(10) Encontrar la representación en serie de Taylor, centrada en $a = 0$, de las siguientes funciones. ¿Para qué valores de x vale la representación?

(a) $f(x) = (1 - x)^2$

$f(x) = (1-x)^2 \Rightarrow f(0) = 1$
$f'(x) = -2(1-x) \Rightarrow f'(0) = -2$
$f''(x) = 2 \Rightarrow f''(0) = 2$
$f'''(x) = 0 \Rightarrow f'''(0) = 0$

Lo mismo para todo $f^{(n)}(0)$ con $n \geq 3$.

$\Rightarrow (1-x)^2 = \frac{1}{0!} + \frac{(-2)x}{1!} + \frac{\cancel{2}x^2}{\cancel{2!}} = x^2 - 2x + 1$

La representación en serie de Taylor de la función es $f(x) = x^2 - 2x + 1$

(b) $f(x) = \ln(1 + x)$

$f(x) = \ln(1+x) \Rightarrow f(0) = 0$
$f'(x) = \frac{1}{1+x} \Rightarrow f'(0) = 1$
$f''(x) = \frac{-1}{(1+x)^2} \Rightarrow f''(0) = -1$
$f'''(x) = \frac{2.\cancel{(1+x)}}{(1+x)^{4-1}} = \frac{2}{(1+x)^3} \Rightarrow f'''(0) = 2$
$f^{IV}(x) = \frac{2(-3)\cancel{(1+x)^2}}{(1+x)^{6-2}} = \frac{-6}{(1+x)^4} \Rightarrow f^{IV}(0) = -6$
$f^{V}(x) = \frac{6.4\cancel{(1+x)^3}}{(1+x)^{8-3}} = \frac{24}{(1+x)^5} \Rightarrow f^{V}(0) = 24$

De forma general,

$$f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots = \frac{x}{1!} + \frac{(-x^2)}{2!} + \frac{2x^3}{3!} + \frac{(-6x^4)}{4!} + \frac{24x^5}{5!} + \dots = \sum_{n=0}^{\infty} (-1)^n \frac{n!\, x^{n+1}}{(n+1)!} = \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{x^{n+1}}{n}}_{\text{Serie de Taylor}}$$

(c) $f(x) = \cos(x)$

$f(x) = \cos(x) \Rightarrow f(0) = 1$
$f'(x) = -\operatorname{sen}(x) \Rightarrow f'(0) = 0$
$f''(x) = -\cos(x) \Rightarrow f''(0) = -1$
$f'''(x) = \operatorname{sen}(x) \Rightarrow f'''(0) = 0$
$f^{IV}(x) = \cos(x) \Rightarrow f^{IV}(0) = 1$

Veo un patrón, en general $f^{(2n)}(x) = (-1)^n$, $f^{(2n+1)}(x) = 0 \quad \forall n \geq 0$.

Luego, $f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots = 1 + \frac{(-x^2)}{2!} + \frac{x^4}{4!} + \dots = \underbrace{\sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!}}_{\text{Serie de Taylor}}$

(d) $f(x) = \operatorname{sen}(5x^2)$

$f(x) = \operatorname{sen}(5x^2) \Rightarrow f(0) = 0$

$f'(x) = 10x\cos(5x^2) \Rightarrow f(0) = 0$

$f''(x) = 10\cos(5x^2) + (10x(-\operatorname{sen}(5x^2)10x))$
$= 10(\cos(5x^2) - 10x^2\operatorname{sen}(5x^2)) \Rightarrow f(0) = 10$

Pendiente a resolver.

(e) $f(x) = e^{5x}$

$f(x) = e^{5x} \Rightarrow f(0) = 1$
$f'(x) = 5e^{5x} \Rightarrow f(0) = 5$
$f''(x) = 25e^{5x} \Rightarrow f(0) = 25$
$f^{IV}(x) = 125e^{5x} \Rightarrow f(0) = 125$

Luego, $f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots = 1 + 5x + \frac{25x^2}{2!} + \frac{125x^3}{3!} + \dots = \sum_{n=0}^{\infty} 5^n \frac{x^n}{n!}$

(f) $f(x) = xe^x$

$f(x) = xe^x \Rightarrow f(0) = 0$
$f'(x) = e^x + xe^x = e^x(1+x) \Rightarrow f(0) = 1$
$f''(x) = e^x(1+x) + e^x = e^x(2+x) \Rightarrow f(0) = 2$
$f'''(x) = e^x(2+x) + e^x = e^x(3+x) \Rightarrow f(0) = 3$

Luego, $f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \frac{f'''(0)x^3}{3!} + \dots = \cancel{0} + x + \frac{2x^2}{2!} + \frac{3x^3}{3!} + \dots = \sum_{n=0}^{\infty} n\frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^n}{(n-1)!}$