Problem 2

What we want is for the Probability Density Function (PDF) of our initial distribution to be always greater than (or equal to) the target distribution (the exponential distribution).

I'll start with the Gaussian G(x), first set up:

$$\frac{Ae^{\frac{-x^2}{2}}}{\lambda e^{-\lambda x}} \overset{?}{\geq} 1 \tag{1}$$

This is can clearly be done for $x = 0$, by letting $A = \lambda$. However notice that as we take the limit to $x \to \infty$

$$\lim_{x \longrightarrow \infty} \frac{Ae^{\frac{-x^2}{2}}}{\lambda e^{-\lambda x}} = \lim_{x \longrightarrow \infty} A' e^{\frac{-x^2+2\lambda x}{2}} = 0 \tag{2}$$

As $x^2 \gg x$ as $x \to \infty$. So clearly Gaussians are the wrong choice.

For the power law we can write:

$$\frac{bx^{-a}}{\lambda e^{-\lambda x}} \overset{?}{\geq} 1 \tag{3}$$

$$\implies \frac{b}{\lambda} \frac{e^{\lambda x}}{x^a} \overset{?}{\geq} 1 \tag{4}$$

$$\implies \sqrt[a]{\frac{b}{\lambda}} \frac{e^{\lambda x/a}}{x} \overset{?}{\geq} 1 \tag{5}$$

Then you can always construct b s.t. $\sqrt[a]{\frac{b}{\lambda}}(\lambda/a) \geq 1$ and be guaranteed the above. This follows from the fact that the derivative of x is fixed, but the exponential function's derivative grows - exponentially - so as long as you match the derivative at 0, you are set.

This then shows Power laws are a viable initial distribution - but are they optimal? That will come later.

Lastly, we have the Lorentzian (the Cauchy distribution as it is often referred to). We can use the same logic as before to write:

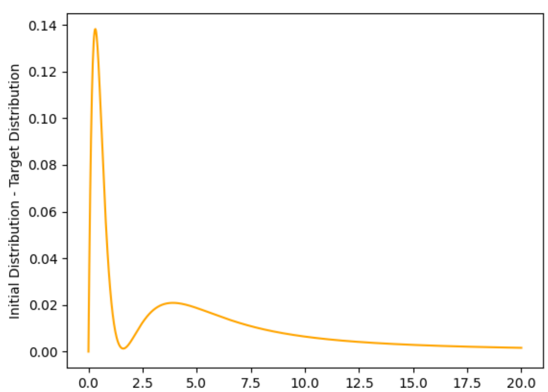
$$\frac{(\frac{x^2}{\gamma^2}+1)^{-1}}{e^{-\lambda x}} \overset{?}{\geq} 1 \tag{6}$$

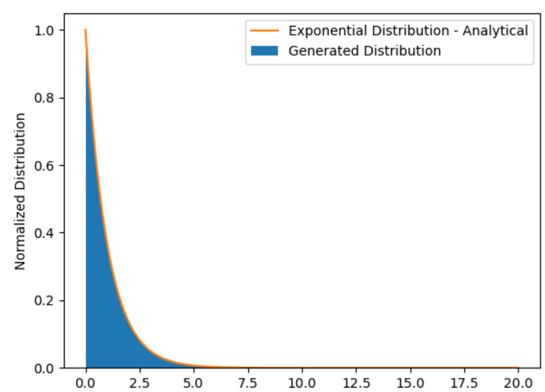
$$\implies \frac{e^{\lambda x}}{\frac{x^2}{\gamma^2}+1} \overset{?}{\geq} 1 \tag{7}$$

Then, if $\lambda \geq \frac{1}{\gamma}$ the above holds. This is easily seen by doing a Taylor transform of the exponential and matching the second degree term with the denominator.

Now for efficiency, we could integrate the target distribution divided by the initial distribution - or compare the areas under the both PDFs, but it's quite obvious the Power Law's divergence makes it much less efficient.

Then, the result with the initial distribution being $\frac{1}{(x/\gamma)^2+1}$ and with $\gamma = 0.808$ is:





Where you can see on the right, the initial distribution is always greater than the target distribution (as required).

On the left, a histogram of the generated 'target' exponential distribution with a 78.8% efficiency (contains 4 million points roughly).