Using Stochastic Dynamic Programming to Simulate Various Consumption Paths in Retirement

Nadia Lucas

Faculty Advisor: James Poterba, Economics

6.UAP Report: May 18th, 2017

1 Abstract

This paper is motivated by a drop in the real interest rate in the United States since the economic downturn of 2008. It uses a stochastic dynamic programming model to predict consumption and savings patterns in retirement in the face of uncertainty in interest rates. By changing the distributions of interest rates, we simulate how different expectations in interest rates affect consumption and savings patterns for people planning for retirement and we simulate what the consumption path of their retirement will be.

2 Introduction

Understanding household saving for retirement is hugely important given that the United States is the 4^{th} highest country in the OECD for elderly poverty rates with 20.6% of the over 65 population being in poverty, and given the approach of the Baby Boom generation to retirement years.

Saving for retirement has historically been one of the biggest problems American households face. To optimally solve for consumption in retirement, factors such as how much you discount future consumption, how much wealth you project to have at retirement, projected life expectancy, and what you expect the interest rate to be in years to come must be taken into consideration. Households control their saving rate, and they may be able to affect their longevity, but they cannot determine the interest rate that they receive on their investments. In fact, people form beliefs about future interest rates to optimally prepare for their own retirements.

The expectation of the long-term real interest rate has a significant effect on households' financial decision-making. Aggregated over all households, this has a major impact on the economy. In recent years in the United States, we have seen consistently low real interest rates. These low interest rates have shifted long-term household expectations in terms of saving for retirement and consumption levels during retirement. Better understanding how this shift affects household consumption can also help to shed light on changing consumption and savings patterns across the country.

To get a feel for how this changes household financial decisions, we can look back on the 10-year treasury inflation protected securities (TIPS) to see how long-term real interest rates have changed. Ten years ago, in 2007, this security yielded a 2.17% return on investment, yet today it's only at 0.48%. In the context of retirement savings, we can think of these returns as being aggregated over the course of many years and see that these low interest rates will dramatically affect the future retirement landscape. With the rates from 2007, investing $1.00 at age 45 would yield $1.71 at age 70, almost doubling in value over the course of 25 years. However, with the returns we are seeing today, $1.00 invested at age 45 yields only $1.13 at age 70. This means that families need to save a lot more now in order to have the same level of wealth at age 70 versus someone planning for retirement a generation ago.

This paper will be exploring how varying interest rates affect consumption patterns today by simulating optimal consumption paths given certain beliefs about the interest rate. By implementing a stochastic dynamic programming simulation, we map optimal consumption paths in the face of uncertainty in the interest rate.

We perform this simulation using many different specifications of a distribution of the interest rates. We begin by using historical levels of corporate stocks to simulate a higher-yield, higher-risk investment, then we increase variance in these returns to see whether or not that affects expected utility, keeping expected yield the same but increasing the risk involved. We perform similar calculations on the 3-month treasury bill rates to simulate a lower-yield but safer investment, one that is more likely used in retirement. Finally we then shift the entire distribution of the interest rate down to a level more reasonably expected of the future given the growth of the economy in recent years. We can thus compare all these subsequent consumption patterns and perform analysis on how expectations of the interest rate affect consumption levels and how they affect retirement outcomes.

3 Data

The only data gathered are the historical levels of various measures of yearly nominal interest rates and inflation rates. These are all gathered from NYU Stern's database of historical annual returns on stocks, bonds, and treasury bills.

Plotted below are different historical distributions of yields used in this paper.

Figure 1: Inflation-adjusted historical returns to treasury bills from 1929 to 2017

We can see that in general historical returns to treasury bill rates have been fairly steady, with the exception of a year right around 1950 where the level of inflation is actually very high, leading to a very low real rate of return. We see the general trend dropping off more steadily in recent years and getting closer and closer to 0. We also perform some simulations involving corporate bond yields.

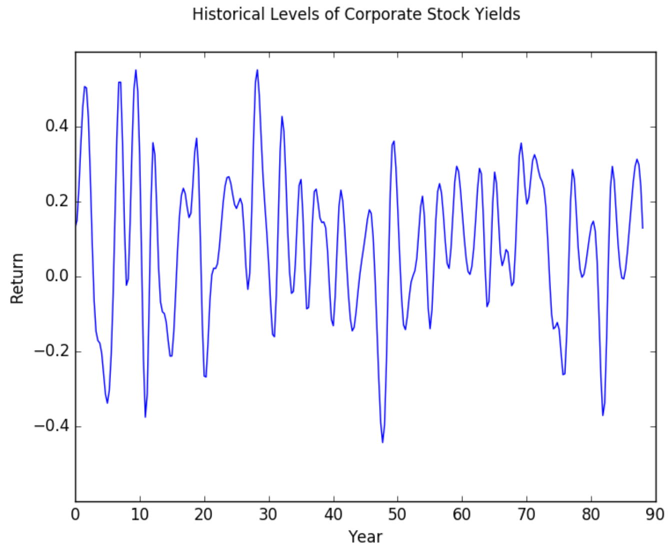

Figure 2: Inflation-adjusted historical returns to corporate stocks from 1929 to 2017

In Figure 2, we see a much more erratic yield from year to year. In general the returns seem somewhat unpredictable..

4 Background

4.1 Problem Setup

In this paper, we consider a stylized model of retirement. The retiree has exactly 20 years of retirement, she has 1000 dollars at the beginning of retirement to allocate across that 20 years as best she can. The money she does not spend each year can be invested and can accumulate interest. The retiree does not value bequests and would like to consume all she has before she dies in year 20.

The simplification of this model leads us to perform an analysis on only one of the factors affecting retirement, the interest rate. We begin by using the above historical levels of the interest rate from the past 89 years. This time frame was chosen to avoid including the interest rates leading to the stock market crash in 1929.

4.2 Deterministic Problem

Before we begin to analyze how the interest rate affects a consumption path throughout retirement, we need to introduce how to solve for an optimal consumption path throughout retirement. In this model, we assume that given a discount factor and a utility function, our agent consumes optimally in each time period in order to maximize her utility over her lifetime.

We define δ to be the discount factor or the amount our agent discounts each subsequent year by. We then construct β such that $\beta = \frac{1}{1+\delta}$. This is the factor we then multiply each subsequent year by to apply the discount factor. In much of this simulation we will use $\delta = 0.01$, $\beta = 0.991$.

We define a utility function to be the utility one gets from how much they consume in time t, c_t as $u(c_t) = log(c_t)$. This utility function is concave and thus captures risk aversion. Using this set up, we can calculate the utility of a consumption pattern in this specific model by the following equation:

$$U(c_0, c_1, ..., c_T) = \sum_{t=0}^{T} \beta^t \log(c_t).$$

Subject to the constraints that:

$$c_t > 0$$
$$w_0 \geq \sum_{t=0}^{T} c_t$$

However, in order to optimally find c_t in each year we must take into account the interest rate, r. Thus we need to start thinking about the wealth we have in each time period, w_t. We have that w_0 is \$1000 since that is our initial wealth. Consumption is always drawn from wealth and in each subsequent time period, we invest all of our wealth and it is accumulated from the interest rate. If we have a constant interest rate, r, we can solve for lifetime utility. In this scenario, we will set $r = 0.02$ and solve for what optimal consumption should be in each time period.

$$U(c_0, c_1, ..., c_n) = \sum_{t=0}^{T} \beta^t \log(c_t).$$

Subject to the constraints:

$$c_t > 0$$
$$w_0 \geq \sum_{t=0}^{T} \frac{c_t}{(1+r)^t}$$

We can solve for this using the Lagrange method. We maximize the following equation and solve for the optimal c_t:

$$L(c_t, \lambda) = \sum_{t=0}^{T} \beta^t \log(c_t) + \lambda \left[\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} - w_0 \right]$$

Taking the first order condition with respect to c_t and solving for the maximum yields:

$$0 = \frac{\beta^t}{c_t} + \frac{\lambda}{(1+r)^t}$$
$$c_t = \frac{\beta^t (1+r)^t}{\lambda}$$

Taking the first order condition with respect to λ yields:

$$\sum_{t=0}^{T} \frac{c_t}{(1+r)^t} - w_0 = 0$$

Substituting what we have for c_t yields:

$$w_0 = \sum_{t=0}^{T} \frac{(1+r)^t \beta^t}{(1+r)^t} \frac{1}{\lambda}$$
$$\lambda = \frac{\sum_{t=0}^{T} \beta^t}{w_0}$$

If we plug in our known values, we can solve for λ, using what we have for interest rate, $r = 0.02$, initial wealth, $w_0 = 1000$, the year we terminate retirement (given that we start in year 0), $T = 19$, and the discount factor, $\delta = 0.01$ giving $\beta = 0.9901$. This yields $\lambda = 0.018266$. From there we can solve for an optimal path of consumption, giving:

$$c_t = [54.75, 55.29, 55.84, 56.39, 56.95, 57.51, 58.08, 58.66, 59.24, 59.82,$$
$$60.42, 61.01, 61.62, 62.23, 62.84, 63.47, 64.09, 64.73, 65.37, 66.01].$$

As we can see from this consumption pattern, because the agent has a very low discount factor, she optimizes by consuming slightly less in the beginning in order to save more money to accumulate with the interest rate so she spends her whole retirement consuming slightly more in each period. The key here is the difference between r and δ and in this case $r - \delta = 1.01$ so the agent values saving money to accumulate more interest over time, thus we see the upward trend in consumption.

4.3 Stochastic Problem

The problem becomes more interesting when we do not know what the interest rate will be. This arises more frequently since it is often the case that the interest rate is uncertain. Thus, people form expectations about what they believe the interest rate will be in the future and plan accordingly. Let's say we now live in a world where the interest rate can be one of two things, it can either be 0 with probability $\frac{1}{2}$, or it can be 0.04 with probability $\frac{1}{2}$. The average interest rate is the same as the above, constant interest rate at 0.02. However, the problem we are solving becomes much more difficult. We then modify our decision problem to maximize:

$$U(c_0, c_1, ..., c_n) = \sum_{t=0}^{T} \beta^t \, \mathbb{E}(\log(c_t))$$

Subject to the constraints:

$$c_t > 0$$

$$w_0 \geq \sum_{t=0}^{T} \frac{c_t}{(1+r)^t}$$

$$w_t \geq 0$$

To illustrate the depth of this new maximization problem, let's take the decision problem for only the second to last period. Because we maximize our consumption, we know that in the final time period, T, we will always consume all we have left. Thus at $T-1$, we given the wealth level at that time, w_{T-1}, we maximize the following expression:

$$\log(c_{T-1}) + \beta\left[\frac{1}{2}\log[(w_{T-1} - c_{T-1})1.00] + \frac{1}{2}\log[(w_{T-1} - c_{T-1})1.04]\right].$$

As we extend this to more and more time periods, this expression quickly gets very hairy. Even though the expected interest rate each period is the same, the expected utility is not. Additionally, at every time period when the interest rate is realized, the wealth we have left to work with changes slightly from our expected wealth in that time period. Thus we have to recalculate what we should consume each time period. The only period of consumption we can explicitly calculate is the first period. This is because at this point, no interest rates are realized and we work only off of the expected utility we get across all future periods. The consumption pattern in all other periods must be simulated. Additionally, there is no nice closed form expression for this equation to include all periods of consumption. Given that the expression quickly gets very complicated given only 2 possible choices for the interest rate, it becomes nearly impossible to write out fully given our uniform distribution of 89 values of the interest rate. Unlike the expression we found above, the only way to calculate the optimal consumption path given realized interest rates is to use stochastic dynamic programming. The best way to do this is with computational power.

5 Stochastic Dynamic Programming Algorithm

In order to solve for optimal consumption patterns in the face of uncertainty, I implement a stochastic dynamic programming simulation using the programming language, Python.

The algorithm works using backwards induction. Starting with the final time period, we can calculate exactly how much the agent will consume given her level of wealth. This is because the agent consumes all she has left in the final time period. Thus we calculate the utility this gives the agent when she has any possible level of wealth in the final time period and memoize these utility levels.

We then can move on to the penultimate time period. In this period we also calculate for any level of wealth, what the optimal consumption path is knowing that whatever the agent doesn't consume today, she must consume in the next period. We have already calculated her utility at any level of consumption in the next period and we have a probability distribution over all possible interest rates so we can calculate what we should consume given that there is a certain probability we accumulate so much in interest over what we do not consume today. From there we now have an expected utility over every level of wealth in the second to last time period. We memoize the expected utilities in this time period along with the optimal consumption level given any level of wealth in this time period.

Moving on to every previous period, we can similarly calculate the expected utility given any possible level of wealth in each time period. Finally, when we are at the very first time period, we calculate the expected utility given the true level of wealth in $t = 0$, and from there we know what our initial consumption level should be. Once we have the initial consumption level, we can begin to run the simulation.

Simulating real life, the agent picks a consumption level based on the optimal levels calculated via the backwards induction method. Then an interest rate will be chosen from the distribution of interest rates and will be implemented on the remaining wealth the agent has in that time period. Her wealth now changes from this period to the next in a different way from what she expected it to. She then adjusts her consumption level in the next period based off of this new information and her new level of wealth using the memoized consumption paths for all levels of wealth in each time period. Continuing like this, the agent then goes through all the periods of retirement and a consumption path is created as every period, a new interest rate is plucked from the distribution and is enacted on the agent's remaining wealth.

We perform this simulation on many different specifications of distributions of the interest rate to simulate how varying factors affect people's consumption behavior, given their expectations about the interest rate.

6 Specifications and Results

We ran many specifications of different distributions of interest rates to simulate how these different distributions would affect consumption and savings behavior among individuals.

We can use any distribution of interests rates in the simulation to calculate a consumption pattern for an agent. Using similar parameters to what we did in the mathematical background above, we use $T = 19$, $w_0 = 1000$, $\delta = 0.01$ for treasury bills, and $\delta = 0.02$ for corporate stocks.

We first begin by a discussion of how changing the variance of the distribution affects the initial level of consumption, c_0. Then we lower the interest rates to see how consumption and savings behavior is affected by the expectation of lower future interest rates.

We begin by looking at what happens if we increase the variance of each distribution. In order to make the variance higher, we can increase the distance between each data point and the mean in a uniform fashion. In the case of Figure 3, the distance was doubled between each data point and the mean, leading to a 4-fold increase in the variance.

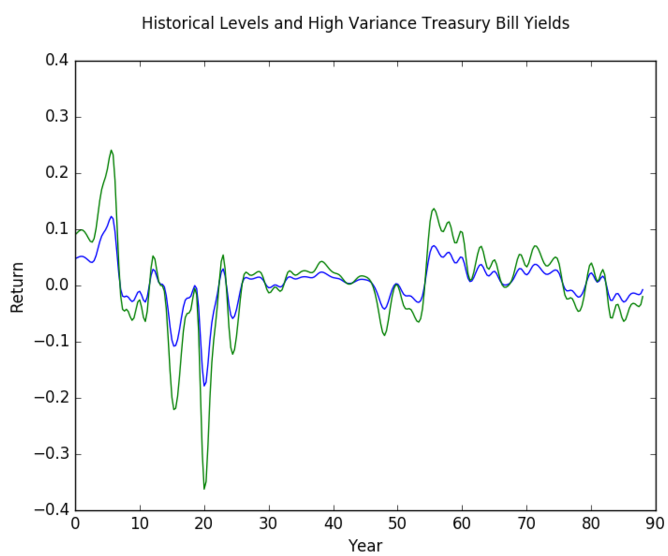

Figure 3: Historical levels of treasury bills (blue) and high variance levels of treasury bills (green)

Using this new distribution, we can compare what consumption patterns are like in both a time of low uncertainty and a time when there is high variance and thus high uncertainty what the interest rate will be in the future. Since treasury bills are quite a low yield to begin with, the increased variance does not actually affect the initial consumption level of the agent. In both cases, we get an initial consumption level, $c_0 = 55$. However, if we look at Figure 4, below, we notice that actually the higher variance distribution causes many consumption paths to be either much lower or much higher than those concentrated at the distribution of historical levels. This gives evidence for increased uncertainty in the face of higher variance returns.

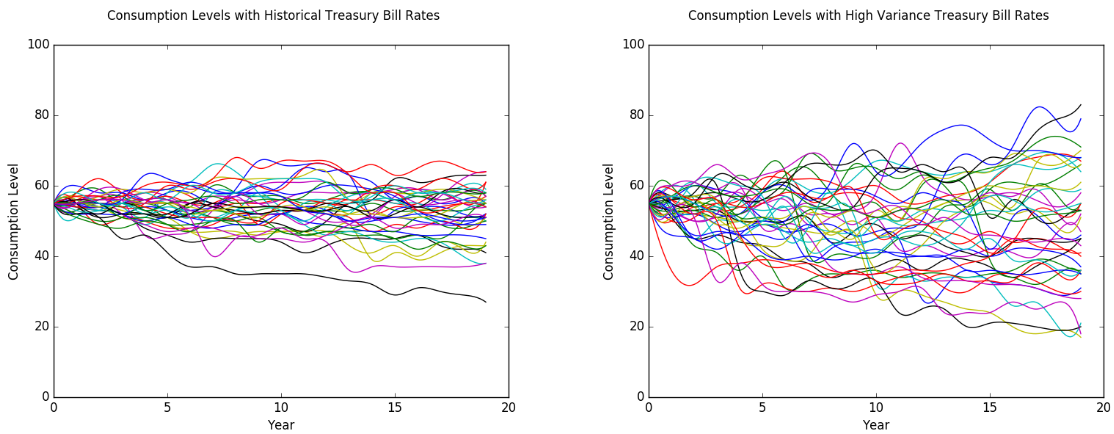

Figure 4: 40 imputations each of consumption paths using historical and high variance treasury bill rates

While the treasury bill distribution may demonstrate how increased variance could lead to uncertainty in future consumption paths, it is very starkly illustrated in the consumption paths we see when using corporate stocks. In Figure 5, we have side-by-side, the consumption paths resulting from historical levels of corporate stock yields and resulting from high variance corporate stock yields. Notice that in the high-variance consumption paths, there are many more paths that fluctuate up very high into the realm of incredibly high consumption. However, there also a huge number of paths that sink to near-zero levels of consumption for most of the time. That sort of near-zero level of consumption is not really present in the less-variance, historical levels of corporate stock yields.

The uncertainty in the high-variance levels is very high. In fact, this is very noticeable when we look at c_0 and what each individual tries to consume in the very first period to maximize expected utility. In the left panel of Figure 5, with historical levels of stocks, $c_0 = 65$ whereas in the right panel, $c_0 = 62$. We then see that rational, risk-averse individuals would cut down on their initial consumption to provide a buffer for themselves in the case of extreme loss if they got unlucky in the market and lost much of their capital.

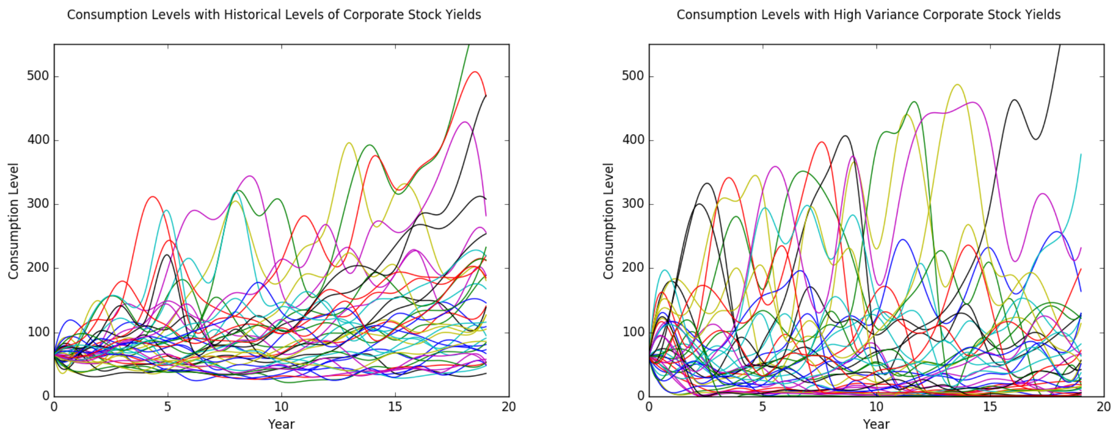

Figure 5: 40 imputations each of consumption paths using historical and high variance corporate stock yields

Finally we can also analyze what happens when we lower the interest rate, say 200 basis points. When this happens, the returns to treasury bills look something like Figure 6.

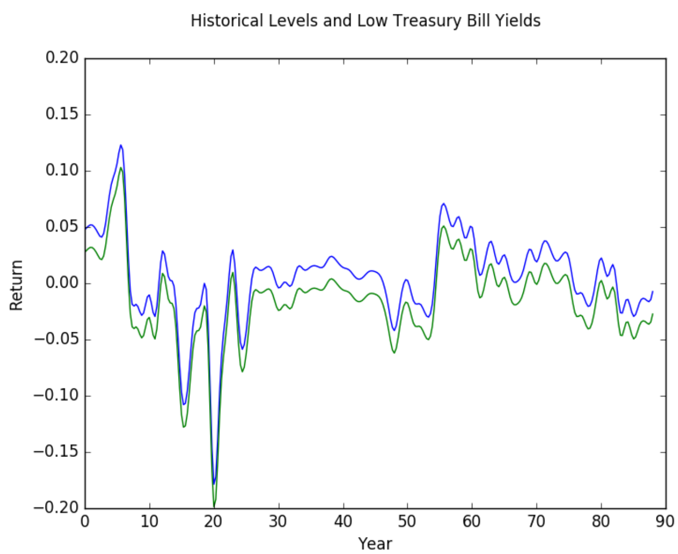

Figure 6: Historical levels of treasury bills (blue) and low levels of treasury bills (green)

In the case of the treasury bill distribution, we end up with consumption paths that look much like the right panel of Figure 7. In this case, we get a very clear downward trend in consumption. And in fact, the consumption in the first period does not change at all, c_0 remain 55. This may seem strange if the agents do not discount the future that much, it seems as if they are losing utility from one period to the next. However, upon closer inspection, it turns out that the interest rate is so low in this new case that it is actually slightly negative. Thus the agent is better off consuming more early on as she loses a small fraction of the money she saves Thus when we get close to a 0% interest rate, we could be faced with many flat or slightly downward trending consumption paths from people in retirement.

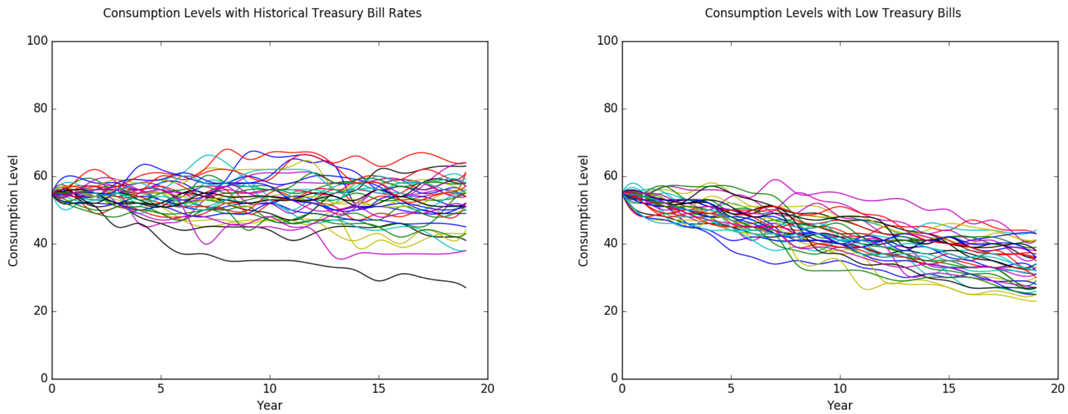

Figure 7: 40 imputations each of consumption levels using historical and low levels of treasury bills

7 Conclusion

In summary, this paper sought to investigate how consumers respond to changes in the interest rate by simulating a rational, risk-averse agent's response to a certain distribution of interest rates. By using stochastic dynamic programming in Python, we were able to model uncertainty in the interest rate and thus provide a robust simulation for visualizing different paths of consumption over the time frame of retirement.

We were able to simulate both added uncertainty and lower yields in the interest rate. In both cases we can explicitly calculate what the effect will be on consumption in the first time period, c_0. Additionally, we can simulate the optimal consumption paths given a randomly chosen set of interest rates to visualize how consumption plays out over time.

In general, added uncertainty leads to more conservative consumption in c_0. This is due to individuals not being able to anticipate getting either very lucky or very unlucky in drawing certain interest rates from the distribution. Because people are in general very loss-averse, they consume conservatively in the first period in order to have a buffer in the face of losing a lot in subsequent periods.

Lowering the levels of the interest rate actually do not make people as conservative as I think I would have predicted when they are choosing consumption in c_0. This is because they are exponential discounters and in our case, the interest rate was slightly negative so it was in their best interest to save as little as possible. We can see this to be a negative ramification on retirement in the future when people expect real interest rates to be as low as they have been around for the past 10 years. This will disincentivize people from saving and will lead to a generally more myopic looking consumption pattern in retirement, leaving less for the end of life. This becomes especially difficult when we also add in the fact that life expectancy is becoming longer and longer.

8 Extensions

In order to make this a more robust model, one could try to extend the simulated decision making process to include the choice between different distributions of interest rates. For example, given lower interest rates, people may decide to switch from safer, lower-yield investments, such as bonds and treasury bills, to higher risk but higher yield investments such as corporate stock.

In retirement, it is very common for people to start drawing out their retirement savings from high-risk investments and put them into safer investments. This is to reduce the risk of losing their nest egg as they draw down their wealth closer and closer to the end of their life. What would be interesting to study is to try and simulate this decision process to see when it is most optimal to make that switch. Given a risk-averse utility function, we could model at what point an individual is better off switching from high-risk, high-yield investments to safer, lower-yield investments. This was beyond the scope of the project but it is an interesting simulation to run. This is because we could simulate whether a lower overall interest rate incentivizes a later shift over to safer investment. This trend could have large ramifications for the safety net of retirement, where the yields are so low that individuals are willing to risk their retirement savings to go for higher-yield investments.

Additionally, people often care about more things than were present in my stylized model. We could add some sort of utility for bequests or add in some sort of uncertainty surrounding life expectancy. These factors would both more accurately reflect the decision-making of retiree today as they often care about those they leave money to and nobody knows exactly when they will die.

Finally, it is important to note that not all people are rational decision-makers. Although this simulation tool is quite powerful, most Americans do not have access to simulations that tell them the optimal amounts to consume and save in retirement in the face of uncertainty in the interest rate. It could be interesting to look at real aggregate savings and consumption data among people of different ages in the United States in different time periods to see if the general trends match up with what we have calculated to be the optimal levels given expectations about the interest rate.

References

[1] “10-Year Treasury Inflation-Indexed Security, Constant Maturity.” *FRED.* Web. 05 Mar. 2017.

[2] ”Annual Returns on Stock, T.Bonds and T.Bills: 1928 - Current.” *Welcome to Pages at the Stern School of Business, New York University.* N.p., n.d. Web. 10 Jan. 2017.

[3] “Inequality – Poverty Rate – OECD Data.” *The OECD.* Web. 05 Mar. 2017.

[4] Sargent, Thomas J., and Rodolfo E. Manuelli. *Dynamic Macroeconomic Theory.* Cambridge, Mass.: Harvard U, 1987. Print.

[5] Stokey, Nancy L., Robert E. Lucas, and Edward C. Prescott. Recursive Methods in Economic Dynamics. Cambridge, MA: Harvard UP, 2004. Print.