Posterior distribution for correlation coeficient. Bivariate case

November 6, 2014

Consider n observations from $Y \sim N_d(0, \Sigma)$ distribution, the likelihood function can be written as follows:

$$p(y|\mu, \Sigma) \propto |\Sigma|^{-n/2} e^{-\frac{1}{2}\sum_{i=1}^{n}(y_i-\mu)^{'}\Sigma^{-1}(y_i-\mu)} = |\Sigma|^{-n/2} e^{-\frac{1}{2}tr(\Sigma^{-1}S_\mu)} \tag{1}$$

where y_i represents the ith observation from the vector Y. Also $S_\mu = \sum_{i=1}^{n}(y_i-\mu)(y_i-\mu)^{'}$, which can be decompose as $S_\mu = \sum_{i=1}^{n}(y_i-\bar{y})(y_i-\bar{y})^{'} + n(bary-\mu)(\bar{y}-\mu)^{'} = A + M$.

1 Derivaitons

To complete model (1) we set uniform prior for μ, $p(\mu) \propto 1$ and we study alternatives for construct a prior on Σ. This implies,

$$\begin{aligned} p(\Sigma|y) &\propto \int p(\Sigma, \mu|y)d\mu \\ &\propto \int |\Sigma|^{-n/2} e^{-\frac{1}{2}tr(\Sigma^{-1}(A+M))} p(\Sigma)d\mu \\ &\propto |\Sigma|^{-n/2} p(\Sigma) e^{-\frac{1}{2}tr(\Sigma^{-1}A)} \int e^{-\frac{1}{2}tr(\Sigma^{-1}M)}d\mu \\ &\propto |\Sigma|^{(-n/2+1/2)} p(\Sigma) e^{-\frac{1}{2}tr(\Sigma^{-1}A)} \end{aligned} \tag{2}$$

For each of these alternatives we derive the posterior distribution of the correlation coefficient ρ.

1.1 Jeffrey prior

A non informative prior for the covariance matrix is $p(\Sigma) \propto |Sigma|^{-(\frac{d+1}{2})}$, then $p(\Sigma|y) \propto |\Sigma|^{-(\frac{d+n}{2})} e^{-\frac{1}{2}tr(\Sigma^{-1}A)}$ implying that $\Sigma|y \sim IW(n-1, A)$. In the bivariate case ($d = 2$) this implies that

$$p(\sigma_1^2, \sigma_2^2, \sigma_{12}) \propto (\sigma_1^2\sigma_2^2 - \sigma_{12}^2)^{-\frac{n+2}{2}} exp\left[-\frac{1}{2(\sigma_1^2\sigma_2^2 - \sigma_{12}^2)}(a_{11}\sigma_2^2 - 2a_{12}\sigma_{12} + a_{22}\sigma_1^2\right]$$

where $a_{kj} = sum_{i=1}^{n}(y_{ik} - \bar{y}_k)(y_{ij} - \bar{y}_j)$. Then Box-Tiao finds the posterior distribution for ρ using a transformation proposed by Fisher to obtain the sampling distribution of the pearson correlation coeficient, $r = a_{12}/\sqrt{a_{11}a22}$

$$x = (\tfrac{\sigma_1\sigma_2}{a_{11}a_{22}})^{1/2} \quad w = (\tfrac{\sigma_1 a_{22}}{a_{11}\sigma_2})^{1/2} \quad \rho = \tfrac{\sigma_{12}}{\sigma_1\sigma_2} \quad J = \tfrac{2x^2(a_{11}a_{22})^{3/2}}{w}$$

$$
\begin{aligned}
p(x,w,\rho) &\propto (x^2(1-\rho^2))^{-\frac{n+2}{2}}\frac{x^2}{w}exp\left[-\frac{1}{2x(1-\rho^2)}(w^{-1}-2r\rho w)\right]\\
p(w,\rho) &\propto \frac{(1-\rho^2)^{-\frac{n+2}{2}}}{w}\int x^{-n}exp\left[-\frac{1}{2x(1-\rho^2)}(w^{-1}-2r\rho w)\right]dx\\
p(w,\rho) &\propto \frac{(1-\rho^2)^{-\frac{n+2}{2}}}{w}(1-\rho^2)^{n-1}(w^{-1}-2r\rho w)^{-(n-1)}
\end{aligned}
$$

finally

$$p(\rho|y)\propto(1-\rho^2)^{n/2-2}\int_0^\infty w^{-1}(w^{-1}-2r\rho+w)^{-(n-1)}dw \tag{3}$$

1.2 Conjugate prior

A similar way can be used for the case when $\Sigma\sim IW(\nu,\Lambda)$, in this case we can include $p(\Sigma)\propto|\Sigma|^{-(\frac{\nu+d+1}{2})}e^{-\frac{1}{2}tr(\Lambda\Sigma^{-1})}$ on equation (2) to get $p(\Sigma|y)\propto|\Sigma|^{-(\frac{\nu+d+n}{2})}e^{-\frac{1}{2}tr(\Sigma^{-1}(A+\Lambda)}$ implying that $\Sigma|y\sim IW(n+\nu-1,A+\Lambda)$ as expected.

Again for bivariate case, posterior distribution of the correlation coefficient can be obtain letting $A^{'}=A+\Lambda$ and applying the same transformation as before to obtain

$$p(\rho|y)\propto(1-\rho^2)^{\frac{n+\nu}{2}-2}\int_0^\infty w^{-1}(w^{-1}-2r^{'}\rho+w)^{-(n+\nu-1)}dw \tag{4}$$

where the only differences with (3) are the effect of ν and we use $r^{'}=\frac{a_{12}+\lambda_{12}}{\sqrt{(a_{11}+\lambda_{11})(a_{22}+\lambda_{22})}}$ instead of the sample correlation.

1.3 Separation strategy

It is known the implied marginal prior distribution from $IW(\nu=d+1,\Lambda=I)$ are $\sigma_i^2\sim IG(1,1/2)$ and $\rho\sim Unif(-1,1)$. A posible separation strategy that mach the marginal prior of the conjugate case is then $p(\Sigma)\propto(\sigma_1^2\sigma_2^2)^{-2}exp\left[-\frac{1}{2\sigma_1^2}-\frac{1}{2\sigma_2^2}\right]$ which gives a posterior for Σ as follows

$$
\begin{aligned}
p(\Sigma|y) &\propto |\Sigma|^{-\frac{n+1}{2}}(\sigma_1^2\sigma_2^2)^{-2}exp\left[-\frac{1}{2}\left(tr(\Sigma^{-1}A)+\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)\right]\\
&\propto (\sigma_1^2\sigma_2^2(1-\rho^2))^{-\frac{n+1}{2}}(\sigma_1^2\sigma_2^2)^{-2}exp\left[-\frac{1}{2}\left(\frac{1}{(1-\rho^2)}\left(\frac{a_{11}}{\sigma_2^2}-\frac{2\rho a_{12}}{\sigma_1\sigma_2}+\frac{a_{22}}{\sigma_1^2}\right)+\frac{1}{\sigma_1^2}+\frac{1}{\sigma_2^2}\right)\right]\\
&\propto (\sigma_1^2\sigma_2^2(1-\rho^2))^{-\frac{n+1}{2}}(\sigma_1^2\sigma_2^2)^{-2}exp\left[-\frac{1}{2(1-\rho^2)}\left(\frac{a_{11}+1-\rho^2}{\sigma_2^2}-\frac{2\rho a_{12}}{\sigma_1\sigma_2}+\frac{a_{22+1-\rho^2}}{\sigma_1^2}\right)\right]
\end{aligned}
$$

applying a similar transformation we can obtain the posterior distribution for ρ, this is

$$x=(\frac{\sigma_1\sigma_2}{a_{11}a_{22}})^{1/2}\quad w=(\frac{\sigma_1 a_{22}}{a_{11}\sigma_2})^{1/2}\quad \rho=\rho\quad J=\frac{2xa_{11}a_{22}}{w}$$

$$
\begin{aligned}
p(x,w,\rho) &\propto (x^2(1-\rho^2))^{-\frac{n+1}{2}}\frac{x^{-3}}{w}exp\left[-\frac{1}{2x(1-\rho^2)}\left(\frac{a_{11}+1-\rho^2}{wa_{11}}-2\rho r+\frac{a_{22+1-\rho^2}}{a_{22}}w\right)\right]\\
p(w,\rho) &\propto (1-\rho^2)^{\frac{n+3}{2}}\frac{1}{w}\left(\frac{a_{11}+1-\rho^2}{wa_{11}}-2r\rho+\frac{a_{22+1-\rho^2}}{a_{22}}w\right)^{-(n+2)}
\end{aligned}
$$

from where we get

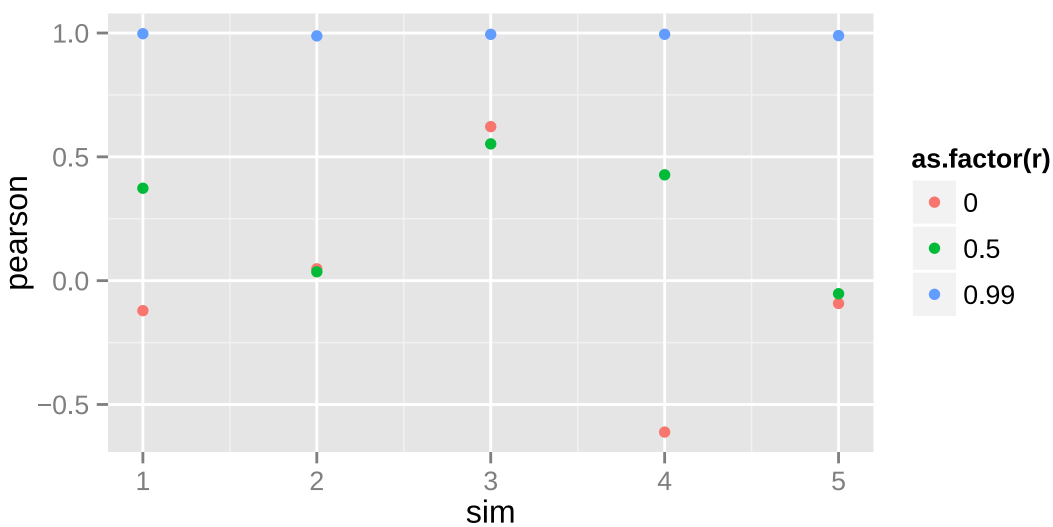
$$p(\rho|y) \propto (1-\rho^2)^{\frac{n+3}{2}} \int \frac{1}{w} \left(\frac{a_{11} + 1 - \rho^2}{w a_{11}} - 2r\rho + \frac{a_{22+1-\rho^2}}{a_{22}} w \right)^{-(n+2)} \tag{5}$$

2 simulations

Here we simulate 5 bivariate normally distributed data set, from the model

$$\begin{pmatrix} Y_1 \\ Y_2 \end{pmatrix} \sim N \left(\begin{pmatrix} 0 \\ 0 \end{pmatrix}, \begin{pmatrix} \sigma_1^2 & \rho\sigma_1\sigma_2 \\ \rho\sigma_1\sigma_2 & \sigma_2^2 \end{pmatrix} \right)$$

with specific values $\sigma_1 = \sigma_2 = 0.01$, and $\rho = 0$, sample size is $n = 10$.



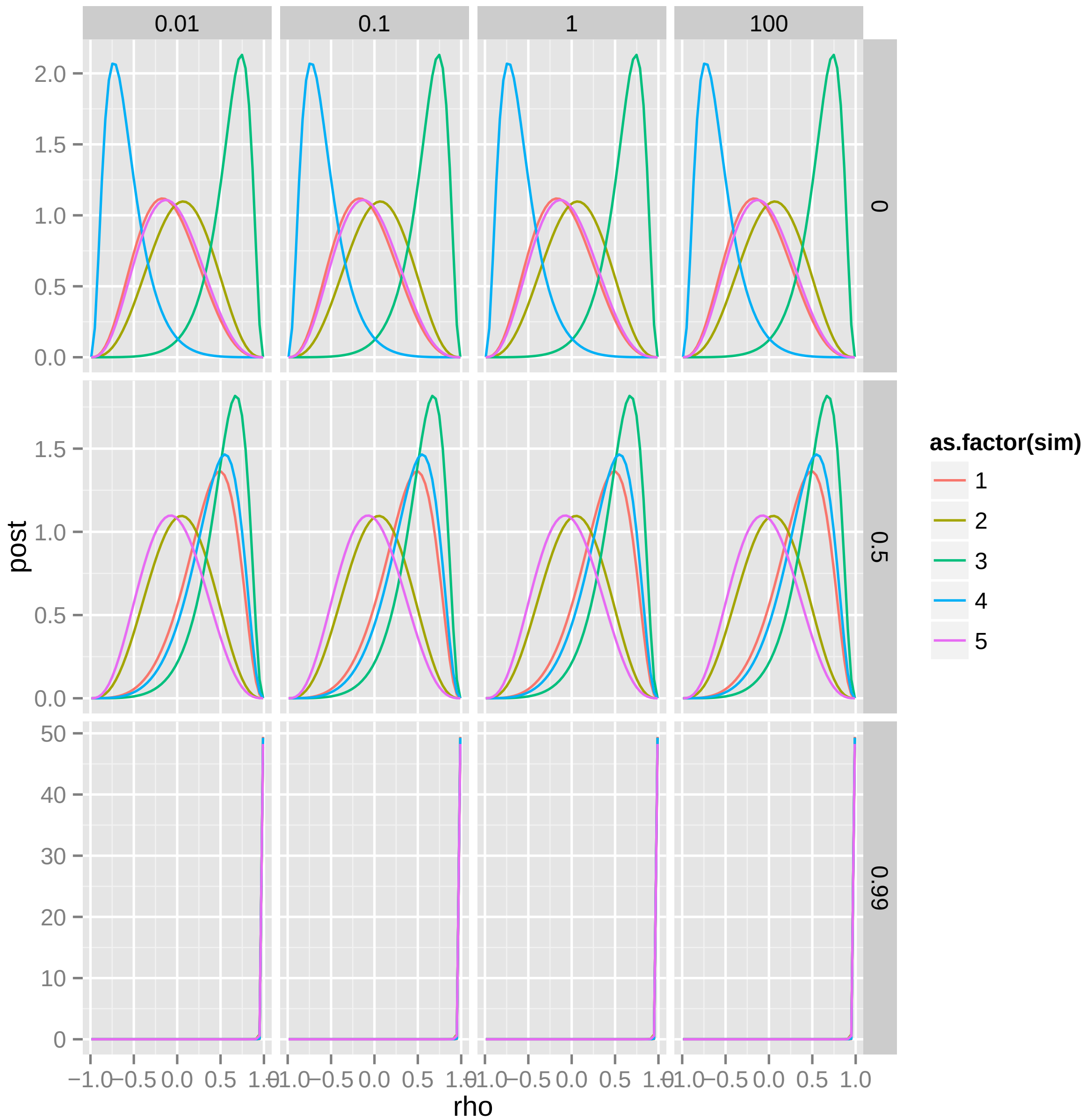


Figure 1: ρ posterior density for Jeffrey prior

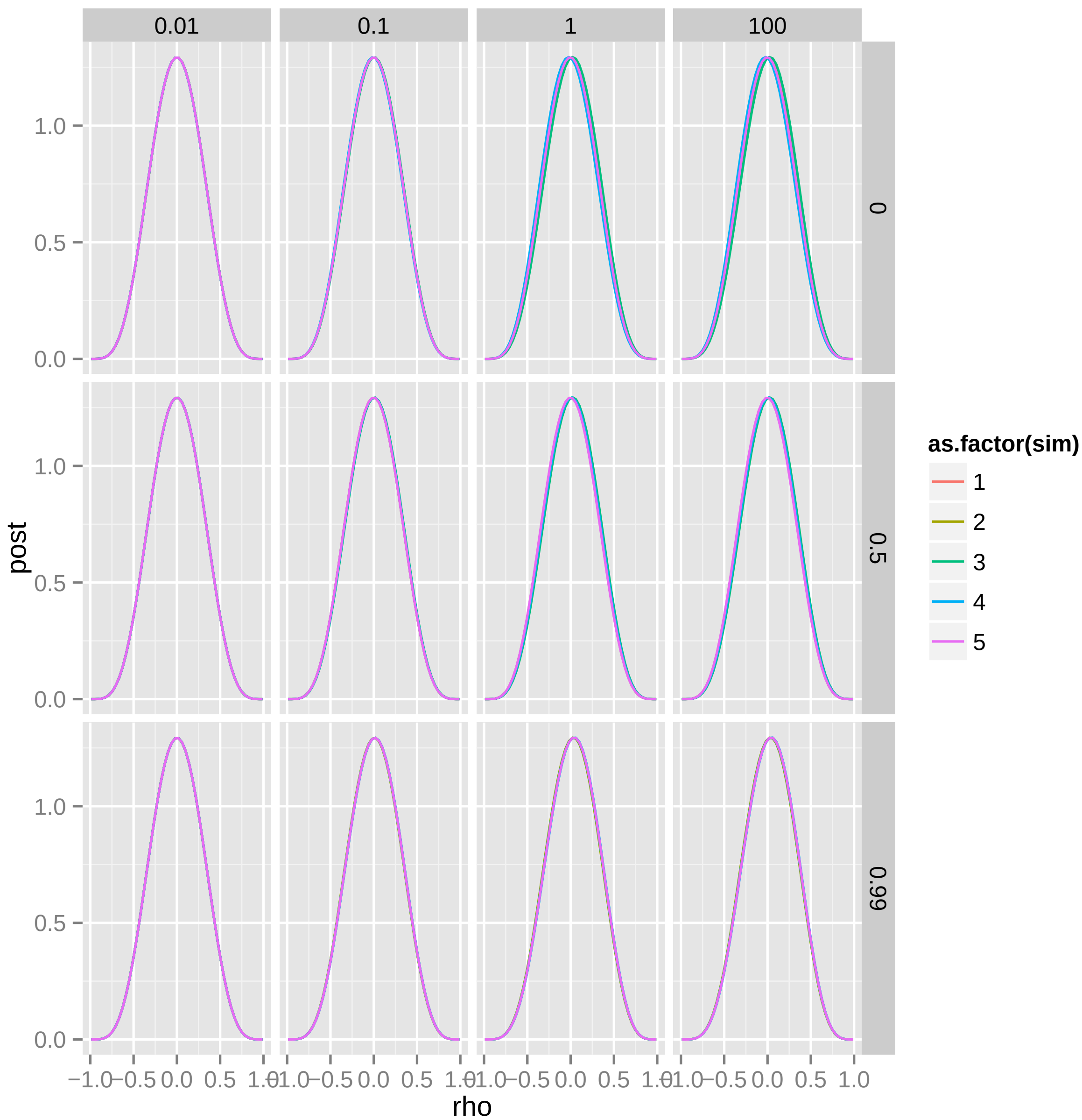


Figure 2: ρ posterior density for IW prior

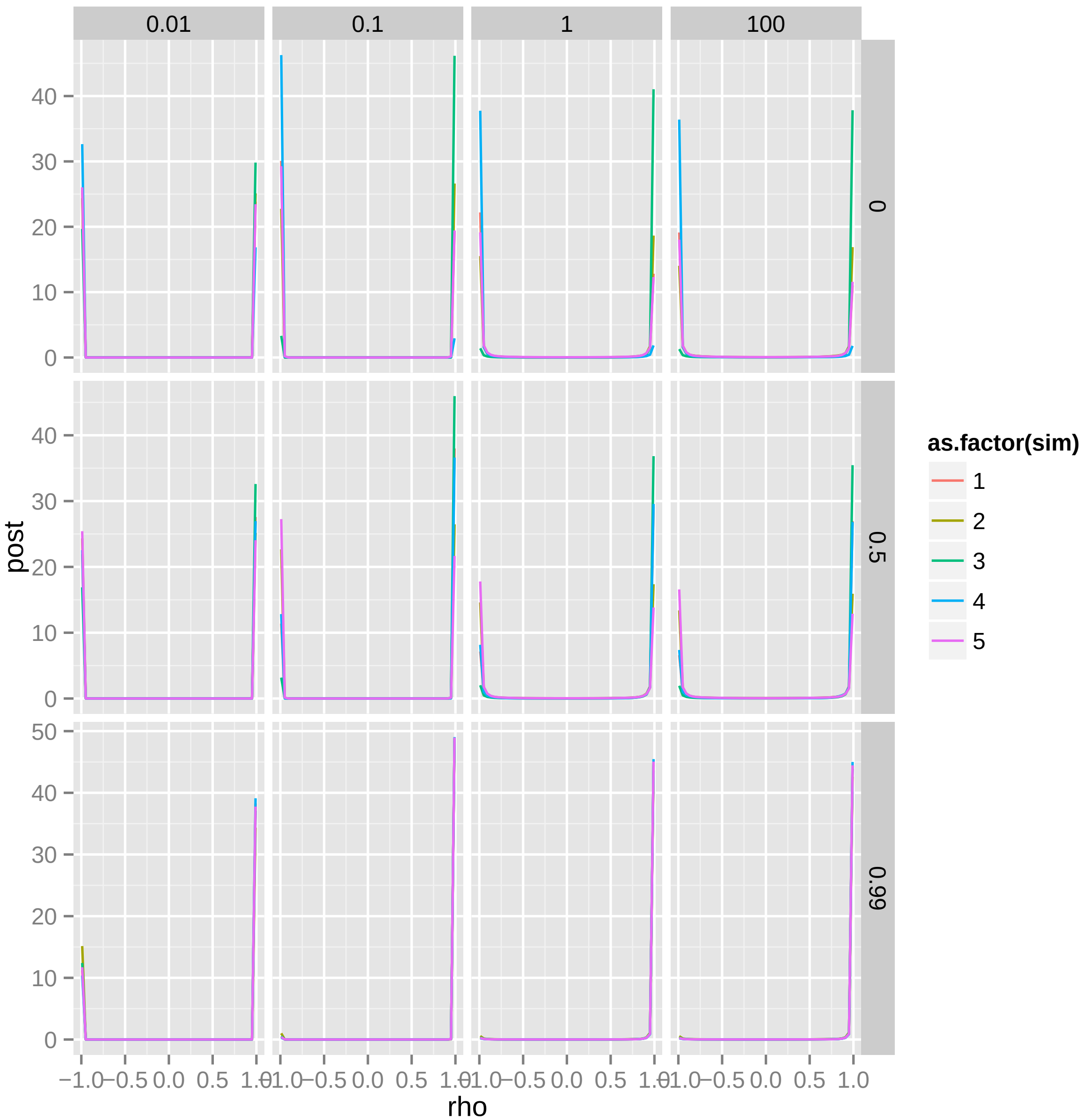


Figure 3: ρ posterior density for $SS[IG, IW]$ prior

3 Low Density Region on IG

One characteristic of the IG density is to have a very low density region for the small values. This produces an overestimation of the variance if the true variance belongs to that low density region. Which is the size of this regioin ? In this section we try to give some answer to this question.

For each $\epsilon > 0$, define δ_ϵ st $f(\delta_\epsilon|\alpha, \beta) = \epsilon$, which implies to find the inverse function of the inverse-gamma density, to solve the equation

$$\frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\frac{\beta}{x}} = \epsilon$$

In order to solve this equation we use *Lambert W function.* In mathematics, the Lambert W function, is a set of functions, namely the branches of the inverse relation of the function xe^x. It can be generalize as the equation solution,

$$\begin{array}{l} log(A + Bx) + Cx = log(D) \\ x = \frac{1}{C} W(\frac{CD}{B} e^{\frac{AC}{B}}) - \frac{A}{B} \end{array}$$

base on this we can find an expresion for δ_ϵ,

$$\begin{aligned}
\frac{\beta^\alpha}{\Gamma(\alpha)} x^{-\alpha-1} e^{-\frac{\beta}{x}} &= \epsilon \\
x^{-\alpha-1} e^{-\frac{\beta}{x}} &= \epsilon \frac{\Gamma(\alpha)}{\beta^\alpha} \\
u^{\alpha+1} e^{-\beta u} = \epsilon_* \quad where & \quad \epsilon_* = \epsilon \frac{\Gamma(\alpha)}{\beta^\alpha} \quad , u = 1/x \\
(\alpha + 1) log(u) - \beta u &= log(\epsilon_*) \\
log(u) - \frac{\beta}{\alpha + 1} u &= \frac{log(\epsilon_*)}{\alpha + 1} \\
then \;\; u &= -\frac{\alpha + 1}{\beta} W\left(-\frac{\beta}{\alpha + 1} \epsilon_*^{\frac{1}{\alpha+1}}\right)
\end{aligned}$$

then noting that $u = 1/\delta_\epsilon$ we obtain the expression

$$\delta_\epsilon^{ig} = -\frac{\alpha + 1}{\beta W\left(-\frac{\beta}{\alpha+1} \epsilon_*^{\frac{1}{\alpha+1}}\right)} \tag{6}$$

We would like to compare δ_ϵ^{ig} values with the corresponding for other distributions used for variance parameters as $LN(\mu, \sigma)$ or $G(\alpha, \beta)$. With a similar procedure we can obtain a close form for the *Gamma* density as $\delta_\epsilon^g = -\frac{\beta W\left(-\frac{\beta}{\alpha-1} \epsilon_*^{\frac{1}{\alpha-1}}\right)}{\alpha-1}$, for the LN there is no close solution (CHECK THIS) but for each ϵ we can find the corresponding δ_ϵ^{ln} numerically.

```r
vals <- mdply(expand.grid(a = seq(1.01, 10, , 20), b = c(0.5, 1), eps = c(0.001,
    0.01, 0.1)), compare.delta)
qplot(data = vals, a, delta, geom = "line", color = dist, facets = b ~ epsilon)
```