1. Basics of Matrices and Vectors

What are Matrices and Vectors?

- **Matrix:** A rectangular grid of numbers arranged in rows and columns. Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

  - Rows represent horizontal slices, columns represent vertical slices.
- **Vector:** A matrix with a single column or row. Example (column vector):

$$v = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Basic Operations

1. **Addition/Subtraction**: Add or subtract corresponding elements. Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} + \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix}$$

2. **Scalar Multiplication**: Multiply each element by a constant. Example:

$$2 \cdot \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 4 \\ 6 & 8 \end{bmatrix}$$

3. **Matrix Multiplication**: Multiply rows of one matrix by columns of another. Example:

$$\begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix} = \begin{bmatrix} 19 & 22 \\ 43 & 50 \end{bmatrix}$$

2. Special Matrices

Diagonal Matrix

- Only non-zero elements are on the diagonal. Example:

$$D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Symmetric Matrix

- A matrix equal to its transpose ($A = A^T$). Example:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 5 \\ 3 & 5 & 6 \end{bmatrix}$$

Skew-Symmetric Matrix

- A matrix where $A = -A^T$. Diagonal elements are always 0. Example:

$$A = \begin{bmatrix} 0 & -2 & -3 \\ 2 & 0 & -4 \\ 3 & 4 & 0 \end{bmatrix}$$

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3. Row Operations and REF

Elementary Row Operations

1. **Row Swap**: Switch two rows.
2. **Row Scaling**: Multiply a row by a non-zero constant.
3. **Row Addition/Subtraction**: Add or subtract multiples of rows.

Row Echelon Form (REF)

1. Convert the matrix into a "stepped" form. Example:

$$\text{Starting Matrix: } \begin{bmatrix} 2 & 4 & 6 \\ 1 & 3 & 5 \\ 0 & 1 & 2 \end{bmatrix}$$

   Step-by-step row operations yield:

$$\text{REF: } \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

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4. Determinants

2×2 Determinants

$$\det(A) = ad - bc \quad \text{for } A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}, \quad \det(A) = (2)(5) - (3)(4) = -2$$

3×3 Determinants

Use cofactor expansion:

$$\det(A) = a_{11} \cdot \det(M_{11}) - a_{12} \cdot \det(M_{12}) + a_{13} \cdot \det(M_{13})$$

Worked example provided in the book will be referenced here.

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5. Matrix Inverses

2×2 Matrix

The inverse of $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is:

$$A^{-1} = \frac{1}{\det(A)} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Example:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}, \quad A^{-1} = \frac{1}{5} \begin{bmatrix} 4 & -3 \\ -1 & 2 \end{bmatrix}$$

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6. Eigenvalues and Eigenvectors

Definition

- Solve $A \cdot v = \lambda \cdot v$, where:
  - λ: Eigenvalue.

- v: Eigenvector.

How to Compute Eigenvalues

1. Subtract λI from A: $A - \lambda I$.
2. Set $\det(A - \lambda I) = 0$.
3. Solve for λ.

Example

For $A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$:

1. $\det(A - \lambda I) = \det \begin{bmatrix} 4 - \lambda & 2 \\ 1 & 3 - \lambda \end{bmatrix} = 0$.
2. Expand determinant: $(4 - \lambda)(3 - \lambda) - 2 = 0$.
3. Solve quadratic equation for λ.

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7. Singular Value Decomposition (SVD)

Definition

Decompose A into:

$$A = U \Sigma V^T$$

- U: Left singular vectors.
- Σ: Diagonal matrix of singular values.

- V^T: Right singular vectors.

Applications

1. **PCA**: Use SVD to reduce dimensions.
2. **Noise Removal**: Filter out small singular values.

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8. Systems of Linear Equations

Solving by Substitution

Example:

$$x + y = 3, \quad 2x + y = 5$$

1. Solve $y = 3 - x$.
2. Substitute into $2x + y = 5$: $2x + (3 - x) = 5$.
3. Solve $x = 2$, $y = 1$.

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9. Applications in Machine Learning

1. **Linear Regression**:
    - Solve $w = (X^T X)^{-1} X^T y$.
2. **PCA**:

- Use eigenvectors of the covariance matrix to reduce dimensions.

3. **Optimization**:
   - Use matrix operations to minimize cost functions.