Detailed Study Notes and Solutions for **Assignment 1**

Q1: Solving a Large Lower-Triangular System

Problem: We need to solve $Lx = b$, where L is a 100000×100000 lower-triangular matrix, and b is a 100000×1 vector. The constraints are:

1. Only individual rows of L can be stored in memory.
2. Writes to memory are more expensive than reads.

Solution Approach: The problem suggests solving $Lx = b$ using **forward substitution** while minimizing memory writes.

Algorithm:

1. Initialize $x = [0] \times 100000$ (initially, all elements are 0).
2. For each row $i = 1$ to 100000:
   - Read row i of L and store temporarily.
   - Compute x_i using:

$$x_i = \frac{b_i - \sum_{j=1}^{i-1} L_{ij} x_j}{L_{ii}}.$$

   - Write x_i to the output vector.
3. Repeat until x is fully computed.

Key Observations:

- The lower-triangular structure ensures $L_{ij} = 0$ for $j > i$, reducing computation.
- Minimizing memory writes is achieved by updating x_i directly after computing it.

Time Complexity:

$$O(n^2)$$

(where $n = 100000$).

Q2: Commutativity of Elimination Matrices

Problem: Given a 50×50 elimination matrix E_{ij} (where the i-th row and j-th column below the diagonal is non-zero), find the set of elimination matrices E_{pq} where $E_{pq}E_{ij} = E_{ij}E_{pq}$. Explain commutativity.

Solution:

1. **Elimination Matrix Structure:** Elimination matrices E_{ij} are designed to zero out specific elements in a matrix A.
2. **Commutativity Analysis:**
   - If $i \neq p$ and $j \neq q$, E_{ij} and E_{pq} operate on different rows and do not interfere, thus $E_{pq}E_{ij} = E_{ij}E_{pq}$.
   - If $i = p$ or $j = q$, the operations affect the same rows, and commutativity may not hold.
3. **Example:** For $i = 5, j = 2$:

$$E_{52} = \text{Matrix with entry at (5,2)} \neq 0.$$

   Calculate commutative pairs E_{pq}.

Conclusion: Commutativity works when elimination matrices act independently on separate rows or columns.

Q3: Finding a 2×2 Matrix with Complex Eigenvalues

Problem: Find a 2×2 matrix A such that:

1. Eigenvalues are complex with modulus 1.
2. Eigenvalues become real for A^{50}.

Solution:

1. **Rotation Matrix:** A 2×2 rotation matrix:

$$A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

2. **Eigenvalues:** For A, eigenvalues are:

$$\lambda = e^{i\theta}, \quad \lambda = e^{-i\theta}.$$

   These eigenvalues have modulus 1.

3. **Condition for Real Eigenvalues:** After $n = 50$, A^{50} aligns with the axes ($\theta = 2\pi/50$).

Result: The rotation matrix meets the requirements.

Q4: Eigenvalues of an Upper-Triangular Matrix

Problem: Is it feasible to compute eigenvalues of an $n \times n$ upper-triangular matrix in $O(n)$?

Solution:

1. **Property of Upper-Triangular Matrices:**
   - Eigenvalues are the diagonal elements.
2. **Algorithm:**
   - Traverse the diagonal of the matrix to collect eigenvalues.
   - Complexity:

$$O(n).$$

Conclusion: The claim is justified. Extracting diagonal elements requires $O(n)$ time.

Q5: Symmetric Matrix Argument

Problem: Analyze the argument that C is symmetric if $C^\top C = CC^\top$.

Solution:

1. **Given Condition:** $C^\top C = CC^\top$.
2. **SVD Analysis:**
   - $C = U\Sigma V^\top$, where U and V are orthogonal matrices, and Σ is diagonal.
   - $C^\top C = V\Sigma^\top \Sigma V^\top$, $CC^\top = U\Sigma\Sigma^\top U^\top$.
   - If $C^\top C = CC^\top$, then $U = V$, and C is symmetric.

Conclusion: The argument is valid.

Key Learnings and Concepts

- **Forward Substitution:** Solves large lower-triangular systems efficiently.
- **Matrix Properties:** Understanding triangular, symmetric, and rotation matrices is critical for eigenvalue problems.
- **Algorithmic Complexity:** Leveraging matrix structure reduces computational overhead.