14) Calcular las siguientes integrales:

a) $\displaystyle\int \frac{x^2+4x+24}{x^2-4x+8}\,dx$

En éste caso puedo hacer división de polinomios:

$$\begin{array}{rl} x^2+4x+24 & \big\lfloor\, x^2-4x+8 \\ -\;x^2-4x+8 & \quad 1 \\ \hline 8x+16 & \end{array}$$

, entonces $\dfrac{x^2+4x+24}{x^2-4x+8} = 1+\dfrac{8x}{x^2-4x+8}$

$$\int \frac{x^2+4x+24}{x^2-4x+8}\,dx = \int 1+\frac{8x}{x^2-4x+8}\,dx$$

$$= \underbrace{\int 1\,dx}_{x+c} + \int \frac{8x}{x^2-4x+8}\,dx$$

Resuelvo la integral restante, dónde $q(x)=x^2-4x+8$ no tiene raíces reales. (Caso 4)

$$\frac{8x+16}{x^2-4x+8} = h_1\cdot\frac{2x-4}{x^2-4x+8} + h_2\cdot\frac{1}{x^2-4x+8}$$

Igualo las coeficientes de los numeradores.

$$8x+16 = h_1(2x-4)+h_2 \quad\Rightarrow \begin{cases} 2h_1 = 8 \\ -4h_1+h_2=16 \end{cases} \Rightarrow \begin{cases} h_1 = 4 \\ h_2 = 32 \end{cases}$$
$$= 2xh_1 - 4h_1 + h_2$$

Completo cuadrado: $x^2-4x+8 = (x-2)^2+4 = (x-2)^2+2^2$

$$\int \frac{x^2+4x+24}{x^2-4x+8}\,dx = \int 4\cdot\frac{2x-4}{\underbrace{x^2-4x+8}_{u}}\,dx + \int \frac{32}{x^2-4x+8}\,dx = 4\int\frac{du}{u} + 32\int\frac{dx}{(x-2)^2+2^2}$$

$$= 4\ln|x^2-4x+8| + 32\cdot\frac{1}{4}\arctan\left(\frac{x-2}{4}\right)+c = 4\ln|x^2-4x+8| + 32\arctan\left(\frac{x-2}{4}\right)+c$$

Finalmente, $\displaystyle\int \frac{x^2+4x+24}{x^2-4x+8}\,dx = x+4\ln|x^2-4x+8|+32\arctan\left(\frac{x-2}{4}\right)+c,\ c\in\mathbb{R}$

b) $\displaystyle\int \frac{2x+1}{x^2+1}\,dx$

El denominador no tiene raíces reales. (Caso 4)

$$\frac{2x+1}{x^2+1} = h_1\cdot\frac{2x}{x^2+1} + h_2\cdot\frac{1}{x^2+1}$$

Igualo las coeficientes de los numeradores.

$$2x+1 = 2xh_1+h_2 \Rightarrow \begin{cases} 2h_1 = 2 \\ h_2 = 1 \end{cases} \Rightarrow \begin{cases} h_1 = 1 \\ h_2 = 1 \end{cases}$$

Entonces, $\dfrac{2x+1}{x^2+1} = \dfrac{2x}{x^2+1} + \dfrac{1}{x^2+1}$ ⊛ No lo ví en un principio, pero puedo ahorrarme los pasos anteriores.

Luego, $\displaystyle\int\frac{2x+1}{x^2+1}\,dx = \int\frac{2x}{x^2+1}+\frac{1}{x^2+1}\,dx = \int\frac{2x}{\underbrace{x^2+1}_{u}}\,dx + \int\frac{dx}{x^2+1} = \int\frac{du}{u}+\int\frac{dx}{x^2+1^2}$

$$= \ln|x^2+1| + \arctan(x)+c,\ c\in\mathbb{R}$$

c) $\displaystyle\int \frac{x-1}{x^2+4}\,dx$

El denominador no tiene raíces reales. (Caso 4)

$$\frac{x-1}{x^2+4} = h_1\cdot\frac{2x}{x^2+4} + h_2\cdot\frac{1}{x^2+4}$$

Igualo las coeficientes de los numeradores:

$$x-1 = 2xh_1+h_2 \Rightarrow \begin{cases} 2h_1 = 1 \\ h_2 = -1 \end{cases} \Rightarrow \begin{cases} h_1 = 1/2 \\ h_2 = -1 \end{cases}$$

Entonces $\dfrac{x-1}{x^2+4} = \dfrac{1}{2}\cdot\dfrac{2x}{x^2+4} + \dfrac{-1}{x^2+4}$

Luego, $\displaystyle\int\frac{x-1}{x^2+4}\,dx = \int\frac{1}{2}\cdot\frac{2x}{x^2+4}+\frac{-1}{x^2+4}\,dx = \frac{1}{2}\int\frac{2x}{\underbrace{x^2+4}_{u}}\,dx - \int\frac{dx}{x^2+4} = \frac{1}{2}\int\frac{du}{u} - \int\frac{dx}{x^2+2^2}$

$$= \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\arctan\left(\frac{x}{2}\right)+c$$

$$= \frac{1}{2}\left(\ln|x^2+4| - \arctan\left(\frac{x}{2}\right)\right)+c,\ c\in\mathbb{R}$$

d) $\displaystyle\int \frac{1}{x^2+3x+2}\,dx$

$x^2+3x+2 = (x+1)(x+2)$. Tengo 2 raíces reales que no se repiten. (Caso 1)

$$\frac{1}{x^2+3x+2} = \frac{A_1}{x+1}+\frac{A_2}{x+2} = \frac{A_1(x+2)+A_2(x+1)}{(x+1)(x+2)} = \frac{(A_1+A_2)x+(2A_1+A_2)}{(x+1)(x+2)}$$

Igualo las coeficientes de los numeradores:

$$1 = (A_1+A_2)x+(2A_1+A_2) \Rightarrow \begin{cases} A_1+A_2 = 0 \\ 2A_1+A_2 = 1 \end{cases} \Rightarrow \begin{cases} A_1 = 1 \\ A_2 = -1 \end{cases}$$

Luego, $\dfrac{1}{x^2+3x+2} = \dfrac{1}{x+1}+\dfrac{-1}{x+2}$, entonces

$$\int\frac{1}{x^2+3x+2}\,dx = \int\frac{1}{x+1}\,dx+\int\frac{-1}{x+2}\,dx = \int\frac{1}{\underbrace{x+1}_{u}}\,dx - \int\frac{1}{\underbrace{x+2}_{v}}\,dx = \int\frac{du}{u}-\int\frac{dv}{v} = \ln|x+1|-\ln|x+2|+c$$

$$= \ln\left|\frac{x+1}{x+2}\right|+c,\ c\in\mathbb{R}$$

e) $\displaystyle\int \frac{x}{x^3-3x+2}\,dx$

$x^3-3x+2 = (x-1)(x-1)(x+2) = (x-1)^2(x+2)$. Tengo 3 raíces reales de las cuáles 2 se repiten. (Caso 3)

$$\frac{x}{x^3-3x+2} = \frac{A_1}{x-1}+\frac{A_2}{(x-1)^2}+\frac{B}{x+2} = \frac{A_1((x-1)(x+2)+A_2(x+2)+B(x-1)^2}{(x-1)^2(x+2)} = \frac{A_1(x^2+x-2)+A_2(x+2)+B(x^2-2x+1)}{(x-1)^2(x+2)}$$

$$= \frac{x^2(A_1+B)+x(A_1+A_2-2B)+(-2A_1+2A_2+B)}{(x-1)^2(x+2)}$$

Igualo las coeficientes de los numeradores:

$$x = x^2(A_1+B)+x(A_1+A_2-2B)+(-2A_1+2A_2+B) \Rightarrow \begin{cases} A_1+B = 0 \\ A_1+A_2-2B = 1 \\ -2A_1+2A_2+B = 0 \end{cases} \Rightarrow \begin{cases} A_1 = 2/9 \\ A_2 = 1/3 \\ B = -2/9 \end{cases}$$

Luego, $\dfrac{x}{x^3-3x+2} = \dfrac{2}{9}\cdot\dfrac{1}{x-1}+\dfrac{1}{3}\cdot\dfrac{1}{(x-1)^2}+\dfrac{-2}{9}\cdot\dfrac{1}{x+2}$, entonces

$$\int\frac{x}{x^3-3x+2}\,dx = \int\frac{2}{9}\cdot\frac{1}{x-1}+\frac{1}{3}\cdot\frac{1}{(x-1)^2}+\frac{-2}{9}\cdot\frac{1}{x+2}\,dx = \int\frac{2}{9}\cdot\frac{1}{x-1}\,dx+\int\frac{1}{3}\cdot\frac{1}{(x-1)^2}\,dx+\int\frac{-2}{9}\cdot\frac{1}{x+2}\,dx$$

$$= \frac{2}{9}\int\frac{1}{x-1}\,dx+\frac{1}{3}\int\frac{1}{(x-1)^2}\,dx-\frac{2}{9}\int\frac{1}{x+2}\,dx = \frac{2}{9}\ln|x-1|+\frac{1}{3}\cdot\left(\frac{-1}{x-1}\right)-\frac{2}{9}\ln|x+2|+c$$

$$= \frac{2}{9}\left(\ln|x-1|-\ln|x+2|\right)-\frac{1}{3(x-1)}+c$$

$$= \frac{2}{9}\ln\left|\frac{x-1}{x+2}\right|-\frac{1}{3(x-1)}+c,\ c\in\mathbb{R}$$

f) $\displaystyle\int \frac{x^3}{(x^2+1)^3}\,dx$

Aplico integración por sustitución, dónde $u=x^2+1 \Rightarrow du=2x\,dx \Rightarrow \dfrac{du}{2x}=dx$

$$\int\frac{x^3}{u^3}\cdot\frac{du}{2x} = \frac{1}{2}\int\frac{x^3}{x}\cdot\frac{du}{u^3} = \frac{1}{2}\int\underbrace{x^2}_{=\,u-1}\cdot\frac{du}{u^3} = \frac{1}{2}\int\frac{u-1}{u^3}\,du = \frac{1}{2}\int\frac{u}{u^3}-\frac{1}{u^3}\,du$$

$$= \frac{1}{2}\left(\int\frac{u}{u^3}\,du-\int\frac{du}{u^3}\right) = \frac{1}{2}\left(\int\frac{du}{u^2}-\int\frac{du}{u^3}\right) = \frac{1}{2}\left(\int u^{-2}\,du-\int u^{-3}\,du\right)$$

$$= \frac{1}{2}\left(\frac{-1}{u}-\left(\frac{-1}{2u^2}\right)\right)+c = \frac{1}{2}\left(\frac{-1}{u}+\frac{1}{2u^2}\right)+c$$

Vuelvo atrás con la sustitución:

$$\int\frac{x^3}{(x^2+1)^3}\,dx = \frac{1}{2}\left(\frac{-1}{x^2+1}+\frac{1}{2(x^2+1)^3}\right)+c,\ c\in\mathbb{R}$$