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Homomesies on permutations - an analysis of maps and statistics in the FindStat database

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Abstract. In this paper, we perform a systematic study of permutation statistics and bijective maps on permutations in which we identify and prove 122 instances of the homomesy phenomenon. Homomesy occurs when the average value of a statistic is the same on each orbit of a given map. The maps we investigate include the Lehmer code rotation, the reverse, the complement, the Foata bijection, and the Kreweras complement. The statistics studied relate to familiar notions such as inversions, descents, and permutation patterns, and also more obscure constructs. Beside the many new homomesy results, we discuss our research method, in which we used SageMath to search the FindStat combinatorial statistics database to identify potential homomesies.

Keywords: Homomesy, permutations, permutation patterns, dynamical algebraic combinatorics, FindStat, Lehmer code, Kreweras complement, Foata bijection

1 Introduction

Dynamical algebraic combinatorics is the study of objects important in algebra and combinatorics through the lens of dynamics. In this paper, we focus on permutations, which are fundamental objects in algebra, combinatorics, representation theory, geometry, probability, and many other areas of mathematics. They are intrinsically dynamical, acting on sets by permuting their components. Here, we study bijections on permutations $f : S_n \to S_n$, so the $n!$ elements being permuted are themselves permutations. In particular, we find and prove many instances of *homomesy* [7], an important phenomenon in dynamical algebraic combinatorics that occurs when the average value of some *statistic* (a meaningful map $g : S_n \to \mathbb{Z}$) is the same over each orbit of the action. Homomesy occurs in many contexts, notably that

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of rowmotion on order ideals of certain families of posets and promotion on various sets of tableaux. See [8, 12, 13] for further discussion.

A prototypical example of our homomesy results is as follows. Consider the Kreweras complement map $\mathcal{K} : S_n \to S_n$ (from Definition 7.1). We show in Proposition 7.10 that the last entry statistic of a permutation exhibits homomesy with respect to the Kreweras complement. See Figure 1 for an example of this result in the case $n = 3$.

orbit of size 1: $\begin{tikzcd} 312 \arrow[loop right, "\mathcal{K}"] \end{tikzcd}$ average of last entry $= 2$

orbit of size 2: $\begin{tikzcd} 123 \arrow[r, "\mathcal{K}"] & 231 \arrow[l, bend left] \end{tikzcd}$ average of last entries $= \frac{3+1}{2}$

orbit of size 3: $\begin{tikzcd} 132 \arrow[r, "\mathcal{K}"] & 213 \arrow[r, "\mathcal{K}"] & 321 \arrow[ll, bend left, "\mathcal{K}"] \end{tikzcd}$ average of last entries $= \frac{2+3+1}{3}$

Figure 1: Orbit decomposition of S_3 under the action of the Kreweras complement. The last entry of each permutation is highlighted. Calculating the averages of these last entries over each orbit, we observe an instance of homomesy.

Rather than pick actions and statistics at random to test for homomesy, we used FindStat [9], the combinatorial statistics database, which (at the time of writing) included 387 permutation statistics and 19 bijective maps on permutations. Using the interface with SageMath [11] computational software, we tested all combinations of these maps and statistics, finding 117 potential instances of homomesy, involving 68 statistics.

We highlight here some of the most interesting results. One initial finding was that homomesies occurred in only nine of the 19 examined maps. Among the maps that do not have any homomesic statistics, we find the well-known inverse map, as well as the first fundamental transform and cactus evacuation. Of the nine maps exhibiting homomesy, four maps (all related to the **Foata bijection** map) have only one homomesic statistic.

Even more intriguing is the large number of homomesies found for the **Lehmer code rotation** map. Despite its presence in FindStat, we could not find any occurrence of the Lehmer code rotation in the literature on combinatorial actions. The study in this paper suggests that this map is worthy of further investigation. Many of the homomesic statistics are related to inversions and descents, but other notable statistics include several *permutation patterns* as well as the *rank* of a permutation.

As we worked through the proofs for the homomesic statistics for the **reverse** and **complement** maps, we found that the global averages are often the same for the two maps. This prompted us to prove a new result which captures the relationship between the reverse and complement. We utilized this result to prove many of the shared homomesies. Given this strong relationship between the two maps, it is also of interest that there are several statistics that are only homomesic for one of the two maps.

In addition to exhibiting homomesies, the action of the **Kreweras complement** map generates an interesting orbit structure on S_n. Examining this orbit structure, we were able to characterize the distribution of all orbits of even size. As a consequence of our homomesy results for the Kreweras complement, there are no orbits of odd size when n is even. Finally, we have a complete characterization of orbit lengths.

Our main results are Theorems 4.6, 5.3, 5.4, 5.5, 6.4 and 7.3, in which we prove all 117 of these homomesies. In addition, we proved homomesy for 5 statistics not in the database, in Theorems 4.7, 5.6, 5.7, and 7.3, for a grand total of 122 homomesic statistics. Furthermore, we found theorems on orbit structure of the maps, chiefly Theorems 4.5 and 7.5.

This paper is organized as follows. In Section 2, we describe in detail our method of searching for potential homomesies. Section 3 contains background material on homomesy and permutations. Sections 4 through 7 contain our main results, namely, homomesies involving one or more related maps.

This is an extended abstract only; for the full version, see [3].

Acknowledgements

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2 Summary of methods

FindStat [9] is an online database of combinatorial statistics developed by Chris Berg and Christian Stump in 2011 and highlighted as an example of a fingerprint database in the Notices of the American Mathematical Society [1]. FindStat not only is a searchable database which collects information, but also yields dynamic information about connections between combinatorial objects. FindStat takes statistics input by a user (via the website [9] or the SageMath interface), uses SageMath to apply combinatorial maps, and outputs corresponding statistics on other combinatorial objects. FindStat has grown expansively to a total of 1787 statistics on 23 combinatorial collections with 249 maps among them (as of April 27, 2022).

For this project, we analyzed all combinations of bijective maps and statistics on one combinatorial collection: permutations. At the time of this investigation, there were 387 statistics and 19 bijective maps on permutations in FindStat. For each map/statistic pair, our empirical investigations either suggested a possible homomesy or provided a counterexample in the form of two orbits with differing averages. We then set about finding proofs for the experimentally identified potential homomesies. These homomesy results are the main theorems of this paper: Theorems 4.6, 5.3, 5.4, 5.5, 6.4 and 7.3.

Thanks to the already existing interface between SageMath [11] and FindStat, we were able to automatically search for pairs of maps and statistics that exhibited homomesic behavior. For each value of $2 \leq n \leq 6$, we ran the following code to identify potential homomesies:

```
sage: from sage.databases.findstat import FindStatMaps, FindStatStatistics
....: findstat()._allow_execution = True  # To run all the code from Findstat
....: for map in FindStatMaps(domain="Cc0001", codomain="Cc0001"):  # Cc0001=Permutations
....:     if map.properties_raw().find('bijective') >= 0:  # Restrict to bijective maps
....:         F = DiscreteDynamicalSystem(Permutations(n), map)   # Fix n ahead of time
....:         for stat in FindStatStatistics("Permutations"):
....:             if F.is_homomesic(stat):
....:                 print(map.id(), stat.id())
```

Note that the choice of running the verification of homomesies for permutations of 2 to 6 elements is not arbitrary: at $n = 6$, the computational results stabilize. We did not find any false positive when using the data from $n = 6$. On the other hand, testing only smaller values of n would have given us many false positives. For example, several statistics in the FindStat database involve the number of occurrences of some permutation patterns of length 5. These statistics evaluate at to 0 for any permutation of fewer than 5 elements, which misleadingly makes them appear 0-mesic if not tested for values of n at least 5.

Descriptions of each map and statistic, including references and important properties, can be found through the FindStat website using the associated FindStat identifier (ID). In this paper, we reference the map IDs. IDs for the homomesic statistics can be found in the longer version of this work [3]. The FindStat database attributes IDs to statistics and maps sequentially; when we did the investigation, the maximum statistic ID in the FindStat database for a statistic on permutations was 1778, and the maximum map ID for a bijective map on permutations was 241. There were four statistics for which we could not disprove homomesy, because the database did not provide values for them on permutations of at least 5 items, nor code to evaluate these statistics; they all correspond to the dimension of some vector spaces.

3 Background

Permutations are a central object in combinatorics, and many statistics on them are well-studied. We refer to standard combinatorics texts for basic definitions.

First defined in 2015 by James Propp and Tom Roby [7], homomesy relates the average of a given statistic over some set, to the averages over orbits formed by a bijective map.

Definition 3.1. Given a set S, an element $x \in S$, and an invertible map $\mathcal{X} : S \to S$, the **orbit** $\mathcal{O}(x)$ is the set $\{y \in S \mid y = \mathcal{X}^i(x) \text{ for some } i \in \mathbb{Z}\}$. That is, $\mathcal{O}(x)$ is the subset of

S reachable from x by applying $\mathcal{X}$ or $\mathcal{X}^{-1}$ any number of times. The **size** of an orbit is its cardinality $|\mathcal{O}(x)|$.

When the set S is finite, each orbit is finite and the invertible function $\mathcal{X}$ is bijective, so S can be realized as a disjoint union of its orbits. Moreover, the **order** of $\mathcal{X}$ is the least common multiple of the cardinalities of the orbits.

Definition 3.2 ([7])**.** Given a finite set S, a bijective map $\mathcal{X} : S \to S$, and a statistic $f : S \to \mathbb{Z}$, we say that $(S, \mathcal{X}, f)$ exhibits **homomesy** if there exists $c \in \mathbb{Q}$ such that for every orbit $\mathcal{O}$, $\frac{1}{|\mathcal{O}|}\sum_{x\in\mathcal{O}} f(x) = c$ where $|\mathcal{O}|$ denotes the number of elements in $\mathcal{O}$. If such a c exists, we say the triple is **c-mesic**.

When the set S is clear from context, we may say a statistic is **homomesic with respect to** $\mathcal{X}$ rather than explicitly stating the triple. When the map $\mathcal{X}$ is also implicit, we may simply say a statistic is **homomesic**. Homomesy may be generalized beyond the realms of bijective actions and integer statistics, but we will not address these in this paper. Note that whenever a statistic is homomesic, the orbit-average value is indeed the global average.

Since the homomesy phenomenon was defined, mathematicians have looked for it on natural combinatorial objects. Permutations indeed arose as such a structure, and some recent work initiated the study of homomesic statistics on permutations. Michael La Croix and Tom Roby [6] focused on the statistic counting the number of fixed points in a permutation, while they looked at compositions of the first fundamental transform with what they call "dihedral actions", a few maps that include the complement, the inverse and the reverse, all discussed in Section 5. Simultaneously, Elizabeth Sheridan-Rossi considered a wider range of statistics for the same maps, as well as for the compositions of dihedral actions with the Foata bijection [10].

Our approach differs from the previous studies by being more systematic. As previously mentioned, we proved or disproved homomesy for all the 7,345 combinations of a bijective map and a statistic on permutations that were in the FindStat database. It is worth noting that the interesting maps described in [6] and [10] are compositions of FindStat maps, but they are not listed as single maps in FindStat. We did not consider compositions of FindStat maps; this would be an interesting avenue for further study.

4 Lehmer code rotation

A famous way to describe a permutation is through its inversions. The Lehmer code of a permutation (defined below) captures this inversion information, providing a bijection between Lehmer codes and permutations. In this section, we describe the Lehmer code rotation map and list 45 statistics that are homomesic with respect to this map. Despite its presence in FindStat, we could not find the Lehmer code rotation map in the literature. We,

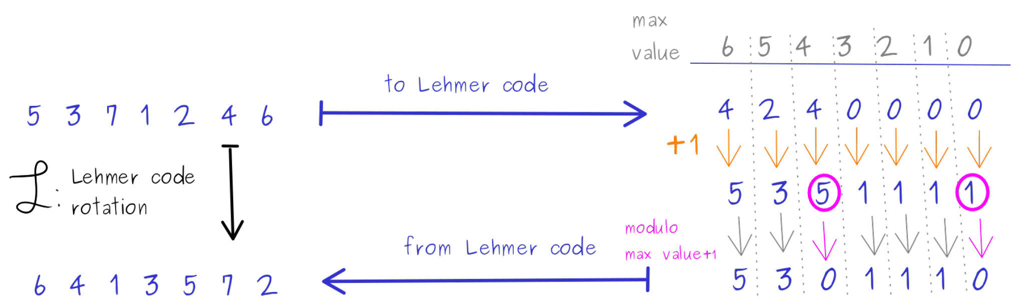

Figure 2: The Lehmer code rotation applied on the permutation 5371246 yields the permutation 6413572. The step-by-step process is illustrated on this picture.

however, found six similar maps in a paper by Vincent Vajnovszki [14]. Those were not in the FindStat database; we tested them anyway and found they did not exhibit interesting homomesies. This makes our findings about the 45 homomesic statistics under the Lehmer code rotation even more intriguing.

Definition 4.1. The **Lehmer code** of a permutation σ is:

$$L(\sigma) = (L(\sigma)_1, \dots, L(\sigma)_n) \quad \text{where} \quad L(\sigma)_i = \#\{j > i \mid \sigma_j < \sigma_i\}.$$

It is well known that there is a bijection between tuples of length n whose entries are integers between 0 and $n - i$ at position i and permutations. Hence, the Lehmer code uniquely defines a permutation. Since the entries of the Lehmer code count the number of inversions that start at each entry of the permutation, the number of inversions in the permutation σ is given by $\sum_{i=1}^{n} L(\sigma)_i$.

Example 4.2. The Lehmer code of the permutation 31452 is $L(31452) = (2, 0, 1, 1, 0)$, whereas the Lehmer code of 42513 is $L(42513) = (3, 1, 2, 0, 0)$.

Definition 4.3. The **Lehmer code rotation** (FindStat map 149) is a map that sends a permutation σ to the unique permutation τ (of the same set) such that every entry in the Lehmer code of τ is cyclically (modulo $n + 1 - i$) one larger than the Lehmer code of σ. This means that the entries of the Lehmer code change from $L(\sigma)_i$ to $L(\tau)_i = L(\sigma)_i + 1 \mod (n - i + 1)$.

An example is illustrated in Figure 2.

Example 4.4. The permutation $\sigma = 31452$ has Lehmer code $L(\sigma) = (2, 0, 1, 1, 0)$. Hence,

$$\begin{aligned} L(\mathcal{L}(\sigma)) &= (2 + 1 \bmod 5,\ 0 + 1 \bmod 4,\ 1 + 1 \bmod 3,\ 1 + 1 \bmod 2,\ 0 + 1 \bmod 1) \\ &= (3, 1, 2, 0, 0). \end{aligned}$$

Because $(3, 1, 2, 0, 0)$ is the Lehmer code of the permutation 42513, $\mathcal{L}(31452) = 42513$.

A noticeable fact about the Lehmer code rotation is that all its orbits have the same size:

Theorem 4.5 (Orbit cardinality)**.** *All orbits of the Lehmer code rotation have size* $\mathrm{lcm}(1, 2, \ldots, n)$.

The main theorem of this section is the following.

Theorem 4.6. *The Lehmer code rotation map exhibits homomesy for* 45 *statistics found in the FindStat database. The full list is in the long version of this paper [3, Theorem 4.6].*

- *Five statistics related to inversions, including the inversion number and the number of inversions of distance at most* 2*, but not the number of inversions of distance at most* 3*.*
- *19 statistics related to descents, including the major index, the number of descents, the number of peaks, the number of valleys, the number of runs, the number of ascents and descents of distance* 2*, and the number of odd and even descents.*
- *11 statistics representing the number of occurrences of some permutation patterns: including the pattern* $21-3$*, the pattern* 123 *or the pattern* 132*, the pattern* $14-2-3$ *or the pattern* $14-3-2$*, and the vincular pattern* $|1-23$*.*
- *Ten miscellaneous statistics, including the number of right-to-left maxima and minima, the rank (among the permutations, in lexicographic order), and the first entry.*

On top of the statistics in the database, we proved homomesy for two classes of statistics that are not in the database.

Theorem 4.7. *The Lehmer code rotation map exhibits homomesy for the following two classes of statistics not found in the FindStat database: The number of inversions starting at the i-th entry of σ is $\frac{n-i}{2}$-mesic, for $1 \leq i < n$; and the number of descents at position i is $\frac{1}{2}$-mesic, for $1 \leq i < n$.*

Notice that the first entry of a permutation is homomesic for the Lehmer code rotation, unlike the last entry (also in the database). Indeed, the first entry of a permutation σ can be read from its Lehmer code, since $\sigma(i) = L(\sigma)_1 + 1$. However, the last entry of a permutation does not have such a simple interpretation in the Lehmer code. Similarly, the number of right-to-left maxima (resp. minima) is homomesic, unlike the number of left-to-right maxima (resp. minima). It is worth mentioning that right-to-left maxima (resp. minima) occur exactly when $L(\sigma)_i = n - i$ (resp. $L(\sigma)_i = 0$), but the left-to-right maxima and minima do not appear obviously in the Lehmer code.

We now turn the spotlight to an interesting statistic homomesic for the Lehmer code rotation:

Definition 4.8. The **rank** of a permutation of $[n]$ is its position among the $n!$ permutations, ordered lexicographically. This is an integer between 1 and $n!$.

To prove homomesy for the rank under the Lehmer code rotation, we describe the connection between the rank and the Lehmer code. This seems to be a known fact, but we could not find a proof in the literature.

Lemma 4.9. *For a permutation σ of $[n]$, the rank of σ is given directly by the Lehmer code $L(\sigma)$ as* $\operatorname{rank}(\sigma) = 1 + \sum_{i=1}^{n-1} L(\sigma)_i (n-i)!$.

The above lemma really is the key to prove homomesy for the rank under the Lehmer code rotation. Interestingly, we also needed this description to prove that the rank is homomesic for the complement map (but it is not for the reverse).

5 Complement and Reverse Maps

The complement and reverse maps have many homomesic statistics in common. The relationship between these provides a way of simultaneously proving many of these shared homomesies. Also of interest are the statistics which exhibit homomesy under the complement map, but not the reverse, or vice versa. We begin by introducing the maps and main theorems.

Definition 5.1. Let $\sigma = \sigma_1 \dots \sigma_n$. The **reverse** (FindStat map 64) is the map that send σ to $\mathcal{R}(\sigma) = \sigma_n \dots \sigma_1$. That is, $\mathcal{R}(\sigma)_i = \sigma_{n+1-i}$. The **complement** (FindStat map 69) is the map that send σ to $\mathcal{C}(\sigma) = (n+1-\sigma_1)\dots(n+1-\sigma_n)$. That is, $\mathcal{C}(\sigma)_i = n+1-\sigma_i$.

Example 5.2. Let $\sigma = 52134$. Then $\mathcal{R}(\sigma) = 43125$ and $\mathcal{C}(\sigma) = 14532$.

The main theorems of this section are as follows. The full theorem statements appear in the long version of this paper [3, Section 5].

Theorem 5.3. *The reverse map and the complement map are both homomesic under:*

- *Nine statistics related to inversions, including the number of inversions of a permutation, the number of even and odd inversions, and the inversion sum;*
- *Seven statistics related to descents, including the number of descents, the number of descents and ascents of distance* 2*, and the number of runs;*
- *Six statistics related to other permutation properties, including the number of inverse descents; and the number of occurrences of one of the patterns* 132*,* 213*, or* 321*.*

Theorem 5.4. *There are 13 statistics where the complement map is homomesic, but the reverse map is not. These include the major index and the rank.*

Theorem 5.5. *There are five statistics where the reverse map is homomesic, but the complement map is not. These include the inverse major index and the disorder.*

The following two theorems are for statistics that are not found in FindStat.

Theorem 5.6. *The number of inversions of the i-th entry of a permutation is $\frac{n-i}{2}$-mesic under the complement.*

Theorem 5.7. *The i-th entry of the permutation is $\frac{n+1}{2}$-mesic under the complement.*

To end this section, we look more closely at a statistic with a proof technique that deviated from those used for the other statistics.

Definition 5.8. Given a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$, cyclically pass through the permutation left to right and remove the numbers $1, 2, \ldots, n$ in order. The **disorder** of the permutation is then defined by counting the number of times a position is not selected and summing that over all the positions.

Example 5.9. Let $\sigma = 12543$. In the first pass, 54 remains. In the second pass only 5 remains. In the third pass, nothing remains. Thus the disorder of σ is 3.

In proving that the disorder of a permutation is homomesic for the reverse, we had to alter our standard method of counting all possibilities of the statistic and dividing by two. Instead of applying that technique to the disorder of a permutation, we instead used the fact that all possible inversion pairs of the form $(i+1, i)$ occur exactly once in either the permutation or its reverse. This was useful because each pass through the permutation to count for disorder ends when encountering an inversion pair of the form $(i+1, i)$. Thus the sum for disorder can be split into parts based on those inversion pairs. Any inversion pair $(i+1, i)$ contributes $n-i$ to the total disorder since when i is removed, $i+1, i+2, \ldots, n$ remain. So, when summing the disorder over both σ and $\mathcal{R}(\sigma)$, every possible inversion pair of the form $(i+1, i)$ occurs exactly once. Thus the disorder over σ and $\mathcal{R}(\sigma)$ can be found by the sum $\sum_{i=1}^{n-1} n - i = 1 + 2 + \ldots + (n-1)$, and the average over the orbit is $\frac{n(n-1)}{4}$.

6 Foata bijection and variations

This section examines homomesies of the following statistic under the Foata bijection and a related map, both of which send the inversion number to the major index.

Definition 6.1. The **Foata bijection** $\mathcal{F}$ (Map 67) is defined recursively on n. Given a permutation $\sigma = \sigma_1\sigma_2\ldots\sigma_n$, compute the image inductively by starting with $\mathcal{F}(\sigma_1) = \sigma_1$. At the i-th step, if $\mathcal{F}(\sigma_1\sigma_2\ldots\sigma_i) = \tau_1\tau_2\ldots\tau_i$, define $\mathcal{F}(\sigma_1\sigma_2\ldots\sigma_i\sigma_{i+1})$ by placing σ_{i+1} at the end of $\tau_1\tau_2\ldots\tau_i$ and breaking into blocks as follows: (1) Place a vertical line to the left of τ_1. (2) If $\sigma_{i+1} \geq \tau_i$, place a vertical line to the right of each τ_k for which $\sigma_{i+1} \geq \tau_k$. (3) If $\sigma_{i+1} < \tau_i$, place a vertical line to the right of each τ_k for which $\sigma_{i+1} < \tau_k$. (4) Then, within each block between vertical lines, cyclically shift the entries one place to the right.

Example 6.2. To compute $\mathcal{F}(31542) = 53412$, the sequence of words is:

$$3 \to 3 \qquad |3|1 \to 31 \qquad |3|1|5 \to 315 \qquad |315|4 \to 5314 \qquad |5|3|14|2 \to 53412.$$

Definition 6.3. The **Lehmer-code-to-major-code** map $\mathcal{M}$ (Map 62) sends a permutation to the unique permutation such that the Lehmer code is sent to the major code.

Theorem 6.4. *The statistic* maj − inv*, which is the difference of the major index and the inversion number, is 0-mesic with respect to each of the following maps and their inverses: the Lehmer-code-to-major-code bijection and the Foata bijection.*

7 Kreweras and inverse Kreweras complements.

The Kreweras complement, introduced in 1972 as a bijection on noncrossing partitions [5], can be understood geometrically as rotating an associated noncrossing matching on $2n$ elements and finding the resulting noncrossing partition [4]. The action of the Kreweras complement may be extended to all permutations as follows.

Definition 7.1. Let σ be a permutation of n elements. The **Kreweras complement** of σ, $\mathcal{K}(\sigma)$, and its inverse (maps 88 and 89 in the FindStat database) are defined as

$$\mathcal{K}(\sigma) = c \circ \sigma^{-1} \qquad \text{and} \qquad \mathcal{K}^{-1}(\sigma) = \sigma^{-1} \circ c,$$

where c is the long cycle $234 \dots 1$.

Example 7.2. Consider $\sigma = 43152$. By definition, $\mathcal{K}(43152) = 23451 \circ 35214 = 41325$.

In [2], the number of disjoint sets in a noncrossing partition of n elements is shown to be $\frac{n+1}{2}$-mesic under a large class of operations which can be realized as compositions of toggles, including the Kreweras complement. Here, we study the generalized action of the Kreweras complement on permutations, proving the following homomesy results.

Theorem 7.3. *The Kreweras complement and its inverse exhibit homomesy for the number of exceedances of a permutation, the number of weak deficiencies of a permutation, the last entry of the permutation, and when n is even, the $\frac{n}{2}$-th element of the permutation.*

Before proving these homomesy results, we describe in detail the orbit structure under the Kreweras complement. From the definition of $\mathcal{K}$ and $\mathcal{K}^{-1}$,

$$\mathcal{K}^j(\sigma) = \begin{cases} c^{\frac{j}{2}} \circ \sigma \circ c^{-\frac{j}{2}} & \text{if } j \text{ is an even integer,} \\ c^{\frac{j+1}{2}} \circ \sigma^{-1} \circ c^{-\left(\frac{j-1}{2}\right)} & \text{if } j \text{ is an odd integer.} \end{cases} \tag{7.1}$$

As a consequence of this observation, we have the following theorem.

Theorem 7.4 (Order)**.** *For all $n > 2$, $\mathcal{K}$ and $\mathcal{K}^{-1}$ have order $2n$ as elements of S_{S_n}.*

The following theorem gives the number of even sized orbits for each even divisior of $2n$.

Theorem 7.5 (Orbit cardinality)**.** *For each even divisor $2k$ of $2n$, the number of orbits of size $2k$ under the action of the Kreweras complement is equal to $\frac{\binom{n}{k}^k k!-T}{2k}$, where $\binom{n}{k}^k k!$ is the number of elements in S_n fixed by $\mathcal{K}^{2k}$ and T is the number of elements in orbits of size d for each proper divisor d of $2k$.*

As we prove in Corollary 7.8, there are no orbits of odd size when n is even. Thus, Theorem 7.5 completely characterizes the distribution of orbits when n is even.

Definition 7.6. Given a permutation $\sigma \in S_n$, an index i is said to be an **exceedance** of σ if $\sigma_i > i$ and a **weak deficiency** or **anti-exceedance** if $\sigma_i \leq i$.

Proposition 7.7. *For the Kreweras complement and its inverse acting on S_n, the number of exceedances is $\frac{n-1}{2}$-mesic while the number of weak deficiencies is $\frac{n+1}{2}$-mesic.*

A key step in the proof of Proposition 7.7 is showing that the number of exceedances for each element of an orbit of odd size is the constant value $\frac{n-1}{2}$. Since the number of exceedances must be an integer, we arrive at a contradiction when n is even.

Corollary 7.8. *If n is even, there are no orbits of odd size under the Kreweras complement or its inverse acting on S_n.*

Definition 7.9. Given a permutation $\sigma \in S_n$, define the lower middle element to be $\sigma_{\frac{n}{2}}$ when n is even, and $\sigma_{\frac{n+1}{2}}$ when n is odd.

Proposition 7.10. *The last entry of a permutation, and when n is even, the lower middle element, are $\frac{n+1}{2}$-mesic under the Kreweras complement and its inverse.*

When we ran the experiment, the only entries of a permutation that were statistics in FindStat were the last entry and the first entry. Our homomesy result for the lower middle element was found analytically. Since then, the upper middle entry of a permutation ($\sigma_{\lceil \frac{n+1}{2} \rceil}$), and the lower middle entry ($\sigma_{\lfloor \frac{n+1}{2} \rfloor}$), have been added to the FindStat database. As noted in the following corollary, the homomesies from Proposition 7.10 are the only i-th entry homomesies possible for the Kreweras complement.

Corollary 7.11. *No entry except the last entry, and the $\frac{n}{2}$-th entry when n is even, is homomesic with respect to Kreweras complement.*

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