Laboratory 2

Moodle quiz: 4/22/24 – 5/6/24

Goal

Learn how to model CT systems with Matlab/Octave.

1 Circuit Analysis

We use the electrical circuit in Fig. 1 as running example for studying LTI systems. The analysis of circuits is based on Kirchhoff's voltage law (KVL) and current law (KCL). To understand the behavior from the systems perspective, we derive two circuit models, a state space model and the unique transfer function.

1.1 RLC Circuit

Analysis of the RLC circuit in Fig. 1 with KVL and KCL yields a time dependent differential equation that represents the circuit behavior independent of the particular choice of input voltage $v_{\text{in}}(t)$.

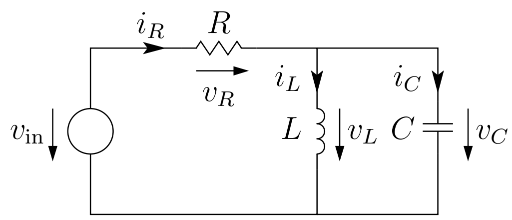

Figure 1: Series composition of resistor R with parallel composition of inductor L and capacitor C.

The device equations of R, L, and C are

$$v_R(t) \;=\; R i_R(t)\,, \qquad v_L(t) \;=\; L\frac{di_L(t)}{dt}\,, \qquad i_C(t) \;=\; C\frac{dv_C(t)}{dt}\,.$$

Applying KVL to the loops to the left and the right of inductor L yields

$$v_{\text{in}}(t) \;=\; v_R(t) + v_L(t)\,, \qquad v_L(t) \;=\; v_C(t)\,,$$

and KCL either at the node above L and C or below yields

$$i_R(t) \;=\; i_L(t) + i_C(t)\,.$$

If we are interested in voltage $v_C(t)$ as function of $v_{\text{in}}(t)$, we may cast the equations above into a single differential equation in $v_C(t)$:

$$\frac{d^2 v_C(t)}{dt^2} + \frac{1}{RC}\frac{dv_C(t)}{dt} + \frac{1}{LC}v_C(t) \;=\; \frac{1}{RC}\frac{dv_{\text{in}}(t)}{dt}\,.$$

This second-order differential equation represents the system with input voltage $v_{in}(t)$ and output voltage $v_C(t)$ in the time domain. Given $v_{in}(t)$, we can solve the differential equation to obtain $v_C(t)$. Analogously, we may deduce a second-order differential equation if we are interested in any other circuit parameter, e.g. resistor voltage $v_R(t)$ or inductor current $i_L(t)$. Finding the solution for a second-order differential equation can be challenging, however, in particular if $v_{in}(t)$ is not continuous such as a unit step. For circuits with more than two energy storage devices, i.e. inductors and capacitors, the order of the differential equation increases accordingly, and we may prefer alternative forms of modeling.

1.2 State-Space Model

A state-space model characterizes a system using state variables. For LTI systems such as the RLC circuit, the $[A, B, C, D]$ representation of the state-space model can be derived in a straightforward fashion. Inductors and capacitors are energy storage devices, and stored energy is the preferred candidate for state. The energy stored in an inductor is reflected by the current flowing through the inductor and the energy stored in a capacitor is reflected by the voltage across the capacitor. Hence, for the RLC circuit in Fig. 1 we define state vector

$$s(t) \;=\; \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix}.$$

The associated CT state-space model consists of the next-state equation, which is a system of two first-order differential equations, and the output equation:

$$\begin{aligned}
\begin{bmatrix} \frac{dv_C(t)}{dt} \\ \frac{di_L(t)}{dt} \end{bmatrix} &= A \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + Bx(t) \\
y(t) &= C \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + Dx(t)
\end{aligned}$$

We determine input $x(t)$ and output $y(t)$ by choosing $v_{\text{in}}(t) = x(t)$ and $v_C(t) = y(t)$, which results in a SISO system:

$$\begin{aligned}
\begin{bmatrix} \frac{dv_C(t)}{dt} \\ \frac{di_L(t)}{dt} \end{bmatrix} &= A \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + Bv_{\text{in}}(t) \\
v_C(t) &= C \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + Dv_{\text{in}}(t)
\end{aligned}$$

Using the circuit analysis in Sec. 1.1 we can identify the $[A, B, C, D]$ matrices immediately, in particular without deriving the second-order differential equation:

$$\begin{aligned}
\frac{dv_C(t)}{dt} &= \frac{1}{C}i_C(t) \;=\; \frac{1}{C}(i_R(t) - i_L(t)) \;=\; \frac{1}{RC}\big(v_{\text{in}(t)} - v_C(t)\big) - \frac{1}{C}i_L(t) \\
\frac{di_L(t)}{dt} &= \frac{1}{L}v_L(t) \;=\; \frac{1}{L}v_C(t)
\end{aligned}$$

The resulting state-space model for the RLC circuit in Fig. 1 with output $v_C(t)$ is:

$$
\begin{aligned}
\begin{bmatrix} \frac{dv_C(t)}{dt} \\ \frac{di_L(t)}{dt} \end{bmatrix} &= \begin{bmatrix} -\frac{1}{RC} & -\frac{1}{C} \\ \frac{1}{L} & 0 \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + \begin{bmatrix} \frac{1}{RC} \\ 0 \end{bmatrix} v_{\text{in}}(t) \\
v_C(t) &= \begin{bmatrix} 1 & 0 \end{bmatrix} \begin{bmatrix} v_C(t) \\ i_L(t) \end{bmatrix} + 0 v_{\text{in}}(t)
\end{aligned}
$$

1.3 Transfer Function

The state-space model depends on the choice of the state variables. In general, this choice is not unique. In contrast, the transfer function of an LTI system is unique, once the input and output are fixed. For a CT system, such as the electrical circuit in Fig. 1, we obtain the transfer function by means of the Laplace transform.

Starting with the circuit analysis in Sec. 1.1, we apply the differentiation property of the Laplace transform

$$\frac{dx(t)}{dt} \quad \overset{\mathcal{L}}{\longleftrightarrow} \quad sX(s)$$

to turn the differential device equations into algebraic equations in complex variable s. The transformed device equations are:

$$V_R \;=\; RI_R\,, \qquad V_L = LsI_L\,, \qquad I_C = CsV_C\,.$$

The KVL and KCL equations transform by linearity (additivity) into:

$$V_{\text{in}} \;=\; V_R + V_L\,, \qquad V_L \;=\; V_C\,, \qquad I_R \;=\; I_L + I_C\,.$$

Choosing output V_C, we deduce the transfer function $H(s)$ algebraically:

$$V_{\text{in}} \;=\; V_R + V_C \;=\; R(I_L + I_C) + V_C \;=\; \frac{RV_C}{sL} + sRCV_C + V_C$$

$$\Rightarrow \qquad H(s) \;=\; \frac{V_C}{V_{\text{in}}} \;=\; \frac{\frac{1}{RC}s}{s^2 + \frac{1}{RC}s + \frac{1}{LC}}$$

The second-order differential equation of Sec. 1.1 comes in handy to verify the derivation of the transfer function or vice versa.

2 Circuit Modeling with Matlab/Octave

The `control` package of Matlab/Octave supports state-space models and transfer functions.

2.1 State-Space Modeling

Let us use the state-space model of Matlab/Octave. We begin by defining the state-space model for the RLC circuit in Fig. 1 using specific device values:

$$R \;=\; 100\,\Omega\,, \qquad L \;=\; 1\,\text{mH}\,, \qquad C \;=\; 10\,\text{nF}\,.$$

Define the matrices of the $[A, B, C, D]$ representation derived in Sec. 1.2:

```
R = 100, L = 1e-3, C = 10e-9;
A = [ -1/(R*C), -1/C; 1/L, 0 ];
B = [ 1/(R*C); 0];
C = [ 1, 0];
D = 0;
sys = ss(A,B,C,D);
```

Variable `sys` stores the state-space model of the circuit, see `help ss`.

We can simulate the system response for various inputs, including the impulse response and step response. For example, the step response is simulated with command

```
step(sys);
```

which plots the step response as shown in Fig. 2. The simulated input is the unit step:

$$v_{in}(t) \;=\; \begin{cases} 1\,\mathrm{V}\,, & t \geq 0\,, \\ 0\,\mathrm{V}\,, & t < 0 \end{cases}$$

For comparison, Fig. 3 shows the screenshot of the Analog Discovery 2 scope recording of the step response of the real circuit.

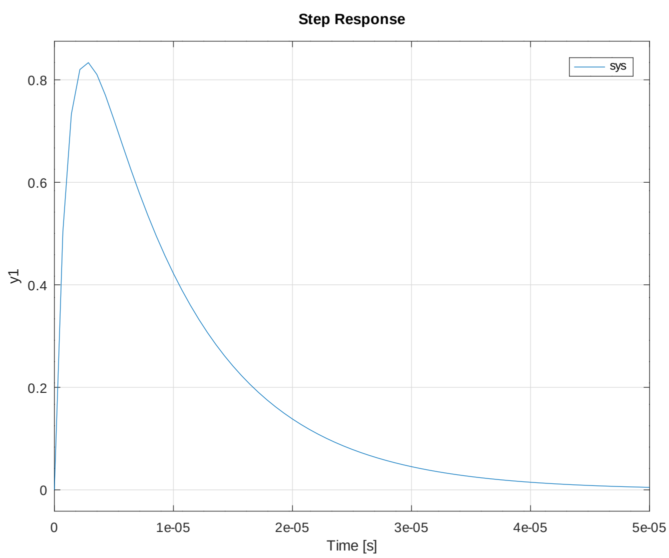

Figure 2: Unit step response of RLC circuit generated by `step(sys)`.

If we wish to apply a step input that jumps to, say, 5 V instead of 1 V, we exploit the linearity (homogeneity) of the LTI system, and plot the step response scaled by factor 5:

```
[y,t,_] = step(sys);
plot(t, 5*y);
xlabel('Time [s]'), ylabel('Step response [V]');
```

Note that Matlab/Octave performs the simulation using the DT state-space model, which it derives from the CT state-space model automatically, without us having to perform the time discretization of the derivative manually.

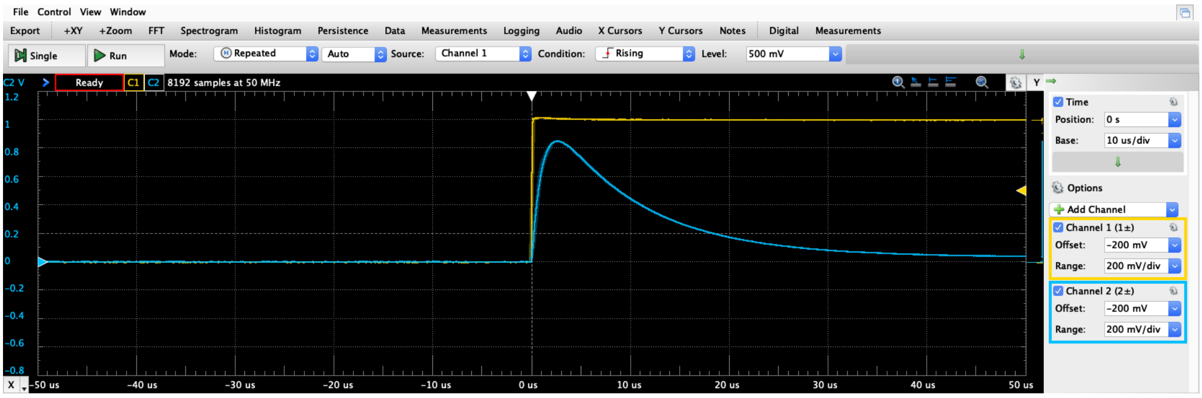

Figure 3: Scope screenshot of unit step response of RLC circuit.

2.2 Transfer-Function Modeling

Analogous to the state-space model, Matlab/Octave permits computing the system response from the transfer function. Given the RLC circuit in Fig. 1, we define the system using the transfer function derived in Sec. 1.3, see `help tf`:

```
R = 100, L = 1e-3, C = 10e-9;
num = [ 1/(R*C), 0 ];
den = [ 1, 1/(R*C), 1/(L*C) ];
sys = tf(num, den);
```

Simulating the unit step response with command

```
step(sys);
```

produces the same plot as shown in Fig. 2. Both models state-space and transfer function are equivalent. You may use whichever you consider more convenient.

It is also possible to convert the transfer function into a state-space model:

```
[A,B,C,D] = tf2ss(num, den);
```

The result is some $[A, B, C, D]$ representation that is not necessarily identical to the one derived in Sec. 1.2. Furthermore, the inverse conversion

```
[num, den] = ss2tf(A,B,C,D);
```

converts the $[A, B, C, D]$ representation into the transfer function, which should be equal to the one derived in Sec. 1.3, due to the uniqueness of the transfer function.

3 Lab Problems

1. Determine the unit step response of the RLC circuit in Fig. 1 with output $v_C(t)$, when choosing different inductors:

   (a) $L = 100\,\mu\text{H}$, and

   (b) $L = 10\,\mu\text{H}$.

2. Use Matlab/Octave function `bode` to find the frequency response of the RLC circuit with output $v_C(t)$. Which filter characteristic does the system exhibit? Explain the step response with the frequency response.

3. Determine the unit step response of the RLC circuit in Fig. 1 when selecting the resistor voltage $v_R(t)$ as output.

4. Use Matlab/Octave function `bode` to find the frequency response of the RLC circuit with output $v_R(t)$. Which filter characteristic does the system exhibit? Can you explain the step response with the frequency response?