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1. Consider the word `unusual`. How many unique subsets of 5 letters (of the 7) exist? How many different strings could be made from 5 of those 7 letters? → How many subsets of those 5 letters?

1 unique subset → {U, N, S, A, L}

unusual
u is already accounted for → only 5 unique letters

SUBSETS where order does matter

_ _ _ _ _
5 · 4 · 3 · 2 · 1 → 5! = 120 different subsets

2. Using a standard deck of playing cards, how many ways are to form a 5-card hand with 2 pairs (i.e. pair of one value, a pair of a different value, and a fifth card of some other value)?

Choose two different values from A – King

Choose any suit of that remaining value selected

$$\binom{13}{2} \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot \binom{11}{1} \cdot \binom{4}{1} = 123552$$

Of all suit variations of that value, choose 2

Of all suit variations of that value, choose 2

Choose one of the 11 remaining values

3. A violinist serenades couples at a romantic restaurant. She will play 16 songs in an hour and there are 7 couples. One couple is having a fight and will allow at most 1 song to be played to them before they ask the violist not to return to their table. If we care only about the number of songs each couple receives, how many ways can the songs be distributed amongst the couples.

Indistinguishable Elements

Distinguishable Boxes: Couples (bars)
Indistinguishable Elements: Songs (stars)

$$16 \cdot \binom{20}{14} = 38760 \text{ ways}$$

16 options for the first couple

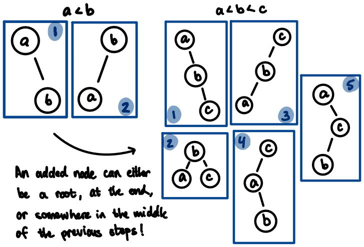
6 − 1 + 15
15 − 1

Stars and bars
15 songs + 6 couples

4. There is a Binary Search Tree with 12 nodes. Each node has a distinct value between 1 and 12. The root has value 3, and its right child has value 9. How many possible Binary Search Trees could this be? Tip: *Try to define how many ways there are to form a BST of 2 nodes. Then try to define how many ways there are to form a BST of 3 nodes (think about the possible insertion order based on rank: smallest, medium, largest)* ***in terms of 2 node trees*** *for certain cases. Continue to do this for 4 node trees (in terms of 3- and 2-node trees for various cases of insertion ordering based on rank) and 5 node trees.*

BST w/ 2-nodes
a<b

BST w/ 3 nodes
a<b<c

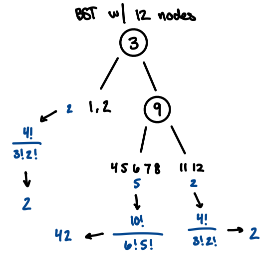


An added node can either be a root, at the end, or somewhere in the middle of the previous steps!

Number of BSTs w/ n nodes

$$\frac{(2n)!}{(n+1)!\,n!}$$

BST w/ 12 nodes



2 · 2 · 42 = 168 different trees

5. 10 friends arrive to get their COVID vaccine during a particular time slot. During that time slot there are 4 identical nurses administering shots, but 1 of the nurses **may** (or **may not**) be scheduled for a break during the time slot in which the friends arrive. Also, how long it takes the nurses to administer a shot varies wildly, so the nurses working during the time slot are guaranteed to serve at least 1 person, but how many additional people they are able to serve is arbitrary. How many different combinations are there for the number of patients served by the nurses?

n = 10 friends → Indistinguishable elements

k = 4 nurses → Distinguishable boxes

$\binom{n-1}{k-1}$ if at least 1 per box

$\binom{10-1}{4-1} = \binom{9}{3} = 84$ if all nurses are working

If one nurse is on break, then $k=3$ → $\binom{10-1}{3-1} = \binom{9}{2} = 36$

$\binom{9}{2} + \binom{9}{3} = 120$ possible combinations

Nurse is on break

All nurses working

Add up the possibilities