Problem 1

a)

$$f_1'(x) = \frac{f(x+\delta) - f(x-\delta)}{2\delta}$$

$$= \frac{1}{2\delta}\left[\sum_{n=0} \frac{\partial_x^n f(x)}{n!} \delta^n - \sum_{n=0} \frac{\partial_x^n f(x)}{n!} (-\delta)^n \right]$$

$$= \frac{2}{2\delta} \sum_{n=0} \frac{\partial_x^{(2n+1)} f(x)}{(2n+1)!} \delta^{(2n+1)}$$

$$\approx \frac{1}{\delta}\left[\partial_x f(x) \cdot \delta + \frac{\partial_x^3 f(x)}{3!} \delta^3 + \frac{\partial_x^5 f(x)}{5!} \delta^5 \right]$$

$$f_2'(x) = \frac{f(x+2\delta) - f(x-2\delta)}{2\delta}$$

$$\approx \frac{1}{\delta}\left[\partial_x f(x) \cdot \delta + 2^2 \frac{\partial_x^3 f(x)}{3!} \delta^3 + 2^4 \frac{\partial_x^5 f(x)}{5!} \delta^5 \right]$$

$$\frac{2^2 f_1'(x) - f_2'(x)}{2^2-1} \approx \left[\partial_x f(x) + \frac{(2^2 - 2^4)}{3} \frac{\partial_x^5 f(x)}{5!} \delta^5 \right]$$

$$\boxed{\approx \left[\partial_x f(x) - \frac{2^2 \partial_x^5 f(x)}{5!} \delta^5 \right]}$$

b)

The error from the Taylor expansion:

$$\frac{4\, \partial_x^5 f(x)}{5!} \delta^5$$

The fractionnal error:

$$\sim \frac{\varepsilon f(x)}{\delta}$$

The variance is:

$$\sigma^2 = \left(\frac{\partial_x^5 f(x)}{30} \right)^2 \delta^{10} + \frac{\varepsilon^2 (f(x))^2}{\delta^2}$$

With derivative w.r.t. δ

$$\frac{\partial (\sigma^2)}{\partial \delta} = \frac{10}{30^2} \left(\partial_x^5 f(x) \right)^2 \delta^9 - \frac{2\varepsilon^2 (f(x))^2}{\delta^3} = 0$$

$$\boxed{\delta = \left[\frac{2 \cdot 30^2}{10} \cdot \varepsilon^2 \cdot \left(\frac{f(x)}{\partial_x^5 f(x)} \right)^2 \right]^{1/12}}$$

Now let's assume $f \sim \exp(ax)$

$$\boxed{\delta = \left[180 \cdot \frac{\varepsilon^2}{a^{10}} \right]^{1/12}}$$

~~[illegible]~~