Nadia Lucas
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# EEE 1 Problem Set 2

## Problem 1 Fishing and Extinction

a. $\dot{x}_t = f(x_t) - y_t$

$x_0 > 0$

$y_t \geq 0$

resource owner $\max_{y_t} \int_0^\infty e^{-rt}(py_t - c(y_t))$ subject to above constraints

$$\mathcal{H} = py_t - c(y_t) + \lambda_t(f(x_t) - y_t)$$

FOCs: $\frac{\partial \mathcal{H}}{\partial y_t}$: $p - c'(y_t) - \lambda_t \leq 0$, $y_t \geq 0$, c.s.

$$-\frac{\partial \mathcal{H}}{\partial x_t} = -\lambda_t f'(x_t) = \dot{\lambda}_t - r\lambda_t$$

TVCs: $\lim_{t\to\infty} e^{-rt} x_t \geq 0$ $\quad \lim_{t\to\infty} e^{-rt}\lambda_t x_t = 0$

b. We want $x_t = 0$ not to be a steady state

A SS requires $\dot{x}_t = 0 \Rightarrow \dot{\lambda}_t = 0 \Rightarrow \lambda_t = \lambda_{ss}$

$\Rightarrow f'(0) = r$ gives extinction is a steady state

So we need $f'(0) > r$ for it not to be driven to extinction (can approach positive SS from any direction)

c. We plot in x-y space, the 2 curves that describe $\dot{x}_t = 0$, $\dot{y}_t = 0$

$\dot{x} = 0 \Rightarrow y_t = f(x_t)$ $\quad (\dot{x} = f(x_t) - y_t)$

$\dot{y} = 0$ curve $\Rightarrow$ $\dot{\lambda}_t = -c''(y_t)\dot{y}_t$, $\quad \dot{\lambda}_t = \lambda_t(r - f'(x_t))$

$= (p - c(y_t))(r - f'(x_t))$

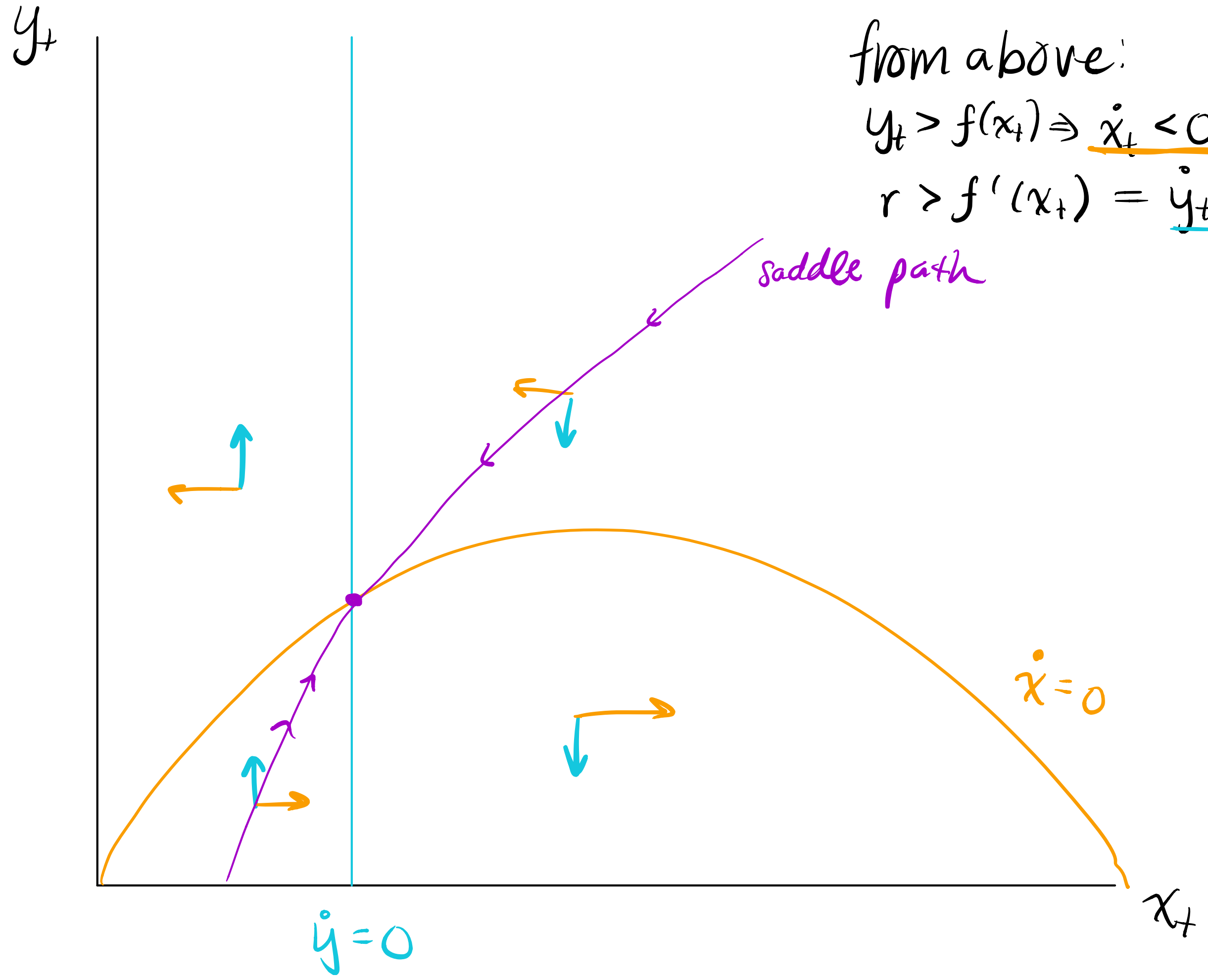
$$(p - c(y_t))(r - f'(x_t)) = -c''(y_t)\dot{y}_t$$

$$\dot{y}_t = \frac{(p - c(y_t))(r - f'(x_t))}{-c''(y_t)},$$

from above:

$y_t > f(x_t) \Rightarrow \dot{x}_t < 0$

$r > f'(x_t) = \dot{y}_t > 0$




d. If the government closes the fishery at time T, then our TVCs change

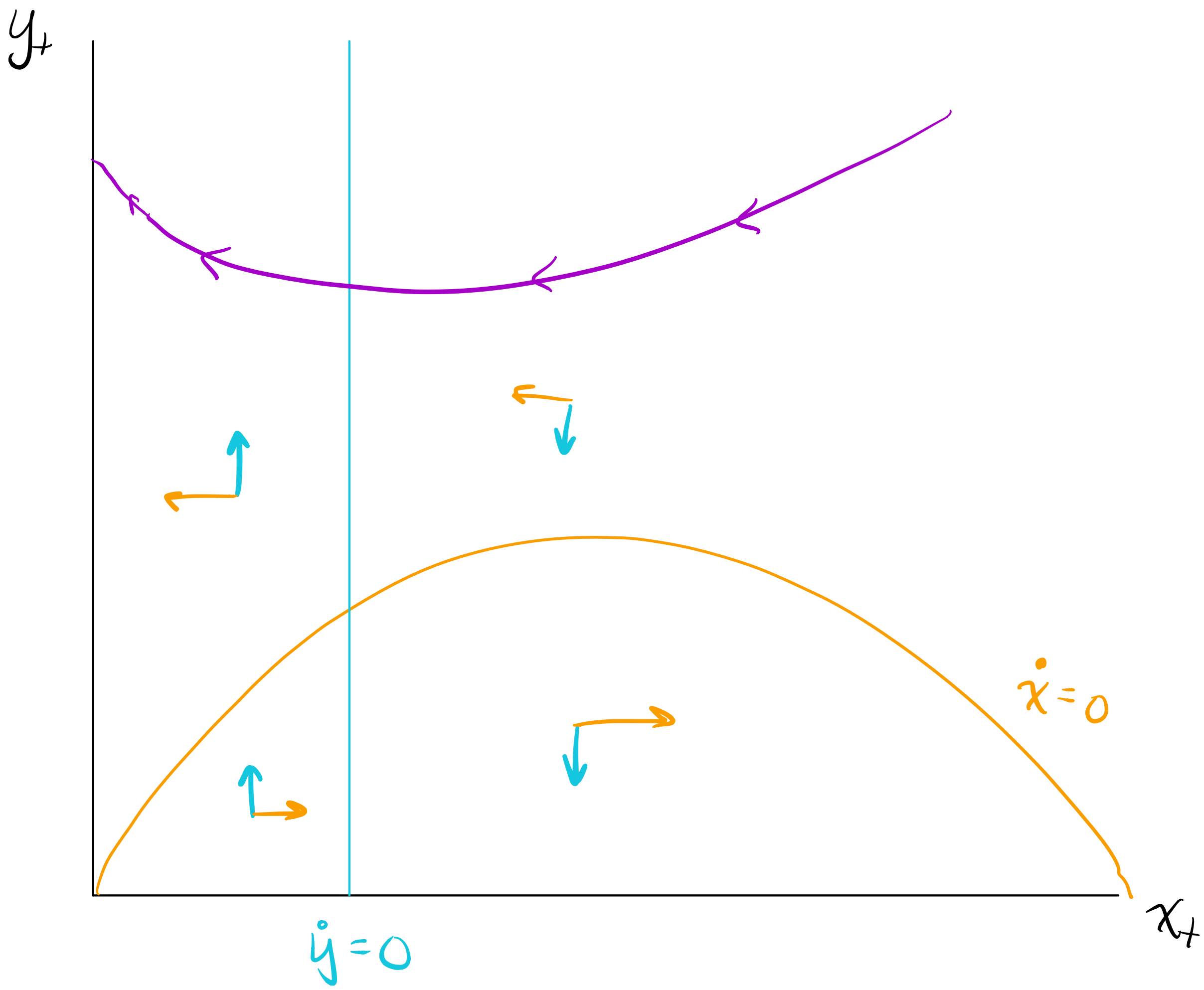
now: $\lambda_T x_T e^{-rT} = 0$

We are given that $p - c'(y_t) > 0 \quad \forall y_t$

$\Rightarrow \lambda_T = p - c'(y_T) > 0$

$\Rightarrow x_T = 0$

e.




We don't hit a steady state here because as T approaches, we no longer care about the shadow value of leaving fish in the fishery ($\lambda_t$ decreasing)

## Problem 2 Common Pool with Exhaustible Resource

Oil reserve with stock $S$, $N$ firms

if one firm extracts $y_{it}$, $Y_t = \sum_i y_{it} = y_{it} + \underbrace{\sum_{j\neq i} y_{jt}(x_t)}_{\text{others' extraction, function of stock}}$

LOM: $\dot{x}_t = -Y_t$

For firm $i$, $\mathcal{H} = p y_{it} - c(y_{it}) - \lambda_t Y_t$

FOCs:

$$\frac{\partial \mathcal{H}}{\partial y_{it}}: \quad p - c'(y_{it}) - \lambda_t \leq 0, \quad y_{it} \geq 0, \quad \text{c.s.}$$

$$-\frac{\partial \mathcal{H}}{\partial x_t} = \lambda_t \sum_{j\neq i} y_{jt}'(x_t) = \dot{\lambda}_t - r\lambda_t$$

TVCs: $\lim_{t\to\infty} e^{-rt} x_t \geq 0, \quad \lim_{t\to\infty} e^{-rt} \lambda_t x_t = 0$

b. In this setting, we now see $\dot{\lambda}_t$, the LOM for the shadow value of oil in the ground changes not only with the rate of interest but now with the extraction of others if I leave an additional unit of oil in the ground. Essentially, laving oil in the ground helps other people a lot more than it helps you so extraction is much faster than in a competitive equilibrium.

## Problem 3 Highway Congestion

GW bridge time: $z = \alpha x$
alternate: 1 hour

a. In a Nash equilibrium, everyone will play a mixed strategy such that the commute times are equal

$$z = 1 = \alpha x$$

$x = 1/\alpha$ → number of drivers on the bridge in N.E.

b. Pareto efficient drivers, social planner solves to mimimize value of lost time on each bridge

$$\min \quad \text{Cost} = \Big( \underbrace{w \cdot 1 \cdot (N - x)}_{\text{val. lost time on alternate}} + \underbrace{w \cdot \alpha x \cdot x}_{\text{val. of lost time on GW}} \Big)$$

$$\frac{\partial \text{cost}}{\partial x} = -w + 2w\alpha x = 0$$

$\boxed{x = \frac{1}{2\alpha}}$ is pareto efficient # of drivers on GW

c. At Pareto optimum, an additional driver causes $\frac{dz}{dx} = \alpha$ longer commute for all drivers. At optimum there are $\frac{1}{2\alpha}$ drivers who lose $w\alpha$ value of time so ~~total~~ externality: $\boxed{\frac{w}{2}}$

d. Imposing a Pigouvian toll of $\frac{w}{2}$ for the GW bridge, the agent now equates value of taking each option $w\alpha x + \frac{w}{2}$ and $w$

$$\Rightarrow \quad w\alpha x + \frac{w}{2} = w$$

$\Rightarrow x = \frac{1}{2\alpha}$, same as Pareto optimum

Drivers still pay $w$ in each setting, they are no better off. The "winners" depend on how the revenue from the toll is spent.

## Problem 4

Benefit of good: $U(q) = aq - \frac{b}{2}q^2$ $\quad a, b > 0$

Social damages: $\tau e$

Production function: $q = f(k, e)$

a. Social Planner:

Note: $k \cdot r = \sum_{i=1}^{n} k_i r_i$

$$\max_{k_i, e} \; U(q) - k \cdot r - \tau e$$

Firms are identical so it suffices to equate all firms' production functions

Social planner's problem can be rewritten with constraint $q = f(k, e)$

$$\max_{k_i, e} \; af(k,e) - \frac{b}{2} f(k,e)^2 - k \cdot r - \tau e$$

for each input:

FOC: $[k_i]$: $\quad a \cdot f_i(k,e) - bf(k,e) f_i(k,e) - r_i = 0$

FOC $[e]$: $\quad a f_e(k,e) - bf(k,e) f_e(k,e) - \tau = 0$

We need to get the firm's problem to match this

Firm solves $\quad \max_{k,e} \; af(k,e) - \frac{b}{2} f(k,e)^2 - k \cdot r - t^* e$

FOC $[k_i]$: same as social planner

FOC $[e]$: $\quad a f_e(k,e) - bq f_e(k,e) - t^* = 0$

$\boxed{t^* = \tau}$ : matches SP objective

b. Now setting up firm problem with an output tax

Firm solves: $\max_{k,e} \; af(k,e) - \frac{b}{2} f(k,e)^2 - k \cdot r - t f(k,e)$

FOC $[k_i]$: $\quad af_i(k,e) - bq f_i(k,e) - r_i - tf_i(k,e)$

FOC $[e]$: $\quad af_e(k,e) - bq f_e(k,e) - tf_e(k,e) = 0$

$\Rightarrow tf_i(k,e) = 0$ and $tf_e(k,e) = \tau$ $\Rightarrow$ $\boxed{t = \tau = 0}$ is the only way to achieve an efficient outcome

c. Now only inputs can be taxed (Similar notation, $t \cdot k = \sum_{i=1}^{n} t_i k_i$)

Firm solves: $\max_{k,e} \; a \cdot f(k,e) - \frac{b}{2} f(k,e)^2 - k \cdot r - t \cdot k$

FOC $[k_i]$: $a f_i(k,e) - bq f_i(k,e) - r_i - t_i = 0$

FOC $[e]$: $a f_e(k,e) - bq f_e(k,e) = 0$

Again, to match the planner's problem, $\boxed{t_i = 0 \text{ and } \tau = 0}$

is the only way to achieve the efficient outcome

d. Now outputs and inputs can be taxed, $\hat{t}$: tax on output

Firm solves: $\max_{k,e} \; a \cdot f(k,e) - \frac{b}{2} f(k,e)^2 - k \cdot r - t \cdot k - \hat{t} f(k,e)$

FOC $[k_i]$: $a f_i(k,e) - bq f_i(k,e) - r_i - t_i - \hat{t} f_i(k,e) = 0$

FOC $[e]$: $a f_e(k,e) - bq f_e(k,e) - \hat{t} f_e(k,e) = 0$

first match FOC $[e]$: $\tau = \hat{t} f_e(k,e)$

$$\Rightarrow \boxed{\hat{t} = \frac{\tau}{f_e(k,e)}}$$

and $t_i - \hat{t} f_i(k,e) = 0$

$$\Rightarrow \boxed{t_i = \frac{\tau f_i(k,e)}{f_e(k,e)}}$$

are the full suite of taxes that match the firm problem with the social planner's problem. However, if we don't observe emissions, we might not be able to observe $f_e(k,e)$ ... But knowing the marginal product of $k_i$ and $e$, you can back out a proxy for emissions and tax inputs and outputs accordingly

e. If $a$ is unknown to the regulator then there is uncertainty in the marginal cost of abatement. $\tau$ is the marginal benefit of one unit of abatement and is always constant. So a tax $= \tau$ will always perfectly work no matter the uncertainty. Setting quantities via cap and trade will yield DWL if the marginal cost of abatement (private benefit) is uncertain

## Problem 5 Permit Market Dynamics

Firms pollute $E_0$ per year over 10 years, cap: $\bar{S} < 10E_0$

permit price: $p(t)$, with the demand $D(p(t))$

marginal abatement costs: $C'(E_0 - D(p))$ $\Rightarrow$ $p(t) = C'(E_0 - D(p(t)))$

with interest rate: $r$

a. Permit prices determined by a no-arbitrage condition and all permits get sold.
So the total quantity demanded $\int_0^{10} D(p(t))\,dt = \bar{S}$

And by no arbitrage, price rises at the rate of interest,

$$p(t) = p(0)e^{rt}$$

$$\Rightarrow \quad \bar{S} = \int_0^{10} D(p(0)e^{rt})\,dt \qquad \text{determines } p(0)$$

b. Now the govt implements a ceiling price $p^c$ that is lower than price at end of time interval in a.

Since now, the goverment provides any number of permits above $\bar{S}$ at $p_c$, $\bar{S}$ permits will be demanded in the time that $t \leq 10 - \tau$

$\bar{S}$ pins down prices up to $t = 10 - \tau$

$$\bar{S} = \int_0^{10-\tau} D(p(0)e^{rt})\,dt \qquad \text{otherwise there will be an arbitrage opportunity}$$

and $P(10-\tau) = p^c$ pins down what $\tau$ is in conjunction with $\bar{S}$

- $D(p^c)$ permits are demanded each period from $10-\tau$ to $10$

So $\Delta = D(p^c)\,\tau$ is the additional permits over $\bar{S}$ sold.

c. If the government only has $G > 0$ permits to defend ceiling at $p^c$

If $G < \Delta$ then we are once again constrained by quantity of permits which will cause prices to rise at rate $r$ after a speculative attack

$$\bar{S} = \int_0^{10-\tau} D(p(0)e^{rt})\,dt \quad \text{with} \quad P(10-\tau) = p^c$$

$$G = \int_{10-\tau}^{10-\varepsilon} D(p^c) + \int_{10-\varepsilon}^{10} D(p^c e^{r(t-10+\varepsilon)})\,dt \qquad \text{with } \varepsilon < \tau$$

Where speculative attack happens at some $10 - \varepsilon$

d. Now if $G$ is sufficiently small, the speculative attack happens right at $\tau$ and prices rise at the rate of interest after.

$$G \leq \int_{10-\tau}^{10} D(p^c e^{r(t-10+\tau)})\,dt$$

and all $G$ permits are bought at $\tau$ at $p^c$ because there are no longer enough permits to cover $D(p^c)$ for any length of time so the no-arbitrage condition leads to a speculative attack

Further, $G + \bar{S} = \int_0^{10} D(p(0)e^{rt})\,dt$ now determines the price path.

e. Now with a floor, $p^f = p(0)$ and prices rise at the rate of interest, otherwise there is an incentive to sell at $p^f$ at time 0 and buy a permit later which wouldn't work out with permit accounting. Prices being higher across the board mean that $\bar{S} - \beta < \bar{S}$ permits are used, in particular

$$\bar{S} - \beta = \int_0^{10} D(p^f e^{rt})\,dt \quad \text{determines } \beta$$

Problem 6 Clean backstop technology

subject to:

$$\begin{cases} \text{initial stock } S_0 > 0 \\ S_{t+1} = S_t - y_t \Rightarrow \dot{S}_t = -y_t \\ y_t \geq 0 \\ b_t \geq 0 \end{cases}$$

a. Social planner's problem : $\max_{y_t, b_t} \int_0^\infty e^{-rt} \left( p y_t + p b_t - c(y_t) - c(b_t) \right) dt$

$$\mathcal{H} = p y_t + p b_t - c_y(y_t) - c_b(b_t) - \lambda_t y_t$$

FOCs:

$$\frac{\partial \mathcal{H}}{\partial y_t} = p - c_y'(y_t) - \lambda_t \leq 0, \quad y_t \geq 0, \quad \text{c.s.}$$

$$\frac{\partial \mathcal{H}}{\partial b_t} = p - c_b'(b_t) \leq 0, \quad b_t \geq 0, \quad \text{c.s.}$$

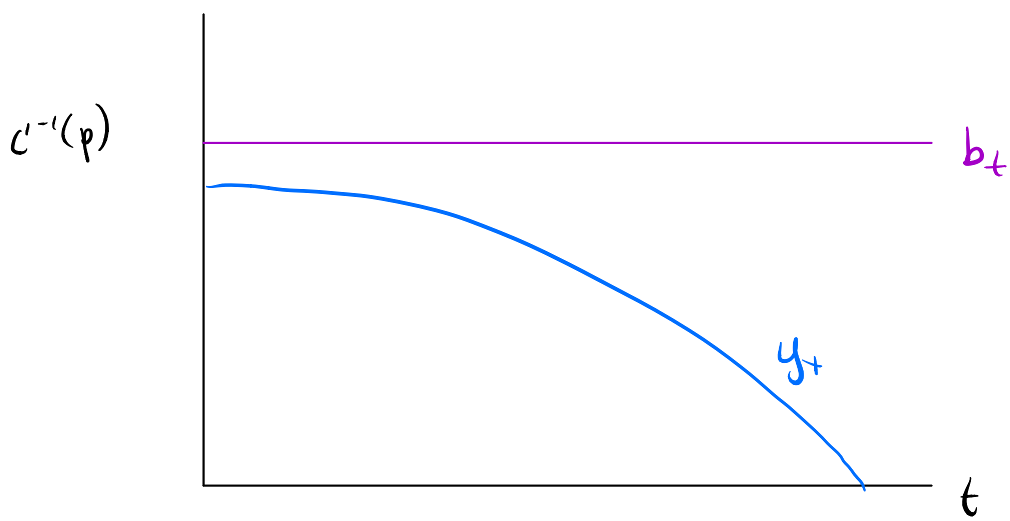
$$-\frac{\partial \mathcal{H}}{\partial S_t} = 0 = \dot{\lambda}_t - r\lambda_t$$

TVCs: $\lim_{t \to \infty} e^{-rt} S_t \geq 0$, $\lim_{t \to \infty} e^{-rt} \lambda_t S_t = 0$

b. Using FOCs, we can find time path of $b_t$, $y_t$

If interior solution: $b_t = c_b'^{-1}(p)$ (constant)

$y_t = c_y'^{-1}(p - \lambda_t)$ $\left( \dot{\lambda}_t = r\lambda_t \Rightarrow \lambda_t = e^{rt}\lambda_0 \right)$




$y_t = c_y'^{-1}(p - e^{rt}\lambda_0)$

we know $c_y'^{-1}$ is str. increasing and $p - e^{rt}\lambda_0$ is exponentially decreasing

c. At time T, all cars banned and nobody demands fuel anymore. If any stock is left at T, then $\lambda_t = 0$ forall $t$ because there is no more scarcity rents on extraction. This implies $y_t = c_y'^{-1}(p)$ is constant so $c_y'^{-1}(p)\, T \leq S_0$ determines how small T should be.

d. $e_y > e_b \geq 0$ $\quad \tau$: damages

$e_y$ is emissions rate, so $e_y y_t$ is total emissions of gas

Planner's problem: $\mathcal{H} = p y_t + p b_t - c_y(y_t) - c_b(b_t) - \lambda_t y_t - \tau e_y y_t - \tau e_b b_t$

Taxing $y_t$ at $\tau e_y$ and $b_t$ at $\tau e_b$ result in the firm problem being the same

$$\begin{pmatrix} \mathcal{H}^y = p y_t - c_y(y_t) - \tau e_y y_t \\ \mathcal{H}^b = p b_t - c_b(b_t) - \tau e_b b_t \end{pmatrix}$$

with the same constraints

e. Suppose instead there is a LCFS $\dfrac{e_y y_t + e_b b_t}{y_t + b_t} \leq \sigma$

Which is an additional constraint in Hamiltonian

Planner now solves

$$\mathcal{H} = p y_t + p b_t - c_y(y_t) - c_b(b_t) - \lambda_t y_t + \mu_t \left( \sigma (y_t + b_t) - e_y y_t - e_b b_t \right)$$

FOCs

$$\frac{\partial \mathcal{H}}{\partial y_t} = p - c_y'(y_t) - \lambda_t + \mu_t \sigma - \mu_t e_y \leq 0, \quad y_t \geq 0, \quad \text{c.s.}$$

$$\frac{\partial \mathcal{H}}{\partial b_t} = p - c_b'(b_t) + \mu_t \sigma - \mu_t e_b \leq 0, \quad b_t \geq 0, \quad \text{c.s.}$$

$$-\frac{\partial \mathcal{H}}{\partial S_t} = 0 = \dot{\lambda}_t - r\lambda_t$$

$\mu_t$ just a constant, $\mu$ because standard binds and doesn't change with time (conceptually it is not a state)

TVCs: $\lim_{t \to \infty} e^{-rt} \lambda_t S_t = 0, \quad \lim_{t \to \infty} e^{-rt} S_t \geq 0$

Matching up with the Social Planner's problem above,

SP: FOC $[y_t]$: $p - c_y'(y_t) - \lambda_t - \tau e_y \geq 0$, $y_t \geq 0$, C.S.

SP: FOC $[b_t]$: $p - c_b'(b_t) - \tau e_b \geq 0$, $b_t \geq 0$, C.S.

We see, $\mu_t(e_b - \sigma) = \tau e_b$

Since $\sigma \in [e_b, e_y]$, we cannot equate the 2 terms unless $e_b = 0$ and $\sigma = 0$

With LCFS, there is always an implicit subsidy on the less polluting fuel, so $e_b > 0$ would mean a negative externality is imposed

$\Rightarrow \boxed{e_b = 0}$ and $\sigma = 0$ implies that no $y_t$ can be used

which is also only first best if all the oil stays in the ground ($\lambda_t = 0 \ \forall t$) and $p - c_y'(0) - \tau e_y \geq 0$

$$\Rightarrow \boxed{c_y'(0) \leq p - \tau e_y}$$

or in other words if 0 production of oil is optimal in the first best where $e_b = 0$ as well.

Problem 7 Mustangs and Civics

a. We denote: $U_f(m_f)$ as utility of faculty

$U_s(m_s)$ as utility of students

Social planner:

$$\max_{m_f, m_s} U_f(m_f) + U_s(m_s) - pe_f m_f - pe_s m_s - \tau\phi_f e_f m_f - \tau\phi_s e_s m_s$$

$$FOC[m_f]: U_f'(m_f) - pe_f - \tau\phi_f e_f = 0$$

$$FOC[m_s]: U_s'(m_s) - pe_s - \tau\phi_s e_s = 0$$

With one tax: $t$

Faculty solves: $\max_{m_f} U_f(m_f) - (p+t)e_f m_f$

$$FOC[m_f]: U_f'(m_f) - (p+t)e_f = 0$$

Student solves $\max_{m_s} U_s(m_s) - (p+t)e_s m_s$

$$FOC[m_s] \quad U_s'(m_s) - (p+t)e_s = 0$$

Matching taxes with extenalities we need

$t = \tau\phi_f$ $\qquad$ $\phi_s > \phi_f > 0$

$t = \tau\phi_s$ $\qquad$ So no tax will solve this

The second best tax can be achieved by minimizing distance between the tax and the externality. Intuitively, the demand response being the same for both faculty and students means that minimizing

$$\min_t |t - \tau\phi_f| + |t - \tau\phi_s|$$

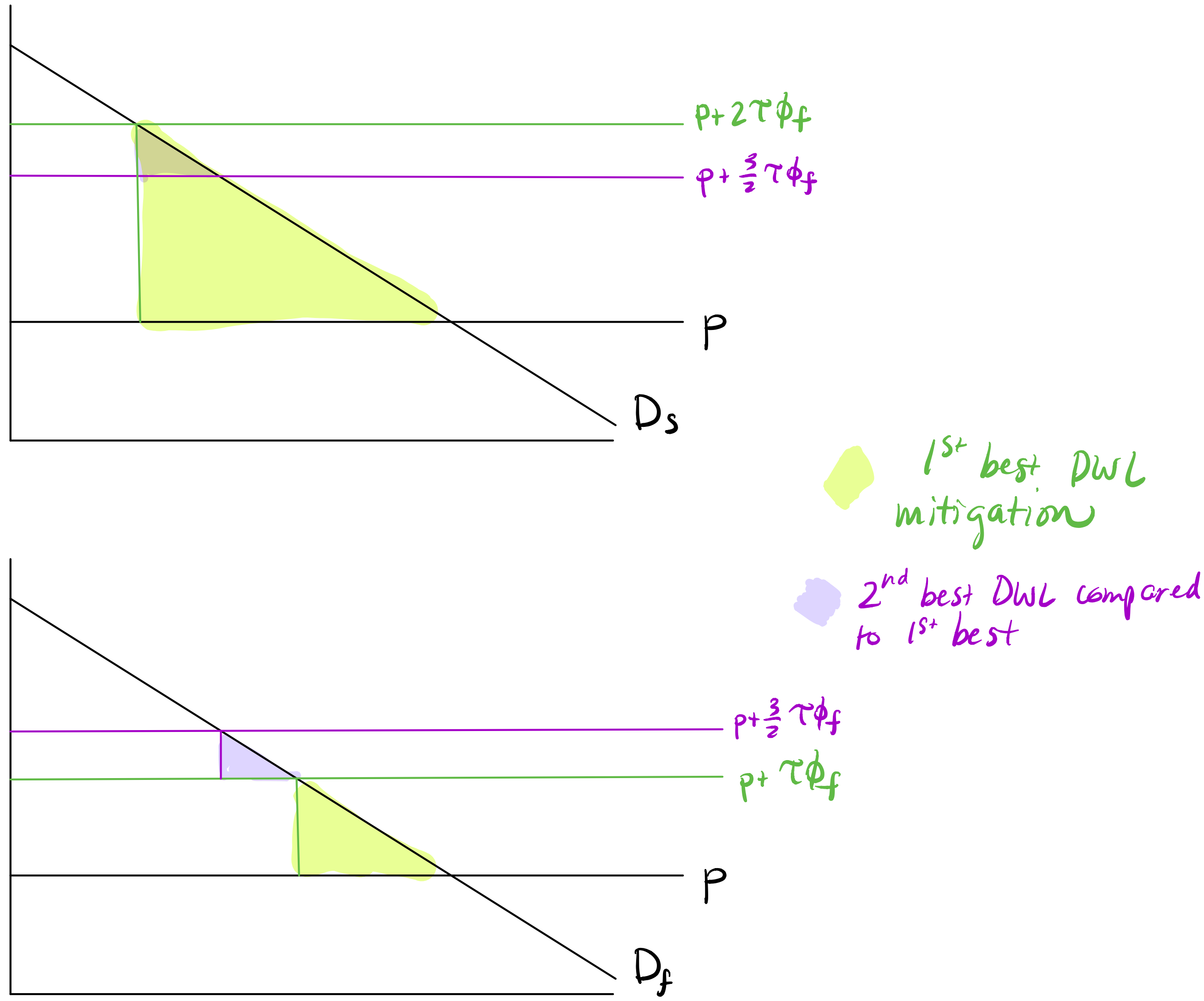
will get the optimal tax. Which means $t^*$ is the mid point

$$\boxed{t^* = \frac{\tau\phi_f + \tau\phi_s}{2}}$$

for the second-best optimal tax

b. We want to compare DWL mitigated by 1st best and 2nd best policy

If $\phi_S = 2\phi_F$, $t^* = \frac{3}{2}\tau\phi_f$




$\frac{dm}{dp}$ gives slope of demand curve (same for faculty and students)
denote this: $\varepsilon$

Finding areas of triangles (summed over each color)

green: $2\tau\phi_f \cdot \frac{2\tau\phi_f}{\varepsilon} \cdot \frac{1}{2} + \frac{\tau\phi_f \cdot \tau\phi_f}{\varepsilon} \cdot \frac{1}{2}$

purple: $\left(.5\tau\phi_f \cdot \frac{.5\tau\phi_f}{\varepsilon} \cdot \frac{1}{2}\right)2$

$\frac{\text{purple}}{\text{green}} = \frac{.25\tau\phi_f/\varepsilon}{2.5\tau\phi_f/\varepsilon} = .1 \Rightarrow$ 2nd best recovers $\boxed{90\%}$ of welfare gain

c. Faculty now have no demand elasticity, if we were to set two different taxes, social planner solves

$$\max_{t} \quad U_f(m_f) + U_s(m_s(p+t)) - (p+\tau\phi_f)e_f m_f - (p+\tau\phi_s) e_s m_s(p+t)$$

where $m_s$ now responds to $t$ as it changes the price around

$$\text{FOC } [t]: \quad U_s'(m_s(p+t))\, m_s'(p+t) - (p+\tau\phi_s) e_s m_s'(p+t) = 0$$

And student FOC gives $U_s'(m_s) = (p+t)e_s$, matching up gives

$$\Rightarrow \boxed{t = \tau\phi_s} \quad \text{optimal}$$

because faculty being inelastic means they are not affected by this or any other tax, the only externalities that can be corrected are the students.