~~is $z = x+iy$ ∴~~

4i/ $z = x+iy$ ∴ $\bar{z} = x - iy$ ∴ $\sqrt{2}\,\bar{z} = \sqrt{2}\,x - \sqrt{2}\,iy$

∴ $\cancel{f(x)} = f(a+ib) = \sqrt{2}\,a - \sqrt{2}\,ib$ ∴

$\lim_{z \to a+ib} f(z) = \cancel{\sqrt{2}} \lim_{z \to a+ib} (\sqrt{2}\,x - \sqrt{2}\,iy) = \cancel{\sqrt{2}}$

$\lim_{x \to a,\, y \to b} (\sqrt{2}\,x - \sqrt{2}\,iy) = \sqrt{2}\,a - \sqrt{2}\,ib = f(a+ib)$ ∴ $f(z)$ is cont

$\forall z \in \mathbb{C}$

$\forall \varepsilon > 0,\ \delta = \varepsilon/\sqrt{2}$ let $0 < |z - z_0| < \delta = \varepsilon/\sqrt{2}$ ∴

$|f(z) - f(z_0)| = |\sqrt{2}\,\bar{z} - \sqrt{2}\,\bar{z}_0| = \sqrt{2}\,|\bar{z} - \bar{z}_0| = \sqrt{2}\,|\overline{(z - z_0)}|$

$= \sqrt{2}\,|z - z_0| < \varepsilon$

4ii/ $\forall \varepsilon > 0$: let $0 < |z - z_0| < \delta$ ∴

$|f(z) - f(z_0)| = |3|z| - 3|z_0|| = 3\big||z| - |z_0|\big| = 3\big||z| + (-|z_0|)\big|$

$\leq 3(|z| + |-|z_0||) = 3(|z| + |z_0|)$

~~$|f(z) - f(z_0)| = |3|z| - 3|z_0|| = 3||z| + \alpha - (|z_0| + \alpha)$~~

Sol 4.ii/ $|(|z| - |w|)| \leq |z - w|$

$|z - w + w| \leq |z - w| + |w| \Rightarrow |z| - |w| \leq |z - w|$

$|z - w - z| \leq |z - w| + |-z| \Rightarrow |w| - |z| \leq |z - w| \geq -(|z| - |w|)$

∴ $\cancel{|(|z| - |w|)}$ $|(|z| - |w|)| \leq |z - w|$

$\forall \varepsilon > 0$ take $\delta = \varepsilon/3$ let $0 < |z - z_0| < \delta = \varepsilon/3$

∴ $|f(z) - f(z_0)| = |3|z| - 3|z_0|| = 3|(|z| - |z_0|)| \leq 3|z - z_0| < \varepsilon$

2/ $z = x+iy$ $w = a+iy$ ∴ $z + w = (x+a) + i(y+b)$ ∴

$|z+w| = \sqrt{(x+a)^2 + (y+b)^2} = ((x^2 + a^2 + 2xa) + (y^2 + b^2 + 2yb))^{1/2}$ ∴

$|z| + |w| = \sqrt{x^2+y^2} + \sqrt{a^2+b^2}$

$|z+w| = \sqrt{(x^2+y^2) + (a^2+b^2) + (2xa + 2yb)}$

2 Sol/ $|z+w|^2 = (z+w)\overline{(z+w)} = (z+w)(\bar{z} + \bar{w}) = z\bar{z} + w\bar{w} + w\bar{z} + z\bar{w}$

$= |z|^2 + |w|^2 + \overline{z\bar{w}} + z\bar{w} = |z|^2 + |w|^2 + 2\mathrm{Re}(z\bar{w}) \leq |z|^2 + |w|^2 + 2|z\bar{w}|$

$= |z|^2 + |w|^2 + 2|z||\bar{w}| = |z|^2 + |w|^2 + 2|z||w| = (|z| + |w|)^2$ ∴

$|z+w| \leq |z| + |w|$

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5a/ ~~$\{z \in \mathbb{C} : \text{to}$~~ $\{x+iy \in \mathbb{C} \ \sqrt{x^2+y^2} < 2\}$

$\therefore$ point $(x,y) = (4,0)$ is a limit point ~~of~~ the set but not in the set so the set is not closed

the set contains all its boundary points set is open

5a sol/ open; not closed open is proved

to see not ~~set~~ closed show the complement is not open

take point 2 let $\varepsilon > 0 \Rightarrow 2 - \frac{\varepsilon}{2} \in D(2,\varepsilon)$ which gives that $D(2,\varepsilon)$ is not contained in the complement

5b/ complement is: $\{z \in \mathbb{C} : |z| = 0, |z| > 1\}$

take point 0 let $\varepsilon > 0$ then $|0 - \frac{\varepsilon}{2} \cancel{\notin}| \in D(0,\varepsilon)$ which gives that $D(0,\varepsilon)$ is not contained in the complement

So set is not closed.

the point 1 is in the set let $\varepsilon > 0$ then

$1 + \varepsilon \notin [0,1]$ so the set is not open

5b sol/ let $S = \{z \in \mathbb{C} : 0 < |z| \le 1\}$

point $1 \in S$ let $\varepsilon > 0$ then $|1 + \frac{\varepsilon}{2} - 1| = \frac{\varepsilon}{2} < \varepsilon$

$\therefore 1 + \frac{\varepsilon}{2} \in D(1,\varepsilon)$ but $|1 + \frac{\varepsilon}{2}| > 1 \therefore$ pt $1 + \frac{\varepsilon}{2} \notin S \therefore$

$D(1,\varepsilon) \not\subset S$ as $\varepsilon > 0$ is arbit, holds $\forall \varepsilon > 0 \therefore$ negating def of open $\therefore S$ is not open

complement: $CS = \{0\} \cup \{z \in \mathbb{C} : |z| > 1\}$

let $1 \ge \varepsilon > 0 \therefore |\frac{\varepsilon}{2} - 0| = \frac{\varepsilon}{2} < \varepsilon \quad \therefore \frac{\varepsilon}{2} \in D(0,\varepsilon)$

since $1 \ge \varepsilon > 0$ have $\frac{\varepsilon}{2} \ne 0$ & $|\frac{\varepsilon}{2}| < 1 \therefore$

$\frac{\varepsilon}{2} \notin CS \quad \therefore D(0,\varepsilon) \not\subset CS \quad \therefore CS$ is not open $\therefore$

S is not closed

\5c/ $S = \{z \in \mathbb{C} : \mathrm{Im}(z) > 1\}$ ~~$S^c = S$~~ $S^c = \mathbb{C} \setminus S$

~~pt $\varepsilon > 0$ where $\varepsilon = \mathrm{Im}(z)$~~ pt $1i$

Complement: $S^c = \{z \in \mathbb{C} : \mathrm{Im}(z) \leq 1\}$ let ~~$\varepsilon > 0$~~ $\varepsilon > 1$

~~let $1 > \varepsilon$~~ $\therefore |\frac{\varepsilon}{2} - 1| \leq \frac{\varepsilon}{2} < \varepsilon \therefore \frac{\varepsilon}{2} \in D(1, \varepsilon)$

but ~~$1 > \varepsilon$~~ $\varepsilon > 1 \therefore \frac{\varepsilon}{2} \neq 1$ & ~~$|\frac{\varepsilon}{2}|$~~

$\therefore$ ~~$\frac{\varepsilon}{2} \notin S$~~ $\frac{\varepsilon}{2} \notin S^c \therefore \frac{\varepsilon}{2} \in S \therefore D(\mathrm{Im}(i), \varepsilon) \not\subset S^c$

$\therefore S^c$ is not open $\therefore S$ is not closed

$\forall z \in \mathbb{C} \setminus \{\mathrm{Im}(z) \leq 1\} : z \in S \therefore S$ is open

point $(\mathrm{Im}(z) = 1) \in S^c$ but $\forall \varepsilon > 0 \quad 1 + \varepsilon \notin S^c \therefore$

S^c is closed $\therefore S$ is open

\5c sol/ open (Ex 6.2 LN)

take pt i in the complement then $i(1 + \frac{\varepsilon}{2}) \in D(i, \varepsilon)$

$\therefore D(i, \varepsilon)$ is not contained in the complement $\therefore$

set is not closed

\5d/ $S^c = \mathbb{C} \setminus S \therefore S^c = \{z \in \mathbb{C} : 1 > |z| \cup |z| > 2\}$

let ~~$2 \leq \varepsilon \geq 1$~~ $2 \geq \varepsilon \geq 1 \therefore |\frac{\varepsilon}{2} - 1| \leq \frac{\varepsilon}{2} < \varepsilon \therefore$

$\frac{\varepsilon}{2} \in D(1, \varepsilon)$ but $2 \geq \varepsilon \geq 1) \therefore \frac{\varepsilon}{2} \neq 1$ & $|\frac{\varepsilon}{2}| < 2$

$\therefore \frac{\varepsilon}{2} \notin S \therefore D(1, \varepsilon) \not\subset S \therefore S$ is not open

pt $1 \in S$ ~~$\forall \varepsilon$~~ $\forall (0 < \varepsilon \leq 1) \quad |1 - \varepsilon| < 1 \therefore |1 - \varepsilon| \notin S$

pt $2 \in S \quad \forall \varepsilon > 0 \quad |2 + \varepsilon| \notin S \therefore S$ is closed

\5d sol/ pt 2 let $\varepsilon > 0 \therefore 2 + \frac{\varepsilon}{2} \in (2, \varepsilon)$

$\therefore D(2, \varepsilon)$ is not contained in set $\therefore$ is not open

to see closed note complement is: $D(0,1) \cup C\bar{D}(0,2)$

ie the union of an open disk with the complement of a closed disk $\therefore$ the complement is open

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\6/ $D^c = \mathbb{C} \setminus D$ $\therefore D^c = \{z \in \mathbb{C} : -2 < Re(z) < \frac{1}{3}\}$

let $\varepsilon > 0$ $\therefore$ ~~let~~ $0 < \varepsilon$ $\therefore (-2 + \varepsilon/2) \in D(-2, \varepsilon)$

$\therefore D(-2, \varepsilon)$ is ~~not~~ contained in the complement

$\therefore D^c$ is not closed $\therefore D$ is not open

not complement is $\therefore D(Re(-2), Re(\frac{1}{3}))$ is an open disk

$\therefore$ the complement is open $\therefore D$ is closed

\6 set/ by def is closed Show $CD = \{z \in \mathbb{C} : -2 < Re(z) < \frac{1}{3}\}$

is open: let $z = x + iy \in CD$ $\therefore -2 < x < \frac{1}{3}$ $\therefore$

$0 < x + 2$ $\quad 0 < \frac{1}{3} - x$ $\quad$ let $0 < \varepsilon < \min(x + 2, \frac{1}{3} - x)$

let $w = a + bi \in D(z, \varepsilon)$ note $w - z = a - x + i(b - y)$ $\therefore$

$|a - x| = |Re(w - z)| \leq |w - z| < \varepsilon$ $\therefore$

$|a - x| < \varepsilon < x + 2$ $\quad |a - x| < \varepsilon < \frac{1}{3} - x$ $\therefore$

$-x - 2 < a - x < \frac{1}{3} - x$ $\therefore -2 < a < \frac{1}{3}$ since $a = Re(w)$ & w is

arbit pt of $D(z, \varepsilon)$ $\therefore D(z, \varepsilon) \subset CD$ as z is an arbit

pt of CD have D is ~~closed~~ not closed $\therefore$

~~D is not open~~

let $\frac{7}{3} > \varepsilon > 0$ $\therefore |(-2 + \frac{\varepsilon}{2}) - (-2)| = \varepsilon/2 < \varepsilon$ $\therefore -2 + \varepsilon/2 \in D(-2, \varepsilon)$

$\therefore -2 + \varepsilon/2 \notin D$ (note: $(-2 + \frac{\varepsilon}{2}) \in \mathbb{R}$ $\therefore Re(-2 + \varepsilon/2) = -2 + \varepsilon/2$)

$\therefore D(-2, \varepsilon) \not\subset D$ holds $\forall \varepsilon > 0$ $\therefore D$ is not open

\7/ the complement: $CD(z_0, r)$ is an closed disk

$\therefore$ point F let $\varepsilon > 0$ $\therefore r - \varepsilon/2 \in D(r, \varepsilon)$ $\therefore D(r, \varepsilon)$ is

not contained in the complement $\therefore$ complement is not

open $\therefore$ D is not closed

$r \notin D$ $\quad \forall \varepsilon > 0 : r + \varepsilon \notin D$ $\quad \forall (z_0 < \varepsilon < r)$

~~r~~ $|r - \varepsilon| < r$ $\therefore |r - \varepsilon| \in D$ $\therefore$ D is an open set

\7 sol/ let $z \in D(z_0, r)$ ie $|z - z_0| < r \therefore \forall \varepsilon > 0$
let $0 < \varepsilon < r - |z - z_0|$ let $\tilde{z} \in D(z, \varepsilon)$ ie $|z - \tilde{z}| < \varepsilon \therefore$
$|\tilde{z} - z_0| = |(\tilde{z} - z) + (z - z_0)| \le |\tilde{z} - z| + |z - z_0| < \varepsilon + |z - z_0|$
$< r - |z - z_0| + |z - z_0| = r$ $\therefore$ by def $\tilde{z} \in D(z_0, r)$ as
$\tilde{z}$ is arbit element in $D(z, \varepsilon)$ $\therefore D(z, \varepsilon) \subset D(z_0, r)$
$\therefore$ an open disk is an open set

\8/ $CD(a, r) = \{z \in \mathbb{C} : a \ge |z| \cup |z| \ge r\}$
$\therefore z \le a$, $z \ge r$ $a < r$ $\therefore |z - a| \le a \le r$
$\therefore |z - a| \le r$ $\therefore \bar{D}(a, r) = \{z \in \mathbb{C} : |z - a| \le r\}$
is a closed set since D is an open set
$\bigcup_{n=1}^{\infty} \bar{D}(0, 2 - \frac{1}{n})$ has limit set $(0, 2 - \frac{1}{1}) = (0, 1)$
is an open set $\therefore$ the set is open

\8 sol/ let $z \in \bar{D}(0, 2 - \frac{1}{n})$ $\therefore |z| \le 2 - \frac{1}{n} < 2$ $\therefore z \in D(0, 2)$
holds $\forall$ integers $n \ge 1$ $\therefore \bigcup_{n=1}^{\infty} \bar{D}(0, 2 - \frac{1}{n}) \subset D(0, 2)$
let $z \in D(0, 2)$ then $|z| < 2$ $\therefore 2 - |z| > 0$ $\therefore \frac{1}{2 - |z|}$
is positive & finite $\therefore$ by archimedean property
of real numbers $\therefore \exists$ a pos integer m st $\frac{1}{2 - |z|} < m$
$\therefore \frac{1}{m} < 2 - |z|$ $-\frac{1}{m} > -(2 - |z|)$ $\therefore 2 - (2 - |z|) < 2 - \frac{1}{m}$
$\therefore |z| < 2 - \frac{1}{m}$ $\therefore z \in \bar{D}(0, 2 - \frac{1}{m}) \subset \bigcup_{n=1}^{\infty} \bar{D}(0, 2 - \frac{1}{n})$
as z is arbit element of $D(0, 2)$ $\therefore D(0, 2) \subset \bigcup_{n=1}^{\infty} (0, 2 - \frac{1}{n})$
$\therefore \bigcup_{n=1}^{\infty} \bar{D}(0, 2 - \frac{1}{n}) = D(0, 2)$ open disks are open sets
$\therefore$ is an open set

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\9/ is $\bar{D}(z_0, r) = \bigcup_{n=1}^{\infty} \bar{D}(0, 2 - \frac{1}{n})$ then
$\bar{D}(z_0, r) = (0, 2)$ $\therefore$ $\bar{D}(z_0, r)$ is an open set
since $z_0 < z < r$ $\therefore$ $\bar{D}$ is not a closed set and

\9 sol/ let $z \in \bar{D}(z_0, r)$ ie $|z - z_0| \leq r$
⟨ for $\bar{D}(z_0, r)$ to be open: $\forall z \in \bar{D}(z_0, r)$; $\exists \varepsilon > 0$:
$D(z, \varepsilon) \subset \bar{D}(z_0, r)$ ⟩
⟨ for $\bar{D}(z_0, r)$ to not be open show:
$\exists z \in \bar{D}(z_0, r)$: $\forall \varepsilon > 0$; $D(z, \varepsilon) \not\subset \bar{D}(z_0, r)$ ⟩
(note $D(z, \varepsilon) \not\subset \bar{D}(z_0, r)$ means $\exists$ (atleast) one pt of $D(z, \varepsilon)$ that is not in $\bar{D}(z_0, r)$)
let $z = z_0 + r$ note: $|z - z_0| = |z_0 + r - z_0| = r$ $\therefore$ z lies on circle $C(z_0, r) \subset \bar{D}(z_0, r)$ pick an $\varepsilon > 0$ let $w = z + \varepsilon/2$
$\therefore$ $|w - z| = \varepsilon/2 < \varepsilon$ $\therefore$ $w \in D(z, \varepsilon)$ but $|w - z_0| =$
$|z_0 + r + \frac{\varepsilon}{2} - z_0| = r + \varepsilon/2 > r$ $\therefore$ $w \notin \bar{D}(z_0, r)$
$\therefore$ $D(z, \varepsilon) \not\subset \bar{D}(z_0, r)$ holds $\forall \varepsilon > 0$ (because $\varepsilon > 0$ is chosen arbitrarily) $\therefore$ no it is not open

\10/ let $z = z_0 + r$ note: $|z - z_0| = |z_0 + r - z_0| = |r| = r$ $\therefore$
z lies on circle $C(z_0, r) \subset \bar{D}(z_0, r)$ for an $\varepsilon > 0$ let
$w = z + \varepsilon/2$ $\therefore$ $|w - z| = |z + \frac{\varepsilon}{2} - z| = |\frac{\varepsilon}{2}| = \varepsilon/2 < \varepsilon$ $\therefore$ $w \in D(z, \varepsilon)$
but $|w - z_0| = |(z + \frac{\varepsilon}{2}) - (z - r)| = |z + \frac{\varepsilon}{2} - z + r| = |\frac{\varepsilon}{2} + r|$
$= r + \varepsilon/2 > r$ $\therefore$ $w \in \bar{D}(z_0, r)$ $\therefore$ though $D(z, \varepsilon) \not\subset D(z_0, r)$
is true but $D(z_0, r) \subset D(z, \varepsilon)$ is a set is closed

\10 sol/ $H = C\bar{D}(z_0, r) = \{z \in \mathbb{C} : |z - z_0| > r\}$
let $z \in H$ & $0 < \varepsilon < |z - z_0| - r$ consider open disk $D(z, \varepsilon)$
let $\tilde{z} \in D(z, \varepsilon)$ $\therefore$ $|z - \tilde{z}| < \varepsilon$ note: $|z - z_0| = \cancel{r}$
$= |(z - \tilde{z}) + (\tilde{z} - z_0)| \leq |z - \tilde{z}| + |\tilde{z} - z_0|$ $\therefore$

$|\tilde{z}-z_0| \geq |z-z_0| - |z-\tilde{z}| > |z-z_0| - \varepsilon > |z-z_0| - |z-z_0| + r = r$

$\therefore |\tilde{z}-z_0| > r \quad \therefore \tilde{z} \in H \quad \therefore D(z,\varepsilon) \subset H$ shows the complement of $\overline{D}(z_0,r)$ is open & $\therefore$ by def of a closed set, that $\overline{D}(z_0,r)$ is closed

\2009 sheet 2/

\1/ $D(z_0,r) \neq \emptyset \qquad \exists z \in \mathbb{C} : z \in D(z_0,r) \qquad z_0 + r \notin D(z_0,r)$

$\exists \varepsilon > 0 : z_0 + \varepsilon \in D(z_0,r) \quad \therefore D(z_0,r)$ Containes Multiple values and is not empty and $D(z_0,r) \subseteq \mathbb{C} \quad \therefore D(z_0,r)$ is an domain

\1 sol/ shown it is open in a previous exercise from 2 previous sheet let $z_1, z_2 \in D(z_0,r) \quad \therefore$ Show $[z_1,z_2] \subset D(z_0,r)$ note 2 line segment $[z_1,z_2]$ can be paramed as:

$[z_1,z_2] = \{z_1 + s(z_2-z_1) : 0 \leq s \leq 1\}$ let $\tilde{z} = z_1 + t(z_2 - z_2)$ for some $0 \leq t \leq 1$ be an arbit pt of $[z_1,z_2]$ have $|\tilde{z}-z_0| =$

$|z_1 + t(z_2-z_1) - z_0| = |z_1 + t(z_2-z_1) - z_0 + t(z_0-z_0)| =$

$|z_1 + tz_2 - tz_1 - z_0 + tz_0 + tz_0| = |tz_2 - tz_0 + z_1 - z_0 + tz_0 - tz_1| =$

$|t(z_2-z_0) + (1-t)(z_1-z_0)| \leq |t(z_2-z_0)| + |(1-t)(z_1-z_0)| \leq$

$|t||z_2-z_0| + |1-t||z_1-z_0| < tr + (1-t)r = r \quad \therefore \hat{z} \in D(z_0,r)$

$\therefore D(z_0,r)$ is a domain

\2/ Continuity def is: $|f^{-1}(U) - f_0^{-1}| < \varepsilon \quad \forall \varepsilon > 0$

U is open $\therefore \exists \delta \ |U - U_0| < \varepsilon \ \forall \varepsilon > 0 \ \forall |U - U_0| \leq \delta \ \therefore \delta < \varepsilon$

$\therefore |f^{-1}(U) - f_0^{-1}(U_0)| = |f^{-1}(U) - f_0^{-1}| < \varepsilon \quad \forall \varepsilon > 0 \ \therefore$

$f^{-1}(U)$ is ~~also~~ open

\2 sol/ $(\Longrightarrow)$ 2 disc $D(f(z),\varepsilon)$ open $\therefore$ 2 condition implies $f^{-1}(D((f(z),\varepsilon)))$ is open it contains 2 pt $z \in f^{-1}f(z)$ since $f^{-1}(D((f(z),\varepsilon)))$ is open, $\exists$ an open disc $D(z,\delta) \subset f^{-1}(D((f(z),\varepsilon))) \ \therefore$ if $\omega \in D(z,\delta) \therefore$

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$f(\omega) \in D(f(z), \varepsilon)$ is equiv to $|z-\omega|<\delta \Rightarrow |f(z)-f(z)|<\varepsilon$

($\Leftarrow$) let U be an open set $\forall z \in f^{-1}(U)$ have that $f(z) \in U$ $\therefore \exists$ an open disc $D(f(z),\varepsilon) \subset U$ the ε & δ-characterisation implies $\exists$ an open disc $D(z,\delta)$ st $f(D(z,\delta)) \subset D(f(z),\varepsilon)$ implies $D(z,\delta) \subset f^{-1}(U)$ as z is an arbit pt in $f^{-1}(U)$ shown $f^{-1}(U)$ is open

\3i/ $\frac{d}{dz}$ $f(z)=(z+1)^2$ $\therefore \frac{df}{dz} = \frac{d}{dz}((z+1)^2) = 2(z+1)$

$2(z+1) \in \mathbb{C}$ $\forall z \in \mathbb{C}$ $\therefore f(z)$ is analytic $\forall z \in \mathbb{C}$ $\therefore$

\3i Sol/ largest set it is analytic is $\mathbb{C}$ $f'(z)=2z+2$

\3ii/ analytic $\forall z^3 \neq 1$ $f(z)=\frac{1}{z^3-1}=(z^3-1)^{-1}$ $\therefore$

$\frac{d}{dz}f(z)=-(z^3-1)^{-2}3z^2$ $z^3=1$ is $z=1$, $\frac{\cancel{1}}{\cancel{\sqrt{2}}}$

$e^{3i\theta}=1=\cos3\theta+i\sin3\theta$ $\cancel{e^{i\theta}}=$ $x+iy=e^z=e^{i\theta}$

$(e^{i\theta})^3=1=e^{i3\theta}=\cos3\theta+i\sin3\theta$ $\therefore$ ~~cos3~~ $\sin3\theta=0$ $\therefore$

$0 \leq 3\theta \leq 6\pi$ $\therefore 3\theta=0$, $3\theta=2\pi$, $3\theta=4\pi$ $\therefore$

$\theta=0$, $\theta=\frac{2\pi}{3}$, $\theta=\frac{4\pi}{3}$ $\therefore z=e^{0i}=1$, $z=e^{\frac{2\pi}{3}i}$

$z=e^{\frac{4\pi}{3}i}$ $\therefore$ largest set is $|z|>1$ for analytic

\3ii Sol / analytic largest set is $\mathbb{C}\setminus \Upsilon_3$ for Υ is 3rd roots of unity $f'(z)=-\frac{3z^2}{(z^3-1)^2}$

\3iii/ largest set it is analytic is $\mathbb{C}$

~~\3iv/~~ $\frac{d}{dz}f(z)=\frac{d}{dz}(3z^2+7z+5)=6z+7$

\3iv/ analytic largest set is $z \neq 3$

$f'(z)=\frac{d}{dz}$ $\frac{(3-z)(3)-(3z-1)(-1)}{(3-z)^2}=\frac{9-3z+3z-1}{(3-z)^2}=\frac{8}{(3-z)^2}$

$\therefore \mathbb{C}\setminus\{3\}$

\3v/ analytical set largest is $\mathbb{C} \setminus \{1\}$

$$\frac{d}{dz} f(z) = \frac{d}{dz}\left(\left(\frac{z^3+1}{z-1}\right)\right) e^{\frac{z^3+1}{z-1}} = \frac{(z-1)(3z^2)-(z^3+1)^1}{(z-1)^2} e^{\frac{z^3+1}{z-1}} =$$

$$e^{\frac{z^3+1}{z-1}} \frac{3z^3-3z^2-z^3-1}{(z-1)^2} = \frac{2z^3-3z^2-1}{(z-1)^2} e^{\frac{z^3+1}{z-1}}$$

\4i/ $\frac{d}{dz} f(z) = 2z+3$ is continuous $\forall z \in \mathbb{C}$ $\therefore f(z)$ is Cauchy-Riemann

\4i Sol/ Set $z = x+iy$ find funcs $u(x,y)$ & $v(x,y)$

$(x+iy)^2+3(x+iy)+2 = x^2-y^2+3x+2+i(2xy+3y) = u(x,y)+iv(x,y)$ $\therefore$

$\frac{\partial u}{\partial x} = 2x+3$ $\quad \frac{\partial v}{\partial x} = 2y$ $\quad \frac{\partial u}{\partial y} = -2y$ $\quad \frac{\partial v}{\partial y} = 2x+3$

\4ii/ $z = x+iy$ $\therefore$ $i(x+iy)^2+2(x+iy) =$

$i(x^2-y^2+2ixy)+2x+2iy = ix^2-iy^2-2xy+2x+i2y =$

$(-2xy+2x)+i(x^2-y^2+2y) = u(x,y)+iv(x,y)$

$u(x,y) = -2xy+2x$ $\therefore \frac{\partial u}{\partial x} = -2y+2$, $\frac{\partial u}{\partial y} = -2x$

$v(x,y) = x^2-y^2+2y$ $\quad \frac{\partial v}{\partial x} = 2x$, $\frac{\partial v}{\partial y} = -2y+2$

\4iii/ $z = x+iy$ $\therefore f(x+iy) = e^{1+x+iy} = e^{1+x}e^{iy} =$

$e^{1+x}(\cos y + i\sin y) = \cancel{\cos^{1+x} \cos y}\ e^{1+x}\cos y + ie^{1+x}\sin y$

$= u(x,y)+iv(x,y)$ $\therefore u(x,y) = e^{1+x}\cos y$ $\quad v(x,y) = e^{1+x}\sin y$

$\frac{\partial u}{\partial x} = e^{1+x}\cos y$ $\quad \frac{\partial v}{\partial x} = e^{1+x}\sin y$

$\frac{\partial u}{\partial y} = -e^{1+x}\sin y$ $\quad \frac{\partial v}{\partial y} = e^{1+x}\cos y$

\5/ $r = \sqrt{x^2+y^2}$ $\quad \theta = \tan^{-1}\left(\frac{y}{x}\right)$ $\quad r\sin\theta = y$ $\quad r\cos\theta = x$ $\therefore$

∂u $\quad u(x,y) = e^{1+r\cos\theta}\cos(r\sin\theta)$

$v(x,y) = e^{1+r\cos\theta}\sin(r\sin\theta)$

\5 Sol/ $x = r\cos\theta$, $y = r\sin\theta$ complex func f $f(x+iy) = u(x,y)+iv(x,y)$ Z Cauchy-Riemann eqns are

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ $\quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$ now x & y are funcs of r & θ

$\therefore \frac{\partial u}{\partial r} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial r} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial r}$ $\quad \frac{\partial u}{\partial \theta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \theta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \theta}$

\2009 Sheet 2/ $\therefore \frac{\partial u}{\partial r} = \frac{\partial v}{\partial x}\cos\theta + \frac{\partial v}{\partial y}\sin\theta$

$\frac{\partial v}{\partial \theta} = -\frac{\partial v}{\partial x} r\sin\theta + \frac{\partial v}{\partial y} r\cos\theta$ which when 2 Cauchy Riemann eqns are applied to it yields $\frac{\partial v}{\partial r} =$

$-\frac{\partial u}{\partial y}\cos\theta + \frac{\partial u}{\partial x}\sin\theta$ $\quad \frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial y} r\sin\theta + \frac{\partial u}{\partial x} r\cos\theta$ $\therefore$

$\frac{1}{r}\frac{\partial v}{\partial \theta} = \frac{\partial u}{\partial r}$ $\quad \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$

\7a/ $\int_\gamma x - y + ix^2 \, dz$ = straight line $\gamma(t) = x(t)$

0 to $1+i$ $\therefore$ $\gamma(t) = z_1 + t(z_2 - z_1) = 0 + t(1+i-0) = t(1+i) = t+it$

$\int_\gamma f(z)dz = \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt \qquad \gamma'(t) = 1+i$

$\therefore \int_\gamma f(z)\,dz = \int_0^{1+i} f(t+it)\cdot(1+i)\,dt = \int_0^{1+i}(t - t + it^2)\cdot(1+i)\,dt =$

$\int_0^{1+i}(it^2)(1+i)\,dt = \int_0^{1+i} it^2 - t^2\,dt = \left[\frac{1}{3}it^3 - \frac{1}{3}t^3\right]_0^{1+i} =$ X

$\frac{1}{3}i((1+i)^3 - 0^3) - \frac{1}{3}((1+i)^3 - 0^3) = \cancel{\frac{1}{3}i}\;(1+i)^3\left(\frac{1}{3}\right)(i-1)$

$= \cancel{(i^2 - 1 + 2i)\frac{1}{3}(i-1-1-i) = \frac{1}{3}(2-i)}(-2)\;(1-1+2i)(1+i)\left(\frac{1}{3}\right)(i-1)$ X

\7a Sol/ $\gamma(t) = t + it,\ 0 \le t \le 1$ $\therefore \int_\gamma x - y + ix^2 = \int_0^1 it^2(1+i)\,dt$

$= \left[-t^3/3 + it^3/3\right]_0^1 = -\frac{1}{3} + i/3$

\6/ $\forall z \in \mathbb{C} : f(z) \in \mathbb{C}$ $\quad$ ~~If~~ $\forall z \in \mathbb{C} : f(z) \in \mathbb{R}$

is $z_0 \in \mathbb{C}$ $\quad z_1 \in \mathbb{C}$ $\quad \therefore \exists z_2 \in (z_0, z_1)$ st $\frac{f(z_0) + f(z_1)}{2}$

$\therefore \forall f(z_3) \in [f(z_0), f(z_1)]\ \exists z_3 \in (z_0, z_1)$ $\therefore$

$f(z_0) = f(z_1) = f(z_2) = f(z_3) = $ constant $\therefore \forall z.\ f(z)$

$\forall z \in \mathbb{C} : f(z) =$ constant

\6 Sol/ imaginary part of f is always zero $\therefore$

$f(x+iy) = u(x,y)$ $\quad \therefore v(x,y) = 0\ \ \forall x, y \therefore \frac{\partial v}{\partial x} = 0 = \frac{\partial v}{\partial y}$

analytic implies CRE is satisfied $\therefore \frac{\partial u}{\partial x} = 0$

$\frac{\partial u}{\partial y} = 0$ implies u constant wrt x & const wrt y $\therefore$ u is constant

\7b/ $\int_\gamma f(z)\,dz = \int_a^b f(\gamma(t))\cdot\gamma'(t)\,dt$

straight line $\gamma(t)=z_1+t(z_2-z_1)=i+t(1+i-i)=i+t$

$0\le t\le 1$ ∴ $\int_\gamma x-y+ix^2\,dz = \int_0^1 f(i+t)\cdot\gamma'(t)\,dt$

$\gamma'(t)=1$ $\quad f(i+t)=f(t+i)=f(t+1i)=t-1+it^2$

∴ $\int_\gamma f(z)\,dz = \int_0^1 (t-1+it^2)\cdot(1)\,dt = \int_0^1 t-1+it^2\,dt = \left[\frac{1}{2}t^2-t+i\frac{1}{3}t^3\right]_0^1$

$=\frac{1}{2}(1^2-0^2)-(1-0)+i\frac{1}{3}(1^3-0^3)=\frac{1}{2}-1+i\frac{1}{3}1=-\frac{1}{2}+\frac{1}{3}i$

\8a/ $\gamma(t)=z_0+re^{it}=2+2e^{it}$ ∴ $z_0=2$, $r=2$ $\quad 0\le t\le 2\pi$ ∴

$\int_\gamma \frac{1}{z-2}\,dz$ ∴ $\gamma'(t)=rie^{it}=2ie^{it}$ ∴

$\int_\gamma \frac{1}{z-2}\,dz=\int_a^b \frac{1}{\gamma(t)-2}\cdot\gamma'(t)\,dt=\int_0^{2\pi}\frac{1}{2+2e^{it}-2}2ie^{it}\,dt$

$=\int_0^{2\pi}\frac{2ie^{it}}{2e^{it}}\,dt=\left[\ln|2e^{it}|\right]_0^{2\pi}=\ln|2e^{2\pi i}|-\ln|2e^0|=$

$\ln|2(\cos 2\pi+i\sin 2\pi)|-\ln|2\cdot 1|=\ln|2(1+0i)|-\ln|2|=\ln|2|-\ln|2|=0$

$\int_0^{2\pi}\frac{ie^{it}}{e^{it}}\,dt=\left[\ln|e^{it}|\right]_0^{2\pi}=\ln|e^{2\pi i}|-\ln|e^{i(0)}|=$ $\quad$ X $2\pi i$

$\ln|\cos 2\pi+i\sin 2\pi|-\ln|e^0|=\ln|1+i(0)|+-\ln|1|=$

$\ln|1|-\ln(0)=0$

\8a/8a/ $\int_0^{2\pi}\frac{ie^{it}}{e^{it}}\,dt=\left[\ln(e^{it})\right]_0^{2\pi}=\ln(e^{2\pi i})-\ln(e^{0i})=2\pi i$

$=\ln(e^{2\pi i})-\ln(e^0)=2\pi i-0=2\pi i$

\8b/ $\gamma(t)=\cancel{z_0+re}\; i+e^{-it}$ $\quad 0\le t\le\frac{\pi}{2}$ ∴ $\gamma'(t)=-ie^{-it}$ ∴

$\int_\gamma \frac{1}{(z-i)^3}\,dz=\int_0^{\pi/2}f(\gamma(t))\cdot\gamma'(t)\,dt=\int_0^{\pi/2}\frac{1}{(i+e^{-it}-i)^3}\cdot -ie^{-it}\,dt$

$=\int_0^{\frac{\pi}{2}}\frac{1}{(e^{-it})^3}(-ie^{-it})\,dt=\int_0^{\pi/2}-i\frac{1}{(e^{-it})^2}\,dt=\int_0^{\pi/2}-ie^{i2t}\,dt=-i\left[\frac{1}{2i}e^{2it}\right]_0^{\pi/2}$

$=-\frac{1}{2}\left[e^{2i\frac{\pi}{2}}-e^{2i(0)}\right]=-\frac{1}{2}\left[e^{i\pi}-e^0\right]=-\frac{1}{2}\left[-1-1\right]=1$

2009 Sheet 2

9/ $\gamma(0) = 4e^{i2\pi(0)} + 3 - 2i = 4e^{0} + 3 - 2i = 4 + 3 - 2i = 7 - 2i$

$\gamma(\tfrac{1}{4}) = 4e^{i2\pi(\frac{1}{4})} + 3 - 2i = 4e^{\frac{1}{2}\pi i} + 3 - 2i =$

$4(\cos\tfrac{1}{2}\pi + i\sin\tfrac{1}{2}\pi) + 3 - 2i = 4(0 + 1i) + 3 - 2i = 4i + 3 - 2i = 3 + 2i$

$|\gamma(\tfrac{1}{4}) - \gamma(0)| = |3 + 2i - (7 - 2i)| = |-4 + 4i| =$

$\sqrt{(-4)^2 + (4)^2} = \sqrt{16 + 16} = \sqrt{32} = \sqrt{4^2 \cdot 2} = \sqrt{4^2}\cdot\sqrt{2} = 4\sqrt{2}$

9 Sol/ def of the length of a smooth curve

~~γ~~ $\gamma : [a, b] \to \mathbb{C}$ is: $L(\gamma) = \int_a^b |\gamma'(t)|\, dt$

note $\gamma'(t) = 8\pi i e^{i2\pi t}$

$L(\gamma) = \int_0^{1/4} |8\pi i e^{i2\pi t}|\, dt = 8\pi \int_0^{1/4} |e^{i2\pi t}|\, dt =$

$8\pi \int_0^{1/4} |\cos(2\pi t) + i\sin(2\pi t)|\, dt = 8\pi \int_0^{1/4} \sqrt{\cos(2\pi t)^2 + \sin(2\pi t)^2}\, dt$

$= 8\pi \int_0^{1/4} dt = 2\pi$

∴ / unit semi circle $\gamma : [a, b] \to \mathbb{C}$ $\gamma(t) = e^{it}$

$0 \le t \le \pi$ ∴ $L(\gamma)$ $\gamma'(t) = ie^{it}$ ∴ $L(\gamma) = \int_0^{\pi} |\gamma'(t)|\, dt =$

$\int_0^{\pi} |ie^{it}|\, dt = \int_0^{\pi} |i||e^{it}|\, dt = \int_0^{\pi} 1 \cdot \sqrt{\cos^2 t + \sin^2 t}\, dt = \int_0^{\pi} 1 \cdot 1\, dt$

$= \int_0^{\pi} 1\, dt = [t]_0^{\pi} = \pi - 0 = \pi$

Sheet 1

6/ $D = \{z \in \mathbb{C} : Re(z) \ge \frac{1}{3} \text{ or } Re(z) \le -2\}$ we need to show

that the complement $CD = \{z \in \mathbb{C} : -2 \le z \le \frac{1}{3}\}$ is open

let ~~$z \in$~~ ~~$-2 < x < 1/3$~~ $z = x + iy \in CD$ ∴ $-2 < x < 1/3$ ∴ $x + 2 > 0$

$x - \frac{1}{3} < 0$ let $\varepsilon > 0$ be st $\varepsilon < \min(x + 2, \frac{1}{3} - x)$ let

$w = a + bi \in D(z, \varepsilon)$ ∴ $w - z = (a + bi) - (x + yi) = a - x + (b - y)i$

$|a - x| = Re(w - z) \le |w - z| < \varepsilon$ ∴ $\left.\begin{array}{l} |a - x| < \varepsilon < x + 2 \\ |a - x| < \varepsilon < \frac{1}{3} - x \end{array}\right| \Longrightarrow$

$-x-2 < a-x < \frac{1}{3} - x \implies -2 < a < \frac{1}{3}$

Since $a \in Re(w)$ and $w \in D(z,\varepsilon)$ $\therefore D(z,\varepsilon) \subset (CD)$ ~~as any~~ ~~point~~ as z is any point in CD we finish our proof. let us now show D is not open:

take for example $\frac{7}{3} > \varepsilon > 0$ $0 < \varepsilon < \frac{7}{3}$ have:

$\left|(-2+\frac{\varepsilon}{2})-(-2)\right| = \frac{\varepsilon}{2} < \varepsilon$ $\therefore -2+\frac{\varepsilon}{2} \in D(-2,\varepsilon)$ but

$-2 < -2+\frac{\varepsilon}{2} < \frac{1}{3}$ $-2+\frac{\varepsilon}{2} \notin D$

$\therefore D(-2,\varepsilon) \not\subset D$

Sheet 3

1/ let $z = x+iy$ use the estimation lemma note the circle's length is its circumference: 2π $\therefore$ using reverse triangle inequality have on the circle:

$|2z^5-1| \geq |2z^5| - |1| = |2z^5| - 1 = 2|z|^5 - 1 = 2(1)^5 - 1 = 2-1 = 1$

$\therefore$ on the circle: $\left|\frac{3z^3 \sin z}{2z^5-1}\right| = \frac{3|z|^3|\sin z|}{|2z^5-1|} \leq 3|\sin z| = 3\left|\frac{e^{iz}-e^{iz}}{2i}\right|$

$\leq \frac{3}{2}(|e^{iz}|+|e^{-iz}|)$

$= \frac{3}{2}(|e^{ix-y}|+|e^{-ix+y}|) = \frac{3}{2}(e^{-y}+e^{y}) \leq 3e$ $\therefore$ the last inequality follows because, on the circle we have $-1 \leq y \leq 1$ which implies $e^{y} \leq e$ and $e^{-y} \leq e$ $\therefore$ the estimation lemma now gives the desired answer: $3e \cdot 2\pi = 6\pi e$

2/ Show the contrapositive statement. Suppose S is not closed then the complement of S, namely CS is not open $\therefore \exists$ a point $z \in CS$ st $\forall \varepsilon > 0$, the open disc $D(z,\varepsilon)$ is not a subset of CS it follows that $\forall n \in \mathbb{N}$ the open disc $D(z, 1/n)$ contains a point of S $\therefore$ may choose a point z_n in this disc from S then $\{z_n\}_{n=1}^{\infty}$ is a sequence of points in S but it's limit z is not

\Sheet 3/

\3a/ ~~Set~~ Let U denote the question set

$\therefore$ U is not a star domain, to prove: show that $\forall z_0 \in U$ the point z_0 is not a star centre for U

let $z_0 \in U$ which implies that $|z_0| > 1$ define $z = -z_0$ then $|z| = |z_0| > 1$ $\therefore$ $z \in U$ the point $0 = \frac{1}{2}z + \frac{1}{2}z_0$ lies on the line between z and z_0 but $0 \notin U$ $\therefore$ z_0 is not a star centre for U

\3b/ this is a star domain show $z_0 = 1$ is a star centre. indeed let $z \in U$ so $Re(z) > 0$ I claim the line segment $[z_0, z]$ lies in U

any point w in $[z_0, z]$ takes the form $w = (1-t)z_0 + tz$ with $0 \le t \le 1$ as $z_0 = 1$, we have $Re(w) = (1-t) + tRe(z) = 1 + t(Re(z) - 1)$ there are two possibilities to consider either $Re(z) \ge 1$ or $Re(z) < 1$ if $Re(z) \ge 1$ then $Re(w) \ge 1 + 0 \times (Re(z) - 1) = 1$ if $Re(z) < 1$ then $Re(w) > 1 + 1 \times (Re(z) - 1) = Re(z) > 0$ either case $Re(w) > 0$ so $w \in U$

\4/ the statement is false. Consider domain ~~$U = D(0,2)\setminus\{0\}$~~
$U = D(0,2)\setminus\{0\}$ the function $f(z) = 1/z$ is holomorphic on U if $\gamma : [0, 2\pi] \to \mathbb{C}$ is the unit circular contour then its image, the unit circle, is completely contained in U ie $\gamma([0,2\pi]) \subset U$ but as shown

$$\int_\gamma f(z)\,dz = 2\pi i \neq 0$$

5a/ Let $f(z)=\sin(\pi z)$ then f is entire (ie analytic everywhere) so may apply Cauchy integral formula

note: $f^{(3)}(z)=-\pi^3\cos(\pi z)$

for every $z\in\mathbb{C}$ the point -1 lies in the interior of Z circle C so Z integral is $\frac{2\pi i}{3!}f^{(3)}(-1)=\frac{2\pi i}{6}\pi^3=\frac{\pi^4 i}{3}$

5b/ let $f(z)=e^{z^2}$ note: $f'(z)=2ze^{z^2}$ & $f''(z)=2e^{z^2}(1+2z^2)$

$f^{(n-1)}(a)=\frac{(n-1)!}{2\pi i}\int_C \frac{f(z)}{(z-a)^n}dz$ as 1 lies in Z interior of C,

Z integral on C is $\frac{2\pi i}{2!}f''(1)=\pi i(2e(1+2))=6e\pi i$

but the integrand is holomorphic on a domain containing the image of γ & its interior ∴ by Cauchy's thm $\int_\gamma \frac{e^{z^2}}{(z-1)^3}=0$

5c/ Can express $f(z)=\frac{2(z^3-z^2-z-1)}{(z^2+1)(z+1)^2}=$

$\frac{f_1(z)}{z-i}+\frac{f_2(z)}{z+i}+\frac{f_3(z)}{z+1}+\frac{f_4(z)}{(z+1)^2}$ say where $f_1(z)=i$, $f_2(z)=-i$,

$f_3(z)=2$ and $f_4(z)=-2$ Can compute the integrals of these four terms seperately via the Cauchy integral formula: $\int_C f(z)dz=2\pi i(f_1(i)+f_2(i)+f_3(-1)+f_4'(-1))=$

$2\pi i(i-i+2)=4\pi i$

6/ know $e^{\pi z}$ is analytic because have shown e^z and πz are analytic let $\gamma(t)=i-ti/2$ $0\le t\le 1$ then $k+\gamma$ is a closed contour Cauchy's thm gives $\int_{k+\gamma}e^{\pi z}=0$

or $\int_k e^{\pi z}=-\int_\gamma e^{\pi z}$ now $e^{\pi\gamma(t)}=e^{i\pi(1-t/2)}$ ∴ $\gamma'(t)=-i/2$ ∴

$\int_\gamma e^{\pi z}=\int_0^1 e^{i\pi(1-t/2)}(-i/2)=-\frac{i}{2}\int_0^1\left(\cos\left(\pi\left(1-\frac{t}{2}\right)\right)+i\sin\left(\pi\left(1-\frac{t}{2}\right)\right)\right)dt=$

Sheet 3

$$= \frac{-i}{2}\left(\frac{\sin(\pi(1-\frac{t}{2}))}{-\pi/2} - i\frac{\cos(\pi(1-\frac{t}{2}))}{-\pi/2}\right)\Big|_0^1 dt =$$

$$\frac{i}{\pi}\left[\sin(\pi(1-\tfrac{t}{2})) - i\cos(\pi(1-\tfrac{t}{2}))\right]\Big|_0^1 dt =$$

$$\frac{i}{\pi}\left[\sin(\tfrac{\pi}{2}) - i\cos(\tfrac{\pi}{2}) - (\sin\pi - i\cos\pi)\right] = \frac{i}{\pi}(1-i) = \frac{1+i}{\pi} \therefore \int_{\gamma_2} e^{\pi z} = -\frac{1+i}{\pi}$$

7/ by Cauchy's integral formula we have:

$f^{(n)}(z_0) = \frac{n!}{2\pi i}\int_{\gamma_R} \frac{f(z)}{(z-z_0)^{n+1}}dz$ where γ_R is a circular contour taken anticlockwise with centre at z_0 and radius R $\therefore$ by estimation lemma:

$$\left|\int_{\gamma_R}\frac{f(z)}{(z-z_0)^{n+1}}dz\right| \le M\times L \therefore \left|\frac{n!}{2\pi i}\int_{\gamma_R}\frac{f(z)}{(z-z_0)^{n+1}}dz\right| = |f^{(n)}(z_0)|$$

$= \left|\frac{n!}{2\pi i}\right|\left|\int_{\gamma_R}\frac{f(z)}{(z-z_0)^{n+1}}dz\right| \le \frac{n!}{2\pi}M\times L$ where $L = 2\pi R$ is the length of the contour (the circumference) and M is the supremum of the integrand on the contour. note: on the contour:

$$\left|\frac{f(z)}{(z-z_0)^{n+1}}\right| = \frac{|f(z)|}{|(z-z_0)^{n+1}|} = \frac{|f(z)|}{R^{n+1}} \therefore$$

$M = \frac{1}{R^{n+1}} \times \sup\{|f(z)| : |z-z_0| = R\} \therefore$

$$|f^{(n)}(z_0)| \le \frac{n!}{2\pi}2\pi R\frac{1}{R^{n+1}}\times\left(\sup\{|f(z)| : |z-z_0| = R\}\right) =$$

$$\frac{n!}{R}\max\{|f(z)| : |z-z_0| = R\}$$

Sheet 4

1a ratio test: Z series $\sum_{n=0}^{\infty} a_n$ converges absolutely if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|<1$ & diverges if $\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|>1$

Set $a_n = n!\,z^n$ $\therefore$ $\left|\frac{a_{n+1}}{a_n}\right| = (n+1)|z|$ tends to infinity as $n\to\infty$, unless $z=0$ then tends to 0 as $n\to\infty$ $\therefore$ Z radius of convergence is 0

1b note: $\forall n \in$, $\left|\frac{\sin(n)}{n^n} z^n\right| \le \frac{|z|^n}{n!}$ Z series $\sum \frac{z^n}{n!}$ converges absolutely $\forall z \in \mathbb{C}$ by the comparison test the original set has infinite radius of convergence

1c let a_n be the summand $\therefore$ $\frac{a_{n+1}}{a_n} = \frac{(3n+3)(3n+2)(3n+1)}{(n+1)(2n+2)(2n+1)} z = \frac{(3+\frac{3}{n})(3+\frac{2}{n})(3+\frac{1}{n})}{(1+\frac{1}{n})(2+\frac{2}{n})(2+\frac{1}{n})} z$ $\therefore$ $\left|\frac{a_{n+1}}{a_n}\right| \Rightarrow \frac{27}{4}|z|$ as $n\to\infty$ $\therefore$

the series converges absolutely if $|z|<\frac{4}{27}$ and diverges if $|z|>\frac{4}{27}$ it follows that Z radius of convergence is $\frac{4}{27}$

1d write $w=\frac{z^2}{9}$ $\therefore$ sum is $\sum w^n$ which converges absolutely if $|w|<1$ & diverges if $|w|>1$ but $|w| = \left|\frac{z^2}{9}\right| = \left(\frac{|z|}{3}\right)^2$ $\therefore$ the original series converges absolutely if $|z|<3$ & diverges if $|z|>3$ $\therefore$ the radius of convergence is 3

2 this power series centred at -2 let a_n be the summand $\therefore$ $\frac{a_{n+1}}{a_n} = \frac{n}{3(n+1)}(z+2) \to \frac{z+2}{3}$ as $n\to\infty$ $\therefore$ the radius of convergence is 3

differentiating term-by-term: $f'(z) = \sum_{n=1}^{\infty} \frac{(z+2)^{n-1}}{3^n} = \frac{1}{3}\sum_{n=0}^{\infty}\frac{(z+2)^n}{3^n}$

$\therefore$ $f'(-1) = \frac{1}{3}\sum_{n=0}^{\infty}\frac{1}{3^n} = \frac{1}{3}\left(\frac{1}{1-\frac{1}{3}}\right) = \frac{1}{2}$ &

$f'(0) = \frac{1}{3}\sum_{n=0}^{\infty}\left(\frac{2}{3}\right)^n = \frac{1}{3}\left(\frac{1}{1-\frac{2}{3}}\right) = 1$

Sheet 4

3a write $w = 3z^2$ ∴ using taylor series for cosh func ∴

$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{w^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{9^n z^{6n}}{(2n)!}$ Z series converges

$\forall w$ & ∴ $\forall z$ ∴ Z radius of convergence is infinity

alternatively: note that Z original func is diffable everywhere ∴ by taylors thm & Z uniqueness thm for power series: Z radius of conver is infinity

3b put $w = -3pz$ ∴ $f(z) = \frac{1}{1-w} = \sum_{n=0}^{\infty} w^n = \sum_{n=0}^{\infty} 3^n p^n z^n$

∴ Z series converges absolutely when $|w|<1$ that is: when $|z| < \frac{1}{3}$ it diverges when $|w|>1$ ∴ Z radius of conver is $\frac{1}{3}$. (alternatively use taylors thm as before)

3c by Z binomial thm: $(1+z)^{-3} =$

$1 - 3z + \frac{(-3)(-4)}{2!} z^2 + \dots + \frac{(-3)(-4)\cdots(-2-n)}{n!} z^n + \dots =$

$\sum_{n=0}^{\infty} (-1)^n \frac{(n+2)!}{2n!} z^n = \sum_{n=0}^{\infty} (-1)^n (n+2)(n+1) z^n$ Z func is holomorphic in Z disc $D(0,1)$ and has a pole at $z=-1$ ∴ by Z taylors thm & Z uniqueness thm of power series, Z power series has radius of converg 1 (alternatively use Z ratio test)

4a begin with partial fracs $\frac{1}{(z-1)^2(z+3)} = \frac{A}{z-1} + \frac{B}{(z-1)^2} + \frac{C}{z+3}$

find $A = -\frac{1}{16}$, $B = \frac{1}{4}$, $C = \frac{1}{16}$ by trying $z=1, z=3, z=0$

∴ as $|z|<1$ ∴ $\frac{1}{z-1} = -\frac{1}{1-z} = -\sum_{n=0}^{\infty} z^n$ ∴ can diff Z series term by term or use binomial thm:

$-\frac{1}{(z-1)^2} = -\sum_{n=1}^{\infty} n z^{n-1} = -\sum_{n=0}^{\infty} (n+1) z^n$ also

$\frac{1}{z+3} = \frac{1}{3(1-(-\frac{z}{3}))} = \frac{1}{3}\sum_{n=0}^{\infty} \frac{(-1)^n}{3^n} z^n$ which is valid as $|\frac{z}{3}| < \frac{1}{3} \leq 1$ $\therefore$ the Laurent Series is:

$\sum_{n=0}^{\infty} \left(\frac{1}{16} + \frac{1}{4}(n+1) + \frac{1}{16}\frac{(-1)^n}{3^{n+1}}\right) z^n = \frac{1}{16}\sum_{n=0}^{\infty}\left(4n+5+\frac{(-1)^n}{3^{n+1}}\right) z^n$ note: in this case, Z Laurent Series is a power series

\4b/ $\frac{1}{(z-1)^2(z+3)} = -\frac{1}{16}\frac{1}{z-1} + \frac{1}{4}\frac{1}{(z-1)^2} + \frac{1}{16}\frac{1}{z+3}$

~~of~~ $1<|z|<3$ $\therefore$ still have $\frac{1}{z+3} = \sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}} z^n$

but now have $|z|>1$ $\therefore$ write $w = \frac{1}{z}$ whence $|w|<1$ $\therefore$

$\frac{1}{z-1} = \frac{w}{1-w} = \frac{1}{\frac{1}{w}-1} = \frac{w}{1-w} = w\frac{1}{1-w} = w\sum_{n=0}^{\infty} w^n = \sum_{n=0}^{\infty} w^{n+1} = \sum_{n=0}^{\infty}\left(\frac{1}{z}\right)^{n+1} =$

$= \sum_{n=1}^{\infty}\left(\frac{1}{z}\right)^{n} = \sum_{n=1}^{\infty} z^{-(n)} = \sum_{n=1}^{\infty} z^{-n} = \sum_{n=-\infty}^{-1} z^n$ to justify this take $z \in D(0,1)$ $\therefore \exists r$ st $|z|<r<1$ by Z M-test, Z Series is unifly convergent on $D(0,r)$ so can be ~~diffsed~~ differed term by term of that disc likewise:

$\frac{1}{(z-1)^2} = \frac{w^2}{(1-w)^2} = w^2\sum_{n=0}^{\infty}(n+1)w^n = \sum_{n=2}^{\infty}(n-1)w^n = \sum_{n=2}^{\infty}(n-1)z^{-n} = -\sum_{n=-\infty}^{-2}(n+1)z^n$

Z Laurent series is: $-\frac{1}{16}\sum_{n=-\infty}^{-1} z^n - \frac{1}{4}\sum_{n=-\infty}^{-2}(n+1)z^n + \frac{1}{16}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}} z^n$

$= -\frac{1}{16}\sum_{n=-\infty}^{-1}(4n+5)z^n + \frac{1}{16}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}} z^n$ its equal valid to use pos vals for Z Summation var & to write Z Series as

$-\frac{1}{16}\sum_{n=1}^{\infty}\frac{(-4n+5)}{z^n} + \frac{1}{16}\sum_{n=0}^{\infty}\frac{(-1)^n}{3^{n+1}} z^n$ note: this Laurent Series contains both neg & pos powers of Z

Sheet 4

4c $|z|>3$ $\quad \dfrac{1}{(z-1)^2(z+3)} = -\dfrac{1}{16}\dfrac{1}{z-1} + \dfrac{1}{4}\dfrac{1}{(z-1)^2} + \dfrac{1}{16}\dfrac{1}{z+3}$

$\therefore$ write $y = \frac{-3}{z}$ $\therefore$ $|y|<1$ $\therefore$

$$\frac{1}{z+3} = \frac{1}{\frac{-3}{y}+3} = \frac{y}{-3+3y} = -\frac{y}{3-3y} = -\frac{y}{3(1-y)} = -\frac{y}{3}\frac{1}{1-y} = -\frac{y}{3}\sum_{n=0}^{\infty} y^n$$

$$= -\frac{1}{3}\sum_{n=0}^{\infty} y^{n+1} = -\frac{1}{3}\sum_{n=1}^{\infty} y^n = -\frac{1}{3}\sum_{n=1}^{\infty}\frac{(-1)^n 3^n}{z^n} = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}3^{n-1}}{z^n} =$$

$$\sum_{n=-\infty}^{-1}(-1)^{1-n}3^{-1-n}\frac{1}{z^{-n}} = \sum_{n=-\infty}^{-1}(-1)^{1-n}3^{-1-n}z^n$$

~~whilst~~ $|z|>3$ $\therefore \frac{1}{z-1}$ doesn't conver $\forall |z|>1$ $\therefore$

let $w = \frac{1}{z}$ $\therefore$ $|w| < \frac{1}{3} < 1$ $\therefore$ $\frac{1}{z-1} = \frac{w}{1-w} = w\sum_{n=0}^{\infty} w^n =$

$$= \sum_{n=0}^{\infty} w^{n+1} = \sum_{n=1}^{\infty} w^n = \sum_{n=-\infty}^{-1} w^{-n} = \sum_{n=-\infty}^{-1}\left(\frac{1}{z}\right)^{-n} = \sum_{n=-\infty}^{-1} z^n$$

$$\frac{1}{(z-1)^2} = \left(\frac{1}{z-1}\right)^2 = \left(\frac{w}{1-w}\right)^2 = \frac{w^2}{(1-w)^2} = w^2\frac{1}{(1-w)^2} = w^2\sum_{n=0}^{\infty}(n+1)w^n =$$

$$\sum_{n=0}^{\infty}(n+1)w^{n+2} = \sum_{n=2}^{\infty}(n-1)w^n = \sum_{n=1}^{\infty}(n-1)w^n = \sum_{n=-\infty}^{-1}(-n-1)w^{-n} = -\sum_{n=-\infty}^{-1}(n+1)w^{-n}$$

$$= -\sum_{n=-\infty}^{-1}(n+1)\left(\frac{1}{z}\right)^{-n} = -\sum_{n=-\infty}^{-1}(n+1)z^n$$ $\therefore$ the laurent series is:

$$-\frac{1}{16}\sum_{n=-\infty}^{-1} z^n - \frac{1}{4}\sum_{n=-\infty}^{-1}(n+1)z^n + \frac{1}{16}\sum_{n=-\infty}^{-1}(-1)^{1-n}3^{-1-n}z^n =$$

$$-\frac{1}{16}\sum_{n=-\infty}^{-1}\left(4n+5+(-1)^{n}3^{-1-n}\right)z^n = -\frac{1}{16}\sum_{n=1}^{\infty}\left(-4n+5+(-1)^n 3^{n-1}\right)z^{-n}$$

in this case the principle part of the laurent series is the laurent series

4d sub $w = z^{-2}$ into power series of $\cos w$ $\therefore$

- $\cos(z^{-2}) = \cos w = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}w^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}(z^{-2})^{2n} = \sum_{n=0}^{\infty}\frac{(-1)^n}{(2n)!}z^{-4n}$

\4e/ product of two power series

$$\frac{e^{-z}}{1-z} = e^{-z} \cdot \frac{1}{1-z} \qquad \frac{1}{1-z} = \sum_{s=0}^{\infty} z^s \qquad e^{-z} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} z^n \therefore$$

$$e^{-z} \cdot \frac{1}{1-z} = \sum_{r=0}^{\infty} \frac{(-1)^r}{r!} z^n \sum_{s=0}^{\infty} z^s = \sum_{n=0}^{\infty} a_n z^n$$ where

$$a_n = \sum_{r=0}^{n} \frac{(-1)^r}{r!} \quad \therefore \quad \frac{e^{-z}}{1-z} = \sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} \left(\sum_{r=0}^{\infty} \frac{(-1)^r}{r!} \right) z^n$$

\5a/ $\sin z$ has an simple zero at $z=0$ the func $1-e^{-z}$ has an single zero at 0 so f has an removable pole at 0

\5b/ $\sin(\frac{1}{z})$ has an essential singularity at 0 (consider the laurent series) let $w = \frac{1}{z} = z^{-1} \therefore \sin(\frac{1}{z}) = \sin w =$

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} w^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} (z^{-1})^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} z^{-2n-1} =$$

$$\sum_{n=-\infty}^{0} \frac{(-1)^{-n}}{(-2n+1)!} z^{2n-1}$$ ~~laurent series has~~ the principle part of the laurent series is the laurent series

\5c/ z^3 has an triple zero at $z=0$ by considering the power series we see the denominator has an zero of order 10 at $z=0$ so $10-3=7$ so f has an pole of order 7 at 0 $\left\{ \frac{d}{dz}(\cos(z^5)-1)^2 = -5z^4 \sin(z^5) = g'(z) \therefore \right.$

$g''(z) = -20z^3 \sin(z^5) - 25z^8 \cos(z^5)$

$g'''(z) = 5(z^2(25z^{10}-12)\sin(z^5) - 60z^7\cos(z^5))$

$g^{(4)}(z) = 5(24(25z^{10}-1)z\sin(z^5) + 5(25z^{10}-96)z^6\cos(z^5))$

$g^{(5)}(z) = 5000z^5(5z^{10}-3)\cos(z^5) - 5(625z^{20}-9000z^{10}+24)\sin(z^5)$

$\therefore g^{(5)}(0) = 0 - 0 = 0 \quad g^{(6)}(0) = 0, \quad g^{(7)}(0) = 0 \quad g^8(0) = 0$

$\frac{d}{dw}(\cos(w)-1) = -\sin w \qquad \frac{d}{dz}(z^5) = 5z^4 \to 20z^3 \to 60z^2 \to 120z \to 120$

$\therefore$ 2nd order and 5th order $\therefore (2)(5) = 10$ th order

Sheet 4

5d $1-\cosh z$ has an double zero at 0

~~$1-\cosh z$~~ $\frac{d}{dz}(1-\cosh z) = -\sinh z$ $\frac{d}{dx}(-\sinh z) = -\cosh z$

$z-\sinh z$ has an triple zero at 0 $\frac{d}{dx}(z-\sinh z) = 1-\cosh z$

$\to -\sinh z \to -\cosh z$ $\therefore 3-2=1$ so f has an simple pole at 0

6 applying the definition show g is differentiable at a indeed: $\lim_{z\to a}\frac{g(z)-g(a)}{z-a} = \lim_{z\to a}\frac{(z-a)^2 f(z)-0}{z-a} = \lim_{z\to a}(z-a)f(z) = 0$

by hypothesis the func g is differentiable (with deriv 0) at a by the product rule it is differentiable at every point $z\neq a$ where f is differentiable $\therefore$ g is holomorphic in an neighborhood of a by taylors thm, we know for z in an disc around a $g(z)=\sum_{n=0}^{\infty} b_n(z-a)^n$ where $b_0 = g(0) = 0$ & $b_1 = g'(0) = 0$ $\therefore$

$g(z) = \sum_{n=2}^{\infty} b_n(z-a)^n = (z-a)^2\sum_{n=0}^{\infty} b_{n+2}(z-a)^n$ but in an punctured disc around a, have $g(z)=(z-a)^2 f(z)$ where on this disc $f(z) = \sum_{n=0}^{\infty} b_{n+2}(z-a)^n$ so f has an removeable singularity at a

7a the hypothesis of the question assure us both that f has an laurent expansion in $\mathbb{C}\setminus\{0\}$ & that the residue of f at 0 is 0

$f(z)=\frac{1}{z^2}$ does not not have an removable singularity

$\therefore$ false

7b true: proof: the residue of f at 0 is 0, so the coeffs of $\frac{1}{z}$ in the laurent expansion is 0

\7c/ false: Counterexample take $f(z)=0$ $\therefore$ f does not have an multiple pole at 0

\7d/ false; Counterexample: take $f(z)=e^{\frac{1}{z}}-1-\frac{1}{z}$ $\therefore$ f has an essential singularity at 0

\\~~week~~ Sheet 5/

\a/ let $f(z)$ be the integrand $\therefore$ $f(z)=\frac{1}{(z-(2+3i))(z-(2-3i))}$ has simple poles at $z=2+3i$ & $z=2-3i$ only the pole $z=2+3i$ lies in the semicircular contour (for large enough R) the residue at this pole is $1/(6i)$ the integral on γ is $\therefore$ $2\pi i/(6i)=\pi/3$

for $z \in \gamma_2^*$ have: $|f(z)| \leq \frac{1}{|z|^2-|4z-13|} \leq \frac{1}{|z|^2-4|z|-|13|} = \frac{1}{R^2-4R-13}$ by the M-L inequality have: $\left|\int_{\gamma_2} f(z)\,dz\right| \leq \frac{\pi R}{R^2-4R-13} = O(1/R) \to 0$ as $R \to \infty$ more simply its acceptable to write immediately that $|f(z)|=O(1/R^2)$ as $R\to\infty$ & then note that the integral is bounded by $\pi R \times O(1/R^2)=O(1/R)\to 0$, as $R\to\infty$

$\therefore$ follows: $\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{\gamma_1} f(z)\,dz = \lim_{R\to\infty}\int_{\gamma} f(z)\,dz = \pi/3$

U is: $\boxed{a \ b \ c \ d \ e \ f \ g}$

$A = \{a, b, c, d, e\}$

$B = \{a, c, e, g\}$

$C = \{b, e, f, g\}$ $\therefore$

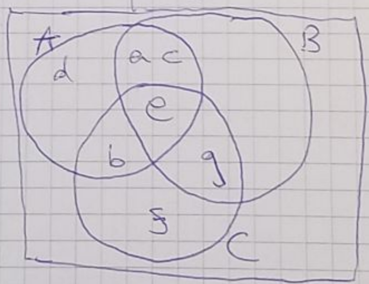
~~A∩B~~ $A \cap B = \{a, c, e\}$

$A \cap C = \{b, e\}$ ~~A~~

$B \cap C = \{e, g\}$ $\therefore$

$A \cap B \cap C = \{e\}$ $\therefore$

U is:



$A = \{a, b, c, d, e\}$, $U = \{a, b, c, d, e, f, g\}$ $\therefore$

$A' = \{f, g\}$ $\therefore$

$A \cap A' = \{a, b, c, d, e\} \cap \{f, g\} = \{\ \} = \varnothing$ $\therefore$

$(A \cap A')' = \varnothing' = \{a, b, c, d, e, f, g\} = U$ ☒

QED ☒

\Sheet 5/

\6/ Compute residues then apply either: semi-circle method or Jordan's inequality or Z slice-of-pie method :) etc. Take R to be a sufficiently large pos real, γ_1 to be contour $[-R,R]$ & $\gamma_2(t) = Re^{it}$ for $0 \le t \le \pi$. Z contour γ will be Z join of γ_1 & γ_2

$\int_{-\infty}^{\infty} \frac{1}{(x^2+4)^2(x^2+9)}\,dx$ $\therefore$ is $f(z) = \frac{1}{(z^2+4)^2(z^2+9)}$ note: there are simple poles at $z = \pm 3i$ & double poles at $z = \pm 2i$ only consider Z poles at $2i$ & $3i$. For Z simple pole at $3i$, Z residue is $\frac{g(3i)}{h'(3i)} = \frac{1}{150i}$ where $g(z) = 1/(z^2+4)^2$ &

$h(z) = z^2+9$ $\therefore \{ f(z) = g(z) \cdot \frac{1}{h(z)}$ $\therefore h'(z) = 2z$ $\therefore$

$g(z) \cdot \frac{1}{h'(z)} = \frac{1}{(z^2+4)^2 \, 2z} \therefore \frac{1}{((3i)^2+4)^2 \, 2 \cdot 3i} = \frac{1}{(-9-4)^2 \cdot 2 \cdot 3i} = \frac{1}{150i} \}$

at $z = 2i$ may write $f(z) = k(z)/(z-2i)^2$ where $k(z) = \frac{1}{(z+2i)^2(z^2+9)}$ is holomorphic & non zero near $2i$. Z residue at $2i$ is $\therefore$

$k'(2i) = \left. \frac{-2(z+2i)(z^2+9) + 2z(z+2i)^2}{[(z+2i)^2(z^2+9)]^2} \right|_{z=2i} = \frac{3}{800} i$ or alternative approach to compute residue is to find Z Laurent series for $f(z)$ about $2i$ by writing $w = z - 2i$ & $f(z) = f(w+2i) =$

$\frac{1}{w^2} \times \frac{1}{(w+2i+2i)^2((w+2i)^2+9)} = \frac{1}{w^2} \times \frac{1}{(w^2+8wi-16)(w^2+4wi+5)} =$

$\frac{1}{w^2} \times \frac{1}{-80-24iw+O(w^2)} = -\frac{1}{80w^2} \times \frac{1}{1-(-\frac{24}{80}iw+O(w^2))} =$

$-\frac{1}{80w^2} \times \left(1 - \frac{24}{80}iw + O(w^2)\right)$. Z coeffs of $1/w$ in this expression is $24i/80^2 = 3i/800$ arguing as before, note that on Z semi-circle Z integrand is $O(1/R^6)$, so Z integral is $O(1/R^5) \to 0$ as $R \to \infty$ it follows that Z integral on Z real line is $2\pi i(\mathrm{Res}(f,2i) + \mathrm{Res}(f,3i)) = 2\pi\left(-\frac{3}{800} + \frac{1}{150}\right) =$:)

$\frac{7\pi}{1200}$

Sheet 5/

c/ $\int_0^\infty \frac{x^2}{(x^2+1)^3}\,dx$ ∴ $f(z)=\frac{z^2}{(z^2+1)^3}$ as Z integrand is an even func, Z desired integral is $I=\frac{1}{2}\int_{-\infty}^{\infty}\frac{x^2}{(x^2+1)^3}\,dx$

Z semicircle method applies to this integral. Z only pole within Z semicircle is a triple pole at $z=i$ there are write Z integrand as $f(z)=k(z)/(z-i)^3$ where $k(z)=z^2/(z+i)^3$ then Z residue at Z double pole is $\mathrm{Res}(f,i)=k''(i)/2!$ something of a chore to compute: $k'(z)=\frac{2z(z+i)^3-3(z+i)^2z^2}{(z+i)^6}=$

$=\frac{-z^2+2iz}{(z+i)^4}$ ∴ $k''(z)=\frac{(-2z+2i)(z+i)^4-4(z+i)^3(-z^2+2iz)}{(z+i)^8}=$

$\frac{(-2z+2i)(z+i)-4(-z^2+2iz)}{(z+i)^5}$ from this we have Z residue

$\mathrm{Res}(f,i)=\frac{k''(i)}{2!}=-\frac{i}{8}\times\frac{1}{2!}=-\frac{i}{16}$ Z integral on Z semicircle is bounded by $\pi R\times O(1/R^2)=O(1/R)\to 0$ as $R\to\infty$ Z integral on Z real line is ∴ $2\pi i\,\mathrm{Res}(f,i)=\pi/8$ Z desired integral is half this quantity or $\pi/16$

d/ $\int_0^\infty \frac{x\sin x}{(x^2+1)^2}\,dx$ ∴ ~~$f(z)=z\sin$~~ Z Residue than to Z even func $\frac{z\sin z}{(z^2+1)^2}$ approach will fail for Z integral on Z semicircle does not tend to 0 as $R\to\infty$

instead write ~~f~~ ~~ze^{iz}~~ $f(z)=\frac{ze^{iz}}{(z^2+1)^2}$ noting Z integrand in Z original problem is Z imaginary part of $f(z)$, if z is real Z only singularity in or on Z contour is a double pole at $z=i$ write $f(z)=k(z)/(z-i)^2$ where $k(z)=\frac{ze^{iz}}{(z+i)^2}$ Z residue at $z=i$ is then $k'(i)$ ∴ $k'(i)=\left.\frac{e^{iz}(1+iz)(z+i)^2-2(z+i)ze^{iz}}{(z+i)^4}\right|_{z=i}$

$=\left.\frac{ie^{iz}(z^2+2iz+1)}{(z+i)^3}\right|_{z=i}=\frac{e^{-1}(-1-2+1)}{8}=\frac{e^{-1}}{4}$ as an alternative could compute residue by finding Z Laurent expansion of f about i Z integrand on Z semicircle is $O(1/R^3)$

(using $|e^{iz}| \le 1$ on Z semicircle) so Z integral on Z semicircle is $O(1/R^2) \to 0$, as $R \to \infty$ ∴ $\int_0^\infty \frac{x\sin x}{(x^2+1)^2}dx =$
$\frac{1}{2} \Im \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{ze^{iz}}{(z^2+1)^2}dz\right) = \frac{1}{2}\frac{\pi e^{-1}}{2} = \frac{\pi e^{-1}}{4}$

\e/ $\int_0^\infty \frac{\sin x}{x(x^2+1)}dx$ apply Z semicircle ~~...~~ method to Z func $f(z) = \frac{e^{iz}}{z(z^2+1)}$ Z imaginary part of which evaluates to our desired integrand, when Z is real now f has simple poles at $z=0$, $z=\pm i$ as Z pole at $z=0$ passes through Z contour of integration, we must use a semicircular indentation of radius ε around Z pole, & let $\varepsilon \to 0$ Said that Z integral $\int_{\gamma_4} f(z)dz$ on this indentation tends to $-\pi i$ as $\varepsilon \to 0$

Z residue at $z=i$ is $Res(f,i) = \frac{g(i)}{h'(i)} = \frac{e^{-1}}{i\times(2i)} = -\frac{e^{-1}}{2}$ where $g(z) = \frac{e^{iz}}{z}$ & $h(z) = z^2+1$ ∴ $\left\{f(z) = g(z)/h(z) = \frac{e^{iz}}{z}\times\frac{1}{z^2+1}\right\}$

Z integrand is $O(1/R^3)$ on Z large semicircle, so Z integral on this semicircle is $O(1/R^2) \to 0$ as $R\to\infty$ don't need to apply Jordan's inequality in this case ∴ $2\pi i Res(f,i) =$
$\int_{\gamma_3+\gamma_1} f(x)dx + \int_{\gamma_4} f(z)dz + \int_{\gamma_4} f(z)dz \to \int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)}dx - \pi i$ ∴

$\int_{-\infty}^{\infty} \frac{e^{ix}}{x(x^2+1)}dx = \pi i - \pi i e^{-1} = \pi i(1-e^{-1})$ ∴ desired integral:

$\int_0^\infty \frac{\sin x}{x(x^2+1)}dx = \frac{1}{2}\Im\left(\int_{-\infty}^{\infty}\frac{e^{ix}}{x(x^2+1)}dx\right) = \frac{\pi}{2}(1-e^{-1})$

\f/ $\int_0^\infty \frac{\cos x}{(x^2+1)^2}dx = I$ desire $f(z) = e^{iz}(z^2+1)^{-2}$ & perform Z semicircle method Z only singularity of f within Z contour occurs at $z=i$ this is a double pole write $f(z) = k(z)/(z-i)^2$ where $k(z) = \frac{e^{iz}}{(z+i)^2}$ then Z residue at Z double pole i is $k'(i) = \left.\frac{ie^{iz}(z+i)^2 - 2(z+i)e^{iz}}{(z+i)^4}\right|_{z=i} = \left.\frac{e^{iz}(iz-3)}{(z+i)^3}\right|_{z=i}$

\sheet 5/ $= -\frac{4e^{-1}}{-8i} = -\frac{ie^{-1}}{2}$ Z integrand on Z semicircle is $O(1/R^4)$ so Z corres integral is $O(1/R^3) \to 0$ as $R \to \infty$

$\therefore I = \frac{1}{2} \Re\left(\int_{-\infty}^{\infty} \frac{e^{iz}}{(z^2+1)^2} dz\right) = \frac{1}{2} \mathrm{Re}\left(\pi e^{-1}\right) = \frac{\pi e^{-1}}{2}$

\g/ $\int_0^\infty \frac{x \sin x}{x^2+1} dx = I$ Z integrand is even set

$f(z) = ze^{iz}/(z^2+1)$ but a naive appli of Z semicircle method will fail for Z integrand is $O(1/R)$ on Z semicircle Z M-L inequality would give a bound of Z integral of $\pi R \times O(1/R) = O(1) \not\to 0$ as $R \to \infty$ this situation Jordan's inequality applies param Z semicircle by $\gamma_2(t) = Re^{it}$ for $0 \le t \le \pi$ now Z integral I_2 on γ_2 has Z bound $|I_2| = \cancel{\left|\int_0^\pi\right.}$ $\left|\int_0^\pi \frac{Re^{it} e^{iRe^{it}}}{R^2 e^{2it}+1} iRe^{it} dt\right| \le \int_0^\pi \frac{R^2}{R^2-1} e^{-R\sin t} dt$ by symmetry of Z sin func $\therefore \int_0^\pi e^{-R\sin t} dt = 2\int_0^{\pi/2} e^{-R\sin t} dt \le \cancel{\int_0^{\pi/2}}$

$2\int_0^{\pi/2} e^{-2Rt/\pi} dt = \frac{\pi}{R}(1-e^{-R})$ where we use Jordan's inequality $\sin t \ge 2t/\pi$ for $0 \le t \le \pi/2$ $\therefore |I_2| = \frac{\pi R}{R^2-1}(1-e^{-R}) \to 0$ as $R \to \infty$

Z only singularity within Z contour is Z simple pole at $z = i$ at this pole, Z residue is $\frac{g(i)}{h'(i)} = \frac{ie^{-1}}{2i} = \frac{e^{-1}}{2}$ where $g(z) = ze^{iz}$ & $h(z) = z^2+1$ $\therefore \{g(z)/h(z) = ze^{iz}/(z^2+1) = f(z)\}$

$\therefore I = \frac{1}{2} \mathrm{Im}\left(\int_{-\infty}^{\infty} f(x) dx\right) = \frac{1}{2} \mathrm{Im}(\pi i e^{-1}) = \frac{\pi e^{-1}}{2}$

\h/ $I = \int_0^\infty \frac{1}{x^{1000}+1} dx$ use Z slice-of-pie approach

note: Z func $f(z) = (z^{1000}+1)^{-1}$ has a single pole at $z = \alpha$ where $\alpha = e^{i\pi/1000}$ this is Z only pole in Z contour γ composed of Z paths γ_1, γ_2 & γ_3 where $\gamma_1 = [0, R]$, $\gamma_2(t) = Re^{it}$ for $0 \le t \le \pi/500$ & γ_3 is Z reversal of $[0, \alpha^2 R]$ which param as $\gamma_3^-(t) = \alpha^2 t$ for $0 \le t \le R$ $\therefore$

$I_3 = -\int_0^R \frac{\alpha^2}{(\alpha^2 t)^{1000}+1}\,dt = -\alpha^2 I_1$ where I_1 is Z integral on γ_1

Z integrand on Z arc γ_2 is $O(1/R^{1000})$ Z length of Z arc is $\pi R/500$ by Z M-L inequality. Z integral I_2 on γ_2 is $O(1/R^{999}) \to 0$ as $R \to \infty$ Z residue of f at α is simply $\mathrm{Res}(f, \alpha) = \frac{1}{1000\alpha^{999}}$ $\therefore \frac{2\pi i}{1000\alpha^{999}} = I_1 + I_2 + I_3 =$

$I_1(1-\alpha^2) + I_2 \to (1-\alpha^2)\int_0^\infty \frac{1}{x^{1000}+1}\,dx$ as $R \to \infty$ $\therefore$ $\cancel{I = \frac{2\pi i}{}}$

$I = \frac{2\pi i}{1000\alpha^{999}(1-\alpha^2)}$ now Z integral must be real (its Z integral of a real-valued func of a real variable), but Z above expression is not evidently real. by rearranging can make it evident: $I = \frac{\pi}{1000} \times \frac{2i}{\alpha^{1000}(\alpha^{-1}-\alpha)} =$

$\frac{\pi}{1000} \times \frac{2i}{\alpha - \alpha^{-1}} = \frac{\pi}{1000}\csc\left(\frac{\pi}{1000}\right)$

\i/ $I = \int_0^\infty \frac{x^{666}}{x^{1000}+1}\,dx$ use Z same slice-of-pie contour with minor modifications can show as before $I_2 \to 0$ as $R \to \infty$ Z integral on γ_3 is $I_3 = -\int_0^R \frac{(\alpha^2 t)^{666}}{(\alpha^2 t)^{1000}+1}\alpha^2\,dt = -\alpha^{2\times 667} I_1$

Z residue at Z simple pole $z = \alpha$ is $\frac{\alpha^{666}}{1000\alpha^{999}}$ by Z residue thm: $2\pi i \frac{\alpha^{666}}{1000\alpha^{999}} = I_1 + I_2 + I_3 =$

$I_1(1-\alpha^{2\times 667}) + I_2 \to (1-\alpha^{2\times 667})\int_0^\infty \frac{x^{666}}{x^{1000}+1}\,dx$ $\therefore$

$I = \int_0^\infty \frac{x^{666}}{x^{1000}+1}\,dx = \cancel{\frac{\pi}{1000}}\ \cancel{2\pi i}\ \frac{\pi}{1000}\,\frac{2i\alpha^{666}}{(1-\alpha^{2\times 667})}$ $\therefore$

$I = \frac{\pi}{1000} \times \frac{2i\alpha^{666}}{\alpha^{999}\alpha^{667}(\alpha^{-667}-\alpha^{667})} = \frac{\pi}{1000} \times \frac{2i}{\alpha^{1000}(\alpha^{-667}-\alpha^{667})} =$

$\frac{\pi}{1000} \times \frac{2i}{\alpha^{667}-\alpha^{-667}} = \frac{\pi}{1000}\csc\left(\frac{667\pi}{1000}\right)$

\Sheet 5/

\a/ $\int_{-\infty}^{\infty} \frac{1}{x^2-4x+13}\,dx$ take R to be a sufficiently large pos real. γ_1 to be contour $[-R,R]$ & $\gamma(t)=Re^{it}$ for $0\le t\le\pi$

Z contour γ will be Z join of γ_1 & γ_2

$f(z)$ be Z integrand: $f(z)=\frac{1}{z^2-4z+13}=\frac{1}{(z-(2+3i))(z-(2-3i))}$

has simple poles at $z=2+3i$ & $z=2-3i$ only Z pole at $z=2+3i$ lies in Z semi-circular contour (for large enough R)

compute Z residue at Z pole $z=2+3i$ with $a=2+3i$ is:

$\mathrm{Res}(f,2+3i)=\lim_{z\to 2+3i}(z-(2+3i))f(z)=$

$\lim_{z\to 2+3i}(z-(2+3i))\frac{1}{(z-(2+3i))(z-(2-3i))}=\lim_{z\to 2+3i}\frac{1}{(z-(2-3i))}=$

$\frac{1}{2+3i-(2-3i)}=\frac{1}{2+3i-2+3i}=\frac{1}{6i}$

$\therefore$ $\mathrm{Res}(f,2+3i)=\frac{1}{6i}$ $\therefore$ by Cauchy's residue thm get:

$\int_\gamma f(z)\,dz=2\pi i\,\mathrm{Res}(f,2+3i)=2\pi i\frac{1}{6i}=\frac{\pi}{3}$

now look at $\gamma_2(t)=Re^{it}$ for $0\le t\le\pi$

for $z\in\gamma_2^*$ have: $|f(z)|=\frac{1}{|z^2-4z+13|}=\frac{1}{|z^2-(4z+13)|}$

$\{\ |z^2-(4z+13)|\ge|z^2|-|4z-13|=|z|^2-(|4z-(13)|)\ge|z|^2-(|4z|-|13|)=$
$|z|^2-4|z|-|13|$ $\therefore$

$|f(z)|=\frac{1}{|z^2-(4z+13)|}\le\frac{1}{|z^2|-4|z|-|13|}=\frac{1}{R^2-4R-13}$ $\{\ z\in\gamma_2^*$ $\therefore$ $z=Re^{it}$

$\therefore$ $|z|=|Re^{it}|=|R||e^{it}|=|R|(1)=|R|=R\ \}$

$\therefore$ by Z M-L inequality: $\left|\int_{\gamma_2}f(z)\,dz\right|\le\pi R\times\frac{1}{R^2-4R-13}=O\left(\frac{1}{R}\right)\to 0$

as $R\to\infty$ $\therefore$

$\int_{-\infty}^{\infty}f(x)\,dx=\lim_{R\to\infty}\left(\int_{\gamma_1}f(z)\,dz+\int_{\gamma_2}f(z)\,dz\right)=\lim_{R\to\infty}\int_\gamma f(z)\,dz=\frac{\pi}{3}$

\d/ $\int_0^\infty \frac{x\sin x}{(x^2+1)^2}\,dx$ $\therefore$ write $f(z)=\frac{ze^{iz}}{(z^2+1)^2}$ $\therefore$

$\int_0^\infty\frac{x\sin x}{(x^2+1)^2}\,dx=\mathrm{Im}\left(\int_\gamma f(z)\,dz\right)$ $\therefore$

Z only singularity in Z contour is at $z=i$ (a double pole)

$\therefore z^2+1=(z-i)(z+i)$ $\therefore$

$$\frac{ze^{iz}}{(z^2+1)^2}=\frac{ze^{iz}}{(z-i)^2(z+i)^2}=\frac{\left(\frac{ze^{iz}}{(z+i)^2}\right)}{(z-i)^2} \quad \therefore$$

$$f(z)=\frac{k(z)}{(z-i)^2} \quad \therefore k(z)=\frac{ze^{iz}}{(z+i)^2} \quad \therefore$$

Z residue at $z=i$ is then $k'(i)$ $\therefore$

$$k'(i)=\left((ze^{iz})'(z+i)^2-ze^{iz}((z+i)^2)'\right)/(z+i)^4=$$

$$\left(e^{iz}(1+iz)(z+i)^2-2(z+i)ze^{iz}\right)/(z+i)^4\Big|_{z=i}=$$

$$\frac{ie^{iz}(z^2+2iz+1)}{(z+i)^3}\Big|_{z=i}=-\frac{e^{-1}(-1-2+1)}{8}=\frac{e^{-1}}{4}$$

Z integrand on Z semicircle is $O\left(\frac{1}{R^3}\right)$ using $|e^{iz}|\le 1$ on Z semicircle) so Z integral on Z semicircle is $O\left(\frac{1}{R^2}\right)\to 0$ as $R\to\infty$ $\therefore$ $\int_\gamma f(z)\,dz\to 0$

{look at $\gamma_2(t)=Re^{it}$ for $0\le t\le\pi$ $\therefore$ for $z\in\gamma_2^*$ have:

$$|(z^2+1)^2|=|(z^2+1)|^2=(|z^2-(-1)|)^2\ge(|z^2|-|-1|)^2=(|z|^2-1)^2$$

$$\therefore |f(z)|=\left|\frac{ze^{iz}}{(z^2+1)^2}\right|\le\frac{|z||e^{iz}|}{|(z^2+1)^2|}\le\frac{|z||e^{iz}|}{(|z|^2-1)^2}=$$

$$\frac{|Re^{it}||e^{iRe^{it}}|}{(|Re^{it}|^2-1)^2}=\frac{|R||e^{it}||e^{iR(\cos t+i\sin t)}|}{(|R|^2|e^{it}|^2-1)^2}=$$

$$\frac{|R||e^{iR\cos t-R\sin t}|}{(|R|^2(1)^2-1)^2}=\frac{R|e^{-R\sin t}e^{iR\sin t}|}{(R^2-1)^2}=\frac{R|e^{-R\sin t}||e^{iR\sin t}|}{(R^2-1)^2}=$$

$$\frac{R}{R^4-2R^2+1}e^{-R\sin t}=O\left(\frac{1}{R^3}\right) \quad \therefore \text{ as } R\to\infty,\ |f(z)|\to 0 \quad \therefore$$

by M-L inequality $\left|\int_{\gamma_2}f(z)\,dz\right|\le\pi R\times\frac{R}{R^4-2R^2+1}e^{-R\sin t}=O\left(\frac{1}{R^2}\right)$

$\to 0$ as $R\to\infty$ $\therefore$ $\int_{\gamma_2}f(z)\,dz\to 0$ as $R\to\infty$ }

$$\therefore \int_{-\infty}^{\infty}f(z)\,dz=\lim_{R\to\infty}\left(\int_{\gamma_1}f(z)\,dz+\int_{\gamma_2}f(z)\,dz\right)=\lim_{R\to\infty}\int_{\gamma_1}f(z)\,dz=2\pi i\,\frac{e^{-1}}{4}=$$

double pole)

$|e^{iz}| \leq 1$ on Z

$(z) \to 0$ as

ve:

$|z|^2-1)^2$

$|e^{it}| =$

$\therefore$

$O\left(\frac{1}{R^2}\right)$

$\frac{e^{-1}}{4} =$

\Sheet 5/ $\pi i \frac{e^{-1}}{2}$ $\therefore$

$$\int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)^2} dx = \operatorname{Im}\left(\int_{-\infty}^{\infty} \frac{z e^{iz}}{(z^2+1)^2} dz = \operatorname{Im}\left(\cancel{2}\pi i \frac{e^{-1}}{2}\right) = \frac{\pi}{2} e^{-1} \therefore\right.$$

$$\int_0^{\infty} \frac{x \sin x}{(x^2+1)^2} = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{(x^2+1)^2} dx = \frac{1}{2} \frac{\pi}{2} e^{-1} = \pi \frac{e^{-1}}{4} \checkmark$$

$$\int_0^{\infty} \frac{x \sin x}{(x^2+1)^2} = \frac{1}{2} \operatorname{Im} \lim_{R\to\infty}\left(\int_{-R}^{R} \frac{z e^{iz}}{(z^2+1)^2} dz\right) = \frac{1}{2} \frac{\pi e^{-1}}{2} = \frac{\pi e^{-1}}{4} \checkmark$$

\h/ $\int_0^{\infty} \frac{1}{x^{1000}+1} dx$ use slice of pie Method

Z since $f(z) = \frac{1}{z^{1000}+1}$ has a simple at $z = \alpha$ where

$\alpha = e^{i\pi/1000}$ {since simple pole at $z^{1000} = -1 \therefore z^{1000} = e^{i\pi} \therefore$

$z = e^{i\pi/1000}$}

Consider Z contour γ Z join of Z paths $\gamma_1, \gamma_2, \gamma_3$ where

$\gamma_1 = [0, R]$ $\quad \gamma_2(t) = Re^{it}$ for $0 \leq t \leq \frac{\pi}{500}$ &

γ_3 is Z reversal of $[0, \alpha^2 R]$ $\{[0, Re^{i\pi/500}]\}$ which we

param as $\gamma_3^-(t) = \alpha^2 t = te^{i\pi/500}$ for $0 \leq t \leq R$

$\therefore \alpha = e^{i\pi/1000}$ is inside Z contour γ

Call $I = I_1 + I_2 + I_3$ where I is Z integral along γ &

I_i for $i = 1, 2, 3$ Z integrals along γ_i

Z integral on γ_3 is $I_3 = -\int_0^R \frac{\alpha^2}{(\alpha^2 t)^{1000}+1} dt = -\int_0^R \frac{e^{i\pi/1000}}{(te^{i\pi/500})^{1000}+1} dt$

$= -\alpha^2 I_1 =$ ~~...~~ $-e^{i\pi/500} I_1$ where I_1 is Z integral on γ_1

recall Z integrand on Z arc γ_2 $O\left(\frac{1}{R^{1000}}\right)$ Z length of Z

arc is $\frac{\pi R}{500}$ By Z M-L inequality, Z integral I_2 on

γ_2 is $O\left(\frac{1}{R^{999}}\right) \to 0$ as $R \to \infty$ $\therefore \int_{\gamma_2} f(z) dz \to 0$ as $R \to \infty$

Z residue of f at α is: $\operatorname{Res}(f, \alpha) = \frac{1}{1000\alpha^{999}}$ $\therefore$

by Cauchy's Residue thm have:

$I = 2\pi i \operatorname{Res}(f, \alpha) = 2\pi i \frac{1}{1000\alpha^{999}}$ ∴

∴ $\underbrace{I}_{\frac{2\pi i}{1000\alpha^{999}}} = I_1 + \underbrace{I_2}_{\to 0} + \underbrace{I_3}_{-\alpha^2 I_1}$ ∴ 1)

$\frac{2\pi i}{1000\alpha^{999}} = I_1 + I_2 + I_3 = I(1-\alpha^2) + I_2 \to (1-\alpha^2)\int_0^\infty \frac{x}{\cancel{x^{1000}}+1}\,dx$

as $R \to \infty$

∴ Z resired integral $I_1 = \frac{2\pi i}{1000\alpha^{999}(1-\alpha^2)} = \int_0^\infty \frac{dx}{x^{1000}+1}$

(integral must be real (real valued func of real variable) ∴

$I_1 = \frac{\pi}{1000} \times \frac{2i}{\alpha^{1000}(\alpha^{-1}-\alpha)} = \frac{\pi}{1000} \times \frac{2i}{\alpha - \alpha^{-1}}$ & since $\alpha^{1000} = -1$ ∴

$I_1 = \frac{\pi}{1000}\csc\left(\frac{\pi}{1000}\right)$

\i/ $\int_0^\infty \frac{x^{666}}{x^{1000}+1}\,dx$ use Z slice of pie Method

consider contour γ Z join of Z paths $\gamma_1, \gamma_2, \gamma_3$ where

$\gamma_1 = [0, R]$, $\gamma_2(t) = Re^{it}$ for $0 \le t \le \frac{\pi}{500}$

γ_3 is Z reversal of $[0, \alpha^2 R]$ param as $\gamma_3^-(t) = \alpha^2 t$

for $0 \le t \le R$

showed $I_2 \to 0$ as $R \to \infty$

Z integral on γ_3 is $I_3 = -\int_0^R \frac{(\alpha^2 t)^{666}}{(\alpha^2 t)^{1000}+1}\,dt = -\alpha^{2\times 667} I_1$

∴ Z residue at Z simple pole $z = \alpha$ is $\frac{\alpha^{666}}{1000\alpha^{999}}$

by Z Cauchy's reside thm:

$2\pi i \frac{\alpha^{666}}{1000\alpha^{999}} = I_1 + I_2 + I_3 = I_1(1-\alpha^{2\times 667}) + I_2 \to \cancel{(1-\alpha^{2\times 667})\int_0^\infty \frac{x^{666}\,dx}{x^{1000}+1}}$

$(1-\alpha^{2\times 667})\int_0^\infty \frac{x^{666}}{x^{1000}+1}\,dx$ ∴ $I_1 = \int_0^\infty \frac{x^{666}}{x^{1000}+1}\,dx = \frac{\pi}{1000}\,\frac{2i\alpha^{666}}{\alpha^{999}(1-\alpha^{2\times 667})}$

∴ $I = \frac{\pi}{1000} \times \frac{2i\alpha^{666}}{\alpha^{999}\alpha^{667}\cancel{\alpha^{2\times 667}}(\alpha^{-667} - \alpha^{667})} =$

$\frac{\pi}{1000} \times \frac{2i}{\alpha^{1000}(\alpha^{-667} - \alpha^{667})} = \frac{\pi}{1000} \times \frac{2i}{\alpha^{667} - \alpha^{-667}} = \frac{\pi}{1000}\csc\left(\frac{667\pi}{1000}\right)$

Sheet 5

i/ $\int_0^\infty \frac{x^{666}}{1+x^{1000}}\,dx$ use slice-of-pie method

fix a real number $R>0$ & take a contour $(\gamma_1, \gamma_2, \gamma_3^{-1})$ where $\gamma_1(t)=t$ & $\gamma_3(t)=te^{2i\pi/1000}$ for $0\le t\le R$ & $\gamma_2(t)=Re^{it}$ for $0\le t\le 2\pi/1000$

let $f(z)=z^{666}/(1+z^{1000})$ then Z only singularity of f within Z contour is at $z=e^{i\pi/1000}$, where Z residue is:

$$Res(f,e^{i\pi/1000}) = \lim_{z\to e^{i\pi/1000}} \frac{(z^{666})}{1000z^{999}} = \frac{(e^{i\pi/1000})^{666}}{1000(e^{i\pi/1000})^{999}} =$$

$$\frac{1}{1000}\frac{e^{666i\pi/1000}}{e^{999i\pi/1000}} = \frac{1}{1000}e^{-333i\pi/1000} \therefore$$

Z integral on Z whole contour is $\frac{2\pi i}{1000}e^{-333i\pi/1000}$

$$\therefore \int_{\gamma_3} f(z)\,dz = \int_0^R \frac{(te^{2i\pi/1000})^{666}}{1+t^{1000}} e^{2i\pi/1000}\,dt =$$

$$\int_0^R \frac{t^{666}e^{1332i\pi/1000}}{1+t^{1000}}e^{2i\pi/1000}\,dt = e^{1334i\pi/1000}\int_0^R \frac{t^{666}}{1+t^{1000}}\,dt$$

Z integral on γ_2 maybe estied using Z M-L inequality

note: Z length of Z contour is $2\pi R/1000$ ∴

$$|f(z)| = \left|\frac{z^{666}}{1+z^{1000}}\right| \le \frac{R^{666}}{1+R^{1000}} \to 0 \text{ as } R\to\infty \therefore$$

$$\left|\int_{\gamma_2}\frac{z^{666}}{1+z^{1000}}\,dz\right| \le \frac{2\pi R}{1000}\times\frac{R^{666}}{1+R^{1000}} \to 0 \text{ as } R\to\infty$$

with sufficiently large R ∴

$$\frac{2\pi i}{1000}e^{-333i\pi/1000} = \int_\gamma f(z)\,dz = (1-e^{1334i\pi/1000})\int_0^R \frac{t^{666}}{1+t^{1000}}\,dt + \int_{\gamma_2} f(z)\,dz$$

$$\to (1-e^{1334i\pi/1000})\int_0^\infty \frac{t^{666}}{1+t^{1000}}\,dt \text{ as } R\to\infty \therefore$$

$$\int_0^\infty \frac{x^{666}}{1+x^{1000}}\,dx = \frac{2\pi i}{1000}\frac{e^{-333i/1000}}{1-e^{1334i\pi/1000}} = \frac{\cancel{2\pi i}}{1000}\frac{2i}{e^{333i\pi/1000}-e^{1667i\pi/1000}}$$

$$= \frac{\pi}{1000}\frac{2i}{e^{333i\pi/1000}-e^{-333i\pi/1000}} = \frac{\pi}{1000}\operatorname{cosec}\left(\frac{333\pi}{1000}\right) =$$

$\frac{\pi}{1000} \operatorname{cosec}\left(\frac{667\pi}{1000}\right)$

\h/ $I = \int_0^\infty \frac{1}{1+x^{1000}}\,dx$ use slice of pie method

fix a real number $R>0$ & take a contour $(\gamma_1, \gamma_2, \gamma_3^{-1})$ where $\gamma_1(t) = t$ & $\gamma_3(t) = te^{2i\pi/1000}$ for $0 \le t \le R$ &

$\gamma_2(t) = Re^{it}$ for $0 \le t \le 2\pi/1000$ $\therefore$

let $f(z) = \frac{1}{1+z^{1000}}$ $\therefore$ Z only singularity of f within Z contour is at $z = e^{i\pi/1000}$ $\therefore$ residue is:

~~Res~~ $f(z) = \frac{h(z)}{k(z)} = \frac{1}{1+z^{1000}}$ $\therefore$ $h(z) = 1$, $k(z) = 1+z^{1000}$ $\therefore$

$\operatorname{Res}(f, e^{i\pi/1000}) = \lim_{z \to e^{i\pi/1000}} \frac{h(z)}{k'(z)} = \lim_{z \to e^{i\pi/1000}} \frac{1}{1000z^{999}} =$

$\frac{1}{1000} \cdot \frac{1}{(e^{i\pi/1000})^{999}} = \frac{1}{1000} e^{-999i\pi/1000}$ $\therefore$

$\int_{\gamma_3} f(z)\,dz = \int_0^R \frac{1}{1+(te^{2i\pi/1000})^{1000}} e^{2i\pi/1000}\,dt =$

$\int_0^R \frac{1}{1+t^{1000}e^{2i\pi}} e^{2i\pi/1000}\,dt = e^{2i\pi/1000} \int_0^R \frac{1}{1+t^{1000}}\,dt$

Z integral on γ_2 be estied via Z M-L inequality

note: Z length of Z contour is $\frac{2\pi R}{1000}$ $\therefore$

$|f(z)| = \left|\frac{1}{1+z^{1000}}\right| \le \frac{1}{-1+R^{1000}}$ $\therefore$ $\left|\int_{\gamma_2} \frac{1}{1+z^{1000}}\,dz\right| \le \left(\frac{2\pi R}{1000}\right) \frac{1}{-1+R^{1000}} \to$

for large R $\therefore$ $\to 0$ as $R \to \infty$ $\left\{ \left|\frac{1}{1+z^{1000}}\right| = \frac{1}{|z^{1000}-(-1)|} \le \right.$

$\left. \frac{1}{|z^{1000}| - |-1|} = \frac{1}{|z|^{1000}-1} = \frac{1}{R^{1000}-1} \right\}$ $\therefore$ Z integral over Z

~~$\frac{1}{1000} e^{-999i\pi/1000}$~~ whole contour is $\frac{2\pi i}{1000} e^{-999i\pi/1000} =$

$(1-e^{2i\pi/1000}) \int_0^R \frac{1}{1+t^{1000}}\,dt + \int_{\gamma_2} f(z)\,dz \to (1-e^{2i\pi/1000}) \int_0^\infty \frac{1}{1+t^{1000}}\,dt$

as $R \to \infty$ $\therefore$ $\int_0^\infty \frac{1}{1+x^{1000}}\,dx = \frac{2\pi i}{1000} \frac{e^{-999i\pi/1000}}{1-e^{2i\pi/1000}} =$

$\frac{\pi}{1000} \frac{2i}{e^{999i\pi/1000} + -e^{1001i\pi/1000}} = \frac{\pi}{1000}$ ~~cosec~~ $\left(\frac{\pi}{1000} \frac{2i}{e^{999i\pi/1000} - e^{-999i\pi/1000}}\right)$

$= \frac{\pi}{1000} \operatorname{cosec}\left(\frac{999\pi}{1000}\right) = \frac{\pi}{1000} \sin\left(\frac{\pi}{1000}\right)$

~~\Qa/ $\frac{dx}{dt} = F(x,t) = \cos(t)$~~ $\therefore$

~~Forward Euler: $\frac{dx}{dt} \approx \frac{x_{n+1} - x_n}{\Delta t} = F(x_n, t_n) = \cos(t_n)$~~ $\therefore$

~~$x_{n+1} - x_n = \Delta t \cos(t_n)$~~ $\therefore$

~~$x_{n+1} = x_n + \Delta t \cos(t_n)$~~

\complex analysis PP2019/

\1a/ Z Cauchy-Riemann eqns for a func $f(x+iy) = u(x,y) + iv(x,y)$ are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

let $z = x+iy$ $\therefore$ our func is:

$f(z) = f(x+iy) = e^{3(x+iy)-2} = e^{3x-2}(\cos 3y + i\sin 3y)$ $\therefore$

set $u = e^{3x-2}\cos 3y$ & $v = e^{3x-2}\sin 3y$ $\therefore$

$\frac{\partial u}{\partial x} = 3e^{3x-2}\cos 3y$ $\quad \frac{\partial v}{\partial y} = 3e^{3x-2}\cos 3y$

$\frac{\partial u}{\partial y} = -3e^{3x-2}\sin 3y$ $\quad -\frac{\partial v}{\partial x} = -3e^{3x-2}\sin 3y$

$\therefore$ Z func is satisfied by Z Cauchy-Riemann eqns

\1bi & 1bii/ Z set D is closed & not open

first show that D is closed. By def of closed, must show Z complement $CD = \{z \in \mathbb{C} : -1 < Re(z) < 5/2\}$ is open. let $z = x+iy \in CD$ $\therefore$ implies: $-1 < x < 5/2$ $\therefore$

$0 < x+1$ $\quad 0 < \frac{5}{2} - x$ $\quad$ let $0 < \varepsilon < \min(x+1, \frac{5}{2} - x)$

let $w = a+bi \in D(z, \varepsilon)$ note: $w - z = a - x + i(b-y)$ $\therefore$

$|a-x| = |Re(w-z)| \le |w-z| < \varepsilon$ $\therefore$

$|a-x| < \varepsilon < x+1$ $\quad |a-x| < \varepsilon < \frac{5}{2} - x$ $\therefore$

$|a-x| < x+1$ $\quad |a-x| < \frac{5}{2} - x$ $\therefore$ $\quad -x-1 < -|a-x|$ $\therefore$

$-x-1 < a-x$ $\quad a-x < \frac{5}{2} - x$ $\therefore$

$-x-1 < a-x < \frac{5}{2} - x$ $\therefore$

$-1 < a < \frac{5}{2}$ $\therefore$ since $a = Re(w)$ & w is an arbit pt of $D(z, \varepsilon)$ have: $D(z, \varepsilon) \subset CD$ $\therefore$

as z is an arbit pt of CD, we have the desired result

Second we show that D is not open

let $\frac{7}{2} > \varepsilon > 0$ have: $\left\{ \frac{5}{2} - -1 = \frac{7}{2} \right\}$ $\therefore$

$\left| \left(-1+\frac{\varepsilon}{2}\right) - (-1) \right| = \frac{\varepsilon}{2} < \varepsilon$ $\left\{ \left(-1+\frac{\varepsilon}{2}\right) - (-1) = -1 + \frac{\varepsilon}{2} + 1 = \frac{\varepsilon}{2} \right\}$ $\therefore$

$-1 + \frac{\varepsilon}{2} \in D(-1, \varepsilon)$ but $\frac{5}{2} > -1 + \frac{\varepsilon}{2} > -1$ $\{ \varepsilon > 0 \therefore \frac{\varepsilon}{2} > 0 \therefore$
$-1 < -1 + \frac{\varepsilon}{2}$ & $\varepsilon < \frac{7}{2} = \frac{5}{2} - -1$ $\therefore \varepsilon < \frac{5}{2} - -1 = \frac{5}{2} + 1$ $\therefore \varepsilon < \frac{5}{2} + 1$
$\therefore$ $\frac{\varepsilon}{2} < \frac{5}{2} + 1$ $\therefore$ $-1 + \frac{\varepsilon}{2} < \frac{5}{2} \}$ $\therefore$

$\therefore -1 < -1 + \frac{\varepsilon}{2} < \frac{5}{2}$ $\therefore$ implies: $-1 + \frac{\varepsilon}{2} \notin D$

(note $-1 + \frac{\varepsilon}{2}$ is a real number & $\therefore Re\left(-1 + \frac{\varepsilon}{2}\right) = -1 + \frac{\varepsilon}{2}$) $\therefore$
$D(-1, \varepsilon) \not\subset D$ moreover this holds $\forall \varepsilon > 0$ this shows the desired result (D is not open)

\1c/ let $w = 3z^3$ using taylor series for $\cos w$ have:
$$f(z) = \sum_{n=0}^{\infty} (-1)^n \frac{w^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{(3z^3)^{2n}}{(2n)!} = \sum_{n=0}^{\infty} (-1)^n \frac{9^n z^{6n}}{(2n)!}$$
z series converges $\forall w$ & $\therefore \forall z$ $\therefore$ z radius of convergence is infinity

\1d/ f has poles when $z = -6$ and when $\frac{\pi z}{4} = \frac{\pi}{2}(2n+1)$ for $n \in \mathbb{Z}$ $\left\{ \cos\left(\frac{\pi}{2}\right) = 0 = \cos\left(\frac{3\pi}{2}\right) \therefore \frac{\pi}{2} + 2\pi n \text{ and } \frac{3\pi}{2} + 2\pi n \therefore \right.$
$\frac{(2n+1)\pi}{2}$ since $\frac{\pi}{2} + 4n\frac{\pi}{2} = (4n+1)\frac{\pi}{2}$ & $\frac{3\pi}{2} + 4n\frac{\pi}{2} = (4n+3)\frac{\pi}{2}$ $\therefore$
$4n+1$ & $4n+3$ is $2n+1$ $\therefore$ $(2n+1)\frac{\pi}{2}$ $\therefore \frac{\pi z}{4} = (2n+1)\frac{\pi}{2}$ $n \in \mathbb{Z}$
$\therefore \frac{z}{4} = (2n+1)\frac{1}{2}$ $\therefore z = 2(2n+1) = 4n+2$ $n \in \mathbb{Z} \}$ $\therefore$ that is when $z = -6$ & when $z = 2(2n+1)$ for $n \in \mathbb{Z}$. the pole at $z = -6$ is a triple pole & z poles at $z = 2(2n+1)$ for $n \neq 2 \in \mathbb{Z}$ are simple poles $\{$ ~~when $z=-6$~~ $\therefore$ if $n = 2$: $2(2(2)+1) = z = 10$
if $n = -2$: $2(2(-2)+1) = z = 2(-3) = -6$ $\therefore$ the $(z+6)^2$ & $\cos\left(\frac{\pi z}{4}\right)$ combine at $z = -6$ to make a triple pole even though

\complex analysis PP2019/ $(z+6)^2$ is just a double pole}

∴ write $g(z)=\frac{z}{(z+6)^2}$ & $h(z)=\cos(\frac{\pi z}{4})$ ∴ Z residue at each

simple pole is { $f(z)=g(z)\frac{1}{h(z)}$ ∴ ~~$h'(z)=-\frac{\pi}{4}$~~

$h'(z)=-\frac{\pi}{4}\sin(\frac{\pi z}{4})$ } $\quad \frac{g(z)}{h'(z)}\Big|_{z=2(2n+1)} = -\frac{4z}{\pi(z+6)^2\sin(\frac{\pi z}{4})}\Big|_{z=2(2n+1)} =$

$-\frac{8(2n+1)}{\pi(4^2(n+2)^2)\sin(\frac{\pi(2n+1)}{2})} = -\frac{16n+8}{\pi 16(n+2)^2\sin(\frac{\pi(2n+1)}{2})} =$

~~$8n$~~ $-\frac{2n+1}{2\pi(n+2)^2\sin(\frac{\pi(2n+1)}{2})} = -\frac{2n+1}{2\pi(n+2)^2(-1)^n} =$

$-1\,\frac{2n+1}{2\pi(n+2)^2}(-1)^n = (-1)^{n+1}\frac{2n+1}{2\pi(n+2)^2}$

\1e/ $0^2=0$ $\quad (e^0-1)=1-1=0$ ∴ ~~triple~~ 3rd order pole

at $z=0$ ∴ write:

$f(z)=\sum_{n=-3}^{\infty} b_n z^n$ ∴ $z+1=(z^3+\frac{z^4}{2!}+\frac{z^5}{3!})\sum_{n=-3} b_n z^n$

{ ~~(z)~~ $z^2(e^z-1)f(z)=z+1=z^2(e^z-1)\sum_{n=-3}^{\infty} b_n z^n$ ∴

~~$z^2(e$~~ $z^2(-1+e^z)f(z)=z+1=z^2(-1+1+z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots)\sum_{n=-3}^{\infty} b_n z^n$ ∴

$z+1=z^2(z+\frac{z^2}{2!}+\frac{z^3}{3!}+\cdots)\sum_{n=-3}^{\infty} b_n z^n=(\frac{z^3}{1!}+\frac{z^4}{2!}+\frac{z^5}{3!}+\cdots)\sum_{n=-3}^{\infty} b_n z^n$ }

∴ comparing coeffs of z^0: $1=b_{-3}$ { $1=b_{-3}\frac{z^3}{1!}z^{-3}=b_{-3}$ }

comparing coeffs of z^1: ~~$+z$~~ $z=\frac{z^4}{2!}b_{-3}z^{-3}+\frac{z^3}{1!}b_{-2}z^{-2}=$

$\frac{1}{2!}b_{-3}z+\frac{1}{1!}b_{-2}z$ ∴ $1=b_{-2}+\frac{1}{2}b_{-3}$ ∴ $1=b_{-2}+\frac{1}{2}(1)$ ∴

$b_{-2}=\frac{1}{2}$

Comparing coeffs of z^2: $0=b_{-1}+b_{-2}/2!+b_{-3}/3!$ ∴

~~$b_{-1}z$~~ $0=b_{-1}+\frac{1}{2}/2+1/6=b_{-1}+\frac{1}{4}+\frac{1}{6}=b_{-1}+\frac{5}{12}=0$ ∴

$b_{-1}=-\frac{5}{12}$

\2a/ Z func is not differentiable at Z pt to see:

compute Z deriv at $z_0=1+i$ let $z=x+iy$ note:

$f(x+iy)=\sqrt{|(x-1)(y-1)|}$ using Z def of Z deriv have:

$$\lim_{z\to z_0}\frac{f(z)-f(z_0)}{z-z_0}=\lim_{z\to z_0}\frac{f(z)-f(1+i)}{z-z_0}=\lim_{z\to z_0}\frac{f(z)}{z-z_0}$$

here Z limit does not exist. to see: compute Z limit along $z=x+1+i(x+1)$ as $x\to 0$ for $x>0$ & for $x<0$ ∴

have: $\frac{f(z)}{z-z_0}=\frac{|x|}{x+ix}\Big\{\cancel{=\lim_{x\to z_0}}\ \lim_{z\to z_0}\frac{f(x+1+i(x+1))}{z-z_0}=$

$$\lim_{z\to z_0}\frac{\sqrt{|(x+1-1)(x+1-1)|}}{x+1+i(x+1)-z_0}=\frac{\sqrt{|(x)(x)|}}{x+1+ix+i-(1+i)}=\frac{\sqrt{|x^2|}}{x+1+ix+i-1-i}$$

$$=\frac{|x|}{x+ix}\Big\}\quad ∴\quad \lim_{x\to 0^+}\frac{|x|}{x+ix}=\lim_{x\to 0^+}\frac{1}{1+i}=\frac{1}{1+i}\quad \&$$

$$\lim_{x\to 0^-}\frac{|x|}{x+ix}=\lim_{x\to 0^-}\frac{-x}{x+ix}=\lim_{x\to 0^-}\frac{-1}{1+i}=\frac{-1}{1+i}\quad ∴$$

Consequently, Z limit & Z deriv does not exist

2b/ Z length of a smooth curve $\gamma:[a,b]\to\mathbb{C}$ $t\mapsto\gamma(t)=x(t)+iy(t)$ is defined by:

$$\text{length}(\gamma)=L(\gamma)=\int_a^b|\gamma'(t)|\,dt=\int_a^b\sqrt{x'(t)^2+y'(t)^2}\,dt$$

using Z product rule:

$\gamma'(t)=t(\cos t+i\sin t)+\sin t-i\cos t+ie^{it}=$

$t(\cos t+i\sin t)+\sin t-i\cos t+i\cos t+(i)(i)\sin t=$

$t(\cos t+i\sin t)+\sin t-i\cos t+i\cos t-\sin t=$

$t(\cos t+i\sin t)$ ∴ $\Big\{|\gamma'(t)|=\sqrt{t^2(\cos^2 t+\sin^2 t)}=\sqrt{t^2}=t\Big\}$

$$|\gamma'(t)|=|t|\quad ∴\quad L(\gamma)=\int_0^{\pi/2}|t|\,dt=\frac{t^2}{2}\Big|_0^{\pi/2}=\frac{\pi^2}{8}$$

2c/ By Cauchy's integral formula have:

$$f^{(n)}(z_0)=\frac{n!}{2\pi i}\int_{\gamma_R}\frac{f(z)}{(z-z_0)^{n+1}}\,dz$$ where γ_R is a circular contour taken anticlockwise, with centre at z_0 & radius R

applying Z estimation lemma have:

\complex analy PP2019/ $|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} M \times L$ (3.1)
where $L = 2\pi R$ is Z length of Z contour (namely Z circumference), & M is Z supremum of Z integrand on Z contour
note on Z contour have: $\left|\frac{f(z)}{(z-z_0)^{n+1}}\right| = \frac{|f|}{|(z-z_0)^{n+1}|} = \frac{|f(z)|}{R^{n+1}}$

$\therefore$ $M = \frac{1}{R^{n+1}} \times \sup\{|f(z)| : |z-a| = R\}$ $\therefore$

putting into (3.1) gives Z desired result:

$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi} \frac{1}{R^{n+1}} \times \sup\{|f(z)| : |z-a| = R\} \times 2\pi R$ $\therefore$

$|f^{(n)}(z_0)| \leq \frac{n!}{R^n} \max\{|f(z) : |z-a| = R\}$

\3a/ use Z estimation lemma: let $z = x+iy$ note: Z circle's length is its circumference: 2π
using Z triangle inequality have on Z circle:
$|3z^4+2| \geq |3z^4| - |2| = 3|z|^4 - 2 = 3(1)^4 - 2 = 1$ $\therefore$
on Z circle: $\left|\frac{4z^3(\sin z + i\cos z)}{3z^4+2}\right| = \frac{|4z^3||(\sin z + i\cos z)|}{|3z^4+2|} =$

$\frac{4|z|^3 \left|\left(\frac{e^{iz}-e^{-iz}}{2i} + i\frac{e^{iz}+e^{-iz}}{2}\right)\right|}{|3z^4+2|} =$

$\left(4|z|^3\left|\left(-\frac{1}{2}ie^{iz} + \frac{1}{2}ie^{-iz} + \frac{1}{2}ie^{iz} + \frac{1}{2}ie^{-iz}\right)\right|\right) / |3z^4+2| =$

$\frac{4|z|^3|ie^{-iz}|}{|3z^4+2|} = \frac{4|z|^3|i||e^{-iz}|}{|3z^4+2|} = \frac{4|z|^3|e^{-iz}|}{|3z^4+2|} \leq \frac{4|z|^3|e^{-iz}|}{|3z^4|-|2|} =$

$\frac{4(1)^3|e^{-iz}|}{1} = 4|e^{-ix+y}| = 4|e^y||e^{-ix}| = 4|e^y| = 4e^y \leq 4e$

because on Z circle have: $-1 \leq y \leq 1$ $\therefore$ $e^y \leq e$
estimation lemma gives desired answer: $4e \cdot 2\pi = 8\pi e$

\3b/ first prove: claim (0.1) $F=\{0\}$ proof:

let $z \in F$ ∴ $|z| = |z-0| < \frac{1}{n}$ $\{ z \in \bigcap_{n=1}^{\infty} D(0,\frac{1}{n})$ ∴

$z \in \bigcap_{n=1}^{\infty}\{z \in \mathbb{C} : |z-0| < \frac{1}{n}\}$ ∴ $|z-0| < \frac{1}{n}$ $\}$

on all integers $n \geq 1$ first consider Z case: $z \neq 0$

then a property of Z norm gives: $|z| \neq 0$ & ∴ $|z| > 0$

By Z Archimedean property, $\exists N$ integer st $N > \frac{1}{|z|}$, ie

$|z| > \frac{1}{N}$ implies: $|z-0| > \frac{1}{N}$ ∴ $|z-0| \not< \frac{1}{N}$ ∴ $z \notin D(0,\frac{1}{N})$

& ∴ $z \notin F$

next consider Z remaining case: $z=0$. Since $|z-0| = |0-0| = |0| = 0 < \frac{1}{n}$ for all integers $n \geq 1$, have that $0 \in D(0,\frac{1}{n})$ $\forall$ integers $n \geq 1$ ∴ have $0 \in F$

consequently have shown that F contains only Z pt 0 □ ∴

let $\varepsilon > 0$ then: $|\frac{\varepsilon}{2} - 0| = \frac{\varepsilon}{2} < \varepsilon$ which shows $\frac{\varepsilon}{2} \in D(0,\varepsilon)$ but $\frac{\varepsilon}{2} \notin F$ because $\frac{\varepsilon}{2} \neq 0$ & Z claim gives that F consists only of Z pt 0 ∴

$D(0,\varepsilon) \not\subset F$ thus holds $\forall \varepsilon > 0$ & ∴ F is not open

\3c/ write $f(z) = \frac{ze^{iz}}{1+z^2}$ ∴ then $f(z)$ has simple poles at $z = \pm i$

consider Z contour $\gamma = (\gamma_0, \gamma_R)$ where γ_0 is Z interval $[-R,R]$ & γ_R is Z semicircle $\gamma_R(t) = Re^{it}$, for $0 \leq t \leq \pi$

∴ Z residue of f at $z=i$ is $\frac{ze^{iz}}{z+i}\Big|_{z=i} = \frac{e^{-1}}{2}$ ∴

by Cauchy's Residue thm: $\int_\gamma f(z)dz = \pi i e^{-1}$

on Z path γ_R write $z = Re^{i\theta}$ ∴ $|e^{iz}| = |e^{iRe^{i\theta}}| =$
$|e^{iR(\cos\theta + i\sin\theta)}| = |e^{-R\sin\theta}||e^{iR\cos\theta}| = |e^{-R\sin\theta}| = e^{-R\sin\theta}$ ∴

\Complex analy PP2019/

$|f(z)| = \left|\frac{Re^{i\theta} e^{iRe^{i\theta}}}{1+z^2}\right| = \frac{|R||e^{i\theta}||e^{iRe^{i\theta}}|}{|1+z^2|} = \frac{Re^{-R\sin\theta}}{|1+z^2|} =$

$\frac{Re^{-R\sin\theta}}{|1+R^2e^{i2\theta}|} \leq \frac{Re^{-R\sin\theta}}{-|1|+|R^2e^{i2\theta}|} = \frac{Re^{-R\sin\theta}}{-1+|R^2||e^{i2\theta}|} = \frac{Re^{-R\sin\theta}}{-1+|R^2|(1)} =$

$\frac{Re^{-R\sin\theta}}{R^2-1} = \frac{R}{R^2-1}e^{-R\sin\theta} = \frac{R}{(R+1)(R-1)}e^{-R\sin\theta} \leq \frac{2}{R}e^{-R\sin\theta}$

$\therefore \left|\int_{\gamma_R} f(z)\,dz\right| = R\left|\int_0^\pi f(\gamma_R(\theta))\,d\theta\right| \leq 2\int_0^\pi e^{-R\sin\theta}d\theta = 4\int_0^{\pi/2} e^{-R\sin\theta}d\theta$

since $\sin\theta \geq \frac{2\theta}{\pi}$ for $0 \leq \theta \leq \frac{\pi}{2}$ $\therefore$ have:

$\left|\int_{\gamma_R} f(z)\,dz\right| \leq 4\int_0^{\pi/2} e^{-2R\theta/\pi}\,d\theta = \frac{2\pi}{R}(1-e^{-R})$

$\left\{\text{since } -R\sin\theta \leq -\frac{2R\theta}{\pi}\right\}$ which tends to 0 as $R \to \infty$

on Z other hand, have: $\mathrm{Im}\left(\int_{\gamma_0} f(z)\,dz\right) = 2\int_0^R \frac{x\sin x}{1+x^2}\,dx$

& $\therefore \int_0^\infty \frac{x\sin x}{1+x^2} = \frac{1}{2}\mathrm{Im}\left(\int_{\gamma_0} f(z)\,dz\right) = \frac{1}{2}\mathrm{Im}(\pi i e^{-1}) = \frac{1}{2}\pi e^{-1} = \frac{\pi}{2e}$

\4a/ if γ is a closed, simple contour, traversed anticlockwise, & f is a func holomorphic on a domain containing Z image & interior of γ except for a finite number of isolated singularities in Z interior (say $a_1, \dots, a_k$), then

$\int_\gamma f(z)\,dz = 2\pi i \sum_{j=1}^{k} \mathrm{Res}(f, a_j)$

\4b/ partial frac form: $f(z) = \frac{1}{(z+2)(z-4)} = \frac{1}{6}\left(\frac{1}{z-4}\right) - \frac{1}{6}\left(\frac{1}{z+2}\right)$

for $|z| < 4$ have: $\frac{1}{z-4} = -\frac{1}{4(1-\frac{z}{4})} = -\frac{1}{4}\frac{1}{1-(\frac{z}{4})} = -\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{z}{4}\right)^n =$

$\cancel{z} - \sum_{n=0}^{\infty} \frac{z^n}{4^{n+1}}$

if $|z| > 2$ write $y = \frac{1}{z}$ & $\therefore$

$$\frac{1}{z+2} = \frac{\left(\frac{1}{z}\right)}{\left(\frac{1}{z}\right)(z+2)} = \frac{\left(\frac{1}{z}\right)}{1+2\frac{1}{z}} = \frac{y}{1+2y} = \frac{y}{1-(-2y)} = y\frac{1}{1-(-2y)} =$$

$$y\sum_{n=0}^{\infty}(-2y)^n = y\sum_{n=0}^{\infty}(-1)^n 2^n y^n = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n 2^n \left(\frac{1}{z}\right)^n =$$

$$\frac{1}{z}\sum_{n=0}^{\infty}(-1)^n 2^n z^{-n} = \sum_{n=0}^{\infty}(-1)^n 2^n z^{-n-1} = \cancel{\sum_{n=0}^{\infty}(-1)^{n+1}2^{n+1}z^{n+1}}$$

$$\sum_{n=0}^{\infty}(-1)^n 2^n z^{-(n+1)} = \sum_{n=-1}^{\infty}(-1)^{n+1}2^{n+1}z^{-(n+2)}$$

$$\left\{ \sum_{n=0}^{\infty}(-1)^n 2^n z^{-(n+1)} \quad \therefore \quad n=0:\ (-1)^0 2^0 z^{-1}\right.$$

$n=1:\ (-1)^1 2^1 z^{-2}$ $\quad n=2:\ (-1)^2 2^2 z^{-3}$ $\quad n=3:\ (-1)^3 2^3 z^{-4}$.

$$\therefore \sum_{n=0}^{\infty}(-1)^n 2^n z^{-(n+1)} = \cancel{\sum_{m=-1}^{-\infty} z^m} \quad \sum_{m=-\infty}^{-1} z^m (-1)^{-m-1} 2^{-m-1} =$$

$$\left.\sum_{m=-\infty}^{-1}(-1)^{-(m+1)}2^{-(m+1)}z^m = \sum_{m=-\infty}^{-1}(-1)^{m+1}2^{-(m+1)}z^m \right\} \quad \therefore$$

$$\frac{1}{z+2} = \frac{1}{z}\sum_{n=0}^{\infty}(-1)^n 2^n z^{-n} = \sum_{m=-\infty}^{-1}(-1)^{m+1}2^{-(m+1)}z^m$$

note this series converges for $|2y|<1$ $\quad \left\{\frac{1}{z+2} = y\sum_{n=0}^{\infty}(-1)^n(2y)^n\right\}$

$\therefore$ get: $f(z) = -\frac{1}{6}\sum_{n=0}^{\infty}\frac{z^n}{4^{n+1}} - \frac{1}{6}\sum_{m=-\infty}^{-1}(-1)^{m+1}2^{-(m+1)}z^m$

simplifying: $f(z) = -\frac{1}{6}\sum_{m=-\infty}^{-1}(-1)^{m+1}\frac{z^m}{2^{m+1}} - \frac{1}{6}\sum_{n=0}^{\infty}\frac{z^n}{4^{n+1}}$

\4e/ let $f(z) = \frac{z^2}{1+z^{12}}$ $\therefore$ $f(z)$ has poles for $z^{12}=-1$, that is, when $z = e^{i\pi/12 + in\pi/6}$ for n integer $\left\{ z = e^{i\theta} \ \therefore\ z^{12} = (e^{i\theta})^{12} =\right.$ $e^{i12\theta} = -1 = e^{i\pi}$ $\quad\therefore\ 12\theta = \pi \quad \therefore \quad 0 \le \theta \le 2\pi \ \therefore$

$0 \le 12\theta \le 24\pi \quad \therefore$

$12\theta = \cancel{23\pi}, \pi, 3\pi, 5\pi, 7\pi, 9\pi, 11\pi, 13\pi, 15\pi, 17\pi, 19\pi, 21\pi$

$\therefore \theta = \frac{\pi}{12}, \frac{\pi}{4}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{3\pi}{4}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{5\pi}{4}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{7\pi}{4}, \frac{23\pi}{12} \ \therefore$

$\theta = \frac{\pi}{12}, \frac{3\pi}{12}, \frac{5\pi}{12}, \frac{7\pi}{12}, \frac{9\pi}{12}, \frac{11\pi}{12}, \frac{13\pi}{12}, \frac{15\pi}{12}, \frac{17\pi}{12}, \frac{19\pi}{12}, \frac{21\pi}{12}, \frac{23\pi}{12}$

$\therefore$ for $n = 0, \ldots, 11$: $\theta = \frac{\pi}{12} + \frac{2n\pi}{12} = \frac{\pi}{12} + n\pi/6 \ \therefore\ z = e^{i\pi/12 + in\pi/6}\}$ $\therefore$

\Complex analy PP2019/ take a contour $\gamma=(\gamma_0,\gamma_R,\gamma_1)$ where γ_0 is Z interval $[0,R]$, γ_R is Z arc $\gamma_R(t)=Re^{it}$ for $0\le t\le\frac{\pi}{6}$ & γ_1 is Z interval $[Re^{i\pi/6}, 0]$

then there is exactly one simple pole of f within Z contour at $e^{i\pi/12}$

recall for $f(z)=h(z)/k(z)$ with an isolated singularity at a have $Res(f,a)=h(a)/k'(a)$

let us calc Z residue at this pole: $f(z)=h(z)/k(z)$

where $h(z)=z^2$ ∴ $k(z)=1+z^{12}$ ∴ $k'(z)=12z^{11}$ ∴

$Res(f, e^{i\pi/12}) = h(z)/k'(z) = \frac{z^2}{12z^{11}}$ ∴

at $z=e^{i\pi/12}$ we get: $Res(f,e^{i\pi/12})=\frac{(e^{i\pi/12})^2}{12(e^{i\pi/12})^{11}}=$

$\frac{e^{i2\pi/12}}{12e^{i11\pi/12}}=\frac{1}{12}e^{\frac{i2\pi}{12}-i\frac{11\pi}{12}}=\frac{1}{12}e^{-i9\pi/12}=\frac{1}{12}e^{-3i\pi/4}=$

$\frac{1}{12}\left(\cos\left(\frac{-3\pi}{4}\right)+i\sin\left(-\frac{3\pi}{4}\right)\right)=\frac{1}{12}\left(-\frac{\sqrt2}{2}-i\frac{\sqrt2}{2}\right)=\frac{1}{12}\left(-\frac{1}{\sqrt2}-i\frac{1}{\sqrt2}\right)=$

$\frac{1}{12}\,\frac{-1-i}{\sqrt2}$ ∴ since $I_R=\int_0^R\frac{x^2}{1+x^{12}}\int_{\gamma_0}f(z)\,dz$ we get

$\int_{\gamma_1}f(z)\,dz=-e^{i\pi/6}\int_0^R f(e^{i\pi/6})\,dt=-e^{i\pi/6}\int_0^R\frac{t^2e^{i\pi/3}}{1+t^{12}}\,dt=$

$-e^{i\pi/6}e^{i\pi/3}\int_0^R\frac{t^2}{1+t^{12}}\,dt=-e^{i\pi/2}I_R=-iI_R$

& by Z Cauchy's Residue thm:

$-\frac{\pi i(1+i)}{6\sqrt2}=I_R(1-i)+\int_{\gamma_R}f(z)\,dz$ ∴

$I_R=\frac{\pi i(1+i)}{6(1-i)\sqrt2}-\frac{1}{1-i}\int_{\gamma_R}f(z)\,dz$

now for z in Z contour γ_R have:

$|f(z)|\le\frac{|z|^2}{|z|^{12}-1}=\frac{R^2}{R^{12}-1}=\frac{R^2}{(R^6-1)(R^6+1)}\le\frac{R^2}{R^{12}-1}\le\frac{2R^2}{R^{12}}=\frac{2}{R^{10}}$

for sufficiently large R ∴ get: $\left|\int_{\gamma_R}f(z)\,dz\right|\le\frac{\pi R}{6}\times\frac{2}{R^{10}}=\frac{\pi}{3R^9}$

∴

$\therefore \int_0^\infty \frac{x^2}{1+x^{12}}\,dx = \lim_{R\to\infty} I_R = \lim_{R\to\infty} \frac{\pi i(1+i)}{6(1-i)\sqrt{2}} - \frac{1}{1-i}\int_{\gamma_R} f(z)\,dz$

$= \frac{\pi i(1+i)}{6(1-i)\sqrt{2}} + -\frac{1}{1-i}(0) = -\frac{\pi i(1+i)^2}{12\sqrt{2}} = \frac{\pi\sqrt{2}}{12}$ which is real

\Complex analy PP2020/

\1ai/ Z formula is $\int_\gamma f = \int_\gamma f(z)dz = \int_a^b f(\gamma(t))\cdot\gamma'(t)dt$

\1aii/

Z line segment can be expressed as

$\gamma(t) = 1+i+t[0-(1+i)] = 1-t+i(1-t)$, $0 \le t \le 1$

let $f(z) = x-y+ix^2$ have: $\gamma'(t) = -1-i$ &

$f(\gamma(t)) = 1-t-(1-t)+i(1-t)^2 = i(1-2t+t^2)$ ∴

$\int_\gamma x-y+ix^2 = \int_0^1 i(1-2t+t^2)(-1-i)dt = (-1-i)\int_0^1 (1-2t+t^2)dt =$

$(1-i)\left(t-t^2+\frac{t^3}{3}\right)\Big|_0^1 = \frac{1}{3}-\frac{i}{3}$

\1bi/ Z Cauchy Riemann eqns for a func $f(x+iy) = u(x,y)+iv(x,y)$ are $\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ & $\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$

\1bii/ first note: $f(z) = 4z^2-20z+25$

let $z = x+iy$ ∴ our func is

$f(z) = f(x+iy) = 4(x+iy)^2 - 20(x+iy)+25 =$

$4x^2-4y^2-20x+25+i(8xy-20y)$

set $u = 4x^2-4y^2-20x+25$ & $v = 8xy-20y$ ∴

$\frac{\partial u}{\partial x} = 8x-20$ $\frac{\partial v}{\partial y} = 8x-20$

$\frac{\partial u}{\partial y} = -8y$ $-\frac{\partial v}{\partial x} = -8y$

\1ci/ Def (open set) let $U \subset \mathbb{C}$ say Z set U is open if $\forall z \in U$, $\exists \varepsilon > 0$ (real number) st $D(z,\varepsilon) \subset U$

\1cii/ Def (closed set) a set $F \subseteq \mathbb{C}$ is called closed if Z complement of F is open, ie $CF = \{z \in \mathbb{C} : z \notin F\}$ is open

\1d/ let a_n be Z summand $\therefore \frac{a_{n+1}}{a_n} = \frac{(3n+3)(3n+2)(3n+1)}{(n+1)(2n+2)(2n+1)} z =$

$\frac{(3+\frac{3}{n})(3+\frac{2}{n})(3+\frac{1}{n})}{(1+\frac{1}{n})(2+\frac{2}{n})(2+\frac{1}{n})} z \quad \therefore \left|\frac{a_{n+1}}{a_n}\right| \to \frac{27}{4}|z|$ as $n \to \infty$

$\therefore$ Z series converges absolutely is $|z| < \frac{4}{27}$ & diverges is $|z| > \frac{4}{27}$ it sollows that Z radius of convergence is $\frac{4}{27}$

\1e/ is $|z|=2$ we approx ß by ~~$g_1(z) = z^5$~~ $g_1(z) = z^5$

$\therefore |ß(z) - g_1(z)| = |-6z^2+3z-1| \le 6|z|^2+3|z|+1 = 24+6+1 = 31$

where as $|g_1(z)| = |z|^5 = 32$ by Rouche's thm, ß & g_1 have Z same number of Zeros in $D(0,2)$. this is 5

is $|z|=1$ approx ß by $g_2(z) = -6z^2$ $\therefore$

$|ß(z) - g_2(z)| = |z^5 - 3z - 1| \le |z|^5 + 3|z| + 1 = 5$ but

$|g_2(z)| = 6|z|^2 = 6$ by Rouches thm ß & g_2 have Z same number of Zeros in $D(0,1)$ this is 2

it sollows that ß has $5-2=3$ zeros in Z annulus

\2a/ Z sundamental thm of contour integration is Z sollowing: let $D \subset \mathbb{C}$ be a domain, $\gamma : [a,b] \to \mathbb{C}$ be a contour contained in D. Let $ß : D \to \mathbb{C}$ be continuous sunc with an antiderivative F on D $\therefore$

$\int_\gamma ß(z)\,dz = F(\gamma(b)) - F(\gamma(a))$

\2b/ Z set D is close show: by Des of closed, must show Z complement: $CD = \{z \in \mathbb{C} : -1 < Re(z) < 1\}$ is open

let $z = x+iy \in CD$ $\therefore -1 < x < 1$ $\therefore 0 < x+1$ $\quad 0 < 1-x$

let $0 < \varepsilon < \min(x+1, 1-x)$ let $w = a+bi \in D(z,\varepsilon)$

note: $w - z = a - x + i(b-y)$ $\therefore |a-x| = |Re(w-z)| \le |w-z| < \varepsilon$

$\therefore |a-x| < \varepsilon < x+1 \quad |a-x| < \varepsilon < 1-x \quad \therefore -x-1 < a-x \therefore$

$-x-1 < a-x < 1-x \quad \therefore -1 < a < 1 \quad \therefore$

\Complex analy PP2020/ since $a = Re(w)$ & w is an arbit pt of $D(z,\varepsilon)$ have $D(z,\varepsilon) \subset CD$

as z is an arbit pt of CD, have Z desired result

\2c/ Z statement is false. counter example:
consider domain $U = D(0,2)\setminus\{0\}$ & since $f(z)=\frac{1}{z}$ is holomorphic on U
if $\gamma:[0,2\pi]\to\mathbb{C}$ is Z unit circlular contour then its image, Z unit circle, is completely in U ie $\gamma([0,2\pi]) \subset U$, but shown by direct computation:

$$\int_\gamma f(z)\,dz = 2\pi i \neq 0$$

\3a/ Z def of Z length of a smooth curve $\gamma:[a,b]\to\mathbb{C}$ is $L(\gamma)=\int_a^b |\gamma'(t)|\,dt$

note: $\gamma'(t)=16\pi i e^{i2\pi t}$ ∴ applying Z def:

$$L(\gamma)=\int_0^{1/4}|16\pi i e^{i2\pi t}|\,dt = 16\pi\int_0^{1/4}|e^{i2\pi t}|\,dt =$$

$$16\pi\int_0^{1/4}|\cos(2\pi t)+i\sin(2\pi t)|\,dt = 16\pi\int_0^{1/4}\sqrt{\cos(2\pi t)^2+\sin(2\pi t)^2}\,dt =$$

$$16\pi\int_0^{1/4} dt = 4\pi$$

\3b/ write expression in partial frac form:

$$f(z)=\frac{1}{(z+i)(z-5i)} = \frac{i}{6}\left(\frac{1}{z+i}\right) - \frac{i}{6}\left(\frac{1}{z-5i}\right)$$

for $|z|<5$ have: $\frac{1}{z-5i} = -\frac{1}{5i(1-\frac{z}{5i})} = -\frac{1}{5i}\sum_{n=0}^{\infty}\left(\frac{z}{5i}\right)^n =$

$$-\sum_{n=0}^{\infty}\frac{z^n}{(5i)^{n+1}}$$

on Z other hand if $|z|>1$ write $y=\frac{1}{z}$ ∴

$$\frac{1}{z+i} = \frac{y}{1+iy} = y\,\frac{1}{1-(-iy)} = y\sum_{n=0}^{\infty}(-i)^n y^n = \frac{1}{z}\sum_{n=0}^{\infty}(-i)^n z^{-n} = \sum_{n=0}^{\infty}(-i)^n z^{-n-1} =$$

$$\sum_{n=0}^{\infty}(-i)^n z^{-(n+1)} = \sum_{m=-\infty}^{-1}(-i)^{m+1} z^m = \sum_{m=-\infty}^{-1}(-1)^{m+1} i^{m+1} z^m = \sum_{m=-\infty}^{-1} i^{m+1} z^m$$

noting: the series converges as $|y|<1$ $\therefore$

$$f(z) = \frac{i}{6}\sum_{m=-\infty}^{-1} i^{m+1} z^m + \frac{i}{6}\sum_{n=0}^{\infty}\frac{z^n}{(5i)^{n+1}}$$ that is,

$$f(z) = -\frac{1}{6}\sum_{m=-\infty}^{-1} i^m z^m + \frac{1}{30}\sum_{n=0}^{\infty}\left(\frac{-i}{5}\right)^n z^n$$

3c/ use the slice-of-pie method: fix a real number $R>0$ & take a contour $(\gamma_1, \gamma_2, \gamma_3^{-1})$, where $\gamma_1(t)=t$ & $\gamma_3(t) = te^{2i\pi/38}$ for $0 \le t \le R$; & $\gamma_2(t) = Re^{it}$ for $0 \le t \le 2\pi/38$

let $f(z) = z^{10}/(1+z^{38})$ then only singularity of f within the contour is at $z = e^{i\pi/38}$, where the residue is

$$\frac{(e^{i\pi/38})^{10}}{38e^{37i\pi/38}} = \frac{1}{38}e^{-27i\pi/38}$$ $\therefore$ the integral on the contour is $\frac{2\pi i}{38}e^{-27i\pi/38}$

now: $$\int_{\gamma_3} f(z)\,dz = \int_0^R \frac{t^{10}e^{20i\pi/38}}{1+t^{38}} e^{2i\pi/38}\,dt = e^{22i\pi/38}\int_0^R \frac{t^{10}}{1+t^{38}}\,dt$$

the integral on γ_2 may be estimated via the M-L inequality first note: the length of the contour is $2\pi R/38$ $\therefore$

$$\left|\int_{\gamma_2}\frac{z^{10}}{1+z^{38}}\right| \le \frac{2\pi R}{38}\frac{R^{10}}{R^{38}-1}$$ if $R>1$ note: this quantity tends to 0 as $R\to\infty$ $\therefore$

$$\frac{2\pi i}{38}e^{-27i\pi/38} = \left(1-e^{22i\pi/38}\right)\int_0^R \frac{t^{10}}{1+t^{38}}\,dt + \int_{\gamma_2} f(z)\,dz \rightarrow$$

$$\left(1-e^{22i\pi/38}\right)\int_0^{\infty}\frac{t^{10}}{1+t^{38}}\,dt$$ as $R\to\infty$ $\therefore$

$$\int_0^{\infty}\frac{t^{10}}{1+t^{38}}\,dt = \frac{2\pi i}{38}\frac{e^{-27i\pi/38}}{1-e^{22i\pi/38}} = \frac{\pi}{38}\frac{2i}{e^{27i\pi/38}-e^{49i\pi/38}} =$$

Complex analy RP2020 / $\frac{\pi}{38}\frac{2i}{e^{27i\pi/38}-e^{-27i\pi/38}} =$

$$\frac{\pi}{38}\left(\frac{e^{27i\pi/38}-e^{-27i\pi/38}}{2i}\right)^{-1} = \frac{\pi}{38}\left(\sin\left(\frac{27\pi}{38}\right)\right)^{-1} =$$

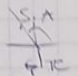
$$\frac{\pi}{38}\csc\left(\frac{27\pi}{38}\right)$$

4a / $f(z)$ has poles when $z = i\pi$ & when $e^z+1=0$

$\therefore$ there are simple poles at $z = i\pi(1+2n)$ for $n \neq 0 \in \mathbb{Z}$

& a triple pole at $z = i\pi$

$\{ e^z+1=0 \quad \therefore e^z = -1 = e^{i\theta} \quad \therefore \tan^{-1}\left(\frac{0}{-1}\right)=0 \quad \therefore$

$y=0, \ x=-1 \quad \therefore$   $\therefore \theta = \pi \}$

write $f(z) = h(z)/k(z)$ where $h(z) = 1/(z-i\pi)^2$ & $k(z) = e^z+1$

for $n \neq 0$ $h(z)$ is holomorphic near $(2n+1)i\pi$ &

$h((2n+1)i\pi) \neq 0$ Z since $k(z)$ has a simple pole

at $(2n+1)i\pi$ so Z residue of f at $(2n+1)i\pi$ is

$$\frac{h((2n+1)i\pi)}{k'((2n+1)i\pi)} = \frac{1}{(2ni\pi)^2 e^{(2n+1)i\pi}} = \frac{1}{4n^2\pi^2}$$

4b / begin by finding Z laurent expansion of $1/(e^z+1)$

about $z = i\pi$ as observed this func has a simple

pole at $z = i\pi$ $\therefore$ may write $1/(e^z+1) = g(z)$ where

$g(z) = \frac{a_{-1}}{w} + a_0 + a_1 w + a_2 w^2 + \cdots$, where $w = z - i\pi$ $\therefore$

$$f(z) = \frac{g(z)}{w^2} = \frac{a_{-1}}{w^3} + \frac{a_0}{w^2} + \frac{a_1}{w} + \cdots \quad \therefore$$

$b_n = a_{n+2}$ $\forall n \geq -3 \in \mathbb{Z}$ in particular $b_{-2} = a_0$ & $b_{-1} = a_1$

now $e^z+1 = \cancel{e^{i\pi}}\, e^{z-i\pi}e^{i\pi}+1 = 1-e^w$ has taylor expansion

$-w - \frac{w^2}{2!} - \frac{w^3}{3!} - \cdots$ $\therefore$ writing $w = z - i\pi$ have:

$$1 = \left(\frac{a_{-1}}{w} + a_0 + a_1 w + a_2 w^2 + \cdots\right)\left(-w - \frac{w^2}{2!} - \frac{w^3}{3!} - \cdots\right)$$

expands to:

$$\cancel{-a_{-1}+(} \ -a_{-1} - \left(\frac{a_{-1}}{2!} + a_0\right)w - \left(\frac{a_{-1}}{3!} + \frac{a_0}{2!} + a_1\right)w^2 - \cdots$$

but this expression must be identically 1 $\therefore$

$a_{-1} = -1$ & the coeffs of all the other powers of w

vanish $\therefore$

$b_{-2} = a_0 = -\frac{a_{-1}}{2!} = \frac{1}{2}$ &

$b_{-1} = a_1 = -\left(\frac{a_{-1}}{3!} + \frac{a_0}{2!}\right) = -\left(-\frac{1}{6} + \frac{1}{4}\right) = -\frac{1}{12}$

thus is $b_{-1} = -\frac{1}{12}$ & $b_{-2} = \frac{1}{2}$

4C) the poles within the contour occur at $z = i\pi$ & $z = -i\pi$

now $Res_{z=-i\pi} f(z) = \frac{1}{4\pi^2(-1)^2} = \frac{1}{4\pi^2}$

& $Res_{z=i\pi} f(z) = b_{-1} = -\frac{1}{12}$ {since triple pole $\therefore$ b_{-3}, b_{-2}, b_{-1} there}

$\therefore$ the integral on γ can be computed via the residue

thm to be: $2\pi i\left(-\frac{1}{12} + \frac{1}{4\pi^2}\right) = \frac{\pi i}{2}\left(\frac{1}{\pi^2} - \frac{1}{3}\right)$

1a/ ~~note |z+w~~ let $z, \omega \in \mathbb{C}$ ∴

$|z+\omega| \le |z|+|\omega|$ and $|z-\omega| \ge |z|-|\omega|$

∴ $|z-w+w| \le |z-w|+|w|$ ∴ $|z-w| \le |z-w|$

and $|z-w-z| \le |z-w|+|-z|$ ∴ $|w|-|z| \le |z-w|$ ∴

~~thus~~ $|(|z|-|w|)| \le |z-w|$ ∴

$||z|-|w|| \le |z-w|$ as required