

Quantum 1 Class Notes 2017/18

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1 March 18th

Introduction:

Quantum Theory falls under the category of the area of physics known as modern physics, which is often defined either chronologically or philosophically:

1. All physics from the start of the 20th Century and onwards is considered modern physics
2. The physics for which we do not have (nor may we ever have) a comfortable intuition for is modern physics.

This philosophical definition connects to the requirement that, in modern physics, we are often required to make a quantum leap - pun intended - or a leap of faith, because Quantum Theory does not allow us to follow our intuitions classical physics; nor our natural intuitions.

Black Body Radiation: Planck's Law

(Reference: pag 4-8, Feynman Lectures Vol 3)

A Black Body object is one which absorbs all incident electromagnetic radiation and when it is at thermal equilibrium with its environment, it emits electromagnetic radiation called black-body radiation.

Let us imagine an object in the shape of a cube with sidelength a , which is metal and hollow such that the wave equation - for electromagnetic waves - holds within it. Thus we know that there will be a wave vector inside:

$$\vec{k} = \frac{2\pi}{2a}(n_x, n_y, n_z) = \frac{2\pi\nu}{c}\hat{n}$$

Where we substituted $2a$ because the wavelength of the fundamental mode.

What will be the wave number in some direction? (Meaning the number of modes until some frequency ν in some direction i)

$$n_i = \frac{2a\nu}{c}$$

Thus the size of the vector \bar{n}

$$n^2 = n_x^2 + n_y^2 + n_z^2 = \left(\frac{2\pi\nu}{c}\right)^2 = \frac{4a^2\nu^2}{c^2}$$

Looking for the number of modes in the cube until some frequency ν , we can assume that $n \gg 1$ in order for the space to be approximately continuous. The vector \bar{n} exists within a sphere in the discrete space $\{n_i\}$.

The number of modes until ν

$$N(\nu) = 2 \cdot \frac{1}{8} \cdot \frac{4\pi}{3} n^3 = \frac{8\pi a^3 \nu^3}{3c^3}$$

Where we have multiplied by 2 because of the 2 polarizations for each axis, and by $\frac{1}{8}$ because only in one-eighth of the sphere are all of the magnitudes positive ($n > 0$ in n space). Essentially we wish to understand the amount of energy contained in the cube, and thus need to first understand the energy in a single mode.

In 1877 Ludwig Boltzmann proposed that the chance to be found at some location and momentum - (p, q) - can be given by a certain probability density:

$$(p, q) \rightarrow E(p, q)$$

$$P(p, q) = \frac{1}{z} e^{-\frac{E(p, q)}{k_B T}}$$

$$[P(E)] = \frac{1}{[E]}$$

Which is known as the **Probability Density Function** and is calculated which the (Hamiltonian form) Lagrangian as a function of p and q - $L = L(q, p, t)$

Where we define z , the partition function, as the constant of normalization:

$$\int dq dp P(p, q) = \frac{1}{z} \int dq dp e^{-\frac{E(q, p)}{k_B T}} = 1$$

Keep in mind that we will be discussing the probability of being in a particular range, **not** being at a point.

Notation: Often we will replace the Boltzmann coefficient with the thermodynamic beta in the following manner:

$$k_B = 1.38 \times 10^{-23} \frac{J}{K} \quad | \quad \beta = (k_B T)^{-1}$$

$$\Rightarrow P(p, q) = \frac{1}{z} e^{-\beta E(q, p)}$$

Thus to write the average energy of system at thermal equilibrium

$$\begin{aligned} \langle E \rangle &= \frac{1}{z} \int dq dp E(q, p) e^{-\beta E(q, p)} = -\frac{1}{z} \frac{\partial}{\partial \beta} \int dq dp e^{-\beta E(q, p)} \\ &= -\frac{1}{z} \frac{\partial}{\partial \beta} z = -\frac{\partial}{\partial \beta} \ln z \end{aligned}$$

Now to find the average energy of a harmonic oscillator at thermal equilibrium we can first calculate the partition function

$$z = \int dq dp e^{-\beta E(q, p)} = \int dq dp e^{-\beta \left(\frac{p^2}{2m} + \frac{m\omega^2}{2} q^2 \right)} = \sqrt{\frac{2\pi m}{\beta}} \sqrt{\frac{2\pi}{m\omega^2 \beta}} = \frac{2\pi}{\omega \beta}$$

Thus

$$\langle E \rangle_{oscillator} = \frac{\partial}{\partial \beta} \left(-\ln \frac{2\pi}{\omega} + \ln \beta \right) = k_B T$$

“A mode of the electromagnetic field is analogous to the mode of a mechanical oscillator”

Now this is the probability of finding the system at some energy, thus the average energy until some frequency ν is

$$U(\nu) = N(\nu) k_B T$$

Meaning that we can define the energy density per unit volume and per unit frequency as

$$u(\nu) = \frac{1}{V} \frac{\partial U}{\partial \nu} = \frac{8\pi\nu^2}{c^3} k_B T$$

Assuming that all of the frequencies are directed/distributed evenly to all directions. Thus we can find the flux of energy per unit volume and unit frequency as

$$R(\nu) = \frac{c}{4\pi} u(\nu) = \frac{2\nu^2}{c^2} k_B T$$

Which is know as the **Rayleigh-Jeans Law**.

This relation works fine for low frequencies but, as a parabolic function, the energy - expressed as the integral - would diverge to infinity as the the frequency reaches and passes ultraviolet, leading to the **Ultraviolet Catastrophe**.

This realization demonstrated that there was a fundamental flaw in the classical theory that the Rayleigh-Jeans law was based upon.

Contemporaneously, Max Planck

Enter Max Planck, who found a partition function that fits under the assumption that not every (continuous) energy works.

$$E_n(\nu) = nh\nu$$

Where h is Planck's constant $h = 6.626 \times 10^{-34} J \cdot s$

Thus because we can only work with specific discrete energies (p and q)

$$\begin{aligned} z &= \sum_{n=0}^{\infty} e^{-\beta nh\nu} \sum_{n=0}^{\infty} (e^{-\beta h\nu})^n = \frac{1}{1 - e^{-\beta h\nu}} \\ \Rightarrow \langle E \rangle &= -\frac{\partial}{\partial \beta} \ln z = -\frac{\partial}{\partial \beta} \ln \left(\frac{1}{1 - e^{-\beta h\nu}} \right) = \frac{h\nu e^{-\beta h\nu}}{1 - e^{-\beta h\nu}} = \frac{h\nu}{e^{\beta h\nu} - 1} \end{aligned}$$

If $h \rightarrow 0$, we find the same result as in classical regime $k_B T$.

Finding the formula for energy flux from Planck

$$R(\nu) = \frac{z}{c^2} \frac{h\nu}{e^{\frac{h\nu}{k_B T}} - 1}$$

What is the overall power emitted?

$$P = \sigma T^4$$

(we'll find σ in homework but I'm pretty sure that it is the radiation constant seen in astrophysics $a = \frac{8\pi^5 k^4}{15c^3 h^3} \dots$ actually the Stefan-Boltzmann constant $\sigma = \frac{2\pi^5 k^4}{15c^3 h^3}$).

Wien's Displacement Law:

$$\lambda_{peak} = \frac{b}{T} = \frac{2.9 \times 10^{-3}}{T} [m]$$

2 March 20th

Photoelectric Effect:

(Messiah 11)

Electron emission from Alkali metal as a result of a light being shone on the metal plate. 1902 observation. 1905 Einstein published 4 groundbreaking papers: special relativity, brownian movement, photoelectric, +?

Light is made of packets of energy $E_{photon} = h\nu$. When this light packet hits a metal the metal releases an electron (1-1)

Work function W the work needed to be done to release an electron from the metal

$$E_{\text{released electron}} : \frac{1}{2}mv^2 = h\nu - W$$

When the wavelength is above a certain threshold this will happen

(Einstein got his Nobel prize for this effect)

h was not necessarily used by Einstein initially, and it was found experimentally that his constant in this context was indeed the same h of Planck

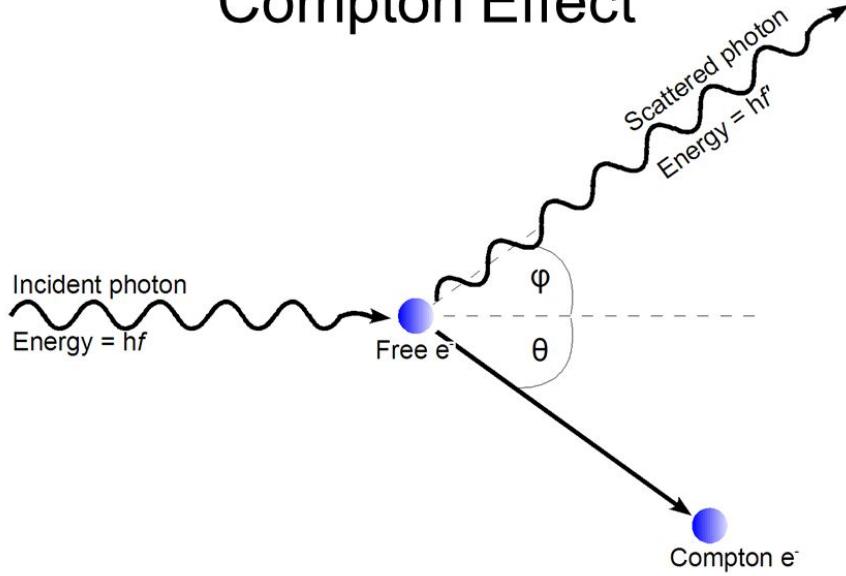
Compton Effect:

(Messiah 13)

When you have EM radiation (short wavelength like x-ray) and it is scattered on a cloud of free (or almost free) electrons (like in conductors).

The effect observed was that some of the light scatters to other angles and the wavelength of the scattered light is longer than the original wavelength.

Compton Effect



Hull and East Yorkshire Hospitals NHS Trust

Figure 1:

This difference is only dependent on the incident angle. Why?

Compton followed Einstein's thinking in that we can view light as particles (photons) with a specific energy, and looked at a consequent momentum which could be associated with photons.

If a photon also has momentum $E_{photon} = h\nu = pc \Rightarrow p = \frac{h\nu}{c} = \frac{h}{\lambda}$.

From conservation of momentum and the cosine theorem

$$\vec{p}'_e = \vec{p} - \vec{p}' \Rightarrow p'_e^2 = p^2 + p'^2 - 2pp' \cos\theta$$

The initial energy will be the resting energy of the electron + the energy of the photon

$$m_e c^2 + pc = \sqrt{m_e^2 c^4 + p_e'^2 c^2} + p' c$$

$$\Rightarrow [m_e c^2 + (p - p')]c]^2 = m_e^2 c^4 + p_e'^2 c^2$$

Substituting p'_e from conservation of momentum

$$m_e^2 c^4 + 2m_e c^3(p - p') + p^2 c^2 + p'^2 c^2 - 2pp' c^2 = m_e^2 c^4 + (p^2 + p'^2 - 2pp' \cos\theta)c^2$$

$$\Rightarrow 2m_e c^3(p - p') - 2pp' c^2 = (-2pp' \cos\theta)c^2$$

$$\Rightarrow m_e c(p - p') = pp'(1 - \cos\theta)$$

So we can find the dependence on the change in wavelength by

$$\frac{p - p'}{pp'} = \frac{\frac{h}{\lambda} - \frac{h}{\lambda'}}{\frac{h^2}{\lambda\lambda'}} = \frac{1}{h}(\lambda' - \lambda) = \frac{2}{m_e c} \sin^2\left(\frac{\theta}{2}\right)$$

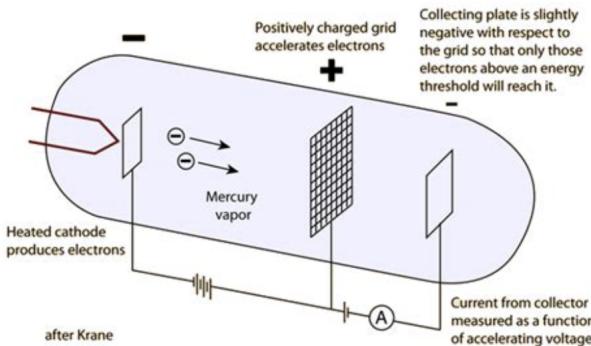


Figure 2:

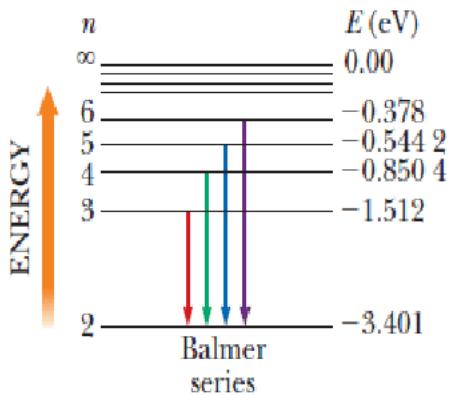


Figure 3:

$$\Rightarrow \Delta\lambda = 2 \cdot \underbrace{\frac{h}{m_e c}}_{\text{compton wavelength}} \cdot \sin^2\left(\frac{\theta}{2}\right)$$

Maybe light transfers the energy slowly, not all at once. That would meant that perhaps the it would require a certain amount of energy to be transferred before the electron “emits”. But then there would be a time difference between transfer and emission (same goes for the photoelectric effect) which is not found.

Compton wavelength is the effective magnitude of the electron.

Atomic Structure:

(Messiah 21)

1911 - Rutherford, in cambridge, scattered alpha particles towards a thin gold plate and found: At the center of an atom there is a positive nucleus surrounded by a negative cloud.

A charge that is accelerating emits radiation.

Circular motion requires constant acceleration therefore there will be constant radiation, meaning that they are losing energy and therefore the electrons will still not be stable in their orbits.

1885 - Johan Balmer found the formula for helium emission lines

$$\nu = R_{yderg} \left(\frac{1}{n^2} - \frac{1}{m^2} \right)$$

and there are the same wavelengths for absorption lines...

Niels Bohr - 1913

Bohr Model:

He suggested that the reason for the behaviour is that the electrons can only have a certain, discrete finite set of energy levels.

And only if the energy of the photon is precisely equal to the energy difference between levels/shells with the jump occur

$$h\nu = E_j - E_i$$

And the energy levels (For hydrogen) must be

$$E_n = -\frac{hR}{n^2}$$

In the same way that the electron is forbidden from going up without a specific energy, it is forbidden from down for the same reason.

1923 - de Broglie Wavelength:

(B, T 18)

Light was a wave phenomenon, while electrons were particle like. We found a way to express energy and momentum and therefore light as a particle, so why can't we look as to whether particles and matter can have wave-like characteristics.

Photons have momentum $p = \frac{h}{\lambda} = \frac{h}{2\pi} \frac{2\pi}{\lambda} = \hbar k$

Thus the wavelength of a particle (DB wavelength) is

$$\lambda = \frac{h}{p}$$

The circumference of the orbital is $2\pi r = n\lambda = n\frac{2\pi}{k}$

Where the radius can be $r = \frac{n}{k}$ ($k = \frac{mv}{\hbar}$)

We find that

$$L = mvr = n\hbar$$

(\hbar and \hbar have units of angular momentum)

The mass written is the mass of the electron (ignoring the fact that it is combined mass of the electron AND the nucleus? because the electron is 2000 times larger)

$$\frac{mv^2}{r} = k_e \frac{e^2}{r^2}$$

Where k_e is coulomb's constant and the velocity is

$$v = \frac{n\hbar}{mr}$$

Thus

$$\begin{aligned} \frac{m}{r} \left(\frac{n\hbar}{mr} \right)^2 &= k_e \frac{e^2}{r^2} \\ \Rightarrow r &= \frac{1}{k_e e^2} \frac{n^2 \hbar^2}{m} = \frac{n^2 \hbar^2}{k_e e^2 m} = n^2 a_0 \end{aligned}$$

$a_0 \approx 0.52 \text{\AA}^\circ$ is the bohr radius which is the radius for hydrogen at something....

Thus

$$v = \frac{n\hbar}{m} \frac{k_e e^2 m}{\hbar^2} = \frac{k_e e^2}{n\hbar}$$

Therefore using the virial theorem

$$E = \frac{mv^2}{2} - k_e \frac{e^2}{r^2} = -\frac{mv^2}{2} = -\frac{m}{2} \frac{k_e^2 e^4}{n^2 \hbar^2} = -\frac{1}{n^2} \frac{mk_e^2 e^4}{2\hbar^2} = -\frac{1}{n^2} E_I$$

Where E_I is the ionization energy and we see that $E_I = hR_{ydb erg}$

March 22th - Tutorial

Reminder - Basic concepts in probability theory

X Random Variable - Discrete:

- Represents a collection of possible states, $x_i \in \{x_1, \dots\}$

Any state like this is obtained at a probability $p_X(x)$

An intuitive definition of p : Imagining that we measured N realizations of X , we will denote N_{x_i} as the number of x_i was realized

1. $0 \leq p_X \leq 1$
2. $\sum_{x_i} p_X(x_i) = 1$ the probability to be in some state...
3. $[p_X(x)] = 1$ - unitless

The expectation value $\mu = \langle X \rangle = E(X) = \sum_{x_i} x_i p(x_i)$ the average value of X obtained after infinite “throws” in the case of dice.

Variance - second moment (concentrated)

$$\sigma^2(X) = \text{Var}(X) = E[(X - \mu)^2] = \sum_{x_i} (x_i - \mu)^2 p(x_i) = E(X^2) - [E(X)]^2$$

This is the average of the deviation from the expectation value squared

Continuous random variable X:

Here we can only define the probability of X existing within a particular range of variables

$$p_X(a \leq x \leq b) = \int_a^b f_X(x) dx$$

where f_X is the density function of probability, probability per unit distance

1. $0 \leq p_X \leq 1$
2. $0 \leq f_X$
3. $\int_{-\infty}^{\infty} f_X(x) dx = 1$
4. $[f] = \frac{1}{[X]}$
5. $E(X) = \int_{-\infty}^{\infty} X f_X(x) dx$
6. $\text{Var}(x) = \dots \int_{-\infty}^{\infty} X^2 f_X(x) dx \dots$

Example - Function of random variables:

The purpose of being able to change the function is best given by the example of changing black body radiation from a function of wavelength to a function of frequency.

Black Body radiation:

$R_\nu d\nu dA d\Omega$ is the power of the energy emitted in the frequency range $[\nu, \nu + d\nu]$ into the solid angle $d\Omega$ from the area element dA

3 March 25th

Correspondence Principle and the (Old) Quantum Theory

(Messiah 27)

This filled the role that is required for every new theory, understanding the previous theory. Just like understanding that Newton was not mistaken, he simply was imprecise in the areas which were disconnected from him. So we are looking for what the old theory covers and where it falls down and there is a need for a new theory.

Quantum number, that counts the energy number.

For high numbers (high energy) we expect the classical theory will converge with the quantum theory. (Pavel gave the wonderful example of special relativity converging with classical mechanics for low velocities)

Density of the levels:

$$\frac{dH}{dn} \xrightarrow{n \rightarrow \infty} h\nu_{classical}(E)$$

The correspondence principle takes the Hamiltonian as a function of n to the limit of $n \rightarrow \infty$ (see correspondence or classical limit). What Bohr did that was special was to check the in the other direction (old quantum theory) and he also provided a rough prescription for the limit: it occurs when the quantum numbers describing the system are large.

(the difference between energy levels)

where H is the classical hamiltonian of the system and n is the quantum number ($\nu_{classical} = \frac{1}{T_{period}}$)

$$\int_{E_{min}}^E \frac{dH}{\nu_{clas}(E)} = nh + Const$$

This is in essence the real formulation of the of the correspondence principle.

Assuming that the system has some minimum energy

$$\begin{aligned} \int_{E_{min}}^E \frac{dH}{\nu_{clas}(E)} &= \int_{E_{min}}^E dH T_{per}(E) = \int_{E_{min}}^E dH \oint \frac{dq}{\dot{q}} = \int_{E_{min}}^E dH \oint \frac{dq}{\frac{\partial H}{\partial p}} = \int_{E_{min}}^E dH \oint dq \frac{\partial p}{\partial E} \\ &= \int_{E_{min}}^E dH \frac{\partial}{\partial E} \oint pdq = \oint_{H(q,p)=E} pdq = \int_{H(q,p) \leq E} dq dp = Action = nh + const \end{aligned}$$

We choose some energy level and take it's path which is described by:

$$\oint_{H(q_i, p_i) = E} p_i dq_i = n_i h$$

This is known as the **Bohr-Sommerfeld Quantization**.

We'll find quantum numbers associated to each degree of freedom. The action is quantized. Recalling that $p = \hbar k$. We shall also recall that the meaning of the path integral is area in phase space, meaning that the area in the phase space is conserved.

This quantity is called the Adiabatic Invariant and is conserved under small parameter changes of the problem according to classical analytical mechanics. It describes a closed path integral on the action in the phase space formulation.

This equation is the mathematical formulation of "old" quantum mechanics.

Light and Polarization Dependent Beam Splitter (PBS):

(Dirac 4, CT 15)

(Fig 1) - 25/3

An electromagnetic planar wave propagating in the \hat{z} direction

$$\bar{E} = \hat{e}_i E_0 e^{i(kz - wt)}$$

Where \hat{e}_i is the vector describing the direction of the light's polarization.

Let's look at three examples: parallel, perpendicular, and let's specifically look at a the 45° polarization:

$$\hat{e}_+ = \frac{1}{\sqrt{2}}(\hat{e}_x + \hat{e}_y)$$

A PBS is a special optical component which transfers polarization in the \hat{x} direction and reflects polarization in the \hat{y} direction. Passing a beam of light through the PBS, half will pass through and half will be reflected, with the beams receiving the respective polarization.

For a single photon we'll have a problem because one of the detectors will have to be "pinged" but the question is which? (because the photon cannot be split)

For a single photon it will be arbitrary, but for $n \rightarrow \infty$ we will see a 50/50 split between the two detectors (determined by the (square) coefficient on the direction of polarization).

A general polarization that we can write is:

$$\hat{e}_i = \hat{e}_x \cos\theta + \hat{e}_y \sin\theta e^{i\varphi}$$

What will happen now with a single photon.

Malus's Law:

$$E^2 = \cos^2\theta \quad I = I_0 \cos^2\theta_i$$

Thus $(\cos\theta)^2$ part will hit the X detector and $(\sin\theta)^2$ part will hit the Y detector

The chance of hitting X: $p_X = |\cos\theta|^2$

The chance of hitting Y: $p_Y = |\sin\theta e^{i\varphi}|^2$

Ten Commandments (Insights)

1. The electric field of light is used at the level of a single photon (quantum) for a probabilistic description of the experiment's outcomes.
2. Just like the electric field holds the linear Maxwell equations - in which the combination of solutions is a solution - so does the probabilistic description hold this principle (exception of the probabilistic description to uphold some linearity) - (superposition).
3. We cannot predict the result of a single experiment, and thus it cannot teach us about the nature of a photon. The romantic and classical determinism must be neglected.
4. In order to predict (measure) in the probabilistic description of the photon, you must repeat the experiment a large number of times, where the process of the system's preparation must be identical each time.
5. Unlike the classical case wherein there is a continuity of possible results as a function of the continuous angle θ , for the lone photon (in this experiment) there are only two options for single measurement. We will call these results the characteristic results (eigenresults) and the values the characteristic values (eigenvalues) of the measurement. (in the PBS case the value is the angle - 0 for X, 90 for Y, and the result is which detector).
6. For states which give an definite result from the eigenvalues, these states are called the characteristics states (eigenstates) of the measurement. (In this case \hat{e}_x and \hat{e}_y).
7. Some general state (not necessarily eigenstate) is written as a linear combination of eigenstates of the measurement, according to the superposition principle. The probability for an eigenresult is given by the absolute value of the magnitude of the amplitude squared of the suitable eigenstate. (just like the p_X and p_Y above).
8. The eigenstates belong to/are associated with the measuring device and the manner of its use. Meaning that different devices will have different eigenstates.
9. After the photon's passage or reflection, the placement of an additional, identical PBS will result in the same result as before. Meaning, after one measurement we will already know with certainty the measurement for the second time. This is because now, all we are introducing into the system are eigenstates. In practice this is the point by which it is determined that the measurement itself influences the outcome of the additional measurement. Measuring that the measurement itself causes significant mathematical changes to our system.

10. Until now we have implied that the whole theory arrived from or relates to photons. However de-Broglie found a wavelength for electrons. Because we see a duality both in electrons and photons, we can expect the logic of quantum, and the formulation of quantum theory, to include and relate to both all wave and particle phenomena.

4 March 27th

State Space and State Vectors: (C.T 109)

Postulate 1a: It is possible to describe the state of a quantum system at a particular moment by a vector in complex Hilbert space. We shall call this space “State Space” and the vector the “State Vector” (Insights 1 & 2). We will denote the Hilbert space \mathcal{H} . A vector in this space will be denoted ψ . Let’s choose an element for example:

$$\psi = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}, \quad \psi' = \begin{bmatrix} c'_1 \\ c'_2 \end{bmatrix}$$

Where $c_1, c_2 \in \mathbb{C}$

Inner product:

$$(\psi', \psi) = \sum_i \bar{c}'_i c_i$$

where the overline (and * in the properties) represents the conjugate.

Properties:

$$(\varphi, \psi) = (\psi, \varphi)^*$$

Linear for the righthand element:

$$(\varphi, c_1\psi_1 + c_2\psi_2) = c_1(\varphi, \psi_1) + c_2(\varphi, \psi_2)$$

and linear for the lefthand element:

$$(c_1\varphi_1 + c_2\varphi_2, \psi) = c_1^*(\varphi_1, \psi) + c_2^*(\varphi_2, \psi)$$

Size of vector:

$$\sqrt{(\psi, \psi)}$$

If $(\varphi, \psi) = 0$ then the vectors are perpendicular (orthogonal).

Let’s now define a new notation for vectors, Dirac’s notation.

$$|\psi\rangle - \text{ket}$$

For example

$$|\psi_1\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, |\psi_2\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow |\psi\rangle = c_1|\psi_1\rangle + c_2|\psi_2\rangle$$

Let’s define: A linear functional χ , which acts on a vector and returns a scalar. $\chi(|\psi\rangle)$

$$\chi(c_1|\psi_1\rangle + c_2|\psi_2\rangle) = c_1\chi(|\psi_1\rangle) + c_2\chi(|\psi_2\rangle)$$

The collection of all the linear functionals which act on a Hilbert space are themselves a Hilbert Space

This Hilbert space of functionals \mathcal{H}^* is called the Dual Hilbert Space.

We shall denote an element in the \mathcal{H}^* space:

$$\langle \chi | - \text{bra}$$

The functional χ acting upon a vector ψ will be denoted

$$\boxed{\langle \chi | \psi \rangle}$$

$$\langle \chi | \psi \rangle = (|\chi\rangle, |\psi\rangle)$$

$$c_1|\psi_1\rangle + c_2|\psi\rangle \Rightarrow c_1^*\langle\psi_1| + c_2^*\langle\psi_2|$$

NOTE: Because both of the spaces are complete spaces (no “points missing” from it - inside or at the boundary - examples include $[0, 1]$, \mathbb{R} , \mathbb{C})

Postulate 1b: When a state vector, whose magnitude is unity (1), is a linear superposition of a number of orthonormal states, the probability of finding the system in either of the states is a squared absolute value quantity of the state (insight 7) (C.T. 253)

$$\begin{aligned} \langle \psi | \psi \rangle &= 1 \\ |\psi\rangle &= c_1|\psi_1\rangle + c_2|\psi_2\rangle \end{aligned} \Rightarrow \begin{aligned} P(\psi_1) &= |c_1|^2 \\ P(\psi_2) &= |c_2|^2 \end{aligned}$$

Why do we require $\langle \psi | \psi \rangle = 1$?

$$\langle \psi | \psi \rangle = \sum_i c_i^* c_i = \sum_i P(\psi_i) = 1$$

We required this in order for the sum of the probabilities of all of the states to be 1.

Assuming now that $c_2 = \rho_2 e^{i\varphi}$. Now $P(\psi_2) = |\rho_2|^2$ which means that the phase does not affect!

$$P(\psi_i) = |\langle \psi_i | \psi \rangle|^2 = |c_1 \langle \psi_i | \psi_1 \rangle + c_2 \langle \psi_i | \psi_2 \rangle|^2 = |c_i|^2$$

The result of this is that this postulate is correct for every vector $|\psi_i\rangle$ of length 1. This is because we can use a unitary (conserves lengths) matrix and thus we need only show that the probability remains the same if the orthonormal base is changed by rotation.

What we are seeing here can effectively be called “Quantum Interference”.

The only state that we will see for two different states which results in the same probabilistic result, will be the state in which the vector is multiplied by some phase.

Meaning that if

$$|\psi'\rangle = e^{i\varphi}|\psi\rangle$$

Thus

$$p(|\psi'\rangle) = p(|\psi\rangle)$$

Meaning that the global phase does not affect anything, because it falls in the conjugation. We can see this explicitly. Assuming that:

$$\begin{aligned} |\psi'\rangle &= \rho_1 e^{i\theta_1} |\psi_1\rangle + \rho_2 e^{i\theta_2} |\psi_2\rangle \\ &= e^{i\theta_1} (\rho_1 |\psi_1\rangle + \rho_2 e^{i(\theta_2 - \theta_1)} |\psi_2\rangle) \end{aligned}$$

Meaning that the only phase that matters is the relative phase between the elements. The probability $P(\psi')$ is obtained by:

$$|\psi'\rangle = c'_1 |\psi_1\rangle + c'_2 |\psi_2\rangle$$

$$\Rightarrow p(\psi') = |\langle \psi' | \psi \rangle|^2 = |(\langle \psi_1 | c_1^* + \langle \psi_2 | c_2^*)(c_1 | \psi_1 \rangle + c_2 | \psi_2 \rangle)|^2 = |c_1'^* c_1 + c_2'^* c_2|$$

Which is where the relative phase fits in.

$$|\varphi\rangle = c'_1 |\psi_1\rangle + c'_2 |\psi_2\rangle$$

$$P(\varphi) = |\langle \varphi | \psi \rangle|^2$$

In two dimensional Hilbert Space we have 4 parameters. It is a complex space, in which each coefficient has a global and relative, phase and magnitude. The degree of freedom of the global phase does not make a physical impact. As such, we have removed another degree of freedom, as we connected between the sizes of the coefficients by requiring the size of the overall vector to be 1. Meaning that at the moment we obtain one of the magnitudes and the relative phase, we have perfectly determined the physical system. In other words there are, in practice, two independent variables required to describe the physical state.

Warning:

We said that the sum of physical states is a physical state. However, postulate 1b works only if the states are normalized. Because the sum of normalized states is not necessarily normalized, we must ensure to normalize them, either in the general formula we will obtain, or by a normalized of the sum in advance.

5 April 10th

Linear Operators:

(C.T 114)

Postulate 2: For every (measurable) physical quantity there is an associated Hermitian linear operator.

An operator is an action upon a vector - vector goes in and vector comes out. (We are working on vectors in Hilbert space that return vectors in the same Hilbert space).

(Capital letters as operators)

$$A|\psi\rangle = |\psi'\rangle$$

And because we require the operator be linear:

$$A(c_1 |\psi_1\rangle + c_2 |\psi_2\rangle) = c_1 A|\psi_1\rangle + c_2 A|\psi_2\rangle$$

At this point it is important to recall that the vectors will be represented by column and row vectors and operators (considering that they are linear transformations) can be represented by matrices. What happens when a linear operator is enacted on another operator? That is in essence the enactment of their product (composition)

$$(AB)|\psi\rangle = A(B|\psi\rangle) \neq (BA)|\psi\rangle$$

Generally the order of operations is relevant and changes the operation.

Writing:\

$$|\psi\rangle\langle\varphi|$$

When organized as $\langle\psi|\psi\rangle$ we found the product of a row vector with a column vector (resulting in a scalar). We recall that the product of a column vector with a row vector from the left does **not** result in a scalar. Therefore we can understand what will now be obtained by this enactment of this operator

While $\langle\varphi||\psi\rangle = \langle\varphi|\psi\rangle$ is the inner product, what is $|\psi\rangle\langle\varphi| |\chi\rangle$?

$$|\psi\rangle\langle\varphi||\chi\rangle = |\psi\rangle\langle\varphi|\chi\rangle = \langle\varphi|\chi\rangle|\psi\rangle$$

Let's define the operator

$$P_\psi = |\psi\rangle\langle\psi|$$

Thus

$$P_\psi|\varphi\rangle = |\psi\rangle\langle\psi||\varphi\rangle = |\psi\rangle\langle\psi|\varphi\rangle = \langle\psi|\varphi\rangle|\psi\rangle$$

Thus this operator is the projection operator as it gives us the length $\langle\psi|\varphi\rangle$ in the $|\psi\rangle$ direction (projection on $|\psi\rangle$). What is the meaning of $|\psi\rangle\langle\psi|$?

$$P_\psi^2 = P_\psi P_\psi = |\psi\rangle\langle\psi||\psi\rangle\langle\psi| = |\psi\rangle\langle\psi| = P_\psi$$

If we take a vector and apply a linear operator and then a linear functional

$$\langle\varphi|(A|\psi\rangle)$$

Which we shall call the Element of the matrix of A between $|\psi\rangle$ and $\langle\varphi|$

This is a mixture of linear actions and actions on a vector, and the final result is a scalar. Meaning that we expect the overall action to be a functional. I.e. there exists a functional $\langle\varphi|A$ in Dual Space which returns the same value which we shall denote $\langle\varphi'|$

$$\langle\varphi'| = \langle\varphi|A$$

We shall define

$$\langle\varphi|(A|\psi\rangle) = \langle\varphi'|\psi\rangle = (\langle\varphi|(A)|\psi\rangle) = \langle\varphi|A|\psi\rangle$$

We can use the linear operator from the right on $\langle\varphi|$ and obtain a vector in Dual Space (functional). I.e. A can be defined also as an operator in Dual Space. we can denote the Dual Space A as A^\dagger . In other words: By isomorphism between *ket* space and *bra* space, we should be able to suit the vector $A|\varphi\rangle$ in *ket* space to the functional $\langle\varphi|A^\dagger$ in *bra* space.

Just as we have the operator A associated to ψ by $A|\psi\rangle = |\psi'\rangle$, we can similarly look for the connection between $\langle\psi|$ and $\langle\psi'|$

$$\langle\psi|A^\dagger = \langle\psi'|$$

Which is known as Hermitian Conjugation.

We shall show that the connection between these elements is indeed connected to the standard meaning of the \dagger symbol in algebra - conjugate transpose.

Characteristics of Hermitian Conjugation:

Recalling that

$$\langle\psi|\varphi\rangle = \langle\varphi|\psi\rangle^*$$

Thus

$$\langle\psi|A^\dagger|\varphi\rangle = \langle\varphi|A|\psi\rangle^*$$

Therefore

$$A^{\dagger\dagger} = A$$

Regarding scalar multiplication $c \in \mathbb{C}$

$$(cA)^\dagger = c^* A^\dagger$$

and also

$$(A + B)^\dagger = A^\dagger + B^\dagger$$

Enacting the dagger on composition of two operators, assuming that

$$|\varphi\rangle = AB|\psi\rangle$$

We know that there must exist $|\chi\rangle$ such that

$$|\chi\rangle = B|\psi\rangle \Leftrightarrow \langle\chi| = \langle\psi|B^\dagger$$

Thus

$$|\varphi\rangle = A|\chi\rangle = AB|\psi\rangle$$

So

$$\langle\varphi| = \langle\chi|A^\dagger = \langle\psi|B^\dagger A^\dagger$$

Therefore it holds that

$$(AB)^\dagger = B^\dagger A^\dagger$$

Multiplying a bra with a ket results in a scalar and a ket with a bra with an operator... wait no actually with a vector.

Definitions of State Space: (C.T 121)

Representation $\{|u_i\rangle\}$ orthonormal whole basis

$$\langle u_i | u_j \rangle = \delta_{ij} \text{ where } \delta_{ij} = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases}$$

$$|\psi\rangle = \sum_i c_i |u_i\rangle$$

Detailed explanation of series representation of vectors:

page 28 of summary.

6 April 15th

Transformation Matrices (C.T. 130)

Assuming that two whole orthonormal bases are given

$$\{u_i\}, \{u'_i\}$$

And some vector

$$c_i = \langle u_i | \psi \rangle$$

also represented as

$$c'_i = \langle u'_i | \psi \rangle$$

And the definition:

$$c_i = \langle u_i | \psi \rangle = \langle u_i | \mathbf{1} | \psi \rangle = \Sigma_j \langle u_i | u'_j \rangle \langle u'_j | \psi \rangle = \Sigma_j t_{ij} c'_j$$

With the matrix elements of the operator T

$$t_{ij} = \langle u_i | u'_j \rangle$$

Thus the elements of the inverse elements are (from the operator T^{-1})

$$t_{ij}^{(-1)} = \langle u'_i | u_j \rangle = \langle u_j | u'_i \rangle^* = t_{ji}^*$$

which are the elements of the operator T^\dagger

Thus we find the unitary operator

$$TT^\dagger = T^\dagger T = \mathbf{1}$$

Measurement Results (C.T. 216)

Postulate 3 : The possible values for the result of a measurement of a physical value are described by the Hermitian Operator are solely its eigenvalues. (commandment 5)

Postulate 4: After the measurement the state of the system changes and collapses to the eigenstate belonging to the measured eigenvalue. (commandment 9)

(Postulate 1b): The chance to find a system in a superposition of two orthonormal (in a particular eigenvalue) is determined by t...

The more degrees of freedom a system has, the faster it loses its “quantumness” and becomes statistical - The Church of the Larger Hilbert Space.

Once again we have to provide the algebra behind these two postulates (3 & 4)

If we have an operator A , its eigenvectors $|\psi_i\rangle$ and the eigenvalues λ_i

$$A|\psi_i\rangle = \lambda_i |\psi_i\rangle$$

As an example let's look at the projection operator $P_\psi = |\psi\rangle\langle\psi|$

$$P_\psi |\varphi\rangle = |\psi\rangle\langle\psi|\varphi\rangle = \lambda |\varphi\rangle$$

Two cases:

$$1. |\psi\rangle = |\varphi\rangle \rightarrow \lambda = 1$$

$$2. |\psi\rangle \perp |\varphi\rangle \rightarrow \lambda = 0$$

The projection operator represents a measurement of whether the state is the ψ state. We are not using the algebraic operation of the projection operator, rather to ask the question whether the system is in the state ψ and to obtain a boolean result.

Finding the eigenvalues/vectors using the representation $\{|u_i\rangle\}$

$$\langle u_i | A | \psi_k \rangle = \Sigma_j \langle u_i | A | u_j \rangle \langle u_j | \psi_k \rangle = \lambda_k \langle u_i | \psi_k \rangle$$

$$\Sigma_j a_{ij} c_j = \lambda_k c_i \Rightarrow \Sigma_j a_{ij} c_j - \lambda_k \delta_{ij} c_j = \Sigma_j (a_{ij} - \lambda_k \delta_{ij}) c_j = 0$$

In vector notation

$$(A - \lambda_k \mathbf{1}) \bar{c} = 0 \Rightarrow |A - \lambda_k \mathbf{1}| = 0$$

Which gives us the characteristic polynomial and the set of eigenvalues

$$\{\lambda_i\}$$

known as the spectrum of A . Which is a non-trivial set.

The probability P to be in a state λ_i is the absolute square of the projection of ψ_i on the initial state

$$P(\lambda_i) = |\langle \psi_i | \psi \rangle|^2 = |c_i|^2$$

The projection operator $P_i = |\psi_i\rangle\langle\psi_i|$ is projected during the measurement

Thus we find

$$|\psi'\rangle = P_i|\psi\rangle = |\psi_i\rangle\langle\psi_i|\psi\rangle = c_i|\psi_i\rangle$$

using the identity $|\psi\rangle = \sum_i c_i |\psi_i\rangle$ for the last step.

If we measure again we obtain the same result as the span of an eigenvector is the same eigenvector. So the same eigenvalue will be obtained.

(C.T. 136)

$$\langle\psi_i|A|\psi_i\rangle = \langle\psi_i|A^\dagger|\psi_i\rangle = \langle\psi_i|A|\psi_i\rangle^* = \lambda_i \langle\psi_i|\psi_i\rangle \Rightarrow \lambda_i \text{ is real}$$

Thus the eigenvalues of hermitian operators are real.

$$\langle\psi_i|A^\dagger = \langle\psi_i|A = \langle\psi_i|\lambda_i^* = \langle\psi_i|\lambda_i$$

The equations of eigenvalues:

$$A|\psi_i\rangle = \lambda_i|\psi_i\rangle$$

$$A|\psi_j\rangle = \lambda_j|\psi_j\rangle$$

Multiplying - projecting the first equation on ψ_j and the second on ψ_i

$$\langle\psi_j|A|\psi_i\rangle = \lambda_i \langle\psi_j|\psi_i\rangle$$

$$\langle\psi_i|A|\psi_j\rangle = \lambda_j \langle\psi_i|\psi_j\rangle$$

Hermitian Conjugation of the second equation.

$$\langle\psi_j|A|\psi_i\rangle = \lambda_j \langle\psi_j|\psi_i\rangle$$

Therefore we find

$$(\lambda_i - \lambda_j) \langle\psi_j|\psi_i\rangle = 0$$

For $\lambda_i \neq \lambda_j$ we see that $|\psi_i\rangle \perp |\psi_j\rangle$

Where $\{|\psi_i\rangle\}$, A - and the basis is orthonormal and whole - are called observable - C.T. 137.

Observable operators - commutating/commuting (C.T. 139)

Where the order of operations doesn't matter - commutator: $AB - BA \equiv [A, B] = 0$

For the basis $\{|\psi_i\rangle\}$

The eigenvalues of A are λ_i and of B are μ_i but have the same eigenvectors.

$$[A, B]|\psi\rangle = (AB - BA)\Sigma_i c_i |\psi_i\rangle = \Sigma_i c_i (\mu_i A - \lambda_i B)|\psi_i\rangle$$

$$= \Sigma_i c_i (\mu_i \lambda_i - \lambda_i \mu_i) |\psi_i\rangle = 0$$

if there exists a whole set... ... two operators are mitchalfim.

If two operators are mitchalfim, it is possible to find a whole set...

Choosing two eigenvectors of A

$$A|\psi_i\rangle = \lambda_i|\psi_i\rangle$$

$$A|\psi_j\rangle = \lambda_j|\psi_j\rangle$$

Doing a similar projection as before in the normality proof

$$\langle\psi_j|BA|\psi_i\rangle = \lambda_i\langle\psi_j|B|\psi_i\rangle$$

$$\langle\psi_i|BA|\psi_j\rangle = \lambda_j\langle\psi_i|B|\psi_j\rangle$$

Taking the hermitian conjugation of the first equation

$$\langle\psi_i|AB|\psi_j\rangle = \lambda_i\langle\psi_i|B|\psi_j\rangle$$

Subtracting

$$\langle\psi_i|AB - BA|\psi_j\rangle = (\lambda_i - \lambda_j)\langle\psi_i|B|\psi_j\rangle = 0$$

$$\Rightarrow \langle\psi_i|B|\psi_j\rangle = \delta_{ij}\mu_i$$

So B is diagonalizable by the same set that A is, which we shall prove.

Multiplying both sides by $\Sigma_i|\psi_i\rangle$

$$\Sigma_i|\psi_i\rangle\langle\psi_i|B|\psi_j\rangle = \Sigma_i\delta_{ij}\mu_i|\psi_i\rangle$$

$$= B|\psi_j\rangle = \mu_j|\psi_j\rangle$$

Thus A and B can be called COMPATIBLE (mtuamim).

This is why the measurement of A will give us information on B . i.e. the measurement of the second one does not give us additional important.

The physical meaning of two operators being mitchalfim and observable - where the observability means that there is a whole set and the commutativity means that the sets are the same one set.

7 April 22nd

Observation Value (C.T. 227)

If we want to learn something about a state we can prepare a series of identical systems for identical measurements, giving us a set spanned by a single vector.

If the initial state is $|\psi\rangle$ and we take N measurements, for λ_i eigenvalues \rightarrow we will get n_i measurements, $\Sigma_i n_i = N$

As $N \rightarrow \infty$ the probability $P_i(\lambda_i) \rightarrow \frac{n_i}{N} = |\langle\psi_i|\psi\rangle|^2$.

It is natural to talk about the average. What is the average value obtained? (not necessarily one of the obtainable values).

$$\langle A \rangle_\psi = \Sigma_i \lambda_i P_i(\lambda_i)$$

A very simple way to calculate this is

$$\langle A \rangle_\psi = \Sigma_i \lambda_i \langle\psi_i|\psi\rangle^2 = \Sigma_i \lambda_i \langle\psi|\psi_i\rangle \langle\psi_i|\psi\rangle$$

$$= \Sigma_i \langle\psi|A|\psi_i\rangle \langle\psi_i|\psi\rangle \underset{\Sigma_i|\psi_i\rangle\langle\psi_i|\equiv 1}{=} \langle\psi|A\mathbf{1}|\psi\rangle = \langle\psi|A|\psi\rangle$$

Which we shall call the observation value (a real value such that $\langle\psi|A|\psi\rangle = \langle\psi|A|\psi\rangle^*$) and is often denoted $\langle A \rangle$.

Standard Deviation and (Heisenberg's) Uncertainty Principle (C.T. 230,286)

First let us look at two non-switchable, measurable operators $[A, B] \neq 0$

If A, B are hermitian, thus their “switch-relation” (commutator) $[A, B]$ is anti-hermitian

$$[A, B]^\dagger = (AB - BA)^\dagger = BA - AB = -[A, B]$$

Therefore we find that the observation value of the anti-hermitian

$$\langle [A, B] \rangle_\psi = i\gamma$$

is a pure imaginary number.

Deviation Operator: $\Delta A = A - \langle A \rangle$, $\Delta B = B - \langle B \rangle$

The standard deviation magnitude (not operator) : $\sigma_A^2 = \langle \Delta A^2 \rangle$, $\sigma_B^2 = \langle \Delta B^2 \rangle$

Let us define the vector $|\varphi\rangle$ which is the linear superposition $|\varphi\rangle = (\Delta A + i\alpha\Delta B)|\psi\rangle$

all we know is that this new vector has a non-negative value $\langle \varphi | \psi \rangle$.

Side note: $[\Delta A, \Delta B] = [A, B]$

$$\langle \varphi | \psi \rangle = \langle \psi | (\Delta A - i\alpha\Delta B)(\Delta A + i\alpha\Delta B)|\psi\rangle$$

$$= \langle \psi | \Delta A^2 | \psi \rangle + \langle \psi | i\alpha(\Delta A \Delta B - \Delta B \Delta A) | \psi \rangle + \langle \psi | \alpha^2 \Delta B^2 | \psi \rangle$$

$$= \langle \Delta A^2 \rangle + i\alpha \langle [\Delta A, \Delta B] \rangle + \alpha^2 \langle \Delta B^2 \rangle$$

$$= \sigma_A^2 - \alpha\gamma + \alpha\sigma_B^2 \geq 0$$

Which we can look at as a parabola of α so we want the discriminant to be non-negative

$$\gamma^2 - 4\sigma_A^2\sigma_B^2 \leq 0 \Rightarrow \sigma_A\sigma_B \geq \frac{\gamma}{2} = \frac{1}{2}|\langle [A, B] \rangle|$$

After the measurement of A we know that this is in the state of minimal uncertainty and maximal certainty in B .

Degenerate Operators (C.T 132, 216, 259)

Non-degenerate operators all have different eigenvalues.

If the eigenvalue λ_i appears g_i times thus we will say that the operator A_i associated with λ_i is degenerate of the order g_i - g_i eigenvectors with the same eigenvalues.

$$A|\psi_i\rangle = A\Sigma_{k=1}^{g_i} c_k |\psi_i^k\rangle$$

$$= \lambda_i \Sigma_{k=1}^{g_i} c_k |\psi_i^k\rangle$$

We still want a whole basis of orthonormal vectors.

$$\langle \psi_i^k | \psi_{i'}^{k'} \rangle = \delta_{ii'} \delta_{kk'}$$

The “wholeness-relation”

$$\Sigma_{i,k} |\psi_i^k\rangle \langle \psi_i^k| = 1$$

What is the meaning here of a measurement?

We said that in a measurement we find an eigenvalue of the operator, which meant that the system was projected onto the eigenstate.

Here, if we measured λ_i , we projected the operator P_i

$$P_i = \sum_{k=1}^{g_i} |\psi_i^k\rangle\langle\psi_i^k|$$

So the measurement is the projection of this operator.

What happens after the measurement?

$$P_i|\psi\rangle = \sum_{k=1}^{g_i} |\psi_i^k\rangle\langle\psi_i^k| \sum_{j,l} c_j^l |\psi_j^l\rangle$$

$$= \sum_{k=1}^{g_i} \sum_{j,l} c_j^l \langle\psi_i^k|\psi_j^l\rangle |\psi_i^k\rangle = \sum_{k=1}^{g_i} \sum_{j,l} c_j^l \delta_{ij} \delta_{kl} |\psi_i^k\rangle$$

$$= \sum_{k=1}^{g_i} c_i^k |\psi_i^k\rangle$$

So after the projection we obtain only the elements belonging to the i^{th} eigenvalue.

We shall see that the length of the vector calculated is

$$|P_i|\psi\rangle|^2 = \langle\psi|P_i^\dagger P_i|\psi\rangle = \langle\psi|P_i|^2\psi\rangle = \langle\psi|P_i|\psi\rangle = \langle|P_i|\rangle_\psi = \sum_{k=1}^{g_i} |c_i^k|^2$$

Which is the observation value of the projection operator.

Revision

If two operators are “switchable”....

For different eigenvalues we found that

$$\langle\psi_i|B|\psi_j\rangle = \delta_{ij}\mu_j$$

which will now not be correct. Now:

$$\langle\psi_i^k|B|\psi_i^l\rangle$$

“whole” set of “switchable”, measurable operators (C.T 143)

Good quantum numbers - describe the system in entirety ?

So we'll need the full set of eigenvalues λ_i and μ_j of the operators A and B respectively.

So we're looking for the minimal set without degeneracy.

And this set will give us a total measurement of the system.

E.G. the atomic states of an electron: energy, angular momentum, spin... we need all 3 in order to precisely known the quantum state of the electron.

8 April 24th

Unitary Operators: (C.T. 176)

Who's hermitian conjugate is equal to its inverse

$$UU^\dagger = U^\dagger U = 1$$

If

$$|\psi'_1\rangle = U|\psi_1\rangle$$

$$|\psi'_2\rangle = U|\psi_2\rangle$$

We find that

$$\langle \psi'_1 | \psi'_2 \rangle = \langle \psi_1 | U^\dagger U | \psi_2 \rangle = \langle \psi_1 | \psi_2 \rangle$$

Doesn't change the inner products - unitary operators - and also conserves length.

Note: This inner product property conservation for finite dimensions is the defining factor for unitary matrices.

If we take an orthonormal basis (whole) and apply a unitary operator, we will obtain a new whole orthonormal basis - conserves lengths and angles.

$$\{|v_i\rangle\}, \text{also the set } \{U|v_i\rangle\}$$

From last lesson

$$(AB)^\dagger = B^\dagger A^\dagger = BA \neq AB$$

Is the hermitian conjugate UV a unitary operator?

$$(UV)^\dagger (UV) = V^\dagger U^\dagger UV = V^\dagger V = 1$$

Thus the product of unitary operators is itself unitary.

Elements of this operator in representation $\{|v_i\rangle\}$: investigating

$$\delta_{ij} = \langle v_i | v_j \rangle = \langle v_i | 1 | v_j \rangle = \langle v_i | U^\dagger U | v_j \rangle = \sum_k \langle v_i | U^\dagger | v_k \rangle \langle v_k | U | v_j \rangle = \sum_k \langle v_k | U | v_i \rangle^* \langle v_k | U | v_j \rangle = \sum_k u_{ki}^* u_{kj}$$

$$\delta_{ij} = \langle v_i | v_j \rangle + \langle v_i | 1 | v_j \rangle = \langle v_i | UU^\dagger | v_j \rangle = \sum_k \langle v_i | U | v_k \rangle \langle v_k | U^\dagger | v_j \rangle = \sum_k \langle v_i | U | v_k \rangle \langle v_j | U | v_k \rangle^* = \sum_k u_{ik} u_{jk}^*$$

Which gives us the inner product of the two columns (in the first case) and the inner product of the two rows (in the second case)

The representation of a unitary operator represents a whole set - because all the rows and separately all of the columns represent a whole set. Every two rows and every two columns are orthonormal to one another.

We shall denote the eigenvectors of U as $|\psi_i\rangle$ and the eigenvalues λ_i .

Special properties:

The size of the eigenvector $\langle \psi_i | \psi_i \rangle = \langle \psi_i | U^\dagger U | \psi_i \rangle = \langle \psi_i | \lambda_i^* \lambda_i | \psi_i \rangle = |\lambda_i|^2 \langle \psi_i | \psi_i \rangle$

Therefore $|\lambda_i|^2 = 1$

$$\lambda_i = e^{i\theta_i}$$

(don't mix the i's of index with imagination)

$$\langle \psi_i | \psi_j \rangle = \langle \psi_i | U^\dagger U | \psi_j \rangle = \langle \psi_i | \lambda_i^* \lambda_j | \psi_j \rangle = \lambda_i^* \lambda_j \langle \psi_i | \psi_j \rangle = e^{i(\theta_i - \theta_j)} \langle \psi_i | \psi_j \rangle$$

If $\theta_i \neq \theta_j (\Leftrightarrow \lambda_i \neq \lambda_j) \Rightarrow \langle \psi_i | \psi_j \rangle = 0$

Rotation or reflection (conserves angles and lengths).

T unitary operator from $\{|v_i\rangle\} \rightarrow \{|v'_i\rangle\}$

definition the representation of the operator elements between the two matrices (tag and no tag) that are the same:

$$\begin{aligned} \langle v'_i | A' | v_j \rangle &\equiv \langle v_i | A | v_j \rangle \\ &= \langle v_i | T^\dagger T A T^\dagger T | v_j \rangle = \langle v'_i | T A T^\dagger | v'_j \rangle \\ &\Rightarrow A' = T A T^\dagger \end{aligned}$$

Thus

$$T^\dagger A' T = A$$

(imaging rotating the PBS or your head)

$$A' |\psi' \rangle = T A T^\dagger T |\psi_i \rangle = T A |\psi_i \rangle = \lambda_i T |\psi_i \rangle = \lambda_i |\psi'_i \rangle$$

Taking some unknown linear operator A - We are only given than it conserves lengths.

$$A(\alpha|u\rangle + \beta|v\rangle)$$

So the length of the vector is

$$(\alpha^* < u | + \beta^* < v |) A^\dagger A (\alpha|u\rangle + \beta|v\rangle) = (\alpha^* < u | + \beta^* < v |)(\alpha|u\rangle + \beta|v\rangle)$$

$$\Rightarrow \alpha^* \alpha < u | A^\dagger A | u \rangle + \beta^* \beta < v | A^\dagger A | v \rangle + \alpha^* \beta < u | A^\dagger A | v \rangle + \beta^* \alpha < v | A^\dagger A | u \rangle = \alpha^* \alpha < u | u \rangle + \beta^* \beta < v | v \rangle + \alpha^* \beta < u | v \rangle + \beta^* \alpha < v | u \rangle$$

Because of conservation of lengths:

$$\alpha^* \beta < u | A^\dagger A | v \rangle + \beta^* \alpha < v | A^\dagger A | u \rangle = +\alpha^* \beta < u | v \rangle + \beta \alpha < v | u \rangle$$

If we choose $(\alpha, \beta) = (1, 1)$

$$1 < u | A^\dagger A | v \rangle + 1 < v | A^\dagger A | u \rangle = 1 < u | v \rangle + 1 < v | u \rangle$$

If we choose $(\alpha, \beta) = (1, i)$

$$i < u | A^\dagger A | v \rangle - i < v | A^\dagger A | u \rangle = i < u | v \rangle + i < v | u \rangle$$

Therefore $\frac{(1)+(2)/i}{2}$

$$< u | A^\dagger A | v \rangle < u | v \rangle$$

Thus it is enough to see that any transformation that conserves lengths, will also conserve angles - thus is unitary.

Functions and Derivatives of Operators (C.T. 169,172)

We already know that $A^{-1} A = 1$ and $A^n = AAA.....A$ (the operation of A , n times)

How can we define $f(A)$?

(Let's image that f is well-behaved and differentiable, thus we can write its Taylor series)

$$f(z) = \sum_{n=0}^{\infty} f_n z^n$$

So let's define

$$f(A) = \sum_{n=0}^{\infty} f_n A^n$$

What are the eigenvalues and eigenvectors of this new operator? (for the eigenstate ψ_i)

$$f(A) |\psi_i \rangle = \sum_{n=0}^{\infty} f_n A^n |\psi_i \rangle = \sum_{n=0}^{\infty} f_n \lambda_i^n |\psi_i \rangle$$

$$= f(\lambda_i) |\psi_i \rangle$$

So $f(A)$ has the same eigenvectors and the eigenvalues are simply the old eigenvalues times the enaction of the function on the eigenvalues

Another definition, if we choose a representation in which A is diagonalized

$$\begin{pmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & & \\ \vdots & & \ddots & \\ 0 & & & \lambda_i \end{pmatrix}$$

In which case $f(A)$ will be

$$\begin{pmatrix} f(\lambda_1) & 0 & \dots & 0 \\ 0 & f(\lambda_2) & & \\ \vdots & & \ddots & \\ 0 & & & f(\lambda_i) \end{pmatrix}$$

Because $(A^n)^\dagger = (A^\dagger)^n$, it also holds that $f(A)^\dagger = f(A^\dagger)$ if and only if f is real.

\Rightarrow if A is hermitian, so is $f(A)$.

$$e^{iA} = \sum_{n=0}^{\infty} \frac{(iA)^n}{n!} = 1 + iA - \frac{1}{2}A^2 - \frac{i}{6}A^3$$

If A is hermitian then

$$(e^{iA})^\dagger = e^{-iA^\dagger} = e^{-iA} = (e^{-iA})^{-1}$$

Therefore e^{-iA} is unitary, with the eigenvalues $e^{i\lambda_i}$.

Any unitary operator can be written as e^{iA} for hermitian A operators.

Another useful property:

The commutator $[A, f(A)] = 0$.

If we have $A(t)$ we can define

$$\frac{d}{dt}(A(t)) = \lim_{\Delta t \rightarrow 0} \frac{A(t + \Delta t) - A(t)}{\Delta t}$$

all of the elements of the operator

$$a_{ij} \xrightarrow{d/dt} \frac{d}{dt}(a_{ij})$$

And we shall see that all of the standard rules for derivation apply to operators.

In order for the operator to be “switchable” with its derivative between any t_1, t_2 we can “switch” $A(t_1), A(t_2)$

$$[A, \frac{dA}{dt}] \stackrel{?}{=} 0 \Rightarrow [A(t_1), A(t_2)] = 0$$

Development of a System in Time: (C.T. 245, 310)

Postulate 5: The development in time of closed, quantum systems is given by a unitary operator that is diagonalized in the representation of energy.

9 April 29th

Development of a System in Time: (C.T. 245, 310)

Postulate 5: The development in time of closed, quantum systems (all the components influencing the system) is given by a unitary operator that is diagonalized in the representation of energy.

- Unitary operator *iff* conserves length

Operator - measurement of the energy of a system, observable, hermitian - denoted H : Quantum Hamiltonian.

Classic hamiltonian is a function but the quantum is an operator.

This has a set of eigenstates defined by the characteristic polynomial:

$$H|E_i\rangle = E_i|E_i\rangle$$

E_i are the states with defined energy.

Always right, even in time-dependent system. Also known as the Time-Independent Schrödinger Equation.

If H is independent of time, the system conserves. (H is an operator that doesn't change in time).

We decided that quantum helps us define the particle-wave duality - now we know that waves have the time dependence

$$e^{-i\omega t}$$

with the energy dependence in the frequency

$$\omega = \frac{E}{k}$$

(frequency = rate of phase change?)

We expect that the (unitary) operator "progression" of time Δt (where $\Delta t = t - t_0$)

$$T(\Delta t)$$

In order to have the "shniut" (duality)

$$T(\Delta t)|E_i\rangle = e^{-i\frac{E}{k}\Delta t}|E_i\rangle$$

Thus we see that the energy states are the eigenstates of the progression of time $T(\Delta t)$ with the eigenvalues $e^{-i\frac{E}{k}\Delta t}$. Therefore we see that it is the function of the hamiltonian operator

$$T(\Delta t) = e^{-i\frac{\hbar}{\hbar}H\Delta t}$$

Taking the initial state:

$$|\psi(t)\rangle = T(\Delta t)|E_i\rangle = e^{-i\frac{\hbar}{\hbar}H(t-t_0)}|E_i\rangle = e^{-i\frac{\hbar}{\hbar}E_i(t-t_0)}|E_i\rangle$$

(recalling that if two operators commute, then a function of the operators commute too...)

Although the state is dynamic (phase) we cannot differentiate the phase state or without phase state so we call this state a "stable" (amid) state.

$$|\psi(t)\rangle = T|\psi(t_0)\rangle = e^{-i\frac{\hbar}{\hbar}H(t-t_0)}\sum_i c_i(t_0)|E_i\rangle = \sum_i c_i(t_0)e^{-i\frac{\hbar}{\hbar}E_i(t-t_0)}|E_i\rangle$$

The magnitude doesn't change

$$c_i(t) = c_i(t_0)e^{-i\frac{\hbar}{\hbar}E_i(t-t_0)}$$

And the probability (and measurement value) doesn't change

$$P(E_i) = |c_i(t)|^2 = |c_i(t_0)|^2$$

So the averages will repeat (over a large number of experiments).

$$\langle H \rangle = \sum_i E_i P(E_i)$$

(A hermitian operator represents a physical quantity)

Unitary operator is always invertible, meaning that we can answer the question of progressing forwards OR backwards in time with such an operator.

For $T(t_0, t)$

$$T^{-1}(t_0, t) = e^{\frac{i}{\hbar}H(t-t_0)} = e^{-\frac{i}{\hbar}H(t_0-t)} = T(t, t_0)$$

What is the rate of change of a hilbert space vector with respective to its propagator($e^{\frac{i}{\hbar}H(t-t_0)}$)?

$$\begin{aligned} \frac{d}{dt}|\psi(t)\rangle &= \frac{d}{dt}\left[e^{-\frac{i}{\hbar}H(t-t_0)}|\psi(t_0)\rangle\right] = -\frac{i}{\hbar}He^{-\frac{i}{\hbar}H(t-t_0)}|\psi(t_0)\rangle \\ &\Rightarrow i\hbar\frac{d}{dt}|\psi(t)\rangle = H|\psi(t)\rangle \end{aligned}$$

Which is the Time Dependent Schrodinger Equation. (Correct also for time dependent H)

It goes without saying that: Because the hamiltonian is observable (represents a real physical quantity) - the set of the states... are a WHOLE set...

any initial state can be written as the spaning of the eigenstates of the hamiltonian ($\sum_i c_i(t_0)|E_i\rangle$)

Meaning that the general solution to the TD SE is

$$\sum_i c_i(t_0)|e^{-\frac{i}{\hbar}E_i(t-t_0)}|E_i\rangle$$

We said that the TD SE is also correct for the time dependent H . In this case - it is often useful - to write if

$$\Delta t \rightarrow 0$$

$$\Rightarrow T(t, t + \Delta t) = 1 - \frac{i}{\hbar}H(t)\Delta t$$

Global Conservation of Probability (CT 237):

We want the dynamics in time to be zero in order to obtain conservation

$$\begin{aligned} \frac{d}{dt}|\psi(t)\rangle|\psi(t)\rangle &= \left(\frac{d}{dt}|\psi(t)\rangle\right)|\psi(t)\rangle + |\psi(t)\rangle\left(\frac{d}{dt}|\psi(t)\rangle\right) \\ &= \frac{i}{\hbar}(|\psi(t)\rangle|H^\dagger|\psi(t)\rangle) + |\psi(t)\rangle\left(-\frac{i}{\hbar}H|\psi(t)\rangle\right) = 0 \end{aligned}$$

because $H^\dagger = H$

Conserves for every component, hence global.

Dynamics of Measured Values (CT 240):

Taking an operator representing some other physical quantity (not energy H), represented by A

$$\langle A \rangle = \langle \psi(t)|A(t)|\psi(t)\rangle$$

Which is just the definition of Measured value but now time dependent.

$$\begin{aligned} \frac{d}{dt}\langle A \rangle &= \left(\frac{d}{dt}\langle \psi(t)|\right)A(t)|\psi(t)\rangle + \langle \psi(t)|A(t)\left(\frac{d}{dt}|\psi(t)\rangle\right) + \langle \psi(t)|\frac{d}{dt}A(t)|\psi(t)\rangle \\ &= \frac{1}{i\hbar}\langle \psi(t)|[AH - HA]|\psi(t)\rangle + \langle \psi(t)|\frac{d}{dt}A(t)|\psi(t)\rangle \end{aligned}$$

The commutator between hermitian operators is antihermitian...

$$= \frac{1}{i\hbar}\langle [A, H]\rangle + \langle \frac{d}{dt}A(t)\rangle$$

If A is not time dependent ($A(t)$) and also A commutes with H ($[A, H] = 0$), A is called a constant of the motion. If we measured A and found some result, we now we're in the durable "amid" state - because it'll be a shared eigenstate between A and H .

(Good quantum numbers).

NEW MATERIAL NOT IN LAST YEAR'S (2016/17) SUMMARY:

Time-Energy Uncertainty According to Mendelsturm-Tam (Griffith 2nd Ed. 114):

Recalling the heisenberg uncertainty with A, H , we can use the uncertainties, σ_A, σ_H

$$\sigma_A \sigma_H \geq \frac{1}{2} | < [A, H] > |$$

Now let's denote $\sigma_H = \Delta E$ and define $\Delta t \equiv \frac{\sigma_A}{|\frac{d}{dt} < A >|}$

This time is the time it takes the measured value $< A >$ to change in one *STD* (standard deviation) σ_A

Thus substituting:

$$\sigma_A \sigma_H \geq \frac{1}{2} | < [A, H] > | \Leftrightarrow \Delta E \Delta t \geq \frac{\hbar}{2}$$

time in quantum mechanics is not a measurable quantity. it is the dynamic variable upon which all the other variables are dependent upon. there is no meaning to the question "what is the time of the system?".

What is the value of a measured quantity AT a time? is more correct.

There are several different versions of this uncertainty equation.

If we look at some system with some uncertainty in energy, the measured values of everything except energy will change faster than $\hbar/2$.

So when the energy goes to zero, the measured value does not change.

One other meaning of this equation: as you want to know a more precise measurement of energy, you must measure the system for a longer period of time.

10 May 1st

Bohr Frequencies (CT 249):

Given H conserved, $A, |\psi(t_0)\rangle$

Expectation value:

$$\langle \psi(t) | A(t) | \psi(t) \rangle$$

$$= \sum_i c_i^*(t_0) e^{\frac{i}{\hbar} E_i (t-t_0)} \langle E_i | A(t) \sum_j c_j(t_0) e^{-\frac{i}{\hbar} E_j (t-t_0)} | E_j \rangle$$

$$= \sum_{i,j} c_i^*(t_0) c_j(t_0) \langle E_i | A(t) | E_j \rangle e^{\frac{i}{\hbar} (E_i - E_j)(t-t_0)}$$

$$\text{So } \omega = \frac{E_i - E_j}{\hbar}$$

In the case of an atom we look at the Hamiltonian of the electron around the nucleus the E_i s are the energy levels

Location and Momentum:

The location is single (discrete) (not continuous)

The functional δ_n and the representation $\{|u_i\rangle\}, |\psi\rangle$

$$\delta_n(|\psi\rangle) = \langle u_n | \psi \rangle = c_n$$

$$\langle \delta_n | = \langle n |$$

This function asking about the value of the vector at n is a functional described by a bra, the associated ket is the eigenvector of the measurement which defines the location of the particle.

\mathbb{X} is the location operator in the discrete space.

$$\langle n | \mathbb{X} | m \rangle = n \delta_{nm}$$

So

$$\mathbb{X}|m\rangle = \sum_n |n\rangle \langle n | \mathbb{X} | m \rangle = \sum_n n \delta_{mn} |n\rangle = m |m\rangle$$

Infinite Continuous Hilbert Space (C.T. 95)

(imagine the space between points goes to zero - like from point charges to uniform continuum of charge)

Using the location operator \mathbb{X} in this continuous space.

$$|x\rangle$$

This is the eigenstate of the operator. After the measurement of the location, the system will collapse to this state and the particle is at $|x\rangle$

If we take a system in the state $|\psi\rangle$ and ask what the probability is of finding the system in the state $|x\rangle$, we must take the projection

$$\langle x | \psi \rangle$$

Which is the probability of being in the SURROUNDING dx next to x and is

$$P(x, x + dx) = |\langle x | \psi \rangle|^2 dx$$

infinite values $\psi(x) = \langle x | \psi \rangle$, known as wave functions.

Probability density: the chance of a particle to be found in the segment $[a, b]$

$$P(a, b) = \int_a^b |\psi(x)|^2 dx$$

Recalling that the probability of being in a specific place (not a region) is zero.

“Yachas HaShelmut”: $\int_{-\infty}^{\infty} dx |x\rangle \langle x| = 1$

$$|\psi\rangle = 1|\psi\rangle = \int_{-\infty}^{\infty} dx |x\rangle \langle x | \psi \rangle = \int_{-\infty}^{\infty} \psi(x) |x\rangle dx$$

The meaning of inner product of two states

$$\langle \varphi | \psi \rangle = \langle \varphi | 1 | \psi \rangle = \int_{-\infty}^{\infty} dx \langle \varphi | x \rangle \langle x | \psi \rangle = \int_{-\infty}^{\infty} dx \langle x | \varphi \rangle^* \langle x | \psi \rangle = \int_{-\infty}^{\infty} dx \varphi^*(x) \psi(x)$$

The meaning of the enactment of an operator A on $|\psi\rangle$

$$\langle x | A | \psi \rangle = \langle x | A 1 | \psi \rangle = \int_{-\infty}^{\infty} dx' \langle x | A | x' \rangle \langle x' | \psi \rangle = \int_{-\infty}^{\infty} dx' a(x, x') \psi(x')$$

Where $\langle x | A | x' \rangle$ is element of the matrix A between x and x' - $a(x, x')$ - representation of A in the location basis.

$$a(x, x') = \langle x | A | x' \rangle = \langle x | A^\dagger | x' \rangle = \langle x' | A | x \rangle^* = a^*(x', x)$$

So

$$\langle \psi | \psi \rangle = \int_{-\infty}^{\infty} dx \psi^*(x) \psi(x) = \int_{-\infty}^{\infty} dx |\psi(x)|^2 = 1$$

All the complex functions on the real, given to normalization L^2 are themselves a Hilbert Space - describes a physical state.

Location Operator and its Eigenstates in Location Representation (Delta Function) (C.T. 100,145, 2nd Ed: 1467)

$$\psi_{x_0}(x) = \psi(x, x_0) = \langle x | x_0 \rangle$$

For $x \neq x_0 \Rightarrow \psi_{x_0}(x) = 0$

Secondly we know that it needs to be normalized...

For $x = x_0 \Rightarrow \psi_{x_0}(x) \neq 0$ and the value needs to be infinite.

$$\int_{-\infty}^{\infty} dx |\langle x | x_0 \rangle|^2 = \int_{-\infty}^{\infty} dx \langle x_0 | x \rangle \langle x | x_0 \rangle = \langle x_0 | x_0 \rangle = \psi_{x_0}(x_0) \rightarrow \infty$$

We see

$$\varphi(x) = \langle x | \varphi \rangle = \int_{-\infty}^{\infty} dx' \langle x | x' \rangle \langle x' | \varphi \rangle = \int_{-\infty}^{\infty} dx' \psi(x', x) \varphi(x')$$

So the eigenstates live only in the bra space (this being infinite). AUGMENTED HILBERT SPACE will include precisely this kind of functions.

Thus we define the function

$$\psi(x, x_0) \equiv \delta(x - x_0)$$

Let's calculate the area under this function (Magnitude L^1)

$$\int_{-\infty}^{\infty} dx \delta(x - x') = \int_{-\infty}^{\infty} dx \delta(x - x') u(x) = u(x') = 1$$

Where $u(x) = 1$

$$\psi(x) = \int_{-\infty}^{\infty} dx' \delta(x - x') \psi(x')$$

11 May 6th

Today we shall discuss the difference between mathematical and physical states

$$\psi(x) = \int_{-\infty}^{\infty} dx' \delta(x - x') \psi(x')$$

We want to see the immediate result that if the delta function is $\langle x | x' \rangle$

$$\langle x | x' \rangle = \langle x | 1 | x' \rangle = \delta(x - x')$$

Let's talk about the location operator itself, looking at it's representation by x

$$\langle x | \mathbb{X} | x' \rangle = a(x, x') = x \delta(x - x')$$

Note that because of Hermitian(ness), we know that there will be symmetry such that:

$$= x \delta(x - x') = x' \delta(x - x')$$

We can check by applying to location operator on an eigenstate (vector)

$$|\mathbb{X}|x_0\rangle = \int_{-\infty}^{\infty} dx |x\rangle \langle x | \mathbb{X} | x_0 \rangle = \int_{-\infty}^{\infty} dx \delta(x - x_0) |x\rangle$$

All of the vectors equal zero except for when $x = x_0$

$$\Rightarrow \int_{-\infty}^{\infty} dx \delta(x - x_0) |x\rangle = x_0 |x_0\rangle$$

We have been given some vector $\psi(x)$ and we want to apply the operator upon it

$$\begin{aligned} \mathbb{X}[\underbrace{\psi(x)}_{=\langle x|\psi\rangle}] &= \langle x|\mathbb{X}\psi\rangle = \int_{-\infty}^{\infty} dx' \langle x|\mathbb{X}|x'\rangle \langle x'|\psi\rangle \\ &= \int_{-\infty}^{\infty} dx' x' \delta(x - x') \psi(x') = x\psi(x) \end{aligned}$$

For a given $\psi(x)$ with a given representation, what is $\psi(x)$ after the application of the location vector? the same $\psi(x)$ vector times x .

For example if we take:

$$\psi(x) = \frac{\sqrt{\frac{a}{\pi}}}{\sqrt{x^2 + a^2}}$$

What are we talking about when we talk about location?

Learning to live with delta functions (understanding the limit of the mathematical world with the physical world).

Mathematician's call the things that are only standalone functionals (not really functions) - distributions.

We will never have a piece of technology with the ability/resolution to measure a MATHEMATICAL point.

$$\tilde{f}(x, \varepsilon) = \frac{1}{\varepsilon} \int_{x - \frac{\varepsilon}{2}}^{x + \frac{\varepsilon}{2}} f(x') dx' = \int_{-\infty}^{\infty} \delta_{\varepsilon}(x', x) f(x') dx'$$

Where

$$\delta_{\varepsilon}(x', x) = \begin{cases} \frac{1}{\varepsilon} & x - \frac{\varepsilon}{2} < x' < x + \frac{\varepsilon}{2} \\ 0 & \text{else} \end{cases}$$

which gives us a Box (Rect) function of width ε , centered at x , with height $\frac{1}{\varepsilon}$.

Any physical quantity, in time or space, is limited by the resolution of the measuring device.

It is clear that at the limit $\lim_{\varepsilon \rightarrow 0} \delta_{\varepsilon}(x', x) \rightarrow \delta(x - x')$

Any function $f(\frac{x}{\varepsilon})$ when $\varepsilon \rightarrow 0$ will work.

E.g. the following three functions:

$$\frac{\sin(\frac{x}{\varepsilon})}{x\pi}, \quad \frac{1}{\sqrt{\pi\varepsilon^2}} e^{-(\frac{x}{\varepsilon})^2}, \quad \frac{1}{\pi} \frac{\varepsilon}{x^2 + \varepsilon^2}$$

Any real, physical state will be in L^2

Momentum Operator and its Eigenstates: (C.T. 100,145)

Recalling DeBroglie and Compton:

$$\lambda = \frac{h}{p}$$

Meaning that if we measure the momentum of a particle, we also know its wavelength.

(Thus the operators for wavelength and momentum measurement must commute and they will have the same eigenstates)

Thus the eigenstate will be $|p\rangle$, which we can represent in time as $\langle x|p\rangle = \psi_p(x)$.

Since it needs a defined wavelength

$$\langle x|p\rangle = \psi_p(x) = \alpha e^{i\frac{2\pi}{\lambda}x} = \alpha e^{\frac{i}{\hbar}px}$$

We see that the absolute value squared of this function (probability density) is

$$|\psi_p(x)|^2 = |\alpha|^2$$

So a particle with a defined momentum, is uniformly distributed across the whole space and has no single location.

$$\psi(x) \rightarrow \tilde{\psi}(p) = \langle p|\psi \rangle$$

So we find the matrix element of the passage matrix

$$\langle p|x\rangle = \langle x|p\rangle^* = \alpha e^{-\frac{i}{\hbar}px}$$

So the translation is

$$\tilde{\psi}(p) = \langle p|\psi \rangle = \int_{-\infty}^{\infty} dx \langle p|x\rangle \langle x|\psi \rangle = \alpha \int_{-\infty}^{\infty} dx e^{-\frac{i}{\hbar}px} \psi(x)$$

Thus for a given $\psi(x)$, we must do a Fourier Transform to obtain $\tilde{\psi}(p)$.

In the same way that x is a whole basis, so is p and therefore

$$\int_{-\infty}^{\infty} dp |p\rangle \langle p| = 1$$

Thus

$$\psi(x) = \langle x|\psi \rangle = \int_{-\infty}^{\infty} dp \langle x|p\rangle \langle p|\psi \rangle = \alpha \int_{-\infty}^{\infty} dp e^{\frac{i}{\hbar}px} \tilde{\psi}(x)$$

Meaning that there is a Fourier relation between the representation of location and the representation of momentum. So x, p - unlike in the classical case - here we require ψ and then can choose the representation according to x, p, E, H or whatever we want.

Just like with x

$$\begin{aligned} \langle p|p'\rangle &= \delta(p-p') = \int_{-\infty}^{\infty} dx \langle p|x\rangle \langle x|\psi \rangle = \alpha^2 \int_{-\infty}^{\infty} dx e^{-\frac{i}{\hbar}px} e^{\frac{i}{\hbar}p'x} \\ &= \alpha^2 \int_{-\infty}^{\infty} dx e^{-\frac{i}{\hbar}(p-p')x} \end{aligned}$$

At the limit

$$\begin{aligned} \lim_{\varepsilon \rightarrow 0} \alpha^2 \int_{-1/\varepsilon}^{1/\varepsilon} dx e^{-\frac{i}{\hbar}(p-p')x} &= \lim_{\varepsilon \rightarrow 0} \alpha^2 \frac{i\hbar}{\Delta p} \left(e^{-\frac{i}{\hbar}\frac{\Delta p}{\varepsilon}} - e^{\frac{i}{\hbar}\frac{\Delta p}{\varepsilon}} \right) \\ &= 2\pi\hbar\alpha^2 \lim_{\varepsilon \rightarrow 0} \frac{\sin \frac{\Delta p}{\hbar\varepsilon}}{\pi\Delta p} \rightarrow 2\pi\hbar\alpha^2 \delta(p-p') = 2\pi\alpha^2 \delta(k-k') \end{aligned}$$

And this requires that

$$\alpha = \frac{1}{\sqrt{2\pi\hbar}}$$

There is algebraic normalization (to 1 or 0) in finite space, whereas in infinite space it is Dirac normalization which normalizes to the delta function.

$$\int_{-\infty}^{\infty} dx e^{ikx} = 2\pi\delta(k)$$

Now let us look at

$$\begin{aligned}\langle \psi | \psi \rangle &= \int_{-\infty}^{\infty} dx \langle \psi | x \rangle \langle x | \psi \rangle = \int_{-\infty}^{\infty} dp \langle \psi | p \rangle \langle p | \psi \rangle \\ &= \int_{-\infty}^{\infty} dx |\psi(x)|^2 = \int_{-\infty}^{\infty} dp |\tilde{\psi}(p)|^2\end{aligned}$$

Which is just the magnitude (and is Parseval's Theorem), and shows us that there is no difference here between location and momentum.

$$\langle p | \mathbb{P} | p' \rangle = p\delta(p - p')$$

$$\mathbb{P}[\tilde{\psi}(p)] = \langle p | \mathbb{P} | \psi \rangle = p\tilde{\psi}(p)$$

Additionally

$$\begin{aligned}\langle x | \mathbb{P} | x' \rangle &= \iint_{-\infty}^{\infty} dp dp' \langle x | p \rangle \langle p | \mathbb{P} | p' \rangle \langle p' | x' \rangle \\ &= \frac{1}{2\pi\hbar} \iint_{-\infty}^{\infty} dp dp' e^{\frac{i}{\hbar}px} p\delta(p - p') e^{-\frac{i}{\hbar}p'x'} = \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp' p' e^{\frac{i}{\hbar}p'(x-x')} \\ &= \frac{1}{2\pi\hbar} \int_{-\infty}^{\infty} dp' \frac{\partial}{\partial x} \frac{\hbar}{i} e^{\frac{i}{\hbar}p'(x-x')} = \frac{-i}{2\pi} \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \hbar dk' e^{\frac{i}{\hbar}k'(x-x')}\end{aligned}$$

Moving from p to k in the last equality

Thus

$$\Rightarrow \frac{-i}{2\pi} \frac{\partial}{\partial x} \int_{-\infty}^{\infty} \hbar dk' e^{\frac{i}{\hbar}k'(x-x')} = -i\hbar \frac{\partial}{\partial x} \delta(x - x') = i\hbar \frac{\partial}{\partial x'} \delta(x - x')$$

12 May 8th

Midterm (25/5) will begin at 10am.

Today's lesson will continue with the momentum operator:

$$\langle x | \mathbb{P} | x' \rangle = -i\hbar \frac{\partial}{\partial x} \delta(x - x') = i\hbar \frac{\partial}{\partial x'} \delta(x - x')$$

What happens when the momentum operator is applied to a state in location representation

$$\begin{aligned}\underbrace{\mathbb{P}[\psi(x)]}_{\langle x | \psi \rangle} &= \langle x | \mathbb{P} | \psi \rangle = \int_{-\infty}^{\infty} dx' \langle x | \mathbb{P} | x' \rangle \langle x' | \psi \rangle \\ &= i\hbar \int_{-\infty}^{\infty} dx' \left[\frac{\partial}{\partial x'} \delta(x - x') \right] \psi(x')\end{aligned}$$

Which we can solve with integration by parts

$$= i\hbar \delta(x - x') \psi(x') \Big|_{-\infty}^{\infty} - i\hbar \int_{-\infty}^{\infty} dx' \delta(x - x') \frac{d\psi(x')}{dx'}$$

$$= -i\hbar \frac{d\psi(x)}{dx}$$

Being that applying the momentum operator will result in multiplication by $-i\hbar$ and derivation.

How will this work on the eigenstate of the momentum operator?

$$\langle x|\mathbb{P}|p\rangle = -\frac{i\hbar}{\sqrt{2\pi\hbar}} \frac{d}{dx} e^{\frac{i}{\hbar}px} = \frac{p}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar}px} = p\langle x|p\rangle$$

What will be the matrix element of the \mathbb{X} operator in the p representation

$$\langle p|\mathbb{X}|p'\rangle = i\hbar \frac{\partial}{\partial p} \delta(p - p')$$

$$\underbrace{\mathbb{X}[\psi(p)]}_{\langle p|\psi\rangle} = \langle p|\mathbb{X}|\psi\rangle = i\hbar \frac{\partial}{\partial p} \tilde{\psi}(p)$$

What are we talking about when we talk about momentum?

(The physical meaning of the momentum states)

Just like with location, we are limited by the size of the measuring device/lab - when they are finite.

In order to see this, let's try to sample some wave function $\psi(x)$, we'll measure (in a system of length $2L$):

$$\begin{aligned} \tilde{\psi}_L(k) &= \frac{1}{\sqrt{2\pi}} \int_{-L}^L dx \cdot e^{-ikx} \psi(x) \\ &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \cdot \text{rect}(x, 2L) \cdot e^{-ikx} \psi(x) \end{aligned}$$

Recalling that the Fourier transformon of the rect function is

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \cdot \text{rect}(x, 2L) \cdot e^{ikx} = \frac{1}{\sqrt{2\pi}} \frac{2}{k} \sin(kL)$$

Thus

$$\begin{aligned} \tilde{\psi}_L(k) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dx \cdot e^{-ikx} \frac{1}{2\pi} \int_{-\infty}^{\infty} dk' \cdot e^{ik'x} \frac{2}{k'} \sin(k'L) \int_{-\infty}^{\infty} dk'' \cdot e^{ik''x} \tilde{\psi}(k'') \\ &= \frac{1}{4\pi^2} \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dk' \sin(k'L) \int_{-\infty}^{\infty} dk'' \cdot e^{i(k''+k'-k)x} \tilde{\psi}(k'') \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} dk' \frac{2}{k'} \sin(k'L) \int_{-\infty}^{\infty} dk'' \cdot \delta(k'' + k' - k) \tilde{\psi}(k'') \\ &= \int_{-\infty}^{\infty} dk' \frac{1}{\pi k'} \sin(k'L) \tilde{\psi}(k - k') \end{aligned}$$

Meaning that we are using the *sinc* function with the length of $\frac{1}{L}$ to sample the frequencies, so only at the limit that $L \rightarrow \infty$ does the sampling function behave like a delta function.

Uncertainty between Location and Momentum of Wave Packet: (C.T. 63)

If a function in location representation is in L^2 so is the function in momentum representation. I.e. if $\psi(x) \in L^2 \Leftrightarrow \tilde{\psi}(p) \in L^2$.

In the test case of a Gaussian normalized in L^2

$$\langle x|\psi\rangle = \psi(x) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} e^{\frac{i}{\hbar}p_0x} e^{-\frac{(x-x_0)^2}{4\sigma^2}}$$

Where σ is the width of the wave packet and $\frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}}$ is the normalization factor in L^2

Where is the average location of this wave packet?

$$\begin{aligned}\langle \mathbb{X} \rangle &= \langle \psi | \mathbb{X} | \psi \rangle = \int_{-\infty}^{\infty} dx \langle \psi | x \rangle \langle x | \mathbb{X} | \psi \rangle = \int_{-\infty}^{\infty} dx \psi^*(x) x \psi(x) \\ &= \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dx \cdot x e^{-\frac{(x-x_0)^2}{2\sigma^2}} = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dx' \cdot (x' + x_0) e^{-\frac{x'^2}{2\sigma^2}} = x_0\end{aligned}$$

Using the substitution $x' = x - x_0$. Note that we obtain two integrals, the anti-symmetric $x' \cdot e^{-x'^2}$ one which zeroes over $(-\infty, \infty)$ and the remaining integral of the constant $x_0 \cdot e^{-x'^2}$.

$$\begin{aligned}\langle \mathbb{X}^2 \rangle &= \int_{-\infty}^{\infty} dx \psi^*(x) x^2 \psi(x) = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dx \cdot x^2 e^{-\frac{(x-x_0)^2}{2\sigma^2}} \\ &= \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dx' \cdot (x'^2 + \underbrace{2x'x_0}_{\text{as above}} + x_0^2) e^{-\frac{x'^2}{2\sigma^2}} \\ &= x_0^2 + \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dx' \cdot x'^2 e^{-\frac{x'^2}{2\sigma^2}} = x_0^2 + \sigma^2\end{aligned}$$

Where $\sigma_x = \sqrt{\langle \mathbb{X}^2 \rangle - \langle \mathbb{X} \rangle^2} = \sigma$

Now the momentum representation of the wave packet state.

$$\begin{aligned}\langle p | \psi \rangle &= \tilde{\psi}(p) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dx \cdot e^{-\frac{i}{\hbar} px} \psi(x) \\ &= \frac{1}{\sqrt{2\pi\hbar}} \frac{1}{(2\pi\sigma^2)^{\frac{1}{4}}} \int_{-\infty}^{\infty} dx \cdot e^{-\frac{i}{\hbar} px} e^{\frac{i}{\hbar} p_0 x} e^{-\frac{(x-x_0)^2}{4\sigma^2}} \\ &= \frac{1}{(8\pi^3\hbar^2\sigma^2)^{\frac{1}{4}}} \int_{-\infty}^{\infty} dx \cdot e^{-\frac{i}{\hbar}(p-p_0)x} e^{-\frac{(x-x_0)^2}{4\sigma^2}}\end{aligned}$$

Using the substitution $x' = x - x_0$

$$= \frac{1}{(8\pi^3\hbar^2\sigma^2)^{\frac{1}{4}}} e^{-\frac{i}{\hbar}(p-p_0)x_0} \int_{-\infty}^{\infty} dx' \cdot e^{-\frac{i}{\hbar}(p-p_0)x'} \cdot e^{-\frac{x'^2}{4\sigma^2}}$$

Using completion of the square

$$\begin{aligned}&= \frac{1}{(8\pi^3\hbar^2\sigma^2)^{\frac{1}{4}}} e^{-\frac{i}{\hbar}(p-p_0)x_0} \int_{-\infty}^{\infty} dx' \cdot e^{-\frac{1}{4\sigma^2} [x'^2 + \frac{2i\sigma^2}{\hbar}(p-p_0)]^2 - \frac{\sigma^2}{\hbar^2}(p-p_0)^2} \\ &= \frac{(16\pi^2\sigma^4)^{\frac{1}{4}}}{(8\pi^3\hbar^2\sigma^2)^{\frac{1}{4}}} e^{-\frac{i}{\hbar}(p-p_0)x_0} e^{-\frac{\sigma^2}{\hbar^2}(p-p_0)^2} = \left(\frac{2\sigma^2}{\pi\hbar^2}\right)^{\frac{1}{4}} e^{\frac{i}{\hbar}p_0x_0} e^{-\frac{i}{\hbar}px_0} e^{-\frac{\sigma^2}{\hbar^2}(p-p_0)^2}\end{aligned}$$

The observation value of the momentum will be

$$\langle \mathbb{P} \rangle = p_0$$

In the same manner as before

$$\langle \mathbb{P}^2 \rangle = p_0^2 + \frac{\hbar^2}{4\sigma^2}$$

$$\sigma_p = \frac{\hbar}{2\sigma}$$

Thus proving

$$\sigma_x \sigma_p = \frac{\hbar}{2}$$

In order to calculate the limit, we can calculate the observation value of the commutator

$$\begin{aligned} & <[\mathbb{X}, \mathbb{P}]> = <\mathbb{X}\mathbb{P} - \mathbb{P}\mathbb{X}> \\ &= -i\hbar \int_{-\infty}^{\infty} dx \psi^*(x) \left[x \frac{d}{dx} - \frac{d}{dx} x \right] \psi(x) \\ &= -i\hbar \int_{-\infty}^{\infty} dx \psi^*(x) [x\psi'(x) - \psi(x) - x\psi'(x)] = i\hbar \int_{-\infty}^{\infty} dx \psi^*(x)\psi(x) \\ &= i\hbar \cdot 1 = i\hbar \end{aligned}$$

Thus the Heisenberg Uncertainty

$$\boxed{\sigma_x \sigma_p \geq \frac{| <[\mathbb{X}, \mathbb{P}]> |}{2} = \frac{\hbar}{2}}$$

(Which stems from symmetries in the commutation of operators)

A gaussian wave packet gives us the greatest physically possible certainty (and it is the only form that gives this limit) (Proof - C.T. 273 or perhaps 283)

We can also repeat the expectation for the expectation value but in the location representation

$$\begin{aligned} \langle \mathbb{P} \rangle &= \frac{\hbar}{i} \int_{-\infty}^{\infty} dx \cdot \psi^*(x) \frac{d\psi(x)}{dx} = \frac{1}{(2\pi\sigma^2)^{\frac{1}{2}}} \frac{\hbar}{i} \int_{-\infty}^{\infty} dx \left[\frac{i}{\hbar} p_0 - \frac{2(x-x_0)}{4\sigma^2} \right] e^{-\frac{(x-x_0)^2}{\sigma^2}} = p_0 \\ \langle \mathbb{P}^2 \rangle &= -\hbar^2 \int_{-\infty}^{\infty} dx \cdot \psi^*(x) \frac{d^2\psi(x)}{dx^2} = \frac{-\hbar^2}{(2\pi\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dx \left[\left(\frac{i}{\hbar} p_0 - \frac{(x-x_0)}{4\sigma^2} \right)^2 - \frac{1}{2\sigma^2} \right] e^{-\frac{(x-x_0)^2}{\sigma^2}} \\ &= p_0^2 + \frac{\hbar^2}{2\sigma^2} + \frac{-\hbar^2}{(2\pi\sigma^2)^{\frac{1}{2}}} \int_{-\infty}^{\infty} dx \frac{(x-x_0)^2}{4\sigma^4} e^{-\frac{(x-x_0)^2}{\sigma^2}} \\ &= p_0^2 + \frac{\hbar^2}{2\sigma^2} + \frac{-\hbar^2}{4\sigma^2} = p_0^2 + \frac{\hbar^2}{4\sigma^2} \end{aligned}$$

MIDTERM ON MATERIAL UNTIL HERE

May 13th

Rules of Quantization for Physical Quantities: (C.T. 222)

If this classical quantity $f(x, p)$ can be written as a function of location and momentum, we will learn how to write the operator.

Rule 1: The quantum operator will be $\mathbb{F} = f(\mathbb{X}, \mathbb{P})$

Examples: Kinetic Energy

$$E_P = \frac{p^2}{2m} \xrightarrow{q} \mathbb{K} = \frac{1}{2m} \mathbb{P}^2$$

For a harmonic oscillator:

$$E_P = \frac{1}{2} kx^2 \xrightarrow{q} \mathbb{V} = \frac{1}{2} k\mathbb{X}^2$$

Problematic

$$px \xrightarrow{q} \mathbb{P}\mathbb{X}$$

Which is not Hermitian because $(\mathbb{X}\mathbb{P})^\dagger = \mathbb{P}^\dagger\mathbb{X}^\dagger = \mathbb{P}\mathbb{X} \neq \mathbb{X}\mathbb{P}$.

So we shall see that

$$px \xrightarrow{q} \frac{1}{2}(\mathbb{P}\mathbb{X} + \mathbb{X}\mathbb{P})$$

Rule 2: Operators are chosen by symmetrization ($x \leftrightarrow p$) such that it will be Hermitian (Check substituting in reverse)

Specifically let us look at the case where the operator is defined solely as a function of x , $\mathbb{F}=f(\mathbb{X})$

$$\begin{aligned} \mathbb{F}[\psi(x)] &= \langle x|\mathbb{F}|\psi\rangle = \langle x|\Sigma_n f_n \mathbb{X}^n|\psi\rangle \\ &= \Sigma_n f_n x^n \psi(x) = f(x)\psi(x) \end{aligned}$$

Similarly if the function is a function solely of momentum $\mathbb{F}=f(\mathbb{P})$, we'll find $f(\frac{\hbar}{i} \frac{d}{dx})\psi(x)$

So for $\mathbb{F} = f(\mathbb{X}, \mathbb{P})$

Looking at the classical Hamiltonian

$$H_{cl} = E_k + E_p = \frac{p^2}{2m} + V(x)$$

$$\xrightarrow{q} \mathbb{H}_{qm} = \frac{\mathbb{P}^2}{2m} + V(\mathbb{X})$$

In location representaiton

$$\begin{aligned} \langle x|\mathbb{H}|\psi\rangle &= \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi(x) \\ \langle p|\mathbb{H}|\psi\rangle &= \left[\frac{p^2}{2m} + V(i\hbar \frac{\partial}{\partial p}) \right] \tilde{\psi}(p) \end{aligned}$$

Ehrenfest Theorem: (C.T. 242)

$$\frac{d}{dt} \langle \mathbb{X} \rangle = \frac{1}{i\hbar} \langle [\mathbb{X}, \mathbb{H}] \rangle = \frac{1}{i\hbar} \langle [\mathbb{X}, \frac{\mathbb{P}^2}{2m}] \rangle = \frac{1}{m} \langle \mathbb{P} \rangle$$

$$\frac{d}{dt} \langle \mathbb{P} \rangle = \frac{1}{i\hbar} \langle [\mathbb{P}, \mathbb{H}] \rangle = \frac{1}{i\hbar} \langle [\mathbb{P}, V(\mathbb{X})] \rangle = -\langle V'(\mathbb{X}) \rangle$$

Where

$$[\mathbb{X}, \frac{\mathbb{P}^2}{2m}] = \frac{1}{2m} [\mathbb{X}, \mathbb{P}^2] = \frac{1}{2m} (\mathbb{X}\mathbb{P}\mathbb{P} - \mathbb{P}\mathbb{P}\mathbb{X}) = \frac{1}{2m} ((\mathbb{P}\mathbb{X} + i\hbar)\mathbb{P} - \mathbb{P}(\mathbb{X}\mathbb{P} - i\hbar)) = \frac{i\hbar}{m} \mathbb{P}$$

and

$$[\mathbb{P}, V(\mathbb{X})] = [\mathbb{P}, \mathbb{X}] V'(\mathbb{X}) = -i\hbar V'(\mathbb{X})$$

This reminds us of the Hamilton-Jacobi Equations

$$\begin{cases} \frac{d}{dt} \langle \mathbb{X} \rangle = \frac{1}{m} \langle \mathbb{P} \rangle & \left(\langle \frac{\partial \mathbb{H}}{\partial \mathbb{P}} \rangle \right) \\ \frac{d}{dt} \langle \mathbb{P} \rangle = -\langle V'(\mathbb{X}) \rangle & \left(-\langle \frac{\partial \mathbb{H}}{\partial \mathbb{X}} \rangle \right) \end{cases}$$

And analogously to Newtonian mechanics

$$m \frac{d^2}{dt^2} \langle \mathbb{X} \rangle = -\langle V'(\mathbb{X}) \rangle$$

When do these results not reflect classical results?

We need the potential $-\langle V'(\mathbb{X}) \rangle$ to look like $V'(\langle \mathbb{X} \rangle)$ (when the wavepacket is very thin - meaning center of mass)

Schrodinger Equation in Location Representation:

(The Time-Dependent Form): (C.T. 184)

For the equation

$$i\hbar \frac{\partial}{\partial t} |\psi\rangle = H|\psi\rangle$$

We'd look at

$$\begin{aligned} \langle x | i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle &= \langle x | H |\psi(t)\rangle \\ \Rightarrow i\hbar \frac{\partial}{\partial t} \psi(x, t) &= \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi(x, t) \end{aligned}$$

Essentially a wave equation, and is the reason why $\psi(x, t)$ is called a wave function.

Writing in the momentum representation

$$\begin{aligned} \langle p | i\hbar \frac{\partial}{\partial t} |\psi(t)\rangle &= \langle p | H |\psi(t)\rangle \\ \Rightarrow i\hbar \frac{\partial}{\partial t} \tilde{\psi}(p, t) &= \left[\frac{p^2}{2m} + V(i\hbar \frac{\partial}{\partial p}) \right] \tilde{\psi}(p, t) \end{aligned}$$

Interesting to see as the relation is a fourier transform between the two expressions.

Using the convolution theorem:

$$V(i\hbar \frac{\partial}{\partial p}) \tilde{\psi}(p, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp' \tilde{V}(p - p') \tilde{\psi}(p', t)$$

The Development in Time of a Free Particle: (C.T. 61)

Free means that the derivative of the potential is zero, meaning the potential is constant (Without the loss of generality $V = 0$)

Looking at the momentum representation under this assumption

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \tilde{\psi}(p, t) &= \frac{p^2}{2m} \tilde{\psi}(p, t) \\ \Rightarrow \tilde{\psi}(p, t) &= e^{-i \frac{p^2}{2m\hbar} t} \tilde{\psi}(p, 0) \end{aligned}$$

Looking at the location representation (Fourier Transform)

$$\psi(x, t) = \frac{1}{\sqrt{2\pi\hbar}} \int_{-\infty}^{\infty} dp e^{i \frac{p}{\hbar} (px - \frac{p^2}{2m} t)} \tilde{\psi}(p, 0)$$

For some wave packet let's write the expectation value at the moment 0

$$\langle \mathbb{X} \rangle(0) = i\hbar \int_{-\infty}^{\infty} dp \tilde{\psi}^*(p, 0) \frac{\partial}{\partial t} \tilde{\psi}(p, 0)$$

So what is the expectation value for any time?

$$\begin{aligned}
\langle \mathbb{X} \rangle(t) &= i\hbar \int_{-\infty}^{\infty} dp \tilde{\psi}^*(p, t) \frac{\partial}{\partial t} \tilde{\psi}(p, t) \\
&= i\hbar \int_{-\infty}^{\infty} dp e^{i \frac{p^2}{2m\hbar} t} \tilde{\psi}^*(p, 0) \frac{\partial}{\partial p} \left(e^{-i \frac{p^2}{2m\hbar} t} \tilde{\psi}(p, 0) \right) \\
&= i\hbar \int_{-\infty}^{\infty} dp e^{i \frac{p^2}{2m\hbar} t} \tilde{\psi}^*(p, 0) \left(e^{-i \frac{p^2}{2m\hbar} t} \frac{\partial}{\partial p} \tilde{\psi}(p, 0) - i \frac{pt}{m\hbar} e^{-i \frac{p^2}{2m\hbar} t} \tilde{\psi}(p, 0) \right) \\
&= i\hbar \int_{-\infty}^{\infty} dp e^{i \frac{p^2}{2m\hbar} t} \tilde{\psi}^*(p, 0) e^{-i \frac{p^2}{2m\hbar} t} \frac{\partial}{\partial p} \tilde{\psi}(p, 0) - i\hbar \int_{-\infty}^{\infty} dp e^{i \frac{p^2}{2m\hbar} t} \tilde{\psi}^*(p, 0) i \frac{pt}{m\hbar} e^{-i \frac{p^2}{2m\hbar} t} \tilde{\psi}(p, 0) \\
&= i\hbar \int_{-\infty}^{\infty} dp \tilde{\psi}^*(p, 0) \frac{\partial}{\partial p} \tilde{\psi}(p, 0) + \frac{t}{m} \int_{-\infty}^{\infty} dp \tilde{\psi}^*(p, 0) p \tilde{\psi}(p, 0) \\
&= \langle \mathbb{X} \rangle(0) + \langle \mathbb{P} \rangle(0) \frac{t}{m}
\end{aligned}$$

So we see the classical results ...

What will be the $\langle \mathbb{P} \rangle(t)$? $\langle \mathbb{P} \rangle(t) = \langle \mathbb{P} \rangle(0)$ Because it will be a number, it will remain constant and be conserved, and will commutate with the hamiltonian (based on \mathbb{P}^2 , which clearly commutes with \mathbb{P})

(free) Gaussian Wave Packet (as a function of time): (C.T. 63)

Taking $\tilde{\psi}(p, 0)$ as the general solution, we will only need to multiply by some phase $e^{-i\cdots}$.

$$= \dots e^{-\frac{\sigma_0^2}{\hbar} p^2} \rightarrow \frac{\sigma_0^2}{\hbar^2} + i \frac{t}{2m\hbar}$$

So the fourier transform, which in the past gave us $\frac{1}{4\sigma_0^2}$ as the coefficient of x , will now give us

$$\frac{1}{4\sigma_0^2} x^2 \rightarrow \frac{1}{4\sigma_0^2} \left(\frac{1}{1 + i \frac{\hbar t}{2m\sigma_0^2}} \right) x^2 = \frac{1 - i \frac{\hbar t}{2m\sigma_0^2}}{4\sigma_0^2 (1 + \left(\frac{\hbar t}{2m\sigma_0^2} \right)^2)} x^2$$

So we see that

$$\sigma_0 \rightarrow \sigma(t) = \sigma_0 \sqrt{1 + \left(\frac{\hbar t}{2m\sigma_0^2} \right)^2}$$

Meaning that the wave feels dispersion as the different components of the wave packet move at different speeds.

So we find the characteristic time for widening

$$\tau = \frac{2m\sigma_0^2}{\hbar}$$

13 May 15th

Time-Independent Schrodinger Equations: (C.T. 67)

$$H|\psi_E\rangle = E|\psi_E\rangle$$

Looking at the projection

$$\langle x|H|\psi_E\rangle = E\langle x|\psi_E\rangle$$

	Finite Potential		Infinite Potential	
(Integral)	Continuous	Non-Continuous	Infinite	Infinite Around Point
ψ''	Continuous	Non-Continuous	0	Undefined
ψ'	Continuous	Continuous	0	Non-Continuous
ψ	Continuous	Continuous	0	Continuous

Table 1: Integral and Wavefunction Derivatives for Different $V(x)$

$$\left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi_E(x) = E \psi_E(x)$$

$$\Rightarrow \frac{\partial^2 \psi_E(x)}{\partial x^2} + \frac{2m}{\hbar^2} [E - V(x)] \psi_E(x) = 0$$

The solutions are the eigenstates of the hamiltonian of the eigenvalues of the energies E .

(These are “amid” states.)

The kinetic component doesn’t change, only the potential part $V(x)$.

The spectrum is all of the eigenvalues which represent the energies (continuous or discrete).

$$\Rightarrow \frac{\partial^2 \psi}{\partial x^2} = \frac{2m}{\hbar^2} [V(x) - E] \psi$$

While $V(x)$ is continuous, so is the second derivative.

When $V(x)$ is not, so the second derivative will also not be continuous.

Integrating around $[x - \epsilon, x + \epsilon]$

$$\Rightarrow \frac{\partial \psi}{\partial x} \Big|_{x+\epsilon} - \frac{\partial \psi}{\partial x} \Big|_{x-\epsilon} = \frac{2m}{\hbar^2} \int_{x-\epsilon}^{x+\epsilon} dx [V(x) - E] \psi(x)$$

If the integral around a point gives a finite number for $V(x)$, so too is the first derivative continuous and therefore the function itself.

Constant Potential (on Segments):

$$\frac{\partial^2 \psi}{\partial x^2} = -\frac{2m}{\hbar^2} [E - V] \psi$$

Will have solutions of the form

$$\psi_E(x) = \begin{cases} E > V & A_+ e^{i\sqrt{\frac{2m}{\hbar^2}(E-V)}x} + A_- e^{-i\sqrt{\frac{2m}{\hbar^2}(E-V)}x} \\ E = V & Ax + B \\ \underbrace{E < V}_{tunnelling} & A_+ e^{\sqrt{\frac{2m}{\hbar^2}(V-E)}x} + A_- e^{-\sqrt{\frac{2m}{\hbar^2}(V-E)}x} \end{cases}$$

From now on we shall denote

$$k = \sqrt{\frac{2m}{\hbar^2}(E - V)}$$

when $E > V$ and

$$\alpha = \sqrt{\frac{2m}{\hbar^2}(V - E)}$$

When $V > E$.

Constant Potential - $V(x) = V$:

If $E > V_0$, the solution is $\psi(x) = A_+e^{ikx} + A_-e^{-ikx}$

We can denote with p , the quantity $p = \pm\sqrt{2m(E - V)}$

Since the operator commutes with the momentum operator...

Meaning we have a superposition of a particle carrying left-right momentum **and** right-left momentum

We shall see, from Dirac's normalization proof, that

$$|A_+|^2 + |A_-|^2 = \frac{1}{2\pi}$$

What happens when $E \leq V_0$? We must have $A_+ = A_- = 0$ meaning that these energy values are prohibited.

Reflection from an Infinite Potential Wall:

$$V(x) = \begin{cases} x > 0, V = 0 & \psi(x) = A_+e^{ikx} + A_-e^{-ikx} \\ x \leq 0, V \rightarrow \infty & \psi(x) = 0 \end{cases}$$

Because of the need for continuity

$$\psi_+(0) = A_+ + A_- = \psi_-(0) = 0$$

$$\Rightarrow A_+ = -A_-$$

For $x > 0$ $\psi(x) = \frac{1}{\sqrt{2\pi}}\sin(kx)$.

Infinite Potential Well: (C.T. 77, 269) (Mashiach 86)

Without the loss of generality, looking at

$$V(x) = \begin{cases} |x| < \frac{L}{2} & V = 0 \\ |x| \geq \frac{L}{2} & V \rightarrow \infty \end{cases}$$

At the boundaries

$$\psi(x = -\frac{L}{2}) = A_+e^{-ik\frac{L}{2}} + A_-e^{ik\frac{L}{2}} = 0$$

$$\psi(x = \frac{L}{2}) = A_+e^{ik\frac{L}{2}} + A_-e^{-ik\frac{L}{2}} = 0$$

When we add the two equations

$$(A_+ + A_-)(e^{ik\frac{L}{2}} + e^{-ik\frac{L}{2}}) = (A_+ + A_-)2\cos\left(\frac{kL}{2}\right) = 0$$

and subtract

$$(A_+ - A_-)(e^{ik\frac{L}{2}} - e^{-ik\frac{L}{2}}) = (A_+ - A_-)2is\sin\left(\frac{kL}{2}\right) = 0$$

In the first case where $A_+ = -A_-$ we see $k = 2n\frac{\pi}{L}$, and the function is $\psi(x) = \sqrt{\frac{2}{L}}\sin(\frac{2n\pi}{L}x)$.

In the second case where $A_+ = A_-$ we see $k = (2n-1)\frac{\pi}{L}$, and the function is $\psi(x) = \sqrt{\frac{2}{L}}\cos(\frac{(2n-1)\pi}{L}x)$.

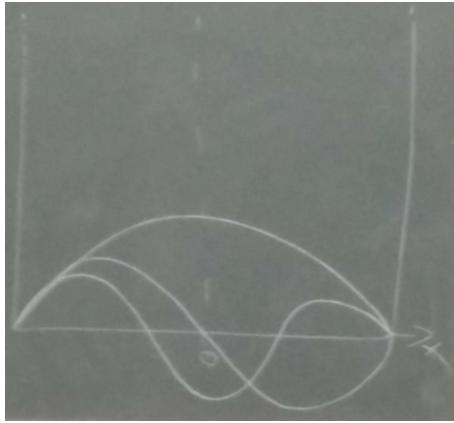


Figure 4: Wave Function in Potential Well

We find solutions (different n)

$$k_n = n \frac{\pi}{L}$$

Giving us the discrete energy levels

$$E_n = \frac{\hbar^2 k_n^2}{2m} = \frac{\hbar^2 \pi^2}{2m L^2} n^2$$

The wells alternate between symmetry and anti-symmetric

Finite Potential Well: (C.T. 74) (Mashiach 88)

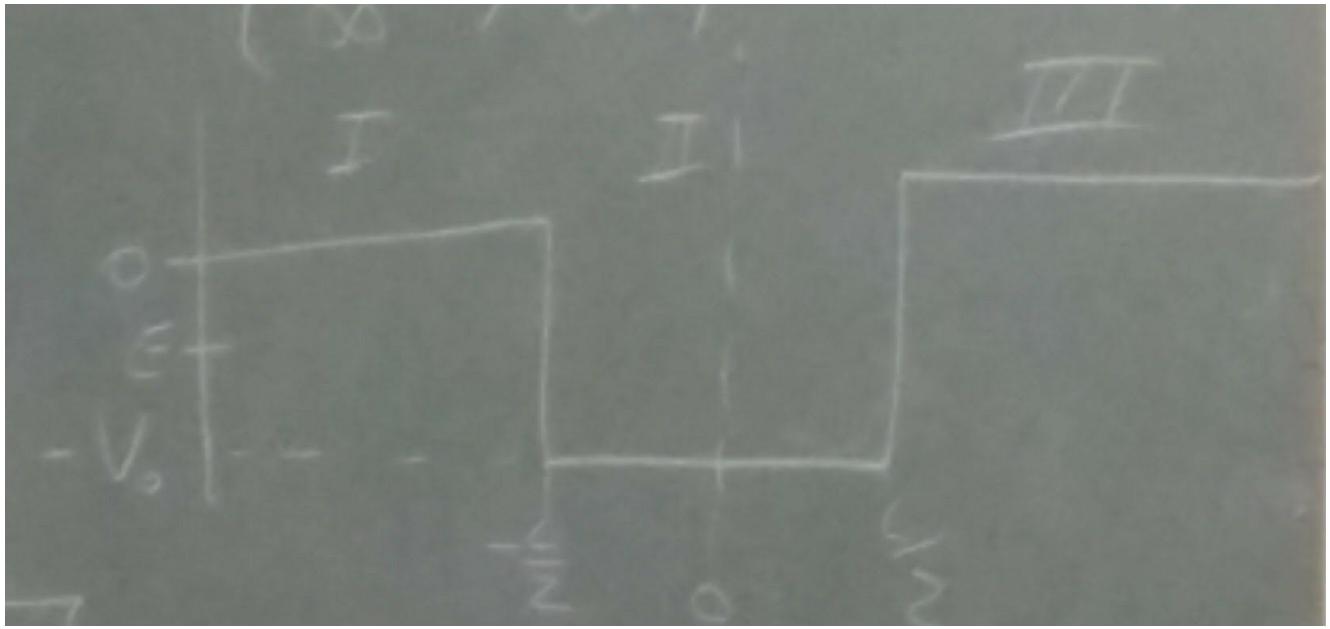


Figure 5: Wave Function in Finite Potential Well

For $-V_0 \leq E \leq 0$

$$\psi_I(x) = A_{1+} e^{\alpha x} + A_{1-} e^{-\alpha x}$$

$$\psi_{II}(x) = A_{2+} e^{ikx} + A_{2-} e^{-ikx}$$

$$\psi_{III}(x) = A_{3+}e^{\alpha x} + A_{3-}e^{-\alpha x}$$

Where

$$\alpha = \sqrt{\frac{2m}{\hbar^2}(-E)}, \quad k = \sqrt{\frac{2m}{\hbar^2}(E + V_0)}, \quad k_0 = \sqrt{\frac{2m}{\hbar^2}V_0} = \sqrt{k^2 + \alpha^2}$$

At the boundary $A_{1-} = A_{3+} = 0$

$$\begin{aligned} A_{1+}e^{-\alpha \frac{L}{2}} &= A_{2+}e^{-ik \frac{L}{2}} + A_{2-}e^{ik \frac{L}{2}} \\ \Rightarrow \alpha A_{1+}e^{-\alpha \frac{L}{2}} &= ikA_{2+}e^{-ik \frac{L}{2}} - ikA_{2-}e^{ik \frac{L}{2}} \\ A_{3-}e^{-\alpha \frac{L}{2}} &= A_{2+}e^{ik \frac{L}{2}} + A_{2-}e^{-ik \frac{L}{2}} \\ \Rightarrow -\alpha A_{3-}e^{-\alpha \frac{L}{2}} &= ikA_{2+}e^{ik \frac{L}{2}} - ikA_{2-}e^{-ik \frac{L}{2}} \end{aligned}$$

We see that

$$\begin{aligned} \alpha A_{1+}e^{-\alpha \frac{L}{2}} &= ikA_{2+}e^{-ik \frac{L}{2}} - ik \left(A_{1+}e^{-\alpha \frac{L}{2}} - A_{2+}e^{-ik \frac{L}{2}} \right) \\ -\alpha A_{3-}e^{-\alpha \frac{L}{2}} &= ikA_{2+}e^{ik \frac{L}{2}} - ik \left(A_{3-}e^{-\alpha \frac{L}{2}} - A_{2+}e^{ik \frac{L}{2}} \right) \end{aligned}$$

Thus

$$\begin{aligned} A_{2+} &= \frac{\alpha + ik}{2ik} A_{1+} e^{(-\alpha + ik) \frac{L}{2}} \\ A_{2+} &= \frac{-\alpha + ik}{2ik} A_{3+} e^{(-\alpha - ik) \frac{L}{2}} \\ \Rightarrow \frac{-\alpha + ik}{\alpha + ik} \frac{A_{3-}}{A_{1+}} &= e^{ikL} \end{aligned}$$

$$V(x) = V(-x), \quad |\psi(x)|^2 = |\psi(-x)|^2, \quad A_{1+} = \pm A_{3-}$$

So

$$\pm \frac{-(\alpha - ik)^2}{\alpha^2 + k^2} = \pm \frac{k^2 - \alpha^2 + 2i\alpha k}{k_0^2} = e^{ikL}$$

Of which the real part is

$$\frac{k^2 - \alpha^2}{k_0^2} = \frac{2k^2}{k_0^2} - 1 = \pm \cos(kL)$$

May 22nd

Last lesson we solved a Finite potential well that was zero everywhere except within a region of size L, at which it is at a potential $-V$. We saw three regions, A_1 , A_2 , A_3 .

We compared the equations for the potential at the \pm edges and found:

$$\frac{-\alpha + ik}{\alpha + ik} \frac{A_{3-}}{A_{1+}} = e^{ikL}$$

And we said A_3 and A_1 describe regions 1 and 3 and found that, for reasons of symmetry (the wave function has to be symmetric or anti-symmetric in order to get characteristic of the symmetry of well in what is the physical characteristic that the probability density function is symmetric).

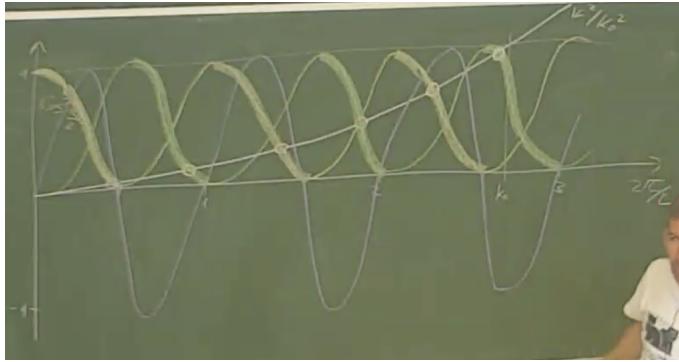


Figure 6: $\cos^2(kL/2)$ and $\sin^2(kL/2)$ - with k^2/k_0^2 , and $\sin(kL)$ in purple, overlapping

Thus the conditions that these coefficients are either equal or simply of opposite sign.

$$A_3 = \pm A_1$$

Plus giving us the symmetric solutions and minus giving us the anti-symmetric solutions.

By multiplication of the denominator's complex conjugate we found the complex equations:

$$\frac{k^2 - \alpha^2 + 2i\alpha k}{k_0^2} = e^{ikL}$$

Where k is the oscillator constant inside the well and α is for outside the well (decreasing).

We found for the real part

$$\frac{k^2}{k_0^2} = \frac{V_0 + E}{V_0} = \begin{cases} \cos^2(kL/2) & \text{symmetric case (+)} \\ \sin^2(kL/2) & \text{anti-symmetric case (-)} \end{cases}$$

We'll also write the imaginary part

$$\pm \frac{2i\alpha k}{k_0^2} = \sin(kL)$$

Now we shall see that we must solve these equations numerically - or in our case graphically.

The horizontal k axis has units of $2\pi/L$ and the vertical axis is the values of the functions (right and left) - where the functions intersect will be the points where the equation holds.

Symmetric solutions only when $\sin(kL)$ has positive values and anti-symmetric solutions when it has negative values.

The *sine* is positive only between 0 and π/L , meaning that the solution of the *cosine* is only relevant in the areas where $\sin(kL)$ is positive - see yellow highlighted areas. The opposite is true for \sin^2 - from the anti-symmetric - will only be relevant when $\sin(kL)$ is negative - see green highlighted areas.

There are only 6 points - solutions to the equation (circled) - where the intersections of k^2/k_0^2 with the relevant functions. There cannot be any more as k^2/k_0^2 overlaps the value 1 at $k = k_0$.

These are the solutions, now what do they mean?

These are solutions of k and there is a connection to the energy, so if we know the k we know the energy in the well. Finite solutions.

We can see a characteristic we saw with the infinite well too, that the solutions have an alternating symmetry. Also solutions of the ground state is always symmetric because of \cos^2 - there must also only be one solution, so at least one solution in any finite symmetric well.

These solutions will, in the potential well, look like

with the higher number of nodes being associated higher k values of the solution.



Figure 7:

Outside of the well the energy is smaller than the potential.

From k_0 we can know the number of solutions, how many times do these functions do until this value? each oscillation gives a solution (half an oscillation for the cosine).

Now we shall write the non-trivial equation for the number of the solutions.

The number of times π goes into $k_0 L / 2 + \pi$ is the number of symmetric modes

$$N_S = \lfloor \frac{k_0 L}{2\pi} + 1 \rfloor = \lfloor \frac{L}{2\pi\hbar} \sqrt{2mV_0} + 1 \rfloor$$

The number of anti-symmetric modes is the number of times π goes into $k_0 L / 2 + \frac{\pi}{2}$

$$N_{AS} = \lfloor \frac{k_0 L}{2\pi} + \frac{1}{2} \rfloor = \lfloor \frac{L}{2\pi\hbar} \sqrt{2mV_0} + \frac{1}{2} \rfloor$$

This is especially confusing when we ask what the total number of modes is

$$N = N_S + N_{AS} = \lfloor \frac{L}{\pi\hbar} \sqrt{2mV_0} + 1 \rfloor = \lceil \frac{L}{\pi\hbar} \sqrt{2mV_0} \rceil$$

Reiterating the fact that there is always at least one solution - the ground state.

So we see that the number of solutions for a well is linearly dependent on the width of the well, and upon the square root of the depth.

When we solve optics in dielectric materials - using the paraxial approximations to the wave equations- it turns into the Helmholtz equation which is essentially Schrodinger's equation.

As the light moves, the index of refraction changes (analogous to the potential (with a negative sign) in Schrodinger's equation).

Total internal reflection occurs when the light ray tries to tilt towards a lower indexed area - used in fiber optics like internet, with single mode fibers.

Let's take these solutions at look at two interesting limits:

Firstly looking at $V_0 \rightarrow \infty$, using the graphical solution.

At this limit, k_0 also goes to infinity meaning that the parabola flattens into a line that approaches the flatness of the k axis where

$$kL = n\pi$$

for all of the nodes, thus

$$\begin{aligned} \sqrt{\frac{2m}{\hbar^2}(E + V_0)} &= \frac{n\pi}{L} \\ \Rightarrow E + V_0 &= \frac{\pi^2\hbar^2}{2mL^2}n^2 \end{aligned}$$

noting that $E + V_0$ is the height/distance from the bottom of the well. The term of the righthand side of the equation is exactly the energy levels for an infinite well. (Sanity Check)

At the limit $V_0 \rightarrow \infty$ and $L \rightarrow 0$, turning the well into something looking like a negative delta function - requiring that $LV_0 = \text{const} = \lambda$ - we see that there will be one solution where

$$k^2/k_0^2 = \frac{E + V_0}{V_0} \rightarrow 1 + \frac{E}{V_0} = \frac{1}{2} (1 + \cos(kL)) \approx 1 - \frac{k^2 L^2}{4}$$

where kL goes to zero (using taylor).

Thus

$$\frac{E}{V_0} = -\frac{1}{4} \frac{2mL^2}{\hbar^2} (V_0 + E) \approx -\frac{1}{4} \frac{2mL^2}{\hbar^2} V_0$$

Between V_0 and E , it is V_0 that wins as it grows to infinity

Thus

$$E = -\frac{mL^2}{2\hbar^2} V_0^2 = -\frac{m\lambda^2}{2\hbar^2}$$

which goes neither zero nor to infinity.

The wave function in the well will look like :

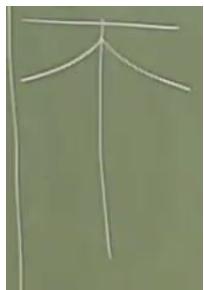


Figure 8:

Noting that the first derivative at this limit is not continuous, but this works fine with the table we prepared because the value of the integral is still finite.

So one limit is taking the depth to infinity and the second retains constant area but takes the width the zero.

If we had some electron caught in this delta type function it would be a really useful limit to use.

Motion Under Constant Force (Free Fall): (Landau-Lipschitz 73)

This is something very trivial for us in classical mechanics, but most basic courses in quantum don't even dare to cover it because of its non-triviality.

For some constant force f acting on a mass m , it could be equal to

$$= mg, \quad = qE$$

or otherwise. We could take a transistor as another example, with a linear potential $V(x) = fx$

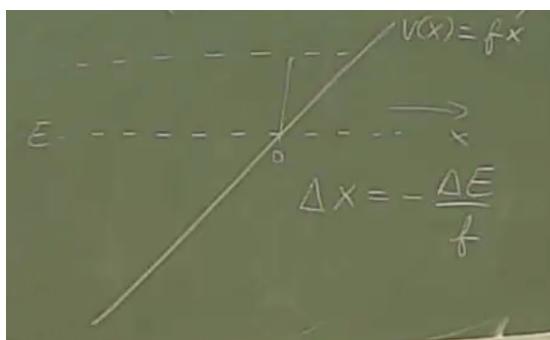


Figure 9: Energy for transistor with linear potential

We can see an interesting characteristic of the solutions.

We expect that movements in the x direction will give us the same solution

$$\Delta x = -\frac{\Delta E}{f}$$

meaning that all the solutions will look the same but will be displaced by a ratio f horizontally - the functional form will be the same.

Now we can say something else, these solutions are solutions that are the eigenfunctions of a Hermitian operator, i.e a whole orthonormal set - perpendicular one to another but displaced horizontally.

One example of such functions is delta, another shall be seen later.

The Schrodinger equation for linear potential

$$-\frac{\hbar^2}{2m} \frac{d^2\psi}{dx^2} + fx\psi - E\psi = 0$$

which is also known as ...

If we define $x' = x - \frac{E}{f}$, moving the function to be such that the zero is defined at the point of intersection between the energy and the potential. This point shall be referred to as the **classical turning point** - as this will be the point at which the particle begins to turn around.

Another unit change will be $y = \left(\frac{2mf}{\hbar^2}\right)^{1/3} x'$ in order obtain a unitless equation:

$$\frac{d^2\psi}{dy^2} - y\psi = 0$$

Let's take a Fourier transform

$$-k^2\tilde{\psi} - i\frac{d\tilde{\psi}}{dk} = 0$$

Which leads us to the solution

$$\tilde{\psi}(k) = A'e^{\frac{ik^3}{3}}$$

The inverse transform (swallowing the $1/2\pi$ in a new coefficient)

$$\psi(y) = A \int_{-\infty}^{\infty} dk e^{\left(\frac{ik^3}{3} + ky\right)} = Ai(y)$$

This function is known as the Airy Function

Physically we see that the area to the right of the zero in the previous figure is a classically forbidden area - of tunnelling.

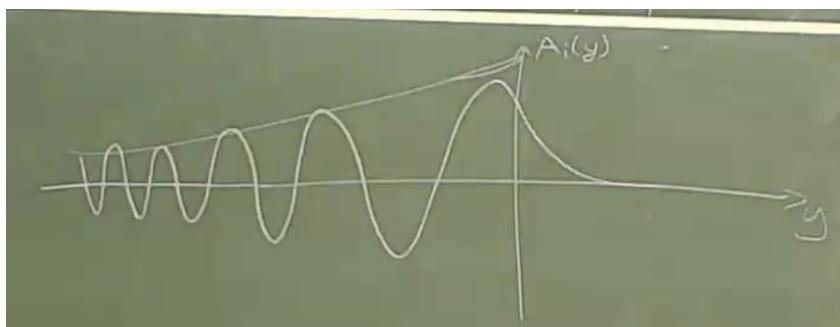


Figure 10: Airy Function

In the positive region the function decreases exponentially, the negative region it does oscillations - with some overall amplitude which decreases towards $-\infty$, though the frequency increases.

When y is much smaller than -1

$$Ai_{y<0}(y) \approx \frac{\sin\left(\frac{2}{3}(-y)^{3/2} + \frac{\pi}{4}\right)}{\sqrt{\pi}(-y)^{1/4}}$$

When y is much larger than +1

$$Ai_{y>0}(y) \approx \frac{e^{-\frac{2}{3}y^{3/2}}}{2\sqrt{\pi}(y)^{1/4}}$$

(We defined the concept of a local frequency as being the derivative of the function in some region - rate of change)

Next lesson we shall give a semi-classical meaning to this concept in the form of “local momentum”.

Sir George Airy was a British (royal) astronomer and the person who defined Greenwich as the zero line (prime meridian).

(One side of a rainbow is more illuminated - inner - than the other)

The “Bairy” function is another independent solution so all solutions will generally be some linear combination of $Ai(y)$ and $Bi(y)$.

Let's multiply two Airy Functions

$$\begin{aligned} & A^2 \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dk' e^{-i\left(\frac{k'^3}{3} + k'(y-y')\right)} \int_{-\infty}^{\infty} dk'' e^{i\left(\frac{k''^3}{3} + k''(y-y'')\right)} \\ &= A^2 \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dk' \int_{-\infty}^{\infty} dk'' e^{iy(k''-k')} e^{i\left(\frac{k''^3}{3} - \frac{k'^3}{3} + k'y' - k''y''\right)} \\ &= 2\pi A^2 \int_{-\infty}^{\infty} dk' \int_{-\infty}^{\infty} dk'' \delta(k'' - k') e^{i\left(\frac{k''^3}{3} - \frac{k'^3}{3} + k'y' - k''y''\right)} \\ &= 2\pi A^2 \int_{-\infty}^{\infty} dk' e^{ik'(y'-y'')} \\ &= 4\pi^2 A^2 \delta(y' - y'') \end{aligned}$$

As we wanted. We took two Airy functions and found that they are perpendicular.

In order for it to be delta we will define $A = \frac{1}{2\pi}$

$$\int_{-\infty}^{\infty} Ai(y) dy = 1$$

So we see that

$$\begin{aligned} \psi_E(x) &= \left(\frac{2m}{\hbar^2 \sqrt{f}}\right)^{1/3} Ai\left[\left(\frac{2m}{\hbar^2 \sqrt{f}}\right)^{1/3} \left(x - \frac{E}{f}\right)\right] \\ \tilde{\psi}(p) &= \frac{1}{\sqrt{2\pi\hbar f}} e^{\frac{i}{\hbar} \left[\frac{p^3}{6mf} - \frac{E}{f}p\right]} \end{aligned}$$

What is interesting about the momentum representation?

There is no dependence on p in the absolute value - in the probability function. Meaning that the a particle in the eigenstate of the Hamiltonian - on this linear potential - exists simultaneously at every possible momentum; with equal probability!

In location this is not the case.

With the Airy function, the points where it intercepts the (horizontal) y axis are the points at which we will never find the particle - because at those points the classical motion is most slow.

How can we see from the Quantum wave function where the particle is moving faster or slower?

A fast classical particle has a high momentum and therefore low deBroglie wavelength.

A real quantum particle can never be in an Airy function.

The characteristic length

$$x_T = \left(\frac{\hbar^2}{2m^2 g} \right)^{1/3}$$

is only for quantum particles, not classical.

14 May 27th

Last lesson we dealt with a finite well and free fall, and the rest of the semester will be finding other solutions to the Schrodinger equation.

One-Dimensional Scattering - Probability Flow: (C.T 238)

The time-dependent Schrodinger equation (location representation)

$$\begin{aligned} i\hbar \frac{\partial}{\partial t} \psi(x, t) &= \left[-\frac{\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + V(x) \right] \psi(x, t) \\ \Rightarrow i\hbar \psi^* \frac{\partial}{\partial t} \psi &= -\frac{\hbar^2}{2m} \psi^* \frac{\partial^2}{\partial x^2} \psi + V(x) \psi^* \psi \\ \Rightarrow -i\hbar \frac{\partial \psi^*}{\partial t} \psi &= -\frac{\hbar^2}{2m} \frac{\partial^2 \psi^*}{\partial x^2} \psi + V(x) \psi \psi^* \end{aligned}$$

Taking these last two equations and subtracting them

$$\begin{aligned} i\hbar \left[\psi^* \frac{\partial}{\partial t} \psi + \frac{\partial \psi^*}{\partial t} \psi \right] &= -\frac{\hbar^2}{2m} \left[\psi^* \frac{\partial^2 \psi}{\partial x^2} - \frac{\partial^2 \psi^*}{\partial x^2} \psi \right] \\ \Rightarrow \frac{\partial |\psi|^2}{\partial t} + \frac{\hbar}{2im} \frac{\partial}{\partial x} \left[\psi^* \frac{\partial}{\partial x} \psi - \frac{\partial \psi^*}{\partial t} \psi \right] &= 0 \end{aligned}$$

For the probability density

$$P = |\psi|^2$$

$$J \equiv \frac{\hbar}{2im} \left[\psi^* \frac{\partial}{\partial x} \psi - \frac{\partial \psi^*}{\partial t} \psi \right]$$

Thus we find the continuity equation

$$\Rightarrow \frac{\partial P}{\partial t} + \frac{\partial J}{\partial x} = 0$$

Where J is the flow of the probability density.

Until now we have had a conservation of global probability, now we find a local conservation of probability.

Let's look at an example

Probability Flow of a Free Particle: (C.T. 240, 280)

moving from $-\infty$ to ∞ with a defined momentum

$$\psi_{p_0}(x) = \frac{1}{\sqrt{2\pi\hbar}} e^{\frac{i}{\hbar} p_0 x}$$

$$P = \frac{1}{2\pi\hbar} = \text{const}$$

Whereas

$$J = |\psi|^2 \frac{p_0}{m} = Pv_g = \text{const}$$

The probability is always moving in both directions and flowing. Similar to the charge density being constant even though the current is flowing.

We can look at the probability flow for different momenta.

Recalling that

$$k = \sqrt{\frac{2m}{\hbar^2}(E - V)} = \frac{p_0}{\hbar}$$

and

$$\psi(x) = A_+ e^{ikx} + A_- e^{-ikx}$$

We can write the probability flow as

$$\begin{aligned} J &= \frac{\hbar}{2im} \left[\psi^* \frac{\partial \psi}{\partial x} - c.c. \right] \\ &= \frac{\hbar}{2im} \left[(A_+^* e^{-ikx} + A_-^* e^{ikx}) (ikA_+ e^{ikx} - ikA_- e^{-ikx}) - c.c. \right] \\ &= \frac{\hbar k}{2im} [|A_+|^2 - |A_-|^2 + A_+ A_-^* e^{2ikx} - A_+^* A_- e^{-2ikx} + c.c.] \\ &= \frac{p_0}{m} [|A_+|^2 - |A_-|^2] \end{aligned}$$

So the local probability flow at a given point has contributions from probability in both directions, where the first term is the same as the previous case.

Let's look at $E < V$, we've seen cases where this holds - quantum tunnelling.

Is there flow in the tunnel?

$$\begin{aligned} J &= \frac{\hbar}{2im} \left[(A_+^* e^{\alpha x} + A_-^* e^{-\alpha x}) (\alpha A_+ e^{\alpha x} - \alpha A_-^* e^{-\alpha x}) - c.c. \right] \\ &= \frac{\hbar\alpha}{2m} [-i|A_+|^2 e^{2\alpha x} + i|A_-|^2 e^{-2\alpha x} - iA_+ A_-^* + iA_+^* A_- + c.c.] \\ &= \frac{\hbar\alpha}{m} [iA_+^* A_- - iA_+ A_-^*] = \frac{\hbar\alpha}{m} [iA_+^* A_- + c.c.] \end{aligned}$$

We saw last lesson that

$$Ai_{y<0}(y) = \frac{\sin(\frac{2}{3}(-y)^{3/2} + \frac{\pi}{4})}{\sqrt{\pi}(-y)^{1/4}}$$

Because, in the case of linear potential, the potential goes to infinity and thus any of the probability that flows into the tunnelling region will be returned/reflected back outwards - meaning that the overall flow is zero.

If ψ is a pure real value (or a pure imaginary value), the flow J will equal zero.

Essentially what we are seeing is that if there is an infinite potential barrier - continuing to $\pm\infty$ - then one of the respective coefficients A_\pm must equal zero in order to avoid the flow "blowing up" and diverging as $x \rightarrow \pm\infty$.

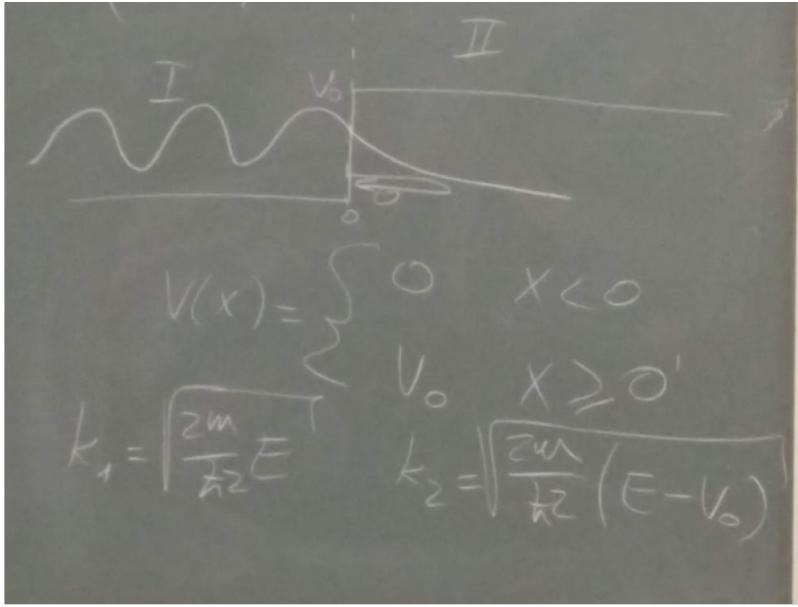


Figure 11:

We will now see an example, in the continuation of the lesson, that it is a necessary (though not sufficient) condition that the classically forbidden region has finite regions on either side; in order for the probability flow J to not equal zero.

Scattering from a Potential (Step): (Mashiach 80, C.T. 69, 281)

$$V(x) = \begin{cases} 0 & x < 0 \\ V_0 & x \geq 0 \end{cases}$$

Defining for the lefthand side $k_1 = \sqrt{\frac{2m}{\hbar^2} E}$ and for the righthand side $k_2 = \sqrt{\frac{2m}{\hbar^2} (E - V_0)}$

We can solve for the case: $E > V_0$

$$\begin{cases} \psi_I(x) = A_+ e^{ik_1 x} + A_- e^{-ik_1 x} \\ \psi_{II}(x) = B_+ e^{ik_2 x} + B_- e^{-ik_2 x} \end{cases}$$

Let's assume that all the particles probabilities come from the lefthand side meaning that we can neglect the B_-

$$\begin{cases} \psi_I(x) = A_+ e^{ik_1 x} + A_- e^{-ik_1 x} \\ \psi_{II}(x) = B_+ e^{ik_2 x} \end{cases}$$

Requiring continuity in equation and first derivate

$$\begin{aligned} \Rightarrow & \begin{cases} A_+ + A_- = B_+ \\ ik_1 A_+ - ik_1 A_- = ik_2 B_+ = ik_2 (A_+ + A_-) \end{cases} \\ \Rightarrow & (k_1 + k_2) A_- = (k_1 - k_2) A_+ \\ \Rightarrow & \frac{A_-}{A_+} = \frac{(k_1 - k_2)}{(k_1 + k_2)} \equiv r \end{aligned}$$

Let's take the continuity condition, multiply it by ik_1 and relate it to the second condition

$$2ik_1 A_+ = (ik_1 + ik_2) B_+$$

$$\Rightarrow \frac{B_+}{A_+} = \frac{2k_1}{(k_1 + k_2)} \equiv t$$

(This is seeming to be analogous to the coefficients of reflection and transmission)

$$J_I = \frac{\hbar k_1}{m} [|A_+|^2 - |A_-|^2]$$

$$J_{II} = \frac{\hbar k_2}{m} [|B_+|^2]$$

So the ratio of the flows can be defined as

$$R = \frac{J_{I_-}}{J_{I_+}} = \frac{|A_-|^2}{|A_+|^2} = |r|^2$$

Where r is the same ratio of amplitudes seen above.

$$\Rightarrow R = |r|^2 = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2} = \frac{(k_1 + k_2)^2 - 4k_1 k_2}{(k_1 + k_2)^2} = 1 - \frac{4k_1 k_2}{(k_1 + k_2)^2}$$

Now the coefficient of the transmission of the flow will be

$$T = \frac{J_{II+}}{J_{I_+}} = \frac{k_2 |B_+|^2}{k_1 |A_+|^2} = \frac{k_2}{k_1} |t|^2 = \frac{4k_1 k_2}{(k_1 + k_2)^2} = 1 - R$$

Note that both t and r (the ratios of amplitudes) are pure real values.

$$R + T = \frac{|A_-|^2}{|A_+|^2} + \frac{k_2}{k_1} \frac{|B_+|^2}{|A_+|^2} = 1$$

$$\Rightarrow |A_-|^2 + \frac{k_2}{k_1} |B_+|^2 = |A_+|^2$$

$$\Rightarrow k_2 |B_+|^2 = k_1 (|A_+|^2 - |A_-|^2)$$

Similar to Kirchoff's law of whatever flows in must flow out.

Let's now look at the case $E < V_0$.

The algebra will be precisely the same except the changing $-\alpha \leftrightarrow ik_2$

$$\Rightarrow \frac{A_-}{A_+} = \frac{(k_1 - i\alpha)}{(k_1 + i\alpha)} \equiv r$$

$$\Rightarrow \frac{B_-}{A_+} = \frac{2k_1}{(k_1 + i\alpha)} \equiv t$$

So we see

$$R = |r|^2 = 1$$

Meaning that there is total reflection and zero flow in the barrier.

There is a relative phase between the amplitude of the wave coming from the left and right

$$\phi = 2\arctan\left(\frac{\alpha}{k_1}\right)$$

which stems from the particle penetrating the barrier - and then exiting.

Finite Potential Barrier: (Mashiach 94, C.T. 72)

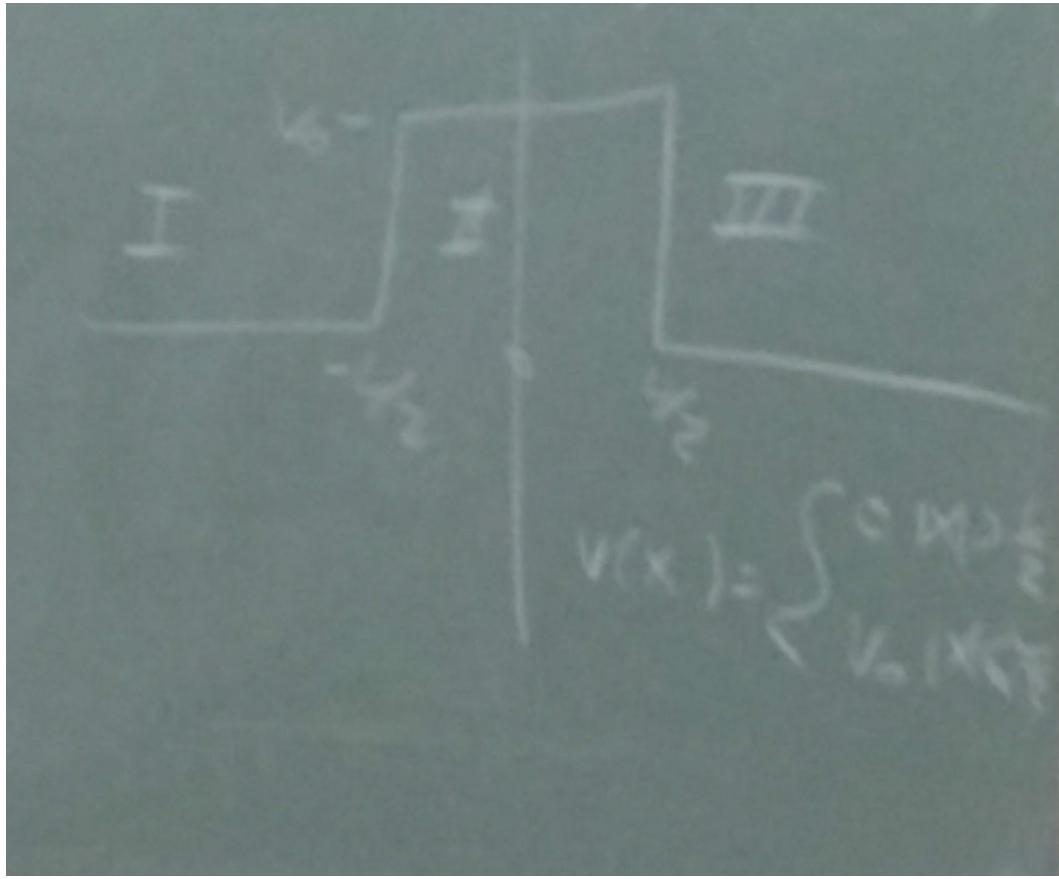


Figure 12:

$$V(x) = \begin{cases} 0 & |x| > L/2 \\ V_0 & |x| \leq L/2 \end{cases}$$

We will see a particle freely propagating towards a barrier, starting with the case $E > V_0$

$$\begin{cases} \psi_I(x) = A_{1+}e^{ik_1 x} + A_{1-}e^{-ik_1 x} \\ \psi_{II}(x) = A_{2+}e^{ik_2 x} + A_{2-}e^{-ik_2 x} \\ \psi_{III}(x) = A_{3+}e^{ik_1 x} + A_{3-}e^{-ik_1 x} \end{cases}$$

We shall look only at flow coming from the left, meaning that we shall neglect the A_{3-} term, but without the loss of generality

15 May 29th

The condition of continuity at the point

$$A_{1+}e^{-ik_1 \frac{L}{2}} + A_{1-}e^{ik_1 \frac{L}{2}} = A_{2+}e^{-ik_2 \frac{L}{2}} + A_{2-}e^{ik_2 \frac{L}{2}}$$

and continuity of the derivative

$$k_1 \left(A_{1+}e^{-ik_1 \frac{L}{2}} - A_{1-}e^{ik_1 \frac{L}{2}} \right) = k_2 \left(A_{2+}e^{-ik_2 \frac{L}{2}} - A_{2-}e^{ik_2 \frac{L}{2}} \right)$$

Recalling that the A_{3-} term cancels because we are assuming that the flow is coming from the left

$$A_{2+}e^{ik_2 \frac{L}{2}} + A_{2-}e^{-ik_2 \frac{L}{2}} = A_{3+}e^{ik_1 \frac{L}{2}}$$

$$\Rightarrow k_2 \left(A_{2+}e^{-ik_2 \frac{L}{2}} - A_{2-}e^{ik_2 \frac{L}{2}} \right) = k_1 A_{3+}e^{ik_1 \frac{L}{2}}$$

So we find that

$$\begin{aligned} t &= \frac{A_{3+}}{A_{1+}} = \frac{4k_1 k_2 e^{-ik_1 L}}{(k_1 + k_2)^2 e^{-ik_2 L} - (k_1 - k_2)^2 e^{ik_2 L}} \\ \Rightarrow T &= |t|^2 = \frac{|A_{3+}|^2}{|A_{1+}|^2} = \frac{4k_1^2 k_2^2}{4k_1^2 k_2^2 + (k_1^2 - k_2^2) \sin^2(k_2 L)} \\ &= \frac{4E(E - V_0)}{4E(E - V_0) + V_0^2 \sin^2 \sqrt{\frac{2mL^2}{\hbar^2}}(E - V_0)} \end{aligned}$$

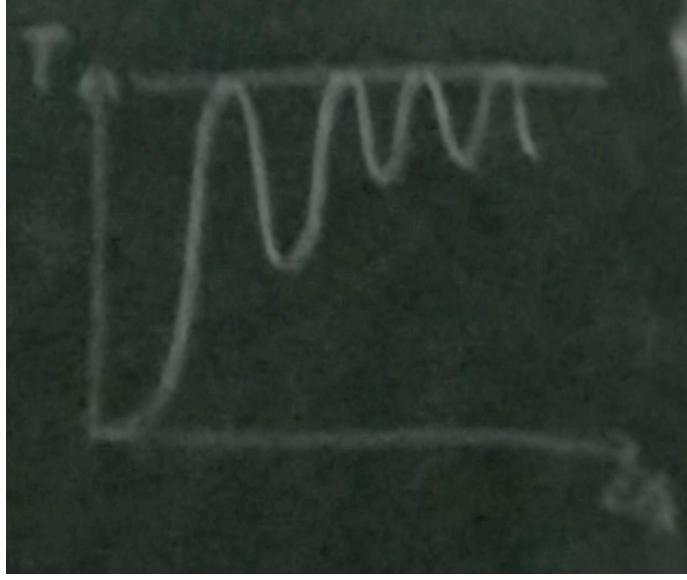


Figure 13: TRANSMISSION PROBABILITY AS FUNCTION OF E/k

Only certain energys will be able to pass through ideally, as they have resonant levels.

Let's look now at $0 < E < V_0$

(we'll see that shall only have to make the substitutions $\alpha \rightarrow ik_2$)

$$\Rightarrow T = \frac{4k_1^2 \alpha^2}{4k_1^2 \alpha^2 + (k_1^2 + \alpha^2)^2 \sinh^2(\alpha L)}$$

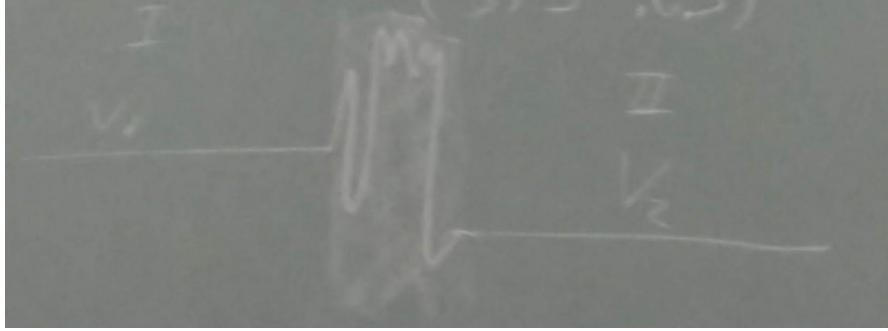


Figure 14:

Looking at the limit (alpha is one over the length of decay)

$$\begin{aligned} \lim_{\alpha L \rightarrow \infty} T &= \frac{1}{1 + \frac{(k_1^2 + \alpha^2)^2}{4k_1^2 \alpha^2} \sinh^2(\alpha L)} \\ &= \frac{16k_1^2 \alpha^2}{(k_1^2 + \alpha^2)^2} e^{-2\alpha L} = \frac{16E(V_0 - E)}{V_0^2} e^{-2\alpha L} \end{aligned}$$

meaning that the longer the length of barrier, the lower the chance for tunnelling.

Transmission and Scattering Matrices: (C.T 359)

$$\begin{cases} \psi_I(x) = A_{1+}e^{ik_1 x} + A_{1-}e^{-ik_1 x} \\ \psi_{II}(x) = B_{2+}e^{ik_2 x} + B_{2-}e^{-ik_2 x} \end{cases}$$

$$\text{where } k_i = \sqrt{\frac{2m}{\hbar^2}(E - V_i)}$$

Let's define 2 vectors made of the coefficients A, B

$$\begin{bmatrix} A_+ \\ B_- \end{bmatrix}$$

Which are entering
and

$$\begin{bmatrix} B_+ \\ A_- \end{bmatrix}$$

which are exiting
Now

$$\begin{bmatrix} B_+ \\ A_- \end{bmatrix} = S_{2 \times 2 \text{ matrix}}(k) \cdot \begin{bmatrix} A_+ \\ B_- \end{bmatrix}$$

Thus we see

$$S_{2 \times 2 \text{ matrix}}(k) = \begin{bmatrix} t_1(k) & r_2(k) \\ r_1(k) & t_2(k) \end{bmatrix}$$

Particle coming from the left (ignoring the solution within the barrier)
the lefthand solution is:

$$\psi_I^+(x) = A_+e^{ik_1 x} + r_1 A_+e^{-ik_1 x}$$

the righthand solution is:

$$\psi_{II}^+(x) = t_1 A_+ e^{ik_2 x}$$

Now a particle coming from the right (plus infinity)

the lefthand solution is:

$$\psi_I^-(x) = t_2 B_- e^{-ik_1 x}$$

and the righthand solution:

$$\psi_{II}^-(x) = r_2 B_- e^{ik_2 x} + B_- e^{-ik_2 x}$$

We can choose to look, without the loss of generality, at $A_+ = B_- = C$

In Schrodinger's equation, if the potential is a pure real value thus for ψ that is a solution, ψ^* (complex conjugate) will also be a solution

$$\psi_I^{+*} = r_1^* C e^{ik_1 x} + C e^{-ik_1 x}$$

$$\psi_{II}^{+*} = t_1^* C e^{-ik_2 x}$$

Since all three of the ψ_I s are solutions, thus also any linear combination of them will be a solution too.

$$\psi_I^- = r_1^* \psi_I^+ - \psi_I^{+*} = (|r|^2 - 1) C e^{-ik_1 x}$$

$$\psi_{II}^- = r_1^* \psi_{II}^+ - \psi_{II}^{+*} = r_1^* t_1 C e^{ik_2 x} - t_1^* C e^{-ik_2 x}$$

So we find that

$$\begin{aligned} t_2 &= \frac{|r|^2 - 1}{-t_1^*} = \frac{k_2}{k_1} |t|^2 \frac{1}{-t_1^*} = \frac{k_2}{k_1} t_1 \\ r_2 &= -r_1^* \frac{t_1}{t_1^*} = -\frac{t_1}{t_1^*} r_1^* \\ \Rightarrow S(k) &= \begin{bmatrix} t_1 & -\frac{t_1}{t_1^*} r_1^* \\ r_1 & \frac{k_2}{k_1} t_1 \end{bmatrix} \end{aligned}$$

(Noting that $\frac{t_1}{t_1^*} = 1$)

Which we can see is a unitary matrix

this matrix connects the amplitudes before the scattering (entering) with those after (leaving the scattering).

Instead of entering and exiting, we can look at lefthand and righthand, and create a transfer matrix which will connect before and after SPATIALLY - while the previous matrix connects before and after in terms of TIME.

$$B_+ = t_1 A_+ - \frac{t_1}{t_1^*} r_1^* B_-$$

$$A_- = r_1 A_+ + \frac{k_2}{k_1} t_2 B_-$$

Isolating B_-

$$B_- = \frac{k_1}{k_2} \frac{1}{t_1} (-r_1 A_+ + A_-)$$

$$\Rightarrow B_+ = t_1 A_+ - \frac{t_1}{t_1^*} r_1^* \frac{k_1}{k_2} \frac{1}{t_1} (-r_1 A_+ + A_-) = t_1 A_+ - \frac{1}{t_1^*} r_1^* \frac{k_1}{k_2} (-r_1 A_+ + A_-)$$

$$= \left(\frac{k_1}{k_2} \frac{|r_1|^2}{t_1^*} + t_1 \right) A_+ - \frac{r_1^*}{t_1^*} \frac{k_1}{k_2} A_- = \frac{k_1}{k_2} \frac{1}{t_1^*} A_+ - \frac{r_1^*}{t_1^*} \frac{k_1}{k_2} A_-$$

Thus

$$\begin{bmatrix} B_+ \\ B_- \end{bmatrix} = T(k) \begin{bmatrix} A_+ \\ A_- \end{bmatrix} = \frac{k_1}{k_2} \begin{bmatrix} \frac{1}{t_1^*} & -\frac{r_1^*}{t_1^*} \\ -\frac{r_1^*}{t_1^*} & \frac{1}{t_1} \end{bmatrix} \begin{bmatrix} A_+ \\ A_- \end{bmatrix}$$

where $\det(T) = 1$.

The meaning of this matrix T ? It can express as a single matrix, the characteristics of series of barriers.

Looking at $T_{22} = \frac{1}{t}$. Helping us solve schrodinger's equation without having to solve a 10X9 matrix (10 equations 9 variables).

For an increasing potential step, for example (for a decreasing example we take the negative of T_{21} and T_{12})

$$r_1 = \frac{k_1 - k_2}{k_1 + k_2}, \quad t_1 = \frac{2k_1}{k_1 + k_2}$$

$$T(k) = \frac{1}{2k_2} \begin{bmatrix} k_1 + k_2 & k_2 - k_1 \\ k_2 - k_1 & k_1 + k_2 \end{bmatrix}$$

For constant potential

$$T(k) = \begin{bmatrix} e^{ikL} & 0 \\ 0 & e^{-ikL} \end{bmatrix}$$

Finite Barrier:

$$T(k) = \frac{1}{2k_1} \begin{bmatrix} k_1 + k_2 & k_1 - k_2 \\ k_1 - k_2 & k_1 + k_2 \end{bmatrix} \begin{bmatrix} e^{ik_2 L} & 0 \\ 0 & e^{-ik_2 L} \end{bmatrix} \frac{1}{2k_2} \begin{bmatrix} k_1 + k_2 & k_2 - k_1 \\ k_2 - k_1 & k_1 + k_2 \end{bmatrix}$$

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$$T(k) = \underbrace{\frac{1}{2k_1} \begin{bmatrix} k_1 + k_2 & k_1 - k_2 \\ k_1 - k_2 & k_1 + k_2 \end{bmatrix}}_{\text{step down}} \underbrace{\begin{bmatrix} e^{ik_2 L} & 0 \\ 0 & e^{-ik_2 L} \end{bmatrix}}_{\text{flat}} \underbrace{\frac{1}{2k_2} \begin{bmatrix} k_1 + k_2 & k_2 - k_1 \\ k_2 - k_1 & k_1 + k_2 \end{bmatrix}}_{\text{step up}}$$

$$= \frac{1}{2k_1 k_2} \begin{pmatrix} \text{duck} = (-i(k_1^2 + k_2^2) \sin(k_2 L) + 2k_1 k_2 \cos(k_2 L))^* & \text{cat} = -(i(k_1^2 - k_2^2) \sin(k_2 L))^* \\ i(k_1^2 - k_2^2) \sin(k_2 L) & -i(k_1^2 + k_2^2) \sin(k_2 L) + 2k_1 k_2 \cos(k_2 L) \end{pmatrix}$$

Calculating it by hand →

$$T = \frac{1}{2k_1 k_2} \begin{pmatrix} 2k_1 k_2 \cos(k_2 L) + i(k_1^2 + k_2^2) \sin(k_2 L) & i(k_2^2 - k_1^2) \sin(k_2 L) \\ i(k_1^2 - k_2^2) \sin(k_2 L) & -i(k_1^2 + k_2^2) \cos(k_2 L) + 2i k_1 k_2 \sin(k_2 L) \end{pmatrix}$$

$$\begin{cases} t_1 = \frac{1}{T_{22}} = \frac{2k_1 k_2 \cdot \text{phase}}{2k_1 k_2 \cos(k_2 L) - i(k_1^2 + k_2^2) \sin(k_2 L)} \\ r_1 = -\frac{T_{21}}{T_{22}} = \frac{-i(k_1^2 - k_2^2) \sin(k_2 L)}{2k_1 k_2 \cos(k_2 L) - i(k_1^2 + k_2^2) \sin(k_2 L)} \end{cases}$$

where the phase is $e^{-ik_1 L}$.

A mathematical trick (with no physical intuition) for

$$\psi_I^+(x) = C e^{-\alpha_1 x} + r_1 C e^{\alpha_1 x}$$

$$\psi_{II}^+(x) = t_1 C e^{-\alpha_2 x}$$

Taking $C \rightarrow 0$ for physical reasons, we'll require the coefficients r_1 and t_1 to approach infinity

$$2iak\cos(kL) - i(k^2 - \alpha^2)\sin(kL) = 0$$

$$\Rightarrow \frac{2\alpha k}{k^2 - \alpha^2} = \tan(kL)$$

Here because there is only the one potential well with a single wave function, there is only one k , the k inside the well.

Because the potential on either side of the well is equal, $\alpha_1 = \alpha_2 = \alpha$.

Tensor Product Space - Multiple Degrees of Freedom: (C.T. 153, 290)

Imagine a particle with a finite, multiple number of DoF, like in the case of the polarizing beam-splitter...

Another case, a photon with polarization next to a similar photon...

For the case of 2 finite DoF denoted a and b - of the polarization and of the location.

Each DoF like this can be described by a Hilbert Space - \mathcal{H}_a and \mathcal{H}_b (DONT GET CONFUSED WITH HAMILTONIANS)

Each of their dimensions will be denoted $\dim(\mathcal{H}_a) = d_a$ and $\dim(\mathcal{H}_b) = d_b$

Vectors of the space $|\alpha\rangle$ and $|\beta\rangle$ respectively.

If for every pair of vectors belonging to \mathcal{H}_a and \mathcal{H}_b respectively, it is possible to associate a vector in \mathcal{H} , thus we shall denote

$$\mathcal{H} = \mathcal{H}_a \otimes \mathcal{H}_b$$

as the tensor product space.

Such that:

$$c|\alpha\rangle \otimes |\beta\rangle = c[|\alpha\rangle \otimes |\beta\rangle] = |\alpha\rangle \otimes [c|\beta\rangle]$$

$$|\alpha\rangle \otimes [|\beta_1\rangle + |\beta_2\rangle] = |\alpha\rangle \otimes |\beta_1\rangle + |\alpha\rangle \otimes |\beta_2\rangle$$

$$[|\alpha_1\rangle + |\alpha_2\rangle] \otimes |\beta\rangle = |\alpha_1\rangle \otimes |\beta\rangle + |\alpha_2\rangle \otimes |\beta\rangle$$

Let's denote the whole, orthonormal sets $\{|u_i\rangle\}$ which spans a and $\{|v_j\rangle\}$ which spans b .

The set $\{|u_i\rangle \otimes |v_j\rangle\}$ spans \mathcal{H} - where there are $d_a \cdot d_b$ different combinations which is equal to $\dim(\mathcal{H})$.

Some vector $|\alpha\rangle$ can be spanned as

$$|\alpha\rangle = \sum_{i=1}^{d_a} a_i |u_i\rangle$$

(with $d_a - 1$ DoF)

And similarly

$$|\beta\rangle = \sum_{j=1}^{d_b} b_j |v_j\rangle$$

(with $d_b - 1$ DoF)

Thus

$$|\alpha\rangle \otimes |\beta\rangle = \sum_{i=1}^{d_a} \sum_{j=1}^{d_b} a_i b_j |u_i\rangle \otimes |v_j\rangle$$

(with $d_a - 1 + d_b - 1 = d_a + d_b - 2$ DoF)

So some general vector $|\psi\rangle$ in the space \mathcal{H}

$$|\psi\rangle = \sum_{i=1}^{d_a} \sum_{j=1}^{d_b} c_{ij} |u_i\rangle \otimes |v_j\rangle$$

(with $d_a \cdot d_b - 1$ DoF)

What is the meaning of the missing DoF?

This characteristic of Hilbert Spaces, where not every state is possible to be written (or even represented!) as the product of states, is the strongest characteristic of Quantum Theory and is the largest difference between Quantum and other theories.

$$|\psi_a\rangle \otimes |\psi_b\rangle \neq c_{11}|u_1\rangle \otimes |v_1\rangle + c_{22}|u_2\rangle \otimes |v_2\rangle$$

These states are known as “Entangled States”...

$$|\psi_a\rangle \otimes |\psi_b\rangle = c_{11}|u_1\rangle \otimes |v_1\rangle + c_{22}|u_2\rangle \otimes |v_2\rangle \underbrace{[+c_{12}|u_1\rangle \otimes |v_2\rangle + c_{21}|u_2\rangle \otimes |v_1\rangle]}_{\text{with this addition it is possible}} = (a_1|u_1\rangle + a_2|u_2\rangle) \otimes (b_1|v_1\rangle + b_2|v_2\rangle)$$

Let's define a shortened notation

$$|\alpha\rangle \otimes |\beta\rangle \equiv |\alpha\rangle|\beta\rangle = |\alpha, \beta\rangle$$

Defining inner product in this space

$$[(\langle \alpha' | \otimes \langle \beta' |) [|\alpha\rangle \otimes |\beta\rangle] = \langle \alpha', \beta' | \alpha, \beta \rangle = \langle \alpha' | \alpha \rangle \langle \beta' | \beta \rangle$$

If $\langle u_i, v_j | u_k, v_l \rangle = \langle u_i | u_k \rangle \langle v_j | v_l \rangle = \delta_{ik} \delta_{jl} = 0$ means that the vectors are perpendicular...

Assuming an operator A_a which acts on vectors in \mathcal{H}_a , we can define A as the extension operator of A_a to \mathcal{H}

$$A|\alpha\rangle \otimes |\beta\rangle = [A_a|\alpha\rangle] \otimes |\beta\rangle$$

For a general state

$$A|\psi\rangle = \sum_{i=1}^{d_a} \sum_{j=1}^{d_b} c_{ij} [A_a|u_i\rangle] \otimes |v_j\rangle$$

Similarly for an operator B_b which acts on vectors in \mathcal{H}_b , we can define B as the extension operator of B_b to \mathcal{H}

$$B|\alpha\rangle \otimes |\beta\rangle = |\alpha\rangle \otimes [B_b|\beta\rangle]$$

For a general state

$$B|\psi\rangle = \sum_{i=1}^{d_a} \sum_{j=1}^{d_b} c_{ij} |u_i\rangle \otimes [B_b|v_j\rangle]$$

Thus

$$D = A_a \otimes B_b$$

$$\Rightarrow D|\alpha\rangle \otimes |\beta\rangle = [A_a|\alpha\rangle] \otimes [B_b|\beta\rangle]$$

Therefore

$$A = A_a \otimes \mathbf{1}, \quad B = \mathbf{1} \otimes B_b$$

This same D can be written as $D = AB = BA$

meaning that operators created from partial operators on different spaces always commute....

If we assume that there are all the eigenvectors known in A_a

$$A_a|u_n\rangle = \lambda_n|u_n\rangle$$

$$\Rightarrow A|u_n\rangle \otimes |\beta\rangle = [A_a|u_n\rangle] \otimes |\beta\rangle = \lambda_n|u_n\rangle \otimes |\beta\rangle$$

If we can write

$$D = A + B = A_a \otimes \mathbf{1} + \mathbf{1} \otimes B_b$$

Where $B_b|v_k\rangle = \mu_k|v_k\rangle$.

The eigenstates can be written as

$$|\psi_{nk}\rangle = |u_n\rangle \otimes |v_k\rangle$$

$$\Rightarrow D|\psi_{nk}\rangle = A|u_n\rangle \otimes |v_k\rangle + B|u_n\rangle \otimes |v_k\rangle = (\lambda_n + \mu_k)|u_n\rangle \otimes |v_k\rangle$$

So the operator D can contain degeneracies.

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$$D = A + B = A_a \otimes \mathbf{1}_a + \mathbf{1}_b \otimes B_b$$

$$A_a|u_n\rangle = \lambda_n|u_n\rangle$$

$$B_b|v_k\rangle = \mu_k|v_k\rangle$$

By applying the operator D

$$D|u_n\rangle \otimes |v_k\rangle = A|u_n\rangle \otimes |v_k\rangle + B|u_n\rangle \otimes |v_k\rangle = (\lambda_n + \mu_k)|u_n\rangle \otimes |v_k\rangle$$

Meaning that we know both the eigenstates and the eigenvalues.

Particles in Three Dimensions: (C.T. 57,150,160,182)

If we have a particle in space, we can independently choose each of the coordinates leading to 3 degrees of freedom (x, y, z) .

coordinate location operator	x X_x	y Y_y	z Z_z
eigenstates states	$ x_0\rangle$ $ \psi_x\rangle$	$ y_0\rangle$ $ \psi_y\rangle$	$ z_0\rangle$ $ \psi_z\rangle$
location representation	$\langle x \psi_x\rangle = \psi_x(x)$	$\langle y \psi_y\rangle = \psi_y(y)$	$\langle z \psi_z\rangle = \psi_z(z)$
location representation after measurement	$\langle x x_0\rangle = \delta(x - x_0)$	$\langle y y_0\rangle = \delta(y - y_0)$	$\langle z z_0\rangle = \delta(z - z_0)$
momentum operator	P_x	P_y	P_z

(Noting that $X = X_x \otimes \mathbf{1}_y \otimes \mathbf{1}_z$)

Defining the Hilbert Space for 3 dimensions

$$\mathcal{H}_r = \mathcal{H}_x \otimes \mathcal{H}_y \otimes \mathcal{H}_z$$

until now these operators have returned a number, but now we want a trio of numbers (not a vector, because we'll save "vectors" for those in Hilbert Space)

In order to be consistent, we shall define the operator

$$\bar{R} = X\hat{x} + Y\hat{y} + Z\hat{z}$$

Similarly

$$\bar{P} = P_x\hat{x} + P_y\hat{y} + P_z\hat{z}$$

So the eigenstates are

$$|\bar{r}_0\rangle = |x_0\rangle \otimes |y_0\rangle \otimes |z_0\rangle = |x_0, y_0, z_0\rangle$$

$$X|\bar{r}_0\rangle = x_0|x_0, y_0, z_0\rangle$$

$$Y|\bar{r}_0\rangle = y_0|x_0, y_0, z_0\rangle$$

$$Z|\bar{r}_0\rangle = z_0|x_0, y_0, z_0\rangle$$

So we see

$$\bar{R}|\bar{r}_0\rangle = (X\hat{x} + Y\hat{y} + Z\hat{z})|\bar{r}_0\rangle = (x_0\hat{x} + y_0\hat{y} + z_0\hat{z})|\bar{r}_0\rangle$$

$$\langle \bar{r}|\bar{r}_0\rangle = [\langle x| \otimes \langle y| \otimes \langle z|] [|x_0\rangle \otimes |y_0\rangle \otimes |z_0\rangle]$$

$$= \langle x|x_0\rangle \langle y|y_0\rangle \langle z|z_0\rangle = \delta(x - x_0)\delta(y - y_0)\delta(z - z_0)$$

We can write the basis $|\bar{r}_0\rangle = |x_0, y_0, z_0\rangle$ for the Hilbert space \mathcal{H}_r .

But for some general ψ in this space, the location representation

$$\langle \bar{r}|\psi\rangle \neq \psi_x(x)\psi_y(y)\psi_z(z) = \psi(x, y, z)$$

Any trio combination of X, Y, Z, P_x, P_y, P_z is a whole set of commuting observable values.

Moving between the location representation to the momentum representation we see that the matrix elements are

$$\langle \bar{p}|\bar{r}\rangle = \langle p_x, p_y, p_z|x, y, z\rangle = \langle p_x|x\rangle \langle p_y|y\rangle \langle p_z|z\rangle$$

$$= \frac{1}{(2\pi\hbar)^{3/2}} e^{-\frac{i}{\hbar}p_x x} e^{-\frac{i}{\hbar}p_y y} e^{-\frac{i}{\hbar}p_z z} = \frac{1}{(2\pi\hbar)^{3/2}} e^{-\frac{i}{\hbar}\bar{p}\cdot\bar{r}}$$

Thus the momentum representation is

$$\tilde{\psi}(\bar{p}) = \tilde{\psi}(p_x, p_y, p_z) = \frac{1}{(2\pi\hbar)^{3/2}} \iiint_{-\infty}^{\infty} dx dy dz e^{-\frac{i}{\hbar}\bar{p}\cdot\bar{r}} \psi(x, y, z) = \frac{1}{(2\pi\hbar)^{3/2}} \iiint_{-\infty}^{\infty} d\bar{r} e^{-\frac{i}{\hbar}\bar{p}\cdot\bar{r}} \psi(\bar{r})$$

and the location representation is

$$\psi(\bar{r}) = \psi(x, y, z) = \frac{1}{(2\pi\hbar)^{3/2}} \iiint_{-\infty}^{\infty} dp_x dp_y dp_z e^{-\frac{i}{\hbar}\bar{p}\cdot\bar{r}} \tilde{\psi}(p_x, p_y, p_z) = \frac{1}{(2\pi\hbar)^{3/2}} \iiint_{-\infty}^{\infty} d\bar{p} e^{-\frac{i}{\hbar}\bar{p}\cdot\bar{r}} \tilde{\psi}(\bar{p})$$

Additionally the momentum operator in the location representation

$$\langle \bar{r} | \bar{P} | \psi \rangle = \langle x, y, z | P_x \hat{x} + P_y \hat{y} + P_z \hat{z} | \psi \rangle$$

$$= \langle x | P_x | \psi \rangle \hat{x} + \langle y | P_y | \psi \rangle \hat{y} + \langle z | P_z | \psi \rangle \hat{z}$$

$$= -i\hbar \left(\hat{x} \frac{\partial}{\partial x} + \hat{y} \frac{\partial}{\partial y} + \hat{z} \frac{\partial}{\partial z} \right) \psi(x, y, z) = -i\hbar \bar{\nabla} \psi(x, y, z)$$

What is the meaning of the operation of some function of the operators

$$\langle \bar{r} | f(X, Y, Z) | \psi \rangle = f(x, y, z) \psi(x, y, z)$$

Looking at some potential, from the Hamiltonian Operator

$$H = -\frac{\hbar^2}{2m} \nabla^2 + V(x, y, z) = -\frac{\hbar^2}{2m} \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) + V(x, y, z)$$

The Schrodinger equation is

$$\nabla^2 \psi_E(x, y, z) + \frac{2m}{\hbar^2} [E - V(x, y, z)] \psi_E(x, y, z) = 0$$

and the continuity equation is

$$\frac{\partial P}{\partial t} + \bar{\nabla} \cdot \bar{J} = 0$$

$$\text{Where } \bar{J}(x, y, z) = \frac{\hbar}{2im} [\psi^* \bar{\nabla} \psi - \psi \bar{\nabla} \psi^*]$$

And we can write the Ehrenfest equations

$$\frac{d}{dt} \langle \bar{R} \rangle = \frac{1}{m} \langle \bar{P} \rangle$$

$$\frac{d}{dp} \langle \bar{P} \rangle = - \langle \bar{\nabla} V(\bar{R}) \rangle$$

Noting that

$$H = H_x + H_y + H_z$$

Where we will look for the individual solutions in the subspace

$$H_x |\psi_n^{(x)}\rangle = E_n^{(x)} |\psi_n^{(x)}\rangle$$

$$H_y |\psi_m^{(y)}\rangle = E_m^{(y)} |\psi_m^{(y)}\rangle$$

$$H_z |\psi_l^{(z)}\rangle = E_l^{(z)} |\psi_l^{(z)}\rangle$$

in order to find the eigenvalues and thus the eigenvectors.

$$H |\psi_{n,m,l}^{(r)}\rangle = \left(E_n^{(x)} + E_m^{(y)} + E_l^{(z)} \right) |\psi_n^{(x)}\rangle |\psi_m^{(y)}\rangle |\psi_l^{(z)}\rangle$$

The location representation being

$$\langle \bar{r} | \psi_{n,m,l}^{(r)} \rangle = \langle x | \langle y | \langle z | |\psi_n^{(x)}\rangle |\psi_m^{(y)}\rangle |\psi_l^{(z)}\rangle$$

$$\Rightarrow \psi_{n,m,l}(x, y, z) = \psi_n(x)\psi_m(y)\psi_l(z)$$

$$V(\bar{r})V_x(x) + V_y(y) + V_z(z)$$

Now we can generalize to multiple dimensions for everything new that we learn.

Harmonic Oscillator, One-Dimensional: (C.T. 481)

The equation and their operators (C.T. 487):

We'll be looking, for the next week, at being able to look at the quantization of harmonic oscillators.

Talking about some system with a returning force $f = -kx$ where the frequency is $\omega = \sqrt{\frac{k}{m}}$

The classical hamiltonian would be

$$H_{cl} = \frac{p^2}{2m} + \frac{m\omega^2}{2}x^2$$

The quantization is trivial

$$H = \frac{1}{2m}P^2 + \frac{m\omega^2}{2}X^2$$

in order to know all the dynamic information we must find the eigenstates of the hamiltonian

$$H|\psi_n\rangle = E_n|\psi_n\rangle$$

To find the allowed energies and the different representations

The differential Schrodinger equation here is

$$\left[-\frac{\hbar^2}{2m}\frac{d^2}{dx^2} + \frac{m\omega^2}{2}x^2 \right] \psi(x) = E\psi(x)$$

Which we will solve in a very elegant manner.

Next we can move to dimensionless equations

The x coordinate and the momentum

$$x = \sqrt{\frac{\hbar}{m\omega}}x' \quad p = \sqrt{m\hbar\omega}p'$$

the operators

$$\begin{aligned} X &= \sqrt{\frac{\hbar}{m\omega}}\tilde{X} & P &= \sqrt{m\hbar\omega}\tilde{P} & H &= \hbar\omega\tilde{H} \\ \Rightarrow \tilde{H} &= \frac{1}{2}(\tilde{X}^2 + \tilde{P}^2) \end{aligned}$$

Where $[\tilde{X}, \tilde{P}] = i$

Since the relation between the two hamiltonians is scalar, they commute and they will have the same eigenvalues

$$\tilde{H}|\psi_n\rangle = \varepsilon_n|\psi_n\rangle$$

So we can write the classic hamiltonian as

$$H_{cl} = \frac{1}{2}(x' - ip')(x' + ip')$$

This leads us to define two new, useful operators

$$\tilde{a} = \frac{1}{\sqrt{2}} (\tilde{X} + i\tilde{P}), \quad \tilde{a}^\dagger = \frac{1}{\sqrt{2}} (\tilde{X} - i\tilde{P})$$

Which are not hermitian

$$[\tilde{a}, \tilde{a}^\dagger] = \frac{1}{2} [\tilde{X} + i\tilde{P}, \tilde{X} - i\tilde{P}] = -\frac{i}{2} [\tilde{X}, \tilde{P}] + \frac{i}{2} [\tilde{P}, \tilde{X}] = 1$$

So we can define the unitless operator

$$\begin{aligned} \tilde{N} &= \tilde{a}^\dagger \tilde{a} = \frac{1}{2} (\tilde{X} - i\tilde{P})(\tilde{X} + i\tilde{P}) = \frac{1}{2} (\tilde{X}^2 + \tilde{P}^2 + i[\tilde{X}, \tilde{P}]) \\ &= \tilde{H} - \frac{1}{2} \end{aligned}$$

Which is the hamiltonian minus a half.

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Last lesson we started to speak about harmonic oscillators. We defined the dimensionless coordinates, change of variables, and new operators like the dimensionless \tilde{X} and \tilde{P} location and momentum operators respectively.

We defined \tilde{a} and \tilde{a}^\dagger , as well as

$$\tilde{N} = \tilde{H} - \frac{1}{2}$$

Thus we can write

$$\tilde{H} = \tilde{a}^\dagger \tilde{a} + \frac{1}{2} = \tilde{N} + \frac{1}{2} = \tilde{a} \tilde{a}^\dagger - \frac{1}{2}$$

which comes from the commutation of \tilde{a}^\dagger and \tilde{a} .

Because the hamiltonian is a real function of the operator \tilde{N} , we know \tilde{N} is a hermitian operator and that they must divide up the eigenstates.

We'll find the eigenstates of \tilde{N} and they will be the same eigenstates of \tilde{H} . Their eigenvalue will have a difference of $\frac{1}{2}$, meaning that the eigenvalues of \tilde{N}

$$\nu_n = \varepsilon_n - \frac{1}{2}$$

where ε_n are the eigenvalues of the hamiltonian tilde operator and E_n of the energy operator.

We shall look at the commutations of \tilde{N} with \tilde{a} and \tilde{a}^\dagger

$$[\tilde{N}, \tilde{a}] = [\tilde{a}^\dagger \tilde{a}, \tilde{a}] = \tilde{a}^\dagger [\tilde{a}, \tilde{a}] + [\tilde{a}^\dagger, \tilde{a}] \tilde{a} = -\tilde{a}$$

and similarly

$$[\tilde{N}, \tilde{a}^\dagger] = [\tilde{a}^\dagger \tilde{a}, \tilde{a}^\dagger] = \tilde{a}^\dagger [\tilde{a}, \tilde{a}^\dagger] + [\tilde{a}^\dagger, \tilde{a}^\dagger] \tilde{a}^\dagger = \tilde{a}^\dagger$$

These are very useful (and appears in most exams)

Energy Spectrum: (C.T. 491)

This is equivalent to finding the spectrum of the operator \tilde{N} , where

$$\tilde{N}|\psi_n\rangle = \nu_n|\psi_n\rangle$$

Such that

$$\varepsilon_n = \nu_n + \frac{1}{2}$$

$$E_n = (\nu_n + \frac{1}{2})\hbar\omega \rightarrow (n + \frac{1}{2})\hbar\omega$$

There are about 10 algebraic steps here to solve the problem:

First we write the vector

$$\tilde{a}|\psi_n\rangle$$

where ψ_n is an eigenstate but also a vector in Hilbert space, it is allowed to apply a non-hermitian operator. (We find a vector in hilbert space when it is applied to a hilbert space vector, nevertheless)
(there is no N nor a without the tilde)

We know about the vector that the size must be non-negative.

$$||\tilde{a}|\psi_n\rangle|| = \langle\psi_n|\tilde{a}^\dagger\tilde{a}|\psi_n\rangle = \langle\psi_n|\tilde{N}|\psi_n\rangle = \nu_n\langle\psi_n|\psi_n\rangle \geq 0$$

Using the fact that ψ_n is an eigenstate of \tilde{N} .

Thus we see that, because $\langle\psi_n|\psi_n\rangle$ is also the size of a vector and thus also non-negative

$$\Rightarrow \nu_n \geq 0$$

The equality occurs when the eigenvalue is equal to zero, which occurs only when the vector itself is the zero vector

$$\tilde{a}|\psi_0\rangle = 0 \Leftrightarrow \nu_0 = 0$$

This teachers us that if \tilde{a} acts on a vector with the eigenvalue 0, the result is 0.

Additionally if we take this equation and enact \tilde{a}^\dagger on it

$$\tilde{a}^\dagger\tilde{a}|\psi_0\rangle = \tilde{N}|\psi_0\rangle = 0$$

Meaning that the application \tilde{N} on any vector resulting in zero demands that that vector be ψ_0 with the eigenvalue 0.

Similarly we can apply \tilde{a}^\dagger on some eigenstate

$$\tilde{a}^\dagger|\psi_n\rangle$$

Whose size will be

$$||\tilde{a}^\dagger|\psi_n\rangle|| = \langle\psi_n|\tilde{a}\tilde{a}^\dagger|\psi_n\rangle = \langle\psi_n|\tilde{a}^\dagger\tilde{a} + 1|\psi_n\rangle = (\nu_n + 1)\langle\psi_n|\psi_n\rangle \geq 0$$

Where the second equality comes from the “yachas hachiluf” (commutator).

But both values are non-negative. However since

$$\nu_n\langle\psi_n|\psi_n\rangle \geq 0 \Rightarrow (\nu_n + 1)\langle\psi_n|\psi_n\rangle > 0$$

Applying the commutators on the eigenstate

$$[\tilde{N}, \tilde{a}]|\psi_n\rangle = -\tilde{a}|\psi_n\rangle$$

$$\Rightarrow \tilde{N}\tilde{a}|\psi_n\rangle = \tilde{a}\tilde{N}|\psi_n\rangle - \tilde{a}|\psi_n\rangle = \tilde{a}\nu_n|\psi_n\rangle - \tilde{a}|\psi_n\rangle$$

$$= (\nu_n - 1)\tilde{a}|\psi_n\rangle$$

and

$$\begin{aligned} [\tilde{N}, \tilde{a}^\dagger]|\psi_n\rangle &= \tilde{a}^\dagger|\psi_n\rangle \\ \Rightarrow \tilde{N}\tilde{a}^\dagger|\psi_n\rangle &= \tilde{a}^\dagger\tilde{N}|\psi_n\rangle + \tilde{a}^\dagger|\psi_n\rangle = \tilde{a}^\dagger\nu_n|\psi_n\rangle + \tilde{a}^\dagger|\psi_n\rangle \\ &= (\nu_n + 1)\tilde{a}^\dagger|\psi_n\rangle \end{aligned}$$

This is the main part of calculating the energy spectrum.

This tells us that, if ψ_n is the eigenstate of the operator \tilde{N} with the eigenvalue ν_n , $\tilde{a}|\psi_n\rangle$ is a vector which is still an eigenvector (a new eigenstate) of the operator \tilde{N} but with a different eigenvalue $\nu_n - 1$.

(The same in the case of $\tilde{a}^\dagger|\psi_n\rangle$ but with $\nu_n + 1$).

Proving by negation that the eigenvalues ν_n are whole numbers.

If ν_n are not whole numbers, there exists a ν_n between the whole numbers n and $n + 1$.

Enacting $\tilde{N}\tilde{a}|\psi_n\rangle$ we get an eigenvalue between $n - 1$ and n . but this wouldn't work because we obtain a negative eigenvalue which would mean that $\nu_n \not\geq 0$ - contradiction.

We find now that

$$\nu_n = n \geq 0$$

We can call the operator \tilde{N} the number operator, as it returns the number...

We see now how the enaction of \tilde{a}^\dagger and \tilde{a} are the ways to go up and down (respectively energy levels which are separated by $\hbar\omega$

Thus they are called:

\tilde{a} the lowering operator - destruction operator

\tilde{a}^\dagger the raising operator - creation operator

Another couple of things

The eigenstates will, from now, be denoted $|\psi_n\rangle = |n\rangle$

(Vacuum state $|0\rangle$, the energy in this state is simply $\frac{1}{2}\hbar\omega$ which causes quantum fluctuations - the vacuum energy is related to the probability of finding the particle at the origin in this state)

additionally, it is important to note that if $\tilde{a}^\dagger|n\rangle$ returns a state larger by 1, if we look at the conjugate $\langle n|\tilde{a}$.

So when the operator \tilde{a} works to the right it lowers the state, when it works to the left it raises the state.

Same applies on the raising operator - when the operator \tilde{a}^\dagger works to the right it raises the state, when it works to the left it lowers the state.

The Eigenstates of the Hamiltonian in the Energy/Number Representation: (C.T. 496)

Because the eigenstates of the hamiltonian and number operator are the same.

We saw at the end of the last lesson that even though we are going from state to, with the lowering and raising operators, we still have to check the normalization.

Just because it is an eigenstate, doesn't mean that they are states of the same length.

So we will calculate what is the length of the vector n found by raising the $n - 1$ vector - i.e. by what factor is $n - 1$ multiplied to get n ?

Similarly we will calculate what is the length of the vector n found by decreasing the $n + 1$ vector - i.e. by what factor is $n + 1$ multiplied to get n ?

$$\begin{aligned}
& ||c|n\rangle|| = ||\tilde{a}^\dagger|n-1\rangle|| \\
\Rightarrow |c|^2 \langle n|n \rangle &= \langle n-1|\tilde{a}\tilde{a}^\dagger|n-1\rangle \\
&= \langle n-1|\tilde{N}+1|n-1\rangle \\
\Rightarrow |c|^2 &= n
\end{aligned}
\qquad
\begin{aligned}
& ||c|n\rangle|| = ||\tilde{a}|n+1\rangle|| \\
\Rightarrow |c|^2 \langle n|n \rangle &= \langle n+1|\tilde{a}^\dagger\tilde{a}|n+1\rangle \\
&= \langle n+1|\tilde{N}|n+1\rangle \\
\Rightarrow |c|^2 &= n+1
\end{aligned}$$

Assuming for the fourth line than $|n\rangle$ is normalized, therefore so is $|n+1\rangle$.

So the conclusion is that the vector $|n\rangle$ obtained is longer by a factor n and in the second case longer by a factor of $n+1$, so the normalizations are:

$$\begin{aligned}
|n\rangle &= \frac{1}{\sqrt{n}}\tilde{a}^\dagger|n-1\rangle & |n\rangle &= \frac{1}{\sqrt{n+1}}\tilde{a}|n+1\rangle \\
\Rightarrow |n+1\rangle &= \frac{1}{\sqrt{n+1}}\tilde{a}^\dagger|n\rangle & \Rightarrow |n-1\rangle &= \frac{1}{\sqrt{n}}\tilde{a}|n\rangle
\end{aligned}$$

If we can raise a state $n-1$ to get to n , we can also raise the previous state $n-2$ to get to $n-1$.

$$|n\rangle = \frac{1}{\sqrt{n}}\tilde{a}^\dagger|n-1\rangle = \frac{1}{\sqrt{n}}\tilde{a}^\dagger\frac{1}{\sqrt{n-1}}\tilde{a}^\dagger|n-2\rangle = \frac{1}{\sqrt{n}}\tilde{a}^\dagger\frac{1}{\sqrt{n-1}}\tilde{a}^\dagger\frac{1}{\sqrt{n-2}}\tilde{a}^\dagger|n-3\rangle = \dots = \frac{1}{\sqrt{n!}}\tilde{a}^{\dagger n}|0\rangle$$

Thus we can write any eigenstate with the ground state.

We will use this to show the orthonormality of the states (even though it is promised by the hermiticity of the operators).

Taking two vectors - recalling that raising for a *bra* is \tilde{a} and **not** \tilde{a}^\dagger

$$\langle m|n\rangle = \frac{1}{\sqrt{m!}}\frac{1}{\sqrt{n!}}\langle 0|\tilde{a}^m\tilde{a}^{\dagger n}|0\rangle = \begin{cases} m > n & \frac{1}{\sqrt{m!}}\langle 0|\tilde{a}^m|n\rangle = \frac{\sqrt{n!}}{\sqrt{m!}}\langle 0|\tilde{a}^{m-n}|0\rangle = 0 \\ m < n & \frac{1}{\sqrt{n!}}\langle m|\tilde{a}^{\dagger n}|0\rangle = \frac{\sqrt{m!}}{\sqrt{n!}}\langle 0|\tilde{a}^{\dagger n-m}|0\rangle = 0 \\ m = n & \frac{1}{\sqrt{m!}}\langle 0|\tilde{a}^m|n\rangle = \langle 0|0\rangle = 1 \end{cases}$$

Also we know that the completeness relation is

$$\Sigma_{n=0}^{\infty}|n\rangle\langle n| = \mathbf{1}$$

What are the matrix elements in the number representation?

$$\langle m|\tilde{a}^\dagger|n\rangle = \sqrt{n+1}\langle m|n+1\rangle = \sqrt{n+1}\delta_{m,n+1}$$

$$\langle m|\tilde{a}|n\rangle = \sqrt{m+1}\langle m+1|n\rangle = \sqrt{m+1}\delta_{m+1,n}$$

Meaning that the only matrix elements that do not zero are the secondary diagonals (lower in the first case, upper in the second case)

$$\begin{vmatrix} 0 & 0 & 0 & 0 & 0 \\ \sqrt{1} & 0 & 0 & 0 & 0 \\ 0 & \sqrt{2} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{3} & 0 & 0 \\ 0 & 0 & 0 & \ddots & 0 \end{vmatrix}
\qquad
\begin{vmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ 0 & 0 & \sqrt{2} & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 \\ 0 & 0 & 0 & 0 & \ddots \\ 0 & 0 & 0 & 0 & 0 \end{vmatrix}$$

And the matrix elements for the location operator:

$$\langle m|\tilde{X}|n\rangle = \frac{1}{\sqrt{2}}\langle m|\tilde{a}^\dagger + \tilde{a}|n\rangle = \frac{1}{\sqrt{2}}(\sqrt{n+1}\delta_{m,n+1} + \sqrt{m+1}\delta_{m+1,n})$$

Meaning

$$\frac{1}{\sqrt{2}} \begin{vmatrix} 0 & \sqrt{1} & 0 & 0 & 0 \\ \sqrt{1} & 0 & \sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & \sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \ddots \\ 0 & 0 & 0 & \ddots & 0 \end{vmatrix}$$

And the matrix elements for the momentum operator:

$$\langle m | \tilde{P} | n \rangle = \frac{i}{\sqrt{2}} \langle m | \tilde{a}^\dagger - \tilde{a} | n \rangle = \frac{i}{\sqrt{2}} (\sqrt{n+1} \delta_{m,n+1} - \sqrt{m+1} \delta_{m+1,n})$$

Meaning

$$\frac{i}{\sqrt{2}} \begin{vmatrix} 0 & -\sqrt{1} & 0 & 0 & 0 \\ \sqrt{1} & 0 & -\sqrt{2} & 0 & 0 \\ 0 & \sqrt{2} & 0 & -\sqrt{3} & 0 \\ 0 & 0 & \sqrt{3} & 0 & \ddots \\ 0 & 0 & 0 & \ddots & 0 \end{vmatrix}$$

Expectation Values of Location and Momentum: (C.T. 503)

If we take the expectation value of the location operator for the eigenstate with the eigenvalue n

$$\langle \tilde{X} \rangle_n = \frac{1}{\sqrt{2}} \langle n | \tilde{a}^\dagger + \tilde{a} | n \rangle = 0$$

and similarly

$$\langle \tilde{P} \rangle_n = \frac{1}{\sqrt{2}} \langle n | \tilde{a}^\dagger - \tilde{a} | n \rangle = 0$$

so the averages over all the states is zero for both location and momentum

For the eigenstate n

$$\sigma_{X_n}^2 = \langle \tilde{X}^2 \rangle_n - \langle \tilde{X} \rangle_n^2 = \frac{1}{2} \langle n | (\tilde{a}^\dagger + \tilde{a})^2 | n \rangle = \frac{1}{2} \langle n | \tilde{a}^{\dagger 2} + \tilde{a}^2 + \tilde{a}^\dagger \tilde{a} + \tilde{a} \tilde{a}^\dagger | n \rangle$$

19 June 12th

Last lesson we were talking about harmonic oscillators and the useful operators \tilde{a} and \tilde{a}^\dagger and with their help \tilde{N} . (in the dimensionless form, the distance between energy levels is 1)

The eigenstates of the hamiltonian in the location and momentum representations: (C.T. 500, 529, 542)

$$\tilde{a}|0\rangle = 0$$

Solving with the dimensionless

$$x' = \sqrt{\frac{m\omega}{\hbar}} x$$

$$\langle x' | \tilde{a} | 0 \rangle = 0$$

$$\Rightarrow \frac{1}{\sqrt{2}} \langle x' | \tilde{X} + i\tilde{P} | 0 \rangle = \frac{1}{\sqrt{2}} (x' + \frac{d}{dx'}) \psi_0(x') = 0$$

We find

$$\psi_0(x') = \pi^{-\frac{1}{4}} e^{-\frac{x'^2}{2}}$$

In the dimensioned form

$$\psi_0(x) = \left(\frac{m\omega}{\pi\hbar}\right)^{\frac{1}{4}} e^{-\frac{m\omega x^2}{2\hbar}}$$

We can apply the raising operator on this ground state to find all the other solutions

$$\begin{aligned} \psi_n(x') &= \frac{1}{\sqrt{n!}} \langle x' | (\tilde{a}^\dagger)^n | 0 \rangle = \frac{1}{\sqrt{n!}} \frac{1}{\sqrt{2^n}} \langle x' | (\tilde{X} - i\tilde{P})^n | 0 \rangle \\ &= \frac{1}{\sqrt{2^n n!}} \langle x' | \left(x' - \frac{d}{dx'} \right)^n | 0 \rangle \\ \Rightarrow \psi_n(x) &= \frac{1}{\sqrt{2^n n!}} \left(\sqrt{\frac{m\omega}{\hbar}} x - \sqrt{\frac{\hbar}{m\omega}} \frac{d}{dx} \right)^n \psi_0(x) \end{aligned}$$

Specifically

$$\psi_1(x') = \sqrt{\frac{2}{\sqrt{\pi}}} x' e^{-\frac{x'^2}{2}}$$

$$\psi_2(x') = \sqrt{\frac{1}{4\sqrt{\pi}}} (2x'^2 - 1) e^{-\frac{x'^2}{2}}$$

The order of the polynomial is the same order n , and this will multiply the gaussian

The solutions will thus be called Gauss-Hermit functions. They are all either symmetric or anti-symmetric according to the evenness or oddness of n .

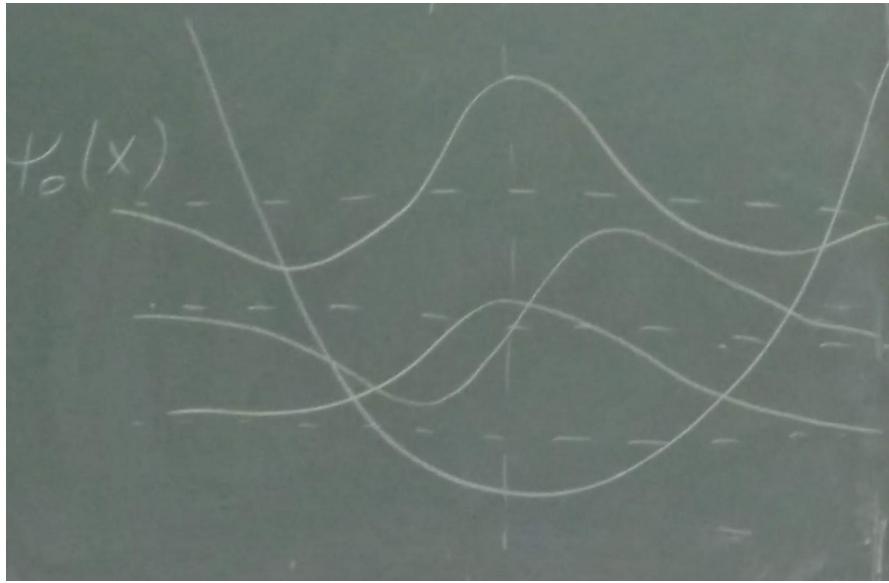


Figure 15:

The number of [real] zeroes (roots) is also n .

If we write the hamiltonian

$$\tilde{H} = \frac{1}{2}(\tilde{X}^2 + \tilde{P}^2)$$

Which wouldn't change if with switched $X \leftrightarrow \pm P$

But we note that we would need to choose the minus for the hamiltonian to remain unchanged.

Thus without doing any algebra we know that we can write the momentum representation

$$\tilde{\psi}_0(p') = \pi^{-\frac{1}{4}} e^{-\frac{p'^2}{2}}$$

And when $X \leftrightarrow -P$, the raising operator is multiplied by a factor of $-i$

$$\psi_n(p') = \frac{(-i)^n}{\sqrt{2^n n!}} \langle p' | \left(p' - \frac{d}{dp'} \right)^n \tilde{\psi}_0(p')$$

So the solutions for a harmonic oscillator **look** the same and are the same functions.

Thus we find that the Gauss-Hermit functions are the eigenvectors of the unitary Fourier operator.

But does this work with the correspondance principle at large values of n

Coherent States: (C.T. 559)

Classically from the Hamilton-Jacobi equations

$$\begin{aligned} \frac{dx}{dt} &= \frac{p}{m}, & \frac{dp}{dt} &= -m\omega^2 x \\ \Rightarrow \sqrt{\frac{\hbar}{m\omega}} \frac{dx'}{dt} &= \frac{\sqrt{m\omega\hbar}}{m} p', & \sqrt{m\omega\hbar} \frac{dp'}{dt} &= -m\omega^2 \sqrt{\frac{\hbar}{m\omega}} x' \\ \frac{dx'}{dt} &= \omega p', & \frac{dp'}{dt} &= -\omega x' \end{aligned}$$

The solutions give cosines and sines which create circles of radius 1 in the x, p plane.

So we'll define a new variable

$$\begin{aligned} \alpha &= \frac{1}{\sqrt{2}}(x' + ip') \\ \Rightarrow \frac{d\alpha}{dt} &= \frac{1}{\sqrt{2}} \left(\frac{dx'}{dt} + i \frac{dp'}{dt} \right) = \frac{1}{\sqrt{2}} (\omega p' - i\omega x') = -i\omega\alpha \end{aligned}$$

So

$$\alpha(t) = \alpha_0 e^{-i\omega t}$$

where α_0 is a complex number whose radius is the size of α_0 .

How do we obtain the classical solutions?

$$x'(t) = \sqrt{2} \operatorname{Re}(\alpha) = \sqrt{2} |\alpha_0| \cos(\omega t + \arg(\alpha_0))$$

$$p'(t) = \sqrt{2} \operatorname{Im}(\alpha) = -\sqrt{2} |\alpha_0| \sin(\omega t + \arg(\alpha_0))$$

The classical energy at any time, for the harmonic oscillator

$$E(t) = \frac{1}{2}(x'^2 + p'^2) = \alpha^* \alpha = |\alpha_0|^2$$

Now quantum. We want to quantize α and we note that the quantization of α is \tilde{a}

(since α is complex, we're not surprised to obtain the corresponding operator that is not Hermitian).
The equation for the expectation value for \tilde{a}

$$\frac{d \langle \tilde{a} \rangle}{dt} = \frac{1}{i\hbar} \langle [\tilde{a}, H] \rangle = -i\omega \langle [\tilde{a}, \tilde{H}] \rangle$$

$$= -i\omega \langle [\tilde{a}, \tilde{N}] \rangle = -i\omega \langle \tilde{a} \rangle$$

Whose solution is

$$\langle \tilde{a} \rangle(t) = \langle \tilde{a} \rangle(0)e^{-i\omega t}$$

We find

$$\langle \tilde{X} \rangle(t) = \sqrt{2} \operatorname{Re}(\langle \tilde{a} \rangle) = \sqrt{2} |\langle \tilde{a} \rangle(0)| \cos(\omega t + \arg(\langle \tilde{a} \rangle(0)))$$

$$\langle \tilde{P} \rangle(t) = \sqrt{2} \operatorname{Im}(\langle \tilde{a} \rangle) = -\sqrt{2} |\langle \tilde{a} \rangle(0)| \sin(\omega t + \arg(\langle \tilde{a} \rangle(0)))$$

When the expectation value at time zero

$$\langle \tilde{a} \rangle(0) = \alpha_0$$

we find the analogous behaviour to the classical case.

This means that the expectation value

$$\langle \tilde{a} \rangle(0) = \langle \psi(0) | \tilde{a} | \psi(0) \rangle = \alpha_0$$

$$\Rightarrow \langle \tilde{a} \rangle(t) = \alpha_0 e^{-i\omega t}$$

One of the cases when this occurs is

$$\tilde{a}|\psi(0)\rangle = \alpha_0|\psi(0)\rangle \equiv \alpha_0|\alpha_0\rangle$$

To find this state, in the energy representation

$$|\alpha_0\rangle = \sum_{n=0}^{\infty} c_n |n\rangle$$

substituting into the eigenvalue equation

$$\tilde{a} \sum_{n=0}^{\infty} c_n |n\rangle = \sum_{n=1}^{\infty} c_n \sqrt{n} |n-1\rangle = \sum_{n=0}^{\infty} c_{n+1} \sqrt{n+1} |n\rangle = \sum_{n=0}^{\infty} \alpha_0 c_n |n\rangle$$

So

$$\Rightarrow c_{n+1} \sqrt{n+1} = \alpha_0 c_n \Rightarrow c_{n+1} = \frac{\alpha_0 c_n}{\sqrt{n+1}}$$

$$\Rightarrow c_n = \frac{\alpha_0}{\sqrt{n}} c_{n-1} = \frac{\alpha_0^2}{\sqrt{n(n-1)}} c_{n-2} = \dots = \frac{\alpha_0^n}{\sqrt{n!}} c_0$$

Thus

$$|\alpha_0\rangle = c_0 \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} |n\rangle$$

So we can normalize by finding

$$\begin{aligned}\langle \alpha_0 | \alpha_0 \rangle &= |c_0|^2 \Sigma_{m=0}^{\infty} \Sigma_{n=0}^{\infty} \frac{(\alpha_0^*)^m}{\sqrt{m!}} \frac{\alpha_0^n}{\sqrt{n!}} \langle m | n \rangle = |c_0|^2 \Sigma_{n=0}^{\infty} \frac{(|\alpha_0|^2)^n}{n!} = |c_0|^2 e^{|\alpha_0|^2} \stackrel{!}{=} 1 \\ &\Rightarrow |\alpha_0\rangle = e^{-\frac{|\alpha_0|^2}{2}} \Sigma_{n=0}^{\infty} \frac{\alpha_0^n}{n!} |n\rangle\end{aligned}$$

and these states are known as coherent states.

$$\langle \tilde{H} \rangle = \langle \alpha_0 | \tilde{a}^\dagger \tilde{a} + \frac{1}{2} | \alpha_0 \rangle$$

(We don't know what happens in the application of \tilde{a}^\dagger on the eigenstates $|\alpha_0\rangle$ - there is no solution!) Instead we can use

$$\tilde{a}|\alpha_0 = \alpha_0|\alpha_0\rangle \Rightarrow \langle \alpha_0 | \tilde{a}^\dagger = \alpha_0^* \langle \alpha_0 |$$

(in general we apply \tilde{a}^\dagger left and \tilde{a} right)

So

$$\langle \tilde{H} \rangle = \langle \alpha_0 | \tilde{a}^\dagger \tilde{a} + \frac{1}{2} | \alpha_0 \rangle = \langle \alpha_0 | \alpha_0^* \alpha_0 + \frac{1}{2} | \alpha_0 \rangle = |\alpha_0|^2 + \frac{1}{2}$$

What is the average excited level (expectation value of \tilde{N})

$$\langle \tilde{N} \rangle = \langle \alpha_0 | \tilde{a}^\dagger \tilde{a} | \alpha_0 \rangle = \langle \alpha_0 | \alpha_0^* \alpha_0 | \alpha_0 \rangle = |\alpha_0|^2$$

So the standard deviation is

$$\begin{aligned}\sigma_N^2 &= \langle \tilde{N}^2 \rangle - \langle \tilde{N} \rangle^2 = \langle \alpha_0 | \tilde{a}^\dagger \tilde{a} \tilde{a}^\dagger \tilde{a} | \alpha_0 \rangle - |\alpha_0|^4 \\ &= |\alpha_0|^2 \langle \alpha_0 | \tilde{a} \tilde{a}^\dagger | \alpha_0 \rangle - |\alpha_0|^4 \\ &= |\alpha_0|^2 \langle \alpha_0 | \tilde{a}^\dagger \tilde{a} + 1 | \alpha_0 \rangle - |\alpha_0|^4 \\ &= |\alpha_0|^2 (|\alpha_0|^2 + 1) - |\alpha_0|^4 = |\alpha_0|^2 = \langle \tilde{N} \rangle\end{aligned}$$

Thus the standard deviation is the root of the expectation value of \tilde{N}

$$\sigma_N = \sqrt{\langle \tilde{N} \rangle}$$

20 June 17th

The last thing we saw in the last lesson was the expectation value of the number operator

$$\langle \tilde{N} \rangle = \langle \alpha_0 | \tilde{a}^\dagger \tilde{a} | \alpha_0 \rangle = \langle \alpha_0 | \alpha_0^* \alpha_0 | \alpha_0 \rangle = |\alpha_0|^2$$

and the standard deviation

$$\sigma_N = \sqrt{\langle \tilde{N} \rangle}$$

Let's calculate the probability of being at any energy level

$$P_n = |\langle n | \alpha_0 \rangle|^2$$

Where $|\alpha_0\rangle = e^{-\frac{|\alpha_0|^2}{2}} \sum_{n=0}^{\infty} \frac{\alpha_0^n}{\sqrt{n!}} |n\rangle$

Thus

$$P_n = e^{-|\alpha_0|^2} \frac{\alpha_0^{2n}}{n!} = e^{-\langle \tilde{N} \rangle} \frac{\langle \tilde{N} \rangle^n}{n!}$$

Which gives us a Poisson Distribution

Until now we dealt with a system at $t = 0$. What will the state be at some t ?

$$|\alpha\rangle(t) = |\alpha(t)\rangle = |\alpha_0 e^{-i\omega t}\rangle$$

Which changes only in phase, not distribution, and can be pictured as a point moving on a circle of radius α_0 . This is why it doesn't matter whether we use the 0 subscript on α , unless there is a specific time dependence. As we saw last lesson

$$\langle \tilde{X} \rangle = \frac{1}{\sqrt{2}} \langle \alpha | \tilde{a}^\dagger + \tilde{a} | \alpha \rangle = \frac{1}{\sqrt{2}} (\alpha^* + \alpha) = \sqrt{2} \operatorname{Re}(\alpha)$$

$$\langle \tilde{P} \rangle = \frac{i}{\sqrt{2}} \langle \alpha | \tilde{a}^\dagger - \tilde{a} | \alpha \rangle = -\frac{i}{\sqrt{2}} (\alpha - \alpha^*) = \sqrt{2} \operatorname{Im}(\alpha)$$

So we can calculate

$$\begin{aligned} \langle X^2 \rangle &= \frac{1}{2} \langle \alpha | (\tilde{a}^\dagger + \tilde{a})^2 | \alpha \rangle = \frac{1}{2} \langle \alpha | \tilde{a}^{\dagger 2} + \tilde{a}^2 + \tilde{a}^\dagger \tilde{a} + \tilde{a} \tilde{a}^\dagger | \alpha \rangle \\ &= \frac{1}{2} \langle \alpha | \tilde{a}^{\dagger 2} + \tilde{a}^2 + 2\tilde{a}^\dagger \tilde{a} + 1 | \alpha \rangle = \frac{1}{2} \langle \alpha | \alpha^{*2} + \alpha^2 + 2\alpha^* \alpha + 1 | \alpha \rangle = \frac{1}{2} (\alpha^* + \alpha)^2 + \frac{1}{2} \end{aligned}$$

and

$$\langle P^2 \rangle = \frac{1}{2} \langle \alpha | (\tilde{a}^\dagger - \tilde{a})^2 | \alpha \rangle = -\frac{1}{2} (\alpha^* - \alpha)^2 + \frac{1}{2}$$

Thus we can calculate the standard deviation in location and momentum

$$\sigma_x^2 = \langle X^2 \rangle - \langle X \rangle^2 = \frac{1}{2}$$

$$\sigma_p^2 = \langle P^2 \rangle - \langle P \rangle^2 = \frac{1}{2}$$

they are not dependent on α nor on the energy of the state.

And this is the maximization of Heisenberg's uncertainty principle, so we know the wave function must be a gaussian. Looking at the location representation

$$\tilde{a}|\alpha\rangle = \alpha|\alpha\rangle$$

$$\langle x' | \tilde{a} | \alpha \rangle = \langle x' | \alpha | \alpha \rangle$$

$$\frac{1}{\sqrt{2}} \langle x' | \tilde{X} + i\tilde{P} | \alpha \rangle = \alpha \langle x' | \alpha \rangle$$

$$\frac{1}{\sqrt{2}} \left(x' + \frac{d}{dx'} \right) \psi_\alpha(x') = \alpha \psi_\alpha(x')$$

$$\Rightarrow \frac{d}{dx'} \psi_\alpha(x') = (-x' + \sqrt{2}\alpha) \psi_\alpha(x')$$

Thus

$$\begin{aligned}\psi_\alpha(x') &= Ce^{-\frac{x'^2}{2} + \sqrt{2}\alpha x'} = Ce^{-\frac{x'^2}{2} + \sqrt{2}[Re(\alpha) + iIm(\alpha)]x'} \\ &= Ce^{-\frac{1}{2}[x' - \sqrt{2}Re(\alpha)]^2 + \sqrt{2}iIm(\alpha)x' + Re^2(\alpha)} \\ &= \pi^{-\frac{1}{4}} e^{-\frac{1}{2}[x' - \langle \tilde{X} \rangle]^2 + i\langle \tilde{P} \rangle x' + Re^2(\alpha)}\end{aligned}$$

Noting that coherent excitations are the closest excitations to classical excitations.

For $n = 0$

$$\tilde{a}|0\rangle = 0$$

$$\tilde{N}|0\rangle = 0$$

Meaning that the ground state is also a coherent state.

Let's look at the orthonormality of coherent states α, β .

$$\begin{aligned}\langle \alpha | \beta \rangle &= e^{-\frac{|\alpha|^2 + |\beta|^2}{2}} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\beta^m}{\sqrt{m!}} \langle n | m \rangle \\ &= e^{-\frac{|\alpha|^2 + |\beta|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha^* \beta)^n}{n!} = e^{-\frac{1}{2}(|\alpha|^2 + |\beta|^2 - 2\alpha^* \beta)} \\ &= e^{iIm(\alpha^* \beta)} e^{-\frac{1}{2}|\alpha - \beta|^2}\end{aligned}$$

So this will never be equal to zero, meaning that there is no orthogonality (only approximately when they are far away)

And the size will be

$$|\langle \alpha | \beta \rangle|^2 = e^{-|\alpha - \beta|^2}$$

If they were a complete, orthonormal set

$$\begin{aligned}\int_{\alpha \in \mathbb{C}} d\alpha |\alpha\rangle \langle \alpha| &= \int_{\alpha \in \mathbb{C}} d\alpha e^{-|\alpha|^2} \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{\alpha^n}{\sqrt{n!}} \frac{\alpha^{*m}}{\sqrt{m!}} |n\rangle \langle m| \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|n\rangle \langle m|}{\sqrt{n!} \sqrt{m!}} \int_{\alpha \in \mathbb{C}} d\alpha e^{-|\alpha|^2} \alpha^n \alpha^{*m} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|n\rangle \langle m|}{\sqrt{n!} \sqrt{m!}} \int_0^{2\pi} d\theta \int_0^{\infty} dr \cdot r e^{-r^2} r^n e^{in\theta} r^m e^{-im\theta} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|n\rangle \langle m|}{\sqrt{n!} \sqrt{m!}} \int_0^{2\pi} d\theta \int_0^{\infty} dr \cdot r e^{-r^2} r^{n+m} e^{i\theta(n-m)} \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{|n\rangle \langle m|}{\sqrt{n!} \sqrt{m!}} \int_0^{\infty} dr \cdot r e^{-r^2} r^{n+m} 2\pi \delta_{nm} \\ &= 2\pi \sum_{n=0}^{\infty} \frac{|n\rangle \langle n|}{n!} \int_0^{\infty} dr \cdot r e^{-r^2} r^{2n}\end{aligned}$$

Substituting $\rho = r^2$

$$= \pi \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} \int_0^{\infty} d\rho \cdot e^{-\rho} \rho^n = \pi \sum_{n=0}^{\infty} \frac{|n\rangle\langle n|}{n!} n! = \pi \sum_{n=0}^{\infty} |n\rangle\langle n|$$

$$= \pi \mathbf{1}$$

This is an over-complete set, as there is more than one representation and any state can be represented by such a set.

What's the meaning of representations?

$$|\psi\rangle = \int d\alpha f(\alpha) |\alpha\rangle$$

For example

$$|\beta\rangle = \frac{1}{\pi} \int_{a \in \mathbb{C}} d\alpha |\alpha\rangle\langle\alpha|\beta\rangle = \frac{1}{\pi} \int_{a \in \mathbb{C}} d\alpha e^{iIm(\alpha^*\beta)} e^{-\frac{1}{2}|\alpha-\beta|^2} |\alpha\rangle$$

What is the representation of an eigenstate of the hamiltonian?

$$|n\rangle = \frac{1}{\pi} \int_{a \in \mathbb{C}} d\alpha |\alpha\rangle\langle\alpha|n\rangle = \int_{a \in \mathbb{C}} d\alpha \frac{e^{-\frac{|\alpha|^2}{2}}}{\pi} \frac{\alpha^{*n}}{\sqrt{n!}} |\alpha\rangle$$

Let's define the operator

$$D(\alpha) = e^{\alpha\tilde{a}^\dagger - \alpha^*\tilde{a}}$$

$$D^\dagger(\alpha) = e^{\alpha^*\tilde{a} - \alpha^*\tilde{a}^\dagger} = D^{-1}(\alpha)$$

Meaning that the operator is unitary!

Becker-Housdorf Equation: (C.T. 174)

If you have e^{A+B} where A, B are operators.

If A, B commute then everything is okay and the exponent can be opened

If they commute with their commutator

$$e^{A+B} = e^{-\frac{1}{2}[A,B]} e^A e^B$$

Checking $D(\alpha)$ on the vacuum state

$$D(\alpha)|0\rangle = e^{-\frac{|\alpha|^2}{2}[\tilde{a}^\dagger, \tilde{a}]} e^{\alpha\tilde{a}^\dagger} e^{-\alpha^*\tilde{a}} |0\rangle = e^{-\frac{|\alpha|^2}{2}} e^{\alpha\tilde{a}^\dagger} |0\rangle = e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha\tilde{a}^\dagger)^n}{n!} |0\rangle$$

$$= e^{-\frac{|\alpha|^2}{2}} \sum_{n=0}^{\infty} \frac{(\alpha)^n}{\sqrt{n!}} |n\rangle = |\alpha\rangle$$

Thus we find a new way of writing

$$D(\alpha)|\beta\rangle = D(\alpha)D(\beta)|0\rangle = e^{\alpha\tilde{a}^\dagger - \alpha^*\tilde{a}} e^{\beta\tilde{a}^\dagger - \beta^*\tilde{a}} |0\rangle$$

Noting that $[\alpha\tilde{a}^\dagger - \alpha^*\tilde{a}, \beta\tilde{a}^\dagger - \beta^*\tilde{a}] = \alpha\beta^* - \alpha^*\beta$

$$\Rightarrow D(\alpha)|\beta\rangle = e^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)} e^{(\alpha + \beta)\tilde{a}^\dagger - (\alpha^* + \beta^*)\tilde{a}} |0\rangle$$

$$= e^{iIm(\alpha\beta^*)} |\alpha + \beta\rangle$$

Thus this is called the displacement operator, taking coherent states and moving by α .

On some general state, it will take the representation and move the amplitude.

21 June 24th

Periodic Systems:

(Finite in their periodicity such that $V(x+d) = V(x)$)

We'll be looking at Ashcroft-Mermin (132) and at the Flokeh-Bloch theorem.

Looking at the translation operator,

$$T_{x_0}|x\rangle = |x - x_0\rangle$$

which is unitary

$$T_{x_0}^\dagger = T_{x_0}^{-1} = T_{-x_0}$$

and hermitian

$$\langle x|T_{x_0}|\psi\rangle = \langle x+x_0|\psi\rangle = \psi(x+x_0)$$

The operator of momentum (and therefore of the kinetic energy) is symmetric to translations - as is any hamiltonian and has the matrix elements

$$\langle x'|T_d H T_d^\dagger|x'\rangle = \langle x'+d|H|x'+d\rangle = \langle x'|H|x'\rangle$$

$$\Rightarrow T_d H T_d^\dagger = H \Rightarrow T_d H T_d^\dagger = T_d H = H T_d$$

Therefore the hamiltonian commutes with the translational operator T_d (including n times d)

$$[H, T_{n \cdot d}] = 0$$

$$T_{md} T_{nd} = T_{(m+n)d} = T_{nd} T_{md}$$

So the eigenstates of the hamiltonian are also the eigenstates of the translation operator

$$H|\varepsilon\rangle = \varepsilon|\varepsilon\rangle$$

$$\Rightarrow T_{nd}|\varepsilon\rangle = \delta_{nd}|\varepsilon\rangle$$

(with no special meaning of the δ_{nd} - not Kronecker) Where $T_{nd} = T_d^n$

Thus

$$\delta_{nd} = \delta_d^n = e^{iA}$$

(because it is unitary)

$$\Rightarrow \delta_d^n = \left(e^{ik(\varepsilon)d}\right)^n = e^{ik(\varepsilon)nd}$$

which are the eigenvalues.

Now we can write the Bloch Theorem:

$$\boxed{\psi_\varepsilon(x+d) = \langle x+d|\psi_\varepsilon\rangle = \langle x|T_d|\psi_\varepsilon\rangle = e^{ikd}\langle x|\psi_\varepsilon\rangle = e^{ikd}\psi_\varepsilon(x)}$$

Another formulation that is more popular is:

$$\boxed{\psi_\varepsilon(x) = e^{ikx}u_\varepsilon(x)}$$

Where $u_\varepsilon(x+d) = u_\varepsilon(x)$ is periodic in x .

Thus

$$\psi_\varepsilon(x+d) = e^{ik(x+d)} u_\varepsilon(x+d) = e^{ikd} e^{ikx} u_\varepsilon(x) = e^{ikd} \psi_\varepsilon(x)$$

showing that the two formulations are equivalent

Scattering from 1D Periodic Potential and Striped Structure: (C.T. 367)

$$\begin{bmatrix} B_+ \\ B_- \end{bmatrix} = T(k) \begin{bmatrix} A_+ \\ A_- \end{bmatrix} = \frac{k_1}{k_2} \begin{bmatrix} \frac{1}{t^*} & -\frac{r^*}{t^*} \\ -\frac{r}{t} & \frac{1}{t} \end{bmatrix} \begin{bmatrix} A_+ \\ A_- \end{bmatrix}$$

What happens when this repeats an infinite number of times (assuming that $t_1 \equiv t$ and $k_1 = k_2$)

$$T_\infty = \lim_{n \rightarrow \infty} T^n$$

$$T = UDU^{-1}$$

where D is a diagonal matrix of T 's eigenvalues

$$D = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

Thus

$$T^n = UDU^{-1}UDU^{-1}UDU^{-1}\dots UDU^{-1} = UD^nU^{-1}$$

$$\Rightarrow T_\infty = \lim_{n \rightarrow \infty} U \begin{bmatrix} \lambda_1^n & 0 \\ 0 & \lambda_2^n \end{bmatrix} U^{-1}$$

Therefore

$$\begin{aligned} |T - \lambda \mathbf{I}| &= \left| \begin{bmatrix} \frac{1}{t^*} - \lambda & -\frac{r^*}{t^*} \\ -\frac{r}{t} & \frac{1}{t} - \lambda \end{bmatrix} \right| = (\lambda - \frac{1}{t^*})(\lambda - \frac{1}{t}) - \frac{|r|^2}{|t|^2} \\ &= \lambda^2 - (\frac{1}{t} + \frac{1}{t^*})\lambda + \frac{1 - |r|^2}{|t|^2} = \lambda^2 - (\frac{1}{t} + \frac{1}{t^*})\lambda + 1 \stackrel{!}{=} 0 \end{aligned}$$

(noting that $k_1 = k_2 \Rightarrow \frac{1 - |r|^2}{|t|^2} = 1$)

If we denote $t = \tau e^{i\varphi}$ where $0 \leq \tau \leq 1$

$$\lambda^2 - 2 \frac{\cos \varphi}{\tau} \lambda + 1 \stackrel{!}{=} 0$$

$$\Rightarrow \lambda_{1,2} = \frac{1}{t} (\cos \varphi \pm \sqrt{\cos^2 \varphi - \tau^2})$$

So if $\cos^2 \varphi < \tau^2$

We can define $\cos \theta \equiv \frac{\cos \varphi}{\tau}$ and substitute to find that

$$\lambda_{1,2} = e^{\pm i\theta}$$

In this case, these eigenvalues have sizes of 1, even when raised to higher powers ($n \rightarrow \infty$) and the matrix D^n will remain a unitary matrix

This is essentially the case of a conductor

and if $\cos^2 \varphi > \tau^2$

We can define $\cosh\alpha \equiv \frac{\cos\varphi}{\tau}$ and substitute to find that

$$\lambda_{1,2} = e^{\pm\alpha}$$

This is essentially the case of an inductor

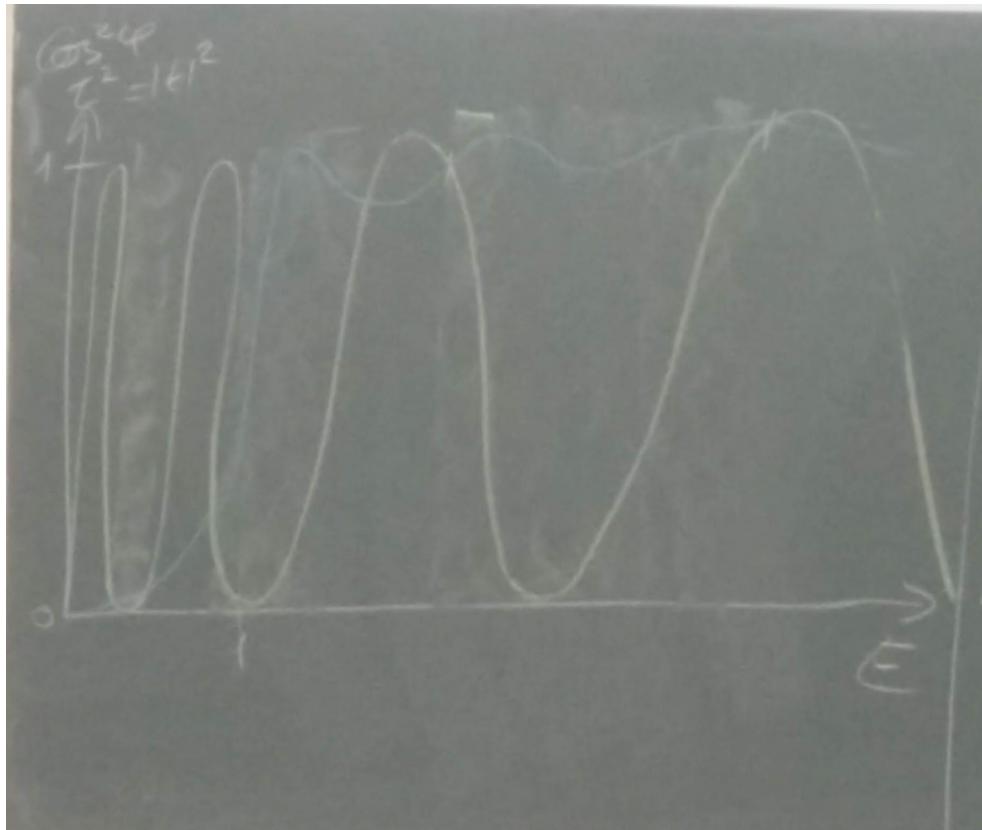


Figure 16: CONDITION VS ENERGY GRAPH: (τ^2 in blue, \cos^2 in green... highlighted area is where the electrons CAN flow)

Where the electrons get to is called a fermi level. And this is the reason behind the Pauli Exclusion principle as we can imagine these electrons as an incompressible fluid.