In-Depth Study Guide for Inner Product Spaces and Orthogonal Projections

This comprehensive study guide consolidates the concepts, examples, and scenarios discussed during an advanced lesson on **inner product spaces**, **orthogonal projections**, and their practical applications. Designed as a one-stop resource, it covers topics like distance and angles between vectors and projections onto lines and subspaces. The guide uses step-by-step explanations and real-world examples to ensure thorough understanding.

Table of Contents

1. Introduction to Inner Product Spaces

An **inner product space** is a vector space equipped with an operation called the **inner product**. This operation allows us to:

- Measure the **angle** between two vectors.
- Measure the **length** (or magnitude) of a vector.

Inner Product Definition

For two vectors $\mathbf{x}$ and $\mathbf{y}$ in an inner product space, the inner product is defined as:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y}$$

Where:

- $\mathbf{x}^T$ is the transpose of $\mathbf{x}$,
- A is a symmetric, positive-definite matrix that defines the space.

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2. Key Properties of Inner Products

1. **Symmetry**:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \mathbf{y}, \mathbf{x} \rangle$$

2. **Linearity**:

$$\langle c\mathbf{x}, \mathbf{y} \rangle = c\langle \mathbf{x}, \mathbf{y} \rangle$$

3. **Positivity**:

$$\langle \mathbf{x}, \mathbf{x} \rangle \geq 0, \quad \text{and} \quad \langle \mathbf{x}, \mathbf{x} \rangle = 0 \iff \mathbf{x} = 0$$

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3. Norms and Distance in Inner Product Spaces

Norms

The **norm** of a vector $\mathbf{x}$ is a measure of its length or magnitude:

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$$

Distance

The distance between two vectors $\mathbf{a}$ and $\mathbf{b}$ is defined as:

$$d(\mathbf{a}, \mathbf{b}) = \|\mathbf{a} - \mathbf{b}\| = \sqrt{\langle \mathbf{a} - \mathbf{b}, \mathbf{a} - \mathbf{b} \rangle}$$

Example: Calculating Distance

Given:

$$\mathbf{a} = [\begin{matrix} 1 \\ 5 \end{matrix}], \quad \mathbf{b} = [\begin{matrix} 2 \\ 7 \end{matrix}], \quad A = [\begin{matrix} 5.5 & -1.5 \\ -1.5 & 5.5 \end{matrix}]$$

1. Compute $\mathbf{a} - \mathbf{b}$:

$$\mathbf{a} - \mathbf{b} = [\begin{matrix} -1 \\ -2 \end{matrix}]$$

2. Compute the inner product:

$$\langle \mathbf{a} - \mathbf{b}, \mathbf{a} - \mathbf{b} \rangle = (-1, -2)^T A (-1, -2)$$

Substituting A:

$$\langle \mathbf{a} - \mathbf{b}, \mathbf{a} - \mathbf{b} \rangle = 21.5$$

3. Compute the distance:

$$d(\mathbf{a}, \mathbf{b}) = \sqrt{21.5} \approx 4.63$$

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4. Angles Between Vectors in Inner Product Spaces

The **angle** between two vectors $\mathbf{a}$ and $\mathbf{b}$ is computed using:

$$\cos(\theta) = \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\|\mathbf{a}\| \|\mathbf{b}\|}$$

Example: Calculating Angles

Given the same vectors and matrix A:

1. Compute $\langle \mathbf{a}, \mathbf{b} \rangle$:

$$\langle \mathbf{a}, \mathbf{b} \rangle = \mathbf{a}^T A \mathbf{b} = 178$$

2. Compute norms:

$$\|\mathbf{a}\| = \sqrt{\langle \mathbf{a}, \mathbf{a} \rangle} = \sqrt{128}, \quad \|\mathbf{b}\| = \sqrt{249.5}$$

3. Compute $\cos(\theta)$:

$$\cos(\theta) = \frac{178}{\sqrt{128} \cdot \sqrt{249.5}} \approx 0.996$$

4. Compute θ:

$$\theta = \arccos(0.996) \approx 5.1^\circ$$

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5. Orthogonal Projections

Projections onto Lines

To project a vector $\mathbf{b}$ onto a line spanned by $\mathbf{v}$:

$$\text{Proj}_{\mathbf{v}}\mathbf{b} = \frac{\langle \mathbf{b}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle}\mathbf{v}$$

Projections onto Subspaces

For a subspace V spanned by columns of a matrix A:

$$\text{Proj}_V \mathbf{b} = A(A^T A)^{-1} A^T \mathbf{b}$$

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6. Practical Applications

1. **Data Science**:
    - Dimensionality reduction (e.g., PCA).
2. **Computer Graphics**:
    - Rendering 3D scenes onto 2D screens.
3. **Signal Processing**:

- Filtering noise from signals.

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7. Summary and Key Formulas

1. **Inner Product**:

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x}^T A \mathbf{y}$$

2. **Norm**:

$$\|\mathbf{x}\| = \sqrt{\langle \mathbf{x}, \mathbf{x} \rangle}$$

3. **Distance**:

$$d(\mathbf{a}, \mathbf{b}) = \sqrt{\langle \mathbf{a} - \mathbf{b}, \mathbf{a} - \mathbf{b} \rangle}$$

4. **Angle**:

$$\cos(\theta) = \frac{\langle \mathbf{a}, \mathbf{b} \rangle}{\|\mathbf{a}\| \|\mathbf{b}\|}$$

5. **Projection**:

$$\text{Proj}_{\mathbf{v}} \mathbf{b} = \frac{\langle \mathbf{b}, \mathbf{v} \rangle}{\langle \mathbf{v}, \mathbf{v} \rangle} \mathbf{v}$$

This guide ensures you master inner product spaces and orthogonal projections with practical, step-by-step examples.