17) Determinar si las siguientes integrales impropias convergen y en tal caso calcularlas.

a) $\displaystyle\int_0^\infty \frac{1}{\sqrt{s+1}}\,ds$

Primero calculo la integral indefinida:

$$\int \frac{1}{\sqrt{s+1}}\,ds = \int \underbrace{(s+1)}_{u}^{-1/2}\,ds = \int u^{-1/2}\,du$$

$$= \frac{u^{1/2}}{\frac{1}{2}} + C = 2u^{1/2} + C \overset{u=x+1}{=} 2(s+1)^{1/2} + C,\ C\in\mathbb{R}$$

Ahora calculo la integral impropia:

$$\int_0^\infty \frac{1}{\sqrt{s+1}}\,ds = \lim_{t\to\infty}\int_0^t \frac{1}{\sqrt{s+1}}\,ds = \lim_{t\to\infty} 2(s+1)^{1/2}\Big|_0^t = 2\lim_{t\to\infty} (s+1)^{1/2}\Big|_0^t$$

$$= 2\lim_{t\to\infty} (t+1)^{1/2} - (0+1)^{1/2} = 2.(\infty - 0) = \infty$$

Por lo tanto, la integral diverge.

b) $\displaystyle\int_0^2 \frac{1}{(1-y)^{2/3}}\,dy = \lim_{t\to 1^-}\int_0^t \frac{1}{(1-y)^{2/3}}\,dy + \lim_{t\to 1^+}\int_t^2 \frac{1}{(1-y)^{2/3}}\,dy$

Calculo la integral indefinida:

$$\int \frac{1}{(1-y)^{2/3}}\,dy = \int \underbrace{(1-y)}_{u}^{-2/3}\,dy$$

$du = -1dy \Rightarrow -du = dy$

$$= \int u^{-2/3}.(-du) = -\int u^{-2/3}\,du$$

$$= \frac{-u^{1/3}}{\frac{1}{3}} = -3\underbrace{u}_{1-y}^{1/3} = -3(1-y)^{1/3} + C,\ C\in\mathbb{R}$$

Entonces, $\displaystyle\int_0^2 \frac{1}{(1-y)^{2/3}}\,dy = \underbrace{\lim_{t\to 1^-} -3(1-y)^{1/3}\Big|_0^t}_{\text{I}} + \underbrace{\lim_{t\to 1^+} -3(1-y)^{1/3}\Big|_t^2}_{\text{II}}$

I $\displaystyle\lim_{t\to 1^-} -3(1-y)^{1/3}\Big|_0^t = \lim_{t\to 1^-} -3(1-t)^{1/3} - \left(-3(1-0)^{1/3}\right) = 0-(-3) = 3$

+

II $\displaystyle\lim_{t\to 1^+} -3(1-y)^{1/3}\Big|_t^2 = \lim_{t\to 1^+} -3(1-2)^{1/3} - \left(-3(1-t)^{1/3}\right) = 3 - 0 = 3$

||

6

Finalmente, $\displaystyle\int_0^2 \frac{1}{(1-y)^{2/3}}\,dy = 6$, por lo tanto la integral converge.

c) $\displaystyle\int_{-\infty}^0 x e^{-x^2}\,dx$

Primero calculo la integral indefinida:

$$\int x e^{-x^2}\,dx$$

Aplico integración por sustitución, donde $u = -x^2 \Rightarrow du = -2x\,dx$

$\Rightarrow \frac{-du}{2x} = dx$

$$\int x e^{-x^2}\,dx = \int \cancel{x} e^u.\left(\frac{-du}{2\cancel{x}}\right) = -\frac{1}{2}\int e^u\,du = \frac{-1}{2}e^u + C$$

Reemplazo u y obtengo $\displaystyle\int x e^{-x^2}\,dx = \frac{-1}{2}e^{-x^2} + C,\ C\in\mathbb{R}$

Ahora calculo la integral impropia:

$$\int_{-\infty}^0 x e^{-x^2}\,dx = \lim_{t\to-\infty}\int_t^0 x e^{-x^2}\,dx$$

$$= \lim_{t\to-\infty} \frac{-1}{2}e^{-x^2}\Big|_t^0 = \left(\frac{-1}{2}\right).\lim_{t\to-\infty} e^{-x^2}\Big|_t^0$$

$$= \left(\frac{-1}{2}\right)\lim_{t\to-\infty} e^{-0^2} - e^{-t^2}$$

$$= \frac{-1}{2}.(1-0) = \frac{-1}{2}$$

Por lo tanto, la integral converge.

d) $\displaystyle\int_{-1}^7 \frac{dx}{\sqrt[3]{x+1}}$

Primero resuelvo la integral indefinida:

$$\int \frac{dx}{\sqrt[3]{x+1}} = \int \underbrace{(x+1)}_{=u\ \Rightarrow\ du = dx}^{-1/3}\,dx$$

$$= \int u^{-1/3}\,du$$

$$= \frac{u^{2/3}}{\frac{2}{3}} + C = \frac{3}{2}u^{2/3} \overset{u=x+1}{=} \frac{3}{2}(x+1)^{2/3} + C,\ C\in\mathbb{R}.$$

Ahora calculo la integral impropia:

$$\int_{-1}^7 \frac{dx}{\sqrt[3]{x+1}} = \lim_{t\to-1}\int_t^7 \frac{dx}{\sqrt[3]{x+1}}$$

$$= \lim_{t\to-1} \frac{3}{2}(x+1)^{2/3}\Big|_t^7 = \frac{3}{2}\lim_{t\to-1} (x+1)^{2/3}\Big|_t^7$$

$$= \frac{3}{2}\lim_{t\to-1} (7+1)^{2/3} - (t+1)^{2/3}$$

$$= \frac{3}{2}(4-0) = 6$$

Por lo tanto, la integral converge.

e) $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2} = \underbrace{\int_{-\infty}^0 \frac{dx}{1+x^2}}_{=\arctan(x)|_{-\infty}^0} + \underbrace{\int_0^\infty \frac{dx}{1+x^2}}_{=\arctan(x)|_0^\infty}$ (I) (II)

I $\displaystyle\int_{-\infty}^0 \frac{dx}{1+x^2} = \lim_{t\to-\infty} \arctan(x)\Big|_{-\infty}^0$

$$= \lim_{t\to-\infty} \arctan(0) - \arctan(-\infty) \approx 0 - \left(\frac{-3}{2}\right) = \frac{3}{2}$$

+

II $\displaystyle\int_0^\infty \frac{dx}{1+x^2} = \lim_{t\to\infty} \arctan(x)\Big|_0^t$

$$= \lim_{t\to\infty} \arctan(t) - \arctan(0) \approx \frac{3}{2} - 0 = \frac{3}{2}$$

|| en realidad $= \pi$

3

Por lo tanto, $\displaystyle\int_{-\infty}^\infty \frac{dx}{1+x^2}$ converge.

f) $\displaystyle\int_0^1 \ln(x)\,dx$

Resuelvo la integral impropia:

$$\int \ln(x)\,dx = \int 1.\ln(x)\,dx$$

Aplico integración por partes, donde: $u = \ln(x)$ $du = \frac{1}{x}$

$v = x$ $dv = 1$

$$= \ln(x).x - \int x.\frac{1}{x}\,dx$$

$$= x\ln(x) - \int 1\,dx$$

$$= x\ln(x) - x + C,\ C\in\mathbb{R}$$

Ahora resuelvo la integral impropia:

$$\int_0^1 \ln(x)\,dx = \lim_{t\to 0^+}\int_t^1 \ln(x)\,dx$$

$$= \lim_{t\to 0^+} x\ln(x) - x\Big|_t^1$$

$$= \lim_{t\to 0^+} (1.\ln(1) - 1) - (t\ln(t) - t) = -1 - 0 = -1$$

Por lo tanto, $\displaystyle\int_0^1 \ln(x)\,dx$ converge.