Problem 3

a) The new set of parameters

We can rewrite our initial equation

$$z - z_0 = a((x - x_0)^2 + (y - y_0)^2) \tag{1}$$

into

$$z = a(x^2 + y^2) - 2ax_0x - 2ay_0y + ax_0^2 + ay_0^2 + z_0$$

and then,

$$z = a(x^2 + y^2) - 2\beta x - 2\gamma y + c \tag{2}$$

where

$$x_0 = \frac{\beta}{a}$$
$$y_0 = \frac{\gamma}{a}$$
$$z_0 = c - ax_0^2 - ay_0^2.$$

b) What are the best-fit parameters?

The best fit parameters are (I re-transformed them into the old ones for convenience):

$$a = 1.6670445477401342 \cdot 10^{-4} \text{ mm}$$

$$x_0 = -1.3604886221978907 \text{ mm}$$

$$y_0 = 58.22147608157876 \text{ mm}$$

$$z_0 = -1512.8772100367871 \text{ mm}$$

c) The uncertainties

By estimating the noise, you can obtain the following values:

$$\sigma_a^2 = 4.1627 \cdot 10^{-15}$$
$$\sigma_\beta^2 = 3.91 \cdot 10^{-9}$$
$$\sigma_\gamma^2 = 3.5551 \cdot 10^{-9}$$
$$\sigma_\beta^2 = 9.7356 \cdot 10^{-2}$$

Using the first degree Taylor expansion we can write for any function f of a, b:

$$\sigma_f^2 = \left|\frac{\partial f}{\partial a}\right|^2 \sigma_a^2 + \left|\frac{\partial f}{\partial b}\right|^2 \sigma_b^2 + \frac{\partial f}{\partial a}\frac{\partial f}{\partial b}\sigma_{ab}.$$

Then, write:

$$\sigma_a = 6.4518 \cdot 10^{-8} \text{ mm}$$
$$\sigma_{x_0} = 0.3751 \text{ mm}$$
$$\sigma_{y_0} = 0.3578 \text{ mm}$$
$$\sigma_{z_0} = 0.3121 \text{ mm}$$

This is enough to find that the focal length is:

$$f = 1.49966 \text{ m} \tag{3}$$

With error bars:

$$\sigma_f = \frac{1}{4a^2}\sigma_a$$
$$\sigma_f = 0.58039 \cdot 10^{-4} \text{ m}$$

Putting it all together, we have:

$$a = (1.6670 \pm 0.0006) \cdot 10^{-4} \text{ mm}$$
$$x_0 = (-1.4 \pm 0.4) \text{ mm}$$
$$y_0 = (58.2 \pm 0.4) \text{ mm}$$
$$z_0 = (-1512.9 \pm 0.3) \text{ mm}$$
$$f = (1.4997 \pm 0.0006) \text{ m}$$

These are all sensible, with the focal length very close (and within error bars) to the true value. Below a nice illustration of the model compared to the data (as a sanity check).