Evolutionary Game Theory

Nadiah Kristensen

July 2023

Sources and further reading

- Evolution and the Theory of Games by John Maynard Smith
- Open Yale Course ‘Game Theory’ by Ben Polack
- Game Theory: Decisions, Interaction and Evolution by James N. Webb

Fitness depends on genotypes and frequencies of others

Fitness is determined not just by an individual's own genotype but by the types and frequencies of other genes in the population

photo BBC

How well my strategy does depends on the strategies of others

Bring game theory over into biology

History of evolutionary game theory

- Maynard Smith & Price recast the game theory into biological context:
  - 'Game' $\rightarrow$ interaction that determines fitness (e.g. snakes fighting for territory)
  - 'Strategy' $\rightarrow$ genetically encoded behaviour or trait
  - 'Player' $\rightarrow$ individual animal, though better to think of as gene
  - 'Payoff' $\rightarrow$ fitness
- Recall the four processes of population genetics:
  1. Selection
  2. Mutation
  3. Genetic drift
  4. Gene flow

  Basic evolutionary game theory only includes the first and, in a limited way, the second.

Hawk Dove

Maynard Smith & Price (1973):

- Males often compete for territory, etc. $\rightarrow$ transmission of genes
- Might expect natural selection to favour maximally effective weapons and fighting styles for a "total war" strategy, battles to the death
- Instead, a "limited war" type is common
- 'Group selection' type explanation was accepted explanation

Hawk-Dove

- Apply ‘hawk-dove’ game to our snakes case-study
- Here ‘hawk’ and ‘dove’ don’t refer to literal species of birds, they are terms from politics and foreign policy
    - e.g. “John Bolton, a known hawk, has advocated for pre-emptive strikes against North Korea”
- A hawkish strategy for snakes is to use their powerful venom against each other

Hawk-Dove

Two animals contesting a favoured territory or other resource with value V. Losing a fight over it has an injury cost C.

1 \ 2	DOVE	HAWK
DOVE	$V/2$, $V/2$	0, V
HAWK	V, 0	$(V-C)/2$, $(V-C)/2$

Assume the cost of losing a fight is greater than the value of the resource itself

$$V < C.$$

Makes sense for our snakes example.

Exercise: Find the pure NE. Is there a symmetric NE (both players play same strategy)?

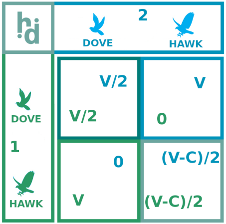
Reminder: Equality of payoffs theorem

Theorem: Let (σ_1^*, σ_2^*) be a Nash equilibrium, and let S_1^* be the set of pure strategies for which σ_1^* specifies $p(s) > 0$. Then $\pi_1(s, \sigma_2^*) = \pi_1(\sigma_1^*, \sigma_2^*)$ for all $s \in S_1^*$.

Exercise: find the mixed-strategy NE

1. Write $\sigma_2^* = (q, 1-q)$ (H, D)
2. Write $\pi_1(H, \sigma_2^*) = \pi_1(D, \sigma_2^*)$
3. Expand (2) q frequencies
4. Sub in payoffs from matrix
5. Solve for q

what's wrong with this analysis?



Evolutionarily stable strategy

- Assumptions:
  - Very large population
  - Randomly matching
  - Play strategy genetically hardwired
  - Payoff determines number of offspring
  - Clonal reproduction
- Goal: find the evolutionarily stable strategy:
  - A genetically-hardwired strategy σ^* that, if almost all members of the population adopt σ^*, then the fitness of these typical members is greater than that of any possible mutant; otherwise, the mutant could invade the population, and it would not be stable.

Evolutionarily stable strategy

Looking for a genetically-hardwired strategy σ^* that, if almost all members of the population adopt σ^*, then the fitness of these typical members is greater than that of any possible mutant; otherwise, the mutant could invade the population, and it would not be stable.

Evolutionarily stable strategy

Looking for a genetically-hardwired strategy σ^* that, if almost all members of the population adopt σ^*, then the fitness of these typical members is greater than that of any possible mutant; otherwise, the mutant could invade the population, and it would not be stable.

An ESS must satisfy, $\forall \sigma$,

$$(1-\varepsilon)\pi(\sigma^*, \sigma^*) + \varepsilon\pi(\sigma^*, \sigma) > (1-\varepsilon)\pi(\sigma, \sigma^*) + \varepsilon\pi(\sigma, \sigma),$$

Evolutionarily stable strategy

Looking for a genetically-hardwired strategy σ^* that, if almost all members of the population adopt σ^*, then the fitness of these typical members is greater than that of any possible mutant; otherwise, the mutant could invade the population, and it would not be stable.

An ESS must satisfy, $\forall \sigma$,

$$(1-\varepsilon)\pi(\sigma^*, \sigma^*) + \varepsilon\pi(\sigma^*, \sigma) > (1-\varepsilon)\pi(\sigma, \sigma^*) + \varepsilon\pi(\sigma, \sigma),$$

which implies

1. $\pi(\sigma^*, \sigma^*) - \pi(\sigma, \sigma^*) > 0$; OR
2. $\pi(\sigma^*, \sigma^*) - \pi(\sigma, \sigma^*) = 0$ AND $\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) > 0$

Evolutionarily stable strategy

An ESS must satisfy

$$(1-\varepsilon)\pi(\sigma^*,\sigma^*)+\varepsilon\pi(\sigma^*,\sigma) > (1-\varepsilon)\pi(\sigma,\sigma^*)+\varepsilon\pi(\sigma,\sigma),$$

which implies

1. $\pi(\sigma^*,\sigma^*)-\pi(\sigma,\sigma^*) > 0$; OR
2. $\pi(\sigma^*,\sigma^*)-\pi(\sigma,\sigma^*) = 0$ AND $\pi(\sigma^*,\sigma)-\pi(\sigma,\sigma) > 0$

Evolutionarily stable strategy

An ESS must satisfy

$$(1-\varepsilon)\pi(\sigma^*,\sigma^*)+\varepsilon\pi(\sigma^*,\sigma) > (1-\varepsilon)\pi(\sigma,\sigma^*)+\varepsilon\pi(\sigma,\sigma),$$

which implies

1. $\pi(\sigma^*,\sigma^*)-\pi(\sigma,\sigma^*) > 0$; OR
2. $\pi(\sigma^*,\sigma^*)-\pi(\sigma,\sigma^*) = 0$ AND $\pi(\sigma^*,\sigma)-\pi(\sigma,\sigma) > 0$

Can rewrite:

1. $\pi(\sigma^*,\sigma^*)-\pi(\sigma,\sigma^*) \geq 0$; AND $\leftarrow$ Nash equilibrium
2. if $\pi(\sigma^*,\sigma^*)-\pi(\sigma,\sigma^*) = 0$, then $\pi(\sigma^*,\sigma)-\pi(\sigma,\sigma) > 0$ $\leftarrow$ extra bit

$$\text{ESS} \implies \text{NE}$$

ESS-NE connection: so useful!

ESS:

1. $\pi(\sigma^*, \sigma^*) - \pi(\sigma, \sigma^*) > 0$; OR
2. $\pi(\sigma^*, \sigma^*) - \pi(\sigma, \sigma^*) = 0$ AND $\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) > 0$

Prisoners' Dilemma

	C	D
C	3, 3	0, 5
D	5, 0	1, 1

Think it through:

- Population of all defectors – average payoff 1
- A few (ε) cooperators invade
- Cooperators mostly encounter defectors – average payoff 0

Hawk dove

We found a mixed-strategy NE $\sigma^* = \frac{V}{C}$. Is it evolutionarily stable?

ESS:

1. $\pi(\sigma^*, \sigma^*) - \pi(\sigma, \sigma^*) > 0$; OR
2. $\pi(\sigma^*, \sigma^*) - \pi(\sigma, \sigma^*) = 0$ AND
   $\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) > 0$

1 \ 2	DOVE	HAWK
DOVE	V/2, V/2	0, V
HAWK	V, 0	(V-C)/2, (V-C)/2

Remember: mixed strategies are weak Nash; we're in case 2.

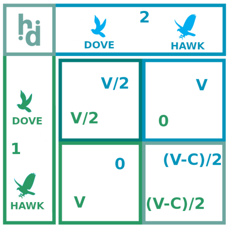
So we need to check: $\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) > 0$

Hawk dove

We found a mixed-strategy NE $\sigma^* = \frac{V}{C}$. Is it evolutionarily stable?

ESS:

1. $\pi(\sigma^*, \sigma^*) - \pi(\sigma, \sigma^*) > 0$; OR
2. $\pi(\sigma^*, \sigma^*) - \pi(\sigma, \sigma^*) = 0$ AND
   $\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) > 0$



Remember: mixed strategies are weak Nash; we're in case 2.

So we need to check: $\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) > 0$

... find condition $(Cp - V)^2 > 0$, therefore ESS

Hawk dove ESS – the achievement

We have shown the mixed strategy $\sigma^* = (V/C, 1 - V/C)$ is evol stable.

Our question was, why is "limited war" common?

- Intuitively, male snakes could fight with venom? ($p_H = 1$?)
- Maybe they don't for the good of the species?
- Evol game theory gives us a reason why 'limited war' is common, from a purely individualistic perspective
    - Occam's Razor

Check the stability of TT+AD mixed strategy

Exercise: Previously, you looked at the tit-for-tat vs all-defect in an iterated PD, which had payoffs:

	TT	all-D
TT	$16, 16$	$1, 11$
all-D	$11, 1$	$2, 2$

And you found a mixed-strategy Nash equilibrium $\sigma^* = (1/6, 5/6)$ (order is: TT, AD). Is this evolutionarily stable?

Steps:

1. Mixed-strategy, so not strict Nash, so we need to check the 'extra' condition: $\pi(\sigma^*, \sigma) - \pi(\sigma, \sigma) > 0$
2. Write out $\pi(\sigma^*, \sigma)$ and $\pi(\sigma, \sigma)$ and sub in $p^* = 1/6$, p, and pure payoffs like $\pi(TT, AD) = 1$ as needed
3. See if you can find a p such that the extra condition is not satisfied

Replicator dynamics

Same kinds of assumptions as before:

- Agents do not have to be conscious or rational: all they need is a strategy that they pass on
- No change in strategy, no mutation to new strategy

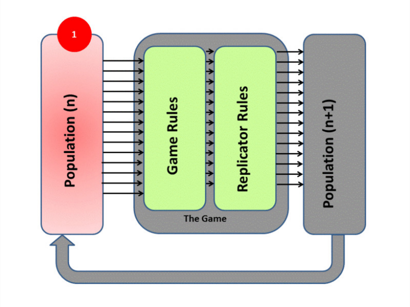
New: Interested in change in distribution of strategies in population

Replicator dynamics

Same kinds of assumptions as before:

- Agents do not have to be conscious or rational: all they need is a strategy that they pass on
- No change in strategy, no mutation to new strategy

New: Interested in change in distribution of strategies in population

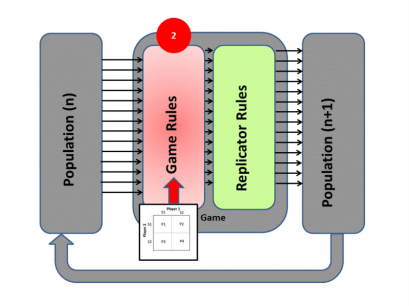


Replicator dynamics

Same kinds of assumptions as before:

- Agents do not have to be conscious or rational: all they need is a strategy that they pass on
- No change in strategy, no mutation to new strategy

New: Interested in change in distribution of strategies in population

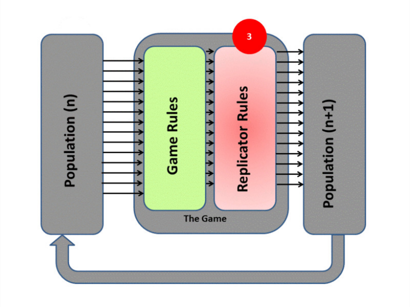


Replicator dynamics

Same kinds of assumptions as before:

- Agents do not have to be conscious or rational: all they need is a strategy that they pass on
- No change in strategy, no mutation to new strategy

New: Interested in change in distribution of strategies in population

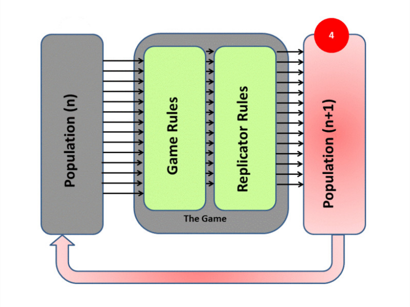


Replicator dynamics

Same kinds of assumptions as before:

- Agents do not have to be conscious or rational: all they need is a strategy that they pass on
- No change in strategy, no mutation to new strategy

New: Interested in change in distribution of strategies in population

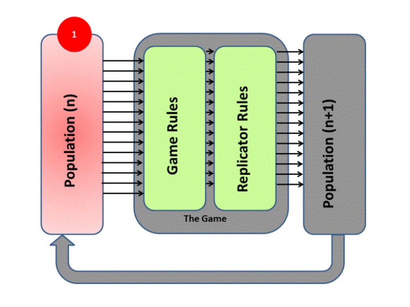


Replicator dynamics

Same kinds of assumptions as before:

- Agents do not have to be conscious or rational: all they need is a strategy that they pass on
- No change in strategy, no mutation to new strategy

New: Interested in change in distribution of strategies in population



Replicator dynamics

The change in the proportion of individuals pursuing strategy i is

$$\dot{p}_i = p_i(f_i(\boldsymbol{p}) - \bar{f}(\boldsymbol{p}))$$

where:

> f_i: the fitness effect of pursuing strategy i, as determined by the game payoffs and strategy frequencies $\boldsymbol{p}$
>
> $\bar{f}$: average f_i in population

For 2-strategy situations, this simplifies to

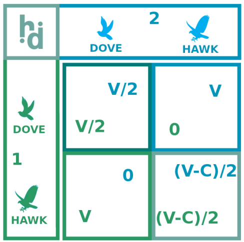
$$\dot{p}_1 = p_1(1 - p_1)(f_1 - f_2)$$

Hawk dove replicator dynamics

- Replicator dynamics

$$\dot{p}_H = p_H(1 - p_H)(f_H - f_D)$$

- Fitness effects (on board)



Let's plot it!

Hawk dove replicator dynamics

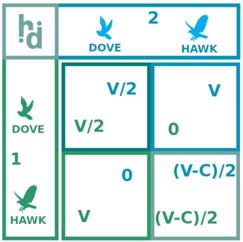
- Replicator dynamics

$$\dot{p}_H = p_H(1 - p_H)(f_H - f_D)$$

- Fitness effects (on board)

$$f_H(\mathbf{p}) = p_H \frac{V - C}{2} + p_D V$$

$$f_D(\mathbf{p}) = p_H\, 0 + p_D \frac{V}{2}$$



- Sub $(1 - p_H) = p_D$, rearrange

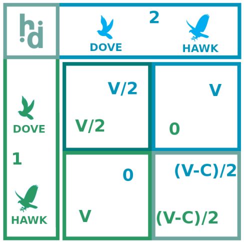
$$\dot{p}_H = p_H(1 - p_H)\left(p_H \frac{V - C}{2} + (1 - p_H)\frac{V}{2}\right)$$

Let's plot it!

Hawk-Dove replicator analysis

$$\dot{p}_H = \frac{dp_H}{dt} = p_H(1 - p_H)\left(p_H \frac{V - C}{2} + (1 - p_H)\frac{V}{2}\right)$$

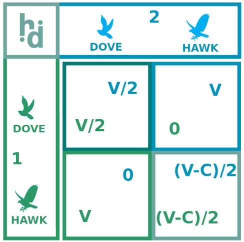
- Equilibrium?



Hawk-Dove replicator analysis

$$\dot{p}_H = \frac{dp_H}{dt} = p_H(1 - p_H)\left(p_H\frac{V-C}{2} + (1-p_H)\frac{V}{2}\right)$$

- Equilibrium? Solve $\frac{dp_H}{dt} = 0$

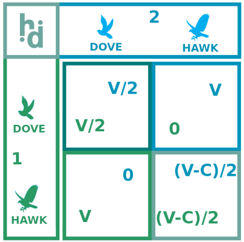


Hawk-Dove replicator analysis

$$\dot{p}_H = \frac{dp_H}{dt} = p_H(1 - p_H)\left(p_H\frac{V - C}{2} + (1 - p_H)\frac{V}{2}\right)$$

- Equilibrium? Solve $\frac{dp_H}{dt} = 0$

$$p_H^* = 0,\ 1,\ \frac{V}{C}$$

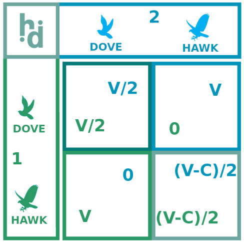


Hawk-Dove replicator analysis

$$\dot{p}_H = \frac{dp_H}{dt} = p_H(1 - p_H)\left(p_H \frac{V - C}{2} + (1 - p_H)\frac{V}{2}\right)$$

- Equilibrium? Solve $\frac{dp_H}{dt} = 0$

$$p_H^* = 0,\ 1,\ \frac{V}{C}$$



- Asymptotic stability?

Hawk-Dove replicator analysis

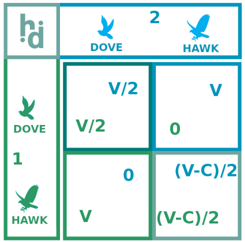
$$\dot{p}_H = \frac{dp_H}{dt} = p_H(1-p_H)\left(p_H\frac{V-C}{2} + (1-p_H)\frac{V}{2}\right)$$

- Equilibrium? Solve $\frac{dp_H}{dt} = 0$

$$p_H^* = 0,\ 1,\ \frac{V}{C}$$

- Asymptotic stability?

$$\left.\frac{d\dot{p}_H}{dp_H}\right|_{p_H=p_H^*} < 0$$



Exercise: Tit-for-tat replicator analysis

Original Prisoners' Dilemma

	C	D
C	$8, 8$	$0, 10$
D	$10, 0$	$1, 1$

When repeated with $\delta = 0.5$, transforms into tit-for-tat vs all-defect with payoffs

	tit-for-tat	all-defect
tit-for-tat	$16, 16$	$1, 11$
all-defect	$1, 11$	$2, 2$

Exercise 1: Plot the replicator dynamics and then we discuss it

Exercise: Tit-for-tat replicator analysis

Original Prisoners' Dilemma

	C	D
C	$8, 8$	$0, 10$
D	$10, 0$	$1, 1$

When repeated with $\delta = 0.5$, transforms into tit-for-tat vs all-defect with payoffs

	tit-for-tat	all-defect
tit-for-tat	$16, 16$	$1, 11$
all-defect	$1, 11$	$2, 2$

Exercise 1: Plot the replicator dynamics and then we discuss it

Exercise 2: Find the steady states and check their stability analytically

Exercise: Tit-for-tat replicator analysis

- Mixed-strategy unstable
- Unstable equilibrium is a separatrix
- Once TT is established in the population, it persists
- But presumably primordial state is (AD, AD), so how did tit-for-tat evolve?