# Game Theory

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# Sources and further reading

- Game Theory: Decisions, Interaction and Evolution by James N. Webb
- Open Yale Course ‘Game Theory’ by Ben Polack
- ‘Game Theory 101’ by William Spaniel

# Motivation

- Many environmental and social problems are cooperation problems
- Social dilemma: conflict between a person's short-term self-interest versus the long-term collective interest
- Examples:
  - climate change: free-riding, distance in time
  - COVID: differences in risks between individuals, social identity signalling
- Cooperation: helping others at a cost to one's self
  - Paradoxical
  - Nevertheless, humans are a very cooperative species

# A brief history of game theory - 1. Cooperative game theory

- John von Neumann and Oskar Morgenstern in 1944
  - Book – *Theory of Games and Economic Behavior*
    - “we wish to find the mathematically complete principles which define ‘rational behaviour’ for the participants in a social economy, and to derive from them the general characteristics of that behaviour”
  - cooperative game theory
    - About humans forming coalitions, making agreements, splitting costs/profits
    - e.g. Several nearby towns want a water supply
  - Not yet so interesting to biologists ...

Morgenstern (left) and von Neumann (right)

# A brief history of game theory - 2. Non-cooperative games

- John Nash - non-cooperative games
  - A more general theory
  - No enforcement mechanisms (e.g. contracts to split costs) outside the game itself
  - Not about coalitions, agreements and side-payments possible between players, but rather individual strategies and payoffs
- Individualistic
  - 'Darwinian' view of the world: each works for themselves and maximises own payoff

# Games

1. A set of players $i$, e.g., $i = 1, 2$
2. A set of pure strategies for each player $S_i$, e.g., $S_1 = \{A, B, C\}$
3. Payoffs, e.g., $\pi_1(A, W) = 3$ and $\pi_2(A, W) = 1$

| | | player 2 | | |
|---|---|---|---|---|
| | | W | X | Y |
| player 1 | A | 3, 1 | 1, 2 | 4, 5 |
| | B | 2, 3 | 2, 5 | 5, 4 |
| | C | 1, 2 | 3, 2 | 2, 1 |

Payoffs: ‘what you care about’

Rational: you want to maximise your own payoff

Given that you are rational, and your goal is to maximise your own payoff, what should you do?

*Exercise*: static, one-shot game on slip of paper

# Social dilemma

**Prisoners' Dilemma**: Two criminals interrogated separately by police. If both stay silent, get a lesser charge and 1 week jail. But police make an offer - testify against the other, and if the other doesn't testify, you go free and they 3 weeks jail. But if both testify against each other, both get 2 weeks jail.

# Social dilemma

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**Pareto optimal** (after Italian economist Vilfredo Pareto): if no player's payoff can be increased without decreasing the payoff to another player.

# Strictly dominated

**Defn**: A strategy for player 1 $s_1$ is strictly dominated by another strategy $s_1'$ if

$$\pi_1(s_1', s_2) > \pi_1(s_1, s_2)\ \forall s_2 \in S_2,$$

and similarly for player 2.

It's saying that $s_1'$ always does better than $s_1$ no matter what the other player does.

A rational player will never choose a strictly dominated strategy.

*Exercise*: empathy game

| | C | D |
|---|---|---|
| C | $-2, -2$ | $-3, -3$ |
| D | $-3, -3$ | $-4, -4$ |

# Iterated deletion of dominated strategies

A rational player will never choose a strictly dominated strategy.

*Common knowledge of rationality*:

1. The players are rational.
2. The players all know that the other players are rational.
3. The players all know that the other players know that they are rational.
4. ... and so on.

*Put yourself in the shoes of others.*

# Iterated deletion of dominated strategies

*Exercise*:

| | L | M | R |
|---|---|---|---|
| U | $1, 0$ | $1, 2$ | $0, 1$ |
| D | $0, 3$ | $0, 1$ | $2, 0$ |

# Iterated deletion of dominated strategies

*Exercise 2*:

| | L | M | R |
|---|---|---|---|
| U | 1, 3 | 4, 2 | 2, 2 |
| C | 4, 0 | 0, 3 | 4, 1 |
| D | 2, 5 | 3, 4 | 5, 6 |

1. Something will go wrong
2. But is there a solution that you thing rational players would land on anyway?

# Iterated deletion of dominated strategies

Iterated deletion of dominated strategies doesn't always work:

1. If there are weakly dominated strategies, where you end up depends on the order
$$\pi_1(s_1', s_2) \geq \pi_1(s_1, s_2) \ \forall s_2 \in S_2,$$
2. Sometimes neither player has a dominated strategy from the start

Can we define a solution in terms of something other than iterated deletion that both identifies obvious solutions and keeps results of dominance techniques?

# Nash equilibrium (pure, 2 strategies)

**Defn**: A strategy for player 1 $\hat{s}_1$ is a pure **best response** to some fixed strategy for player 2, $s_2$, if

$$\hat{s}_1 \in \underset{s_1 \in S_1}{\operatorname{argmax}} \, \pi_1(s_1, s_2).$$

Similarly, a strategy for player 2 $\hat{s}_2$ is a pure best response to some fixed strategy for player 1, $s_1$, if

$$\hat{s}_2 \in \underset{s_2 \in S_2}{\operatorname{argmax}} \, \pi_2(s_1, s_2).$$

**Defn 1**: A pair of strategies $(s_1^*, s_2^*)$ is a pure **Nash equilibrium** if

$$s_1^* \in \underset{s_1 \in S_1}{\operatorname{argmax}} \, \pi_1(s_1, s_2^*),$$

and

$$s_2^* \in \underset{s_2 \in S_2}{\operatorname{argmax}} \, \pi_1(s_1^*, s_2).$$

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*Ex. Use best response to find NE*

| | | player 2 | | |
|---|---|---|---|---|
| | | W | X | Y |
| player 1 | A | 3, 1 | 1, 2 | 4, 5 |
| | B | 2, 3 | 2, 5 | 5, 4 |
| | C | 1, 2 | 3, 2 | 2, 1 |

# Nash equilibrium (pure, 2 strategies)

**Defn 1**: A pair of strategies $(s_1^*, s_2^*)$ is a pure **Nash equilibrium** if

$$s_1^* \in \operatorname*{argmax}_{s_1 \in S_1} \pi_1(s_1, s_2^*),$$

and

$$s_2^* \in \operatorname*{argmax}_{s_2 \in S_2} \pi_1(s_1^*, s_2).$$

**Defn 2**: A pure 2-player **Nash equilibrium** is a pair of strategies $(s_1^*, s_2^*)$ such that

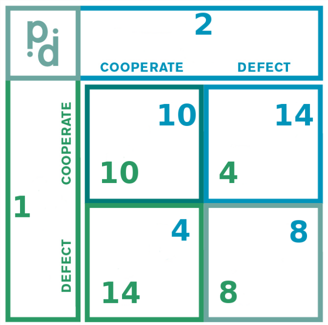
$$\pi_1(s_1^*, s_2^*) \geq \pi_1(s_1, s_2^*) \; \forall s_1 \in S_1$$

and

$$\pi_2(s_1^*, s_2^*) \geq \pi_2(s_1^*, s_2) \; \forall s_2 \in S_2.$$

# Climate dilemma

- Two countries.
- A fossil-fuel economy is worth 20 units.
- A switch to renewables reduces economic benefit to 10 units.
- But cost of $CO_2$ emissions paid by both countries via climate change.
- Cost is 6 units per polluting country.

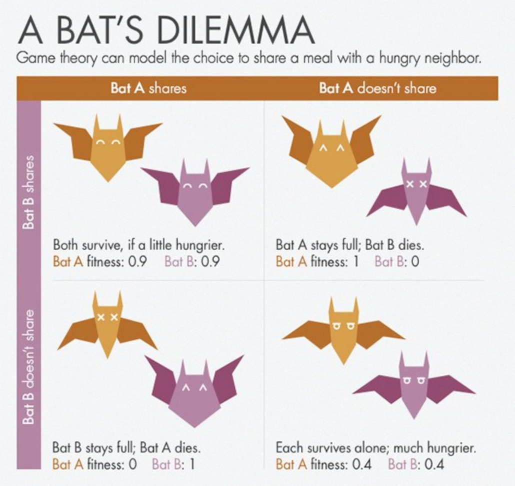



It is in the interests of each country, *regardless of what the other is doing*, to keep on polluting.

# Seeming counter-examples

- Cleaner fish
- Vampire bat
  - Few nights without food, will die
  - Roost together in caves
  - Regurgitate food for each other

# Vampire bat's dilemma

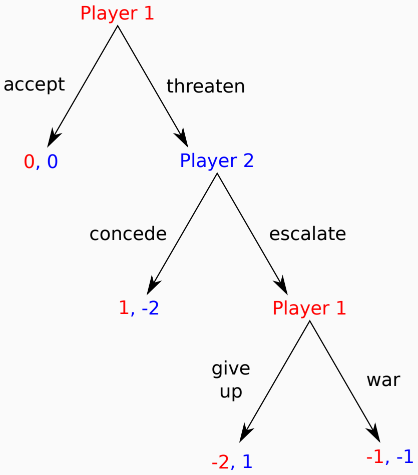



Olena Shmahalo/Quanta Magazine

*How do you behave in real life?*

# Backwards induction

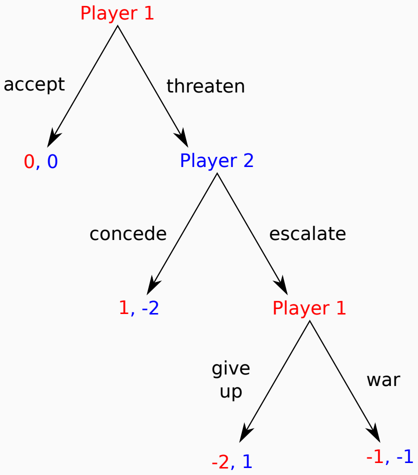
Extensive form game.



# Backwards induction

Extensive form game.

The optimal sequence of actions can be determined by working backwards in time, from the end of the problem to the start.



# Iterated Prisoners' Dilemma

*Exercise*: Let's play Prisoners' Dilemma

| | C | D |
|---|---|---|
| C | 3, 3 | 0, 5 |
| D | 5, 0 | 1, 1 |

# Iterated Prisoners' Dilemma (finite)

Consider a PD repeated for just 2 periods:

- In stage 2, all players must play defect. Why?
  - All prior payoffs are already locked in, sunk costs.
  - No future interactions.
  - Only goal is to maximise final-stage payoffs.
- In stage 1, the choice made does not influence payoffs in stage 2. Only goal is to maximise payoffs in stage 1. Therefore, play defect.
- Can extend this argument to any finite number of iterations.

Empty promises and non-credible threats.

*Any ideas how to fix this?*

# Iterated Prisoners' Dilemma

Fixed probability of playing again $\delta = 0.75$

Payoffs:

| | C | D |
|---|---|---|
| C | 3, 3 | 0, 5 |
| D | 5, 0 | 1, 1 |

# Discounted payoff

$\delta$: discount factor, future interaction might not happen

If player $i$'s payoff at time $t$ is $a_i(t)$, then the player's total expected payoff is

$$\sum_{t=0}^{\infty} \delta^t a_i(t).$$

**Reminder**: geometic series

If $|\delta| < 1$,

$$a + a\delta + a\delta^2 + \ldots = \sum_{t=1}^{\infty} a\delta^t = \frac{a}{1-\delta}.$$

# Grim Trigger

| | C | D |
|---|---|---|
| C | 3, 3 | 0, 5 |
| D | 5, 0 | 1, 1 |

Grim Trigger:

1. First, play C.
2. Continue playing C until anyone plays D, then play D forever.

**Q**: If we consider only three pure strategies — Grim, Cooperate, and Defect — is (Grim, Grim) a pure Nash equilibrium?

Reminders:

- NE: $\pi_1(G, G) \geq \pi_1(s_1, G)$ for $s_1 = C, D$
- Payoff: $\sum_{t=1}^{\infty} a\delta^t = \frac{a}{1-\delta}$

# Grim Trigger

Condition for (Grim, Grim) to be the Nash equilibrium:

$$\frac{3}{1-\delta} \geq 5 + \frac{\delta}{1-\delta},$$
$$\delta \geq \frac{1}{2}$$

Interpretation: Grim Trigger is an NE provided $\delta$ is high enough.

# Aside: subgame perfect equilibrium

You might have noticed something odd about my Grim rule:

1. First, play C.
2. Continue playing C until **anyone** plays D, then play D forever.

I did this to make sure that my version of Grim satisfied the 'one-stage deviation condition' and thus satisfied a strong condition, subgame perfect Nash equilibrium.

# Grim trigger

Grim Trigger:

1. First, play C.
2. Continue playing C until anyone plays D, then play D forever.

*Discuss: does anything seem a bit extreme about Grim?*

# The Axelrod Tournaments

- 1980s Robert Axelrod held a series of tournaments
  - Scientists could submit their code to play PD
  - Each algorithm would be played against each other for multiple rounds of PD
- Many clever algorithms submitted

R. Axelrod (left)

# The Axelrod Tournaments

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- Many clever algorithms submitted

R. Axelrod (left)

Winner - Anatol Rapoport - with a very simple program `tit-for-tat`:

1. First play is ‘cooperate’
2. Then repeat:
    2.1 If opponent played ‘defect’, next play is ‘defect’
    2.2 If opponent played ‘cooperate’, next play is ‘cooperate’

# Tit-for-tat exercise

*Exercise*: Consider only the pure strategies 'tit-for-tat' versus 'all-defect' in the iterated PD with payoffs below. Let $\delta = 0.5$.

Original PD

| | C | D |
|---|---|---|
| C | $8, 8$ | $0, 10$ |
| D | $10, 0$ | $1, 1$ |

Tit-for-tat versus all-defect

| | TT | all-D |
|---|---|---|
| TT | ? | ? |
| all-D | ? | ? |

1. Calculate payoffs, write in normal form
2. Find NE

Tips. NE $\pi_1(G, G) \geq \pi_1(s_1, G)$ for $s_1 = C, D$, Payoff $\sum_{t=1}^{\infty} a\delta^t = \frac{a}{1-\delta}$

# Tit-for-tat exercise

*Exercise*

Tit-for-tat versus all-defect

| | TT | all-D |
|---|---|---|
| TT | $16, 16$ | $1, 11$ |
| all-D | $11, 1$ | $2, 2$ |

2 Nash equilibria: TT and all-D.

# The Folk Theorem

**Folk Theorem**: If the Nash equilibrium in a static game is socially sub-optimal, players can always do better if the game is repeated and the discount factor is high enough.

Reasoning:

- If the NE is socially sub-optimal, there are some actions that can lead to a higher payoff pair.
- Use the rule:
  1. First, play the higher-payoff action
  2. Continue playing action until anyone deviates, then play Nash action forever.
- A deviator may gain a higher payoff in that stage, but that benefit is weighed against future loss

# Tit-for-tat

1. First play is ‘cooperate’
2. Then repeat:
    2.1 If opponent played ‘defect’, next play is ‘defect’
    2.2 If opponent played ‘cooperate’, next play is ‘cooperate’

# Tit-for-tat

1. First play is ‘cooperate’ - be nice
2. Then repeat:
    2.1 If opponent played ‘defect’, next play is ‘defect’
    2.2 If opponent played ‘cooperate’, next play is ‘cooperate’

# Tit-for-tat

1. First play is ‘cooperate’ - be nice
2. Then repeat:
    2.1 If opponent played ‘defect’, next play is ‘defect’ - be provokable
    2.2 If opponent played ‘cooperate’, next play is ‘cooperate’

# Tit-for-tat

1. First play is ‘cooperate’ - be nice
2. Then repeat:
    2.1 If opponent played ‘defect’, next play is ‘defect’ - be provokable
    2.2 If opponent played ‘cooperate’, next play is ‘cooperate’ - be forgiving

# Mixed strategies

A **mixed strategy** $\sigma$ specifies the probabilities $p(s)$ with which the pure strategies $s \in S$ are used, $\sigma = (p(s_a), p(s_b), \ldots)$.

**Example**: Matching coins. Each player places a 50c coin either heads or tails up. If the coins match, player 1 wins both coins; if the coins differ, player 2 wins.

| | H | T |
|---|---|---|
| H | $+1, -1$ | $-1, +1$ |
| T | $-1, +1$ | $+1, -1$ |

# Equality of payoffs

**Theorem**: Let $(\sigma_1^*, \sigma_2^*)$ be a Nash equilibrium, and let $S_1^*$ be the set of pure strategies for which $\sigma_1^*$ specifies $p(s) > 0$. Then $\pi_1(s, \sigma_2^*) = \pi_1(\sigma_1^*, \sigma_2^*)$ for all $s \in S_1^*$.

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Matching coins example:

| | H | T |
|---|---|---|
| H | $+1, -1$ | $-1, +1$ |
| T | $-1, +1$ | $+1, -1$ |

# Mixed strategies

*Exercise*: Find the mixed strategy NE given the payoffs below.

| | TT | all-D |
|---|---|---|
| TT | $16, 16$ | $1, 11$ |
| all-D | $11, 1$ | $2, 2$ |

# Updates previous definitions (replace $s_i$ with $\sigma_i$)

**Defn 1**: A pair of strategies $(\sigma_1^*, \sigma_2^*)$ is a **Nash equilibrium** if

$$\sigma_1^* \in \operatorname*{argmax}_{\sigma_1 \in \Sigma_1} \pi_1(\sigma_1, \sigma_2^*) \text{ and } \sigma_2^* \in \operatorname*{argmax}_{\sigma_2 \in \Sigma_2} \pi_1(\sigma_1^*, \sigma_2).$$

(mutual best response)

**Defn 2**: A 2-player **Nash equilibrium** is a pair of strategies $(\sigma_1^*, \sigma_2^*)$ such that

$$\pi_1(\sigma_1^*, \sigma_2^*) \geq \pi_1(\sigma_1, \sigma_2^*) \, \forall \sigma_1 \in \Sigma_1 \text{ and } \pi_2(\sigma_1^*, \sigma_2^*) \geq \pi_2(\sigma_1^*, \sigma_2) \, \forall \sigma_2 \in \Sigma_2.$$

(no profitable deviation)

**Nash's Theorem (1950)**: Every game that has a finite number of players with a finite number of pure strategies for each player has at least one Nash equilibrium, which may involve pure or mixed strategies.