^\circ = S^c$

is S^c is open, ~~since~~ $0 \notin S \Rightarrow 0 \in S^c$

so there exists $\varepsilon > 0$ with $\varepsilon \in \mathbb{R}$ so since $0 \notin S$

$\Rightarrow \exists \varepsilon > 0; \varepsilon \notin S$ and $\varepsilon \in S^c$ so $\cancel{(0,\varepsilon) \in S^c}$ $(0,\varepsilon) \subset S^c$

but $\cancel{\exists n \in \mathbb{N}}$ $\cancel{\exists 2n \in \mathbb{N}; \frac{1}{2n} < \varepsilon}$ $\cancel{\exists n \in \mathbb{N}; \frac{1}{2n+1} < \varepsilon}$

$\cancel{\text{so } \frac{1}{2n} \in (0,\varepsilon) \subset}$ $\cancel{\text{so } \frac{1}{2n+1} \in (0,\varepsilon)}$

$\exists n \in \mathbb{N}; \frac{1}{2n} < \varepsilon$ so $\frac{1}{2n} \in (0,\varepsilon) \subset S^c$ so $\frac{1}{2n} \in S^c$

but $\forall n \in \mathbb{N}; \frac{1}{2n} \in S$ so $\frac{1}{2n} \in S$ and $\frac{1}{2n} \in S^c$ is a contradiction so S^c is not open

$\Rightarrow (S^c)^c = S$ is not closed

so S is neither ~~&~~ closed nor open

~~\1/ $\bar{S} = S \cup \partial S = S$ but $S \subseteq \partial S$~~

~~so for $S \cup \partial S = S \Rightarrow \partial S = S$~~

~~since $\forall x \in S ; x \in \partial S$ but $\exists x \in \partial S ; x \notin S$~~
~~there can exist $x \in \partial S ; x \notin S$ is S is an open set or neither open nor closed so is~~
~~so $\bar{S} = S \cup \partial S = S \cup S = S = \partial S$ is assumed to be true~~
~~then since $\bar{S} = \partial S$ $\partial S \subseteq S ; \forall x \in \partial S ; x \in S$~~
~~so S contains all its boundary points~~

~~so S contains all its limit points also~~

\2/

is S is closed $\Rightarrow S^c$ is open: ~~$\forall x \in S^c$~~
so $\forall x_0 \in S^c$ there exists an ε-neighborhood ~~contained~~
contained in S^c
so there exists ~~of~~ an ~~ε-neighborhood~~ deleted
ε-neighborhood about x_0 that does not contain
a point in S
so not every deleted ε-neighborhood about x_0
contains a point in S so x_0 is not a limit
point of S
so $\forall x_0 \in S^c ; x_0$ is not a limit point of S
so since there exists an ε-neighborhood about
x_0 that is contained in S^c then not every
neighborhood of x_0 contains a point in S so
x_0 is not a boundary point of S
so $\forall x_0 \in S^c ; x_0 \notin \partial S$
and since this original statement is equivalent

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to its contrapositive and the contrapositive of $(\forall x_0 \in S^c); (x_0 \notin \partial S)$ is $\neg(x_0 \notin \partial S); \neg(\forall x_0 \in S^c)$

$\Rightarrow \forall x_0 \in \partial S; x_0 \notin S^c \Rightarrow \forall x_0 \in \partial S; x_0 \in S$

$\Rightarrow \partial S \subset S$

and since $\bar{S} = S \cup \partial S$ and $\partial S \subset S$

so $S \cup \partial S = S$ so $\bar{S} = S$ as required.

if $S = \bar{S} = S \cup \partial S$ then $(S \cup \partial S) \subset S$ so $\partial S \subset S$ so S contains all its boundary points

so for all points x_0, of which every neighborhood of x_0, contains at least one point in S and one not in S then $x_0 \in S$

so for all x_0 that has every deleted ~~neighbor~~ neighborhood of x_0 contain a point in S then ~~$S \subset S$~~ $x_0 \in S$

so S contains all its limit points

$\Rightarrow$ no point of S^c is a limit point of S

so every point in the complement has a neighborhood which is contained completely with in the complement

so then every point $x_0 \in S^c$ must have a neighborhood contained in S^c which means S^c is open, therefore $(S^c)^c$ is closed $(S^c)^c = S$ so S is closed. as required.

so if $S \subset \mathbb{R}$ is closed $\Rightarrow S = \bar{S}$

and if $S = \bar{S} \Rightarrow S \subset \mathbb{R}$ is closed

so $S \subset \mathbb{R}$ is closed if and only if $S = \bar{S}$ QED

\3/ so $S \subset \bigcup_{H \in \mathcal{H}} H$ with $\mathcal{H} = \{H_1, H_2, H_3, \ldots\}$

where each H_i is open

and $S = \left\{ \frac{(-1)^n}{n} \middle| n \in \mathbb{N} \right\} = \left\{ \frac{-1}{1}, \frac{1}{2}, \frac{-1}{3}, \frac{1}{4}, \frac{-1}{5}, \ldots \right\}$

$= \left\{ \frac{-1}{1}, \frac{-1}{3}, \frac{-1}{5}, \frac{-1}{7}, \ldots, \ldots, \frac{1}{8}, \frac{1}{6}, \frac{1}{4}, \frac{1}{2} \right\}$

is $\mathcal{H} = \left\{ \left(\frac{(-1)^n}{n+(\frac{1}{2})}, \frac{(-1)^n}{n-0.5} \right) \middle| n \in \mathbb{N} \right\}$ then $\mathcal{H} =$

$(\frac{-2}{3}, \frac{-2}{1}) \cup (\frac{2}{5}, \frac{2}{3}) \cup (\frac{-2}{7}, \frac{-2}{5}) \cup (\frac{2}{9}, \frac{2}{7}) \cup (\frac{-2}{11}, \frac{-2}{9}) \cup (\frac{2}{13}, \frac{2}{11}) \cup \ldots$

$= (\frac{-2}{1}, \frac{-2}{3}) \cup (\frac{-2}{5}, \frac{-2}{7}) \cup (\frac{-2}{9}, \frac{-2}{11}) \cup \ldots \cup \ldots \cup (\frac{2}{13}, \frac{2}{11}) \cup (\frac{2}{9}, \frac{2}{7}) \cup (\frac{2}{5}, \frac{2}{3})$

then $\mathcal{H}$ is an open covering of S

since $\forall x \in S; x \in \mathcal{H}$ so $S \subset \mathcal{H}$ as required.

from question \1.(vi)/ S is neither closed nor open

so S is not closed

$S \subset [-1, \frac{1}{2}]$ so $\forall x \in S; x \in [-1, \frac{1}{2}]$ so S is bouned

$\Rightarrow$ S is bounded but not closed

$\Rightarrow$ S is not compact

So Heine-Borel Theorem does not apply to S, so for S ~~As~~ there exists no open covering ~~of S~~ consisting ~~consisten~~ of finitely many open sets belonging to $\mathcal{H}$.

\4/ for $\varepsilon > 0$ and $|x-1| < \delta$: $|f(x) - 0| =$

$|x^2 - 2x + 1 - 0| = |x^2 - 2x + 1| = |(x-1)(x-1)|$

$= |(x-1)^2| = |x-1|^2 < \varepsilon$

so $|x-1| < +\sqrt{\varepsilon}$ so is $+\sqrt{\varepsilon} = \delta$

$\Rightarrow |x-1| < \delta$ as required so $\lim_{x \to 1} f(x) = 0$

so $\lim_{x \to 1^+} f(x) \in \mathbb{R}$ and $\lim_{x \to 1^-} f(x) \in \mathbb{R}$ so $\lim_{x \to 1} f(x) \in \mathbb{R}$

~~and $\lim_{x \to 1} f(x) \in \mathbb{R}$~~ so since $f(x)$ converges to a limit as x approaches 1 from both the left and right side, ~~the limit at $x_0 = 1$ exists so $f(x)$ is continuous at $x_0 = 1$ as required~~ and $f(x_0) = f(1) = 1^2 - 2(1) + 1 = 0 \in \mathbb{R}$, so $f(x_0)$ exists and the limit at $x_0 = 1$ exists ~~so $f(x)$ is continuous at $x_0 = 1$ as required.~~ and since above it shows $\lim_{x \to 1} f(x) = 0$

and $f(1) = 0$ so $\lim_{x \to 1} f(x) = f(1)$ so $f(x)$ is continuous at $x_0 = 1$ as required

\5/ $\forall x \in \mathbb{R}; |\sin(x)| \leq 1 \Rightarrow -1 \leq \sin(x) \leq 1$

So $\mathrm{Max}(\sin(x)) = 1$ and $(1)(0) = 0$ so $\lim_{x \to 0^+} [(1)x^3] = 0$

and $\mathrm{Min}(\sin(x)) = -1$ and $(-1)(0) = 0$ so $\lim_{x \to 0} [(-1)x^3] = 0$

So $\lim_{x \to 0^+} x^3 \sin\left(\frac{1}{x}\right) = 0$

and $\lim_{x \to 0^-} x^3 \sin\left(\frac{1}{x}\right) = 0$ so $\lim_{x \to 0} x^3 \sin\left(\frac{1}{x}\right) = 0$

so $\lim_{x \to 0} g(x) = 0$ and $g(0) = 0$ so $g(x)$ is continuous $\forall x \in \mathbb{R}$

~~so for~~

so $\lim_{x \to x_0} \frac{g(x) - g(x_0)}{x - x_0} = g'(x_0) = \lim_{x_0+h \to x_0} \frac{g(x_0+h) - g(x_0)}{x_0 + h - x_0}$

$\{\text{for } x = x_0 + h\}$ $= \lim_{x_0+h \to x_0} \frac{g(x_0+h) - g(x_0)}{h} = \lim_{h \to 0} \frac{g(x_0+h) - g(x_0)}{h}$

$\forall x_0 \in \mathbb{R}$

So $g'(x_0) = \lim_{h \to 0} \frac{g(x_0+h) - g(x_0)}{h} =$

$\lim_{h \to 0} \frac{(x_0+h)^3 \sin\left(\frac{1}{x_0+h}\right) - x_0^3 \sin\left(\frac{1}{x_0}\right)}{h} = 3x_0^2 \sin\left(\frac{1}{x_0}\right) - x^3\left(\frac{1}{x_0^2}\right)\cos\left(\frac{1}{x_0}\right)$

~~$= 3x_0^2 - x_0 \cos\left(\frac{1}{x_0}\right)$ so $g'(x) = 3x^2 - x\cos\left(\frac{1}{x}\right)$~~

~~$= 3x_0^2 \sin\left(\frac{1}{x_0}\right) - x^3\left(\cos\left(\frac{1}{x}\right)\right)$~~

$= 3x_0^2 \sin\left(\frac{1}{x_0}\right) - x\cos\left(\frac{1}{x}\right)$ so $g'(x) = 3x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right)$

for $\varepsilon > 0$ and ~~$|x - x_0| < \delta: |g'(x) -$~~ ~~$x_1, x_2 \in \mathbb{R}; |x_1 - x_2| < \delta$:~~

~~$|g'(x_1) - g'(x_2)| <$~~

~~$|3x_1^2 \sin\left(\frac{1}{x_1}\right) - x_1\cos\left(\frac{1}{x_1}\right) - 3x_2^2 \sin\left(\frac{1}{x_2}\right) + x_2\cos\left(\frac{1}{x_2}\right)|$~~

~~$= |3\left(x_1^2 \sin\left(\frac{1}{x_1}\right) - x_2^2 \sin\left(\frac{1}{x_2}\right)\right) - \left(x_1\cos\left(\frac{1}{x_1}\right) - x_2\cos\left(\frac{1}{x_2}\right)\right)|$~~

~~$\leq |3(x_1^2 \sin\left(\frac{1}{x_1}\right) - x_2^2 \sin\left(\frac{1}{x_2}\right))| + |x_1\cos\left(\frac{1}{x_1}\right) - x_2\cos\left(\frac{1}{x_2}\right)|$~~

So $g'(x \neq 0) = 3x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right)$

and $\cancel{\lim_{x\to 0}}$ for $g(0) = 0$ if $f(x) = 0$

then $f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{0-0}{h}$

$= \lim_{h\to 0} \frac{0}{h} = \lim_{h\to 0} 0 = 0$

$\therefore$ since $\cancel{g'(0)}$ $g(0) = 0 \Rightarrow g'(0) = \frac{d}{dx}(0) = 0$

So $g'(0) = 0$ and $g'(x \neq 0) = 3x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right)$

$$\text{so } g'(x) = \begin{cases} 3x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right), & \text{if } x \neq 0, \\ 0, & \text{if } x = 0 \end{cases}$$

So $3x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right) \in \mathbb{R}$ for $x \neq 0$

and for $x = 0$; $g'(0) = 0 \in \mathbb{R}$ so

$\forall x \in \mathbb{R}; g'(x) \in \mathbb{R}$ so g is differentiable on $\mathbb{R}$

and $\lim_{x\to x_0} g'(x) = \lim_{x\to x_0}\left[3x^2 \sin\left(\frac{1}{x}\right) - x\cos\left(\frac{1}{x}\right)\right]$

$= \lim_{x\to x_0}\left[3x^2 \sin\left(\frac{1}{x}\right)\right] - \lim_{x\to x_0}\left[x\cos\left(\frac{1}{x}\right)\right]$

$= 3\lim_{x\to x_0}(x^2) \lim_{x\to x_0}\left(\sin\left(\frac{1}{x}\right)\right) - \lim_{x\to x_0}[x] \lim_{x\to x_0}\left[\cos\left(\frac{1}{x}\right)\right]$

$= (3x_0^2)\lim_{x\to x_0}\left[\sin\left(\frac{1}{x}\right)\right] - (x_0)\lim_{x\to x_0}\left[\cos\left(\frac{1}{x}\right)\right]$

$= (3x_0^2)\sin\left(\frac{1}{x_0}\right) - x_0\cos\left(\frac{1}{x_0}\right) = g'(x_0)$

$\therefore \lim_{x\to x_0}\left[g'(x)\right] = g'(x_0)$ and $\lim_{x\to 0}\left[g'(x)\right] =$

$g'(0) = 0 = g'(0)$ so $g'(x)$ is continuous on $\mathbb{R}$

So g is differentiable on $\mathbb{R}$ and g' is continuous on $\mathbb{R}$ as required