Problem 1

a)

$$f_1'(x) = \frac{f(x+\delta) - f(x-\delta)}{2\delta}$$

$$= \frac{1}{2\delta}\left[\sum_{n=0} \frac{\partial_x^n f(x)}{n!}\delta^n - \sum_{n=0} \frac{\partial_x^n f(x)}{n!}(-\delta)^n\right]$$

$$= \frac{2}{2\delta}\sum_{n=0} \frac{\partial_x^{(2n+1)} f(x)}{(2n+1)!}\delta^{(2n+1)}$$

$$\approx \frac{1}{\delta}\left[\partial_x f(x)\cdot\delta + \frac{\partial_x^3 f(x)}{3!}\delta^3 + \frac{\partial_x^5 f(x)}{5!}\delta^5\right]$$

$$f_2'(x) = \frac{f(x+2\delta) - f(x-2\delta)}{2\delta}$$

$$\approx \frac{1}{\delta}\left[\partial_x f(x)\cdot\delta + 2^2\frac{\partial_x^3 f(x)}{3!}\delta^3 + 2^4\frac{\partial_x^5 f(x)}{5!}\delta^5\right]$$

$$\frac{2^2 f_1'(x) - f_2'(x)}{2^2 - 1} \approx \left[\partial_x f(x) + \frac{(2^2 - 2^4)}{3} \frac{\partial_x^5 f(x)}{5!} \delta^5 \right]$$

$$\boxed{\approx \left[\partial_x f(x) - \frac{2^2 \partial_x^5 f(x)}{5!} \delta^4 \right]}$$

b)

The error from the Taylor expansion:

$$\frac{4\,\partial_x^5 f(x)}{5!}\,\delta^4$$

The fractionnal error:

$$\sim \frac{\varepsilon f(x)}{\delta}$$

The variance is:

$$\sigma^2 = \left(\frac{\partial_x^5 f(x)}{30}\right)^2 \delta^8 + \frac{\varepsilon^2 (f(x))^2}{\delta^2}$$

With derivative w.r.t. δ

$$\frac{\partial(\sigma^2)}{\partial\delta} = \frac{8}{30^2}\left(\partial_x^5 f(x)\right)^2 \delta^7 - 2\frac{\varepsilon^2 (f(x))^2}{\delta^3} = 0$$

$$\boxed{\delta = \left[\frac{2 \cdot 30^2}{8} \cdot \varepsilon^2 \cdot \left(\frac{f(x)}{\partial_x^5 f(x)} \right)^2 \right]^{1/10}}$$

Now let's assume $f \sim \exp(ax)$

$$\boxed{\delta = \left[225 \cdot \frac{\varepsilon^2}{a^{10}} \right]^{1/10}}$$

~~[illegible]~~

$$\delta \approx \left(\frac{\varepsilon}{a^5} \right)^{1/5}$$