Derangements: Solving Problems by Counting (Certain Types Of) Permutations

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Comment for the interpreters: I will start with an anecdote about Secret Santa and derangement. The only specific vocabulary here will be conditional probability.

Fixed points and derangements

A function f has a fixed point x if $f(x) = x$. This is also the case for permutations.

x	1	2	3	4	5	6	7	8	9	10
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A permutation without fixed point is a *derangement*.

x	1	2	3	4	5	6	7	8	9	10
$f(x)$	3	1	6	2	8	5	10	9	4	7

Notation for permutations

There are three main ways to write permutations:

Two-lines $$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\ 3 & 1 & 6 & 2 & 8 & 5 & 10 & 9 & 4 & 7 \end{pmatrix}$$

One-line $$(3, 1, 6, 2, 8, 5, 10, 9, 4, 7)$$

Cycle $$(1, 3, 6, 5, 8, 9, 4, 2)(7, 10)$$

In the cycle notation, a fixed point is a cycle of length 1.

The Secret Santa problem

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That is $\frac{d_n}{n!}$, where d_n is the number of derangements of n objects.

Counting derangements

How big is d_n?

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One can construct a derangement of n from a derangement of $n-2$ items or from a derangement of $n-1$ items:

- From a derangement of $\{1, 2, \ldots, n-2\}$, add two fix points and swap them.
- From a derangement of $\{1, 2, \ldots, n-1\}$, insert an element at one of the $n-1$ positions in an existing cycle.

These are all the ways to create a derangement.

Counting derangements

Example ($\sigma = (1345)(26)$)

The item 6 is in a 2-cycle, and (1345) *is a derangement of 4 objects.*

Example ($\sigma = (134)(265)$)

The item 6 is not in a 2-cycle, but it is appended to the cycle (52)*.* $(134)(25)$ *is a derangement of 5 objects.*

Counting derangements

Theorem (Recursive formula for derangements)

The number of derangements is $d_n = (n-1)(d_{n-1} + d_{n-2})$.

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Again, how does that solve the derangement problem?

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Theorem

The proportion of derangements is $\frac{d_n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k}$.

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Proof (by induction).

Base case: $n = 0$, $d_0 = 1$ ✓

Induction step:

$$\begin{aligned}
\frac{d_{n+1}}{(n+1)!} &= \frac{(n+1)d_n + (-1)^{n+1}}{(n+1)!} \\
&= \frac{d_n}{n!} + \frac{(-1)^{n+1}}{(n+1)!} \\
&= \sum_{k=0}^{n} \frac{(-1)^k}{k} + \frac{(-1)^{n+1}}{(n+1)!} = \sum_{k=0}^{n+1} \frac{(-1)^k}{k}
\end{aligned}$$

□

How is that helpful?

The sum is annoying, but we can remember this identity:

$$\lim_{n\to\infty} \sum_{k=0}^{n} \frac{x^k}{k} = e^x.$$

At $x = -1$, that tells us that $\lim_{n\to\infty} \frac{d_n}{n} = \frac{1}{e} \approx 0.37$.

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Conclusion: No matter how many people participate in your gift exchange, your Secret Santa drawing has roughly 37% chances of succeeding!

Recap

Recursive formula

$\downarrow$

(more useful) Recursive formula

$\downarrow$

Ratio (generating function)

$\downarrow$

Asymptotic formula

$\downarrow$

Closed formula

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Recursive formula

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Ratio (generating function)

$\downarrow$

Asymptotic formula

$\downarrow$

Closed formula

Theorem

The number of derangements is $d_n = \lfloor n! \cdot e^{-1} \rceil$, *if* $n \geq 1$.

A bijection

Definition

An *ascent* in a permutation is a value i such that $\sigma_i < \sigma_{i+1}$. n is always an ascent in a permutation of n.

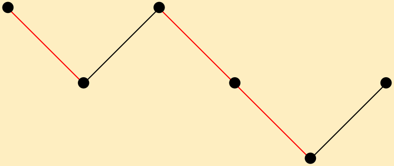

Figure: Ascents and descents of the permutations 435216 and 316524, among others

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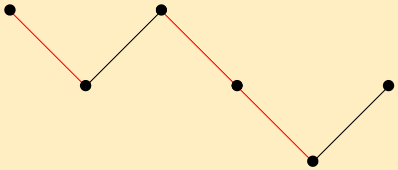

Figure: Ascents and descents of the permutations 435216 and 316524, among others

Theorem (Désarménien, 1982)

Derangements are in bijection with permutations whose first ascent is even.

Désarménien's claim

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Derangements are in bijection with permutations whose first ascent is even.

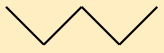
Caveat! This does not mean permutations whose first ascent is even are derangements. As a counter-example 435216 (from last page) is (14235)(6), and has a fixed point.

Désarménien's claim: the bijection

From derangements to desarrangements (i.e. permutations with first ascents even):

- Write the permutations in cycles (of length at least 2).
- Write the smallest item in each cycle in second position.
- Order the cycles in decreasing order ot their smallest item.
- Remove the parenthesis (concatenate the numbers, to go from cycle notation to one-line notation).

Example

24513 *has no fixed point, and* $24513 = (124)(35) = (53)(412)$*.*
The permutation 53412 *is a desarrangement (*  *).*

Désarménien's claim: the bijection

From desarrangements to derangements (i.e. the other way around):

- Read the permutation from right to left until you find 1. He is in a second position of the cycle going until the end.
- Repeat with the rest of the permutation, while looking at the smallest element not in the cycles already listed.

Example

53412 *is a desarrangement. The cycle containing 1 is* (412), *and* (53) *is the other cycle. So* 53412 *is sent to* $(53)(412) = (124)(35)$, *a derangement.*

Why bother?

Ascents, descents and valleys are interesting to people studying occurrences of patterns in permutations.

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My favorite example: card shuffling!

Random-to-top shuffling

Pick any card, put it on top, repeat... We can write a (transition) matrix with the probabilities to get from one permutation of the cards to the other:

$$\mathrm{R2T}_3 = \begin{array}{c|c} & \begin{array}{cccccc} [123] & [132] & [213] & [231] & [312] & [321] \end{array} \\ \begin{array}{c} [123] \\ [132] \\ [213] \\ [231] \\ [312] \\ [321] \end{array} & \begin{pmatrix} w_1 & 0 & w_2 & 0 & w_3 & 0 \\ 0 & w_1 & w_2 & 0 & w_3 & 0 \\ w_1 & 0 & w_2 & 0 & 0 & w_3 \\ w_1 & 0 & 0 & w_2 & 0 & w_3 \\ 0 & w_1 & 0 & w_2 & w_3 & 0 \\ 0 & w_1 & 0 & w_2 & 0 & w_3 \end{pmatrix} \end{array}$$

Theorem (Phatarfod, 1991)

The eigenvalues of the transition matrix of random-to-top are the partial sums of w_i's. For a sum of k terms, the eigenvalue has multiplicity d_{n-k}, the number of derangements.

If you like derangements (and number sequences, in general)

THE ON–LINE ENCYCLOPEDIA OF INTEGER SEQUENCES ®

founded in 1964 by N. J. A. Sloane

Derangements are sequence A000166

Thank you!