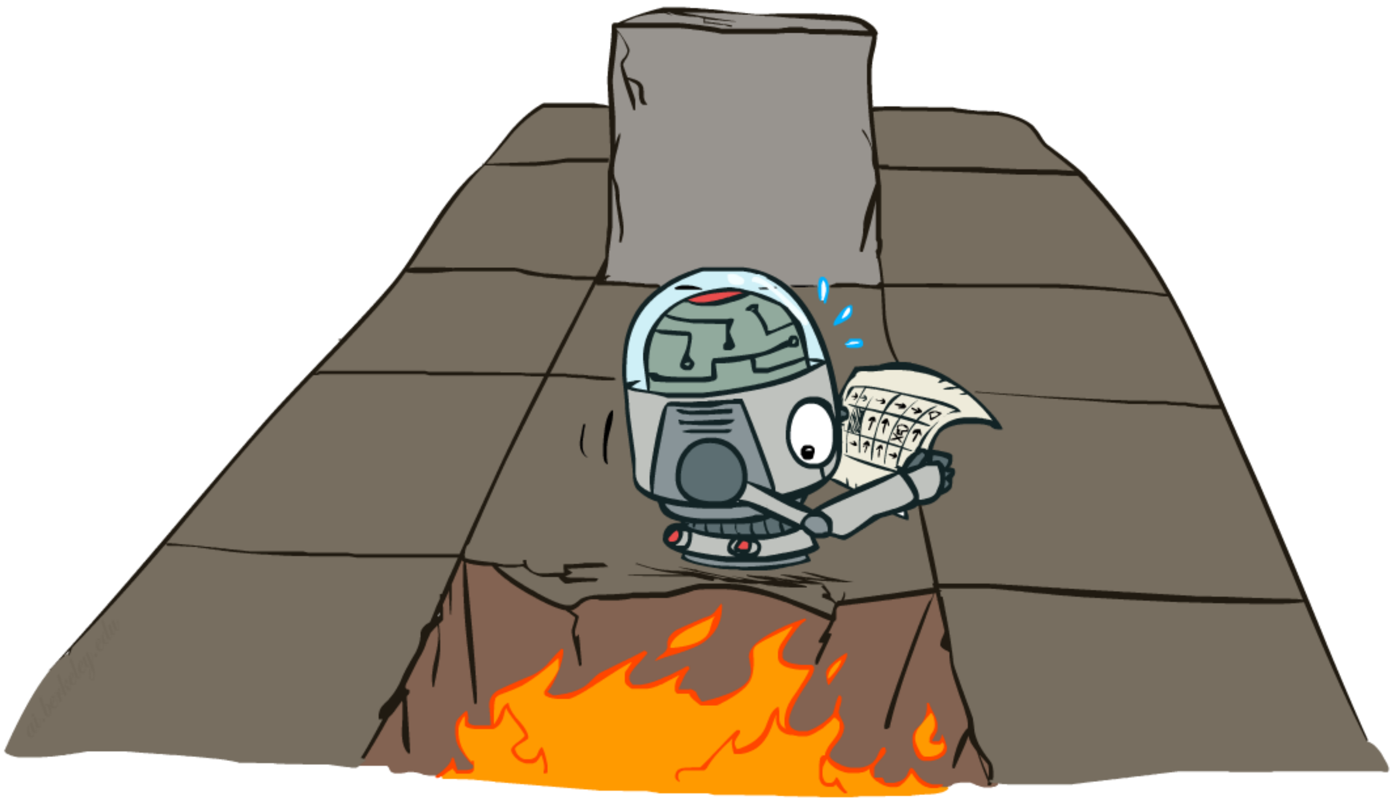
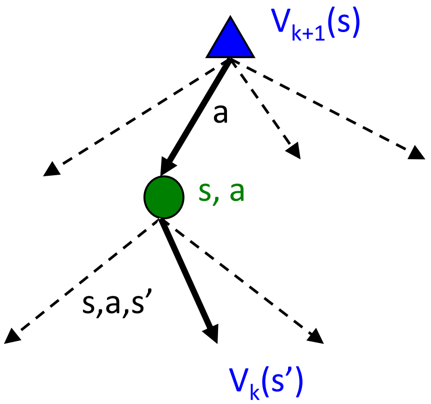
Markov Decision Processes II



The Bellman Equations

- Definition of "optimal utility" via Expectimax recurrence gives a simple one-step lookahead relationship amongst optimal utility values



$$V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

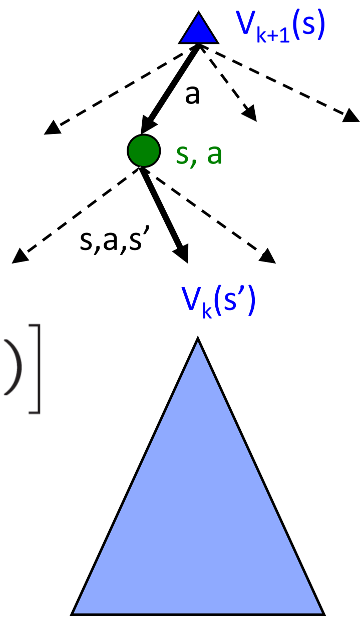
$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

- These are the Bellman equations, and they characterize optimal values in a way they can be used over and over

Value Iteration

- Bellman equations characterize the optimal values:

$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s') \right]$$



- Value iteration computes them:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s') \right]$$

- Value iteration is just a fixed point solution method
  - … though the V_k vectors are also interpretable as time-limited values

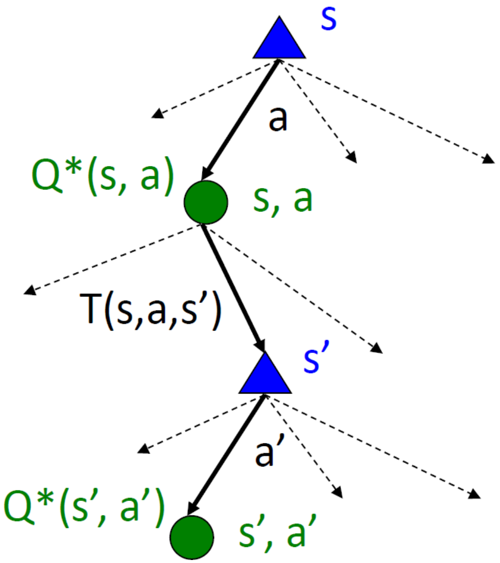
Quiz: Bellman equation for Q values?

- We saw Bellman equation that characterized optimal V*(s)

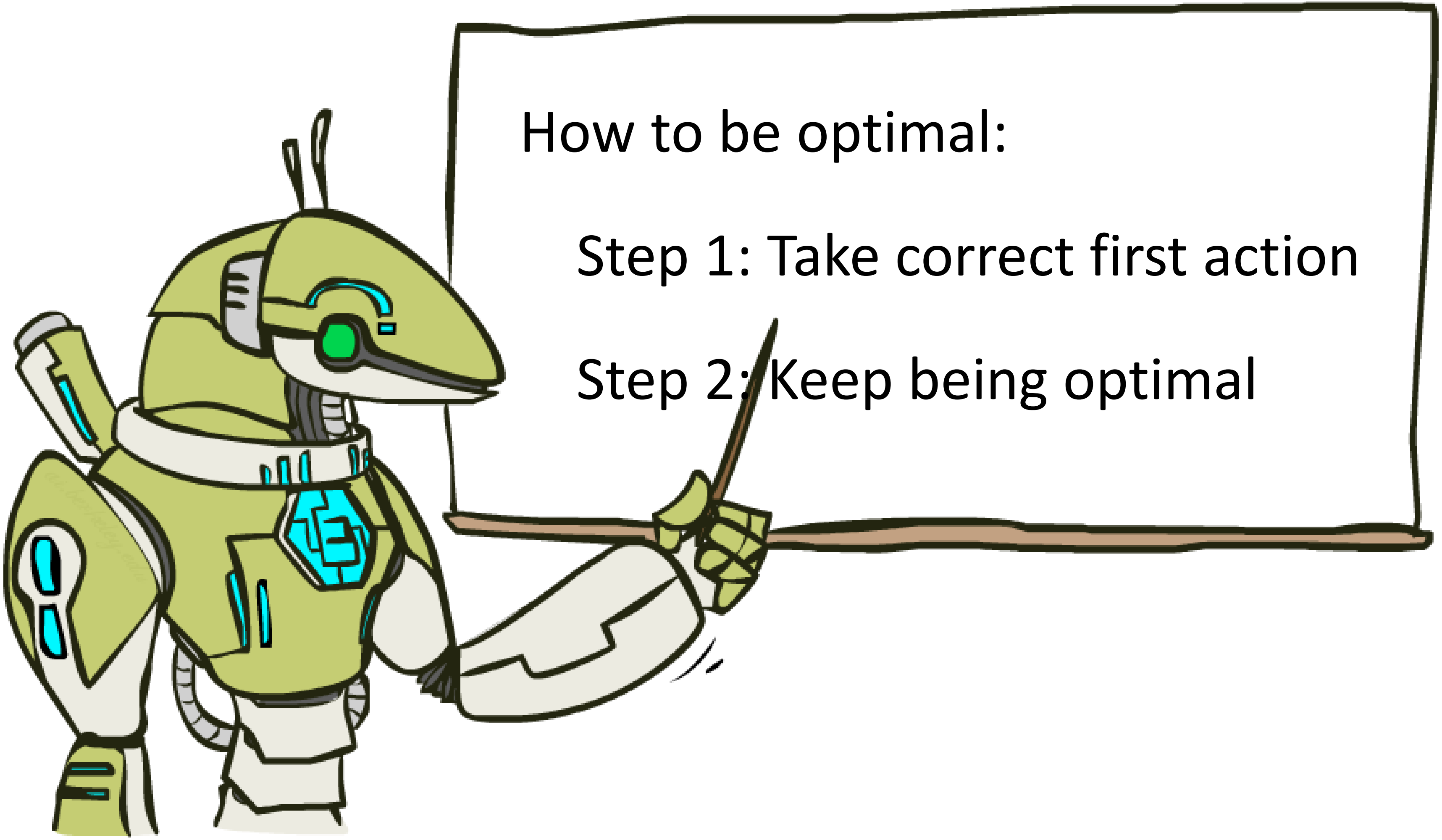
$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s')\right]$$

- Can we write down Bellman equation for Q*(s,a)?

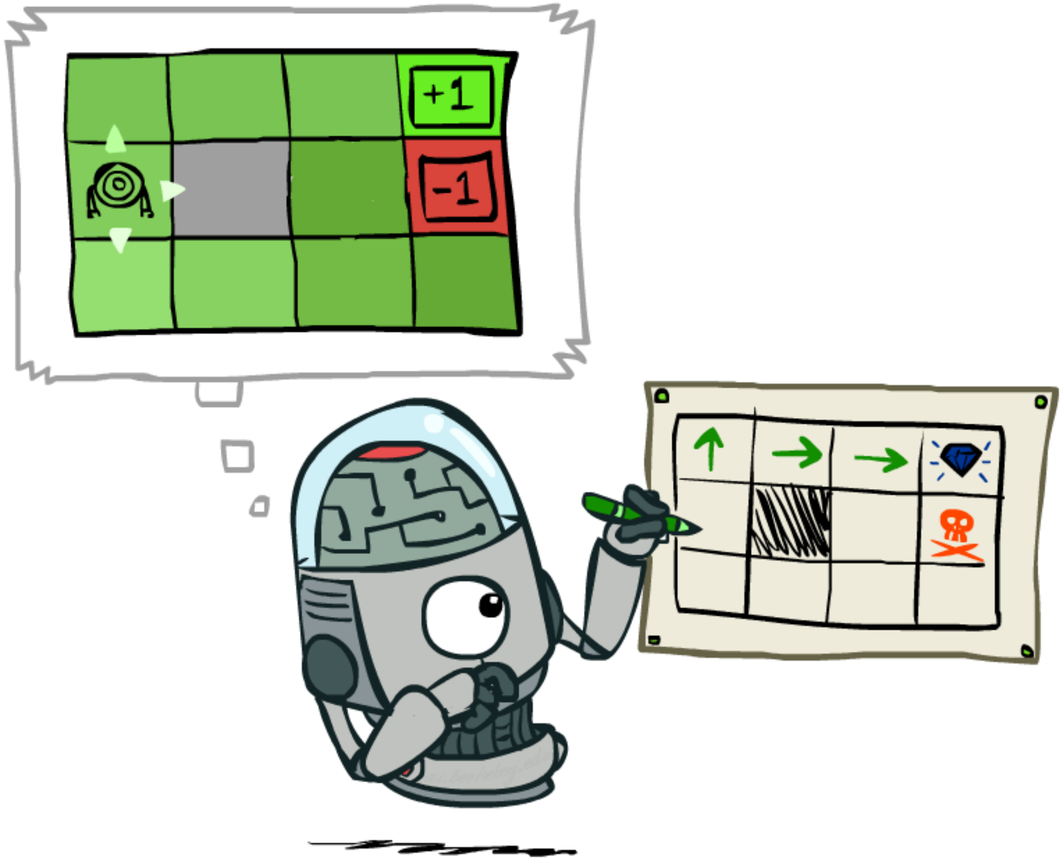
$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma \max_{a'} Q^*(s',a') \right]$$



The Bellman Equations

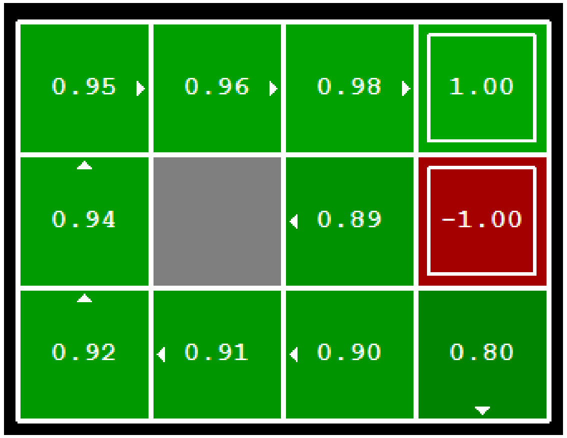


Policy Extraction



Computing Actions from Values

- Let’s imagine we have the optimal values V*(s)



- How should we act?
  - It’s not obvious!
- We need to do a mini-Expectimax (one step)

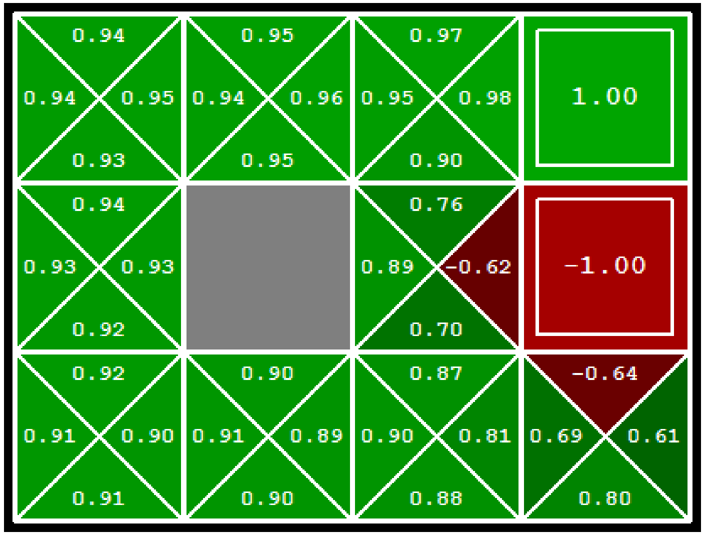
$$\pi^*(s) = \arg\max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

- This is called policy extraction, since it gets the policy implied by the values

Computing Actions from Q-Values

- Let's imagine we have the optimal q-values:
- How should we act?
  - Completely trivial to decide!

$$\pi^*(s) = \arg\max_a Q^*(s,a)$$

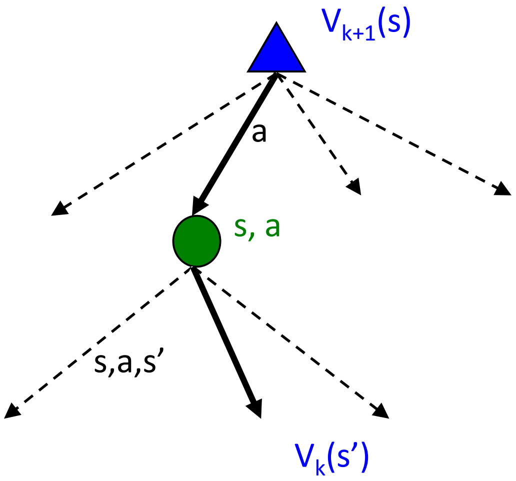


- Important lesson: actions are easier to select from q-values than values!

Problems with Value Iteration

- Value iteration repeats the Bellman updates:

$$V_{k+1}(s) \leftarrow \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V_k(s')\right]$$



- Problem 1: It's slow – $O(S^2A)$ per iteration

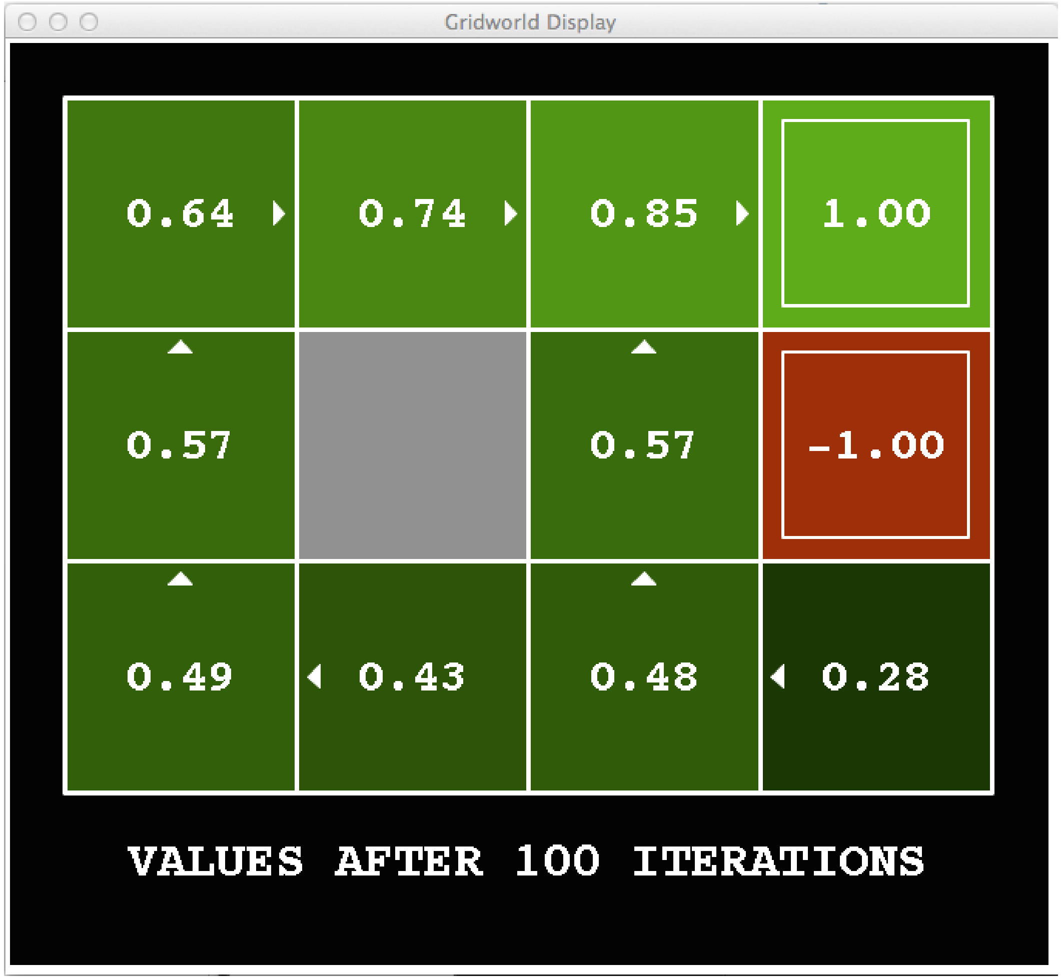
- Problem 2: The "max" at each state rarely changes

- Problem 3: The policy often converges long before the values

k=12

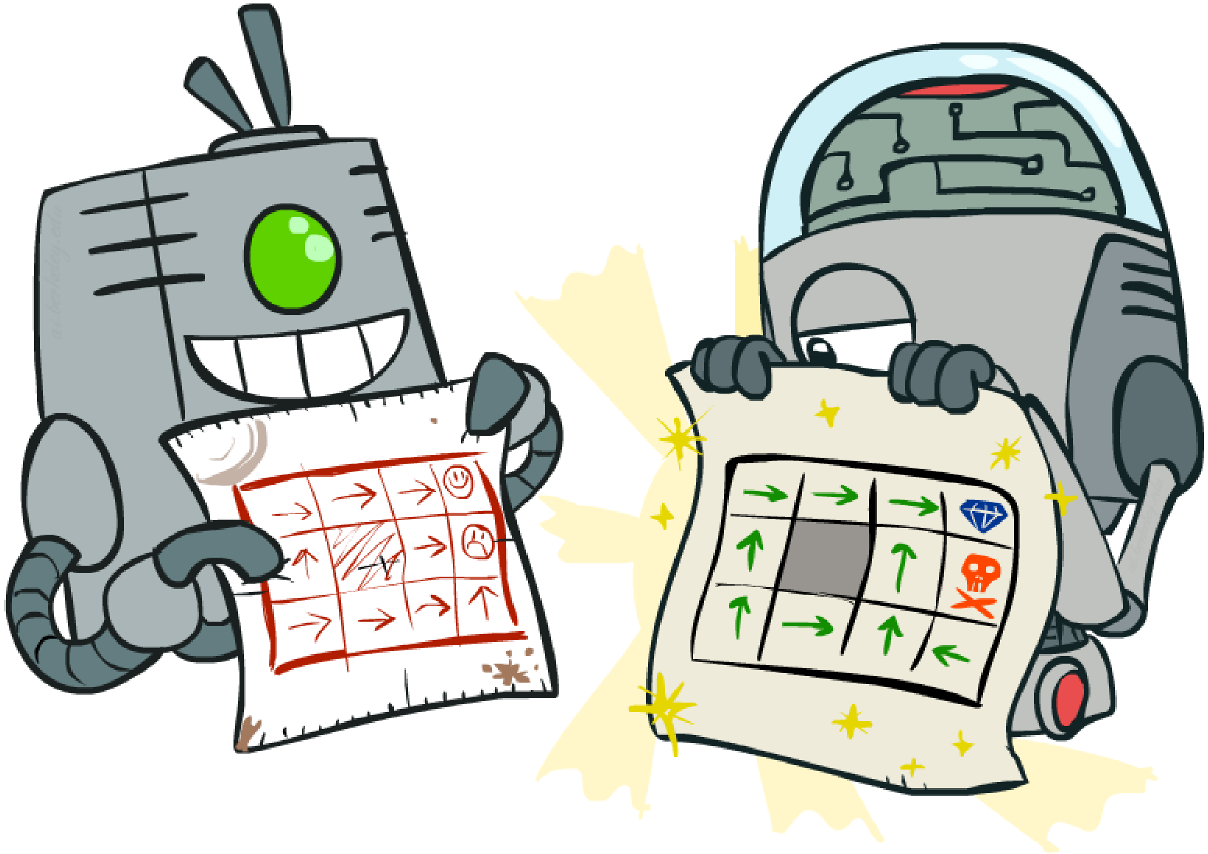
Noise = 0.2
Discount = 0.9
Living reward = 0

k=100

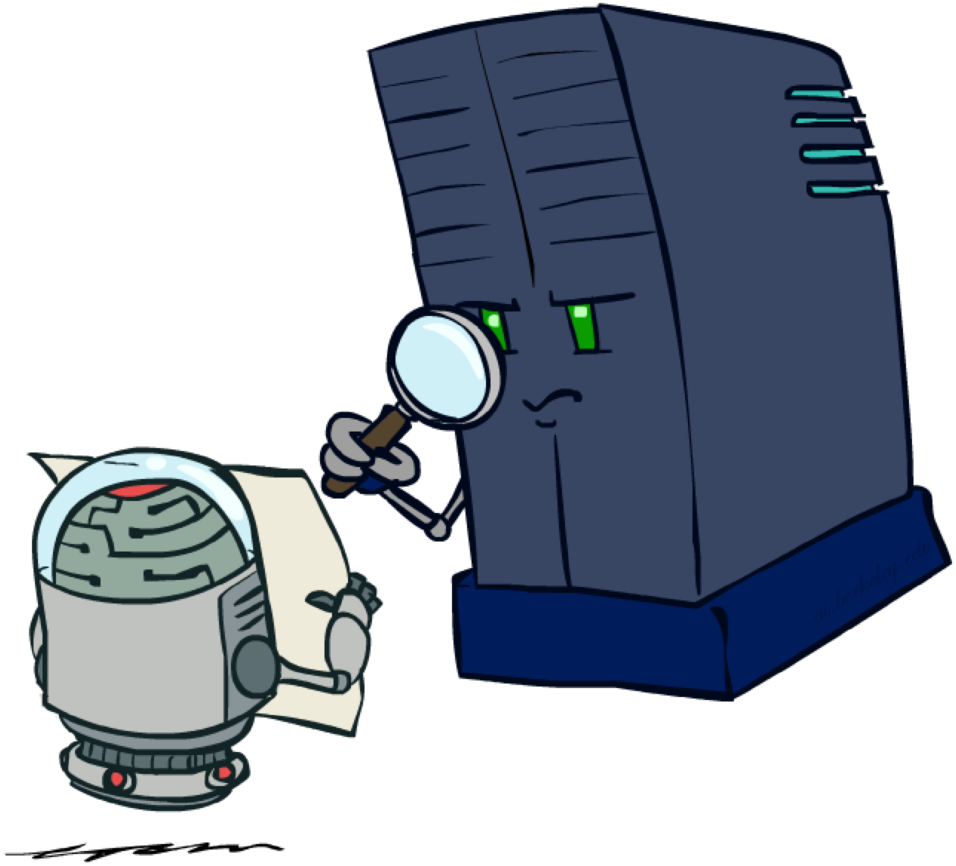


Noise = 0.2
Discount = 0.9
Living reward = 0

Policy Methods



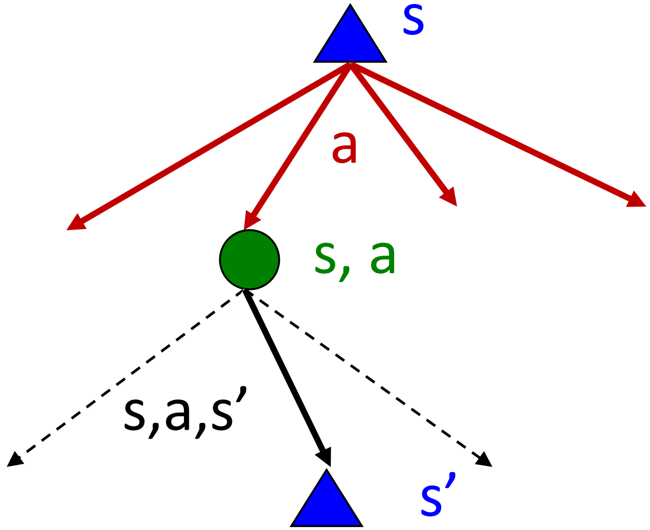
Policy Evaluation

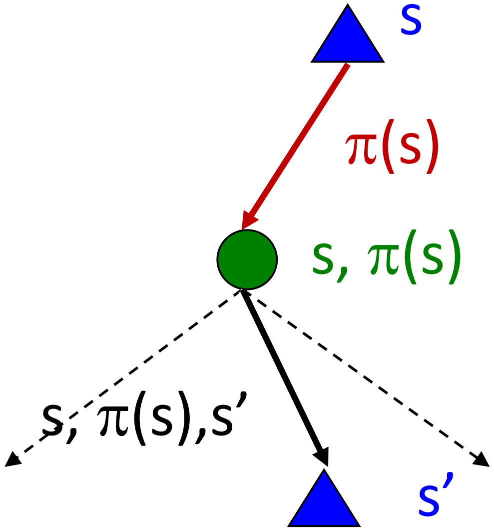


Fixed Policies

Do the optimal action $V^*(s)$

Do what π says to do $V^\pi(s)$



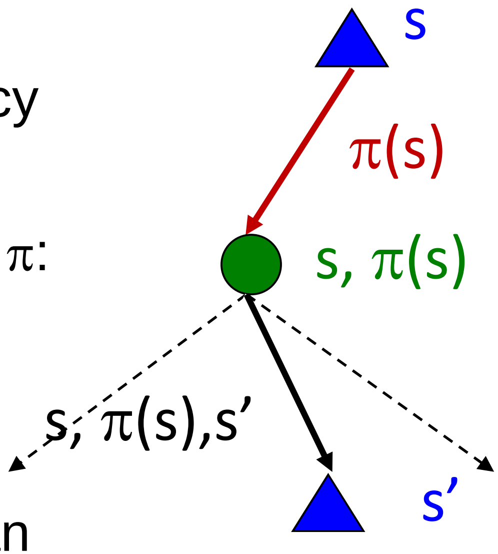


- Expectimax trees max over all actions to compute the optimal values
- If we fixed some policy $\pi(s)$, then the tree would be simpler – only one action per state
  - … though the tree's value would depend on which policy we fixed

Utilities for a Fixed Policy

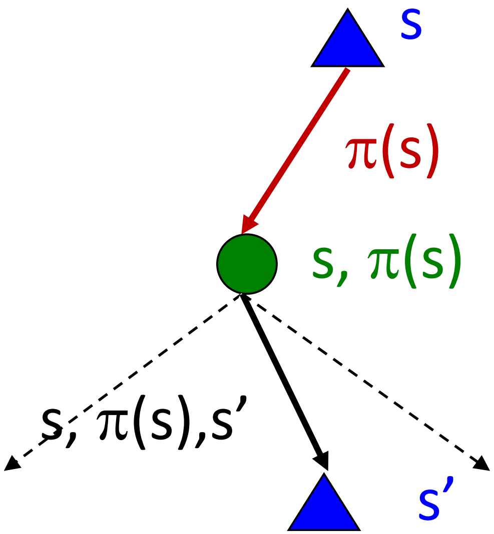
- Another basic operation: compute the utility of a state s under a fixed (generally non-optimal) policy
- Define the utility of a state s, under a fixed policy π:

  $V^{\pi}(s)$ = expected total discounted rewards starting in s and following π
- Recursive relation (one-step look-ahead / Bellman equation):



$$V^{\pi}(s) = \sum_{s'} T(s,\pi(s),s')[R(s,\pi(s),s') + \gamma V^{\pi}(s')]$$

Policy Evaluation

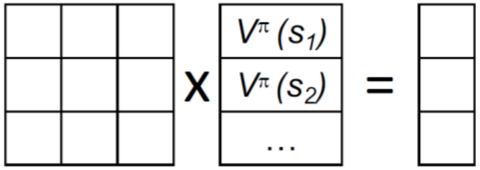


- How do we calculate the V's for a fixed policy π?
- Idea 1: Turn recursive Bellman equations into updates (like value iteration)

$$V_0^\pi(s) = 0$$

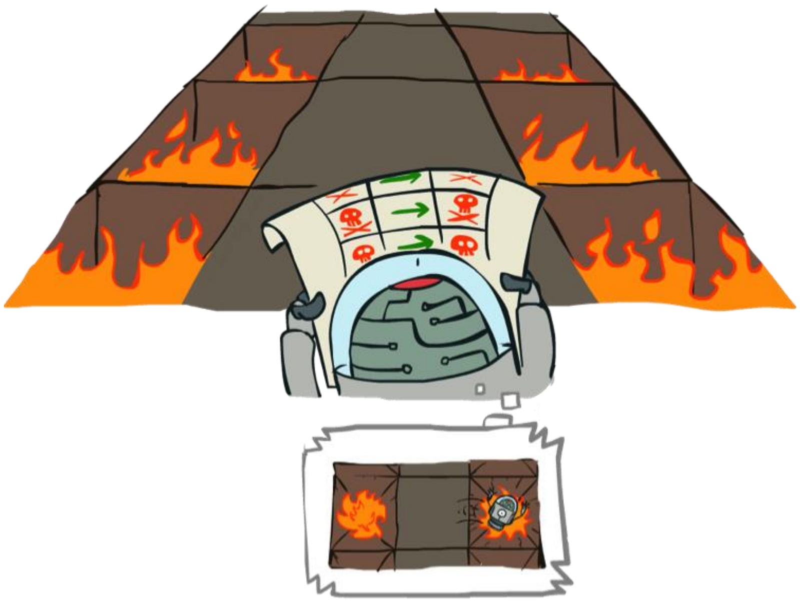
$$V_{k+1}^\pi(s) \leftarrow \sum_{s'} T(s,\pi(s),s')[R(s,\pi(s),s') + \gamma V_k^\pi(s')]$$

- Efficiency: $O(S^2)$ per iteration
- Idea 2: Without the maxes, the Bellman equations are just a linear system
  - Solve with your favorite linear system solver (e.g. Matlab)

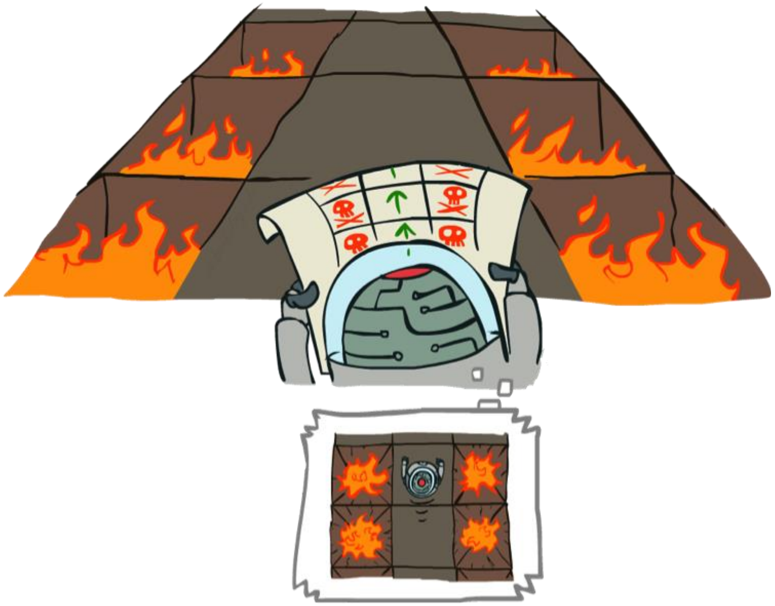


Example: Policy Evaluation

Always Go Right



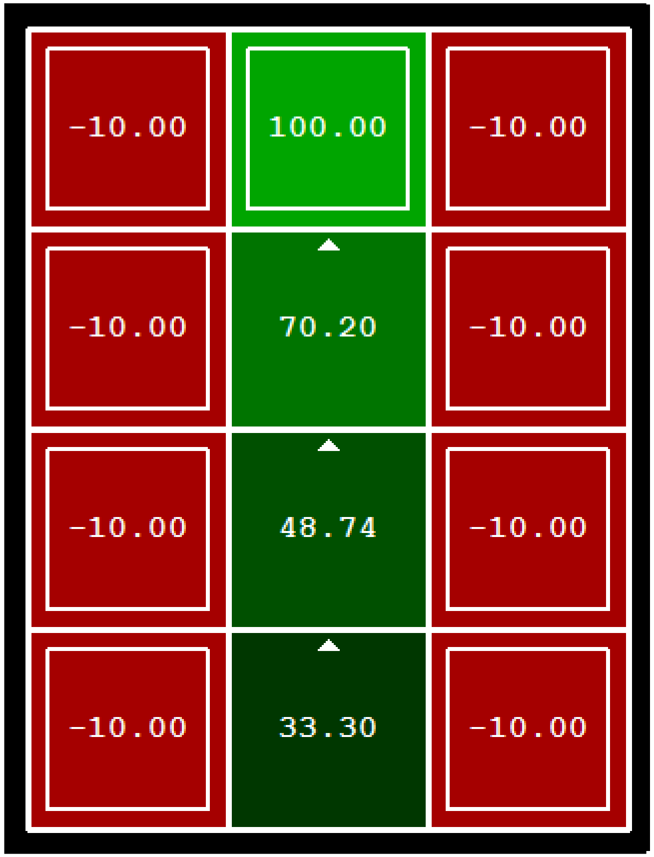
Always Go Forward



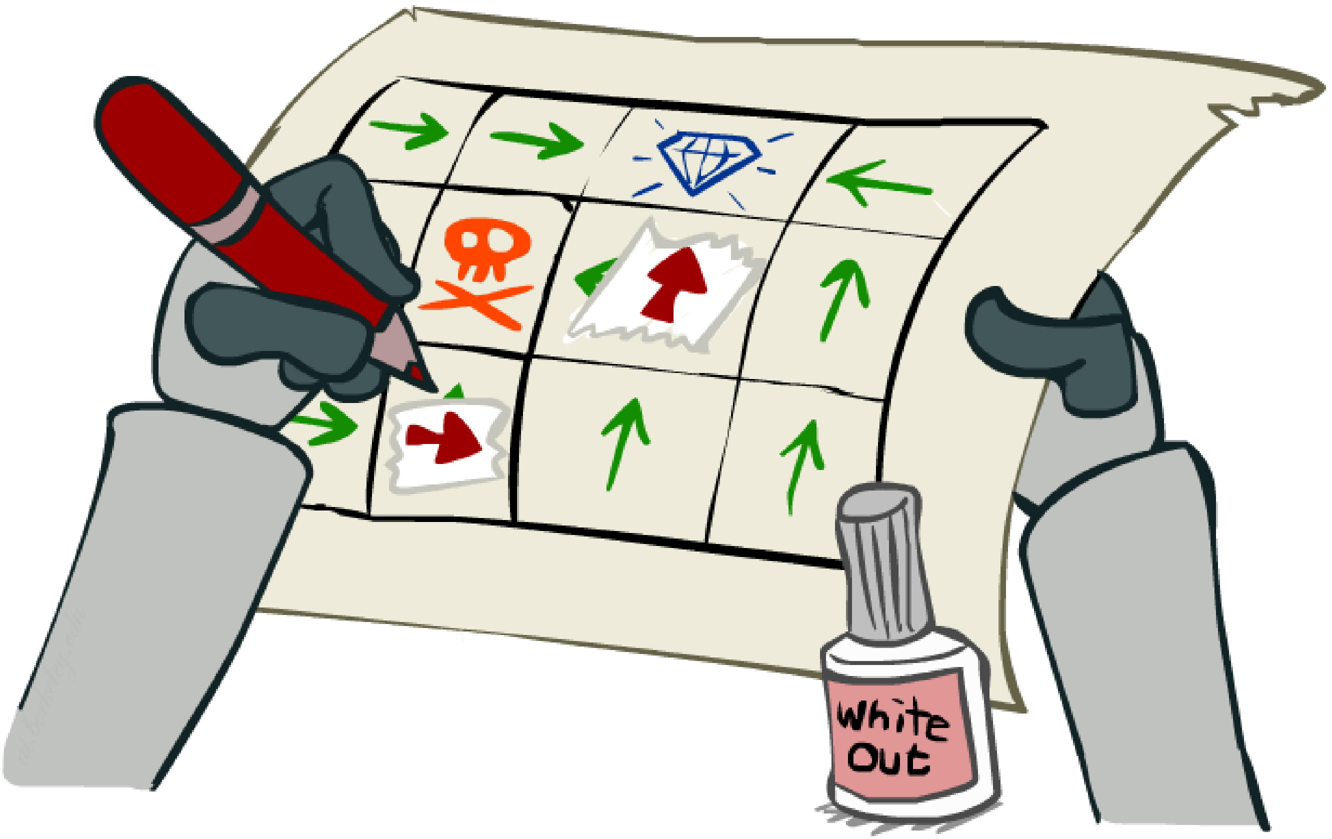
Example: Policy Evaluation

Always Go Right

Always Go Forward



Policy Iteration



Policy Iteration

- Alternative approach for optimal values:
  - Step 1: Policy evaluation: calculate utilities for some fixed policy (not optimal utilities!) until convergence
  - Step 2: Policy improvement: update policy using one-step look-ahead with resulting converged (but not optimal!) utilities as future values
  - Repeat steps until policy converges

- This is policy iteration
  - It's still optimal!
  - Can converge (much) faster under some conditions

Policy Iteration

- Evaluation: For fixed current policy π, find values with policy evaluation:
  - Iterate until values converge:

$$V_{k+1}^{\pi_i}(s) \leftarrow \sum_{s'} T(s, \pi_i(s), s') \left[R(s, \pi_i(s), s') + \gamma V_k^{\pi_i}(s')\right]$$

- Improvement: For fixed values, get a better policy using policy extraction
  - One-step look-ahead:

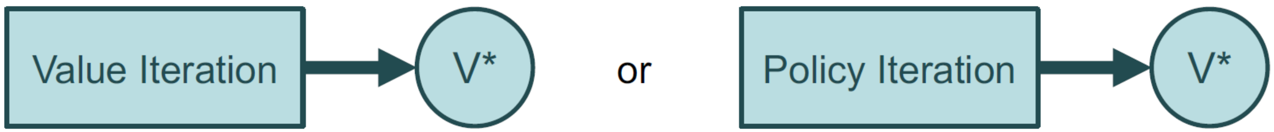
$$\pi_{i+1}(s) = \arg\max_a \sum_{s'} T(s, a, s') \left[R(s, a, s') + \gamma V^{\pi_i}(s')\right]$$

Comparison

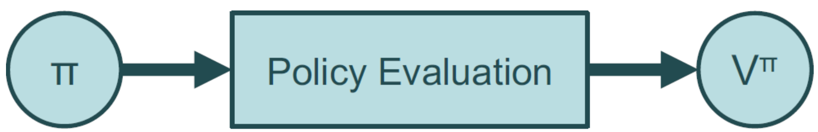
- Both value iteration and policy iteration compute the same thing (all optimal values)
- In value iteration:
  - Every iteration updates both the values and (implicitly) the policy
  - We don't track the policy, but taking the max over actions implicitly recomputes it
- In policy iteration:
  - We do several passes that update utilities with fixed policy (each pass is fast because we consider only one action, not all of them)
  - After the policy is evaluated, a new policy is chosen (slow like a value iteration pass)
  - The new policy will be better (or we're done)
- Both are dynamic programs for solving MDPs

Summary: MDP Algorithms

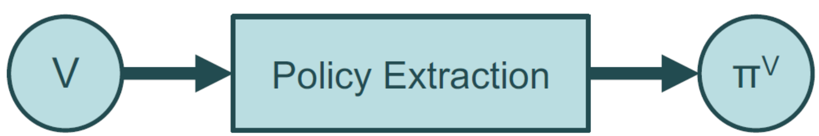
- So you want to….
  - Compute optimal values: use value iteration or policy iteration



  - Compute values for a particular policy: use policy evaluation



  - Turn your values into a policy: use policy extraction (one-step lookahead)



Summary: MDP Algorithms

- So you want to….
  - Compute optimal values: use value iteration or policy iteration
  - Compute values for a particular policy: use policy evaluation
  - Turn your values into a policy: use policy extraction (one-step lookahead)

- These all look the same!
  - They basically are – they are all variations of Bellman updates
  - They all use one-step lookahead Expectimax fragments
  - They differ only in whether we plug in a fixed policy or max over actions

Summary: Bellman Equation Zoo!

- Optimal V and Q value functions:

$$V^*(s) = \max_a \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma V^*(s') \right] \qquad V^*(s) = \max_a Q^*(s,a)$$

$$Q^*(s,a) = \sum_{s'} T(s,a,s') \left[R(s,a,s') + \gamma \max_{a'} Q^*(s',a') \right]$$

- Value function for fixed policy p:

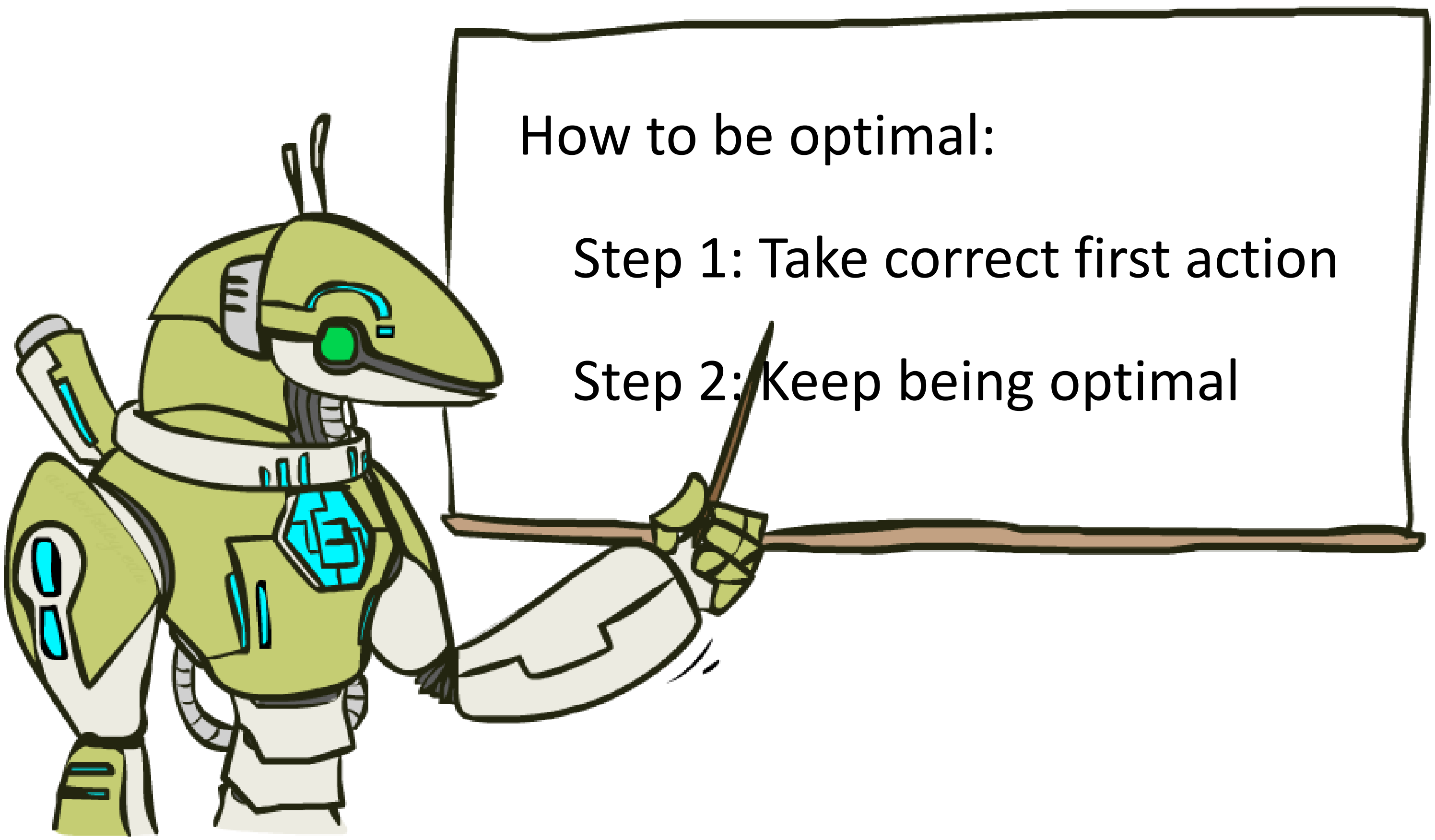
$$V^\pi(s) = \sum_{s'} T(s,\pi(s),s')[R(s,\pi(s),s') + \gamma V^\pi(s')]$$

- Policy p for V and Q value functions:

$$\pi^*(s) = \arg\max_a \sum_{s'} T(s,a,s')[R(s,a,s') + \gamma V^*(s')]$$

$$\pi^*(s) = \arg\max_a Q^*(s,a)$$

The Bellman Equations



"Journey of a thousand optimal steps begins with a first optimal step"