MTH2009 CW1

1/ by M-L lemma: is there is an constant $M \geq 0$ such that $|f(z)| \leq M$ for all points z on γ
then $\left|\int_\gamma f(z)\right| \leq M \cdot L(\gamma) = L(\gamma) \cdot M$

So for $z \in \gamma(t)$ then $t \in [a,b]$ then
$|f(z)| = |f(\gamma(t))| \leq \sup_{t\in[a,b]} |f(\gamma(t))| = M = \text{constant}$ $\therefore$

$|f(z)| \leq M$
and $L(\gamma)$ is the length of γ $\therefore$
$\left|\int_\gamma f(z)\,dz\right| \leq L(\gamma) \sup_{t\in[a,b]} |f(\gamma(t))|$ as required $\square$

\2/ $z = x + iy$ for $x, y \in \mathbb{R}$ and $f(z) = u(x,y) + iv(x,y)$

$\therefore f(z) = f(x+iy) = 2i(x+iy)^2 + \sqrt{2}\pi(x+iy) + 4\sqrt{2} =$

$2i(x^2 + 2ixy - y^2) + \sqrt{2}\pi x + \sqrt{2}\pi iy + 4\sqrt{2} =$

$2ix^2 - 4xy - 2iy^2 + \sqrt{2}\pi x + 4\sqrt{2} + \sqrt{2}\pi iy =$

$(-4xy + \sqrt{2}\pi x + 4\sqrt{2}) + i(2x^2 - 2y^2 + \sqrt{2}\pi y) =$

$u(x,y) + iv(x,y) \quad \therefore \cancel{u(x,y) =}$

$u(x,y) = (-4xy + \sqrt{2}\pi x + 4\sqrt{2})$ and

$v(x,y) = (2x^2 - 2y^2 + \sqrt{2}\pi y) \quad \therefore$

$$\frac{\partial u}{\partial x} = \frac{\partial u(x,y)}{\partial x} = \frac{\partial}{\partial x}(-4xy + \sqrt{2}\pi x + 4\sqrt{2}) =$$

$-4y + \sqrt{2}\pi$ and

$$\frac{\partial v}{\partial x} = \frac{\partial v(x,y)}{\partial x} = \frac{\partial}{\partial x}(2x^2 - 2y^2 + \sqrt{2}\pi y) =$$

$4x$ and

$$\frac{\partial u}{\partial y} = \frac{\partial u(x,y)}{\partial y} = \frac{\partial}{\partial y}(-4xy + \sqrt{2}\pi x + 4\sqrt{2}) = -4x$$

and

$$\frac{\partial v}{\partial y} = \frac{\partial v(x,y)}{\partial y} = \frac{\partial}{\partial y}(2x^2 - 2y^2 + \sqrt{2}\pi y) = -4y + \sqrt{2}\pi \quad \therefore$$

$$\frac{\partial u}{\partial x} = -4y + \sqrt{2}\pi = \frac{\partial v}{\partial y} \quad \therefore \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and}$$

$$-\frac{\partial v}{\partial x} = -(4x) = -4x = \frac{\partial u}{\partial y} \quad \therefore -\frac{\partial u}{\partial y} = \frac{\partial v}{\partial x} \quad \therefore$$

those are the Cauchy-Riemann equations of the function as required $\square$

3/ length of γ is $L(\gamma)$ and for $\gamma:[a,b]\to\mathbb{C}$

then $L(\gamma)=\int_a^b|\gamma'(t)|\,dt$ where

$\gamma'(t)=\frac{d}{dt}(\gamma(t))$

$t=0$ to $t=\frac{\pi}{2}$ and $0<\frac{\pi}{2}$ $\therefore$ $a=0$ and ~~b=~~

$b=\frac{\pi}{2}$ $\therefore$ $\gamma(t):[0,\frac{\pi}{2}]\to\mathbb{C}$ and $\gamma'(t)=\frac{d}{dt}(\gamma(t))=$

$\frac{d}{dt}(e^{it}+t(\sin t-i\cos t))=$

$ie^{it}+(\sin t-i\cos t)+t(\cos t+i\sin t)=$

$i\cos t-\sin t+\sin t-i\cos t+t\cos t+it\sin t=$

$t\cos t+it\sin t=\gamma'(t)$ $\therefore$

$|\gamma'(t)|=\sqrt{(t\cos t)^2+(t\sin t)^2}=\sqrt{t^2\cos^2 t+t^2\sin^2 t}=$

$\sqrt{t^2(\cos^2 t+\sin^2 t)}=\sqrt{t^2\cdot 1}=\sqrt{t^2}=t$ $\therefore$

$L(\gamma)=\int_a^b|\gamma'(t)|\,dt=\int_0^{\pi/2}t\,dt=\left[\frac{1}{2}t^2\right]_0^{\pi/2}=\frac{1}{2}\left(\left(\frac{\pi}{2}\right)^2-0^2\right)$

$=\frac{1}{2}\left(\frac{\pi^2}{4}\right)=\frac{1}{8}\pi^2=L(\gamma)$

is the length of the Smooth Curve

\4/ by M-L lemma: if there is an constant $M \geq 0$ such that $|f(z)| \leq M$ for all points z on γ then $\left|\int_\gamma f(z)\right| \leq M \cdot L(\gamma)$ where $L(\gamma)$ is the length of the path γ

and $f(z) = \frac{3z^3 \sin(2z)}{3z^5+1}$, ~~and 2 |z|=1~~ $z = x+iy$

and unit circle $\gamma : [a,b] \to \mathbb{C}$ $\therefore$ $\gamma(t) = e^{it}$
$0 \leq t \leq 2\pi$ $\therefore$ $|z| = 1$ and $x, y \in \mathbb{R}$ $\therefore$

$\gamma'(t) = \frac{d}{dt}(\gamma(t)) = \frac{d}{dt}(e^{it}) = ie^{it}$ $\therefore$ $\gamma : [0, 2\pi] \to \mathbb{C}$

~~$L(\gamma$~~ $\therefore$ ~~γ~~ $L(\gamma) = \int_0^{2\pi} |\gamma'(t)| dt = \int_0^{2\pi} |ie^{it}| dt = \int_0^{2\pi} |i||e^{it}| dt$

$= \int_0^{2\pi} \sqrt{0^2 + (1)^2} \cdot |\cos t + i \sin t| dt = \int_0^{2\pi} 1 \cdot \sqrt{\cos^2 t + \sin^2 t}\, dt =$

$= \int_0^{2\pi} 1 \cdot \sqrt{1}\, dt = \int_0^{2\pi} 1\, dt = [t]_0^{2\pi} = 2\pi - 0 = 2\pi = L(\gamma)$

{note: $|z - w| \geq |z| - |w|$ is reverse triangle inequality}

and $|f(z)| = \left|\frac{3z^3 \sin(2z)}{3z^5+1}\right| = \frac{|3z^3 \sin(2z)|}{|3z^5+1|}$

$= \frac{|3z^3 \sin(2z)|}{|3z^5 - (-1)|} \leq \frac{|3z^3 \sin(2z)|}{|3z^5| - |-1|} = \frac{3|z|^3 |\sin(2z)|}{3|z|^5 - 1}$

$= \frac{3(1)^3 \left|\frac{e^{i2z} - e^{-i2z}}{2i}\right|}{3(1)^5 - 1} = \frac{3}{2} \cdot \frac{1}{|2i|} |e^{2iz} - e^{-2iz}| =$

$\frac{3}{2} \cdot \frac{1}{2} |e^{2zi} + (-e^{-2zi})| \leq \frac{3}{4}(|e^{2zi}| + |-e^{-2zi}|) =$

$\frac{3}{4}(|e^{2zi}| + |e^{-2zi}|) = \frac{3}{4}(|e^{2(x+iy)i}| + |e^{-2(x+iy)i}|) =$

$$\frac{3}{4}\left(\left|e^{2xi-2y}\right|+\left|e^{-2xi+2y}\right|\right)=\frac{3}{4}\left(\left|e^{-2y}\right|\left|e^{2xi}\right|+\left|e^{2y}\right|\left|e^{-2xi}\right|\right)=$$

$$\frac{3}{4}\left(\left|e^{-2y}\right|\sqrt{\cos^2(2x)+\sin^2(2x)}+\left|e^{2y}\right|\sqrt{\cos^2(-2x)+\sin^2(-2x)}\right)$$

$$=\frac{3}{4}\left(e^{-2y}\sqrt{1}+e^{2y}\sqrt{1}\right)=\frac{3}{4}\left(e^{-2y}+e^{2y}\right)=$$

$$\frac{3}{4}\left(\left|e^{-2y}\right|+\left|e^{2y}\right|\right)\leq\frac{3}{4}\left(e^{2\cdot 1}+e^{2\cdot 1}\right)\quad\left\{\text{as } -1\leq y\leq 1\right\}$$

$$=\frac{3}{4}\left(e^2+e^2\right)=\frac{3}{4}\left(2e^2\right)=\frac{3}{2}e^2=M=\text{constant}$$

$$\therefore\left|\int_\gamma f(z)\,dz\right|\leq M\cdot L(\gamma)=\frac{3}{2}e^2\cdot 2\pi=3\pi e^2\ \therefore$$

$$\left|\int_\gamma f(z)\,dz\right|=\left|\int_\gamma\frac{3z^3\sin(2z)}{3z^5+1}\,dz\right|\leq 3\pi e^2$$