(7) Sean $z = 1 + i$ y $w = \sqrt{2} - i$. Calcular:

a) z^{-1}; $1/w$; z/w; w/z.

b) $1 + z + z^2 + z^3 + \cdots + z^{2019}$.

c) $(z(z + w)^2 - iz)/w$.

a) $z^{-1} = \dfrac{\bar{z}}{|z|^2} = \dfrac{1-i}{2}$

$\dfrac{1}{w} = \dfrac{1}{\sqrt{2}-i} = \dfrac{\sqrt{2}+i}{(\sqrt{2}-i)(\sqrt{2}+i)} = \dfrac{\sqrt{2}+i}{3}$

$\dfrac{z}{w} = \dfrac{1+i}{\sqrt{2}-i} = \dfrac{(1+i)(\sqrt{2}+i)}{(\sqrt{2}-i)(\sqrt{2}+i)} = \dfrac{\sqrt{2}+1+i(\sqrt{2}+1)}{3}$

$\dfrac{w}{z} = \dfrac{\sqrt{2}-i}{1+i} = \dfrac{(\sqrt{2}-i)(1-i)}{(1+i)(1-i)} = \dfrac{\sqrt{2}-1-i(\sqrt{2}+1)}{2}$

b) Por un lado $1+z+z^2+z^3+\cdots+z^{2019} = \dfrac{1-z^{2020}}{1-z} = \dfrac{1-(1+i)^{2020}}{-i} = i(1-z^{2020})$

Por el otro lado tenemos que $1+i = \sqrt{2}e^{i\pi/4}$, luego $z = \sqrt{2}e^{i\pi/4}$, y por lo tanto
$z^{2020} = 2^{1010}e^{i1010\pi/4} = 2^{1010}e^{i252\pi} = 2^{1010}e^{i0} = 2^{1010}$

Por lo tanto, $1+z+z^2+z^3+\cdots+z^{2019} = i(1-z^{2020}) = i(1-2^{1010})$

c) Primero calculemos el numerador por partes:

$$\begin{aligned}
z(z+w)^2 &= (1+i)(1+i+\sqrt{2}+i)^2 = (1+i)(1+\sqrt{2})^2 \\
&= (1+i)(1+2\sqrt{2}+2) = (1+i)(3+2\sqrt{2}) \\
&= 3+2\sqrt{2}+3i+2i\sqrt{2}
\end{aligned}$$

Luego,
$$\begin{aligned}
z(z+w)^2 - iz &= 3+2\sqrt{2}+3i+2i\sqrt{2} - i(1+i) \\
&= 3+2\sqrt{2}+3i+2i\sqrt{2} - i - i^2 \\
&= 3+2\sqrt{2}+3i+2i\sqrt{2} - i + 1 \\
&= 4+2\sqrt{2}+2i+2i\sqrt{2}
\end{aligned}$$

Dividir por w es multiplicar por $\dfrac{\bar{w}}{|w|^2} = \dfrac{\sqrt{2}+i}{3}$, y por lo tanto,

$$\begin{aligned}
\frac{z(z+w)^2 - iz}{w} &= \frac{(4+2\sqrt{2}+2i+2i\sqrt{2})(\sqrt{2}+i)}{3} \\
&= \frac{4\sqrt{2}+4+2\sqrt{2}+2i\sqrt{2}+2\sqrt{2}+2i-2+2\sqrt{2}\,i+2i^2}{3} \\
&= \frac{6\sqrt{2}+4+4i}{3}
\end{aligned}$$