Slide 11-12: Elementary Row Operations

What are Row Operations?
Row operations are tools used to simplify matrices and solve systems of linear equations. By manipulating the rows of a matrix, we can transform it into simpler forms like Row Echelon Form (REF) or Reduced Row Echelon Form (RREF), which make solving equations much easier.

Types of Row Operations

1. **Row Swap (Swapping Rows)**
   This operation exchanges two rows in the matrix.

   Example:
   Start with matrix A:

   $$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

   Swap R_1 and R_2:

   $$A = \begin{bmatrix} 3 & 4 \\ 1 & 2 \end{bmatrix}$$

   Why is this useful?
   Sometimes, we need to position a row with a leading 1 (pivot) at the top to simplify the matrix.

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2. **Row Scaling (Multiplying a Row by a Non-Zero Constant)**
   Multiply all elements of a row by a constant.

Example:
Start with:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

Multiply R_1 by 2:

$$R_1 \rightarrow 2 \cdot R_1 = \begin{bmatrix} 2 & 4 \\ 3 & 4 \end{bmatrix}$$

Why is this useful?
Scaling a row can help us create a pivot (1) or align elements for elimination.

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3. **Row Addition/Subtraction (Adding/Subtracting a Multiple of One Row to Another)**
   Replace one row with the sum or difference of itself and a multiple of another row.

   Example:
   Start with:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

   Subtract $3 \cdot R_1$ from R_2:

$$R_2 \rightarrow R_2 - 3 \cdot R_1 = \begin{bmatrix} 1 & 2 \\ 0 & -2 \end{bmatrix}$$

   Why is this useful?
   Row addition/subtraction is used to eliminate variables or simplify matrices into triangular forms.

Why Are These Operations Important?

1. **Solving Linear Equations**: Simplify matrices to forms that make solving equations straightforward.
2. **Finding Determinants**: Transform matrices into triangular forms to compute determinants efficiently.
3. **Computing Inverses**: Use row operations to reduce a matrix to its identity matrix while transforming another matrix into its inverse.

Slide 13-15: Row Echelon Form (REF)

Definition: A matrix is in Row Echelon Form if:

1. All rows with only zeros are at the bottom.
2. The first non-zero number in each row (pivot) is to the right of the pivot in the row above.
3. All entries below each pivot are zeros.

Example: Convert a Matrix to REF

Start with:

$$A = \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

Step 1: Make the first pivot 1
Divide R_1 by 2:

$$\begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

Step 2: Eliminate the first column from R_2
Subtract R_1 from R_2:

$$R_2 \to R_2 - R_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 1 & 2 \end{bmatrix}$$

Step 3: Eliminate the second column from R_3
Subtract R_2 from R_3:

$$R_3 \to R_3 - R_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Final Result:
This is the Row Echelon Form (REF):

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$$

Importance of REF:

1. It simplifies solving systems of linear equations using back-substitution.
2. It's the first step in computing matrix rank or inverses.

Slide 16-17: Rank of a Matrix

Definition:
The rank of a matrix is the number of non-zero rows in its Row Echelon Form (REF).

- Rank tells us the number of **linearly independent rows or columns** in the matrix.
- It shows how much "independent information" is contained in the matrix.

Example: Find the Rank of a Matrix

Start with:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

Step 1: Convert to REF
Perform row operations:

1. Subtract $4 \cdot R_1$ from R_2:

$$R_2 \rightarrow R_2 - 4 \cdot R_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 7 & 8 & 9 \end{bmatrix}$$

2. Subtract $7 \cdot R_1$ from R_3:

$$R_3 \rightarrow R_3 - 7 \cdot R_1 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & -6 & -12 \end{bmatrix}$$

3. Eliminate the second column from R_3:

$$R_3 \rightarrow R_3 - 2 \cdot R_2 = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

Step 2: Count Non-Zero Rows
The REF is:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

Non-zero rows: 2.
Rank = 2.

Why Is Rank Important?

1. Rank tells us if the system of linear equations is consistent (has solutions).
2. It determines if a matrix is invertible:
   - If $\text{Rank} = \text{Number of Rows}$, the matrix is invertible.