Astronomical Image Processing

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Set Up

Before We Start

Tutorial materials and link to online Binder interface available at:

https://github.com/sfarrens/euroscipy

If you plan to run the tutorial via Binder now would be a good time to launch it!

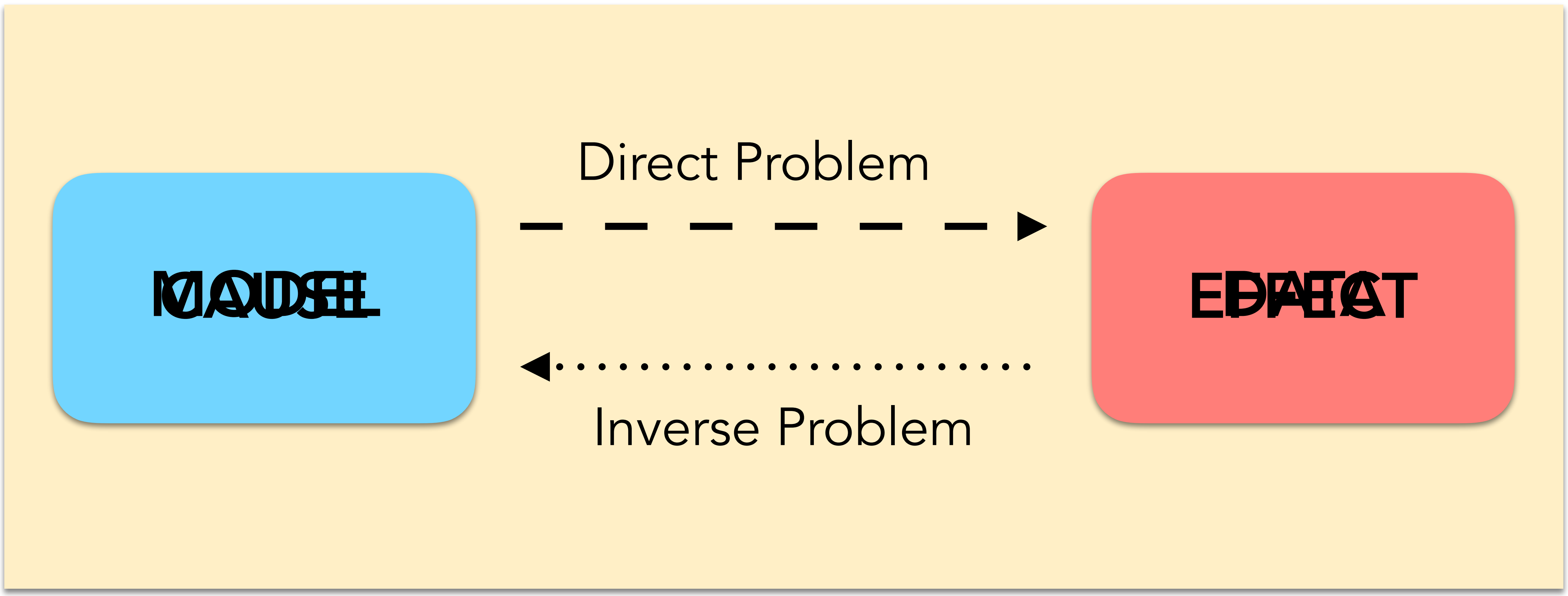
Otherwise, if you are running the tutorial locally, make sure you:
- create the conda environment
- then activate the environment

```
conda env create -f environment.yml
conda activate euroscipy-astro
```

Outline

- **Inverse Problems**
  - Linear Regression
  - Ill-posed Problems
- **Regularisation**
  - Sparsity
- **Jupyter Notebooks**

Inverse Problems



With an inverse problem one attempts to obtain information about a physical system from observed measurements.

Inverse Problems

Linear Inverse Problem

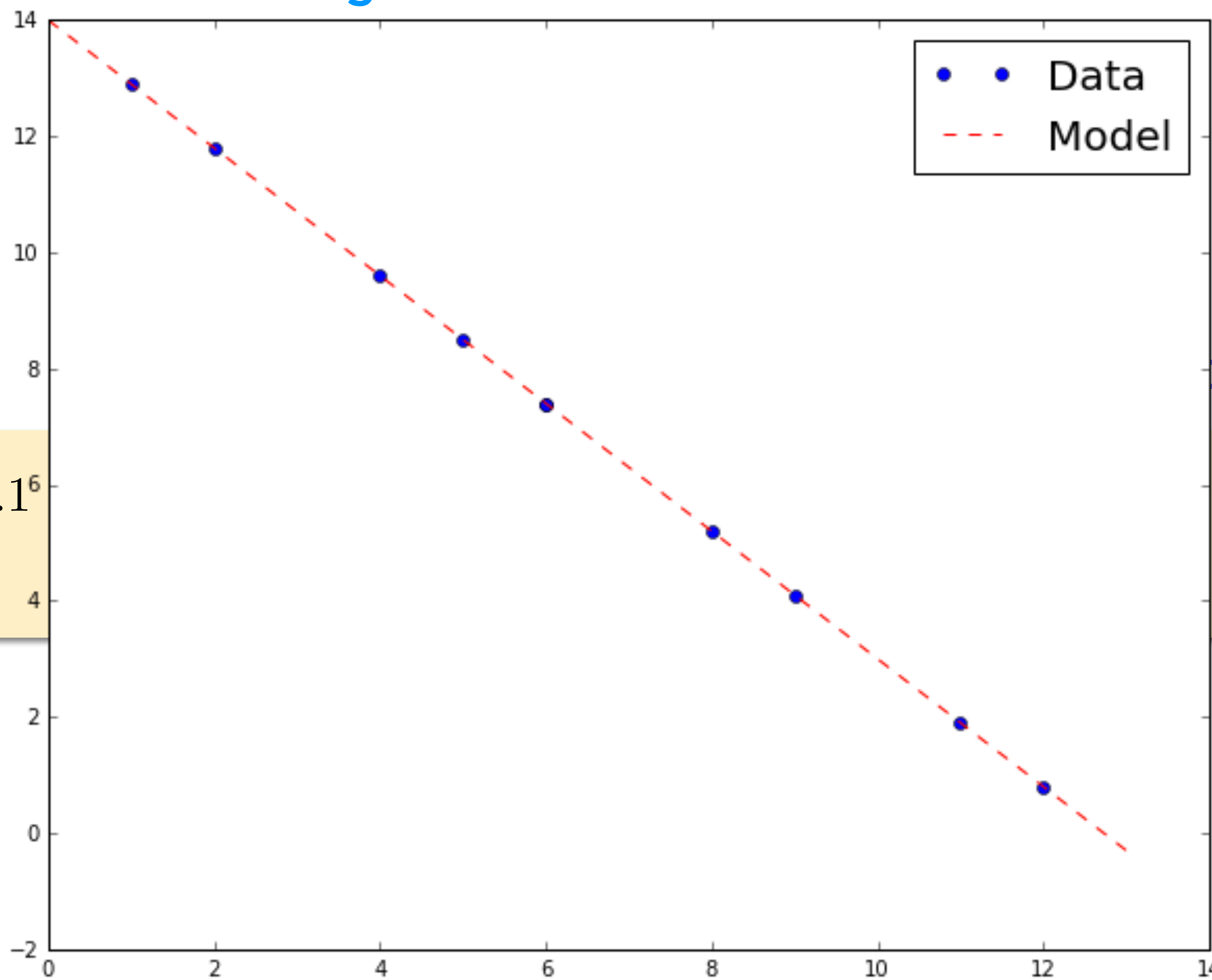
$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

$\mathbf{y} \in \mathbb{R}^m$ | $\mathbf{A} \in \mathbb{R}^{m \times n}$ | $\mathbf{x} \in \mathbb{R}^n$

Data Vector | Operation Matrix | Model Vector

Linear Regression

Straight Line : Direct Problem



Model

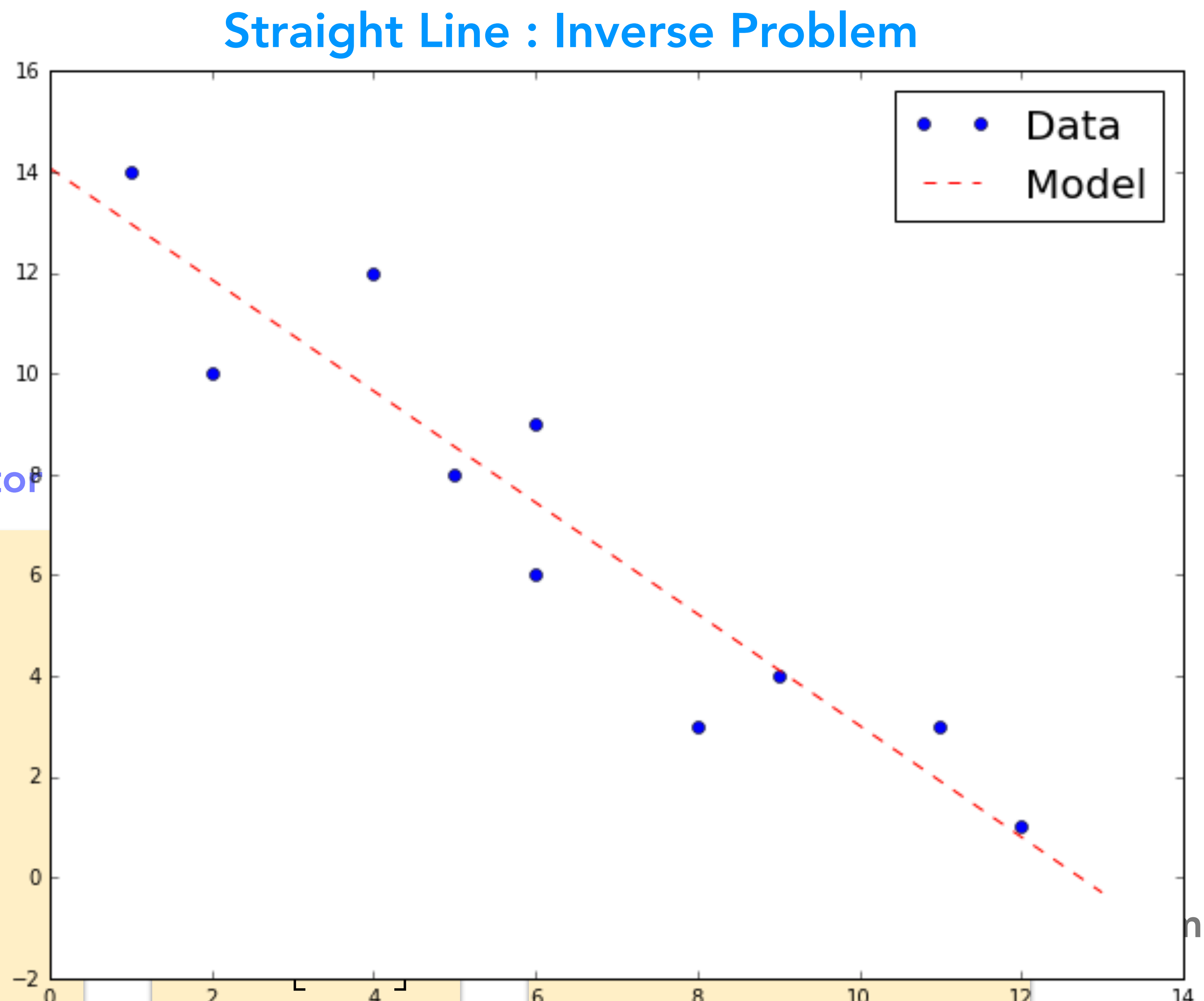
$$m = -1.1$$

$$b = 14$$

Data Vector

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

Linear Regression



Data Vector

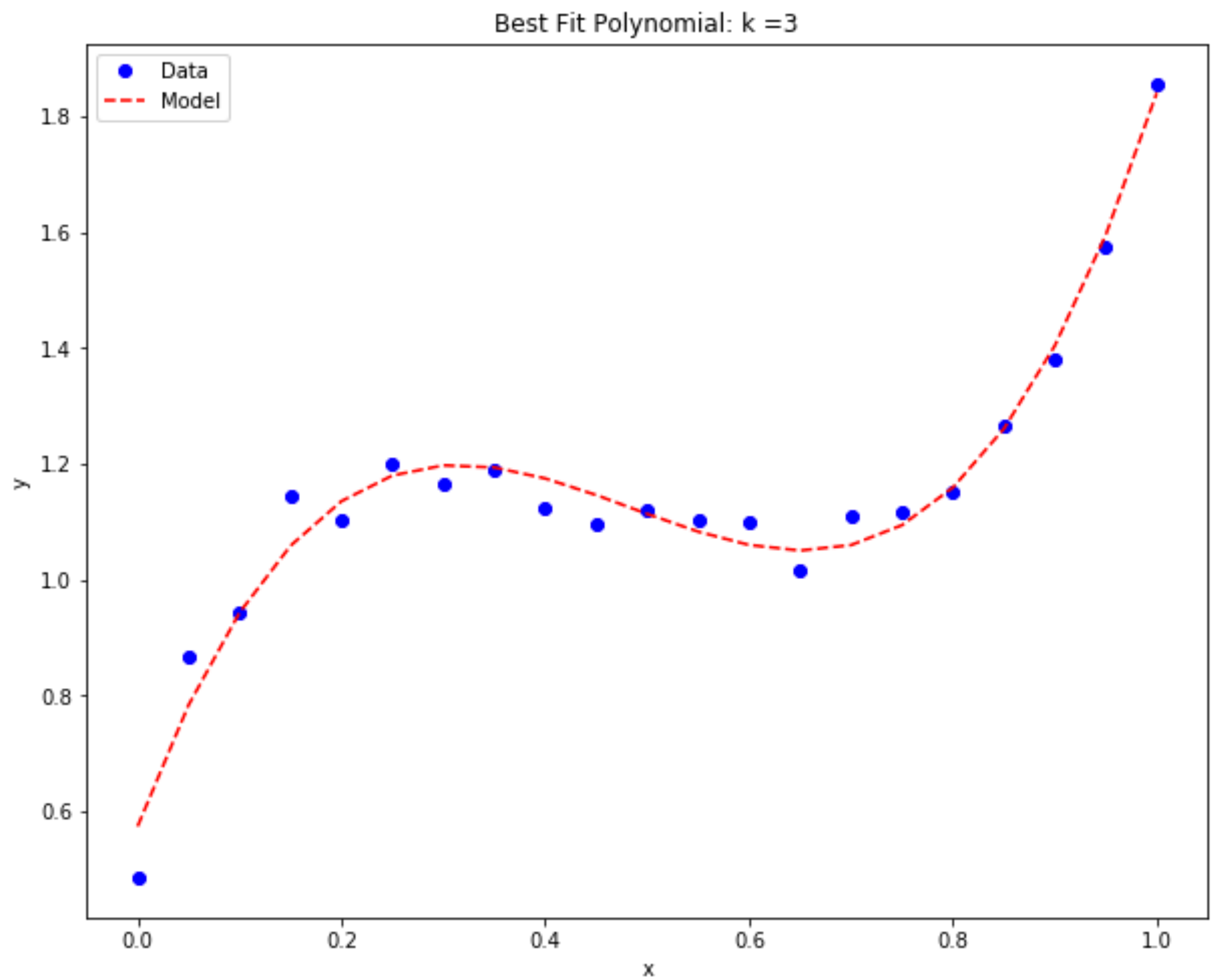
$$\mathbf{y} = \begin{bmatrix} 3 \\ 10 \\ 3 \\ 6 \\ 8 \\ 12 \\ 1 \\ 4 \\ 9 \\ 14 \end{bmatrix}$$

nal Equation)

Linear Regression

Polynomial Line : Inverse Problem

$$y = a_0 + a_1 x + a_2 x^2 + ... + a_k x^k$$



Ill-Posed Problem

Well-Posed Problem

1. A solution exists
2. The solution is unique
3. The solution's behaviour changes continuously with the initial conditions

Ill-Posed Problem

1. No solution exists
2. The solution is not unique
3. The problem is ill-conditioned

Ill-Posed Problem

Well-Conditioned Problem

$$\begin{array}{ccc} \mathbf{y} & \mathbf{A} & \mathbf{x} \\ \begin{bmatrix} 4 \\ 7 \end{bmatrix} = & \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} & \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{array} \quad \rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 4 \\ 7 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2.01 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1.96 \\ 1.02 \end{bmatrix}$$

Ill-Conditioned Problem

$$\begin{bmatrix} 3 \\ 1.47 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0.48 & 0.99 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 3 \\ 1.47 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0.49 & 0.99 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad \rightarrow \quad \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

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Regularisation

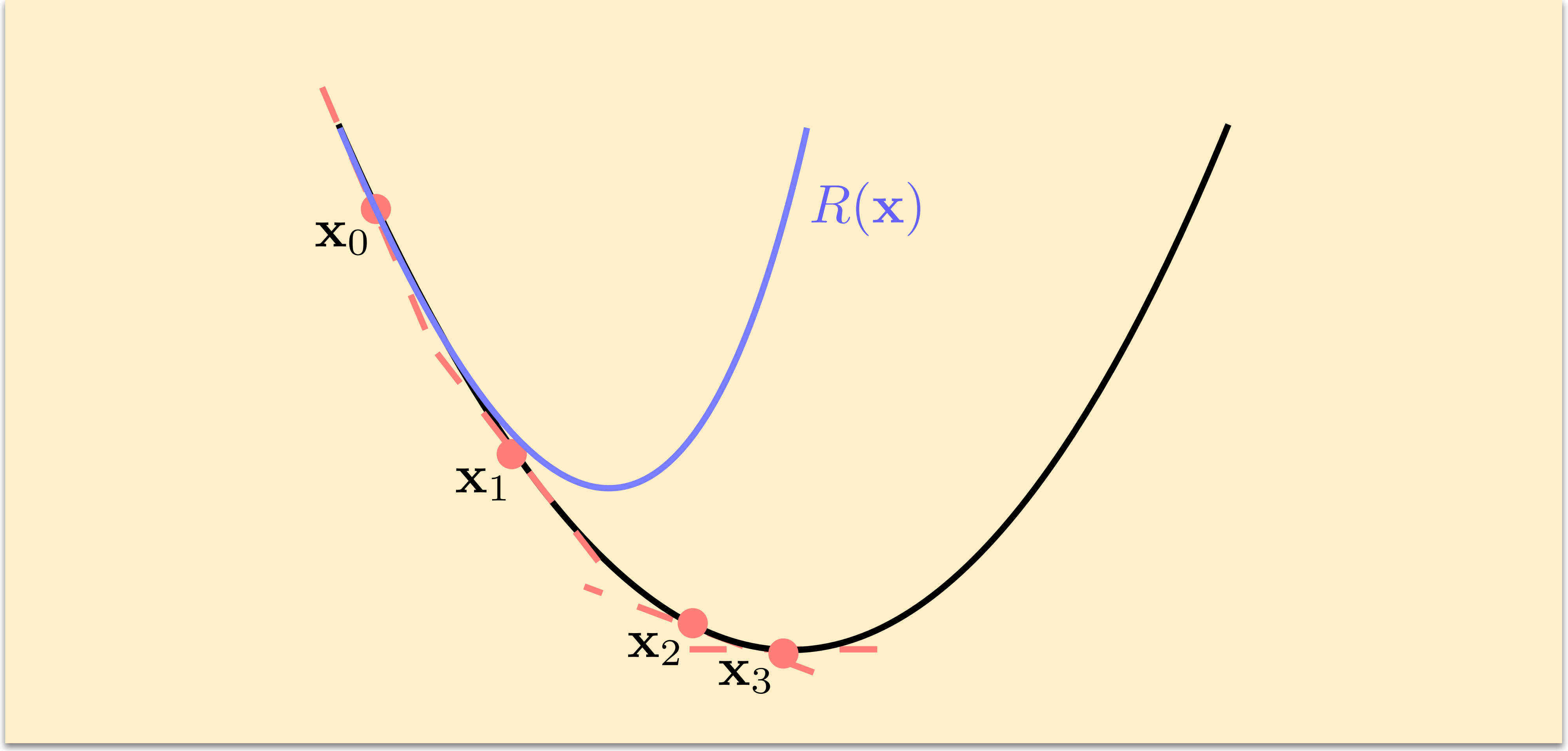
$$\underset{\mathbf{x}}{\operatorname{argmin}} \quad \frac{1}{2}\|\mathbf{y} - A\mathbf{x}\|_2^2 + \lambda R(\mathbf{x})$$

1. Find **x** such that **y-Ax** is small
2. We have some prior knowledge about the properties of **x** given by R(**x**)

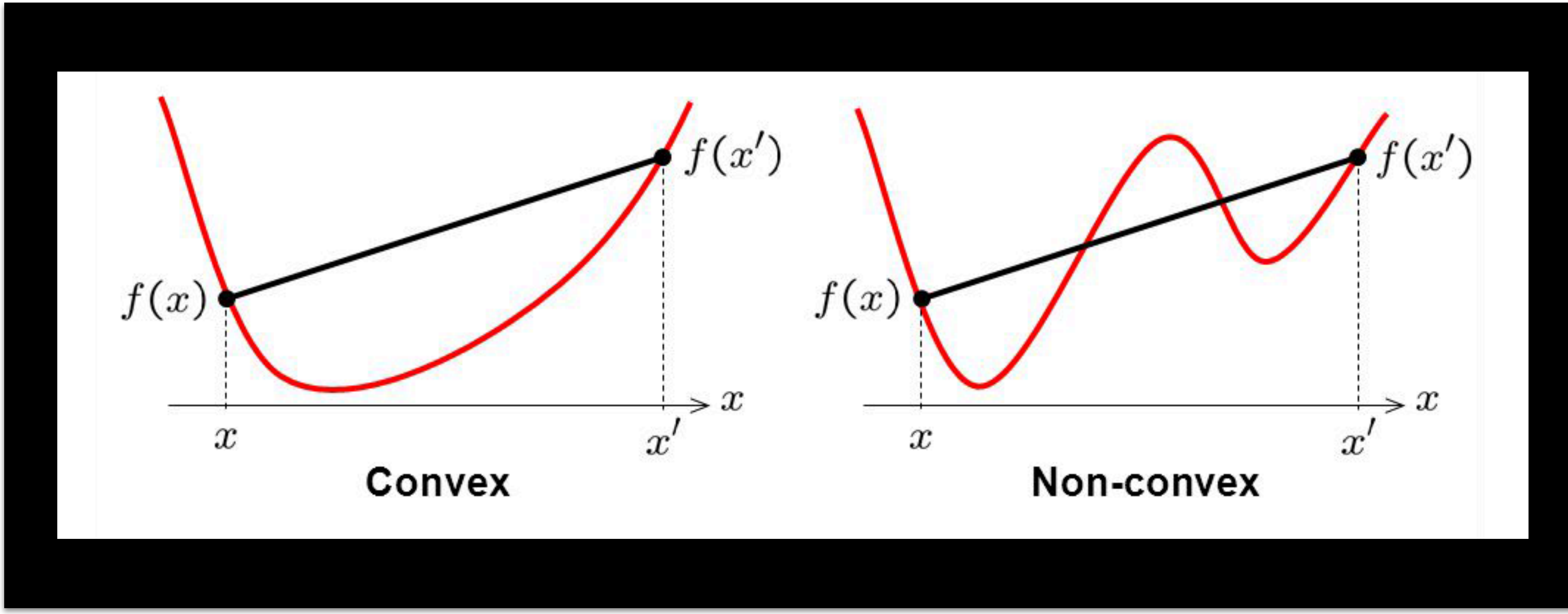
Regularisation

$$F(\mathbf{x}) = \frac{1}{2}\|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2$$

$$\nabla F(\mathbf{x}) = \mathbf{A}^T(\mathbf{y} - \mathbf{A}\mathbf{x})$$



Convexity

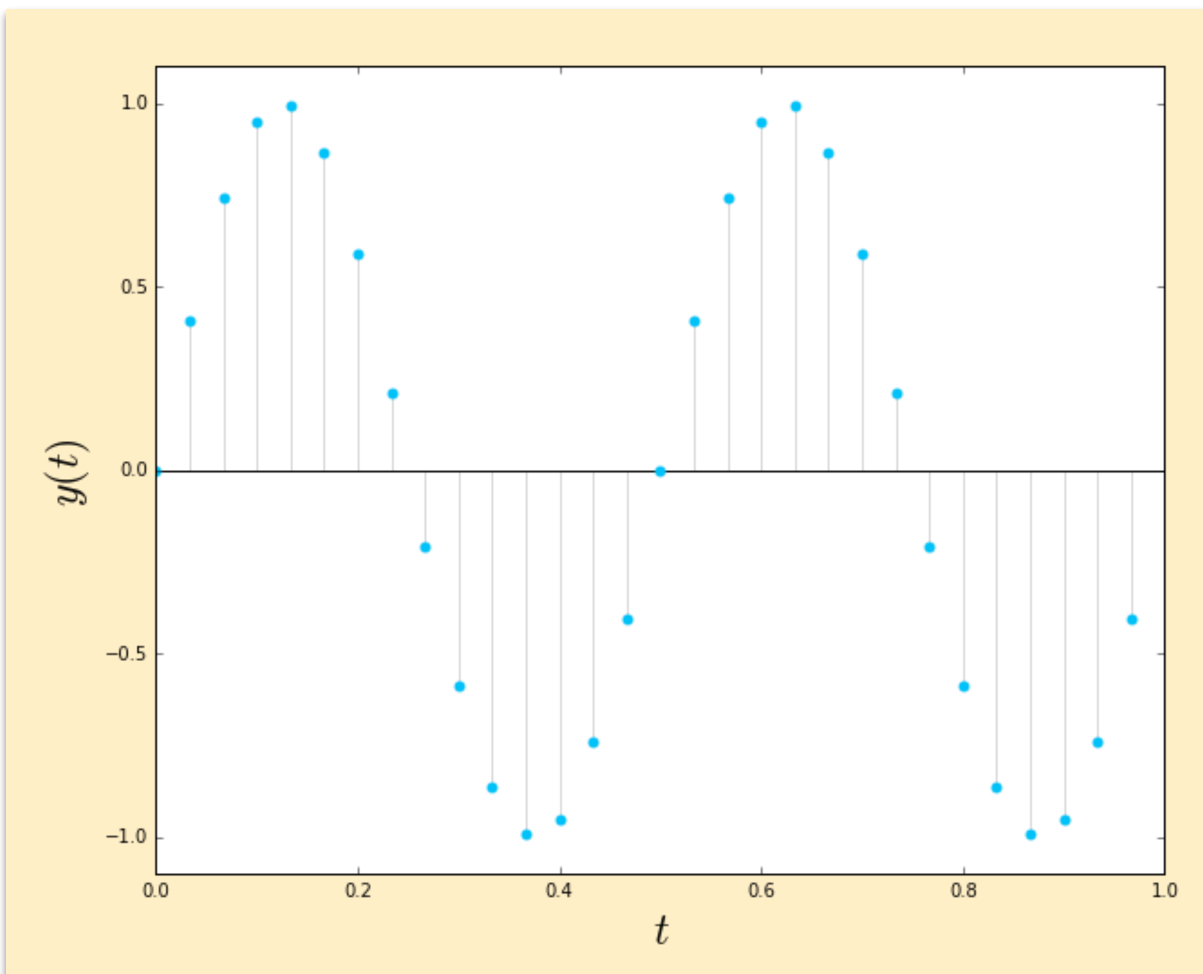


In general we want to preserve convexity

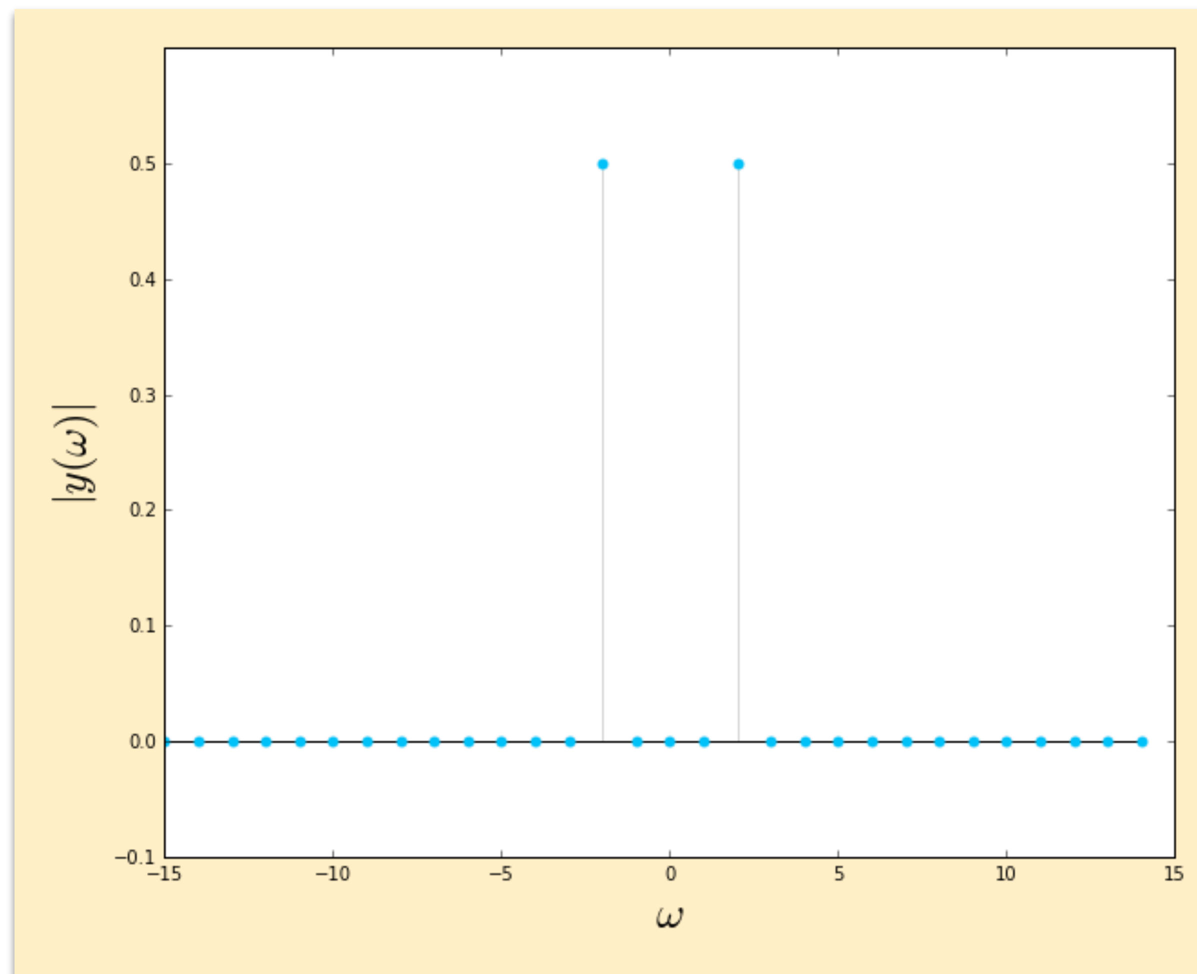
Sparsity

A sparse signal is one that is comprised mostly of zeros when expressed in the appropriate basis.

Direct Space



Sparse Space



Sparsity

$$\mathbf{x} = \phi\alpha = \sum_{i=1}^{n} \phi_i \alpha_i$$

ϕ is the dictionary that converts the signal to a sparse representation. (*e.g.* Fourier transform, wavelet transform, *etc.*)

Measuring Sparsity

$$\|\alpha\|_0 \rightarrow \|\alpha\|_1 = \sum_{i=1}^{n} |\alpha_i|$$

Not convex

Compressive Sensing Theorem

This theorem demonstrates that, under certain conditions regarding the signal and the operation matrix, a perfect reconstruction can be achieved through l_1 minimisation.

No such theorem exists for any other regularisation technique.

Sparsity

Sparse Minimisation

$$\underset{\alpha}{\operatorname{argmin}} \quad \frac{1}{2}\|\mathbf{y} - A\phi\alpha\|_2^2 + \lambda\|\alpha\|_1$$

Applications

- **Denoising**
- Deconvolution
- Component Separation
- Inpainting
- Blind Source Separation
- Minimisation algorithms
- Compressed Sensing

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Jupyter Notebooks

Let's Get Started!

Reminder, tutorial materials available at:

https://github.com/sfarrens/euroscipy

If you have not already done so, please:

- activate the conda environment
- run jupyter notebook

```
conda activate euroscipy-astro
jupyter notebook
```

Please open the file **sparsity.ipynb**