Variations on the Missionaries and Cannibals Problem

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Plan

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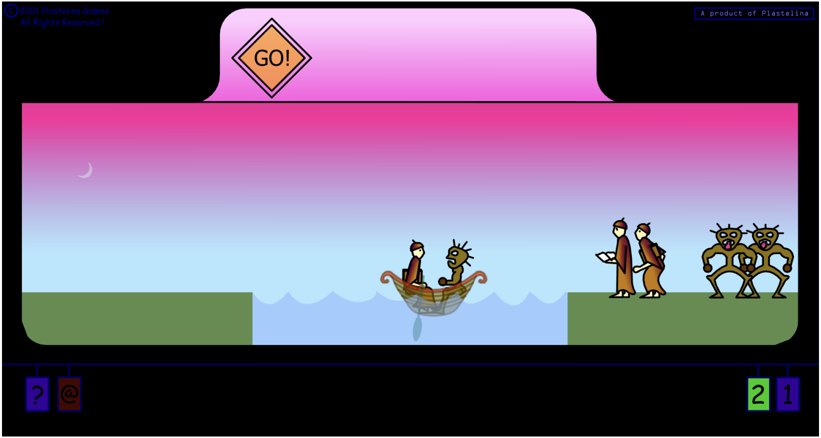
Wolf, goat and cabbage

Missionaries and cannibals



Jealous husbands

The missionaries and cannibals become three married couples, with the constraint that no woman can be in the presence of another man unless her husband is also present.

There cannot be both women and men present on a bank with women outnumbering men; therefore any solution to the jealous husbands problem will also become a solution to the missionaries and cannibals problem

General problem statement - I

- $A_1, A_2 \ldots, A_k$ – initial number of objects from each object class
- $C_{bank}(a_1, a_2, ..., a_k)$ set of conditions for both river banks, where $(a_1, a_2, ..., a_k)$ – number of objects from each class at the first bank
- $C_{boat}(b_1, b_2, ..., b_k)$ set of conditions for the boat

We may want to determine

- minimal number of moves required to transfer all object from one river bank to another
- number of (optimal) solutions
- at least one solution
- all solutions

General problem statement II

Missionaries and cannibals

- parameterized with $(\mathrm{M}, \mathrm{C}, \mathrm{B}, \mathrm{d})$ – total number of missionaries, cannibals, boat capacity, and the 'safety margin' – missionaries should outnumber cannibals by at least d
- classical puzzle – $(3, 3, 2, 0)$
- when $\mathrm{d} = 0$ we know the exact solution for any values of M, C and B
- when $\mathrm{d} > 0$ in general – open problem

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Graph representation

Directed graph with the vertices labelled by triples $(\mathrm{m}, \mathrm{c}, \mathrm{b})$

- m – the number of missionaries currently at the first bank
- c – the number of cannibals currently at the first bank,
- $\mathrm{b} = 1$ – the boat is currently at the first bank, $\mathrm{b} = 0$ – at the second bank

Graph representation

Constrains

- $m > 0 \Rightarrow m \geq c$
- $3 - m > 0 \Rightarrow 3 - m \geq 3 - c$

Edges

- $(m, c, 1) \Rightarrow (m - e_1, c - e_2, 0)$
- $(m, c, 0) \Rightarrow (m + e_1, c + e_2, 1)$

where $1 \leq e_1 + e_2 \leq 2$ and both vertices are legal

Dijkstra algorithm

DFS, BFS, other algorithms – also possible

Dijkstra – standard graph search algorithm, finds the number of solutions. Original version traces back only one solution

Can be improved by storing all the predecessors with the same best g-value

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Adjacency matrix - I

$$A_{(i,j)} = \begin{cases} 1, & \text{if there is a directed edge between i and j} \\ 0, & \text{otherwise} \end{cases} \tag{1}$$

- $A^{k}_{(i,j)}$ is the number of paths of length k between i and j
- Doesn't tell us how to actually solve the riddle

Adjacency matrix - II

$$A_{(i,j)} = \begin{cases} x_{ij}, & \text{if there is a directed edge between i and j} \\ 0, & \text{otherwise} \end{cases} \quad (2)$$

- x_{ij} – formal variable
- $A^k_{(i,j)}$ – polynomial in the indeterminates a_{ij} where each monomial corresponds to a path that is easy to reconstruct

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Crossing polynomial

$$P(x_1, x_2, \ldots, x_k) = \sum_{(b_1, b_2, \ldots, b_k):\ C_{river}(b_1, b_2, \ldots, b_k)} x_1^{b_1} x_2^{b_2} \ldots x_k^{b_k}$$

i.e for the original missionaries and cannibals riddle,

$$P(m, c) = m + m^2 + c + c^2 + mc$$

Clean-up linear operator

for monomials

$$T(x_1^{a_1} x_2^{a_2} \dots x_k^{a_k}) = \begin{cases} x_1^{a_1} x_2^{a_2} \dots x_k^{a_k}, & \text{if } C_{\text{bank}}(a_1, a_2, \dots, a_k) \\ 0, & \text{otherwise} \end{cases} \tag{3}$$

Positions representation

- initial position: $f_0(x_1, x_2, \ldots, x_k) = x_1^{A_1} x_2^{A_2} \ldots x_k^{A_k}$
- crossing in forward direction – $g_i(x_1, x_2, \ldots, x_k) = T[f_{i-1}(x_1, x_2, \ldots, x_k) \cdot P(x_1^{-1} x_2^{-1} \ldots x_k^{-1})]$
- crossing in backward direction – $f_i(x_1, x_2, \ldots, x_k) = T[g_{i-1} P(x_1, x_2, \ldots, x_k) \cdot x_1 x_2 \ldots x_k]$
- final position: non-zero constant term of g_i, $2i - 1$ – total number of moves

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References I

- Original paper
- Wolf, goat and cabbage interactive
- Missionaries and cannibals interactive