Home Assignment - 1

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November 14, 2016

1 Introduction

2 Theory

Let $\mathbf{Y}_k$ represent the n known observations of spatial data and $\mathbf{Y}_u$ represent the m data points that needs to be estimated. For ordinary Kriging ($\mu = \mathbf{1}\beta$), the optimal predictions are given by

$$\hat{\mathbf{Y}}_u = \mathbf{1}_u\hat{\beta} + \Sigma_{uk}\Sigma_{kk}^{-1}\left(\mathbf{Y}_k - \mathbf{1}_k\hat{\beta}\right)$$

$$\hat{\beta} = \left(\mathbf{1}_k^T\Sigma_{kk}^{-1}\mathbf{1}_k\right)^{-1}\mathbf{1}_k^T\Sigma_{kk}^{-1}\mathbf{Y}_k,$$

where

$$\begin{bmatrix}\mathbf{Y}_k \\ \mathbf{Y}_u\end{bmatrix} \in \mathbf{N}\left(\begin{bmatrix}\mathbf{1}_k\beta \\ \mathbf{1}_u\beta\end{bmatrix}, \begin{bmatrix}\Sigma_{kk} & \Sigma_{ku} \\ \Sigma_{uk} & \Sigma_{uu}\end{bmatrix}\right).$$

Firstly, we show that the predictions are linear in the observations, i.e., $\hat{\mathbf{Y}}_u = \boldsymbol{\lambda}^T\mathbf{Y}_k$ for some $\boldsymbol{\lambda}$. We may re-write $\hat{\beta}$ as

$$\hat{\beta} = \boldsymbol{\gamma}^T\mathbf{Y}_k$$

$$\boldsymbol{\gamma} = \left(\left(\mathbf{1}_k^T\Sigma_{kk}^{-1}\mathbf{1}_k\right)^{-1}\mathbf{1}_k^T\Sigma_{kk}^{-1}\right)^T.$$

We note that the vector $\boldsymbol{\gamma}^T$ is of dimensions $1 \times n$, and therefore, $\hat{\beta}$ is a scalar. Substituting the above in the expression for $\hat{\mathbf{Y}}_u$, we obtain

$$\begin{aligned}\hat{\mathbf{Y}}_u &= \mathbf{1}_u\boldsymbol{\gamma}^T\mathbf{Y}_k + \Sigma_{uk}\Sigma_{kk}^{-1}\mathbf{Y}_k - \Sigma_{uk}\Sigma_{kk}^{-1}\mathbf{1}_k\boldsymbol{\gamma}^T\mathbf{Y}_k \\ &= \left(\mathbf{1}_u\boldsymbol{\gamma}^T + \Sigma_{uk}\Sigma_{kk}^{-1} - \Sigma_{uk}\Sigma_{kk}^{-1}\mathbf{1}_k\boldsymbol{\gamma}^T\right)\mathbf{Y}_k \\ &= \boldsymbol{\lambda}^T\mathbf{Y}_k,\end{aligned}$$

where $\boldsymbol{\lambda} = \left(\mathbf{1}_u\boldsymbol{\gamma}^T + \Sigma_{uk}\Sigma_{kk}^{-1} - \Sigma_{uk}\Sigma_{kk}^{-1}\mathbf{1}_k\boldsymbol{\gamma}^T\right)^T$.

Secondly, we show that the predictions are unbiased, i.e., $\mathbf{E}(\hat{\mathbf{Y}}_u) = \mathbf{E}(\mathbf{Y}_u)$. The expected value of $\hat{\beta}$ is given by

$$\begin{aligned}\mathbf{E}\left(\hat{\beta}\right) &= \left(\mathbf{1}_k^T\Sigma_{kk}^{-1}\mathbf{1}_k\right)^{-1}\mathbf{1}_k^T\Sigma_{kk}^{-1}\,\mathbf{E}(\mathbf{Y}_k) \\ &= \left(\mathbf{1}_k^T\Sigma_{kk}^{-1}\mathbf{1}_k\right)^{-1}\mathbf{1}_k^T\Sigma_{kk}^{-1}\,\mathbf{1}_k\beta \\ &= \beta.\end{aligned}$$

Therefore,

$$\begin{aligned}
\mathbf{E}(\hat{\mathbf{Y}}_u) &= \mathbf{1}_u \mathbf{E}\left(\hat{\beta}\right) + \Sigma_{uk}\Sigma_{kk}^{-1}\left(\mathbf{E}(\mathbf{Y}_k) - \mathbf{1}_k\mathbf{E}\left(\hat{\beta}\right)\right) \\
&= \mathbf{1}_u\beta + \Sigma_{uk}\Sigma_{kk}^{-1}\left(\mathbf{1}_k\beta - \mathbf{1}_k\beta\right) \\
&= \mathbf{1}_u\beta.
\end{aligned}$$

3 Data

4 Least-Squares

5 Universal Kriging

6 Conclusions