

Homework #1: Prove the Gradients of the Sigmoid and Tanh Functions

In this exercise, you are tasked with deriving the gradients of two common activation functions in deep learning: the sigmoid function and the tanh function. Prove each gradient and simplify your answers as much as possible.

1. Sigmoid Function Gradient

The sigmoid function is defined as:

$$\sigma(x) = \frac{1}{1 + e^{-x}}$$

Task

1. Differentiate $\sigma(x)$ with respect to x .
2. Show that the derivative of $\sigma(x)$ can be expressed in terms of $\sigma(x)$ itself:

$$\frac{d\sigma(x)}{dx} = \sigma(x)(1 - \sigma(x))$$

Hint: To simplify your work, start by applying the chain rule to the expression $\sigma(x) = (1 + e^{-x})^{-1}$.

2. Tanh Function Gradient

The hyperbolic tangent function is defined as:

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

Task

1. Differentiate $\tanh(x)$ with respect to x .
2. Show that the derivative of $\tanh(x)$ can be expressed as:

$$\frac{d \tanh(x)}{dx} = 1 - \tanh(x)^2$$

Hint: You may want to express $\tanh(x)$ in terms of $\sigma(x)$, noting that $\tanh(x) = 2\sigma(2x) - 1$, as an alternative approach.

Submission

1. Write out the derivations step-by-step.
2. Simplify each result as much as possible.

3. Confirm that your results mat4. Complete the derivations in a Jupyter notebook. Use markdown cells containing LaTeX code.
4. Submit your notebook by uploading it to either Google Colab or GitHub.
5. Share the link to your notebook with idan.tobis@gmail.com.ch the expected gradient forms given above.

Solution 1: Sigmoid Function Gradient

$$\begin{aligned}
 \frac{d\sigma(x)}{dx} &= (-1) * (1 + e^{-x})^{-2} e^{-x} * (-1) = \frac{e^{-x}}{(1 + e^{-x})^{-2}} = \frac{e^{-x}}{(1 + e^{-x})(1 + e^{-x})} \\
 &= \left(\frac{1}{1 + e^{-x}}\right) * \left(\frac{e^{-x}}{1 + e^{-x}} - 1 + 1\right) = \sigma(x) * \left(\frac{e^{-x}}{1 + e^{-x}} - \frac{1 + e^{-x}}{1 + e^{-x}} + 1\right) \\
 &= \sigma(x) * \left(\frac{e^{-x} - 1 - e^{-x}}{1 + e^{-x}} + 1\right) = \sigma(x) * \left(\frac{-1}{1 + e^{-x}} + 1\right) = \sigma(x) * (1 - \sigma(x))
 \end{aligned}$$

Solution 2. Tanh Function Gradient

$$\begin{aligned}
 \frac{d \tanh(x)}{dx} &= \frac{\frac{d(e^x - e^{-x})}{dx} * (e^x + e^{-x}) - \frac{d(e^x + e^{-x})}{dx} * (e^x - e^{-x})}{(e^x + e^{-x})^2} \\
 &= \frac{(e^x + e^{-x}) * (e^x + e^{-x}) - (e^x - e^{-x}) * (e^x - e^{-x})}{(e^x + e^{-x})^2} \\
 &= \frac{(e^x + e^{-x})^2}{(e^x + e^{-x})^2} - \frac{(e^x - e^{-x})^2}{(e^x + e^{-x})^2} \\
 &= 1 - \left(\frac{e^x - e^{-x}}{e^x + e^{-x}}\right)^2 = 1 - (\tanh(x))^2
 \end{aligned}$$