Public goods games

Nadiah Kristensen

July 2023

Sources and further reading

- Peña, J., Lehmann, L., & Nöldeke, G. (2014). Gains from switching and evolutionary stability in multi-player matrix games. Journal of Theoretical Biology, 346, 23-33.
- Archetti, M., Scheuring, I., & Yu, D. (2020). The non-tragedy of the non-linear commons. (Preprints.org)

The Tragedy of the Commons

Hardin (1968, *Science*)

- William Forster Lloyd's *Tragedy of the Commons*
    - Herders share a common, limited-size, pasture
    - Utility to an individual herder of adding one more sheep is greater than the cost to themselves, a cost shared by all
    - Rational herders will put as many sheep on the commons as possible

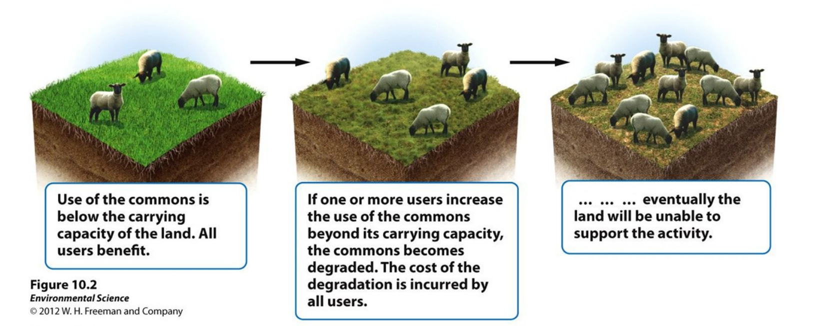

Figure 10.2
Environmental Science
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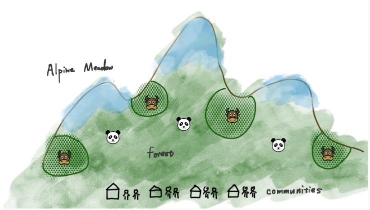
The Tragedy of the Commons in real life!

Thanks, Zhou!



Linear versus ‘provision point’

Public goods game

Recall replicator equation

$$\dot{p} = p(1-p)(f_C - f_D)$$

Previously, in the two-player game, we would write

$$f_C = p_C \pi(C, C) + p_D \pi(C, D)$$

where p_C is the probability to be grouped with a cooperator and p_D is the probability to be grouped with a defector.

In PGG, games are now played in groups of size n.

Let k denote the no. of cooperators among the $n-1$ others in your group. What's the probability that you'll be grouped with k cooperators?

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$$P[k \text{ cooperators among } n-1 \text{ others}] = \binom{n-1}{k} p_C^k (1-p)^{n-1-k}$$

Public goods game

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Now we have

$$P[k \text{ cooperators among } n-1 \text{ others}] = \binom{n-1}{k} p_C^k (1-p_C)^{n-1-k}$$

Let $\pi_C(k)$ be the payoff to a cooperator grouped with k cooperators.
What should the fitness effect of playing cooperate f_C be?

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Let $\pi_C(k)$ be the payoff to a cooperator grouped with k cooperators. What should the fitness effect of playing cooperate f_C be?

$$f_C = \sum_{k=0}^{n-1} \underbrace{\binom{n-1}{k} p_C^k (1-p_C)^{n-1-k}}_{P(k)} \pi_C(k)$$

Linear public goods game

Also known as the n-player Prisoners' Dilemma

Each cooperator produces benefit B, and total benefit is split between everyone, including defectors, $b_i = iB/n$

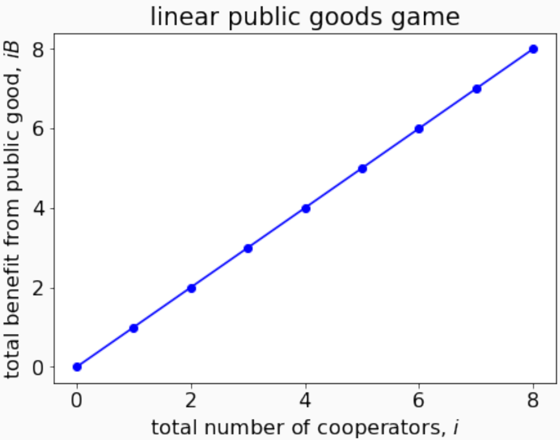
$$f_C(p) = \sum_{k=0}^{n-1} \underbrace{\binom{n-1}{k} p^k (1-p)^{n-1-k}}_{\text{Pr grouped with } k \text{ cooperators}} \, b_{k+1} - c$$

$$f_D(p) = \sum_{k=0}^{n-1} \underbrace{\binom{n-1}{k} p^k (1-p)^{n-1-k}}_{\text{Pr grouped with } k \text{ cooperators}} \, b_k$$

Plot it in Python

Linear public goods game

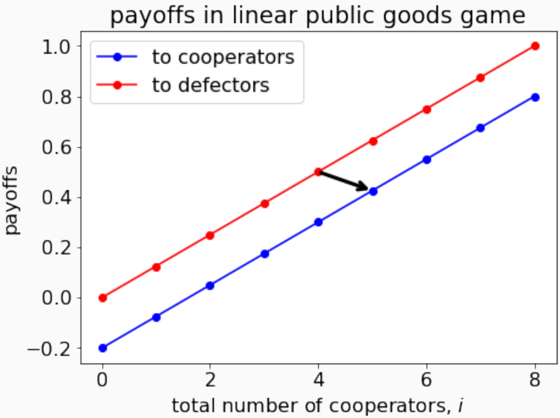
Setting $b_i = iB/n$, each additional cooperator creates the same increment in total benefit



Linear public goods game

We chose costs and benefits so defectors always get a higher payoff

More importantly, it never increases your payoff to switch from defector to cooperator

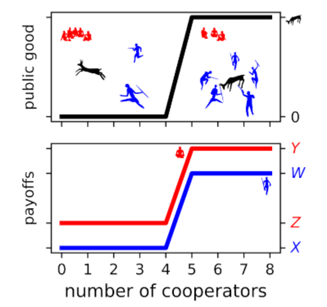


Most public goods games are not linear

- almost certainly nonlinear in nature
- examples from Archetti et al. (2020)
  - biological effect of molecules (e.g., growth factors) on cell fitness generally a sigmoidal function
  - herd immunity in vaccination, threshold like shape
- economies of scale
- mass hunting techniques (Balme 2018)

Threshold game

- Example scenario:
  - 8 individuals in the group
  - it takes 5 or more to surround prey
  - prey shared (non-excludable PG)
  - defectors get a higher payoffs than cooperators (red line above blue)

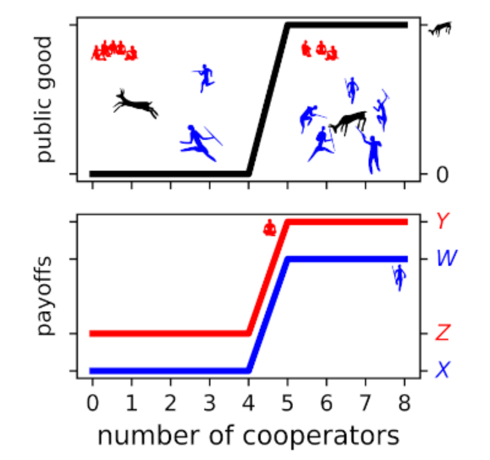


Threshold game

Discussion:

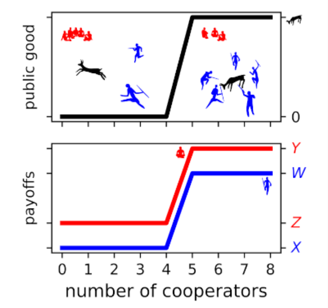
Can you find the (asymmetric) pure strategy Nash equilibrium?

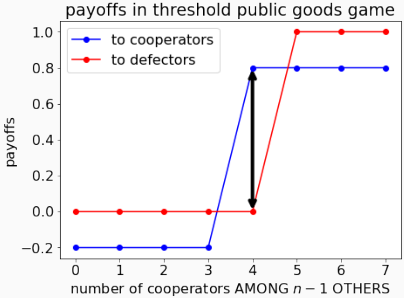
Think about scenarios for players $i = 1, 2, \ldots, 8$, e.g., C, C, D, C, ... Is there a scenario in which no one would want to switch strategy?



Threshold game

Asymmetric pure-strategy NE when 5 or cooperating and 3 are defecting.





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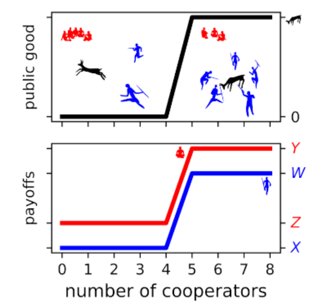
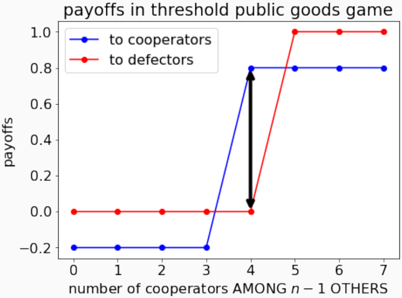




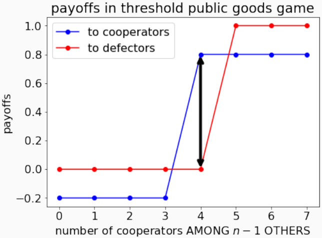
Maybe it makes sense to be hardwired to cooperate if your probability of being pivotal is high enough.

Threshold game

Let k be the number of cooperators among $n-1$ OTHER players.

Payoff for cooperators

$$\pi_C(k) = \begin{cases} b-c & \text{if } k \geq \tau - 1, \\ -c & \text{otherwise.} \end{cases}$$



Payoff for defectors

$$\pi_D(k) = \begin{cases} b & \text{if } k \geq \tau, \\ 0 & \text{otherwise.} \end{cases}$$

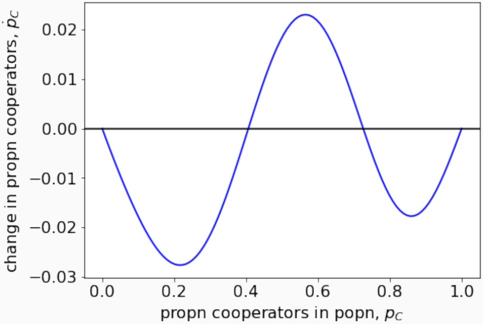
Replicator equation $\dot{p} = p(1-p)(f_C - f_D)$ now with

$$f_C(p) = \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i}\, \pi_C(k)$$

$$f_D(p) = \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i}\, \pi_D(k)$$

Threshold game example

I chose a group of size $n = 8$, a threshold $\tau = 5$, and a cost of contributing $c = 0.2$, and the plot of $\dot{p}_C$ versus p_C looks like:



Discuss dynamics:

Threshold game example

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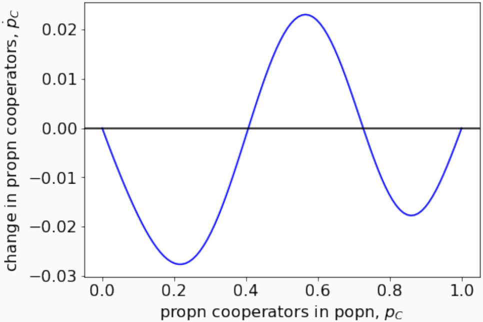

Discuss dynamics: Cooperation can persist in the population in coexistence with defection. But it cannot invade.

Let's explore the threshold game together

Split you into groups:

1. What happens when cost of contributing is increased? Try $n = 8$, $\tau = 5$, $c = 0.3$.
2. What happens when $\tau = 1$? Try $n = 8$, $\tau = 1$, $c = 0.5$.
3. What happens when $\tau = n$? Try $n = 8$, $\tau = 8$, $c = 0.5$.

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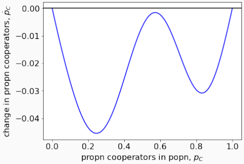
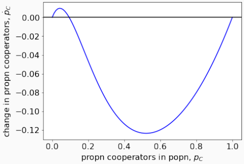
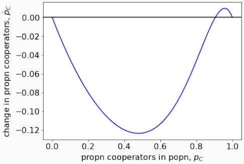
increase cost



threshold $\tau = 1$



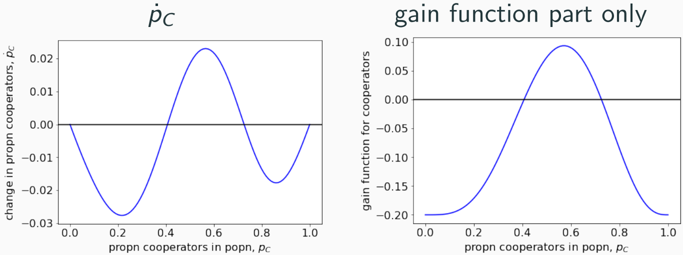
threshold $\tau = n$



The gain function determines steady states and stability

In the replicator equation, the gain function is the part that determines the existence of interior equilibria and their stability

$$\dot{p} = p(1-p)\underbrace{(f_C - f_D)}_{\text{gain function}}$$



If we had insights into the gain function, we might be able to describe the dynamics qualitatively without solving for the steady state (difficult analytically).

Bernstein polynomial: The Bezier curve

A polynomial of the form:

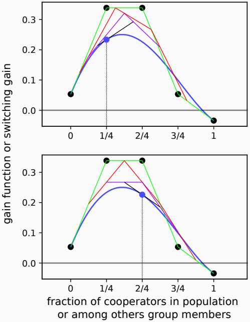
$$B(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} P_i$$

where P_i are the control points.

Show examples in web browser. Notice:

- *Start- and end-point match*
- *Shape preserving*

The gain function is a Bernstein polynomial



Peña, Lehmann, & Nöldeke (2014)

Bernstein:

$$B(t) = \sum_{i=0}^{n} \binom{n}{i} t^i (1-t)^{n-i} \, P_i$$

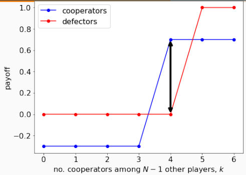
Gain function

$$g(p) = f_C - f_D = \sum_{i=0}^{n-1} \binom{n-1}{i} p^i (1-p)^{n-1-i} \, \underbrace{(\pi_C(k) - \pi_D(k))}_{d_k}$$

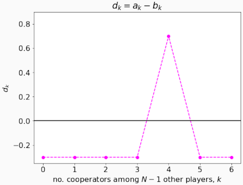
and the control points are the switching gains.

The gain function is a Bernstein polynomial, example

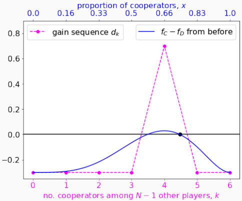
- Payoffs when $n-1$ other members have k cooperators



- Switching gain:
    - Given k cooperators, how much payoff would you gain if you switched from defector to cooperator?

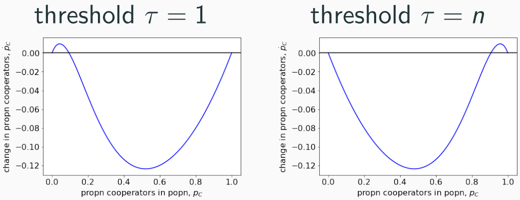


- Gain function:
    - A Bernstein polynomial with the switching gains as its control points
    - Therefore, general shape of switching gains indicates existence and stability of evolutionary steady states



Use Bernstein polynomial theory to make sense of

Sketch the switching gains for the two cases below, and see if you can immediately discern qualitative dynamics plotted below are general



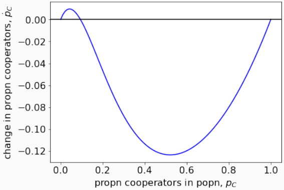
Recall, control points are $d_k = \pi_C(k) - \pi_D(k)$ where payoffs are

$$\pi_C(k) = \begin{cases} b - c & \text{if } k \geq \tau - 1, \\ -c & \text{otherwise.} \end{cases} \text{ and } \pi_D(k) = \begin{cases} b & \text{if } k \geq \tau, \\ 0 & \text{otherwise.} \end{cases}$$

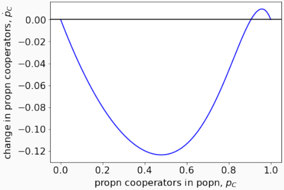
Assume $b > c$, and remember the Bernstein polynomial's start- and end-points match the first and last control points.

Use Bernstein polynomial theory to make sense of

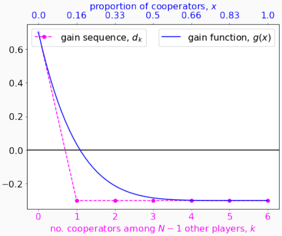
threshold 1



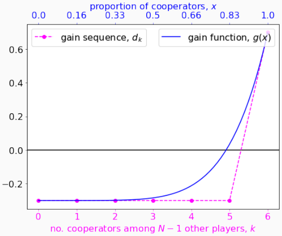
threshold n



threshold 1



threshold n



Reference

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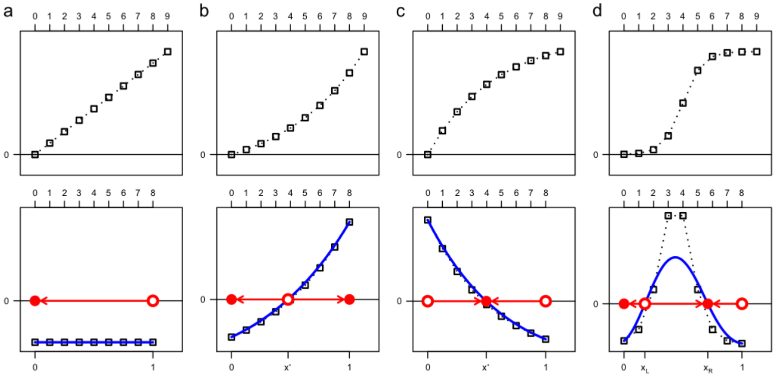

Fig. 3. Examples of constant cost games with $N=n+1=9$ and $c=1/2$ for different benefit sequences. The first row shows the benefit sequence r_j; the second row shows the gain sequence **d** (squares, dotted line; top axis), and corresponding gain function $g(x)$ (solid line; bottom axis) and phase portrait (circles, arrows). (a) Linear benefits (see Section 4.3.1) with $r=5$ and $c=1$. (b) Convex benefits (see Section 4.3.2) as given by (13) with $r=5$ and $w=1.2$. (c) Concave benefits (see Section 4.3.2) as given by (13) with $r=20$ and $w=0.8$. (d) Sigmoid benefits (see Section 4.3.3) as studied by Archetti and Scheuring (2011) with $r_j=r/[1+\exp(-s(j-m))]$, $r=20$, $m=4$, and $s=1.5$.