# MTH2009 CW2

\1/ the Cauchy-Riemann equations for an function $f(z)$ ~~is~~ where $f(z) = Re(f(z)) + i Im(f(z))$ $= u + iV = u(x,y) + iV(x,y)$ where $z = x + iy$ for $x, y \in \mathbb{R}$ is:

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ $\therefore$

its given $u(x,y) = 2x^3 - 6xy^2 + 2xy = u$ $\therefore$

$\frac{\partial u}{\partial x} = \frac{\partial}{\partial x}(2x^3 - 6xy^2 + 2xy) = 6x^2 - 6y^2 + 2y = \frac{\partial v}{\partial y} = \frac{\partial V(x,y)}{\partial y}$

and $\frac{\partial u}{\partial y} = \frac{\partial}{\partial y}(2x^3 - 6xy^2 + 2xy) = -12xy + 2x$ $\therefore$

$-\frac{\partial u}{\partial y} = -(-12xy + 2x) = 12xy - 2x = \frac{\partial v}{\partial x} = \frac{\partial V(x,y)}{\partial x}$

$\therefore \frac{\partial v}{\partial x} = 12xy - 2x$ $\therefore$ $V(x,y) = \frac{1}{2} \cdot 12x^2y - \frac{1}{2} \cdot 2x^2 + g(y)$

$= 6x^2y - x^2 + g(y)$ where $g$ is an function of $y$ $\therefore$

$\frac{\partial v}{\partial y} = 6x^2 - 6y^2 + 2y$ $\therefore$ $V(x,y) = 6x^2y - \frac{1}{3} \cdot 6y^3 + \frac{1}{2} \cdot 2y^2 + h(x)$

$= 6x^2y - 2y^3 + y^2 + h(x)$ where $h$ is an function of $x$ $\therefore$

$V(x,y) = 6x^2y - 2y^3 + y^2 + h(x) = 6x^2y - x^2 + g(y)$ $\therefore$

$-2y^3 + y^2 + h(x) = -x^2 + g(y)$ $\therefore$ $h(x) = -x^2$ and

$g(y) = -2y^3 + y^2$ $\therefore$

$V(x,y) = 6x^2y - x^2 - 2y^3 + y^2 = V$ $\therefore$

$f(z) = f(x+iy) = Re(f(x+iy)) + i\,Im(f(x+iy)) =$
$u(x+iy) + iV(x+iy) = u(x,y) + iV(x,y) = u + iV =$
$2x^3 - 6xy^2 + 2xy + i(6x^2y - x^2 - 2y^3 + y^2) = f(x+iy) = f(z)$

\2/ The Cauchy-Riemann equations for an function $f(z)$ where $f(z) = Re(f(z)) + i\,Im(f(z)) =$
$u(z) + iv(z) = u(x+iy) + iV(x+iy) = u(x,y) + iV(x,y) =$
$u+iv$ where $z = x+iy$ for $x, y \in \mathbb{R}$ is:

$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}$ and $-\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x}$ $\therefore$

in D $|z|<1$ $\therefore$ $|x|<1$ and $|y|<1$ $\therefore$

$Re(f(z)) = u(z)$, $Im(f(z)) = V(z)$ $\therefore$

$u(z) + V(z) = 10$ $\therefore$ $u+v = 10$ $\therefore$

$$\frac{\partial}{\partial x}(u+v) = \frac{\partial}{\partial x}(10) = \frac{\partial u}{\partial x} + \frac{\partial v}{\partial x} = 0 \quad \therefore$$

$$\frac{\partial v}{\partial x} = -\frac{\partial u}{\partial x}$$

$$\frac{\partial}{\partial y}(u+v) = \frac{\partial}{\partial y}(10) = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial y} = 0 \quad \therefore$$

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial y} \quad \therefore$$

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial y} \qquad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial x} \quad \therefore$$

$$\frac{\partial v}{\partial y} = -\frac{\partial u}{\partial x} \text{ and } \frac{\partial v}{\partial y} = \frac{\partial u}{\partial x} \quad \therefore \quad -\frac{\partial u}{\partial x} = \frac{\partial u}{\partial x} \quad \therefore$$

$$2\frac{\partial u}{\partial x} = 0 = \frac{\partial u}{\partial x} \quad \therefore$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 0 \quad \therefore \quad \frac{\partial v}{\partial y} = -\frac{\partial u}{\partial y} = 0 = \frac{\partial u}{\partial y} \quad \therefore$$

$$-\frac{\partial u}{\partial y} = 0 = \frac{\partial v}{\partial x} \quad \therefore$$

$\frac{\partial u}{\partial x} = 0 \therefore u = A(y)$ where $A$ is an function of $y$

$\frac{\partial u}{\partial y} = 0 \therefore u = B(x)$ where $B$ is an function of $x$

$\therefore u = A(y) = B(x) \therefore A(y) = \text{Constant} = B(x) = C_1$ $\frac{1}{2}$

$\frac{\partial v}{\partial x} = 0 \therefore V = h(y)$ where $h$ is an function of $y$

$\frac{\partial v}{\partial y} = 0 \therefore V = g(x)$ where $g$ is an function of $x$ $\therefore$

$V = g(x) = h(y)$ $\therefore$ $g(x) = \text{Constant} = h(y) = C_2$ $\therefore$

$f(z) = Re(f(z)) + Im(f(z)) = u + iV = u(x,y) + iV(x,y) =$
$u(z) + iV(z) = C_1 + iC_2 = C_3 = \text{Constant} = f(z)$
where $C_1, C_2, C_3$ are all constants $\therefore$
$f$ is constant in the open disc $D$

3/ by the M-L lemma: if there is an constant $M \geq 0$ such that $|f(z)| \leq M$ for all points $z$ on $\gamma$ then $\left|\int_\gamma f(z)\,dz\right| \leq M \cdot L(\gamma)$

where $L(\gamma)$ is the length of the path $\gamma$

and $f(z) = \frac{1}{2+z^2}$ , $z = x+iy$ with $x, y \in \mathbb{R}$

and upper half of the unit circle so unit semi circle: $\gamma : [a,b] \to \mathbb{C}$ $\therefore$ $\gamma(t) = e^{it}$ $0 \leq t \leq \pi$ going anti-clockwise $\therefore$ $|z| = 1$ $\therefore$ $|x| < 1$ and $|y| < 1$ as $-1 \leq x \leq 1$ and $0 \leq y \leq 1$ $\therefore$

$\gamma'(t) = \frac{d}{dt}(\gamma(t)) = \frac{d}{dt}(e^{it}) = ie^{it}$ $\therefore$ $\gamma : [0, \pi] \to \mathbb{C}$

$\therefore$ $L(\gamma) = \int_0^\pi |\gamma'(t)|\,dt$ $\therefore$ $|\gamma'(t)| = |ie^{it}| =$

$|i||e^{it}| =$ $|0+1i||\cos t + i\sin t| =$

$\sqrt{0^2+1^2} \cdot \sqrt{\cos^2 t + \sin^2 t} = \sqrt{1} \cdot \sqrt{1} = 1$ $\therefore$

$$L(\gamma) = \int_0^\pi 1\,dt = \Big[t\Big]_0^\pi = \pi - 0 = \pi = L(\gamma)$$

$\{$ note: $|z - w| \geq |z| - |w|$ is reverse triangle inequality

$\therefore \frac{1}{|z-w|} \leq \frac{1}{|z|-|w|}$ $\}$ $\therefore$

$|f(z)| = \left|\frac{1}{2+z^2}\right| = \frac{1}{|2+z^2|} = \frac{1}{|(2)-(-z^2)|} \leq \frac{1}{|2|-|-z^2|} =$

$\frac{1}{2-|z^2|} = \frac{1}{2-|z|^2} = \frac{1}{2-(1)^2} = \frac{1}{2-1} = \frac{1}{1} = 1$ $\therefore$

$|f(z)| \leq 1 = M =$ Constant $\therefore$ $\left|\int_\gamma f(z)\,dz\right| =$

$\left|\int_\gamma \frac{1}{2+z^2}\,dz\right| \leq M \cdot L(\gamma) = 1 \cdot \pi = \pi$ $\therefore$ $\left|\int_\gamma \frac{dz}{2+z^2}\right| \leq \pi$

$\square$

4/ The Cauchy-Hadamard theorem states:

the series $\sum_{n=0}^{\infty} a_n (z-\gamma)^n$ with $\gamma, a_n \in \mathbb{C}$

converges for all $z$ such that $|z-\gamma| < R$ where

$\frac{1}{R} = \limsup_{n\to\infty} |a_n|^{\frac{1}{n}} = \lim_{n\to\infty} \left( \sup_{m\geq n} |a_m|^{\frac{1}{m}} \right)$

$R$ is called the radius of convergence and can take values on $\mathbb{R} \cup \{\infty\}$

for $z = x+iy$ for $x, y \in \mathbb{R}$ $\therefore$

for $\sum_{n=0}^{\infty} a_n z^n = \sum_{n=0}^{\infty} a_n (z-0)^n$ converges if $|z| < R$

with $R = \frac{1}{\limsup_{n\to\infty} |a_n|^{\frac{1}{n}}}$ $\therefore$

$\frac{1}{R} = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$ $\therefore$ $R = \frac{1}{\lim_{n\to\infty} |a_n|^{\frac{1}{n}}}$

$a_n \in \mathbb{C}$ $\therefore$ for $\alpha, \beta \in \mathbb{R}$ : $a_n = \alpha + i\beta = Re(a_n) + i Im(a_n)$

$\therefore$ $\alpha = Re(a_n)$ and $\beta = Im(a_n)$ $\therefore$

$|a_n| = ((Re(a_n))^2 + (Im(a_n))^2)^{\frac{1}{2}}$ and $Im(a_n) \in \mathbb{R}$ $\therefore$

$(Im(a_n))^2 \geq 0$ and $|Re(a_n)| = \sqrt{(Re(a_n))^2}$ $\therefore$

$|a_n| \geq |Re(a_n)|$ $\therefore$ $\lim_{n\to\infty} |a_n| \geq \lim_{n\to\infty} |Re(a_n)|$ $\therefore$

$\lim_{n\to\infty} |a_n|^{\frac{1}{n}} \geq \lim_{n\to\infty} |Re(a_n)|^{\frac{1}{n}}$ $\therefore$

$\frac{1}{\lim_{n\to\infty} |a_n|^{\frac{1}{n}}} \leq \frac{1}{\lim_{n\to\infty} |Re(a_n)|^{\frac{1}{n}}}$ and

let $r$ be the radius of convergence of the series: $\sum_{n=0}^{\infty} (Re(a_n)) z^n = \sum_{n=0}^{\infty} (Re(a_n))(z-0)^n$

$\therefore$ $\frac{1}{r} = \lim_{n\to\infty} |Re(a_n)|^{\frac{1}{n}}$ $\therefore$

$r = \frac{1}{\lim |Re(a_n)|^{\frac{1}{n}}}$ $\therefore$

$R \leq r$ $\therefore$

the radius of convergence of the series:
$\sum_{n=0}^{\infty} (Re(a_n)) z^n$ ~~has~~ is greater than or equal
to $R$ as required $\square$