BUSN 33921: Problem Set 2

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Problem 1

From class, the non-sequential Stigler search model gives that the expected minimum price from n searches is

$$E_n[\min\{p\}] = \int_{\underline{p}}^{\bar{p}} np(1-F(p))^{n-1}f(p)dp$$

Actually solving out the integral using integration by parts

$$\begin{aligned}
E_n[\min\{p\}] &= [-p(1-F(p)^n]_{\underline{p}}^{\bar{p}} - \int_{\underline{p}}^{\bar{p}} -(1-F(p))^n dp \\
&= \underline{p} + \int_{\underline{p}}^{\bar{p}} (1-F(p))^n dp
\end{aligned}$$

We are asked whether the marginal benefit is decreasing in the number of searches. The marginal benefit of an additional search when you have searched n times is

$$\frac{\partial E_n[\min\{p\}]}{\partial n}$$

and we show that this is decreasing over time. In other words that

$$\frac{\partial[E_{n-1}[\min\{p\}] - E_n[\min\{p\}]}{\partial n} < 0$$

Solving for this:

$$\begin{aligned}
\frac{\partial[E_{n-1}[\min\{p\}] - E_n[\min\{p\}]}{\partial n} &= \frac{\partial}{n}\left[\int_{\underline{p}}^{\bar{p}} (1-F(p))^{n-1} - (1-F(p))^n dp\right] \\
&= \frac{\partial}{\partial n}\left[\int_{\underline{p}}^{\bar{p}} (1-F(p))^n \frac{F(p)}{1-F(p)} dp\right] \\
&= \int_{\underline{p}}^{\bar{p}} \ln(1-F(p))(1-F(p))^{n-1}F(p)dp
\end{aligned}$$

$(1-F(p))F(p) \geq 0$ for all p and $\ln(1-F(p)) \leq 0$ for all p with some $p \in (\underline{p}, \bar{p})$ such that those inequalities are strict. This gives that the entire term is negative, giving that the marginal value of additional searching (n) is strictly decreasing over time. This makes sense because the probability that you hit a price that is worse than the best price you've seen before increases as you uncover more price and your best price gets better.

Problem 2

a

With no revisiting, the Bellman for the optimal sequential search is as follows

$$V(p) = \min\left\{p, k + \int_{\underline{p}}^{\bar{p}} V(p)f(p)dp\right\}$$

b

The value of option ii.) is a positive constant because the distribution is unchanging and there is no discounting. Intuitively, the value of 'waiting' each period is just the same luck of the draw of drawing a new price independently next period. This implies there is a cricial price z where the value function has the same value under both i.) and ii.). Since every period we choose between our price draw, p, and the same constant, $\int_{\underline{\mathrm{p}}}^{p} V(p)f(p)dp$ this means that at some $p = z$, the agent is indifferent between taking the draw and waiting, implying

$$z = k + \int_{\underline{p}}^{\bar{p}} V(p)f(p)dp$$

c

$$V(p) = \begin{cases} p & \text{if } p \leq z \\ z & \text{if } p > z \end{cases}$$

d

Substituting c into b

$$\begin{aligned}
z &= k + \int_{\underline{p}}^{\bar{p}} V(p)f(p)dp \\
&= k + \int_{\underline{p}}^{z} pf(p)dp + \int_{z}^{\bar{p}} zf(p)dp \\
z\left(\int_{\underline{p}}^{\bar{p}} f(p)dp - \int_{z}^{\bar{p}} f(p)dp\right) &= k + \int_{\underline{p}}^{z} pf(p)dp \\
z\int_{\underline{p}}^{z} f(p)dp &= k + \int_{\underline{p}}^{z} pf(p)dp \\
\implies k &= \int_{\underline{p}}^{z} (z-p)f(p)dp
\end{aligned}$$

The optimal policy is the same whether or not revisit is allowed because essentially if we haven't found a good enough price, even when revisit is allowed, the value of searching again is higher than revisiting any price that didn't previously make you cease the search.

Problem 3

a

The demand function from the paper is

$$q_j = \frac{1}{n}G(x_{n+1}) - \frac{1}{j}G(x_j) + \sum_{i=j+1}^{n} \frac{1}{i(i-1)}G(x_i)$$

Taking the derivative with respect to own-price gives

$$\frac{\partial q_j}{\partial p_j} = -\frac{1}{j}g(x_j)\frac{\partial x_j}{\partial p_j} + \sum_{i=j+1}^{n} \frac{1}{i(i-1)}g(x_i)\frac{\partial x_i}{\partial p_j}$$

reminder that from equation (4) in the paper we have

$$\frac{\partial x_k}{\partial p_j} = \begin{cases} -1/n & j < k \\ (j-1)/n & j = k \\ 0 & j > k \end{cases}$$

Using this to plug in we get

$$\begin{aligned}\frac{\partial q_j}{\partial p_j} &= -\frac{1}{j}g(x_j)(j-1)/n + \sum_{i=j+1}^{n} \frac{1}{i(i-1)}g(x_i)(-1/n) \\ &= -\left[\frac{j-1}{jn}\right]g(x_j) - \frac{1}{n}\sum_{i=j+1}^{n} \frac{1}{i(i-1)}g(x_i)\end{aligned}$$

Since $g()$ is a pdf, it will always be positive. $j-1 \geq 0$, $n > 0$, $i(i-1) > 0$ so the entire term $\frac{\partial q_j}{\partial p_j} \geq 0$.

b

Now consider the special case where there is a continuum of consumers of mass M and search costs uniformly distributed between 0 and some positive number K. the expressions for demand and own-price demand derivative.

Ignoring the mass, we now have an expression for $G(x) = \frac{x}{K}$, $G(x_{n+1}) = 1$ and $g(x) = \frac{1}{K}$so demand is now

$$\begin{aligned}q_j &= \frac{1}{n} - \frac{1}{j}\frac{x_j}{K} + \sum_{i=j+1}^{n} \frac{1}{i(i-1)}\frac{x_i}{K} \\ &= \frac{1}{K}\left[\frac{K}{n} - \frac{x_j}{j} + \sum_{i=j+1}^{n} \frac{1}{i(i-1)}x_i\right]\end{aligned}$$

Plugging in using equation (3) from the paper we have

$$
\begin{aligned}
x_j &= \sum_{i=1}^{j-1} (p_j - p_i) f(p_i) \\
&= \left[p_j - \sum_{i=1}^{j-1} \frac{p_i}{j-1} \right] \frac{j-1}{n}
\end{aligned}
$$

we now have

$$
\begin{aligned}
q_j &= \frac{1}{K} \left[\frac{K}{n} - \frac{j-1}{jn} \left[p_j - \sum_{i=1}^{j-1} \frac{p_i}{j-1} \right] + \sum_{i=j+1}^{n} \frac{1}{i(i-1)} \frac{i-1}{n} \left(p_i - \sum_{k=1}^{i-1} \frac{p_k}{i-1} \right) \right] \\
&= \frac{1}{Kn} \left[K - \frac{j-1}{j} p_j + \frac{1}{j} \sum_{i=1}^{j-1} p_i + \sum_{i=j+1}^{n} \left[\frac{p_i}{i} - \frac{1}{i(i-1)} \sum_{k=1}^{i-1} p_k \right] \right] \\
&= \frac{1}{Kn} \left[K - p_j + \frac{p_j}{j} + \frac{1}{j} \sum_{i=1}^{j} p_i - \frac{p_j}{j} + \sum_{i=j+1}^{n} \left[\frac{p_i}{i} - \left(\frac{1}{i-1} - \frac{1}{i} \right) \sum_{k=1}^{i-1} p_k \right] \right]
\end{aligned}
$$

Isolating just this last term we have

$$
\begin{aligned}
&\sum_{i=j+1}^{n} \left[\left(\frac{1}{i-1} - \frac{1}{i} \right) \sum_{k=i}^{i-1} p_k \right] \\
&= \left(\frac{1}{j} - \frac{1}{j+1} \right) \sum_{k=1}^{j} p_k + \left(\frac{1}{j+1} - \frac{1}{j+2} \right) \sum_{k=1}^{j+1} p_k + \left(\frac{1}{j+2} - \frac{1}{j+3} \right) \sum_{k=1}^{j+2} p_k + ... \\
&= \frac{1}{j} \sum_{k=1}^{j} p_k - \frac{1}{j+1} p_{j+1} + \frac{1}{j+2} p_{j+2} + ... \frac{1}{n-1} p_{n-1} - \frac{1}{n} \sum_{i=1}^{n} p_i \\
&= \frac{1}{j} \sum_{k=1}^{j} p_k + \sum_{i=j+1}^{n-1} \frac{p_i}{i} - \frac{1}{n} \sum_{i=1}^{n} p_i
\end{aligned}
$$

and substituting back in and then cancelling

$$
\begin{aligned}
q_j &= \frac{1}{Kn} \left[K - p_j + \frac{p_j}{j} + \frac{1}{j} \sum_{i=1}^{j} p_i - \frac{p_j}{j} + \sum_{i=j+1}^{n} \frac{p_i}{i} - \frac{1}{j} \sum_{k=1}^{j} p_k - \sum_{i=j+1}^{n-1} \frac{p_i}{i} + \frac{1}{n} \sum_{i=1}^{n} p_i \right] \\
&= \frac{1}{Kn} \left[K - p_j + \frac{1}{n} \sum_{i=1}^{n} p_i \right]
\end{aligned}
$$

Rewriting this considering the mass of consumers we have

$$
q_j = \frac{M}{Kn} \left[K - p_j + \frac{1}{n} \sum_{i=1}^{n} p_i \right]
$$

and the derivative is

$$
\frac{\partial q_j}{\partial p_j} = \frac{-M}{Kn} + \frac{M}{Kn^2}
$$

c

If firms have constant but heterogeneous marginal costs c_j, the implied equilibrium price in terms of c_j and $\bar{c}$. Firms (implicitly) maximize

$$\begin{aligned}
&\max_{p_j}(p_j - c_j)q_j \\
\implies\ & q_j + (p_j - c_j)\frac{\partial q_j}{\partial p_j} = 0 \\
\implies\ & p_j = c_j - \frac{q_j}{\partial q_j/\partial p_j}
\end{aligned}$$

Plugging in from (b) and solving out for the summation

$$\begin{aligned}
p_j &= c_j - \frac{M}{Kn}\left[K - p_j + \frac{1}{n}\sum_{i=1}^{n} p_i\right] * \frac{Kn^2}{M(1-n)} \\
&= c_j - \left[K - p_j + \frac{1}{n}\sum_{i=1}^{n} p_i\right]\frac{n}{1-n} \\
&= c_j - \frac{n}{1-n}K + \frac{n}{1-n}p_j - \frac{n}{1-n}\bar{p} \\
\frac{1-2n}{1-n}p_j &= c_j - \frac{n}{1-n}K - \frac{n}{1-n}\bar{p} \\
p_j &= \frac{1-n}{1-2n}c_j - \frac{n}{1-2n}K - \frac{n}{1-2n}\bar{p} \\
\implies \bar{p} &= \frac{1-n}{1-2n}\bar{c} - \frac{n}{1-2n}K - \frac{n}{1-2n}\bar{p} \\
\implies \frac{1-n}{1-2n}\bar{p} &= \frac{1-n}{1-2n}\bar{c} - \frac{n}{1-2n}K \\
\implies \bar{p} &= \bar{c} - \frac{n}{n-1}K
\end{aligned}$$

Plugging this back into p_j we now have it all in terms of c_j and $\bar{c}$ and constants

$$p_j = \frac{1-n}{1-2n}c_j - \frac{n}{1-2n}K - \frac{n}{1-2n}\bar{c} - \frac{n}{n-1}\frac{n}{1-2n}K$$

4

Say we have advertising data by product for all products in a certain industry. Level of advertising is positively correlated with total sales of the industry, but ads for a particular product don't seem to influence the demand share of that product. Some reasons this could be:

- This could be in an industry where firms can't sell their own products, they have to use large retail vendors (such as electronics firms all selling their products at Best Buy or Fry's). Advertising gets the consumer in the door of the retail store but once the consumer is there, they see all the competing products and don't make a decision based on advertising anymore. Instead they compare on observable characteristics and the effect of advertising for a product of a particular brand goes away at that point.
- There could be some sort of equilibrium where each firm in the industry advertises for all the same products as the other firms. So one firm advertising more for a product leads to

all other firms advertising more for that product and demand for that product across firms stays the same. We can think of this as some equilibrium where if a firm does not advertise, they will lose substantial market share so the firms advertise for products to keep them afloat and demand stays constant. This endogenous response could lead to us under-predicting how much advertising at a particular firm affects demand for that product.

- Finally, there could be a story here about prestige effects. This could be that advertising a particular product from a brand doesn't necessarily boost demand for that product but boosts how consumers view the brand across the board. Say an airline company starts producing luxury private jet experiences and advertises for those in flashy, fun ads. Regular consumers can't afford that experience, and it might do nothing to boost demand in the luxury private jet market. However, normal people might go and purchase and economy ticket on that airline the next time they buy a plane ticket because they associate that brand with luxury and flashy and fun experiences, even though that's not the good they are purchasing themselves.

5

a

The firm's profits per consumer of advertising is

$$\pi(p) = (p - c)[1 - \alpha + \alpha(1 - F(p))]^{N-1}$$

If the consumer sees the ad and it is the lowest price ad, the firm will get profit $p-c$. The probability that the ad reaches the consumer is α and the probability that if the price reached the consumer, some other price was lower is $F(p)$. Therefore

$$P(p < p_i \mid \text{seen}) = 1 - P(p_i < p \mid \text{seen}) = 1 - \alpha F(p)$$

And since there are $N - 1$ other prices we have

$$\begin{aligned} \pi(p) &= (p - c)[1 - \alpha F(p]^{N-1} \\ &= (p - c)[1 - \alpha + \alpha(1 - F(p))]^{N-1} \end{aligned}$$

b

If consumers never pay more than m, the per-consumer profit of sending out an ad with $p = m$ is

$$\pi(m) = (m - c)(1 - \alpha)^{N-1}$$

Just plugging in that $F(m) = 1$ since effectively, setting a price above m guarantees you do not sell.

c

In a mixed-strategy equilibrium, firms will be indifferent between advertising any of the prices (which implies that setting price at m would be the same as setting any other price in the distribution. This implies $\pi(p) = \pi(m)$

$$\begin{aligned}
(p-c)[1-\alpha+\alpha(1-F(p)]^{N-1} &= (m-c)(1-\alpha)^{N-1} \\
\frac{p-c}{m-c} &= \left(\frac{1-\alpha}{1-\alpha+\alpha(1-F(p))}\right)^{N-1} \\
\left(\frac{p-c}{m-c}\right)^{\frac{1}{N-1}} &= \frac{1-\alpha}{1-\alpha+\alpha(1-F(p))} \\
(1-\alpha)\left(\frac{m-c}{p-c}\right)^{\frac{1}{N-1}} &= 1-\alpha+\alpha(1-F(p)) \\
F(p) &= 1-\frac{1-\alpha}{\alpha}\left(\frac{m-c}{p-c}\right)^{\frac{1}{N-1}}+\frac{1-\alpha}{\alpha} \\
&= \frac{1}{\alpha}-\frac{1-\alpha}{\alpha}\left(\frac{m-c}{p-c}\right)^{\frac{1}{N-1}}
\end{aligned}$$

The distribution changes in α as we can see that as α increases, the distribution for any given p is lower. Intuitively, the mass of the CDF shifts down because the more people who can receive ads, the more firms compete on price and the equilibrium price distribution will go down.

d

The lowest price offer in this distribution will be $\underline{p}$ such that $F(\underline{p}) = 0$. Knowing that a firm is indifferent from setting this price and m, we have

$$\begin{aligned}
&(\underline{p}-c) = (m-c)(1-\alpha)^{N-1} \\
\implies &\underline{p} = (m-c)(1-\alpha)^{N-1}+c
\end{aligned}$$

The intuition behind the markup above c is that it depends on α. We see that as α goes to zero, there will be no markup over marginal cost because there will be perfect competition. But as long as there is an outside shot that the ad you send will reach consumers who won't see any other ads, there will always be an incentive to add a small markup to get those consumers to buy the product at a markup.

e

The distribution of prices paid is not the same as the distribution of prices offered because consumers take the minimum of the prices they are offered over all N draws and we can plug in the distribution found in (c)

$$G(p) = 1-(1-F(p)^N$$