2008 real number system Real analysis

Ex 1a/ $S=[0,1]=\{x\in\mathbb{R}: 0\le x\le 1\}$

$\cancel{8}$ $\sup(S)=1$ 1st check $x\le 1\ \forall x\in S$ ✓

2nd $\forall b<1, \exists x\in S\ x>b$ ✓

1b/ $T=(0,1)=\{x\in\mathbb{R}: 0<x<1\}$ ~~sup~~ $\cancel{\sup(S)\ge 1}$

$\sup(T)=1$ $\cancel{\forall x\in T}$ $x\le 1\ \forall x\in T$

$\forall b<1, \exists x\in T; x>b$ supremum may ~~of~~ may not be an element of the set

infimum $a\in\mathbb{R}$ s.t $x\ge a\ \forall x\in S$ then S is bounded below and a is an lower bound of S

if α is an lower bound of S but no numbers greater than α is then α is the infimum of S denoted $\alpha=\inf(S)$ infimum is greatest lower bound

Ex 1.8/ Q rational numbers $S=\{r\in\mathbb{Q}: r^2<2\}$

$=\{r\in\mathbb{Q}: -\sqrt{2}<r<\sqrt{2}\}$ so supremum doesn't exist so rational numbers don't have a completeness proper

Ex 1.11/ $S=\{x\in\mathbb{R}: x<2\}$ $\sup(S)=2$ $\inf(S)=-\infty$

2008 real analysis Exercise sheet 1

1i/ $S=\left\{1+\frac{(-1)^n}{n}\ \middle|\ n\in\mathbb{N}\right\}=0, \frac{3}{2}, \frac{2}{3}, \frac{5}{4}, \frac{4}{5}, \frac{7}{6}, \frac{6}{7}\dots$

~~as $n\to\infty$, $1+(-1)$~~ $\lim_{n\to\infty}\left(1+\frac{(-1)^n}{n}\right)=1+\lim_{n\to\infty}\frac{(-1)^n}{n}=1+0=1$

$\frac{3}{2}>1$ so $\frac{3}{2}$ is the biggest value it takes since all others after will be closer to 1 so $\sup(S)=3/2$ and $\inf(S)=0$

1ii/ ~~$S=\{e^n|n\in\mathbb{Z}\}$~~ $S=\{e^n|n\in\mathbb{N}\cup\{0\}\}$ note

$n\ge 0$ and $e>1$ so ~~$x\in\mathbb{N}$~~ for $x\in\mathbb{N}$

$e^x\ge e$ so $S=1, e, e^2, e^3, e^4, \dots$ $\lim_{n\to\infty}e^n=\infty$

so S has no desired suprema and $\inf(S)=1$

12/ so ~~x~~ ~~for $x \in X$~~, ~~so~~ $\forall x \in X, (x \in \mathbb{R}^+)$ ~~$x > 0$~~ so $x > 0$

so $x^2 > x, \forall x \in X$. so $x^2 + \sqrt{3} > x, \forall x \in X$

so $\sup(Y) = (\sup(X))^2 + \sqrt{3} = \alpha^2 + \sqrt{3}$

13/ $a, b \in \mathbb{R}$, $a < b \Rightarrow \frac{a}{b} < 1$ and $1 < \frac{b}{a}$

so $\frac{a}{b} < 1 < \frac{b}{a}$ so $\frac{a^2}{b} < a < b$

is $\frac{p}{q}$ is irrational then $\frac{p}{q}$ are not coprime

// $qa < p < bq$ $\frac{qa}{p} < 1 < \frac{qb}{p}$ $\frac{p}{q}$ was rational so $\frac{q}{p}$ is too

unless $p = 0$ so ~~$\frac{q}{pb}$~~ $\frac{qa}{p} < \frac{qb}{p}$ so $\frac{q}{pb} < \frac{q}{pa}$

14i/ $a, b \in \mathbb{R}$ $|a+b| \leq |a| + |b|$

$|a-b| = |a-(+b)| =$ ~~$||a| - |b||$~~ note $|a-b| \leq |a+b|$

$= |-(-a+b)| = |-a+b| \geq |-|a|-|b|| = |-(|a|+|b|)| = ||a|+|b||$

$|a+b| \leq |a| + |b|$ so $|a-b| \leq$ ~~$|a|-|b| \leq$~~ $|a-b| \geq |a| - |b|$

but $|a-b| \geq 0$ $|a|-|b| \leq ||a|-|b||$

14ii/ $|a+b| \leq |a| + |b|$ $|a|-|b| \leq ||a|-|b||$

$|a|+|b| \geq |a|-|b|$

so $|a+b| \leq |a|+|b| \geq |a|-|b| \leq ||a|-|b||$

so $x \leq y \geq z \leq t$

15i/ $I = \bigcap_{F \in \mathcal{F}} F$ and $U = \bigcup_{F \in \mathcal{F}} F$

so $I^c \cup I = \mathbb{R} \Rightarrow \mathbb{R} \setminus I = I^c$

$I^c = \left(\bigcap_{F \in \mathcal{F}} F\right)^c = \left(\bigcup_{F \in \mathcal{F}} F^c\right)$ $(F)^c = F^c$ and $F^c \cap F = \mathbb{R}$

$(F_1 \cap F_2)^c = F_1^c \cup F_2^c \Rightarrow I^c = \left(\bigcup_{F \in \mathcal{F}} F^c\right)$

15ii/ ~~$\forall$~~ $U^c \cup U = \mathbb{R} \Rightarrow U^c = \mathbb{R} \setminus U$

$F \cup F^c = \mathbb{R} \Rightarrow F^c = \mathbb{R} \setminus F$

so $(F_1 \cup F_2)^c = F_1^c \cap F_2^c$

and also $(F_1 \cup F_2 \cup F_3)^c = F_1^c \cap F_2^c \cap F_3^c$

and so $\left(\bigcup_{F \in \mathcal{F}} F\right)^c = U^c = F_1^c \cap F_2^c \cap F_3^c \cap \ldots = \bigcap_{F \in \mathcal{F}} (F^c)$

16i/ ε-neighb is $x_0 - \varepsilon \leq x_0 \leq x_0 + \varepsilon$

so $\forall x \in S$, $\frac{1}{2} \leq x \leq 1$ $x_0 = \frac{3}{4}$ $1 - \frac{3}{4} = \frac{1}{4}$, $\frac{3}{4} - \frac{1}{2} = \frac{1}{4}$

so an ε-neighb of S is $x_0 - \frac{1}{4} < x_0 < x_0 + \frac{1}{4}$

for largest ε, $\varepsilon = \frac{1}{4}$

16ii/ $\forall x \in S$, $0 < x < 2$ $x_0 = 1$ so $2 - 1 = 1$, $1 - 0 = 1$

so $x_0 - 1 < x_0 < x + 1$ is a ε neighb contained in S

for largest ε of $\varepsilon = 1$

17i/ $S^c = (-\infty, -1] \cup [2, 3)$

limit points of S^c are $(-\infty, -1] \cup [2, 3]$

the limit points of S are $[-1, 2] \cup [3, \infty)$

~~S does~~ $2 \notin S$ so S doesn't contain all its limit points

so its not closed

$S = (-1, 2) \cup [3, \infty)$ 3 is not an interior point of S

but -1 is not an interior point of S^c so S is not open

and S^c is not open so S is neither closed nor open

17ii/ $S^c = [1, 2]$ limit points of S are $(-\infty, 1] \cup [2, \infty)$

but $1 \notin S$ so S is not closed

interior points of S are $(-\infty, 1) \cup (2, \infty)$

~~which are all in S hence~~ ~~so S contain~~ so all points in

S are interior points so S is open

\7iii/ $S^c=(-\infty,-3)\cup(-2,7)\cup(8,\infty)$

limit points of S are $[-3,-2]\cup[7,8]$ so S contains all its limit points so S is closed

\7iv/ for $S=\{1+\frac{(-1)^n}{n} \mid n\in\mathbb{N}\}$ or $S=\{e^n \mid n\in\mathbb{N}\cup\{0\}\}$

now $S=\{x\in\mathbb{R} \mid x\in\mathbb{N}\}=\{1,2,3,4,5,\dots\}$

so $\cancel{S=[1,\infty)}$ so $S=\{1\}\cap\{2\}\cap\{3\}\cap\dots$

is a finite set so has no limit points

so it does contain its limit points so its closed

\8i/ $S=\bigcap_{F\in\mathcal{F}} F$ for $\mathcal{F}$ collection on $\mathbb{R}$ all F are

open ~~if a<b~~ $a<b<c<d$ if $F_1=(a,c)$ $F_2=(b,d)$

$F_1\cap F_2=(a,c)\cap(b,d)=(b,c)$ which is open

so $\bigcap_{F\in\mathcal{F}} F$ is open

\8ii/ ~~(−∞~~ if $\bigcap_{F\in\mathcal{F}} F=(-\infty,\infty)$ after the intersection of all infinitely many sets spanning $\mathbb{R}$

but $(-\infty,\infty)$ is a closed set since it contains all its limit points

\9i/ $S=\bigcup_{F\in\mathcal{F}} F$ for $\mathcal{F}$ is a collection on $\mathbb{R}$ all F are closed

if $a<b<c<d$ if $F_1=[a,b]$ a is not an interior point hence F_1 is not open, limit points of F_1 are $[a,b]$

so F_1 contains all its limit points so F_1 is close

so $F_2=[c,d]$ is also closed $F_1\cup F_2=[a,b]\cup[c,d]$

which ~~has limit~~ contains all its limit points so is closed so $\bigcup_{F\in\mathcal{F}} F$ is closed by induction

2008 ex sheet 1

\q1ii/ is $\bigcup_{F\in\mathcal{F}} F = [-100, 100]$ after the union of infinitely many sets of $[-100,0]$, $[0,100]$ and sets of $\{y + \frac{(-1)^n}{n} \mid n \in \mathbb{N}\}$ for $y \in \mathbb{R}, -1 < y < 1$

which means since y is infinitely many numbers that is infinitely many sets unioned to make a closed set

union of infinitely many sets to form $(-\infty, \infty)$ which is $\mathbb{R}$ set which is open

$a < p/q < b$ $\quad p = 1 \quad \varepsilon = b - a \quad q$ s.t. $q(b-a) > 1 \quad j$ s.t. $> qa$

$j \in \mathbb{N}$ obvious is $a \le 0$ with $\varepsilon = 1$ & $p = qa$ is $a > 0$ let p be smallest integer s.t. $p > qa \Rightarrow p - 1 \le qa \Rightarrow$

$qa < p \le qa + 1$ since $1 < q(b-a) \Rightarrow qa < p < qa + q(b-a) = qb$

so $qa < p < qb \Rightarrow a < p/q < b$

q s.t. $q(b-a) > 1 \quad p$ s.t. $p > qa \Rightarrow p - 1 \le qa$

with $qa < p$ so $qa < p < qa + 1 \quad 1 < q(b-a)$

$qa < p < qa + q(b-a) = qb \quad a < p/q < q$

$|f(x) - L| < \varepsilon$ new def $\Rightarrow$ original def take $k = 1$

original def $\Rightarrow$ new def $|f(x) - L| < \varepsilon$ for $0 < |x - x_0| < \delta$, then let $\varepsilon' = \varepsilon/k$ then $|f(x) - L| < k\varepsilon' = \varepsilon$

def 2.19: $\lim_{x\to x_0^-} f(x) = \infty$ or $f(x_0^-) = \infty \quad (a, x_0)$

$f(x) > M$ if $x_0 - \delta < x < x_0$ reflected in line x_0

$x_0 < x < x_0 + \delta \quad \lim_{x\to x_0} f(x) = \infty \quad (a,b)$ with $x_0 \in (a,b)$

$f(x) > M$ if $0 < |x - x_0| < \delta$

15i/ $\mathcal{F}=\{A,B\}$ claim $I^c \subset \bigcup\{F^c \mid F\in\mathcal{F}\}$ take arbit point $x\in I^c$ then $x\notin I=\bigcap\{F \mid F\in\mathcal{F}\}$ this means there exists an $F\in\mathcal{F}$ s.t. $x\notin F$ this implies that $x\in\bigcup\{F^c \mid F\in\mathcal{F}\}$

$I^c \supset \bigcup\{F^c \mid F\in\mathcal{F}$

$\lim_{x\to\infty}(x\sin x)\neq\infty$

1.10.i/ $\mathbb{Z}=\{\ldots,-5,-4,-3,-2,-1,0,1,2,3,4,5,\ldots\}$

is $x_0=0$ a deleted ε-neighb with $\varepsilon<1$ is not ~~in~~ S so 0 is not a limit point same for all integers so S has no limit points so set of limit points of S is $\emptyset$

0 is a boundary ~~of S~~ point of S and so are all the integers so the boundary of S is ∂S $=\{\ldots,-5,-4,-3,-2,-1,0,1,2,3,4,5,\ldots\}=\mathbb{Z}$

$\overline{S}=S\cup\partial S=\mathbb{Z}\cup\mathbb{Z}=\mathbb{Z}$

0 is a isolated point of S since for $\varepsilon<1$ the ε neighb of $x_0=0$ contains only ~~of~~ $0\in S$ same for all other integers so set of isolated points of S is $\mathbb{Z}$

~~$S^c=\{\mathbb{R}\setminus\mathbb{Z}\mid$~~ $S^c=\{x\in(\mathbb{R}\setminus\mathbb{Z}) \mid x\in\mathbb{R}\}$

$\forall x, 0<x<1$ x is a interior of S^c same for all other points inbetween integers so the interior of S^c is $\ldots\cup(-5,-4)\cup(-4,-3)\cup(-3,-2)\cup(-2,-1)\cup(-1,0)\cup(0,1)\cup(1,2)\cup\ldots$

So the exterior of S is

$\ldots\cup\{-3\}\cup\{-2\}\cup\{-1\}\cup\{0\}\cup\{1\}\cup\{2\}\cup\{3\}\cup\ldots=\mathbb{Z}$

\10ii/ $S = \bigcup_{n\in\mathbb{Z}} (n, n+1) =$

$\cdots \cup (-3,-2) \cup (-2,-1) \cup (-1,0) \cup (0,1) \cup (1,2) \cup (2,3) \cup \cdots$

$[-1,0]$ are limit points of S so are $[0,1]$ and so on

so set of limit points of S are

$\cdots \cup [-3,-2] \cup [-2,-1] \cup [-1,0] \cup [0,1] \cup [1,2] \cup [2,3] \cup \cdots$

-1 is a boundary point of S, so is 0, so is 1 and so on

so the set of boundary points of S is

$\partial S = \{\ldots -3, -2, -1, 0, 1, 2, 3, \ldots\}$

the closure of S is $\overline{S} = \partial S \cup S =$

$\cdots \cup (-2,-1) \cup (-1,0) \cup (0,1) \cup (1,2) \cup \cdots \cup \{\ldots -3,-2,-1,0,1,2,3 \ldots\}$

$= \{\ldots, -3, -2, -1, 0, 1, 2, 3\} \cup \left(\bigcup_{n\in\mathbb{Z}} (n, n+1) \right) =$

$\cdots \cup [-2,-1] \cup [-1,0] \cup [0,1] \cup [1,2] \cup \cdots = (-\infty, \infty) = \mathbb{R}$

S has no isolated points so the interior of S is $\emptyset$

$S^c = \cdots \cup \{-2\} \cup \{-1\} \cup \{0\} \cup \{1\} \cup \{2\} \cup \cdots = \mathbb{Z}$

no points are interior to S^c so $(S^c)^\circ = \emptyset$

and this therefore is the exterior of S

\10iii/ $S = \{x \mid x = 1/n,\ n = \mathbb{N}\}$ so $x = \{\frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \ldots\}$

so $S = \{1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots\}$

1 is not a limit point of S neither are any other points

so S has no limit points so set of limit points of S is $\emptyset$

the neighb of $\varepsilon < \frac{1}{2}$ for $x_0 = 1$ only contains 1

so 1 is a boundary point of S and so are all its

other points so $\partial S = S$ and $\overline{S} = \partial S \cup S = S \cup S = S$

ε neighb of $\varepsilon < \frac{1}{2}$ for $x_0 = 1$ only contains 1 ~~so 1 is~~ in S
so 1 is an isolated point of S same all its other points
so the set of isolated points of S is

$$\left\{1, \frac{1}{2}, \frac{1}{3}, \dots\right\} = S$$

$$S^c = (-\infty, 1) \cup (1, \tfrac{1}{2}) \cup (\tfrac{1}{2}, \tfrac{1}{3}) \cup (\tfrac{1}{3}, \tfrac{1}{4}) \cup \dots$$

$(-\infty, 1)$ is interior to S^c and so is $(1, \frac{1}{2})$ and so on
so the interior of S^c is $(S^c)^o =$
$(-\infty, 1) \cup (1, \frac{1}{2}) \cup (\frac{1}{2}, \frac{1}{3}) \cup (\frac{1}{3}, \frac{1}{4}) \cup \dots = S^c$
which is therefore the exterior of S

iii/ S is therefore closed so S contains all its limit points so S has no isolated points and S is bounded so it is a finitely large set that doesn't extend to $-\infty$ or $+\infty$

2008 Sheet 2

12 i/ so as $x \to \infty$, $f(x) \to \infty$ f is defined on an inter (a, ∞) and $\forall M\ \exists \beta > 0$ s.t. if $x > \beta$ then $f(x) > M$

12 ii/ so as $x \to \infty$, $f(x) \to -\infty$ so f is defined on an interval (a, ∞) and $\forall M\ \exists \beta > 0$ S.t if $x > \beta$ then $f(x) < M$ ✓

12 iii/ so as $x \to -\infty$, $f(x) \to \infty$ so f is defined on an interval $(-\infty, b)$ and $\forall M\ \exists \beta < 0$ s.t. if $x < \beta$ then $f(x) > M$ ✓

12 iv/ so as $x \to -\infty$, $f(x) \to -\infty$ so f is defined on an interval $(-\infty, b)$ and $\forall M\ \exists \beta < 0$ S.t. if $x < \beta$ then $f(x) < M$ ✓

13 i/ $|x - x_0| < \delta$ $\qquad |f(x) - f(x_0)| = \left| \frac{1}{x^2+1} - \frac{1}{x_0^2+1} \right|$

$= \left| \frac{1}{x^2+1} + t - \frac{1}{x_0^2+1} - t \right| = \left| \left(\frac{1}{x^2+1} + t\right) + \left(-\frac{1}{x_0^2+1} - t\right) \right|$

$\leq \left| \frac{1}{x^2+1} + t \right| + \left| -\frac{1}{x_0^2+1} - t \right| = \left| \frac{1}{x^2+1} + t \right| + \left| -\left(\frac{1}{x_0^2+1} + t\right) \right|$

$= \left| \frac{1}{x^2+1} + t \right| + \left| \frac{1}{x_0^2+1} + t \right|$

or $\left| \frac{1}{x^2+1} - \frac{1}{x_0^2+1} \right| \leq \left| \frac{1}{x^2+1} \right| + \left| \frac{1}{x_0^2+1} \right| = \frac{1}{x^2+1} + \frac{1}{x_0^2+1} = \frac{x_0^2+1+x^2+1}{(x^2+1)(x_0^2+1)}$

$= \frac{x_0^2 + x^2 + 2}{x^2 x_0^2 + x^2 + x_0^2 + 2}$

or $\left| \frac{1}{x^2+1} - \frac{1}{x_0^2+1} \right| = \left| \frac{x_0^2+1-(x^2+1)}{x^2x_0^2+x^2+x_0^2+2} \right| = \left| \frac{-(x^2 - x_0^2)}{(x^2+1)(x_0^2+1)} \right| = \left| \frac{-(x+x_0)(x-x_0)}{(x^2+1)(x_0^2+1)} \right|$

$\lim_{x\to\infty} \frac{1}{x^2+1} = 0$ indeed $\forall \varepsilon > 0$ choose $\beta = \sqrt{\frac{1}{\varepsilon}}$

so if $x > \beta$ then $\frac{1}{x^2+1} < \frac{1}{x^2} < \frac{1}{\beta^2} = \varepsilon$

so $f(x) = \frac{1}{x^2+1}$ for $\lim_{x \to \infty}$ so $\frac{1}{x^2+1} < \frac{1}{x^2} < \frac{1}{\beta^2}$ for $x > \beta$

so $\frac{1}{\beta^2} = \varepsilon$ so $\beta = \frac{1}{\sqrt{\varepsilon}} = \beta$ so $\frac{1}{x^2+1} < \varepsilon$ for $x > \beta$

$\forall \varepsilon > 0$ so $\lim_{x \to \infty} \frac{1}{x^2+1} = 0$

3ii/ $\alpha > 0$, note $|\sin(x)| \leq 1$, $|x|^\alpha \geq 0$ so

so $\left| \frac{\sin x}{|x|^\alpha} \right| \leq 1$ for $\forall \varepsilon > 0$, $\exists \beta$ $x > \beta$ then

$\cancel{\frac{\sin x}{|x|^\alpha}}$ $\frac{\sin x}{|x|^\alpha} \leq \frac{1}{|x|^\alpha} < \frac{1}{|\beta|^\alpha} = \frac{1}{\beta^\alpha}$

so $\frac{1}{\varepsilon} = \beta^\alpha$ so $\left(\frac{1}{\varepsilon}\right)^{1/\alpha} = \beta$ so $\frac{1}{\beta^\alpha} = \varepsilon$ so $\frac{\sin x}{|x|^\alpha} < \varepsilon$

$\forall \varepsilon > 0$ so $\lim_{x \to \infty} \frac{\sin x}{|x|^\alpha} = \lim_{x \to \infty} f(x) = 0$

$\forall \varepsilon > 0$ choose $\beta = \left(\frac{1}{\varepsilon}\right)^{1/\alpha}$ if $x > \beta \Rightarrow \frac{1}{|x|^\alpha} < \frac{1}{\beta^\alpha}$

so $\frac{\sin x}{|x|^\alpha} \leq \frac{1}{|x|^\alpha} < \frac{1}{\beta^\alpha} = \varepsilon$

3iii / $\forall \varepsilon > 0$ if $x > \beta \Rightarrow \frac{\sin x}{|x|^\alpha} \leq \frac{1}{|x|^\alpha} \not< \frac{1}{|\beta|^\alpha} = \frac{1}{\beta^\alpha} = \varepsilon$

for $\beta = \left(\frac{1}{\varepsilon}\right)^{1/\alpha}$ since $\alpha \leq 0$ so if $\alpha = 0$

$\frac{1}{|x|^0} = \frac{1}{1} = 1 = \frac{1}{1} = \frac{1}{|\beta|^0}$ so $\lim_{x \to \infty} f(x)$ is not defined

pick $\varepsilon_0 = 1$, $\forall \beta \in \mathbb{R}$ choose $n > \beta$ obtain $x_1 = \frac{\pi}{2} + 2n\pi$

$x_2 = \frac{\pi}{2} + (2n+1)\pi$, $x_1 > \beta$ & $x_2 > \beta$ so f at x_1 & x_2 find

$\left| \frac{\sin x_1}{|x_1|^\alpha} - \frac{\sin x_2}{|x_2|^\alpha} \right| = \left| \left(\frac{\pi}{2} + 2n\pi\right) \right|^c + \left| \left(\frac{\pi}{2} + (2n+1)\pi\right) \right|^c$ $\quad c = -\alpha \geq 0$

but $\left| \left(\frac{\pi}{2} + 2n\pi\right) \right|^c + \left| \left(\frac{\pi}{2} + (2n+1)\pi\right) \right|^c \geq 2 > 1$ conclude limit

doesn't exist and cannot be ∞ {not exist in the extended reals}

by contradic, $M = 1$, $\beta \in \mathbb{R}$ s.t. $f(x) > M$ if $x > \beta$ but $n > \beta$ so $\pi n > \beta$

and $f(\pi n) = \frac{\sin(\pi n)}{|\pi n|^\alpha} = 0 < 1$ similarly limit isn't $-\infty$

2008 Sheet 2

\3iv/ $\forall \varepsilon > 0$ if $x > \beta \Rightarrow e^x \not< e^\beta = \varepsilon$ when $\beta = \ln(\varepsilon)$

so limit is not defined so limit $\lim_{x\to\infty} f(x) = \lim_{x\to\infty} e^x = \infty$

$\forall M$ choose $\beta = \log_e |M|$ and $\forall x > \log_e |M|$ have $f(x) = e^x > M$

hence $\lim_{x\to\infty} e^x = \infty$

$\forall x > \beta$ $e^x > e^\beta = \varepsilon$ ~~when~~ since $\beta = \ln(\varepsilon)$, $\forall \varepsilon > 0$

so $f(x) > \varepsilon$ so $\lim_{x\to\infty} f(x) = \infty$

\5i/ $|x_1 - x_2| > \delta$ $\quad |f(x_1) - f(x_2)| = \left|\frac{1}{x_1} - \frac{1}{x_2}\right| = \left|\frac{x_2 - x_1}{x_1 x_2}\right|$

$= \left|\frac{1}{x_1 x_2}\right| \left|-(x_1 - x_2)\right| = \left|\frac{1}{x_1 x_2}\right| |x_1 - x_2| > \left|\frac{1}{x_1 x_2}\right| \delta = \varepsilon_0$

when $\delta = \varepsilon_0 |x_1 x_2|$ so for $\varepsilon_0 > 0$ so $|f(x_1) - f(x_2)| > \varepsilon_0 > 0$

so $\forall \varepsilon_0$ $|f(x_1) - f(x_2)| > 0$ so $\lim_{x\to 0} \frac{1}{x}$ is undefined

contradic: suppose $\exists L \in \mathbb{R}$ s.t. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon$

if $0 < |x - x_0| < \delta$ hence if $0 < |x_1 - x_0| < \delta$ & $0 < |x_2 - x_0| < \delta \Rightarrow$

$|f(x_1) - f(x_2)| = |(f(x_1) - L) + (L - f(x_2))| \le |f(x_1) - L| + |f(x_2) - L| < \varepsilon + \varepsilon = 2\varepsilon$

$\exists \varepsilon_0$ s.t. $\exists x_1$ & x_2 s.t. $0 < |x_1 - x_0| < \delta$ & $0 < |x_2 - x_0| < \delta$ &

$|f(x_1) - f(x_2)| > \varepsilon_0$ but $\varepsilon = \frac{\varepsilon_0}{2}$ get $|f(x_1) - f(x_2)| < 2\varepsilon = \varepsilon_0$

which is a contradic for deleted δ-neighb of 0 let $x_1 = \left\lceil \frac{1}{\delta} \right\rceil + 1$

& $x_2 = -x_1 \Rightarrow 0 < |x_{1,2}| < \delta$ now $\varepsilon_0 = 1$ have $\left|\frac{1}{x_1} - \frac{1}{x_2}\right| \ge 2 > \varepsilon_0$

\5ii/ ~~$\exists \varepsilon_0$ s.t. $\exists x_1$ &~~ for deleted δ-neighb of 0 let

$x_1 = \cos\left(\frac{1}{\delta}\right) + 1$ & $x_2 = -x_1 \Rightarrow 0 \le |x_{1,2}| < \delta$ now $\varepsilon_0 = 1$

have $\left|\cos\left(\frac{1}{x_1}\right) - \cos\left(\frac{1}{x_2}\right)\right| \ge 2 > \varepsilon_0$

~~$|x_1 - x_2|$~~ $0 < |x_1 - x_2| < \delta$ $|f(x_1) - f(x_2)| = |\cos(\frac{1}{x_1}) - \cos(\frac{1}{x_2})| \leq 2$

$|\cos(\frac{1}{x_1}) - \cos(\frac{1}{x_2})| = |\cos(\frac{1}{x_1}) - L - \cos(\frac{1}{x_2}) + L|$

$\leq |\cos(\frac{1}{x_1}) - L| + |\cos(\frac{1}{x_2}) + L| = \varepsilon + \varepsilon = 2\varepsilon = \varepsilon_0$ when $\frac{\varepsilon_0}{2} = \varepsilon$

deleted δ-neighb of 0 $n = [\frac{1}{\delta}] + 1$ & $x_1 = \frac{1}{n\pi}$ &

$x_2 = \frac{1}{(n+1)\pi}$ $\Rightarrow 0 < |x_{1,2}| < \delta$ now if ε_0 have $|\cos(\frac{1}{x_1}) - \cos(\frac{1}{x_2})| = 2 > \varepsilon_0$

$L \in \mathbb{R}$ $\forall \varepsilon > 0 \exists \delta > 0$ s.t. $|f(x) - L| < \varepsilon$ if $0 < |x - x_0| < \delta$ if

if $0 < |x_1 - x_0| < \delta$ & $0 < |x_2 - x_0| < \delta \Rightarrow$

~~$|f(x_1 - x_0|$~~ $|f(x_1) - f(x_2)| \leq |f(x_1) - L| + |f(x_2) - L| < \varepsilon + \varepsilon = 2\varepsilon$

$\exists \varepsilon_0$ s.t. $\exists x_1$ & x_2 s.t. $0 < |x_1 - x_0| < \delta$ & $0 < |x_2 - x_0| < \delta$ &

$|f(x_1) - f(x_2)| > \varepsilon_0$ for $\varepsilon = \frac{\varepsilon_0}{2}$ get $|f(x_1) - f(x_2)| < 2\varepsilon = \varepsilon_0$

$\int f$ iii/ deleted δ-neighb of 0 $x_1 = [\frac{1}{\delta}] + 1$, $x_2 = -x_1$

$\Rightarrow 0 < |x_{1,2}| < \delta$. $\varepsilon_0 = 1 \Rightarrow |f(x_1) - f(x_2)| = |x - x| = |0| = 0 < \varepsilon_0$

or $|f(x_1) - f(x_2)| = |0 - 0| = |0| = 0 < \varepsilon_0$ so only continuous

at $x = 0$

continuity at 0 $\forall \varepsilon > 0$ & $\forall 0 < |x - 0| < \delta$ with $\delta \in \varepsilon$

$\Rightarrow |f(x) - f(0)| = |f(x)| \leq |x| < \delta = \varepsilon$ hence

$\lim_{x \to 0} f(x) = 0 = f(0)$ & f is continuous at 0 consider $x_0 \neq 0$

given deleted δ-neighb $(x_0 - \delta, x_0 + \delta)$ of x_0 if x_0 is rational

take $x_1 = x_0$ & x_2 as any irrational in $(x_0 - \delta, x_0 + \delta)$

which does exist let $\varepsilon_0 = |x_0|/2 \Rightarrow |f(x_1) - f(x_2)| = |x_0 - 0| > \varepsilon_0$

if x_0 is irrational take $x_2 = x_0$ & take $\varepsilon_0 = |x_0|/2 \Rightarrow$

$\exists x_1$ rational in $(x_0 - \delta, x_0 + \delta)$ with $|x_1| > |x_0|/2$ (which does exist)

$\Rightarrow |f(x_1) - f(x_2)| = |x_1 - 0| > \varepsilon_0$

2008 Sheet 2

\6/ deleted δ-neighb s.t. $\forall x \in (x_0-\delta, x_0+\delta)$, $\exists f(x) \in \mathbb{R}$

So for $x > x_0 \Rightarrow f(x) > f(x_0) \geq \mu \Rightarrow f(x) > \mu$

since continuous

let $\varepsilon = \frac{f(x_0)-\mu}{2} > 0$ by continuity $\exists \delta > 0$ s.t. $\forall x \in (x_0-\delta, x_0+\delta)$

$>\varepsilon_0$ have $f(x) \in (f(x_0)-\varepsilon, f(x_0)+\varepsilon)$ & since

$f(x_0) - \varepsilon > \mu$, $(x_0-\delta, x_0+\delta)$ is a suitable such neighb

also for $f(x_0) < \mu$

\7 i/ $\lim_{x\to 0} \frac{1}{x}$ is undefined so not continuous

$\lim_{x\to 0^-} \frac{1}{x} = -\infty$ so it is continuous from the left

$\lim_{x\to 0^+} \frac{1}{x} = +\infty$ so it is continuous from the right X

not continuous from left or right since its not continuous at 0

\7 ii/ $f(x) \geq 0$ is continuous $\forall x$ so it is cont for all x from right & 2 left

product of $g(x) = x$ & $h(x) = x$ which are cont

\7 iii/ $\lim_{x\to 0} \sin(\frac{1}{x})$ is undefined so

$\sin(\frac{1}{x})$ is not continuous so $f(x)$ is not continuous

not cont from right at 0 $f(0) = 1$ but

$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+} \frac{x}{x} + \frac{x}{x}(1+x)\sin\frac{1}{x} = \lim_{x\to 0^+} 1 + \sin(\frac{1}{x}) + x\sin\frac{1}{x}$

doesn't exist since $\lim_{x\to 0^+} \sin\frac{1}{x}$ doesn't exist

so also not continuous from left at 0

since $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-} \frac{x}{x} - \frac{x}{x}(1+x)\sin\frac{1}{x} = \lim_{x\to 0^-} 1 - \sin\frac{1}{x} + x\sin\frac{1}{x}$ doesnot exist

18/ $\lim_{x \to x_0} \left(\frac{f}{g}\right)(x) = \lim_{x \to x_0} \frac{f(x)}{g(x)} = \frac{\lim_{x \to x_0} f(x)}{\lim_{x \to x_0} g(x)} = \frac{L_1}{L_2}$

and $L_2 \neq 0$ so $\frac{L_1}{L_2}$ can be determined

$\forall (L_1, L_2) \in \mathbb{R} \times (\mathbb{R} \setminus \{0\})$ and $L_1, L_2 = \pm\infty$

case $L_1 = \infty$ & $L_2 \in \mathbb{R} \setminus 0$ suppose $L_2 > 0$ fix $M \in \mathbb{R}$

$\lim_{x \to x_0} f(x) = \infty \Rightarrow \exists\, \delta_1 > 0$ s.t. if $0 < |x - x_0| < \delta_1 \Rightarrow f(x) > M'$

where $M' = \frac{ML_2}{2}$ similarly for $\varepsilon = \frac{L_2}{2}$ $\exists\, \delta_2$ s.t. if

$0 < |x - x_0| < \delta_2 \Rightarrow g(x) > L_2 - \varepsilon = \frac{L_2}{2}$ now let $\delta = \min\{\delta_1, \delta_2\} \Rightarrow$

if $0 < |x - x_0| < \delta$ obtain $\frac{f(x)}{g(x)} > M$ if $L_2 < 0$

same choosing $M' = \frac{ML_2}{2}$ & $\varepsilon = -\frac{L_2}{2}$

19/ if $\lim_{x \to x_0} g(x)$ does not exist but $\lim_{x \to x_0} (f(x) + g(x))$ does exist then if $\lim_{x \to x_0} f(x)$ did exist then $\lim_{x \to x_0} (f(x) + g(x)$ still wouldn't exist so by contradiction in this cas $\lim_{x \to x_0} f(x)$ doesn't exist so by ~~ch~~ $\lim_{x \to x_0} f(x)$ does exist for $\lim_{x \to x_0} (f(x) + g(x))$ to not exist.

or if $\lim_{x \to x_0} f(x)$ doesn't exist then $\lim_{x \to x_0} (f(x) + g(x))$ does exist $\perp$ so $\lim_{x \to x_0} f(x)$ does exist

taking $g = -f \Rightarrow \lim_{x \to x_0} g(x)$ doesn't exist but $\lim_{x \to x_0} (f(x) + g(x)) = \lim_{x \to x_0} 0 = 0$ which exists so by $\perp$ ~~fo~~ f does exist

2008 Sheet 2

\10/ if $f^{-1}((a,b))$ is a closed set $\Rightarrow$ is not continuous

$\exists c \in \mathbb{R}, c \notin f^{-1}((a,b)) \Rightarrow f((a,b))$ is not continuous

$\forall x \in (a,b)$ so by $\perp$

$f^{-1}((a,b))$ must be an open set

let x_0 be in $\mathbb{R}$ for $\varepsilon > 0$ consider ε-neighb

by $(f(x_0)-\varepsilon, f(x_0)+\varepsilon)$ so

$f^{-1}((f(x_0)-\varepsilon, f(x_0)+\varepsilon)) = \{x \in \mathbb{R} \mid f(x) \in (f(x_0)-\varepsilon, f(x_0)+\varepsilon)\}$

is an open set & is coincides with

$\{x \in \mathbb{R} \mid |f(x)-f(x_0)| < \varepsilon\}$ point x_0 belongs to

open set $f^{-1}((f(x_0)-\varepsilon, f(x_0)+\varepsilon))$ hence $\exists \delta > 0$ & a

δ-neighb $(x_0-\delta, x_0+\delta)$ which is contained inside

$f^{-1}((f(x_0)-\varepsilon, f(x_0)+\varepsilon))$ so proved given $x_0 \in \mathbb{R}$ $\forall \varepsilon > 0$

$\exists \delta > 0$ s.t. if $|x-x_0| < \delta \Rightarrow |f(x)-f(x_0)| < \varepsilon$ so f is cont in x_0

\11/ $\psi_T(x) = \begin{cases} 1 & \text{if } x \in T \\ 0 & \text{if } x \in T^c \end{cases}$ interior is neighb in set

if $x_0 \in T^\circ$, $T^\circ \subseteq T \Rightarrow \psi_T((x_0-\delta, x_0+\delta)) = 1$

for δ-neighb about x_0 so at $x_0 \in T^\circ$ $\psi_T(x)$ is cont

if $x_0 \in (T^c)^\circ$, $(T^c)^\circ \subseteq T^c \Rightarrow$ for δ-neighb about x_0 of

$(x_0-\delta, x_0+\delta)$, $\psi_T((x_0-\delta, x_0+\delta)) = 0$

so at $x_0 \in (T^c)^\circ$ $\psi_T(x)$ is continuous

so for $x_0 \in T^\circ \cup (T^c)^\circ$ $\psi_T(x)$ is continuous

if $x_0 \in T^\circ$, $\exists$ a neighb (a,b) of x_0 s.t. $(a,b) \subseteq T$

$\Rightarrow \lim_{x \to x_0} \psi_T(x) = \lim_{x \to x_0} 1 = \psi_T(x_0) = 1$

if $x_0 \in (T^c)^\circ$, $\exists$ a neighb (c,d) of x_0 s.t. $(c,d) \subseteq T^c$

hence $\lim_{x\to x_0} \psi_T(x) = \lim_{x\to x_0} 0 = \psi_T(x_0) = 0$

So proved if $x_0 \in T^\circ \cup (T^c)^\circ \Rightarrow \psi_T$ is cont at x_0

for only if: suppose ψ_T is cont at x_0 ∴ means f is defined in a neighb of x_0 if $x_0 \in T$ have $\psi_T(x_0) = 1$ &

continuity assumption means $\lim_{x\to x_0} \psi_T(x) = 1$

this implies $\exists$ a neighb (a,b) of x_0 s.t. $\psi_T((a,b)) = 1$

suppose by contradic that $\forall$ neighb I of x_0

$\exists$ at least a point $x_1 \in I$ s.t. $\psi_T(x_1) = 0$ & choose

$\varepsilon_1 = \frac{1}{2}$ the existance of the above limit ensures

$\exists\, \delta > 0$ s.t. $|\psi_T(x) - \psi_T(x_0)| < \varepsilon_1 \ \forall x \in (x_0 - \delta, x_0 + \delta)$ but

interval $I = (x_0 - \delta, x_0 + \delta)$ contains x_1 where $\psi_T(x_1) = 0$

at this point $|\psi_T(x) - \psi_T(x_0)| = 1 > \varepsilon_1$ so $\perp$

hence $(a,b) \subseteq T$ & $x_0 \in T^\circ$

if $x_0 \notin T \Rightarrow \lim_{x\to x_0} \psi_T(x) = 0$ so same as above $\Rightarrow$

$\exists$ a neighb (c,d) of x_0 s.t. $(c,d) \subseteq T^c$ hence $x_0 \in (T^c)^\circ$

\12/ continuous at 0 so $f(0) = \lim_{x\to 0} f(x)$

and $\cancel{f(0) = f(0) +}$ $f(0) + f(0+0) = f(0) + f(0) = 2f(0)$

$= f(x-x) = f(x) + f(-x) \quad \forall x \in \mathbb{R}$

$2f(0) = f(0) \Rightarrow f(0) = 0 \Rightarrow f(x) + f(-x) = 0 \Rightarrow f(-x) = -f(x)$

since ~~f(x) continuous at 0 then~~ $f(x_1 + x_2)$ is cont

at 0 then for $f(-x) = -f(x)$, $f(x)$ is continuous

for all $x \in \mathbb{R}$ ~~since &~~ $f(0) = f(0+0) = f(0) + f(0) \Rightarrow f(0) = 0$

continuity means $\lim_{h\to 0} f(h) = f(0) = 0$ arb $x_0 \in \mathbb{R}$

$\lim_{x\to x_0} f(x) = \lim_{h\to 0} f(x_0 + h) = \lim_{h\to 0}[f(x_0) + f(h)] = \lim_{h\to 0} f(x_0) + \lim_{h\to 0} f(h)$

$= f(x_0) + 0 = f(x_0)$ f cont $\forall x_0 \in \mathbb{R}$

2008 Sheet 2

\13i/ $1-x^2 \geq 0 \Rightarrow 1 \geq x^2 \Rightarrow -1 \leq x \leq 1$ so

cont for $x \in [-1, 1]$ $\quad 1-x^2 > 0$ when $-1 < x < 1$

$\sqrt{1-x^2}$ cont on $(-1, 1)$

$\lim_{x \to 1^-} \sqrt{1-x^2} = 0$ & $\lim_{x \to -1^+} \sqrt{1-x^2} = 0$ showing this

\13ii/ $\log(1+\sin x) = \ln(1+\sin x) \Rightarrow 1+\sin x > 0$

$\Rightarrow \sin x > -1 \quad \sin x = -1$ for $\frac{3\pi}{2} + 2\pi(n)$

so continuous for $x \in \mathbb{R} \setminus \left\{ \frac{3\pi}{2} + 2\pi n \mid n \in \mathbb{Z} \right\}$

$1+\sin x > 0$ when $x \neq \frac{3}{2}\pi + 2\pi n$ with $n \in \mathbb{Z}$ follows

$\log(1+\sin x)$ cont on $\bigcup_{n \in \mathbb{Z}} \left(\frac{3}{2}\pi + 2\pi n, \frac{3}{2}\pi + 2\pi(n+1) \right)$

\13iii/ $\frac{-1}{1-2x} \in \mathbb{R} \Rightarrow 1-2x \neq 0 \Rightarrow 1 \neq 2x \Rightarrow \frac{1}{2} \neq x$

so continuous on $x \in (-\infty, \frac{1}{2}) \cup (\frac{1}{2}, +\infty)$

cont on $(-\infty, \frac{1}{2}) \cup (\frac{1}{2}, \infty)$

\14/ so $\lim_{x \to x_0} f(x) = f(x_0)$ so $|f(x_0)| = \left| \lim_{x \to x_0} f(x) \right| = \lim_{x \to x_0} |f(x)|$

so $|f(x_0)|$ is cont if $f(x_0)$ is cont

if $f(x) = \begin{cases} x, & 2 \leq x, \\ -x, & 2 > x \end{cases}$ so $|f(x)| = \begin{cases} |x|, & 2 \leq x, \\ |-x|, & 2 > x \end{cases}$

$= \begin{cases} x, & 2 \leq x, \\ x, & 2 > x \end{cases} = \{ x \mid \forall x \in \mathbb{R} \}$ so $|f(x)|$ is cont at x_0

but $f(x)$ is not

$f(x)$ is cont at x_0, $g(x) = |x|$ is cont on $\mathbb{R}$ so $f(x_0)$ is an interior of $D_g \Rightarrow (g \circ f)(x) = |f(x)|$ is cont at x_0

if $f(x) = \begin{cases} -1 & \text{if } x \geq 0 \\ 1 & \text{if } x < 0 \end{cases}$ not cont at 0 while $|f(x)| = 1$

is cont on $\mathbb{R}$

if $\ln(x)$ $x \in (0, \infty)$

$f(x_1) = \min\limits_{x \in (0,\infty)} f(x)$ $\quad f(x_2) = \max\limits_{x \in (0,\infty)} f(x)$

$x_1 = \lim\limits_{x \to 0} x$, $x_2 = \lim\limits_{x \to \infty} x$ $\quad$ ~~$f(x) = 0$~~

$f(x) = 0 \ \forall x \in (0, \infty)$ min is 0 & max is 0 & attained at any $x \in (0, \infty)$

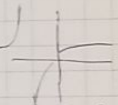
\1 5ii/ $f(\infty) = \infty$ $\quad f(0) = -\infty$

if $f(x) = \ln(x)$ $\quad x > 0$ so $(0, \infty) \subseteq [0, \infty)$

if $\lim\limits_{x \to 0^+} \ln(x) = -\infty$ $\quad \lim\limits_{x \to \infty} \ln(x) = \infty$ so $f(x)$ has

cannot attain its max or min

take $f(x) = \left(-\frac{1}{x+2} + 1\right) \sin x$ $\quad$ {plot $y = \left(-\frac{1}{x+2} + 1\right)$



cont as formed from sums and products of cont fns on $[0, \infty]$ is also bounded on $[0, \infty)$

$\left|\left(-\frac{1}{x+2} + 1\right) \sin x\right| \leq \left|-\frac{1}{x+2} + 1\right| \leq 1$ $\quad f(x) \neq 1$ since $\sin x \leq 1$ &

$-\frac{1}{x+2} + 1 < 1 \ \forall x \in [0, \infty)$ $\quad \lim\limits_{x \to \infty} -\frac{1}{x+2} + 1 = 1 \Rightarrow \forall \varepsilon > 0, \exists \beta \in \mathbb{R}$ s.t. if $x > \beta$

$\Rightarrow -\frac{1}{x+2} > -\varepsilon$ so $-\frac{1}{x+2} + 1 > 1 - \varepsilon$ $\quad$ let $n \in \mathbb{N}$ s.t $n > \beta$ & consider

$x_0 = \frac{\pi}{2} + 2n\pi$ so $x_0 > \beta$ so $f(x_0) = \sin(x_0)\left(-\frac{1}{x_0+2} + 1\right) = \left(-\frac{1}{x_0+2} + 1\right) > 1 - \varepsilon$

$\Rightarrow \exists x_0$ s.t. $f(x_0)$ is arbit close to 1 but $f(x) \neq 1 \ \forall x \in [0, \infty)$

so $f(x)$ doesn't attain a max in $[0, \infty)$ same way

doesn't attain min in $(0, \infty)$

2008 Sheet 2

|16/ $f(x) \neq C$, $\forall x \in \mathbb{R}$. C is a constant

i) $I = (a,b) \Rightarrow f((a,b)) = (f(a), f(b))$ is an interval

if I is closed so $I = [a,b]$ and finite so $a, b \in \mathbb{R}$

so $f([a,b]) = [f(a), f(b)]$

$\alpha = \inf\{f(x) \mid x \in I\}$ $\beta = \sup\{f(x) \mid x \in I\}$

α, β in extended reals $\forall y, \alpha < y < \beta$; $y \in S$

$\Rightarrow (\alpha, \beta) \subseteq S$ hence S can be $[\alpha, \beta]$, $(\alpha, \beta]$, $[\alpha, \beta)$ or (α, β)

let y s.t. $\alpha < y < \beta$ by def of supremum

$\exists y_1 \in S$ s.t. $y_1 > y$ & $y_1 \leq \beta$ since $y_1 \in S$ $\exists x_1 \in I$ s.t.

$f(x_1) = y_1$ by def of infimum $\exists y_2 \in S$ s.t. $y_2 < y$

since $y_2 \in S$ $\exists x_2 \in I$ s.t. $f(x_2) = y_2$ by construction

$f(x_2) < y < f(x_1)$ can suppose $x_1 < x_2$ hence by IVT

$\exists c \in [x_1, x_2]$ s.t. $f(c) = y$ showing $y \in S$

|17/ $f(x) = \begin{cases} x, & x > 0 \\ 0, & x = 0 \end{cases}$ $\lim_{x \to 0} f(x) = 0 = f(0)$

so continuous $\forall x \in [0, \infty)$ but not uniformly cont

|18/ $I = (\alpha, \beta)$ $\alpha, \beta \in \mathbb{R}$ $x_0 \in I$

$\lim_{x \to x_0} f(x) = f(x_0)$ $\lim_{x \to x_0} g(x) = g(x_0)$

so f, g are uniformly cont

so $\lim_{x \to x_0} (\cancel{f\Delta}(f+g)(x)) = \lim_{x \to x_0} f(x) + \lim_{x \to x_0} g(x) = \cancel{f(x) + g(x)}$

$\lim_{x \to x_0} f(x) + g(x_0) + f(x_0) + g(x_0)$ so $f+g$ is uniformly cont

$$\lim_{x \to x_0} (fg)(x) = \lim_{x \to x_0} f(x) \lim_{x \to x_0} g(x) = f(x_0) \lim_{x \to x_0} g(x) = f(x_0) g(x_0)$$

So fg is uniformly cont

2008 Sheet 3

note: $\frac{d}{dx}(x^2e^{7x}\sin x) = \cancel{2xe^{7x}\sin x}$

$2xe^{7x}\sin x + 7x^2e^{7x}\sin x + x^2e^{7x}\cos x$

\1i/ $\frac{d}{dx}(x\sin(x)\sin(\frac{1}{x})) = \frac{d}{dx}[x\sin(x)\sin(x^{-1})] =$

$\sin(x)\sin(\frac{1}{x}) + x\cos(x)\sin(\frac{1}{x}) + x\sin(x)(-1x^{-2})\cos(\frac{1}{x})$

$= \sin(x)\sin(\frac{1}{x}) + x\cos(x)\sin(\frac{1}{x}) - \frac{1}{x}\sin(x)\cos(\frac{1}{x})$

$\lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0} = \lim_{x\to x_0}\frac{x\sin(x)\sin(\frac{1}{\cancel{x_0}}) - x_0\sin(x_0)\sin(\frac{1}{x_0})}{x-x_0}$

for $x_0 = 0$: $x_0\sin(x_0)\sin(\frac{1}{x_0}) = 0\sin(0)\sin(\frac{1}{0})$ is undefined so

$\lim_{x\to 0}\frac{x\sin(x)\sin(\frac{1}{x}) - 0\sin(0)\sin(\frac{1}{0})}{x-0} = \lim_{x\to 0}\frac{f(x)-f(0)}{x-0}$

is undefined so $\cancel{\nexists}$ $\frac{d}{dx}[f_1(x)]$ exists

$\forall x \in (\mathbb{R}\setminus\{0\})$ so $\forall x \in ((-\infty,0)\cup(0,\infty))$

\1ii/ $\frac{d}{dx}(|x|)$ at $x=0$ is undefined

and $\frac{d}{dx}\left(\frac{1}{x-1}\right)$ at $x=1$ is undefined

so $\frac{d}{dx}[f_2(x)]$ is undefined for $x=0$, $x=1$

so $\frac{d}{dx}[f_2(x)]$ exists $\cancel{for}$ $\forall x \in (\mathbb{R}\setminus\{0,1\})$

so $\forall x \in ((-\infty,0)\cup(0,1)\cup(1,\infty))$

note: $\frac{d}{dx}(|x|) = \frac{d}{dx}(\mathrm{sgn}(x)x) = \mathrm{sgn}(x)\frac{d}{dx}(x) = \mathrm{sgn}(x)(1)$

and $|x| = \mathrm{sgn}(x)x \Rightarrow \mathrm{sgn}(x) = \frac{|x|}{x}$

$\Rightarrow \mathrm{sgn}(x)(1) = \mathrm{sgn}(x) = \frac{|x|}{x} = \frac{d}{dx}(|x|)$

So $\frac{d}{dx}\left(|x|\sqrt{x^2-2x+1}\right) = \frac{d}{dx}\left(|x|(x^2-2x+1)^{1/2}\right)$

$= \frac{|x|}{x}\sqrt{x^2-2x+1} + |x|\left(\frac{1}{2}\right)(x^2-2x+1)^{-1/2}$

$\frac{d}{dx}(x-1) = 1 \implies$

$\frac{d}{dx}\left[f_2(x)\right] = \frac{d}{dx}\left(\frac{|x|\sqrt{x^2-2x+1}}{x-1}\right)$

$= \frac{(x-1)\left(\frac{|x|}{x}\sqrt{x^2-2x+1} + |x|\left(\frac{1}{2}\right)(x^2-2x+1)^{-1/2}\right) - |x|\sqrt{x^2-2x+1}}{(x-1)^2}$ (1)

$= \frac{(x-1)\frac{|x|}{x}\sqrt{x^2-2x+1} + \frac{1}{2}|x|(x^2-2x+1)^{-1/2} - |x|\sqrt{x^2-2x+1}}{(x-1)^2}$

for $x \neq 0,\ x \neq 1$

2/ $\frac{d}{dx}(f(x)) = \frac{d}{dx}\left(x^2\sin\left(\frac{1}{x}\right)\right) = \frac{d}{dx}\left(x^2\sin(x^{-1})\right)$

$= 2x\sin\left(\frac{1}{x}\right) + x^2(-1x^{-2})\cos(x^{-1})$

$= 2x\sin\left(\frac{1}{x}\right) - \cos\left(\frac{1}{x}\right)$

~~$f'(x_0) =$~~ for $x = x_0 + h \implies$, $f'(x_0) = \lim_{x\to x_0}\frac{f(x)-f(x_0)}{x-x_0}$

$= \cancel{\lim_{x\to x_0}} \lim_{x_0+h\to x_0}\frac{f(x_0+h)-f(x_0)}{h} = \lim_{h\to 0}\frac{f(x_0+h)-f(x_0)}{h}$

$= \lim_{h\to 0}\frac{(x_0+h)^2\sin\left(\frac{1}{x_0+h}\right) - x_0^2\sin\left(\frac{1}{x_0}\right)}{h}$ which ~~exists for $x_0 \in \mathbb{R}$~~

~~so is differentiable~~ which does approach a limit $\forall x_0 \in \mathbb{R}$ so is differentiable on $\mathbb{R}$

$\lim_{x\to 0}\left(\frac{1}{x}\right)$ is undefined

$\lim_{x\to 0}\sin\left(\frac{1}{x}\right)$ is undefined so $\lim_{x\to 0}x^2\sin\left(\frac{1}{x}\right)$ is undefined

so f' is not continuous at 0

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3i/ $\lim_{x\to 0^-} \sin x = -0 = 0$

$\lim_{x\to 0^-} \frac{1}{x} = -\infty \Rightarrow \lim_{x\to 0^-} \frac{\sin x}{x} = \cancel{\lim_{x\to 0} \frac{\frac{d}{dx}(\sin(x))}{\frac{d}{dx}(x)}}$

$\cancel{= \lim_{x\to 0^-} \frac{\cos(x)}{1} = \frac{1}{1} = 1} \quad = +\infty$

$\lim_{x\to 0^+} \sin x = +0 = 0$, $\lim_{x\to 0^+} \frac{1}{x} = +\infty \Rightarrow \lim_{x\to 0^+} \frac{\sin x}{x} = +\infty$

$\lim_{x\to 0} \frac{\sin x}{x} = \lim_{x\to 0} \frac{\frac{d}{dx}(\sin x)}{\frac{d}{dx}(x)} = \lim_{x\to 0} \frac{\cos x}{1} = \frac{\cos 0}{1} = \frac{1}{1} = 1$

So define $f(0) = 1 \Rightarrow f(x) = \begin{cases} \frac{\sin x}{x}, & x \neq 0 \\ 1, & x = 0 \end{cases}$

3ii/ $f(x) = x^{-1}\sin(x) \Rightarrow \frac{d}{dx}(f(x)) = \frac{d}{dx}(x^{-1}\sin x)$

$= -x^{-2}\sin(x) + x^{-1}\cos(x) = -\frac{\sin x}{x^2} + \frac{\cos x}{x} = 0$

$\Rightarrow -\sin x + x\cos x = 0 \quad \Rightarrow x\cos x = \sin x$

$\Rightarrow x = \frac{\sin x}{\cos x} = \tan x \qquad x \neq 0$

note: $+\frac{\sin(-x)}{-x} = \frac{-\sin(x)}{-x} = +\frac{\sin x}{x} \Rightarrow f(-x) = +f(x)$

so f is symmetrical so local extreme point has a real conjugate so ~~$\bar{x} - 0 = 0 - \bar{x}$~~

$\bar{x}_1 - 0 = 0 - \bar{x}_2$ so $-\bar{x}_1 = \bar{x}_2$ so $|f(\bar{x})| = \frac{1}{\sqrt{(1+\bar{x}^2)}}$

for $\bar{x} = 0$ $\quad |f(0)| = \frac{1}{\sqrt{1+0^2}} = \frac{1}{\sqrt{1}} = \frac{1}{1} = 1$

~~sin~~ $f'(x) = -\frac{\sin x}{x^2} + \frac{\cos x}{x}$

$f(x) = \frac{\sin x}{x} \Rightarrow$ ~~$x \sin$~~ $x f(x) = \sin x \Rightarrow$

$f(x) + x f'(x) = \cos x \qquad \sin^2 x = (x f(x))^2 = x^2 f^2(x)$

$\cos^2(x) = (f + x f')^2 = f^2 + x^2 (f')^2 + 2x f f'$

$\cos^2(x) + \sin^2(x) = 1 = f^2 + x^2 (f'(x))^2 + 2x f f'(x) + x^2 f^2(x)$

$= x^2 (f^2 + (f')^2) + 2x f f'$

\3iii/ $|f(x)| = (1+x^2)^{-1/2}$

Max $f(x) = 1$ ~~lim~~ Min $f(x) > -1$

$\Rightarrow |\text{Min} f(x)| < +1 \Rightarrow |f(x)| \leq 1$

f is symmetrical so $f(-x) = +f(x)$ so all $f(x)$ ~~is~~ attained by $x \in \mathbb{R}$ is allso attained by $-x$

\4i/ $f'(x) = \frac{d}{dx}(x^3+ax+b) = 3x^2+a = 0$

$\Rightarrow x^2 = -\frac{a}{3}$ ~~$\neq 0$~~

$\lim_{x\to+\infty} f(x) = \lim_{x\to\infty}(x^3+ax+b) = (\infty)^3 + a(\infty) + b = \infty^3 + \infty = \infty$

Since $a, b \in \mathbb{R}$ so $a, b \neq \infty$ and $a, b \neq -\infty$

$\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(x^3+ax+b) = (-\infty)^3 + a(-\infty) + b$

$= (-\infty)^3 - a\infty = -\infty$ $\forall a \in \mathbb{R}$ even for $a \leq 0$ or $a \geq 0$

So $\exists x_1 \in \mathbb{R}, f(x_1) < 0$ and $\exists x_2 \in \mathbb{R}, f(x_2) > 0$

$\Rightarrow$ by mean value theorem $\exists C \in \mathbb{R}; f(C) = 0$

\4ii/ $\lim_{x\to-\infty} f(x) = \lim_{x\to-\infty}(x^4+ax+b) = (-\infty)^4 + a(-\infty) + b$

$= (\infty)^4 - a\infty = +\infty$ for $a \leq 0$ and $a \geq 0$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty}(x^4+ax+b) = \infty^4 + a\infty + b = \infty$

$\Rightarrow f(0) = 0^4 + a(0) + b = b$

$\Rightarrow \exists x_1 > b; f(x_1) > f(b)$

$\text{Min}(f(x)) = b$ so $f(x) \geq b$

for $x_1 \in (b, \infty)$ $f(x_1) > f(b)$

for ~~$x_2 \in (-\infty, b)$~~ for $x_2 = b$, $f(x_2) = f(b)$

$\Rightarrow f(x_2) < f(x_1)$ $\exists f(x_3) < b$, $f(x_3) \geq f(x_1) = g(x_1)$

$\{g(x) = f(x)\} \Rightarrow$ by MVT $\exists x_3 < x_4 < x_2; g(x_3) > g(x_4)$ $g(x_2) < g(x_4)$

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and $g(x_4) = g(x_1) \Rightarrow g(\alpha_1) = g(\alpha_2)$

for $\alpha_1 > b$ g is increasing so only one root in $\alpha \in (b, \infty)$ and $g(\alpha_1) = g(\alpha_2)$ so other root is $g(\alpha_2)$ so only 2 roots

\3ii redo/ $f'(x) = -\frac{\sin x}{x^2} + \frac{\cos x}{x}$

$f'(\bar{x}) = 0 = -\frac{\sin(\bar{x})}{(\bar{x})^2} + \frac{\cos\bar{x}}{\bar{x}} \Rightarrow 0 = -\sin\bar{x} + \bar{x}\cos\bar{x}$

$f(\bar{x}) = \frac{\sin\bar{x}}{\bar{x}} \Rightarrow \bar{x}f(\bar{x}) = \sin\bar{x} \Rightarrow$

$0 = -x f(\bar{x}) + \bar{x}\cos\bar{x} \Rightarrow 0 = -f(\bar{x}) + \cos\bar{x}$

$\Rightarrow f(\bar{x}) = \cos\bar{x}$

$\Rightarrow (f(\bar{x}))^2 = \cos^2(\bar{x})$, $\sin^2(\bar{x}) = (\bar{x}f(\bar{x}))^2 = \bar{x}^2(f(\bar{x}))^2$

$\Rightarrow \cos^2(\bar{x}) + \sin^2(\bar{x}) = 1 = (f(\bar{x}))^2 + \bar{x}^2(f(\bar{x}))^2$

$= (f(\bar{x}))^2(1 + \bar{x}^2) \Rightarrow$

$\frac{1}{1+\bar{x}^2} = (f(\bar{x}))^2 \Rightarrow f(\bar{x}) = \pm\sqrt{\frac{1}{1+\bar{x}^2}} = \pm\frac{1}{\sqrt{1+\bar{x}^2}}$

$\Rightarrow |f(\bar{x})| = \left|\pm\frac{1}{\sqrt{1+\bar{x}^2}}\right| = \frac{1}{\sqrt{1+\bar{x}^2}}$

\5/ so $f'(0) \in \mathbb{R} \Rightarrow f(x)$ is continuous at $x = 0$

$f(0+0) = f(0)f(0) = 2f(0) = f(0) \Rightarrow f(0) = 0$

$f(4+0) = f(4)f(0) = f(4)(0) = 0 = f(4)$

$f(x+0) = f(x)f(0) = f(x)0 = 0 = f(x)$

$\Rightarrow \forall x \in \mathbb{R};\ f(x) = 0 \Rightarrow f$ is continuous $\forall x \in \mathbb{R}$

and f exists $\forall x \in \mathbb{R}$ with $f(x) = 0$

and $\frac{d}{dx}(0) = 0 \Rightarrow f'(x) = 0\ \forall x \in \mathbb{R}$ therefore

$f'(x)$ exists $\forall x \in \mathbb{R}$

16/ $\cancel{\lim_{x\to x_0}\frac{f(x)}{g(x)}}$ so $\lim_{x\to x_0} f(x) = f(x_0) = \lim_{x\to x_0} g(x) = g(x_0)$

$\Rightarrow \lim_{x\to x_0}\frac{f(x)}{g(x)} = \lim_{x\to x_0}\frac{f'(x)}{g'(x)}$ {exists since $\lim_{x\to x_0} g'(x)$

$= g'(x_0) \neq x_0$} so $\lim_{x\to x_0}\frac{f'(x)}{g'(x)} = \frac{f'(x_0)}{g'(x_0)}$

17/ GMVT is: if f & g are cont on $[a,b]$

& diffable on $(a,b) \Rightarrow$

$[g(b)-g(a)]f'(c) = [f(b)-f(a)]g'(c)$, $c\in(a,b)$

MVT is: if f is cont on $[a,b]$ & diffable on (a,b)

$\Rightarrow f'(c) = \frac{f(b)-f(a)}{b-a}$ $c\in(a,b)$

$\lim_{x\to x_0} f(x)=0$ & $\lim_{x\to x_0} g(x)=0$ so f & g are cont at x_0

$\lim_{x\to x_0}\frac{f'(x)}{g'(x)}$ exists so f & g are diffable at x_0

so by MVT $f'(x_0) = \frac{f(x_2)-f(x_1)}{x_2-x_1}$ $x_0\in(x_1,x_2)$

and $g'(x_0) = \frac{g(x_4)-g(x_3)}{x_4-x_3}$. $x_0\in(x_3,x_4)$

by GMVT: $[g(x_4)-g(x_3)]f'(x_0) = [f(x_2)-f(x_1)]g'(x_0)$

{$(x_4-x_3)g'(x_0) = g(x_4)-g(x_3)$, $(x_2-x_1)f'(x_0) = f(x_2)-f(x_1)$ }

so $[(x_4-x_3)g'(x_0)]f'(x_0) = [(x_2-x_1)f'(x_0)]g'(x_0)$

$\Rightarrow (x_4-x_3)=(x_2-x_1) \Rightarrow x_3=x_1$, $x_2=x_4 \Rightarrow (x_3,x_4)=(x_1,x_2)$

$\lim_{x\to x_0}\frac{f'(x)}{g'(x)}\in\mathbb{R}$ so $\frac{f'(x_0)}{g'(x_0)}\in\mathbb{R}$ so $\frac{f'(x_0)}{g'(x_0)} = \frac{\left[\frac{f(x_2)-f(x_1)}{x_2-x_1}\right]}{\left[\frac{g(x_2)-g(x_1)}{x_2-x_1}\right]}$

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so $\dfrac{f'(x_0)}{g'(x_0)} = \dfrac{f(x_2)-f(x_1)}{g(x_2)-g(x_1)}$

so by GMVT, $[g(x_2)-g(x_1)]f'(x_0) = [f(x_2)-f(x_1)]g'(x_0)$

gives $\dfrac{f'(x_0)}{g'(x_0)} = \dfrac{f(x_2)-f(x_1)}{g(x_2)-g(x_1)}$ which is in agreement

with MVT so

f & g are cont on $[x_1, x_2]$ and differentiable on (x_1, x_2)

with $x_0 \in (x_1, x_2)$ so

$\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to x_0} \dfrac{f'(x)}{g'(x)} \in \mathbb{R}$ so $\lim\limits_{x \to x_0} \dfrac{f(x)}{g(x)}$ exists also

18/ prove: if $f'(x)=0$ $\forall x \in (a,b)$ then f is const on (a,b)

if f' exists & does not change sign on (a,b) then f is monotonic on (a,b): increasing, nondecreasing, decreasing, nonincreasing as $f'(x)>0$, $f'(x)\geq 0$, $f'(x)<0$, $f'(x)\leq 0$ respectively $\forall x \in (a,b)$

if $|f'(x)| \leq M$ $a<x<b$ then $|f(x)-f(x')| \leq M|x-x'|$
$x, x' \in (a,b)$

MVT is: if f is cont on $[a,b]$ & differentiable on (a,b)
$\Rightarrow f'(c) = \dfrac{f(b)-f(a)}{b-a}$ $c \in (a,b)$

~~18~~/ if $f'(x)=0$ $\forall x \in (a,b) \Rightarrow 0 = \dfrac{f(b)-f(a)}{b-a}$ $\forall x \in (a,b)$

$\Rightarrow 0 = f(b)-f(a) \Rightarrow f(b)=f(a)$ $\forall x \in (a,b)$

so $f(x)$ is constant ~~for~~ $\forall x \in (a,b)$

8ii/ f' exists so $f'(x) \in \mathbb{R}$ $\forall x \in (a,b)$

f' does not change sign in (a,b) so is $f'(x) > 0$

$\forall x(a,b)$ so $f'(x) = \frac{f(b) - f(a)}{b-a} > 0 \Rightarrow f(b) - f(a) > 0$

$\Rightarrow f(b) > f(a)$ ~~and~~ $\forall x \in (a,b)$ and $f'(x) > 0$ $\forall x \in (a,b)$

so $f(x)$ is monotonically increasing $\forall x \in (a,b)$

f' exists so $f'(x) \in \mathbb{R}$ $\forall x \in (a,b)$

f' does not change sign in (a,b) so is $f'(x) \leq 0$

$\forall x(a,b)$ so $f'(x) = \frac{f(b) - f(a)}{b-a} \leq 0 \Rightarrow f(b) - f(a) \leq 0$

$\Rightarrow f(b) \leq f(a)$ $\forall x \in (a,b)$ & $f'(x) \leq 0$ $\forall x \in (a,b)$

so $f(x)$ is monotonically non increasing $\forall x \in (a,b)$

8iii/ $|f'(x)| \leq M$ $\forall x \in (a,b)$ ~~the~~ so ~~for~~ $\forall (x, x') \in (a,b)$

$\left|\frac{f(b) - f(a)}{b-a}\right| \leq M$ ~~and~~ so ~~if~~ ~~for~~ for $x, x' \in (a,b)$

~~$f'(x) = \frac{f(x)}{}$~~ $f'(x) = \frac{f(x) - f(x')}{x - x'}$ ~~&~~ so $|f'(x)| \leq M$

so $\left|\frac{f(x) - f(x')}{x - x'}\right| \leq M$ so $\frac{|f(x) - f(x')|}{|x - x'|} \leq M$

so $\frac{|f(x) - f(x')|}{|x - x'|} |x - x'| \leq M |x - x'|$

so $|f(x) - f(x')| \leq M|x - x'|$ for $x, x' \in (a,b)$ $\square$

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19) $f'(x_0) = 0 \in \mathbb{R}$ so x_0 is a extreme point and f is cont at x_0 and $f''(x_0) > 0$

so for deleted δ neighborhood of $(x_0-\delta, x_0+\delta)$

$f((x_0-\delta, x_0+\delta)) > f(x_0)$ so $f(x_0)$ is a local Minimum for $x_0 \in (x_0-\delta, x_0+\delta)$

if $f''(x_0) < 0$

so δ neighborhood of $(x_0-\delta, x_0+\delta)$

$f((x_0-\delta, x_0+\delta)) < f(x_0)$ so $f(x_0)$ is a local Maximum for $x_0 \in (x_0-\delta, x_0+\delta)$

so the statement is $A \Rightarrow B$ converse is $B \Rightarrow A$

$A \Rightarrow B$ inverse is $\neg A \Rightarrow \neg B$

$A \Rightarrow B$ contrapositive is $\neg B \Rightarrow \neg A$

$A \Rightarrow B$ negation is $A \wedge \neg B$ $(A \Rightarrow \neg B)$ since original is $A \wedge B$

statement is: is $f'(x_0) = 0 \wedge$ is $f''(x_0) > 0$; $Min(f(x)) = x_0$

partial converse is:

is $f''(x_0) \in \mathbb{R}$, $\wedge$ is $Min(f(x)) = x_0$; $f''(x_0) \geq 0$

$Min(f(x)) = x_0$ so $f'(x_0) = 0$ and for δ-neighborhood about x_0 of $(x_0-\delta, x_0+\delta)$; $f((x_0-\delta, x_0+\delta)) \geq f(x_0)$

$\Rightarrow f''(x_0) \geq 0$

$f''(x_0) \in \mathbb{R} \wedge Max(f(x)) = x_0$; $f''(x_0) \leq 0$

$Max(f(x)) = x_0$ so $f'(x_0) = 0$ and for δ-neighborhood about x_0 of $(x_0-\delta, x_0+\delta)$; $f((x_0-\delta, x_0+\delta)) \leq f(x_0)$

$\Rightarrow f''(x_0) \leq 0$

converse ~~is~~ : $f'(x_0) = 0 \wedge f''(x_0) > 0$; $Min(f(x)) = x_0$

is: $Min(f(x)) = x_0$; $f'(x_0) = 0 \wedge f''(x_0) > 0$

~~but is~~

but is ~~α~~ $f(x)=x^4 \Rightarrow f(0)=0^4=0$ so 0 is the local minimum of f so $f(x)\geq 0 \ \forall x \in \mathbb{R}$

$f'(x)=4x^3$ and so $f'(0)=4(0)^3=0$

so for $x_0=0$, $f'(x_0)=0$

but $f''(x)=(4)3x^2=12x^2$

so $f''(x_0)=f''(0)=12(0)^2=0 \not> 0$ so statement is false by counter example

\1©i/ $f'(x_0)>0$ so for $\delta>0$ so $f'(x_0)\in\mathbb{R}$ so $f(x)$ is continuous at x_0

~~so by MVT:~~ IVT for derivatives is:

f is differentiable on $[a,b]$ $f'(a)\neq f'(b)$ ~~& μ~~ &

~~$f'(a)<\mu<f'(b)$~~ μ is inbetween $f'(a)$ & $f'(b)$ then

$f'(c)=\mu$ $c\in(a,b)$

~~so by IVT for derivatives:~~

~~$f'(x_0$~~ IVT is: f is cont on $[a,b]$, $f(a)\neq f(b)$ & μ is between $f(a)$ & $f(b)$ then $f(c)=\mu$ $c\in(a,b)$

so for $\delta>0 \Rightarrow x_0-\delta<x_1<x_0$, $f'(x_0)>0 \Rightarrow$ by IVT:

$f(x_0-\delta)<f(x_1)<f(x_0)$

and for $x_0<x_2<x_0+\delta \Rightarrow f(x_0)<f(x_2)<f(x_0+\delta$

so $f(x_1)<f(x_0)$ for $x_1<x_0$ and $f(x_0)<f(x_2)$ for $x_0<x_2$

so f is increasing at x_0

\1©ii/ for $f(x)=\begin{cases} x^2\sin(\frac{1}{x}) & , x\neq 0 \\ 0 & , x=0 \end{cases}$ so $\delta>0$,

$x_0-\delta<x_1<x_2<x_0+\delta$, for $x_0=0$; $-\delta<x_1<x_2<+\delta$,

$f(x_0)=f(0)=0$ but $\exists x_1\in(-\delta,x_2)$ and

$\exists x_2\in(x_1,+\delta)$; $f(x_1)\geq f(x_2)$ so $f(x_2)\not> f(x_1)$

so ~~f is interval~~ f increasing on interval $(x_0-\delta, x_0+\delta)$

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\\|10iii/ $f'(x_0)>0$ so $f'(x_0)\in\mathbb{R}$ so f exists at x_0 and is continuous and differentiable at x_0 for $x_0\in(a,b)$ and continuous for $x_0\in[a,b]$

so $\delta>0$; $f(x_0)<f(x_0+\delta)$ and $f(x_0-\delta)<f(x_0)$

Since derivative is continuous ~~for~~ at x_0

so $f(x_0-\delta)<f(x_0)<f(x_0+\delta)$. So for $x_0-\delta=a$, $x_0+\delta=b$

f is increasing in ~~th~~ for $x_0\in(x_0-\delta, x_0+\delta)$

VI/ ~~so $f'([a,b])\in\mathbb{R}$~~ so f is continuous in $[a,b]$

so for $x_0\in[a,b]$; $f'(x_0)\notin\mathbb{R}$

by IVT: for $f(a)\neq f(b)$ ~~and μ between $f(a)$ & $f(b)$~~

~~then $f(x_0)=\mu$ for $x_0\in(a,b)$~~

but because f' is discontinuous at x_0

so $\mu\notin\mathbb{R}$ and μ is not between $f(a)$ & $f(b)$

for $f(x_0)=\mu$ so $\lim_{x\to x_0} f(x)=\mu\notin\mathbb{R} \Rightarrow \lim_{x\to x_0} f(x)\notin\mathbb{R}$

so $\lim_{x\to x_0} f(x)$ does not exist so

$\lim_{x\to x_0^+} f(x)$ does not exist & $\lim_{x\to x_0^-} f(x)$ also does not exist

2008 Sheet 4

1/ ~~$\frac{1}{n+1}$~~ is $S_n = \frac{1}{n}$ $\quad \lim_{n\to\infty} S_n = \infty \quad$ so for $S_n = 1+\frac{1}{n+1}$

$\frac{1}{n} \le 1 \quad \therefore 1+\frac{1}{n+1} > \frac{1}{n} \quad \therefore \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right) > \lim_{n\to\infty}\frac{1}{n}$

$\therefore \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right) = \infty$ X sequences not series

$\lim_{n\to\infty} S_n = \lim_{n\to\infty}\left(1+\frac{1}{n+1}\right) = 1+0 = 1 \quad \forall \varepsilon > 0 ; |S_n - 1|$

$= \left|1+\frac{1}{n+1} - 1\right| = \left|\frac{1}{n+1}\right| = \frac{1}{n+1} \le \frac{1}{N+1} < \frac{\varepsilon}{1+\varepsilon} < \varepsilon \quad \forall n \ge N$ with

$N > \frac{1}{\varepsilon} \quad \Big\{ \frac{1}{N} < \frac{1}{\varepsilon}$ for $N > \varepsilon$, $\frac{1}{N+1} < \frac{1}{1+\varepsilon}$ for $N > \varepsilon$

is $N > \frac{1}{\varepsilon}$; $\frac{1}{N} < \frac{\varepsilon}{1} \quad \therefore \frac{1}{N+1} < \frac{\varepsilon}{1+1}$.

~~$\frac{1}{N+1} < \frac{1}{1+\varepsilon} \le \frac{\varepsilon}{1+\varepsilon}$~~ $\quad$ ~~$\frac{1}{N+1} < \frac{1}{\varepsilon+1}$ for $N > \varepsilon$~~ $\Big\}$

1iii/ $\lim_{n\to\infty}\frac{\sin n}{\sqrt{n}} = 0$ by inspection $\quad |S_n - 0| =$

$\left|\frac{\sin n}{\sqrt{n}} - 0\right| = \left|\frac{\sin n}{\sqrt{n}}\right| \le \frac{1}{\sqrt{n}} \le \frac{1}{\sqrt{N}} < \varepsilon$

$\forall n \ge N$ with $\frac{1}{\varepsilon} < \sqrt{N}$ with so $\frac{1}{\varepsilon^2} < N$

1iv/ $\lim_{n\to\infty} S_n = \lim_{n\to\infty}\frac{2n^2}{1-n^2} = \cancel{\frac{2n^2}{1-1}} \lim_{n\to\infty}\left(\frac{2n^2}{1-n^2} \div \frac{n^2}{n^2}\right)$

$= \lim_{n\to\infty}\left(\frac{(2n^2/n^2)}{((1-n^2/n^2)}\right) = \lim_{n\to\infty}\frac{2}{\left(\frac{1}{n^2}-1\right)} = \frac{2}{0-1} = -2$ by inspection

$\forall \varepsilon > 0$

$|S_n - (-2)| = |S_n + 2| = \left|\frac{2n^2}{1-n^2} + 2\right| \cancel{\le \left|\frac{2n^2}{1-n^2}\right| + |2|}$

$= \left|\frac{2n^2+2-2n^2}{1-n^2}\right| = \left|\frac{2}{1-n^2}\right| < \left|\frac{1}{1-n^2}\right| \le \left|\frac{1}{n^2}\right|$ $\{$since $|1-n^2| \le |n^2|\}$

$\le \frac{1}{n^2} \le \frac{1}{N^2} \le \frac{1}{N} < \varepsilon \quad \forall n \ge N$

with $N > \frac{1}{\varepsilon} \quad \left|\frac{2}{1-n^2}\right| = \left|-\left(\frac{2}{1-n^2}\right)\right| = \left|\frac{2}{n^2-1}\right| = \frac{2}{n^2-1} < \frac{1}{(n+1)(n-1)}$

$< \frac{1}{(n-1)^2} < \frac{1}{(N-1)^2} < \varepsilon \quad \forall n \ge N$ with $\left\{N-1 > \frac{1}{\sqrt{\varepsilon}}\right\}$ $N > \frac{1}{\sqrt{\varepsilon}} + 1$

2008 Sheet 4

1iii/ $\lim_{n\to\infty} s_n = \lim_{n\to\infty} \frac{2n^2}{n^2+1} = \cancel{2} \lim_{n\to\infty} \frac{2}{1+\left(\frac{1}{n^2}\right)} = \frac{2}{1+0} = \frac{2}{1} = 2$

by inspection

~~for~~ $\forall \varepsilon > 0 \; \exists N$; $|s_n - 2| = \left|\frac{2n^2}{1+n^2} - 2\right| = \cancel{\left|\frac{2n^2+2+2n^2}{1+n^2}\right|}$

$= \left|\frac{2n^2 - 2 - 2n^2}{1+n^2}\right| = \left|\frac{-2}{1+n^2}\right| = \left|\frac{2}{1+n^2}\right| = \frac{2}{1+n^2} \leq \frac{2}{1+N^2} < \cancel{2}\,\varepsilon$

$\forall n \geq N$ with $1+N^2 > \frac{2}{\varepsilon} \therefore N^2 > \frac{2}{\varepsilon} - 1 \quad N > \sqrt{\frac{2}{\varepsilon} - 1}$

or $\frac{2}{1+N^2} \cancel{\leq \frac{2}{(1+N)^2} \leq \varepsilon} < \cancel{\sqrt{\frac{2}{N^2}}} < \frac{2}{N^2} < \varepsilon \quad \forall n \geq N$ with

$N^2 > \frac{2}{\varepsilon} \therefore N > \sqrt{\frac{2}{\varepsilon}}$ $\quad \left\{ \frac{2}{n^2+1} < \frac{2}{n^2} < \frac{1}{N^2} < \varepsilon \; \forall n \geq N \text{ with} \right.$

$\left. N > \frac{1}{\sqrt{\varepsilon}} \right\}$

2i/ $\lim_{n\to\infty} \sqrt{\ln(n)} = \infty$ by inspection

$\forall \varepsilon > 0 \; \exists \quad \lim_{n\to\infty} \ln(n) = \infty \quad \therefore \lim_{n\to\infty} s_n = \infty \qquad \forall \delta > 0$

$\forall \varepsilon > 0 \; |s_n - \delta| = |\sqrt{\ln(n)} - \delta| \leq |\sqrt{\ln(n)}| + |-\delta| = |\sqrt{\ln(n)}| + |\delta|$

$= \sqrt{\ln(n)} + \delta \leq \cancel{\sqrt{\ln(N)}} \geq \sqrt{\ln(N)} + \delta \geq \varepsilon \quad \cancel{\text{for}} \; \forall n \geq N$

with $\sqrt{\ln(N)} < \varepsilon - \delta \therefore \ln(N) < (\varepsilon - \delta)^2 \therefore N = e^{(\varepsilon-\delta)^2}$

$\therefore \sqrt{\ln(n)} + \delta > \varepsilon$ for large enough n $\lim_{n\to\infty} s_n = \delta \therefore$

$\lim_{n\to\infty} s_n = \infty$ Sol: $\forall M \in \mathbb{R} \quad \sqrt{\ln(n)} > \sqrt{\ln(N)} > M$

$\forall n \geq N$ with $N > e^{M^2}$ hence $\lim_{n\to\infty} s_n = \infty$

2ii/ $\lim_{n\to\infty} s_n$ is undesired as it oscillates by inspection

$\forall M \in \mathbb{R} \quad \cancel{\lim_{n\to\infty} \sin(} \qquad \left|\sin\left(\frac{n\pi}{2}\right)\right| \leq \sin\left(\frac{n\pi}{2}\right)$

~~for~~ $\sin\left(\frac{n\pi}{2}\right) \geq \sin\left(\frac{N\pi}{2}\right) \qquad \sin\left(\frac{n\pi}{2}\right) = \sin\left(\frac{(n+4)\pi}{2}\right) =$

$\cancel{\sin\left(\frac{n\pi}{2} + \frac{2\pi}{2}\right) = \sin\left(\frac{n\pi}{2} + \pi\right) = \sin\left(\frac{(n+2)\pi}{2}\right)} \quad \sin\left(\frac{n\pi + 4\pi}{2}\right)$

$\cancel{\sin\left(\frac{n+4\pi}{}\right)} = \sin\left(\frac{n\pi}{2} + 2\pi\right) \therefore$ oscillates

5.1: $\sin\left(\frac{n\pi}{2}\right) = \begin{cases} 0, & n=2k, \\ 1, & n=4k+1, \\ -1, & n=4k+3 \end{cases}$ hence we have three

convergent ~~sequences~~ subsequences with different lims & ∴ S_n cannot converge since S_n is bounded by 1 it does not diverge to $\pm\infty$ & ∴ oscillates

\2iii/ there are infinite primes bigger than 0 ∴ infinite $S_n(n) = n$ and $S_n(n) = 0$

∴ S_n have ~~two~~ one convergent subsequence $S_n = \{0$, if n is prime. and one divergence subsequence $S_n = \{n$, if n is not prime which diverges to infinity ∴ S_n cannot converge ∴ diverges bounded below 0 ∴ doesnt diverge to $-\infty$ unbounded above doesnt diverge to ∞ since $\forall M>0$ & $N>0$ $\exists\, n \geq N$ s.t n is prime & ∴ $S_n = 0 < M$ ∴ sequence oscillates

\3i/ for $n=2k$ $\lim_{k\to\infty} S_{2k} = \lim_{k\to\infty} (-1)^{2k}(2k) = \lim_{k\to\infty} 1^k(2k)$

$= \lim_{k\to\infty}(2k) = \infty$ so $\lim_{k\to\infty} S_{2k} = \infty = L_{2k}$

for $n=2k-1$ $\lim_{k\to\infty} S_{2k-1} = \lim_{k\to\infty} (-1)^{2k-1}\cancel{(2k)} = \lim_{k\to\infty}(-1)(2k-1)$

$= -\infty = L_{2k-1}$, $\lim_{k\to\infty} S_{2k} = \infty$ $\lim_{k\to\infty} S_{2k+1} = -\infty$

\3ii/ $\cos\left(\frac{(2k-1)\pi}{2}\right) = 0$; $\cos\left(\frac{(4k-2)\pi}{2}\right) = -1$

$\cos\left(\frac{(4k)\pi}{2}\right) = 1$ ∴ $\lim_{k\to\infty} S_{2k-1} = 0$, $\lim_{k\to\infty} S_{4k} = 1$,

$\lim_{k\to\infty} S_{4k-2} = -1$

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\3iii/ $\lim_{n\to\infty} \frac{1}{n} = 0 = \lim_{n\to\infty} S_n$ $\therefore \lim_{k\to\infty} S_{n_k} = 0$ is convergent

~~for~~ \4i/ $\therefore \exists n_1 < n_2$ s.t. $x_{n_1} < x_{n_2}$

$\therefore$ Subsequence $\{x_{n_2}\}$ has $\lim_{k\to 2} x_{n_2} = \infty$

but x_{n_2} is arbit $\therefore n_2$ is arbit $\therefore 2$ is arbit $\therefore$

$\exists \{x_{n_k}\}; \lim_{k\to\infty} x_{n_k} = \infty$ since $\{x_n\}$ is sequence of reals unbounded above $\therefore \forall j \in \mathbb{N}\ \exists n_j$ s.t. $x_{n_j} > j$ choose subsequence n_t of n_j st. $\{x_{n_t}\}$ is increasing & another subsequence n_k of n_t s.t. n_k is increasing now $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ and $\forall M \in \mathbb{R}$ $\exists K \in \mathbb{N}$ s.t $K > M$ & $x_{n_K} > K > M$, $\forall k > K \therefore n_k > n_K$ & $x_{n_k} > x_{n_K} > M$ $\therefore \lim_{k\to\infty} x_{n_k} = \infty$

\4ii/ since $\{x_n\}$ is sequence of real numbers unbounded below, then $\forall j \in \mathbb{N}, \exists n_j; x_{n_j} < -j$ choose a subsequence n_t of n_j s.t. $\{x_{n_t}\}$ is decreasing & another subsequence n_k of n_t s.t. n_k is decreasing now $\{x_{n_k}$ is a subsequence of $\{x_n\}$ & $\forall M \in \mathbb{R}, \exists K \in \mathbb{N}$ s.t $K < M$ & $x_{n_K} < K < M$ & $\forall k > K \Rightarrow n_k < n_K$ & $x_{n_k} < x_{n_K} < M$ $\therefore \lim_{k\to\infty} x_{n_k} = -\infty$

\5/ $S_n = \begin{cases} M, & n = 2k, \\ 0, & n = 2k+1 \end{cases}$

subsequence $n = 2k$ so S_{2k} has $\lim_{k\to\infty} S_{2k} = M$

but $\lim_{n\to\infty} S_n \neq M$ Sol: $\{S_n\} = \{1, 1, 2, 1, 2, 3, 1, 2, 3, 4, 1, 2, 3, 4, 5, \ldots\}$

$\forall M \in \mathbb{N}$ set $n_k = M + \sum_{i=M-1}^{k} i$ $\therefore$ subsequence $S_{n_k} = M$ $\forall k \in \mathbb{N}$ & $\therefore \lim_{k\to\infty} S_{n_k} = M$

\6i/ $\{S_n\}$ is Cauchy $\therefore$ $\forall \varepsilon>0, \exists N\in\mathbb{N}; (n\geq N \wedge m\geq N),$

$(n,m\geq N); |S_n - S_m| < \varepsilon$

$\therefore 0 < |S_n - S_m| < \varepsilon > 0 \quad \forall \varepsilon > 0$

$\therefore \lim_{N\to\infty} |S_n - S_m| = 0 \quad \therefore \lim_{n\to\infty} S_n = \lim_{m\to\infty} S_m$ so $S_n - S_m \to 0$ as $N\to\infty$

so as $N\to\infty$, $S_n \to S_m$ $\therefore$ $S_n < \infty$ $\forall N\in\mathbb{N}$ $\therefore$ S_n is bounded

$\forall \varepsilon>0, \exists N\in\mathbb{N}$ s.t if $m,n\geq N$; $|S_n - S_m| < \varepsilon$ for $\varepsilon = 1$

obtain that $\forall n\geq N$ $|S_n| = |S_n - S_N + S_N| \leq |S_n - S_N| + |S_N|$

$< 1 + |S_N|$ $\therefore$ ~~if M = max~~ $|S_n| < 1 + |S_n|$

$\therefore$ if $M = \max\{|S_1|, \dots, |S_N|, 1+|S_N|\}$ then $|S_n| \leq$

$M = 1 + |S_N|$ $\therefore$ $|S_N| \leq M$ $\forall n\in\mathbb{N}$ $\therefore$ $\{S_n\}$ is bounded

\6ii/ $\{S_n\}$ is bounded so $\forall n\in\mathbb{N}$ $|S_n| \leq M$

$\therefore M = 1 + |S_N|$ for $M = \max\{|S_1|, \dots, |S_N|, 1+|S_N|\}$

so $|S_N| < 1 + |S_N|$ because $1 + |S_N| \leq |S_N| + |S_m - S_N|$

$= |S_n - S_N + S_N| = |S_n|$ for $\forall n \geq N$ for $\varepsilon = 1$ so $|S_n - S_m| < \varepsilon$

so $\{S_n\}$ is cauchy

so $\forall N\in\mathbb{N}$ $|S_N| < M+1$ where $\max\{|S_1|, \dots, |S_N|, 1+|S_N|\} = M$

$\therefore |S_N| < 1 + |S_N|$ let $\varepsilon > 0$ $\therefore$ $\exists N$ s.t. if $n\geq N, m\geq N$

$\Rightarrow |S_n - S_m| < \varepsilon/2$ $\quad |S_n - S_m| = |(S_n - S_N) - (S_m - S)|$

$\leq |S_n - S_N| + |S_m - S_N| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$ $\quad\therefore$ $\{S_n\}$ is cauchy

Sol: no, its not true. eg $\{S_n\} = \{(-1)^n\}$ is bounded by 1 but it is not Cauchy because it is not convergent

\7/ So $\{S_n\}$ and $\{t_n\}$ are Cauchy so $\lim_{n\to\infty} S_n = S$

$\lim_{n\to\infty} t_n = t$ so they are convergent so

$\lim_{n\to\infty}(t_n + S_n) = \lim_{n\to\infty} S_n + \lim_{n\to\infty} t_n = S + t$ $\therefore$ $\{S_n + t_n\}$ is convergent

$\therefore$ $\{S_n + t_n\}$ is Cauchy $\varepsilon > 0$. $\exists N_1 \in \mathbb{N}$ s.t. $\forall n, m \geq N_1$;

$|S_n - S_m| < \varepsilon/2$ & $\exists N_2 \in \mathbb{N}$ s.t. $\forall n,m \geq N_2$; $|t_n - t_m| < \varepsilon/2$ choose

$N = \max\{N_1, N_2\}$ $\forall m,n\geq N$ $|S_n + t_n - S_m - t_m| = |(S_n - S_m) - (t_n - t_m)| \leq |S_n - S_m| + |t_n - t_m|$

$< \varepsilon/2 + \varepsilon/2 + \varepsilon$

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18/ ~~~~ $\{S_n\}$ Cauchy so ~~$\exists_n$~~ $\lim_{n\to\infty} S_n = S$ since

$\{S_n\}$ converges $\lim_{n\to\infty} \frac{S_n + S_{n-1}}{2} = \frac{S+S}{2} = \frac{2S}{2} = S$

$\lim_{n\to\infty} S_{n+1} = S$ $\therefore \{S_n\}$ converges if S_n is bounded then S_n converges note if $S_0 = S_1$ then have constant sequence & convergence is trivial

let $0<r<1$ & $S_{n+1} = r(S_n + S_{n-1})$, $n \geq 1$ have $S_{n+1} - S_n$

$= r(S_{n-1} - S_n)$ $\{ S_{n+1} = r(S_n + S_{n-1}) \therefore S_{n+1} - S_n = r(S_n + S_{n-1}) - S_n$

$= rS_n + rS_{n-1} - S_n = S_n(r-1) + rS_{n-1}$ $\quad r(S_{n-1} - S_n) = r(S_{n-1}) - rS_n$

$S_{n+1} - S_n + rS_n = rS_{n-1} = S_{n+1} + (r-1)S_n \}$

$S_{n+1} = r(S_n + S_{n-1}) \therefore S_{n+1} - S_n = r(S_{n-1} - S_n) = (r)r(S_{n-1} - S_{n-2})$

$= \cdots = (-r)^n (S_1 - S_0) \therefore |S_m - S_n| \leq |S_m - S_{m-1}| + |S_{m-1} - S_{m-2}| + \cdots +$

$|S_{n+1} - S_n| = \sum_{i=n}^{m-1} |(-r)^i (S_1 - S_0)| = |S_1 - S_0| \sum_{i=1}^{m-1} r^i = |S_1 - S_0| \left(\frac{1-r^m}{1-r} - \frac{1-r^n}{1-r} \right)$

$= |S_1 - S_0| \left(\frac{r^n(1-r^{m-n})}{1-r} \right) \leq |S_1 - S_0| \frac{r^n}{1-r}$ since $\lim_{n\to\infty} r^n = 0$ $\forall \varepsilon > 0$

$\exists N$ s.t. $r^n < \frac{\varepsilon(1-r)}{|S_1 - S_0|}$ $\forall n \geq N$ $\therefore \forall m \geq n > N$; $|S_m - S_n| \leq$

$|S_1 - S_0| \frac{r^n}{1-r} < \varepsilon$ $\therefore$ Cauchy

19/ $S_{n+1} = 1/((1/S_n)+1) = \frac{1}{\left(\frac{1}{S_n} + 1\right)}$ Cauchy's convergence criterion for series: a series $\sum a_n$ converges if & only if for every $\varepsilon > 0$ there is an integer N s.t.

$|a_n + a_{n+1} + \cdots a_m| < \varepsilon$ if $m \geq n \geq N$

so a series $\sum a_n$ converges iff $\forall \varepsilon > 0$ $\exists N$;

$|a_n + a_{n+1} + \cdots + a_m| < \varepsilon$ if $m \geq n \geq N$

note if $\sum_{n=1}^{\infty} \frac{1}{n} = a = \frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \cdots$

$\therefore \frac{a}{2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \cdots$

& $\therefore \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \cdots > a/2$

because $1 > \frac{1}{2}$

so $\frac{a}{2} = \frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \cdots$

& $\quad \frac{1}{1} + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13} + \cdots = b$

because $\frac{1}{1} > \frac{1}{2}$, $\frac{1}{3} > \frac{1}{4}$, $\frac{1}{5} > \frac{1}{6}$

so $\frac{a}{2} = \sum_{n=1}^{\infty} \frac{1}{2n}$ and $b = \sum_{n=1}^{\infty} \frac{1}{2n-1}$

and since $\frac{1}{2n-1} > \frac{1}{2n}$ $\forall n \in \mathbb{N}$ $\therefore$ $b > a/2$

but $a = \sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} \frac{1}{2n} + \sum_{n=1}^{\infty} \frac{1}{2n-1} = \frac{a}{2} + b$

$\therefore$ $a - \frac{a}{2} = b = \frac{a}{2}$ but $b > \frac{a}{2}$ $\therefore$

$a = \frac{a}{2} + \frac{a}{2} > a$ $\therefore$ $\cancel{\Sigma}$ $\perp$ so $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges to ∞

$\therefore \forall \varepsilon > 0$ $\quad \sum_{n=1}^{\infty} \frac{1}{n} = \sum_{n=1}^{\infty} S_n$ for $\{S_n\} = \left\{\frac{1}{n}\right\}$

for $m \geq n \geq N$

$|S_1 + S_2 + \cdots + S_m| = \left|\frac{1}{1} + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{m}\right| > \varepsilon$ $\therefore$

by Cauchys convergence criterion $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges

Sol: show $\exists \varepsilon_0 > 0$ s.t $\forall N \in \mathbb{N}$, $\exists m \geq n \geq N$ s.t.

$\left|\frac{1}{n} + \frac{1}{n+1} + \cdots + \frac{1}{m}\right| \geq \varepsilon_0$ let $\varepsilon_0 = \frac{1}{2}$ then given N let

$n = N+1$ & $m = 2N$ then $\sum_{i=n}^{m} \frac{1}{i} \geq$

$\frac{1}{n} + \frac{1}{n+1} + \frac{1}{n+2} + \cdots + \frac{1}{m} \geq N \cdot \frac{1}{2N} \left\{= (n-1) \cdot \frac{1}{m}\right\} = \frac{1}{2} = \varepsilon_0$

(there are N terms in the sum $\{m-(n-1) = 2N-(N+1-1) = 2N-N = N\}$

each less then or equal to $\frac{1}{m} = \frac{1}{2N}$) $\{$Arz

$\{$ $m = \text{Max}\{n, n+1, \ldots, m\}$ $\therefore$ $\frac{1}{m} = \min\left\{\frac{1}{n}, \frac{1}{n+1}, \ldots, \frac{1}{m}\right\}$

$\therefore$ $\frac{1}{m} < \frac{1}{n+1} < \frac{1}{n}\}$ hence the haromin series does not satisfy the Cauchy Convergence criterion and there fore diverges

2008 Sheet 4

\10/ is $\lim_{n\to\infty}(a_n + a_{n+1} + \ldots + a_{n+r}) \neq 0$, $r \geq 0$

$\Rightarrow \lim_{n\to\infty} a_n \neq 0 \therefore$ ~~$\sum_{n\to\infty}$~~ ~~$\sum_{n=1}^{\infty} a_n \Rightarrow$~~ $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{n-1} a_n + \sum_{n=n}^{\infty} a_n$

~~$= \sum_{n=1}^{n-1} a_n \pm \beta \geq \pm\beta$~~

$= \sum_{n=1}^{n-1} a_n + \beta$ for $\sum_{n=1}^{n-1} a_n \in \mathbb{R}$ for $n \in \mathbb{N}$ and $\beta \notin \mathbb{R}$

$\therefore \sum_{n=1}^{n-1} a_n + \beta \notin \mathbb{R} \therefore \sum_{n=1}^{\infty} a_n \notin \mathbb{R}$ is $\lim_{n\to\infty}(a_n + a_{n+1} + \ldots + a_{n+r}) \neq 0$

~~$\therefore$ to converge~~ $\therefore \perp \therefore$ to converge ~~lim~~

$\lim_{n\to\infty}(a_n + a_{n+1} + \ldots + a_{n+r}) = 0$, $r \geq 0$ must be true

the converse is: is $\lim_{n\to\infty}(a_n + a_{n+1} + \ldots + a_{n+r}) = 0$, $r \geq 0$ then $\sum a_n$ converges but

for $a_n = \frac{1}{n}$ then $\sum_{n=1}^{\infty} a_n = \sum_{n=1}^{\infty} \frac{1}{n} = \infty$ so $\sum_{n=1}^{\infty} a_n$ diverges

so $\sum_{n=1}^{\infty} a_n$ does not converge even though

$\lim_{n\to\infty} a_n = \lim_{n\to\infty} \frac{1}{n} = 0 \therefore \lim_{n\to\infty}(a_n + a_{n+1} + \ldots + a_{n+r}) = 0$, $r \geq 0$

therefore this statement is false $\therefore$ the converse of the original statement in question \10/ is false

~~Sol~~: by Cauchy convergence criterion for series is $\sum a_n$ converges then $\forall \varepsilon > 0 . \exists N \in \mathbb{N}$ s.t $\forall m \geq n \geq N$

$|a_n + a_{n+1} + \ldots + a_m| < \varepsilon \therefore$ setting $m = n + r$ (r fixed) have $\forall \varepsilon > 0, \exists N \in \mathbb{N}$ s.t. $\forall n \geq N$ $|a_n + a_{n+1} + \ldots + a_{n+r}| < \varepsilon$

the converse is false ex let $a_n = \frac{1}{n}$ so harmonic series

$\therefore \forall \varepsilon > 0$ choose $N > \frac{r+1}{\varepsilon} \therefore \forall n \geq N$ have

$a_n + a_{n+1} + \ldots + a_{n+r} = \frac{1}{n} + \frac{1}{n+1} + \ldots + \frac{1}{n+r} < (r+1)\frac{1}{n}$ $\left\{\text{since } \frac{1}{n+r} < \frac{1}{n}\right\}$

$\leq (r+1)\frac{1}{N} < \varepsilon \therefore N > \frac{r+1}{\varepsilon} \therefore \lim_{n\to\infty}(a_n + a_{n+1} + \ldots + a_{n+r}) = 0$ but we know the harmonic series diverges

2008 Sheet 5

\1i/ $f_1(x) = \frac{1}{1+1x} = \frac{1}{1+x}$ $\quad f_2(x) = \frac{1}{1+2x}$ $\quad f_3(x) = \frac{1}{1+3x}$

$f_n(x) = \frac{1}{1+nx}$ $\therefore \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{1}{1+nx} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)+x} = \lim_{n\to\infty} \frac{\left(\frac{1}{n}\right)}{x} = 0$

$= \lim_{n\to\infty} \frac{1}{nx} = 0 \;\; \forall x \in [0,1]$

$\therefore f_n(x) = \frac{1}{1+nx}$, $f(x) = 0$ $\therefore \|F_n(x) - F(x)\|_{[0,1]}$

$= \sup_{x\in[0,1]} |F_n(x) - F(x)| = \sup_{x\in[0,1]} \left|\frac{1}{1+nx} - 0\right| = \cancel{\sup}$

$\sup_{x\in[0,1]} \left|\frac{1}{1+nx}\right| = \sup_{x\in[0,1]} \left(\frac{1}{1+nx}\right) = \sup\left[\frac{1}{1+n1}, \frac{1}{1+(n)(0)}\right] = \sup\left[\frac{1}{1+n}, 1\right]$

$= 1$

$\lim_{n\to\infty} \|f_n(x) - f(x)\|_{[0,1]} = \lim_{n\to\infty} \left\|\frac{1}{1+nx} - 0\right\|_{[0,1]} = \lim_{n\to\infty} \left\|\frac{1}{1+nx}\right\|_{[0,1]}$

$= \lim_{n\to\infty} \sup_{x\in[0,1]} \left|\frac{1}{1+nx}\right| = \lim_{n\to\infty} \sup\left\{\left|\frac{1}{1+nx}\right| \;\middle|\; x\in[0,1]\right\}$

$= \lim_{n\to\infty} \sup\left\{\frac{1}{1+nx} \;\middle|\; x \in [0,1]\right\} = \lim_{n\to\infty} \sup\left[\frac{1}{1+n(1)}, \frac{1}{1+n(0)}\right]$

$= \lim_{n\to\infty} \sup\left[\frac{1}{1+n}, 1\right] = \lim_{n\to\infty} (1) = 1 \neq 0$ $\therefore$ the sequence $\{f_n\}$ of functions defined on a set $[0,1]$ does not converge uniformly to the limit function

note $\lim_{n\to\infty} \sup\left[\frac{1}{1+n}, \frac{1}{1+np}\right] = \lim_{n\to\infty} \left(\frac{1}{1+np}\right) = 0 \;\; \forall p \in \mathbb{R}_{\geq 0}$

~~for~~ for $0 < p < 1$ $\therefore x \in [p,1]$ $\therefore \lim_{n\to\infty} \sup\left\{\frac{1}{1+nx} \;\middle|\; x \in [p,1]\right\}$

$= \lim_{n\to\infty} \sup\left[\frac{1}{1+n}, \frac{1}{1+np}\right] = \lim_{n\to\infty} \frac{1}{1+np} = 0$ so the largest subdomain on which $\{f_n\}$ uniformly converges is $x \in [p,1]$ for $0 < p < 1$ so $x \in (0,1]$

<u>Sol:</u> if $x = 0$ $\therefore f_n(x) = 1 \;\forall n$ while for each $x \in (0,1)$ & $\forall \varepsilon > 0$

$\left|\frac{1}{1+nx}\right| \leq \frac{1}{1+Nx} = \frac{\varepsilon}{1+\varepsilon} < \varepsilon \;\; \forall n \geq N$ with $N > \frac{1}{x\varepsilon}$ $\Big\{ 1+\varepsilon = \varepsilon + \varepsilon N x$

$\therefore 1 = \varepsilon N x \therefore \frac{1}{\varepsilon} = Nx \Big\}$ $\therefore \lim_{n\to\infty} f_n(x) = 0$ $\therefore f_n(x)$ converges pointwise on $[0,1]$

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1) a) the function $f = \begin{cases} 1 & \text{if } x=0 \\ 0 & x \in (0,1] \end{cases}$

i) if 2 $\begin{cases} \therefore \text{ if } x=0 \quad f_n(x) = \frac{1}{1+n(0)} = \frac{1}{1} = 1 \quad \therefore \text{ for } \varepsilon > 0 \end{cases}$

& if $x \in (0,1]$ & $\left|\frac{1}{nx}\right| = \frac{1}{1+nx} \leq \frac{1}{Nx+1} < \frac{\varepsilon}{1+\varepsilon} < \varepsilon$

for $1+\varepsilon = \varepsilon Nx + \varepsilon \therefore 1 = \varepsilon Nx \therefore$ if $\frac{1}{1+Nx} < \frac{\varepsilon}{1+\varepsilon} \therefore 1+\varepsilon < \varepsilon + Nx\varepsilon$

$\therefore 1 < \varepsilon Nx \therefore N > \frac{1}{\varepsilon x}$ for $\forall n \geq N$ and $N > \frac{1}{\varepsilon x} \therefore$

$\lim_{n\to\infty} f_n(x) = 0 \therefore f_n(x)$ pointwise $x \in [0,1] \therefore f(x) = \left\{ \begin{matrix} 1, & x=0 \\ 0, & x \in (0,1] \end{matrix} \right\}$

ii) sequence cannot converge uniformly on $[0,1]$ because a uniform limit of continuous funcs is cont but f_n is cont $\forall n$ but the limit f is not cont

$f_n(0) = f(0) = 1 \ \forall n$ so $\{f_n\}$ uniformly converges at $x=0$ if interval is truncated to $[a,1]$ for $0<a<1$ then let $S = \{0\} \cup [a,1]$ since we have uniform conver for $[a,1]$ also $\therefore \|f_n - f\|_S = \left|\frac{1}{1+na}\right| \leq \frac{1}{1+Na} = \frac{\varepsilon}{1+\varepsilon} < \varepsilon$

$\forall n \geq N$ with $N > \frac{1}{a\varepsilon} \therefore \lim_{n\to\infty} \|f_n - f\|_S = 0 \therefore f_n(x)$ convers unifsly on S to the func $f|_S$ (restiriction of f to subdomain S)

b) ii/ note $|\sin(nx)| \leq 1 \therefore \left|\frac{1}{n}\sin(nx)\right| \leq \left|\frac{1}{n}(1)\right| \leq \frac{1}{n} \leq \frac{1}{N} < \varepsilon$

$\forall n > N$ with $N > \frac{1}{\varepsilon} \therefore f(x) = 0$ is the pointwise limit of $\{f_n(x)\}$ for $x \in \mathbb{R}$

$\|f_n(x) - f(x)\|_{\mathbb{R}} = \|\frac{1}{n}\sin(nx) - 0\|_{\mathbb{R}} = \|\frac{1}{n}\sin(nx)\|_{\mathbb{R}} = \sup_{x\in\mathbb{R}} \left|\frac{1}{n}\sin(nx)\right|$

$= \sup\{|\frac{1}{n}\sin(nx)| \mid x \in \mathbb{R}\} = \sup\{\frac{1}{n}\sin(nx) \mid x \in \mathbb{R}\} = \infty$

but $\{f_n(x)\}$ uniformly converges for $x = \pi k$ for $k \in \mathbb{Z} \therefore x = \{\pi k \mid k \in \mathbb{Z}\}$

\\ Sol/ let $f(x)=0 \ \forall x\in\mathbb{R}$ $\therefore \|f_n - f\|_{\mathbb{R}} = \frac{1}{n} < \varepsilon \ \forall n \geq N$ with $N > \frac{1}{\varepsilon}$ $\therefore$ f_n converges uniformly to ~~0~~ the zero func and uniform convergence implies point wise conver

$$\Big\{ f(x)=0 \ \therefore \|f_n - f\| = \left\|\left(\tfrac{1}{n}\sin(nx)\right) - 0\right\|_{\mathbb{R}}$$

$$= \sup_{x\in\mathbb{R}} \left|\tfrac{1}{n}\sin(nx)\right| = \sup_{x\in\mathbb{R}}\left(\left|\tfrac{1}{n}\right|\left|\sin(nx)\right|\right) = \sup_{x\in\mathbb{R}}\left(\left|\tfrac{1}{n}\right|\right) \ (1)$$

$$= \sup_{x\in\mathbb{R}}\left|\tfrac{1}{n}\right| = \tfrac{1}{n} \leq \tfrac{1}{N} < \varepsilon \ \text{ for } n \geq N \text{ with } N > \tfrac{1}{\varepsilon} \Big\}$$

\\ iii/ $f_1(x) = \begin{cases} 0 & \text{if } x \leq 1 \\ x-1 & \text{if } x > 1 \end{cases}$ $\quad f_2(x) = \begin{cases} 0 & \text{if } x \leq 2 \\ x-2 & \text{if } x > 2 \end{cases}$

$f_3(x) = \begin{cases} 0 & \text{if } x \leq 3 \\ x-3 & \text{if } x > 3 \end{cases}$ $\quad x\in\mathbb{R}$

$$\therefore \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\left(\begin{cases} 0 & \text{if } x \leq n \\ x-n & \text{if } x > n \end{cases}\right) = \begin{cases} 0 & \text{if } x \leq \infty \\ x-n & \text{if } x > \infty \end{cases}$$

$$\{\text{but } x\in\mathbb{R} \ \therefore\} = \{0 \text{ if } x \leq \infty \ \therefore 0 = f(x) = 0$$

but for $x \leq n$ $\quad f(x) = \lim\limits_{n\to\infty} 0 = 0$

but for $x > n$ $\quad f(x) = \lim\limits_{n\to\infty}(x-n)$

so $\lim\limits_{n\to\infty} f_n(x) = 0 = f(x)$

$\forall x\in\mathbb{R} \ \exists n\in\mathbb{N}: x\leq n \ \therefore \lim\limits_{n\to\infty} f_n(x) = 0$ $\therefore$ the point wise limit of $\{f_n\}$ is the function $f(x)=0 \ \forall x\in\mathbb{R}$

$f(x)=0 \ \therefore \|f_n - f\|_{\mathbb{R}} = \|f_n(x) - 0\|_{\mathbb{R}} = \|f_n(x)\|_{\mathbb{R}} = \sup\limits_{x\in\mathbb{R}} |f_n(x)|$ $\quad \{f_n(x) \geq 0\}$

$= \sup\limits_{x\in\mathbb{R}} (f_n(x)) = x - n \leq x - N < \varepsilon \quad \forall n \geq N$ with $N > x - \varepsilon$

$\therefore \{f_n\}$ converges uniformly

Convergence is not uniform since $\forall \varepsilon > 0$, $n\in\mathbb{N}$:

$\exists x_1 > n+\varepsilon \ \therefore \|f - f_n\|_{\mathbb{R}} \geq |0 - f_n(x_1)| = |-f_n(x_1)| = f_n(x_1) = x_1 - n$
$> ((n+\varepsilon) - n) = \varepsilon$

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for $x \le n$ $\quad f_n(x) = 0 \;.\; f(x) = 0 \;\therefore$

$$\|f_n - f\|_{[-\infty,n]} = \|0-0\| = \cancel{\sup_{[-\infty}} \sup_{x\in[-\infty,n]} |0| = 0$$

$\therefore \{f_n\}$ converges uniformly for $x \le n$

~~for~~ ~~for~~ ~~for~~ $\forall n \in \mathbb{N}$, $\|f - f_n\|_{\mathbb{R}} = \sup_{x\in\mathbb{R}} |f_n(x)|$

$= \cancel{\sup} = \cancel{\{|x-n| \mid x\in\mathbb{R}\}} = \sup\{|f_n(x)| \mid x \in \mathbb{R}\}$ ~~xx~~

$= \cancel{\{|f_n} = \sup\{|x-n| \mid x \ge n\} = \infty \;\therefore$ the sup norm cannot be less than any $\varepsilon > 0$ for $\forall n \in \mathbb{N}$ but convergence is unif on any interval of form $S = (-\infty, b]$ with finite b : $\forall \varepsilon > 0$ have $\|f - f_n\|_S = 0 < \varepsilon \quad \forall n > b$

$\{\therefore x \in [-\infty, b] \quad n > b \;\therefore x \le b \;\therefore \cancel{x} \; x < n\}$

\iv/ $f_1(x) = \begin{cases} 1, & 0 < x \le 1 \\ \frac{1}{x^4}, & 1 < x \le 1 \end{cases} = 1 \qquad f_2(x) = \begin{cases} 2, & 0 < x \le \frac{1}{2} \\ \frac{1}{x^4}, & \frac{1}{2} < x \le 1 \end{cases}$

$f_3(x) = \begin{cases} 3 & 0 < x \le \frac{1}{3} \\ \frac{1}{x^4} & \frac{1}{3} < x \le 1 \end{cases} \qquad \therefore$ for $x \in (0, \frac{1}{n}]$, $\varepsilon > 0$, $n \in \mathbb{N}$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} (n) = \infty$ and for $x \in (\frac{1}{n}, 1]$

$\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \left(\frac{1}{x^4}\right) = \frac{1}{x^4}$ but $\frac{1}{x^2} \ne 0$ but

for $x \in (0, \frac{1}{n}]$ for $n \to \infty \Rightarrow x \in (0, 0]$ and

$\lim_{n\to\infty} \left(\frac{1}{n}\right) = 0 \Rightarrow$ as $n \to \infty$ for $x \in (\frac{1}{n}, 1]$; ~~f_n~~ $\lim_{n\to\infty} f_n(x) = \frac{1}{x^4}$

and as $n \to \infty$ $\therefore$ for, $x \in (0, 1]$ $\lim_{n\to\infty} f_n(x) = \frac{1}{x^4} = f(x)$

• $(\forall x \in (0,1], \cancel{\forall} \exists N \in \mathbb{N} : (\forall n \ge N ; x > \frac{1}{n}) : \lim_{n\to\infty} f_n(x) = \frac{1}{x^4} = f(x)$

for $x \in (0, \frac{1}{n}]$ $\forall \varepsilon > 0$, $n \in \mathbb{N}$:
$\|f_n - f\|_{(0,\frac{1}{n}]} = \|n - 0\|_{(0,\frac{1}{n}]} = \sup_{x\in(0,\frac{1}{n}]} |n| = n \geq N \leq \varepsilon$

$\forall n \geq N$ with $N \leq \varepsilon$

but for $x \in (\frac{1}{n}, 1]$ $\forall \varepsilon > 0$, $n \in \mathbb{N}$: $\|f_n - f\|_{(\frac{1}{n},1]}$
$= \|\frac{1}{x^4} - 0\|_{(\frac{1}{n},1]} = \sup_{x\in(\frac{1}{n},1]} |\frac{1}{x^4}| = \sup_{x\in(\frac{1}{n},1]} \frac{1}{x^4} = \frac{1}{(\frac{1}{n})^4} = n^4$

$\geq N^4 > N = \varepsilon$ $\forall n \geq N$ with $\varepsilon \geq N$

$\therefore$ since it does not converge uniformly

does not conver uniformly in $(0,1]$ because
$\|f_n - f\|_{(0,1]} = \sup_{x\in(0,1]} |f_n(x) - f(x)| = \sup_{0<x\leq\frac{1}{n}} |n - f(x)|$

$= \sup_{x\in(0,\frac{1}{n}]} |n - \frac{1}{x^4}| = \infty$ because if $x \in (\frac{1}{n}, 1]$

then $f_n(x) - f(x) = \frac{1}{x^4} - \frac{1}{x^4} = 0$ and the set $\left\{|n - \frac{1}{x^4}| : x \leq \frac{1}{n}\right\}$

is not bounded to see this suppose by contradiction:

that $\exists M > 0 : |n - \frac{1}{x^4}| \leq M$ $\forall x \leq \frac{1}{n}$ then for $x = \frac{1}{n}$ we get

that $|n - n^4| = n|1 - n^3| \leq M$ $\forall n$ but $n|1-n^3| = n|n^3 - 1| \leq M$ $\forall n$

is false $\therefore$ by de $\forall M > 0 : |n - \frac{1}{x^4}| \geq M$ $\forall x \in (0, \frac{1}{n}]$

\iv/ for $n \in \mathbb{N}$, $\varepsilon > 0$ for $x < 0$

$f_1(x) = \frac{e^x}{x}$, $f_2(x) = \frac{e^x}{x^2}$, $f_3(x) = \frac{e^x}{x^3}$

$\therefore$ $\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} \frac{e^x}{x^n}$

$\therefore$ for $x > 1$ $\lim_{n\to\infty} \frac{e^x}{x^n} = 0$, for $x = 1$: $\lim_{n\to\infty} \frac{e^x}{x^n} = \lim_{n\to\infty} \frac{e}{1^n} = e$

for $0 < x < 1$: $\lim_{n\to\infty} \frac{e^x}{x^n} = \infty$

for $x < 1$: $\lim_{n\to\infty} \frac{e^x}{x^n} = 0$

for $-1 \leq x < 0$ $\lim_{n\to\infty} \frac{e^x}{x^n}$ is undefined $\therefore$

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$$f(x) = \begin{cases} 0 & , x \in (1,\infty) \\ e & , x = 1 \\ 0 & , x \in (-\infty,-1) \\ \text{undesired} & ; x \in [-1,0] \\ \infty & , x \in (0,-1) \end{cases}$$

$\left\{ \lim_{n\to\infty} |x^n| = \infty \right.$ for $|x|>1 \therefore \forall |x|>1 : \lim_{n\to\infty} \frac{e^x}{x^n} = 0$

for $|x|<1$: $\lim_{n\to\infty} |x^n| = 0 \therefore 0<|x|<1 \quad \lim_{n\to\infty} \frac{e^x}{x^n} = \infty$

for $x=1 \quad \lim_{n\to\infty} \frac{e^1}{1^n} = e$

for $x=-1$: $\cancel{\lim_{n\to\infty}} \frac{e^{-1}}{(-1)^n} = \frac{1}{(-1)^n e} = \frac{(-1)^n}{e} \therefore \lim_{n\to\infty} \frac{(-1)^n}{e}$ does not converge

$\therefore$ sequence $\{f_n\}$ converges pointwise on $(-\infty,-1) \cup [1,\infty)$ to function $f(x) = \begin{cases} 0 & , x<-1 \\ e & , x=1 \\ 0 & , x>1 \end{cases}$

$\therefore$ for $x=1$: $\|f_n - f\|_{x=1} = \sup_{x=1} |f_n - f| = \sup_{x=1} \left| \frac{e^x}{x^n} - e \right|$

$= \cancel{\sup \left| \frac{e^x}{1^n} \right|} \quad = \sup \left| \frac{e^1}{1^n} - e \right| = |e-e| = |0| = 0 < \varepsilon \quad \forall \varepsilon > 0$

$\therefore$ converges unif for $x=1$ &

for $x>1$: $\forall \varepsilon>0$: $\|f_n - f\|_{(1,\infty)} = \sup_{x\in(1,\infty)} |f_n - f|$

$= \sup_{x\in(1,\infty)} \left| \frac{e^x}{x^n} - 0 \right| = \sup_{x\in(1,\infty)} \left| \frac{e^x}{x^n} \right| = \sup_{x\in(1,\infty)} \frac{e^x}{x^n} \cancel{\leq}$

to converge uniformly the limit function must be continuous but here the limit function is a pointwise function that is not continuous therefore ~~the limit~~ it does not converge uniformly

$\sup_{x\in(1,\infty)} \frac{e^x}{x^n} \neq 0 < \varepsilon \; \forall \varepsilon > 0$

for $x<-1$: $\|f_n - f\|_{(-\infty,-1)} = \sup_{-\infty<x<-1} \left| \frac{e^x}{x^n} - 0 \right| = \sup_{-\infty<x<-1} \left| \frac{e^x}{x^n} \right|$

$\leq \sup_{1<x<\infty} \left| \frac{e^x}{x^n} \right| = \sup_{1<x<\infty} \frac{e^x}{x^n} = 0 < \varepsilon \; \forall \varepsilon > 0 \;\therefore$

~~the~~ f_n converges uniformly for the subdomain S:
$S = (-\infty, -1) \cup [1, \infty)$

{ since each $\{f_n\}$ is cont but f is not the convergence cannot be unif but convergence is unif on any set of the form $S = (-\infty, a] \cup [b, c]$ with $a < -1$ and $1 < b < c$:

$\|f_n - f\|_S = \sup_{x \in S} |f_n(x) - f(x)| \leq \max\left\{\left|\frac{e^a}{a^n}\right|, \left|\frac{e^b}{b^n}\right|, \left|\frac{e^c}{c^n}\right|\right\}$ which follows because f is increasing on $(-\infty, a]$ and convex on $[b, c]$ since $|a|, |b|, |c|$ are all greater than 1 $\therefore$ this converges to 0 as $n \to \infty$

\5/ for $\sum_{n=1}^{\infty} M_n < \infty$, $n \geq k$ $\therefore$ $\sum f_n = \sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$

$\therefore f_n = \frac{x}{n(1+nx^2)}$ $\therefore \|f_n\|_{\mathbb{R}} = \sup_{x \in \mathbb{R}} |f_n| = \sup_{x \in \mathbb{R}} \left|\frac{x}{n(1+nx^2)}\right|$

$= \frac{1}{n} \sup_{x \in \mathbb{R}} \left|\frac{x}{1+nx^2}\right| = \frac{1}{n} \sup_{x \in \mathbb{R}} \left[|x| \frac{1}{1+nx^2}\right] = \sup_{x \in (0,\infty)} \frac{x}{n(1+nx^2)}$

$= \sup_{0<x<\infty} \frac{\left(\frac{x}{x}\right)}{n(1+nx^2)\left(\frac{1}{x}\right)} = \sup_{0<x<\infty} \frac{(1)}{n\left(\frac{1}{x}+nx\right)}$ ~~= (18 min~~

$= (1) \frac{1}{\inf_{0<x<\infty} n\left(\frac{1}{x}+nx\right)}$ $\therefore \frac{d}{dx}\left[n(x^{-1}+nx)\right] = n\left[-x^{-2}+n\right]$

$= n\left[-\frac{1}{x^2}+n\right] = 0 \Rightarrow -\frac{n}{x^2}+n$ $\therefore n = \frac{n}{x^2}$ $\therefore 1 = \frac{1}{x^2}$ $\therefore x^2 = 1$

$0 < x < \infty$ $\therefore x = 1$ $\therefore \sup_{x \in (0,\infty)} \frac{x}{n(1+nx^2)} = \frac{1}{n(1+n(1)^2)} = \frac{1}{n(1+n)} = \frac{1}{n+n^2}$

$= \frac{1}{n(1+n)} = M_n$ $\therefore \|f_n\|_{\mathbb{R}} \leq \frac{1}{n(1+n)}$

and $\sum_{n=1}^{\infty} \frac{1}{n(1+n)} = \sum_{n=1}^{\infty} \frac{1}{n^2+n} \leq \sum_{n=1}^{\infty} \frac{1}{n^2} < \infty$ $\therefore$ ~~$\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$~~

$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} < \infty$ $\therefore$ by the weierstrass test

$\sum_{n=1}^{\infty} \frac{x}{n(1+nx^2)}$ converges uniformly

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denote $f_n(x) = \frac{x}{n(1+nx^2)}$ $\therefore$ $f'(x) = \frac{d}{dx}\left(\frac{x}{n+n^2x^2}\right)$

$= \frac{d}{dx}\left[(x)(n+n^2x^2)^{-1}\right] = (n+n^2x^2)^{-1} - x(n+n^2x^2)^{-2}(2n^2x)$

$= \frac{n+n^2x^2}{(n+n^2x^2)^2} - \frac{x(2n^2x)}{(n+n^2x^2)^2} = \frac{n+n^2x^2-2n^2x^2}{(n+n^2x^2)^2} = \frac{n-n^2x^2}{n^2(1+nx^2)^2}$

$\therefore$ find the critical points ~~at~~ $\{$by $f_n'(x)=0\}$

$\left\{\frac{n-n^2x^2}{n^2(1+nx^2)^2} = 0 \doteq n-n^2x^2 \therefore n^2x^2 = n \therefore x^2 = \frac{1}{n} \therefore x = \pm\sqrt{\frac{1}{n}}\right\}$

at $\bar{x}_n = \frac{1}{+\sqrt{n}}$ and $\underline{x}_n = -\frac{1}{\sqrt{n}}$ and also have for small

$\delta > 0$: $f_n'(\bar{x}_n + \delta) < 0$ and $f_n'(\bar{x}_n - \delta) > 0$ whilst

$f_n'(\underline{x}_n + \delta) > 0$ and $f_n'(\underline{x}_n + \delta) < 0$ indicating that

$\bar{x}_n$ is the global maximum and $\bar{x}_n$ is the global

minimum and we have $\{|\cancel{f_n(x)}| \leq |f_n'(\bar{x}_n)|$

$= \left|\frac{n-n^2\left(\frac{1}{\sqrt{n}}\right)}{n^2\left(1+n\left(\frac{1}{\sqrt{n}}\right)^2\right)^2}\right| = \left|\frac{n-n^{3/2}}{n^2\left(1+n\left(\frac{1}{n}\right)\right)^2}\right| = \left|\frac{n-n^{3/2}}{n^2(1+1)^2}\right| = \left|\frac{n-n^{3/2}}{4n^2}\right| = \left|\frac{n^{3/2}-n}{4n^2}\right|\}$

$\therefore$ and $|f_n(x)| \leq |f_n(\bar{x}_n)| = \left|\frac{\bar{x}_n}{n(1+n\bar{x}_n^2)}\right| = \left|\frac{\left(\frac{1}{\sqrt{n}}\right)}{n\left(1+n\left(\frac{1}{\sqrt{n}}\right)^2\right)}\right|$

$= \left|\frac{n^{-1/2}}{n\left(1+n\left(\frac{1}{n}\right)\right)}\right| = \left|\frac{1}{n^{1/2}n(1+1)}\right| = \left|\frac{1}{4n^{3/2}}\right| = \frac{1}{4n^{3/2}} < \frac{1}{n^{3/2}}$

since ~~$\sum \frac{1}{n^{3/2}} < \sum \frac{1}{n^2}$~~ $\sum \frac{1}{n^{3/2}}$ converges $\therefore$ by the weirstrass

test $\sum \frac{x}{n(1+nx^2)}$ converges unif'ly $\forall x \in \mathbb{R}$

6/ $\therefore \sum_{n=1}^{\infty} M_n < \infty$, $\sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \frac{1}{n^{3/2}}\left(\frac{x}{1+x}\right)^n$

$\therefore \|f_n\|_{\mathbb{R}} = \sup_{x\in\mathbb{R}} |f_n| = \sup_{x\in\mathbb{R}} \left|\frac{1}{n^{3/2}}\left(\frac{x}{1+x}\right)^n\right| = \frac{1}{n^{3/2}} \sup_{x\in\mathbb{R}} \left|\left(\frac{x}{1+x}\right)^n\right|$

$= \frac{1}{n^{3/2}}\left(\sup_{x\in\mathbb{R}}\left|\frac{x}{1+x}\right|\right)^n$ $\therefore \frac{d}{dx}\left(\frac{x}{1+x}\right) = \frac{d}{dx}\left[x(1+x)^{-1}\right]$

$= (1+x)^{-1} - x(1+x)^{-2} = \frac{1+x}{(1+x)^2} + \frac{-x}{(1+x)^2} = \frac{1}{(1+x)^2} = 0 \therefore$

$$\lim_{x\to -1^-}\left|\frac{x}{1+x}\right| = \lim_{x\to-1^-}\left|\frac{-1}{1+x}\right| = \lim_{x\to 0^-}\left(\frac{-1}{x}\right) = +\infty ,$$

$$\lim_{x\to -1^+}\left(\frac{x}{1+x}\right) = \lim_{x\to-1^+}\left(\frac{-1}{1+x}\right) = \lim_{x\to 0^+}\left(\frac{-1}{x}\right) = -\infty$$

$$\therefore \lim_{x\to -1^+}\left|\frac{x}{1+x}\right| = |-\infty| = +\infty \therefore$$

$$\lim_{x\to -1^-}\left|\frac{x}{1+x}\right| = \lim_{x\to -1^+}\left|\frac{x}{1+x}\right| = \infty = \lim_{x\to 1}\left(\left|\frac{x}{1+x}\right|\right) \therefore$$

$$\frac{1}{n^{3/2}}\left(\sup_{x\in\mathbb{R}}\left|\frac{x}{1+x}\right|\right)^n = \frac{1}{n^{3/2}}\left(\sup_{x=-1}\left|\frac{x}{1+x}\right|\right)^n = \|f_n\|_{x=-1} \therefore$$

$$\frac{1}{n^{3/2}}\left(\sup_{x\neq -1}\left|\frac{x}{1+x}\right|\right)^n < \|f_n\|_{x=-1} \therefore \|f_n\|_{x\neq -1} < \|f_n\|_{x=-1}$$

but $\|f_n\|_{x=-1} = \infty \therefore \|f_n\|_{x\neq-1} < \infty$; - by the weierstrass test $\sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\left(\frac{x}{1+x}\right)^n$ convergesunifsly for $x \neq -1$

with $\sup_{x\neq -1}\frac{1}{n^{3/2}}\left(\left|\frac{x}{1+x}\right|\right)^n = M_n < \infty$ and $\sum_{n=1}^{\infty} M_n < \infty$;.

$\{n^a b^n \therefore \ln(n^a b^n) = \ln(n^a) + \ln(b^n) = a\ln(n) + n\ln(b)\}$

Sol: note $\left|\frac{x}{1+x}\right| < 1$ if $x \in (-\frac{1}{2}, \infty)$ $\{$if $\frac{x}{1+x} = 1 \therefore x = 1+x$

$\therefore$ if $x < 1+x \therefore$ if $\frac{x}{1+x} = -1 \therefore x = -1-x \therefore 2x = -1 \therefore x = -\frac{1}{2}$,

$\lim_{x\to\infty}\frac{x}{1+x} = \lim_{x\to\infty}\frac{1}{0+1} = \lim_{x\to\infty} 1 = 1 \therefore -1 < \frac{x}{1+x} < 1$ for $x \in (-\frac{1}{2}, \infty)$

$\therefore \left|\frac{x}{1+x}\right| < 1$ for $x \in (-\frac{1}{2}, \infty)\}$;. for $x \in (-\frac{1}{2}, \infty)$

$\frac{1}{n^{3/2}}\left(\frac{x}{1+x}\right)^n < \frac{1}{n^{3/2}} \ \forall n \therefore$ by weirestrass test since

$\sum_{n=1}^{\infty}\frac{n}{n^{3/2}} < \infty \therefore \sum_{n=1}^{\infty}\frac{1}{n^{3/2}}\left(\frac{x}{1+x}\right)^n$ ~~convs~~ unifsly $\forall x \in (-\frac{1}{2}, \infty)$

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1/ so $\sum_{n=1}^{\infty}|a_n|<\infty$ $\therefore$ $\lim_{n\to\infty}|a_n|=0$

$|\cos(nx)|\le 1$ $\therefore$ $-1\le\cos(nx)\le 1$, $-1\le\sin(nx)\le 1$

$|a_n|\ge 0$ $\therefore$ $0\le\sum_{n=1}^{\infty}|a_n|<\infty$ $\therefore$

$|a_n|\cos(nx)\le|a_n|$ $\therefore$ $\sum_{n=1}^{\infty}|a_n|\cos(nx)\le\sum_{n=1}^{\infty}|a_n|<\infty$

$\therefore$ by the weierstrass test $\sum_{n=1}^{\infty}|a_n|\cos(nx)$

and $\left\{\sum_{n=1}^{\infty}|a_n|\sin(nx)\le\sum_{n=1}^{\infty}|a_n|<\infty\right\}$ and also $\sum_{n=1}^{\infty}|a_n|\sin(nx)$

conv unif $\therefore$ since $\sum_{n=1}^{\infty}a_n\le\sum_{n=1}^{\infty}|a_n|$

$\therefore$ $-\infty<\sum_{n=1}^{\infty}a_n<\infty$ $\therefore$ $\left|\sum_{n=1}^{\infty}a_n\cos(nx)\right|\le\left|\sum_{n=1}^{\infty}a_n\right|$

and $\left|\sum_{n=1}^{\infty}a_n\sin(nx)\right|\le\left|\sum_{n=1}^{\infty}a_n\right|$ $\therefore$ both $\sum a_n\cos nx$, $\sum a_n\sin nx$

conv unif and are cont funcs

Sol: $f_n(x)=a_n\sin(nx)$ $\therefore$ $\|f_n\|_{\mathbb{R}}=\sup_{x\in\mathbb{R}}|a_n\sin nx|\le|a_n|$

$\therefore$ $\sum|a_n|<\infty$ $\therefore$ by weierstrass test $\therefore$ $\sum_{n=1}^{\infty}f_n(x)$ conv unif $\forall x\in\mathbb{R}$ that is to say the sequence of partial sums $F_N=\sum_{n=1}^{N}f_n(x)$ conv unif to $F(x)=\sum_{n=1}^{\infty}f_n(x)$ since each func f_n is cont, the partial sums F_N are cont as a sum of cont funcs & since the convergence to F is unif this implies that $F(x)=\sum_{n=1}^{\infty}f_n(x)$ is cont (Cor 3.43) analogous for $\cos(nx)$ also

3/ $\therefore$ $-\infty<f_n<\infty$ $\forall n$ and $x\in U$

$\therefore$ $\|f_n\|_U\le M_n$ for $\sum_{n=1}^{\infty}M_n<\infty$ $\quad |f_n(x)|\le M_n$

$\therefore$ $\sup_{x\in U}|f_n(x)|\le M_n$ $\quad$ $\{f_n\}$ conv unif on U to f $\therefore$

$\|p\|^2 + 2\langle p, h\rangle + \|h\|^2$

$Df(p) = (D_1 f(p) \cdots D_n f(p)) = (2p_1 \cdots 2p_n) = 2p^T$

$Df(p) \cdot h = 2p^T h = 2\langle p, h\rangle$

$$\lim_{h\to 0} \frac{\| \|p\|^2 + 2\langle p,h\rangle + \|h\|^2 - \|p\|^2 - 2\langle p,h\rangle \|}{\|h\|} = \lim_{h\to 0} \|h\| = 0$$

\ex 4.8/: $g(x) = f(p) + Df(p) \circ (x-p)$ linear approx at p

$f: \mathbb{R}^2 \to \mathbb{R}$, $f(x) = e^{-(x_1^2+x_2^2)}$ $\quad Df(p) = (D_1 f(p) \quad D_2 f(p))$

$= (-2p_1 e^{-(x_1^2+x_2^2)} \quad -2p_2 e^{-(x_1^2+x_2^2)})$

Z linear approx is $g(x) = e^{-(p_1^2+p_2^2)} + (-2p_1 e^{-(p_1^2+p_2^2)} \quad -2p_2 e^{-(p_1^2+p_2^2)})\begin{pmatrix} x_1-p_1 \\ x_2-p_2 \end{pmatrix}$

$= e^{-(p_1^2+p_2^2)}(1 - 2p_1(x_1-p_1) - 2p_2(x_2-p_2))$ $\quad$ let $x_3 = g(x) \Rightarrow$

$2p_1 x_1 + 2p_2 x_2 + e^{(p_1^2+p_2^2)} x_3 = 1 + 2p_1^2 + 2p_2^2$ is eqn of plane in $\mathbb{R}^3$

Z tangent plane to Z surface described by f in $\mathbb{R}^3$

<u>Differentiability</u> <u>implies</u> <u>continuity</u>:

\thm 4.9/ Differentiability implies continuity/

Let $\Omega \subset \mathbb{R}^n$ be an open set if Z func $f: \Omega \to \mathbb{R}^m$ is differentiable at $p \in \Omega$ then it is continuous at p

<u>proof</u>:/ since f is differentiable at p

$\lim_{h\to 0} \|f(p+h) - f(p) - A\cdot h\| / \|h\| = 0$ implies

$\lim_{h\to 0} \|f(p+h) - f(p) - A\cdot h\| = 0$ since A is a linear map it

is cont & $\therefore \lim_{h\to 0} A\cdot h = A\cdot 0 = 0$ $\therefore \lim_{h\to 0} \|f(p+h) - f(p)\| = 0$ $\therefore$

$\lim_{h\to 0} f(p+h) = f(p)$ & so f is cont at p $\quad \square$

Z <u>existence</u> of <u>partial</u> <u>derives</u> <u>does</u> <u>not</u> imply

<u>diffability</u>: Consider func $f: \Omega \to \mathbb{R}^m$ with $\Omega \subset \mathbb{R}^n$

an open set if Z <u>func</u> f is diffable at $p \in \Omega$

then Z derive $Df(p)$ is given by Z Jacobian Matrix:

$$J = \begin{bmatrix} D_1 f_1(p) & D_2 f_1(p) & \cdots & D_n f_1(p) \\ \vdots & & \ddots & \vdots \\ D_1 f_m(p) & D_2 f_m(p) & \cdots & D_n f_m(p) \end{bmatrix}$$ ie $Df(p) = J$ but

∃ examples of funcs where all partial derivs at a pt p exist (so Z Jacobian Matrix exists) but Z func is not diffable at p (so that certainly $Df(p) \neq J$ since $Df(p)$ doesnt exist)

$\left\{ \lim_{h \to 0} \|f(p+h) - f(p) - Df(p) \cdot h\| / \|h\| = 0 \right\}$ {$Df(p)$ doesnt exist}

\thm 4.11/ continuity of partial derivs implies diffability/

let $\Omega \subset \mathbb{R}^n$ be open & $f: \Omega \to \mathbb{R}^m$ suppose for each $i = 1, \dots, m$ $j = 1, \dots, n$ & all $p \in \Omega$ Z partial deriv mappings $D_j f_i(p)$ exist and are cont then f is diffable at all $p \in \Omega$

∃ funcs where Z partial derivs are discont but Z func is diffable ∴ converse of this Thm isnt true

\ex 4.12/ let $\Omega = \mathbb{R}^2$ $f: \Omega \to \mathbb{R}^2$ $f(x) = \begin{pmatrix} x_1 \sin(x_2) \\ x_1 \cos(x_2) \end{pmatrix}$

$D_1 f_1(x) = \sin(x_2)$ $D_2 f_1(x) = x_1 \cos(x_2)$ $D_1 f_2(x) = \cos(x_2)$

$D_2 f_2(x) = -x_1 \sin(x_2)$ these are all cont as products & compositions of cont funcs ∴ f is diffable $\forall x \in \Omega$ & $Df(x) = \begin{pmatrix} \sin(x_2) & x_1 \cos(x_2) \\ \cos(x_2) & -x_1 \sin(x_2) \end{pmatrix}$

\week 11/

\chain rule for higher dimensions funcs from $\mathbb{R}^n$ to $\mathbb{R}^m$/

Similiar with chain rule in one dimen to find derivs of Z composition of funcs: is $f, g: \mathbb{R} \to \mathbb{R}$ where g is diffable at some point p & f is diffable at $g(p)$ then with $h = f \circ g$ have $h'(p) = f'(g(p))g'(p)$ have a generalisation of this result in higher dimens {so $f, g: \mathbb{R}^1 \to \mathbb{R}^1$}

\thrm 4.13/ \chain rule/ let $g: \mathbb{R}^n \to \mathbb{R}^m$ be diffsable at p & $f: \mathbb{R}^m \to \mathbb{R}^l$ be diffsable at $g(p)$ the Z composition $h = f \circ g$ is diffsable at p $\{h: \mathbb{R}^n \to \mathbb{R}^l\}$ with derivative ~~$Dh(p) = Df$~~

$$Dh(p) = Df(g(p)) \circ Dg(p)$$

using Z jacobians Can write Chain rule as Z following Mat Product:

$$Dh(p) = \begin{bmatrix} D_1 f_1(g(p)) & \cdots & D_m f_1(g(p)) \\ D_1 f_2(g(p)) & \cdots & D_m f_2(g(p)) \\ \vdots & & \vdots \\ D_1 f_l(g(p)) & \cdots & D_m f_l(g(p)) \end{bmatrix} \begin{bmatrix} D_1 g_1(p) & \cdots & D_n g_1(p) \\ D_1 g_2(p) & \cdots & D_n g_2(p) \\ \vdots & \ddots & \vdots \\ D_1 g_m(p) & \cdots & D_n g_m(p) \end{bmatrix}$$

resulting $l \times n$ Mat $(l \times m) \times (m \times n)$

\Local Maxima & Minima for Funcs from $\mathbb{R}^n$ to $\mathbb{R}$/

\Def 4.14/ let $\Omega \subset \mathbb{R}^n$ be open & $f: \Omega \to \mathbb{R}$ a pt $p \in \Omega$ s.t. $Df(p) = 0$ is called a <u>Stationary point</u>

$\{$So have $(D_1 f(p)\ D_2 f(p) \cdots D_n f(p)) = Df(p) = (0 \cdots 0)$ $\}$

So 0 represents Z $1 \times n$ Zero Mat $(0 \cdots 0)$

\lemma 4.15/ let $\Omega \subset \mathbb{R}^n$ be open & $f: \Omega \to \mathbb{R}$ be diffsable at each pt in Ω if a point $p \in \Omega$ is a local extremum pt of f then p is a Stationary pt

\<u>Proof</u>/ take an arbit vec $v \in \mathbb{R}^n \setminus \{0\}$ & define Z func $g: T \to \Omega$, $g(t) = p + tv$ where $T = (-\varepsilon, \varepsilon) \subset \mathbb{R}$ with $\varepsilon > 0$ sufficiently small s.t. $g(T) \subset \Omega$ now we consider ~~fg~~ $f \circ g: T \to \mathbb{R}$

Since f has a local extremum at p, $f(g(t))$ has a local extremum at $t = 0$ Since g is diffsable $\{g(t) = p + tv \therefore g'(t) = v\}$ by the Chain rule,

$0 = (f\circ g)'(0) = Df(g(0))\,Dg(0) = Df(p)v$ since vector v is arbit $\Rightarrow Df(p) = (0 \dots 0)$ $\square$

The converse of this lemma is not true. if the deriv is zero at pt a this doesn't imply that pt a is a local extremum of f. a simple example is func $f(x) = x^3$

\aside/ • for $f: \mathbb{R} \to \mathbb{R}$ have three types of stationary pts: local min, local max, point of inflection

• for $f: \mathbb{R}^n \to \mathbb{R}$, $n>1$ we have infinitely many types of stationary pts

• for $f: \mathbb{R}^2 \to \mathbb{R}$ there are three main types: local min, local max, saddle {saddle has infinitely many subtypes}

Hessian mat provides a "second deriv test"

\Ex 4.16:/ find stationary points of $f: \mathbb{R}^3 \to \mathbb{R}$

$f(x) = x_1^2 + x_2^2 + x_3^2 - 4x_3$ have

$Df(x) = (D_1 f(x) \quad D_2 f(x) \quad D_3 f(x)) = (2x_1 \quad 2x_2 \quad 2x_3 - 4)$

$= (0 \quad 0 \quad 0)$ so $x_1 = 0$, $x_2 = 0$, $x_3 = 2$

$\therefore$ stationary pt at $p = (0, 0, 2)^T$

not $f(p) = -4$ & $f(x) = x_1^2 + x_2^2 + (x_3 - 2)^2 - 4 \geq -4$

$\therefore f(x) \geq -4$ hence p is a global minimum