Chapter 08.04
Runge-Kutta 4th Order Method for Ordinary Differential Equations-More Examples
Chemical Engineering

Example 1

The concentration of salt x in a home made soap maker is given as a function of time by

$$\frac{dx}{dt} = 37.5 - 3.5x$$

At the initial time, $t = 0$, the salt concentration in the tank is 50 g/L Using Runge-Kutta 4th order method and a step size of, $h = 1.5\ \text{min}$, what is the salt concentration after 3 minutes?

Solution

$$\frac{dx}{dt} = 37.5 - 3.5x$$

$$f(t,x) = 37.5 - 3.5x$$

$$x_{i+1} = x_i + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

For $i = 0$, $t_0 = 0$, $x_0 = 50$

$$\begin{aligned}
k_1 &= f(t_0, x_0) \\
&= f(0,50) \\
&= 37.5 - 3.5(50) \\
&= -137.5
\end{aligned}$$

$$\begin{aligned}
k_2 &= f\left(t_0 + \frac{1}{2}h, x_0 + \frac{1}{2}k_1 h\right) \\
&= f\left(0 + \frac{1}{2}1.5, 50 + \frac{1}{2}(-137.5)1.5\right) \\
&= f(0.75, -53.125) \\
&= 37.5 - 3.5(-53.125) \\
&= 223.44
\end{aligned}$$

$$\begin{aligned}
k_3 &= f\left(t_0 + \frac{1}{2}h, x_0 + \frac{1}{2}k_2 h\right) \\
&= f\left(0 + \frac{1}{2}1.5, 50 + \frac{1}{2}(223.44)1.5\right) \\
&= f(0.75, 217.58) \\
&= 37.5 - 3.5(217.58)
\end{aligned}$$

$$
\begin{aligned}
&= -724.02 \\
k_4 &= f(t_0 + h, x_0 + k_3 h) \\
&= f(0 + 1.5, 50 + (-724.03)1.5) \\
&= f(1.5, -1036.0) \\
&\ 37.5 \quad 3.5(\quad 1036.0) \\
&= 3663.6 \\
x_1 &= x_0 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h \\
&= 50 + \frac{1}{6}(-137.5 + 2(223.44) + 2(-724.02) + (3663.6))1.5 \\
&= 50 + \frac{1}{6}(2525.0)1.5 \\
&= 681.24 \text{ g/L}
\end{aligned}
$$

x_1 is the approximate concentration of salt at

$$t = t_1 = t_0 + h = 0 + 1.5 = 1.5$$

$$x(1.5) \approx x_1 = 681.24 \text{ g/L}$$

For $i = 1,\ t_1 = 1.5,\ x_1 = 681.24$

$$
\begin{aligned}
k_1 &= f(t_1, x_1) \\
&= f(1.5, 681.24) \\
&= 37.5 - 3.5(681.24) \\
&= -2346.8 \\
k_2 &= f\left(t_1 + \frac{1}{2}h, x_1 + \frac{1}{2}k_1 h\right) \\
&= f\left(1.5 + \frac{1}{2}1.5, 681.24 + \frac{1}{2}(-2346.8)1.5\right) \\
&= f(2.25, -1078.9) \\
&= 37.5 - 3.5(-1078.9) \\
&= 3813.6 \\
k_3 &= f\left(t_1 + \frac{1}{2}h, x_1 + \frac{1}{2}k_2 h\right) \\
&= f\left(1.5 + \frac{1}{2}1.5, 681.24 + \frac{1}{2}(3813.6)1.5\right) \\
&= f(2.25, 3541.4) \\
&= 37.5 - 3.5(3541.4) \\
&= -12358
\end{aligned}
$$

$$
\begin{aligned}
k_4 &= f(t_1 + h, x_1 + k_3 h) \\
&= f(1.5 + 1.5, 681.24 + (-12358)1.5) \\
&= f(3, -17855) \\
&= 37.5 - 3.5(-17855) \\
&= 62530
\end{aligned}
$$

$$x_2 = x_1 + \frac{1}{6}(k_1 + 2k_2 + 2k_3 + k_4)h$$

$$= 681.24 + \frac{1}{6}(-2346.8 + 2(3813.6) + 2(-12358) + 62530)1.5$$

$$= 681.24 + \frac{1}{6}(43096)1.5$$

$$= 11455 \text{ g/L}$$

x_2 is the approximate concentration of salt at

$$t_2 = t_1 + h = 1.5 + 1.5 = 3 \text{min}$$

$$x(3) \approx x_2 = 11455 \text{ g/L}$$

The exact solution of the ordinary differential equation is given by

$$x(t) = 10.714 + 39.286e^{-3.5t}$$

The solution to this nonlinear equation at $t = 3\text{min}$ is

$$x(3) = 10.715 \text{ g/L}$$

Figure 1 compares the exact solution with the numerical solution using Runge-Kutta 4$^{\text{th}}$ order method using different step sizes.

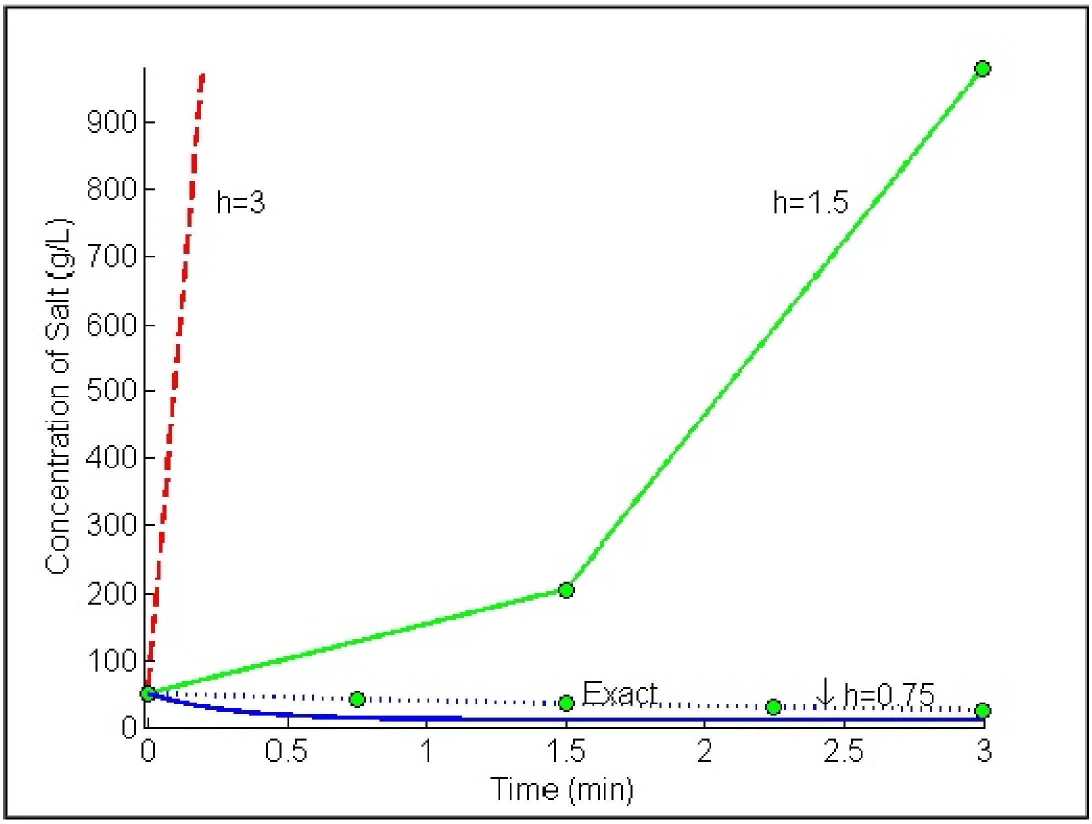

Figure 1 Comparison of Runge-Kutta 4$^{\text{th}}$ order method with exact solution for different step sizes.

Table 1 and Figure 2 show the effect of step size on the value of the calculated temperature at $t = 3$ min.

Table 1 Value of concentration of salt at 3 minutes for different step sizes.

Step size, h	$x(3)$	E_t	$\|\epsilon_t\|\%$
3	14120	−14109	131680
1.5	11455	−11444	106800
0.75	25.559	−14.843	138.53
0.375	10.717	−0.0014969	0.013969
0.1875	10.715	−0.00031657	0.0029544

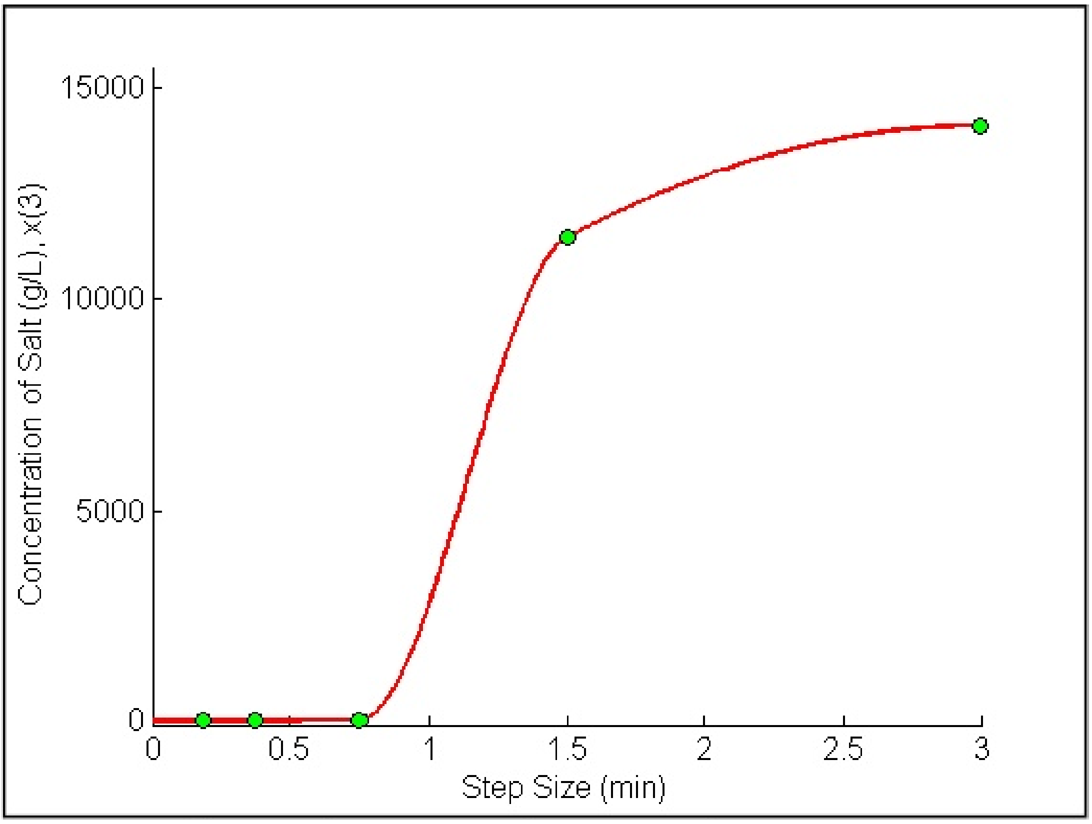

Figure 2 Effect of step size in Runge-Kutta 4th order method.

In Figure 3, we are comparing the exact results with Euler's method (Runge-Kutta 1st order method), Heun's method (Runge-Kutta 2nd order method) and Runge-Kutta 4th order method.

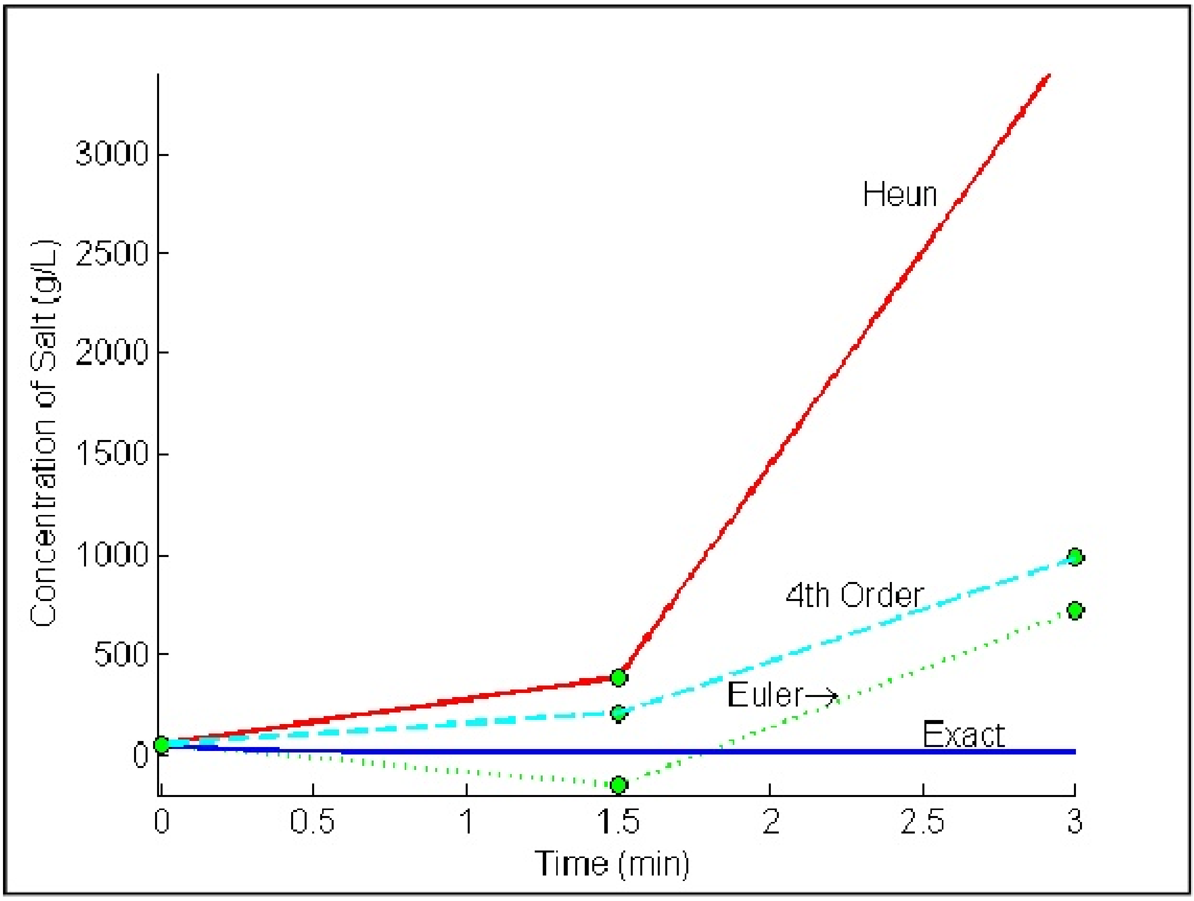

Figure 3 Comparison of Runge-Kutta methods of 1st, 2nd, and 4th order.