Problem 6

a)

I tried to obtain this as rigorously as I could (without the Cauchy Principal Value trick). First for any function $f(x)$:

$$F(\omega) = \int_{-\infty}^{\infty} f(x)\exp(-i\omega x)dx \tag{1}$$

$$F(\omega) = \int_{-\infty}^{\infty} (f(x) - c + c)\exp(-i\omega x)dx \tag{2}$$

$$F(\omega) = c\delta(\omega) + \int_{-\infty}^{\infty} (f(x) - c)\exp(-i\omega x)dx \tag{3}$$

Now, let c be s.t.

$$\int_{-\infty}^{\infty} (f(x) - c)dx \tag{4}$$

Then clearly,

$$c = F(0) \tag{5}$$

Now if we play with this,

$$f(x) = \int_{-\infty}^{\infty} F(\omega)\exp(i\omega x)d\omega \tag{6}$$

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty} F(\omega)\exp(i\omega x)d\omega \tag{7}$$

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\int_{-\infty}^{\infty}\left[c\delta(\omega) + \int_{-\infty}^{\infty}(f(\tilde{x}) - c)\exp(-i\omega\tilde{x})d\tilde{x}\right]\exp(i\omega)d\omega \tag{8}$$

$$f'(x) = \frac{\mathrm{d}}{\mathrm{d}x}\Big(c + \int_{-\infty}^{\infty}\Big[\int_{-\infty}^{\infty}(f(\tilde{x}) - c)\exp(-i\omega\tilde{x})d\tilde{x}\Big]\exp(i\omega x)d\omega\Big) \tag{9}$$

We can now apply the Leibniz integral rule and differentiate under the integral sign.

$$f'(x) = i\omega \int_{-\infty}^{\infty} \Big[\int_{-\infty}^{\infty} (f(\tilde{x}) - c) \exp(-i\omega\tilde{x}) d\tilde{x} \Big] \exp(i\omega x) d\omega \Big) \tag{10}$$

$$f'(x) = i\omega \int_{-\infty}^{\infty} (F(\omega) - c\delta(\omega)) \exp(i\omega x) d\omega \tag{11}$$

Then by bijectivity of the Fourier transform:

$$\mathcal{F}\Big(f'(x)\Big) = i\omega(\mathcal{F}\Big(f(x)\Big) - c\delta(\omega)) \tag{12}$$

$$\boxed{\mathcal{F}\Big(f(x)\Big) = \frac{1}{i\omega}\mathcal{F}\Big(f'(x)\Big) + c\delta(\omega)} \tag{13}$$

Where, $\omega = 2\pi k$. Note, coming up with this is **by far** the most challenging part of this proof/derivation. Also, this holds in discrete form as well.

Now, we can write (for heaviside function $u(x)$):

$$\frac{\mathrm{d}}{\mathrm{d}x} u(x) = \delta(x). \tag{14}$$

Since,

$$\mathcal{F}\Big(\delta(x)\Big) = 1 \tag{15}$$

$$\therefore \tag{16}$$

$$\boxed{\mathcal{F}\Big(u(x)\Big) = \frac{1}{i\omega} + \frac{1}{2}\delta(\omega).} \tag{17}$$

Again, (15) (16) (17) also hold in the discrete context.

Now, notice we can write (using the Leibniz rule again):

$$\mathcal{F}(xf(x)) = i\frac{\partial}{\partial\omega}\mathcal{F}(f(x)) \tag{18}$$

and by substitution we can also write:

$$\mathcal{F}(f(-x)) = F(-\omega). \tag{19}$$

These are trivial enough (for both DFT and Countinous Fourier transforms) that a proof does not seem necessary.

Now let's write $|x| = xu(x) - xu(-x)$, then:

$$\mathcal{F}(|x|) = \mathcal{F}(xu(x) - xu(x)) \tag{20}$$
$$\mathcal{F}(xu(x)) = i\frac{\partial}{\partial\omega}\Big(\frac{1}{i\omega} + \frac{1}{2}\delta(\omega)\Big) \tag{21}$$
$$\mathcal{F}(xu(x)) = \Big(-\frac{1}{\omega^2} + \frac{i}{2}\delta'(\omega)\Big) \tag{22}$$
$$\therefore$$
$$\mathcal{F}(-xu(-x)) = \Big(-\frac{1}{\omega^2} + \frac{i}{2}\delta'(-\omega)\Big) \tag{23}$$
$$\implies \mathcal{F}(|x|) = \Big(-\frac{2}{\omega^2} + \frac{i}{2}\delta'(\omega) + \frac{i}{2}\delta'(-\omega)\Big). \tag{24}$$

Finally,

$$\delta'(\omega) = \lim_{h\to 0}\frac{\delta(\omega+h) - \delta(\omega-h)}{2h} \tag{25}$$
$$\delta'(-\omega) = \lim_{h\to 0}\frac{\delta(-\omega+h) - \delta(-\omega-h)}{2h} \tag{26}$$
$$\delta(\omega) = \delta(-\omega) \tag{27}$$
$$\therefore$$
$$\delta'(-\omega) = \lim_{h\to 0}\frac{\delta(\omega-h) - \delta(\omega+h)}{2h} \tag{28}$$
$$\implies \delta'(\omega) + \delta'(-\omega) = 0 \tag{29}$$
$$\implies \boxed{\mathcal{F}(|x|) = -\frac{2}{\omega^2}.} \tag{30}$$

Now we can define Brownian motion/Random walks s.t.:

$$\left\langle \Big(f(x) - f(x+\Delta) \Big)^2 \right\rangle \propto |\Delta|. \tag{31}$$

$$\left\langle \Big(f(x)^2 + 2f(x)f(x+\Delta) + f(x+\Delta)^2 \right\rangle \propto |\Delta|. \tag{32}$$

after a long time,

$$\left\langle f(x)^2 \right\rangle = \left\langle f(x+\Delta)^2 \right\rangle \tag{33}$$

$$\therefore$$

$$\langle f(x)f(x+\Delta) \rangle \sim N - |\Delta| \tag{34}$$

Then, as seen in class for a stationary distribution we can write:

$$\mathcal{F}\Big(\langle f(x)f(x+\Delta) \rangle \Big) \sim \mathcal{F}(N - |\Delta|) \tag{35}$$

$$\mathcal{F}(N - |\Delta|) = N\delta(\omega) - \mathcal{F}(|\Delta|) \tag{36}$$

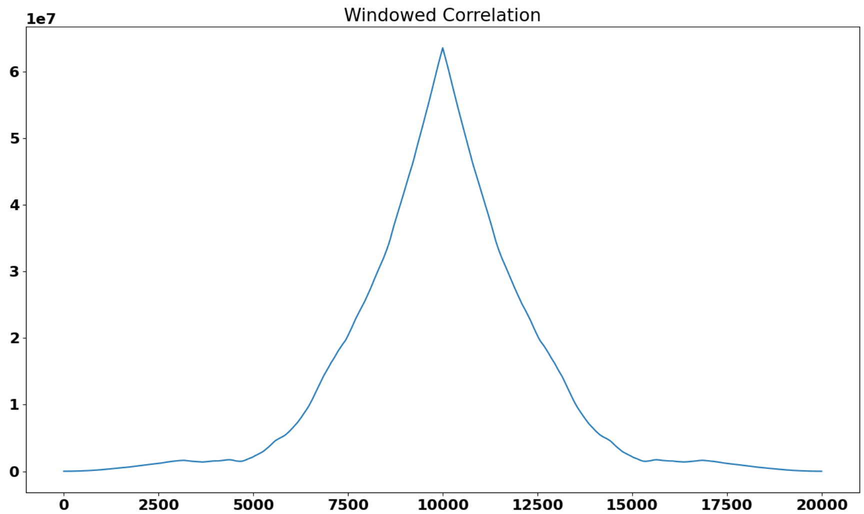
plugging back our previous result in here,

$$\boxed{\mathcal{F}\Big(\langle f(x)f(x+\Delta) \rangle \Big) \sim N\delta(\omega) + \frac{2}{\omega^2}.} \tag{37}$$

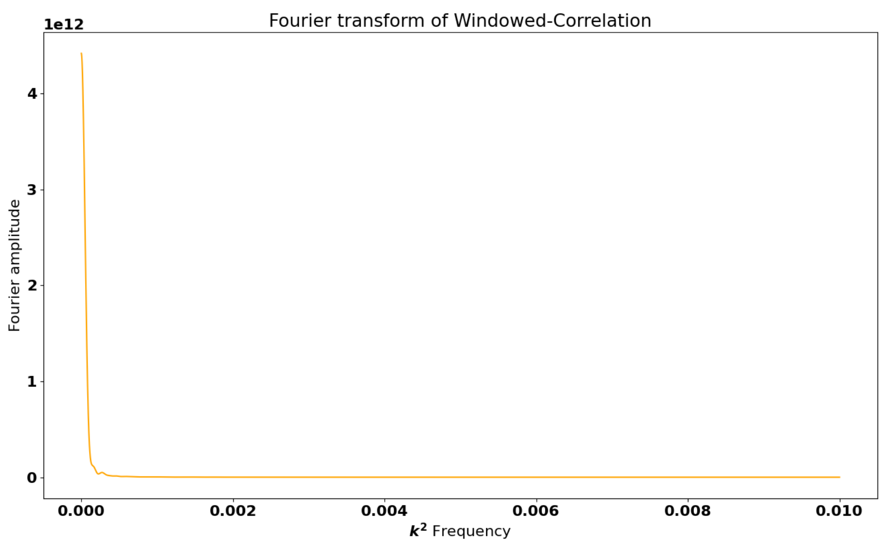
Which completes our proof and shows that the power spectrum of a random walk goes like ω^{-2} (or equivalently like k^{-2}) - again this holds for DFT as much as for CFT.

b)

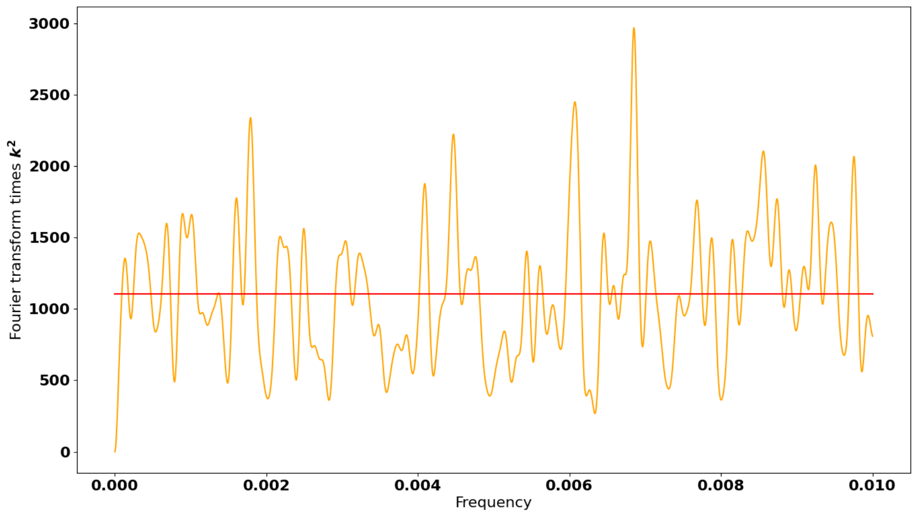
The (windowed) correlation function of the random walk looks as expected like $|x|$ as shown below,



The Fourier transform of the (windowed) correlation function then looks like:



Which if it indeed goes like k^{-2} it should behave like a constant when multiplied by k^2. This is shown in the plot below:



This confirms that the random walk has a k^{-2} power spectrum.