Minium Fuel Optimal Control

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We have a linear dynamical system with states $x(t) \in \mathbf{R}^n$, $t = 0, 1, \ldots, N-1$, and actuator signal $u(t) \in \mathbf{R}$. The dynamics of the system is given by the recursion

$$x(t+1) = Ax(t) + bu(t) \tag{1}$$

where $A \in \mathbf{R}^{n\times n}$ and $b \in \mathbf{R}^n$ are given. We assume $x(0) = 0$. We need to choose the inputs $u(0), \ldots, u(N-1)$ so as to minimize the total fuel consumed, which is given by

$$F = \sum_{i=0}^{N-1} f(u(t)) \tag{2}$$

subject to the constraint that $x(N) = x_{\text{des}}$, where N is the desired time horizon and x_{des} is the desired final state. The function $f : \mathbf{R} \to \mathbf{R}$ is the fuel use map which gives the fuel usage as a function of the actuator input and is given by

$$f = \begin{cases} |a| & |a| \leq 1 \\ 2|a| - 1 & |a| > 1 \end{cases} \tag{3}$$

From the initial condition and the recursion in (1), it follows that

$$\begin{aligned} x(1) &= bu(0) \\ x(2) &= Abu(0) + bu(1) \\ x(3) &= A^2bu(0) + Abu(1) + bu(2) \\ &\vdots \\ x_{\text{des}} = x(N) &= A^{N-1}bu(0) + \cdots + b \\ &= \begin{bmatrix} A^{N-1}b & A^{N-2}b & \ldots & b \end{bmatrix} u. \end{aligned}$$

To express the cost function as a *linear* cost function, we use the epigraph trick. Minimizing the function f (in (3)) is the same as minimizing a linear objective in t given by

$$\begin{aligned} \text{minimize} \quad & t \\ \text{subject to} \quad & a \leq t \\ & -a \leq t \\ & 2a - 1 \leq t \\ & -2a - 1 \leq t \end{aligned}$$

Thus, we may express the cost function (and the minimization problem) as:

$$\begin{array}{ll} \text{minimize} & \mathbf{1}^T t \\ \text{subject to} & Du = x_{\text{des}} \\ & u \preceq t \\ & -u \preceq t \\ & 2u - 1 \preceq t \\ & -2u - 1 \preceq t \end{array} \tag{4}$$

where $D = \begin{bmatrix} A^{N-1}b & A^{N-2}b & \dots & b \end{bmatrix}$.

For the example in A3.17 with the following data

$$A = \begin{bmatrix} -1 & 0.4 & 0.8 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \qquad b = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix}, \qquad x_{\text{des}} = \begin{bmatrix} 7 \\ 2 \\ -6 \end{bmatrix}, \qquad N = 30$$

the optimal actuator input (solved both directly and as the above LP) is shown in the figure below.