12_String_Problem

March 28, 2016

1 12 Linear Algebra

1.1 Motivating problem: Two masses on three strings

Two masses M_1 and M_2 are hung from a horizontal rod with length L in such a way that a rope of length L_1 connects the left end of the rod to M_1, a rope of length L_2 connects M_1 and M_2, and a rope of length L_3 connects M_2 to the right end of the rod. The system is at rest (in equilibrium under gravity).

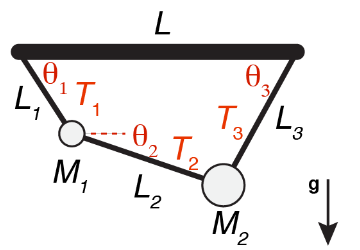


Figure 1: Schematic of the 1 rod/2 masses/3 strings problem.

Find the angles that the ropes make with the rod and the tension forces in the ropes.

1.2 Theoretical background

Treat $\sin\theta_i$ and $\cos\theta_i$ together with T_i, $1 \le i \le 3$, as unknowns that have to simultaneously fulfill the nine equations

$$-T_1\cos\theta_1 + T_2\cos\theta_2 = 0 \tag{1}$$
$$T_1\sin\theta_1 - T_2\sin\theta_2 - W_1 = 0 \tag{2}$$
$$-T_2\cos\theta_2 + T_3\cos\theta_3 = 0 \tag{3}$$
$$T_2\sin\theta_2 + T_3\sin\theta_3 - W_2 = 0 \tag{4}$$
$$L_1\cos\theta_1 + L_2\cos\theta_2 + L_3\cos\theta_3 - L = 0 \tag{5}$$
$$-L_1\sin\theta_1 - L_2\sin\theta_2 + L_3\sin\theta_3 = 0 \tag{6}$$
$$\sin^2\theta_1 + \cos^2\theta_1 - 1 = 0 \tag{7}$$
$$\sin^2\theta_2 + \cos^2\theta_2 - 1 = 0 \tag{8}$$
$$\sin^2\theta_3 + \cos^2\theta_3 - 1 = 0 \tag{9}$$

Consider the nine equations a vector function $\mathbf{f}$ that takes a 9-vector $\mathbf{x}$ of the unknowns as argument:

$$\mathbf{f}(\mathbf{x}) = 0 \tag{10}$$

$$\mathbf{x} = \begin{pmatrix} x_0 \\ x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \\ x_8 \end{pmatrix} = \begin{pmatrix} \sin\theta_1 \\ \sin\theta_2 \\ \sin\theta_3 \\ \cos\theta_1 \\ \cos\theta_2 \\ \cos\theta_3 \\ T_1 \\ T_2 \\ T_3 \end{pmatrix} \tag{11}$$

$$\mathbf{L} = \begin{pmatrix} L \\ L_1 \\ L_2 \\ L_3 \end{pmatrix}, \quad \mathbf{W} = \begin{pmatrix} W_1 \\ W_2 \end{pmatrix} \tag{12}$$

Solve with generalized Newton-Raphson:

$$\mathsf{J}(\mathbf{x})\Delta\mathbf{x} = -\mathbf{f}(\mathbf{x})$$

and

$$\mathbf{x} \leftarrow \mathbf{x} + \Delta\mathbf{x}.$$

1.3 Problem setup

Set the problem parameters and the objective function $\mathbf{f}(\mathbf{x})$

```
In [1]: import numpy as np

        # problem parameters
        W = np.array([10, 20])
        L = np.array([8, 3, 4, 4])

        def f_2masses(x, L, W):
            return np.array([
                    -x[6]*x[3] + x[7]*x[4],
                     x[6]*x[0] - x[7]*x[1] - W[0],
                    -x[7]*x[4] + x[8]*x[5],
                     x[7]*x[1] + x[8]*x[2] - W[1],
                     L[1]*x[3] + L[2]*x[4] + L[3]*x[5] - L[0],
                    -L[1]*x[0] - L[2]*x[1] + L[3]*x[2],
                    x[0]**2 + x[3]**2 - 1,
                    x[1]**2 + x[4]**2 - 1,
                    x[2]**2 + x[5]**2 - 1,
                ])

        def fLW(x):
            return f_2masses(x, L, W)
```

1.3.1 Initial values

Guess some initial values (they don't have to fullfil the equations!):

```
In [2]: # initial parameters
        theta0 = np.deg2rad([45, 45, 90])
        T0 = np.array([1, 1, 2])
        x0 = np.concatenate([np.sin(theta0), np.cos(theta0), T0])
```

```
In [3]: x0
```

```
Out[3]: array([  7.07106781e-01,   7.07106781e-01,   1.00000000e+00,
                 7.07106781e-01,   7.07106781e-01,   6.12323400e-17,
                 1.00000000e+00,   1.00000000e+00,   2.00000000e+00])
```

```
In [4]: f_2masses(x0, L, W)
```

```
Out[4]: array([  0.        , -10.        ,  -0.70710678, -17.29289322,
                -3.05025253,  -0.94974747,   0.        ,   0.        ,   0.        ])
```

1.3.2 Visualization

Plot the positions of the 2 masses and the 3 strings for any solution vector **x**:

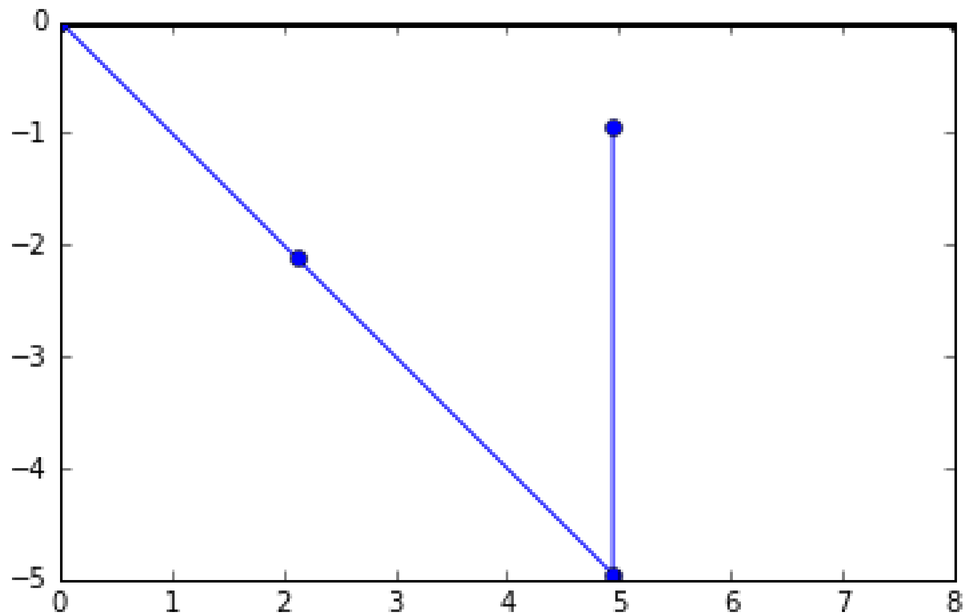
```
In [5]: import matplotlib
        import matplotlib.pyplot as plt
        %matplotlib inline
```

```
In [6]: def plot_2masses(x, L, W, **kwargs):
            """Plot 2 mass/3 string problem for parameter vector x and parameters L and W"""

            kwargs.setdefault('linestyle', '-')
            kwargs.setdefault('marker', 'o')
            kwargs.setdefault('linewidth', 1)

            r0 = np.array([0, 0])
            r1 = r0 + np.array([L[0], 0])
            rod = np.transpose([r0, r1])

            L1 = r0 + np.array([L[1]*x[3], -L[1]*x[0]])
            L2 = L1 + np.array([L[2]*x[4], -L[2]*x[1]])
            L3 = L2 + np.array([L[3]*x[5], L[3]*x[2]])
            strings = np.transpose([r0, L1, L2, L3])

            ax = plt.subplot(111)
            ax.plot(rod[0], rod[1], color="black", marker="d", linewidth=4)
            ax.plot(strings[0], strings[1], **kwargs)
            ax.set_aspect(1)
            return ax
```

What does the initial guess look like?

```
In [7]: plot_2masses(x0, L, W)
```

```
Out[7]: <matplotlib.axes._subplots.AxesSubplot at 0x7fc9701a86d8>
```



1.4 Jacobian

Write a function `Jacobian(f, x, h=1e-5)` that computes the Jacobian matrix numerically (use the central difference algorithm).

```
In [8]: def Jacobian(f, x, h=1e-5):
            """df_i/dx_j with central difference (f(x+h/2)-f(x-h/2))/h"""
            J = np.zeros((len(f(x)), len(x)), dtype=np.float64)
            hvec = np.zeros_like(x, dtype=np.float64)
            for j in range(len(x)):
                hvec *= 0
                hvec[j] = 0.5*h
                J[:, j] = (f(x + hvec) - f(x - hvec))/h
            return J
```

Test Jacobian on

$$\mathbf{f}(\mathbf{x}) = \begin{pmatrix} x_0^2 - x_1 \\ x_0 \end{pmatrix}$$

with analytical result

$$\mathsf{J} = \frac{\partial f_i}{\partial x_j} = \begin{pmatrix} 2x_0 & -1 \\ 1 & 0 \end{pmatrix}$$

```
In [9]: def ftest(x):
            return np.array([
                    x[0]**2 - x[1],
                    x[0]
                ])
        x0test = np.array([1, 0])
        J = Jacobian(ftest, x0test)
        print(J)
```

```
[[ 2. -1.]
 [ 1.  0.]]
```

Test that it also works for our starting vector:

```
In [10]: Jacobian(fLW, x0)
```

```
Out[10]: array([[ 0.        ,  0.        ,  0.        , -1.        ,  1.        ,
                  0.        , -0.70710678,  0.70710678,  0.        ],
                [ 1.        , -1.        ,  0.        ,  0.        ,  0.        ,
                  0.        ,  0.70710678, -0.70710678,  0.        ],
                [ 0.        ,  0.        ,  0.        ,  0.        , -1.        ,
                  2.        ,  0.        , -0.70710678,  0.        ],
                [ 0.        ,  1.        ,  2.        ,  0.        ,  0.        ,
                  0.        ,  0.        ,  0.70710678,  1.        ],
                [ 0.        ,  0.        ,  0.        ,  3.        ,  4.        ,
                  4.        ,  0.        ,  0.        ,  0.        ],
                [-3.        , -4.        ,  4.        ,  0.        ,  0.        ,
                  0.        ,  0.        ,  0.        ,  0.        ],
                [ 1.41421356,  0.        ,  0.        ,  1.41421356,  0.        ,
                  0.        ,  0.        ,  0.        ,  0.        ],
                [ 0.        ,  1.41421356,  0.        ,  0.        ,  1.41421356,
                  0.        ,  0.        ,  0.        ,  0.        ],
                [ 0.        ,  0.        ,  2.        ,  0.        ,  0.        ,
                  0.        ,  0.        ,  0.        ,  0.        ]])
```

1.5 n-D Newton-Raphson Root Finding

Write a function `newton_raphson(f, x, Nmax=100, tol=1e-8, h=1e-5)` to find a root for a vector function `f(x)=0`. (See also 11 Root-finding by trial-and-error and the 1D Newton-Raphson algorithm in 11_Root_finding.ipynb.) As a convergence criterion we demand that the length of the vector `f(x)` (the norm — see `np.linalg.norm`) be less than the tolerance.

```
In [11]: def newton_raphson(f, x, Nmax=100, tol=1e-8, h=1e-5):
             """n-D Newton-Raphson: solves f(x) = 0.

             Iterate until |f(x)| < tol or nmax steps.
             """
             x = x.copy()
             for istep in range(Nmax):
                 fx = f(x)
                 if np.linalg.norm(fx) < tol:
                     break
                 J = Jacobian(f, x, h=h)
                 Delta_x = np.linalg.solve(J, -fx)
                 x += Delta_x
             else:
                 print("Newton-Raphson: no root found after {0} iterations (eps={1}); "
                       "best guess is {2} with error {3}".format(Nmax, tol, x, fx))
             return x
```

1.5.1 Solve 2 masses/3 strings string problem

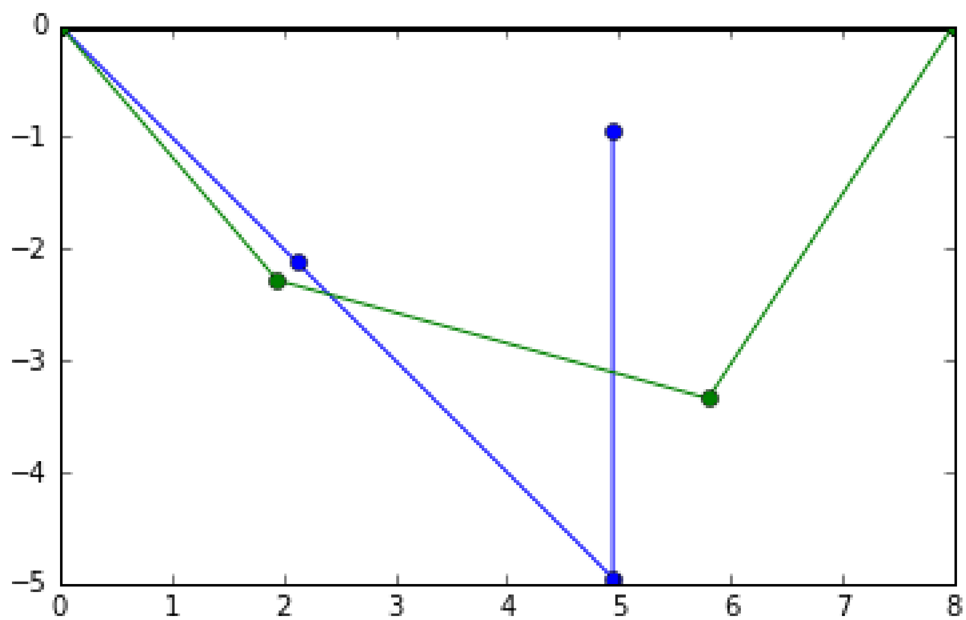
Solution

```
In [12]: x = newton_raphson(fLW, x0)
         print(x0)
         print(x)
```

```
[  7.07106781e-01   7.07106781e-01   1.00000000e+00   7.07106781e-01
   7.07106781e-01   6.12323400e-17   1.00000000e+00   1.00000000e+00
   2.00000000e+00]
[  0.76100269   0.26495381   0.83570583   0.64874872   0.9642611
   0.54917735  17.16020978  11.54527968  20.27152804]
```

Plot the starting configuration and the solution:

```
In [13]: plot_2masses(x0, L, W)
         plot_2masses(x, L, W)

Out[13]: <matplotlib.axes._subplots.AxesSubplot at 0x7fc970195a20>
```



Pretty-print the solution (angles in degrees):

```
In [14]: def pretty_print(x):
             theta = np.rad2deg(np.arcsin(x[0:3]))
             tensions = x[6:]
             print("theta1 = {0[0]:.1f} \t theta2 = {0[1]:.1f} \t theta3 = {0[2]:.1f}".format(theta))
             print("T1     = {0[0]:.1f} \t T2     = {0[1]:.1f} \t T3     = {0[2]:.1f}".format(tensions)

In [15]: print("Starting values")
         pretty_print(x0)
         print()
         print("Solution")
         pretty_print(x)
```

```
Starting values
theta1 = 45.0          theta2 = 45.0          theta3 = 90.0
```

```
T1     = 1.0          T2     = 1.0          T3     = 2.0

Solution
theta1 = 49.6          theta2 = 15.4          theta3 = 56.7
T1     = 17.2          T2     = 11.5          T3     = 20.3
```

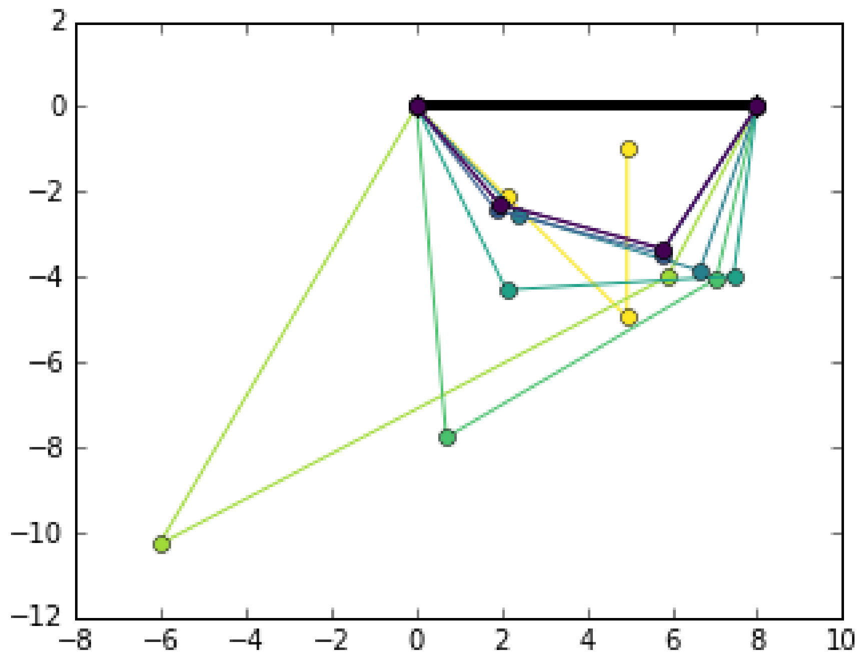
Show intermediate steps Create a new function `newton_raphson_intermediates()` based on `newton_raphson()` that returns all trial `x` values including the last one.

```
In [16]: def newton_raphson_intermediates(f, x, Nmax=100, tol=1e-8, h=1e-5):
             """n-D Newton-Raphson: solves f(x) = 0.

             Iterate until |f(x)| < tol or nmax steps.

             Returns all intermediates.
             """
             intermediates = []
             x = x.copy()
             for istep in range(Nmax):
                 fx = f(x)
                 if np.linalg.norm(fx) < tol:
                     break
                 J = Jacobian(f, x, h=h)
                 Delta_x = np.linalg.solve(J, -fx)
                 intermediates.append(x.copy())
                 x += Delta_x
             else:
                 print("Newton-Raphson: no root found after {0} iterations (eps={1}); "
                     "best guess is {2} with error {3}".format(Nmax, tol, x, fx))
             return np.array(intermediates)
```

Visualize the intermediate configurations:

```
In [17]: x_series = newton_raphson_intermediates(fLW, x0)
```

```
In [18]: ax = plt.subplot(111)
         ax.set_prop_cycle("color", [plt.cm.viridis_r(i) for i in np.linspace(0, 1, len(x_series))])
         for x in x_series:
             plot_2masses(x, L, W)
```



It's convenient to turn the above plotting code into a function that we can reuse:

```
In [19]: def plot_series(x_series, L, W):
             """Plot all N masses/strings solution vectors in x_series (N, 9) array"""
             ax = plt.subplot(111)
             ax.set_prop_cycle("color", [plt.cm.viridis_r(i) for i in np.linspace(0, 1, len(x_series))]
             for x in x_series:
                 plot_2masses(x, L, W)
             return ax
```

1.6 Additional work

Try different masses, e.g. M1 = M2 = 10, or M1= 0 , M2 = 10.

1.6.1 M1 = M2 = 10

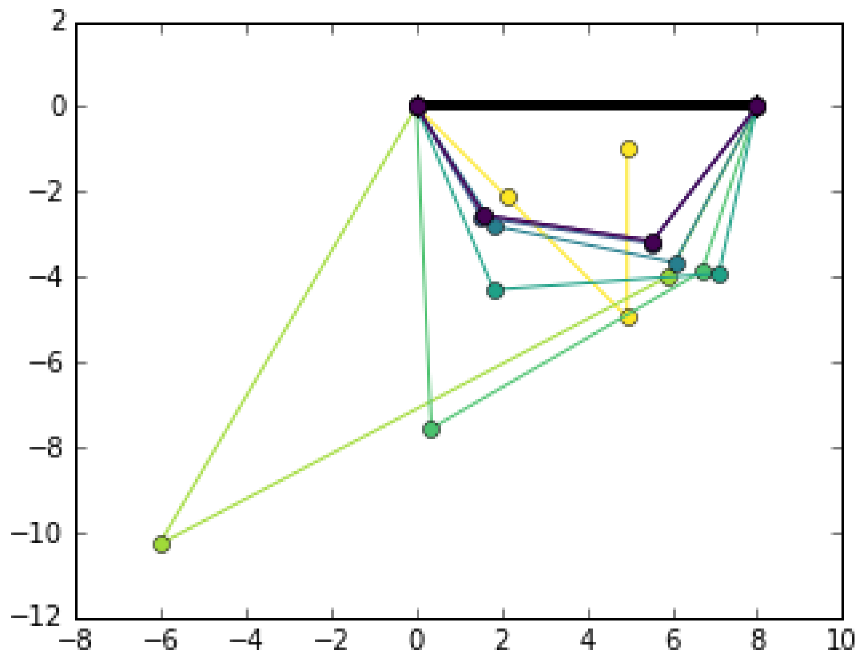
```
In [20]: W_2 = np.array([10, 10])
         def fLW_2(x):
             return f_2masses(x, L, W_2)
```

```
In [21]: x_series_2 = newton_raphson_intermediates(fLW_2, x0)
         pretty_print(x_series_2[-1])
```

```
theta1 = 57.9           theta2 = 8.8           theta3 = 52.1
T1     = 13.1           T2     = 7.0           T3     = 11.3
```

```
In [22]: plot_series(x_series_2, L, W_2)
```

```
Out[22]: <matplotlib.axes._subplots.AxesSubplot at 0x7fc970002dd8>
```



1.6.2 M1 = 0, M2 = 10

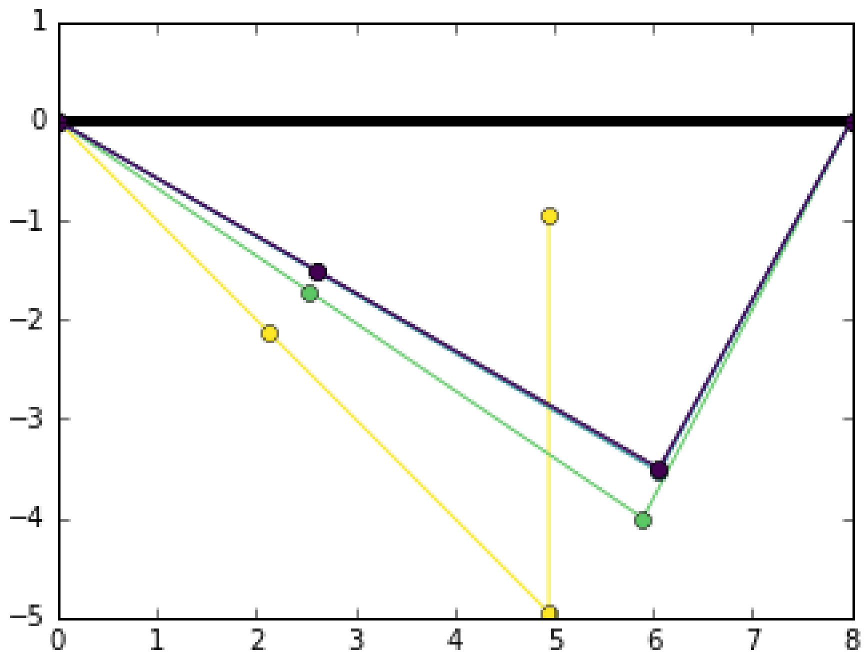
In [23]:
```
W_3 = np.array([0, 10])
def fLW_3(x):
    return f_2masses(x, L, W_3)
```

In [24]:
```
x_series_3 = newton_raphson_intermediates(fLW_3, x0)
pretty_print(x_series_3[-1])
```

```
theta1 = 30.0           theta2 = 30.0           theta3 = 61.0
T1     = 4.8           T2     = 4.8           T3     = 8.7
```

In [25]:
```
plot_series(x_series_3, L, W_3)
```

Out[25]: <matplotlib.axes._subplots.AxesSubplot at 0x7fc96ff8bf98>




```
In [ ]:
```