Laboratory 4

Moodle quiz: 4/28/25 – 5/5/25

Goal

Experiment with aliasing using Matlab/Octave.

1 Sampling Sinusoids

If we sample sinusoid $x(t) = \cos(2\pi f_0 t)$ with sampling rate $f_s = 1/T_s$ then the samples are:

$$\forall n \in \mathbb{Z}\,, \quad x[n] \;=\; x(nT_s) \;=\; \cos(2\pi f_0 n T_s)\,.$$

Aliasing refers to the phenomenon that frequency $f_k = f_0 + kf_s$ is indistinguishable from f_0 after sampling for all $k \in \mathbb{Z}$. In other words, in DT, frequency f_0 has infinitely many aliases. This fact is easy to see by means of trigonometry, $\cos(x + y) = \cos(x)\cos(y) - \sin(x)\sin(y)$:

$$\begin{aligned}
\forall k, n \in \mathbb{Z}\,,\ \cos(2\pi(f_0 + kf_s)nT_s) &= \cos(2\pi f_0 nT_s + 2\pi kn) \\
&= \cos(2\pi f_0 nT_s)\underbrace{\cos(2\pi kn)}_{=1} - \sin(2\pi f_0 nT_s)\underbrace{\sin(2\pi kn)}_{=0} \\
&= \cos(2\pi f_0 nT_s)
\end{aligned}$$

We can experience aliasing with sampled audio signals using Matlab/Octave. Let us choose audio sampling rate $f_a = 28\,\text{kHz}$:

```
fa = 28000;                 % audio sampling rate
ta = [0:1/fa:1];            % sampling times for 1 second of sound
```

The sampling rate must match the capabilities of your audio hardware to produce a sound. Common values for f_a supported by hardware are in range $8\,\text{kHz} \le f_a \le 44.1\,\text{kHz}$. The following script plays pure cosines with frequencies $f_k = f_0 + kf_s$:

```
fs = 700;                   % sampling rate
f0 = 200;                   % frequency of pure tone

for k = 4:-1:-4
  fk = f0 + k*fs
  x  = cos(2*pi*fk*ta);     % sample cos with audio sampling rate fa
  sound(x, fa);             % play sampled cos
endfor
```

Each cosine of frequency f_k is sampled with audio sampling rate f_a at time points `ta`. The sampling theorem predicts that there is no aliasing if $f_a > 2f_k$. This is the case for all k in the `for` loop. Therefore, each frequency f_k produces a unique pure tone.

In contrast, if we sample the cosine with sampling rate $f_s = 700\,\text{Hz}$ rather than f_a, we expect all frequencies f_k to be indistinguishable. The next script samples each cosine with sampling rate f_s at time points `ts`, even though f_s is too low for common audio hardware:

```
ts = [0:1/fs:1];          % sampling times for 1 second of sound
for k = 4:-1:-4
  fk = f0 + k*fs
  x  = cos(2*pi*fk*ts);   % sample cos with sampling rate fs
  sound(upconv(x, 1/fs, fa/fs), fa);  % play upconverted cos
endfor
```

This script deals with the complication that f_s is probably too low for your audio hardware to produce an audible sound. If you call

```
sound(x, fs)
```

then Matlab/Octave complains with an error message or fails silently. We solve this problem by up-converting the sampled cosine `x` to audio sampling rate f_a with function `upconv`:

```
function y = upconv(x, Ts, M)
  % upconversion of signal x by M
  % x  = sampled signal
  % Ts = sampling period of x
  % M  = upsampling factor (must be integer)
  L = 42 * M;                % L = one-sided sinc length
  N = length(x);             % N = samples in x
  y = zeros(1, (N-1)*M+1);   % make space for M-1 zeros between samples
  y(1:M:end) = x;            % upsampling: insert zeros

  s = sinc([-L:L] / M);      % LPF with truncated sinc
  y = conv(y, s, 'same');    % mode 'same' clips the convolution
endfunction
```

Save this function in file `upconv.m`. Now, playing the cosines with frequencies f_k produces indistinguishable pure tones of frequency $f_0 = 200\,\text{Hz}$. This is the effect of aliasing.

Exercises

1. Replace the cosine with a sine function. Do you hear the difference?

2. Explain why negative frequencies f_k produce audible tones by expressing the corresponding cosines and sines with positive frequencies using trigonometric identities.

3. Which pure-tone frequency do you hear when changing f_0 to 300 Hz, 400 Hz, and 500 Hz? Explain why you hear each frequency.

2 Lab Problem

Consider a composite signal of two sinusoids:

$$x(t) \;=\; \frac{1}{2}\big(\cos(2\pi f_0 t) + \sin(2\pi(2f_0)t)\big)\,.$$

Implement a script that plays the sound of signal $x(t)$ for $f_0 = 400\,\mathrm{Hz}$ when sampled with different sampling rates:

$$f_s \;\in\; \{1800\,\mathrm{Hz},\; 1200\,\mathrm{Hz},\; 900\,\mathrm{Hz},\; 720\,\mathrm{Hz},\; 600\,\mathrm{Hz}\}$$

For each of the sampling frequencies, explain why you hear the sound you hear.