UNIVERSITY OF BIRMINGHAM

2. Kinematics, statics, and dynamics of robotic arm

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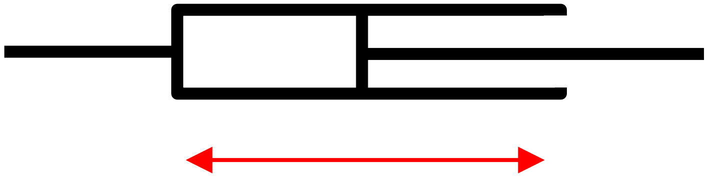
Department of Electronic, Electrical and Systems Engineering

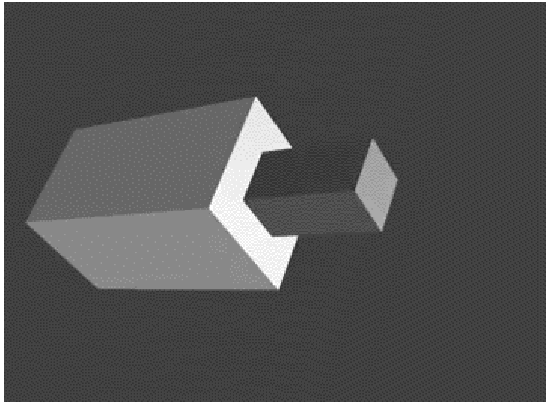
Content

2.1 Robot Mechanisms

Prismatic joint

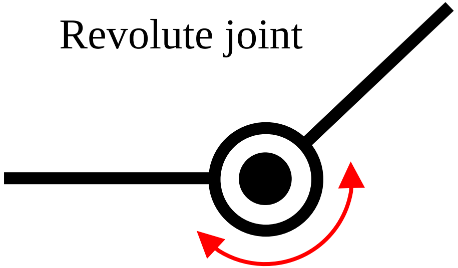


Allows a pair of links makes a translational displacement along a fixed axis

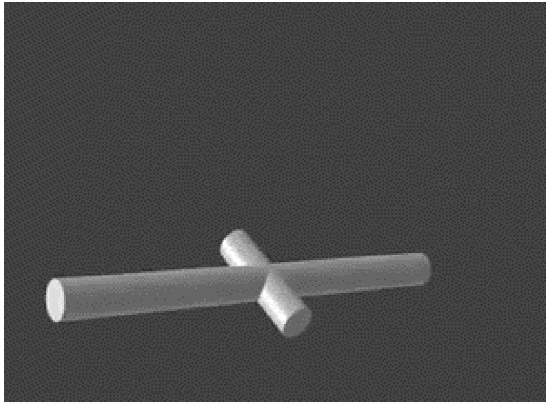


https://www.youtube.com/watch?v=ih3oXigeY-U&list=PLrIVgT56nVQ5arwjkuRbyw6lBshrRW9PG&index=2

Revolute joint



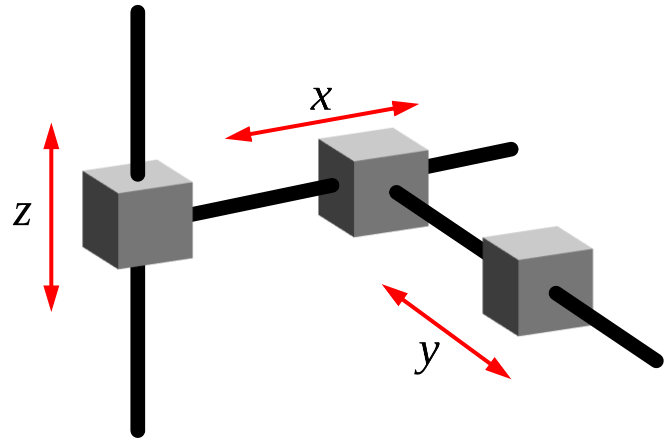
Allows a pair of links rotates about a fixed axis



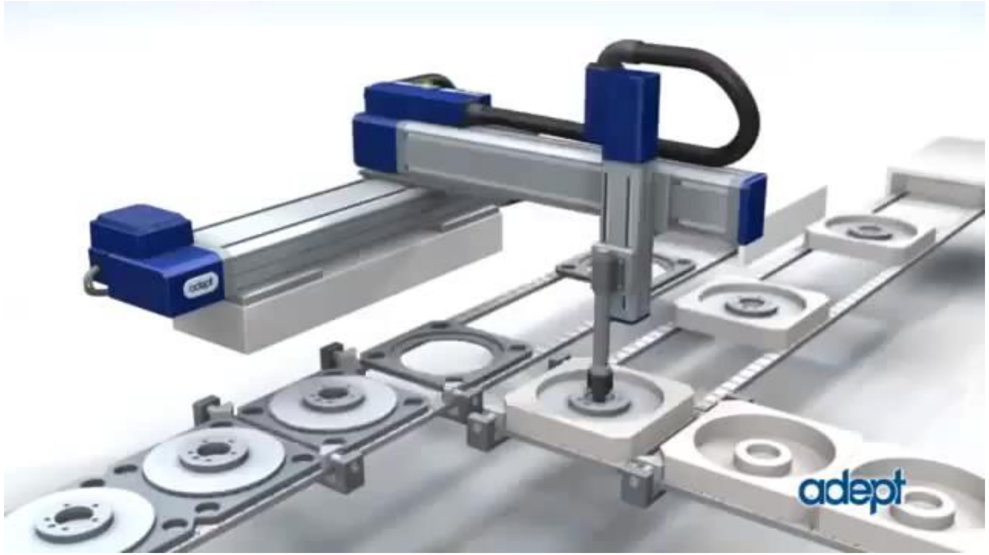
https://www.youtube.com/watch?v=wwyJS9X3WvE&list=PLrIVgT56nVQ5arwjkuRbyw6lBshrRW9PG&index=1

2.1 Robot Mechanisms

- Cartesian coordinate robot



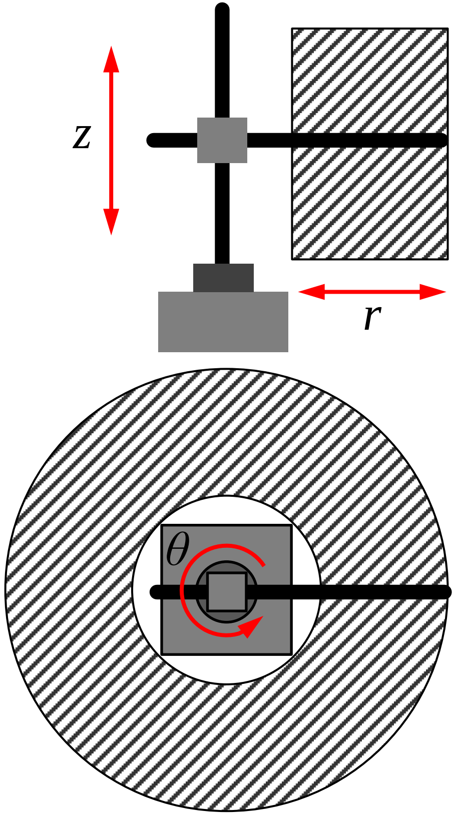
3 prismatic joints

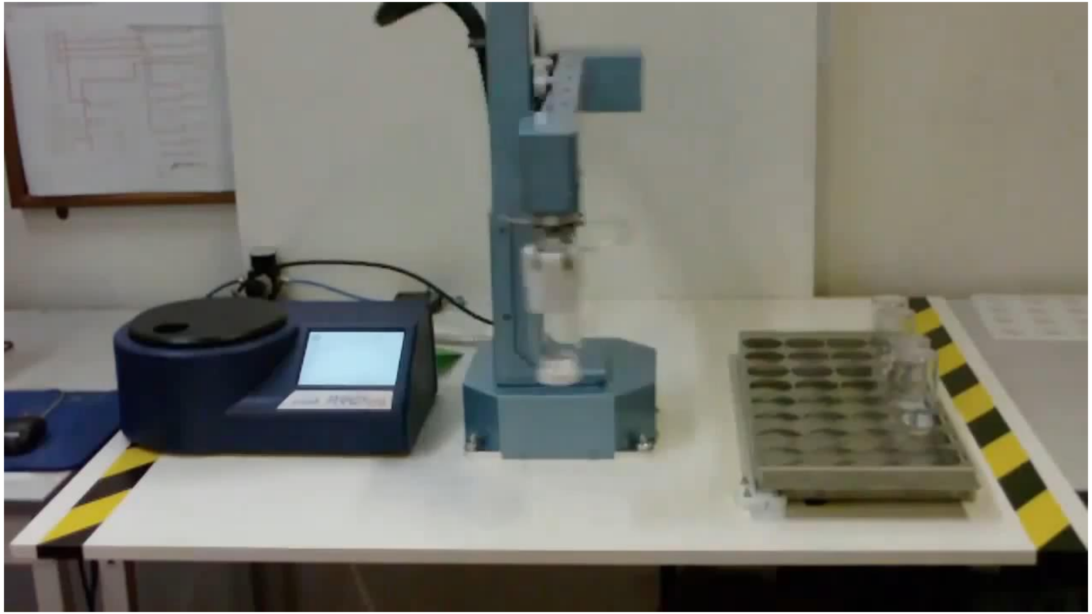


https://www.youtube.com/watch?v=ci_mpRERMog

2.1 Robot Mechanisms

- Cylindrical coordinate robot



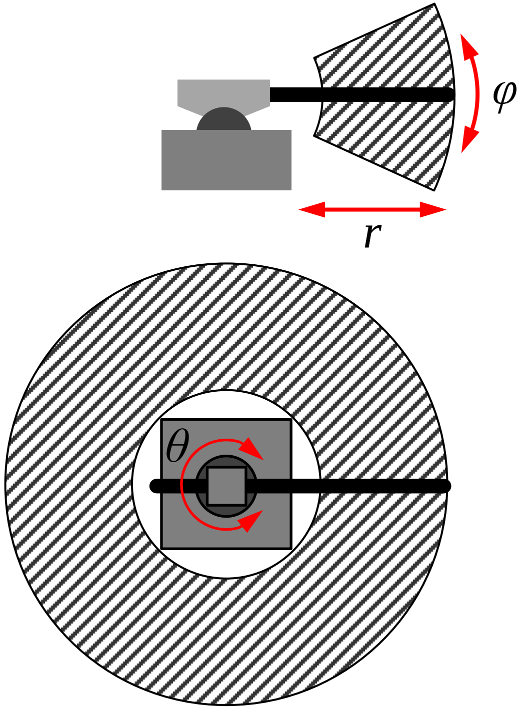


https://www.youtube.com/watch?v=Hj7PxjeH5y0

2 prismatic joints + 1 revolute joint

2.1 Robot Mechanisms

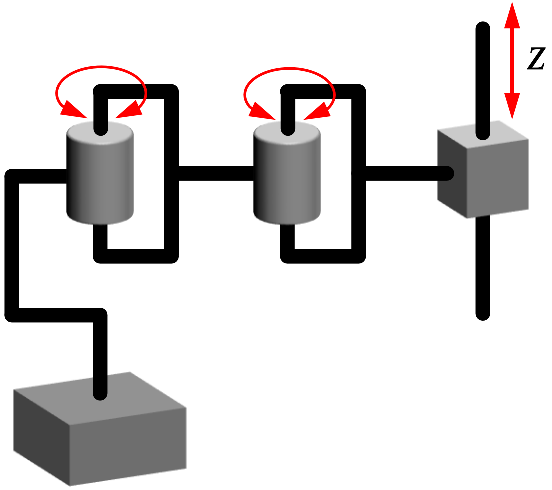
- Spherical coordinate robot



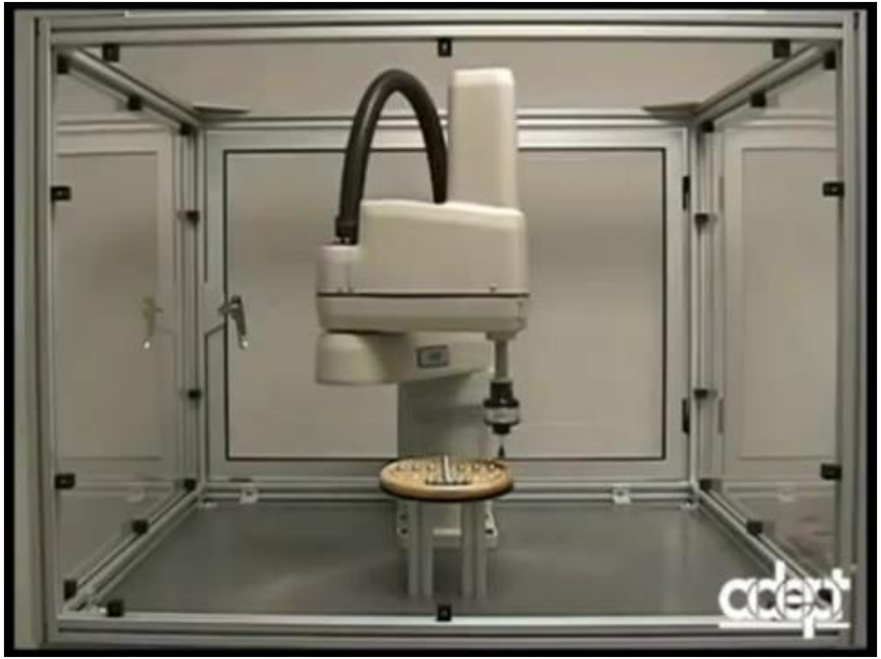
https://www.youtube.com/watch?v=3cW7D8c5DTc

1 prismatic joint + 2 revolute joints

- SCARA robot



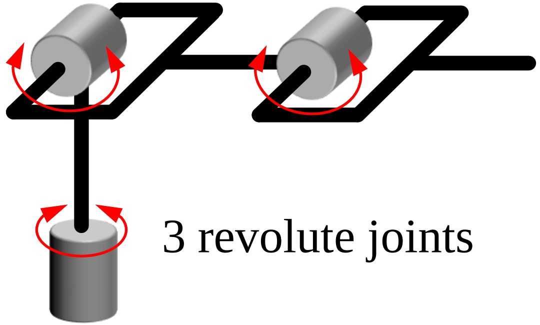
1 prismatic joint + 2 revolute joints



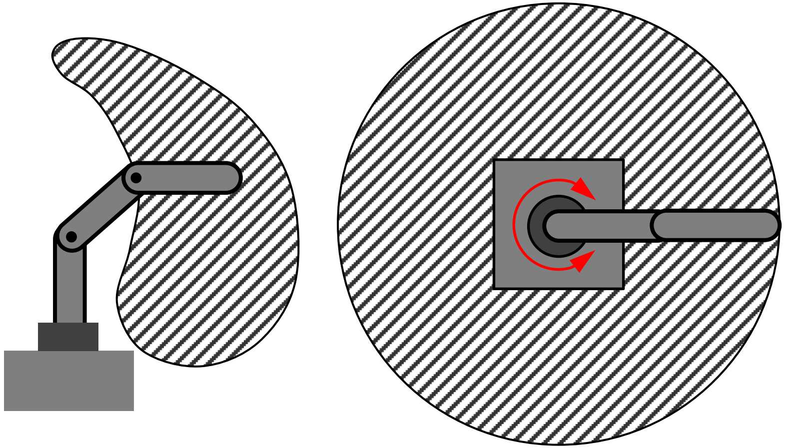
https://www.youtube.com/watch?v=vKD20BTkXhk

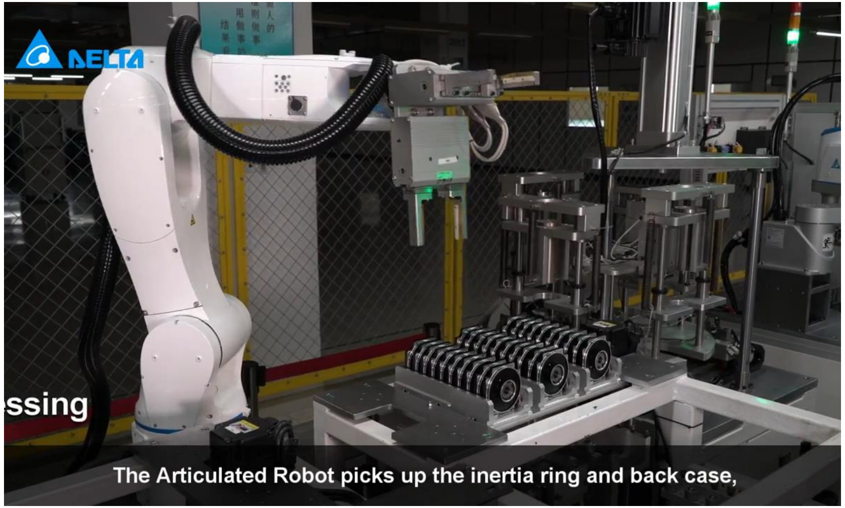
2.1 Robot Mechanisms

- Articulated robot



3 revolute joints

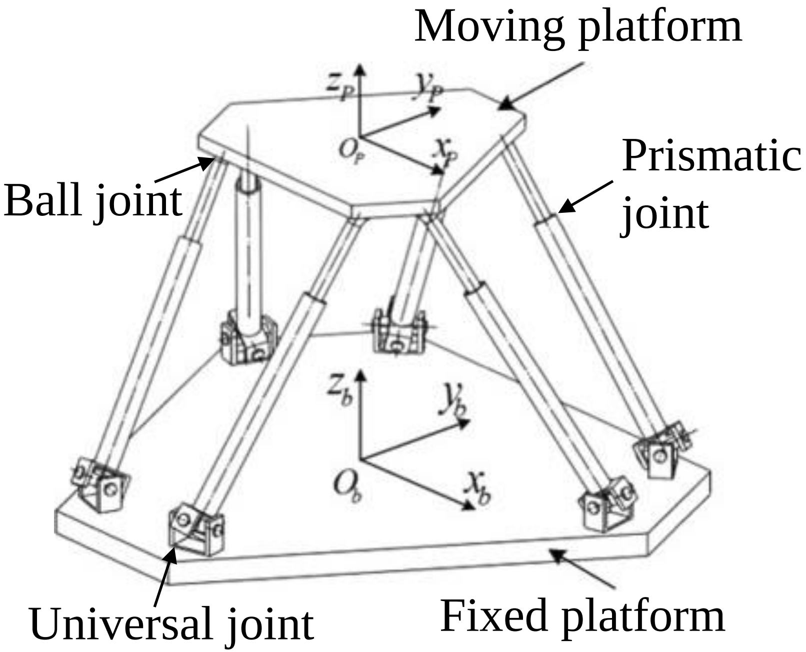


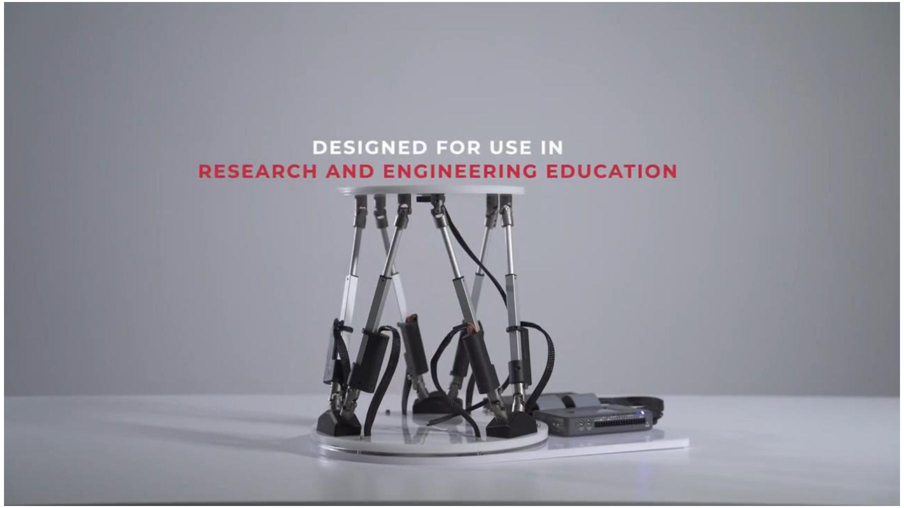


https://www.youtube.com/watch?v=HgDEqlhjhrE

2.1 Robot Mechanisms

- Stewart mechanism parallel-link robot

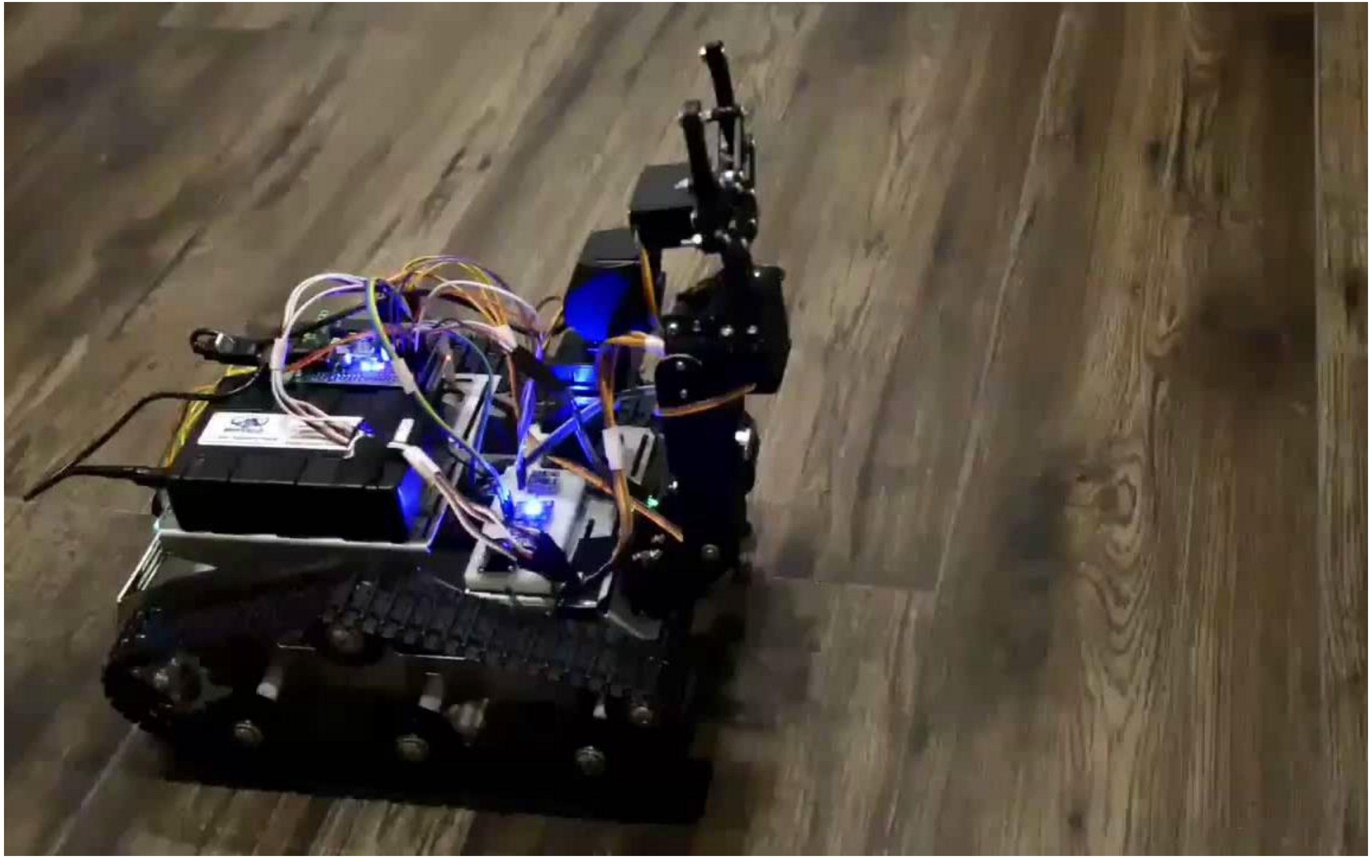




https://www.youtube.com/watch?v=vlCH4zhIqmM

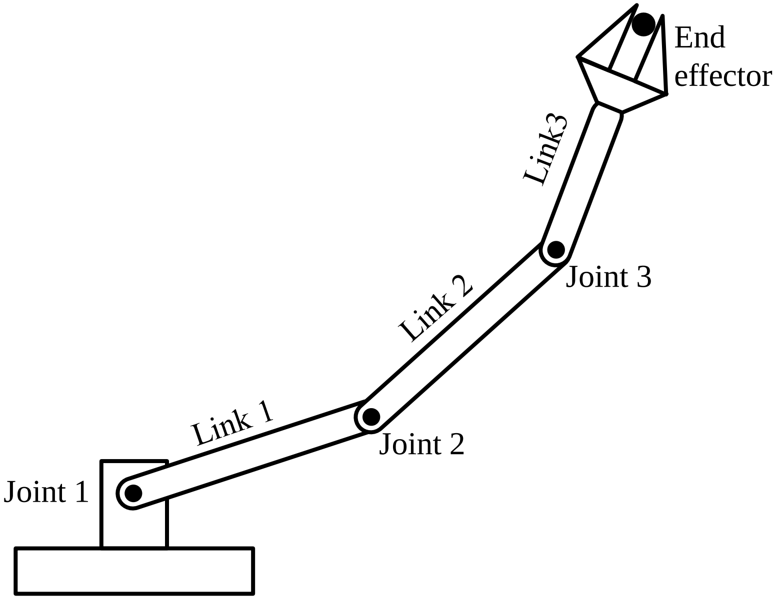
Kumar, P. R., & Bandyopadhyay, B. (2013, July). In 2013 11th IEEE International Conference on Industrial Informatics (INDIN) (pp. 343-348). IEEE.

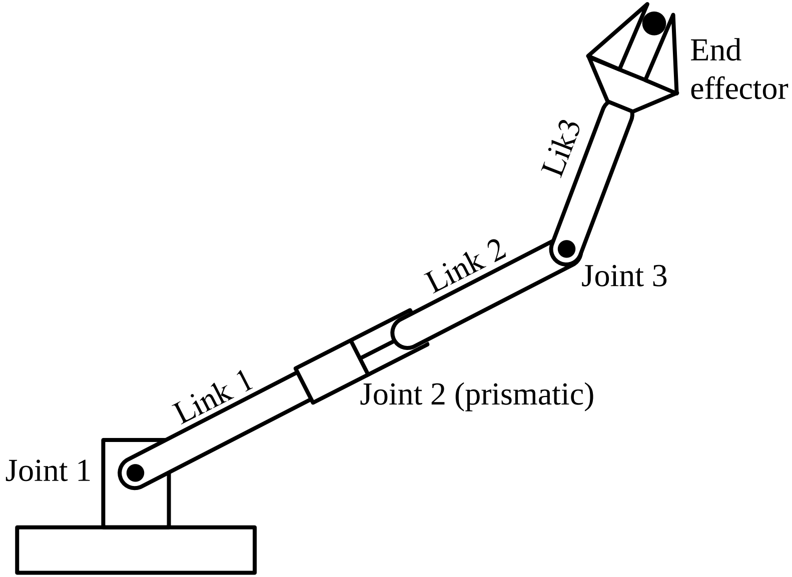
2.1 Robot Mechanisms



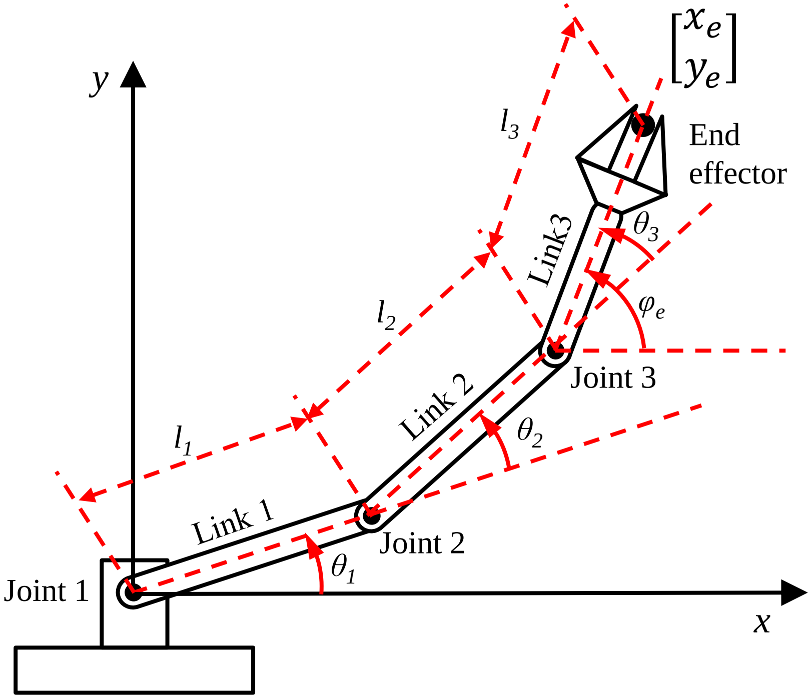
2.2 Planar Kinematics

Kinematics is Geometry of Motion, which describes the motion of points, bodies (objects), and systems of bodies (groups of objects) without considering the forces that cause them to move. (*Wikipedia*)





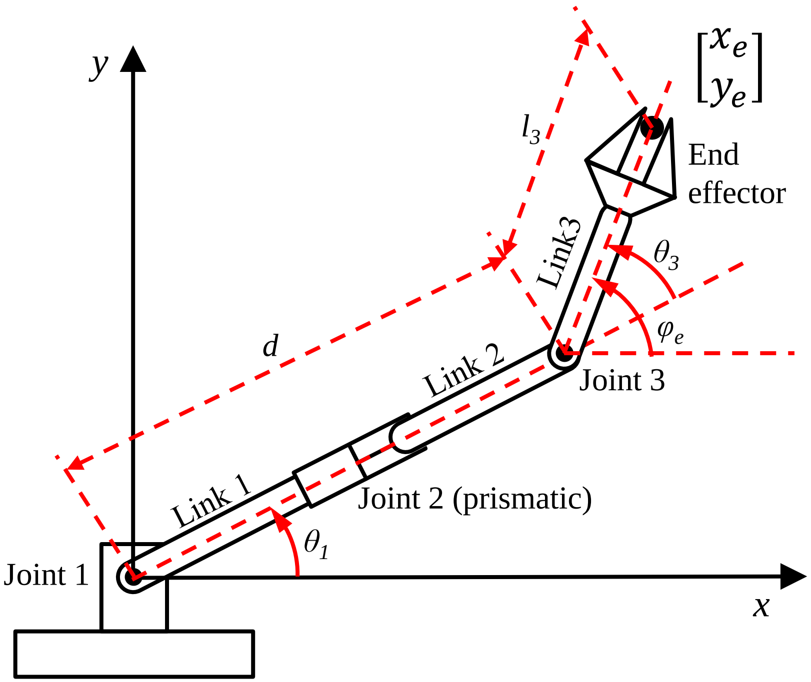
2.2 Planar Kinematics



$$
\begin{cases}
x_e = l_1 cos\theta_1 + l_2 cos(\theta_1 + \theta_2) + l_3 cos(\theta_1 + \theta_2 + \theta_3) \\
y_e = l_1 sin\theta_1 + l_2 sin(\theta_1 + \theta_2) + l_3 sin(\theta_1 + \theta_2 + \theta_3) \\
\varphi_e = \theta_1 + \theta_2 + \theta_3
\end{cases}
$$

A set of algebraic equations relating the position and orientation of a robot end-effecter to actuator or active joint displacements, is called **Forward Kinematic Equations**

2.2 Planar Kinematics



$$
\begin{cases}
x_e = d\cos\theta_1 + l_3\cos(\theta_1 + \theta_3) \\
y_e = d\sin\theta_1 + l_3\sin(\theta_1 + \theta_3) \\
\varphi_e = \theta_1 + \theta_3
\end{cases}
$$

2.2 Planar Kinematics

For formal expression, joint displacement q_i of the *i-th* joint represents either distance d_i or angle θ_i depending on the type of joint.

$$q_i = \begin{cases} d_i \\ \theta_i \end{cases}$$

The displacements involved in a robot mechanism with n joints can be represented use a column vector $\boldsymbol{q}$.

$$\boldsymbol{q} = [q_1 \quad q_2 \quad q_3 \cdots \quad q_n]^T$$

The end-effecter position and orientation by vector $\boldsymbol{p}$ is described by three variables:

$$\boldsymbol{p} = \begin{bmatrix} x_e \\ y_e \\ \varphi_e \end{bmatrix} = f(\boldsymbol{q})$$

2.2 Planar Kinematics

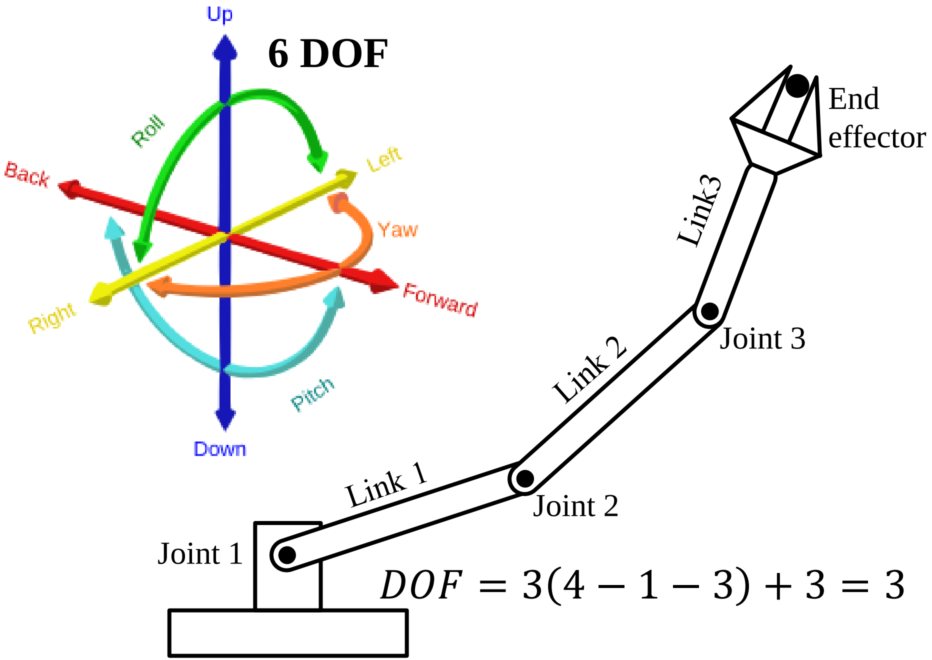
Joints are independent for a serial link mechanism. Therefore, vector ***q*** is a collector of joint displacements, which uniquely determines the robot configuration. These joint displacements are *generalized coordinates* that locate the robot mechanism uniquely and completely. Formally, the number of generalized coordinates is called **degrees of freedom (DOF)**.[1]

Grübler's formular:

$$DOF = m(N - 1 - J) + \sum_{i=1}^{J} f_i$$

N = number of bodies, including ground
J = number of joints
m = 6 for spatial bodies, 3 for planar
f_i = DOF of the i-th joint

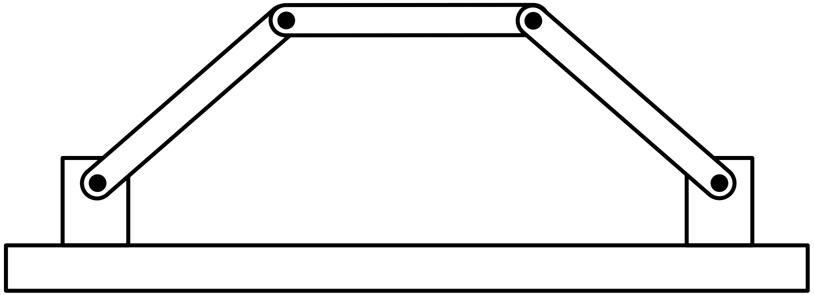
6 DOF



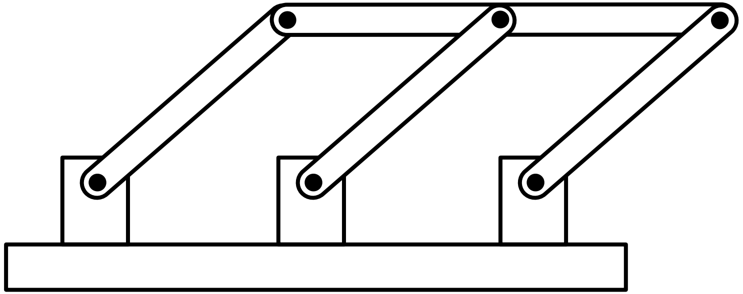
$$DOF = 3(4 - 1 - 3) + 3 = 3$$

([1]*Introduction to Robotics, H. Harry Asada*)

2.2 Planar Kinematics



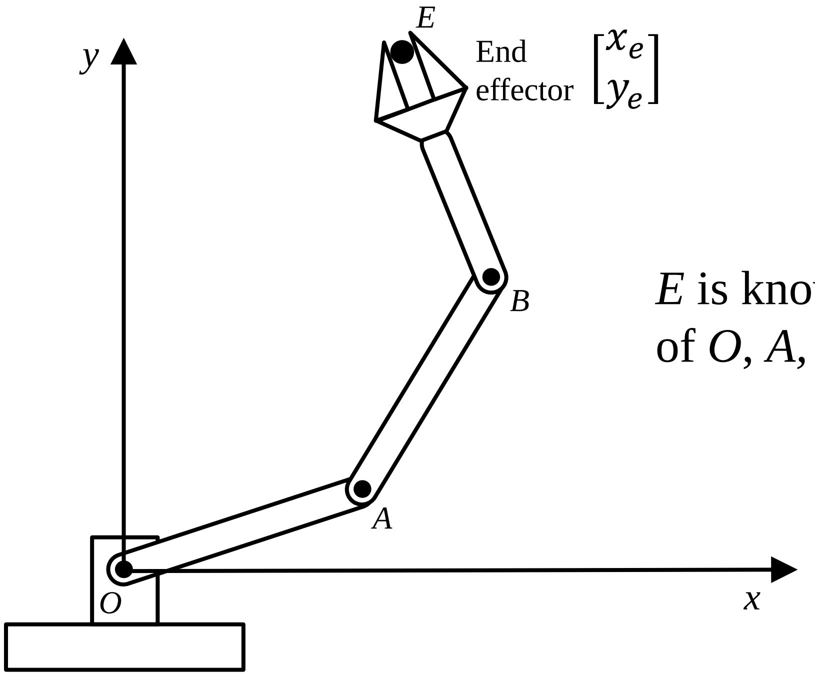
$DOF = 3(4-1-4)+4 = 1$



$DOF = 3(5-1-6)+6 = 0$????

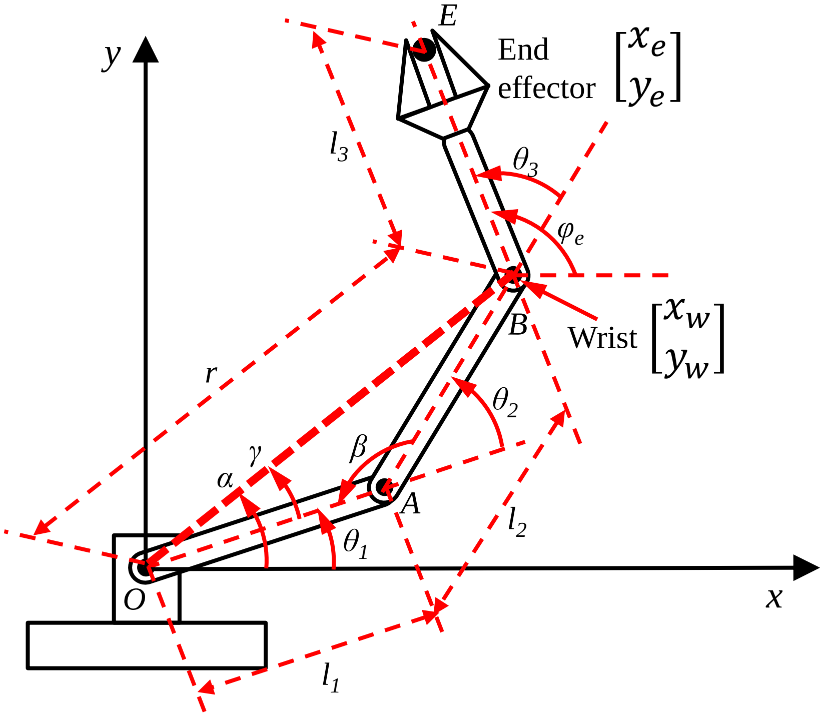
2.2 Planar Kinematics

The **inverse kinematics** problem is referred as finding the joint displacements that lead the end-effecter to the specified position and orientation. The kinematic equation must be solved for joint displacements, given the end-effecter's position and orientation.



E is known, what are the angles of O, A, and B ?

2.2 Planar Kinematics



$$x_w = x_e - l_3 cos(\varphi_e)$$
$$y_w = y_e - l_3 sin(\varphi_e)$$

$$l_1^2 + l_2^2 - 2l_1 l_2 cos\beta = r^2 = x_w^2 + y_w^2$$

$$\boxed{\theta_2 = \pi - \beta = \pi - cos^{-1}\frac{l_1^2 + l_2^2 - x_w^2 - y_w^2}{2l_1 l_2}}$$

$$r^2 + l_1^2 - 2rl_1 cos\gamma = l_2^2$$

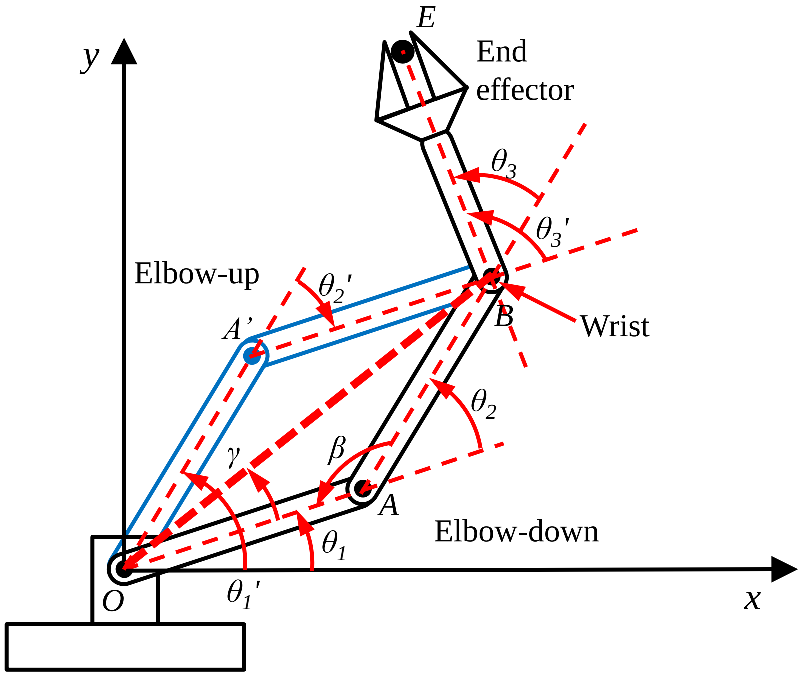
$$\alpha = tan^{-1}\frac{y_w}{x_w}$$

$$\boxed{\theta_1 = \alpha - \gamma = tan^{-1}\frac{y_w}{x_w} - cos^{-1}\frac{x_w^2 + y_w^2 + l_1^2 - l_2^2}{2l_1\sqrt{x_w^2 + y_w^2}}}$$

$$\boxed{\theta_3 = \varphi_e - \theta_1 - \theta_2}$$

2.2 Planar Kinematics

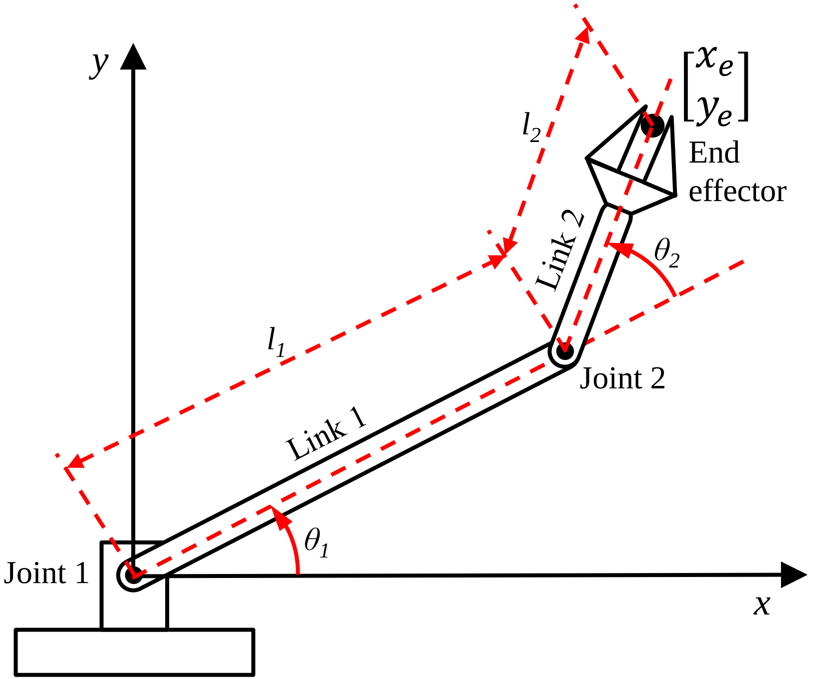
Inverse kinematics problems often have multiple solutions, as they are nonlinear. Specifying end-effecter position and orientation does not uniquely determine the whole configuration of the system.



$$\theta_1{}' = \theta_1 + 2\gamma$$
$$\theta_2{}' = -\theta_2$$
$$\theta_3' = \varphi_e - \theta_1{}' - \theta_2{}' = \theta_3 + 2\theta_2 - 2\gamma$$

2.2 Planar Kinematics

Differential relationship between the joint displacements and the end-effecter location coordinates joint motions, i.e. coordination of the motion of the individual joints moves the end-effecter in a specified direction at a specified speed.



$$x_e(\theta_1, \theta_2) = l_1 cos\theta_1 + l_2 cos(\theta_1 + \theta_2)$$
$$y_e(\theta_1, \theta_2) = l_1 sin\theta_1 + l_2 sin(\theta_1 + \theta_2)$$

Multivariable chain rule:

$$\frac{d}{dt} f\big(\boldsymbol{v}(t)\big) = \nabla f\big(\boldsymbol{v}(t)\big) \cdot \dot{\boldsymbol{v}}(t)$$

where $\boldsymbol{v}(t) = \begin{bmatrix} v_x(t) \\ v_y(t) \\ v_z(t) \end{bmatrix}$ is a vector field

2.2 Planar Kinematics

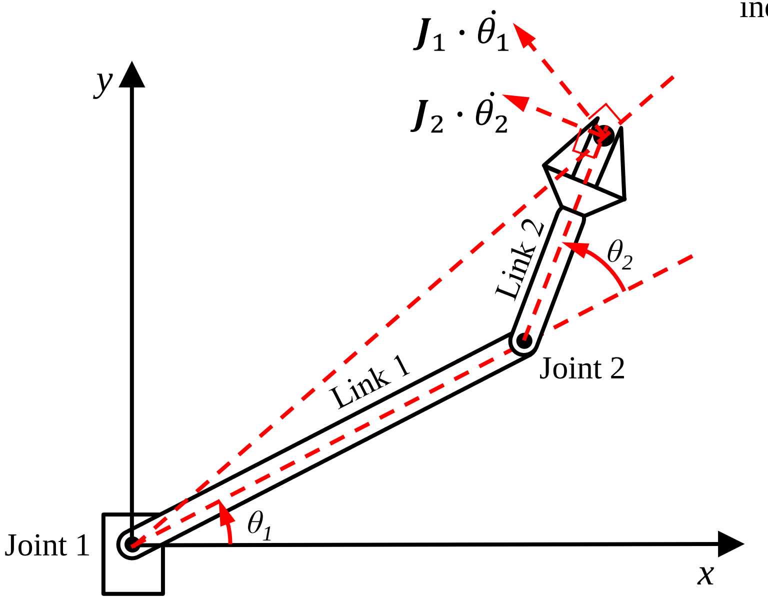
$$x_e(\theta_1,\theta_2) = l_1 cos\theta_1 + l_2 cos(\theta_1+\theta_2)$$
$$y_e(\theta_1,\theta_2) = l_1 sin\theta_1 + l_2 sin(\theta_1+\theta_2)$$

$$\frac{dx_e(\theta_1,\theta_2)}{dt} = \frac{\partial x_e(\theta_1,\theta_2)}{\partial\theta_1}\frac{d\theta_1}{dt} + \frac{\partial x_e(\theta_1,\theta_2)}{\partial\theta_2}\frac{d\theta_2}{dt}$$
$$\frac{dy_e(\theta_1,\theta_2)}{dt} = \frac{\partial y_e(\theta_1,\theta_2)}{\partial\theta_1}\frac{d\theta_1}{dt} + \frac{\partial y_e(\theta_1,\theta_2)}{\partial\theta_2}\frac{d\theta_2}{dt}$$

$$\boldsymbol{v_e} = \begin{bmatrix} \frac{dx_e(\theta_1,\theta_2)}{dt} \\ \frac{dy_e(\theta_1,\theta_2)}{dt} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x_e(\theta_1,\theta_2)}{\partial\theta_1} & \frac{\partial x_e(\theta_1,\theta_2)}{\partial\theta_2} \\ \frac{\partial y_e(\theta_1,\theta_2)}{\partial\theta_1} & \frac{\partial y_e(\theta_1,\theta_2)}{\partial\theta_2} \end{bmatrix}}_{\boldsymbol{J}} \cdot \underbrace{\begin{bmatrix} \frac{d\theta_1}{dt} \\ \frac{d\theta_2}{dt} \end{bmatrix}}_{\dot{\boldsymbol{q}}} = \boldsymbol{J}\cdot\dot{\boldsymbol{q}}$$

Jacobian Matrix: $\boldsymbol{J} = \begin{bmatrix} -l_1 sin\theta_1 - l_2 sin(\theta_1+\theta_2) & -l_2 sin(\theta_1+\theta_2) \\ l_1 cos\theta_1 + l_2 cos(\theta_1+\theta_2) & l_2 cos(\theta_1+\theta_2) \end{bmatrix}$

$$\boldsymbol{J} = \begin{bmatrix} \dfrac{\partial x_e(\theta_1,\theta_2)}{\partial\theta_1} & \dfrac{\partial x_e(\theta_1,\theta_2)}{\partial\theta_2} \\ \dfrac{\partial y_e(\theta_1,\theta_2)}{\partial\theta_1} & \dfrac{\partial y_e(\theta_1,\theta_2)}{\partial\theta_2} \end{bmatrix} = [\boldsymbol{J}_1 \quad \boldsymbol{J}_2] \Rightarrow \boldsymbol{v_e} = \underbrace{\boldsymbol{J}_1 \cdot \frac{d\theta_1}{dt}}_{\text{The end-effecter velocity induced by the first joint only}} + \underbrace{\boldsymbol{J}_2 \cdot \frac{d\theta_2}{dt}}_{\text{The end-effecter velocity induced by the second joint only}} = \boldsymbol{J}_1 \cdot \dot{\theta}_1 + \boldsymbol{J}_2 \cdot \dot{\theta}_2$$



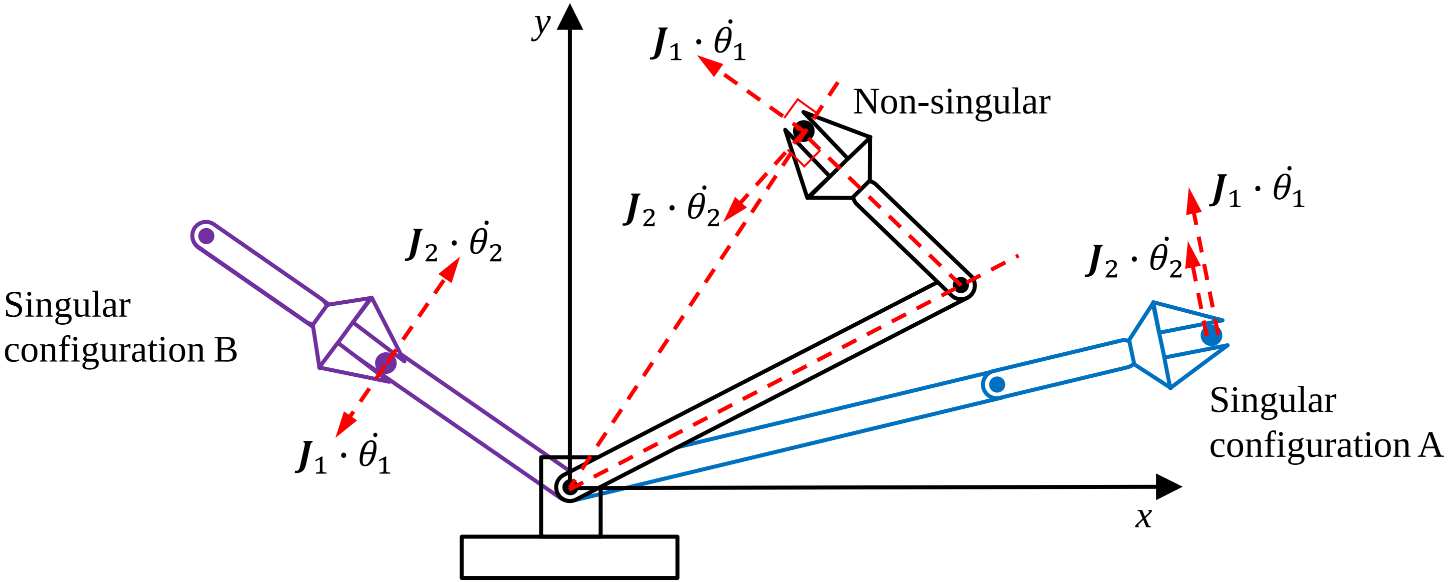
The end-effecter velocity is given by a linear combination of the Jacobian column vectors weighted by the individual joint velocities.

$$\boldsymbol{v_e} = \dot{\boldsymbol{p}} = \boldsymbol{J}_1 \cdot \dot{\theta}_1 + \boldsymbol{J}_2 \cdot \dot{\theta}_2 + \cdots + \boldsymbol{J}_n \cdot \dot{\theta}_n$$

2.2 Planar Kinematics

If the two vectors point in different directions, the whole two-dimensional space is covered with the linear combination of the two vectors.
If the two Jacobian column vectors are aligned (θ_2 is 0 or 180º), the end-effecter cannot be moved in an arbitrary direction. The Jacobian matrix becomes **singular** at these positions.

$$det\,\boldsymbol{J} = det\begin{bmatrix} -(l_1 \pm l_2)sin\theta_1 & \mp l_2 sin\theta_1 \\ (l_1 \pm l_2)cos\theta_1 & \pm l_2 cos\theta_1 \end{bmatrix} = 0$$



2.2 Planar Kinematics

Inverse kinematics needs to be formulated for obtaining the joint velocities that allow the end-effecter to move at a given desired velocity.

$$\dot{\boldsymbol{p}} = \boldsymbol{J} \cdot \dot{\boldsymbol{q}} \Rightarrow \dot{\boldsymbol{q}} = \boldsymbol{J}^{-\boldsymbol{1}} \cdot \dot{\boldsymbol{p}} \Rightarrow \begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix} = \boldsymbol{J}^{-\boldsymbol{1}} \begin{bmatrix} v_x \\ v_y \end{bmatrix}$$

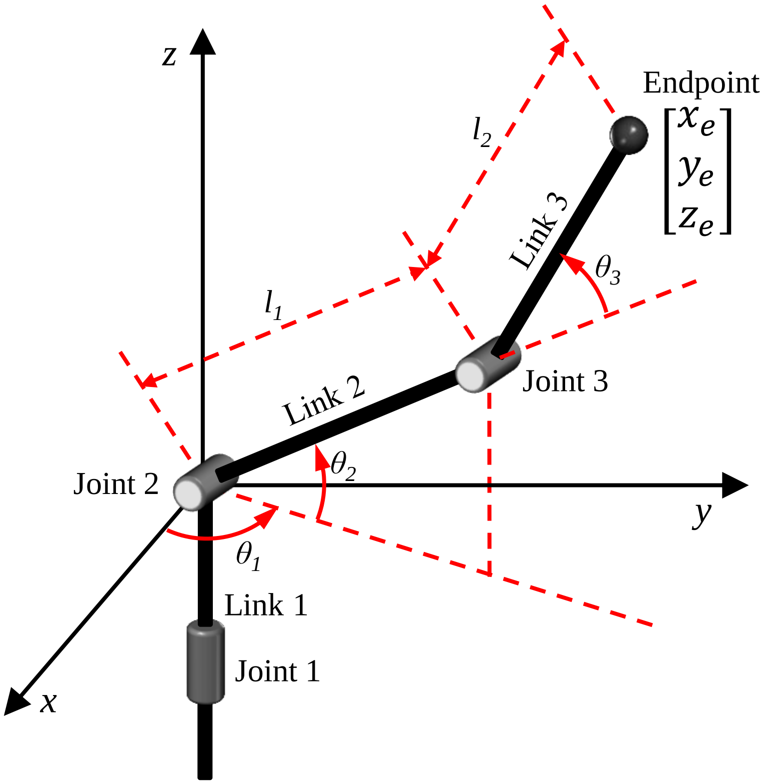
The differential kinematics problem has a unique solution as long as the Jacobian is non-singular.

$$\boldsymbol{J}^{-\boldsymbol{1}} = \frac{1}{det\,\boldsymbol{J}} adj(\boldsymbol{J}) = \frac{1}{l_1 l_2 sin(\theta_2)} \begin{bmatrix} l_2 cos(\theta_1 + \theta_2) & l_2 sin(\theta_1 + \theta_2) \\ -l_1 cos\theta_1 - l_2 cos(\theta_1 + \theta_2) & -l_1 sin\theta_1 - l_2 sin(\theta_1 + \theta_2) \end{bmatrix}$$

$$\dot{\theta}_1 = \frac{v_x cos(\theta_1 + \theta_2) + v_y sin(\theta_1 + \theta_2)}{l_1 sin(\theta_2)}$$

$$\dot{\theta}_2 = -\frac{v_x [l_1 cos\theta_1 + l_2 cos(\theta_1 + \theta_2)] + v_y [l_1 sin\theta_1 + sin(\theta_1 + \theta_2)]}{l_1 l_2 sin(\theta_2)}$$

2.2 Planar Kinematics



$$\begin{cases} x_e(\theta_1, \theta_2, \theta_3) = [l_1 cos\theta_2 + \ l_2 cos(\theta_2 + \theta_3)]cos\theta_1 \\ y_e(\theta_1, \theta_2, \theta_3) = [l_1 cos\theta_2 + \ l_2 cos(\theta_2 + \theta_3)]sin\theta_1 \\ z_e(\theta_2, \theta_3) = l_1 sin\theta_2 + \ l_2 sin(\theta_2 + \theta_3) \end{cases}$$

2.2 Planar Kinematics

$$\frac{dx_e(\theta_1,\theta_2,\theta_3)}{dt} = \frac{\partial x_e(\theta_1,\theta_2,\theta_3)}{\partial\theta_1}\frac{d\theta_1}{dt} + \frac{\partial x_e(\theta_1,\theta_2,\theta_3)}{\partial\theta_2}\frac{d\theta_2}{dt} + \frac{\partial x_e(\theta_1,\theta_2,\theta_3)}{\partial\theta_3}\frac{d\theta_3}{dt}$$

$$\frac{dy_e(\theta_1,\theta_2,\theta_3)}{dt} = \frac{\partial y_e(\theta_1,\theta_2,\theta_3)}{\partial\theta_1}\frac{d\theta_1}{dt} + \frac{\partial y_e(\theta_1,\theta_2,\theta_3)}{\partial\theta_2}\frac{d\theta_2}{dt} + \frac{\partial y_e(\theta_1,\theta_2,\theta_3)}{\partial\theta_3}\frac{d\theta_3}{dt}$$

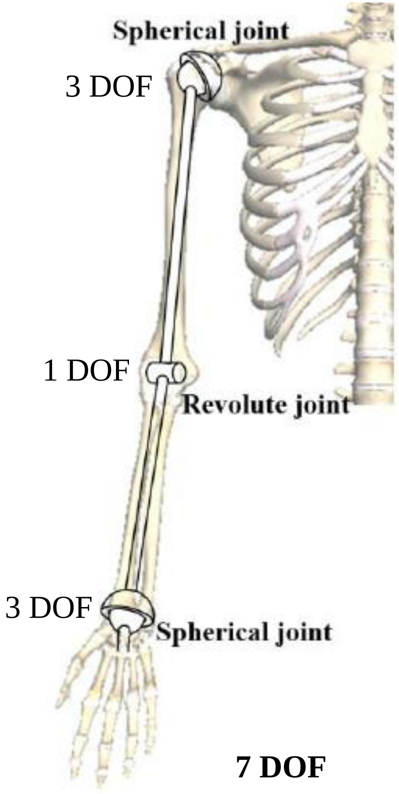
$$\frac{dz_e(\theta_2,\theta_3)}{dt} = \frac{\partial z_e(\theta_2,\theta_3)}{\partial\theta_2}\frac{d\theta_2}{dt} + \frac{\partial x_e(\theta_2,\theta_3)}{\partial\theta_3}\frac{d\theta_3}{dt}$$

$$\dot{\boldsymbol{p}} = \begin{bmatrix} -l_1 sin\theta_1 cos\theta_2 - \ l_2 sin\theta_1 cos(\theta_2+\theta_3) & -l_1 cos\theta_1 sin\theta_2 - \ l_2 cos\theta_1 sin(\theta_2+\theta_3) & -\ l_2 cos\theta_1 sin(\theta_2+\theta_3) \\ l_1 cos\theta_1 cos\theta_2 + l_2 cos\theta_1 cos(\theta_2+\theta_3) & -l_1 sin\theta_1 sin\theta_2 - \ l_2 sin\theta_1 sin(\theta_2+\theta_3) & -\ l_2 sin\theta_1 sin(\theta_2+\theta_3) \\ 0 & l_1 cos\theta_2 + l_2 cos(\theta_2+\theta_3) & l_2 cos(\theta_2+\theta_3) \end{bmatrix} \cdot \begin{bmatrix} \frac{d\theta_1}{dt} \\ \frac{d\theta_2}{dt} \\ \frac{d\theta_3}{dt} \end{bmatrix} = \boldsymbol{J} \cdot \dot{\boldsymbol{q}}$$

2.2 Planar Kinematics

If the arm configuration is non-singular, a unique solution exists to the differential kinematic equation. However, when a planar (spatial) robot arm has more than three (six) degrees of freedom, we can find an infinite number of solutions that provide the same motion at the end-effecter. Such a robot is referred to as a **redundant robot**.

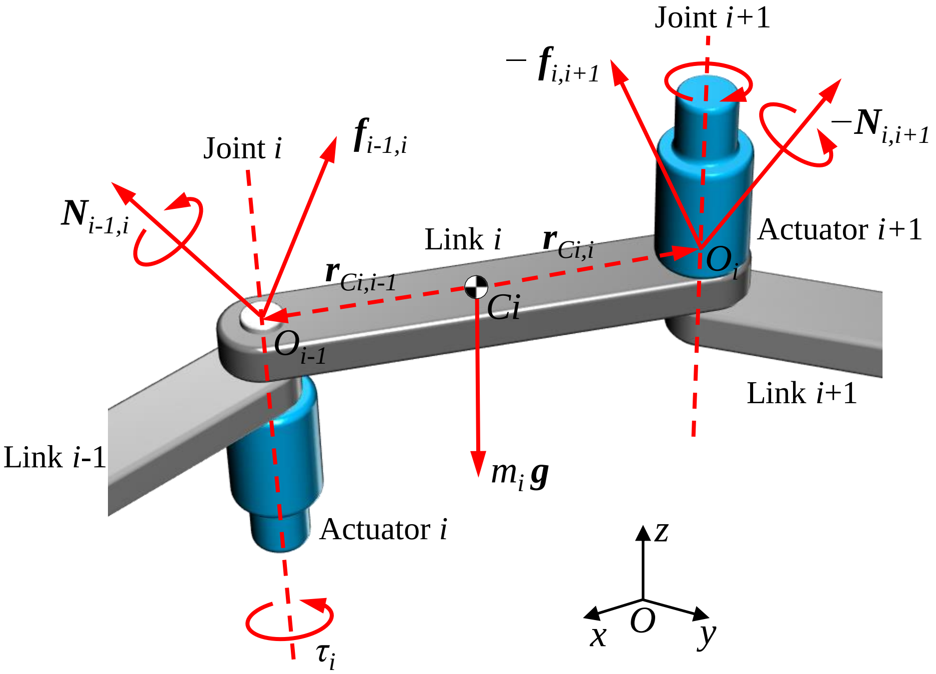
Consider for instance the human arm, which has seven degrees of freedom excluding the joints at the fingers. When the hand is placed on a desk and fixed in its position and orientation, the elbow position can still vary continuously without moving the hand.



Moon, H.,et al., Experimental observations on human reaching motion planning with and without reduced mobility. https://doi.org/10.1115/IMECE2012-87763

2.3 Statics

A robot generates a force and a moment at its end-effecter by controlling individual actuators. To generate a desired force and moment, the torques of the multiple actuators must be coordinated.



Through the connections with the adjacent links, link i receives forces and moments from both sides of the link.

$\boldsymbol{f}_{i-1,i}$: a 3D vector representing the linear force acting from link i-1 to link i

$\boldsymbol{f}_{i,i+1}$: the force from link i to link i+1

$m_i\,\boldsymbol{g}$: the gravity force acting at the mass centroid Ci, where $\boldsymbol{g}$ is the vector representing the acceleration of gravity

$\boldsymbol{N}_{i-1,i}$: the moment applied to link i by link i-1

$\boldsymbol{N}_{i,i+1}$: the moment applied to link i+1 by link I

$\boldsymbol{r}_{Ci,i-1}$; $\boldsymbol{r}_{Ci,i}$: position vectors from Ci to joints

Note that all the vectors are defined with respect to the **base coordinate system** *O-xyz*.

Balance of linear forces:

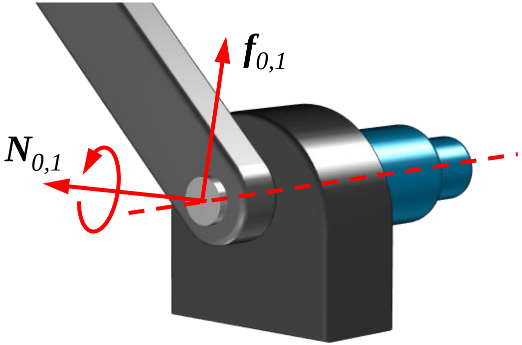
$$\boldsymbol{f}_{i-1,i} - \boldsymbol{f}_{i,i+1} + m_i \boldsymbol{g} = \boldsymbol{0}, \qquad i = 1, \cdots, n$$

See **Appendix 1** for the definition of the Moment of a force

Balance of moments with respect to the centroid *Ci* (take *O* as the origin):

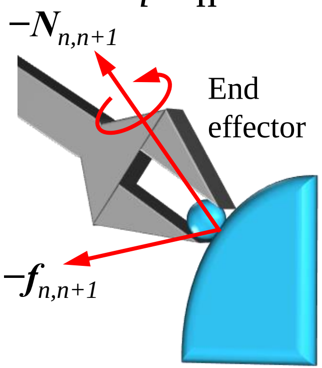
$$\boldsymbol{N}_{i-1,i} - \boldsymbol{N}_{i,i+1} + \boldsymbol{r}_{Ci,i-1} \times \boldsymbol{f}_{i-1,i} + \boldsymbol{r}_{Ci,i} \times \left(-\boldsymbol{f}_{i,i+1}\right) = \boldsymbol{0}, \qquad i = 1, \cdots, n$$

$i = 1$



Base link

i = n



Link n contacts the environment, which is regarded as an additional link, numbered n+1. The final coupling force $\boldsymbol{f}_{n,n+1}$ and moment $\boldsymbol{N}_{n,n+1}$ applied to the end-effecter perform a given task. We define the vector $\boldsymbol{F}$ as the endpoint force and moment vector:

$$\boldsymbol{F} = \begin{bmatrix} \boldsymbol{f}_{n,n+1} \\ \boldsymbol{N}_{n,n+1} \end{bmatrix}, \qquad \boldsymbol{F} \in \Re^{6\times1}$$

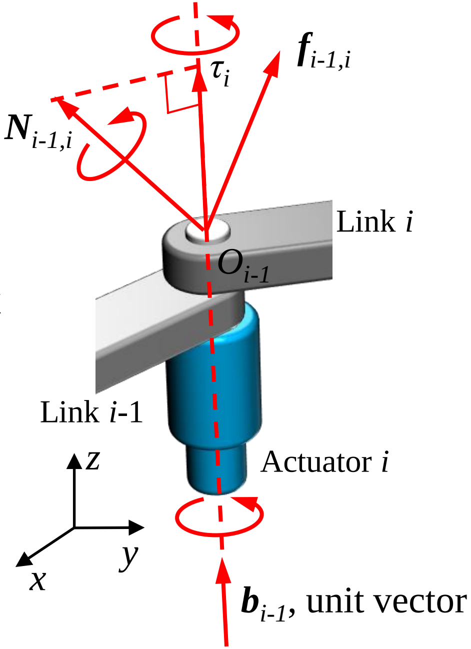
2.3 Statics

Since the simultaneous equations based on the balance of forces and moments are complex and difficult to solve, the **energy method** is the ideal choice when dealing with complex robotic systems. In the energy method, a system is described with respect to energy and work. Terms associated with forces and moments that do not produce, store, or dissipate energy are eliminated.

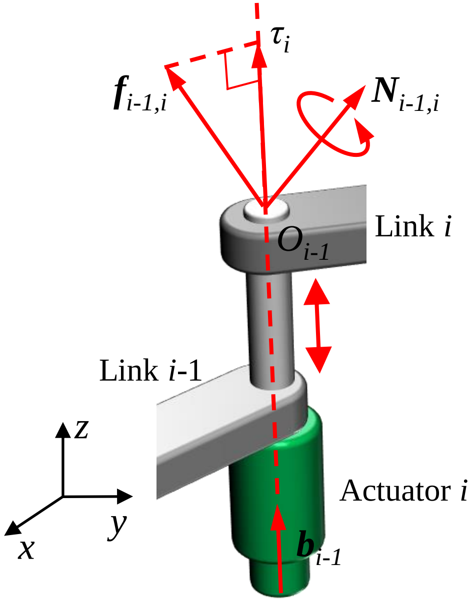
Revolute joint

$$\tau_i = \boldsymbol{b}_{i-1}{}^{T} \cdot \boldsymbol{N}_{i-1,i}$$

torque



Prismatic joint

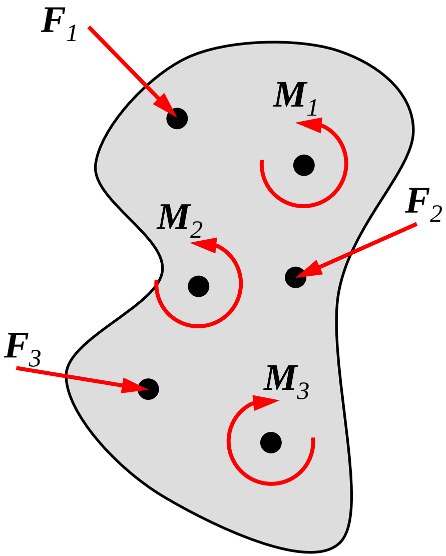


$$\tau_i = \boldsymbol{b}_{i-1}{}^{T} \cdot \boldsymbol{f}_{i-1,i}$$

force

Principle of Virtual Work

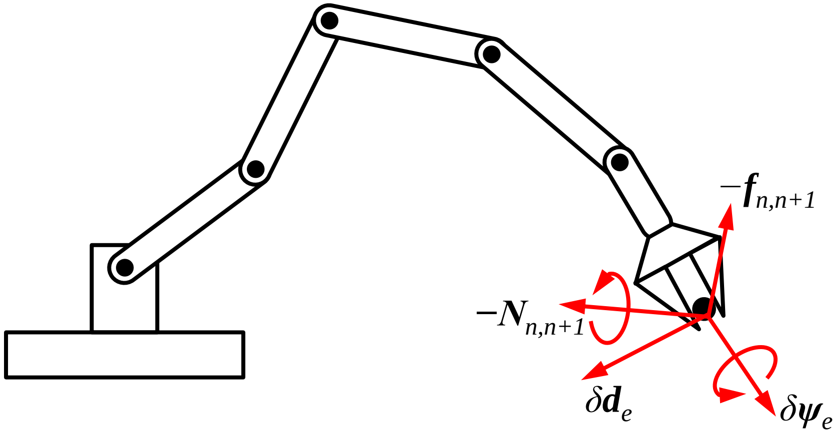
- Virtual displacements are **arbitrary infinitesimal** displacements of a mechanical system that conform to the geometric constraints of the system.
- The work done by all the forces acting on an **equilibrium** system, during a small virtual displacement is **ZERO**.



There is no displacement when the body is in equilibrium, so:

$$\delta work = F_1 \cdot \delta x_1 + F_2 \cdot \delta x_2 + F_3 \cdot \delta x_3 + M_1 \cdot \delta\theta_1 + M_2 \cdot \delta\theta_2 + M_3 \cdot \delta\theta_3 = 0$$

2.3 Statics



Using the *Principle of Virtual Work*. Consider virtual displacements $\delta\boldsymbol{q} = [\delta q_1 \ \ \delta q_2 \ \ \delta q_3 \cdots \delta q_n]^T$ at individual joints, and at the end-effecter $\delta\boldsymbol{p} = [\delta\boldsymbol{d}_e^T \ \ \delta\boldsymbol{\psi}_e^T]^T$.

The joint torques from joint 1 through joint n define the $n \times 1$ joint torque vector $\boldsymbol{\tau} = [\tau_1 \ \ \tau_2 \ \ \tau_3 \cdots \tau_n]^T$. The joint torque $\boldsymbol{\tau}$ and endpoint force and moment, $-\boldsymbol{F}$, act on the serial linkage system.

The virtual work done by the forces and moments is given by:

$$\begin{aligned}\delta work &= \tau_1 \cdot \delta q_1 + \tau_2 \cdot \delta q_2 + \cdots + \tau_n \cdot \delta q_n + \left(-\boldsymbol{f}_{n,n+1}\right)^T \cdot \delta\boldsymbol{d}_e + \left(-\boldsymbol{N}_{n,n+1}\right)^T \cdot \delta\boldsymbol{\psi}_e \\ &= \boldsymbol{\tau}^T \cdot \delta\boldsymbol{q} + (-\boldsymbol{F})^T \cdot \delta\boldsymbol{p}\end{aligned}$$

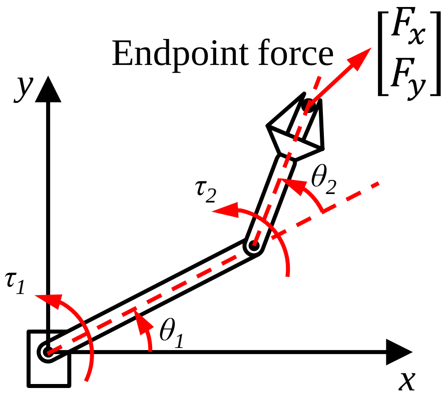
2.3 Statics

$$\boldsymbol{v_e} = \dot{\boldsymbol{p}} = \frac{\delta \boldsymbol{p}}{\delta t} = \boldsymbol{J} \cdot \dot{\boldsymbol{q}} = \boldsymbol{J} \cdot \frac{\delta \boldsymbol{q}}{\delta t} \Longrightarrow \delta \boldsymbol{p} = \boldsymbol{J} \cdot \delta \boldsymbol{q}$$

$$\delta work = \boldsymbol{\tau}^T \cdot \delta \boldsymbol{q} + (-\boldsymbol{F})^T \cdot \boldsymbol{J} \cdot \delta \boldsymbol{q} = (\boldsymbol{\tau}^T - \boldsymbol{F}^T \cdot \boldsymbol{J}) \cdot \delta \boldsymbol{q} = (\boldsymbol{\tau} - \boldsymbol{J}^T \cdot \boldsymbol{F})^T \cdot \delta \boldsymbol{q}$$

In a static situation, for the virtual work ($\delta work$) to vanish for arbitrary virtual displacements, we must have:

$$\boldsymbol{\tau} - \boldsymbol{J}^T \cdot \boldsymbol{F} = \boldsymbol{0} \Longrightarrow \boxed{\boldsymbol{\tau} = \boldsymbol{J}^T \cdot \boldsymbol{F}}$$



$$\boldsymbol{\tau} = \begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \boldsymbol{J}^T \cdot \boldsymbol{F} =$$

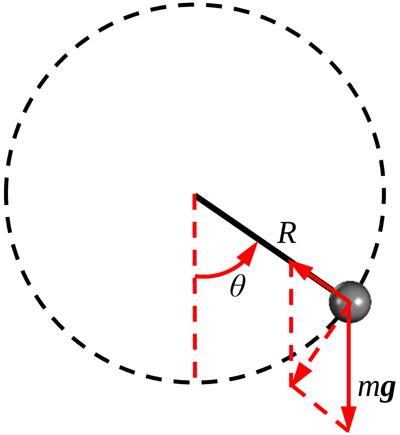
$$\begin{bmatrix} -l_1 sin\theta_1 - l_2 sin(\theta_1 + \theta_2) & l_1 cos\theta_1 + l_2 cos(\theta_1 + \theta_2) \\ -l_2 sin(\theta_1 + \theta_2) & l_2 cos(\theta_1 + \theta_2) \end{bmatrix} \cdot \begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

2.4 Dynamics

The dynamic behavior is described in terms of the time rate of change of the robot configuration in relation to the joint torques exerted by the actuators. This relationship can be expressed by a set of differential equations, called *equations of motion*. Two methods can be used in order to obtain the equations of motion: the *Newton-Euler formulation*, and the *Lagrangian formulation*.

D'Alembert's principle is the principle of virtual work with the inertial forces ($\boldsymbol{F}^*_j$) added to the list of forces that do work ($\boldsymbol{F}_i$):

$$\delta work = \sum_i \boldsymbol{F}_i \cdot \delta \boldsymbol{s}_i + \sum_j \boldsymbol{F}_j^* \cdot \delta \boldsymbol{s}_j = 0$$



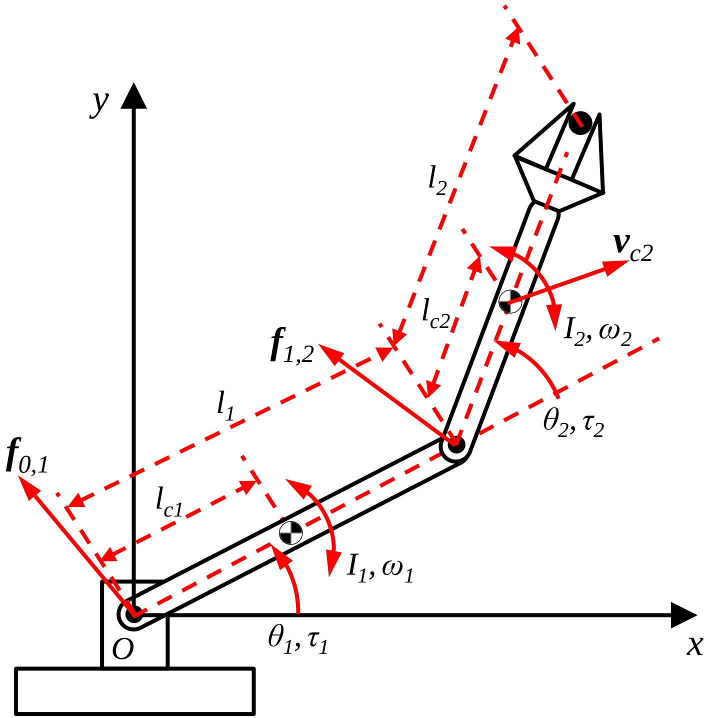
Force that do work: $m\boldsymbol{g}sin\theta$

Inertial force: $-mR\dfrac{d^2\theta}{dt^2}$

$$\left(m\boldsymbol{g}sin\theta - mR\frac{d^2\theta}{dt^2}\right)R\delta\theta = 0$$

$$\frac{\boldsymbol{g}sin\theta}{R} - \frac{d^2\theta}{dt^2} = 0$$

2.4 Dynamics



$\boldsymbol{v}_{ci}$: velocity of the centroid of link i
I_i: moment of inertia of link i
ω_i: angular velocity of the centroid of link i
m_i: mass of link i

Newton-Euler formulation:

$$\text{L1}\begin{cases} \boldsymbol{f}_{0,1} - \boldsymbol{f}_{1,2} + m_1\boldsymbol{g} - m_1\dot{\boldsymbol{v}}_{c1} = \boldsymbol{0}, \\ \boldsymbol{N}_{0,1} - \boldsymbol{N}_{1,2} + \boldsymbol{r}_{c1,O} \times \boldsymbol{f}_{0,1} + \boldsymbol{r}_{c1,1} \times (-\boldsymbol{f}_{1,2}) - I_1\dot{\omega}_1 = 0 \end{cases}$$

$$\text{L2}\begin{cases} \boldsymbol{f}_{1,2} + m_2\boldsymbol{g} - m_2\dot{\boldsymbol{v}}_{c2} = \boldsymbol{0}, \\ \boldsymbol{N}_{1,2} + \boldsymbol{r}_{c2,1} \times \boldsymbol{f}_{1,2} - I_2\dot{\omega}_2 = 0 \end{cases}$$

For the planar manipulator: $\boldsymbol{N}_{0,1} = \tau_1$ *and* $\boldsymbol{N}_{1,2} = \tau_2$

$\boldsymbol{f}_{1,2} = m_2\dot{\boldsymbol{v}}_{c2} - m_2\boldsymbol{g}$ and $\boldsymbol{f}_{0,1} = \boldsymbol{f}_{1,2} - m_1\boldsymbol{g} + m_1\dot{\boldsymbol{v}}_{c1}$

$$\tau_2 + \boldsymbol{r}_{c2,1} \times m_2\dot{\boldsymbol{v}}_{c2} - \boldsymbol{r}_{c2,1} \times m_2\boldsymbol{g} - I_2\dot{\omega}_2 = 0$$

$$\begin{aligned} &\tau_1 - \tau_2 + \boldsymbol{r}_{c1,O} \times (m_2\dot{\boldsymbol{v}}_{c2} - m_2\boldsymbol{g} - m_1\boldsymbol{g} + m_1\dot{\boldsymbol{v}}_{c1}) \\ &+ \boldsymbol{r}_{c1,1} \times (m_2\boldsymbol{g} - m_2\dot{\boldsymbol{v}}_{c2}) - I_1\dot{\omega}_1 \\ &= \tau_1 - \tau_2 + \boldsymbol{r}_{O,1} \times (m_2\boldsymbol{g} - m_2\dot{\boldsymbol{v}}_{c2}) - \boldsymbol{r}_{c1,O} \\ &\times (m_1\boldsymbol{g} - m_1\dot{\boldsymbol{v}}_{c1}) - I_1\dot{\omega}_1 = 0 \end{aligned}$$

2.4 Dynamics

$\omega_1 = \dot{\theta}_1$ and $\omega_2 = \dot{\theta}_1 + \dot{\theta}_2$

$$\boldsymbol{c}1 = \begin{bmatrix} l_{c1}cos\theta_1 \\ l_{c1}sin\theta_1 \end{bmatrix} \Longrightarrow \boldsymbol{v}_{c1} = \begin{bmatrix} -l_{c1}\dot{\theta}_1 sin\theta_1 \\ l_{c1}\dot{\theta}_1 cos\theta_1 \end{bmatrix} \Longrightarrow \dot{\boldsymbol{v}}_{c1} = \boxed{\begin{bmatrix} -l_{c1}\ddot{\theta}_1 sin\theta_1 - l_{c1}\dot{\theta}_1{}^2 cos\theta_1 \\ l_{c1}\ddot{\theta}_1 cos\theta_1 - l_{c1}\dot{\theta}_1{}^2 sin\theta_1 \end{bmatrix}}$$

$$\boldsymbol{c}2 = \begin{bmatrix} l_1 cos\theta_1 + l_{c2}cos(\theta_1+\theta_2) \\ l_1 sin\theta_1 + l_{c2}sin(\theta_1+\theta_2) \end{bmatrix} \Longrightarrow \boldsymbol{v}_{c2} = \begin{bmatrix} -\{l_1 sin\theta_1 + l_{c2}sin(\theta_1+\theta_2)\}\dot{\theta}_1 - l_{c2}sin(\theta_1+\theta_2)\dot{\theta}_2 \\ \{l_1 cos\theta_1 + l_{c2}cos(\theta_1+\theta_2)\}\dot{\theta}_1 + l_{c2}cos(\theta_1+\theta_2)\dot{\theta}_2 \end{bmatrix} \Longrightarrow$$

$$\boxed{\dot{\boldsymbol{v}}_{c2} = \begin{bmatrix} -l_1\left(\ddot{\theta}_1 sin\theta_1 + \dot{\theta}_1{}^2 cos\theta_1\right) - l_{c2}(\ddot{\theta}_1 + \ddot{\theta}_2)sin(\theta_1+\theta_2) - l_{c2}\left(\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + 2\dot{\theta}_1\dot{\theta}_2\right)cos(\theta_1+\theta_2) \\ l_1\left(\ddot{\theta}_1 cos\theta_1 - \dot{\theta}_1{}^2 sin\theta_1\right) - l_{c2}\left(\dot{\theta}_1{}^2 + \dot{\theta}_2{}^2 + 2\dot{\theta}_1\dot{\theta}_2\right)sin(\theta_1+\theta_2) + l_{c2}(\ddot{\theta}_1 + \ddot{\theta}_2)cos(\theta_1+\theta_2) \end{bmatrix}}$$

$$\boxed{\begin{aligned} \tau_1 &= H_{11}\ddot{\theta}_1 + H_{12}\ddot{\theta}_2 - h\dot{\theta}_2{}^2 - 2h\dot{\theta}_1\dot{\theta}_2 + G_1 \\ \tau_2 &= H_{22}\ddot{\theta}_2 + H_{21}\ddot{\theta}_1 - h\dot{\theta}_1\dot{\theta}_2 + G_2 \end{aligned}}$$

$$H_{11} = m_1 l_{c1}{}^2 + I_1 + m_2\left(l_1{}^2 + l_{c2}{}^2 + 2l_1 l_{c2}cos\theta_2\right) + I_2$$

$$H_{22} = m_2 l_{c2}{}^2 + I_2$$

$$H_{12} = H_{21} = m_2\left(l_{c2}{}^2 + l_1 l_{c2}cos\theta_2\right) + I_2$$

$$h = m_2 l_1 l_{c2} sin\theta_2$$

$$G_1 = m_1 l_{c1} g cos\theta_1 + m_2 g\{l_{c2}cos(\theta_1+\theta_2) + l_1 cos\theta_1\}$$

$$G_2 = m_2 g l_{c2} cos(\theta_1+\theta_2)$$

2.4 Dynamics

In the *Newton-Euler formulation*, the equations of motion relates force and momentum, as well as torque and angular momentum. In the Newton-Euler formulation, the equations are not expressed in terms of independent variables, and do not include input joint torques explicitly. Complex arithmetic operations are needed to derive the closed-form dynamic equations.

An alternative to the Newton-Euler formulation of manipulator dynamics is the ***Lagrangian formulation***, which describes the behavior of a dynamic system in terms of work and energy stored in the system. Let $q_1, \cdots, q_n$ be generalized coordinates that completely locate a dynamic system. Let T and U be the total kinetic energy and potential energy stored in the dynamic system. We define the Lagrangian L by:

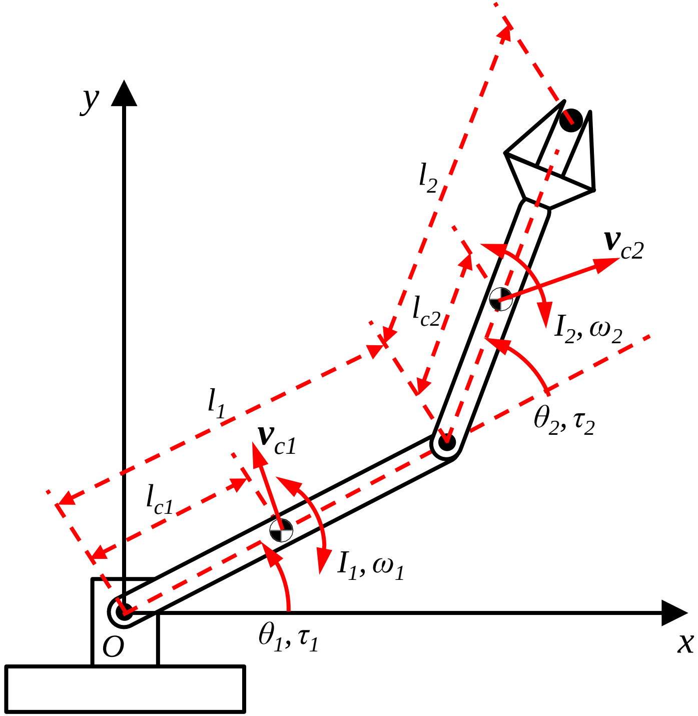
$$L(q_i, \dot{q}_i) = T(q_i, \dot{q}_i) - U(q_i)$$

See **Appendix 2** for the deduction of the *Lagrangian dynamic equations*

Equations of motion of the dynamic system are given by:

$$\frac{d}{dt}\frac{\partial L(q_i, \dot{q}_i)}{\partial \dot{q}_i} - \frac{\partial L(q_i, \dot{q}_i)}{\partial q_i} = Q_i, \qquad i = 1, \cdots, n$$

where Q_i is the generalized force corresponding to the generalized coordinate q_i.



The total kinetic energy stored in the two links moving at linear velocity $\boldsymbol{v}_{ci}$ and angular velocity ω_i at the centroids is given by:

$$T = \sum_{i=1}^{2} \left(\frac{1}{2} m_i |\boldsymbol{v}_{ci}|^2 + \frac{1}{2} I_i {\omega_i}^2 \right)$$

where $\omega_1 = \dot{\theta}_1, \qquad \omega_2 = \dot{\theta}_1 + \dot{\theta}_2$

$$|\boldsymbol{v}_{c1}|^2 = \left(l_{c1}\dot{\theta}_1\right)^2, \quad |\boldsymbol{v}_{c2}|^2 = |\boldsymbol{J}_{c2}\dot{\boldsymbol{q}}|^2 = (\boldsymbol{J}_{c2}\dot{\boldsymbol{q}})^T \boldsymbol{J}_{c2}\dot{\boldsymbol{q}}$$

$$\boldsymbol{v}_{c2} = \underbrace{\begin{bmatrix} -l_1 sin\theta_1 - l_{c2} sin(\theta_1+\theta_2) & -l_{c2} sin(\theta_1+\theta_2) \\ l_1 cos\theta_1 + l_{c2} cos(\theta_1+\theta_2) & l_{c2} cos(\theta_1+\theta_2) \end{bmatrix}}_{\boldsymbol{J}_{c2}} \underbrace{\begin{bmatrix} \dot{\theta}_1 \\ \dot{\theta}_2 \end{bmatrix}}_{\dot{\boldsymbol{q}}}$$

2.4 Dynamics

$$T=\frac{1}{2}H_{11}\dot{\theta}_1^{\,2}+H_{12}\dot{\theta}_1\dot{\theta}_2+\frac{1}{2}H_{22}\dot{\theta}_2^{\,2}=\frac{1}{2}\begin{bmatrix}\dot{\theta}_1 & \dot{\theta}_2\end{bmatrix}\begin{bmatrix}H_{11} & H_{12}\\ H_{21} & H_{22}\end{bmatrix}\begin{bmatrix}\dot{\theta}_1\\ \dot{\theta}_2\end{bmatrix}$$

$$H_{11}=m_1 l_{c1}^{\,2}+I_1+m_2\left(l_1^{\,2}+l_{c2}^{\,2}+2l_1 l_{c2}cos\theta_2\right)+I_2$$

$$H_{22}=m_2 l_{c2}^{\,2}+I_2$$

$$H_{12}=H_{21}=m_2\left(l_{c2}^{\,2}+l_1 l_{c2}cos\theta_2\right)+I_2$$

The potential energy stored in the two links is given by:

$$U=m_1 g l_{c1}sin\theta_1+m_2 g[l_1 sin\theta_1+l_{c2}sin(\theta_1+\theta_2)]$$

Since we know *T* and *U*, now we are ready to obtain Lagrange's equations of motion. For the first link:

$$\boxed{\frac{\partial L}{\partial q_1}=-\frac{\partial U}{\partial \theta_1}=-[m_1 l_{c1} g cos\theta_1+m_2 g\{l_{c2}cos(\theta_1+\theta_2)+l_1 cos\theta_1\}]=-G_1}$$

$$\frac{\partial L}{\partial \dot{q}_1}=\frac{\partial T}{\partial \dot{\theta}_1}=H_{11}\dot{\theta}_1+H_{12}\dot{\theta}_2$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right)=H_{11}\ddot{\theta}_1+H_{12}\ddot{\theta}_2+\frac{\partial H_{11}}{\partial \theta_2}\dot{\theta}_1\dot{\theta}_2+\frac{\partial H_{12}}{\partial \theta_2}\dot{\theta}_2^{\,2}}$$

2.4 Dynamics

For the second link:

$$\boxed{\frac{\partial L}{\partial q_2} = -\frac{\partial U}{\partial \theta_2} = -m_2 g l_{c2} cos(\theta_1 + \theta_2) = -G_2}$$

$$\frac{\partial L}{\partial \dot{q}_2} = \frac{\partial L}{\partial \dot{\theta}_2} = H_{22}\dot{\theta}_2 + H_{12}\dot{\theta}_1$$

$$\boxed{\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) = H_{12}\ddot{\theta}_1 + H_{22}\ddot{\theta}_2 + \frac{\partial H_{12}}{\partial \theta_2}\dot{\theta}_1\dot{\theta}_2}$$

Substituting the above equations for joints and 1 and 2 into Lagrangian equations of motion, we can obtain:

$$\tau_1 = Q_1 = H_{11}\ddot{\theta}_1 + H_{12}\ddot{\theta}_2 - m_2 l_1 l_{c2}\dot{\theta}_2^{\,2} sin\theta_2 - 2m_2 l_1 l_{c2}\dot{\theta}_1\dot{\theta}_2 sin\theta_2 + G_1$$

$$\tau_2 = Q_2 = H_{22}\ddot{\theta}_2 + H_{21}\ddot{\theta}_1 - m_2 l_1 l_{c2}\dot{\theta}_1\dot{\theta}_2 sin\theta_2 + G_2$$

The results are identical to the equations obtained using Newton-Euler formulation. However, the Lagrangian Formulation is simpler and more systematic than the Newton-Euler Formulation. The difference between the two methods is more significant when the degrees of freedom increase.

2.4 Dynamics

When extend Lagrange's equations of motion to the ones for a general n d.o.f. robot:

$$T = \sum_{i=1}^{n} \left(\frac{1}{2} m_i \boldsymbol{v}_{ci}{}^T \boldsymbol{v}_{ci} + \frac{1}{2} \boldsymbol{\omega}_i{}^T I_i \boldsymbol{\omega}_i \right), \qquad i = 1, \cdots, n$$

where $\boldsymbol{\omega}_i$ and $\boldsymbol{I}_i$ are, respectively, the 3×1 angular velocity vector and the 3×3 inertia matrix of the i-th link viewed from the base coordinate frame, i.e. inertial reference. We next use the Jacobian matrix relating the centroid velocity to joint velocities for rewriting the centroidal velocity and angular velocity for three-dimensional multi-body systems.

$$\boldsymbol{v}_{ci} = \boldsymbol{J}_i^L \dot{\boldsymbol{q}}, \qquad \boldsymbol{\omega}_i = \boldsymbol{J}_i^A \dot{\boldsymbol{q}} \qquad \text{where} \qquad \boldsymbol{q} = [q_1, \cdots, q_n]^T$$

where $\boldsymbol{J}_i^L$ and $\boldsymbol{J}_i^A$ are, respectively, the $3 \times n$ Jacobian matrices relating the centroid linear velocity and the angular velocity of the i-th link to joint velocities.

$$\boxed{T = \frac{1}{2} \sum_{i=1}^{n} \left\{ m_i \dot{\boldsymbol{q}}^T \boldsymbol{J}_i^{L^T} \boldsymbol{J}_i^L \dot{\boldsymbol{q}} + \dot{\boldsymbol{q}}^T \boldsymbol{J}_i^{A^T} I_i \boldsymbol{J}_i^A \dot{\boldsymbol{q}} \right\} = \frac{1}{2} \dot{\boldsymbol{q}}^T \boldsymbol{H} \dot{\boldsymbol{q}}} \quad \text{where} \quad \boldsymbol{H} = \underbrace{\sum_{i=1}^{n} \left(m_i \boldsymbol{J}_i^{L^T} \boldsymbol{J}_i^L + \boldsymbol{J}_i^{A^T} I_i \boldsymbol{J}_i^A \right)}_{\textit{Multi-Body Inertia Matrix}}, \boldsymbol{H} \in \Re^{n \times n}$$

2.4 Dynamics

The quadratic form associated with the *multi-body inertia matrix* ***H*** represents kinetic energy, which is a *symmetric matrix* and is always strictly positive unless the system is at rest. ***H*** involves Jacobian matrices, which vary with linkage configuration and is configuration-dependent. ***H*** can be rewritten as $H(\boldsymbol{q})$, a function of joint coordinates $\boldsymbol{q}$. Using the components of the multi-body inertia matrix $\boldsymbol{H}=\{H_{ij}\}$, we can write the total kinetic energy in scalar quadratic form:

$$T = \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} H_{ij}\dot{q}_i\dot{q}_j$$

To formulate Lagrange's equations of motion:

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} = \frac{d}{dt}\underbrace{\left(\sum_{j=1}^{n} H_{ij}\dot{q}_j\right)}_{\textit{Since } \boldsymbol{H} \textit{ is symmetric}} = \sum_{j=1}^{n} H_{ij}\ddot{q}_j + \sum_{j=1}^{n}\frac{dH_{ij}}{dt}\dot{q}_j \quad \text{where} \quad \frac{dH_{ij}}{dt} = \sum_{k=1}^{n}\frac{\partial H_{ij}}{\partial q_k}\dot{q}_k$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} = \sum_{j=1}^{n} H_{ij}\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} \frac{\partial H_{ij}}{\partial q_k}\dot{q}_k\dot{q}_j \quad \text{and} \quad \frac{\partial T}{\partial q_i} = \frac{\partial}{\partial q_i}\left(\frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n} H_{jk}\dot{q}_j\dot{q}_k\right) = \frac{1}{2}\sum_{j=1}^{n}\sum_{k=1}^{n}\frac{\partial H_{jk}}{\partial q_i}\dot{q}_j\dot{q}_k$$

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}_i} - \frac{\partial T}{\partial q_i} = \sum_{j=1}^{n} H_{ij}\ddot{q}_j + \sum_{j=1}^{n}\sum_{k=1}^{n} \underbrace{\left(\frac{\partial H_{ij}}{\partial q_k} - \frac{1}{2}\frac{\partial H_{jk}}{\partial q_i}\right)}_{C_{ijk}}\dot{q}_j\dot{q}_k$$

C_{ijk}: Christoffel's Three-Index Symbol

Conservative forces is given by partial derivatives of potential energy U in Lagrange's equations. If gravity is the only conservative force, the total potential energy stored in n links is given by:

$$U = -\sum_{i=1}^{n} m_i \boldsymbol{g}^T \boldsymbol{r}_{o,ci}$$

where $\boldsymbol{r}_{o,ci}$ is the position vector of the centroid Ci that is dependent on joint coordinates

2.4 Dynamics

Substituting this potential energy U into Lagrange's equations of motion yields the following gravity torque seen by the i-th joint:

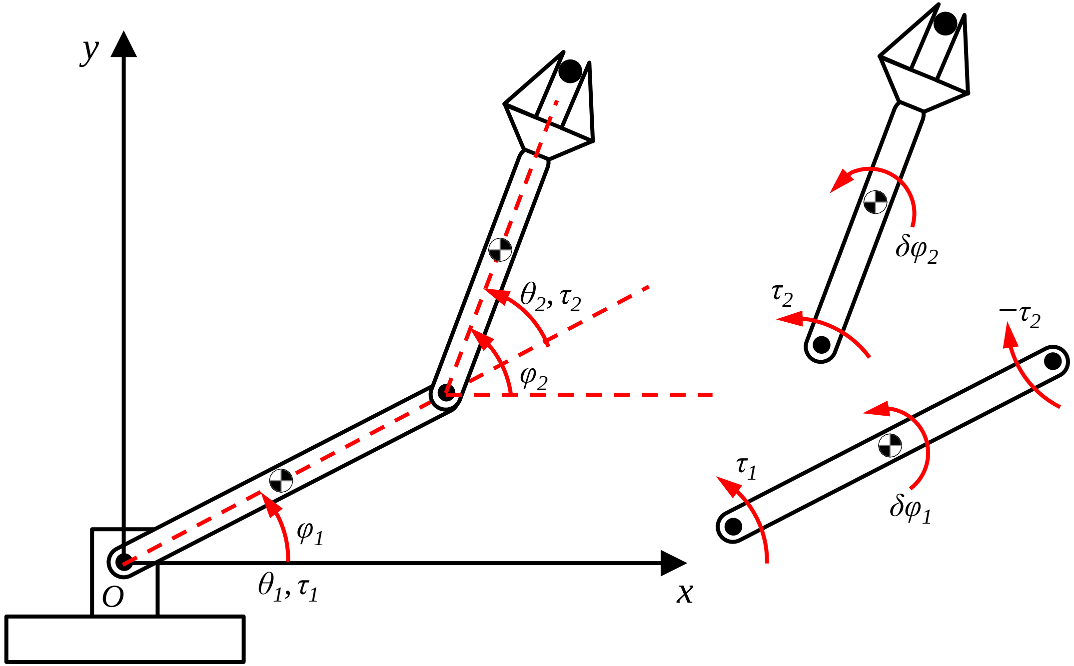
$$G_i = \frac{\partial U}{\partial q_i} = -\sum_{j=1}^{n} m_j \boldsymbol{g}^T \frac{\partial \boldsymbol{r}_{o,ci}}{\partial q_i} = -\sum_{j=1}^{n} m_j \boldsymbol{g}^T \boldsymbol{J}_{j,i}^L$$

where $\boldsymbol{J}_{j,i}^L$ is the i-th column vector of the 3×1 Jacobian matrix relating the linear centroid velocity of the j-th link to joint velocities.

Non-conservative forces acting on the robot mechanism are represented by generalized forces Q_i in Lagrangian formulation. Let $\delta work$ be virtual work done by all the non-conservative forces acting on the system. Generalized forces Q_i associated with generalized coordinates q_i, e.g. joint coordinates, are defined by:

$$\delta work = \sum_{i=1}^{n} Q_i \delta q_i$$

If $\delta work$ is given by the inner product of joint torques and virtual joint displacements, $\tau_1\delta_1 + \cdots + \tau_n\delta_n$, the joint torque itself is the generalized force corresponding to the joint coordinate.

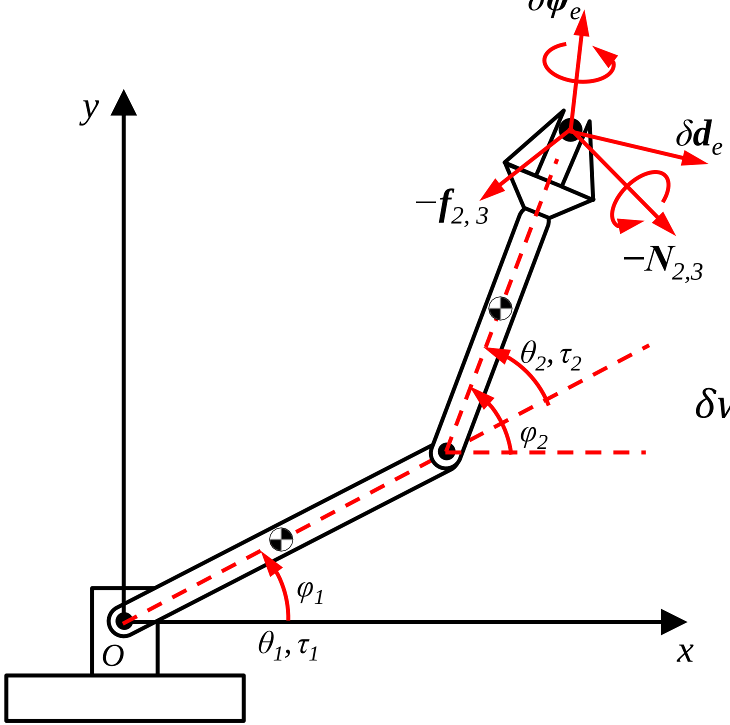


$$\delta work = (\tau_1 - \tau_2)\delta\varphi_1 + \tau_2\delta\varphi_2$$

$$\varphi_1 = \theta_1, \qquad \varphi_2 = \theta_1 + \theta_2$$

$$\delta work = \tau_1\delta\theta_1 + \tau_2\delta\theta_2$$

This example shows that the generalized forces associated with the original generalized coordinates, i.e. joint coordinates, are τ_1 and τ_2.



Non-conservative forces acting on a robot mechanism include not only these joint torques but also any other external force $\boldsymbol{F}_{ext}$ (acts on the endpoint in this example). Consider virtual displacements $\delta\boldsymbol{q} = [\delta q_1 \;\; \delta q_2]^{\mathrm{T}}$ at individual joints, and at the end-effecter $\delta\boldsymbol{p} = [\delta\boldsymbol{d}_e^{\mathrm{T}} \;\; \delta\boldsymbol{\psi}_e^{\mathrm{T}}]^{\mathrm{T}}$, The virtual work is obtained as:

$$\delta work = \tau_1 \cdot \delta q_1 + \tau_2 \cdot \delta q_2 + \left(-\boldsymbol{f}_{2,3}\right)^T \cdot \delta\boldsymbol{d}_e + \left(-\boldsymbol{N}_{2,3}\right)^T \cdot \delta\boldsymbol{\psi}_e$$

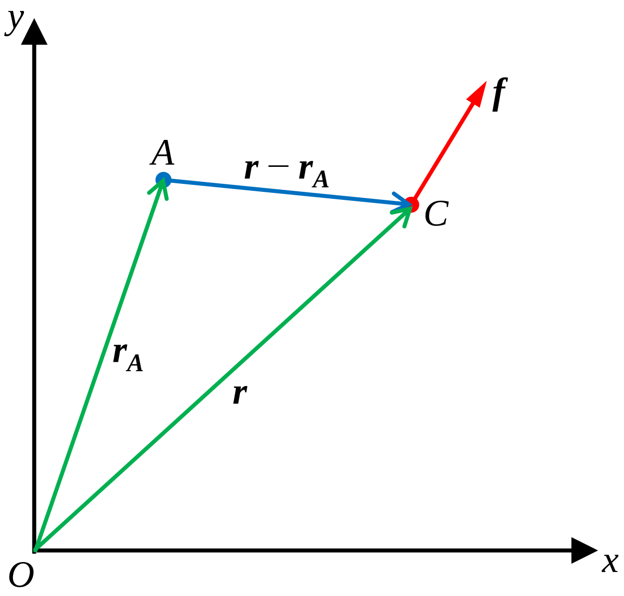
In a more general form with n links:

$$\delta work = \boldsymbol{\tau}^T \cdot \delta\boldsymbol{q} + \boldsymbol{F}_{ext}{}^T \cdot \delta\boldsymbol{p} = \left(\boldsymbol{\tau} + \boldsymbol{J}^T\boldsymbol{F}_{ext}\right)^T \delta\boldsymbol{q} = \boldsymbol{Q}^T\delta\boldsymbol{q}$$

where $\boldsymbol{F}_{ext} = \left(-\boldsymbol{f}_{n,n+1}\right)^T \cdot \delta\boldsymbol{d}_e + \left(-\boldsymbol{N}_{n,n+1}\right)^T \cdot \delta\boldsymbol{\psi}_e$

$$\boxed{\boldsymbol{Q} = \boldsymbol{\tau} + \boldsymbol{J}^T\boldsymbol{F}_{ext}}$$

Appendix 1: Moment of a force

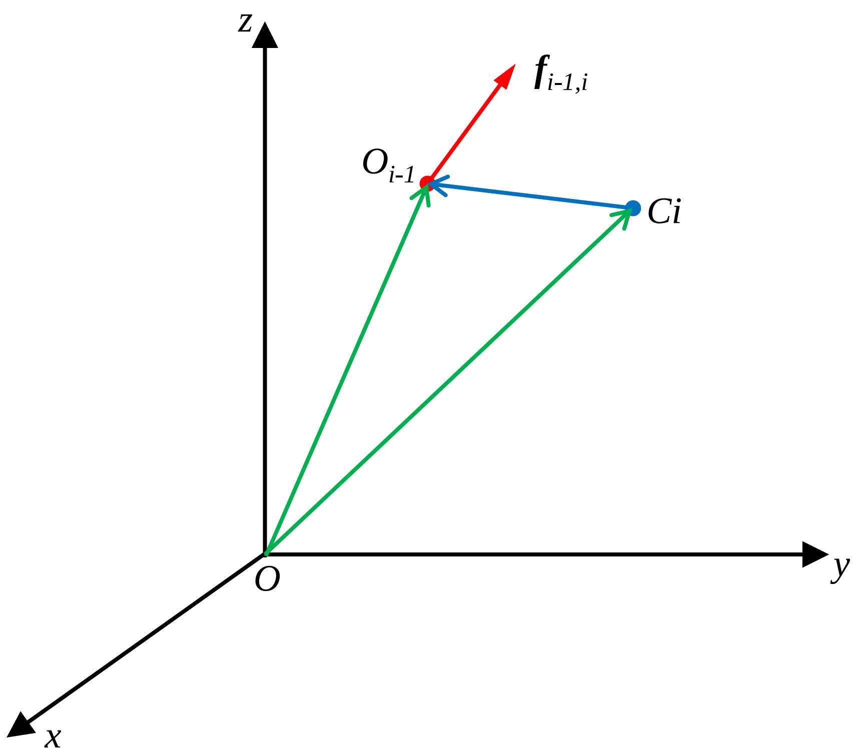


The moment of $\boldsymbol{f}$ about point A is then defined as:

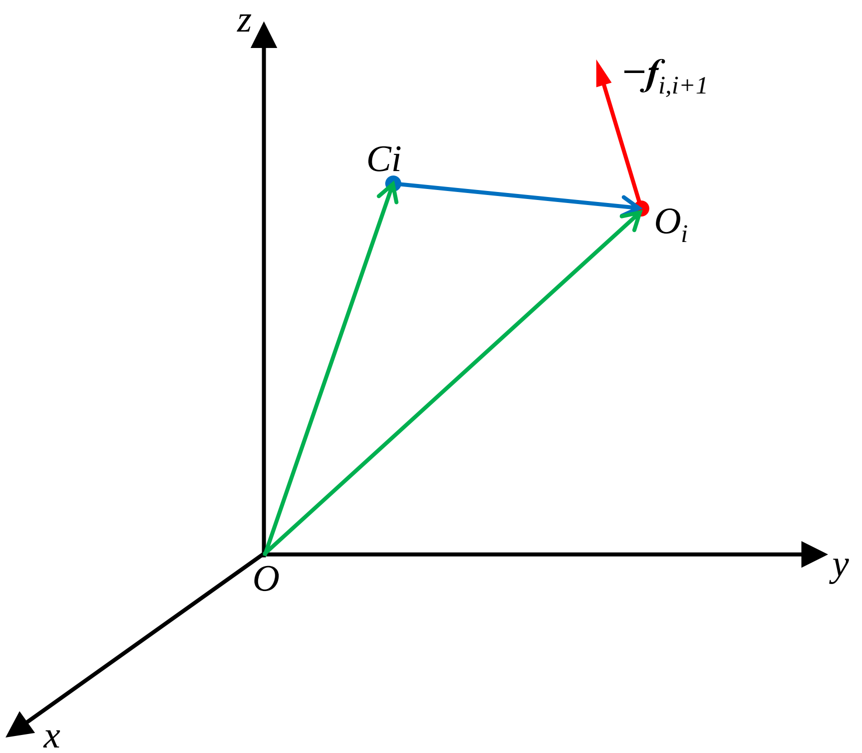
$$\boldsymbol{M}_A = (\boldsymbol{r} - \boldsymbol{r}_A) \times \boldsymbol{f}$$

Appendix 1: Moment of a force

Moment of $\boldsymbol{f}_{i\text{-}1,i}$ about point Ci:



Moment of $-\boldsymbol{f}_{i,i+1}$ about point Ci:



Appendix 2: Lagrange's equations

Treating each robotic arm as a point mass, *D'Alembert's principle* can be rewritten as:

$$\delta work = \sum_j \left(F_j + F_j^*\right) \cdot \delta p_j + \sum_i (\tau_i + \tau_i^*) \cdot \delta q_i = 0$$

where p_j is the position of j-th link, which is a function of generalized coordinates q_i and time t.

$$p_j = p_j(q_i, t)$$

Assuming virtual changes take place at a fixed time, thus:

$$\delta p_j = \sum_i \frac{\partial p_j}{\partial q_i} \delta q_i \quad \text{and} \quad \sum_{j,i} F_j \frac{\partial p_j}{\partial q_i} \delta q_i + \sum_i \tau_i \delta q_i = \sum_i Q_i \delta q_i \quad \text{where} \quad Q_i = \tau_i + \sum_j F_j \frac{\partial p_j}{\partial q_i}$$

Again, Q_i is the *generalized force* corresponding to the generalized coordinate q_i. Turning to the inertial forces in *d'Alembert's principle*, we note that:

$$\sum_j F_j^* \delta p_j = -\sum_j m_j \frac{d^2 p_j}{dt^2} \delta p_j = -\sum_{j,i} m_j \frac{d^2 p_j}{dt^2} \frac{\partial p_j}{\partial q_i} \delta q_i \quad \text{and} \quad \sum_i \tau_i^* \delta q_i = -\sum_i I_i \frac{d^2 q_i}{dt^2} \delta q_i$$

Appendix 2: Lagrange's equations

Using the product rule backwards, we see that:

$$\sum_j m_j \underbrace{\frac{d^2 p_j}{dt^2}}_{u'} \underbrace{\frac{\partial p_j}{\partial q_i}}_{v} = \sum_j m_j \left[\frac{d}{dt}\underbrace{\left(\dot{p}_j \frac{\partial p_j}{\partial q_i}\right)}_{uv} - \dot{p}_j \underbrace{\frac{d}{dt}\left(\frac{\partial p_j}{\partial q_i}\right)}_{v'}\right] \quad \text{where} \quad \dot{p}_j = \sum_i \frac{\partial p_j}{\partial q_i}\dot{q}_i + \frac{\partial p_j}{\partial t} \Rightarrow \frac{\partial \dot{p}_j}{\partial \dot{q}_i} = \underbrace{\frac{\partial p_j}{\partial q_i}}_{v}$$

$$\left.\begin{aligned} \frac{\partial \dot{p}_j}{\partial q_i} &= \sum_k \frac{\partial^2 p_j}{\partial q_i \partial q_k}\dot{q}_k + \frac{\partial^2 p_j}{\partial q_i \partial t} \\ \frac{d}{dt}\left(\frac{\partial p_j}{\partial q_i}\right) &= \sum_k \frac{\partial^2 p_j}{\partial q_i \partial q_k}\dot{q}_k + \frac{\partial^2 p_j}{\partial t \partial q_i} \end{aligned}\right] \Rightarrow \underbrace{\frac{d}{dt}\left(\frac{\partial p_j}{\partial q_i}\right)}_{v'} = \frac{\partial \dot{p}_j}{\partial q_i}$$

$$\sum_j m_j \frac{d^2 p_j}{dt^2}\frac{\partial p_j}{\partial q_i} = \sum_j m_j \left[\frac{d}{dt}\left(\dot{p}_j \frac{\partial \dot{p}_j}{\partial \dot{q}_i}\right) - \dot{p}_j \frac{\partial \dot{p}_j}{\partial q_i}\right] = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\sum_j \frac{m_j \dot{p}_j^{\,2}}{2} - \frac{\partial}{\partial q_i}\sum_j \frac{m_j \dot{p}_j^{\,2}}{2}$$

Also, $\displaystyle\sum_i I_i \frac{d^2 q_i}{dt^2} = \frac{d}{dt}\sum_i I_i \dot{q}_i = \frac{d}{dt}\sum_i I_i \left(\dot{q}_i \frac{\partial \dot{q}_i}{\partial \dot{q}_i}\right) = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\sum_i \frac{I_i \dot{q}_i^{\,2}}{2}$

Appendix 2: Lagrange's equations

$$\sum_j m_j \frac{d^2 p_j}{dt^2}\frac{\partial p_j}{\partial q_i} + \sum_i I_i \frac{d^2 q_i}{dt^2} = \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\sum_j \frac{m_j {\dot{p}_j}^2}{2} - \frac{\partial}{\partial q_i}\sum_j \frac{m_j {\dot{p}_j}^2}{2} + \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\sum_i \frac{I_i {\dot{q}_i}^2}{2} - \underbrace{\frac{\partial}{\partial q_i}\sum_i \frac{I_i {\dot{q}_i}^2}{2}}_{0}$$

$$= \frac{d}{dt}\frac{\partial}{\partial \dot{q}_i}\left(\sum_j \frac{m_j {\dot{p}_j}^2}{2} + \sum_i \frac{I_i {\dot{q}_i}^2}{2}\right) - \frac{\partial}{\partial q_i}\left(\sum_j \frac{m_j {\dot{p}_j}^2}{2} + \sum_i \frac{I_i {\dot{q}_i}^2}{2}\right) = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}$$

where $T = \sum_j \frac{m_j {\dot{p}_j}^2}{2} + \sum_i \frac{I_i {\dot{q}_i}^2}{2}$, which is the total kinetic energy

$$\left.\begin{aligned}
&\delta work = \sum_j (F_j + F_j^*)\cdot \delta p_j + \sum_i (\tau_i + \tau_i^*)\cdot \delta q_i = 0\\
&\sum_j F_j \delta p_j + \sum_i \tau_i \delta q_i = \sum_i Q_i \delta q_i\\
&\sum_j F_j^* \delta p_j + \sum_i \tau_i^* \delta q_i = -\sum_i \left\{\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}\right\}\delta q_i
\end{aligned}\right\} \sum_i \left\{Q_i - \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) + \frac{\partial T}{\partial q_i}\right\}\delta q_i = 0 \Rightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = Q_i$$

Appendix 2: Lagrange's equations

Often forces are conservative and possible to represent as the gradient of a potential energy:

$$F_j = -\frac{\partial W}{\partial p_j}, \qquad \tau_i = -\frac{\partial X}{\partial q_i}$$

$$Q_i = \tau_i + \sum_j F_j \frac{\partial p_j}{\partial q_i} =- -\frac{\partial X}{\partial q_i} - \sum_j \frac{\partial W}{\partial p_j}\frac{\partial p_j}{\partial q_i} = -\frac{\partial X + \partial W}{\partial q_i} = -\frac{\partial U}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i}$$

If U is not an explicit function of time or of the generalized velocities

$$\frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i}\right) - \frac{\partial T}{\partial q_i} = -\frac{\partial U}{\partial q_i} \Longrightarrow \frac{d}{dt}\left(\frac{\partial T}{\partial \dot{q}_i} - \underbrace{\frac{\partial U}{\partial \dot{q}_i}}_{0}\right) - \frac{\partial (T-U)}{\partial q_i} = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right) - \frac{\partial L}{\partial q_i} = 0$$

where $L = T - U$ is called the *Lagrangian*, which is a function of both q_i and $\dot{q}_i$. If some of the forces are conservative and others are not, then the more general form and be obtained:

$$\boxed{\frac{d}{dt}\frac{\partial L(q_i,\dot{q}_i)}{\partial \dot{q}_i} - \frac{\partial L(q_i,\dot{q}_i)}{\partial q_i} = Q_i}$$