Problem 5

a)

$$\sum_{x=0}^{N-1} \exp(-2\pi i k x/N) = \sum_{x=0}^{N-1} (\exp(-2\pi i k/N))^x$$

Now this is a geometric series, and as you well know:

$$\begin{aligned}
\alpha \sum_{x=0}^{N-1} \alpha^x = \sum_{x=1}^{N} \alpha^x = \sum_{x=0}^{N-1} \alpha^x - 1 + \alpha^N \\
\sum_{x=0}^{N-1} \alpha^x - \alpha \sum_{x=0}^{N-1} \alpha^x = 1 - \alpha^N \\
\sum_{x=0}^{N-1} \alpha^x = \frac{1 - \alpha^N}{1 - \alpha}
\end{aligned}$$

Then,

$$\boxed{\sum_{x=0}^{N-1} \exp(-2\pi i k x/N) = \frac{1 - \exp(-2\pi i k)}{1 - \exp(-2\pi i k/N)}}$$

$\square$

b)

First when $k \to 0$

$$\lim_{k\to 0} \frac{1-\exp(-2\pi ik)}{1-\exp(-2\pi ik/N)}$$
$$= \lim_{k\to 0} \frac{2\pi ik + O(k^2)}{2\pi ik/N}$$
$$= \lim_{k\to 0} \frac{2\pi ik + O(k^2)}{2\pi ik/N + O(k^2/N^2)}$$
$$= N$$

$$\boxed{\lim_{k\to 0} \sum_{x=0}^{N-1} \exp(-2\pi ikx/N) = N.} \tag{1}$$

Second, if we look at cases where k $\in \mathbb{N} \setminus \{\mathrm{N}\}$, then

$$\exp(-2\pi ik) \to 1$$
$$\exp(-2\pi ik/N) < 1$$

and hence,

$$\sum_{x=0}^{N-1} \exp(-2\pi ikx/N) = \frac{1-\exp(-2\pi ik)}{1-\exp(-2\pi ik/N)} = \frac{1-1}{1-\exp(-2\pi ik/N)}$$
$$\sum_{x=0}^{N-1} \exp(-2\pi ikx/N) = 0.$$

Third, when $k \to N$

$$\sum_{x=0}^{N-1} \exp(-2\pi ikx/N) = \sum_{x=0}^{N-1} \exp(-2\pi ix) = \sum_{x=0}^{N-1} 1 = N. \tag{2}$$

c)

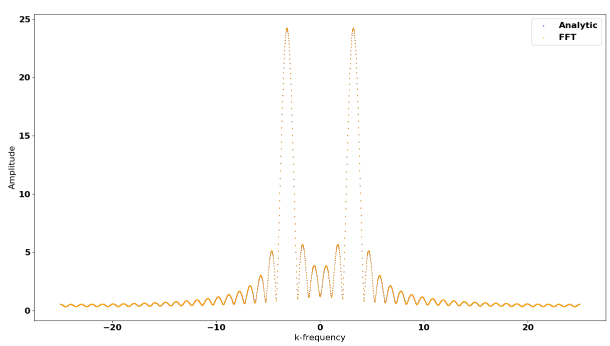
We can start by writing:

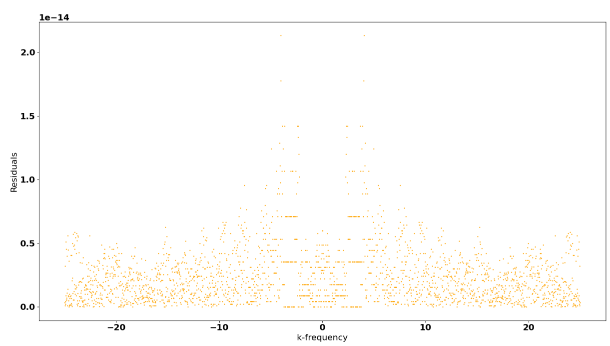
$$sin(2\pi k_s x/N) = \frac{1}{2i}\Big(\exp(2\pi k_s x/N) - \exp(-2\pi k_s x/N)\Big)$$

then

$$\begin{aligned}
&\frac{1}{2i}\sum_{x=0}^{N-1}\Big(\exp(2\pi k_s x/N) - \exp(-2\pi k_s x/N)\Big)\exp(-2\pi i k x/N) \\
&= \frac{1}{2i}\sum_{x=0}^{N-1}\Big(\exp(-2\pi[k-k_s]x/N) - \exp(-2\pi[k+k_s]x/N)\Big) \\
&= \frac{1}{2i}\Big(\frac{1-\exp(-2\pi i[k-k_s])}{1-\exp(-2\pi i[k-k_s]/N)} - \frac{1-\exp(-2\pi i[k+k_s])}{1-\exp(-2\pi i[k+k_s]/N)}\Big)
\end{aligned}$$

Below a plot comparing the analytical prediction and the FFT. They agree within machine precision.

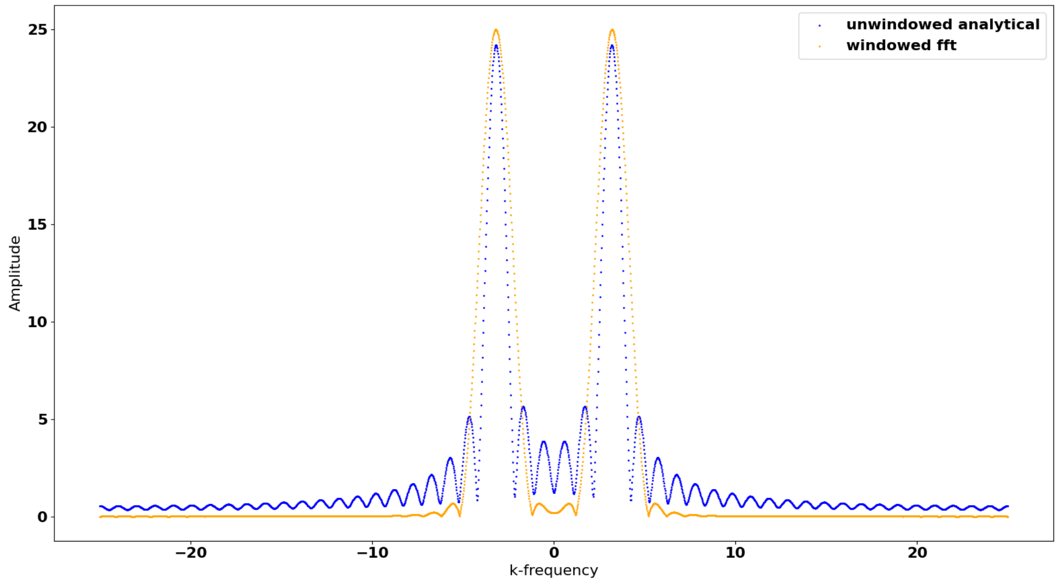




We indeed get something close to a delta function near $k_S = 3.2$ with considerable spectral leakage.

d)

Now if we multiply by the window function, we get:



which leads a dramatic drop in the spectral leakage.

e)

The Fourier transform of the window function $W = 0.5(1 - \cos(2\pi x/N))$ is then:

$$\mathcal{F}(0.5(1 - \cos(2\pi x/N))) = 0.5(\mathcal{F}(1) - \mathcal{F}(\cos(2\pi x/N)))$$

where $\mathcal{F}(1) = N\delta(k)$ and almost trivially, $\mathcal{F}(\cos(2\pi x/N)) = N\delta(k-1) + N\delta(k+1)$. Then,

$$\mathcal{F}(W) = \frac{N}{2}\Big(\delta(k) - \frac{1}{2}\Big(\delta(k-1) + \delta(k+1)\Big)\Big) \tag{3}$$

Now let's assume our unwindowed function is $f(x)$ with fourier transform $F(k)$, then:

$$f_w(x) = f(x)w(x)$$

then by the convolution theorem

$$F_w(k) = F(k) \circledast W(k)$$

$$F_w(k) = \sum_{\tilde{k}} W(\tilde{k}) F(\tilde{k} - k)$$

$$F_w(k) = \sum_{\tilde{k}} \frac{N}{2}\Big(\delta(\tilde{k}) - \frac{1}{2}\Big(\delta(\tilde{k}-1) + \delta(\tilde{k}+1)\Big)\Big) F(\tilde{k} - k)$$

$$\boxed{F_w(k) = \frac{N}{2}\Big(F(-k) - \frac{1}{2}\Big(F(-k-1) + F(-k+1)\Big)\Big)} \tag{4}$$

Now if we use our results from c), we can obtain a closed form for the windowed function using only the unwindowed form. Below a plot of the analytical windowed Fourier transform.