1 Independent Set on Paths

Let $G = (V, E)$ be an undirected graph with n nodes. A subset of the nodes is called an **independent set** if no two of them are joined by an edge. Finding large independent sets is difficult in general; but here we'll see that it can be done efficiently if the graph is "simple" enough.

Call a graph $G = (V, E)$ a **path** if its nodes can be written as $v_1, v_2, \ldots, v_n$, with an edge between v_i and v_j if and only if the numbers i and j differ by exactly 1. With each node v_i, we associate a positive integer *weight* w_i.

The goal in this exercise is to solve the following problem:

Find an independent set in a path G whose total weight is as large as possible.

- Draw the five-node path with node weights (in order): 1, 8, 6, 3, 6.
- Give an example to show that the following algorithm *does not* always find an independent set of maximum total weight:

```
(The "heaviest-first" greedy algorithm)
  Start with S equal to the empty set
  While some node remains in G:
    Pick a node v_i of maximum weight
    Add v_i to S
    Delete v_i and its neighbors from G

  Return S
```

- Give an example to show that the following algorithm also *does not* always find an independent set of maximum total weight.

```
Let S_1 be the set of all v_i where i is an odd number.
Let S_2 be the set of all v_i where i is an even number.
(Note that both S_1 and S_2 are independent sets.)
Determine the greater of the sum of weights in S_1 or S_2
    and return that set.
```

- Describe a recursive backtracking algorithm that takes an n-node path G with weights and returns an independent set of maximum total weight.

2 Magic Squares

You may use the Internet to search for references, explanations, proposed solutions to this exercise. Explain answers to the questions below in your own words.

A magic square of order n is an arrangement of the integers from 1 to n^2 in an $n \times n$ matrix, with each number occurring exactly once, so that each row, each column, and each main diagonal has the same sum.

- For a magic square of order n, what does the sum of each row equal? Prove that.
- Describe a backtracking algorithm (in the form of how backtracking algorithms are described in our textbook and how we've covered in class) to generate *a* magic square.

3 Addition Chain

Exercise 3(a), Chapter 2 (page 94) in the textbook.