18) Determinar si cada una de las siguientes integrales impropias converge o no.

a) $\int_4^{\infty} \frac{1}{\sqrt{s}-1}\, ds$

Primero calculo la integral indefinida:

$\int \frac{1}{\sqrt{s}-1}\, ds$ Aplico integración por sustitución, donde:

$u = \sqrt{s}-1 \Rightarrow du = \frac{1}{2\sqrt{s}}\, ds \Rightarrow 2\sqrt{s}\, du = ds$

(*) $2\sqrt{s} = 2(u+1)$

$$\int \frac{1}{\sqrt{s}-1}\, ds = \int \frac{1}{u} \cdot 2(u+1)\, du = 2\int \frac{u+1}{u}\, du = 2\int \frac{u}{u} + \frac{1}{u}\, du$$

$$= 2\int 1 + \frac{1}{u}\, du = 2\left(u + \ln|u|\right) + C$$

$$\overset{u=\sqrt{s}-1}{=} 2\left(\sqrt{s}-1 + \ln|\sqrt{s}-1|\right) + C,\ C \in \mathbb{R}$$

Ahora resuelvo la integral impropia:

$$\int_4^{\infty} \frac{1}{\sqrt{s}-1}\, ds = \lim_{t\to\infty} \int_4^t \frac{1}{\sqrt{s}-1}\, ds$$

$$= \lim_{t\to\infty} 2\left(\sqrt{s}-1+\ln|\sqrt{s}-1|\right)\Big|_4^t = 2\lim_{t\to\infty}\left(\sqrt{s}-1+\ln|\sqrt{s}-1|\right)\Big|_4^t$$

$$= 2\lim_{t\to\infty}\left(\sqrt{t}-1+\ln|\sqrt{t}-1|\right) - \left(\sqrt{4}-1+\ln|\sqrt{4}-1|\right)$$

$$= 2(\infty - 1) = \infty$$

Por lo tanto, $\int_4^{\infty} \frac{1}{\sqrt{s}-1}\, ds$ diverge.

b) $\int_0^{\infty} e^{-x}\cos(x)\, dx$

Primero calculo la integral indefinida:

$\int e^{-x}\cos(x)\, dx$

Aplico integración por partes, dónde $u = \cos(x)$ $du = -\operatorname{sen}(x)$

$v = -e^{-x}$ $dv = e^{-x}$

$$\int e^{-x}\cos(x)\, dx = \cos(x)(-e^{-x}) - \int -e^{-x}\cdot(-\operatorname{sen}(x))\, dx$$

$$= -e^{-x}\cos(x) - \int e^{-x}\operatorname{sen}(x)\, dx$$

Vuelvo a aplicar integración por partes: $u = \operatorname{sen}(x)$ $du = \cos(x)$

$v = -e^{-x}$ $dv = e^{-x}$

$$\overbrace{\int e^{-x}\cos(x)\, dx}^{(*)} = -e^{-x}\cos(x) - \left(\operatorname{sen}(x)(-e^{-x}) - \int -e^{-x}\cos(x)\, dx\right)$$

$$= -e^{-x}\cos(x) + e^{-x}\operatorname{sen}(x) - \underbrace{\int e^{-x}\cos(x)\, dx}_{=(*)}$$

$$2\int e^{-x}\cos(x)\, dx = e^{-x}\left(\operatorname{sen}(x) - \cos(x)\right) + c$$

$$\int e^{-x}\cos(x)\, dx = \frac{1}{2}e^{-x}\left(\operatorname{sen}(x)-\cos(x)\right) + c,\ c \in \mathbb{R}$$

Ahora calculo la integral impropia:

$$\int_0^{\infty} e^{-x}\cos(x)\, dx = \lim_{t\to\infty}\int_0^t e^{-x}\cos(x)\, dx = \lim_{t\to\infty}\frac{1}{2}e^{-x}\left(\operatorname{sen}(x)-\cos(x)\right)\Big|_0^t$$

$$= \frac{1}{2}\lim_{t\to\infty} e^{-x}\left(\operatorname{sen}(x)-\cos(x)\right)\Big|_0^t$$

$$= \frac{1}{2}\lim_{t\to\infty}\left(e^{-t}(\operatorname{sen}(t)-\cos(t))\right) - \left(e^{-0}(\operatorname{sen}(0)-\cos(0))\right)$$

$$= \frac{1}{2}\left(0-(-1)\right) = \frac{1}{2}$$

Por lo tanto, $\int_0^{\infty} e^{-x}\cos(x)\, dx$ Converge.

c) $\int_0^4 \frac{dx}{(x-3)^{2/3}}$

Primero resuelvo la integral indefinida:

$$\int \frac{dx}{(x-3)^{2/3}} = \int \underbrace{(x-3)}_{u}{}^{-2/3}\, dx = \int u^{-2/3}\, du = \frac{u^{1/3}}{\frac{1}{3}} + C \overset{u=x-3}{=} 3(x-3)^{1/3} + C,\ C \in \mathbb{R}$$

Ahora calculo la integral impropia:

$$\int_0^4 \frac{dx}{(x-3)^{2/3}} = \underbrace{\lim_{t\to 3^-}\int_0^t \frac{dx}{(x-3)^{2/3}}}_{\mathrm{I}} + \underbrace{\lim_{t\to 3^+}\int_t^4 \frac{dx}{(x-3)^{2/3}}}_{\mathrm{II}}$$

I $\lim_{t\to 3^-}\int_0^t \frac{dx}{(x-3)^{2/3}} = \lim_{t\to 3^-} 3(x-3)^{1/3}\Big|_0^t = 3\lim_{t\to 3^-}(x-3)^{1/3}\Big|_0^t$

$$= 3\lim_{t\to 3^-}(t-3)^{1/3} - (0-3)^{1/3} = 3(0-\sqrt[3]{-3}) = -3\sqrt[3]{-3} = 3\sqrt[3]{3}$$

II $\lim_{t\to 3^+}\int_t^4 \frac{dx}{(x-3)^{2/3}} = \lim_{t\to 3^+} 3(x-3)^{1/3}\Big|_t^4 = 3\lim_{t\to 3^+}(x-3)^{1/3}\Big|_t^4$

$$= 3\lim_{t\to 3^+}(t-3)^{1/3} - (4-3)^{1/3} = 3(0-1) = 3$$

Luego, $\int_0^4 \frac{dx}{(x-3)^{2/3}} = 3\sqrt[3]{3} + 3$, por lo tanto Converge.

d) $\int_0^1 x\ln(x)\, dx$

Primero calculo la integral indefinida:

$\int x\ln(x)\, dx$

Aplico integración por partes, dónde: $u = \ln(x)$ $du = \frac{1}{x}$

$v = \frac{x^2}{2}$ $dv = x$

$$\int x\ln(x)\, dx = \frac{x^2}{2}\ln(x) - \int \frac{x^{\cancel{2}}}{2}\cdot\frac{1}{\cancel{x}}\, dx = \frac{x^2}{2}\ln(x) - \frac{1}{2}\int x\, dx$$

$$= \frac{x^2}{2}\ln(x) - \frac{1}{2}\cdot\frac{x^2}{2} + c = \frac{x^2}{2}\left(\ln(x) - \frac{1}{2}\right)$$

Ahora calculo la integral impropia:

$$\int_0^1 x\ln(x)\, dx = \lim_{t\to 0^+}\int_t^1 x\ln(x)\, dx = \lim_{t\to 0^+}\frac{x^2}{2}\left(\ln(x)-\frac{1}{2}\right)\Big|_t^1$$

$$= \lim_{t\to 0^+}\left(\frac{1^2}{2}\left(\ln(1)-\frac{1}{2}\right)\right) - \left(\frac{t^2}{2}\left(\ln(t)-\frac{1}{2}\right)\right)$$

$$= \frac{1}{2}\cdot\left(\frac{-1}{2}\right) - 0\left(-\infty-\frac{1}{2}\right) = \frac{-1}{4}$$

Por lo tanto, $\int_0^1 x\ln(x)\, dx$ converge.

e) $\int_0^4 \frac{dx}{x^2-x-2}$

Primero calculo la integral indefinida:

$$\int \frac{dx}{x^2-x-2} = \int \frac{dx}{(x-2)(x+1)}$$

Luego, $\frac{1}{(x-2)(x+1)} = \frac{A_1}{x-2} + \frac{A_2}{x+1} = \frac{A_1(x+1)+A_2(x-2)}{(x-2)(x+1)}$

Igualo los numeradores:

$$1 = x(A_1+A_2) + (A_1 - 2A_2) \Rightarrow \begin{cases} A_1 + A_2 = 0 \\ A_1 - 2A_2 = 1 \end{cases} \Rightarrow \begin{cases} A_1 = \frac{1}{3} \\ A_2 = \frac{-1}{3} \end{cases}$$

Entonces, $\frac{1}{(x-2)(x+1)} = \frac{\frac{1}{3}}{x-2} + \frac{\frac{-1}{3}}{x+1} = \frac{1}{3(x-2)} - \frac{1}{3(x+1)}$

$$\int \frac{dx}{x^2-x-2} = \int \frac{1}{3(x-2)} - \frac{1}{3(x+1)}\, dx = \frac{1}{3}\int \frac{dx}{\underbrace{x-2}_{u}} - \frac{1}{3}\int \frac{dx}{\underbrace{x+1}_{v}}$$

$$= \frac{1}{3}\left(\int \frac{du}{u} - \int \frac{dv}{v}\right) = \frac{1}{3}\cdot\left(\ln|u| - \ln|v|\right) + c = \frac{1}{3}\ln\left|\frac{u}{v}\right| + c$$

$$= \frac{1}{3}\ln\left|\frac{x-2}{x+1}\right| + c,\ c \in \mathbb{R}$$

Ahora resuelvo la integral impropia:

$$\int_0^4 \frac{dx}{x^2-x-2} = \underbrace{\lim_{t\to 2^-}\int_0^t \frac{dx}{x^2-x-2}}_{\mathrm{I}} + \underbrace{\lim_{t\to 2^+}\int_t^4 \frac{dx}{x^2-x-2}}_{\mathrm{II}}$$

I $\lim_{t\to 2^-}\int_0^t \frac{dx}{x^2-x-2} = \lim_{t\to 2^-}\frac{1}{3}\ln\left|\frac{x-2}{x+1}\right|\Big|_0^t = \frac{1}{3}\lim_{t\to 2^-}\ln\left|\frac{x-2}{x+1}\right|\Big|_0^t$

$$= \frac{1}{3}\lim_{t\to 2^-}\ln\left|\frac{t-2}{t+1}\right| - \ln\left|\frac{0-2}{0+1}\right| = \frac{1}{3}\Big(\underbrace{\ln 0 - \ln 2}_{\ln\left(\frac{0}{2}\right)\ \text{sin definir.}}\Big)$$

Como I diverge, $\int \frac{dx}{x^2-x-2}$ diverge.