Comprehensive Study Notes and Solutions for **Assignment 2**

Q1: Gradient and Quadratic Approximation

Problem: Compute the gradient ∇f and the quadratic approximation for $f(x, y, z) = \ln(xyz)$ around $(x_0, y_0, z_0) = (1, 1, 1)$.

Solution:

1. **Gradient Computation:** The gradient of $f(x, y, z) = \ln(xyz)$ is:

$$\nabla f = [\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}].$$

   Using partial derivatives:

$$\frac{\partial f}{\partial x} = \frac{1}{x}, \quad \frac{\partial f}{\partial y} = \frac{1}{y}, \quad \frac{\partial f}{\partial z} = \frac{1}{z}.$$

   At $(x_0, y_0, z_0) = (1, 1, 1)$:

$$\nabla f = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

2. **Quadratic Approximation:** Using the second-order Taylor expansion:

$$f(x, y, z) \approx f(x_0, y_0, z_0) + \nabla f \cdot \Delta x + \frac{1}{2} \Delta x^\top H \Delta x,$$

   where H is the Hessian matrix.

   - **Hessian Matrix:** The Hessian is a diagonal matrix:

$$H = \begin{bmatrix} -\frac{1}{x^2} & 0 & 0 \\ 0 & -\frac{1}{y^2} & 0 \\ 0 & 0 & -\frac{1}{z^2} \end{bmatrix}.$$

At $(1, 1, 1)$, the Hessian is:

$$H = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

- **Resulting Quadratic Approximation:**

$$f(x, y, z) \approx 0 + \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}^\top \begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix}^\top \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} x-1 \\ y-1 \\ z-1 \end{bmatrix}.$$

Simplify to:

$$f(x, y, z) \approx (x-1) + (y-1) + (z-1) - \frac{1}{2} \left[(x-1)^2 + (y-1)^2 + (z-1)^2\right].$$

Q2: Hessian and Gradient Analysis

Problem: Analyze $f(x, y) = x^3 - 3xy^2 + y^3$. Compute:

1. The gradient ∇f.
2. The Hessian matrix H.
3. The quadratic approximation around $(0, 0)$.

Solution:

1. **Gradient Computation:**

$$\nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix}.$$

$$\frac{\partial f}{\partial x} = 3x^2 - 3y^2, \quad \frac{\partial f}{\partial y} = -6xy + 3y^2.$$

At $(x, y) = (0, 0)$:

$$\nabla f = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

2. **Hessian Matrix:** The Hessian matrix is:

$$H = \begin{bmatrix} \frac{\partial^2 f}{\partial x^2} & \frac{\partial^2 f}{\partial x \partial y} \\ \frac{\partial^2 f}{\partial y \partial x} & \frac{\partial^2 f}{\partial y^2} \end{bmatrix}.$$

Compute second derivatives:

$$\frac{\partial^2 f}{\partial x^2} = 6x, \quad \frac{\partial^2 f}{\partial x \partial y} = -6y,$$

$$\frac{\partial^2 f}{\partial y^2} = 6x.$$

At $(x, y) = (0, 0)$:

$$H = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

3. **Quadratic Approximation:** Around $(0, 0)$, the Taylor expansion simplifies to:

$$f(x, y) \approx f(0, 0) + \nabla f \cdot \Delta x + \frac{1}{2} \Delta x^\top H \Delta x.$$

Since $f(0,0) = 0$, $\nabla f = 0$, and $H = 0$, we conclude:

$$f(x, y) \approx 0.$$

Q3: Taylor Series for Nonlinear Functions

Problem: Approximate $f(x, y) = e^{x+y}$ around $(x, y) = (0, 0)$ using a Taylor series.

Solution:

1. **Taylor Expansion:** The Taylor series of $f(x, y)$ around (x_0, y_0) is:

$$f(x, y) \approx f(x_0, y_0) + \nabla f \cdot \Delta x + \frac{1}{2} \Delta x^\top H \Delta x + \dots$$

2. **Compute Terms:**
   - $f(0,0) = e^0 = 1.$
   - \nabla f = \begin{bmatrix} \frac{\partial f}{\partial x} \\ \frac{\partial f}{\partial y} \end{bmatrix} = \begin{bmatrix} e^{x+y} \\ e^{x+y} \end{bmatrix}. \] At \((x, y) = (0, 0):

$$\nabla f = [\begin{matrix} 1 \\ 1 \end{matrix}].$$

   - Hessian matrix:

$$H = [\begin{matrix} e^{x+y} & e^{x+y} \\ e^{x+y} & e^{x+y} \end{matrix}].$$

   At $(x, y) = (0, 0)$:

$$H = [\begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix}].$$

3. **Approximation:** Substituting these values:

$$f(x,y) \approx 1 + (1)(x) + (1)(y) + \frac{1}{2}\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Simplify to:

$$f(x,y) \approx 1 + x + y + x^2 + xy + y^2.$$