2. Probability of Events and Axioms

2.1 Probability of Events

What is Probability?

Probability is a way of measuring how likely it is for something to happen. Think of it as guessing the chance of rain, the odds of winning a game, or the likelihood of pulling a red card from a deck.

Formula

The probability of an event E happening is given by:

$$P(E) = \frac{\text{Number of favorable outcomes}}{\text{Total number of outcomes}}$$

Real-World Example: Rolling a Die

Scenario: What is the probability of rolling a 5 on a six-sided die?

1. **Step 1: Count the total outcomes**
   A standard die has 6 faces: {1, 2, 3, 4, 5, 6}.
   So, the total number of outcomes = **6**.

2. **Step 2: Identify the favorable outcomes**
   Rolling a 5 → Only **one favorable outcome**.
3. **Step 3: Apply the formula**

$$P(\text{Rolling a 5}) = \frac{1}{6}$$

Answer: The probability of rolling a 5 is $\frac{1}{6}$, or approximately 16.67%.

2.2 Axioms of Probability

The **three rules of probability** are the foundations of how probabilities work.

1. **Non-Negativity**:
   The probability of any event E is always ≥ 0. For example:

$$P(\text{Rolling a die}) \geq 0$$

2. **Total Probability**:
   The sum of probabilities for all possible outcomes equals 1. For example:

$$P(1) + P(2) + P(3) + P(4) + P(5) + P(6) = 1$$

3. **Addition Rule**:
   If two events A and B cannot happen at the same time (disjoint events), then:

$$P(A \cup B) = P(A) + P(B)$$

Real-World Example: Drawing Cards

Scenario: You draw a card from a deck. What is the probability of it being either a **red card** or a **face card**?

1. **Step 1: Count favorable outcomes for each event**
   - Red cards: 26 cards.
   - Face cards: 12 cards.
2. **Step 2: Adjust for overlap**
   Red face cards = 6 cards (3 hearts + 3 diamonds).
3. **Step 3: Use the addition rule**

$$P(\text{Red or Face}) = P(\text{Red}) + P(\text{Face}) - P(\text{Red and Face})$$

$$P(\text{Red or Face}) = \frac{26}{52} + \frac{12}{52} - \frac{6}{52} = \frac{32}{52} = \frac{8}{13}$$

Answer: $P(\text{Red or Face}) = \frac{8}{13}$, or approximately 61.54%.

2.3 Complement Rule

What is the Complement Rule?

The complement rule helps calculate the probability of an event **not happening**:

$$P(\text{Not E}) = 1 - P(E)$$

Real-World Example: Not Rolling a 6

Scenario: What is the probability of not rolling a 6 on a die?

1. **Step 1: Calculate $P(6)$**

$$P(6) = \frac{1}{6}$$

2. **Step 2: Use the complement rule**

$$P(\text{Not } 6) = 1 - P(6) = 1 - \frac{1}{6} = \frac{5}{6}$$

Answer: $P(\text{Not } 6) = \frac{5}{6}$, or approximately 83.33%.

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3. Conditional Probability and Bayes Theorem

3.1 Conditional Probability

What is Conditional Probability?

Conditional probability is the probability of event A, given that event B has already occurred:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

Real-World Example: Drawing a Red Card Given it's a Face Card

Scenario: A deck has 52 cards, with 12 face cards (3 per suit). What is the probability of drawing a **red card**, given it's a **face card**?

1. **Step 1: Identify probabilities**
   - Red face cards = 6 (3 hearts + 3 diamonds).
   - Total face cards = 12.
2. **Step 2: Use the formula**

$$P(\text{Red} \mid \text{Face}) = \frac{P(\text{Red Face})}{P(\text{Face})} = \frac{6}{12} = \frac{1}{2}$$

Answer: $P(\text{Red} \mid \text{Face}) = \frac{1}{2}$, or 50%.

3.2 Bayes Theorem

What is Bayes Theorem?

Bayes theorem updates the probability of an event based on prior knowledge:

$$P(A|B) = \frac{P(B|A) \cdot P(A)}{P(B)}$$

Real-World Example: Medical Test

Scenario: A disease affects 1 in 1000 people. The test has:

- $P(\text{Positive} \mid \text{Disease}) = 0.99$,

- $P(\text{Positive} \mid \text{No Disease}) = 0.01$.

What is $P(\text{Disease} \mid \text{Positive})$?

Step-by-Step Solution:

1. **Step 1: Compute** $P(\text{Positive})$:

$$P(\text{Positive}) = P(\text{Positive} \mid \text{Disease}) \cdot P(\text{Disease}) + P(\text{Positive} \mid \text{No Disease}) \cdot P(\text{No Disease})$$

$$P(\text{Positive}) = (0.99 \cdot 0.001) + (0.01 \cdot 0.999) = 0.01098$$

2. **Step 2: Use Bayes Theorem:**

$$P(\text{Disease} \mid \text{Positive}) = \frac{P(\text{Positive} \mid \text{Disease}) \cdot P(\text{Disease})}{P(\text{Positive})}$$

$$P(\text{Disease} \mid \text{Positive}) = \frac{0.99 \cdot 0.001}{0.01098} \approx 0.0901$$

Answer: $P(\text{Disease} \mid \text{Positive}) \approx 9.01\%$.

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4. Random Variables (Single)

4.1 Discrete Random Variables

What is a Discrete Random Variable?

A discrete random variable takes specific values (e.g., 0, 1, 2, etc.). It's often used in situations like flipping coins or rolling dice.

Real-World Example: Flipping Coins

Scenario: Flip a coin 3 times. Let X represent the number of heads.

1. **Step 1: List outcomes**
   Outcomes = {HHH, HHT, HTH, HTT, THH, THT, TTH, TTT}.
2. **Step 2: Assign values to X:**
   - $X = 0$ (no heads): {TTT}.
   - $X = 1$ (1 head): {HTT, THT, TTH}.
   - $X = 2$ (2 heads): {HHT, HTH, THH}.
   - $X = 3$ (3 heads): {HHH}.
3. **Step 3: Compute probabilities:**
   - $P(X = 0) = \frac{1}{8}$,
   - $P(X = 1) = \frac{3}{8}$,
   - $P(X = 2) = \frac{3}{8}$,
   - $P(X = 3) = \frac{1}{8}$.