---

1. Vectors: Operations and Representation

Definition:

A vector is an ordered list of numbers, representing quantities such as position, velocity, or features in data.

Vector Addition Example:

Add two vectors $\mathbf{v} = [2, 3]$ and $\mathbf{w} = [4, -1]$.

Step 1: Element-Wise Addition

Add corresponding elements:

$$\mathbf{v} + \mathbf{w} = [2 + 4, 3 + (-1)] = [6, 2]$$

Final Solution:

The result is $[6, 2]$.

Dot Product Example:

Find the dot product of $\mathbf{v} = [1, 2, 3]$ and $\mathbf{w} = [4, -5, 6]$.

Step 1: Multiply Corresponding Elements

$$1 \cdot 4 + 2 \cdot (-5) + 3 \cdot 6 = 4 - 10 + 18$$

Step 2: Simplify

$$4 - 10 + 18 = 12$$

Final Solution:

The dot product is 12.

---

2. Systems of Linear Equations

Definition:

A system of linear equations consists of multiple equations with shared variables, typically written in the form $Ax = b$.

Example:

Solve the system of equations:

$$x + y = 3, \quad 2x + y = 5$$

Step 1: Solve for y in terms of x

From the first equation:

$$y = 3 - x$$

Step 2: Substitute $y = 3 - x$ into the second equation

$$2x + (3 - x) = 5$$

Step 3: Simplify

$$2x - x + 3 = 5 \quad \Rightarrow \quad x + 3 = 5 \quad \Rightarrow \quad x = 2$$

Step 4: Solve for y

Substitute $x = 2$ into $y = 3 - x$:

$$y = 3 - 2 = 1$$

Verification:

Substitute $x = 2$ and $y = 1$ into both equations:

1. $x + y = 3 \quad \Rightarrow \quad 2 + 1 = 3$ (True)
2. $2x + y = 5 \quad \Rightarrow \quad 2(2) + 1 = 5$ (True)

Final Solution:

The solution is $x = 2, y = 1$.

---

3. Matrices: Representation and Operations

Definition:

A matrix is a rectangular array of numbers, used to represent transformations or systems of equations.

Matrix Multiplication Example:

Multiply $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$ with $B = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$.

Step 1: Multiply Rows by Columns

For the first row:

$$1 \cdot 5 + 2 \cdot 6 = 5 + 12 = 17$$

For the second row:

$$3 \cdot 5 + 4 \cdot 6 = 15 + 24 = 39$$

Final Solution:

The product is:

$$\begin{bmatrix} 17 \\ 39 \end{bmatrix}$$

---

4. Gaussian Elimination

Definition:

Gaussian elimination transforms a system of linear equations into row-echelon form for easy solution.

Example:

Solve:

$$x + y + z = 6, \quad 2x + y - z = 1, \quad 3x - y - z = 1$$

Step 1: Write the Augmented Matrix

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 2 & 1 & -1 & 1 \\ 3 & -1 & -1 & 1 \end{bmatrix}$$

Step 2: Eliminate Below the Pivot

- Subtract $2 \times \text{Row } 1$ from Row 2.
- Subtract $3 \times \text{Row } 1$ from Row 3.

Result:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & -1 & -3 & -11 \\ 0 & -4 & -4 & -17 \end{bmatrix}$$

Step 3: Eliminate Below the New Pivot

Divide Row 2 by -1 and eliminate below Row 2.

Result:

$$\begin{bmatrix} 1 & 1 & 1 & 6 \\ 0 & 1 & 3 & 11 \\ 0 & 0 & -7 & -27 \end{bmatrix}$$

Step 4: Back-Substitution

1. Solve $-7z = -27$ for $z = 27/7$.

2. Substitute z into Row 2 to find y.
3. Substitute y and z into Row 1 to find x.

---

5. Eigenvalues and Eigenvectors

Definition:

Eigenvalues (λ) and eigenvectors ($\mathbf{v}$) satisfy:

$$A\mathbf{v} = \lambda\mathbf{v}$$

Example:

Find eigenvalues and eigenvectors of:

$$A = \begin{bmatrix} 4 & 1 \\ 2 & 3 \end{bmatrix}$$

Step 1: Solve $\det(A - \lambda I) = 0$

$$\det \begin{bmatrix} 4-\lambda & 1 \\ 2 & 3-\lambda \end{bmatrix} = 0$$

$$(4-\lambda)(3-\lambda) - 2 = 0 \quad \Rightarrow \quad \lambda^2 - 7\lambda + 10 = 0$$

Step 2: Factorize

$$\lambda = 5, \lambda = 2$$

Step 3: Solve for Eigenvectors

For $\lambda = 5$:

$$\begin{bmatrix} -1 & 1 \\ 2 & -2 \end{bmatrix} \mathbf{v} = 0$$

Eigenvector: $\mathbf{v}_1 = [1, 1]$.

For $\lambda = 2$:

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \end{bmatrix} \mathbf{v} = 0$$

Eigenvector: $\mathbf{v}_2 = [1, -2]$.

---

6. Singular Value Decomposition (SVD)

Definition:

SVD decomposes a matrix A into:

$$A = U \Sigma V^T$$

Example:

For $A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$, SVD produces:

- U: Left singular vectors.

- Σ: Singular values.
- V^T: Right singular vectors.

Applications:

- Dimensionality reduction.
- Image compression.

---

7. Optimization Techniques

Gradient Descent:

Minimizes a function $f(x)$ by iteratively updating x in the direction of the steepest descent.

Example:

Minimize $f(x) = x^2 - 4x + 4$.

1. Compute gradient: $f'(x) = 2x - 4$.
2. Update rule: $x = x - \eta f'(x)$ (learning rate $\eta = 0.1$).
3. Iterate until $f'(x) = 0$.

This detailed explanation provides a thorough understanding of each concept