$\lim_{n\to\infty} \|f_n - f\|_u = 0 = \lim_{n\to\infty} \sup_{x\in U} |f_n - f|$

$\sup_{x\in U} |f_n - f| \le \sup_{x\in U} |f_n| = \|f_n\|_u \le M_n \therefore 0 \le M_n$

$\therefore \lim_{n\to\infty} \sup_{x\in U} |f_n - f| \le \lim_{n\to\infty} M_n$ but $\left|\sum_{n=1}^{\infty} M_n\right| < \infty \therefore$

$\lim_{n\to\infty} M_n = 0 \therefore \lim_{n\to\infty} \sup_{x\in U} |f_n - f| \le 0$ but

$\sup_{x\in U} |f_n - f| \ge 0 \therefore$ ~~lim~~ $0 \le \lim_{n\to\infty} \sup_{x\in U} |f_n - f| \le 0 \therefore$

$\lim_{n\to\infty} \sup_{x\in U} |f_n - f| = 0$ as required $\therefore f$ is bounded on U

~~for~~ ~~for $\{f_n\}$ uos~~ f is $\{f_n\}$ convs unisly on U

<u>Sol</u>: ~~$\varepsilon > 0$~~ fix $\varepsilon > 0 \therefore \exists N \in \mathbb{N}$ s.t. $\forall n \ge N \sup_{x\in U} |f_n(x) - f(x)| < \varepsilon$

let $\bar{n} > N$ & let $x \in U \therefore$ ~~$|f(x)| = |f(x) - f_{\bar{n}}(x) + f($~~

$|f(x)| = |f(x) - f_{\bar{n}}(x) + f_{\bar{n}}(x)| \le |f_{\bar{n}}(x)| + |f(x) - f_{\bar{n}}(x)|$

$\le M_{\bar{n}} + \sup_{x\in U} |f_{\bar{n}}(x) - f(x)| < M_{\bar{n}} + \varepsilon$ in other words $f(x)$ is bounded on U

$\Big\{ \lim_{n\to\infty} \|f_n - f\|_u = 0 \therefore \lim_{n\to\infty} \sup_{x\in U} |f_n(x) - f(x)| = 0 \therefore \varepsilon > 0$

$\exists N \in \mathbb{N}$ s.t. ~~$\sup_{x\in U} |f_n(x)$~~ $\forall n \ge N : \sup_{x\in U} |f_n(x) - f(x)| < \varepsilon \Big\}$

$\Big\{$ let $\bar{n} > N$, $x \in U \therefore |f(x)| < M_{\bar{n}} + \varepsilon \therefore$ ~~for~~ $x \in U$ f is bounded

$\Big\backslash$ 3:ii / So $\lim_{n\to\infty} f_n(x) = f(x)$ $\quad |f_n(x)| \le M_n$

for $\varepsilon > 0$, $N \in \mathbb{N}$ and $\forall \bar{n} \ge N$: ~~$|f(x)|$~~

$|f_{\bar{n}}(x) - f(x)| < \varepsilon$ $\quad \bar{n} > N$ let $x \in U \therefore |f(x)| =$

$|f(x) - f_{\bar{n}}(x) + f_{\bar{n}}(x)| \le |f_{\bar{n}}| + |f(x) - f_{\bar{n}}(x)|$ ~~$= |f(x$~~

$= |f_{\bar{n}}(x)| + |f_{\bar{n}}(x) - f(x)| \le M_{\bar{n}} + |f_{\bar{n}}(x) - f(x)| < M_{\bar{n}} + \varepsilon \therefore f$ is bounded on U

not true like (ex 1.iv) $f_n(x) = \begin{cases} n & 0 < x \le 1/n \\ 1/x^4 & \frac{1}{n} < x \le 1 \end{cases}$

2008 Sheet 6

\1i/ $f:[a,b]\to\mathbb{R}$ f is C^0 ~~then~~ f is not C^1

$\therefore f(x)$ is C^0, $f(x)$ is not C^1

• if $f(x)=1$ $\therefore (C)^0=1$ and $C'=C\neq 1$

Sol: if $f(x)=|x|$, cont but not diff'able at $x=0$

• if $f(x)=\begin{cases} x^2\sin(\frac{1}{x}) & x\neq 0 \\ 0 & x=0\end{cases}$ is diff'able on $\mathbb{R}$ ($\therefore$ cont)

but ~~the~~ the diff. f' is not not at $x=0$ so it is not C^1

\1ii/ if $f(x)=\ln(x^2)$ $\therefore$ ~~$f'=2x\ln x$~~ $f'=2x\left(\frac{1}{x^2}\right)=2\left(\frac{1}{x}\right)$

if $f(x)=\ln\left(\frac{1}{x}\right)\therefore f'=-x^{-2}\frac{1}{1/x}=-x^{-1}$

Sol: $f(x)=x^{4/3}$ on $\mathbb{R}$ $\therefore f'(x)=\frac{4}{3}x^{1/3}$ $\therefore f''(x)=\frac{4}{9}x^{-2/3}$

$\therefore f'$ is cont but f'' is not defined at 0 (so isn't cont)

$\therefore f\in C^1$ but $f\notin C^2$

\1iii/ if $f=x^{7/3}$ on $\mathbb{R}$ $\therefore f'=\frac{7}{3}x^{4/3}$ $\therefore$ $f''=\frac{28}{9}x^{1/3}$

$\therefore f'''=\frac{28}{27}x^{-2/3}$ $\therefore f''$ is cont but f''' is not defined at 0

$\therefore$ isn't cont $\therefore f\in C^2$ but $f\notin C^3$

\1iv/ $f(x)=0$ $\therefore \forall x\in\mathbb{R}$ $f(x)=0$ $\therefore f$ is cont

$\therefore f'(x)=\frac{d}{dx}(0)=0$ $\therefore f'=0\ \forall x\in\mathbb{R}$ $\therefore f'$ is cont $\therefore$

$f^{(\infty)}(x)=\frac{d}{dx^{\infty}}(0)=0$ $\therefore f^{(\infty)}=0\ \forall x\in\mathbb{R}$ $\therefore f^{(\infty)}$ is cont

$\therefore f\in C^{\infty}$ $f=$ poly $a_0, a_1, \ldots, a_n$

$f=e^x$, $\pm\sin x$, $\pm\cos(x)$

\2i/ $L(f;P)=\sum_{i=1}^{n} m_i\Delta x_i$ $U(f;P)=\sum_{i=1}^{n} M_i\Delta x_i$

$m_i=\inf_{x\in[x_{i-1},x_i]} f(x)$ $M_i=\sup_{x\in[x_{i-1},x_i]} f(x)$ $\therefore$

m.i. $P = (x_0, x_1, \ldots, x_n)$ of $[a, b] = [1, 3]$

$\therefore x_i - x_{i-1} = \frac{3-1}{n} = \frac{2}{n}$ $\therefore x_i = x_{i-1} + \frac{2}{n}$

$\therefore m_i = \inf_{x_{i-1} \le x \le x_{i-1} + \frac{2}{n}} (x) = \cancel{(x_{i-1} + \frac{2}{n})} = (x_{i-1})$

$M_i = \sup_{x_{i-1} \le x \le x_{i-1} + \frac{2}{n}} (x) = (x_{i-1} + \frac{2}{n})$ $\therefore \left\{ \int_a^b x \, dx = \left[\frac{1}{2}x^2\right]_a^b = \frac{1}{2}(b^2 - a^2) \right.$

$\lim_{n \to \infty} \sum_{i=1}^{\infty} M_i \Delta x_i = \cancel{\lim_{n\to\infty} \sum_{i=1} (x_{i-1}} \lim_{n \to \infty} \sum_{i=1}^{\infty} (x_{i-1} + \frac{2}{n}) \Delta x_i$

$= \sum_{i=1}^{\infty} (x_{i-1}) \Delta x_i$

$\lim_{n \to \infty} \sum_{i=1}^{n} m_i \Delta x_i = \lim_{n \to \infty} \sum_{i=1}^{n} (x_{i-1}) \Delta x_i = \cancel{\lim_{n\to\infty}} \sum_{i=1}^{\infty} (x_{i-1}) \Delta x_i$

~~Sol:~~ $\frac{3-1}{n} = \frac{2}{n}$ uniform partion of $[1, 3]$, n subintervals

$P_n = (1, 1 + \frac{2}{n}, \ldots, 1 + \frac{2}{n} i, \ldots, 3)$ with the $\Delta x_i = \frac{2}{n}$

$\therefore$ LRS: $L(f; P_n) = \sum_{i=1}^{n} \left(1 + \frac{2}{n}(i-1)\right) \frac{2}{n} = \sum_{i=1}^{n} \left(\frac{2}{n} + \frac{2^2}{n^2}(i-1) \right) =$

$\frac{2}{n}\left(\sum_{i=1}^{n} 1 + \frac{2}{n} \sum_{i=1}^{n} (i-1) \right) = \frac{2}{n}(n) + \frac{2}{n}\left(\frac{2}{n} \sum_{i=1}^{n} i - \frac{2}{n} \sum_{i=1}^{n} 1 \right)$

$= 2 + \frac{4}{n^2} \left(\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1 \right) = 2 + \frac{4}{n^2} \left(\frac{1}{2}(n)(n+1) - n \right) = 2 + 2(n^2)(n^2 + n) - 4 \frac{1}{n}$

$= 4 - \frac{2}{n}$

and the URS: $U(f; P_n) = \sum_{i=1}^{n} (1 + \frac{2}{n} i) \frac{2}{n} = \frac{2}{n} \left(\sum_{i=1}^{n} 1 + \frac{2}{n} \sum_{i=1}^{n} i \right)$

$= 2 + \frac{4}{n^2} \sum_{i=1}^{n} i = 2 + \frac{4}{n^2} \frac{n(n+1)}{2} = 2 + \frac{2(n^2+n)}{n^2} = 4 + \frac{2}{n}$

$\therefore \underline{\int_1^3} f(x)\,dx = \sup_P L(f; P) = \cancel{\sup_P (4 - \frac{2}{n})} \ge \sup_n L(f; P_n) = \sup_n (4 - \frac{2}{n}) = 4$

$\overline{\int_1^3} f(x)\,dx = \inf_P U(f; P) \le \inf_n U(f; P_n) = \inf_n f(4 + \frac{2}{n}) = 4$ and since

$4 \le \underline{\int_1^3} f(x)\,dx \le \overline{\int_1^3} f(x)\,dx \le 4 \therefore \underline{\int_1^3} f(x)\,dx = \overline{\int_1^3} f(x)\,dx = \int_1^3 f(x)\,dx = 4$

2008 Sheet 6

2 ii) $\frac{b-a}{n} = \frac{1-0}{n} = \frac{1}{n} = \Delta x_i$ $\therefore$ uniform partition of $[0,1]$

n subintervals $P_n = (0, 0+\frac{1}{n}, \ldots, 0+\frac{1}{n}i, \ldots, 1)$

$= (0, \frac{1}{n}, \ldots, \frac{1}{n}i, \ldots, 1)$ width $\Delta x_i = \frac{1}{n}$ $\therefore$ LRS:

$L(f;P_n) = \sum_{i=1}^{n} \left((\frac{1}{n}(i-1))^2 + \frac{1}{n}(i-1) \right) \frac{1}{n}$

$= \frac{1}{n} \sum_{i=1}^{n} \left(\frac{1}{n^2}(i^2-2i+1) + \frac{1}{n}(i-1) \right) = \frac{1}{n^3} \sum_{i=1}^{n}(i^2-2i+1) + \frac{1}{n^2}\sum_{i=1}^{n}(i-1)$

$= \frac{1}{n^3}\sum_{i=1}^{n} i^2 - 2\left(\frac{1}{n^3}\right)\sum_{i=1}^{n} i + \frac{1}{n^3}\sum_{i=1}^{n} 1 + \frac{1}{n^2}\sum_{i=1}^{n} i - \frac{1}{n^2}\sum_{i=1}^{n} 1$

$= \frac{1}{n^3}\left(\frac{1}{6}\right)n(n+1)(2n+1) - \frac{2}{n^3}\left(\frac{n}{2}\right)(n+1) + \frac{1}{n^3}(n) + \frac{1}{n^2}\left(\frac{n}{2}\right)(n+1) - \frac{1}{n^2}(n)$

$= \frac{1}{6}\left(\frac{1}{n^2}\right)(2n^2+1+3n) - \frac{1}{n^2}(n+1) + \frac{1}{n^2} + \left(\frac{1}{2}\right)\frac{1}{n}(n+1) - \frac{1}{n}$

$= \frac{1}{6}\left(2 + \frac{1}{n^2} + \frac{3}{n}\right) - \frac{1}{n} - \frac{1}{n^2} + \frac{1}{n^2} + \left(\frac{1}{2}\right)\left(1+\frac{1}{n}\right) - \frac{1}{n}$

$= \frac{1}{3} + \frac{1}{6}\frac{1}{n^2} + \frac{1}{2}\frac{1}{n} - 2\frac{1}{n} + \frac{1}{2} + \frac{1}{2}\frac{1}{n}$

$= \frac{1}{6}\frac{1}{n^2} - \frac{1}{n} + \frac{5}{6}$ and the URS: $U(f;P_n) =$

$\sum_{i=1}^{n} \left((\frac{1}{n}(i))^2 + \frac{1}{n}i \right)\frac{1}{n} = \frac{1}{n}\sum_{i=1}^{n}\left(\frac{1}{n^2}i^2 + \frac{1}{n}i\right) = \frac{1}{n^3}\sum_{i=1}^{n} i^2 + \frac{1}{n^2}\sum_{i=1}^{n} i$

$= \frac{1}{n^3}\frac{1}{6}n(n+1)(2n+1) + \frac{1}{n^2}\frac{1}{2}n(n+1)$

$= \frac{1}{6}\frac{1}{n^2}(2n^2+1+3n) + \frac{1}{2}\frac{1}{n^2}(n^2+n) =$

$\frac{1}{6}\left(2+\frac{1}{n^2}+3\frac{1}{n}\right) + \frac{1}{2}\left(1+\frac{1}{n}\right) = \frac{1}{3} + \frac{1}{6}\frac{1}{n^2} + \frac{1}{2}\frac{1}{n} + \frac{1}{2} + \frac{1}{2}\frac{1}{n}$

$= \frac{1}{6}\frac{1}{n^2} + \frac{1}{n} + \frac{5}{6}$ $\therefore$ $\underline{\int_0^1} f(x)\,dx = \underline{\int_0^1} x^2+x\,dx = \sup_P L(f;P) \geq \sup_n L(f;P_n)$

$= \sup_n\left(\frac{1}{6}\frac{1}{n^2} - \frac{1}{n} + \frac{5}{6}\right) = \frac{5}{6}$ and $\overline{\int_0^1} f(x) = \overline{\int_0^1} x^2+x\,dx$

$= \inf_P U(f;P) \leq \inf_n U(f;P_n) = \inf_n\left(\frac{1}{6}\frac{1}{n^2} + \frac{1}{n} + \frac{5}{6}\right) = \frac{5}{6}$ $\therefore$

$\frac{5}{6} \le \underline{\int_0^1} f(x) \, dx \le \overline{\int_0^1} f(x) dx \le \frac{5}{6} \therefore \overline{\int_0^1} f(x) \, dx \; \underline{\int_0^1} f(x) \, dx$

$= \int_0^1 f(x) \, dx = \int_0^1 x^2 + x \, dx = \frac{5}{6}$

~~3i~~ $\therefore f:[0,1] \to \mathbb{R}$ riemann integrabal is

$\underline{\int_0^1} f(x) \, dx = \overline{\int_0^1} f(x) \, dx$ for any partition P

$m_i = \inf_{x \in [x_{i-1}, x_i]} f(x) = \inf_{x \in [x_{i-1}, x_i]} (x^2) = (x_{i-1})^2 = x_{i-1}^2$

$M_i = \sup_{x \in [x_{i-1}, x_i]} f(x) = \sup_{x \in [x_{i-1}, x_i]} (x^2) = (x_i)^2 = x_i^2$

$\therefore \frac{1-0}{n} = \frac{1}{n} = \Delta x_i$ is width $P_n = (0 + \frac{1}{n}, \dots, \frac{1}{n} i, \dots, 1)$

$\therefore L(f; P_n) = \sum_{i=1}^{n} (\frac{1}{n}(i-1))^2 \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^{n} (i^2 - 2i + 1) =$

$\frac{1}{n^3} \sum_{i=1}^{n} i^2 - 2 \frac{1}{n^3} \sum_{i=1}^{n} i + \frac{1}{n^3} \sum_{i=1}^{n} 1 =$

$\frac{1}{n^3} \frac{1}{6} n(n+1)(2n+1) - 2 \frac{1}{n^3} \frac{1}{2} n(n+1) \frac{1}{n^3}(n) =$

$\frac{1}{6} \frac{1}{n^2}(2n^2 + 3n + 1) - \frac{1}{n^2}(n+1) + \frac{1}{n^2} = \frac{1}{6}(2 + 3\frac{1}{n} + \frac{1}{n^2}) - \frac{1}{n} - \frac{1}{n^2} + \frac{1}{n^2}$

$= \frac{1}{3} + \frac{1}{2}\frac{1}{n} + \frac{1}{6}\frac{1}{n^2} - \frac{1}{n} = \frac{1}{6}\frac{1}{n^2} - \frac{1}{n} + \frac{1}{3}$

and $U(f; P_n) = \sum_{i=1}^{n} (\frac{1}{n} i)^2 \frac{1}{n} = \frac{1}{n^3} \sum_{i=1}^{n} i^2 =$

$\frac{1}{n^3} \frac{1}{6} n(n+1)(2n+1) = \frac{1}{n^2} \frac{1}{6} (2n^2 + 3n + 1) = \frac{1}{6}(2 + 3\frac{1}{n} + \frac{1}{n^2}) = \frac{1}{6}\frac{1}{n^2} + \frac{1}{2}\frac{1}{n} + \frac{1}{3}$

$\therefore \underline{\int_0^1} f(x) \, dx = \sup_P L(f; P) \ge \sup_n L(f; P_n) = \sup_n (\frac{1}{6}\frac{1}{n^2} - \frac{1}{n} + \frac{1}{3}) = \frac{1}{3}$

and $\overline{\int_0^1} f(x) \, dx = \inf_P (U(f; P) \le \inf_n (U(f; P_n) = \inf_n (\frac{1}{6}\frac{1}{n^2} + \frac{1}{2}\frac{1}{n} + \frac{1}{3}) = \frac{1}{3}$

$\therefore \frac{1}{3} \le \underline{\int_0^1} f(x) \, dx \le \overline{\int_0^1} f(x) \le \frac{1}{3} \therefore \underline{\int_0^1} f(x) \, dx = \overline{\int_0^1} f(x) \, dx \therefore$

$\underline{\int_0^1} x^2 \, dx = \overline{\int_0^1} x^2 \, dx \therefore$ by riemanns criterion for integrability
$\int_0^1 f(x) \, dx = \int_0^1 x^2 \, dx$ is integrable

2008 Sheet 6

uniform partition ~~P_n~~ $P_n = (0, \frac{1}{n}, \dots, \frac{1}{n}i, \dots, 1)$

$\therefore$ since f is increasing on $[0,1]$ we have the lower riemann sums $L(f; P_n) = \sum_{i=1}^{n}((\frac{1}{n}(i-1))^2)\frac{1}{n}$

& upper Riemann Sums $U(f; P_n) = \sum_{i=1}^{n}(\frac{1}{n}i)^2\frac{1}{n}$

$\therefore U(f; P_n) - L(f; P_n) = \sum_{i=1}^{n}(\frac{1}{n}i)^2\frac{1}{n} - \sum_{i=1}^{n}(\frac{1}{n}(i-1))^2\frac{1}{n}$

$= \frac{1}{n^3}\sum_{i=1}^{n}(i^2-(i-1)^2)$ {telescopic sum}

$= \frac{1}{n^3}n^2 = \frac{1}{n}$ now for $n \geq N$ with $N > \frac{1}{\varepsilon}$ have

$|U(f; P_n) - L(f; P_n)| = \frac{1}{n} < \frac{1}{N} < \varepsilon$ & $\therefore$ riemanns integrability criterion is satisfied

\4/ f is non decreasing $\therefore \inf_{[a,b]} f(x) = f(a)$,

$\sup_{[a,b]} f(x) = f(b)$ $\therefore$ $\inf_{[x_{i-1},x_i]} f(x) = f(x_{i-1})$, $\sup_{[x_{i-1},x_i]} f(x) = f(x_i)$

$\therefore$ ~~for P_n~~ $\frac{b-a}{n} = \Delta x_i$ is width $\therefore P_n = (a, a+\frac{b-a}{n}, \dots, a+\frac{b-a}{n}i, \dots, b)$

$\therefore L(f; P_n) = \cancel{\sum_{i=1}^{n}(a+\frac{b-a}{n}i}$ $\sum_{i=1}^{n}\left(f(a+\frac{b-a}{n}(i-1))\frac{b-a}{n}\right)$

$= \frac{b}{n}\sum_{i=1}^{n}f(a+\frac{b-a}{n}(i-1)) - \frac{a}{n}\sum_{i=1}^{n}f(a+\frac{b-a}{n}(i-1))$

$U(f; P_n) = \sum_{i=1}^{n}\left(f(a+\frac{b-a}{n}i)\frac{b-a}{n}\right) =$

$\frac{b}{n}\sum_{i=1}^{n}f(a+\frac{b-a}{n}i) - \frac{a}{n}\sum_{i=1}^{n}f(a+\frac{b-a}{n}i)$, $f(a+\frac{b-a}{n}(i-1)) \leq f(a+\frac{b-a}{n}i)$

$\therefore \sum_{i=1}^{n}f(a+\frac{b-a}{n}(i-1)) \leq \sum_{i=1}^{n}f(a+\frac{b-a}{n}i)$ $\therefore$

$\left|\frac{b-a}{n}\left[\sum_{i=1}^{n}f(a+\frac{b-a}{n}(i-1)) - \sum_{i=1}^{n}f(a+\frac{b-a}{n}i)\right]\right| < \varepsilon$ for $\varepsilon > 0$

and $n \geq N$ for $N \in \mathbb{N}$ therefore riemanns integrability criterion is satisfied so f is riemann integrable

Sol: Let $P=(x_0,\dots,x_n)$ be a unif partition of $[a,b]$

$x_i = a+\frac{(b-a)}{n}i \therefore \cancel{\Delta x_i = \frac{1}{n}}$ so that $\Delta x_i = \frac{1}{n} \therefore$

$U(f;P)-L(f;P) = \sum_{i=1}^{n} \sup_{[x_{i-1},x_i]}(f(x))\Delta x_i - \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]}(f(x))\Delta x_i$

$= \frac{1}{n}\sum_{i=1}^{n}\left(\sup_{[x_{i-1},x_i]} f(x) - \inf_{[x_{i-1},x_i]} f(x)\right) = \frac{1}{n}\sum_{i=1}^{n}(f(x_i)-f(x_{i-1}))$

{telescoping sum} $= \frac{1}{n}(f(x_n)-f(x_0)) = \frac{1}{n}(f(b)-f(a))$

used the fact that f is nondecreasing to obtain

$\sup_{[x_{i-1},x_i]} f(x) = f(x_i)$ & $\inf_{[x_{i-1},x_i]} f(x) = f(x_{i-1}) \therefore$ taking $n \geq N$

with $N > \frac{f(b)-f(a)}{\varepsilon}$ $\left\{ \frac{1}{n}(f(b)-f(a)) \leq \frac{1}{N}(f(b)-f(a)) < \varepsilon \right.$

$\therefore \frac{f(b)-f(a)}{\varepsilon} < N \}$ we have $U(f;P)-L(f;P) = \frac{1}{n}(f(b)-f(a))$

$\leq \frac{1}{N}(f(b)-f(a)) < \varepsilon \therefore$ riemann's integrability criterion

is satisfied

\5/ $P=(ca,\dots,cb)=(x_0,\dots,x_n)$ of $[ca,cb]$

$\therefore x_i = ca+\frac{cb-ca}{n}i$ so that $\cancel{\Delta}$ $\Delta x_i = \frac{1}{n} \therefore$

$U(f;P)-L(f;P) = \sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f(x)\Delta x_i - \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f(x)\Delta x_i =$

$\frac{1}{n}\left(\sum_{i=1}^{n}\left(\sup_{[x_{i-1},x_i]} f(x) - \inf_{[x_{i-1},x_i]} f(x)\right) = \frac{1}{n}\left(\sum_{i=1}^{n}(f(x_i)-f(x_{i-1}))\right)\right.$ {telescoping sum}

$= \frac{1}{n}(f(x_n)-f(x_0)) = \frac{1}{n}(f(cb)-f(ca)) \leq \frac{1}{N}(f(cb)-f(ca)) < \varepsilon$

for $\varepsilon>0$, $n \geq N$ with $N > \frac{f(cb)-f(ca)}{\varepsilon}$

and for $P_2=(a,\dots,b)=(x_0,\dots,x_n)$ of $[a,b] \therefore x_i = a+\frac{b-a}{n}i$

so that $\Delta x_i = \frac{1}{n} \therefore U(f;P_2)-L(f;P_2)$

2008 Sheet 6

$$= c\left(\sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f(x)\Delta x_i - \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f(x)\Delta x_i\right) =$$

$$c\frac{1}{n}\left(\sum_{i=1}^{n} f(x_i) - \sum_{i=1}^{n} f(x_{i-1})\right) = c\frac{1}{n}\sum_{i=1}^{n}\left(f(x_i) - f(x_{i-1})\right)$$ {telescoping sum}

$$= c\frac{1}{n}\left(f(x_n) - f(x_0)\right) = c\frac{1}{n}\left(f(cb) - f(ca)\right) \leq \frac{1}{N}c(f(b) - f(a)) < \varepsilon$$

for $\varepsilon > 0$, $n \geq N$ with $N > \frac{c(f(b) - f(a))}{\varepsilon}$

$\therefore$ for $P(ca, \ldots, cb) = (x_0, \ldots, x_n)$ of $[ca, cb]$ $\therefore x_i = ca + \frac{cb - ca}{n}i$

so that $\Delta x_i = \frac{1}{n}$ so $a = ca + \frac{cb - ca}{n}i$ $\therefore an = can + cb - cai$

$\therefore cai = can + cb - an$ $\therefore i = \frac{can + cb - an}{ca} = n + \frac{b}{a} - \frac{n}{c}$

and $b = ca + \frac{cb - ca}{n}i$ $\therefore bn = can + cb - cai$ $\therefore cai = can + cb - bn$

$\therefore i = \frac{can + cb - bn}{ca} = n + \frac{b}{c} - \frac{b}{ca}n$ $\therefore$

$a = x_{n + \frac{b}{a} - \frac{n}{c}}$, $b = x_{n + \frac{b}{c} - \frac{b}{ca}n}$

Sol: if $c = 0$ then is trivial, assume $c > 0$ let $g(x) = f(x)$

$g(x) = f(cx)$ consider partition $P = (x_0, \ldots, x_n)$ of $[a, b]$ $\therefore$

$Q = (cx_0, \ldots, cx_n)$ is a partition of $[ca, cb]$ the relationship

between P & Q demonstrates a one to one correspondance

between all partions of $[a, b]$ & all partions of $[ca, cb]$

$\therefore \Delta cx_i = c\Delta x_i$ $\therefore$ $L(f, Q) = \sum_{i=1}^{n} \inf_{x \in [cx_{i-1}, cx_i]} f(x)\Delta x_i$

$$= \sum_{i=1}^{n} \inf_{[x_{i-1}, x_i]} f(cx)\Delta cx_i = c\sum_{i=1}^{n} \inf_{[x_{i-1}, x_i]} f(cx)\Delta x_i = cL(g, P)$$

since one to one correspondence between partions of $[a, b]$ & $[ca, cb]$ must have that the set of all lower sums of f on $[a, b]$ is equal to the set of all lower sums of g on $[ca, cb]$ times c that is $\{L(f, R) \mid R \text{ is a partition of } [a, b]\} = \{cL(g, R) \mid R \text{ is a partition of } [ca, cb]\}$ $\therefore$

$$\sup_R L(f, R) = \sup_R cL(g, R) = c\sup_R L(g, R)$$

$\therefore \underline{\int_{ca}^{cb}} f(x)\,dx = c\underline{\int_{ca}^{cb}} f(cx)\,dx$ similar result holds for all upper sums but don't need to consider since the sums are Riemann integrable. Must have the lower Riemann integrals are equal to the integrals $\therefore \int_{ca}^{cb} f(x)\,dx = c\int_a^b f(cx)\,dx$ the case for $c<0$ is similar but order of partition pts changes

\6iv $\therefore \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$ and

PF $P = (a, \dots, b) = (x_0, \dots, x_n)$ $\quad x_i = a + \frac{b-a}{n} i$

width $\Delta x_i = \frac{1}{n}$ $\quad \therefore U(f,P) = \sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f(x)\Delta x_i$

$= \frac{1}{n}\sum_{i=1}^{n} \sup_{x\in[x_{i-1},x_i]} f(x) \geq 0$

$L(f,P) = \sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f(x)\Delta x_i = \frac{1}{n}\sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f(x) \geq 0$

$\therefore U(f;P) - L(f,P) = \frac{1}{n}\left(\sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f(x) - \inf_{[x_{i-1},x_i]} f(x)\right) \geq 0$

$\therefore \overline{\int_a^b} f(x)\,dx \geq \underline{\int_a^b} f(x)\,dx \geq 0$ but $\int_a^b f(x)\,dx$ is Riemann integrable $\therefore \underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx$

$\therefore \int_a^b f(x)\,dx \geq 0$ as required

Sol: if $f(x) \geq 0\ \forall x \in [a,b]$ $\therefore \inf_{x\in[a,b]} f(x) \geq 0$ since

~~$(b-a)\inf$~~ $(b-a)\inf_{x\in[a,b]} f(x) \leq \int_a^b f(x)\,dx \leq (b-a)\sup_{x\in[a,b]} f(x)$

$\left\{\text{because } \int_a^b f(x)\,dx = \sum_{i=1}^{n} f(x_i)\Delta x_i = \frac{b-a}{n}\sum_{i=1}^{n} f(x_i)\right\}$ conclude that $\int_a^b f(x)\,dx \geq 0$

~~Vms~~ $g = 9.81 ms^{-2}$ $a = 0.3 ms^{-2}$ $S = 50 m$

$u = 0.5 ms^{-1}$ $V ms^{-1}$

a, u, S V

$V^2 = u^2 + 2aS = (0.5)^2 + (2)(0.3)(50) = V^2 = 30.25$

$V = \pm\sqrt{30.25}$ $\{u > 0, a > 0 \therefore V > 0\}$

$\therefore V = +\sqrt{30.25} = 5.5 ms^{-1}$

2008 Sheet 6

\6ii/ is $f(x) \leq g(x) \ \forall x \in [a,b] \therefore \inf_{x\in[a,b]} f \leq g(x)$, $\sup_{x\in[a,b]} f(x) \leq g(x)$

$\therefore$ since $(\int_a^b f(x)\,dx = \sum_{i=1}^{\infty} f(x_i)\Delta x_i = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) = \lim_{n\to\infty} \frac{b-a}{n} \sum_{i=1}^{n} f(a + \frac{(b-a)}{n}$

for $P_n = (a, \ldots, b) = (x_0, \ldots, x_n) = (x_0, \ldots, a + \frac{(b-a)}{n}i, \ldots, x_n)$

$= (a, \ldots, a + \frac{(b-a)}{n}i, \ldots, b) \therefore$ ~~$\sup_{x\in[a,b]}$~~

$(b-a) \inf_{x\in[a,b]} f \leq \int_a^b f(x)\,dx \leq (b-a) \sup_{x\in[a,b]} f(x)$

$\therefore \int_a^b f(x)\,dx \leq (b-a) \sup_{x\in[a,b]} f(x) \leq \int_a^b g(x)\,dx$ since $g(x)$ is riemann integrable $\therefore \int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$

Sol: $g(x) - f(x) \geq 0 \ \forall x \in [a,b] \therefore \inf_{x\in[a,b]} (g(x) - f(x)) \geq 0$

since $(b-a) \inf_{x\in[a,b]} (g(x) - f(x)) \leq \int_a^b g(x) - f(x)\,dx \leq (b-a) \sup_{x\in[a,b]} (g(x) - f(x))$

$\therefore \int_a^b (g(x) - f(x))\,dx \geq 0 \therefore \int_a^b g(x)\,dx - \int_a^b f(x)\,dx \geq 0 \therefore$

$\int_a^b f(x)\,dx \leq \int_a^b g(x)\,dx$ as required

7/ $\therefore \int_a^b f(x)\,dx = \int_a^b f(x)\,dx \qquad f(x) \geq 0 \;\forall x \in [a,b]$

$\therefore \lim_{x\to x_0} f(x) = f(x_0) \;\forall x \in [a,b]$

Partion $P = (a, \ldots, b) = (x_0, \ldots, x_i, \ldots, x_n)$ width $\Delta x_i = \frac{1}{n}$ for $x_i = a + \frac{b}{n} i \;\therefore$

$L(f;P) = \sum_{i=1}^{n} \inf_{x\in[x_{i-1},x_i]} f(x)\Delta x_i = \frac{1}{n}\sum_{i=1}^{n} \inf_{[x_{i-1},x_i]} f(x)$

$U(f;P) = \sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f(x)\Delta x_i = \frac{1}{n}\sum_{i=1}^{n} \sup_{[x_{i-1},x_i]} f(x) \;\therefore$

$\int_a^b f(x)\,dx = \sum \lim_{n\to\infty} \sum_{i=1}^{n} f(x_i)\Delta x_i \qquad P = (x_0, \ldots, x_n) = (a, \ldots, b)$

$\therefore (b-a)\inf_{x\in[a,b]} f(x) \leq \int_a^b f(x)\,dx \leq (b-a)\sup_{x\in[a,b]} f(x)$

$\therefore 0 \leq (b-a)\inf_{x\in[a,b]} f(x) \leq \int_a^b f(x)\,dx \;\therefore\; 0 \leq \int_a^b f(x)\,dx$

$= \int_a^{x_0} f(x)\,dx + \int_{x_0}^b f(x)\,dx + \lim_{n\to\infty}\left(f(x_0)\frac{b-a}{n}\right) \qquad f(x_0) > 0 \;\therefore$

$\int_a^{x_0} f(x)\,dx > 0,\; \int_{x_0}^b f(x_0)\,dx > 0 \;\therefore\; \int_a^{x_0} f(x)\,dx + \int_{x_0}^b f(x)\,dx > 0 \;\therefore$

$\int_a^b f(x)\,dx > 0$ for $x_0 \in [a,b]$ Sol: f is cont at x_0 with $f(x_0) > 0$ taking $\varepsilon = \frac{f(x_0)}{2} > 0 \;\exists\; \delta > 0$ s.t $\forall |x - x_0| < \delta$: have $|f(x) - f(x_0)| > \varepsilon \;\therefore\; f(x) > \varepsilon > 0$ on interval $(x_0 - \delta, x_0 + \delta) \;\therefore$ Considering upper & lower Rieman Sums for partition Q containing the interval $[x_0 - \frac{\delta}{2}, x_0 + \frac{\delta}{2}]$ of width δ since $f(x) > 0$ on this interval & also $f(x) \geq 0$ everywhere $\therefore L(f;Q) \geq \varepsilon\delta > 0 \;\therefore$

$0 < L(f;Q) \leq \sup_P (L;P) = \int_a^b f(x)\,dx$

2008 Sheet 6

18/ partitions $P_n = (0, \frac{1}{n}, 1)$ $n \in \mathbb{N}$ ∴

$L(f; P_n) = (\frac{1}{n} - 0)(0) + (1 - \frac{1}{n})(\frac{1}{x}) = \frac{1}{x} - \frac{1}{n}\frac{1}{x} = (1-\frac{1}{n})(\frac{1}{1}) = 1 - \frac{1}{n}$

$U(f; P_n) = (\frac{1}{n} - 0)(\frac{1}{x}) + (1 - \frac{1}{n})(\frac{1}{x}) = (\frac{1}{x})(1) + \frac{1}{x}(\frac{1}{n} - \frac{1}{n}) = \frac{1}{x} = \infty$

∴ $\underline{\int_0^1} f(x)\,dx = \sup_p L(f;P) \geq \sup_n L(f;P_n) = \sup_n (1 - \frac{1}{n}) = \sup_n (1) = 1$

$\overline{\int_0^1} f(x)\,dx = \inf_p U(f;P) \leq \inf_n U(f;P_n) = \inf_n (\frac{1}{x}) = \inf_n (\infty) = \infty$

∴ $1 \neq \infty$ ∴ $\underline{\int_0^1} f(x)\,dx \neq \overline{\int_0^1} f(x)\,dx$ ∴ $f(x)$ is does not follow the riemann the riemann integrability criterion ∴ not riemann integrable

Sol: when $x = 0$ the upper & lower Riemann Sums are 0 so dont contribute to the Riemann integral

$\cos x \in [0,1]$ analyse $\int_0^1 \frac{1}{x}\,dx$ consider partition $P = (0, x_1, x_2, ..., x_{n-1}, 1)$ ∴ for 1st subinterval $[0, x_1]$ have $\sup_{[0,x_1]} f = \infty$ ∴ the upper riemann sum is not defined

∴ Corresponding riemann integral does not exist

∴ f is not riemann integral

also if P is a partition of $[0,1]$ and I is any subinterval in P then $\sup_{x \in I} = 1$ & $\inf_{x \in I} f(x) = 0$

∴ $M_i = 1$ & $m_i = 0$ ∴ $L(f;P) = 0$ & $U(f;P) = 1$ ∴ the lower & upper riemann integrals are not equal ∴ this func is not riemann integral

19/ $\forall a, b \in \mathbb{R}$, $a < b$ $\exists c \in [a,b)$, $c \notin \mathbb{Q}$

$\forall a, b \notin \mathbb{Q}$, $a < b$ $\exists c \in (a,b)$, $c \in \mathbb{Q}$

∴ for partition $P_n = (0, x_1, x_2, ..., x_{n-1}, 1)$ ∴ $L(f;P_n) = 0$,

$U(f;P_n) = 1$ ∴ for $\overline{\int_0^1} f(x)\,dx = \inf_p U(f;P) \leq \inf_n U(f;P_n) = 1$

$$\underline{\int_0^1} f(x)\,dx = \sup_P \cancel{U}L(f;P) \geq \sup_n L(f;P_n) = 0$$

$\therefore 0 \leq \underline{\int_0^1} f(x)\,dx \leq \overline{\int_0^1} f(x)\,dx \leq 1$ $\underline{\text{(b)}}$: Consider

$c \in (0,1)$ Partition $P = (0, c, 1)$ take a tagging $T = (a, b)$ where $a \in \mathbb{Q} \cap (0, c)$ and $b \in (c,1) \setminus \mathbb{Q}$ $\therefore a \in \mathbb{Q}, b \notin \mathbb{Q}$

$\therefore$ Riemann Sum: $S = (f; P; T) =$

$f(a)(c-0) - f(b)(1-c) = 1c + 0(1-c) = c$ if $c = 0$ then take Partition $P = (0,1)$ and take $T = (a)$ with $a \in (0,1) \setminus \mathbb{Q}$

$\therefore S(f, P; T) = f(a)(1-0) = 0 = c$ and if $c = 1$ take $a \in (0,1) \cap \mathbb{Q}$ $\therefore S(f; P; T) = f(a)((1-0)) = 1 = c$

$\backslash 10 /$ ~~to~~ $n\log(n) = \log(n^n)$ $\log(e) = 1$ $\log(1) = 0$

$n = \log(e^n)$ ~~$\therefore$ to~~ $\therefore n\log(n) - n + 1 = \log(n^n) - \log(e^n) + \log(e)$

$= \log\left(\frac{n^n e}{e^n}\right) = \log\left(\frac{n^n}{e^{n-1}}\right) = \log(n^n e^{+1-n})$

~~$\forall i \in \mathbb{N}$~~ $\forall i$ $\log(1) = 0 < \log(2)$ $\therefore \forall i \in \mathbb{N}: \log(i) \geq 0 \therefore$

~~$\sum_{i=1}^{n} \log(i) = \log \sum_{i=2}^{n} \log(i)$~~ $\therefore \log(n^n e^{-n+1}) \leq \sum_{i=2}^{n} \log(i) = \sum_{i=1}^{n} \log(i)$

$((-n+1+1) = -n$ $\therefore (n+1)\log(n+1) - (n+1) + 1 = \log((n+1)^{n+1} e^{-n})$

$\sum_{i=1}^{n-1} \log(i) = \sum_{i=1}^{n} \log(i) - \log(n)$ $\therefore \sum_{i=1}^{n} \ln(i) + \ln(n) = \sum_{i=1}^{n} \ln(i)$

$\therefore n\ln(n) - n + 1 = \ln(n^n e^{-n+1}) \leq \sum_{i=1}^{n} \ln(i) + \ln(n)$

$\sum_{i=1}^{n-1} \ln(i) \leq \ln(n^n e^{-n+1}) \leq \sum_{i=1}^{n} \ln(i) = \ln\left(\prod_{i=1}^{n} i\right) = \ln((n)!)$

~~$\ln((i)!)$~~ $\therefore \ln((n-1)!) \leq \ln(n^n e^{-n+1}) \leq \ln(n!)$

$\therefore (n-1)! \leq n^n e^{-n+1} \leq n!$

$(n+1)\ln(n+1) - (n+1) + 1 = \ln((n+1)^{n+1} e^{-n}) \geq \sum_{i=1}^{n-1+1} \ln i = \sum_{i=1}^{n} \ln i = \ln(n!))$

$\therefore (n+1)^{n+1} e^{-n} \geq n! \therefore n^n e^{-n+1} \leq n! \leq$ ~~$(n+1)e$~~ $(n+1)^{n+1} e^{-n}$ $\left\{ \sum_{i=1}^{n-1} \ln i = \ln((n-1)!) \right.$

as required

2008 Sheet 7

2/ $$H'(x)=\frac{d}{dx}\left[\int_{f(x)}^{g(x)} h(t)\,dt\right]=\frac{d}{dx}\left(\int h(g(x))\,\cancel{dx}-\int h(f(x))\,\cancel{dx}\right)$$

$$=h(g(x))\,g'(x)-h(f(x))\,f'(x)$$ ✓

4i/ $$\int_a^b f_n(x)\,dx=\cancel{\int_a^b nx^{2n}\,dx-\int_a^{-1} nx^{2n}}$$

$$=\int_a^{-1} f_n(x)\,dx+\int_{-1}^{1} f_n(x)+\int_1^b f_n(x)=\int_a^{-1} 0\,dx+\int_{-1}^{1} nx^{2n}\,dx+\int_1^b 0\,dx$$

$$=\int_{-1}^{1} nx^{2n}\,dx=n\int_{-1}^{1} x^{2n}\,dx=n\left[\frac{1}{2n+1}x^{2n+1}\right]_{-1}^{1}=$$

$$\frac{n}{2n+1}\left(1^{2n+1}-(-1)^{2n+1}\right)=\frac{n}{2n+1}\left[1-(-1)^{2n}(-1)\right]=\frac{n}{2n+1}\left[1-(1)(-1)\right]=\frac{2n}{2n+1}$$

$$\therefore \lim_{n\to\infty}\int_a^b f_n(x)\,dx=\lim_{n\to\infty}\frac{2n}{2n+1}=\lim_{n\to\infty}\frac{(2)}{2+\left(\frac{1}{n}\right)}=\frac{2}{2}=1$$

$$\lim_{n\to\infty} f_n(x)=\lim_{n\to\infty} nx^{2n}=\infty \quad \therefore \quad \int_a^b \lim_{n\to\infty} f_n(x)\,dx=\int_a^b \infty\,dx=\infty$$

i/ $$f(x)=\left(\left(\int \frac{1}{1+\sin^2 x}\,dx\right)\circ(x^3)\right)-\left(\left(\int\frac{1}{1+\sin^2 x}\,dx\right)\circ(a)\right)$$

$$\therefore f'(x)=\left(\left(\frac{1}{1+\sin^2 x}\right)\circ(x^3)\right)\cdot(3x^2)+0=\frac{3x^2}{1+\sin^2(x^3)}$$

ii/ $$f'(x)\,\frac{d}{dx}\left(\int_{x^3}^{a}\frac{1}{1+\sin^2(t)}\,dt\right)=\frac{d}{dx}\left(-\int_a^{x^3}\frac{1}{1+\sin^2(t)}\,dt\right)$$

$$=\cancel{\frac{-3x^2}{1}}\quad -\frac{d}{dx}\left(\int_a^{x^3}\frac{1}{1+\sin^2(t)}\,dt\right)=\frac{-3x^2}{1+\sin^2(x^3)}$$

iii/ $$f'(x)=\frac{d}{dx}\left(\left(\left(\left(\int\frac{1}{1+\sin^2(t)}\right)\circ(x)\right)-\left(\left(\int\frac{1}{1+\sin^2(t)}\right)\circ(a)\right)\right)^3\right)=$$

$$3\left(\int_a^x\frac{1}{1+\sin^2(t)}\,dt\right)^2\left(\left(\left(\frac{1}{1+\sin^2(t)}\right)\circ(x)\right)(1)-\left(\left(\frac{1}{1+\sin^2(t)}\right)\circ(a)\right)(0)\right)=3\left(\int_a^x\frac{1}{1+\sin^2(t)}\,dt\right)^2\left(\frac{1}{1+\sin^2(x)}\right)$$

\iv/ $f(x) = \left(\left(\int \frac{1}{1+\sin^2 t} dt\right) \circ \left(\int_a^x \frac{1}{1+\sin^2(t)} dt\right)\right) - \left(\left(\int \frac{1}{1+\sin^2 t} dt\right) \circ (a)\right)$

$\frac{d}{dx}\left(\int_a^x \frac{1}{1+\sin^2 t} dt\right) = \frac{1}{1+\sin^2(x)}$ $\therefore$ $f'(x) =$

$\frac{1}{1+\sin^2\left(\int_a^x \frac{1}{1+\sin^2(t)} dt\right)} \left(\frac{1}{1+\sin^2(t)} dt\right) - \frac{1}{1+\sin^2(a)} (0)$

$= \frac{1}{\left(\int_a^x \frac{1}{1+\sin^2(t)} dt\right)\left(1+\sin^2\left(\int_a^x \frac{1}{1+\sin^2 t} dt\right)\right)}$

\4ii/ $\int_a^b f_n(x) dx = \int_0^1 \frac{e^{-x^2/n} + x^3}{1+\ln\left(\frac{1}{n}+1\right)} dx$ $\left\{\frac{d}{dx}\left(1+\ln\left(\frac{1}{n}+1\right)\right) = 0\right\}$

$= \left(1+\ln\left(\frac{1}{n}+1\right)\right)^{-1} \int_0^1 e^{-x^2/n} + x^3 dx$ $\left\{\frac{d}{dx}\left(e^{-x^2/n}\right) = -2x\frac{1}{n}e^{-x^2/n}\right\}$

$\left\{\frac{d}{dx}\left(-x^2\frac{1}{n}\right) = \left(-2x\frac{1}{n}\right)\right\}$ $= \left(1+\ln\left(\frac{1}{n}+1\right)\right)^{-1}\left(\int_0^1 e^{-x^2/n} dx + \int_0^1 x^3 dx\right)$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(\left(1+\ln\left(\frac{1}{n}+1\right)\right)^{-1}\left(e^{-x^2/n}+x^3\right)\right)$

$= (1+\ln(0+1))^{-1}(e^0+x^3) = (1+\ln(1))^{-1}(1+x^3) = (1+0)(1+x^3) = 1+x^3$

$\therefore \int_a^b \lim_{n\to\infty} f_n(x) dx = \int_0^1 1+x^3 dx = \left[x+\frac{1}{4}x^4\right]_0^4$

$= (\cancel{4}-0) + \frac{1}{4}(4^4-0^4) = 4+\frac{1}{4}(256) = 4+64 = 68$

\4iii/ $\int_0^1 f_n(x) \cancel{dx} = \int_0^1 x^n dx = \left[\frac{1}{n+1}x^{n+1}\right]_0^1 =$

$\frac{1}{n+1}\left[1^{n+1} - 0^{n+1}\right] = \frac{1}{n+1}[1-0] = \frac{1}{n+1}$ $\therefore \lim_{n\to\infty}\int_0^1 f_n(x) = \lim_{n\to\infty}\frac{1}{n+1} = 0$

$\cancel{\lim_{n\to\infty}}$ for $x \in [0,1]$, $\therefore x \geq 0 \therefore \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} x^n = \infty \therefore$

$\cancel{\lim_{n\to\infty}}$ $\int_0^1 \lim_{n\to\infty} f_n(x) dx = \int_0^1 \infty dx = \infty \neq 0$

5/ 2008 Sheet 7

$\lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} nx(1-x)^n = \lim_{n\to\infty} \frac{nx}{(1-x)^{-n}}$

$= \lim_{n\to\infty} \frac{n}{-n(1-x)^{-n-1}(-1)} = \lim_{n\to\infty} \frac{1}{(1-x)^{-n-1}} = \lim_{n\to\infty} (1-x)^{n+1} = 0$

for $x \in (0,1]$, for $x=0$ $\therefore \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} nx(1-x)^n$

$= \lim_{n\to\infty} n(0)(1-0)^n = \lim_{n\to\infty} (0) = 0 \therefore \forall x \in [0,1] \lim_{n\to\infty} g_n(x) = 0$

$\therefore \{g_n\}_{n=1}^{\infty}$ converges $\quad \int_0^1 g_n(x)\,dx = \int_0^1 nx(1-x)^n\,dx$

$\left\{ \int (1-x)^n\,dx = \frac{-1}{n+1}(1-x)^{n+1}, \quad \frac{d}{dx}(nx) = n \right\}$

$= \left[nx\left(\frac{-1}{n+1}(1-x)^{n+1}\right)\right]_0^1 - \int_0^1 n \frac{-1}{n+1}(1-x)^{n+1}\,dx$

$= \frac{-n}{n+1}\left[1(1-1)^{n+1} - (0)(1-0)^{n+1}\right] - \left[\frac{n}{n+1}\frac{1}{n+2}(1-x)^{n+2}\right]_0^1$

$= \frac{-n}{n+1} 1(0)^{n+1} - \frac{n}{n^2+2+3n}\left[(1-1)^{n+2} - (1-0)^{n+2}\right]$

$= \frac{-n}{n+1}(0) - \frac{n}{n^2+2+3n}\left[0^{n+2} - (1)^{n+2}\right]$

$= 0 - \frac{n}{n^2+2+3n}[-1] = \frac{n}{n^2+2+3n} = \frac{1}{n+\frac{2}{n}+3}$

$\therefore \lim_{n\to\infty} \int_0^1 g_n(x)\,dx = \lim_{n\to\infty}\left[\frac{1}{n+3+2(\frac{1}{n})}\right] = \lim_{n\to\infty}\left[\frac{1}{n+3}\right] = 0$

$\therefore \left\{\int_0^1 g_n(x)\,dx\right\}_{n=1}^{\infty}$ converges

6/ is $f_n(x) = e^{\frac{1}{n}x}$ $\therefore \int_a^b f_n(x)\,dx = \int_a^b e^{\frac{1}{n}x}\,dx = \left[n e^{\frac{1}{n}x}\right]_a^b$

$= ne^{\frac{1}{n}b} - ne^{\frac{1}{n}a} \therefore \lim_{n\to\infty}\int_a^b f_n(x)\,dx = \lim_{n\to\infty} n\left(e^{\frac{1}{n}b} - e^{\frac{1}{n}a}\right) =$

$\lim_{n\to\infty} n\left(e^0 - e^0\right) = \lim_{n\to\infty} n(0) = 0$

\7/ dirichlet func is $\delta(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

So is $\delta_n(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ \frac{1}{n}, & x \notin \mathbb{Q} \end{cases}$

$\therefore \lim_{n\to\infty} \delta_n(x) = \lim_{n\to\infty} \left(\begin{cases} 1, & x \in \mathbb{Q} \\ \frac{1}{n}, & x \notin \mathbb{Q} \end{cases} \right) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

is not riemann integrable

but $\delta_1(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 1, & x \notin \mathbb{Q} \end{cases}$ is riemann integrable

2008 Sheet 8

\1| i / $x_n = \begin{pmatrix} \frac{\sin(n)}{n} \\ \frac{\cos n}{n} \end{pmatrix}$, $\cancel{\frac{\cos(n)}{n}}$ $\therefore \varepsilon > 0$ is $|x_n - 0| = \begin{pmatrix} \left|\frac{\sin n}{n}\right| \\ \left|\frac{\cos n}{n}\right| \end{pmatrix} \leq$

$\begin{pmatrix} \left|\frac{\sin N}{N}\right| \\ \frac{\cos N}{N} \end{pmatrix} \leq \varepsilon = \begin{pmatrix} \varepsilon_1 \\ \varepsilon_2 \end{pmatrix}$ $\quad \frac{1}{\varepsilon_1} \leq N$, $\frac{1}{\varepsilon_2} \leq N$ & $\varepsilon_3 = \max(\varepsilon_1, \varepsilon_2) \therefore$

& by inspection limit is $x = (0,0)^T$ $\therefore \forall \varepsilon > 0$

$\|x_n - x\| = \|x_n\| = \sqrt{\frac{\sin^2(n)}{n^2} + \frac{\cos^2(n)}{n^2}} = \sqrt{\frac{1}{n}} = \frac{1}{n} \leq \frac{1}{N} < \varepsilon$ for

$\forall n \geq N$ with $N > \frac{1}{\varepsilon}$

\ iii / by inspection limit is $x = (2, 3, 4)^T$ $\therefore \forall \varepsilon > 0$

$\|x_n - x\| = \|x_n\| = \sqrt{\left(\frac{2n+1}{n}\right)^2 + \left(\frac{3n^2+1}{n^2}\right)^2 + 4^2} =$

$\left(\frac{4n^2+1+4n}{n^2} + \frac{9n^4+1+6n^2}{n^4} + \frac{16n^4}{n^4}\right)^{1/2} = \left(\frac{4n^4+n^2+4n^3}{n^4} + \frac{25n^4+6n^2+1}{n^4}\right)^{1/2} =$

$\sqrt{\frac{29n^4+4n^3-7n^2-1}{n^4}}$

$\|x_n - x\| = \left\|\left(\frac{2n+1}{n} - 2, \frac{3n^2+1}{n^2} - 3, 4-4\right)^T\right\| =$

2008 Sheet 8

$\|(\frac{1}{n}, \frac{1}{n^2}, 0)^T\| = \sqrt{(\frac{1}{n})^2 + (\frac{1}{n^2})^2 + 0^2} = \sqrt{\frac{1}{n^2} + \frac{1}{n^4}} < \sqrt{\frac{1}{n^2} + \frac{1}{n^2}} =$

$\sqrt{\frac{2}{n^2}} \leq \sqrt{\frac{1}{N^2}} = \frac{1}{N} < \varepsilon \quad \forall n \geq \sqrt{2}N$ with $N > \frac{1}{\varepsilon}$

$\sqrt{\frac{2}{n^2}} = \frac{\sqrt{2}}{n} \leq \frac{\sqrt{2}}{N} < \varepsilon \quad \forall n \geq N$ with $N > \frac{\sqrt{2}}{\varepsilon}$

\2/ if $\{x_n\}$ converges in $\mathbb{R}^p$ then $\forall n: x_n$ converges $\mathbb{R}^p$

$\therefore$ ~~$\|x_n - x$~~ $\forall n: \|x_n - x\| < \varepsilon \quad \forall \varepsilon > 0 \quad \therefore$

all elements of x_n must converge $\therefore$

x_n^i converges in $\mathbb{R}$ $\therefore$ ~~lim~~ $|x_n^i - x^i| < \varepsilon \quad \forall \varepsilon > 0, \forall i \in P$

and if $\forall i \in P: |x_n^i - x^i| < \varepsilon \quad \therefore \lim_{n \to \infty} (x_n^i) = x^i \in \mathbb{R} \therefore$

$\lim_{n \to \infty} \{x_n\} = x \therefore \|x_n - x\| < \varepsilon \quad \forall \varepsilon > 0,$ ~~$n \in \mathbb{Z}$~~ $\forall x_n \in \{x_n\} \therefore$

$\lim_{n \to \infty} \{x_n\} = \{x\} \in \mathbb{R}^p \therefore \{x_n\}$ converges $\therefore \{x_n\}$ converges

if and only if $\{x_n^i\}$ converges in $\mathbb{R}$

\2 sol/: (if) assume each sequence $\{x_n^i\}$ converges in $\mathbb{R}$ and

denote limit by $x^i = \lim_{n \to \infty} x_n^i \therefore \forall \varepsilon > 0 \; \exists N^i \in \mathbb{N}: \|x_n^i - x\| < \frac{\varepsilon}{\sqrt{p}} \; \forall n \geq N^i$

show $\lim_{n \to \infty} x_n = x = (x^1, \ldots, x^n)^T$ let $N = \max_{i=1,\ldots,n} N^i \therefore \forall n \geq N$

$\|x_n - x\| = \sqrt{(x_n^1 - x^1)^2 + \cdots + (x_n^p - x^p)^2} < \sqrt{\frac{\varepsilon^2}{p} + \cdots + \frac{\varepsilon^2}{p}} = \varepsilon$

(only if) assume $\lim_{n \to \infty} x_n$ exists and denote limit by:

$x = (x^1, \ldots, x^n) \therefore \forall \varepsilon > 0 \; \exists N \in \mathbb{N}: \|x_n - x\| < \varepsilon \; \forall n \geq N$ but

$\forall n \geq N: |x_n^i - x^i| = \sqrt{(x_n^i - x^i)^2} <$

$\sqrt{(x_n^1 - x^1)^2 + \cdots + (x_n^i - x^i)^2 + \cdots + (x_n^p - x^p)^2} = \|x_n - x\| < \varepsilon \therefore$

$\lim_{n \to \infty} x_n^i = x^i$ exists $\forall i = 1, \ldots, p$

\3/ Cauchy sequence: $\forall \varepsilon, \exists N; n>N \wedge m>N: |x_n - x_m| < \varepsilon$

$\forall \varepsilon>0, \exists N; n,m>N: |x_n - x_m| < \varepsilon$

$\therefore$ for vectors in $\mathbb{R}^n$ big:

$\forall \varepsilon>0, \exists N_i; \cancel{n}, m>N_i: ||x_n - x_m|| < \varepsilon$

is cauchy $\therefore$ for $\mathbb{R}^n$: $\{x_n\}$

$\forall \varepsilon>0, \exists N; \cancel{n}, m>N: ||x_n - x_m|| < \varepsilon$

$\therefore \lim_{n\to\infty}\{x_n\} - \underline{x} = \underline{0} \therefore \lim_{n\to\infty}\{x_n\} \in \mathbb{R}^n \therefore \{x_n\}$ converges

and if $\{x_n\}$ converges: $\lim_{n\to\infty}\{x_n\} \in \mathbb{R}^n \therefore \lim_{n\to\infty}\{x_n\} - \underline{x} = 0 \therefore$

~~$\forall n: \mathbb{R}$ $\forall n; \mathbb{R}$ $\forall n. \forall \varepsilon>0; ||x$~~ $\forall n; \forall \varepsilon>0: ||x_n - x|| < \varepsilon \therefore$

$\forall n; \forall \varepsilon>0, \exists N; n,m>N: ||x_n - x_m|| < \varepsilon \therefore$

$\forall \varepsilon>0, \exists N; n,m>N: ||x_n - x_m|| < \varepsilon \therefore \{x_n\}$ is cauchy $\therefore$

if sequence is cauchy it converges

\3.S.1/ sequence $\{x_n\}$ of vectors $x_n \in \mathbb{R}^p$ is a cauchy sequence if $\forall \varepsilon>0, \exists N \in \mathbb{N}; n,m \geq N: ||x_n - x_m|| < \varepsilon$

(if): assume sequence $\{x_n\}$ of vecs $x_n \in \mathbb{R}^p$ is cauchy

show each component sequence $\{x_n^i\}$ $i=1,\dots,p$ is a cauchy sequence in $\mathbb{R} \therefore$ converges by \Thm 3.17/ $\therefore \{x_n\}$ conv in $\mathbb{R}^p$

since $\{x_n\}$ is cauchy: $\forall \varepsilon>0, \exists N \in \mathbb{N}; \forall n,m>N: ||x_n - x_m|| < \varepsilon$

$\therefore |x_n^i - x_m^i| = \sqrt{(x_n^i - x_m^i)^2} <$

$((x_n^1 - x_m^1)^2 + \dots + (x_n^i - x_m^i)^2 + \dots + (x_n^p - x_m^p)^2)^{1/2} = ||x_n - x_m|| < \varepsilon$

$\therefore$ each component sequence $\{x_n^i\}$ $i=1,\dots,p$ is a cauchy sequence in $\mathbb{R}$

(only if): if sequence $\{x_n\}$ of vecs $x_n \in \mathbb{R}^p$ conv to $x \in \mathbb{R}^p$

$\therefore \forall \varepsilon>0, \exists N;$ if $n \geq N \Rightarrow ||x_n - x|| < \frac{\varepsilon}{2}$ $\therefore$ suppose $n,m \geq N \therefore$

$||x_n - x_m|| =$

2083 Sheet 8

$\|(x_n - x) - (x_m - x)\| \le \|x_n - x\| + \|x_m - x\|$ {triangle inequality}

$< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ ∴ $\{x_n\}$ is Cauchy

\4i/ $f'(x) = \frac{d}{dx} f(x) = \frac{d}{dx}((x,0,\dots,0)^T) = (1,0,\dots,0)^T \in \mathbb{R}^n$

$\forall x \in \mathbb{R}$ ∴ $\cancel{\|f'(x)}$ $\forall \varepsilon>0,\ \|f'(x) - (1,0,\dots,0)^T\| < \varepsilon$

∴ $\forall x\in\mathbb{R}: f(x) \in \mathbb{R}^n$ ∴ $f(x)$ is cont $\forall x \in \mathbb{R}$

\4i sol/ let $a\in\mathbb{R}$ fix $\varepsilon>0$ ∴ $\|f(x)-f(a)\| =$

$\|(x-a,0,0,\dots,0)^T\| = \sqrt{(x-a)^2} = |x-a| < \delta = \varepsilon$ holds $\forall |x-a|<\delta$

where $\delta=\varepsilon$ ∴ $\lim_{x\to a} f(x) = f(a)$ $\forall a\in\mathbb{R}$ and f is cont on $\mathbb{R}$

~~\4~~ ii/ is $\varepsilon>0$ say $\forall\delta>0: |x-0|<\delta$ ∴

$\|f(x)-f(0)\| = \|f(x) - 0\| = \|f(x)\| < \sqrt{\left(\frac{x_1^2x_2^2}{x_1^2+x_2^2}\right)^2} =$

$\left|\frac{x_1^2x_2^2}{x_1^2+x_2^2}\right| = |x-0| < \delta = \varepsilon$ ∴ holds $\forall |x-0|<\delta$ with $\delta = \varepsilon$

∴ $\lim_{x\to0} f(x) = f(0)$ $\forall\varepsilon$ and f is cont on $\mathbb{R}$

\4ii sol/ $\|f(x)-f(0)\| = \frac{x_1^2x_2^2}{x_1^2+x_2^2} \le \frac{(x_1^2)^2+(x_2^2)^2+2x_1^2x_2^2}{x_1^2+x_2^2} =$

$\frac{(x_1^2+x_2^2)^2}{x_1^2+x_2^2} = x_1^2+x_2^2 < \delta^2 < \varepsilon$ holds $\forall 0<\|x-0\| = \sqrt{x_1^2+x_2^2} < \delta$ with $\delta = \sqrt{\varepsilon}$ use $x_1^2 \le x_1^2+x_2^2$ and $x_2^2 \le x_1^2+x_2^2$

∴ $\lim_{x\to0} f(x) = f(0)$ and f is cont at 0

\4iii/ $\|f(x)-f(0)\| = \sqrt{(2x_2\sin(x_1))^2+(3x_1\sin x_2)^2}$

$= \|f(x) - (2(0)\sin(0), 3(0)\sin(0))^T\| = \|f(x) - (0,0)^T\| = \|f(x)\|$

$= \sqrt{4x_2^2\sin^2(x_1)+9x_1^2\sin^2(x_2)} \le \sqrt{4x_2^2+9x_1^2} \le \sqrt{4x_2^2+12x_1x_2+9x_1^2} =$

$\sqrt{(2x_2^2+3x_1^2)^2} = |2x_2^2+3x_1^2| = 2x_2^2+3x_1^2 < \delta^2 = \varepsilon$ holds

$\forall 0<\|x-0\| = \sqrt{2x_2^2+3x_1^2} < \delta$ with $\delta = \sqrt{\varepsilon}$ ∴ $\lim_{x\to0} f(x) = f(0)$

and f is cont at 0

\4iii Sol/ $f(0)=(0,0)^T$ $\therefore$ $\|f(x)-f(0)\|=$

$\sqrt{(2x_2\sin x_1)^2+(3x_1\sin x_2)^2}=\sqrt{4x_2^2\sin^2(x_1)+9x_1^2\sin^2(x_2)}<$

$\sqrt{4x_2^2+9x_1^2}<\sqrt{9x_2^2+9x_1^2}=3\sqrt{x_2^2+x_1^2}<3\delta=\varepsilon$ holds

$\forall\, 0<\|x-0\|=\sqrt{x_1^2+x_2^2}<\delta$ with $\delta=\frac{\varepsilon}{3}$ $\therefore$

$\lim_{x\to 0}f(x)=f(0)$ and f is cont at 0

\5i/ $f(0)=0$ $\therefore$ $\|f(x)-f(0)\|=\left\|\frac{x_1x_2}{x_1^2+x_2^2}\right\|=\sqrt{\left(\frac{x_1x_2}{x_1^2+x_2^2}\right)^2}=$

$\sqrt{\frac{x_1^2x_2^2}{(x_1^2+x_2^2)^2}}\leq\sqrt{\frac{(x_1^2+x_2^2)^2}{(x_1^2+x_2^2)^2}}=\sqrt{1}=\sqrt{1}=1$

$\|f(x)-f(0)\|=\frac{x_1^2x_2^2}{x_1^2+x_2^2}\geq\frac{x_1x_2}{(x_1+x_2)^2}=\frac{x_1x_2}{x_1^2+2x_1x_2+x_2^2}$

$\leq\frac{x_1^2+2x_1x_2+x_2^2}{(x_1+x_2)^2}=\frac{(x_1+x_2)^2}{(x_1+x_2)^2}=1>\cancel{\delta}\,\varepsilon$ $\therefore$

$\lim_{x\to 0}f(x)\neq f(0)$

\5i Sol/ $\forall\varepsilon>0,\ \exists\delta>0;\ 0<\|x\|<\delta$ $\therefore$

$\|f(x)-f(0)\|=\left|\frac{x_1x_2}{x_1^2+x_2^2}\right|<\varepsilon$ but consider vectors

$x=(x_1,x_2)^T\in\mathbb{R}$ Satisfying $0<\|x\|<\delta$, $x_1=x_2$ and $x_1x_2\neq 0$

$\therefore\left|\frac{x_1x_2}{x_1^2+x_2^2}\right|=\frac{x_1^2}{2x_1^2}=\frac{1}{2}$ $\forall\delta>0$ $\therefore$ f is not cont at 0

\5ii/ $f(0)=0$ $\therefore$ $\forall\varepsilon>0,\ \exists\delta>0;\ 0<\|x\|<\delta$ $\therefore$

$\|f(x)-f(0)\|=\|f(x)\|=\left|\frac{x_1+x_2}{x_1-x_2}\right|<\varepsilon$ but consider vectors

$x=(x_1,x_2)^T\in\mathbb{R}$ satisfying $0<\|x\|<\delta$, $\frac{1}{2}x_1=x_2$ and $x_1\neq 0$,

$x_2\neq 0$ $\therefore$ $\left|\frac{x_1+x_2}{x_1-x_2}\right|=\left|\frac{x_1+\frac{1}{2}x_1}{x_1-\frac{1}{2}x_1}\right|=\left|\frac{\frac{3}{2}x_1}{\frac{1}{2}x_1}\right|=|3|=3$ $\forall\delta>0$ $\therefore$

f is not cont at 0 ✓

2008 Sheet 8

6/ continuous at a: $\|f(x)-f(a)\|<\varepsilon \quad \forall \varepsilon>0$ holds $\forall |x-a|<\delta \quad \therefore \lim_{x\to a} f(x) = f(a)$

if $\lim_{x\to a} f_i(x) = f_i(a) \quad \therefore \forall\varepsilon>0 \quad \|f_i(x)-f_i(a)\|<\varepsilon$

$\forall |x-a|<\delta \quad \therefore \quad \|f(x)-f(a)\| =$

$\sqrt{(f_1(x)-f_1(a))^2+\cdots+(f_m(x)-f_m(a))^2}$

$\leq \sqrt{Max((f_i(x)-f_i(a))^2)} = Max(f_i(x)-f_i(a)) < \varepsilon \quad \forall\varepsilon>0$:

and if $\|f(x)-f(a)\|<\varepsilon \quad \forall |x-a|<\delta$

$\therefore$ ~~lim~~ $\|(f_1(x)-f_1(a))+\cdots+(f_m(x)-f_m(a))\|<\varepsilon$

$\forall |x-a|<\delta \quad \therefore$ ~~if~~ f is cont at $a\in A$ iff f_i is cont at a for $i=1,\ldots,m$.

6 Sol:/ ($\Leftarrow$) suppose each map $f_i:\mathbb{R}\to\mathbb{R}$ is cont at p for $i=1,\ldots,m$ show f is cont at $p\in A$

fix $\varepsilon>0 \quad \therefore$ ~~for~~ for each i, $\exists\delta_i>0$; for $x\in A$ with $0<\|x-p\|<\delta_i$ have $|f_i(x)-f_i(p)|<\varepsilon/n$ now let $\delta = Min_{i=1,\ldots,m}(\delta_i)$ if $x\in A$, $0\|x-p\|<\delta$ have: $\|f(x)-f(p)\|\leq$

$n \max_{i=1,\ldots,m} |f_i(x)-f_i(p)| < n\frac{\varepsilon}{n} = \varepsilon \quad \therefore f$ is cont at p

($\Rightarrow$) now suppose f is cont at p fix $\varepsilon>0$ $\therefore$

$\exists\delta>0$; $\forall x\in A$, $0<\|x-p\|<p$ have: $\|f(x)-f(p)\|<\varepsilon$

fix $j\in\{1,\ldots,m\}$ have: $|f_j(x)-f_j(p)| < \max_{i=1,\ldots,m}|f_i(x)-f_i(p)| \leq$

$\|f(x)-f(p)\|<\varepsilon \quad \therefore f_j$ is cont at p

6 redo/ if $f_i:\mathbb{R}\to\mathbb{R}$ is cont at a for $i=1,\ldots,m$

fix $\varepsilon>0 \quad \therefore$ for each i, $\exists\delta_i>0$; for $x\in A$ with $0<\|x-a\|<\delta_i$ have: $|f_i(x)-f_i(a)|<\varepsilon/n$ let $\delta = Min_{i=1,\ldots,m}(\delta_i)$ if $x\in A$, $0<\|x-a\|<\delta$ have $\|f(x)-f(a)\|<n \max_{i=1,\ldots,m}|f_i(x)-f_i(a)| < n\varepsilon/n = \varepsilon \quad \therefore f$ is cont at a

if f is cont at a fix $\varepsilon>0$ $\therefore \exists \delta>0;\ \forall x\in A,\ 0<\|x-a\|<a$ have ~~fix~~ $i\in\{1,\dots,m\}$ have $|f_i(x)-f_i(a)|\le$

$\max_{i=1,\dots,m}|f_i(x)-f_i(a)|\le\|f(x)-f(a)\|<\varepsilon$ $\therefore$ f_i is cont at a

\6 redo/ if $f_i:\mathbb{R}\to\mathbb{R}$ is cont at a for $i=1,\dots,m$

fix $\varepsilon>0$ $\therefore$ for each i, $\exists\delta_i>0;$ for $x\in A$ with $0<\|x-a\|<\delta_i$

have: $|f_i(x)-f_i(a)|<\varepsilon/n$ let $\delta=\min_{i=1,\dots,m}(\delta_i)$ if $x\in A$ $0<\|x-a\|<\delta$

have: $\|f(x)-f(a)\|<n\max_{i=1,\dots,m}|f_i(x)-f_i(a)|<n\,\varepsilon/n=\varepsilon$ $\therefore$ f is cont at a

if f is cont at a fix $\varepsilon>0$ $\therefore \exists\delta>0;\ \forall x\in A,\ 0<\|x-a\|<a$ have:

fix $i\in\{1,\dots,m\}$ have: $|f_i(x)-f_i(a)|\le\max_{i=1,\dots,m}|f_i(x)-f_i(a)|\le$

$\|f(x)-f(a)\|<\varepsilon$ $\therefore$ f_i is cont at a

\7/ fix $i\in\{1,\dots,m\}$ have: $|\pi_i(x)-\pi_i(a)|\le\max_{i=1,\dots,m}|\pi_i(x)-\pi_i(a)|\le$

~~$\|f$~~ $\|\pi(x)-\pi(a)\|<\varepsilon$ $\therefore$ π_i is cont at $a\in\mathbb{R}$

$\pi_i((a_1,\dots,a_n)^T)=a_i$ $\therefore$ $|\pi_i(x)-\pi_i(a)|=|x_i-a_i|\le$

$n\max_{i=1,\dots,n}|x_i-a_i|\le n\,\frac{\varepsilon}{n}=\varepsilon$ $\therefore$ π_i is cont $\forall a\in\mathbb{R}$

\7 sol:/ show for any $x=(x_1,\dots,x_n)^T\in\mathbb{R}^n$, $\lim_{y\to x}\pi_i(y)=\pi(x)$

for any $\varepsilon>0$ have: $\|\pi_i(y)-\pi_i(x)\|=|y_i-x_i|=\sqrt{(y_i-x_i)^2}$

$<\sqrt{(y_1-x_1)^2+\dots+(y_i-x_i)^2+\dots+(y_n-x_n)^2}=\|y-x\|<\delta=\varepsilon$

for all $0<\|y-x\|<\delta$ with $\delta=\varepsilon$ $\therefore$ Cont $\forall x\in\mathbb{R}^n$

\8 i/ \thm 3.15/ \continuity in terms of coord funcs/ let $A\subset\mathbb{R}^n$

and $f:A\to\mathbb{R}^m$ $\therefore$ f is cont at a point $a\in A$ iff each

coord func $f_i:\mathbb{R}^n\to\mathbb{R}$ $i=1,\dots,m$ is cont at a ||

\thm 3.16/ \contity of 2 compost of two cont funcs/

let $A\subset\mathbb{R}^n$, $B\subset\mathbb{R}^m$ suppose $f:A\to B$ is cont at a

and $g:B\to\mathbb{R}^l$ is cont at $f(a)$ $\therefore$ $g\circ f:A\to\mathbb{R}^l$ is cont at a ||

2008 Sheet 8

\thm 3.17/ \properties of cont funcs in $\mathbb{R}^n$ / $A \subset \mathbb{R}^n$

- and $f, g : A \to \mathbb{R}$ let $\lim_{x \to a} f(x) = f(a)$ $\lim_{x \to a} g(x) = g(a)$

$\therefore \lim_{x \to a} f(x) + g(x) = f(a) + g(a)$

$\lim_{x \to a} f(x)g(x) = f(a)g(a)$ $\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f(a)}{g(a)}$ if $g(a) \neq 0$ //

\thm 3.18/ projection func ~~$\pi_i : \mathbb{R}^n \to \mathbb{R}$ given by~~

$\pi_i : \mathbb{R}^n \to \mathbb{R}$ given by $\pi_i(x) = x_i$ is cont //

\8i/ $f(a) = \frac{e^{a_1 a_2}}{1 + \cos^2(a_2)}$ $\therefore \| f_i(x) - f_i(a) \| =$

$\| (f_1(x) - f_1(a), f_2(x) - f_2(a)) \| = \sqrt{}$

$\frac{e^{(x_1 - a_1)(x_2 - a_2)}}{1 + \cos^2(x_2 - a_2)}$ $\lim_{x \to a} x = a$ $\therefore \lim_{x \to a} e^x = e^a$

$\lim_{x \to a} \frac{1}{1 + \cos^2(x)} = \frac{1}{1 + \cos^2(a)}$ $\therefore$ is $1 + \cos^2(a) \neq 0$

$\lim_{x \to a} \frac{e^{x_1 x_2}}{1 + \cos^2(x_2)} = \frac{e^{a_1 a_2}}{1 + \cos^2(a_2)}$ $\therefore \| f(x) - f(a) \| < \varepsilon, \forall \varepsilon > 0$

holds $|x - a| < \delta$

\8i sol/ $p, q, r : \mathbb{R} \to \mathbb{R}$ be cont funcs defined:

- $p(x) = e^x$ $q(x) = \cos(x)$ $r(x) = x^2$ $s(x) = 1$ $\therefore$

write $f(x)$ as $\frac{p(\pi_1(x)\pi_2(x))}{s(x) + r(q(\pi_2(x)))}$ $\therefore f$ is cont by applying thms

\8ii/ $f(a) = (a_1 a_2, a_1 + a_2, a_1^2)^T$

is $p : \mathbb{R} \to \mathbb{R}$ $p(x) = x^2$ $\therefore$ write $f(x)$ as:

$(\pi_1(x)\pi_2(x), \pi_1(x) + \pi_2(x), p(\pi_1(x)))^T$

\8ii sol/ by thm 3.15 f is cont iff each coord func

$f_i : \mathbb{R}^2 \to \mathbb{R}$ $f_1(x) = x_1 x_2$ $f_2(x) = x_1 + x_2$ $f_3(x) = x_1^2$ are cont

- can write $f_i(x)$ as: $f_1(x) = \pi_1(x)\pi_2(x)$

$f_2(x) = \pi_1(x) + \pi_2(x)$ $f_3(x) = \pi_1(x)\pi_1(x)$ by applying thms 3.16–3.18

makes it clear f is cont

2008 Sheet 9

\i/ $\therefore f_i : \mathbb{R} \to \mathbb{R}$, $f_i(x_i) = x_i$ where $x = (x_1, x_2, \ldots x_n)^T$

$\therefore f_i(x_i) \in \mathbb{R}$ $\forall x_i \in \mathbb{R}$ $\therefore$ $f(x_i)$ is continuous $\therefore$

$\frac{d}{dx_i} f(x_i) = f'(x_i)$ $\forall$ $i = 1, 2, \ldots, n$ $\therefore$

$\frac{d}{dx} f(x) = f'(x)$ $\forall x \in \mathbb{R}^n$ since $f(x)$ is continuous

\i(a)/ note the Jacobian is the $n \times n$ identity matrix I

$\therefore$ is $Df(p) = I$ $\therefore$

$f(p+h) - f(p) - Df(p) \cdot h = (p+h) - p - h = 0$

$\{ f(p+h) - f(p) - I \cdot h = 0 = f(p+h) - f(p) - h =$

$(p+h) - (p) - h = p + h - p - h = 0 \}$

$\therefore \lim_{h \to 0} \frac{\|f(p+h) - f(p) - Df(p) \cdot h\|}{\|h\|} = 0$ $\{ \lim_{h \to 0} \frac{\|0\|}{\|h\|} = \lim_{h \to 0} 0 = 0$

$\{$ is $\lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$ $\therefore$ for vectors in $\mathbb{R}^n \}$

$\{ J = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix} = \begin{bmatrix} \partial f_1/\partial x_1 & \cdots & \frac{\partial f_1}{\partial x_n} \\ & \ddots & \\ \frac{\partial f_n}{\partial x_1} & & \frac{\partial f_n}{\partial x_n} \end{bmatrix}$ $\therefore f_i = f(x_i)$ $i = 1, \ldots, n$

$\therefore \frac{\partial f_1}{\partial x_1} = \frac{\partial}{\partial x_1} f_1(x_1) = \frac{\partial}{\partial x_1}(x_1) = 1$ $\therefore \frac{\partial f_i}{\partial x_i} = \frac{\partial}{\partial x_i} f_i(x_i) = \frac{\partial}{\partial x_i}(x_i) = 1$

$i = 1, \ldots, n$ but $\frac{\partial f_i}{\partial x_j} = \frac{\partial}{\partial x_j} f_i(x_i) = \frac{\partial}{\partial x_j}(x_i) = 0$ $\forall i \neq j$ $i, j = 1, \ldots, n$

$\therefore J = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & 0 \\ 0 & & 1 & \ddots & \vdots \\ \vdots & & & & 0 \\ 0 & 0 & \cdots & 0 & 1 \end{bmatrix} = I_{n \times n} = J_{n \times n} \}$

$\therefore$ f is differentiable at any $p \in \mathbb{R}^n$

\ii/ $\therefore f(x_1, x_2) = x_1^2 + x_2^2$ for $x = (x_1, x_2)^T$ $\therefore$

$\frac{\partial f(x)}{\partial x_1} = \frac{\partial f}{\partial x_1} = \frac{\partial}{\partial x_1} f(x_1, x_2) = \frac{\partial}{\partial x_1}(x_1^2 + x_2^2) = 2x_1$ $\therefore \frac{\partial f}{\partial x_2} = 2x_2$

$\therefore \frac{\partial^2 f}{\partial x_2^2} = 2$ $\therefore \frac{\partial^2 f}{\partial x_1^2} = 2$ $\therefore \frac{\partial^2 f}{\partial x_2 \partial x_1} = \frac{\partial}{\partial x_2}(2x_1) = 0 = \frac{\partial^2 f}{\partial x_1 \partial x_2}$ $\therefore$

2008 Sheet 9

$$\therefore \begin{bmatrix} \partial^2 f/\partial x_1^2 & \partial^2 f/\partial x_1\partial x_1 \\ \frac{\partial^2 f}{\partial x_2 \partial x_1} & \frac{\partial^2 f}{\partial x_2^2} \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I \quad \therefore \text{ is } Df(p) = 2I \therefore$$

$$f(p+h) - f(p) - Df(p)\cdot h = (p_1+h_1)^2 + (p_2+h_2)^2 - (p_1^2+p_2^2) - 2I\cdot h =$$

$$p_1^2 + h_1^2 + 2p_1h_1 + p_2^2 + h_2^2 + 2p_2h_2 - p_1^2 - p_2^2 - 2h_1 - 2h_2 =$$

$$h_1^2 + 2p_1h_1 + h_2^2 + 2p_2h_2 - 2h_1 - 2h_2 = h_1(h_1 + 2p_1 - 2) + h_2(h_2 + 2p_2 - 2) \therefore$$

$$\lim_{h\to 0} \frac{\|f(p+h) - f(p) - Df(p)\cdot h\|}{\|h\|} = \lim_{h\to 0} \left\| \frac{h(h_1+2p_1-2) + h_2(h_2+2p_2-2)}{h} \right\|$$

$$= \lim_{h\to 0} \|(h_1 + 2p_1 - 2) + h_2 + 2p_2 - 2\| = 0 + 2p_1 - 2 + 0 + 2p_2 - 2 = 2(p_1+p_2) - 4$$

$$= 2(p_1 + p_2 - 2).$$

\iii sol/ set $h = (h_1, h_2)^T \therefore f(p+h) - f(p) - Df(p)\cdot h =$

$$(p_1+h_1)^2 + (p_2+h_2)^2 - (p_1^2+p_2^2) - (2p_1 \;\; 2p_2)\begin{pmatrix} h_1 \\ h_2 \end{pmatrix} =$$

$$p_1^2 + h_1^2 + 2p_1h_1 + p_2^2 + h_2^2 + 2p_2h_2 - p_1^2 - p_2^2 - (2p_1h_1 + 2p_2h_2) =$$

$$h_1^2 + 2p_1h_1 + h_2^2 + 2p_2h_2 - 2p_1h_1 - 2p_2h_2 = p_1^2 + p_2^2 = h_1^2 + h_2^2$$

$$\frac{\partial f}{\partial p_1} = \frac{\partial f(p)}{\partial p_1} = \frac{\partial}{\partial p_1}(p_1^2 + p_2^2) = 2p_1 \quad \therefore \frac{\partial f}{\partial p_2} = 2p_2 \quad \therefore$$

$$Df(p) = \begin{bmatrix} \frac{\partial f}{\partial p_1} & \frac{\partial f}{\partial p_2} \end{bmatrix} = \begin{bmatrix} 2p_1 & 2p_2 \end{bmatrix}, \quad h = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} \therefore$$

$$Df(p)\cdot h = \begin{bmatrix} 2p_1 & 2p_2 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = 2p_1h_1 + 2p_2h_2 \quad \therefore$$

$$\frac{\|f(p+h) - f(p) - Df(p)\cdot h\|}{\|h\|} = \frac{\|h_1^2 + h_2^2\|}{\|(h_1, h_2)\|} = \frac{\sqrt{(h_1^2+h_2^2)^2}}{\sqrt{h_1^2+h_2^2}} = \frac{(h_1^2+h_2^2)}{(h_1^2+h_1^2)^{1/2}} =$$

$$\sqrt{h_1^2 + h_2^2} = \|h_1, h_2\| = \|h\| \quad \therefore \lim_{h\to 0} \frac{\|f(p+h) - f(p) - Df(p)\cdot h\|}{\|h\|} =$$

$\lim_{h\to 0}\|h\| = \|\lim_{h\to 0} h\| = \|0\| = \sqrt{0^2} = 0 \quad \therefore f$ is differentiable at any $p \in \mathbb{R}^2$

\iii/ $x = (x_1, x_2)^T$, $h = (h_1, h_2)^T \therefore Df(x) = D(x_1x_2) =$

$$\begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x_1}(x_1x_2) & \frac{\partial}{\partial x_2}(x_1x_2) \end{bmatrix} = \begin{bmatrix} x_2 & x_1 \end{bmatrix} \therefore Df(x)\cdot h =$$

$\begin{bmatrix} x_2 & x_1 \end{bmatrix}\begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = x_2h_1 + x_1h_2 \quad \therefore$

$f(x+h) - f(x) - Df(x)\cdot h = t$

$(x_1+h_1)(x_2+h_2) - (x_1x_2) - (x_1h_1 + x_1h_2) =$

$x_1x_2 + h_1h_2 + x_1h_2 + x_2h_1 - x_1x_2 - x_2h_1 - x_2h_2 =$

$h_1h_2 + x_1h_2 + x_2h_1 - x_1h_2 - x_2h_1 = h_1h_2$

$\cancel{x_1h_2 + x_2h_1 =} \quad \therefore$

$\lim_{h\to 0} \dfrac{\|f(x+h) - f(x) - Df(x)\cdot h\|}{\|h\|} = \cancel{\lim_{h\to 0} \dfrac{\|x_1h_2 + x_2h_1\|}{\|h\|}} = \lim_{h\to 0} \dfrac{\|h_1h_2\|}{\|h\|} =$

$\cancel{\lim_{h\to 0} \dfrac{\sqrt{(x_1h_2 + x_2h_1)^2}}{\sqrt{h_1^2+h_2^2}}} \cancel{= \lim_{h\to 0} \dfrac{x_1h_2 + x_2h_1}{\sqrt{h_1^2+h_2^2}}} \quad \lim_{h\to 0} \dfrac{\sqrt{(h_1h_2)^2}}{\sqrt{h_1^2+h_2^2}} \cancel{= \lim_{h\to 0} \dfrac{h_1h_2}{\sqrt{h_1^2+h_2^2}}}$

$= \lim_{h\to 0} \dfrac{|h_1h_2|}{\sqrt{h_1^2+h_2^2}} \le \lim_{h\to 0} \dfrac{\frac{1}{2}(h_1^2+h_2^2)}{(h_1^2+h_2^2)^{1/2}} = \lim_{h\to 0} \frac{1}{2}\sqrt{h_1^2+h_2^2} =$

$\cancel{\lim_{h_1\to 0, h_2\to 0} \left(\tfrac{1}{2}\sqrt{h_1^2+h_2^2}\right) = \tfrac{1}{2}\sqrt{0+0} = 0} \quad = \lim_{h\to 0} \frac{1}{2}\|h\| = 0 \quad \therefore$

f is differentiable $\forall x \in \mathbb{R}^2$ $\quad \therefore$

$0 \le \lim_{h\to 0} \dfrac{\|f(x+h) - f(x) - Df(x)\cdot h\|}{\|h\|} = LHS \le 0 \quad \therefore$

$LHS = 0 \quad \therefore f$ is diffable $\forall x \in \mathbb{R}^2$

2/ $\therefore$ ~~f is~~ $f: \Omega \to \mathbb{R}^m$ is differentiable at $p \in \Omega \quad \therefore$

$\lim_{x\to 0} \dfrac{\|f(p+x) - f(p) - Df(p)\cdot x\|}{\|x\|} = 0 \quad x = (x_1, \dots, x_n)^T$ for $p \in \Omega$

for $g: \Omega \to \mathbb{R}^n$

~~and~~ $\lim_{x\to 0} \dfrac{\|g(p+x) - g(p) - Dg(p)\cdot x\|}{\|x\|} \quad \forall p \in \Omega \quad \therefore$

$\therefore \lim_{x\to 0} \dfrac{\|h(p+x) - h(p) - Dh(p)\cdot x\|}{\|x\|} =$

$\lim_{x\to 0} \dfrac{\|f(p+x) + g(p+x) - (f(p) + g(p)) - D(f(p) + g(p))\cdot x\|}{\|x\|} =$

$\lim_{x\to 0} \dfrac{\|f(p+x) - f(p) - Df(p)\cdot x\|}{\|x\|} + \lim_{x\to 0} \dfrac{\|g(p+x) - g(p) - Dg(p)\cdot x\|}{\|x\|}$

$= 0 + 0 = 0 \quad \therefore h$ is differentiable $\forall p \in \Omega$

2008 Sheet 9 $p=[p_1, p_2, \dots, p_n]$

• $Dh(p) = \left[\frac{\partial h}{\partial p_1} \ \frac{\partial h}{\partial p_2} \ \dots \ \frac{\partial h}{\partial p_n}\right] = \left[\frac{\partial}{\partial p_1} h(p) \ \dots \ \frac{\partial}{\partial p_n} h(p_n)\right]$ ∴

$Dh(p) = \left[\frac{\partial}{\partial p_1}(f(p)+g(p)) \ \dots \ \frac{\partial}{\partial p_n}(f(p_n)+g(p_n))\right] =$

$\left[\frac{\partial}{\partial p_1} f(p) + \frac{\partial}{\partial p_1} g(p) \ \dots \ \frac{\partial}{\partial p_n} f(p) + \frac{\partial}{\partial p_n} g(p)\right] =$

$\left[\frac{\partial}{\partial p_1} f(p) \ \dots \ \frac{\partial}{\partial p_n} f(p)\right] + \left[\frac{\partial}{\partial p_1} g(p) \ \dots \ \frac{\partial}{\partial p_n} g(p)\right] = Df(p) + Dg(p)$

2 ∃ linear maps $Df(p), Dg(p) : \mathbb{R}^n \to \mathbb{R}^m$;

$\lim_{x\to p} \frac{\|f(x)-f(p)-Df(p)\cdot(x-p)\|}{\|x-p\|}, \ \lim_{x\to p} \frac{\|g(x)-g(p)-Dg(p)\cdot(x-p)\|}{\|x-p\|} = 0$

∴ $\frac{\|h(x)-h(p)-Dh(p)\cdot(x-p)\|}{\|x-p\|} = \cancel{\|f(x)} =$

$\frac{\|f(x)+g(x)-f(p)-g(p)-Df(p)\cdot(x-p)-Dg(p)\cdot(x-p)\|}{\|x-p\|} \le$

$\frac{\|f(x)-f(p)-Df(p)\cdot(x-p)\|}{\|x-p\|} + \frac{\|g(x)-g(p)-Dg(p)\cdot(x-p)\|}{\|x-p\|}$ ∴

$\lim_{x\to p} \frac{\|h(x)-h(p)-Dh(p)\cdot(x-p)\|}{\|x-p\|} = 0 \left\{ = \lim_{x\to 0} \frac{\|h(p+x)-h(p)-Dh(p)\cdot x\|}{\|x\|} \right\}$

∴ h is diffable at p

3 ∴ ∃ linear maps $Df(p), Dg(p) : \mathbb{R}^n \to \mathbb{R}^m$,

$\lim_{x\to p} \frac{\|f(x)-f(p)-Df(p)\cdot(x-p)\|}{\|x-p\|}, \ \lim_{x\to p} \frac{\|g(x)-g(p)-Dg(p)\cdot(x-p)\|}{\|x-p\|} = 0$ ∴

$\lim_{x\to p} \frac{\|h(x)-h(p)-Dh(p)\cdot(x-p)\|}{\|x-p\|} =$

$\lim_{x\to p} \frac{\|f(x)\cdot g(x)-f(p)\cdot g(p)-D(f(p)\cdot g(p))\cdot(x-p)\|}{\|x-p\|}$

3 sol / let $S:\mathbb{R}^2\to\mathbb{R}$ be $S(x)=x_1x_2$ and $T:\mathbb{R}^n\to\mathbb{R}^2$ be $T(x)=(f(x),g(x))^T$

then note: $h=S\circ T$, T is diffable at p (lemma 4.10) and S is diffable at $T(p)$ $\therefore h=S\circ T$ is diffable at p

$\{h=f(x)g(x)\}$ $\{h(p)=S(T(p))=S\circ T(p) \therefore\}$

$\therefore Dh(p)=DS(T(p))\circ DT(p)=\begin{pmatrix}g(p) & f(p)\end{pmatrix}\begin{pmatrix}Df(p)\\ Dg(p)\end{pmatrix}=$
$g(p)Df(p)+f(p)Dg(p)$

4i / $h=(h_1,h_2)^T$ $\therefore f(x+h)-f(x)-Df(x)\cdot h=$

$(x_1+h_1)^2+(x_2+h_2)^2-(x_1+h_1)-(x_1+h_1)(x_2+h_2)-(x_1^2+x_2^2-x_1-x_1x_2)-Df(x)\cdot h$

$\left\{Df(x)=\begin{bmatrix}\frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2}\end{bmatrix}=\begin{bmatrix}2x_1-1-x_2 & 2x_2-x_1\end{bmatrix} \therefore Df(x)\cdot h=\right.$

$\left.\begin{bmatrix}2x_1-1-x_2 & 2x_2-x_1\end{bmatrix}\begin{bmatrix}h_1\\ h_2\end{bmatrix}=2x_1h_1-h_1-x_2h_1+2x_2h_2-x_1h_2\right\}\therefore$

$=x_1^2+h_1^2+2x_1h_1+\cancel{x_2^2}+h_2^2+2x_2h_2-\cancel{x_1}-h_1-\cancel{x_1x_2}-h_1h_2-x_1h_2-x_2h_1-\cancel{x_1^2}-\cancel{x_2^2}+\cancel{x_1}+\cancel{x_1x_2}-Df(x)\cdot h$

$=h_1^2+2x_1h_1+2x_2h_2-h_1h_2-x_1h_2-x_2h_1-Df(x)\cdot h=$

$h_1^2+\cancel{2x_1h_1}+\cancel{2x_2h_2}-h_1h_2-x_1h_2-\cancel{x_2h_1}-\cancel{2x_1h_1}+h_1+\cancel{x_2h_1}-\cancel{2x_2h_2}+x_1h_2=$

$h_1^2-h_1h_2-\cancel{x_1h_2}+h_1+\cancel{x_1h_2}=h_1^2-h_1h_2+h_1=h_1(h_1-h_2+1)$

$\therefore \lim_{h\to 0}\frac{\|f(x+h)-f(x)-Df(x)\cdot h\|}{\|h\|}=\lim_{h\to 0}\frac{\|h_1(h_1-h_2+1)\|}{\sqrt{h_1^2+h_2^2}}=$

$\lim_{h\to 0}\frac{|h_1(h_1-h_2+1)|}{\sqrt{h_1^2+h_2^2}}$

4ii sol / $D_1f(x)=2x_1-1-x_2$, $D_2f(x)=2x_2-x_1$ Clearly these are continuous $\forall x\in\mathbb{R}^2$ deduce by thm 4.11: f is everywhere differentiable and moreover:

$Df(x)=(2x_1-1-x_2 \quad 2x_2-x_1)$

4iii / $D_1f(x)=\frac{\partial}{\partial x_1}f(x)=5x_1^4x_2^2$, $D_2f(x)=2x_1^5x_2$ are both continuous $\forall x\in\mathbb{R}^2$ $\therefore f$ is differentiable everywhere

$\therefore Df(x)=\begin{bmatrix}5x_1^4x_2^2 & 2x_1^5x_2\end{bmatrix}$

2008 Sheet 9

4iii) $D_1 f(x) = \frac{\partial}{\partial x_1} f(x) = \frac{\partial}{\partial x_1}\begin{pmatrix} x_1^2 x_2^2 \\ \cos(x_1)\sin(x_2)\end{pmatrix} = \begin{pmatrix} 2x_1 x_2^2 \\ -\sin(x_1)\sin(x_2)\end{pmatrix}$

$D_2 f(x) = \frac{\partial}{\partial x_2}\begin{pmatrix} x_1^2 x_2^2 \\ \cos(x_1)\sin(x_2)\end{pmatrix} = \begin{pmatrix} 2x_1^2 x_2 \\ \cos(x_1)\cos(x_2)\end{pmatrix}$

$\therefore Df(x) = \begin{bmatrix} 2x_1x_2^2 & 2x_1^2x_2 \\ -\sin(x_1)\sin(x_2) & \cos(x_1)\cos(x_2)\end{bmatrix}$ = Jacobian

$D_1 f(x)$ and $D_2 f(x)$ are continuous $\forall x \in \mathbb{R}^2$ and $\therefore$ f is differentiable everywhere

2008 Sheet 10

1) ~~$h = f\circ g = f$~~ $h(x) = (f\circ g)(x) = f(g(x)) = f\begin{pmatrix} x_1x_2 \\ x_1+x_2\end{pmatrix}$

$= \begin{pmatrix} e^{x_1x_2} \\ \sin(x_1+x_2)\end{pmatrix}$ $\therefore$ ~~$Dh(x) = \begin{pmatrix} \frac{\partial}{\partial x_1}(e^{x_1x_2}) \\ \frac{\partial}{\partial x_2}(\sin(x_1+x_2))\end{pmatrix} = \begin{pmatrix} x_2e^{x_1x_2} \\ \cos(x_1+x_2)\end{pmatrix}$~~

~~$Dh(x) = D(f(g(x))) = D(f(g(x)) + Dg(x)$~~

$Dh(x) = D(f(g(x)) \circ D(g(x)) = \begin{pmatrix} x_2e^{x_1x_2} \\ \cos(x_1+x_2)\end{pmatrix} \circ \begin{pmatrix} \frac{\partial}{\partial x_1}(x_1x_2) \\ \frac{\partial}{\partial x_2}(x_1+x_2)\end{pmatrix} =$

$\begin{pmatrix} x_2e^{x_1x_2} \\ \cos(x_1+x_2)\end{pmatrix} \circ \begin{pmatrix} x_2 \\ 1\end{pmatrix}$

Sol: $\therefore Dh(x) = \begin{pmatrix} x_2e^{x_1x_2} & x_1e^{x_1x_2} \\ \cos(x_1+x_2) & \cos(x_1+x_2)\end{pmatrix}$

$Df(x) = \begin{pmatrix} e^{x_1} & 0 \\ 0 & \cos(x_2)\end{pmatrix}$, $Dg(x) = \begin{pmatrix} x_2 & x_1 \\ 1 & 1\end{pmatrix}$

$\therefore Dh(x) = Df(g(x))\,Dg(x) = \begin{pmatrix} e^{x_1x_2} & 0 \\ 0 & \cos(x_1+x_2)\end{pmatrix}\begin{pmatrix} x_2 & x_1 \\ 1 & 1\end{pmatrix} = \begin{pmatrix} x_2e^{x_1x_2} & x_1e^{x_1x_2} \\ \cos(x_1+x_2) & \cos(x_1+x_2)\end{pmatrix}$

2i) $Df(x) = [2x_1x_2 + 2 \quad x_1^2 - 3x_2^2]$ $\therefore D_1f(x)$ and $D_2f(x)$ are continuous $\forall x \in \mathbb{R}^2$ $\therefore$ $f(x)$ is differentiable $\forall x \in \mathbb{R}^2$

$Df(x) = 0 = \begin{bmatrix} 2x_1x_2+2 & x_1^2-3x_2^2 \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix} \therefore$

$2x_1x_2+2=0$, $x_1^2-3x_2^2=0$ $\therefore x_1^2 = 3x_2^2 \therefore$ ~~x_1^2~~ $x_1 = \pm\sqrt{3}\,x_2 \therefore$

$2(\pm\sqrt{3}\,x_2)x_2+2=0=\pm 2\sqrt{3}\,x_2^2+2 \quad \therefore$

$-2 = \pm 2\sqrt{3}\,x_2^2 \therefore \pm\frac{1}{\sqrt{3}} = x_2^2 \therefore \quad x_2^2 \neq -\frac{1}{\sqrt{3}} \therefore x_2^2 = \frac{1}{+\sqrt{3}} = 3^{-1/2}$

$\therefore x_2 = \pm(3^{-1/2})^{\frac{1}{2}} = \pm 3^{-1/4}$ if $x_2 = +3^{-1/4} \therefore$

$\pm\sqrt{3}\,(3^{-1/4}) = x_1 = \pm 3^{-3/4} \qquad (\pm 3^{-3/4}, +3^{-1/4})$

if $x_2 = -3^{-1/4} \therefore \pm\sqrt{3}\,(-3^{-1/4}) = x_1 = \mp 3^{-3/4} \therefore (\mp 3^{-3/4}, -3^{-1/4})$

$2x_1x_2 = -2 \quad \therefore x_1x_2 = -1 \quad \therefore x_1 = \frac{-1}{x_2} \therefore \quad \frac{1}{x_2^2} - 3x_2^2 = 0$

$\therefore 1 = 3x_2^4 \quad \therefore \frac{1}{3} = x_2^4 \quad \therefore x_2^2 = \pm\sqrt{\frac{1}{3}} \quad \therefore$ ~~x_2~~ $x_2 \in \mathbb{R} \therefore$

$x_2^2 \neq -\sqrt{\frac{1}{3}} \therefore \quad x_2^2 = +\sqrt{\frac{1}{3}} \quad \therefore x_2 = \pm 3^{-1/4}$

if $x_2 = +3^{-1/4} \therefore -\frac{1}{3^{-1/4}} = -3^{1/4} = x_1 \therefore (-3^{1/4}, +3^{-1/4})^T = x$

if $x_2 = -3^{-1/4} \therefore \frac{-1}{-3^{-1/4}} = x_1 = 3^{1/4} \therefore (3^{1/4}, -3^{-1/4}) = x$

2ii Sol/ $Df(x) = \begin{bmatrix} -2x_1x_2^2e^{-x_1^2x_2^2} & -2x_2x_1^2e^{-x_1^2x_2^2} \end{bmatrix} \therefore D_1f(x)$,

$D_2f(x)$ cont $\forall x \in \mathbb{R}^2 \quad \therefore f(x)$ diffable $\forall x \in \mathbb{R}^2$

if $Df(x) = 0 \therefore -2x_1x_2^2e^{-x_1^2x_2^2} = 0$, $-2x_1x_2^2e^{-x_1^2x_2^2} = 0$

which are where $x_1 = 0$ or $x_2 = 0$ (infinitely many solutions)

2iii/ ~~$Df(x)$~~ Jacobian is: $Df(x) = (\cos(x_1)\sin(x_2) \quad \sin(x_1)\cos(x_2))$

$Df(x) = 0 \therefore \cos(x_1)\sin(x_2) = 0 \quad \sin(x_1)\cos(x_2) = 0$

which are $x = \begin{pmatrix} 2n\pi + m\frac{\pi}{2} \\ 2p\pi + q\frac{\pi}{2} \end{pmatrix}$ arbitrary $n, m, p, q \in \mathbb{Z}$

$\cos((2k+1)\frac{\pi}{2}) = 0 =$ ~~cost~~, $\sin(2k\pi) = 0$

$\therefore \cos(2n\pi + m\frac{\pi}{2})\cos(2n\pi + m\frac{\pi}{2}) = 0 \quad 2n\pi + m\frac{\pi}{2} = \begin{cases} 2k\pi \\ (2k+1)\pi \end{cases}$

2iv

2008 CW1 sols

proof 2: $\Rightarrow$ if S is closed $\therefore S^c$ is open $\therefore S^c$ is a neighb of every pt in S^c $\therefore$ no pt of S^c can be a boundary pt of S (since $S^c \cap S = \emptyset$) $\therefore$ S contains all its boundary pts $\therefore \partial S \subset S$ $\therefore$ hence $\bar{S} \subset S$ ($S \subset \bar{S}$ clear from $\bar{S} = S \cup \partial S$) $\therefore S = \bar{S}$

$\Leftarrow$ if now if $\bar{S} = S$ $\therefore \bar{S} = S \cup \partial S \subset S$ $\therefore \partial S \subset S$ $\therefore$ no pt $x_0 \in S^c$ is a boundary pt $\therefore \forall$ pt in S^c has a neighb $U \subset S^c$ $\therefore$ is an interior pt of S^c and S^c is open $\therefore S = (S^c)^c$ is closed

3 sol / following open covering is valid:

$\mathcal{H} = \{(-\frac{3}{2}, \frac{1}{2})\} \cup \{I_k \mid k \in \mathbb{N}\} \cup \{J_k \mid k \in \mathbb{N} \cup \{0\}\}$ where

$I_k = (-\frac{1}{2k}, -\frac{1}{2k+2})$ and $J_k = (\frac{1}{2k+3}, \frac{1}{2k+1})$ $\therefore$

$\mathcal{H} = \{(-\frac{3}{2}, \frac{1}{2})\} \cup \{(-\frac{1}{2k}, -\frac{1}{2k+2}) \mid k \in \mathbb{N}\} \cup \{(\frac{1}{2k+3}, \frac{1}{2k+1}) \mid k \in \mathbb{N} \cup \{0\}\}$

each interval in $\mathcal{H}$ covers one pt only of S:

① $(-\frac{3}{2}, \frac{1}{2})$ covers -1

② interval $I_k \{(-\frac{1}{2k}, -\frac{1}{2k+2})\}$ covers pt $\frac{-1}{2k+1}$

interval $J_k = (\frac{1}{2k+3}, \frac{1}{2k+1})$ covers pt $\frac{1}{2k+2}$

there is no finite subcovering, $\forall$ finite subcollection from $\mathcal{H}$ let k be the max natural number st I_k or J_k is in the collection then the pts $\frac{-1}{2i+1}$ & $\frac{1}{2i+2}$ are not covered for $i > k$

2 The converse to Heine Borel thm (is true) say ~~the~~ non-compact (ie not closed and bounded) set there must ~~exist~~ exist an open covering which has no finite subcovering applies here (S is not compact: not close cause 0 is a limit pt that's not in set) (but S is bounded)

(this result doesn't say every infinite open covering does not have a finite subcovering only that $\exists$ an infinite open covering that doesn't have a finite subcovering) (the example open covering doesn't have a finite subcovering)

\4 sol/ $f(1)=0$ $\therefore$ verify $\lim_{x\to 1} f(x)=0$: $\therefore$

$\forall \varepsilon>0 \quad |f(x)-0| = |x^2-2x+1| = |(x-1)^2| = |x-1|^2 < \delta^2 = \varepsilon$

$\forall x: 0<|x-1|<\delta$ with $\delta=\sqrt{\varepsilon}$ that is $\lim_{x\to 1} f(x) = 0 = f(1)$

$\therefore f$ is continuous at $x_0=1$

{ ~~with~~ $\forall |x-1|<\delta$ and $\delta=\sqrt{\varepsilon}$ }

{note if δ st $\delta=\varepsilon/2$ doesn't strictly work since $\delta^2=\varepsilon^2/4$ less than ε if $\varepsilon<4$ doesn't hold if $\varepsilon\geq 4$ ~~but~~ but since only interested in arbit small vals of ε sort of okay if $\forall(0<\varepsilon<4)$ rather than $\forall \varepsilon>0$ }

\5 sol/ func g is diffable at every $x\neq 0$ cause its a product and composition of diffable funcs that is:

$g'(x) = 3x^2\sin(\frac{1}{x}) - x\cos(\frac{1}{x}),\ x\neq 0$ $\therefore$

$$g'(x)=\begin{cases} 3x^2\sin(\frac{1}{x}) - x\cos(\frac{1}{x}) & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

and g is diffable on $\mathbb{R}$

{ now check differentiability of $x_0=0$: $g'(0)=\lim_{x\to 0}\frac{g(x)-g(0)}{x-0}$

$=\lim_{x\to 0}\frac{x^3\sin(\frac{1}{x})-0}{x-0} = \lim_{x\to 0} x^2\sin\frac{1}{x} = 0$ $\therefore$

$$g'(x)=\begin{cases} 3x^2\sin\frac{1}{x} - \cos\frac{1}{x} & \text{if } x\neq 0 \\ 0 & \text{if } x=0 \end{cases}$$

and g is diffable on $\mathbb{R}$ }

$g'(x)$ is cont $\forall x\neq 0$ as a sum, product and compos of cont funcs. g' is also cont at $x_0=0$ because $\lim_{x\to 0} g'(x) = \lim_{x\to 0} 3x^2\sin(\frac{1}{x}) - x\cos(\frac{1}{x}) = 0 = g'(0)$ $\therefore g'$ is cont on $\mathbb{R}$

2008 Sheet 9

\2iv/ $D f(x) = [x_2+x_3 \quad x_1+x_3 \quad x_2+x_1]$

$\therefore Df(x)=0 \quad \therefore x_2+x_3=0, \; x_1+x_3=0 \quad x_2+x_1=0 \therefore$

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \therefore \text{REF}: \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ which indicates

a unique sol, being the trivial sol $x = (0,0,0)^T$

\2v/ $D f(x) = (2x_1x_2 \quad x_1^2-2 \quad 2x_3)$

if $Df(x)=0$, i $\quad 2x_1x_2=0, \; x_1^2-2=0, \; 2x_3=0$

which are $x = (\sqrt{2}, 0, 0)^T$ and $x = (-\sqrt{2}, 0, 0)^T$

{ $x_1^2=2 \therefore x_1 = \pm\sqrt{2}$ if $x_1 = +\sqrt{2} \therefore x_2 = 0$

if $x_1 = -\sqrt{2} \therefore x_2 = 0 \therefore (\sqrt{2}, 0, 0)^T = x, \; x = (-\sqrt{2}, 0, 0)^T$ }

2008 CW1 Sols

\i sol/ $S^o = \emptyset$ since for any ε-neighb U of a point in S, $U \cap S^c \neq \emptyset$

\ii sol/ complement is: $S^c = \cancel{(-\infty, -1)}$

$$(-\infty,-1) \cup \left(\bigcup_{n=0}^{\infty}\left(-\frac{1}{2n+1}, -\frac{1}{2n+3}\right)\right) \cup \{0\} \cup \left(\bigcup_{n=1}^{\infty}\left(\frac{1}{2n+2}, \frac{1}{2n}\right)\right) \cup \left(\frac{1}{2}, \infty\right) \therefore$$

exterior is: $(S^c)^o =$

$$(-\infty,-1) \cup \left(\bigcup_{n=0}^{\infty}\left(-\frac{1}{2n+1}, -\frac{1}{2n+3}\right)\right) \cup \left(\bigcup_{n=1}^{\infty}\left(\frac{1}{2n+2}, \frac{1}{2n}\right)\right) \cup \left(\frac{1}{2}, \infty\right)$$

\iii sol/ set of limit points is $\{0\}$

\iv sol/ set of boundary points is $\partial S = S \cup \{0\}$ and the closure is $\bar{S} = S \cup \partial S = S \cup \{0\}$

\v sol/ set of isolated points is S (can be seen using the open intervals defined in H in the sol of \Q3/)

\1 ii Set/ set is neither open nor closed [the set is not open since not every point of S is an interior point (in fact no point is an interior point) the set is not close since it does not contain all of its limit points

\2 Set/ recall: ① if a set S contains an ε-neighb of x_0 then S is a neighb of x_0
② if S is a neighb of x_0 then x_0 is an interior point of S
③ if every point of S is an interior point, then S is open
④ S is closed if S^c is open
⑤ x_0 is a limit point of S if every deleted neighb of x_0 contains a point of S
⑥ A point x_0 is a boundary point of S if every neighb of x_0 contains a point of S
⑦ the closure of a set S is the set $\bar{S} = S \cup \partial S$

also recall: to prove two sets A & B are equal must show both $A \subset B$ & $B \subset A$

Proof 1: $\Rightarrow$ if S is closed $\therefore$ S includes all its limit points (Coroll 1.26) let $x_0 \in \partial S$ $\therefore$ $\forall$ neighb U of x_0 contains at least one point in S $\therefore$ x_0 is a limit point of S $\therefore x_0 \in S$ since S is close $\therefore \partial S \subset S$ & $\bar{S} \subset S$ ($x_0 \in \partial S \Rightarrow x_0 \in S$) (note: it is not the case that every boundary pt is a limit pt. in fact a boundary pt that is not a limit pt must be an isolated pt). The inclusion $S \subset \bar{S}$ from $\bar{S} = S \cup \partial S$ $\therefore \bar{S} = S$
$\Leftarrow$ now if $\bar{S} = S$ $\therefore \bar{S} = S \cup \partial S \subseteq S$ $\therefore \partial S \subset S$ $\therefore$ if there were a limit pt x_0 not included in S then it cannot be a boundary pt $\therefore \exists$ neighb U of x_0 s.t. $U \cap S^c = \emptyset$ ie $U \subset S$ $\therefore x_0 \in S$ after all (in fact is an interior pt) $\therefore$ S includes all its limit pts $\therefore$ is closed

2008 CW2 Sols

)1 Sols/ by inspection limit is 1
(for large n, $\sqrt{n} << n \therefore \frac{n}{n+\sqrt{n}} \approx \frac{n}{n} = 1$)

$\therefore |S_n - 1| = \left|\frac{n}{n+\sqrt{n}} - 1\right| = \left|\frac{-\sqrt{n}}{n+\sqrt{n}}\right| = \left|\frac{-1}{\sqrt{n}+1}\right| = \frac{1}{\sqrt{n}+1} \leq \frac{1}{\sqrt{N}+1}$

$< \frac{1}{\left(\frac{1}{\varepsilon}\right)+1} = \frac{\varepsilon}{1+\varepsilon} < \varepsilon \quad \forall n \geq N$ with $N > \frac{1}{\varepsilon^2}$ (any possibility and any suitable N is okay) $\left\{ \frac{1}{\sqrt{N}+1} < \varepsilon \therefore \frac{1}{\varepsilon} < \sqrt{N}+1 \right.$

$\left. \therefore \cancel{\frac{1}{\varepsilon}} < \frac{1}{\varepsilon} - 1 < \sqrt{N} \quad \frac{1}{\varepsilon} - 1 < \frac{1}{\varepsilon} \right\}$ $\Big\{$ if $N > \frac{1}{\varepsilon^2} \therefore \varepsilon^2 > \frac{1}{N} \therefore$

$\varepsilon > \frac{1}{\sqrt{N}} \therefore \varepsilon + 1 > \frac{1}{\sqrt{N}} + 1 = \frac{1+\sqrt{N}}{\sqrt{N}} \therefore \frac{1}{\varepsilon+1} < \frac{\sqrt{N}}{1+\sqrt{N}} \Big\}$

$\cancel{\frac{1}{\sqrt{N}}} \left\{ \frac{1}{\sqrt{n}+1} < \frac{1}{\sqrt{n}} \leq \frac{1}{\sqrt{N}} < \varepsilon \quad \forall n \geq N \text{ with } \frac{1}{\varepsilon^2} < N \right\}$

)2 Sol/ have: $\sin(x) \geq \frac{1}{\sqrt{2}} \quad \forall x \in H_k$ with

$H_k = \left[2\pi k + \frac{\pi}{4}, 2\pi k + \frac{3\pi}{4}\right], k \in \mathbb{N}$

and $\sin(x) \leq -\frac{1}{\sqrt{2}} \quad \forall x \in I$, $I_j = \left[2\pi j - \frac{3\pi}{4}, 2\pi j - \frac{\pi}{4}\right], j \in \mathbb{N}$

$\left\{ -\frac{3\pi}{4} < -\frac{\pi}{4} \therefore 2\pi j - \frac{3\pi}{4} < 2\pi j - \frac{\pi}{4} \right\}$

Since $\left|\left(2\pi k + \frac{3\pi}{4}\right) - \left(2\pi k + \frac{\pi}{4}\right)\right| = \frac{\pi}{2} > 1 \quad \therefore H_k \cap \mathbb{N} \neq \emptyset \ \forall k \in \mathbb{N} \therefore$
$\forall k \in \mathbb{N}$ can choose a natural number $n_k \in H_k$ [note may be a max of 2 choices since H_k has width $1 < \frac{\pi}{2} < 2$ choose one arbitly] similarly choose $n_j \in I_j$
(these increasing sequences of natural numbers n_k and n_j exist because width of intervals where $\sin(x) \geq \frac{1}{\sqrt{2}}$ and $\sin(x) \leq -\frac{1}{\sqrt{2}}$ are greater than 1) $\therefore$ sequences:

$S_{n_k} = \sin(n_k) \geq \frac{1}{\sqrt{2}}$ & $S_{n_j} = \sin(n_j) \leq -\frac{1}{\sqrt{2}}$ ($\therefore$ these sequences, vals in S_{n_k} & S_{n_j} are bounded away from each other by a distance of $2\frac{1}{\sqrt{2}} = \sqrt{2}$) $\therefore$ sequence $S_n\}$ converges if and only if its Cauchy, that is,

$\forall \varepsilon > 0, \exists N > 0$ s.t. $\forall m, n \geq N$ have: $|S_n - S_m| < \varepsilon$

$\{\forall \varepsilon > 0, \exists N > 0$ such that for all $m, n \geq N$ have: $|S_n - S_m| < \varepsilon\}$

since have $\forall k, j \in \mathbb{N}$ $|S_{n_k} - S_{n_j}| \geq 2\frac{1}{\sqrt{2}} = \sqrt{2}$ and for any N can find k and j st $n_k, n_j \geq N$ means that S_n is not Cauchy $\therefore$ S_n does not converge

\3 sol/ $\forall x \in \mathbb{R}$ have $\lim_{n\to\infty} \frac{x}{n} = 0$ and since $\sin(x)$ is continuous; point wise limit is.

$f(x) = \lim_{n\to\infty} f_n(x) = \sin(0) = 0 \quad \forall x \in \mathbb{R}$

$\forall n \in \mathbb{N}$ consider the pt $x_n = \frac{n\pi}{2}$ for which

$f_n(x_n) = \sin\left(\frac{n\pi}{2n}\right) = 1 \quad \therefore$

$\|f_n - f\|_{\mathbb{R}} = \sup_{x\in\mathbb{R}} |\sin(\frac{x}{n}) - 0| \geq 1 \quad \forall n \in \mathbb{N}$ [sufficient to simply recognise and state that $\|f_n - f\|_{\mathbb{R}} = 1 \ \forall n \in \mathbb{N}$] so cannot make the ~~sup~~ sup norm less than arbitrary $\varepsilon > 0$ by considering $\forall n \geq N$ for some $N > 0$ $\therefore$ convergence is not uniform

\4 sol/ consider uniform partition of $[-1, 1]$ in n $\left\{\frac{1 - -1}{n} = \frac{2}{n}\right.$ subintervals: $P_n = (-1, -1 + \frac{2}{n}, \ldots, -1 + \frac{2i}{n}, \ldots, 1)$ each ~~of~~ width $\Delta x_i = \frac{2}{n}$ since f is increasing obtain lower Riemann sums: $L(f, P_n) = \sum_{i=1}^{n} \inf_{x\in[x_{i-1}, x_i]} f(x)\Delta x_i = \sum_{i=1}^{n} f(x_{i-1})\Delta x_i =$

$\sum_{i=1}^{n} \left(2\left(-1 + \frac{2}{n}(i-1)\right) + 1\right)\frac{2}{n} = \frac{2}{n}\sum_{i=1}^{n}\left(-1 + \frac{4(i-1)}{n}\right) =$

$-2 + \frac{8}{n^2}\left(\sum_{i=1}^{n} i - \sum_{i=1}^{n} 1\right) = -2 + \frac{8}{n^2}\left(\frac{n(n+1)}{2} - n\right) = 2 - \frac{4}{n}$

upper Riemann sums: $U(f, P_n) = \sum_{i=1}^{n} \sup_{x\in[x_{i-1}, x_i]} f(x)\Delta x_i = \sum_{i=1}^{n} f(x_i)\Delta x_i =$

$\sum_{i=1}^{n}\left(2\left(-1 + \frac{2i}{n}\right) + 1\right)\frac{2}{n} = \frac{2}{n}\sum_{i=1}^{n}\left(-1 + \frac{4i}{n}\right) = -2 + \frac{8}{n^2}\left(\sum_{i=1}^{n} i\right) =$

$-2 + \frac{8}{n^2}\left(\frac{n(n+1)}{2}\right) = 2 + \frac{4}{n}$ $\therefore$ following bound on lower Riemann

2008 ~~Sheet~~ CW2 sols

1) integral: $\underline{\int_{-1}^{1}} f(x)\,dx = \sup_P L(f;P) \geq \sup_n L(f;P_n) = 2$

$\therefore \underline{\int_{-1}^{1}} f(x)\,dx \geq 2$

and following bound on upper ~~bar~~ Riemann integral

$\overline{\int_{-1}^{1}} f(x)\,dx = \inf_P U(f;P) \leq \inf_n U(f;P_n) = 2$

$\therefore \overline{\int_{-1}^{1}} f(x) \leq 2 \quad \therefore \quad 2 \leq \underline{\int_{-1}^{1}} f(x) \leq \overline{\int_{-1}^{1}} f(x)\,dx \leq 2$ $\therefore$ lower Riemann

equals upper Riemann integral with a val of 2 $\therefore$

f is Riemann integrable with definite integral val of 2

$\therefore \underline{\int_{-1}^{1}} f(x)\,dx = \overline{\int_{-1}^{1}} f(x)\,dx = \int_{-1}^{1} f(x)\,dx = 2$

5 sol/ note: $f(x) - g(x) = 0 \;\forall x \neq x_0$ and $f(x_0) - g(x_0) > 0$ say

$f(x_0) - g(x_0) = C$ assume $x_0 \in (a,b)$

consider following sequence of partitions of $[a,b]$:

$P_n = (a, x_0 - \frac{1}{2^n}, x + \frac{1}{2^n}, b)$ where n is chosen large enough

so that $x_0 - \frac{1}{2^n} > a$ & $x_0 + \frac{1}{2^n} < b$ labelling intervals of partition

$I_{n,1}, I_{n,2}\; I_{n,3}$ note $x_0 \in I_{n,2}$ have

$\inf_{x \in I_{n,i}} (f(x) - g(x)) = 0 \quad i = 1,2,3 \quad \therefore$

$L(f-g;P_n) = \sum_{i=1}^{3} \inf_{x \in I_{n,i}} (f(x) - g(x)) \Delta I_{n,i} = 0$ &

$\sup_{x \in I_{n,1}} (f(x) - g(x)) = \sup_{x \in I_{n,3}} (f(x) - g(x)) = 0 \quad \therefore \sup_{x \in I_{n,2}} (f(x) - g(x)) = C$ and

$\Delta I_{n,2} = \frac{1}{2^n} + \frac{1}{2^n} = \frac{1}{2^{n-1}} \therefore U(f-g;P_n) = \sum_{i=1}^{3} \sup_{x \in I_{n,i}} (f(x) - g(x)) \Delta I_i = \frac{C}{2^{n-1}}$

$\therefore \overline{\int_a^b} (f(x) - g(x))\,dx = \inf_P U(f-g;P) \leq \inf_n U(f-g;P_n) = \inf_n \left\{ \frac{C}{2^{n-1}} \right\} = 0$ &

$\underline{\int_a^b} (f(x) - g(x))\,dx = \sup_P L(f-g;P) \geq \sup_n L(f-g;P_n) = 0 \;\therefore$

$0 \leq \underline{\int_a^b} (f(x) - g(x))\,dx \leq \overline{\int_a^b} (f(x) - g(x))\,dx \leq 0 \quad \therefore f - g$ is integrable &

$\int_a^b (f(x)-g(x))\,dx = 0$ $\therefore$ by linearity of integral:

$\int_a^b (f(x)-g(x))\,dx = \int_a^b f(x)\,dx - \int_a^b g(x)\,dx = 0 \therefore$

$\int_a^b f(x)\,dx = \int_a^b g(x)\,dx$ case where $x_0 = a$ or $x_0 = b$

taking partition of form: $P_n = (a, a+\frac{1}{2^n}, b)$ or $\cancel{P_n = (a, b-\frac{1}{2^n}, b)}$

$P_n = (a, b-\frac{1}{2^n}, b)$ respectively

2008 PP2020

1ai sols/ the set S is not open because $x_0 = 2020$ is not an interior point of S. It is not closed because it does not contain all its limit pts ex it doesn't contain 5 which is a limit pt

1aii sol/ set of limit pts of S is given by $(-\infty, 5]$

the set of interior pts is given by ~~(-\infty,5~~ $(-\infty, 5)$

1b sol/ fix $\varepsilon > 0$ then for $\beta = \sqrt[3]{\frac{1}{\varepsilon}}$ we get that if $x > \beta$ then
$\left|\frac{1}{x^3+2x+1}\right| \leq \left|\frac{1}{x^3}\right| < \frac{1}{\beta^3} = \varepsilon$ as we wanted to show

1c sol/ let $A \subset \mathbb{R}$ & $f, g: A \to \mathbb{R}$ and let $\lim_{x\to a} f(x) = f(a)$, ~~and~~ $\lim_{x\to a} g(x) = g(a)$

$\therefore$ fix $\varepsilon > 0$ by def: $\lim_{x\to a} f(x) = f(a)$ $\therefore$
$\exists\, \delta_1$ st $\forall x \in A$, with $0 < \|x-a\| < \delta_1$ have: $|f(x) - f(a)| < \frac{\varepsilon}{2}$
for $g(x)$ $\exists\, \delta_2$ st for $x \in A$ with $0 < \|x-a\| < \delta_2$ have
$|g(x) - g(a)| < \frac{\varepsilon}{2}$ now take $\delta = \min\{\delta_1, \delta_2\}$ say:
if $x \in A$ with $0 < \|x-a\| < \delta$ $\therefore$ triangle inequality have:
$|f(x)+g(x) - (f(a)+g(a))| \leq |f(x)-f(a)| + |g(x)-g(a)| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\therefore$ by def of a limit: $\lim_{x\to a}(f(x)+g(x)) = f(a)+g(a)$

1di sol/ if $g: \mathbb{R}^n \to \mathbb{R}^n$ is contly diffable in an open set containing p & 2 differential $Dg(p)$ is invertible then $\exists$ opensets $U, V \subset \mathbb{R}^n$ with $p \in U$ & $g(p) \in V$ st $g: U \to V$ is bijective also 2 inverse func $g^{-1}: V \to U$ is contly diffable & $Dg^{-1}(g(p)) = (Dg(p))^{-1}$

1dii/ note: $h(0) = 0$ & all partial derivs $D_i h^j(p)$ for $i,j = 1,2$ exist and are cont in P $\therefore$ h is contly diffable.

everywhere in $\mathbb{R}^2$ (by a result from the course) not setting $p=(x,y)^T$ have: (making usual identification of a linear map with its matrix with respect to the canonical basis) $Dh(p)=\begin{pmatrix} 1 & 25y^4 \\ 7y & 7x-1 \end{pmatrix}$

$\therefore Dh(0)=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ clearly is invertable $\therefore$ apply inverse func thm deduce $\exists$ open sets $U, V \subset \mathbb{R}^2$ with $0 \in U$ & $0 \in V$ st $h: U \to V$ is a bijection since h maps U bijectively onto V $\therefore \exists p \in U$ with $h(p)=q$ with $q \in V$

$\{ D_1 h(p) = [1 \quad 25y^4] \quad D_2 h(p) = [7y \quad 7x-1)$

\2a/ if $n=0$ then $p(x)=a_0$ which is bounded

suppose if $n \neq 0$ then $\lim_{x\to\infty} p(x)=\infty$ so $p(x)$ is not bounded

\2b/ $\lim_{x\to\infty} p(x)=\infty$ and $\lim_{x\to-\infty} p(x)=-\infty$ and f is continuous

on $\mathbb{R}$ then $\exists x_1, x_2 \in \mathbb{R}$ st $f(x_1)<\alpha$ and $f(x_2)>\alpha$ $\therefore$ by the intermediate value thm: $\exists x_0$ between x_1 and x_2 st $f(x_0)=\alpha$

\2c/ since n is even $\lim_{x\to\infty} p(x)=\infty$ and $\lim_{x\to-\infty} p(x)=\infty$

consider $p(0)$ since limit $\lim_{x\to\infty} p(x)=\infty$ and $\lim_{x\to-\infty} p(x)=\infty$ then by the intermediate value thm we get: $\exists x_0<0$ st $p(x_0)=p(0)+1$ and $x_1>0$ st $p(x_1)=p(0)+1$

\3a/ a func $f:[a,b]\to\mathbb{R}$ is Riemann integrable if and only if $\forall \varepsilon>0$, $\exists$ a partition P st $U(f;P)-L(f;P)<\varepsilon$ where $U(f;P)$ and $L(f;P)$ denote the upper and lower Riemann sums respectively

\3b/ given $n\in\mathbb{N}$ let $P=\left(a+\frac{i(b-a)}{n}\right)_{i=0,\ldots,n}$

then $L(f;P)=\sum_{i=0}^{n-1}\frac{b-a}{n}f\left(a+\frac{i(b-a)}{n}\right)$ and

$U(f;P)=\sum_{i=1}^{n}\frac{b-a}{n}f\left(a+\frac{i(b-a)}{n}\right)$ so then $U(f;P)-L(f;P)=\frac{b-a}{n}(f(b)-f(a))$

this tends to 0 as $n\to\infty$ so using Riemann criterion for integrability f is Riemann integrable

2B38 PP 2020

\3c/ observe: $\lim_{n\to\infty}\left(\frac{\sum_{k=0}^{n-1}\sin\left(\frac{k\pi}{2n}\right)}{n}\right) = \frac{2}{\pi}\sum_{k=1}^{n}\sin\left(\frac{k\pi}{2n}\right)\frac{\pi}{2n} = \frac{2}{\pi}U(f;P)$

where $U(f;P)$ is the upper Riemann sum for the func f and P is the partition of $[0,\frac{\pi}{2}]$ into n subintervals of equal length. Since f is interchangeable it follows that: $\lim_{n\to\infty}U(f;P) = \int_0^1 \sin x\,dx$ then by the fundamental thm of calculus and the fact $(-\cos x)' = \sin x$ it follows that $\int_0^1 \sin x\,dx = \cos(0) - \cos(\frac{\pi}{2}) = 1$ $\therefore$

$$\lim_{n\to\infty}\frac{\sum_{k=0}^{n-1}\sin\left(\frac{k\pi}{2n}\right)}{n} = \lim_{n\to\infty}\frac{2}{\pi}U(f;P) = \frac{2}{\pi}\int_0^1 \sin(x)\,dx = \frac{2}{\pi}$$

\4a/ compute $f'(x) = g(x) = e^x - 2\lambda x$ since $g \in C^1([0,1])$ it is Lipschitz cont on $[0,1]$ & the smallest Lipschitz const is $L = \sup_{x\in[0,1]} g'(x) = \sup_{x\in[0,1]} e^x - 2\lambda = e - 2\lambda$ because $g'(x)$ is strictly increasing in $[0,1]$ ($g''(x) = e^x > 0$ for $x \in [0,1]$)

\4b/ if $\lambda = 0$ then $\lim_{x\to\infty} e^x - \lambda x^2 = \infty$ and $\lim_{x\to-\infty} e^x - \lambda x^2 = \infty$

if $0 < \lambda < \frac{e}{2}$ then $\lim_{x\to\infty} e^x - \lambda x^2 = \infty$ and $\lim_{x\to-\infty} e^x - \lambda x^2 = -\infty$

\4c/ if $\lambda = 0$ the eqn has no sols

consider case $0 < \lambda < \frac{e}{2}$ in this case $\lim_{x\to\infty} e^x - \lambda x^2 = \infty$ and $\lim_{x\to-\infty} e^x - \lambda x^2 = -\infty$ we want to study the derivative $f'(x) = e^x - 2\lambda x$

if $0 < \lambda < \frac{e}{2}$ then $\lim_{x\to\infty} f'(x) = \infty$ and $\lim_{x\to-\infty} f'(x) = \infty$ $\therefore$ $f''(x) = e^x - 2\lambda$ and $f''(x) = 0$ iff $x = \log 2\lambda$ and $f'(\log(2\lambda)) = 2\lambda - 2\lambda\log 2\lambda = 2\lambda(1 - \log 2\lambda)$

so $f'(\log(2\lambda)) > 0$ $\{0 < \lambda < \frac{e}{2}$ can conclude $f'(x) > 0\ \forall x \in \mathbb{R}$

since $\lim_{x\to\infty} e^x - \lambda x^2 = \infty$ and $\lim_{x\to-\infty} e^x - \lambda x^2 = -\infty$ and $f'(x) > 0\ \forall x \in \mathbb{R}$

can conclude the eqn $e^x - \lambda x^2 = 0$ has one sol

consider case $\lambda < 0$ denote $f(x) = e^x - \lambda x^2$ this case $f(x) > 0\ \forall x \in \mathbb{R}$ so the above eqn has no sols

~~2008~~ ~~Sheet~~ PP 2019 Sols

\1a Sols/ set S is not open because ex pt 1 is not an interior pt. any ε-neighb of 1 contains at least a point $x>1$ hence it is not contained in S the set S is also not closed because it does not contain, which is a limit pt of S because every deleted neighb of 3 contains a pt of S and a pt in S^c

\1b Sol/ ~~complement~~ complement of S is given by $S^c = (-\infty, -18) \cup (1, 3]$ the set of interior pts is $(S^c)^\circ = (-\infty, -18) \cup (1, 3)$

\2 Sol/ func $x\cos(\frac{1}{x^3})$ is defined in a deleted neighb of 0 and that $\forall x \neq 0$ $|x\cos(\frac{1}{x^3})| \leq |x|$ $\therefore$ $\forall \varepsilon > 0, \exists \delta = \varepsilon$ st if $0<|x|<\delta$, then $|x\cos(\frac{1}{x^3})| \leq |x| < \delta = \varepsilon$

\3 Sol/ a func $f : [a,b] \to \mathbb{R}$ is lipschitz cont if $\exists L>0$ st $|f(x)-f(y)| \leq L|x-y|$, $\forall x,y \in [a,b]$ the constant L is called a lipschitz const for f

\3b Sol/ have $f'(x) = e^{-x} - (x+1)e^{-x} = -xe^{-x}$ & $f''(x) = xe^{-x} - e^{-x} = (x-1)e^{-x}$ $\therefore$ f' has a local turning pt at $x=1 \in [0,4]$ $\therefore$ $L = \max\{|f'(0)|, |f'(1)|, |f'(4)|\} = \max\{0, e^{-1}, 4e^{-4}\} = \frac{1}{e}$

\4a Sol/ sufficient that sequence of funcs, $\{f_n\}_{n=1}^{\infty}$ is a sequence of Riemann integrable funcs defined on a finite interval $[a,b]$ and f_n converges uniformly to f as $n \to \infty$

\4b Sol/ $f_n(x) = e^{-nx^2} \leq e^{-n}$ on $[1,2]$ $\therefore$ sequence of funcs $\{f_n\}_{n=1}^{\infty}$ converges uniformly to 0 on $[1,2]$ $\therefore$ limit and integral can be exchange so the limit is 0

\5a Sol/ the func $f:\mathbb{R}^n \to \mathbb{R}^m$ is diffable at p is f $\exists A \in L(\mathbb{R}^n, \mathbb{R}^m)$ st if $g(x) = A(x-p) + f(p)$ then

$\lim_{x \to p} \frac{\|f(x) - g(x)\|}{\|x-p\|} = 0$ where $A \in L(\mathbb{R}^n, \mathbb{R})$ is a linear map

$\mathbb{R}^n \to \mathbb{R}^m$ write $A = Df(p)$ $\therefore \lim_{x \to p} \|f(x) - f(p) - Df(p)\|$

Call $Df(p)$ the differential of the map f at pt p

\5b Sol/ consider map $f:\mathbb{R}^n \to \mathbb{R}$ given by $f(x) = \langle x, b \rangle$ to see that f is diffable at each $p \in \mathbb{R}^n$, with $Df(p)[h] = \langle h, b \rangle$ note: $f(x) - Df(p)(x-p) - f(p) = \langle x, b \rangle - \langle x-p, b \rangle - \langle p, b \rangle = 0$ by linearity of the inner product $\therefore$ have

$\lim_{x \to p} \frac{\|f(x) - g(x)\|}{\|x-p\|} = 0$

\6a Sol/ say $f:[a,b] \to \mathbb{R}$ is cont from the right at a if $\lim_{x \to a^+} f(x) = f(a)$ means that f is defined in $[a, c)$ where $c \in \mathbb{R}$ and $\forall \varepsilon > 0 : \exists \delta > 0$ st if $a \le x < x + \delta$ $|f(x) - f(a)| < \varepsilon$

\6b Sol/ the func $f(x)$ is a cont func on $[1,2]$ because product of composition of cont funcs ($0 < 1 < 2 < \pi$ } so $\sin x \neq 0$ $\forall x \in [1,2]$) and we proved that a cont func on a closed and bounded and bounded interval is bounded

\6c Sol/ func f is cont in $[a,b]$ which is closed and bounded $\therefore \exists c \in [a,b]$ st $f(c) \le f(x)$ $\forall x \in [a,b]$ in particular $f(c) \le f(x_0)$ but $f(x_0) \le f(x)$ $\forall x \in (-\infty, a) \cup (b, \infty)$ by hypothesis $\therefore$ $f(c) \le f(x)$ $\forall x \in \mathbb{R}$ $\therefore f(c) = m$ where $m = \min\{f(x) \mid x \in \mathbb{R}$

\7a/ define: $M_i = \sup_{x \in [x_{i-1}, x_i]} f(x)$ $m_i = \inf_{x \in [x_{i-1}, x_i]} f(x)$

$\Delta x_i = x_i - x_{i-1}$ $\therefore$ lower Riemann sum of f with respect to partition P is $L(f; P) = \sum_{i=1}^{n} m_i \Delta x_i$ upper Riemann sum of f with respect to partition P is: $U(f; P) = \sum_{i=1}^{n} M_i \Delta x_i$

\7b/ lower Riemann integral of f over $[a,b]$ is:

$\underline{\int_a^b} f(x)\,dx = \sup_P L(f, P)$ upper Riemann integral for f over $[a,b]$ is $\overline{\int_a^b} f(x)\,dx = \inf_P U(f, P)$

2008 PP 2019 sols

the func f is Riemann integrable iff $\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx$

14c sol/ given partition $\hat{P} = (0, \frac{1}{n}, \frac{2}{n}, \dots, \frac{n-1}{n}, 1)$ so $x_i = \frac{i}{n}$ for $i = 0, 1, 2, \dots, n$ now applying formula for Riemann sums:

$$L(f;P) = \sum_{i=1}^{n} (x_i - x_{i-1}) \inf_{x\in[x_{i-1},x_i]} f(x) = \sum_{i=1}^{n} \left(\frac{i}{n} - \frac{i-1}{n}\right) \inf_{x\in[\frac{i-1}{n},\frac{i}{n}]} x^2 =$$

$$\frac{1}{n}\sum_{i=1}^{n}\left(\frac{i-1}{n}\right)^2 = \frac{1}{n^3}\frac{(n-1)n(2n-1)}{6} = \frac{(n-1)(2n-1)}{6n^2}$$ also

$$U(f;P) = \sum_{i=1}^{n} (x_i - x_{i-1}) \sup_{x\in[x_{i-1},x_i]} f(x) = \sum_{i=1}^{n} \left(\frac{i}{n} - \frac{i-1}{n}\right) \sup_{x\in[\frac{i-1}{n},\frac{i}{n}]} x^2$$

$$= \frac{1}{n}\sum_{i=1}^{n}\left(\frac{i}{n}\right)^2 = \frac{(n+1)(2n+1)}{6n^2}$$ $\therefore$ taking limits:

$$\lim_{n\to\infty} L(f;P) = \lim_{n\to\infty} \frac{(n-1)(2n-1)}{6} = \frac{1}{3} \quad \& \quad \lim_{n\to\infty} U(f;P) = \lim_{n\to\infty}\frac{(n+1)(2n+1)}{6} = \frac{1}{3}$$

$\therefore$ by def $f(x) = x^2$ is integrable on $[0,1]$

MTH2008 EXAM

690011024 SECTION A

1ai/ $3 \in S$ but is not an ~~a~~ interior point of S
so S is not open

~~2 is~~ $x_0 = 2$ is a limit point of S but $2 \notin S$ so
S is not closed
so S is neither open nor closed

1aii/ the set of interior points of S is:

$$S^\circ = (-1,2) \cup (3,\infty)$$

the set of boundary points of S is:

$$\{-1\} \cup \{2\} \cup \{3\}$$

\1b/ have: $f(0) = 0$ but

$\forall \varepsilon > 0$: $|f(x) - 0| = |f(x)| = \left|\frac{1}{x^2}\right| = \frac{1}{x^2} > \varepsilon$

$\forall (0 < |x - 0| < \delta)$ with $\delta = \frac{1}{\sqrt{\varepsilon}}$

$\therefore \lim_{x \to 0} f(x) \neq 0$ $\therefore f(x)$ is not continuous

at $x_0 = 0$ because $\lim_{x \to 0} f(x) \neq f(0)$

\1C/ note: $\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & , n=2k, \\ 1 & , n=4k+1, \\ -1 & , n=4k+3. \end{cases}$ $k \in \mathbb{Z}$

$\therefore \cancel{\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & , n=2k, \\ (-1)^n & , n=2k+1. \end{cases}}$ $\therefore$

$\lim_{n\to\infty} \sin\left(\frac{n\pi}{2}\right)$ ~~diverges~~ does not converge

and $\cancel{\cos(n\pi) = (-1)^n}$ $\cos(n\pi) = (-1)^n$

$\cancel{\text{and } \sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0 & , n=2k, \\ (-1)^{n+1} & , n=2k+1 \end{cases}}$

$\cancel{S_{2k} \geq S}$

$\sin\left(\frac{2k\pi}{2}\right) = \sin(\pi k) = 0$

$\cos(2k\pi) = \cos(2\pi) = \cos(0) = 1$

$\therefore S_{2k} = \sin\left(\frac{2k\pi}{2}\right) + \cos(2k\pi) = 0 + 1 = 0$ $\therefore$

$\lim_{k\to\infty} S_{2k} = \lim_{k\to\infty} 0 = 0 \in \mathbb{R}$ $\therefore$ the subsequence

$\{S_{2k}\}$ converges since

$\forall \varepsilon > 0: |S_{2k} - 0| < \varepsilon$ for $N \in \mathbb{N}$ and $k \geq N$

$\sin\left(\frac{(2k+1)\pi}{2}\right) = (-1)^k$, $\cos((2k+1)\pi) = -1$

$\therefore S_{2k+1} = \sin\left(\frac{(2k+1)\pi}{2}\right) + \cos((2k+1)\pi) = (-1)^k - 1$

$\therefore S_{2k+1}$ does not converge

1d/ $0 \le x \le 1$ $\therefore x^n \le x, \forall n \in \mathbb{N}$

$\therefore \lim_{n\to\infty} \frac{x^n}{n} = 0$ and since x^n is a continuous function $\therefore$

the pointwise limit is:

$f(x) = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{x^n}{n} = 0 \quad \forall x \in \mathbb{R}$

$\|f_n - f\|_{x\in[0,1]} = \sup_{x\in[0,1]} \left|\frac{x^n}{n} - 0\right| = \sup_{x\in[0,1]} \left|\frac{x^n}{n}\right| = \cancel{x^n} \left|\frac{1^n}{n}\right| = \left|\frac{1}{n}\right| = \frac{1}{n}$

$\therefore \lim_{n\to\infty} \|f_n - f\|_{x\in[0,1]} = \lim_{n\to\infty} \frac{1}{n} = 0 \quad \therefore$

$\forall \varepsilon > 0, \ \|f_n - f\|_{x\in[0,1]} < \varepsilon \quad \forall n \ge N$ for $N \in \mathbb{N}$

$\therefore \{f_n\}$ converges uniformly

1e/ $D_1 f(x) = \frac{\partial}{\partial x_1} f(x) = \frac{\partial}{\partial x_1}(x_1^3 + x_1x_3 + x_2^2 - 2x_2) =$

$3x_1^2 + x_3$

$D_2 f(x) = \frac{\partial}{\partial x_2} f(x) = \frac{\partial}{\partial x_2}(x_1^3 + x_1x_3 + x_2^2 - 2x_2) = 2x_2 - 2$

$D_3 f(x) = \frac{\partial}{\partial x_3} f(x) = \frac{\partial}{\partial x_3}(x_1^3 + x_1x_3 + x_2^2 - 2x_2) = x_1$ $\therefore$

$D_1 f(x)$, $D_2 f(x)$, $D_3 f(x)$ are all products and compositions of continuous functions so $Df(x)$ is continuous $\forall x \in \mathbb{R}^3$

with:

$Df(x) = (3x_1^2 + x_3 \quad 2x_2 - 2 \quad x_1)$ $\therefore$ $f(x)$ is differentiable everywhere

Stationary points when:

$Df(x) = 0 = (0 \quad 0 \quad 0) = (3x_1^2 + x_3 \quad 2x_2 - 2 \quad x_1)$ $\therefore$

$3x_1^2 + x_3 = 0$, $2x_2 - 2 = 0$, $x_1 = 0$ $\therefore$

$x_1 = 0 \implies 3(0)^2 + x_3 = 0 = 0 + x_3 = x_3 = 0$

$\therefore$ $x_3 = 0$, $x_1 = 0 \implies 2x_2 - 2 = 0$ $\therefore$ $2x_2 = 2$

$\therefore$ $x_2 = 1$ $\therefore$ Solutions are:

$x = (0 \quad 1 \quad 0)^T$ is the stationary point of f

(7)

\2ai/ x_0 is a limit point of S if every deleted neighborhood of x_0 contains a point of S

$\therefore$ if a point x_0 is a limit point of S:

~~For ε-ne For deleted ε-neighb~~

~~∀deleted~~ ∀ deleted ε-neighborhoods of x_0 being $(x_0-\varepsilon, x_0+\varepsilon) \implies \exists x \in S: x \in (x_0-\varepsilon, x_0+\varepsilon)$

$\therefore$ all interior points of S, being S° with $x_1 \in S^\circ$ then $\exists \varepsilon > 0: (x_1-\varepsilon, x_1+\varepsilon) \in S$

$\therefore \exists x_2 \in (x_1-\varepsilon, x_1+\varepsilon)$ ~~x_2~~ such that: $x_2 \in S$

$\therefore$ indeed the limit point of S: x_0 could be a interior point of S.

if x_0 is a boundary point of S then:

~~$\forall \varepsilon > 0, \exists (x_0-\varepsilon,$~~

$\forall \varepsilon > 0, \exists x_3 \in S$ such that $x_3 \in (x_0-\varepsilon, x_0+\varepsilon)$ and $\exists x_4 \in S^c$ (where $S^c = \mathbb{R} \setminus S$) such that

~~$\exists x_{?}$~~ $x_4 \in (x_0-\varepsilon, x_0+\varepsilon)$ $\therefore$

~~$\exists x_3$ such that x~~ ∀deleted ε-neighborhoods of x_0 it can be true that $\exists x_3 \in S$ such that x_3 is with this deleted ε-neighborhood of x_0 $\therefore$ it can be true that x_0 is a boundary point of ~~x_0~~ S $\therefore$

a limit point of S is either a interior point or a boundary point of S since

any other points that are neither a boundary or interior point of S would not be in S and so not be a limit point of S

2aii/ note: the set $\emptyset$ has ~~not~~ no limit points but it has no points
$\therefore$ it is true that all its points are isolated
$\therefore$ if a set S has not limit points, all its points must be isolated.

note: the set ~~is~~ $V=\{3\}$ has no limit points since ~~the dele~~ all deleted neighborhoods of $x_0=3$ do not contain a point in 3 $\therefore$
~~if all a sets points are isolated~~
where ~~if~~ $x_0=3$ was an isolated point of V $\therefore$

if all a sets points are isolated then it has no limit points $\therefore$

the original statement from the question is true that a set has no limit points if and only if all its points are isolated

\2aiii/ if x_0 is a boundary point of ~~x_0~~ S then ∃an deleted ε-neighborhood of x_0 ~~such that~~ denoted set V such that: ~~$\exists x_1 \in S : x_1 \in V$ and $\exists x_2 \notin S : x_2 \in V$ would be true if $x_0 \in S$~~

$\exists x_1 \in V : x_1 \in S$ and $\exists x_2 \in V : x_2 \notin S$ would still ~~be~~ be true even if $x_0 \notin S$ ~~at~~ as long as x_0 is a limit point of S ∴ Since the set of boundary points of contains points x_0 where it can be true that $x_0 \notin S$ and that a set of isolated points ~~for~~ is a closed set means that: the boundary of S is closed.

2b/ if f is not a constant function then $\exists x_1, x_2 \in [a,b]$ such that:
$f(x_1) \neq f(x_2)$ but f is continuous
$\therefore$ by the intermediate value theorem:
$\exists x_3 \in (a,b)$ such that: $f(x_3) = \frac{f(x_1)+f(x_2)}{\sqrt{2}}$

where x_1 and x_2 are chosen such that $f(x_3)$ is rational even if $f(x_1)$ and $f(x_2)$ are ~~are~~ irrational ~~such that~~ for example if
$f(x_1) = c\sqrt{2}$ for $c \in \mathbb{Q}$ and
$f(x_2) = d\sqrt{2}$ for $d \in \mathbb{Q}$ $\therefore$ $\frac{c\sqrt{2}+d\sqrt{2}}{\sqrt{2}} =$
$f(x_3) = \frac{(c+d)\sqrt{2}}{\sqrt{2}} = c+d \in \mathbb{Q}$ $\therefore$ $f(x_3) \in \mathbb{Q}$
which is a contradiction since $f(x) \notin \mathbb{Q}$ $\forall x \in [a,b]$ $\therefore$ by proof by contradiction:

it is not ~~true~~ possible that f is not a constant function $\therefore$

f must be a constant function

4a $\frac{1-(0)}{n} = \frac{1}{n}$ is width $\therefore \Delta x_i = \frac{1}{n}$

for $n \in \mathbb{N}$, $i \in \mathbb{N}$ and $i \leq n$

for uniform partition:

$P_n = (0, 0+\frac{1}{n}, \ldots, 0+\frac{1}{n}i, \ldots 1) = (x_0, \ldots, x_i, \ldots, x_n)$

so $x_0 = 0$, $x_n = 1$, $x_i = 0 + \frac{1}{n}i = \frac{1}{n}i$, $x_{i-1} = 0 + \frac{1}{n}(i-1) = \frac{1}{n}(i-1)$

note: $f(x) = \sqrt{x} = x^{1/2}$ $\therefore$ $f'(x) = \frac{d}{dx}(x^{1/2}) = \frac{1}{2}x^{-1/2} = \frac{1}{2}\cdot\frac{1}{x^{1/2}} > 0$

$\forall x \in (0,1]$ $\therefore$ $f(x)$ is an increasing function $\therefore$

$f(x_i) > f(x_{i-1})$ $\therefore$

for $x \in [x_{i-1}, x_i]$: $\sup\limits_{x\in[x_{i-1},x_i]} f(x) = f(x_i) = (x_i)^{1/2}$

and ~~for~~ $\inf\limits_{x\in[x_{i-1},x_i]} f(x) = f(x_{i-1}) = (x_{i-1})^{1/2}$

$\therefore$ have lower Riemann sum: $L(f;P_n) =$

$\sum\limits_{i=1}^{n}\left(\inf\limits_{x\in[x_{i-1},x_i]} f(x)\right)\Delta x_i = \sum\limits_{i=1}^{n}(x_{i-1})^{1/2}\frac{1}{n} = \frac{1}{n}\sum\limits_{i=1}^{n}\left(\frac{1}{n}(i-1)\right)^{1/2} =$

$\frac{1}{n}\sum\limits_{i=1}^{n}\frac{1}{n^{1/2}}(i-1)^{1/2} = \frac{1}{n^{3/2}}\sum\limits_{i=1}^{n}(i-1)^{1/2} = L(f;P_n)$

and $U(f;P_n) = \sum\limits_{i=1}^{n}\left(\sup\limits_{x\in[x_{i-1},x_i]}(f(x))\,\Delta x_i\right) = \frac{1}{n}\sum\limits_{i=1}^{n}(x_i)^{1/2} =$

$\frac{1}{n}\sum\limits_{i=1}^{n}\left(\frac{1}{n}i\right)^{1/2} = \frac{1}{n}\sum\limits_{i=1}^{n}\frac{1}{n^{1/2}}i^{1/2} = \frac{1}{n^{3/2}}\sum\limits_{i=1}^{n}i^{1/2} = U(f;P_n)$

$\therefore U(f;P_n) - L(f;P_n) = \frac{1}{n^{3/2}}\sum\limits_{i=1}^{n}i^{1/2} - \frac{1}{n^{3/2}}\sum\limits_{i=1}^{n}(i-1)^{1/2} =$

$\frac{1}{n^{3/2}}\left(\sum\limits_{i=1}^{n}\left(i^{1/2} - (i-1)^{1/2}\right)\right) = \frac{1}{n^{3/2}}\left((n)^{1/2} + \sum\limits_{i=1}^{n-1}(i^{1/2}) - \left((1-1)^{1/2} + \sum\limits_{i=1}^{n-1}i^{1/2}\right)\right)$

$= \frac{1}{n^{3/2}}\left(n^{1/2} - 0^{1/2} + \sum\limits_{i=1}^{n}(i^{1/2} - i^{1/2})\right) =$

$\frac{1}{n^{3/2}}(n^{1/2} - 0 + 0) = \frac{1}{n^{3/2}}(n^{1/2}) = n^{-1} = \frac{1}{n}$ $\therefore$

$\frac{1}{n} = U(f; P_n) - L(f; P_n)$ $\therefore$ so for $N \in \mathbb{N}$

$\therefore$ for $\varepsilon > 0$: $\frac{1}{n} \leq \frac{1}{N} < \varepsilon$ $\forall n \geq N$ with $\frac{1}{\varepsilon} < N$

$\therefore \lim_{n \to \infty} (U(f; P_n) - L(f; P_n)) = 0$ $\therefore$ the Riemann integrability criterion is satisfied $\therefore$ $f(x)$ is Riemann integrable

4b if $H(x)=\int_{f(x)}^{g(x)} h(t)\,dt$ then:

- $\frac{d}{dx}(H(x)) = H'(x) = h(g(x))g'(x) - h(f(x))f'(x)$

So if $f(x)=0$, $g(x)=\sin(x)$, $h(t)=e^{-t^2}$

$\therefore f'(x)=0$, $g'(x)=\cos(x)$,

$h(g(x)) = e^{-\sin^2(x)}$, $h(f(x)) = e^{-0^2} = e^0 = 1$

$\therefore$ if $H(x) = \int_0^{\sin(x)} e^{-t^2}\,dt$ $\therefore$

$$H'(x) = e^{-\sin^2(x)}\cos(x) - 1(0) = e^{-\sin^2(x)}\cos(x) \quad \therefore$$

and $\frac{d}{dx}(f(x)) = f'(x) = \frac{d}{dx}\left(\int_0^{\sin(x)} e^{-t^2}\,dt\right)\left(-\sin\left(\int_0^{\sin(x)} e^{-t^2}\,dt\right)\right)$

$= \cos(x)e^{-\sin^2(x)}\left(-\sin\left(\int_0^{\sin(x)} e^{-t^2}\,dt\right)\right) =$

$\cancel{= \cos(x)e^{-\sin^2(x)}\int_0^{\sin(x)} e^{-t^2}\,dt}$

$- \cos(x)e^{-\sin^2(x)}\sin\left(\int_0^{\sin(x)} e^{-t^2}\,dt\right) = f'(x) \quad \forall x \in [0,\infty)$

4 c) have $x_0 = (0,0)^T \therefore f(x_0) = 0 \quad \therefore \forall (0 < \|x - 0\| < \delta)$ (c)

- $\forall \varepsilon > 0: \|f(x) - f(x_0)\| = \|f(x) - 0\| = \|f(x)\| =$

$$\left\| \frac{2x_1x_2}{\sqrt{x_1^2+x_2^2}} \right\| = \sqrt{\left(\frac{2x_1x_2}{\sqrt{x_1^2+x_2^2}}\right)^2} = \left|\frac{2x_1x_2}{\sqrt{x_1^2+x_2^2}}\right| = \frac{|2x_1x_2|}{\sqrt{x_1^2+x_2^2}}$$

$$= \frac{|2x_1x_2|}{\|x\|} = \frac{2|x_1x_2|}{\|x\|} \leq \frac{2 \cdot \frac{1}{2}(x_1^2+x_2^2)}{\|x\|} \quad \{\text{by young's inequality}\}$$

$$= \frac{(x_1^2+x_2^2)}{\|x\|} = \frac{\left(\sqrt{x_1^2+x_2^2}\right)^2}{\|x\|} = \frac{\|x\|^2}{\|x\|} = \|x\| = \|x-0\| < \delta = \varepsilon$$

$\forall \|x - 0\| < \delta$, $\delta > 0$ with $\delta = \varepsilon$

$\therefore \lim_{x \to 0} f(x) = 0 \quad \therefore f(x_0)$ is continuous for $x_0 = (0,0)^T$

for $x \neq (0,0)^T$: $f(x) = \dfrac{2x_1x_2}{\sqrt{x_1^2+x_2^2}}$ which is a product and composition of continuous functions

so $f(x)$ is continuous $\forall x \neq (0,0)^T \therefore$

$f(x)$ is continuous $\forall x \in \mathbb{R}^2$