Differential equations as mathematical models

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ENGR 213 Lecture 6
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Mathematical models

The mathematical description of a social, physical or economic phenomenon is called a **mathematical model**, and is generally made of equations.
It is a simplification of reality, even though it can be very accurate if we take into account the right variables.

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7. **Solve** the differential equation.
8. If needed, **adjust your model** by taking more variables into account. Example: Friction, or air resistance in physical models.

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- Springs - Newton's second law of motion and Hooke's law

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- Remark: the simplified model with only births is useful to model populations that reproduce fast over a short period of time, such as bacteria or invasive species

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- Adjustments: We could still take migration into account.

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- Adjustments: This assumes that the object is small enough compared to the medium as not to alter the temperature of the medium.

Example problem (on the board)

When a cake is removed from an oven, its temperature is measured at 300°F. Three minutes later its temperature is 200°F. How long will it take for the cake to cool off to 100°F in a room at a temperature of 70°F?

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  Issue: x and y both vary over time. A hypothesis one can make is that, in a small town where none of the n people who live there is infected comes an infected person. Then $x + y = n + 1$, so $y = n + 1 - x$, and we reduce to a single variable.

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- The new equation $\frac{dx}{dt} = R_0 x(n + 1 - x)$ is a Bernoulli equation: $(\frac{dx}{dt} - R_0(n + 1)x = -R_0 x^2)$, so we can solve it through substitution.

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- Adjustments can be made for taking into account that the virus is evolving, or that people don't interact with each other uniformly in a community.

Mixtures

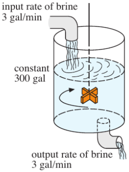

FIGURE 1.3.3 Mixing tank

Picture credit: Zill's textbook.

Problem

A solution containing some concentration of salt is slowly poured into another one with a different concentration into a leaky tank. What is leaking is the mixed solution. What is the new concentration of the solution?

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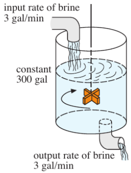

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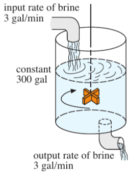

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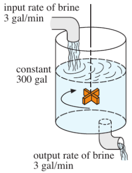

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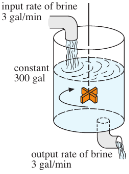

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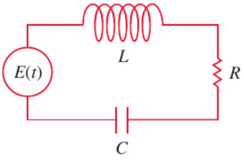
Example problem (solved on the board)

A tank contains 200 liters of fluid in which 30 grams of salt is dissolved. Brine containing 1 gram of salt per liter is then pumped into the tank at a rate of 4 L /min; the well-mixed solution is pumped out at the same rate. Find the number $A(t)$ of grams of salt in the tank at time t.

Electrical circuits



(a) *LRC*-series circuit

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A series circuit contains an inductor, a resistor and a capacitor. What is the charge on the capacitor, $q(t)$*, at a given time?*

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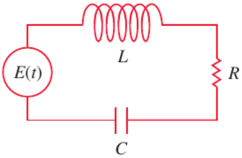

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Hypothesis

Kirchoff's laws:

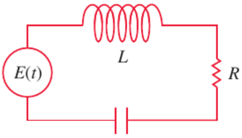
$$i = i_L = i_R = i_C, \qquad E(t) = u_L + u_R + u_C.$$

Also,

$$u_L = L\frac{d_i}{dt}, u_R = Ri, u_C = \frac{1}{C}q,$$

where L is the inductance, R the resistance and C the capacitance. The charge is related to the current through $\frac{dq}{dt} = i(t)$.

Electrical circuits (continued)



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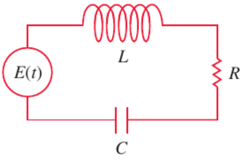

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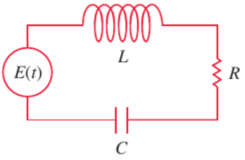
- Plugging $i = \frac{dq}{dt}$ into $E(t)$:

$$E(t) = L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q,$$

which is a second-order differential equation.

Electrical circuits (continued)



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  which is a second-order differential equation.
- Solving second-order differential equations is hard, but we can easily solve LR-circuits of RC-circuits, since they give rise to the first-order equations

$$E(t) = L\frac{di}{dt} + Ri \text{ and } E(t) = R\frac{dq}{dt} + \frac{1}{C}q.$$

Drainage of a tank

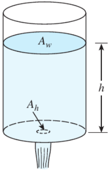

FIGURE 1.3.4 Water draining from a tank

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A tank (such as a bath tub) is emptying itself through a sharp hole. If the water is filled up to height h, how is h changing over time?

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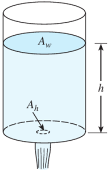

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Hypothesis (Torricelli's law)

The speed v of efflux of water through a sharp-edged hole at the bottom of a tank filled to a depth h is the same as the speed that a body (in this case a drop of water) would acquire in falling freely from a height h; that is, $v = \sqrt{2gh}$*, where g is the acceleration due to gravity.*

Drainage of a tank (continued)

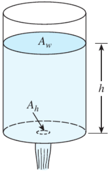

FIGURE 1.3.4 Water draining from a tank

Picture credit: Zill's textbook.

Hypothesis (Torricelli's law)

The speed of efflux of water is $v = \sqrt{2gh}$, where g is the acceleration due to gravity.

Drainage of a tank (continued)

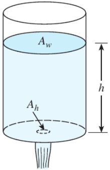

FIGURE 1.3.4 Water draining from a tank

Picture credit: Zill's textbook.

Hypothesis (Torricelli's law)

The speed of efflux of water is $v = \sqrt{2gh}$, where g is the acceleration due to gravity.

- Variable(s): h is changing. The area of the hole, A_h, impacts the volume that goes out. The volume of the tank, V, helps to measure the height of the water that is left through $V = hA_w$, where A_w is the area of a cross-section of the tank (on the picture here, a cylinder).

Drainage of a tank (continued)

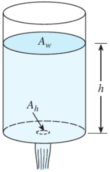

FIGURE 1.3.4 Water draining from a tank

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- Equation: On one hand, $\frac{dV}{dt} = \frac{dh}{dt}A_w$, if A_w is constant (if the tank is a cube or a cylinder). Also, $\frac{dV}{dt} = -A_h v = -A_h\sqrt{2gh}$. Solving for $\frac{dh}{dt}$, we get $\frac{dh}{dt} = -\frac{A_h\sqrt{2gh}}{A_w}$, which is a separable equation (when the volume is proportional to the height.)