Gentzen–Prawitz Natural deduction

$$\begin{array}{c} A^x \\ \vdots \end{array} \qquad \frac{\begin{array}{c}\vdots \\ B\end{array}}{A \to B}\ I[x]\!\to \qquad \frac{\begin{array}{c}\vdots \\ A \to B\end{array} \qquad \begin{array}{c}\vdots \\ A\end{array}}{B}\ E\!\to$$

$$\frac{\begin{array}{c}\vdots \\ A\end{array} \quad \begin{array}{c}\vdots \\ B\end{array}}{A \land B}\ I\land \qquad \frac{\begin{array}{c}\vdots \\ A \land B\end{array}}{A}\ E_L\land \qquad \frac{\begin{array}{c}\vdots \\ A \land B\end{array}}{B}\ E_R\land$$

$$\frac{\begin{array}{c}\vdots \\ A \lor B\end{array} \qquad \begin{array}{c}\vdots \\ A \to C\end{array} \qquad \begin{array}{c}\vdots \\ B \to C\end{array}}{C} \qquad \frac{\begin{array}{c}\vdots \\ \bot\end{array}}{A}$$

x is not in uncancelled assumptions

$$\frac{A}{\forall x.\, A}\ I\forall \qquad \frac{\forall x.\, A}{A[x := M]}\ E\forall$$

$$\frac{A[x := M]}{\exists x.\, A}\ I\exists \qquad \frac{\exists x.\, A \qquad \forall x.\, A \to B}{B}\ E\exists \quad x \notin FV(B)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{B \to C^{x} \qquad \dfrac{A \to B^{y} \qquad A^{x}}{B}}{C}}{A \to C}\ I[x]}{(A \to B) \to A \to C}\ I[y]}{(B \to C) \to (A \to B) \to A \to C}\ I[x]$$

→ What we usually use: $\dfrac{\Gamma \vdash A \to B \quad \Gamma \vdash A}{\Gamma \vdash B}$

is called Natural Deduction in Sequent-style though it is not a sequent calculus

Fitch Style Natural Deduction

$A \to B,\ B \to C,\ A \vdash C$

1	$A \to B$	hypotheses
2	$B \to C$	
3	A	
4	B	E 1,3
5	C	E 2,4

conclusion

$A \to B,\ B \to C \vdash A \to C$

1	$A \to B$		
2	$B \to C$		
3	A		implication introduction
4	B	E 1,3	
5	C	E 2,4	
6	$A \to C$	I 3-5	

Rules

m | A
⋮
n | B
⋮
p | $A \land B$ $I\land$ m,n

m | B
⋮
n | A
⋮
p | $A \land B$ $I\land$ m,n

m | $A \land B$
⋮
n | A $E_L\land$ m

m | $A \land B$
⋮
n | B $E_R\land$ n

m | A
⋮
n | $A \lor B$ $I_L\lor$

m | B
⋮
n | $A \lor B$ $I_R\lor$

m | $A \lor B$
⋮
n | A
⋮
p | C
q | B
⋮
r | C
s | C $E\lor$ m, n-p, q-r

m | A
⋮
n | B
n+1 | $A \to B$ I m-n

m | $A \to B$
⋮
n | A
⋮
p | B $E\to$ m,n

m | A
⋮
n | $A \to B$
⋮
p | B $E\to$ m,n

m | A
⋮
n | A R m

m | $\neg A$
⋮
n | $\bot$
n+1 | A PC m-n

m | u ← guard
⋮   no formula containing u can be repeated here
n | $A[u/x]$
n+1 | $\forall x.A$ $I\forall$ m-n

A modal logic Σ is normal if it contains:

- $\Box(p \to q) \to (\Box p \to \Box q)$ K
- $\Diamond p \longleftrightarrow \neg\Box\neg p$ DUAL

$A \in \Sigma \Rightarrow \Box A \in \Sigma$

And it is closed under modus ponens and necessitation

imply:

→ A normal modal logic is well-behaved in the sense that a formula that is valid in all modal logics is derivable:

$$\models A \Rightarrow \vdash A$$

→ The problem with the necessitation rule is that it only holds at the model instead of at the world level:

$$M \models_w A \not\Rightarrow M \models_w \Box A$$

But

$$M \models A \Rightarrow M \models \Box A$$

translated to derivation systems it says:

$A \in \Sigma \Rightarrow \Box A \in \Sigma$

but not

$A \to \Box A \in \Sigma$

→ In Hilbert-style derivation systems each line represents a formula in our logic and so we can have $\frac{A}{\Box A}$

→ In natural deduction systems this is not the case:

$$\dfrac{\dfrac{[A]^x}{\Box A}\,\text{NEC}}{A \to \Box A}\,I[x]$$

1	A	NEC 1
2	$\Box A$	
3	$A \to \Box A$	I 1-2

When we start a proof we assume an arbitrary world. Every subordinate proof inherits this world so in essence we are saying:

$A^w \vdash \Box A^w$ and so $(A \to \Box A)^w$

since w is arbitrary

$\vdash A \to \Box A$

Fitch-Style Natural Deduction for K

The key idea is to introduce a new 'strict subordinate proof':

	$\boxed{\Box}$	
	$\vdots$	
m	A	
m+1	$\Box A$	$I\Box$ m

m	$\Box A$	one-level of $\boxed{\Box}$
	$\vdots$	
	$\boxed{\Box}$	
	$\vdots$	
n	A	$E\Box$ m

$$\frac{\Gamma, \blacksquare \vdash t : A}{\Gamma \vdash \mathrm{box}\ t : \Box A}$$

$$\frac{\Gamma \vdash t : \Box A}{\Gamma, \blacksquare, \Gamma' \vdash \mathrm{close}\ t : A}$$

Proof of $K : \Box(A \to B) \vdash (\Box A \to \Box B)$

1	$\Box(A \to B)$	
2	$\Box A$	
3	$\boxed{\Box}$	
4	$A \to B$	$E\Box$ 1
5	A	$E\Box$ 2
6	B	$E\to$ 4,5
7	$\Box B$	$I\Box$ 6
8	$\Box A \to \Box B$	$I\to$ 2-7

but not $A \nvdash \Box A$

1	A	
2	$\boxed{\Box}$	
3		? can't access A here

Additional import rules corresponding to other axioms

4-import ($\Box A \to \Box\Box A$)

	$\vdots$	
m	$\Box\varphi$	
	$\vdots$	
	$\boxed{\Box}$	
	$\vdots$	
n	$\Box\varphi$	4-import m

T-export ($\Box A \to A$)

	$\boxed{\Box}$	
	$\vdots$	
m	φ	
m+1	φ	5-export m

$\Box A \vdash \Box\Box A$

1	$\Box A$	
2	$\boxed{\Box}$	
3	$\Box A$	4-import 1
4	$\Box\Box A$	I$\Box$ 3

$\Box A \to A$

1	$\Box A$	
2	$\boxed{\Box}$	
3	A	$\Box$O 1
4	A	T-export 3