Posterior distribution for correlation coeficient. Bivariate case

September 26, 2014

Consider n observations from $Y \sim N_d(0, \Sigma)$ distribution, the likelihood function can be written as follows:

$$p(y|\mu, \Sigma) \propto |\Sigma|^{-n/2} e^{-\frac{1}{2}\sum_{i=1}^n y_i^{'}\Sigma^{-1}y_i} = |\Sigma|^{-n/2} e^{-\frac{1}{2}tr(\Sigma^{-1}S_0)} \tag{1}$$

where y_i represents the ith observation from the vector Y, and $S_0 = \sum_{i=1}^n y_i y_i^{'}$.

Here we simulate 5 bivariate normally distributed data set, from the model

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N\left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix}\right)$$

with specific values $\sigma_1 = \sigma_2 = 0.01$, and $\rho = 0$, sample size is $n = 10$.

	sim	pearson
1	1	-0.12
2	2	0.05
3	3	0.62
4	4	-0.61
5	5	-0.09

1 Conjugate prior, $IW(3, I)$

Conjugate prior for the covariance matrix in the normal model is the IW, also $\Sigma \sim IW(3, I)$ implies IG on the variances and uniform priors con correlation.

Prior density is $p(\Sigma) \propto |\Sigma|^{-(\nu+d+1)/2} e^{-\frac{1}{2}tr(\Lambda\Sigma^{-1})}$ and the full conditional for Σ is $\Sigma|y, \mu \sim IW(n+3, I+S_\mu)$. If $\mu|\Sigma \sim N(\mu_0, \Sigma/\kappa_0)$, then the marginal posterior for Σ,$\Sigma|y$, has an IW distribution.

From the posterior of Σ it is posible to derive the posterior density of the correlation coefficient (matt pdf). Specifically when $\Sigma|y \sim IW(n+3, I+S_0)$ we have

$$p(\rho|y) \propto (1-\rho^2)^{(n-3)/2} exp\left[-\frac{\rho}{1-\rho^2}\sum y_{1i}y_{2i}\right]$$

2 SS $[IG(1, 1/2), IW(3, I)]$ prior

The prior for covariance matrix is $\Sigma \sim \text{SS}[IG(1, 1/2), IW(3, I)]$, which means that $\sigma_i^2 \overset{iid}{\sim} IG(1, 1/2)$ and $R = \Delta Q \Delta$ where $Q \sim IW(3, I)$ and Δ is a diagonal matrix with i^{th} diagonal element $Q_{ii}^{-1/2}$. This implies

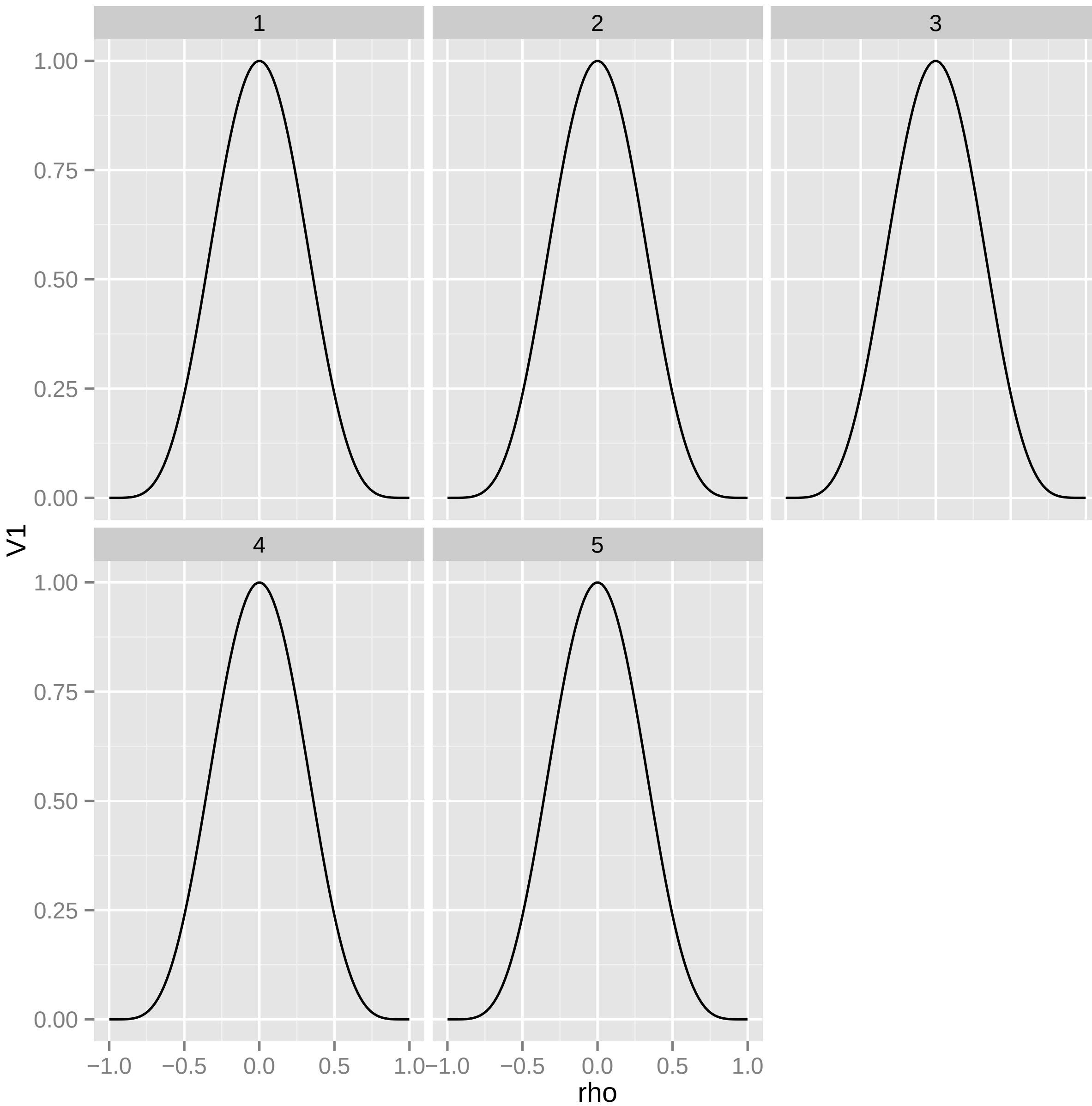

Figure 1: ρ posterior density for IW prior

that each correlation is uniformly distributed.

The prior marginals for variances and correlation are identical to the implied by the conjugate model, but here variances and correlations are consider to independent in the prior.

In the bivariate case there is only one correlation coefficient so we can directly use $\rho \sim unif(-1, 1)$.

$$
\begin{aligned}
p(\sigma_1^2, \sigma_1^2, \rho|y) \quad & \propto p(y|\mu, \Sigma) \prod_i (\sigma_i^2) e^{\frac{-1}{2\sigma_i^2}} \\
& \propto (\sigma_1^2 \sigma_2^2 (1-\rho^2))^{\frac{-n}{2}} e^{\frac{-1}{2} \frac{\sigma_2^2 \sum y_{1i}^2 + \sigma_1^2 \sum y_{2i}^2 - 2\sigma_1\sigma_2\rho \sum y_{1i}y_{2i}}{\sigma_1^2\sigma_2^2(1-\rho^2)}} \prod_i (\sigma_i^2) e^{\frac{-1}{2\sigma_i^2}} \\
& \propto ((1-\rho^2))^{\frac{-n}{2}} \sigma_1^{-n/2-2} exp\left[-\frac{1}{2\sigma_1^2}(1 + \frac{\sum y_{i1}^2}{(1-\rho^2)})\right] \sigma_2^{-n/2-2} exp\left[-\frac{1}{2\sigma_2^2}(1 + \frac{\sum y_{i2}^2}{(1-\rho^2)})\right] exp\left[-\frac{1}{2}\frac{\rho}{1-\rho^2}\frac{\sum y_{1i}y_{2i}}{\sigma_1\sigma_2}\right]
\end{aligned}
$$

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dinvgamma_log <- function(x, a = 1, b = 0.5) a * log(b) - lgamma(a) - (a + 1) *
    log(x) - b/x

post_igiw <- function(dts, sig1, sig2, rho, dt) {
    # prior for variances is IG(1, 1/2) prior for correlations is Unif(0,1),
    # althogh it came from an IW likelyhood : N(0, Sigma), where Sigma= D*R*D
    R <- diag(c(1, 1))
    R[2, 1] <- rho
    R[1, 2] <- rho
    D <- diag(c(sig1, sig2))
    Sigma <- D %*% R %*% D
    d <- subset(dt, sim == dts)[, c("X1", "X2")]
    like <- sum(dmvnorm(d, sigma = Sigma, log = TRUE))
    like + dinvgamma_log(sig1^2) + dinvgamma_log(sig2^2)
}


# post_igiw(sig1=.01, sig2=.01, rho=0, d=dd)

vals <- expand.grid(dts = 1:5, sig1 = seq(0.001, 0.5, , 20), sig2 = seq(0.001,
    0.5, , 20), rho = seq(-1, 1, , 30))
post.vals <- mdply(vals, post_igiw, dt = dd)
post.vals$jointpost <- with(post.vals, exp(V1))

# get marginals
post.rhopr <- ddply(post.vals, .(dts, rho), summarise, post.pr = sum(jointpost))
post.rho <- ddply(post.rhopr, .(dts), transform, post = post.pr/sum(post.pr))

post.s1pr <- ddply(post.vals, .(dts, sig1), summarise, post.pr = sum(jointpost))
post.s1 <- ddply(post.s1pr, .(dts), transform, post = post.pr/sum(post.pr))
```

3 SS $[G(3/2, 1/2), W(3, I)]$ prior

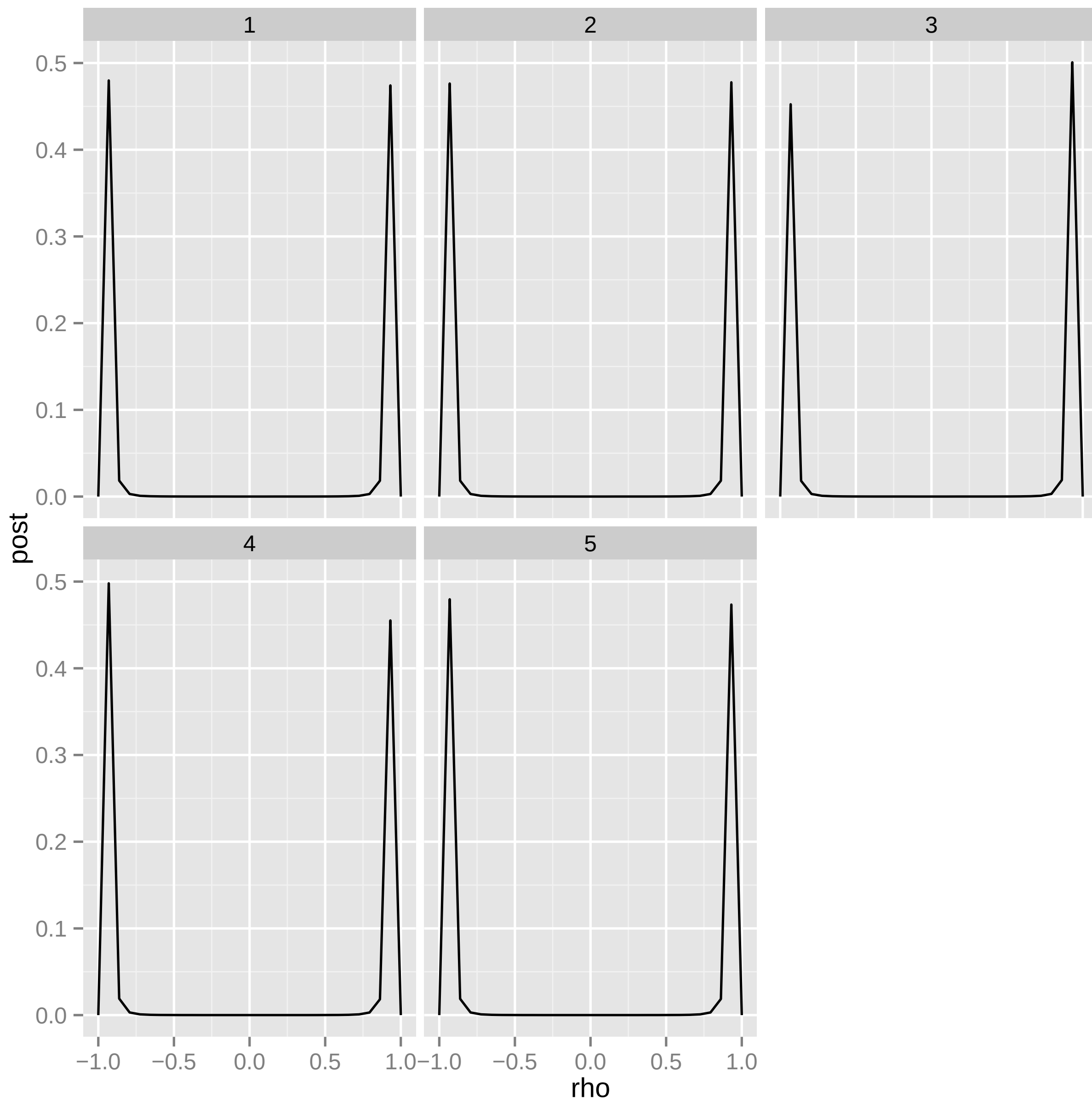

Figure 2: ρ psoterior density for SS $[IG, IW]$ prior

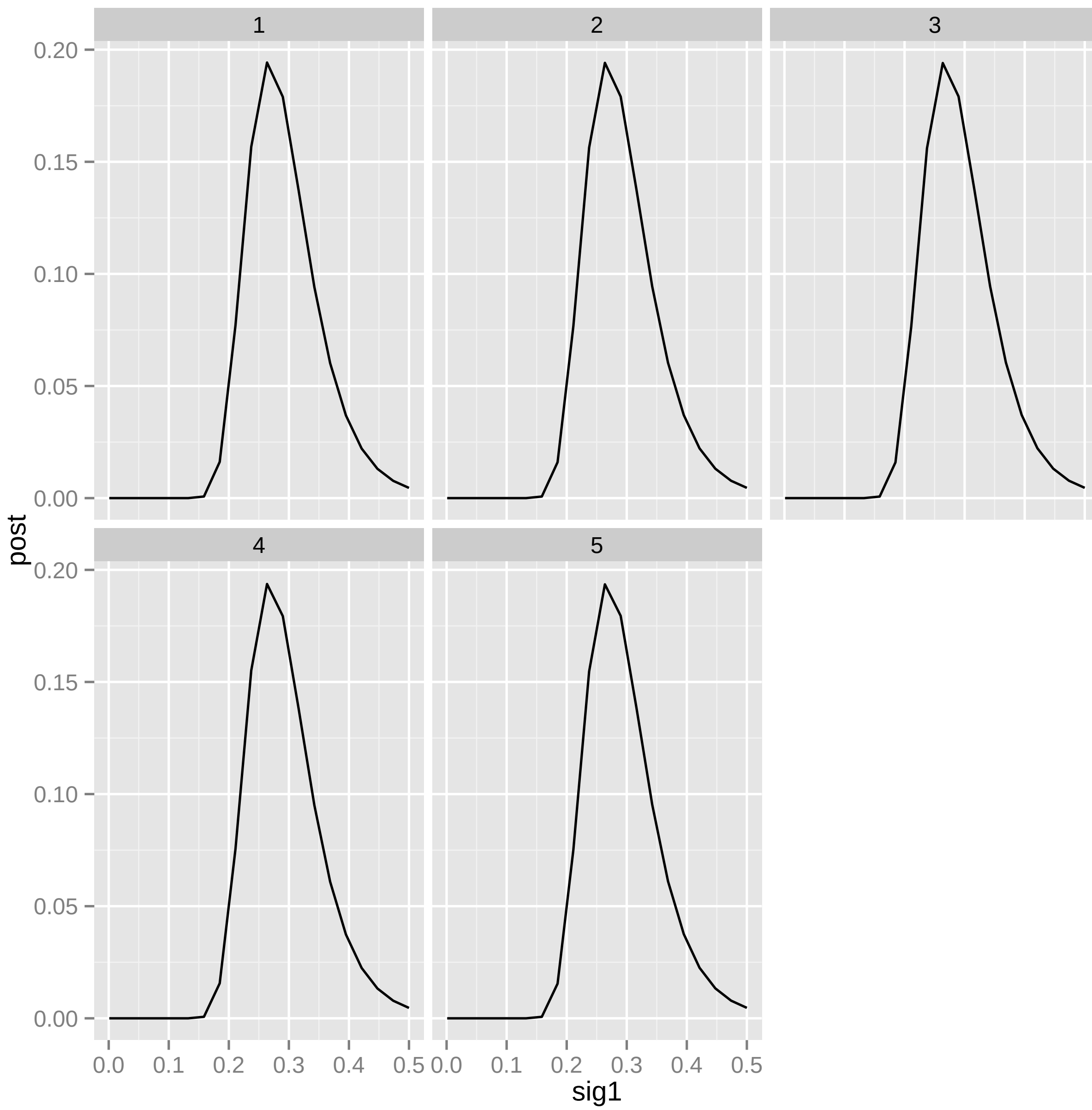

Figure 3: σ_1 psoterior density for SS $[IG, IW]$ prior

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## [1] 57.28
```

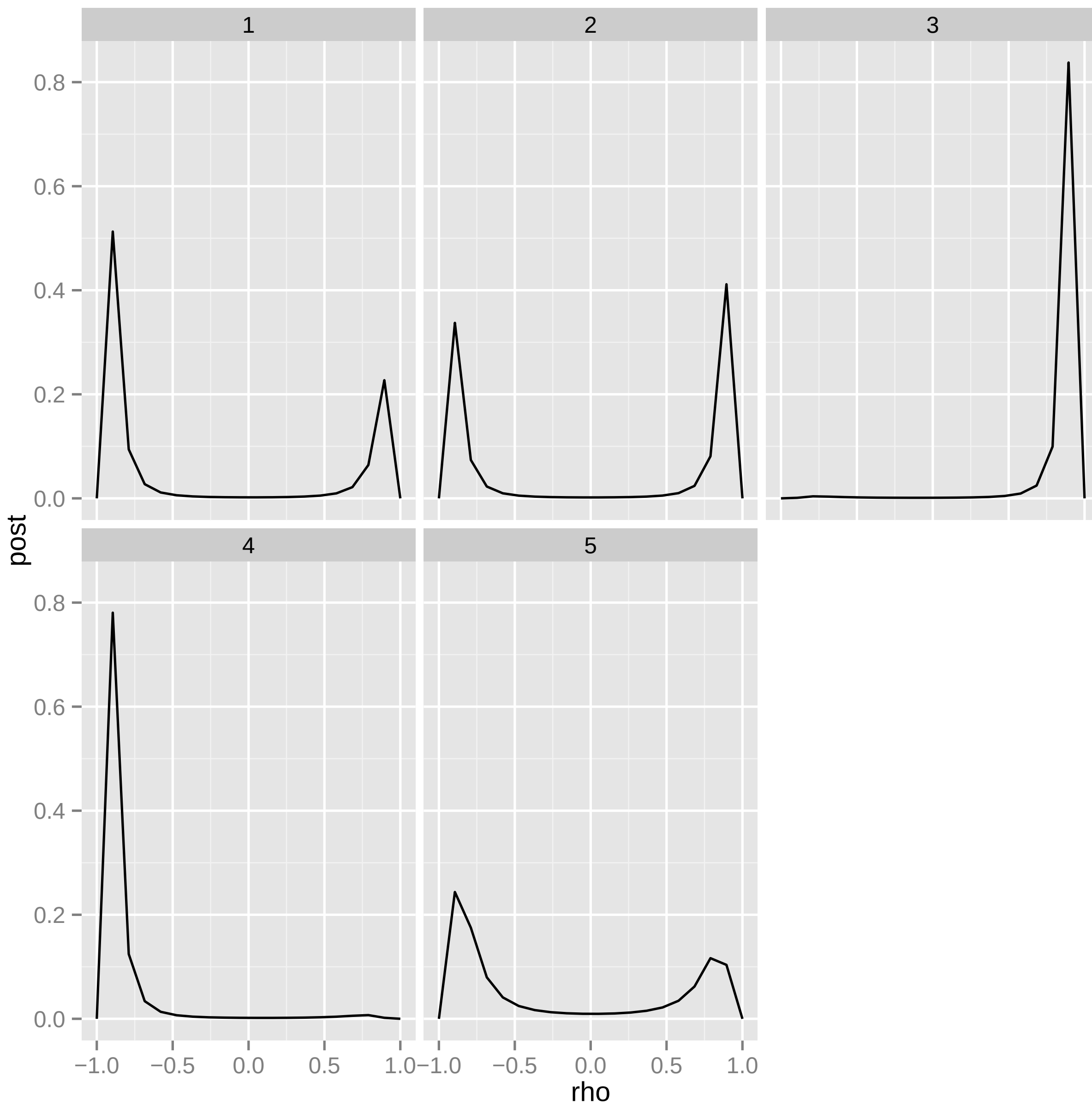

Figure 4: ρ psoterior density for SS $[G(3/2,1/2),W(3,I)]$ prior

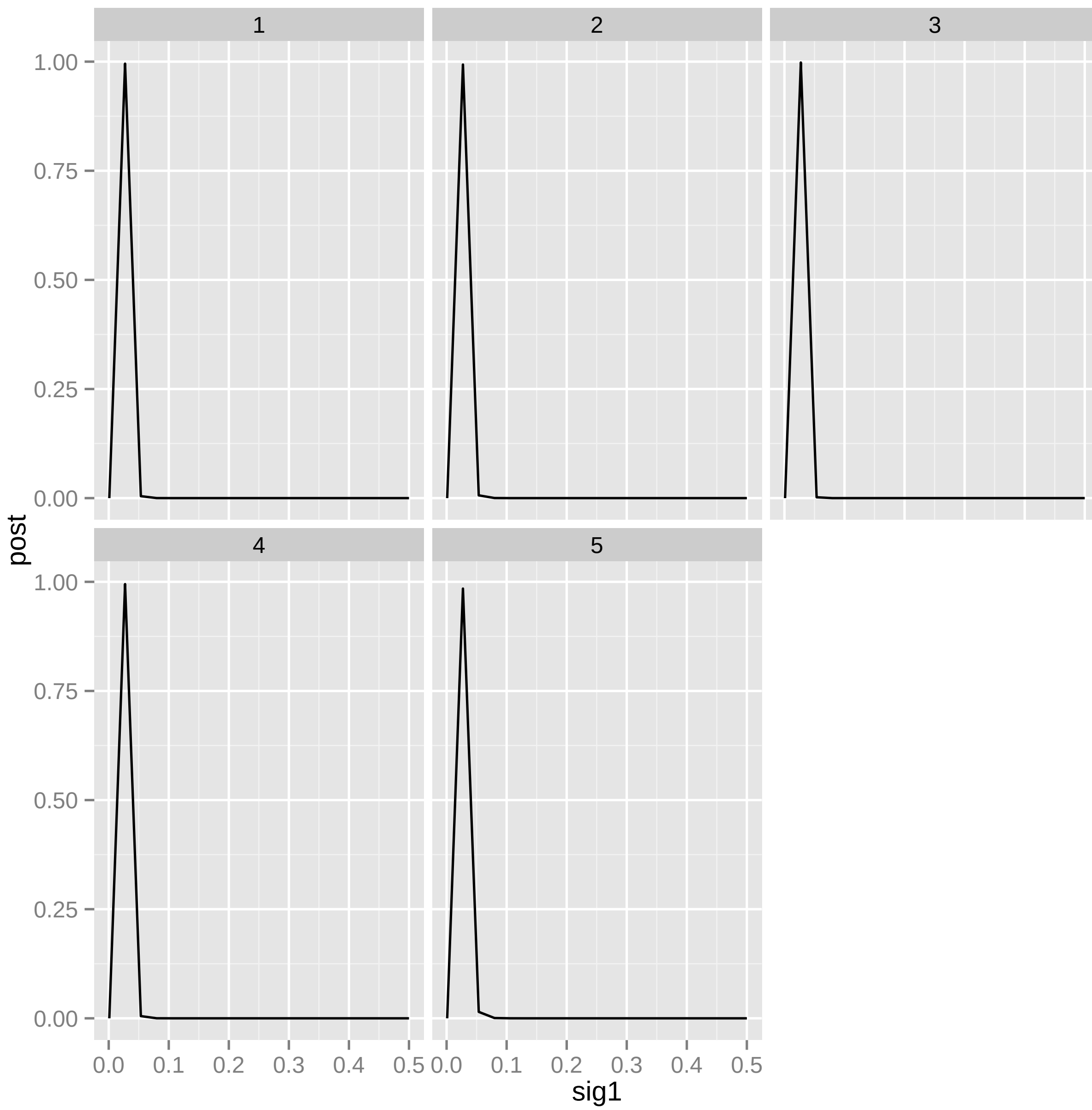

Figure 5: σ_1 psoterior density for SS $[G(3/2, 1/2), W(3, I)]$ prior

4 $\Sigma \sim W_2(3, I)$ prior