# Slide 9-10: Special Matrices

### Symmetric Matrix

- **Definition**: A matrix is symmetric if the element at row $i$, column $j$ ($a_{ij}$) is the same as the element at row $j$, column $i$ ($a_{ji}$) for all $i$ and $j$.
- In simpler terms, if you "flip" the matrix across its diagonal, it remains unchanged.

**Example**:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

**Step 1**: Verify symmetry using notation and numbers:

- Look at $a_{12}$ (row 1, column 2) and $a_{21}$ (row 2, column 1):
  - $a_{12} = 2$, $a_{21} = 2$ (they are equal).
- Look at $a_{13}$ (row 1, column 3) and $a_{31}$ (row 3, column 1):
  - $a_{13} = 3$, $a_{31} = 3$ (they are equal).
- Look at $a_{23}$ (row 2, column 3) and $a_{32}$ (row 3, column 2):
  - $a_{23} = 5$, $a_{32} = 5$ (they are equal).

Since all $a_{ij} = a_{ji}$, the matrix $A$ is symmetric.

**Step 2**: Importance of symmetric matrices:

- **Simplify computations**: Symmetric matrices reduce the number of independent elements, making them easier to analyze.
- **Applications**:
  - In **statistics**, covariance matrices are symmetric because the relationship between two variables is mutual.
  - In **physics**, symmetric matrices appear in calculations like moment of inertia.

## Diagonal Matrix

- **Definition**: A diagonal matrix is a square matrix where all non-diagonal elements are zero. The main diagonal contains all the non-zero values.
- In simpler terms, only the values along the diagonal (top-left to bottom-right) matter.

**Example**:

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}$$

**Step 1**: Verify the structure:

- Main diagonal: $1, 2, 3$.
- All other elements: $0$.

**Step 2**: Why are diagonal matrices useful?

- They are simple to compute with:
  - **Addition/Subtraction**: Add or subtract the diagonal elements directly.
  - **Multiplication**: Each diagonal element multiplies independently.
- **Example**: Multiply $A$ by a vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$:

$$A \cdot \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 1 \cdot x \\ 2 \cdot y \\ 3 \cdot z \end{bmatrix}$$

Result: $\begin{bmatrix} x \\ 2y \\ 3z \end{bmatrix}$.

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## Skew-Symmetric Matrix

- **Definition**: A matrix is skew-symmetric if the transpose of the matrix is equal to the negative of the matrix ($A = -A^T$).
- All diagonal elements in a skew-symmetric matrix are $0$.

**Example**:

$$A = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$$

**Step 1**: Verify using notation and numbers:

- Check $a_{12}$ and $a_{21}$: $a_{12} = -2$, $a_{21} = 2$ (they are negatives of each other).
- Check $a_{13}$ and $a_{31}$: $a_{13} = -3$, $a_{31} = 3$ (they are negatives of each other).
- Check $a_{23}$ and $a_{32}$: $a_{23} = -4$, $a_{32} = 4$ (they are negatives of each other).
- Diagonal elements $a_{11}, a_{22}, a_{33} = 0$.

**Step 2**: Importance:

- Used in **physics** for angular momentum calculations and in **computer graphics** for rotations.

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## Slide 11-12: Elementary Row Operations

**What are they?** Row operations are used to manipulate matrices into simpler forms, such as **Row Echelon Form (REF)** or **Reduced Row Echelon Form (RREF)**, making it easier to solve systems of equations.

### Three Types of Row Operations

1. **Row Swap**:
   - Swap two rows in the matrix.
   - **Example**:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

2. **Row Scaling**:
   - Multiply all elements in a row by a non-zero constant.
   - **Example**:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad R_1 \rightarrow 2 \cdot R_1 = \begin{bmatrix} 2 & 4 \\ 3 & 4 \end{bmatrix}$$

3. **Row Addition/Subtraction**:
   - Replace one row with the sum or difference of itself and a multiple of another row.
   - **Example**:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad R_2 \rightarrow R_2 - 3 \cdot R_1 = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

### Why are these important?

- These operations are the foundation of solving linear systems, finding determinants, and computing matrix inverses.

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## Slide 13-15: Row Echelon Form (REF)

**Definition**: A matrix is in REF if:

1. All zero rows are at the bottom.
2. The first non-zero entry in each row (called the **pivot**) is to the right of the pivot in the row above.
3. All entries below a pivot are zero.

**Example**:

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

**Step 1**: Make the first pivot 1 by dividing $R_1$ by 2:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

**Step 2**: Eliminate the first column from $R_2$:

$$R_2 \to R_2 - R_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

**Step 3**: Eliminate the second column from $R_3$:

$$R_3 \to R_3 - R_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

This is now in Row Echelon Form (REF).

## Slide 16-17: Rank of a Matrix

**Definition**: The rank of a matrix is the number of non-zero rows in its Row Echelon Form. It tells us how much "independent information" the matrix contains.

**Example**:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

**Step 1**: Convert $A$ to REF:

$$\begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

**Step 2**: Count the non-zero rows: **Rank = 2**.