QNMs

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Contents

1 Scalar field in a curved background

The Lagrangian density of a complex scalar field propagating in a curved spacetime is given by

$$\mathcal{L}_\phi = g^{\mu\nu}(D_\mu\phi)^*(D_\nu\phi) + m^2\,\phi^*\phi + \xi R\,\phi^*\phi, \tag{1}$$

where ϕ denotes the complex scalar field and m its mass. The term proportional to $\xi R\phi^*\phi$ represents the non-minimal coupling between the scalar field and the gravitational field, with ξ being the corresponding coupling constant.

The operator

$$D_\nu = \nabla_\nu - iqA_\nu \tag{2}$$

is the extended covariant derivative, which introduces minimal coupling to the electromagnetic field A_ν, where q is the electric charge of the scalar field.

1.1 Field equations

Applying the variational principle with respect to ϕ^*, the variation of the action reads

$$\delta S_\phi = -\int d^4x\sqrt{-g}\Big[g^{\mu\nu}(\nabla_\mu\delta\phi^* + iqA_\mu\delta\phi^*)(D_\nu\phi) + m^2\,\delta\phi^*\phi + \xi R\,\delta\phi^*\phi\Big]. \tag{3}$$

Integrating by parts, this expression can be rewritten as

$$\begin{aligned}\delta S_\phi = -\int d^4x\sqrt{-g}\Big[&\nabla_\mu(g^{\mu\nu}\delta\phi^* D_\nu\phi)\\ &+\Big(-\nabla_\mu(g^{\mu\nu}D_\nu\phi) + iqA_\mu g^{\mu\nu}D_\nu\phi + m^2\phi + \xi R\phi\Big)\delta\phi^*\Big].\end{aligned} \tag{4}$$

The first term corresponds to a boundary contribution and vanishes by assuming $\delta\phi^* = 0$ on $\partial\mathcal{M}$. Therefore, the variation reduces to

$$\delta S_\phi = -\int d^4x\sqrt{-g}\Big(-D_\mu(g^{\mu\nu}D_\nu\phi) + m^2\phi + \xi R\phi\Big)\delta\phi^*. \tag{5}$$

Imposing $\delta S_\phi = 0$ for arbitrary variations $\delta\phi^*$ yields the equation of motion

$$D_\mu D^\mu\phi - m^2\phi - \xi R\phi = 0. \tag{6}$$

1.2 Energy–momentum tensor

The contribution of the scalar field to the energy–momentum tensor is obtained by varying the action with respect to the metric $g^{\mu\nu}$,

$$\delta S_\phi = -\frac{1}{2}\int d^4x\sqrt{-g}\,T^\phi_{\mu\nu}\,\delta g^{\mu\nu}. \tag{7}$$

The relevant variations are

$$\delta\sqrt{-g} = -\frac{1}{2}\sqrt{-g}\,g_{\mu\nu}\delta g^{\mu\nu}, \tag{8}$$

$$\delta R = R_{\mu\nu}\delta g^{\mu\nu} + \nabla_\mu\nabla_\nu\delta g^{\mu\nu} - \nabla_\beta\nabla^\beta(g_{\mu\nu}\delta g^{\mu\nu}). \tag{9}$$

Using these expressions, the variation of the action becomes

$$\delta S_\phi = -\int d^4x\Big[\delta(\sqrt{-g})\mathcal{L}_\phi + \sqrt{-g}\,\delta\mathcal{L}_\phi\Big] \tag{10}$$

$$= -\int d^4x\sqrt{-g}\Big[-\frac{1}{2}g_{\mu\nu}\mathcal{L}_\phi\,\delta g^{\mu\nu} + (D_\mu\phi)^*(D_\nu\phi)\,\delta g^{\mu\nu} + \xi\phi^*\phi\,\delta R\Big] \tag{11}$$

$$= -\int d^4x\sqrt{-g}\,\delta g^{\mu\nu}\Big[-\frac{1}{2}g_{\mu\nu}\mathcal{L}_\phi + (D_\mu\phi)^*(D_\nu\phi) + \xi\left(R_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\nabla_\beta\nabla^\beta\right)\phi^*\phi\Big]. \tag{12}$$

Comparing with the definition of the energy–momentum tensor, one finally obtains

$$\begin{aligned} T^\phi_{\mu\nu} = &-\frac{1}{2}g_{\mu\nu}\Big[(D_\mu\phi)^*(D^\mu\phi) + m^2\phi^*\phi\Big] + (D_\mu\phi)^*(D_\nu\phi) \\ &+ \xi\left(G_{\mu\nu} - \nabla_\mu\nabla_\nu + g_{\mu\nu}\nabla_\beta\nabla^\beta\right)\phi^*\phi, \end{aligned} \tag{13}$$

where $G_{\mu\nu}$ is the Einstein tensor. At first order, this contribution does not enter the total energy-momentum tensor; therefore, the scalar field does not backreact on the background geometry.

1.3 Background of the Black Hole

For the background of a static, spherically symmetric black hole, and to first order in the scalar field, the Einstein equations read

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}\left(R - 2\Lambda\right) = 8\pi T_{\mu\nu}. \tag{14}$$

From these equations it follows that $R = 4\Lambda$. The resulting field equations admit a family of solutions of the form

$$ds^2 = g_{\alpha\beta}\,dx^\alpha dx^\beta \tag{15}$$

$$= -f(r)\,dt^2 + \frac{1}{f(r)}\,dr^2 + r^2 d\Omega^2. \tag{16}$$

Among these solutions are the Schwarzschild, Reissner–Nordström, and their (A)dS generalizations. The values of r for which $f(r) = 0$ correspond to the horizons of the black hole.

The tortoise coordinate r_* is defined by

$$\frac{dr_*}{dr} = \frac{1}{f(r)}, \tag{17}$$

which maps the horizons to asymptotic regions.

2 Radial Equation for a Charged Scalar Field

2.1 Separation of variables

For a charged scalar field, the equation of motion is given by

$$D_\mu D^\mu\phi - m^2_{\text{eff}}\,\phi = 0, \tag{18}$$

where the parameter m^2_{eff} denotes the effective (quadratic) mass of the field, defined as $m^2_{\text{eff}} = m^2 + 4\xi\Lambda$.

The electromagnetic potential is chosen as $A_\mu = \left(-\frac{Q}{r}, 0, 0, 0\right)$, which corresponds to the background electromagnetic field of a charged black hole, where Q denotes its electric charge.

With this choice, the equation of motion can be written as

$$g^{\mu\nu}\nabla_\mu\nabla_\nu\phi - \frac{1}{f(r)}\left(2i\frac{Qq}{r}\,\partial_t\phi - \frac{q^2Q^2}{r^2}\,\phi\right) - m_{\text{eff}}^2\,\phi = 0. \tag{19}$$

A relevant property is

$$g^{\mu\nu}\nabla_\mu\nabla_\nu = \frac{1}{\sqrt{-g}}\partial_\mu\left(\sqrt{-g}\,g^{\mu\nu}\partial_\nu\right), \quad \text{with} \quad \sqrt{-g} = r^2\sin\theta. \tag{20}$$

Then, the equation explicitly reads

$$-\frac{\partial_t^2\phi}{f(r)} + \frac{1}{r^2}\partial_r\left(r^2 f(r)\partial_r\phi\right) + \frac{1}{r^2}\nabla_s^2\phi - \frac{1}{f(r)}\left(2i\frac{Qq}{r}\partial_t\phi - \frac{q^2Q^2}{r^2}\phi\right) - m_{\text{eff}}^2\phi = 0, \tag{21}$$

where ∇_s^2 denotes the angular Laplacian on the unit two-sphere. Given the spherical symmetry of the background spacetime, it is convenient to expand the scalar field in spherical harmonics $Y_{l,m}(\theta,\phi)$ and satisfies $\nabla_s^2 Y_{l,m}(\theta,\phi) = -l(l+1)$,

$$\phi = \frac{1}{r}\sum_{l,m}\int\frac{d\omega}{2\pi}X_{l,m}(r,\omega)\,Y_{l,m}(\theta,\phi)\,e^{-i\omega t}. \tag{22}$$

Substituting this decomposition into the equation of motion, one obtains the radial equation

$$\frac{1}{r}\partial_r\left(r^2 f(r)\partial_r\frac{X}{r}\right) + \left[\frac{1}{f(r)}\left(\omega^2 - 2\omega\frac{qQ}{r} + \frac{q^2Q^2}{r^2}\right) - \frac{l(l+1)}{r^2} - m_{\text{eff}}^2\right]X = 0. \tag{23}$$

By introducing the tortoise coordinate r_* in the radial derivative, the equation can be cast into a Schrödinger-like form,

$$\partial_{r_*}^2 X + \left[\omega^2 - 2\omega\frac{qQ}{r} - V(r)\right]X = 0, \tag{24}$$

where the effective potential is given by

$$V(r) = f(r)\left(m_{\text{eff}}^2 + \frac{l(l+1)}{r^2} + \frac{\partial_r f(r)}{r}\right) - \frac{q^2Q^2}{r^2}. \tag{25}$$

The effective potential encodes the effects of spacetime curvature, angular momentum, and electromagnetic interaction.

2.2 Asymptotically flat case ($\Lambda = 0$)

A general example of a static, spherically symmetric spacetime in the absence of a cosmological constant is provided by the Reissner–Nordström geometry. In this case, the metric function takes the form

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{26}$$

For this background, the effective mass reduces to $m_{\text{eff}}^2 = m^2$. The spacetime admits two horizons located at

$$r_\pm = M \pm \sqrt{M^2 - Q^2}. \tag{27}$$

Here, r_+ corresponds to the event horizon, while r_- denotes the inner (Cauchy) horizon. The asymptotic behavior of the effective potential near the event horizon and at spatial infinity is given by

$$V(r) \sim \begin{cases} -\frac{q^2Q^2}{r_+^2} & , \quad r \to r_+ \\ m^2 & , \quad r \to +\infty \end{cases} \tag{28}$$

Accordingly, the radial solution exhibits the following asymptotic form:

$$X(r) \sim \begin{cases} e^{-i\left(\omega - \frac{qQ}{r_+}\right)r_*} & , \quad r \to r_+ \\ e^{+i\sqrt{\omega^2-m^2}\,r_*} & , \quad r \to +\infty \end{cases} \tag{29}$$

With the tortoise coordinate r_*

$$r_* \sim r + \frac{1}{2\kappa_+}\ln(r-r_+) - \frac{1}{2\kappa_-}\ln(r-r_-), \tag{30}$$

where the constants $\kappa_\pm$ are given by

$$\kappa_\pm = \frac{1}{2}f'(r)\bigg|_{r=r_\pm}, \tag{31}$$

To compactify the radial coordinate, we introduce the dimensionless variable $u \in [0,1]$

$$u = 1 - \frac{r_+}{r}. \tag{32}$$

From this, the divergent terms of the ingoing mode as $u \to 0$ (i.e. $r \to r_+$),

$$e^{-i\left(\omega-\frac{qQ}{r_+}\right)r_*} \sim e^{-\frac{ir_+}{1-u}\left(\omega-\frac{qQ}{r_+}\right)} r_+ u^{-\frac{i}{2\kappa_+}\left(\omega-\frac{qQ}{r_+}\right)} (1-u)^{\frac{i}{2\kappa_+}\left(\omega-\frac{qQ}{r_+}\right)} \left(\frac{r_+}{1-u} - r_-\right)^{\frac{i}{2\kappa_-}\left(\omega-\frac{qQ}{r_+}\right)}. \tag{33}$$

Similarly, the divergent terms of the outgoing mode as $u \to 1$ (i.e. $r \to \infty$),

$$e^{i\sqrt{\omega^2-m^2}r_*} \sim e^{\frac{ir_+}{1-u}\sqrt{\omega^2-m^2}} r_+ u^{\frac{i}{2\kappa_+}\sqrt{\omega^2-m^2}} (1-u)^{-\frac{i}{2\kappa_+}\sqrt{\omega^2-m^2}} \left(\frac{r_+}{1-u} - r_-\right)^{-\frac{i}{2\kappa_-}\sqrt{\omega^2-m^2}}. \tag{34}$$

Factoring out the divergent behavior, the remaining function $\tilde{X}(u)$ is regular

$$X(u) = u^{-\frac{i}{2\kappa_+}\left(\omega-\frac{qQ}{r_+}\right)} e^{\frac{ir_+}{1-u}\sqrt{\omega^2-m^2}} (1-u)^{-\frac{i}{2\kappa_+}\sqrt{\omega^2-m^2}} \tilde{X}(u). \tag{35}$$

2.3 Asymptotically de Sitter case ($\Lambda > 0$)

A general example of a static, spherically symmetric spacetime with a positive cosmological constant is provided by the Reissner–Nordström–de Sitter geometry. In this case, the metric function takes the form

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{\Lambda r^2}{3}. \tag{36}$$

The spacetime admits three horizons located at r_-, r_+, and r_c. Here, r_+ corresponds to the event horizon, while r_- denotes the inner (Cauchy) horizon and r_c denotes the cosmological horizon. The asymptotic behavior of the effective potential near the event horizon and at spatial infinity is given by

$$V(r) \sim \begin{cases} -\frac{q^2Q^2}{r_+^2} & , \quad r \to r_+ \\ -\frac{q^2Q^2}{r_c^2} & , \quad r \to r_c \end{cases} \tag{37}$$

Accordingly, the radial solution exhibits the following asymptotic form:

$$X(r) \sim \begin{cases} e^{-i\left(\omega - \frac{qQ}{r_+}\right) r_*} & , \quad r \to r_+ \\ e^{+i\left(\omega - \frac{qQ}{r_c}\right) r_*} & , \quad r \to r_c \end{cases} \tag{38}$$

With the tortoise coordinate r_*

$$r_* \sim \frac{1}{2\kappa_+} \ln(r - r_+) - \frac{1}{2\kappa_c} \ln(r_c - r) - \frac{1}{2\kappa_-} \ln(r - r_-) \tag{39}$$

where the constants $\kappa_i, \quad i = \{-, +, c\}$ are given by

$$\kappa_i = \frac{1}{2} f'(r)\Big|_{r=r_i}. \tag{40}$$

We introduce the variable u,

$$u = \frac{1}{r}. \tag{41}$$

From this, the divergent terms of the incoming mode at the event horizon ($r \to r_+$),

$$e^{-i\left(\omega - \frac{qQ}{r_+}\right) r_*} \sim \frac{e^{-\frac{i}{u}\left(\omega - \frac{qQ}{r_+}\right)}}{u^3 u_+ u_- u_c} (u_+ - u)^{-\frac{i}{2\kappa_+}\left(\omega - \frac{qQ}{r_+}\right)} (u_- - u)^{\frac{i}{2\kappa_-}\left(\omega - \frac{qQ}{r_+}\right)} (u - u_c)^{\frac{i}{2\kappa_c}\left(\omega - \frac{qQ}{r_+}\right)} \tag{42}$$

Similarly, the divergent terms of the outgoing mode at the cosmological horizon ($r \to r_c$),

$$e^{i\left(\omega - \frac{qQ}{r_c}\right) r_*} \sim \frac{e^{\frac{i}{u}\left(\omega - \frac{qQ}{r_c}\right)}}{u^3 u_+ u_- u_c} (u_+ - u)^{\frac{i}{2\kappa_+}\left(\omega - \frac{qQ}{r_+}\right)} (u_- - u)^{-\frac{i}{2\kappa_-}\left(\omega - \frac{qQ}{r_c}\right)} (u - u_c)^{-\frac{i}{2\kappa_c}\left(\omega - \frac{qQ}{r_c}\right)} \tag{43}$$

Factoring out the divergent behavior, the remaining function $\tilde{X}(u)$ is regular

$$X(u) = (u_+ - u)^{-\frac{i}{2\kappa_+}\left(\omega - \frac{qQ}{r_+}\right)} (u - u_c)^{-\frac{i}{2\kappa_c}\left(\omega - \frac{qQ}{r_c}\right)} \tilde{X}(u) \tag{44}$$

3 Method AIM Improved

The radial equation recast is the form as

$$\tilde{X}''(u) = \lambda_0(u)\tilde{X}'(u) + s_0(u)\tilde{X}(u), \tag{45}$$

The key idea of the asymptotic iteration method is that successive derivatives of this equation preserve the same functional structure, namely a linear combination of $\tilde{X}'(u)$ and $\tilde{X}(u)$. Taking the derivative of the above equation with respect to u, one obtains

$$\begin{aligned} \tilde{X}^{(3)}(u) =& \lambda_0'(u)\tilde{X}'(u) + \lambda_0(u)\tilde{X}''(u) + s_0'(u)\tilde{X}(u) + s_0(u)\tilde{X}'(u), && (46) \\ =& \lambda_0'(u)\tilde{X}'(u) + \lambda_0(u)\left(\lambda_0(u)\tilde{X}'(u) + s_0(u)\tilde{X}(u)\right) + s_0'(u)\tilde{X}(u) + s_0(u)\tilde{X}'(u), && (47) \\ =& \left[\lambda_0'(u) + s_0(u) + \lambda_0^2(u)\right]\tilde{X}'(u) + \left[s_0'(u) + \lambda_0(u)s_0(u)\right]\tilde{X}(u) && (48) \end{aligned}$$

This shows explicitly that the third derivative retains the same structure as the original equation. For convenience, the coefficients are defined as

$$\lambda_1(u) = \lambda_0'(u) + s_0(u) + \lambda_0^2(u) \quad \text{and} \quad s_1(u) = s_0'(u) + \lambda_0(u)s_0(u). \tag{49}$$

In general, after i successive differentiations, one finds that the $(n+1)$-th derivative of the function can be expressed as

$$\tilde{X}^{(n+1)}(u) = \lambda_{n-1}(u)\tilde{X}'(u) + s_{n-1}(u)\tilde{X}(u), \tag{50}$$

where $\lambda_{n-1}(u)$ and $s_{n-1}(u)$ are functions generated iteratively. Derivating this expression with respect to u we have

$$\begin{aligned}
\tilde{X}^{(n+2)}(u) =& \lambda'_{n-1}(u)\tilde{X}'(u) + \lambda_{n-1}(u)\tilde{X}''(u) + s'_{n-1}(u)\tilde{X}(u) + s_{n-1}(u)\tilde{X}'(u), && (51)\\
=& \lambda'_{n-1}(u)\tilde{X}'(u) + \lambda_{n-1}(u)\left(\lambda_0(u)\tilde{X}'(u) + s_0(u)\tilde{X}(u)\right) + s'_{n-1}(u)\tilde{X}(u) + && (52)\\
& + s_{n-1}(u)\tilde{X}'(u), &&\\
=& \left[\lambda'_{n-1}(u) + s_{n-1}(u) + \lambda_0(u)\lambda_{n-1}(u)\right]\tilde{X}'(u) + \left[s'_{n-1}(u) + s_0(u)\lambda_{n-1}(u)\right]\tilde{X}(u) && (53)
\end{aligned}$$

Therefore, the functional form is preserved under differentiation, and the coefficients satisfy the recurrence relations

$$\lambda_n(u) = \lambda'_{n-1}(u) + s_{n-1}(u) + \lambda_0(u)\lambda_{n-1}(u), \qquad s_n(u) = s'_{n-1}(u) + s_0(u)\lambda_{n-1}(u). \tag{54}$$

For asymptotically large values of n, the asymptotic iteration method assumes that the ratio of the coefficients stabilizes, leading to the condition

$$\frac{s_n(u)}{\lambda_n(u)} = \frac{s_{n-1}(u)}{\lambda_{n-1}(u)} \equiv \beta(u), \tag{55}$$

where the quasinormal modes are obtained from the condition

$$\delta_n(u) = s_n(u)\lambda_{n-1}(u) - s_{n-1}(u)\lambda_n(u) = 0. \tag{56}$$

For a fixed expansion point u_0, the quantities depend parametrically on the complex frequency $\omega = \omega_{Re} - i\omega_{Im}$ through the radial equation. Consequently, the above condition becomes an equation for ω. Only discrete values of the frequency satisfy this relation, and these values correspond to the quasinormal modes of the system. In this context, the AIM condition plays the role of a quantization condition, selecting a discrete spectrum compatible with the imposed boundary conditions.

By expanding λ_n and s_n in a Taylor's series around the point which u_0:

$$\lambda_n(u) = \sum_{i=0}^{\infty} c_n^i (u-u_0)^i, \tag{57}$$

$$s_n(u) = \sum_{j=0}^{\infty} d_n^j (u-u_0)^j, \tag{58}$$

From the definition of $\lambda_n(u)$

$$\begin{aligned}
\sum_{i=0}^{\infty} c_n^i (u-u_0)^i &= \sum_{j=0}^{\infty} j c_{n-1}^j (u-u_0)^{j-1} + \sum_{m=0}^{\infty} d_{n-1}^m (u-u_0)^m + \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} c_0^l c_{n-1}^k (u-u_0)^{l+k}, && (59)\\
&= \sum_{j=0}^{\infty} (j+1) c_{n-1}^{j+1} (u-u_0)^j + \sum_{m=0}^{\infty} d_{n-1}^m (u-u_0)^m + \sum_{p=0}^{\infty}\sum_{l=0}^{p} c_0^l c_{n-1}^{p-l} (u-u_0)^p. && (60)
\end{aligned}$$

In the second term, the index is changed by defining it $j \to j+1$; in the third term, the new index is defined $p = l + k$, which allows rewriting the double sum.

$$c_n^i = (i+1)c_{n-1}^{i+1} + d_{n-1}^i + \sum_{l=0}^{i} c_0^l c_{n-1}^{i-l}. \tag{61}$$

Similarly, $s_n(u)$

$$\sum_{j=0}^{\infty} d_n^j (u-u_0)^j = \sum_{j=0}^{\infty} j d_{n-1}^j (u-u_0)^{j-1} + \sum_{l=0}^{\infty}\sum_{k=0}^{\infty} d_0^l c_{n-1}^k (u-u_0)^{l+k}, \tag{62}$$

$$= \sum_{j=0}^{\infty} (j+1) d_{n-1}^{j+1} (u-u_0)^j + \sum_{p=0}^{\infty}\sum_{l=0}^{\infty} d_0^l c_{n-1}^{p-l} (u-u_0)^p. \tag{63}$$

Then,

$$d_n^j = (j+1) d_{n-1}^{j+1} + \sum_{l=0}^{j} d_0^l c_{n-1}^{j-l}. \tag{64}$$

Thus, the quantization condition of the quasinormal modes depends exclusively on the coefficients of the series of λ_n and s_n

$$\delta_n(u) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} (u-u_0)^{i+j} \left(d_n^i c_{n-1}^j - d_{n-1}^i c_n^j \right) = 0, \tag{65}$$

when the condition is evaluated at the expansion point u_0, we obtain

$$\delta_n(u_0) = d_n^0 c_{n-1}^0 - d_{n-1}^0 c_n^0 = 0. \tag{66}$$

4 Numerical Implementation

After deriving the explicit forms of $\lambda_n(u)$ and $s_n(u)$ from the general formalism, the numerical scheme is implemented by fixing the expansion point at the maximum of the effective potential, whenever such a maximum is present.

4.1 Example: Schwarzschild black hole and a massive scalar field

As a concrete application, we consider a massive scalar field propagating in the Schwarzschild spacetime. The corresponding radial equation reads

$$\frac{1}{r}\partial_r \left(r^2 f(r) \partial_r \frac{X}{r} \right) + \left[\frac{\omega^2}{f(r)} - \frac{l(l+1)}{r^2} - m^2 \right] X = 0, \tag{67}$$

with the metric function

$$f(r) = 1 - \frac{2M}{r}. \tag{68}$$

The form of the effective potential is given by

$$V(r) = \left(1 - \frac{2M}{r}\right) \left(\frac{l(l+1)}{r^2} + \frac{2M}{r^3} + m^2 \right), \tag{69}$$

We introduce the compact radial coordinate

$$u = 1 - \frac{2M}{r}, \tag{70}$$

We put the solution form as:

$$X(u) = u^{-2iM\omega} e^{\frac{2iM}{1-u}\sqrt{\omega^2-m^2}} (1-u)^{-2iM\sqrt{\omega^2-m^2}} \tilde{X}(u), \tag{71}$$

as consequence, the radial equation is reduced to a second-order differential equation for $\tilde{X}(u)$. where the coefficient functions are explicitly given by

$$\lambda_0(u) = \frac{-1 + \big(4 + 4ikM(u-2) - 3u\big)u + 4iM(1-u)^2\omega}{(1-u)^2 u}, \tag{72}$$

$$s_0(u) = \frac{1}{(-1+u)^4 u}[-(-7+u)(-1+u)^2 + 4k^2M^2(-2+u)^2u + 4kM(-1+u)^2(I(-1+u) \tag{73}$$
$$+ 2M(-2+u)\omega) + 4M\left(m^2M + \omega\left(I(-1+u)^3 + M(-2+u)(2+(-2+u)u)\omega\right)\right)],$$

and we have defined

$$k = \sqrt{\omega^2 - m^2}. \tag{74}$$

The numerical implementation can be found inin `https://github.com/nabvargasp/QNMs-BH_AIM/tree/main/Schwarzschild%20AIM`.

The implementation requires an expansion point u_0, usually selected at the potential's maximum. Figure 1 shows that increasing the mass modifies the potential, leading to the disappearance of the maximum at a critical mass m_0.

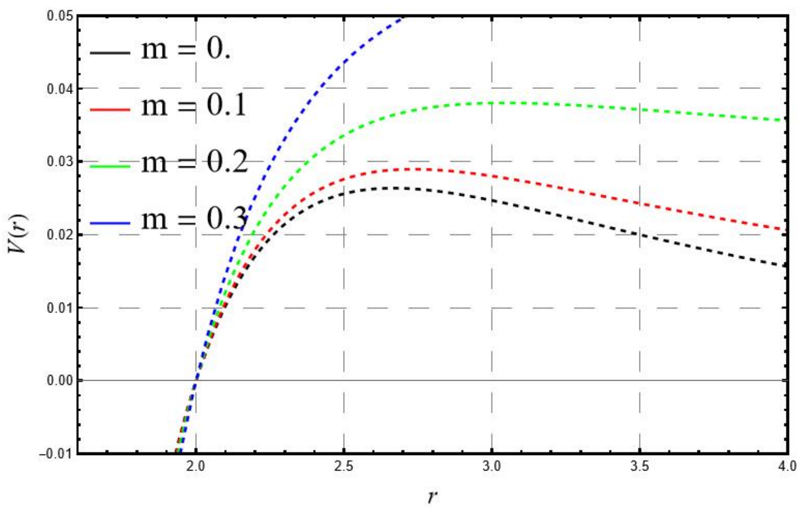


Figure 1: Effective potential for a massive scalar field in a Schwarzschild black hole (fundamental mode, l=0) for different values of the mass.

4.2 Reissner–Nordström black hole and a massive charged scalar field

The corresponding radial equation reads

$$\frac{1}{r}\partial_r\left(r^2 f(r)\partial_r \frac{X}{r}\right) + \left[\frac{1}{f(r)}\left(\omega^2 - 2\omega\frac{qQ}{r} + \frac{q^2Q^2}{r^2}\right) - \frac{l(l+1)}{r^2} - m^2\right] X = 0, \tag{75}$$

with the metric function

$$f(r) = 1 - \frac{2M}{r} + \frac{Q^2}{r^2}. \tag{76}$$

The form of the effective potential is given by

$$V(r) = \left(1 + \frac{Q^2}{r^2} - \frac{2M}{r}\right)\left(\frac{l(l+1)}{r^2} - \frac{2Q^2}{r^4} + \frac{2M}{r^3} + m^2\right) - \frac{q^2Q^2}{r^2}. \tag{77}$$

We introduce the compact radial coordinate

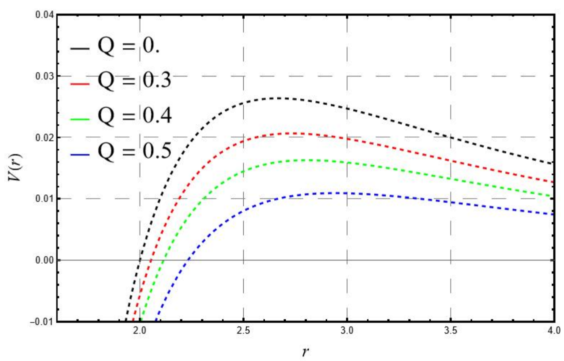
$$u = 1 - \frac{r_+}{r}. \tag{78}$$

Using the solution form

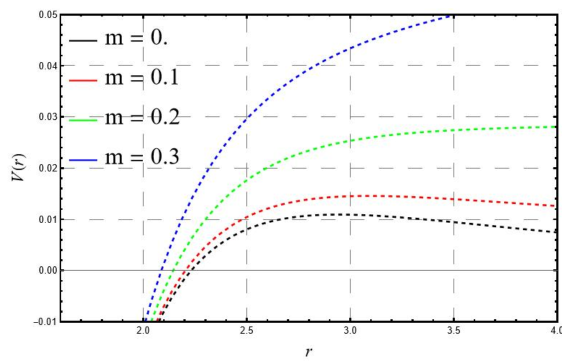
$$X(u) = u^{-\frac{i}{2\kappa_+}\left(\omega - \frac{qQ}{r_+}\right)} e^{\frac{ir_+}{1-u}\sqrt{\omega^2 - m^2}} (1-u)^{-\frac{i}{2\kappa_+}\sqrt{\omega^2-m^2}} \tilde{X}(u), \tag{79}$$

The radial equation reduces to a smooth second-order differential equation for $\tilde{X}(u)$. The explicit form of λ_0 and s_0, and the numerical implementation can be found inin `https://github.com/nabvargasp/QNMs-BH_AIM/tree/main/Reissner%20Nordstrom%20AIM`.

For the Reissner–Nordström case, the effective potential includes a term $-q^2Q^2/r^2$ that can become dominant for sufficiently large couplings qQ. As a result, the potential maximum can be significantly reduced or even vanish, as illustrated in Figure 2A. Moreover, the combined effect of the field's mass and charge further enhances this behavior, as shown in Figure 2B, limiting the applicability of the AIM method for computing quasi-normal modes.



(a) Effect of the coupling qQ for a massless field ($m = 0$) with charge $q = 0.7$.



(b) Combined effect of mass and charge for $Q = 0.5$ and $q = 0.7$.

Figure 2: Effective potential for a charged massive scalar field in a Reissner Nördstrom black hole (fundamental mode, l=0) for different values of the mass, q and Q.