Problem 2

$$\frac{f(x+dx)-f(x-dx)}{2dx} \approx f'(x) + \frac{f'''(x)\,dx^2}{3!}$$

Then,

$$\sigma^2 \sim \left(\frac{f'''(x)\,dx^2}{3!}\right)^2 + \left(\frac{\varepsilon f(x)}{dx}\right)^2$$

$$= \frac{(\partial_x^3 f(x))^2\,dx^4}{36} + \frac{\varepsilon^2 f^2(x)}{dx^2}$$

$$\frac{\partial(\sigma^2)}{\partial(dx)} = \frac{(\partial_x^3 f(x))^2\,dx^3}{9} - \frac{2\varepsilon^2 f^2(x)}{dx^3} = 0$$

$$dx = \left[18\,\varepsilon^2\left(\frac{f(x)}{\partial_x^3 f(x)}\right)^2\right]^{1/6}$$

$$dx = \left[\sqrt{18}\, \varepsilon \left(\frac{f(x)}{\partial_x^3 f(x)} \right) \right]^{1/4} .$$

Then, we can approx.

$$\partial_x^3 f(x) \approx \frac{f(x+3dx) - 3\big(f(x+dx) - f(x-dx)\big) - f(x-3dx)}{8\, dx^3}$$

$$= \partial_x^3 f(x) + \frac{\partial_x^5 f(x)}{2} dx^2 + \dots$$