# PPHA 44330: Problem Set 1

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## Problem 1

### a

State variables: $r_t$, $x_t$

Controls: $q_t$, $w_t$

### b

Problem setup:

$$\begin{aligned}
&\max_{q_t, w_t} \sum_{t=1}^{T} \delta^t (pq_t - c(q_t, r_t) - k(w_t)) \\
&r_{t+1} \leq r_t + f(w_t, x_t) - q_t \\
&x_{t+1} \geq x_t + w_t
\end{aligned}$$

Short form Lagrangian:

$$\mathcal{L} = \sum_{t=1}^{T} \delta^t \left[ (pq_t - c(q_t, r_t) - k(w_t)) \right] + \lambda_t (r_t + f(w_t, x_t) - q_t - r_{t+1}) + \mu_t (x_t + w_t - x_{t+1})$$

KKT conditions:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial q_t} &= \delta^t [p - c_q(q_t, r_t)] - \lambda_t \leq 0, \quad q_t \geq 0, \quad \text{c.s.} \\
\frac{\partial \mathcal{L}}{\partial w_t} &= -\delta^t k'(w_t) - \lambda_t f_w(w_t, x_t) + \mu_t \leq 0, \quad w_t \geq 0, \quad \text{c.s.} \\
\frac{\partial \mathcal{L}}{\partial x_t} &= \lambda_t f_x(w_t, x_t) + \mu_t - \mu_{t-1} = 0 \\
\frac{\partial \mathcal{L}}{\partial r_t} &= -\delta^t c_r(q_t, r_t) + \lambda_t - \lambda_{t-1} = 0
\end{aligned}$$

TVC (which makes redundant the non-negativity constraint on $r_t$):

$$r_{T+1} \geq 0, \quad \lambda_T \geq 0, \quad \lambda_T r_{T+1} = 0$$

### c

The second FOC condition determines $w_t$ and can be rewritten

$$\delta^t k'(w_t) \geq \lambda_t f_w(w_t, x_t) - \mu_t$$

The discounted marginal cost of depleting the mine an extra foot must be greater than or equal to the marginal benefit of digging that extra foot. The marginal benefit is a combination of adding to reserves now (which provides payoff later on) offset by the shadow value of an extra unit of $x_t$

The inequality is due to the non-negativity constraint, which binds when the marginal cost of beginning to deepen the mine outweighs the marginal benefit so no deepening occurs.

### d

The fourth condition determines that $\lambda_t$ decreases over time, rewritten as

$$\lambda_t - \lambda_{t-1} = \delta^t c_r(q_t, r_t)$$

We are given that $c_r < 0$ so we know that $\lambda_t$ is decreasing over time ($\lambda_t < \lambda_{t-1}$) in present value. This is because the shadow value of and extra unit of $r_t$ discounted back to $t = 0$ which will decrease as $t$ increases

Similarly, from the third FOC

$$\mu_t - \mu_{t-1} = \lambda_t f_x(w_t, x_t)$$

Since we know from the TVC that $\lambda_T \geq 0$, and that $\lambda_t$ decreases over time, this implies $\lambda_t \geq 0$. And since we are given $f_x < 0$, we know that $\mu_t - \mu_{t-1} < 0$, and thus we have that $\mu_t$ is decreasing over time as well. This also makes sense because $\mu_t$ is the shadow value of an extra unit of $x_t$ discounted back to $t = 0$ which similarly decreases as $t$ increases.

## Problem 2

### a

Gross revenues: $P$ is unchanging, drilling bring $X$ barrels out of the ground next period and $\lambda \in (0, 1)$ less each period. The revenue from one well, discounted by $\delta$ each period

$$\begin{aligned}
&\delta \left[ PX + \sum_{t=1}^{\infty} \delta^t P(1-\lambda)^t X \right] \\
&= \sum_{t=1}^{\infty} \delta^t P(1-\lambda)^{t-1} X \\
&= PX\delta \sum_{t=0}^{\infty} [\delta(1-\lambda)]^t \\
&= \frac{PX\delta}{1 - \delta(1-\lambda)}
\end{aligned}$$

## b

In equilibrium, drilling must cease in finite time because each period, there is some positive $\{R_t\}$ rig rental cost that is taken as given by the extractor. Since there are a finite number of wells $W$ to be drilled. The only way the rig rental market clears is for $R_t$ to be such that at any time $t$, $R_t = \delta^t \frac{PX\delta}{1-\delta(1-\lambda)}$. Thus in each period, the extractor is indifferent between drilling and not. If this is not the case for $\{R_t\}$, then all the oil will be extracted in any period where revenue exceeds rental rates. As $R(a_t)$ is strictly upward-sloping, we have that the rig rental rate is at least $R(0)$. The value of not drilling a well is bounded by $R(0)$ so eventually, drilling will stop in finite time.

## c

In equilibrium, both the number of drilling rigs rented and rental rate decreases over time. In each period, the value of a barrel of oil is $P$ which can increase at the rate of $r$. Thus, $R_t$ monotonically decreases so future oil extract increases in value at the same rate $r$. $R(a_t)$ is strictly upward-sloping, thus $a_t$ also decreases over time.

## d

$F_t$ is the amount of oil flowing from all opened wells at time $t$. Initial flow is $F_0 = 0$ with $F_{t+1} = f_t(1-\lambda) + a_tX$. For $F_{t+1} > F_t$ in the initial periods of drilling, the newly drilled wells still produce so if $X * a_{t+1} > X * (1-\lambda) * a_t$ then we can see flow potentially increasing over time in the beginning, even after the first period. This wouldn't be possible if $R(a_t)$ was not upward-sloping but constant. If $R(a_t)$ were constant, by similar logic to b, all wells would be drilled immediately, so we couldn't see an increase in flow over time.

## e

The finite time version of the problem:

$$\begin{aligned}
\max_{a_i} \sum_{t=0}^{T} \delta^t \left(PF_t - a_tR_t\right) \text{ subject to:}& \\
F_{t+1} = F_t(1-\lambda) + a_tX& \\
w_{t+1} = w_t - a_t& \\
w_0 = W, F_0 = 0& \\
w_{T+1} \geq 0, \mathrm{F}_{T+1} \geq 0, a_t \geq 0&
\end{aligned}$$

Lagrangian:

$$\mathcal{L} = \sum_{t=0}^{T} \delta^t(PF_t - a_tR_t) + \theta_t(F_t(1-\lambda) + a_tX - F_{t+1}) + \gamma_t(w_t - a_t - w_{t+1})$$

FOCs:

$$\begin{aligned}
\frac{\partial \mathcal{L}}{\partial a_t} &= -\delta^t R_t + \theta_t X - \gamma_t \leq 0, \quad a_t \geq 0, \quad \text{c.s.} \\
\frac{\partial \mathcal{L}}{F_t} &= \delta^t P + \theta_t(1-\lambda) - \theta_{t-1} = 0 \\
\frac{\partial \mathcal{L}}{w_t} &= \gamma_t - \gamma_{t-1} = 0
\end{aligned}$$

TVCs:

$$F_{T+1} \geq 0, \quad \theta_T \geq 0, \quad F_{T+1}\theta_T = 0$$
$$w_{T+1} \geq 0, \quad \gamma_T \geq 0, \quad w_{T+1}\gamma_T = 0$$

### f

The FOC that determines the number of wells $a_t$ is the first one. We have the shadow value of the extra flow of an additional well drilled net of discounted drilling costs has to equal the shadow value of an additional well in reserves. When we look at the $w_t$ FOC we see that $\gamma_t = \gamma_{t-1}$ and that $w_T + 1\gamma_T = 0$. This implies that $\gamma$ is constant so it must be that in every period the marginal value of drilling given costs and the discount rate are constant.

### g

The aggregate wealth of extraction, discounted to $t = 0$ is $\gamma W$. As described above, the present value, $\gamma_t$, is constant throughout so we denote it as $\gamma$ which is the value of any undrilled well. Thus the total wells are W implying the total value of all undrilled wells discounted to $t = 0$ is $\gamma W$.

### h

$\theta_T = 0$ because $F_{T+1} > 0$ since $T$ is some finite number and we can express it as

$$\begin{aligned}
F_{T+1} &= F_T(1-\lambda) + a_T X \\
&= (F_{T-1}(1-\lambda) + a_{T-1}X)(1-\lambda) + a_T X \\
&= \sum_{t=0}^{T} F_t(1-\lambda)^{T-t+1} + a_t X(1-\lambda)^{T-t}
\end{aligned}$$

Since $\lambda > 0$, $F_{T+1} > 0$ as a fraction of all wells drilled before it. Thus it must be that $\theta_T = 0$ by the TVC on $F_{T+1}$

The difference equation that governs $\theta_t$ is the second FOC. We have

$$\begin{aligned}
\theta_{t-1} &= \delta^t P + \theta_t(1-\lambda) \\
\implies \theta_t &= \delta^{t+1} P + \theta_{t+1}(1-\lambda) \\
&= \delta^{t+1} P + \delta^{t+2} P(1-\lambda) + \theta^{t+2}(1-\lambda)^2 \\
&= \delta^{t+1} P + \delta^{t+2} P(1-\lambda) + \delta^{t+3} P(1-\lambda)^2 + \theta^{t+3}(1-\lambda)^3 \\
&= \sum_{i=t+1}^{T} \delta^i P(1-\lambda)^{i-t-1} \\
&= P\delta^{t+1} \sum_{i=0}^{T} \delta^i (1-\lambda)^i
\end{aligned}$$

Now taking limits

$$\begin{aligned}
\lim_{T\to\infty} P\delta^{t+1}\sum_{i=0}^{T}\delta^i(1-\lambda)^i \\
&= P\delta^{t+1}\sum_{i=0}^{\infty}\delta^i(1-\lambda)^i \\
&= \frac{P\delta^{t+1}}{1-\delta(1-\lambda)}
\end{aligned}$$

If we plug in $t = 0$, this is the same expression derived for revenue in part (a).

## Problem 3

### a

The monopolist controls quantities and prices are a function of those quantities and consumer demand, denoted $p(q_t)$. The monopolist problem is therefore

$$\begin{aligned}
\max_{y_t} & p(y_t)y_t \\
\text{s.t.} \quad & x_t \geq 0 \\
& \dot{x_t} = -y_t
\end{aligned}$$

The (current value) Hamiltonian is

$$\mathcal{H} = p(y_t)y_t - \lambda_t y_t$$

FOCs:

$$\begin{aligned}
\frac{\partial \mathcal{H}}{\partial y_t} &= p'(y_t)y_t + p(y_t) - \lambda_t = 0 \\
-\frac{\partial \mathcal{H}}{\partial x_t} &= 0 = \dot{\lambda}_t - r\lambda_t
\end{aligned}$$

TVCs:

$$\begin{aligned}
\lim_{t\to\infty} e^{-rt}x_t\lambda_t = 0 \\
\lim_{t\to\infty} e^{-rt}\lambda_t \geq 0
\end{aligned}$$

Interpreting these conditions, we have that the marginal benefit of extraction (denoted by increase in sales offset by the change in price) is equal to the shadow value of keeping that marginal amount of the resource in the ground ($\lambda_t$).

### b

With constant elasticity, we have from the demand side that

$$\begin{aligned}
\frac{dy_t/y_t}{dp_t/p_t} &= \epsilon \\
\implies \frac{dp_t}{dy_t}y_t &= \frac{1}{\epsilon}p_t
\end{aligned}$$

And can be rewritten as

$$p'(y_t)y_t = \frac{1}{\epsilon}p_t$$

So we can rewrite our FOC as

$$\lambda_t = \left(\frac{1}{\epsilon} + 1\right)p_t$$
$$\implies \frac{\lambda_t}{p_t} = \frac{1}{\epsilon} + 1$$

Seeing as the ratio is a constant, we are on a balanced growth path where $\lambda_t$ and $p_t$ both grow at rate $r$.

### c

Similarly we can use the properties of linear demand to write the functional form for price in a general sense

$$y_t = a + bp_t$$
$$p_t = \frac{y_t - a}{b}$$
$$p_t' = \frac{1}{b}$$

Putting this all together

$$\frac{a + bp_t}{b} + p_t = \lambda_t$$
$$\frac{a}{b} + 2p_t = \lambda_t$$
$$p_t = \frac{a}{2b} + \frac{\lambda_t}{2}$$

We can always denote $\lambda_t = \lambda_0 e^{rt}$. Given the intercept term, we see that $p_t$ grows at a slower rate than $\lambda_t$.

### d

CES demand means that the monopolist faces the same price/quantity tradeoff at any point along the demand curve. However with linear demand, demand is more inelastic in the beginning of the extraction period so prices are higher relatives to CES demand in the beginning and rise more slowly.

## Problem 4

### a

Social Planner solves

$$\max_{y_t}\{u(y_t) - c(y_t)\}$$
$$\text{s.t.} \quad R_t \geq 0, \quad R_0 \text{ given}, \quad \dot{R}_t = -y_t$$

Giving the Hamiltonian

$$\mathcal{H} = u(y_t) - c(y_t) - \mu_t y_t$$

**b**

FOCs:

$$\frac{\partial \mathcal{H}}{\partial y_t} = u'(y_t) - c'(y_t) - \mu_t \leq 0, \quad y_t \geq 0, \quad \text{c.s.}$$
$$-\frac{\partial \mathcal{H}}{\partial R_t} = 0 = \dot{\mu}_t - \rho\mu_t$$

TVC:

$$\lim_{t\to\infty} e^{-\rho t}\mu_t \geq 0, \quad \lim_{t\to\infty} e^{-\rho t} R_t \mu_t = 0$$

**c**

Thinking about how to sketch the time paths of $y_t, u'(y_t), \mu_t$. We know the time path of $\mu_t$ given FOC2. We can use that coupled with the fact that $u'(y_t)$ is decreasing in $y_t$ and $c'(y_t)$ is increasing in $y_t$. Thus we have that $\lambda_t$ is some decreasing function of $y_t$. It's unclear whether this function is concave or convex.

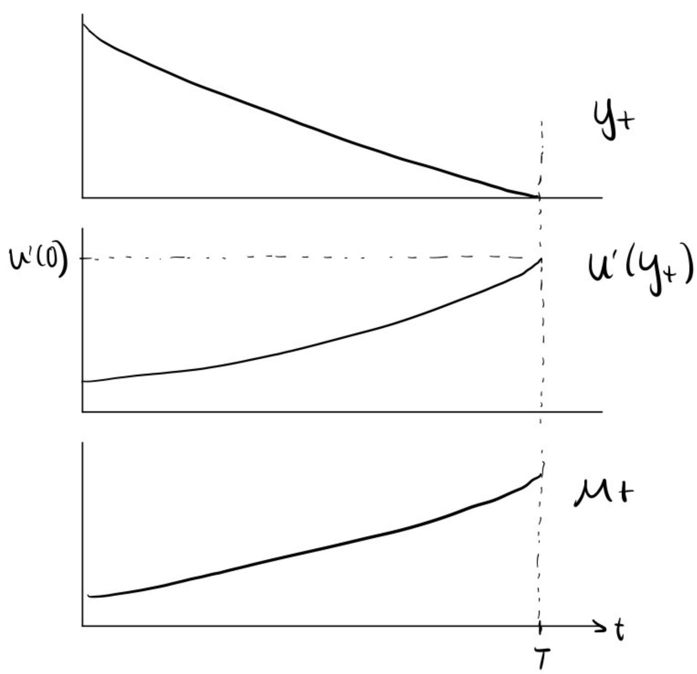


Figure 1: Time path

**d**

At some $t^* < T$ there is an unexpected increase in resource demand so $v'(y_t) = u'(y_t) + \alpha$. There will be an immediate jump in the shadow value at time $t^*$ that persists because uranium is uniformly more valuable. Similarly, $u'(y_t)$ jumps because it's now of a different functional form. Finally, $y_t$ jumps because it is all of a sudden more valuable to extract, but the curve flattens out so all of $R_t$ still gets extracted by time $T$ and the choke price is hit by $v'(0)$.

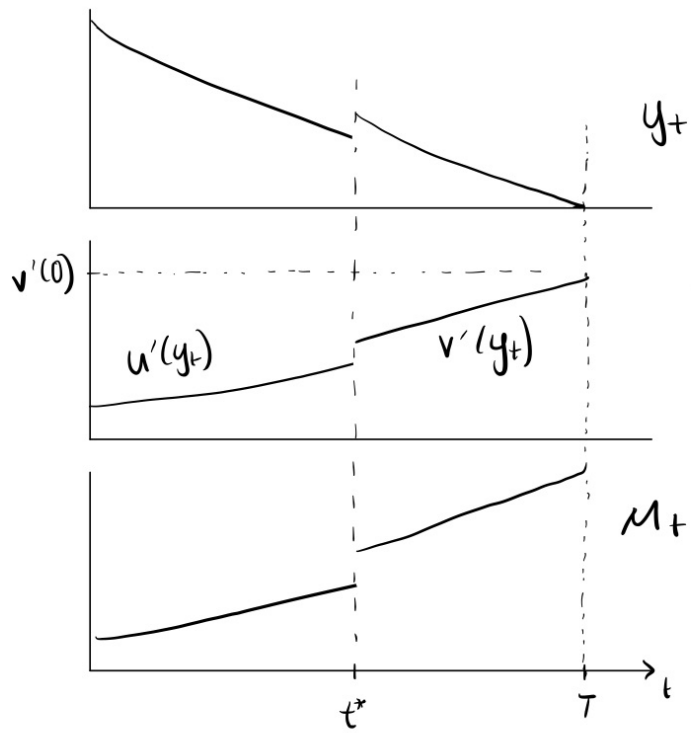


Figure 2: Time path with unanticipated utility shock

### e

Now suppose the demand shock is anticipated. The shadow value will be a smooth line because it is determined by the arbitrage equation. Now for $y_t$, less is extracted prior to the shift and more is extracted afterwards when it becomes more valuable than how extraction occurs in c. Mechanically as a result, $u'(y_t)$ jumps at $t^*$.

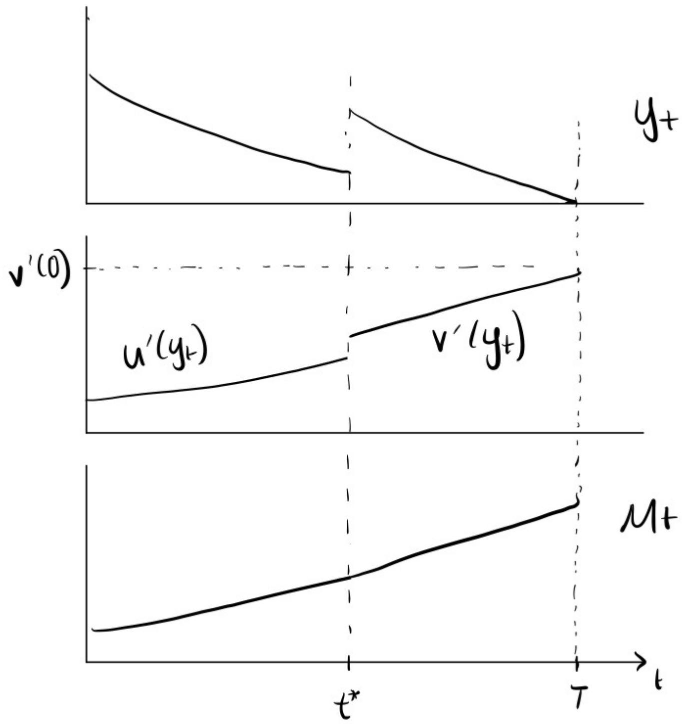


Figure 3: Time path with anticipated utility shock

### f

Investments in production capacity are required to mine the uranium, so $y_t \leq K_t$. The new problem setup:

$$\max_{y_t, a_t} \int_0^\infty e^{-\rho t} u(y_t) - c(y_t) - f(a_t)$$

$$\begin{aligned} \text{subject to: } & y_t \leq K_t, \quad R_t \geq 0 \\ & \dot{K}_t = a_t \\ & \dot{R}_t = -y_t \\ & R_0 \text{ given}, \quad K_0 = 0 \end{aligned}$$

The planner's CV Hamiltonian

$$\mathcal{H} = u(y_t) - c(y_t) - f(a_t) + -\mu_t(y_t) + \theta_t(a_t) + \phi_t K_t - y_t$$

### g

FOCs:

$$
\begin{aligned}
\frac{\partial \mathcal{H}}{\partial y_t} &= u'(y_t) - c'(y_t) - \mu_t - \phi_t \leq 0, \quad y_t \geq 0, \quad \text{c.s.} \\
\frac{\partial \mathcal{H}}{\partial a_t} &= -f'(a_t) + \theta_t \leq 0, \quad a_t \geq 0, \quad \text{c.s.} \\
-\frac{\partial \mathcal{H}}{\partial R_t} &= 0 = \dot{\mu}_t - \rho\mu_t \\
-\frac{\partial \mathcal{H}}{\partial K_t} &= -\phi_t = \dot{\theta}_t - \rho\theta_t
\end{aligned}
$$

TVCs:

$$
\begin{aligned}
\lim_{t\to\infty} e^{-\rho t}\mu_t \geq 0, &\quad \lim_{t\to\infty} e^{-\rho t} R_t \mu_t = 0 \\
\lim_{t\to\infty} e^{-\rho t}\theta_t \geq 0, &\quad \lim_{t\to\infty} e^{-\rho t} K_t \theta_t = 0
\end{aligned}
$$

### h

As $t \to \infty$, the only way for the transversality conditions to hold is if $\theta_t = 0$ or $K_t = 0$. Since there is no depreciation in capacity, this implies that $\lim_{t\to\infty} e^{-\rho t}\theta_t = 0$. Intuitively this makes sense because production will increase until you hit peak capacity (if we think about adding capacity as an additional cost for production in the beginning of the time horizon). Once production starts drawing down from that peak, capacity will still remain there forever, and at that point, the shadow value of capacity falls to 0.

### i

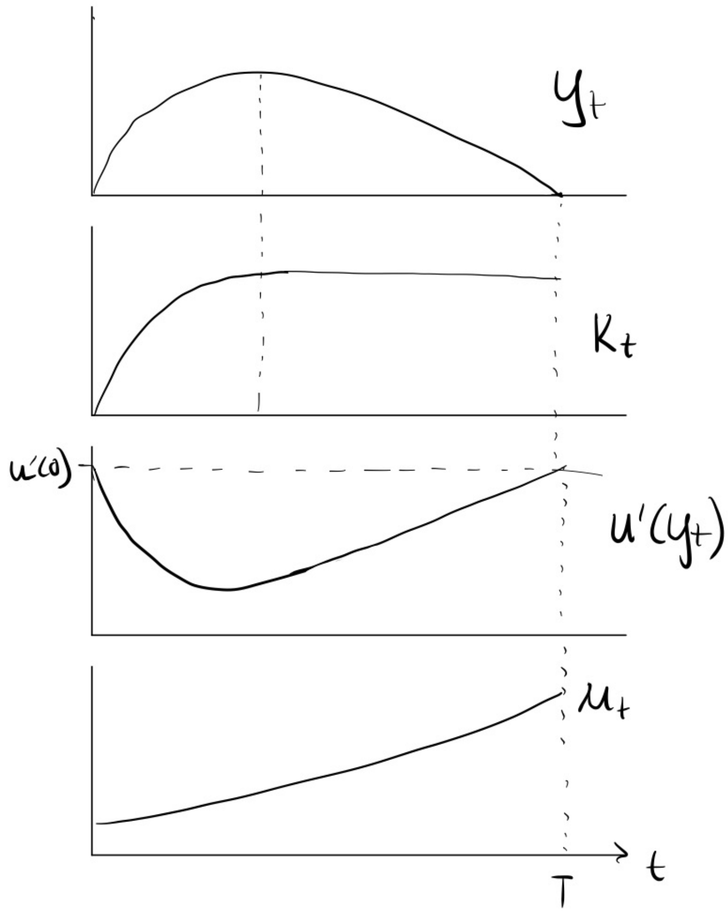


Figure 4: Time path with capacity constraints

## Problem 5

### a

Total demand is $P(Q)$, denote the time path of prices as $\{P_t\}$ and is taken as given each period. Probability that unobtanium is valued at $t$ is $e^{-at}$ and discount rate is $r$. Stockpile is $s_t$. We have that the law of motion for the state variable is:

$$\dot{s}_t = -q_t$$

Current value Hamiltonian

$$\mathcal{H} = e^{-at} P_t q_t - \lambda_t q_t$$

FOCs:

$$\frac{\partial \mathcal{H}}{\partial q_t} = e^{-at} P_t - \lambda_t = 0$$
$$-\frac{\partial \mathcal{H}}{\partial s_t} = 0 = \dot{\lambda}_t - r\lambda_t$$

TVC:

$$s_T \geq 0, \quad e^{-rT}\lambda_T \geq 0 \quad e^{-rT}\lambda_T s_T = 0$$

### b

We can rewrite the first FOC as

$$P_t = e^{\alpha t}\lambda_t$$

Knowing the growth rate of $\lambda_t$ is $r$, we see that the growth rate of $P_t$ if no substitute is discovered is $\alpha + r$

### c

Given the probability, we have

$$E[P_t] = e^{-\alpha t} P_t$$

which means it starts in the same place and grows at rate $\alpha$. This is because the growth rate with no uncertainty is $\alpha + r$ but uncertainty removes the $\alpha$ essentially because every period it could fall to zero with probability $\alpha$ and that probability compounds every period

### d

Now, $\alpha = 0$ means that the price grows at rate $r$. Because of the finite pool of unobtanium though, the extraction path starts at a lower spot because there is no risk of a substitute being found. Intuitively, unobtanium is more valuable in this case than in b in the beginning since there is no possibility of a substitute being found but near the end of the period, if it becomes clear there is no substitute, the unobtanium in part b will have been mined too much so the value will be higher as it becomes more scarce.

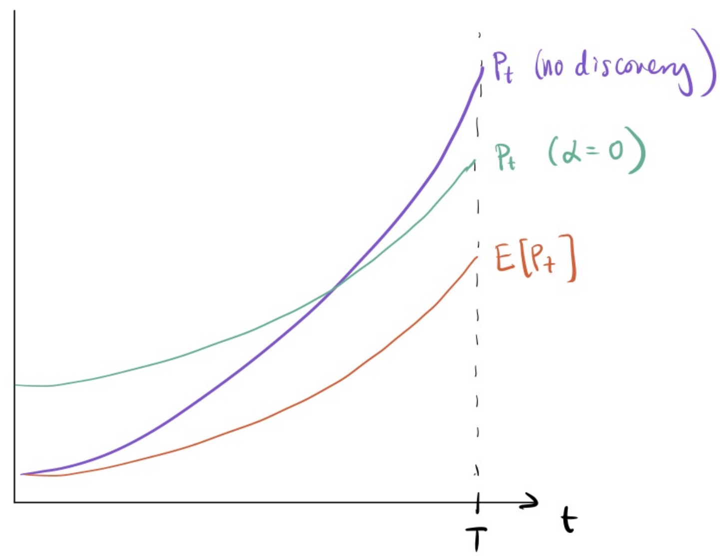


Figure 5: Price paths for b (purple), c (orange), and d(green)