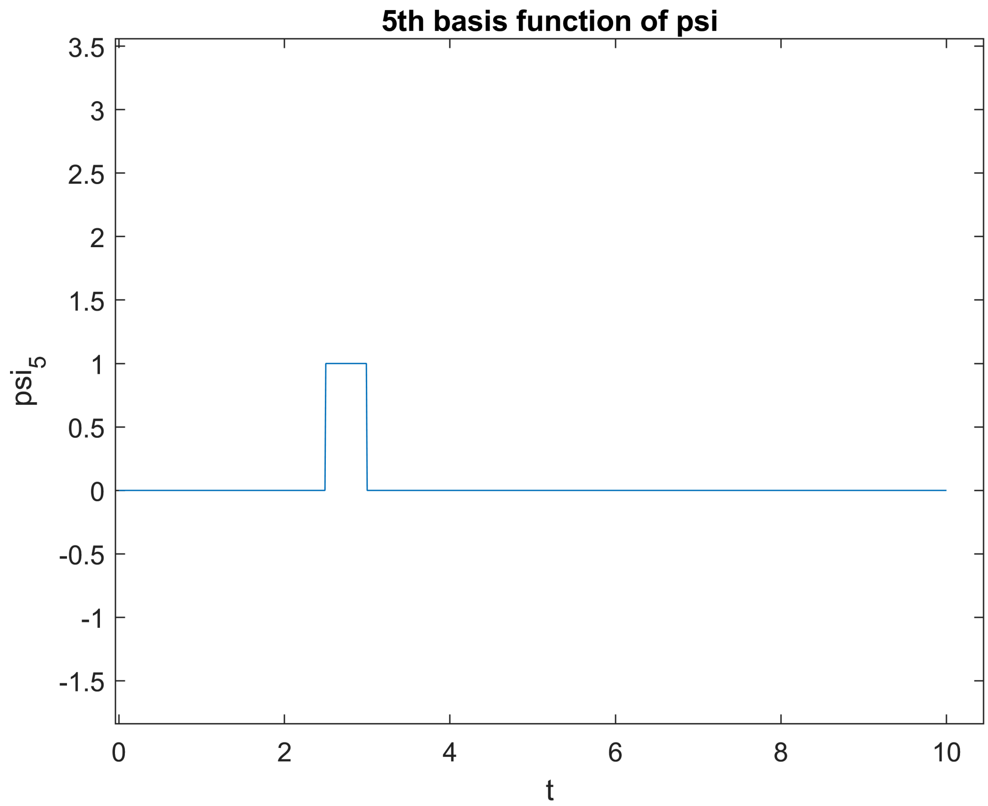
Sampling and recovery using different orthonormal basis functions

Lets assume two given priodic signals: $f(t)$ and $g(t)$:

$$f(t) = 4\cos\left(4\frac{\pi t}{T}\right) + \sin\left(\frac{10\pi t}{T}\right)$$

$$g(t) = 2sign\left(\sin\left(\frac{6\pi}{T}t\right)\right) - 4sign\left(\sin\left(\frac{4\pi t}{T}\right)\right)$$

```
T = 10;
t = 0:1/100:10;


g = @(t) 2*sign(sin(6*pi*t/T)) -4*sign(sin(4*pi*t/T));

f = @(t) 4*cos(4*pi*t/T) + sin(10*pi*t/T);

n = -20:20;
phi =  exp(1i*2*pi*(t.').*n/T);
f_phi = @(t) exp(1i*2*pi*(t.').*n/T);


n = 0:19;
psi = rectpuls(t.'/(T/20) -(n+0.5),2*(T/20));
f_psi =@(n,t) rectpuls(t.'/(T/20) -(n+0.5),2*(T/20));

plot(t,f_psi(5,t)); title("5th basis function of psi") ;xlabel("t"); ylabel('psi_5')

xlim([-0.04 10.45])
ylim([-1.84 3.56])
```



nowthat everything is set up, it is time to define our function that will take in the basis functions $\psi_n(t)$ or perheps $\phi_n(t)$ and in addition a signal $g(t)$ and output a_n, the projection coefficients. such that:

$$c_n = \frac{\int_0^T x(t)\phi_n^*(t)\mathrm{dt}}{||\phi_n(t)||}$$

so we will:

1.calculate the norm, squared.

2.calculate the projection.

3.devide them for each n, to get the n'th coefficient.

the function is declared at the **end of the document, and is as follows:

```
% function [a_hat] = Projection_coef(x,PSI,T)
```

```
% %INPUT: x -continuos time x(t) values
% %       PSI - a matrix wit basis functions
% %       T period duration
% %
%     PSI_CONJ = conj(PSI);
%     norm = trapz(abs(PSI.^2));
%     proj = trapz(x.'.*PSI_CONJ);
%     a_hat = proj/norm;
% end
```

---

let us calculate the coefficients c_n for $f(t)$ using $\phi_n(t)$ basis functins.

a quick reminder:

$$f(t) = 4\cos\left(4\frac{\pi t}{T}\right) + \sin\left(\frac{10\pi t}{T}\right)$$

```
cn_of_f_using_phi = Projection_coef(f(t), phi , T);
TAB =table(cn_of_f_using_phi)
```

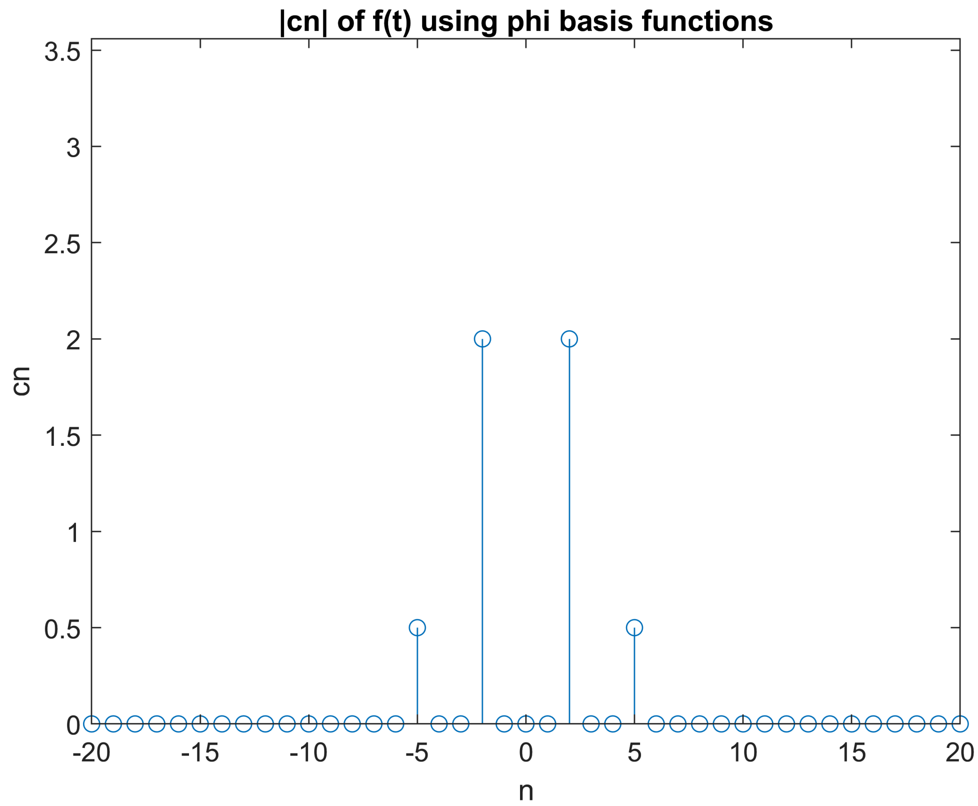
TAB = 1×1 table

...

	cn_of_f_using_phi			
1	2.5224e-16 - 3.42...	-1.4255e-16 + 2.9...	-5.1514e-17 + 1.6...	-1.9540e-17 - 4.2...

let's plot it just to understand what we see:

```
stem(-20:20,abs(cn_of_f_using_phi) ); title('|cn| of f(t) using phi basis functions'),xlabel('r
ylim([0 3.56])
```



explaination:

we got exactly what we shuld have expected. f consists of 2 frequancies exactly (of sine and cosine) which means 4 spikes total, conjugates inclouded. $(\pm\omega_a, \pm i \cdot \omega_b)$

now we shall move to calculate the coefficients c_n for $f(t)$ using $\psi_n(t)$ basis functins.

```
cn_of_f_using_psi = Projection_coef(f(t), psi , T);
TAB =table(cn_of_f_using_psi);
```

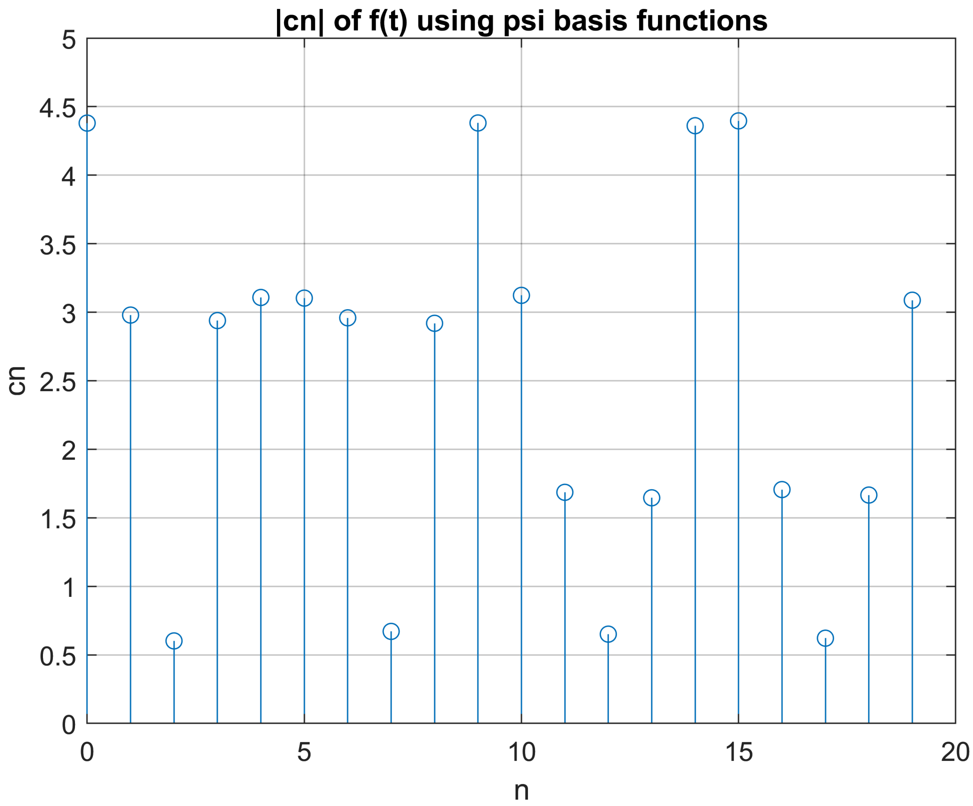
again plotting will help us a great del to understand the table:

```
stem(0:19,abs(cn_of_f_using_psi) ); title('|cn| of f(t) using psi basis functions'),xlabel('n')
ylim([0 5])
```

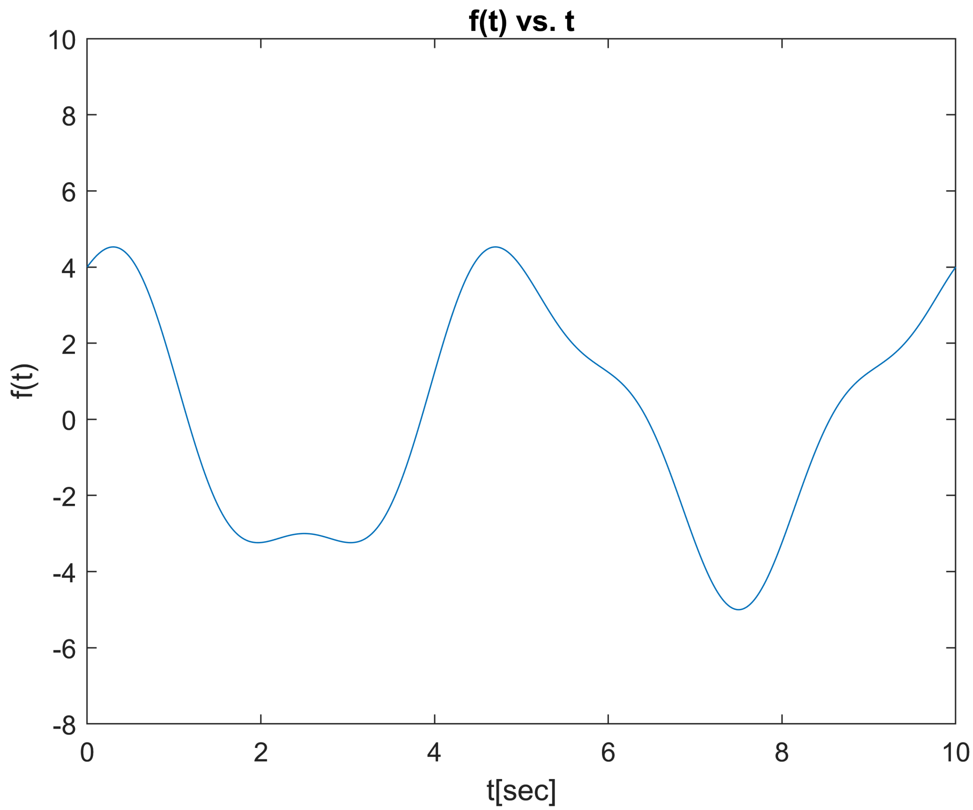
explaination:

to understand the coefficients graph shape we shuld take a look at the signal:

```
grid on;
```




```
plot( t,f(t)); title ('f(t) vs. t') ; xlabel('t[sec]');ylabel('f(t)')
ylim([-8 10])
```



the projection is exactly proportional to the integral of $f(t)$ in the respective 0.05T section! so when we look at each c_n we see a normalized sum (average) of the $f(t)$ signal at each time section respectively.

---------------- $g(t)$ ----------------

now is the time to calculate the coefficients c_n for $g(t)$ using $\phi_n(t)$ basis functins.

we defined:

$$g(t) = 2sign\left(\sin\left(\frac{6\pi}{T}t\right)\right) - 4sign\left(\sin\left(\frac{4\pi t}{T}\right)\right)$$

```
cn_of_g_using_phi = Projection_coef(g(t), phi , T);
TAB =table(cn_of_g_using_phi)
```

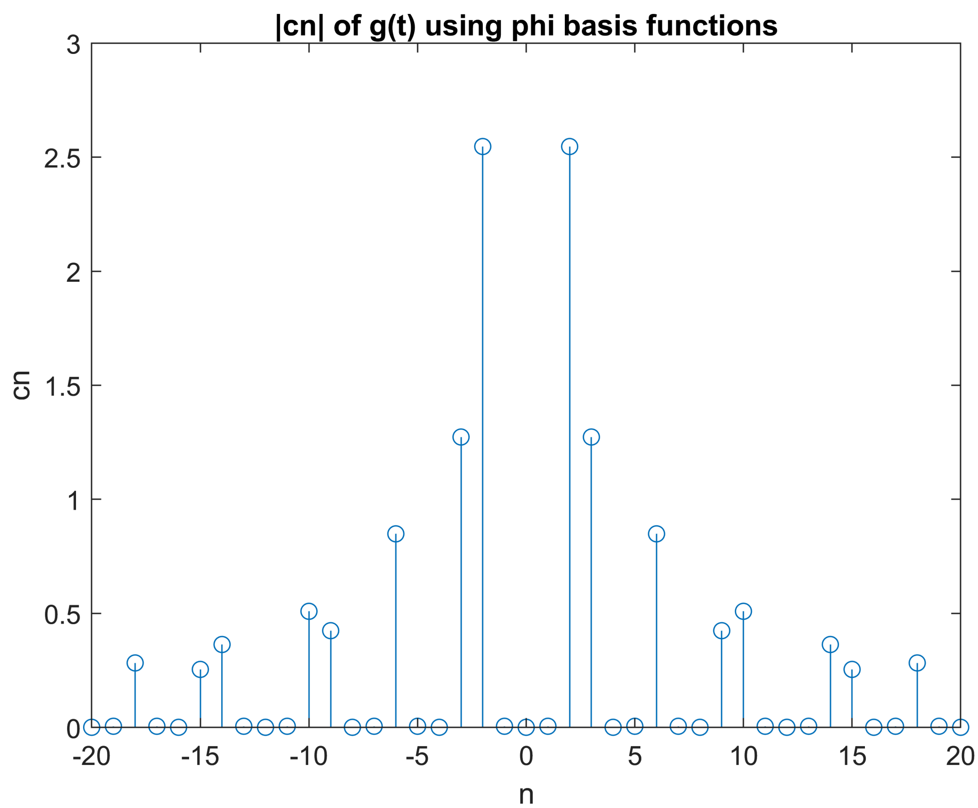
TAB = 1×1 table

...

	cn_of_g_using_phi			
1	-1.0000e-03 + 1.2...	-0.0050 - 0.0024i	0.0150 - 0.2826i	-0.0050 + 0.0022i

let's plot it just to understand what we see:

```
stem(-20:20,abs(cn_of_g_using_phi) ); title('|cn| of g(t) using phi basis functions'),xlabel('r
```



explaination:

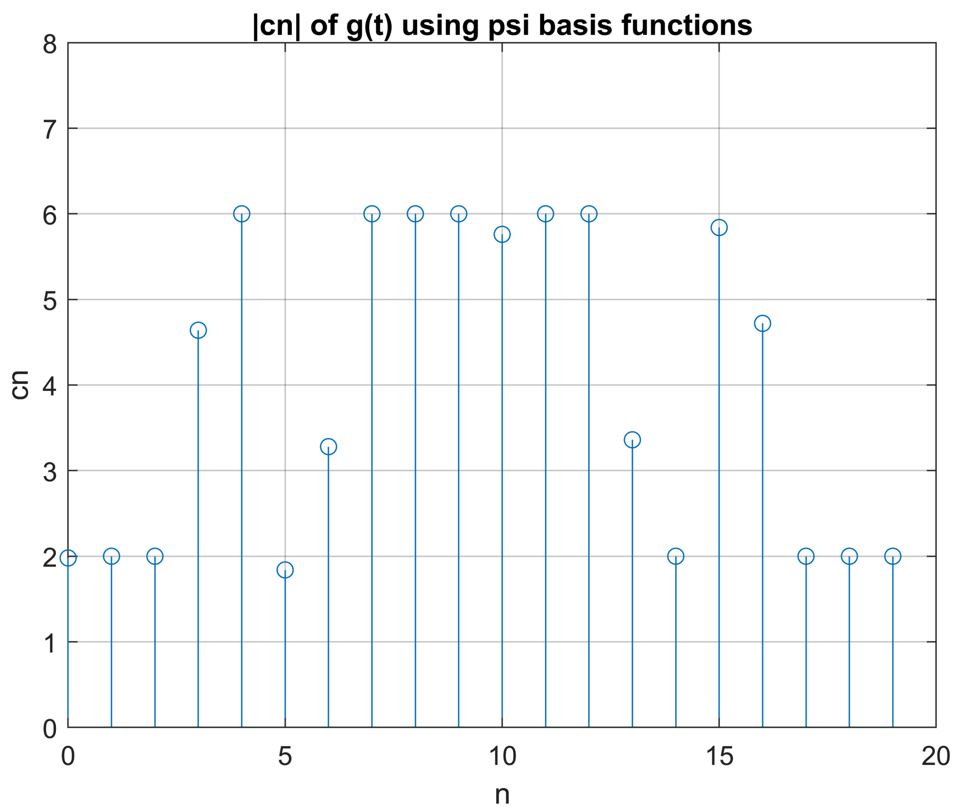
once again we got exactly what we deserve. this time g consists of infinity frequancies (althogh it seems to be but an innocent trig function) because of the underivable sign function. later we will plot the signal and see that it is not even continuos, meaning, from intro to fourier analysis, that it is the reason to the "ripples" in the coefficients.

now we shall move to calculate the coefficients c_n for $g(t)$ using $\psi_n(t)$ basis functins.

```
cn_of_g_using_psi = Projection_coef(g(t), psi , T);
TAB =table(cn_of_g_using_psi);
```

again plotting will help us a great del to understand the table:

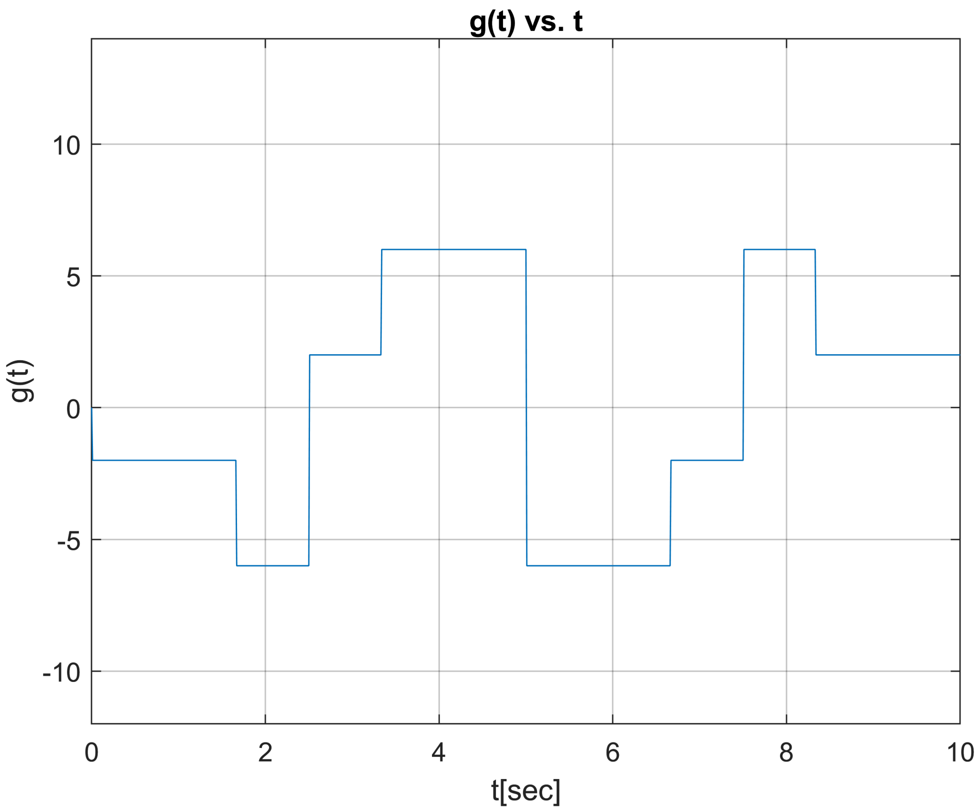
```
stem(0:19,abs(cn_of_g_using_psi) ); title('|cn| of g(t) using psi basis functions'),xlabel('n')
ylim([0 8])
grid on
```



explaination:

like before , to understand the coefficients graph shape we shuld take a look at the signal:

```
plot( t,g(t)); title ('g(t) vs. t') ; xlabel('t[sec]');ylabel('g(t)')
ylim([-12 14])
grid on;
```



once again, we get what we deserve: the projection is exactly proportional to the integral of $g(t)$ in the respective 0.05T section! so when we look at each c_n we see a normalized sum (average) of the $g(t)$ signal at each time section respectively.

to recover the original signals we will use the recover formula:

$$\hat{x}(t) = \sum c_n \phi_n(t)$$

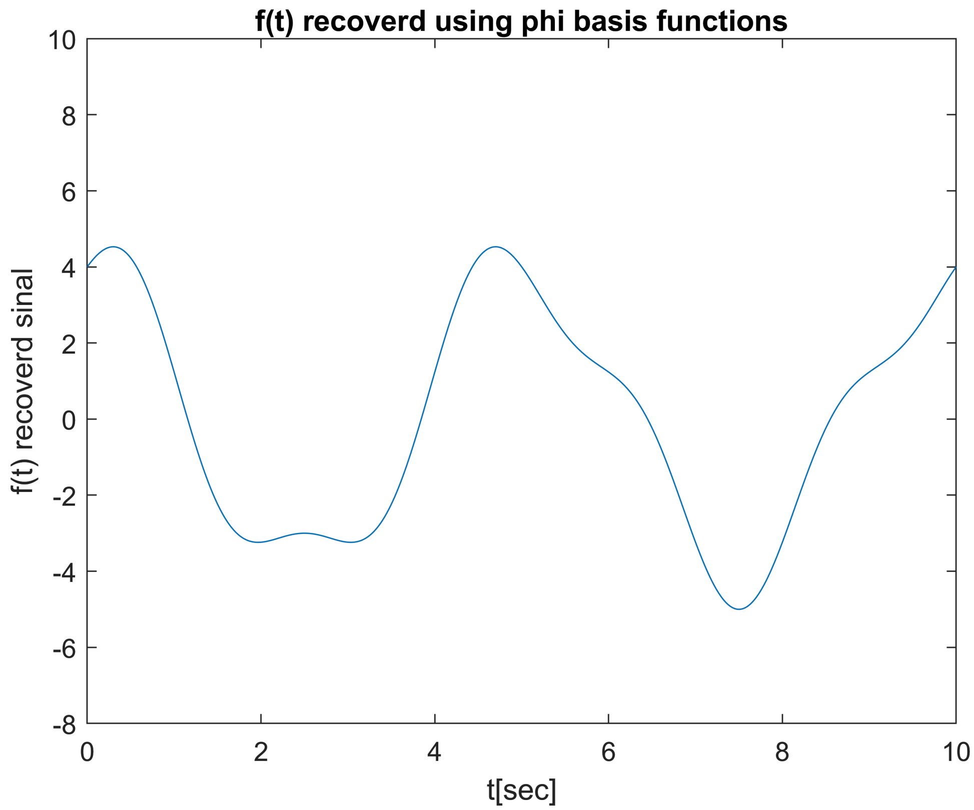
here we calculate-

$$\widehat{f}(t) = \sum_{n=-20}^{20} c_n \phi_n(t)$$

```
f_hat = sum((cn_of_f_using_phi.*phi).');
plot(t,f_hat); title('f(t) recoverd using phi basis functions');xlabel('t[sec]');ylabel('f(t) r
```

```
Warning: Imaginary parts of complex X and/or Y arguments ignored
```

```
ylim([-8 10])
```



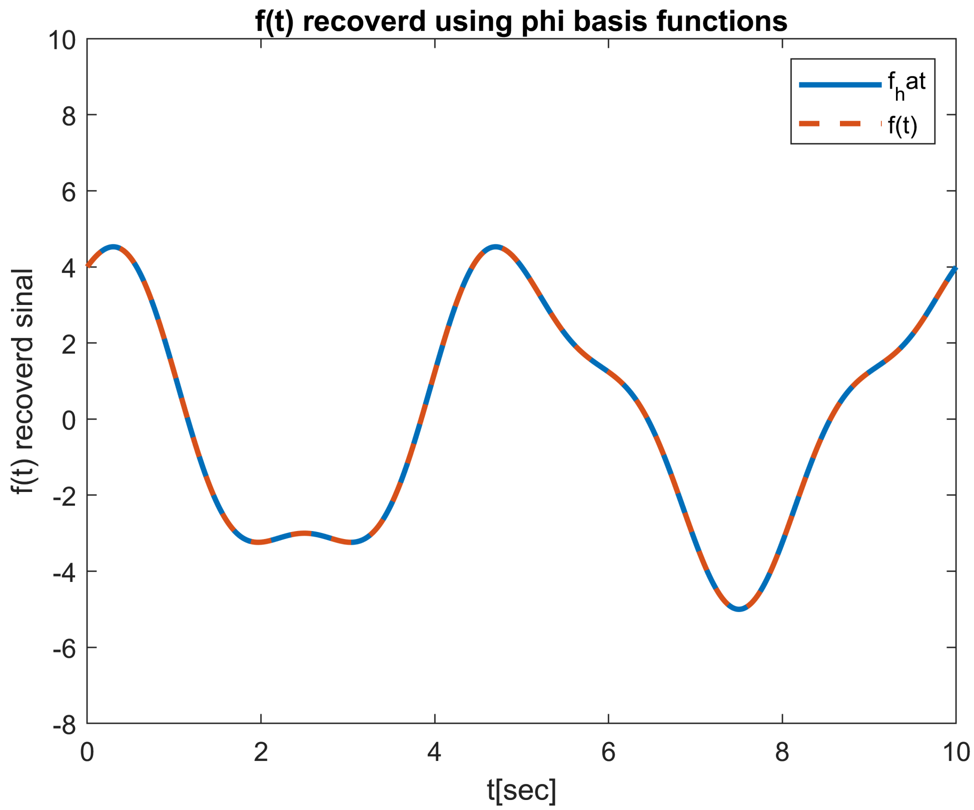

```

```

looking good so far. out ofcuriosity, we must check if the recovery is perfect.

```
plot(t,f_hat, t,f(t),'--',"linewidth",2); title('f(t) recoverd using phi basis functions');xlab
```

```
Warning: Imaginary parts of complex X and/or Y arguments ignored
```

```
ylim([-8 10])
```

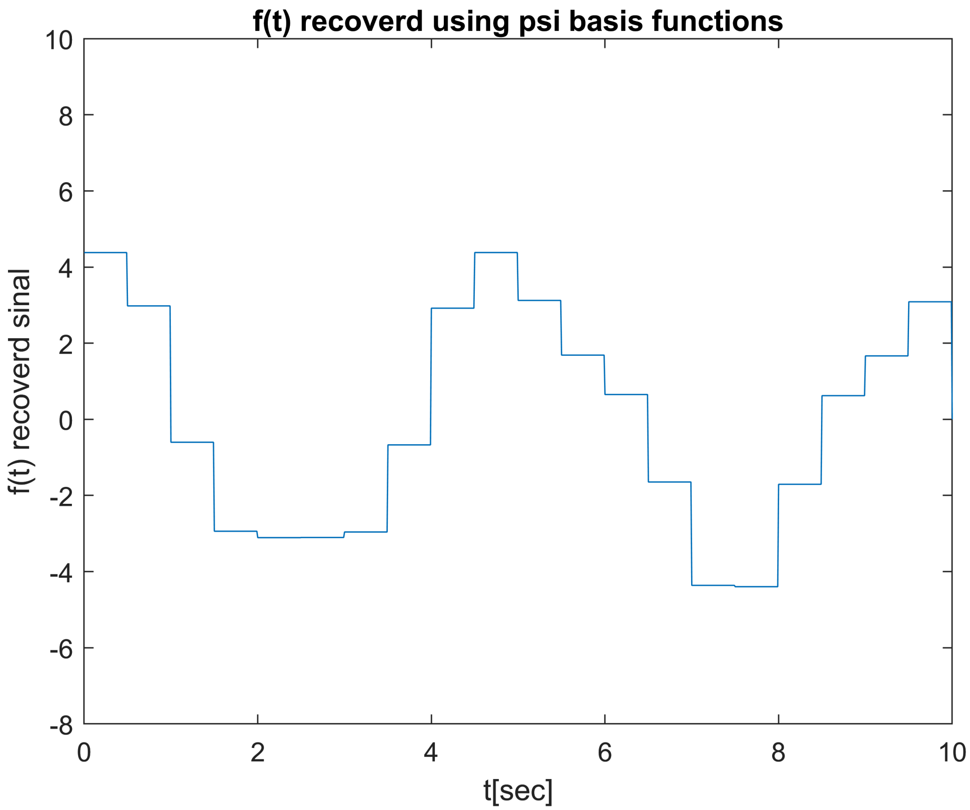


the recovery is perfect. this is obvious, it happens because $\{\phi_n(t)\}$ spans the hibertspace to which f(t) belongs.

we want to also calculate-

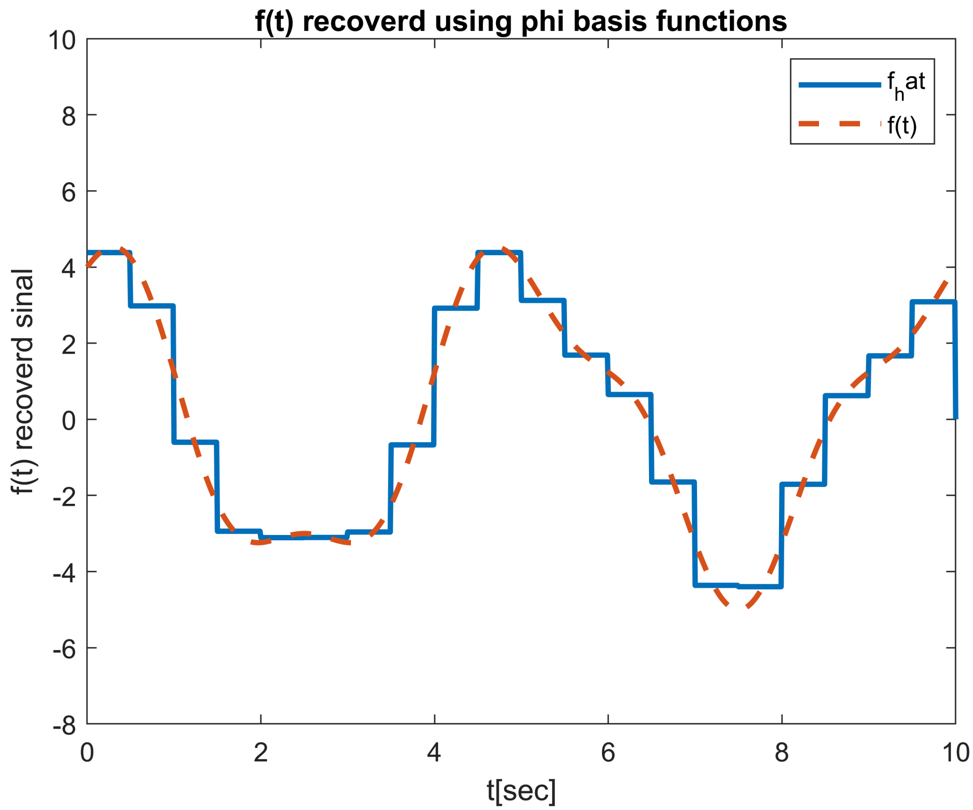
$$\hat{f}(t) = \sum_{n=-20}^{20} c_n \psi_n(t)$$

```
f_hat_psi = sum((cn_of_f_using_psi.*psi).');
plot(t,f_hat_psi); title('f(t) recoverd using psi basis functions');xlabel('t[sec]');ylabel('f(
ylim([-8 10])
```



looking simmilar also but surely, the recovery is **not** perfect.

```
plot(t,f_hat_psi, t,f(t),'--',"linewidth",2); title('f(t) recoverd using phi basis functions');
ylim([-8 10])
```



this is because of curse, $\{\psi_n(t)\}$ does not span the space in which f exists, and thus a perfect recovery will be impossible. more precision can be achived using a larger number of (narrower pulse) basis functions.

$g(t)$

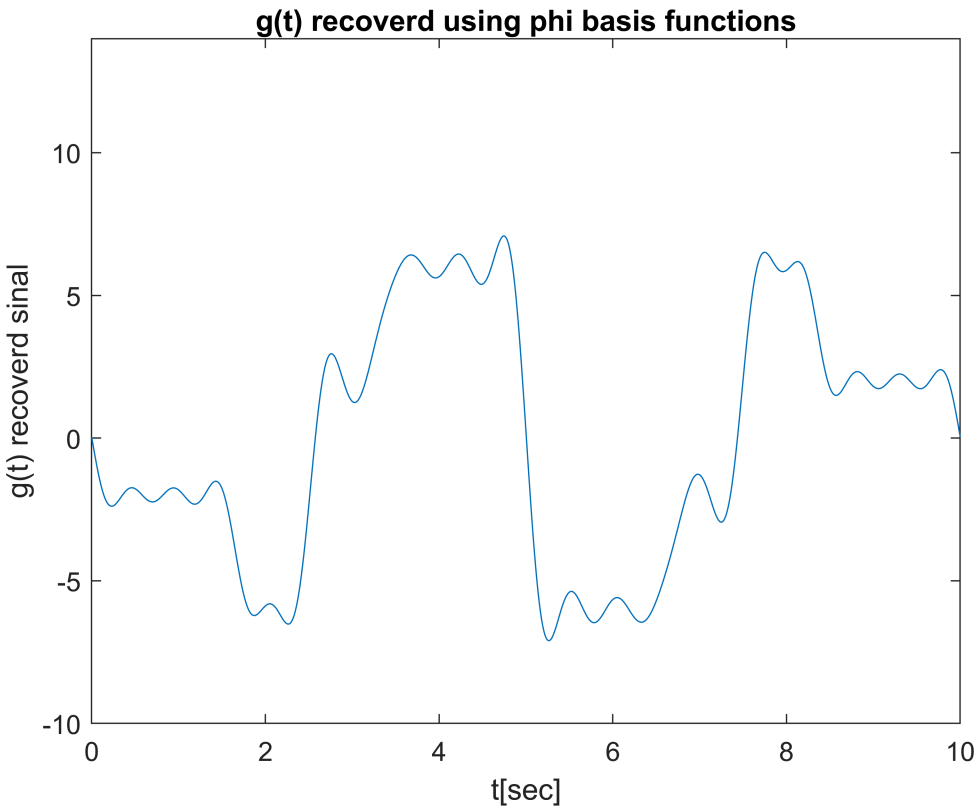
now we calculate the same for g:

$$\hat{g}(t) = \sum_{n=-20}^{20} c_n \phi_n(t)$$

```
g_hat = sum((cn_of_g_using_phi.*phi).');
plot(t,g_hat); title('g(t) recoverd using phi basis functions');xlabel('t[sec]');ylabel('g(t) r
```

```
Warning: Imaginary parts of complex X and/or Y arguments ignored
```

```
ylim([-10 14])
```

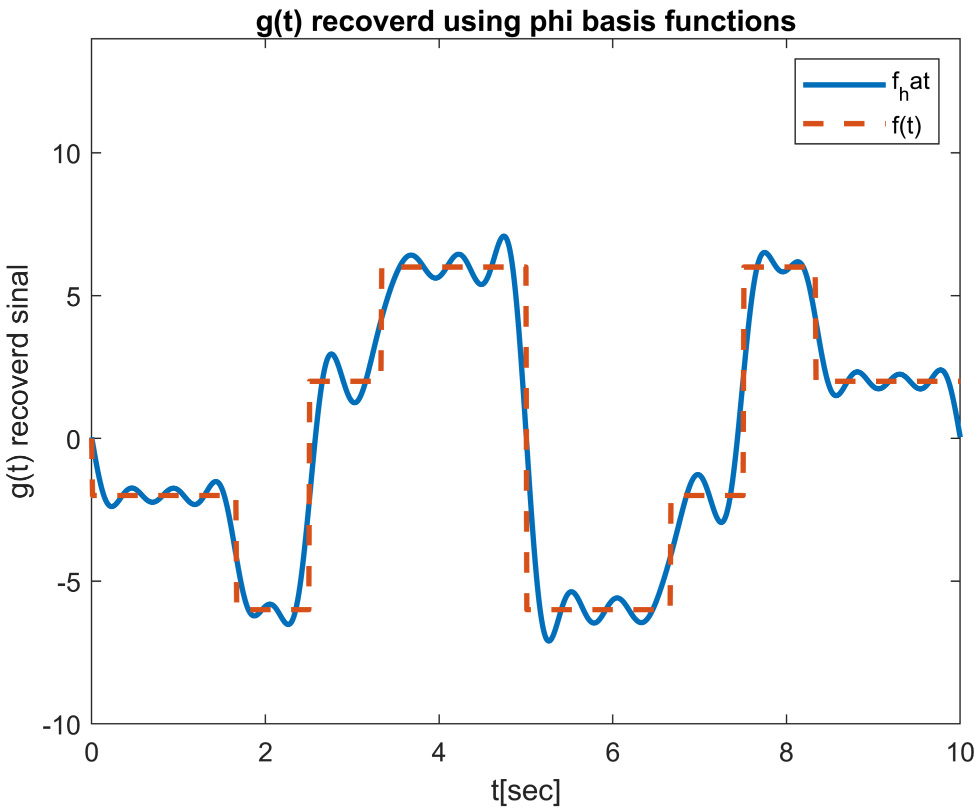


this resembles the shape but this is not a good recovery. let;s compare it with g(t):

```
plot(t,g_hat, t,g(t),'--',"linewidth",2); title('g(t) recoverd using phi basis functions');xlab
```

```
Warning: Imaginary parts of complex X and/or Y arguments ignored
```

```
ylim([-10 14])
```



this is because again, $\{\phi_n(t)\}$ does not span the space in which g exists.

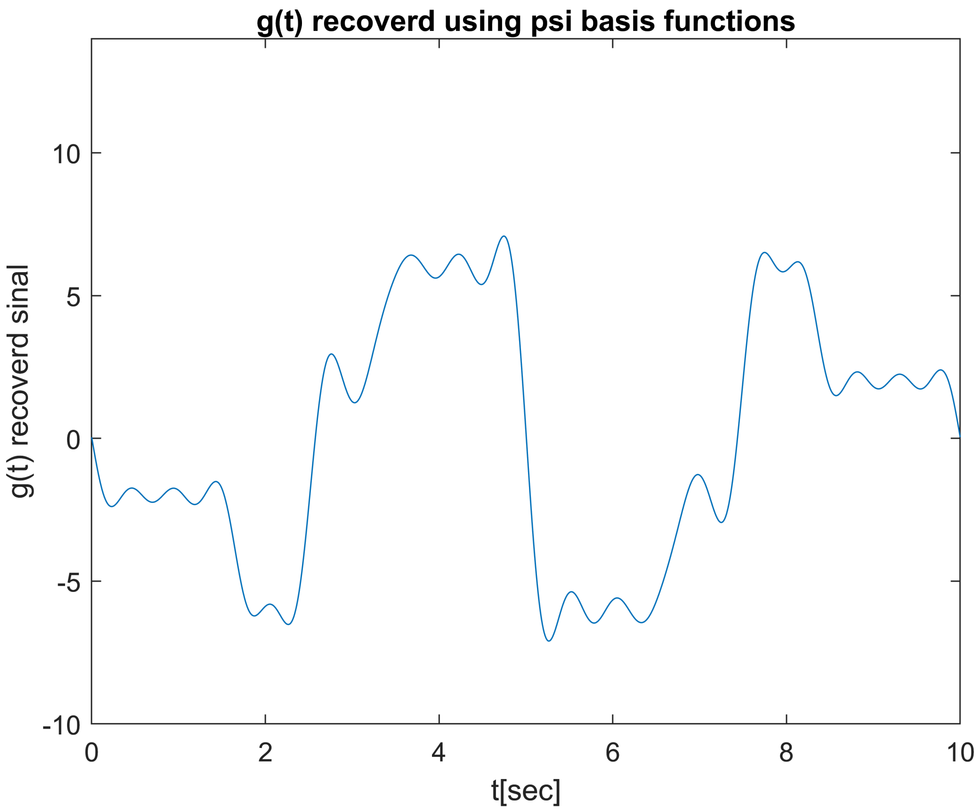
we want to also calculate-

$$\hat{g}(t) = \sum_{n=0}^{20} c_n \psi_n(t)$$

```
g_hat_psi = sum((cn_of_g_using_psi.*psi).');
plot(t,g_hat); title('g(t) recoverd using psi basis functions');xlabel('t[sec]');ylabel('g(t) r
```

```
Warning: Imaginary parts of complex X and/or Y arguments ignored
```

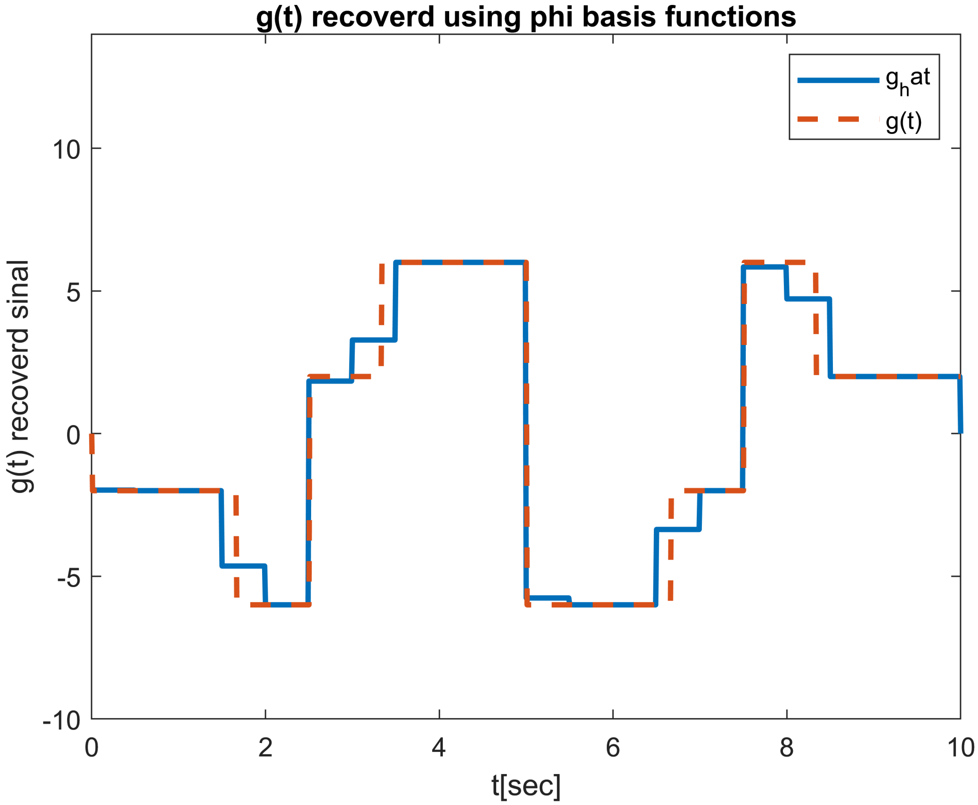
```
ylim([-10 14])
```



looking simmilar also but again, the recovery is **not** perfect.

a perfect recovery won't be possible, and as discussed, more precision will require more basis functions, as close as possible to infinity.

```
plot(t,g_hat_psi, t,g(t),'--',"linewidth",2); title('g(t) recoverd using phi basis functions');
ylim([-10 14])
```



this is because of curse, $\{\psi_n(t)\}$ does not span the space in which g exists, they may resemble eachothers characteristics, but no series of α_n in the complex plane gives us $g(t) = \sum_{n=0}^{20} \alpha_n \psi_n(t)$

so a perfect recover wouldn't be possible. it may be perfect for other $\psi_n(t)$ choises though, perheps more pulses, wth narrower dutycycles.(s.t at least every rise and falls of g will aline with one rise and fall respectively of $\psi_n(t)$)

```
function [a_hat] = Projection_coef(x,PSI,T)
%INPUT: x -continuos time x(t) values
%       PSI - a matrix wit basis functions
%       T period duration
%
    PSI_CONJ = conj(PSI);
    norm = trapz(abs(PSI.^2));
    proj = trapz(x.'.*PSI_CONJ);
```

```
    a_hat = proj./norm;
end
```