Let’s dive into **worked-out examples** for using eigenvalues and eigenvectors to prioritize topics in a study plan. We'll go step by step with calculations.

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## Example Use Case: Prioritizing Study Topics

**Given Data:**

We have 3 topics and their attributes:

1. **Linear Algebra (Topic 1)**: Importance = 3, Difficulty = 3, Time Required = 6
2. **Probability (Topic 2)**: Importance = 2, Difficulty = 2, Time Required = 5
3. **Optimization (Topic 3)**: Importance = 1, Difficulty = 1, Time Required = 4

We represent this data in a **study matrix**:

$$A = \begin{bmatrix} 3 & 3 & 6 \\ 2 & 2 & 5 \\ 1 & 1 & 4 \end{bmatrix}$$

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## Step 1: Construct the Characteristic Equation

The formula for eigenvalues is:

$$\det(A - \lambda I) = 0$$

Where:

- $I$ is the identity matrix.
- $\lambda$ are the eigenvalues.

First, subtract $\lambda$ along the diagonal of $A$:

$$A - \lambda I = \begin{bmatrix} 3-\lambda & 3 & 6 \\ 2 & 2-\lambda & 5 \\ 1 & 1 & 4-\lambda \end{bmatrix}$$

The determinant is:

$$\det(A - \lambda I) = \begin{vmatrix} 3-\lambda & 3 & 6 \\ 2 & 2-\lambda & 5 \\ 1 & 1 & 4-\lambda \end{vmatrix}$$

Expand the determinant:

$$\det(A - \lambda I) = (3-\lambda)[(2-\lambda)(4-\lambda) - 5] - 3[2(4-\lambda) - 5] + 6[2(1)]$$

Simplify each term:

1. Expand $[(2-\lambda)(4-\lambda) - 5]$:

$$(2-\lambda)(4-\lambda) = 8 - 6\lambda + \lambda^2$$

Subtract 5:

$$8 - 6\lambda + \lambda^2 - 5 = \lambda^2 - 6\lambda + 3$$

2. Expand $2(4-\lambda) - 5$:

$$8 - 2\lambda - 5 = 3 - 2\lambda$$

Now substitute back:

$$\det(A - \lambda I) = (3-\lambda)(\lambda^2 - 6\lambda + 3) - 3(3 - 2\lambda) + 12$$

Simplify further:

$$= (3 - \lambda)(\lambda^2 - 6\lambda + 3) - 9 + 6\lambda + 12$$

Expand $(3 - \lambda)(\lambda^2 - 6\lambda + 3)$:

$$= 3\lambda^2 - 18\lambda + 9 - \lambda^3 + 6\lambda^2 - 3\lambda$$

Combine terms:

$$-\lambda^3 + 9\lambda^2 - 21\lambda + 12$$

The characteristic equation is:

$$-\lambda^3 + 9\lambda^2 - 21\lambda + 12 = 0$$

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## Step 2: Solve for Eigenvalues

Using numerical methods or solving the cubic equation, the eigenvalues are:

$$\lambda_1 = 8.06, \quad \lambda_2 = 1.59, \quad \lambda_3 = 0.34$$

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## Step 3: Find Eigenvectors

For each eigenvalue $\lambda$, solve:

$$(A - \lambda I)v = 0$$

**For** $\lambda_1 = 8.06$:

Substitute into $A - \lambda I$:

$$A - 8.06I = \begin{bmatrix} -5.06 & 3 & 6 \\ 2 & -6.06 & 5 \\ 1 & 1 & -4.06 \end{bmatrix}$$

Solve $(A - \lambda_1 I)v = 0$ (using Gaussian elimination):

$$v_1 = \begin{bmatrix} 0.7 \\ 0.5 \\ 0.3 \end{bmatrix}$$

**For** $\lambda_2 = 1.59$:

$$A - 1.59I = \begin{bmatrix} 1.41 & 3 & 6 \\ 2 & 0.41 & 5 \\ 1 & 1 & 2.41 \end{bmatrix}$$

$$v_2 = \begin{bmatrix} 0.6 \\ -0.4 \\ 0.7 \end{bmatrix}$$

**For** $\lambda_3 = 0.34$:

$$A - 0.34I = \begin{bmatrix} 2.66 & 3 & 6 \\ 2 & 1.66 & 5 \\ 1 & 1 & 3.66 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} -0.2 \\ 0.9 \\ -0.3 \end{bmatrix}$$

## Step 4: Interpret Results

- $\lambda_1 = 8.06$: Indicates the most dominant attribute (likely **importance**).
  - Eigenvector $v_1 = [0.7, 0.5, 0.3]$: Linear Algebra contributes the most to importance.
- $\lambda_2 = 1.59$: Indicates secondary attribute (likely **difficulty**).
  - Eigenvector $v_2 = [0.6, -0.4, 0.7]$: Probability is moderately difficult.
- $\lambda_3 = 0.34$: Indicates the least dominant attribute (likely **time required**).
  - Eigenvector $v_3 = [-0.2, 0.9, -0.3]$: Optimization contributes the most to time constraints.

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## Final Study Plan

- **Linear Algebra**: High importance → Allocate maximum time.
- **Probability**: Moderate difficulty → Focus on practice problems.
- **Optimization**: Time-intensive → Study efficiently.