Expanding and Providing Additional Worked-Out Examples for Key Concepts

Below, I'll expand on the sections from earlier, adding more detailed examples and step-by-step walkthroughs to ensure a clear understanding of the topics.

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4. Systems of Linear Equations

Revisiting the Concept

Systems of linear equations involve multiple equations that are solved simultaneously. These systems can be written in standard form:

$$Ax = b$$

where:

- A is the coefficient matrix,
- x is the column vector of variables,
- b is the column vector of constants.

Additional Example

Solve the system:

$$x + 2y + z = 6$$

$$2x + 3y + 3z = 14$$

$$y + 2z = 8$$

Step 1: Represent as an Augmented Matrix

$$\begin{bmatrix} 1 & 2 & 1 & | & 6 \\ 2 & 3 & 3 & | & 14 \\ 0 & 1 & 2 & | & 8 \end{bmatrix}$$

Step 2: Apply Gaussian Elimination Eliminate the first column in row 2 (R_2) by performing:

$$R_2 \to R_2 - 2R_1$$

$$\begin{bmatrix} 1 & 2 & 1 & | & 6 \\ 0 & -1 & 1 & | & 2 \\ 0 & 1 & 2 & | & 8 \end{bmatrix}$$

Next, eliminate the second column in row 3:

$$R_3 \to R_3 + R_2$$

$$\begin{bmatrix} 1 & 2 & 1 & | & 6 \\ 0 & -1 & 1 & | & 2 \\ 0 & 0 & 3 & | & 10 \end{bmatrix}$$

Step 3: Back-Substitution Start from the last row:

$$3z = 10 \quad \Rightarrow \quad z = \frac{10}{3}$$

Substitute $z = \frac{10}{3}$ into row 2:

$$-y + \frac{10}{3} = 2 \quad \Rightarrow \quad y = \frac{4}{3}$$

Finally, substitute $y = \frac{4}{3}, z = \frac{10}{3}$ into row 1:

$$x + 2\left(\frac{4}{3}\right) + \frac{10}{3} = 6 \quad \Rightarrow \quad x = \frac{2}{3}$$

Final Solution:

$$x = \frac{2}{3}, y = \frac{4}{3}, z = \frac{10}{3}$$

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5. Gaussian Elimination

Revisiting the Concept

Gaussian elimination transforms a system into row-echelon form, making it easier to solve via back-substitution.

Additional Worked-Out Example

Solve the system:

$$x + y + z = 6, \quad 2x + 3y + 5z = 4, \quad 4x + 5y + 6z = 7$$

Step 1: Write the Augmented Matrix

$$\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 2 & 3 & 5 & | & 4 \\ 4 & 5 & 6 & | & 7 \end{bmatrix}$$

Step 2: Eliminate Below the Pivot First, eliminate the first column in rows 2 and 3:

$$R_2 \to R_2 - 2R_1, \quad R_3 \to R_3 - 4R_1$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 3 & | & -8 \\ 0 & 1 & 2 & | & -17 \end{bmatrix}$$

Next, eliminate the second column in row 3:

$$R_3 \rightarrow R_3 - R_2$$

$$\begin{bmatrix} 1 & 1 & 1 & | & 6 \\ 0 & 1 & 3 & | & -8 \\ 0 & 0 & -1 & | & -9 \end{bmatrix}$$

Step 3: Back-Substitution Start from the last row:

$$-z = -9 \quad \Rightarrow \quad z = 9$$

Substitute $z = 9$ into row 2:

$$y + 3(9) = -8 \quad \Rightarrow \quad y = -35$$

Finally, substitute $y = -35, z = 9$ into row 1:

$$x - 35 + 9 = 6 \quad \Rightarrow \quad x = 32$$

Final Solution:

$$x = 32, y = -35, z = 9$$

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6. Inverse of a Matrix

Revisiting the Concept

The inverse of a matrix A exists only if $\det(A) \neq 0$. It satisfies $A \cdot A^{-1} = I$, where I is the identity matrix.

Additional Worked-Out Example

Find the inverse of:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

Step 1: Compute the Determinant

$$\det(A) = (2)(4) - (3)(1) = 8 - 3 = 5$$

Step 2: Write the Adjoint

$$\text{Adj}(A) = \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

Step 3: Compute the Inverse

$$A^{-1} = \frac{1}{\det(A)} \cdot \text{Adj}(A)$$

$$A^{-1} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 0.8 & -0.6 \\ -0.2 & 0.4 \end{bmatrix}$$

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8. Singular Value Decomposition (SVD)

Revisiting the Concept

SVD decomposes a matrix A into three components:

$$A = U\Sigma V^T$$

where:

- U: Left singular vectors,

- Σ: Diagonal matrix of singular values,
- V^T: Right singular vectors.

Additional Example

Let:

$$A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

Step 1: Compute $A^T A$

$$A^T A = \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 10 & 6 \\ 6 & 10 \end{bmatrix}$$

Step 2: Solve for Eigenvalues Solve $\det(A^T A - \lambda I) = 0$:

$$\det \begin{bmatrix} 10 - \lambda & 6 \\ 6 & 10 - \lambda \end{bmatrix} = 0$$

$$(10 - \lambda)^2 - 36 = 0 \quad \Rightarrow \quad \lambda = 16, 4$$

Step 3: Compute Singular Values Singular values are $\sigma_i = \sqrt{\lambda_i}$:

$$\sigma_1 = \sqrt{16} = 4, \quad \sigma_2 = \sqrt{4} = 2$$

Step 4: Compute U and V The eigenvectors of $A^T A$ give V, and U is computed as:

$$u_i = \frac{1}{\sigma_i} A v_i$$

Application: SVD is used in Principal Component Analysis (PCA) for dimensionality reduction.