

Gentzen - Prawitz Natural deduction

$$\begin{array}{c}
 A^x \\
 \vdots \\
 \frac{B}{A \rightarrow B} I(x) \rightarrow \\
 \vdots
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \\
 \frac{A \rightarrow B \quad A}{B} E \rightarrow \\
 \vdots
 \end{array}$$

$$\begin{array}{c}
 \vdots \quad \vdots \\
 \frac{A \quad B}{A \wedge B} I \wedge \\
 \vdots
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \\
 \frac{A \wedge B}{A} E \wedge \\
 \vdots
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \\
 \frac{A \wedge B}{B} E \wedge \\
 \vdots
 \end{array}$$

$$\begin{array}{c}
 \vdots \\
 A \vee B \\
 \vdots
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \\
 A \rightarrow C \quad B \rightarrow C \\
 \hline
 C
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \\
 \frac{\perp}{A}
 \end{array}$$

x is not
in an undischarged
assumption

$$\frac{A}{\forall x. A} IV$$

$$\frac{\forall x. A}{A[x:=M]} EV$$

$$\frac{A[x:=M]}{\exists x. A} I\exists$$

$$\frac{\exists x. A \quad \forall x. A \rightarrow B}{B} E\exists \quad (x \neq FV(B))$$

$$\begin{array}{c}
 \vdots \\
 B \rightarrow C \\
 \vdots
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \\
 A \rightarrow B \\
 \vdots
 \end{array}
 \quad
 \begin{array}{c}
 \vdots \\
 A^x \\
 \vdots
 \end{array}$$

$$\frac{C}{A \rightarrow C} I(x)$$

$$\frac{(A \rightarrow B) \rightarrow A \rightarrow C}{(B \rightarrow C) \rightarrow (A \rightarrow B) \rightarrow A \rightarrow C} I(x)$$

→ What we usually use: $\frac{\Gamma \vdash A \rightarrow B \quad \Gamma \vdash A}{\Gamma \vdash B}$

is called Natural Deduction in sequent-style
though it's not a sequent calculus

Fitch Style Natural Deduction

$$A \rightarrow B, B \rightarrow C, A \vdash C$$

1		$A \rightarrow B$	
2		$B \rightarrow C$	
3		A	
<hr/>			
4		B	$\rightarrow E 1,3$
5		C	$\rightarrow E 2,4$
<hr/>			
			conclusion

} hypotheses

$$A \rightarrow B, B \rightarrow C \vdash A \rightarrow C$$

1		$A \rightarrow B$	
2		$B \rightarrow C$	
<hr/>			
3			A
4		B	$\rightarrow E 1,3$
5		C	$\rightarrow E 2,4$
6		$A \rightarrow C$	$\rightarrow I 3-5$

} implication introduction

Rules

m		A	
...			
n		B	
...			
p		$A \wedge B$	$\wedge I m, n$

m		B	
...			
n		A	
...			
p		$A \wedge B$	$\wedge I m, n$

m		$A \wedge B$	
...			
n		A	$\wedge E m$

m		$A \wedge B$	
...			
n		B	$\wedge E n$

m		A	
...			
n		$A \vee B$	$\vee I$

m		B	
...			
n		$A \vee B$	$\vee I$

m		$A \vee B$	
...			
n			A
...			
p		C	
...			
q			B
...			
r		C	
s		C	$\vee E m, n-p, q-r$

m			A
...			
n			B
n+1		$A \rightarrow B$	$\rightarrow I m-n$

m		$A \rightarrow B$	
...			
n		A	
p		B	$\rightarrow E m, n$

m		A	
...			
n		$A \rightarrow B$	
p		B	$\rightarrow E m, n$

no formula containing u can be repeated here

m		A	
...			
n			A
n+1		$\neg A$	$\neg I m-n$

m		$\neg A$	
...			
n		\perp	
n+1		A	$\neg E m-n$

m		u	
...			
n		$A(u/x)$	
n+1		$\forall x. A$	$\forall I m-n$

A modal logic Σ is normal if it contains:

- $\Box(p \rightarrow q) \rightarrow (\Box p \rightarrow \Box q)$ K
- $\Box p \leftrightarrow \neg \Box \neg p$ dual

$$A \in \Sigma \Rightarrow \Box A \in \Sigma$$

And it is closed under modus ponens and necessitation
 imply:
 → A normal modal logic is well-behaved in the sense that
 a formula that is valid in all modal logics is derivable:

$$\models A \Rightarrow \vdash A$$

→ The problem with the necessitation rule is that it only
 holds at the model instead of at the world level:

$$M \models_w A \not\Rightarrow M \models_w \Box A$$

But

$$M \models A \Rightarrow M \models \Box A$$

translated to derivation systems
 it says:
 $A \in \Sigma \Rightarrow \Box A \in \Sigma$
 but not
 $A \rightarrow \Box A \in \Sigma$

→ In Hilbert-style derivation systems each line represents a
 formula in our logic and so we can have $\frac{A}{\Box A}$

→ In natural deduction systems this is not the case:

$\frac{(A)^x}{\Box A} \text{ NEC}$ $\frac{\Box A}{A \rightarrow \Box A} \text{ I}[x]$	$\frac{A \rightarrow \Box A}{A \rightarrow \Box A} \text{ I}[x]$	<table border="0"> <tr> <td style="padding-right: 5px;">1</td> <td style="border-left: 1px solid black; padding-left: 5px;">A</td> <td style="padding-left: 10px;"></td> </tr> <tr> <td style="padding-right: 5px;">2</td> <td style="border-left: 1px solid black; padding-left: 5px;">\Box A</td> <td style="padding-left: 10px;">NEC 1</td> </tr> <tr> <td style="padding-right: 5px;">3</td> <td style="border-left: 1px solid black; padding-left: 5px;">A \rightarrow \Box A</td> <td style="padding-left: 10px;">I 1-2</td> </tr> </table>	1	A		2	\Box A	NEC 1	3	A \rightarrow \Box A	I 1-2
1	A										
2	\Box A	NEC 1									
3	A \rightarrow \Box A	I 1-2									

When we start a proof we
 assume an arbitrary world.
 Every subordinate proof inherits
 this world so in essence we
 are saying:

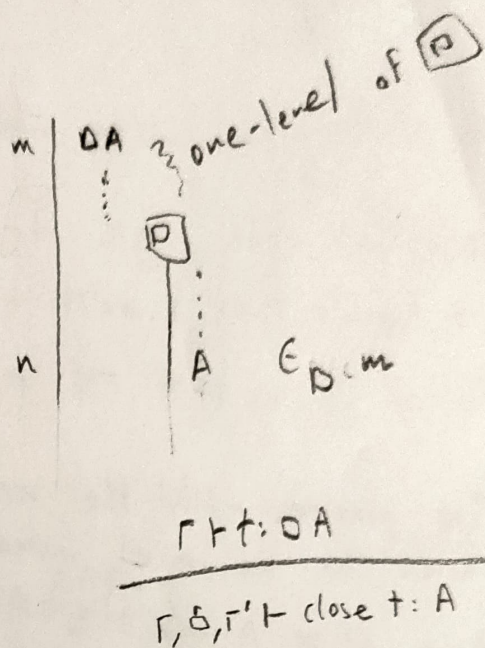
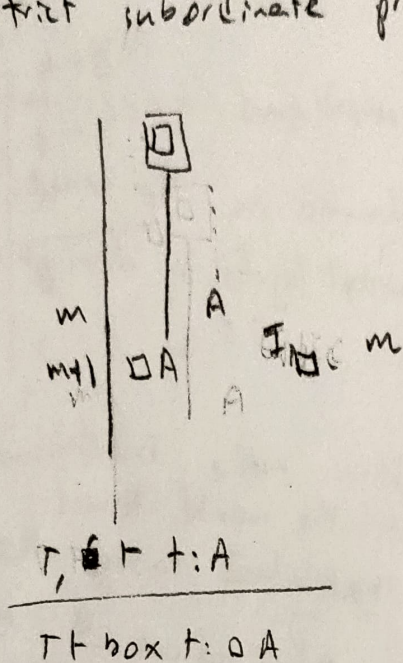
$$A^w \vdash \Box A^w \text{ and so } (A \rightarrow \Box A)^w$$

since w is arbitrary

$$\vdash A \rightarrow \Box A$$

Fitch-Style Natural Deduction for \mathcal{K}

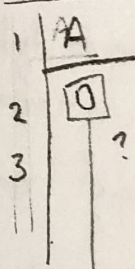
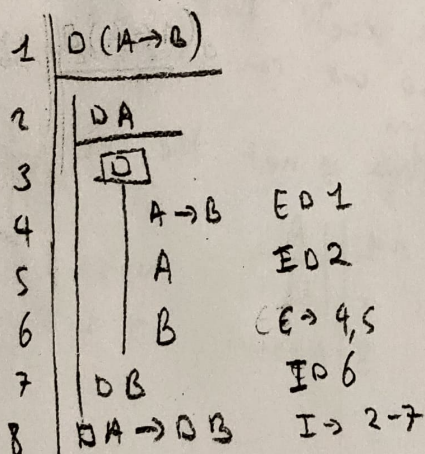
The key idea is to introduce a new 'strict subordinate proof':



Proof of $\mathcal{K} : \Box(A \rightarrow B) \vdash (\Box A \rightarrow \Box B)$

but not

~~$\Box A \rightarrow \Box A$~~

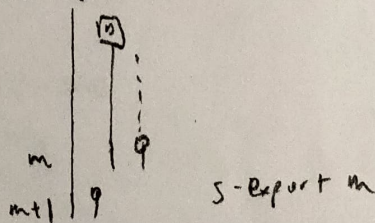
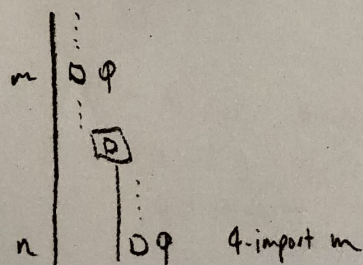


? can't access A here

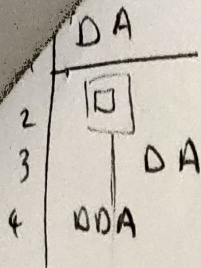
Additional import rules corresponding to other axioms

4-import ($\Box A \rightarrow \Box \Box A$)

T-export ($\Box A \rightarrow A$)



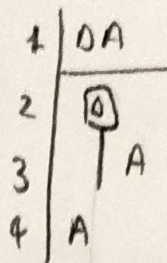
DA + DDA



4-import 1

TD 3

DA → A



30 1

T-Export 3