Analytical Methods

Boundary-value problems (II)

Part A – Eigenvalue problems

(general considerations)

- In the ***preceding*** we focused on ***inhomogeneous*** boundary-vale problems.

- In these problems - the *associated homogeneous* problem had only a ***trivial solution***.

Example

$$y'' + y = 1$$

$$y(0) = 0 \quad , y(\pi/2) = 0$$

$$\longrightarrow \boxed{\varphi(x) = 1 - cos(x) - sin(x)}$$

- The *associated homogeneous* problem:

$$y'' + y = 0$$

$$y(0) = 0 \quad , y(\pi/2) = 0$$

$$\longrightarrow \boxed{\varphi(x) = 0}$$

Eigenvalue problems

- In the following we focus on ***homogeneous*** boundary-vale problems.

A simple example

$$y'' + \lambda \cdot y = 0$$

$$y(0) = 0 \quad , y(1) = 0$$

$\boxed{\lambda = 1} \rightarrow \boxed{\varphi(x) = 0}$ *Trivial solution*

$\boxed{\lambda_n = (n \cdot \pi)^2} \rightarrow \boxed{\varphi_n(x) = \sin(n \cdot \pi \cdot x)}$ ***Non-trivial solutions***

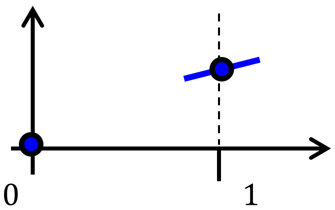
Eigenvalues ***Eigenfunctions***

- These problems will have a ***free parameter*** λ.

- For certain λ values → ***non-trivial solutions*** **may** be obtained.

What λ values will produce non-trivial solutions ?

Example 1- (Regular B.C)



$$y'' + \lambda \cdot y = 0 \qquad (0 \leq x \leq 1)$$

$$y(0) = 0 \quad , \quad y(1) + k \cdot y'(1) = 0 \qquad (k = 1)$$

- The possibilities for the λ values (***all must be tested!***):

$\boxed{\lambda = 0}$ $\downarrow$ $\varphi(x) = A + B \cdot x$

Nontrivial solution cannot satisfy the B.C.
(Show at home)

$\boxed{\lambda = \mu^2 > 0}$ $\downarrow$ $\varphi(x) = A \cdot \sin(\mu \cdot x) + B \cdot \cos(\mu \cdot x)$

$\boxed{\lambda = -\mu^2 < 0}$ $(\mu > 0)$ $\downarrow$ $\varphi(x) = A \cdot \sinh(\mu \cdot x) + B \cdot \cosh(\mu \cdot x)$

Non-trivial solution cannot satisfy the B.C.
(Show at home)

Case $\lambda = \mu^2 > 0$

$$y'' + \mu^2 \cdot y = 0 \qquad (0 \le x \le 1)$$

$$\downarrow$$

$$\varphi(x) = A \cdot \sin(\mu \cdot x) + B \cdot \cos(\mu \cdot x)$$

- Imposing the B.C.:

$$\varphi(0) = 0 \longrightarrow \boxed{B = 0} \longrightarrow \boxed{\varphi(x) = A \cdot \sin(\mu \cdot x)}$$

$$\varphi(1) + k \cdot \varphi'(1) = 0 \longrightarrow A \cdot [\underbrace{\sin(\mu) + k \cdot \mu \cdot \cos(\mu)}_{= 0\ (?)}] = 0$$

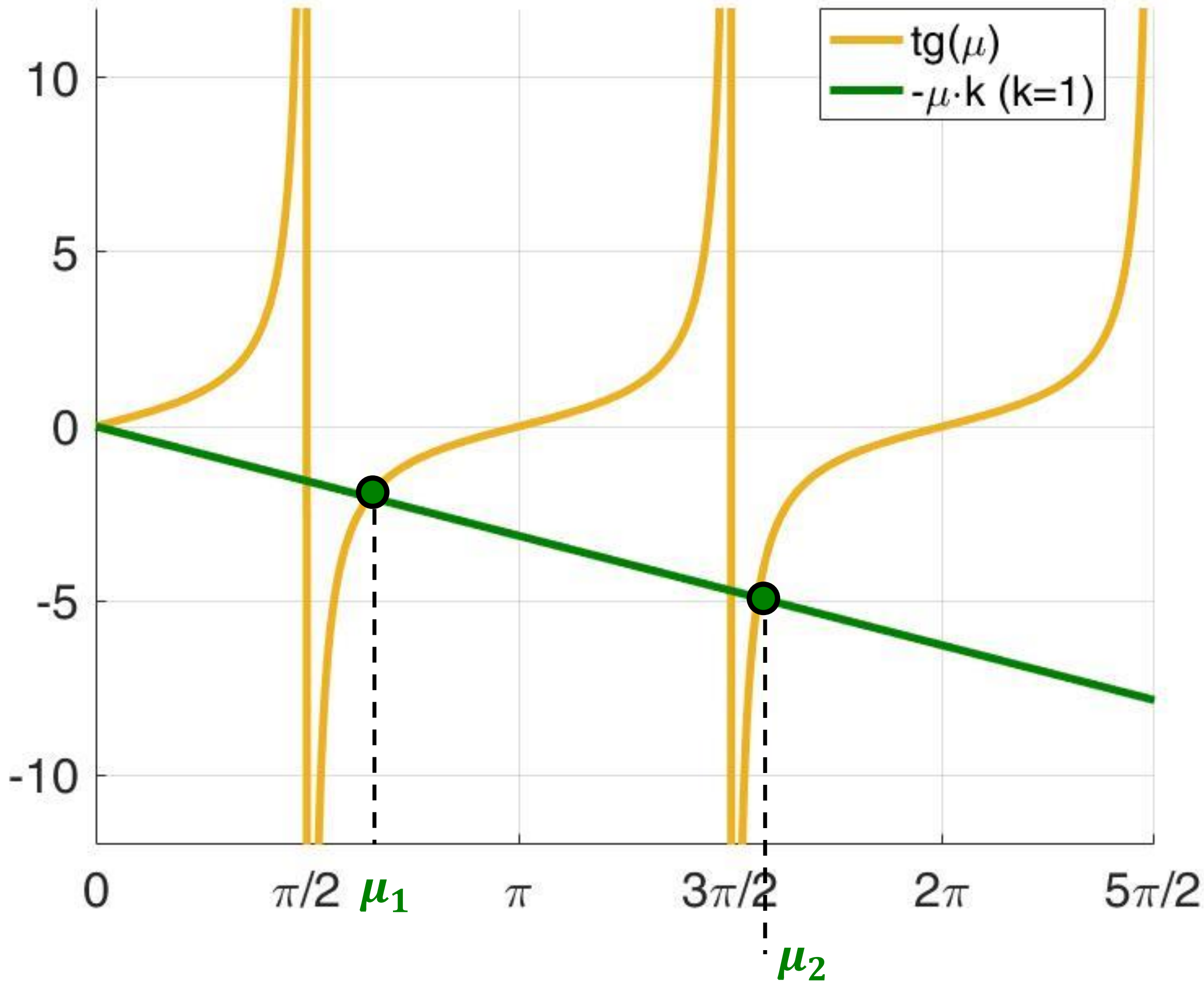
$$\downarrow$$

$$\boxed{tan(\mu) = -\mu \cdot k} \quad \checkmark$$

Non-trivial solution that satisfies the B.C.

$$tan(\mu) = -\mu \cdot k$$

$$k = 1 \rightarrow \ \mu_1 = 2.03\,, \mu_2 = 4.93 \ \ldots$$



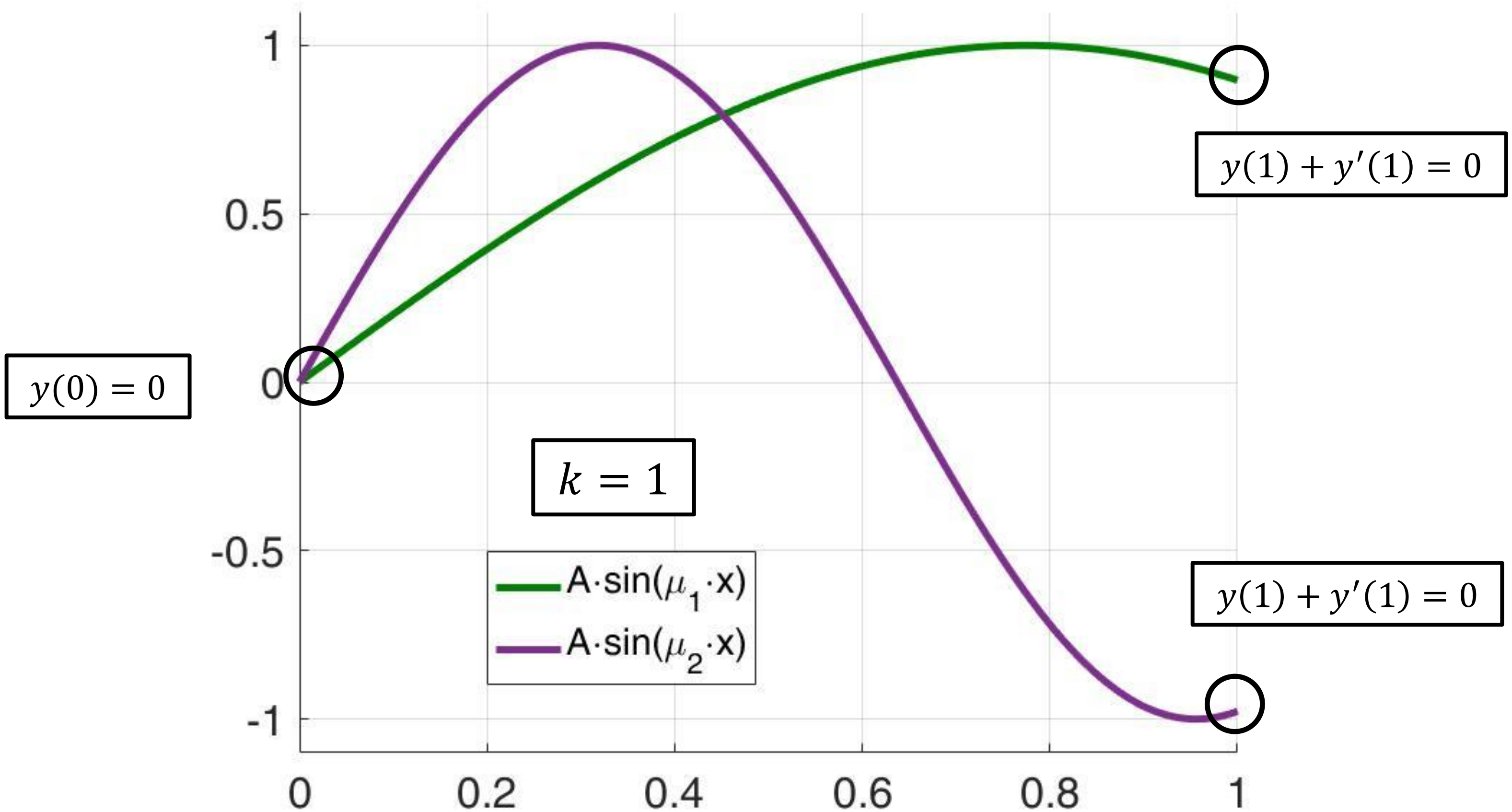
The eigenvalues:

$$\boxed{\lambda_n = \mu_n^2}$$

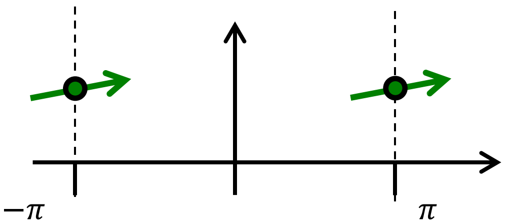
$k = 1 \rightarrow \mu_1 = 2.03 , \mu_2 = 4.93 \; \ldots$

The eigenfunction:

$$\boxed{\varphi_n(x) = \sin(\mu_n \cdot x)}$$



Example 2 (Periodic B.C.)



$$y'' + \lambda \cdot y = 0 \qquad (-\pi \le x \le \pi)$$

$$y(-\pi) = y(\pi) \quad , \quad y'(-\pi) = y'(\pi)$$

- The possibilities for the λ values (***all must be tested!***):

$$\boxed{\lambda = 0} \qquad \boxed{\lambda = \mu^2 > 0} \qquad \boxed{\lambda = -\mu^2 < 0} \qquad (\mu > 0)$$

$$\downarrow \qquad \downarrow \qquad \downarrow$$

$$\varphi(x) = A + B \cdot x \qquad \varphi(x) = A \cdot \sin(\mu \cdot x) + B \cdot \cos(\mu \cdot x) \qquad \varphi(x) = A \cdot \sinh(\mu \cdot x) + B \cdot \cosh(\mu \cdot x)$$

Non-trivial solution cannot satisfy the B.C.
(Show at home)

Case I - $\lambda = 0$

$$y'' = 0 \qquad (0 \le x \le 1)$$

$$\downarrow$$

$$\varphi(x) = A + B \cdot x$$

- Imposing the B.C.:

$$\varphi(-\pi) = \varphi(\pi) \longrightarrow B = 0 \longrightarrow \varphi(x) = A$$

$$\varphi'(-\pi) = \varphi'(\pi) \longrightarrow 0 = 0 \quad \checkmark$$

Non-trivial solution that satisfies the B.C.

The eigenvalues: $\boxed{\lambda = 0}$

The eigenfunction: $\boxed{\phi(x) = 1}$

***Case II* - $\lambda = \mu^2 > 0$**

$$y'' + \mu^2 \cdot y = 0 \qquad (0 \le x \le 1)$$

$$\downarrow$$

$$\varphi(x) = A \cdot \sin(\mu \cdot x) + B \cdot \cos(\mu \cdot x)$$

- Imposing the B.C.:

$$\varphi(-\pi) = \varphi(\pi) \longrightarrow \boxed{2 \cdot A \cdot \underbrace{\sin(\mu \cdot \pi)}_{=0} = 0}$$

$$\varphi'(-\pi) = \varphi'(\pi) \longrightarrow \boxed{2 \cdot B \cdot \overbrace{\sin(\mu \cdot \pi)}^{=0} = 0}$$

$$\downarrow$$

$$\boxed{\mu_1 = 1\,, \mu_2 = 2, \dots, \mu_n = n, \dots} \quad \checkmark$$

Non-trivial solution that satisfies the B.C.

The eigenvalues:

$$\lambda_n = \mu_n^2 = n^2$$

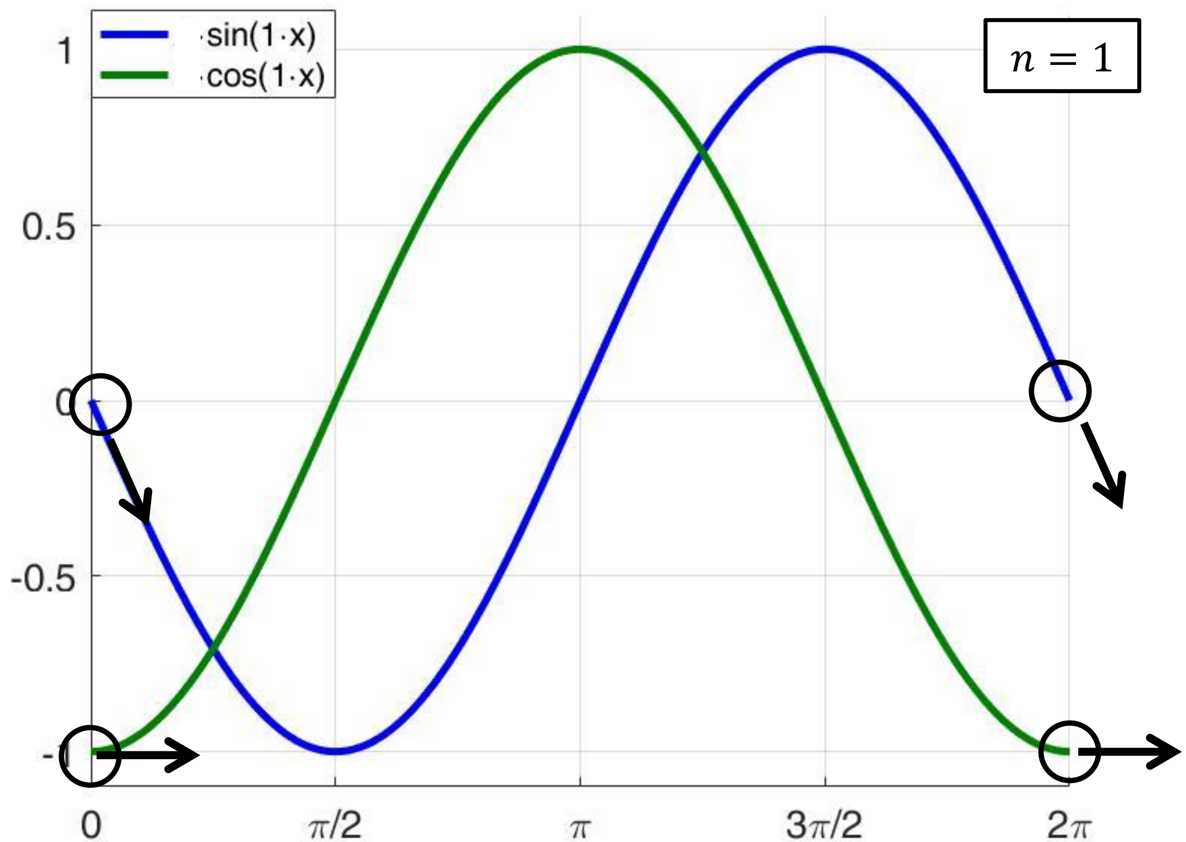
Two eigenfunctions for each eigenvalue (!)

The eigenfunction:

$$\varphi_n(x) = \sin(n \cdot x) \qquad \psi_n(x) = \cos(n \cdot x)$$

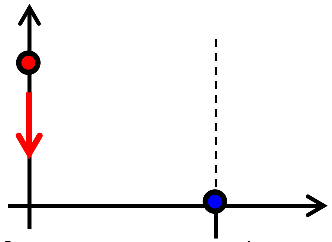
$$y(-\pi) = y(\pi)$$

$$y'(-\pi) = y'(\pi)$$



Example 3 - Bessel's equation (Singular B.C.)

Friedrich Wilhelm Bessel



$$x^2 \cdot y'' + x \cdot y' + (\lambda \cdot x^2 - \nu^2) \cdot y = 0 \qquad (\nu \geq 0)$$

$$(0 \leq x \leq 1)$$

$$\lim_{x \to 0} |y(x)| < \infty$$

$$y(1) = 0$$

➢ The possibilities for the λ values:

$\boxed{\lambda = -\mu^2 < 0}$ ✗

$\boxed{\lambda = \mu^2 = 0}$ ✗

$\boxed{\lambda = \mu^2 > 0}$ $(\mu > 0)$ ✓

__Cannot__ satisfy the B.C. ! (seen in a minute…)

__Can__ satisfy the B.C. ! (see in a minute…)

Case I - $\lambda = 0$

$$x^2 \cdot y'' + x \cdot y' - \nu^2 \cdot y = 0$$

Euler equation

$$\downarrow$$

$$r \cdot (r-1) + r - \nu^2 = 0$$

$r_1 = \nu$
$r_2 = -\nu$

$$\downarrow$$

$$\varphi(x) = A \cdot x^{\nu} + B \cdot x^{-\nu}$$

- Imposing the B.C.:

$$\lim_{x \to 0} |\varphi(x)| < \infty \longrightarrow \boxed{B = 0} \longrightarrow \boxed{\varphi(x) = A \cdot x^{\nu}}$$

$$\varphi(1) = 0 \longrightarrow \boxed{A = 0}$$

Non-trivial solution cannot satisfy the B.C.
(Show at home)

Case II - $\lambda = -\mu^2 < 0$

$$x^2 \cdot y'' + x \cdot y' - (\mu^2 \cdot x^2 + \nu^2) \cdot y = 0$$

- Introducing the transformation:

$$x^2 \cdot \mu^2 \quad \rightarrow \quad \tilde{x}^{\,2} \qquad\qquad x \cdot \frac{dy(x)}{dx} \rightarrow \tilde{x} \cdot \frac{dy(\tilde{x})}{d\tilde{x}} \qquad\qquad x^2 \cdot \frac{d^2y(x)}{dx^2} \rightarrow \tilde{x}^{\,2} \cdot \frac{d^2y(\tilde{x})}{d\tilde{x}^2}$$

$$\downarrow$$

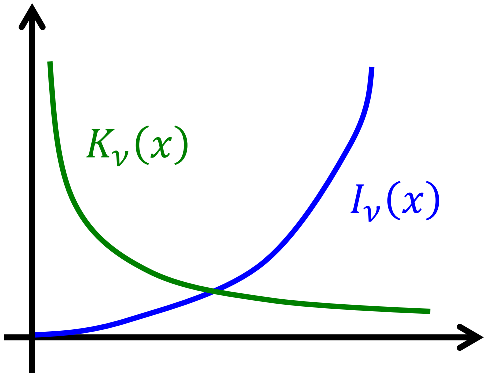
$$\tilde{x}^2 \cdot y'' + \tilde{x} \cdot y' - (\tilde{x} + \nu^2) \cdot y = 0$$

Modified Bessel equation

$$\downarrow$$

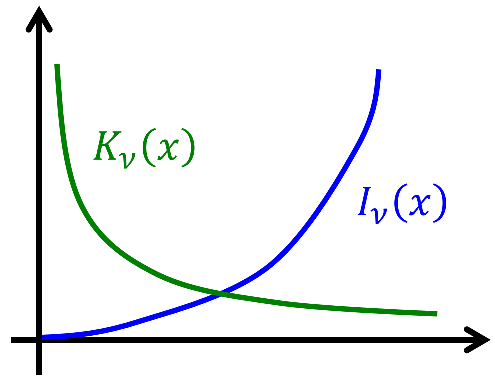
Modified Bessel functions of the first and second kinds of order ν

$$\varphi(\tilde{x}) = A \cdot I_\nu(\tilde{x}) + B \cdot K_\nu(\tilde{x})$$



Modified Bessel functions of the ***first and second kinds*** of *order* ν

$$\varphi(\tilde{x}) = A \cdot I_\nu(\tilde{x}) + B \cdot K_\nu(\tilde{x})$$



- Imposing the B.C.:

$$\lim_{x\to 0}|\varphi(x)| = \lim_{\tilde{x}\to 0}|\varphi(\tilde{x})| < \infty \longrightarrow \boxed{B = 0} \longrightarrow \boxed{\varphi(\tilde{x}) = A \cdot I_\nu(\tilde{x})}$$

$$\boxed{x \cdot \mu \to \tilde{x}}$$

$$\varphi(x = 1) = \varphi(\tilde{x} = 1 \cdot \mu) = 0 \longrightarrow A \cdot \underbrace{I_\nu(\mu)}_{\neq 0} = 0 \longrightarrow \boxed{A = 0}$$

Non-trivial solution cannot satisfy the B.C.
(Show at home)

Case III - $\lambda = \mu^2 > 0$

$$x^2 \cdot y'' + x \cdot y' + (\mu^2 \cdot x^2 - \nu^2) \cdot y = 0$$

$$\underbrace{x^2 \cdot y''}_{\tilde{x}^2 \cdot y''} \qquad \underbrace{x \cdot y'}_{\tilde{x} \cdot y'} \qquad \underbrace{\mu^2 \cdot x^2}_{\boxed{\tilde{x} = x \cdot \mu}}$$

$$\downarrow$$

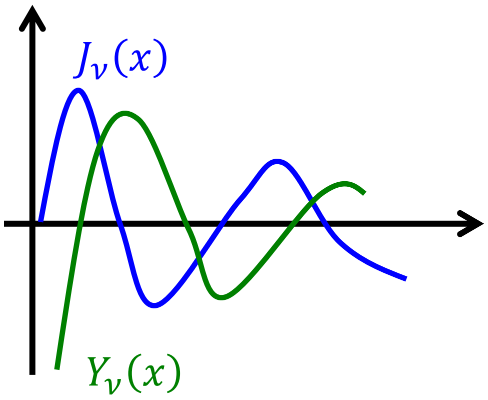
$$\tilde{x}^2 \cdot y'' + \tilde{x} \cdot y' + (\tilde{x} - \nu^2) \cdot y = 0$$

Bessel equation

$$\downarrow$$

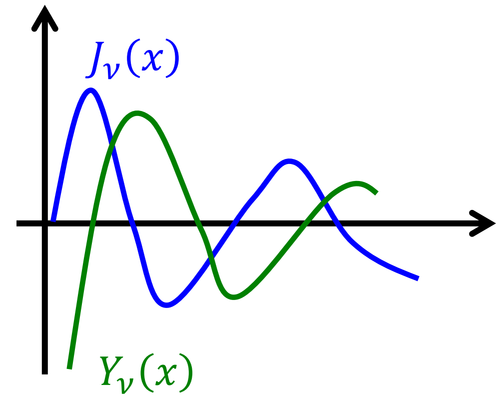
*Bessel functions of the **first and second kinds** of order* ν

$$\varphi(\tilde{x}) = A \cdot J_\nu(\tilde{x}) + B \cdot Y_\nu(\tilde{x})$$



Bessel functions of the ***first and second kinds*** *of order* ν

$$\varphi(\tilde{x}) = A \cdot J_\nu(\tilde{x}) + B \cdot Y_\nu(\tilde{x})$$



- Imposing the B.C.:

$$\lim_{x\to 0}|\varphi(x)| = \lim_{\tilde{x}\to 0}|\varphi(\tilde{x})| < \infty \longrightarrow \boxed{B = 0} \longrightarrow \boxed{\varphi(\tilde{x}) = A \cdot J_\nu(\tilde{x})}$$

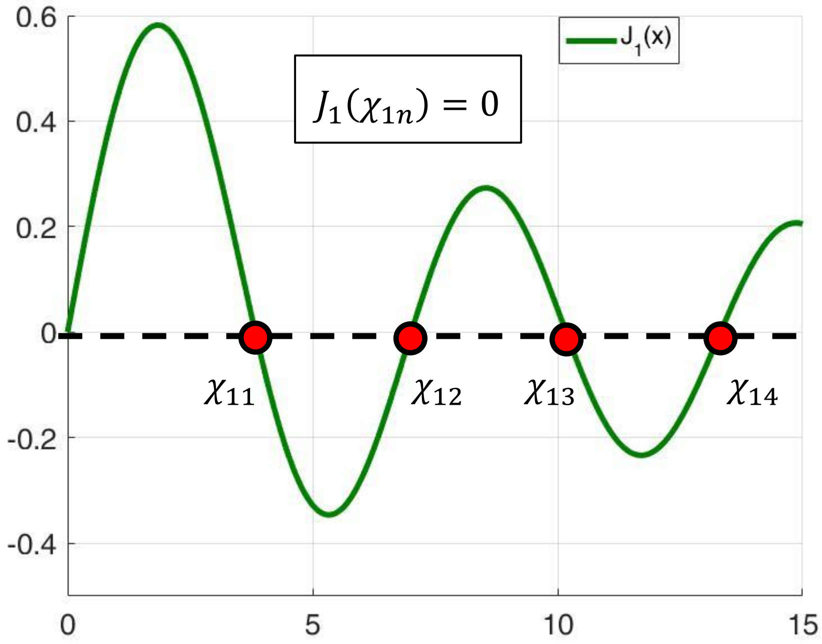
$$\boxed{x \cdot \mu \to \tilde{x}}$$

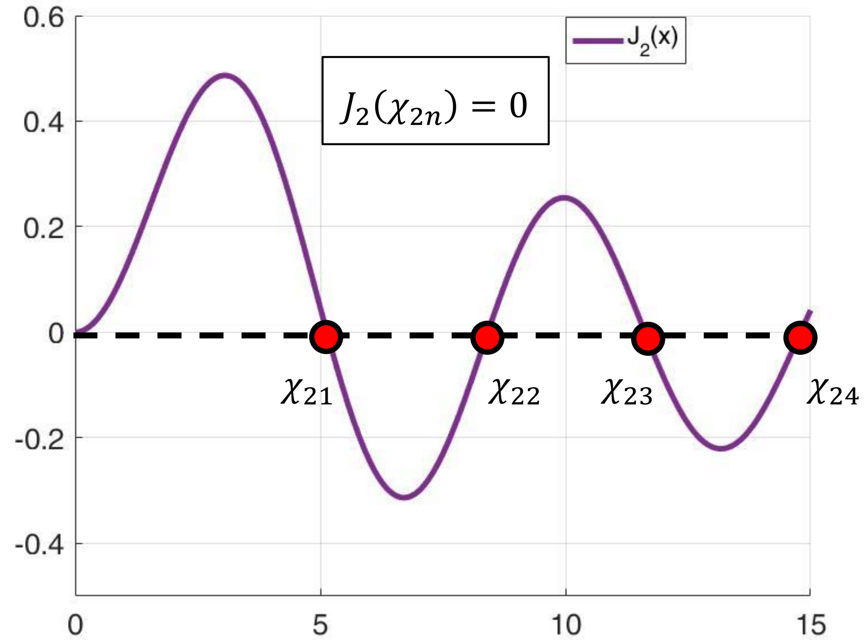
$$\varphi(x = 1) = \varphi(\tilde{x} = 1 \cdot \mu) = 0 \longrightarrow A \cdot \underbrace{J_\nu(\mu)}_{= 0\ (?)} = 0 \longrightarrow \boxed{\mu_n = \chi_{\nu n}}$$

✓ *Non-trivial solution that satisfies the B.C.*

Reminder: Zeroes of Bessel functions

$$\boxed{J_\nu(\chi_{\nu n}) = 0 \quad (n = 1,2,3\ldots)}$$





The eigenvalues:

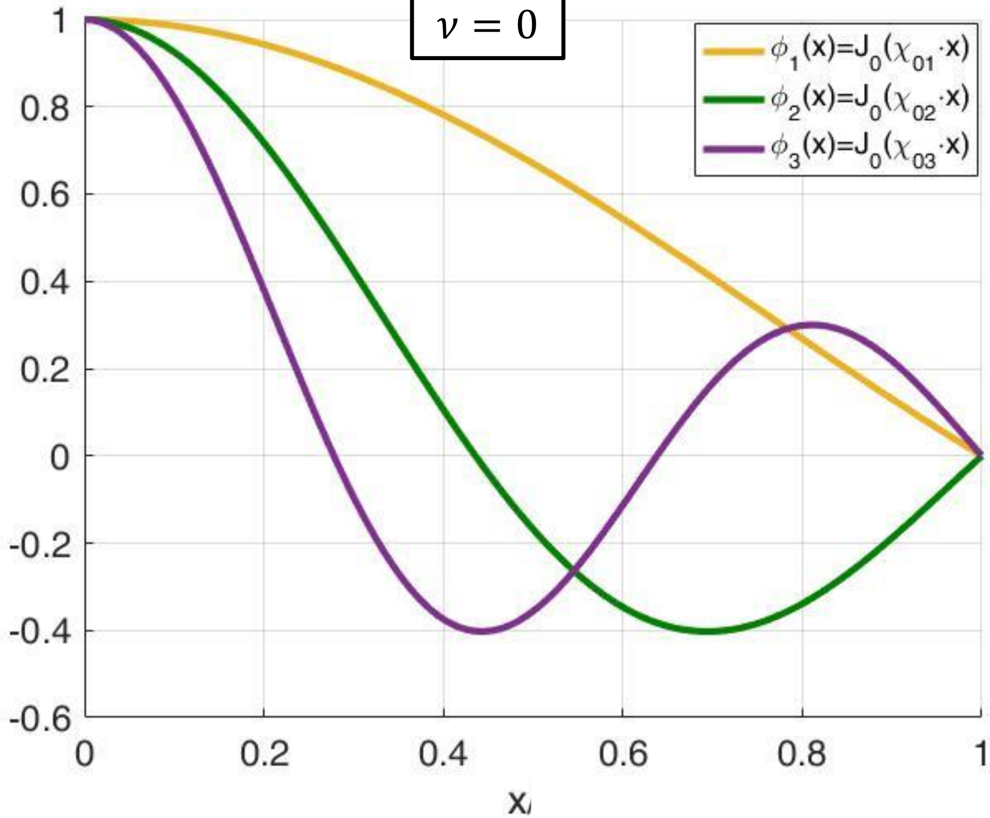
$$\boxed{\lambda_n = \mu_n^2 = {\chi_{\nu n}}^2} \qquad (n = 1,2,3 \dots)$$

The eigenfunction:

$$\boxed{\varphi_n(x) = J_\nu(\chi_{\nu n} \cdot x)}$$

Example #1 ($\nu = 0$)

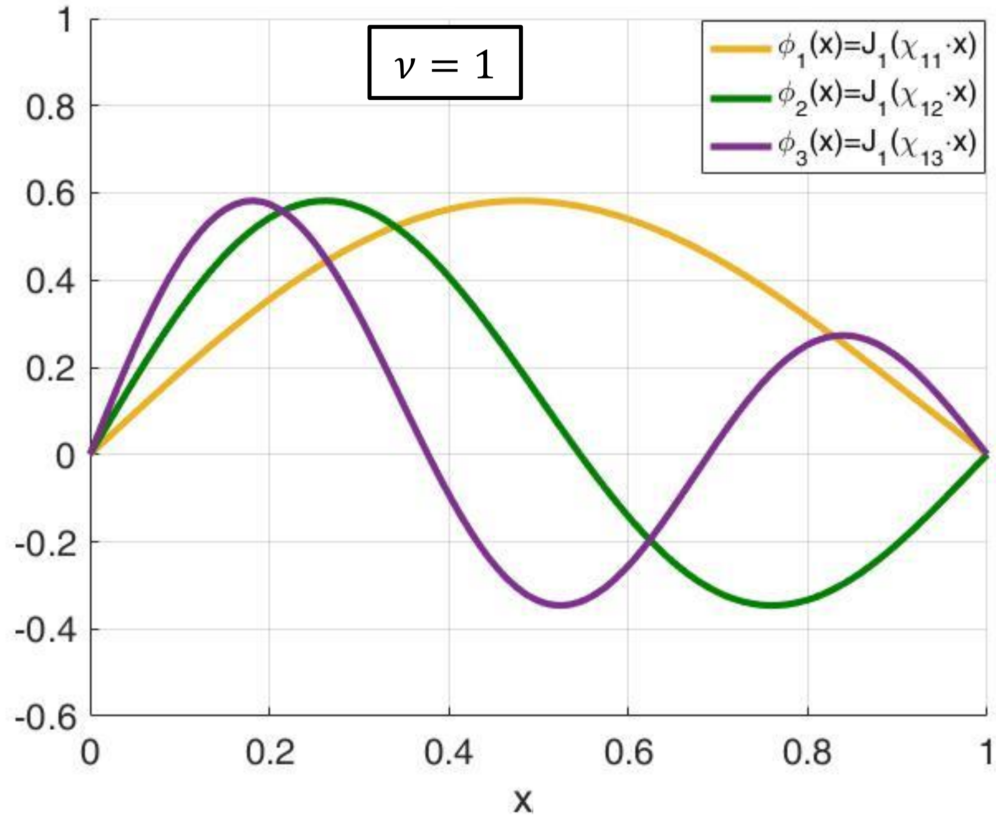
$$\boxed{\nu = 0}$$



The eigenvalues: $\boxed{\lambda_n = \mu_n^2 = {\chi_{\nu n}}^2}$ $(n = 1,2,3 \dots)$

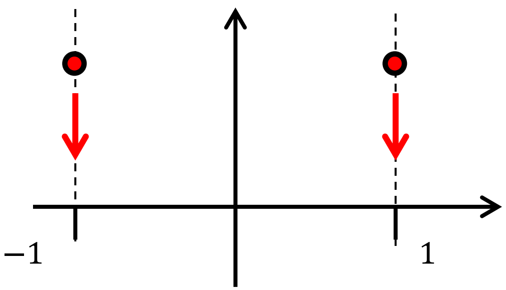
The eigenfunction: $\boxed{\varphi_n(x) = J_\nu(\chi_{\nu n} \cdot x)}$

Example #2 $(\nu = 1)$



Example 4 – Legendre equation (Singular B.C.)

Adrien-Marie Legendre



$$(1-x^2)\cdot y'' - 2x\cdot y' + \lambda\cdot y = 0 \qquad (-1\le x\le 1)$$

$$\lambda = l\cdot(1+l) \qquad (l \ge 0)$$

$$\lim_{x\to -1}|y(x)| < \infty \qquad \lim_{x\to 1}|y(x)| < \infty$$

$$\downarrow$$

$$\varphi(x) = A\cdot \underbrace{P_l(x)}_{\text{Legendre polynomials}} + B\cdot \underbrace{Q_l(x)}_{\text{Legendre functions}}$$

Refresh your memory...

(Lecture 2)

➢ Imposing the B.C.:

$$\left.\begin{aligned}&\lim_{x\to -1}|\varphi(x)| < \infty\\ &\lim_{x\to 1}|\varphi(x)| < \infty\end{aligned}\right] \longrightarrow \quad \boxed{l = 0,1,2\ ...} \quad \boxed{B = 0} \quad \longrightarrow \quad \boxed{\varphi(x) = A\cdot P_l(x)}$$

The eigenvalues:

$$\boxed{\lambda_l = l \cdot (1 + l)}$$

$(l = 0,1,2 \ldots)$

The eigenfunction:

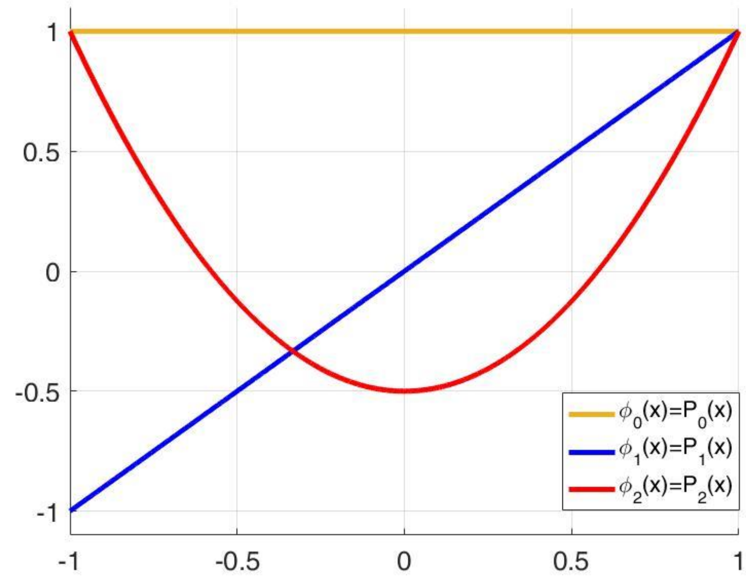
$$\boxed{\varphi_l(x) = p_l(x)}$$

The Legendre polynomials

$$P_{l=0}(x) = 1$$

$$P_{l=1}(x) = x$$

$$P_{l=2}(x) = \frac{1}{2} \cdot (3x^2 - 1)$$



Part B *– Sturm-Liouville systems*

Jacques Charles
François Sturm

Joseph Liouville

(I) Definition: Strum-Liouville (S-L) systems

The differential equation

$$\boxed{\frac{d}{dx}\left(p(x)\cdot\frac{dy}{dx}\right)+\left(q(x)+\lambda\cdot s(x)\right)\cdot y=0}$$

Strum-Liouville equation

- The functions $p(x)$, $q(x)$ **and** $s(x)$ are *real valued functions* of x

- The *parameter* λ is ***independent*** of x

- Using the differential operator:

$$L=\tilde{L}=\frac{d}{dx}\left(p(x)\cdot\frac{d}{dx}\right)+q(x)$$

Strum-Liouville operator

- The equation takes the form of:

$$\boxed{L[y]+\lambda\cdot s(x)\cdot y=0}$$

Strum-Liouville equation

Strum-Liouville equation

$$L[y] + \lambda \cdot s(x) \cdot y = 0 \qquad (a \leq x \leq b)$$

<u>The boundary-conditions</u>

$$\begin{cases} a_1 \cdot y(a) + a_2 \cdot y'(a) = 0 \\ b_1 \cdot y(b) + b_2 \cdot y'(b) = 0 \end{cases}$$

<u>Regular B.C.</u>

$$\begin{cases} y(a) = y(b) \; ; \; y'(a) = y'(b) \\ p(x = a) = p(x = b) \end{cases}$$

<u>Periodic B.C.</u>

$$\begin{cases} \lim_{x \to a} |y(x)| < \infty \; ; \; p(x = a) = 0 \\ b_1 \cdot y(b) + b_2 \cdot y'(b) = 0 \end{cases}$$

<u>Singular B.C.</u>

"Self-adjoint" boundary conditions

Self-adjoint <u>operator</u>

self-adjoint B.V.P (!)

<u>Strum-Liouville systems</u>

Example 1

$$y'' + \lambda \cdot y = 0 \qquad (0 \le x \le 1)$$

Regular B.C. $\quad y(0) = 0 \ , \quad y(1) + k \cdot y'(1) = 0 \qquad (k > 0)$

The differential equation:

$$L[y] + \lambda \cdot s(x) \cdot y = 0$$

$$L[y] = \frac{d}{dx}\left(p(x) \cdot \frac{d}{dx}\right) + q(x)$$

$\longleftrightarrow$

$$p(x) = 1 \ , \quad q(x) = 0, \quad s(x) = 1$$

$\downarrow$

A ***regular*** S-L system

Example 2

$$y'' + \lambda \cdot y = 0 \qquad (-\pi \le x \le \pi)$$

Periodic B.C. $\quad y(-\pi) = y(\pi) \quad , \quad y'(-\pi) = y'(\pi)$

The differential equation:

$$L[y] + \lambda \cdot s(x) \cdot y = 0$$

$$L[y] = \frac{d}{dx}\left(p(x) \cdot \frac{d}{dx}\right) + q(x)$$

$\longleftrightarrow$

$$p(x) = 1 \quad , \quad q(x) = 0, \quad s(x) = 1$$

$$p(-\pi) = p(\pi)$$

$\downarrow$

A ***periodic*** S-L system

Example 3

$$x^2 \cdot y'' + x \cdot y' + (\lambda \cdot x^2 - \nu^2) \cdot y = 0 \qquad (\nu \geq 0) \qquad (0 \leq x \leq 1)$$

Friedrich Wilhelm Bessel

Singular B.C. $\qquad \lim_{x\to 0}|y(x)| < \infty \qquad y(1) = 0$

The differential equation:

- **Not** presented via a S-L form !! (**not** shown by a self-adjoint operator…)
- ***Transformation*** into a Self-adjoint operator:

$$\boxed{(x \cdot y')' - \frac{\nu^2}{x} \cdot y + \lambda \cdot x \cdot y = 0} \longleftrightarrow \boxed{p(x) = x \;,\quad q(x) = -\nu^2/x, \quad s(x) = x}$$

Refresh your memory…
(Lecture 1)

$$\boxed{p(x = 0) = 0}$$

$$\downarrow$$

A ***Singular*** S-L system

Example 4


Adrien-Marie Legendre

$$(1-x^2)\cdot y''-2x\cdot y'+\lambda\cdot y=0 \qquad (-1\le x\le 1)$$

$$\lambda=l\cdot(1+l) \qquad (l\ge 0)$$

Singular B.C. $\qquad \lim_{x\to -1}|y(x)|<\infty \qquad \lim_{x\to 1}|y(x)|<\infty$

The differential equation:

- ***Transformation*** into a Self-adjoint operator:

$$\boxed{((1-x^2)\cdot y')'+\lambda\cdot y=0} \quad \longleftrightarrow \quad \boxed{p(x)=1-x^2,\quad q(x)=0,\quad s(x)=1}$$

$$\boxed{p(x=\pm 1)=0}$$

$$\downarrow$$

A ***Singular*** S-L system

(II) The eigenvalue problem – S-L system

$$L[y] + \lambda \cdot s(x) \cdot y = 0$$

$$U_a[y] = 0 \qquad U_b[y] = 0$$

Regular/*Periodic*/ *singular*

Objective: We **seek** for λ_n ***parameters*** and ***non-trivial*** $\varphi_n(x)$ ***solutions*** – such that:

$$L[\varphi_n] + \lambda_n \cdot s(x) \cdot \varphi_n = 0 \quad \longleftrightarrow \quad \varphi_n(x; \lambda_n) \neq 0$$

$\downarrow$ *eigenvalues* $\qquad$ $\downarrow$ *eigenfunction*

Methodology

Step 1: ***Transform*** the equation into S-L form and identify $p(x)$, $q(x)$ and $s(x)$

$$\underbrace{\frac{d}{dx}\left(p(x)\cdot\frac{dy}{dx}\right)+q(x)\cdot y}_{L[y]}+\lambda\cdot s(x)\cdot y=0$$

(Important for later…)

Step 2: *Find the* ***general solution*** *of the equation* (λ – yet unknown).

$$\varphi(x;\lambda)=A\cdot\varphi_1(x;\lambda)+B\cdot\varphi_2(x;\lambda)$$

Step 3: ***Impose B.C.*** and identify the set of λ ***eigenvalues*** that produce non-trivial solutions.

$$\lambda=\{\lambda_1,\lambda_2,\dots,\lambda_n\}$$

Step 4: Find the corresponding ***eigenfunctions*** φ_n for ***each*** λ_n (can be more than one..)

$$\varphi_n(x;\lambda_n)\neq 0$$

(III) Characteristics of eigenvalues and eigenfunctions of S-L system

$$\begin{cases} a_1 \cdot y(a) + a_2 \cdot y'(a) = 0 \\ b_1 \cdot y(b) + b_2 \cdot y'(b) = 0 \end{cases}$$

Theorem – regular S-L systems

*A regular S-L system has an **infinite sequence** of **real** and **distinct** eigenvalues.*

$$\lambda_0 < \lambda_1 < \lambda_2 < \cdots$$

with

$$\lim_{n\to\infty} \lambda_n = \infty$$

*For **each** eigenvalue λ_n - the corresponding eigenfunction φ_n is **real** and **uniquely** determined.*

$$\lambda_n \leftrightarrow \varphi_n$$

*The eigenfunction φ_n has **exactly** $\mathbf{n}$ **zeros** in $a < x < b$.*

Theorem – periodic S-L systems

$$\begin{cases} y(a) = y(b) \;\; ; \;\; y'(a) = y'(b) \\ p(x=a) = p(x=b) \end{cases}$$

The eigenvalues of a periodic S-L system form a sequence:

$$\lambda_0 < \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \cdots$$

There exists a unique eigenvalue λ_0 with a unique eigenfunction φ_0.

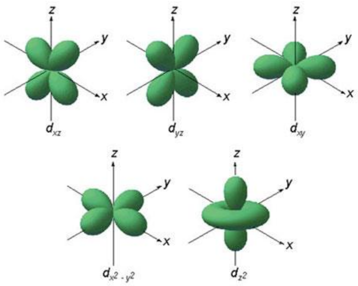
$$\lambda_0 \leftrightarrow \varphi_0$$

If $\lambda_{k+1} < \lambda_{k+2}$ then the eigenfunctions φ_{k+1} and φ_{k+2} are distinct.

$$\lambda_{k+1} \leftrightarrow \varphi_{k+1} \qquad \lambda_{k+2} \leftrightarrow \varphi_{k+2}$$

*If $\lambda_{k+1} = \lambda_{k+2}$ the the eigenfunctions φ_{k+1} and φ_{k+2} are yet **linear independent** – but share the same eigenvalue:*

$$\lambda_{k+1} \leftrightarrow \varphi_{k+1} \qquad \lambda_{k+1} \leftrightarrow \varphi_{k+2}$$



e.g. degenerated states in atomic orbitals

$$\begin{cases} \lim_{x \to a} y(x) < \infty \ ; \ p(x=a) = 0 \\ b_1 \cdot y(b) + b_2 \cdot y'(b) = 0 \end{cases}$$

A few comments - Singular S-L system

- The *eigenvalue* "***spectrum***" of a singular S-L system may be ***discrete*** and/or **continues**.

- For the ***discrete*** case, we have a set of eigenvalues and eigenfunctions – as before.

$$\lambda_0 < \lambda_1 \le \lambda_2 \le \lambda_3 \le \cdots \qquad\qquad \lambda_n \leftrightarrow \varphi_n(x)$$

- For the ***continues*** case, the eigenvalues λ take a certain ***range*** – generating a set of "two-variable" functions:

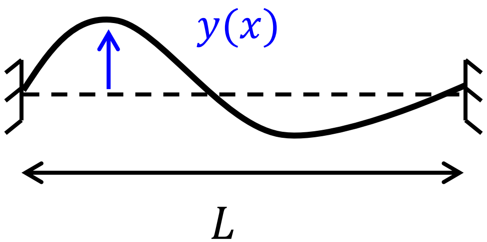
$$\lambda_i \le \lambda \le \lambda_j \qquad\qquad \lambda \leftrightarrow \varphi(x, \lambda)$$

Much less intuitive…

- Let's see this through an example….

Examples – Standing waves in a string

(1) Finite-length string (warm-up)



$$y'' + \lambda \cdot y = 0 \qquad (0 \leq x \leq L)$$

$$y(0) = 0 \quad , \quad y(L) = 0$$

- This is a ***regular*** S-L system – for which the general solution is:

$$\varphi(x) = A \cdot \sin(\kappa \cdot x) + B \cdot \cos(\kappa \cdot x) \qquad \lambda = \kappa^2$$

- Imposing the B.C.:

$$y(0) = 0 \longrightarrow B = 0$$

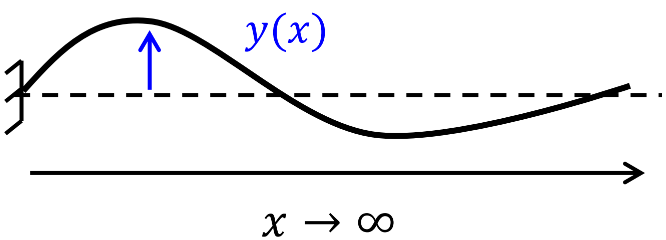
"wavenumber"

$$y(L) = 0 \longrightarrow \sin(\kappa \cdot L) = 0 \longrightarrow \kappa_n = n \cdot \frac{\pi}{L}$$

- The eigenfunctions and eigenvalues are thus:

$$\boxed{\varphi_n(x) = \sin(\kappa_n \cdot x)} \qquad \boxed{\lambda_n = \kappa_n^2 = \left(n \cdot \frac{\pi}{L}\right)^2} \qquad n = 1,2, \ldots$$

(2) A semi-infinite string ($L \to \infty$)



$$y'' + \lambda \cdot y = 0 \qquad (0 \le x < \infty)$$

$$y(0) = 0 \quad , \quad y(x \to \infty) = ?$$

- To obtain a ***singular*** S-L system, the B.C term at $x \to \infty$ must be:

$$\lim_{x \to \infty} |y(x)| < \infty$$ ✓ *Can be satisfied*

$$\lim_{x \to \infty} p(x) = 0$$ ✗ *Cannot be satisfied* $p(x) = 1$

- Thus - it is a ***not well-defined singular*** S-L system – but yet an ***eigenvalue problem***

Any ideas for an "informal" analysis ???

Adaptation of the finite-length solution for $L \to \infty$.

- For the finite-length case we obtained:

$$\varphi_n(x) = \sin(\kappa_n \cdot x) \qquad \lambda_n = \kappa_n^2 = \left(n \cdot \frac{\pi}{L}\right)^2 \qquad n = 1,2,\dots$$

- When taking $L \to \infty$, the eigenvalues λ_n approaches to a positive ***continues*** parameter (λ):

$$\lambda_n = \left(n \cdot \frac{\pi}{L}\right)^2 \longrightarrow 0 < \lambda$$

Eigenvalues range

- Similarly, the half-wavenumbers κ_n also approaches to a positive ***continues*** parameter (κ):

$$\kappa_n = \sqrt{\lambda_n} \longrightarrow 0 < \sqrt{\lambda} = \kappa$$

Wavenumber range

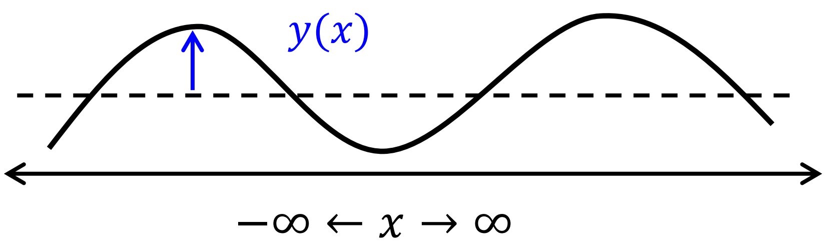
- The eigenfunction becomes a ***continues*** function of both x and κ:

$$\varphi_n(x) = \sin(\kappa_n \cdot x) \longrightarrow \varphi_\kappa(x) = \varphi(x,\kappa) = \sin(\kappa \cdot x)$$

*"**Descrete**" eigenfunction set*

*"**Continues**" eigenfunction set*

(3) An infinite string



$$y'' + \lambda \cdot y = 0 \qquad (-\infty \le x < \infty)$$

$$\lim_{x\to-\infty} |y(x)| < \infty, \quad \lim_{x\to\infty} |y(x)| < \infty$$

- This is also a ***not*** ***well-defined*** ***singular*** S-L system – but yet an ***eigenvalue problem***

- The ***general*** solution is:

$$\varphi(x) = A \cdot \sin(\kappa \cdot x) + B \cdot \cos(\kappa \cdot x) \qquad \lambda = \kappa^2$$

- A ***non-trivial solution*** is obtained for ***any*** κ value (eigenvalue) that is:

$$0 < k \in \mathcal{R}$$

Continues *eigenvalues* ***range***

- Thus, each eigenvalue (κ) has ***two*** eigenfunctions:

$$\varphi_k(x) = \varphi(x, \kappa) = \sin(\kappa \cdot x) \qquad \psi_k(x) = \psi(x, \kappa) = \cos(\kappa \cdot x)$$

Continues *eigenfunction* ***sets***