MTH2008 Real Analysis

Coursework 2

1 / note: $\frac{n}{n+\sqrt{n}} \div \frac{n}{n} = \frac{\left(\frac{n}{n}\right)}{\left(\frac{n+\sqrt{n}}{n}\right)} =$

$$\frac{(1)}{\left(\frac{n}{n}\right)+\left(\frac{n^{1/2}}{n}\right)} = \frac{1}{1+\left(\frac{1}{\sqrt{n}}\right)}$$

$\therefore \lim_{n\to\infty}\left(\frac{1}{1+\left(\frac{1}{\sqrt{n}}\right)}\right) = \frac{1}{1+0} = 1$ for $n \in \mathbb{N}$

So for $\varepsilon > 0$: $|S_n - 1| = \left|\frac{n}{n+\sqrt{n}} - 1\right|$

$$= \left|\frac{n}{n+\sqrt{n}} - \frac{n+\sqrt{n}}{n+\sqrt{n}}\right| = \left|\frac{n-n-\sqrt{n}}{n+\sqrt{n}}\right| = \left|\frac{-\sqrt{n}}{n+\sqrt{n}}\right|$$

$$= \left|\frac{\sqrt{n}}{n+\sqrt{n}}\right| \cancel{\le \frac{\sqrt{N}}{N+\sqrt{N}}} = \frac{+\sqrt{n}}{n+\sqrt{n}} \le \frac{\sqrt{N}}{N+\sqrt{N}}$$

$$\cancel{\frac{\sqrt{N}}{N+\sqrt{N}} < \varepsilon \Rightarrow \sqrt{N} < \varepsilon N + \varepsilon\sqrt{N}}$$

$$\cancel{\frac{\sqrt{N}}{N+\sqrt{N}} < \sqrt{\varepsilon} \quad \frac{N}{N^2+N+2N^{3/2}} < \varepsilon}$$

$\Big\{$ if $\frac{\sqrt{N}}{N+\sqrt{N}} < \frac{\sqrt{\varepsilon}}{\varepsilon+\sqrt{\varepsilon}} \Rightarrow \varepsilon\sqrt{N} + \sqrt{\varepsilon}\sqrt{N} < \sqrt{\varepsilon}N + \sqrt{\varepsilon}\sqrt{N}$

$\Rightarrow \varepsilon\sqrt{N} < \sqrt{\varepsilon}N \Rightarrow \frac{\varepsilon}{\sqrt{\varepsilon}} < \frac{N}{\sqrt{N}} \Rightarrow \sqrt{\varepsilon} < \sqrt{N}$

$\Rightarrow \varepsilon < N \Big\}$

$\therefore \frac{\sqrt{N}}{N+\sqrt{N}} < \frac{\sqrt{\varepsilon}}{\varepsilon+\sqrt{\varepsilon}} < \varepsilon \quad \therefore |S_n - 1| < \varepsilon$

for $n \ge N$ for $N \in \mathbb{N}$ with $N > \varepsilon$

$\therefore \lim_{n\to\infty} S_n = 1$

~~2/ note $\sin\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} = \sin\left(\frac{3\pi}{4}\right)$~~

~~and $\sin\left(\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} = \sin\left(\frac{7\pi}{4}\right)$~~

~~and $\sin(x) = \sin(x+2\pi) = \sin(x+2\pi l)$ for all $l \in \mathbb{Z}$~~

~~$\therefore \sin\left(\frac{\pi}{4}\right) = \sin\left(\frac{\pi}{4}+2\pi l\right) = \frac{1}{\sqrt{2}} = \sin\left(\frac{3\pi}{4}+2\pi l\right) = \sin\left(\frac{3\pi}{4}\right)$~~

~~and $\sin\left(\frac{5\pi}{4}\right) = \sin\left(\frac{5\pi}{4}+2\pi l\right) = -\frac{1}{\sqrt{2}} = \sin\left(\frac{7\pi}{4}+2\pi l\right) = \sin\left(\frac{7\pi}{4}\right)$~~

~~$\forall l \in \mathbb{Z}$~~

~~$\therefore \forall x \in \left(\frac{\pi}{4}, \frac{3\pi}{4}\right) : \sin(x) > \frac{1}{\sqrt{2}}$~~

~~$\therefore \sin\left(\frac{2\pi}{4}\right) = \sin\left(\frac{\pi}{2}\right) > \frac{1}{\sqrt{2}}$~~

2/ for degrees note: $\sin(45) = \frac{1}{\sqrt{2}} = \sin(135)$

and $\sin(225) = -\frac{1}{\sqrt{2}} = \sin(315)$

and $\forall x \in \mathbb{R} : \sin(x+360) = \sin(x)$ ~~$= \sin(x+360$~~

$\therefore \sin(x) = \sin(x+360l) \quad \forall l \in \mathbb{Z}$

~~$\therefore \sin(\mathbb{Z}$~~

$\therefore \sin(45) = \sin(45+360l) = \frac{1}{\sqrt{2}} = \sin(135) = \sin(135+360l)$

and $\sin(225) = \sin(225+360l) = -\frac{1}{\sqrt{2}} = \sin(315) = \sin(315+360l)$

$\therefore \forall x \in [45, 135] : \sin(x) \geq \frac{1}{\sqrt{2}} \quad \therefore$

$\sin(90) \geq \frac{1}{\sqrt{2}} \quad \therefore \sin(90+360l) \geq \frac{1}{\sqrt{2}}$

and ~~$\forall x \in [225, 3$~~ $\forall x \in [225, 315] \leq -\frac{1}{\sqrt{2}}$

$\therefore \sin(270) \leq -\frac{1}{\sqrt{2}} \quad \therefore \sin(270+360l) \leq -\frac{1}{\sqrt{2}}$

since $\sin(90) = 1$ and $\sin(270) = -1$

③

$\therefore$ for degrees let sequence $S_n = \sin(n)$
for $n \in \mathbb{N}$
have subsequence $\sin(n_k) = S_{n_k}$
for $k \in \mathbb{N}$ and $n_k = 90 + 360k$
$\therefore$ ~~sl~~ $S_{n_k} = \sin(n_k) = \cancel{\sin(90+360)\geq}$
$\sin(90+360k) \geq \frac{1}{\sqrt{2}}$ $\forall k \in \mathbb{N}$

and also let S_n have a subsequence
$\sin(n_j) = S_{n_j}$ for $j \in \mathbb{N}$ and $270 + 360j = n_j$

$\therefore S_{n_j} = \sin(n_j) = \sin(270 + 360j) \leq -\frac{1}{\sqrt{2}}$

$\therefore$ for $\varepsilon > 0$ $\exists N_1 \in \mathbb{N}$ suchthat ~~$\forall N_1$~~ $\forall a_1, b_1 \geq N_1$
and ~~$a, b \in$~~ ~~$a \in n_k, b \in n_{k_2} \Rightarrow$~~ $a_1 = n_{k_1}$, $b_1 = n_{k_2} \Rightarrow$
$|S_{a_1} - S_{b_1}| = |\sin(a_1) - \sin(b_1)| \cancel{<1}$

$= |\sin(n_{k_1}) - \sin(n_{k_2})| =$
~~$|\sin(90+360k_1) - \sin(90$~~
$= |\sin(90+360k_1) - \sin(90+360k_2)|$ $\{k_1, k_2 \in k\}$
$= |\sin(90) - \sin(90)| = |0| = 0 < \varepsilon$

$\therefore$ the subsequence ~~n_k~~ S_{n_k} of S_n converges

now for $\varepsilon > 0$ $\exists N_2 \in \mathbb{N}$ suchthat $\forall a_2, b_2 \geq N_2$
and $a_2 = n_{j_1}$, $b_2 = n_{j_2} \Rightarrow$
$|S_{a_2} - S_{b_2}| = |\sin(a_2) - \sin(b_2)|$

$= |\sin(n_{j_1}) - \sin(n_{j_2})| =$
$|\sin(270+360j_1) - \sin(270+360j_2)|$

$= |\sin(270) - \sin(270)| = |0| = 0 < \varepsilon$
$\therefore$ the subsequence S_{n_j} of S_n converges

but since ~~sin(270~~
$\sin(90+360k) \geq \frac{1}{\sqrt{2}}$

the sequence S_{n_k} converging means it must converge only to a value bigger than $\frac{1}{\sqrt{2}}$

and since $\sin(270+360j) \leq -\frac{1}{\sqrt{2}}$
the sequence S_{n_j} converging means it must converge only to a value smaller than $-\frac{1}{\sqrt{2}}$

$\therefore$ S_{n_k} and S_{n_j} converge to different values

with n_j and n_k being different ~~subsequences~~ subsequences of n $\therefore$ sequence S_n is not

Cauchy $\therefore$
sequence S_n does not fit the Cauchy convergence criterion for sequences $\therefore$
S_n does not converge for $S_n = \sin(n)$
for $n \in \mathbb{N}$ and n is an angle in degress

but because an angle x in degress is $(x)(\frac{\pi}{180})$ in radians which is a linear translation $\therefore$
$S_n = \sin(n)$ also does not converge for $n \in \mathbb{N}$ and n is an angle in radians
$\therefore$ $S_n = \sin(n)$ diverges ~~QED~~
since the sin function is the same in both degrees and radians

3/ $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \sin\left(\frac{x}{n}\right) = \sin(0) = 0 = f(x)$

$\cancel{\lim_{k\to\infty} f_n}$ $\therefore f(x) = 0$ is the pointwise limit of $f_n(x)$ for $n \in \mathbb{N}$, $\forall x \in \mathbb{R}$

So if this is true then for $\varepsilon > 0$:

$|f_n(x) - f(x)| = \left|\sin\left(\frac{x}{n}\right) - 0\right| = \left|\sin\left(\frac{x}{n}\right)\right| \leq 1$

$\therefore 0 \leq \left|\sin\left(\frac{x}{n}\right)\right| \leq 1$

and since $\sin(0) = 0$ and $\cancel{\sin(x)}$ $\sin\left(\frac{\pi}{2}\right) = 1$ and for $x \in [0, 1]$: $\sin(x)$ is a non-decreasing function

$\therefore \cancel{|\sin(\frac{x}{n}}$

but also note $\|f_n(x) - f(x)\|_{\mathbb{R}}$

$= \cancel{\|f_n(x)} = \left\|\sin\left(\frac{x}{n}\right) - 0\right\|_{\mathbb{R}} = \left\|\sin\left(\frac{x}{n}\right)\right\|_{\mathbb{R}}$

$= \sup_{x\in\mathbb{R}} \left|\sin\left(\frac{x}{n}\right)\right|$

$\therefore$ since $\left|\frac{x}{n} - 0\right| = \left|\frac{x}{n}\right| = \frac{|x|}{n} \leq \frac{|x|}{N} < \varepsilon$

for $\varepsilon > 0$ and $\cancel{\forall}$ $N \in \mathbb{N}$ and $n \geq N$ with $\frac{|x|}{\varepsilon} < N$

$\therefore \lim_{n\to\infty} \left(\frac{x}{n}\right) = 0$

$\therefore \lim_{n\to\infty} \left(\sin\left(\frac{x}{n}\right)\right) = \cancel{\lim_{n\to\infty}} \sin\left(\lim_{n\to\infty}\left(\frac{x}{n}\right)\right) = \sin(0) = 0$

$\therefore \lim_{n\to\infty} \left|\sin\left(\frac{x}{n}\right)\right| = |0| = 0$

$\therefore \lim_{n\to\infty} \sup_{x\in\mathbb{R}} \left|\sin\left(\frac{x}{n}\right)\right| = \sup_{x\in\mathbb{R}} \left(\lim_{n\to\infty} \left|\sin\left(\frac{x}{n}\right)\right|\right)$

$= \sup_{x\in\mathbb{R}} (0) = 0 < \varepsilon \therefore \left\|\sin\left(\frac{x}{n}\right) - 0\right\|_{\mathbb{R}} < \varepsilon$

$\therefore \|f_n(x) - f(x)\|_{\mathbb{R}} < \varepsilon$

for $f(x) = 0$ $\therefore$ $f_n(x)$ converges uniformly to $f(x) = 0$

$\therefore$ Since $f_n(x)$ has uniform convergence it also has pointwise convergence

$\therefore$

$f_n(x)$ converges pointwise to $f(x) = 0$

(7)

4/ $\frac{1-(-1)}{n} = \frac{2}{n}$ is width $\therefore \Delta x_i = \frac{2}{n}$

for $n \in \mathbb{N}$, $i \in \mathbb{N}$ and $i \leq N$

for uniform partition

$P_n = (-1, -1+\frac{2}{n}, \ldots, -1+\frac{2}{n}i, \ldots, 1) = (x_0, \ldots, x_i, \ldots, x_n)$

so $x_0 = -1$, $x_n = 1$, $x_i = -1+\frac{2}{n}i$, $x_{i-1} = -1+\frac{2}{n}(i-1)$

note also $f'(x) = \frac{d}{dx}(2x+1) = 2 > 0 \ \forall x \in \mathbb{R}$

$\therefore f(x)$ is an increasing function

$\therefore f(x_i) > f(x_{i-1})$

$\therefore$ for $x \in [x_{i-1}, x_i]$: $\sup_{x \in [x_{i-1}, x_i]} f(x) = f(x_i)$

and $\inf_{x \in [x_{i-1}, x_i]} f(x) = f(x_{i-1})$

$\therefore$ have Lower Riemann Sum:

$L(f; P_n) = \sum_{i=1}^{n} \left(\inf_{x \in [x_{i-1}, x_i]} (f(x)) \Delta x_i \right) =$

$\Delta x_i \sum_{i=1}^{n} \left(\inf_{x \in [x_{i-1}, x_i]} f(x) \right) = \frac{2}{n} \sum_{i=1}^{n} f(x_{i-1})$

$= \frac{2}{n} \sum_{i=1}^{n} f(-1+\frac{2}{n}(i-1)) = \frac{2}{n} \sum_{i=1}^{n} (2(-1+\frac{2}{n}(i-1))+1) = L(f; P_n)$

and $U(f; P_n) = \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1}, x_i]} (f(x)) \Delta x_i \right)$

$= \Delta x_i \sum_{i=1}^{n} \left(\sup_{x \in [x_{i-1}, x_i]} f(x) \right) = \frac{2}{n} \sum_{i=1}^{n} f(x_i) = \frac{2}{n} \sum_{i=1}^{n} f(-1+\frac{2}{n}i)$

~~$= U(f; P_n = 2$~~

$= \frac{2}{n} \sum_{i=1}^{n} (2(-1+\frac{2}{n}i)+1) = U(f; P_n) \therefore$

$U(f; P_n) - L(f; P_n) =$

$\frac{2}{n} \sum_{i=1}^{n} (2(-1+\frac{2}{n}i)+1) - \frac{2}{n} \sum_{i=1}^{n} (2(-1+\frac{2}{n}(i-1))+1)$

$$= \frac{2}{n}\sum_{i=1}^{n}\left(2\left(-1+\frac{2}{n}i\right)+1-\left(2\left(-1+\frac{2}{n}(i-1)\right)+1\right)\right)$$

$$\cancel{= \frac{2}{n}\sum_{i=1}^{n}(2}$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left(-2+\frac{4}{n}i+1-\left(-2+\frac{4}{n}(i-1)+1\right)\right)$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left(-2+\frac{4}{n}i+1+2-\frac{4}{n}(i-1)-1\right)$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left(\frac{4}{n}i-\frac{4}{n}(i-1)+(-2+2-1+1)\right)$$

$$= \frac{2}{n}\sum_{i=1}^{n}\left(\frac{4}{n}(i-(i-1)\right) =$$

$$\frac{8}{n^2}\sum_{i=1}^{n}(i-(i-1)) \quad \text{(telescopic sum)}$$

$$= \frac{8}{n^2}\left((1-(1-1))+(2-(2-1))+(\cdots)+(n-(n-1))\right)$$

$$= \frac{8}{n^2}\left((1-0)+(2-1)+\cdots+(n-(n-1))\right)$$

$$= \frac{8}{n^2}\left(-0+(1-1)+(2-2)+\cdots+(n-1-(n-1))+n\right)$$

$$= \frac{8}{n^2}(-0+n) = \frac{8}{n^2}(n) = \frac{8}{n} = U(f;P_n)-L(f;P_n)$$

$\therefore$ for $\varepsilon > 0$, $\exists$ $N \in \mathbb{N}$:

$$|U(f;P_n)-L(f;P_n)| = \left|\frac{8}{n}\right| = 8\frac{1}{n} \leq 8\frac{1}{N} < \varepsilon$$

for $n \geq N$ with $N > 8\frac{1}{\varepsilon}$. So the Riemann intergrability criterion is satisfied $\therefore$ $f(x)$ is Riemann integrable and $U(f;P_n) = \frac{2}{n}\sum_{i=1}^{n}\left(2\left(-1+\frac{2}{n}i\right)+1\right)$

$= \frac{2}{n}\sum_{i=1}^{n}\left(-2+\frac{4}{n}i+1\right) = \frac{2}{n}\sum_{i=1}^{n}\left(-1+\frac{4}{n}i\right)$

$= \frac{2}{n}\sum_{i=1}^{n}(-1) + \frac{2}{n}\sum_{i=1}^{n}\left(\frac{4}{n}i\right) = -\frac{2}{n}\sum_{i=1}^{n}(1) + \frac{8}{n^2}\sum_{i=1}^{n} i$

$= -\frac{2}{n}(n) + \frac{8}{n^2}\left(\frac{1}{2}\right)(n)(n+1) = -2 + \frac{4}{n}(n+1)$

$= -2+4+\frac{4}{n} = \frac{4}{n}+2 = U(f;P_n)$ and

$L(f;P_n) = \frac{2}{n}\sum_{i=1}^{n}\left(2\left(-1+\frac{2}{n}(i-1)\right)+1\right)$

$= \frac{2}{n}\sum_{i=1}^{n}\left(-2+\frac{4}{n}(i-1)+1\right) = \frac{2}{n}\sum_{i=1}^{n}\left(-1+\frac{4}{n}i-\frac{4}{n}\right)$

$= \frac{2}{n}\sum_{i=1}^{n}(-1) + \frac{2}{n}\sum_{i=1}^{n}\left(\frac{4}{n}i\right) + \frac{2}{n}\sum_{i=1}^{n}\left(-\frac{4}{n}\right)$

$= -\frac{2}{n}\sum_{i=1}^{n}(1) + \frac{8}{n^2}\sum_{i=1}^{n} i + \frac{2}{n}\left(-\frac{4}{n}\right)\sum_{i=1}^{n}(1) =$

$-\frac{2}{n}(n) + \frac{8}{n^2}\left(\frac{1}{2}\right)(n)(n+1) - \frac{8}{n^2}(n)$

$= -2 + \frac{4}{n}(n+1) - 8\frac{1}{n} = -2+4+4\frac{1}{n}-8\frac{1}{n}$

$= -4\frac{1}{n}+2 = L(f;P_n)$

and $\underline{\int_{-1}^{1}} f(x)\,dx = \sup_P L(f;P) \ge \sup_n L(f;P_n)$

$= \sup_n\left(-4\frac{1}{n}+2\right) = \lim_{n\to\infty}\left(-4\frac{1}{n}+2\right) = 0+2 = 2$

$\therefore \underline{\int_{-1}^{1}} f(x)\,dx \ge 2$

and $\overline{\int_{-1}^{1}} f(x)\,dx = \inf_P U(f;P) \le \inf_n U(f;P_n)$

$= \inf_n\left(\frac{4}{n}+2\right) = \lim_{n\to\infty}\left(\frac{4}{n}+2\right) = 0+2 = 2$

$\therefore \overline{\int_{-1}^{1}} f(x)\,dx \le 2$

but $\underline{\int_{-1}^{1}} f(x)\,dx \le \overline{\int_{-1}^{1}} f(x)\,dx$

$\therefore \quad 2 \le \underline{\int_{-1}^{1}} f(x)\,dx \le \overline{\int_{-1}^{1}} f(x)\,dx \le 2$

$\therefore \quad \underline{\int_{-1}^{1}} f(x)\,dx = \overline{\int_{-1}^{1}} f(x)\,dx = 2 = \int_{-1}^{1} f(x)\,dx$

$= \int_{-1}^{1} (2x+1)\,dx = 2$

5/ f and g are Riemann integral

So $\underline{\int_a^b} f(x)dx = \overline{\int_a^b} f(x)dx$ and $\underline{\int_a^b} g(x)dx = \overline{\int_a^b} g(x)dx$

and $(b-a)\inf_{x\in[a,b]} f(x) \le \int_a^b f(x)dx \le (b-a)\sup_{x\in[a,b]} f(x)$

and ~~(b-a) inf g~~

$(b-a)\inf_{x\in[a,b]} g(x) \le \int_a^b g(x)dx \le (b-a)\sup_{x\in[a,b]} g(x)$ and

$(b-a)\inf_{x\in[a,b]} (f(x)-g(x)) \le \int_a^b f(x)-g(x)dx \le (b-a)\sup_{x\in[a,b]} (f(x)-g(x))$

So let $h(x) = f(x) - g(x)$ $\therefore$ ~~$h(x)$~~ $h:[a,b] \to \mathbb{R}$

and ~~for $x_0\in[a,b]$ $f(x)$~~ $f(x)-g(x) = 0$ $\forall x\in[a,b]$

except one point $x_0\in[a,b]$ where $f(x_0)-g(x_0) > 0$

$\therefore$ ~~$h(x) \ge$~~ So let for arbitrary value ~~$\beta\in\mathbb{R}_{>0}$~~

~~$f(x_0) - g(x_0) = \beta$~~ $\gamma\in\mathbb{R}_{>0}$: $f(x_0)-g(x_0) = \gamma > 0$

$$\therefore h(x) = \begin{cases} \gamma, & x = x_0, \\ 0, & x \ne x_0. \end{cases}$$

for partition $P_k = (a, x_0 - \frac{1}{2^k}, x_0 + \frac{1}{2^k}, b)$ $k\in\mathbb{N}$

$\therefore L(f;P_k) =$

$(x_0 - \frac{1}{2^k} - a)(0) + (x_0 + \frac{1}{2^k} - (x_0 - \frac{1}{2^k}))(0) + (b - (x_0 + \frac{1}{2^k}))(0)$

$= 0 + 0 + 0 = 0 = L(f;P_k)$ and

$U(f;P_k) =$

$(x_0 - \frac{1}{2^k} - a)(0) + (x_0 + \frac{1}{2^k} - (x_0 - \frac{1}{2^k}))(\gamma) + (b - (x_0 + \frac{1}{2^k}))(0)$

$= 0 + \left(x_0 + \frac{1}{2^k} - x_0 + \frac{1}{2^k}\right)(\gamma) + 0$

$= \left(\frac{1}{2^k} + \frac{1}{2^k}\right)(\gamma) = \left(\frac{1}{2^{k-1}}\right)(\gamma) = U(\delta; P_k)$

$\therefore \underline{\int_a^b} h(x)\,dx = \sup_P L(\delta; P) \geq \sup_k L(\delta; P_k)$

$= \sup_k (0) = 0 \quad \therefore \quad \underline{\int_a^b} h(x)\,dx \geq 0$ and

$\overline{\int_a^b} h(x)\,dx = \inf_P U(\delta; P) \leq \inf_k U(\delta; P_k)$

$= \inf_k \left(\left(\frac{1}{2^{k-1}}\right)(\gamma)\right) = \lim_{k \to \infty} \left(\left(\frac{1}{2^{k-1}}\right)(\gamma)\right)$

$= \gamma \lim_{k \to \infty} \left(\frac{1}{2^{k-1}}\right) = \gamma(0) = 0 \quad \therefore$

$\overline{\int_a^b} h(x)\,dx \leq 0$ but $\underline{\int_a^b} h(x)\,dx \leq \overline{\int_a^b} h(x)\,dx$

$\therefore 0 \leq \underline{\int_a^b} h(x) \leq \overline{\int_a^b} h(x)\,dx \leq 0 \quad \therefore$

$\underline{\int_a^b} h(x) = \overline{\int_a^b} h(x)\,dx = 0 = \int_a^b h(x)\,dx$

but $h(x) = \delta(x) - g(x) \therefore$

$\int_a^b h(x)\,dx = \int_a^b (\delta(x) - g(x))\,dx = 0 \quad \therefore$

$\int_a^b \delta(x)\,dx - \int_a^b g(x)\,dx = 0$

$\therefore \int_a^b \delta(x)\,dx = \int_a^b g(x)\,dx$

as required