2008 Jan exams each week new slides check box if youre done
if some need to be done in order
teams MTH2008 class note book

{ Study injective
Surjective }

- one note has notebook

$5 \in X$ is false of $X = \{1, 2, -8, y, \{5,6\}, apple\}$

$\{-8\} \in X$ of $X = \{1, 2, -8, y, \{5,6\}, apple\}$ is true ✗ (apple

if $A \Rightarrow B$ then converse is $\neg B \Rightarrow \neg A$ ✗ $B \Rightarrow A$ is the converse

$A \Rightarrow B$ contrapositive $\neg A \Rightarrow \neg B$

$X = \{1, 2, -8, y, \{5,6\}, apple\}$ $5 \notin X$

$A \Rightarrow B$ negation $B \Rightarrow A$ ✗ $A \wedge \neg B$ since $A \wedge B \Leftrightarrow A \Rightarrow B$??

$A_n = [0, 1/n)$ $n \in \mathbb{N}$ $\frac{1}{10} < 1$ $[0, 1]$ $\frac{1}{1} = 1 \in \mathbb{N}$ so $[0, 1]$ ✗

$\bigcup [0, 1/n) \rightarrow [0, 1]$ sets $[0, 1/2)$, $[0, 1/3)$, $[0, 1/4)$ and joining all these intervals ~~[0,1)~~

$A_n = [0, 1/n)$ $n \in \mathbb{N}$ $\bigcap_{n=1}^{\infty} A_n$ so $[0, 1/2) \cap [0, 1/3) \cap [0, 1/4) \ldots \rightarrow \{0\}$

$f: \mathbb{R} \rightarrow \mathbb{R}$ $f(x) = x^2$ not injective neither is $f(x) = \arctan(x)$

$f(x) = x\sin(x)$ is injective ✗ $f(x) = \arctan(x)$ is injective

$f(x) = x\sin(x)$ is surjective $f(x_1), f(x_2)$

$f(x) = x^3$ is bijective since it is injective and surjective

surjective $\Rightarrow$ {every y has atleast one x

$$a_n = \frac{n(3+n)}{n(1+4n)} = \frac{(3+n)}{1+4n} = \frac{3}{1+4n} + \frac{n}{1+4n} \lim_{n\to\infty} \Rightarrow 0 + \frac{n}{1+4n}$$

Lopitals $\frac{1}{0+4} = \frac{1}{4}$

$$\frac{3n+n^2}{n+4n^2} = \frac{n^2\left(\frac{3}{n}+1\right)}{n^2\left(\frac{1}{n}+4\right)} = \frac{\left(\frac{3}{n}+1\right)}{\left(\frac{1}{n}+4\right)} = \frac{\left(\frac{3}{n}\right)}{\left(\frac{1}{n}+4\right)} + \frac{1}{\left(\frac{1}{n}+4\right)}$$

$$\frac{\left(\frac{3}{n}+1\right)}{\left(\frac{1}{n}+4\right)} \lim_{n\to\infty} \Rightarrow \frac{\left(\frac{3}{\infty}+1\right)}{\left(\frac{1}{\infty}+4\right)} = \frac{0+1}{0+4} = \frac{1}{4}$$

limit of a seque existing is unique

if exists $\lim_{n\to\infty}(a_n + b_n) = \lim_{n\to\infty} a_n + \lim_{n\to\infty} b_n$ is true if convergent
untrue if one of them diverges

is converges then it is bounded

is $\lim_{n\to\infty} a_n = 0$ then $\sum_{n=1}^{\infty} a_n$ converges is false since $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

is $\lim_{n\to\infty} a_n = 0$ to converge

$\lim_{n\to\infty} \frac{1}{n!} = 0$ $\quad \lim_{n\to\infty} \frac{1}{\pi^n} = 0 \; \{\pi > 1\}$

$\lim_{n\to\infty} \frac{(-1)^n}{n} = 0$ $\quad \lim_{n\to\infty} \frac{1+n^2}{n+2n^2} = \lim_{n\to\infty} \frac{n^2(\frac{1}{n^2}+1)}{n^2(\frac{1}{n}+2)} = \lim_{n\to\infty} \frac{\frac{1}{n^2}+1}{\frac{1}{n}+2} = \frac{1}{2} \neq 0$

so it as a sequence doesnt converge

$\lim_{x\to a} f(x) = f(a) \; \forall a$ is f is Continuous

is diffent then cont
is a funt is cont doesnt nessesarly mean its differtiable

biology mathmatical medicine
Stats and modelling geology climate theory

2008 real analysis {properties of the real number system

- ~~Field~~ the real numbers form a field - thats a set equipped with two binary operations - addition, multiplication

the real numbers are an ordered field we can say one element is bigger than another

- real number system is complete - no gaps or holes in that line

Supremum, {symmetrical notion called the infimum} }

$\frac{\infty}{\infty}$ Extended real number system $\frac{\infty}{\infty}$ undefined

$a \in \mathbb{R}$ $a + \infty = \infty$

$\mathbb{R} \cup \{\infty\} \cup \{-\infty\}$ $-\infty \cdots$ ——— $\cdots \infty$

real number systems from axioms by recalling their fundamental properties Dedekind's cuts Cauchy sequences geometrically with all approaches giving same object so theyre all isomorphic so theres only one set of objects that has the properties of real numbers

field properties real numbers ~~set~~ with + and multiplication form a field

0 additive identity 1 multi identity $a + 0 = a$ $a1 = a$

commutative associative distributive

additive inverse $-a$ multiplicative inverse $\frac{1}{a}$ if $a \neq 0$

order relation $a, b \in \mathbb{R}$ $a = b \vee a < b \vee b < a$

if $a < b \wedge b < c \Rightarrow a < c$ (so $<$ is transitive

if $a < b \Rightarrow a + c < b + c$ $\forall c \in \mathbb{R}$ ~~if $0 < c$~~

and if $0 < c$ then $ac < bc$

supremum $S \subset \mathbb{R}$ if ~~$\exists b \in \mathbb{R}$~~ $\exists b \in \mathbb{R}$ s.t. $x \leq b$ $\forall x \in S$

$\Rightarrow S$ is bounded above ~~and~~ and is an upper bound of S

if β is an upper bound of S but no number less than β is then β is called the supremum of S denoted

$\beta = \sup S$ supremum is least upper bound

for infimum $\alpha = \inf(S)$

the <u>completeness</u> <u>axiom</u>

$\mathbb{R} \subset \mathbb{R}$ is not bounded above

completeness axiom: if a non empty set of real numbers is bounded above then it has a supremum

altogether we say the real number system is a <u>complete</u> <u>ordered</u> <u>field</u>

completeness axiom is a fundamental and distinguishing property of the real number system so can show it is the only complete ordered field

so completeness meaning the real number line has "no gaps" in contrast the rationals numbers do not possess the property of the completeness axiom and the rational number line has a gap where each irrational number would appear

<u>extended</u> <u>real</u> <u>number</u> <u>system</u>

extend real number system by adjoining two fictitious points at infinity ∞ and $-\infty$ defining ~~$x \in \mathbb{R}$~~

$-\infty < x < \infty$ now if a set S is unbounded above write $\sup S = \infty$ if S is unbounded below write $\inf S = -\infty$

the extended real number system may commonly be denoted by $\bar{\mathbb{R}}$ $[-\infty, \infty]$ or $\mathbb{R} \cup \{-\infty, \infty\}$

$\mathbb{R} = (-\infty, \infty)$

if $a \in \mathbb{R} \Rightarrow a + \infty = \infty + a = \infty \quad a - \infty = -\infty + a = -\infty$

$\frac{a}{\infty} = \frac{a}{-\infty} = 0$

if $a > 0$ then $a\infty = \infty a = \infty$

$a(-\infty) = ((-\infty)a) = -\infty$

if $a < 0$ then $a\infty = \infty a = -\infty$

$a(-\infty) = (-\infty a) = \infty$

$\infty + \infty = \infty\infty = (-\infty)(-\infty) = \infty \quad -\infty - \infty = \infty(-\infty) = (-\infty)\infty = -\infty$

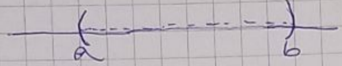
$|\infty| = |-\infty| = (\infty)$

not used to define $\infty - \infty$, $0 \cdot \infty$, $\frac{\infty}{\infty}$, $0/0$

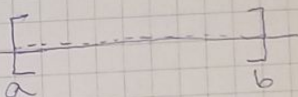
call <u>indeterminate</u> <u>forms</u> left undefined

$a, b \in \bar{\mathbb{R}}$ with $a<b$ define an open interval as

$(a,b) = \{x \mid a<x<b\}$ $\bar{\mathbb{R}} \to \mathbb{R} \cup \{-\infty, \infty\}$

so 

$(-\infty, 1)$:  $(-\infty, \infty) = \mathbb{R}$

closed interval $[a,b] = \{x \mid a \le x \le b\}$: 

<u>Def 1.16</u>: (a) if x_0 is a real number & $\varepsilon>0$ the Z open interval $(x_0-\varepsilon, x_0+\varepsilon)$ is an <u>ε-neighborhood</u> of x_0

(b) if a set S contains an ε-neighborhood of x_0 then S is a neighborhood of x_0

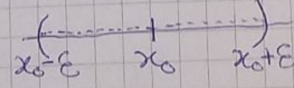
(c) if S is a neighborhood of x_0 then x_0 is an <u>interior point</u> of S

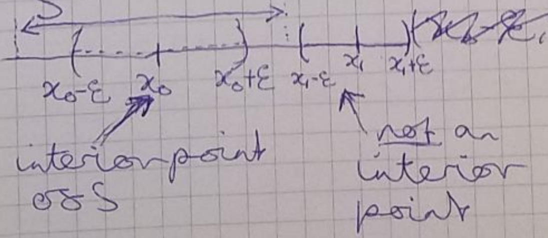
(d) Z set of interior points of S is the interior of S denoted by S^0

(e) if every point of S is an interior point (that is $S^0 = S$) then S is <u>open</u>

(f) A set S is <u>closed</u> if S^c is open

for (a): think ε is a "small number"

$(x_0-\varepsilon, x+\varepsilon)$ 

(b): 

$(x_0-\varepsilon, x_0+\varepsilon) \subset S$

ⓒ in other words $(x_1-\varepsilon, x_1+\varepsilon) \not\subset S$

ⓔ also said as every point in S has an ε-neighb contained in S

\eg1.17/ • any open interval $S=(a,b)$ is open $(a,b\in\overline{\mathbb{R}})$

need to show that $\forall x_0\in(a,b), \exists\, \varepsilon>0$ s.t. $(x_0-\varepsilon, x_0+\varepsilon) \subset (a,b)$

first assume $a,b\in\mathbb{R}$ let $x_0\in(a,b)$ and let $\varepsilon=\min\{x_0-a,\ b-x_0\}$ so (a x_0 b) it takes the minimum distance

then clearly ~~$(x_0-\varepsilon)$~~ $(x_0-\varepsilon, x_0+\varepsilon)\subset(a,b)$

\ex check cases $a=-\infty$ or $b=\infty$/

• $\mathbb{R}=(-\infty,\infty)$ is open

• $S^c=(-\infty,a]\cup[b,\infty)$ is closed

• $\mathbb{R}^c=\emptyset$ is closed

• however $\emptyset$ is also open therefore $\mathbb{R}$ is also closed

these are the only sets with this property

the concept of a neighb is used to talk about "closeness" between points

in particular a neighb of a point x_0 is a set that must contain all points "sufficiently close" to x_0 (since it contains an ε-neighbourh of x_0)

theorem 1.18: ⓐ Z union of open sets is open

ⓑ Z intersection of closed sets is closed

these statements apply to arbit collections finite or not of open or closed sets

proof:
(a) let $\mathcal{G}$ be a collection of open sets and let
$S = \bigcup_{G \in \mathcal{G}} G$ {now want to show that S is open}

if $x_0 \in S$ then $x_0 \in G_0$ for some $G_0 \in \mathcal{G}$
Since G_0 is open it must contain an ε-neighb of x_0
the ε-neighb $(x_0 - \varepsilon, x_0 + \varepsilon)$ is in S hence S is neighb of x_0, and x_0 is an interior point of S
Since x_0 was arbit, all points in S are interior points hence S is open

(b) proof. $\mathcal{F}$ be a collection of closed sets and let
$T = \bigcap_{F \in \mathcal{F}} F$ then (see exercise sheet) $T^c = \bigcup_{F \in \mathcal{F}} F^c$
Since each set F^c is open, so T^c is open by (a)
{since taking the union of open sets} therefore
T is closed □

|ex1.19| • for $a, b \in \mathbb{R}$ the set $[a,b] = \{x \mid a \leq x \leq b\}$ is closed since $[a,b]^c = (-\infty, a) \cup (b, \infty)$ is a union of open intervals, it is open. $[a,b]$ is called a <u>closed interval</u>

• Sets of the form $[a,b)$ and $(a,b]$ for $a, b \in \mathbb{R}$ are called half-open or half-closed intervals and these are neither open nor closed
take $[a,b)$ the point a is not an interior point (why?), hence its not open
, $[a,b)^c = (-\infty, a) \cup [b, \infty)$ and now b is not an interior point hence $[a,b)^c$ is not open

• what about intervals of the form $(-\infty, a]$ or $[a, \infty)$? where $a \in \mathbb{R}$

it can be shown that the intersection of finitely many open sets is open, & that the union of finitely many closed sets is closed however the intersection of infinitely many open sets need not be open and the union of infinitely many closed sets need not be close {see ex sheet}

\ in Ex 1.17 / saw $\mathbb{R}$ and $\emptyset$ are both open and closed

\ in Ex 1.19 / saw $[a,b)$ and $(a,b]$ are neither open nor closed

the concept of open and closed sets is not a dichotomy

Def (P.16): A deleted neighborhood of a point x_0 is a set that contains every point of some neighborhood of x_0 except x_0 itself eg:

$S = \{x \mid 0 < |x - x_0| < \varepsilon\} \subset (x_0 - \varepsilon, x_0 + \varepsilon)$ {since it excludes $x = x_0$ since if $x = x_0 \Rightarrow x - x_0 = 0 \not> 0$ since $0 = 0$}

so S is a deleted neighborhood of x_0 we also say that it is a deleted ε-neighb of x_0

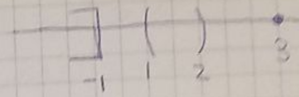
every neighb of a point x_0 by def contains an ε-neighb of x_0 for some ε

therefore every deleted neighb of a point x_0 contains a deleted ε-neighb of a point x_0

Def 1.21: S be a subset of $\mathbb{R}$ then

ⓐ x_0 is a limit point of S if every deleted neighb of x_0 contains a point of S

\eg 1.22/ let $S=(-\infty,-1]\cup(1,2)\cup\{3\}$



(a) for any point $x_0<-1$ every
• deleted ε-neighb will contain a point in S

for point $x_0=-1$ every deleted ε neighb will contain a point in S {namely to the left of x_0}

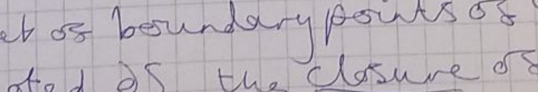
for point $x_0=1$ every deleted ε neighb will contain a point in S {namely to the right of x_0} and the same argument applies to $x_0=2$ but to the left of x_0

any point between 1 and 2 has same argument

for $x_0<-1$
so all the points above are limit points

but for $x_0=3$ if you take a deleted ε neighb its not true that every ε in that neighb contains a point from S {since if you take $\varepsilon<1$ then no longer intersect ~~poi~~ any point in S so 3 is not a limit point

So



so the limit points of the set S are $(-\infty,-1]\cup[1,2]$

(b) x_0 is a boundary point of S if every neighb of x_0 contains at least one point in S and ~~at~~ one not in S
the set of boundary points of S is the boundary of S denoted ∂S the closure of S denoted by $\bar{S}$ is $S\cup\partial S$

So $\cancel{\lim_{x\to 0^+}[(-1)x^3]=0}$

So $\cancel{\lim_{x\to 0^-}[(-1)x^3]=0}$

(b) for $x_0 < -1$ it is not true that x_0 is a boundary point since with a small enough ε a ~~deleted~~ ε neighb of x_0 will not contain a point outside of S {since it will only contain points in S for $x < -1$}

but for $x_0 = -1$ it is true that every ε neighb intersects points in S and not in S {because points to left will intersect points in S and points to its right won't} so -1 is a boundary point

similar argument for $x_0 = 1$ {~~but~~ it will have points in S to its right and won't to its left} and the exact same argument for $x_0 = 2$ so $x_0 = 1$ and $x_0 = 2$ are boundary points

for x_0 is a point in between 1 and 2 same argument ~~for~~ as for $x_0 < -1$ as to why its not a boundary point

for $x_0 = 3$ an ε neighb will always intersect 3 which is in S and a point next to 3 which is not in S so points either side of 3 so 3 is a boundary point of S

So over-all the boundary of S: ∂S

$\partial S = \{-1, 1, 2, 3\}$, $\bar{S} = S \cup \partial S = (-\infty, -1] \cup [1, 2] \cup \{3\}$

(c) x_0 is an isolated point of S is $x_0 \in S$ and there is a neighb of x_0 that contains no other point of S

but any point from $-\infty$ to -1 will have a point from S in their ε-neighb similarly for 1 to 2 but for $x_0 = 3$ the ε neighb for small enough ε will only contain ~~the 3 which the~~ point in S of 3 with all others in the ε neighb not being in 3 so the only isolated point of S is 3

(d) x_0 is exterior to S if x_0 is the interior of S^c
the collection of such points is the exterior of S

- $S^c = (-1,1] \cup [2,3) \cup (3,\infty)$

$(S^c)^0$ {the interior of the complement and the interior is every point where there exists a ε neighb of that point which is contained within the set}
so $(S^c)^0 = (-1,1) \cup (2,3) \cup (3,\infty)$ and therefore this is the exterior of S

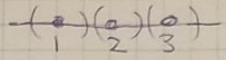
theorem 1.25: a set S is closed if and only if no point of S^c is a limit point of S

proof: {if and only if $\Leftrightarrow$} so start with "$\Rightarrow$" {only if}

if S is closed, S^c is open and then for any $x_0 \in S^c$ there exists an ε-neighb contained in S^c
but here we found an εneighb that is completely contained within the complement so x_0 cannot be a limit point of ~~x_0~~ S
for "$\Leftarrow$" {if} if no point of S^c is a limit point of S
~~then every poi~~ so every point in the complement has a ~~ε~~ neighb which is contained completely within the complement so then every point $x_0 \in S^c$ must have a neighb contained in S^c which means ~~that~~ S^c is open, therefore $(S^c)^c$ is closed but $(S^c)^c = S$
So S is closed $\square$

Corollary 1.26: A set is closed if and only if it contains all its limit points

- but note that a closed set might not have any limit points. If a set has no limit points, it is closed.

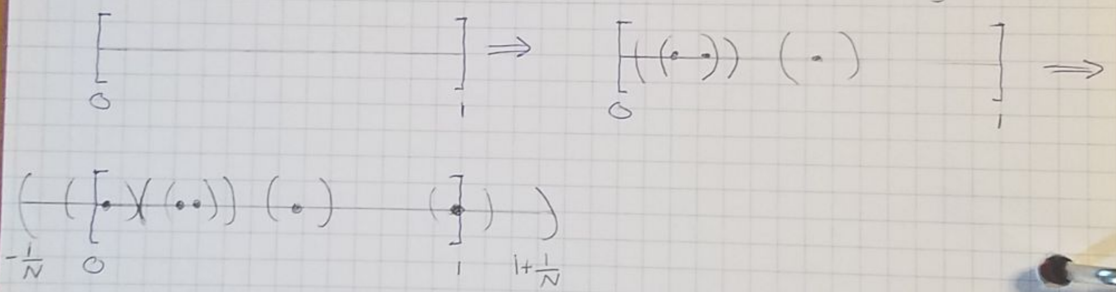
\ex/ take finite set $S=\{1,2,3\}$ 

this set has no limit points {since ~~any~~ x_0 in S has a deleted ε-neighb that doesn't contain a point in S with small enough ε} ~~so thus~~ So its closed

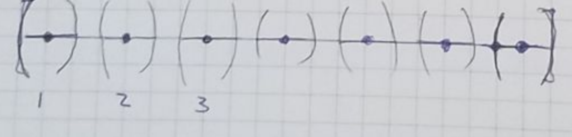
open coverings: (def(P.18)): A collection $\mathcal{H}$ of open sets is an <u>open covering</u> of a set S if every point in S is contained in a set H belonging to $\mathcal{H}$; that is if $S \subset \bigcup_{H\in\mathcal{H}} H$

{a collection of open sets means: $\mathcal{H}=\{H_1, H_2, H_3, \dots\}$ where each H_i is open}

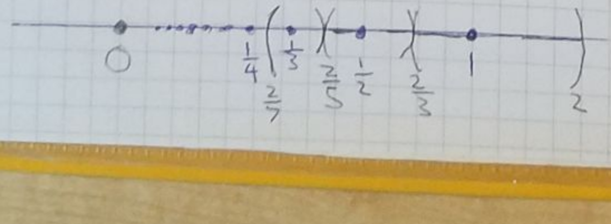
\eg 1.27/ $S_1=[0,1]$ $\mathcal{H}_1=\{(x-\frac{1}{N}, x+\frac{1}{N}) \mid x\in(0,1)\}$ $N\in\mathbb{N}$ fixed



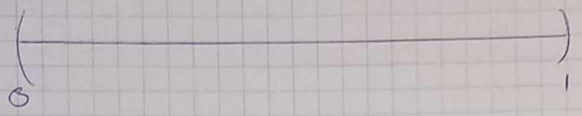
$S_2=\mathbb{N}$ $\mathcal{H}_2=\{(n-\frac{1}{4}, n+\frac{1}{4}) \mid n\in\mathbb{N}\}$

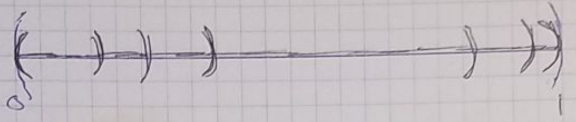


$S_3=\{\frac{1}{n} \mid n\in\mathbb{N}\}$ $\mathcal{H}_3=\{(\frac{1}{n+\frac{1}{2}}, \frac{1}{n-\frac{1}{2}}) \mid n\in\mathbb{N}\}$



$S_4=(0,1)$ $\mathcal{H}_4=\{(0,\rho) \mid 0<\rho<1\}$





so is an open covering

<u>Thm 1.28</u> Heine-Borel thm: if $\mathcal{H}$ is an open covering of a <u>closed</u> <u>and</u> <u>bounded</u> <u>subset</u> $\underline{S}$ of the real line then S has an open covering $\tilde{\mathcal{H}}$ consisting of <u>finitely many</u> open sets belonging to $\underline{\mathcal{H}}$

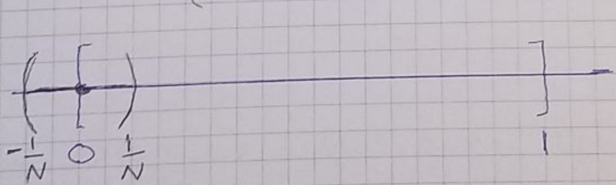
{$\mathcal{H}$ could be infinite}

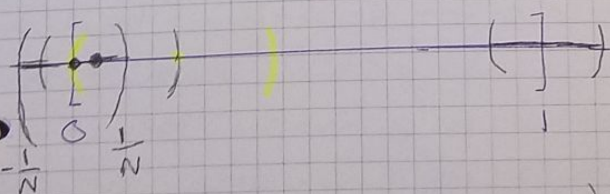
say $\tilde{\mathcal{H}}$ is a finite subcovering of S

eg 1.29/ $S_1=[0,1]$ $\mathcal{H}_1=\{(x-\frac{1}{N}, x+\frac{1}{N}) \mid x\in(0,1)\}$ $N\in\mathbb{N}$ fixed

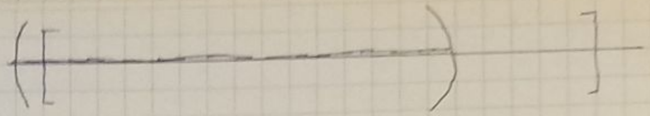
S_1 is closed and bounded

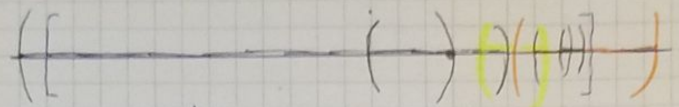
take $\tilde{\mathcal{H}}_1=\{(x_k-\frac{1}{N}, x_k+\frac{1}{N}) \mid x_k=\frac{k}{2N}, 0\le k\le 2N-1\}$





{$\tilde{\mathcal{H}}_1$ only contains $2N$ intervals}

Heine-Borel thm: 



only need a finite sub cover

[eg1.29] the H-B thm doesnt apply to S_2, S_3 and S_4 {applies to a closed and bounded subset}

$S_2 = \mathbb{N}$ closed but not bounded

$S_3 = \{\frac{1}{n} \mid n \in \mathbb{N}\}$ bounded but not closed

$S_4 = (0,1)$ is bounded but not closed

if set is closed and bounded is <u>Compact</u>
so they dont apply because theyre not compact
so we should not expect there to be a finite subcovering from $\mathcal{H}_2$, $\mathcal{H}_3$ and $\mathcal{H}_4$ respectively and indeed there isnt

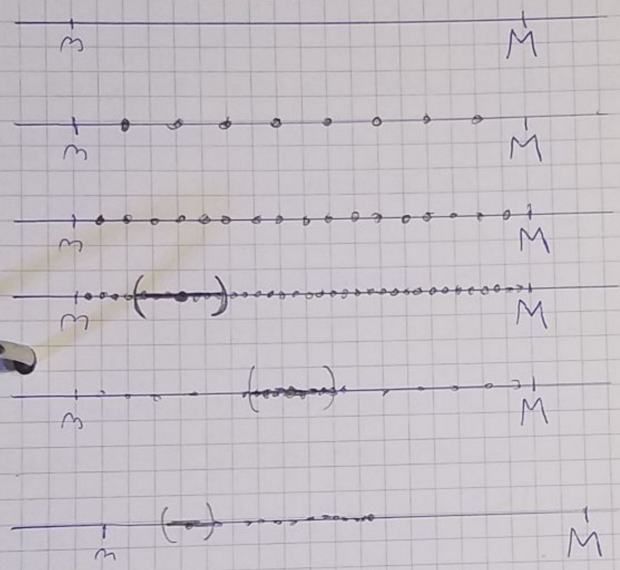
for $S_2 = \mathbb{N}$: $\mathcal{H}_2' = \{(0,\infty)\}$

for $S_3 = \{\frac{1}{n} \mid n \in \mathbb{N}\}$: $\mathcal{H}_3' = \{(0, \frac{3}{2})\}$

for $S_4 = (0,1)$: $\mathcal{H}_4' = \{(0,1)\}$

A closed and bounded set is called a <u>compact set</u>
it turns out the converse to the Heine-Borel thm is also true so we could just as well define a set S of reals to be compact if it has the Heine-Borel property; that is if every open covering of S contains a finite subcovering

thm 1.3: Bolzano-Weierstrass thm: every bounded infinite set of real numbers has at least one limit point



like $S_3 = \{ \frac{1}{n} \mid n \in \mathbb{N} \}$

proof: will show that a bounded non empty set without a limit point can contain only finitely many points (why is this equivalent?) if S has no limit point S is closed (thm 1.25) and every point $x \in S$ has an open neighb N_x that contains no point of S other than x. The collection $H = \{ N_x \mid x \in S \}$ is an open covering of S since S is compact the H-B thm implies there's a finite subcovering

$N_{x_1}, \ldots, N_{x_n}$ since N_{x_i} only contains x_i,

$S = \{ x_1, \ldots, x_n \}$ QED

(1) fundamentals $\lim_{x\to x_0} f(x)$ (2) one sided limits $\lim_{x\to x_0^+} f(x)$

$\lim_{x\to x_0^-} f(x)$ (3) limits at $\pm\infty$ $\lim_{x\to+\infty} f(x)$, $\lim_{x\to-\infty} f(x)$

(4) infinite limits "$\lim_{x\to x_0} f(x)=\infty$"

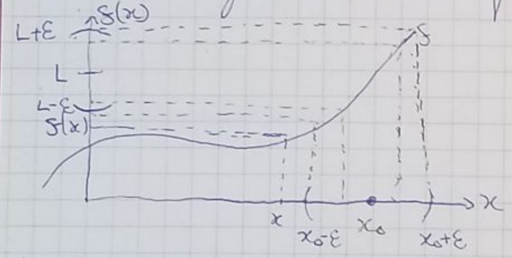
def of a limit: here consider limits of real functions that is real-valued functions of a real variable or in symbols, functions f of the form

$f: X\to\mathbb{R}$, with $X\subset\mathbb{R}$

def (2.6): say that $f(x)$ approaches the limit L as x approaches x_0, and write $\lim_{x\to x_0} f(x)=L$

if f is defined on some deleted neighborhood of x_0 and for every $\varepsilon>0$ there is a $\delta>0$ such that $|f(x)-L|<\varepsilon$ if $0<|x-x_0|<\delta$

deleted neighborhood – equivalently "deleted ε-neighb"



eg/ let $f(x)=2x$ Claim: $\lim_{x\to 1} f(x)=2$

$\delta=0.01$ so $0.99<1<1.01$ $\varepsilon=0.1$ so $1.9<2<2.1$

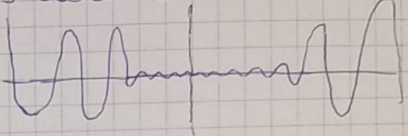
proof: let $\delta=\frac{\varepsilon}{10}$ then $\forall x$ s.t. $0<|x-1|<\delta$ we have

$|f(x)-2|=|2x-2|=2|x-1|<2\delta=\frac{2}{10}\varepsilon<\varepsilon$ QED

so $|f(x)-2|$ is $|f(x)-L|$ and $|x-1|=|x-x_0|$

eg 2.8/ note the def does not involve $f(x_0)$ and does not require that f is defined at x_0

let $f(x) = x\sin\left(\frac{1}{x}\right)$, $x \neq 0$

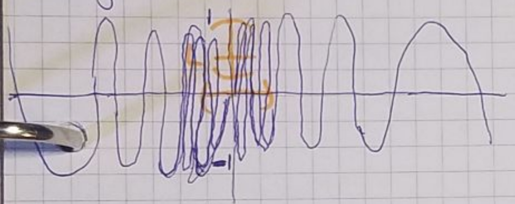


claim $\lim_{x\to 0} f(x) = 0$

proof: let $\delta = \varepsilon$ then $\forall x$ s.t. $0 < |x-0| < \delta$ ~~$|f(x)| < \varepsilon$~~

$|f(x) - 0| = \left|x\sin\left(\frac{1}{x}\right)\right| = |x|\left|\sin\left(\frac{1}{x}\right)\right| \leq |x| < \delta = \varepsilon$ QED

let $g(x) = \sin\left(\frac{1}{x}\right)$, $x \neq 0$ the $\lim_{x\to 0} g(x)$ does not exist



~~let $\delta = \varepsilon$ $\forall x$ s.t. $0 < |x-0| < \delta$~~

~~$|f(x) - 0| = \left|\sin\left(\frac{1}{x}\right)\right| \leq |x| < \delta = \varepsilon$~~ X wrong

$\left|\sin\left(\frac{1}{x}\right)\right| \not\leq |x|$ $\forall x$ so limit not 0

thm 2.9: if $\lim_{x\to x_0} f(x)$ exists then it is unique that is if

$\lim_{x\to x_0} f(x) = L_1$ and $\lim_{x\to x_0} f(x) = L_2$ then $L_1 = L_2$

thm 2.10: if $\lim_{x\to x_0} f(x) = L_1$ and $\lim_{x\to x_0} g(x) = L_2$ then

$\lim_{x\to x_0} (f+g)(x) = L_1 + L_2$, $\lim_{x\to x_0} (f-g)(x) = L_1 - L_2$, $\lim_{x\to x_0} (fg)(x) = L_1 L_2$

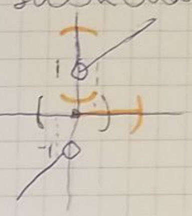
and if $L_2 \neq 0$ $\lim_{x\to x_0} \left(\frac{f}{g}\right)(x) = \frac{L_1}{L_2}$

here L_1 & L_2 are finite but they can hold for infinite limits

def (2.12): (a) say that $f(x)$ approaches the left-hand limit L as x approaches x_0 from the left and write $\lim_{x\to x_0^-} f(x) = L$ if f is defined on some open interval (a, x_0) and for each $\varepsilon > 0$ there is a $\delta > 0$ s.t. $|f(x) - L| < \varepsilon$ if $x_0 - \delta < x < x_0$

(b) we say that $f(x)$ approaches the right hand limit L as x approaches x_0 from the right and write $\lim_{x\to x_0^+} f(x) = L$

if f is defined on some open interval (x_0, b) and for each $\varepsilon>0$ there is a $\delta>0$ such that $|f(x)-L|<\varepsilon$ if $x_0<x<x_0+\delta$



(eg p.29) $g(x)=\begin{cases} x+1, & x>0, \\ x-1, & x<0 \end{cases}$

so $\lim_{x\to 0} g(x)$ does not exist but

$\lim_{x\to 0^+} g(x)=1$ $\quad \lim_{x\to 0^-} g(x)=-1$

often simplify the notation by writing $\lim_{x\to x_0^-} f(x)=f(x_0^-)$

& $\lim_{x\to x_0^+} f(x)=f(x_0^+)$ analogous results to the "uniqueness of limits" (thm 2.9) & the "algebra of limits" (thm 2.10) hold for one sided limits. also have the following result connecting limits & one sided limits.

Thm 2.16: A function f has a limit at x_0 if and only if it has left & right hand limits x_0 & they are equal, more specifically $\lim_{x\to x_0} f(x)=L$ if and only if $f(x_0+)=f(x_0-)=L$

Def (2.17): say that $f(x)$ approaches the limit L as x approaches ∞ and write $\lim_{x\to\infty} f(x)=L$ if f is defined on an interval (a,∞) and for each $\varepsilon>0$ there is a number β s.t. $|f(x)-L|<\varepsilon$ if $x>\beta$

(eg 2.8) $g(x)=\frac{2|x|}{1+x}$ claim $\lim_{x\to\infty} g(x)=2$

proof: $|g(x)-2| = \left|\frac{2|x|-2(1+x)}{1+x}\right| = \left|\frac{2|x|-2-2x}{1+x}\right| = \left|\frac{-2}{1+x}\right| = \frac{2}{1+x} < \frac{2}{x}$

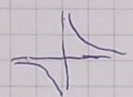
{need to make this less than ε} so for $x>\beta$ with $\beta=\frac{2}{\varepsilon}$

QED

Def 2.19: Say that $f(x)$ approaches ∞ as x approaches x_0 from the left and write $\lim_{x\to x_0^-} f(x) = \infty$ or $f(x_0^-) = \infty$ if f is defined on an interval (a, x_0) and for each real number M there is a $\delta > 0$ such that $f(x) > M$ if $x_0 - \delta < x < x_0$

and similarly for $x \to x_0^+$, $x \to x_0$, $x \to \infty$, $x \to -\infty$

leg/ try define infinite limits for these cases

{eg/ $f(x) = \frac{1}{x}$  $x \neq 0$ claim: $\lim_{x\to 0^+} f(x) = \infty$

$\frac{1}{x} > M$ $\forall x$ s.t. $0 < x < \delta$ with $\delta = \frac{1}{M}$ QED

important: "$\lim_{x\to x_0} f(x)$ exists" means that $\lim_{x\to x_0} f(x) = L$ where L is finite to leave open the possibility that $L = \pm\infty$ we will say that $\lim_{x\to x_0} f(x)$ exists in the extended reals this convention also applies to one sided limits and limits as x approaches $\pm\infty$

{$\lim_{x\to x_0} f(x) = \infty$ is shorthand}

the "uniqueness of limits" (thm 2.9) and the "Algebra of limits" (thm 2.10) are also valid with x_0 replaced by $\pm\infty$ Moreover the counterparts of thm 2.10 all remain valid if either or both of L_1 & L_2 are infinite provided that their right sides are not indeterminate & in the quotient case additionally $L_2 \neq 0$

Continuity: def of $f(x_0^-) = \lim_{x\to x_0^-} f(x)$, $f(x_0^+) = \lim_{x\to x_0^+} f(x)$ & $\lim_{x\to x_0} f(x)$ do not involve $f(x_0)$ or even require that it be defined however for continuity we are interested in the case where $f(x_0)$ is defined and equal to one or More of these quantities

def(2.26): (a) we say that f is Continuous at x_0 if f is defined on an open interval (a,b) containing x_0 and $\lim_{x\to x_0} f(x) = f(x_0)$

(b) we say that f is Continuous from the left at x_0 if ~~if~~ f is defined on an open interval (a, x_0) and $f(x_0^-) = f(x_0)$

(c) we say that f is Continuous from the right at x_0 if f is defined on an open interval (x_0, b) and $f(x_0+) = f(x_0)$

{intuitive idea of continuity: graph is a connected curve}

thm2.27: (a) function f is Continuous at x_0 if & only if f is defined on an open interval (a,b) containing x_0 & for each $\varepsilon > 0$ there is a $\delta > 0$ s.t. $|f(x) - f(x_0)| < \varepsilon$ whenever $|x - x_0| < \delta$ (*)

(b) A function f is continuous from Z right at x_0 iff f is defined on an inter $[x_0, b)$ & for each $\varepsilon > 0$ there is a $\delta > 0$ s.t. (*) holds whenever $x_0 \le x < x_0 + \delta$

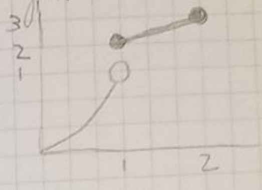
(c) A function f is continuous from Z left at x_0 iff f is defined on an inter $(a, x_0]$ & for each $\varepsilon > 0$ there is a $\delta > 0$ s.t. (*) holds whenever $x_0 - \delta < x \le x_0$

eg/ $f(x) = x \sin\left(\frac{1}{x}\right)$ is not contin at $x_0 = 0$ (why?)

$$g(x) = \begin{cases} x \sin\left(\frac{1}{x}\right), & x \neq 0, \\ 0, & x = 0 \end{cases}$$ is contin at $x_0 = 0$

since $\lim_{x\to 0} g(x) = 0 = g(0)$

note that f is contin at x_0 iff $f(x_0^-)=f(x_0^+)=f(x_0)$
or equivalently iff it is contin from the right & the left
at x_0 {why? exercise in notes}



eg 2.28/ $f:[0,2]\to\mathbb{R}$ $f(x)=\begin{cases} x^2, & 0\le x<1, \\ x+1, & 1\le x\le 2 \end{cases}$

$f(0^+)=0=f(0)$
contin from the right at 0
$f(0^-)=1\neq f(1)=2$ not continuous from the left at 1

$f(1+)=2=f(1)=2$ contin from the right at 1
$f(2^-)=3=f(2)=3$ contin from the left at 2

if $x_0,x\in(0,1)$ $\quad |f(x)-f(x_0)|=|x^2-x_0^2|=|(x+x_0)(x-x_0)|$

$=|x(x-x_0)+x_0(x-x_0)|=x|x-x_0|+x_0|x-x_0|<|x-x_0|+|x-x_0|$

{since $x_0,x\in(0,1)$} $=2|x-x_0|<2\delta=\varepsilon$

$\forall x$ s.t. $|x-x_0|<\delta$ with $\delta=\frac{\varepsilon}{2}$ so $2\delta=\varepsilon$

if $x_0,x\in(1,2)$ $\quad |f(x)-f(x_0)|=|(x+1)-(x_0+1)|=|x-x_0|<\delta$

$\forall x$ s.t. $|x-x_0|<\delta$ with $\delta=\varepsilon$

{so f is contin in $(0,1)\cup(1,2)$}

Def 2.29: A func f is contin on an open inter (a,b) if it is cont at every point in (a,b) if in addition, $f(b^-)=f(b)$ or $f(a^+)=f(a)$ then f is cont on $(a,b]$ or $[a,b)$ respectively if f is cont on (a,b) & $f(b^-)=f(b)$ and $f(a^+)=f(a)$ both hold, then f is cont on $[a,b]$

More generally if S is a subset of D_f consisting of finitely many ~~distinct~~ disjoint inters, then f is cont on S if f is cont on every inter in S

(henceforth, in connection with funcs of one variable whenever we say f is cont on S" we mean that S is a set of this kind)

eg 2.30/ $f(x)=\sqrt{x}$, $0 \le x \le \infty$ <u>claim</u>: f is cont $\forall x_0 \in D_f$

$\{D_f$ is domain of $f\}$

<u>proof</u>: for $x_0 \neq 0$ $|f(x)-f(x_0)| = |\sqrt{x}-\sqrt{x_0}|$

$$= |\sqrt{x}-\sqrt{x_0}| \cdot \frac{|\sqrt{x}+\sqrt{x_0}|}{|\sqrt{x}+\sqrt{x_0}|} = \frac{|x-x_0|}{|\sqrt{x}+\sqrt{x_0}|} = \frac{|x-x_0|}{\sqrt{x}+\sqrt{x_0}} \le \frac{|x-x_0|}{\sqrt{x_0}} < \frac{\delta}{\sqrt{x_0}} = \varepsilon$$

$\forall x$ s.t. $|x-x_0|<\delta$ with $\underline{\delta = \sqrt{x_0}\,\varepsilon}$ so $\frac{\delta}{\sqrt{x_0}} = \varepsilon$

for $x_0=0$: $|f(x)-f(x_0)| = |\sqrt{x}-\sqrt{0}| = \sqrt{x-0} < \sqrt{\delta} = \varepsilon$

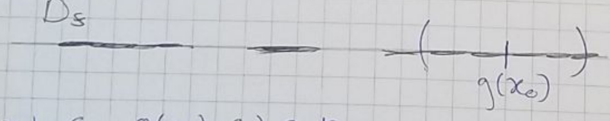
so this can be true $\forall x$ s.t. $0<x<\delta$ with $\underline{\delta=\varepsilon^2}$

so $f(0^+) = f(0) = 0$ QED

<u>thm 2.35</u>: if f & g are cont on a set S then so are $f+g$, $f-g$ and fg in addition f/g is cont at each x_0 in S s.t. $g(x_0) \neq 0$

<u>proof</u>: eg {use algebra of limit rules}

<u>thm 2.40</u>: Suppose that g is cont at x_0, $g(x_0)$ is an interior point of D_f & f is cont at $g(x_0)$ then $f \circ g$ is cont at x_0



so

$(g(x_0)-\varepsilon, g(x_0)+\varepsilon) \subset D_f$ or $\lim_{x \to x_0} f(g(x)) = f(g(x_0))$

<u>proof</u>: Suppose $\varepsilon>0$ since $g(x_0) \in D_f^\circ$ {the interior of the domain of f} and f is cont at $g(x_0)$, that means $\exists \delta_1>0$ s.t. $f(t)$ is defined & $|f(t)-f(g(x_0))|<\varepsilon$ if $|t-g(x_0)|<\delta_1$

since g is cont at x_0, $\exists \delta_2 > 0$ s.t. ~~$g(x)$~~

$g(x)$ is defined (why?) and $|g(x)-g(x_0)|<\delta_1$ if $|x-x_0|<\delta_2$

then $|t-g(x_0)|<\delta_1$ and $|x-x_0|<\delta$ together imply that $|f(g(x))-f(g(x_0))|<\varepsilon$ if $|x-x_0|<\delta_2$

bounded functions: def (p.46): A func f is bounded below on a set S if there is a real number m s.t. $f(x)\geq m \ \forall x\in S$ in this case, the set $V=\{f(x) \mid x\in S\}$ has an infimum α & we write

$\alpha=\inf_{x\in S} f(x)$ if there is a point x_1 in S s.t. $f(x_1)=\alpha$

we say that α is the minimum of f on S and write

$$\alpha=\min_{x\in S} f(x)=\min V$$

$$\left\{\alpha=\inf_{x\in S} f(x)=\inf V\right\}$$

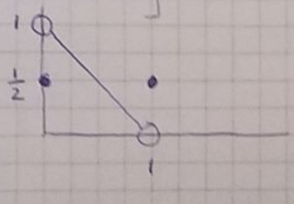
Similarly f is bounded above on S if there is a real number M s.t. $f(x)\leq M \ \forall x$ in S in this case V has a supremum β and we write $\beta=\sup_{x\in S} f(x)$

if there is a point x_2 in S s.t. $f(x_2)=\beta$ we say that β is the maximum of f on S and write $\beta=\max_{x\in S} f(x)$

if f is bounded above and below on a set S we say that f is bounded on S

$$\left\{\beta=\sup_{x\in S} f(x)=\sup V\right\} \qquad \left\{\beta=\max_{x\in S} f(x)=\max V\right\}$$

\eg 4.42/ $g(x)=\begin{cases}\frac{1}{2}, & x=0,\ x=1,\\ 1-x, & 0<x<1\end{cases}$



g is bounded on $[0,1]$

$$V=\{g(x) \mid x\in[0,1]\}=(0,1)$$

so $\sup_{x\in[0,1]} g(x)=1$ $\quad \inf_{x\in[0,1]} g(x)=0$

g does not attain a Max or Min value on $[0,1]$

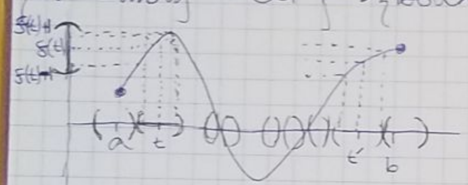
$h(x)=1-x, 0\le x\le 1$

~~so~~ $\sup_{x\in[0,1]} h(x)=1=h(0)=\max_{x\in[0,1]} h(x)$

$\inf_{x\in[0,1]} h(x)=0=h(1)=\min_{x\in[0,1]} h(x)$

thm (2.44) - boundedness thm: if f is cont on a finite closed inter $[a,b]$ then f is bounded on $[a,b]$

{Continuity "local"} $\xrightarrow{H-B}$ {bounded "globally"} {H-B Heine borel thm}



$\varepsilon=1$ so ~~$f(t)\pm$~~ $f(t)-1<f(t)<f(t)+1$

can be done at every point t in a and b

so δ neighb of different widths can find a finite number of them that cover a to b {since it forms an open cover} so it must form a finite sub cover so n finite intervals so

$\max_{1\le i\le n} |f(t_i)| = M$ so bounded in $+M$, $-M$

proof: suppose $t\in[a,b]$ since f is cont at t, $\exists$ an open inter I_t with $t\in I_t$, s.t. $|f(x)-f(t)|<1$ if $x\in I_t\cap[a,b]$ $(*)$ {$1=$"ε"} ~~so~~ {$I_t\cap[a,b]$ effectively the δ neighb "$(t-\delta, t+\delta)$"}

the collection $H=\{I_t | a\le t\le b\}$ is open cover of $[a,b]$ since $[a,b]$ is compact by the H-B thm, $\exists$ a finite subcover made up of inters $I_{t_1},\dots,I_{t_n}$, by $(*)$ with $t=t_i$, $|f(x)-f(t_i)|<1$ if $x\in I_{t_i}\cap[a,b]$

therefor $|f(x)|=|f(x)-f(t_i)+f(t_i)|\le |f(x)-f(t_i)|+|f(t_i)|$

{by triangle ineq} $\leq 1 + |f(t_i)|$ if $x \in I_{t_i} \cap [a,b]$ (**)

now let $M = 1 + \max_{1 \leq i \leq n} |f(t_i)|$ since $[a,b] \subset \bigcup_{i=1}^{n} (I_{t_i} \cap [a,b])$

then by (**) implies that $|f(x)| \leq M$ $\forall x \in [a,b]$ QED

thm 2.45: Extreme value thm: suppose that f is cont on a finite close inter $[a,b]$ let $\alpha = \inf_{a \leq x \leq b} f(x)$ and $\beta = \sup_{a \leq x \leq b} f(x)$ then α & β ~~are~~ respectively the minimum & maximum of f on $[a,b]$; that is there are points x_1 & x_2 in $[a,b]$ s.t. $f(x_1) = \alpha$ & $f(x_2) = \beta$

proof: we'll show that x_1 exists {proof for x_2 is similar ex}

suppose for a contradiction that there is no point $x_1 \in [a,b]$ s.t. $f(x_1) = \alpha$ then $f(t) > \alpha$ $\forall t \in [a,b]$, so $f(t) > \frac{f(t)+\alpha}{2} > \alpha$

since f is cont at t there is an open inter I_t about t {see ex sheet} s.t. $f(x) > \frac{f(t)+\alpha}{2}$ if $x \in I_t \cap [a,b]$

then the collection $H = \{I_t \mid a \leq t \leq b\}$ is an open covering of $[a,b]$ since $[a,b]$ is compact the H-B thm implies that there is a finite sub-covering using open inters $I_{t_1}, \dots, I_{t_n}$ around points $t_1, \dots, t_n$

define $\alpha_1 = \min_{1 \leq i \leq n} \frac{f(t_i)+\alpha}{2}$ then $f(t) > \alpha_1$ ~~$\forall t \in$~~

$\forall t \in \bigcup_{i=1}^{n} (I_{t_i} \cap [a,b]) = [a,b]$ since $\alpha_1 > \alpha$ we have a contradiction therefore $f(x_1) = \alpha$ for some $x_1 \in [a,b]$ QED

thm 2.47: intermediate value thm: suppose that f is cont on $[a,b]$, $f(a) \neq f(b)$ & μ is between $f(a)$ & $f(b)$

then $f(c)=\mu$ for some c in (a,b)

proof: suppose that $f(a)<\mu<f(b)$ the set

$S=\{x \mid a\le x\le b$ and $f(x)\le\mu\}$ is bounded and nonempty

let $c=\sup S$ we will show that $f(c)=\mu$

if $f(c)>\mu$ then $c>a$ and since f is cont at c,

$\exists\,\varepsilon>0$ s.t. $f(x)>\mu$ if $c-\varepsilon<x\le c$ {ex sheet}

therefore $c-\varepsilon$ is an upperbound for S, contradicting the def of c

if $f(c)<\mu$ then $c<b$ & $\exists\,\varepsilon>0$ s.t. $f(x)<\mu$ for $c\le x<c+\varepsilon$, so c is not an upperbound for S contradicting the def of c therefore $f(c)=\mu$

the proof for $f(b)<\mu<f(a)$ is obtained by applying the above to $-f$ QED

uniform continuity: def 2.48: A func f is uniformly cont on a subset S of its domain if for every $\varepsilon>0$ there is a $\delta>0$ s.t. $|f(x)-f(x')|<\varepsilon$ whenever $|x-x'|<\delta$ and $x,x'\in S$

~~continuous~~

func is cont: for every $\varepsilon>0$ & each $x_0\in S$, there is a $\delta>0$ s.t. $|f(x)-f(x_0)|<\varepsilon$ whenever $|x-x_0|<\delta$ & $x\in S$

{δ may depend on x_0 as well as ε}

{in uniform continuity δ does not depend on x or x'}

~~but in~~

lex 2.49 / $f(x)=2x$ is uniformly cont on $(-\infty,\infty)$

$|f(x)-f(x')|=|2x-2x'|=2|x-x'|<2\delta$

if $|x-x'|<\delta$ with $\delta=\frac{\varepsilon}{2}$ so $2\delta=\varepsilon$

{$\delta=\frac{\varepsilon}{2}$ so δ is independent of x,x'} hence is uniformly cont

|ex 2.50| $g(x)=x^2$ is uniformly cont on $[-r,r]$, $r>0$

$|g(x)-g(x')| = |x^2-(x')^2| = |x-x'||x+x'| \leq 2r|x-x'| < 2r\delta$

if $|x-x'|<\delta$, with $\delta = \frac{\varepsilon}{2r}$ and $x, x' \in [-r,r]$

$\{\delta = \frac{\varepsilon}{2r}$ independent of x and $x'\}$

often a concept is clarified by considering its negation: a func f is <u>not</u> uniformly cont on S if there is an $\varepsilon_0>0$ s.t. if δ is any pos num there are points x & x' in S s.t. $|x-x'|<\delta$ but $|f(x)-f(x')| \geq \varepsilon_0$

|ex 2.51| $g(x)=x^2$ is <u>not</u> uniformly con on $(-\infty,\infty)$

we'll show that for $\delta>0$, $\exists\, \varepsilon_0>0$, $x, x' \in (-\infty,\infty)$ s.t. $|x-x'|<\delta$ and $|g(x)-g(x')| \geq \varepsilon_0$

let $\varepsilon_0=1$ we have $|g(x)-g(x')| = |x^2-(x')^2| = |x-x'||x+x'|$

if $|x-x'| = \frac{\delta}{2}$ and $x, x' > \frac{1}{\delta}$ then $|x-x'||x+x'|$

$= \frac{\delta}{2}\left(\frac{1}{\delta}+\frac{1}{\delta}\right) = 1 = \varepsilon_0$ QED {closed, but not bounded}

|ex 2.52| $f(x)=\cos\left(\frac{1}{x}\right)$ is cont on $(0,1]$ but is <u>not</u> uniformly cont on $(0,1]$

$\left|f\left(\frac{1}{n\pi}\right) - f\left(\frac{1}{(n+1)\pi}\right)\right| = 2 = \varepsilon_0$ {bounded, but not closed}

<u>thm 2.53</u>: if f is cont on a closed & bounded inter $[a,b]$ then f is uniformly cont on $[a,b]$

<u>proof</u>: suppose that $\varepsilon>0$ since f is cont on $[a,b]$ for each $t\in[a,b]$ $\exists\, \delta_t>0$ s.t. $|f(x)-f(t)|<\varepsilon/2$ if $|x-t|<2\delta_t$ $x\in[a,b]$ $(*)$ $\{t=x_0$, $\varepsilon/2=\varepsilon$, $2\delta_t=\delta\}$

let $I_t = (t-\delta_t, t+\delta_t)$, then collection

$\mathcal{H} = \{I_t \mid t\in[a,b]\}$ is an open cover of $[a,b]$

since $[a,b]$ is compact {closed & bounded} then it follows from the heine-borel thm that it implies there is a finite subcovering $I_{t_1}, \dots, I_{t_n}$ desire

$\delta = \min\{\delta_{t_1}, \dots, \delta_{t_n}\}$ we will show that if

$|x - x'| < \delta$ and $x, x' \in [a, b]$

then $|f(x) - f(x')| < \varepsilon$

we have $|f(x) - f(x')| = |f(x) - f(t_i) + f(t_i) - f(x')|$

{by triangle ineqn} $\leq |f(x) - f(t_i)| + |f(t_i) - f(x')|$ (**)

now $x \in I_{t_i}$ for some i, so $|x - t_i| < \delta_{t_i}$

then by (*) with $t = t_i$,

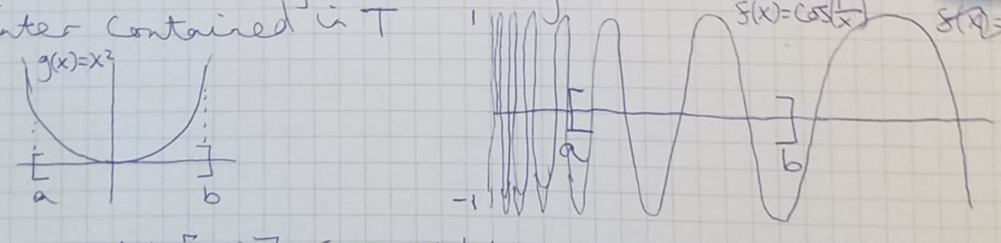
$|f(x) - f(t_i)| < \frac{\varepsilon}{2}$

hence $|x' - t_i| = |x' - x + x - t_i| \leq |x' - x| + |x - t_i| < \delta + \delta_{t_i}$

$\leq 2 \cdot \delta_{t_i}$ therefore $|f(x') - f(t_i)| < \varepsilon/2$

by (**), $|f(x) - f(x')| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ □

Corollary 2.54: if f is cont on a set T then f is uniformly cont on any finite closed inter contained in T



bound to $[a,b]$ it would be uniformly cont

Def of a derivative: Def 2.59:
A function f is differentiable at an interior point x_0 of its domain if the difference quotient $\frac{f(x)-f(x_0)}{x-x_0}$, $x \neq x_0$ approaches a limit as x approaches x_0, in which case the limit is called the derivative of f at x_0, and is denoted by $f'(x_0)$; thus $f'(x_0) = \lim_{x \to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ (*)

it is sometimes convenient to let $x = x_0 + h$ and write (*) as

$$f'(x_0) = \lim_{h \to 0} \frac{f(x_0+h)-f(x_0)}{h}$$

Extreme values: def p.63: we say that $f(x_0)$ is a local extreme value of f if there is a $\delta > 0$ such that $f(x) - f(x_0)$ does not change sign on $(x_0-\delta, x_0+\delta) \cap D_f$ (†)
More specifically, $f(x_0)$ is a local maximum value of f if $f(x) \le f(x_0)$ or a local minimum value of f if $f(x) \ge f(x_0)$
for all x in the set (†) the point x_0 is called a local extreme point of f or, more specifically a local maximum or local minimum point of f

thm 2.70: ~~if~~
if f is differentiable at a local point $x_0 \in D_f^0$
then $f'(x_0) = 0$
if $f'(x_0) = 0$ we say that x_0 is a critical point of f the above thm says that every local extreme point of f at which f is differentiable is a

critical point of f the converse is false for example 0 is a critical point of $f(x)=x^3$ but not a local extreme point

__proof__: consider case where x_0 is a local max then $\exists \delta > 0$ s.t. $(x_0-\delta, x_0+\delta) \subset D_f$ and $f(x_0) \geq f(x)$ for all $x \in (x_0-\delta, x_0+\delta)$ we have

$\cancel{\lim_{x \to x_0^-} f(x-\dots}$ $\lim_{x \to x_0^-} \frac{f(x)-f(x_0)}{x-x_0} \geq 0$ and

$\lim_{x \to x_0^+} \frac{f(x)-f(x_0)}{x-x_0} \leq 0$ since $f'(x_0)$ exists,

$f'(x_0) \geq 0$ and $f'(x_0) \leq 0$, hence $f'(x_0) = 0$

the case of a local min x_0 is obtained by applying the above to $-f$ $\square$

__thm 2.71__ rolles thm: suppose that f is continuous on ~~the~~ closed interval $[a,b]$ and differentiable on the open interval (a,b), and $f(a)=f(b)$ then $f'(c)=0$ for some c in the open interval (a,b)

__proof__: since f is continuous on $[a,b]$, f attains both minimum & maximum values,
$\alpha = \min_{x \in [a,b]} f(x)$, $\beta = \max_{x \in [a,b]} f(x)$ $\{$by EVT (2.45)$\}$

if $\alpha = \beta$ then f is constant on (a,b) and $f'(x)=0$ $\forall x \in [a,b]$ if $\alpha \neq \beta$ then atleast one of α or β is attained at a point $c=(a,b)$ $\{$since $f(a)=f(b)\}$, & hence $f'(c)=0$ $\square$

__thm 2.72__: intermediate value thm for derivatives (darboux) thm suppose that f is differentiable on $[a,b]$, $f'(a) \neq f'(b)$ and μ is between $f'(a)$ & $f'(b)$ then $f'(c)=\mu$ for some c in (a,b)

note: this is not simply the IVT applied to f' since f may not be continuous

proof: suppose that $f'(a) < \mu < f'(b)$ and define $g(x) = f(x) - \mu x \Rightarrow g'(x) = f'(x) - \mu$, $a \leq x \leq b$, and have $g'(a) < 0$ and $g'(b) > 0$ (*)

Since g is continuous on $[a,b]$, g attains a min at some point $c \in [a,b]$ {by the EVT}

$\Rightarrow$ by (*) implies $\exists$ a $\delta > 0$ s.t. $g(x) < g(a)$, $a < x < a+\delta$ and $g(x) < g(b)$, $b - \delta < x < b$ {ex sheet}

$\Rightarrow c \neq a$ & $c \neq b \Rightarrow a < c < b$ & ~~there~~fore $g'(c) = 0$ since c is a min in D_g^o

that is $f'(c) = \mu$

the proof when $f'(b) < \mu < f'(a)$ is obtained by applying the above to $-f$ $\square$

thm (2.73) generalized mean value thm, ~~Cauchys~~ Cauchys MVT

if f & g are continuous on the closed inter $[a,b]$ & differentiable on the open inter $(a,b) \Rightarrow$
$[g(b) - g(a)] f'(c) = [f(b) - f(a)] g'(c)$ for some c in (a,b)

proof: let $h(x) = (g(b) - g(a)) f(x) - (f(b) - f(a)) g(x)$

h is continuous on $[a,b]$ and differentiable on (a,b)

$\Rightarrow$ Rolle's thm implies that $h'(c) = 0$ for some $c \in (a,b)$ that is $h'(c) = (g(b) - g(a)) f'(c) - (f(b) - f(a)) g'(c) = 0$ $\square$

thm (2.74): Mean value thm: if f is cont on Z closed int $[a,b]$ & diffable on Z open int $(a,b) \Rightarrow f'(c) = \frac{f(b) - f(a)}{b - a}$ for some c in (a,b)

proof: Apply the C-MVT with $g(x) = x$ $\square$

{could be multiple c's or even every point in (a,b)}

Consequence of the Mean value thm:

thm (2.75) if $f'(x) = 0$ $\forall x$ in $(a,b) \Rightarrow f$ is const ~~on~~ on (a,b)

thm (2.76): if f' exists & doesn't change sign on (a,b) the f is Monotonic on (a,b): increasing, nondecreasing, decreasing or nonincreasing as $f'(x) > 0$, $f'(x) \geq 0$, $f'(x) < 0$ or $f'(x) \leq 0$ respectively $\forall x$ in (a,b)

So increasing is $f'(x) > 0$,
nondecreasing is $f'(x) \geq 0$, decreasing is $f'(x) < 0$,
nonincreasing is $f'(x) \leq 0$ $\forall x \in (a,b)$

thm (2.77): if $|f'(x)| \leq M$ $\forall x \in (a,b) \Rightarrow$
$|f(x) - f(x')| \leq M|x - x'|$ $\forall x, x' \in (a,b)$

limits Subsequences Cauchy's convergence criterions

Def 3.1: A sequence $\{S_n\}$ converges to a limit S if for every $\varepsilon > 0$ there is an integer N s.t. $|S_n - S| < \varepsilon$ if $n \geq N$

in this case we say that $\{S_n\}$ is convergent and write $\lim_{n\to\infty} S_n = S$

a sequence that does not converge ~~is~~ diverges or is divergent //

sequence is $f: \mathbb{N} \to \mathbb{R}$ $f(n) = S_n$ so $\lim_{n\to\infty} S_n = S$ so $\lim_{n\to\infty} f(n) = S$

~~Def 3.~~ sequences diverging to $\pm\infty$ Def P.27:

say that $\lim_{n\to\infty} S_n = \infty$ if for any real number a, $S_n > a$ for large n $(n \geq N)$ similarly, $\lim_{n\to\infty} = -\infty$ if for any real number a, $S_n < a$ for large n however we do not regard $\{S_n\}$ as convergent unless $\lim_{n\to\infty} S_n$ is finite to emphazise this distinction we say that $\{S_n\}$ diverges to $\infty(-\infty)$ if $\lim_{n\to\infty} S_n = \infty(-\infty)$ //

~~thm 3.5~~: sequence of function values

thm 3.5: let $\lim_{x\to\infty} f(x) = L$ where L is in the extended reals, & suppose that $S_n = f(n)$ for large n. then $\lim_{n\to\infty} S_n = L$ //

Def 3.9: A sequence $\{t_k\}$ is a subsequence of a sequence $\{S_n\}$ if $t_k = S_{n_k}$, $k \geq 0$ where $\{n_k\}$ is an increasing finite sequence of integers in the domain of $\{S_n\}$ we denote the ~~seque~~ subsequence $\{t_k\}$ by $\{S_{n_k}\}$ //

note that $\{S_n\}$ is a subsequence of itself as can be seen by taking $n_k = k$. All other subsequences of $\{S_n\}$ are obtained by deleting terms from $\{S_n\}$ and leaving those remaining in their original relative order

\ex 3.10/ $\{S_n\} = \{\frac{1}{n}\} = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \cdots\}$ convergent $\lim_{n\to\infty} S_n = 0$

$n_k = 2k$ $\quad \{t_k\} = \{S_{n_k}\} = \{S_{2k}\} = \{\frac{1}{2k}\} = \{\frac{1}{2}, \frac{1}{4}, \cdots\}$ convergent to 0

$n_k = 2k+1$ $\quad \{u_k\} = \{S_{n_k}\} = \{S_{2k+1}\} = \{\frac{1}{2k+1}\} = \{1, \frac{1}{3}, \cdots\}$ convergent to 0

\ex / $S_n = (-1)^n$ $\quad S_{2k} = (-1)^{2k} = 1$ converges

$S_{2k+1} = (-1)^{2k+1} = -1$ converges

but $S_n = (-1)^n$ diverges

$S_{3k} = (-1)^{3k} = (-1)^k$ diverges

thm 3.12: if $\lim_{n\to\infty} S_n = S$ $(-\infty \le S \le \infty)$ then $\lim_{k\to\infty} S_{n_k} = S$ for every subsequence $\{S_{n_k}\}$ of $\{S_n\}$ //

proof:

Consider the case where S is finite (ex. infinite case)

$\forall \varepsilon > 0$, $\exists$ an integer N s.t. $|S_n - S| < \varepsilon$ if $n \ge N$

since $\{n_k\}$ is an increasing sequence, there exists an integer K s.t. $n_k \ge N$ if $k \ge K$ therefore $|S_{n_k} - S| < \varepsilon$ if $k \ge K$ $\square$

thm 3.13: A point $\bar{x}$ is a limit point of a set S if and only if there is a sequence $\{x_n\}$ of points in S s.t. $x_n \ne \bar{x}$ for $n \ge 1$, & $\lim_{n\to\infty} x_n = \bar{x}$ //

proof: ("if") Suppose such a sequence $\{x_n\}$ exists. then for all $\varepsilon>0$ there is an integer N s.t. $0<|x_n-\bar{x}|<\varepsilon$ for all $n\geq N$ therefore every ε-neighb of $\bar{x}$ contains infinitely many points of S, hence $\bar{x}$ is a limit point of S
("only if") Now let $\bar{x}$ be a limit point of S. therefore every integer $n\geq 1$ the interval $(\bar{x}-\frac{1}{n},\bar{x}+\frac{1}{n})$ contains some point $x_n\in S$ with $x_n\neq\bar{x}$
since $|x_m-\bar{x}|\leq\frac{1}{n}$ if $m\geq n$, $\lim_{n\to\infty}x_n=\bar{x}$ $\square$

thm 3.14: (a) if $\{x_n\}$ is bounded then $\{x_n\}$ has a convergent subsequence
(b) if $\{x_n\}$ is unbounded above then $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ s.t. $\lim_{k\to\infty}x_{n_k}=\infty$

(c) if $\{x_n\}$ is unbounded below then $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ s.t. $\lim_{k\to\infty}x_{n_k}=-\infty$ //

proof: (a) let S be the set of distinct numbers of $\{S_n\}$ if S is finite; then there exists $\bar{x}\in S$ which occures infinitely often in $\{x_n\}$, i.e. there exists a subsequence $\{x_{n_k}\}$ s.t. $x_{n_k}=\bar{x}$ for all k then $\lim_{k\to\infty}x_{n_k}=\bar{x}$ and we are done
in the case S is infinite then since S is bounded the Bolzano-Weierstrass thm ~~ap~~ implies that S has a limit point $\bar{x}$. By our previous thm there is a sequence of points $\{y_j\}$ in S with $y_j\neq\bar{x}$ s.t. $\lim_{j\to\infty}y_j=\bar{x}$
however $\{y_j\}$ may not be a subsequence of $\{x_n\}$ i.e. it may not correspont to terms $y_j=x_{n_j}$ where $\{n_j\}$

is increasing but we can take an increasing subsequence of $\{n_j\}$, call it $\{n_{j_k}\}$, then $\{y_{j_k}\} = \{S_{n_{j_k}}\}$ is a subsequence of $\{y_j\}$ and must therefore have the same limit as $\{y_j\}$.

$\lim_{k\to\infty} S_{n_{j_k}} = \bar{x}$ $\square$

Def 3.15: a sequence $\{S_n\}$ of real numbers is said to be a Cauchy sequence (or, more simply, is said to be Cauchy) if for any $\varepsilon > 0$, there exists a $N \in \mathbb{N}$ s.t. if $n \geq N$ and $m \geq N$ (briefly $n, m \geq N$) then $|S_n - S_m| < \varepsilon$ //

Ex 3.16/ $\{S_n\} = \{\frac{1}{n}\}$ is Cauchy. let $\varepsilon > 0$ for any $N > 0$, let $m, n \geq N$. then $|S_n - S_m| = |\frac{1}{n} - \frac{1}{m}|$ $\leq \frac{1}{n} + \frac{1}{m} \leq \frac{2}{N}$ take $N = \left[\frac{2}{\varepsilon}\right] + 1$ then $|S_n - S_m|$ $\leq \frac{2}{N} < \varepsilon$

a Cauchy sequence is always convergent and a convergent sequence is always a Cauchy sequence

Convergent implies Cauchy

Lemma 3.17: let $\{S_n\}$ be a convergent sequence of real numbers. then $\{S_n\}$ is Cauchy //

Cauchy implies convergent

Thm 3.18:- Cauchy convergence criterion
let $\{S_n\}$ be a Cauchy sequence. then $\{S_n\}$ is convergent //

Lemma 3.17: proof: suppose $S_n \to S$ as $n \to \infty$ let $\varepsilon > 0$ then $\exists$ an N s.t. if $n \geq N$ then $|S_n - S| < \varepsilon/2$ now take $m, n \geq N$ then $|S_n - S_m| = |(S_n - S) - (S_m - S)|$ $\leq |S_n - S| + |S_m - S| < \frac{\varepsilon}{2} + \varepsilon/2 = \varepsilon$ $\square$

thm 3.18: Proof: Let $\{S_n\}$ be Cauchy then $\{S_n\}$ is bounded (see ex sheet) by thm 3.14(a) there is $\therefore$ a convergent subsequence $S_{n_k} \to S$ as $k \to \infty$ for some $S \in \mathbb{R}$. we claim $S_k \to S$ as $k \to \infty$ $[S_n = S_k]$
Let $\varepsilon > 0$, then $\exists$ N_1 s.t. if $k \geq N_1$, $|S_{n_k} - S| < \varepsilon/2$
there also exists N_2 s.t. if $m, n \geq N_2$ then $|S_m - S_n| < \varepsilon/2$ if $k \geq \max\{N_1, N_2\}$ then $|S_k - S|$
$= |S_k - S_{n_k} + S_{n_k} - S| \leq |S_k - S_{n_k}| + |S_{n_k} - S| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\square$

Series of constants: series

Def 3.21: if $\{a_n\}_k^\infty$ is an infinite sequence of real numbers the symbol $\sum_{n=k}^\infty a_n$ is an **infinite series**, and a_n is the **n^{th} term** of the series. we say that $\sum_{n=k}^\infty a_n$ **converges to the sum** A, and write $\sum_{n=k}^\infty a_n = A$ if the sequence $\{A_n\}_k^\infty$ defined by $A_n = a_k + a_{k+1} + \dots + a_n$, $n \geq k$ converges to A the finite sum A_n is the **n^{th} partial sum** of $\sum_{n=k}^\infty a_n$

or by changing indexes:
$\sum_{n=k}^\infty a_n = A = \lim_{N \to \infty} A_N = \lim_{n \to \infty} \sum_{n=k}^{N} a_n$ if the sequence $\{A_N\}_k^\infty$ defined by $A_N = a_k + a_{k+1} + \dots + a_N$, $N \geq k$ converges to A the finite sum A_N is the N^{th} partial sum of $\sum_{n=k}^\infty a_n$

if $\{A_n\}_k^\infty$ diverges, we say that $\sum_{n=k}^\infty a_n$ **diverges**: in particular, if $\lim_{n \to \infty} A_n = \infty$ or $-\infty$ we say that $\sum_{n=k}^\infty a_n$ **diverges to ∞ or $-\infty$**, and write $\sum_{n=k}^\infty a_n = \infty$ or $\sum_{n=k}^\infty a_n = -\infty$ a divergent infinite series that does

not diverge to $\pm\infty$ is said to **oscillate**, or **be** not necessarily periodic
oscillatory \\

thm 3.23 - Cauchy's Convergence criterion for series
a series $\sum a_n$ converges if and only if for every
$\varepsilon > 0$ there is an integer N s.t.
$|a_n + a_{n+1} + \dots + a_m| < \varepsilon$ if $m \geq n \geq N$ (*) \\

proof: let $\{a_n\}$ denote the sequence of partial sums of $\sum a_n$ then $A_m - A_{n-1} = a_n + \dots + a_m$ therefore,
$|A_m - A_{n-1}| < \varepsilon$ if $m \geq n \geq N$ (**) (restatement of (*))
to say $\sum a_n$ is convergent is to say $\{A_n\}$ is convergent
this is equivalent to $\{A_n\}$ being Cauchy which is statement (**) $\square$

Corollary 3.25: if $\sum a_n$ converges then $\lim_{n\to\infty} a_n = 0$ \\
proof: taking $m = n$ in thm 3.23, for all $\varepsilon > 0$, $\exists N > 0$, s.t. $|a_n| < \varepsilon$ if $n \geq N$. that is, $\lim_{n\to\infty} a_n = 0$ $\square$
Corollary: (Divergence test)
the contrapositive is: if $\lim_{n\to\infty} a_n \neq 0$, then $\sum a_n$ diverges \\

the converse is: if $\lim_{n\to\infty} a_n = 0$, then $\sum a_n$ converges
THIS IS **FALSE!!!** {standard counter example is the harmonic series: $\sum_{n=1}^{\infty} \frac{1}{n} = \infty$ even though $\lim_{n\to\infty} \frac{1}{n} = 0$ }

pointwise convergence \Def 3.28/ Suppose that $\{F_n\}$ is a sequence of funcs on D & a sequence of vals $\{F_n(x)\}$ converges for each x in some subset S of D then say that $\{F_n\}$ converges pointwise on S to a limit func F defined by $F(x) = \lim_{n\to\infty} F_n(x)$, $x \in S$ \\
So sequence $\{F_1(x), F_2(x), \dots\}$ has a limit $\forall x \in S$ to give F

\ex/ $F_n(x) = \frac{1}{n}$, $x \in \mathbb{R}$ So $F_1(x) = \frac{1}{1} = 1$, $F_2(x) = \frac{1}{2}$
$F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \frac{1}{n} = 0$ $\forall x \in \mathbb{R}$ \\

\ex/ $F_n(x) = \frac{1}{n}x$, $x \in \mathbb{R}$ $F(x) = \lim_{n\to\infty} F_n(x) = \lim_{n\to\infty} \frac{1}{n}x = 0$ $\forall x \in \mathbb{R}$
\\

\ex/ $F_n(x) = x^n$, $x \in \mathbb{R}$ if $x > 1$ $\lim_{n\to\infty} F_n(x) = \infty$
if $x = 1$ $\lim_{n\to\infty} F_n(1) = 1$ if $0 \le x < 1$ $\lim_{n\to\infty} F_n(x) = 0$

if $0 < x < -1$; $\lim_{n\to\infty} F_n(x) = 0$
if $x \le -1$, $\lim_{n\to\infty} F_n(x)$ does not exist $\therefore F(x) = \lim_{n\to\infty} F(x) = \begin{cases} 1, & x = 1 \\ 0, & -1 < x < 1 \end{cases}$

$S = (-1, 1]$
notation $\|g\|_S = \sup_{x \in S} |g(x)| = \sup\{|g(x)| \mid x \in S\}$
{supremum norm} $f(x) = 2x$ $S = (-1, \frac{1}{2}]$
$\therefore \|f\|_S = \sup\{|2x| \mid x \in (-1, \frac{1}{2}]\} = \sup[0, 2] = 2$
{"length", "size", e.g. length of a vector $\|\underline{v}\| = \sqrt{\underline{v} \cdot \underline{v}}$}
Lemma 3.33: if g & h are defined on S, then
$\|g + h\|_S \le \|g\|_S + \|h\|$ {triangle ineq}
& $\|gh\|_S \le \|g\|_S \|h\|_S$ Moreover if either g or h is bounded on S then $\|g - h\|_S \ge |\|g\|_S - \|h\|_S|$ {reverse triangle ineq} \\

- Def 3.34: a sequence $\{F_n\}$ of funcs defined on a set S converges uniformly to a limit func F if $\lim_{n\to\infty} \|F_n - F\|_S = 0$ thus $\{F_n\}$ converges uniformly on S

to F on S if for each $\varepsilon > 0$ there is an integer N s.t.
$\|F_n - F\|_S < \varepsilon$ if $n \geq N$
for $\|F_n - F\|_S = \sup\{|F_n(x) - F(x)| \mid x \in S\}$ //
<u>uniform convergence</u>: \ex / $F_n(x) = \frac{1}{n}$ $x \in \mathbb{R}$
$\|F_n - F\|_{\mathbb{R}} = \sup_{x \in \mathbb{R}} |\frac{1}{n} - 0| = \frac{1}{n} \leq \frac{1}{N}$ $\forall n \geq N$ when $N > \frac{1}{\varepsilon}$
$\therefore F(x) = 0$, $x \in \mathbb{R}$ is uniformly convergent //
\ex / $F_n(x) = \frac{1}{n}x$ $\forall x \in \mathbb{R}$
$F(x) = 0$, $\forall x \in \mathbb{R}$
$\|F_n - F\|_{\mathbb{R}} = \sup_{x \in \mathbb{R}} |F_n(x) - F(x)| = \sup_{x \in \mathbb{R}} |\frac{1}{n}x - 0| = \infty$
hence $\{F_n\}$ does not converge uniformly to F //

\ex / $F_n(x) = x^n$ $x \in (-1, 1]$ $\quad F(x) = \begin{cases} 1, & x = 1, \\ 0, & -1 < x < 1 \end{cases}$
for $n \in \mathbb{N}$ consider $x_n = (2\varepsilon)^{1/n}$
with $0 < \varepsilon < \frac{1}{2}$ $\|F_n - F\|_{(-1,1]} = \sup_{x \in (-1,1]} |x^n - F(x)| \geq |x_n^n - 0| = 2\varepsilon > \varepsilon$
//
<u>thm 3.36</u>: let $\{F_n\}$ be defined on S then
ⓐ $\{F_n\}$ converges pointwise to F on S if & only if there is, for each $\varepsilon > 0$ & $x \in S$, an integer N (which may depend on x as well as ε) s.t. $|F_n(x) - F(x)| < \varepsilon$ if $n \geq N$
ⓑ $\{F_n\}$ converges uniformly to F on S if & only if there is for each $\varepsilon > 0$ an integer N (which depends only on ε & not on any particular x in S) s.t. $|F_n(x) - F(x)| < \varepsilon$ $\forall x$ in S if $n \geq N$ //
<u>thm 3.38</u>: if $\{F_n\}$ converges uniformly to F on S then $\{F_n\}$ converges pointwise to F on S. The converse is false; that is pointwise convergence does not imply uniform convergence //
<u>thm 3.40 - Cauchy's uniform convergence criterion)</u>: a sequence of funcs $\{F_n\}$ converges uniformly on a set S if & only if for each $\varepsilon > 0$ there is an integer N s.t. $\|F_n - F_m\|_S < \varepsilon$ if $n, m \geq N$ //

Proof: (only if) suppose $\{F_n\}$ converges uniformly on S then for $\varepsilon > 0$, $\exists N > 0$ s.t. $\|F_k - F\|_S < \varepsilon/2$ if $k \geq N$ $\therefore$ $\|F_n - F_m\|_S = \|F_n - F + F - F_m\|_S$
$\leq \|F_n - F\| + \|F_m - F\|_S = \varepsilon/2 + \varepsilon/2 = \varepsilon$ if $m, n \geq N$

(if) suppose $\|F_n - F_m\| < \varepsilon$ if $m, n \geq N$ (*)
thm 3.18 (Cauchys convergence for sequences) implies that $\{F_n(x)\}$ converges for each $x \in S$ ie pointwise convergence holds for some func F on S to see that the convergence is uniform we have for $n, m \geq N$

$|F_m(x) - F(x)| = |F_m(x) - F_n(x) + F_n(x) - F(x)|$
$\leq |F_m(x) - F_n(x)| + |F_n(x) - F(x)| \leq$
$\|F_m - F_n\|_S + |F_n(x) - F(x)| \leq \varepsilon + |F_n(x) - F(x)|$ {by (*)}

Since $\lim_{n\to\infty} F_n(x) = F(x)$ $|F_n(x) - F(x)| < \varepsilon$ for some $n \geq N$
hence $|F_m(x) - F(x)| < \varepsilon + \varepsilon = 2\varepsilon$ if $m \geq N$ But x was completely arbit. So this holds $\forall x \in S$ hence $\|F_m - F\|_S \leq 2\varepsilon$ if $m \geq N$ & since $\varepsilon > 0$ is arbit $\{F_n\}$ converges uniformly on S □

Thm 3.42: Properties preserved by uniform convergence
if $\{F_n\}$ converges uniformly to F on S & each F_n is cont at a point x_0 in S then so is F similar statements hold for continuity from the right & left
//

Proof: suppose that each F_n is cont at $x_0 \in S$ if $x \in S$ & $n \geq 1$ then $|F(x) - F(x_0)| \leq$
$\leq |F(x) - F_n(x)| + |F_n(x) - F_n(x_0)| + |F_n(x_0) - F(x_0)|$ {double triangle ineq} $\leq |F_n(x) - F_n(x_0)| + 2\|F_n - F\|_S$ (†)

suppose $\varepsilon > 0$ since $\{F_n\}$ converges uniformly to F on S we can choose a fixed $\bar{n}$ so that $\|F_{\bar{n}} - F\|_S < \varepsilon$ by (†) $|F(x) - F(x_0)|$

$< |F_{\bar{n}}(x) - F_{\bar{n}}(x_0)| + 2\varepsilon \quad , x \in S$ (†)

{in (†) take from $|F_n(x) - F_n(x_0)|$ $n = \bar{n}$

& from $\|F_n - F\|_S < \varepsilon$} hence since $F_{\bar{n}}$ is cont at x_0

$\exists \delta > 0$ s.t. $|F_{\bar{n}}(x) - F_{\bar{n}}(x_0)| < \varepsilon$ if $|x - x_0| < \delta$ by (†)

$|F(x) - F(x_0)| < 3\varepsilon$ if $|x - x_0| < \delta$, $\therefore$ F is cont at x_0 $\square$

Corollary 3.43: if $\{F_n\}$ converges uniformly to F on S & each F_n is cont on S, then so is F; that is a uniform limit of cont funcs is cont //

pointwise & uniform convergence

Def 3.44: if $\{f_j\}_k^\infty$ is a sequence of real-valued funcs defined on a set D of reals, then $\sum_{j=k}^{\infty} f_j$ is an infinite series (or simply a series) of funcs on D & partial sums of $\sum_{j=k}^{\infty} f_j$ are defined by $F_n = \sum_{j=k}^{n} f_j$, $n \geq k$ if $\{F_n\}_k^\infty$ converges pointwise to a func F on a subset S of D we say that $\sum_{j=k}^{\infty} f_j$ converges pointwise to the sum F on S, & write

$F(x) = \sum_{j=k}^{\infty} f_j(x) = \lim_{n\to\infty} F_n(x) \quad x \in S$ if $\{F_n\}$ converges uniformly to F on S, we say that $\sum_{j=k}^{\infty} f_j$ converges uniformly to F on S //

|ex 3.45/ $f_j(x) = x^j$, $j \geq 0$, $x \in \mathbb{R}$

$\sum_{j=0}^{\infty} f_j(x)$ has partial sums $F_n(x) = 1 + x + x^2 + \dots + x^n$

$= \begin{cases} \frac{1 - x^{n+1}}{1 - x}, & x \neq 1, \\ n+1, & x = 1 \end{cases}$ $\therefore \{F_n\}$ converges pointwise to

$F(x) = \frac{1}{1-x}$ if $x \in (-1, 1)$ & diverges if $|x| \geq 1$ is pointwise convergence

Consider now $\|F - F_n\|_{(-1,1)} = \sup_{x \in (-1,1)} \left| \frac{x^{n+1}}{1-x} \right| = \infty$ hence we do not have uniform convergence on (-1,1) convergence is

uniform on $[-r,r]$ where $0<r<1$ since then
$$\|F-F_n\|_{[-r,r]}=\frac{r^{n+1}}{1-r}\xrightarrow{n\to\infty}0 \quad \backslash\backslash$$

tests for uniform converg of series

thm3.46: (Cauchy's uniform converg criterion)
a series $\sum f_n$ converges uniformly on a set S if & only if for each $\varepsilon>0$ there is an integer N s.t. $\|f_n+f_{n+1}+\dots+f_m\|_S<\varepsilon$ if $m\geq n\geq N$ \\ $\circledast$ {if $n=m \Rightarrow f_n$}

proof: apply thm3.40 (Cauchy uniform conver for sequences of funcs) to the partial sums of $\sum f_n$ noting that $f_n+f_{n+1}+\dots+f_m=F_m-F_{n-1}$ $\square$

Corollary 3.47: if $\sum f_n$ converges uniformly on S then $\lim_{n\to\infty}\|f_n\|_S=0$ \\

proof: set $n=m$ in $\circledast$ in thm3.46 $\square$

contrapositive: if $\lim_{n\to\infty}\|f_n\|_S\neq 0$ then $\sum f_n$ does not converge uniformly "not uniformly convergent test" (analogous to the divergence test" for series of consts)

thm3.48 (weierstrass's test): the series $\sum f_n$ converges uniformly on S if $\|f_n\|_S\leq M_n$ $n\geq k$ $\oplus$
where $\sum M_n<\infty$ \\

proof: from Cauchy's convergence criterion for series of consts, for each $\varepsilon>0$ $\exists N>0$ s.t. $M_n+M_{n+1}+\dots+M_m<\varepsilon$ if $m\geq n\geq N$, so,
$\|f_n+f_{n+1}+\dots+f_m\|_S\leq\|f_n\|_S+\|f_{n+1}\|_S+\dots+\|f_m\|_S<\varepsilon$ if $m\geq n\geq N$ {by $\oplus$} hence by thm3.46, $\sum f_n$ converges uniformly on S $\square$

ex 3.49 taking $M_n = 1/n^2$ $\sum \frac{1}{n^2} < \infty$ we have both $\frac{1}{x^2+n^2} < \frac{1}{n^2}$ & $\frac{\sin(nx)}{n^2} < \frac{1}{n^2}$ for all $x \in \mathbb{R}$ hence by Z weierstrass test, the series $\sum \frac{1}{x^2+n^2}$ & $\sum \frac{\sin(nx)}{n^2}$ both converge uniformly on $\mathbb{R}$ //

ex 3.50 let $f_n(x) = \left(\frac{x}{1+x}\right)^n$ then if $x \in S$ satisfies $\left|\frac{x}{1+x}\right| \leq r$ with $0<r<1$, then $\|f_n\|_S \leq r^n$, hence since $\sum r^n$ {with $0<r<1$} $\sum r^n < \infty$ $\therefore$ $\sum f_n$ ~~converges~~ CU (converges uniformly) by Z weierstrass test (with $M_n = r^n$) could equivalently $\frac{-r}{1+r} \leq x \leq \frac{r}{1-r}$ so $\sum f_n$ CU on any compact subset of $(-\frac{1}{2}, \infty)$ but does not CU on $(-1/2, b)$, $b < \infty$ or $[a, \infty)$, $a > -\frac{1}{2}$ (why?)

Riemann integrals: changes depending on whether you choose height from right or left hand side {could also take any point inside those sub intervals like the midpoint}

lower reemann sum is when you always take the infemum of the height for all partition points and upper reemann sum is when you always take Z supremum

so all heighs are between the infimum and supremum so reemann integral could be the common value obtained for Z upper & lower reimann integral

Def (2.1): Partition: let $[a,b]$ be a closed set. A partition P of $[a,b]$ is a finite ordered sequence of real numbers $P = (x_0, x_1, x_2, \ldots, x_n)$ with $a = x_0 < x_1 < x_2 < \cdots < x_n = b$ the points $x_0, \ldots, x_n$ are called the partition points of P. So, P divides the interval $[a,b]$ into n subintervals where n is a positive integer

so $a = x_0$ then x_1 to x_2 is I_2 ... then ... x_{n-1}
then $b = x_n$ and $I_i = [x_{i-1}, x_i]$
so partition notation: $P = (x_0, x_1, x_2, \dots, x_n)$
$\Delta x_i = x_i - x_{i-1}$ {width of I_i} > 0 so $\Delta x_i > 0$ since ordered
$\|P\| = \max_i \Delta x_i$ so looking into ideas of $\lim_{n\to\infty} \|P_n\| = 0$
so if for ex $\Delta x_2 \neq \Delta x_4$ it could be $\|P\| = \Delta x_2 > \Delta x_4$
Tags $T = (t_1, t_2, \dots, t_n)$ $t_i \in [x_{i-1}, x_i]$ so
(P, T) is a tagged partition
Riemann Sums: a general riemann sum takes the form
$S(f;P;T) = \sum_{i=1}^{n} f(t_i)\Delta x_i$
Riemann Sums: Lower & upper Riemann Sums:
Def 2.2: let $f: [a,b] \to \mathbb{R}$ be a bounded func on the int $[a,b]$, & $P = (x_0, x_1, \dots, x_n)$ be a partition of $[a,b]$
let $m_i = \inf_{x\in[x_{i-1},x_i]} f(x)$ $M_i = \sup_{x\in[x_{i-1},x_i]} f(x)$

① the Lower Riemann sum of f with respect to a partition P is $L(f;P) = \sum_{i=1}^{n} m_i \Delta x_i$
② the upper Riemann sum of f with respect to partition P is
$U(f;P) = \sum_{i=1}^{n} M_i \Delta x_i$

boundedness: • for a given partition P & any valid tagging T we have $m_i \le f(t_i) \le M_i \ \forall i = 1, 2, \dots, n$ &
$L(f;P) \le S(f;P,T) \le U(f;P)$
• letting $m = \inf_{x\in[a,b]} f(x)$ & $M = \sup_{x\in[a,b]} f(x)$ {exist since f is bounded} $m(b-a) \le L(f;P) \le U(f,P) \le M(b-a)$ for any partition P { $L(f;P) \le S(f;P;T) \le U(f;P)$ }

refinement: A partition Q is a ~~f~~ refinement of a partition P if for all points $x \in P$ then also $x \in Q$ & Q contains more points than P i.e. ~~if~~ P contains $|P| = n$ points then Q contains $|Q| = m$ points with $m > n$

refining a partition decreases the upper Riemann sums and increases the lower Riemann sums

Thm 2.3: Convergence of Riemann Sums: let $f : [a,b] \to \mathbb{R}$ & let P be a partition of the int $[a,b]$ & let Q be a refinement of P then $L(f;P) \leq L(f;Q) \leq U(f;Q) \leq U(f;P)$

proof: let $P = (x_0, x_1, \dots, x_n)$ let $h \in \{1, \dots, n-1\}$ & consider $[x_h, x_{h+1}]$ let $c \in (x_h, x_{h+1})$ & $Q = (x_0, \dots, x_h, c, x_{h+1}, \dots, x_n)$ then $\inf\limits_{x \in [x_h, c]} f(x) \geq \inf\limits_{x \in [x_h, x_{h+1}]} f(x)$ & $\inf\limits_{x \in [c, x_{h+1}]} f(x) \geq \inf\limits_{x \in [x_h, x_{h+1}]} f(x)$ now consider

$$L(f;Q) - L(f;P) = (c - x_h) \inf_{x \in [x_h, c]} f(x) + (x_{h+1} - c) \inf_{x \in [c, x_{h+1}]} f(x) - (x_{h+1} - x_h) \inf_{x \in [x_h, x_{h+1}]} f(x) \geq \left[(c - x_h) + (x_{h+1} - c) - (x_{h+1} - x_h)\right] \inf_{x \in [x_h, x_{h+1}]} f(x)$$

$\{\text{all cancels to } 0\} = 0 \quad \therefore L(f;Q) \geq L(f;P)$

the demonstration that $U(f;Q) \leq U(f;P)$ is similar ex

so we have $L(f;P) \leq L(f;Q) \leq U(f;Q) \leq U(f;P)$ this is for a refinement Q with one extra point if Q has k extra points then repeat the above argument k times $\square$

Corollary (2.5) let $f : [a,b] \to \mathbb{R}$ be bounded let P_1 & P_2 be two partitions of $[a,b]$ then $L(f;P_1) \leq U(f;P_2)$

proof: let Q be a common refinement of P_1 & P_2 that is to say all $x \in P_1$ are in Q and all $x \in P_2$ are in Q then thm 2.3 says that $L(f;P_1) \leq L(f;Q) \leq U(f;Q) \leq U(f;P_2)$

$\square$

Riemann integral: lower & upper Riemann integrals:

Def 2.7: let $f:[a,b]\to\mathbb{R}$ be a bounded func on Z int $[a,b]$ & P ranges over all partitions of $[a,b]$

then ① Z lower Riemann integral of f over $[a,b]$ is

$\underline{\int_a^b} f(x)\,dx = \sup_P L(f;P)$ {exists since f is bounded}

② Z upper Riemann integral of f over $[a,b]$ is

$\overline{\int_a^b} f(x)\,dx = \inf_P U(f;P)$ {exists since f is bounded}

Proposition (2.8) if $f:[a,b]\to\mathbb{R}$ is bounded then

$$\underline{\int_a^b} f(x)\,dx \le \int_a^b f(x)\,dx$$

proof: let $L = \{L(f;P) \mid P \text{ is a partition of } [a,b]\}$

$U = \{U(f;P) \mid P \text{ is a partition of } [a,b]\}$ then for all $l \in L$ & $u \in U$ by Cor 2.5; $l \le u$, $\therefore \sup L \le \inf U$ $\square$

Z Riemann integral

Def 2.9:

a bounded func f is said to be Riemann integrable on Z int $[a,b]$ if Z lower & upper Riemann integrals are equal. When this common value exists, we call it Z riemann integral & denote it by $\int_a^b f(x)\,dx$

$= \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$ if they are not equal Z func is not Riemann integrable

ex 2.11/ $f:[a,b]\to\mathbb{R}$ $f(x) = C$ for any partition P,

$m_i = \inf_{x\in[x_{i-1},x_i]} f(x) = C$ $\quad M_i = \sup_{x\in[x_{i-1},x_i]} f(x) = C$ $\therefore$

$L(f;P) = U(f;P) = \sum_{i=1}^{n} C\,\Delta x_i = C\sum_{i=1}^{n}\Delta x_i = C(b-a)$ $\quad \therefore \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b}$ $= C(b-a)$

$= \int_a^b f(x)\,dx = \int_a^b f(x)\,dx$

\ex2.12/ $f:[-1,1]\to\mathbb{R}$ $f(x)=\begin{cases}1 & x=0\\ 0 & x\neq 0\end{cases}$

consider partitions $P_h=$

$=(-1,-\frac{1}{2^h},\frac{1}{2^h},1)$ $h\in\mathbb{N}$ $\therefore L(f,P_h)=$

$0\times(-\frac{1}{2^h}-(-1))+0\times(\frac{1}{2^h}-(-\frac{1}{2^h}))+0\times(1-\frac{1}{2^h})=0$

$U(f;Ph)=0\times(-\frac{1}{2^h}-(-1))+1\times(\frac{1}{2^h}-(-\frac{1}{2^h}))+0\times(1-\frac{1}{2^h})=\frac{1}{2^h}+\frac{1}{2^h}$

$=\frac{2}{2^h}=\frac{1}{2^{h-1}}$ $\therefore$

$\underline{\int_{-1}^{1}} f(x)dx=\sup_P L(f;P)\geq\sup_h L(f;P_h)=0 = \alpha$ but

$\overline{\int_{-1}^{1}} f(x)dx=\inf_P U(f;P)\leq\inf_h U(f;P_h)=\inf_h\left\{\frac{1}{2^{h-1}}\right\}=0=\beta$

$\therefore \alpha\geq 0$, $\beta\leq 0$ but from Prop 2.8: $\alpha\leq\beta$ so

$0\leq\alpha\leq\beta\leq 0$ $\therefore \alpha=\beta=0$ & $\int_{-1}^{1}f(x)\,dx=0$

\ex/ dirichlet function $f:[0,1]\to\mathbb{R}$

$f(x)=\begin{cases}1, & x \text{ is rational}\\ 0, & x \text{ is irrational}\end{cases}$ $\therefore L(f;P)=0$ {since irrational numbers are dense in real line no matter how small the interval there will be a irrational number}

$U(f;P)=1$ {since theres always a ration number in these sub intervals no matter how small} $\therefore$

$\underline{\int_0^1} f(x)dx=0\neq 1=\overline{\int_0^1} f(x)dx$ $\therefore f$ is not Riemann integrable

any continuous func is itegrable

any cont func is riemman integrable $F(x)=\int_a^x f(t)dt$ is a cont func

linearity \thm2.16/: linearity of Σ rieman integral

let $f:[a,b]\to\mathbb{R}$ & $g:[a,b]\to\mathbb{R}$ be Riemann integrable & let $c\in\mathbb{R}$ then $f+cg$ is intergrable & $\int_a^b f(x)+cg(x)dx=\int_a^b f(x)dx+c\int_a^b g(x)dx$

so $\int_a^b f(x)+g(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$

& $\int_a^b c f(x)\,dx = c\int_a^b f(x)\,dx$ since $\underline{A}(\underline{u}+\underline{v}) = \underline{A}\underline{u} + \underline{A}\underline{v}$

$\underline{A}(c\underline{v}) = c\underline{A}\underline{v}$

linear integration fits into class of operations called: "linear functions"

<u>additivity</u>

\thm: 2.17/. <u>additivity of Σ</u> <u>riemann integral</u>

let $f:[a,b]\to\mathbb{R}$ & let $c\in(a,b)$ then f is integrable iff $f|_{[a,c]}$ & $f|_{[c,b]}$ are integrable &

$\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$

$f|_{[a,c]}:[a,c]\to\mathbb{R}$ $\quad f|_{[a,c]}(x) = f(x)$

<u>Continuity</u>

\thm 2.18:/ <u>Continuous</u> <u>funcs</u> <u>are</u> <u>integrable</u>

let $f:[a,b]\to\mathbb{R}$ be cont on $[a,b]$ then f is integrable

<u>proof</u>: we show that Riemann integrability criterion holds (thm 2.15) ∴ since f is cont on $[a,b]$ it is uniformly cont on $[a,b]$ (part 1 thm 2.53) ∴ ~~$\forall\varepsilon>0$~~ $\forall\varepsilon>0, \exists\delta>0$ s.t. $|f(x)-f(y)| < \frac{1}{2(b-a)}\varepsilon \quad \forall |x-y|<\delta$

let ~~PA~~ $P=(x_0,\dots,x_k)$ be a partition of $[a,b]$ s.t. $\Delta x_i<\delta$ $\forall i=1,\dots,k$ eg. let $x_i = a+\frac{(b-a)i}{k}$ for $i=0,\dots,k$ with $k>\frac{(b-a)}{\delta}$ now for $x,y\in[x_{i-1},x_i]$, T8

$|f(x)-f(x)| < \frac{1}{2(b-a)}\varepsilon$ ∴ $\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x) \le \frac{1}{2(b-a)}\varepsilon$

$< \frac{1}{(b-a)}\varepsilon$ now the difference between upper & lower Riemann sums:

$U(f;P) - L(f;P) = \sum_{i=1}^{k} \left[\sup_{x\in[x_{i-1},x_i]} f(x) - \inf_{x\in[x_{i-1},x_i]} f(x) \right] \Delta x_i$

$< \sum_{i=1}^{k} \frac{1}{(b-a)} \varepsilon \, [x_i - x_{i-1}] = \frac{\varepsilon}{(b-a)} (-x_0 + x_k) = \frac{\varepsilon}{(b-a)} (-a+b) = \varepsilon$

$\therefore$ by Z Riemann criterion for integrability f is Riemann integrable $\square$

Continuity

\thm 2.19: / indefinite integrals are continuous

let $f:[a,b]\to\mathbb{R}$ be integrable on $[a,b]$ & F defined on $[a,b]$ as $F(x) = \int_a^x f(t)\,dt$ then F is cont on $[a,b]$

so is $F(x_0) = \int_a^{x_0} f(t)\,dt$ need $\lim_{x\to x_0} F(x) = F(x_0)$

note that f being cont so it is integrable (thm 2.18) is sufficient but not necessary so is a func is cont then it is integrable but not all integrable funcs are continuous

like $f(x) = \begin{cases} 1, & x=0 \\ 0, & x\neq 0 \end{cases}$ so for $x\in[-1,1]$ $\int_{-1}^{1} f(x)\,dx = 0$

but it is not cont (ex 2.12)

First FTC

\thm 2.20: / the first fundamental thm of calculus

let $f:[a,b]\to\mathbb{R}$ be a cont func & let's define Z func $F(x) = \int_a^x f(t)\,dt$ then F is cont on $[a,b]$ diffable on (a,b) & we have $F'(x) = f(x)$ $\forall x\in(a,b)$

proof: since f is cont on $[a,b]$ it is also uniformly cont on $[a,b]$: $\forall\varepsilon>0, \exists\delta>0$ s.t. for $|x-y|<\delta$,

$|f(x) - f(y)| < \varepsilon$ we will show $f(x) = F'(x) = \lim_{y\to x} \frac{F(y)-F(x)}{y-x}$

we have $\left| \frac{F(y)-F(x)}{y-x} - f(x) \right| = \left| \frac{1}{y-x} \int_x^y f(t)\,dt - f(x) \right|$

$\left\{ \text{since } F(y) - F(x) = \int_a^y f(t)\,dt - \int_a^x f(t)\,dt \right\}$ $= \left| \frac{1}{y-x} \int_x^y (f(t) - f(x))\,dt \right|$

$\left\{ \text{since } \frac{1}{y-x}\int_x^y f(x)\,dt = \frac{y-x}{y-x} f(x) \right\}$

$\therefore$ since $x<t<y$, $|x-t|<\delta$ & $\therefore |f(t)-f(x)|<\delta$

$\therefore \left| \frac{1}{y-x}\int_x^y (f(t)-f(x))\,dt \right| \leq \frac{1}{|y-x|} \sup_{t\in[x,y]} |f(t)-f(x)||y-x|$

$= \sup_{t\in[x,y]} |f(t)-f(x)| < \varepsilon \quad \therefore \left| \frac{F(y)-F(x)}{y-x} \right| < \varepsilon \quad \therefore F'(x)=f(x)$ $\square$

Second FTC:

Thm 2.21: The Second fundamental thm of Calculus
let $f:[a,b]\to\mathbb{R}$ be cont & $F:[a,b]\to\mathbb{R}$ be cont on $[a,b]$
& diffable on (a,b) where $F'(x)=f(x)$ then

$$\int_a^b f(t)\,dt = F(b)-F(a)$$

proof: define $G:[a,b]\to\mathbb{R}$ by $G(x)=\int_a^x f(t)\,dt$ using
the first FTC: $G'(x)=f(x)$ now consider
$H(x)=G(x)-F(x)$ $\therefore H'(x)=G'(x)-F'(x)=f(x)-f(x)=0$
$\therefore$ this implies $H(x)$ is const on $[a,b]$ say $H(x)=A$
however, $G(a)=\int_a^a f(t)\,dt=0$ so $H(a)=G(a)-F(a)$
$=-F(a)=A$ finally $G(x)=F(x)+A=F(x)-F(a)$ $\forall x\in[a,b]$
and taking $x=b$: $G(b)=\int_a^b f(t)\,dt = F(b)-F(a)$ $\square$

Substitution formula:

Thm 2.22: Substitution formula: let $I_1=[\alpha,\beta]$ &
$I_2=[\gamma,\delta]$ be finite intervals let $g: I_1\to(\gamma,\delta)$ be a
diffable func with g' cont on (α,β)
suppose $f: I_2\to\mathbb{R}$ is cont then if $[a,b]\subset(\alpha,\beta)$
{so $[a,b]$ is contained in (α,β)} $\int_{g(a)}^{g(b)} f(t)\,dt = \int_a^b f(g(t))g'(t)\,dt$

proof: by the 1st FTC since $f(t)$ is cont by the chain
$\exists F:[\gamma,\delta]\to\mathbb{R}$ with $F'(x)=f(x)$ $\forall x\in(\gamma,\delta)$

rule $F \circ g : [\alpha, \beta] \to \mathbb{R}$ is diffable on (α, β) with
$(F\circ g)'(x) = F'(g(x))g'(x) \quad \forall x \in (\alpha, \beta)$ now using 2

2nd FTC: $\int_a^b f(g(t))g'(t)\,dt = \int_a^b (F\circ g)'(t)\,dt$

$= (F\circ g)(b) - (F\circ g)(a) = F(g(b)) - F(g(a)) = \int_{g(a)}^{g(b)} f(t)\,dt$ $\square$

ex 2.23/ given 2 change variable $g(t) = \frac{1}{t}$ show for, $0 < a < b$ $\int_a^b \frac{1}{1+t^2}\,dt = \int_{1/b}^{1/a} \frac{1}{1+t^2}\,dt$

let $f(t) = \frac{1}{1+t^2}$ & take $I_1 = [\frac{a}{2}, 2b]$ $I_2 = [\frac{1}{2b}, \frac{2}{a}]$

$\therefore g(t) : I_1 \to I_2$ & $g'(t) = -\frac{1}{t^2}$ is cont on $(\frac{a}{2}, 2b)$

$\therefore$ since $f : I_2 \to \mathbb{R}$ is cont $\therefore \int_{g(a)}^{g(b)} f(t)\,dt = \int_{1/a}^{1/b} \left(\frac{1}{1+t^2}\right)dt$

$= \int_a^b \frac{1}{1+(\frac{1}{t})^2}\left(-\frac{1}{t^2}\right)dt = -\int_a^b \frac{1}{1+t^2}\,dt \quad \therefore \int_{1/b}^{1/a} \frac{1}{1+t^2}\,dt = \int_a^b \frac{1}{1+t^2}\,dt$

if $f(x) = \lim_{n\to\infty} f_n(x)$ (pointwise convergence) then does $\lim_{n\to\infty} \int_a^b f_n(x) = \int_a^b f(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx$ {when can we interchange the limit & integral sign}

answer/ only pointwise convergence: sometime the same
if uniform convergence: Yes! always the same

point wise convergence

ex/ $f_n(x) = \begin{cases} 0, & x \in [\frac{1}{n}, 1], x = 0 \\ n, & x \in (0, \frac{1}{n}] \end{cases}$

$\therefore f_1(x) = \begin{cases} 0, & x \in [1, 1] \\ 1, & x \in (0, 1] \end{cases}$ $\quad f_2(x) = \begin{cases} 0, & x \in [\frac{1}{2}, 1] \\ 2, & x \in (0, \frac{1}{2}) \end{cases}$

$f_3(x) = \begin{cases} 0, & x \in [\frac{1}{3}, 1] \\ 3, & x \in (0, \frac{1}{3}) \end{cases}$ $\quad \dots \quad \therefore \{ f(x) = \lim_{n\to\infty} f_n(x) = 0,$ for $x \in [0, 1]\}$ but does not converge uniformly on $[0, 1]$

$\int_0^1 f_n(x)\,dx = n\cdot\frac{1}{n} = 1 \quad \therefore \lim_{n\to\infty}\int_0^1 f(x)\,dx = 1 \neq \int_0^1 f(x)\,dx = 0$

$= \int_0^1 0\,dx$

ex/: $f_n(x) = \begin{cases} 0 & x\in[\frac{1}{n^2},1] \quad x=0 \\ n & x\in(0,\frac{1}{n^2}) \end{cases}$ $\therefore$

$f_1(x) = \begin{cases} 0 & x\in[1,1] \\ 1 & x\in(0,1) \end{cases}$ $\quad f_2(x) = \begin{cases} 0 & x\in[\frac{1}{4},1] \\ 2 & x\in(0,\frac{1}{4}) \end{cases}$

$f_3(x) = \begin{cases} 0 & x\in[\frac{1}{9},1] \\ 3 & x\in(0,\frac{1}{9}) \end{cases}$ $\quad \therefore f(x) = \lim_{n\to\infty} = 0 \;,\; x\in[0,1]$

but does not converge uniformly {since for all n there is still a value outside all ε-neighborhoods of that func} $\therefore \int_0^1 f_n(x)\,dx = n\cdot\frac{1}{n^2} = \frac{1}{n}$ $\therefore$

$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = 0 = \int_0^1 f(x)\,dx$

<u>uniform convergence</u>

thm 2.27:/ <u>uniform convergence & riemann integrability</u>

let $\{f_n\}_{n=1}^{\infty}$ be a sequence of riemann integrable funcs defined on a finite int $[a,b]$ if f_n converges uniformly to f as $n\to\infty$ then

- f is riemann integrable &
- $\lim_{n\to\infty}\int_a^b f_n(x)\,dx = \int_a^b f(x)\,dx$

ex/ $f_n(x) = \frac{1}{n}\sin(nx) \;,\; x\in[0,1]$

$\lim_{n\to\infty} f_n(x) = 0 = f(x)$ and uniformly converges

since $f_1(x) = \sin(x)$, $f_2(x) = \frac{1}{2}\sin(2x)$, $f_3(x) = \frac{1}{3}\sin(3x)$

$\therefore$ since there exists for all ε a big enough n to be with that ε neighb for $f_n(x)$ $\therefore$

$\int_0^1 \frac{1}{n}\sin(nx)\,dx = \left[\frac{-\cos(nx)}{n^2}\right]_0^1 = \frac{1-\cos(nx)}{n^2}$ $\therefore$

$\lim_{n\to\infty}\int_0^1 f_n(x)\,dx = 0 = \int_0^1 f(x)\,dx$

<u>the set</u> $\mathbb{R}^n$: the set $\mathbb{R}^n$ consists of Z column vecs with n real components $\mathbb{R}^n = \{(x_1, x_2, \dots, x_n)^T \mid x_i \in \mathbb{R}, i = 1, 2, \dots, n\}$

$\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix} = (x_1, x_2, \dots, x_n)^T$ here we are looking at vecs & components of vecs. let $x, y \in \mathbb{R}^n$ be two vecs in $\mathbb{R}^n$ with $x = (x_1, x_2, \dots, x_n)^T$ $y = (y_1, y_2, \dots, y_n)^T$ then Z sum of two vecs is given by $x + y = (x_1 + y_1, x_2 + y_2, \dots, x_n + y_n)^T$ also if $\lambda \in \mathbb{R}$ we have $\lambda x = (\lambda x_1, \lambda x_2, \dots, \lambda x_n)^T$ these defs tell us that $\mathbb{R}^n$ has Z structure of a vec space over $\mathbb{R}$ {do need to satisfy more axioms from linear algebra}

<u>norms on</u> $\mathbb{R}^n$ \def 3.1:/ a <u>norm</u> is a func $\|\cdot\| : \mathbb{R}^n \to [0, \infty)$ satisfying Z following properties for all $x, y \in \mathbb{R}^n$ and $c \in \mathbb{R}$: {note norm is "length of vec"}

①: $\|x\| \geq 0$ & $\|x\| = 0$ iff $x = 0 = \underline{0} = (0, 0, \dots, 0)^T$

②: $\|cx\| = |c|\|x\|$

③: $\|x + y\| \leq \|x\| + \|y\|$ (triangle inequality)

in $\mathbb{R}$ we've used the norm $\|\cdot\| = |\cdot|$ given by the modulus func we know that $|\cdot|$ satisfies 1, 2, & 3

so ①: $|x| \geq 0$, $|x| = 0$ iff $x = 0$

②: $|cx| = |c||x|$ ③: $|x + y| \leq |x| + |y|$

• the <u>euclidean norm</u>: $\|x\| = \sqrt{\langle x, x \rangle}$ {$\langle x, x \rangle$ is the inner product aka dot product $\therefore \langle x, y \rangle = \sum_{i=1}^{n} x_i y_i$}

$\therefore \|x\| = \sqrt{\langle x, x \rangle} = \sqrt{\sum_{i=1}^{n} x_i^2}$ $\therefore$ for $n = 1$ $\sqrt{x^2} = |x|$

• the 1-norm: $\|x\|_1 = |x_1| + |x_2| + \dots + |x_n|$

• the <u>max-norm</u>: $\|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\}$

• the p-norm $\|x\|_p = \left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}$ $\quad \therefore \|x\|_2 = \|x\| = \sqrt{\langle x, x\rangle}$

$\therefore \lim_{p\to\infty} \|x\|_p = \|x\|_\infty = \max\{|x_1|, |x_2|, \dots, |x_n|\}$

\Thm 3.3:/ <u>the</u> <u>Cauchy-Schwartz</u> <u>inequality</u>: let x, y be ~~vects~~ vecs in $\mathbb{R}^n$ then $\langle x, y\rangle \le \|x\|\|y\|$ $= \sqrt{\langle x,x\rangle}\sqrt{\langle y,y\rangle}$ equivalently can write this using sums:

$$\left(\sum_{i=1}^{n} x_i y_i\right)^2 \le \left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right)$$

$$\{\langle x,y\rangle^2 \le \langle x,x\rangle\langle y,y\rangle\}$$

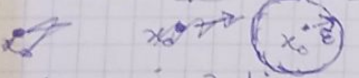
\proof:/ consider $\sum_{i=1}^{n} (ax_i + y_i)^2 \ge 0 \ \forall a \in \mathbb{R}$ expanding this we get: $\sum_{i=1}^{n} (ax_i + y_i)^2 = a^2 \sum_{i=1}^{n} x_i^2 + 2a \sum_{i=1}^{n} x_i y_i + \sum_{i=1}^{n} y_i^2$

considering this as a quadratic in a, it has either one root or no roots hence

$\left(2\sum_{i=1}^{n} x_i y_i\right) - 4\left(\sum_{i=1}^{n} x_i^2\right)\left(\sum_{i=1}^{n} y_i^2\right) \le 0$ which after rearranging gives the C-S ineq. $\square$

<u>topology in $\mathbb{R}^n$</u>: \Def 3.5:/

①: if $x_0 \in \mathbb{R}^n$ & $\varepsilon > 0$ then the set $U(x_0, \varepsilon) = \{x \in \mathbb{R}^n \mid \|x - x_0\| < \varepsilon\}$ is the ε-<u>neighbourhood</u> of x_0 or the <u>open ball of radius</u> ε <u>centred</u> <u>at</u> x_0 {in $\mathbb{R}$ $U = (x_0 - \varepsilon, x_0 + \varepsilon)$}



{not including the line since <}

sphere in 3 dimensions and in 1 dimension

②: a set $S \subset \mathbb{R}^n$ is called an open set if for each $x_0 \in S$ there is an $\varepsilon > 0$ s.t. the ε-neighb of x_0 is contained in S i.e. $U(x_0, \varepsilon) \subset S$

③: a set $S \subset \mathbb{R}^n$ is <u>closed</u> if S^c is open

④: A set ~~SE~~ $S \subset \mathbb{R}^n$ is <u>bounded</u> if $\exists M > 0$ s.t. $\|x\| < M$ for all $x \in S$

⑤: a set $S \subset \mathbb{R}^n$ is <u>compact</u> if it is closed & bounded

Sequences in $\mathbb{R}^n$ \Def/: the sequence $\{x_n\}_{n=1}^{\infty}$ with with $x_n \in \mathbb{R}^p$ $\{$so $x_n = (y_1, y_2, \dots, y_p)^T\}$ **converges to the vector** $x \in \mathbb{R}^p$ if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ we have $\|x_n - x\| < \varepsilon$ we write $\lim_{n\to\infty} x_n = x$

\ex 3.8/ let $x_n \in \mathbb{R}^p$ (p fixed) be given by $x_n = \left(\frac{1}{n}, \frac{1}{n^2}, \dots, \frac{1}{n^p}\right)^T$ then $\lim_{n\to\infty} x_n = 0 = \underbrace{(0, 0, \dots, 0)^T}_{p \text{ components}}$ ~~{p components}~~

let $\varepsilon > 0$ then have $\|x_n - 0\| = \|x_n\| = \cancel{\sum_{i=1}^{p}}$

$$= \sqrt{\sum_{i=1}^{p} \frac{1}{n^{2i}}} = \sqrt{\frac{1}{n^2} + \frac{1}{n^4} + \dots + \frac{1}{n^{2p}}} < \sqrt{\frac{p}{n^2}} = \frac{\sqrt{p}}{N} < \varepsilon$$

for all $n \geq N$ with $N > \frac{\sqrt{p}}{\varepsilon}$

Continuity of $f: \mathbb{R} \to \mathbb{R}$

\Def/: limit of a func: we say that $\lim_{x\to x_0} f(x) = L$ if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. for $0 < |x - x_0| < \delta$ have $|f(x) - L| < \varepsilon$

\Def/ func $f(x)$ is **continuous** at $x = x_0$ if $\lim_{x\to x_0} f(x) = f(x_0)$ equivalently if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. for $0 < |x - x_0| < \delta$ we have $|f(x) - f(x_0)| < \varepsilon$

Funcs of the form $f: \mathbb{R}^n \to \mathbb{R}^m$: a func $f: \mathbb{R}^n \to \mathbb{R}^m$ has as input a vec $x \in \mathbb{R}^n$ & the output $f(x)$ is a vec in $\mathbb{R}^m$ this output vec can be written in terms of the **coordinate funcs** $f_i: \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$ so that $f(x)$ is the column vec $f(x) = (f_1(x) \;\; f_2(x) \;\; \cdots \;\; f_m(x))^T$

~~Th~~ **Limits**: \Def/: let $A \subset \mathbb{R}^n$ & suppose $f: A \to \mathbb{R}^m$ we say that $y_0 \in \mathbb{R}^m$ is the **limit of** f **as** x **tends to** x_0 if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$ with $0 < \|x - x_0\| < \delta$ $\{\|x - x_0\|$ is $\|\cdot\|: \mathbb{R}^n \to [0, \infty)\}$ we have $\|f(x) - y_0\| < \varepsilon$ $\{\|f(x) - y_0\|$ is $\|\cdot\|: \mathbb{R}^m \to [0, \infty)\}$ we write $\lim_{x\to x_0} f(x) = y_0$

Limits & continuity for $f:\mathbb{R}^n \to \mathbb{R}^m$: Continuity:

\Def f/: let $A \subset \mathbb{R}^n$ & $f: A \to \mathbb{R}^m$ we say that f is

- continuous at $x_0 \in A$ if $\lim_{x \to x_0} f(x) = f(x_0)$

equivalently if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$ with $\|x - x_0\| < \delta$ we have $\|f(x) - f(x_0)\| < \varepsilon$ if f is continuous $\forall x_0 \in A$ then we say that f is continuous on A

\ex3.14/ let $f:\mathbb{R}^n \to \mathbb{R}$ be defined by $f(x) = \|x\|$ take an arbitrary point $x_0 \in \mathbb{R}^n$ then $\forall \|x - x_0\| < \delta$

$\|f(x) - f(x_0)\| = \| \|x\| - \|x_0\| \| \leq \|x - x_0\|$ {reverse triangle inequal}

$< \delta = \varepsilon$ with $\delta = \varepsilon$ hence f is cont on $\mathbb{R}^n$

useful thms \thm3.15/: continuity in terms of coordinate funcs: let $A \subset \mathbb{R}^n$ & $f: A \to \mathbb{R}^m$ then f is cont at a pt $a \in A$ iff each coord func $f_i : \mathbb{R}^n \to \mathbb{R}$, $i = 1, \dots, m$ is cont at a

~~proof~~ $f(x) = (f_1(x), \dots, f_m(x))^T$

\proof/ {see ex sheet}

\thm3.16/ Continuity of the composition of two cont funcs

let $A \subset \mathbb{R}^n$, $B \subset \mathbb{R}^m$ suppose that ~~$f: A \to B$~~ $f: A \to B$ is cont at a and $g: B \to \mathbb{R}^l$ is cont at $f(a)$ then $g \circ f: A \to \mathbb{R}^l$ is cont at a

proof/ fix $\varepsilon > 0$ by continuity of f at a,

$\forall \varepsilon' > 0, \exists \delta > 0$, s.t. $\forall x \in A$ with $\|x - a\| < \delta$, we have $\|f(x) - f(a)\| < \varepsilon'$ similarly by continuity of g at $f(a)$, $\exists \delta' > 0$ s.t. $\forall y \in B$ with $\|y - f(a)\| < \delta'$, we have $\|g(y) - g(f(a))\| < \varepsilon$ now with $y = f(x)$ & $\varepsilon' = \delta'$

we have that if $\|x - a\| < \delta$ then $\|g(f(x)) - g(f(a))\| < \varepsilon$ {so $\|f(x) - f(a)\| < \varepsilon' = \delta'$} that is $f \circ g$ is cont at a $\square$

\thm 3.17/: propertise of Cont funcs in $\mathbb{R}^n$:
let $A \subset \mathbb{R}^n$ & $f,g: A \to \mathbb{R}$ & let $\lim_{x\to a} f(x) = f(a)$
$\lim_{x\to a} g(x) = g(a)$ {ie. f & g are both cont at a}

①: $\lim_{x\to a} f(x)+g(x) = f(a)+g(a)$ {f+g is cont at a}

②: $\lim_{x\to a} f(x)g(x) = f(a)g(a)$ {f·g is cont at a}

③: $\lim_{x\to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$ provided $g(a) \neq 0$ {$\frac{f}{g}$ is cont at a}

{proof in notes}

\thm 3.18/: the projection function $\pi_i : \mathbb{R}^n \to \mathbb{R}$
given by $\pi_i(x) = x_i$ is continuous
so $\pi_3 : \mathbb{R}^4 \to \mathbb{R}$ $\quad x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4 \therefore \pi_3(x) = x_3$
{proof in ex sheet}

\ex 3.19/ consider $f: \mathbb{R}^3 \to \mathbb{R}$, $f(x) = \frac{\sin(x_1 x_2)}{e^{x_3}}$
by the previous thms this is cont as a combination of cont funcs: let $g,h: \mathbb{R} \to \mathbb{R}$ by $g(x) = \sin(x)$, $h(x) = e^x$
(which are both cont) then $f(x) = \frac{g(\pi_1(x)\pi_2(x))}{h(\pi_3(x))}$
{$\pi_i : \mathbb{R}^3 \to \mathbb{R}$} then f is cont!

topology in $\mathbb{R}^n$ week 9:

\Def 3.5/: ① if $x_0 \in \mathbb{R}^n$ & $\varepsilon > 0 \Rightarrow$ set $U(x_0, \varepsilon) =$
$\{x \in \mathbb{R}^n \mid \|x - x_0\| < \varepsilon\}$ is the ε-neigh of x_0 or the open ball radius ε centred at x_0 (like $U(x_0-\varepsilon, x_0+\varepsilon)$)

② set $S \subset \mathbb{R}^n$ called an open set if for each $x_0 \in S$ there $\varepsilon > 0$ s.t. ε-neigh of x_0 is contained in S ie $U(x_0, \varepsilon) \subset S$

③ set $S \subset \mathbb{R}^n$ is closed if S^c is open

④ set $S \subset \mathbb{R}^n$ is bounded if $\exists M > 0$ s.t. $\|x\| \leq M \; \forall x \in S$

⑤ set $S \subset \mathbb{R}^n$ is compact if it is closed & bounded

Sequences in $\mathbb{R}^n$ \Def/ Sequence $\{x_n\}_{n=1}^{\infty}$ with $x_n \in \mathbb{R}^p$ converges to vec $x \in \mathbb{R}^p$ if $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ have $\|x_n - x\| < \varepsilon$ write. $\lim_{n\to\infty} x_n = x$

$x_n = (y_1, y_2, y_3, \dots, y_p)^T$

\Ex3.8/ $x_n \in \mathbb{R}^p$ (p fixed) be given by $x_n = (\frac{1}{n}, \frac{1}{n^2}, \dots, \frac{1}{n^p})^T$

$\Rightarrow \lim_{n\to\infty} x_n = 0 = (0, 0, \dots, 0)^T \leftarrow \{p \text{ components}\}$

let $\varepsilon > 0 \Rightarrow \|x_n - 0\| = \|x_n\| = \sqrt{\sum_{i=1}^{p} (\frac{1}{n^i})^2} = \sqrt{\frac{1}{n^2} + \frac{1}{n^4} + \dots + \frac{1}{n^{2p}}} < \sqrt{\frac{p}{n^2}}$
$= \frac{\sqrt{p}}{n}$ $\therefore$ for all $n \geq N$ with $N > \frac{\sqrt{p}}{\varepsilon}$ $\therefore \frac{\sqrt{p}}{n} \leq \frac{\sqrt{p}}{N} < \varepsilon$

\Thm 3.15/ Continuity in terms of coordinate funcs: let $A \subset \mathbb{R}^n$ & $f: A \to \mathbb{R}^m \Rightarrow f$ is cont at pt $a \in A$ iff each coordinate func $f_i: \mathbb{R}^n \to \mathbb{R}$ $i = 1, \dots, m$ is cont at a
$f(x) = (f_1(x), \dots, f_m(x))^T$

Continuity \Def/: $A \subset \mathbb{R}^n$ & $f: A \to \mathbb{R}^m$ say f is cont at $x_0 \in A$ if $\lim_{x \to x_0} f(x) = f(x_0)$: equivalently if $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\forall x \in A$ with $\|x - x_0\| < \delta$ have $\|f(x) - f(x_0)\| < \varepsilon$ if f is cont $\forall x_0 \in A \Rightarrow f$ is cont on A

Derivative of $f: \mathbb{R} \to \mathbb{R}$: let $g(x)$ be tangent line to the func $f(x)$ at pt p: $g(x) = f(p) + A(x - p)$ where $A = f'(p)$
$\therefore$ reformulate: \Def 4.3/ a func $f: (a, b) \to \mathbb{R}$ is differentiable at the pt $p \in (a, b)$ if $\forall \varepsilon > 0, \exists \delta > 0$ & a number $A \in \mathbb{R}$ s.t. for $0 < |x - p| < \delta$ have
$\left|\frac{f(x) - g(x)}{x - p}\right| = \left|\frac{f(x) - f(p) - A(x - p)}{x - p}\right| < \varepsilon$ number A is called the derivative of f at p & denoted by $f'(p)$ this def makes it clear g is a "good approx" to the func f at p call g the linear approx to f at p

Linear Maps Def 4.4: A func $f: \mathbb{R}^n \to \mathbb{R}^m$ is a <u>linear map</u> if $\forall x, y \in \mathbb{R}^n$ & $\alpha \in \mathbb{R}$ have

- $f(x+y) = f(x) + f(y)$ $A(x+y) = Ax + Ay$
- $f(\alpha x) = \alpha f(x)$ $A(\alpha x) = \alpha A x$

denote collection of linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ by $L(\mathbb{R}^n, \mathbb{R}^m)$

$e_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$, $e_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}$ $A = \begin{pmatrix} | & | & & | \\ f(e_1) & f(e_2) & \cdots & f(e_n) \\ | & | & & | \end{pmatrix}$ {$m \times n$ matrix}

Derivative of $f: \mathbb{R}^n \to \mathbb{R}^m$ Def (4.5) let $\Omega \in \mathbb{R}^n$ be an open set. Z func $f: \Omega \to \mathbb{R}^m$ is differentiable at $p \in \Omega$ if $\exists A \in L(\mathbb{R}^n, \mathbb{R}^m)$ s.t. with $g(x) = f(p) + A(x-p)$ have $\lim_{x \to p} \frac{\|f(x) - g(x)\|}{\|x - p\|} = 0$ write $Df(p) := A$ & call $Df(p)$ Z <u>derivative</u> of f at Z pt p

equivalently replace with $\lim_{h \to 0} \frac{\|f(p+h) - f(p) - A(h)\|}{\|h\|} = 0$

$g(x)$ is linear approx $h \in \mathbb{R}^n$

Z derive is a func $Df: \mathbb{R}^n \to L(\mathbb{R}^n; \mathbb{R}^m)$ {$f': \mathbb{R} \to \mathbb{R}$}
using standard bases in $\mathbb{R}^n$ & $\mathbb{R}^m$ will identify $Df(p)$ with Z corres matrix of size $m \times n$ shown that:

~~$\frac{\partial f_i}{\partial x_j}$~~ $f(x) = \begin{pmatrix} f_1(x) \\ \vdots \\ f_m(x) \end{pmatrix}$ $(Df(p))_{ij} = D_j f_i(p) = \frac{\partial f_i}{\partial x_j}(p)$

Z <u>partial derivative</u> of Z coordinate func f_i at Z pt p with respect to coord $x_j \implies$ $Df(p) =$

$\begin{pmatrix} D_1 f_1(p) & D_2 f_1(p) & \cdots & D_n f_1(p) \\ D_1 f_2(p) & D_2 f_2(p) & \cdots & D_n f_2(p) \\ \vdots & & & \vdots \\ D_1 f_m(p) & D_2 f_m(p) & \cdots & D_n f_m(p) \end{pmatrix}$ $f_i: \mathbb{R}^n \to \mathbb{R}$ this matrix is called Z <u>jacobian</u>

Z existence of Z jacobian does not guarantee differentiability

ex 4.6: $f: \mathbb{R}^n \to \mathbb{R}$, $f(x) = \|x\|^2$ show
$\lim_{h \to 0} \frac{\|f(p+h) - f(p) - Df(p) \cdot h\|}{\|h\|} = 0 \ \forall p \in \mathbb{R}^n$

$f(p) = \|p\|^2 = p_1^2 + p_2^2 + \cdots + p_n^2$ $f(p+h) = (p_1+h_1)^2 + (p_2+h_2)^2 + \cdots + (p_n+h_n)^2 =$