Collective risk

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Contents

Statement

The clients of an insurance company generate claims according to independent (homogeneous) Poisson processes with common rate λ. The claims are independent and the amount of each of them follows a distribution with cdf $F(x) = 1 - (100/x)^{2.5}$ if $x > 100$. New clients enroll the company according to another (homogeneous) Poisson process with rate ν and the time they stay in the company follows an exponential distribution with rate μ. Each policyholder pays a fixed amount a every year to the company. If the policyholder stays in the company only during a fraction of the year, she will pay the corresponding fraction of a which is assumed to be paid on a continuous manner.

If the initial capital is c_0 and the initial number of clients is n_0, we want to compute the probability that the capital of the company remains positive during some given time t_l.

Introduction

Each policyholder has a fixed amount a to pay every year, so the capital of the company will increase by a times the number of clients every fraction of year.

On the other hand, they also hold the right to claim the premium on the policy, which happens according to a Poisson process with common rate λ. The amount of is this premium is distributed according to a Pareto distribution with parameters $\alpha = 2.5$ and $\beta = 100$.

The clients of the company can leave the company at any time, and the time they stay in the company follows an exponential distribution with rate μ.

Clients come according to an independent (homogeneous) Poisson processes with rate ν, and they stay according to an exponentially distributed time with rate μ.

So in general, the capital of the company at any time t will be:

$$C(t) = c_0 + at(n_0 + N_A(t) - N_D(t)) - \sum_{j=1}^{N_C(t)} X_j$$

where $N_A(t)$ is the number of clients that arrive by time t, $N_D(t)$ is the number of clients that leave by time t, $N_C(t)$ is the number of claims that arrive by time t, X_j is the amount of the j-th claim, and $n(t)$ is the number of clients at time t.

The project

Project development including the code and emphasizing the main difficulties and most important parts.

First, to simulate the amount of the claims we will define a function with the inverse of the cdf of the Pareto distribution. Mathematically, this function is:

$$F^{-1}(x) = \frac{\beta}{\sqrt[\alpha]{(1-x)}} = \frac{100}{\sqrt[2.5]{(1-x)}}$$

```r
inverse.claim <- function(x) {
  return(max((100 / (1 - x)^(1 / 2.5)), 0))
}
```

The algorithm we will follow is, for each path:

```
1. Initialize variables $t = 0, n = n_0, c = c_0$.
2. Compute the time of the first event: $dt = rexp(1, \lambda * n + \mu * n + \nu)$
3. Update the time: $t = t + dt$
4. While $t < tl$:
   4.1. Update capital with the proportional part of the payments: $c = c + n * a * dt$
   4.2. Decide the type of event and update in each case:
      4.2.1. If $u < \lambda * n/(\lambda * n + \mu * n + \nu)$, claim event so update capital $= c - X_j$
      4.2.2. If $u < (\lambda * n + \mu * n)/(\lambda * n + \mu * n + \nu)$, departure event so $n = n - 1$
      4.2.3. Else, enrollment event, so $n = n + 1$
   4.3. Compute the time of the next event: $dt = rexp(1, \lambda * n + \mu * n + \nu)$
   4.4. Update the time: $t = t + dt$
```

Thus for one simulation we have:

```r
simulate_one_insurance <- function(c0, n0, a, tl, lambda, mu, nu) {
  # Initialize variables
  t <- 0
  n <- n0
  c <- c0
  # Update capital with annuity payments
  dt <- rexp(1, lambda * n + mu * n + nu)
  t <- t + dt
  while (t < tl) {
    # Update the capital with the proportional part of the payments
    c <- c + n * a * dt
    # Decide the type of event
    u <- runif(1)
    if (u < lambda * n / (lambda * n + mu * n + nu)) {
      # Claim event
      claim_amount <- inverse.claim(runif(1))
      c <- c - claim_amount
    } else if (u < (lambda * n + mu * n) / (lambda * n + mu * n + nu)) {
      # Departure event
      n <- max(n - 1, 0)
    } else {
      # Enrollment event
      n <- n + 1
    }
    # Stop if capital becomes negative
    if (c < 0) break
    # Time of next event
    dt <- rexp(1, lambda * n + mu * n + nu)
    t <- t + dt
  }
  return(c)
}
```

And for multiple:

```r
simulate_insurance <- function(c0, n0, a, tl, lambda, mu, nu, MC) {
  final_capitals <- vector("numeric", MC)
  # Simulate MC paths
  final_capitals <- replicate(MC, simulate_one_insurance(c0, n0, a, tl, lambda, mu, nu))
  # Compute and return outputs
  fraction <- sum(final_capitals > 0) / MC
  pos_fc <- final_capitals[final_capitals > 0]
  mean_final_capitals <- mean(pos_fc)
  sd_final_capitals <- sd(pos_fc)
  return(list(fraction = fraction, mean_final_capital = mean_final_capitals,
              sd_final_capital = sd_final_capitals, pos_fc = pos_fc))
}
```

Antithetic variates

We try a variance reduction technique called antithetic variates. The idea is to simulate pairs of antithetic paths, that is, paths with the same initial conditions but with the sign of the increments changed. This way, the variance of the final capital is reduced by a factor of 2. The algorithm is the same as before, but we will simulate $MC/2$ pairs of antithetic paths. The final capital of the i-th path will be the sum of the final capital of the i-th antithetic path and the final capital of the $(MC/2 + i)$-th antithetic path.

```r
simulate_one_insurance_antithetic <- function(c0, n0, a, tl, lambda, mu, nu) {
  # Initialize variables
  t <- 0
  n <- n0
  c <- c0
  c_anti <- c0
  # Update capital with annuity payments
  dt <- rexp(1, lambda * n + mu * n + nu)
  t <- t + dt
  while (t < tl) {
    # Update the capital with the proportional part of the payments
    c <- c + n * a * dt
    c_anti <- c_anti + n * a * dt
    # Decide the type of event
    u <- runif(1)
    if (u < lambda * n / (lambda * n + mu * n + nu)) {
      # Claim event
      u <- runif(1)
      if (c > 0) { # If the capital is positive, we can perform a claim
        claim_amount <- inverse.claim(u)
        c <- c - claim_amount
      }
      if (c_anti > 0) {
        # If the capital is positive for antithetic,
        # we can perform a claim else keep it negative
        claim_anti <- inverse.claim(1-u)
        c_anti <- c_anti - claim_anti
      }
    } else if (u < (lambda * n + mu * n) / (lambda * n + mu * n + nu)) {
      # Departure event
      n <- max(n - 1, 0)
    } else {
      # Enrollment event
      n <- n + 1
    }
    # Stop if both capitals becomes negative, if only one does,
    # we keep going with the other
    if (c < 0 && c_anti < 0) break
    # Time of next event
    dt <- rexp(1, lambda * n + mu * n + nu)
    t <- t + dt
  }
  return(c(c, c_anti))
}
```

```r
simulate_insurance_antithetic <- function(c0, n0, a, tl, lambda, mu, nu, MC) {
  final_capitals <- vector("numeric", MC)
  # Simulate MC pairs of antithetic paths
  # for (i in 1:(MC/2)) {
  #   final_capitals[i] <- simulate_one_insurance(c0, n0, a, tl, lambda, mu, nu)
  #   final_capitals[i + (MC/2)] <- simulate_one_insurance(c0, n0, a, tl, lambda, mu, nu) * -1
  # }
  # Compute and return outputs
  # antithetic_final_capitals <- c(final_capitals[1:(MC/2)], -final_capitals[(MC/2+1):MC])
  antithetic_final_capitals <- replicate(MC/2, simulate_one_insurance_antithetic(
                                          c0, n0, a, tl, lambda, mu, nu))
  fraction <- sum(antithetic_final_capitals > 0) / MC
  pos_fc <- antithetic_final_capitals[antithetic_final_capitals > 0]
  mean_final_capitals <- mean(pos_fc)
  sd_final_capitals <- sd(pos_fc)
  return(list(fraction = fraction, mean_final_capital = mean_final_capitals,
              sd_final_capital = sd_final_capitals, pos_fc = pos_fc))
}
```

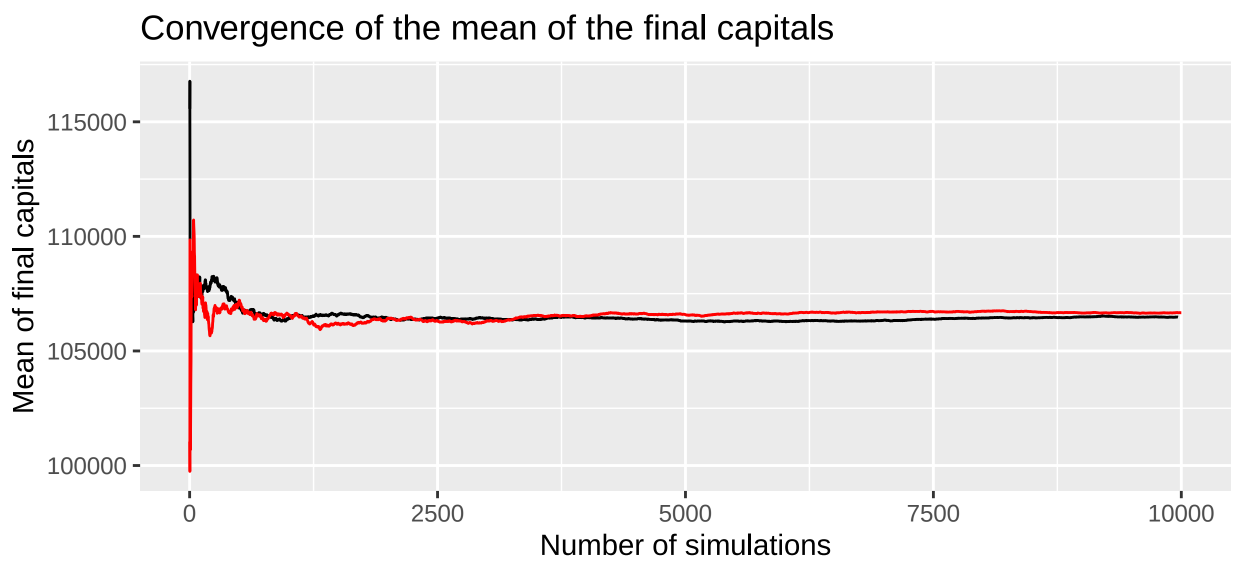
Results

```
set.seed(42)
res_normal <- simulate_insurance(c0 = 1000, n0 = 100, a = 100, tl = 100,
                          lambda = 0.1, mu = 0.1, nu = 0.3, MC = 10000)
res_anti <- simulate_insurance_antithetic(c0 = 1000, n0 = 100, a = 100, tl = 100,
                          lambda = 0.1, mu = 0.1, nu = 0.3, MC = 10000)
kable(rbind(res_normal[1:3], res_anti[1:3]),
      caption = "Results of the simulation with and without antithetic variates")
```

Table 1: Results of the simulation with and without antithetic variates

fraction	mean_final_capital	sd_final_capital
0.9967	106475.971675903	10516.1123920357
1	106666.637870849	10361.4480989

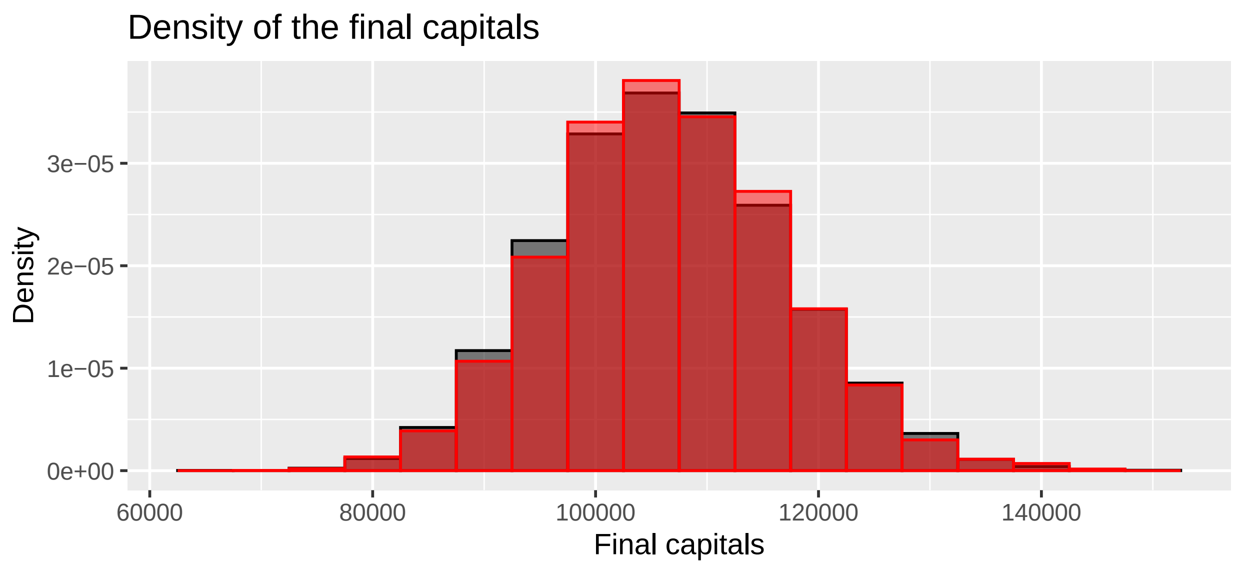
```
num.normal <- as.numeric(unlist(res_normal['pos_fc']))
num.anti <- as.numeric(unlist(res_anti['pos_fc']))
p1 <- ggplot() +
      geom_line(mapping=aes(x = 1:length(num.normal),
                            y = cumsum(num.normal)/(1:length(num.normal))),
                color = "black") +
      geom_line(mapping=aes(x=1:length(num.anti),
                            y = cumsum(num.anti)/(1:length(num.anti))),
                color = "red") +
      labs(x = "Number of simulations", y = "Mean of final capitals",
           title = "Convergence of the mean of the final capitals")
plot(p1)
```



```
p2 <- ggplot() +
      geom_histogram(mapping= aes(x = num.normal, y=after_stat(density)),
                     binwidth=5000, color = "black", fill = "black", alpha = .5) +
      stat_function(fun=dnorm, args=list(mean=mean(num.normal),
                    sd = sd(num.normal), color = "black")) +
      geom_histogram(mapping= aes(x = num.anti, y=after_stat(density)),
                     binwidth=5000, color = "red", fill = "red", alpha = .5) +
      stat_function(fun=dnorm, args=list(mean=mean(num.anti),
                    sd = sd(num.anti), color = "red")) +
      labs(x = "Final capitals", y = "Density",
            title = "Density of the final capitals")
plot(p2)
```

```
## Warning: Computation failed in `stat_function()`
## Caused by error in `fun()`:
## ! unused argument (color = "black")
```

```
## Warning: Computation failed in `stat_function()`
## Caused by error in `fun()`:
## ! unused argument (color = "red")
```



Numerical results, quality assessment of the approximations, and time efficiency of the algorithms.

Conclusions

About the results, how the difficulties were solved, and possible alternative approaches. Keep the focus, the conclusions must be as brief as possible.

References

Including textbooks, webpages, and class notes.