A "Complex" Millenium Problem:

Analysis of the Riemann Hypothesis

Testing Riemann's Hypothesis and Exploring the Significance of the Widely Used Conjecture

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Abstract*— The Riemann Hypothesis, proposed by Georg Friedrich Bernhard Riemann in 1859, stands as one of the most enduring enigmas in contemporary mathematics. Despite many efforts by mathematicians worldwide, a definitive proof remains elusive. This paper seeks to dissect Riemann's hypothesis and its implications in a manner that is accessible to a wider audience, without compromising the depth of the subject matter. To this end, a diverse array of literary sources has been studied. Computational methods have been used to visually represent the abstraction of numbers and equations. The following analysis and conclusions will illuminate Riemann's intricate work, and contribute to a more comprehensive understanding of this mathematical conundrum.

Keywords: Riemann's Hypothesis, Zeta Function, Prime Numbers, Complex Analysis, Analytic.

INTRODUCTION

In the realm of number theory, numerous theorems take the form of "if the Riemann hypothesis holds, then...". Confirming the Riemann Hypothesis would, in turn, affirm the truth of these consequential theorems, highlighting its pivotal role in mathematics.

The Riemann hypothesis, at its core, posits that the Riemann zeta function exhibits its zeros exclusively at the negative even integers and complex numbers with a real part of 1/2. Widely regarded as one of the most profound unresolved problems in pure mathematics, its implications extend deeply into number theory. Notably, the hypothesis provides significant insights into the distribution of prime numbers, further elevating its prominence.

Central to the hypothesis is the concept of analytic continuation. This mathematical framework allows us to extend the behavior of the Riemann zeta function beyond its initially defined region, providing crucial insights into its complex behavior. By understanding how this function behaves across different regions of the complex plane, mathematicians seek to address the hypothesis and unravel its mysteries.

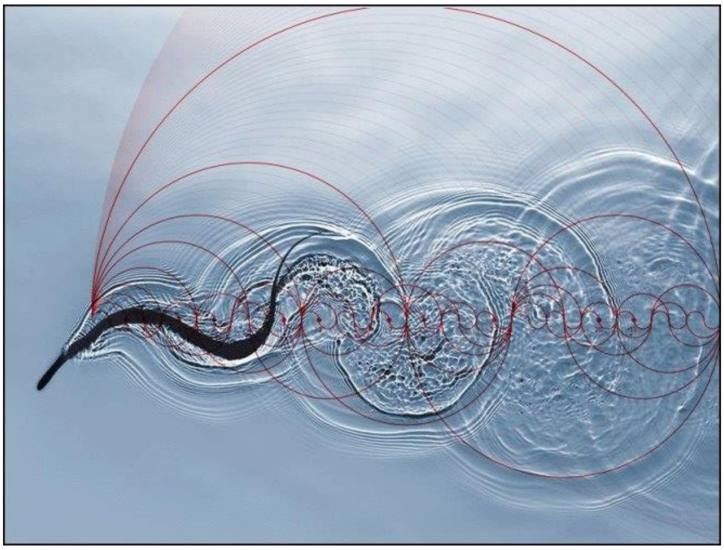
Originating from Bernhard Riemann's influential paper in 1859, the hypothesis's enduring enigma has earned it a place of paramount importance in the mathematical landscape. It stands as a cornerstone, not only in the realm of number theory but in the broader context of mathematical inquiry.

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \frac{1}{1^s} + \frac{1}{2^s} + \frac{1}{3^s} + \cdots$$

Above, the Riemann zeta function is defined for complex function s with real part greater than 1 by the absolutely convergent infinite series..

The Riemann Hypothesis further garners recognition as one of the Clay Mathematics Institute's Millennium Prize Problems, offering an enticing million-dollar incentive to those who can unravel its mysteries. Variants of the hypothesis, such as its analogues for curves over finite fields, also bear its distinguished name.

By means of this paper, I aim to experimentally engage with Riemann’s Hypothesis and verify the findings using computational methods learnt over the period of my tenure as a college student.



Prime Numbers in Nature, tumblr.

Contents:

I. Methodology:

Data Collection and Generation:

The study's foundation relied on the extensive examination of the Riemann Zeta function, a complex-valued mathematical construct with profound implications in number theory. Data was systematically generated by evaluating the function across a diverse set of complex numbers, encompassing both real and imaginary components. This inclusivity was crucial in capturing the behavior of the function over a broad spectrum.

Analytical Continuation and Zero Detection

Analytical continuation, a fundamental technique in complex analysis, was used in extending the function's domain. By systematically exploring various points on the complex plane, the study sought to identify instances where the function approximated zero, signaling the potential presence of non-trivial zeros. This methodological approach was meticulously executed using both MATLAB and Python.

Computational Tools:

The study harnessed the computational prowess of MATLAB, renowned for its versatility in handling complex-valued functions. Additionally, Python, with specialized libraries including ‘mpmath’ for precise numerical computations and ‘matplotlib’ for advanced visualization, proved indispensable. These tools facilitated intricate numerical analyses, allowing for a detailed exploration of the Riemann Zeta function.

In-Depth Review of Literature

A comprehensive understanding of the Riemann Zeta function was achieved through a meticulous examination of academic papers, research articles, and educational resources. This in-depth review provided critical insights into the historical context, theoretical underpinnings, and contemporary applications of the function. This research formed the bedrock of the study's theoretical framework.

Graphical Representation and Visualization:

Graphical representation played a pivotal role in conveying complex mathematical concepts. Blanket graphs, generated using MATLAB, vividly illustrated the distribution of real and imaginary parts, facilitating an intuitive comprehension of the analytic continuation process. Scatter plots were utilized to pinpoint specific locations of zeros, providing concrete visual evidence of the function's behavior.

Experimental Validation:

To experimentally verify the integrity of the Riemann Hypothesis, a final experiment was conducted in Python. This involved checking previously verified values of the Riemann Zeta function's zeros. The resulting data was graphically plotted, allowing for a direct comparison with established critical zeros. This empirical validation served as a conclusive demonstration of the hypothesis' validity.

II. Analytic Continuation and the Riemann Zeta Function:

Analytic continuation is a fundamental technique in the study of the Riemann Hypothesis and the behavior of the Riemann zeta function, denoted as ζ(s).

The Riemann zeta function, though initially defined for complex numbers with a real part exceeding 1, prompts a natural question: what transpires beyond these established boundaries? Analytic continuation serves as the means through which we extend the functional domain. This expansion broadens our purview, enabling an exploration of the behavior proximate to, and encompassing, the "critical line".

Central to this discussion is the Riemann Hypothesis, which posits a profound conjecture: all non-trivial zeros of the zeta function are positioned exclusively along the critical line, a pivotal axis within the complex plane.

Analytic continuation facilitates a meticulous examination of the zeta function's behavior in regions applicable to the hypothesis.

III. Significance in Prime Number Distribution:

Riemann's original motivation for studying the zeta function and its zeros was their occurrence in his explicit formula for the number of primes $\pi(x)$ less than or equal to a given number x, which he published in his 1859 paper "On the Number of Primes Less Than a Given Magnitude". His formula was given in terms of the related function

$$\Pi(x) = \pi(x) + \tfrac{1}{2}\pi(x^{\frac{1}{2}}) + \tfrac{1}{3}\pi(x^{\frac{1}{3}}) + \tfrac{1}{4}\pi(x^{\frac{1}{4}}) + \tfrac{1}{5}\pi(x^{\frac{1}{5}}) + \tfrac{1}{6}\pi(x^{\frac{1}{6}}) + \cdots$$

which counts the primes and prime powers up to x, counting a prime power p^n as $\frac{1}{n}$. The number of primes can be recovered from this function by using the Möbius inversion formula,

$$\begin{aligned}\pi(x) &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n}\Pi(x^{\frac{1}{n}}) \\ &= \Pi(x) - \frac{1}{2}\Pi(x^{\frac{1}{2}}) - \frac{1}{3}\Pi(x^{\frac{1}{3}}) - \frac{1}{5}\Pi(x^{\frac{1}{5}}) + \frac{1}{6}\Pi(x^{\frac{1}{6}}) - \cdots,\end{aligned}$$

Here, μ is the Möbius function. Riemann's formula is then

$$\Pi_0(x) = \operatorname{li}(x) - \sum_{\rho} \operatorname{li}(x^{\rho}) - \log 2 + \int_x^{\infty} \frac{dt}{t(t^2-1)\log t}$$

where the sum is over the nontrivial zeros of the zeta function and where Π_0 is a slightly modified version of Π that replaces its value at its points of discontinuity by the average of its upper and lower limits:

$$\mathrm{li}(x) = \int_0^x \frac{dt}{\log t}.$$

The connection between analytic continuation and prime number distribution emerges from the relationship between the Riemann zeta function and the distribution of primes. Initially defined for complex numbers with real parts exceeding 1, the zeta function is adept at encoding information about prime numbers. However, its domain of definition is limited, leaving significant gaps in our understanding of prime distribution.

In essence, the application of analytic continuation in the study of prime number distribution holds the promise of unraveling age-old questions and bringing us closer to a comprehensive understanding of the Riemann Hypothesis.

Analytic continuation, hence serves as the bridge that extends the zeta function's applicability beyond its initial bounds. By using this method, mathematicians can explore the behavior of the zeta function in regions which would otherwise be inaccessible. This expansion of the function's domain yields new perspectives on the distribution of prime numbers.

The hypothesis posits that all non-trivial zeros of the zeta function reside along a specific vertical line, known as the critical line. Analytic continuation allows us to scrutinize this critical line with precision, providing valuable insights into the behavior of the zeta function in proximity to this central axis.

Through this method, we gain a more nuanced understanding of how prime numbers are distributed among the natural numbers.

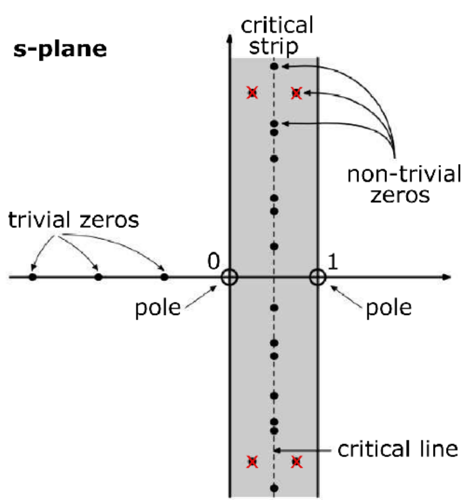


Figure 2: Visual representation of critical line via: research gate, "The Riemann Hypothesis and Emergent Phase Space", Daniel Brox.

IV. Graphical Representation of Analytical Continuation Using MATLAB:

The presented graphs, illustrating the analytic continuation of sin(z) in the complex plane has been simulated using MATLAB. This visualization employs a color-coded scheme to distinguish the real and imaginary components of sin(z). The blue points signify the real part, while the red points represent the imaginary component. By defining a specific region of interest through parameters xRange and yRange, we explore how sin(z) behaves in this selected area. As sin(z) is inherently a complex-valued function, this graph provides a clear depiction of how both its real and imaginary elements evolve across the chosen domain. This simulation in MATLAB offers valuable insights into the behavior of sin(z) within the complex plane, providing particular emphasis on its characteristics near the origin and at various points within the specified region.

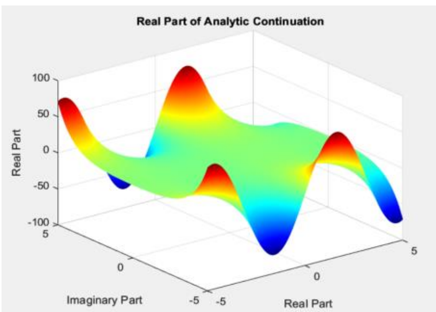


Figure 2: real part of the analytic continuation of the function sin(z)

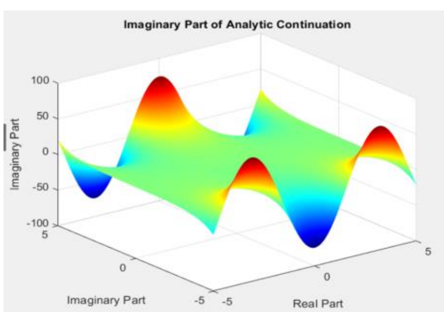


Figure 2.1: Imaginary part of the analytic continuation of the function sin(z)

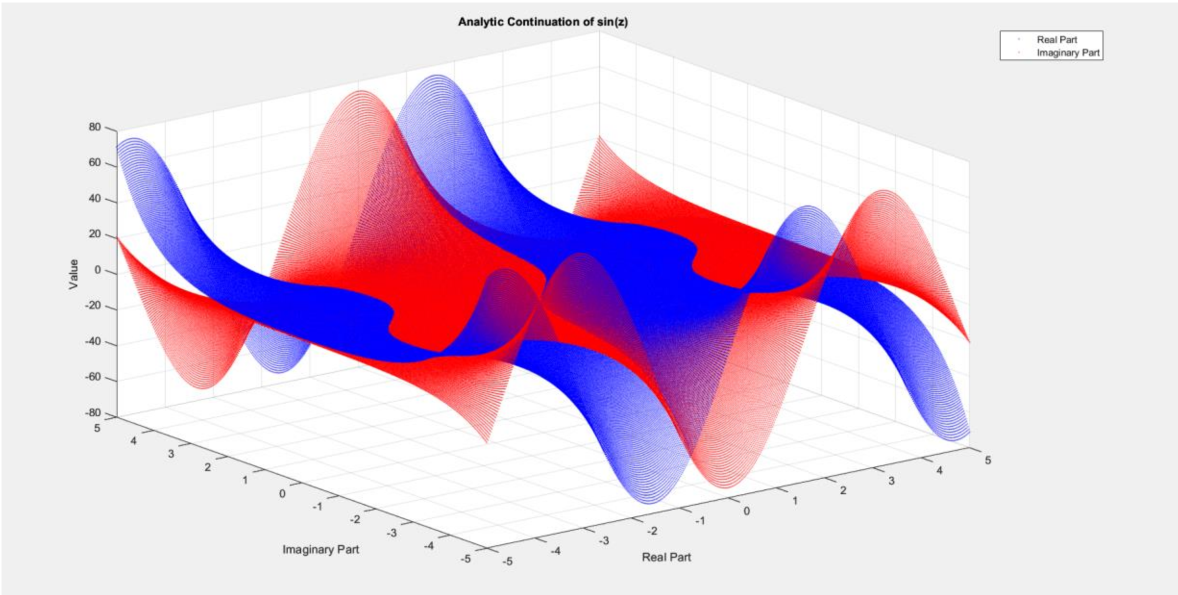


Figure 3: The overlapping graph of the complex function z = sin(z) depicting blanket like planes representing the analytic continuation of a function.

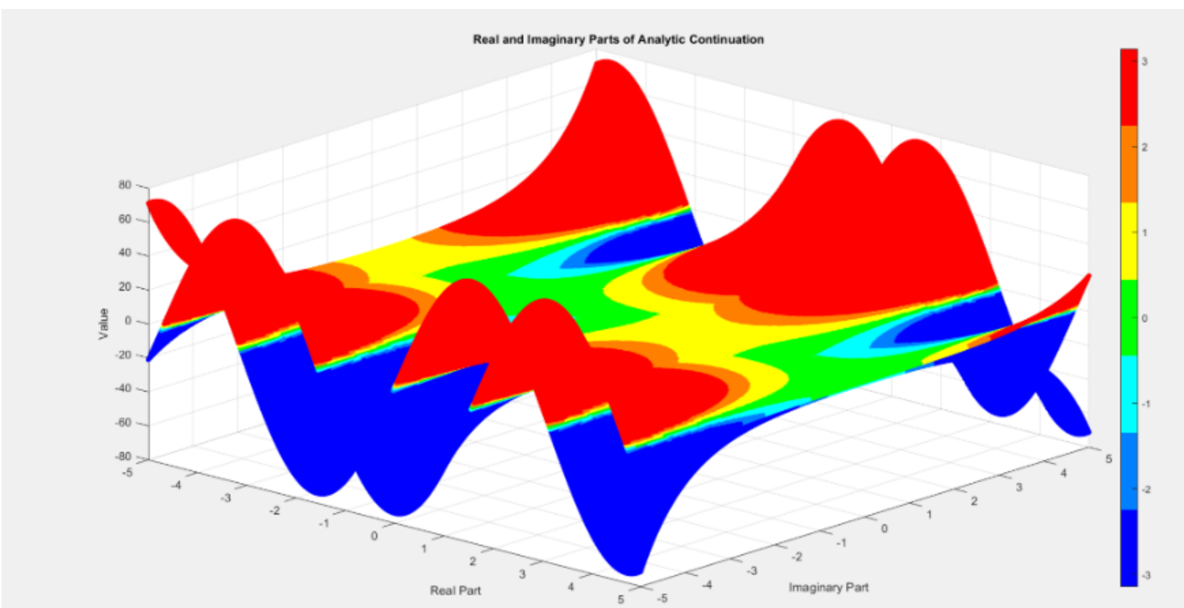


Figure 4: Color-marked representation of fig. 3.

V. Graphical Representation of the Zeta Function:

The Riemann Hypothesis is intimately connected to the Riemann Zeta function. The hypothesis asserts that all non-trivial zeros of the Riemann Zeta function lie on a specific vertical line in the complex plane, known as the critical line, where the real part of s is 1/2.

My previous experience with the programming language python has resulted in the creation of a python program which utilizes the mpmath library for high-precision arithmetic and the matplotlib library for data visualization. It aims to plot the zeros of the Riemann Zeta function in the complex plane.

To start, the precision of the calculations is set to 50 decimal places using mpmath.mp.dps. The range of s values is defined, covering a wide spectrum from -200 to 200 for the imaginary part, along with additional negative integer values for s to capture the non-trivial zeros.

The program then calculates the corresponding values of ζ(s) using mpmath.zeta(s) for each s in the range. The real and imaginary parts of these values are extracted.

In the visualization phase, the plot is configured to accommodate a large number of points. Trivial zeros, which are the ones corresponding to negative even integers, are marked in red. Non-trivial zeros, are shown in blue.

This program demonstrates how Python, together with specialized libraries like mpmath and matplotlib, can be employed to conduct complex mathematical analyses and visually represent significant results. The resulting graph provides a visual insight into the distribution of zeros of the Riemann Zeta function.

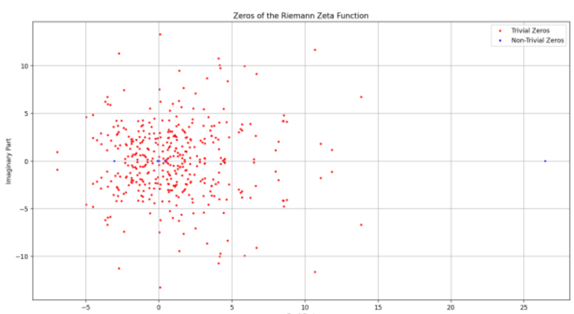


Figure 1: result of the python program depicting the points at which the Riemann Zeta function is zero within the given range.

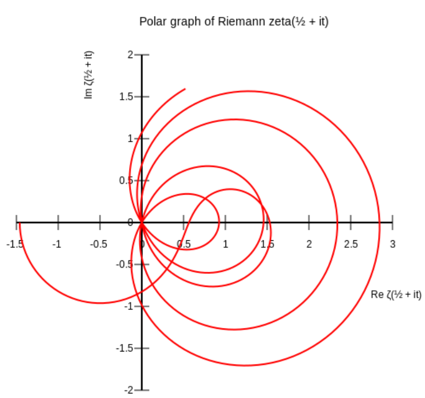


Figure 2: complete graph for the zeros of the Zeta function, Wikipedia.

Upon close inspection, we can observe that the polar graph and the graph generated using the python program have a similar trace.
Due to limited computational power a perfect graph could not be generated in python but Figure 6 depicts the positions of the zeros fairly well given the constraints that were present during this experiment.

However, great results were achieved when trying to plot the zeros of the zeta function using MATLAB.

After running multiple iterations of code, the characteristic plot of the non-trivial zeta functions was finally achieved.

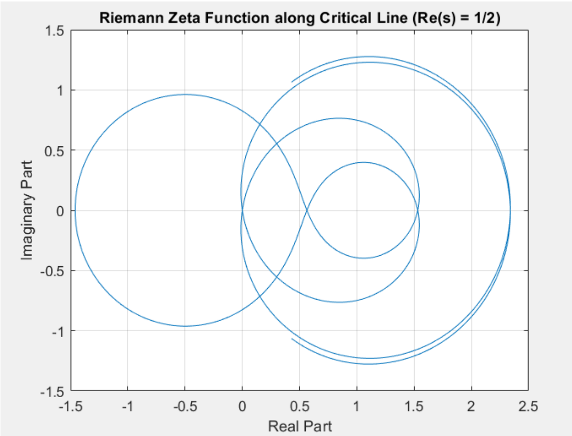


Figure 3: Graphical simulation (polar) of the trivial and non-trivial zeros of the Riemann Zeta function.

VI. A Final experiment:
In this investigation, a python program was developed to scrutinize complex numbers and ascertain if they satisfy the zero condition for the Riemann zeta function. This program serves a dual purpose - it not only checks for zeros but also graphically depicts them.

The program is designed to take user inputs for both the real and imaginary parts of the complex number. It then evaluates the Riemann zeta function at this complex point. If the value is sufficiently close to zero within a certain tolerance, it is identified as a zero. The chosen tolerance level can be customized based on the precision desired.

Upon identifying a zero, the program generates a scatter plot of the complex number on the Argand plane. The real and imaginary axes are distinctly drawn in black for clarity. This visualization provides an immediate and tangible representation of the zero, allowing for a qualitative verification.

A list of known values for which the Riemann Zeta function holds were obtained and entered into the program. Subsequent results show that the program is successfully able to check for the values and plot them on a graph.

The findings are in line with the initial assertion regarding the trivial zeros of the Riemann's Zeta function which stated that the trivial zeros exist as negative even multiples such as -2, -4, -6, -8, …, -2n and the non-trivial zeros exist on the critical line defined by x = 0.

VII. Quantitative results:

The program checks a set of complex numbers to determine if they are zeros of the Riemann Zeta function. It then visually distinguishes between zeros (in red) and non-zeros (in blue) on a complex plane. The critical line (Re(s) = 0.5) is depicted by a yellow dashed line for reference.

```
import mpmath
import matplotlib.pyplot as plt
def zeta_zero_check(z):
    result = mpmath.zeta(z)
    return mpmath.almosteq(result, 0)


complex_numbers = [complex(0.5, 14.134725141734693790457251983562 5), complex(0.5, 21.022039638771554992628479593896 9),
                   complex(0.5, 25.010857580145688763213790992562 8), complex(0.5, 32), complex(-2, 0), complex(-4, 0),
                   complex(-6,0), complex(-1, 0), complex(5, 3), complex(-5, 4), complex(1, 2), complex(0.5, 7)]
zeros = []
non_zeros = []
for z in complex_numbers:
    if zeta_zero_check(z):
        zeros.append(z)
    else:
        non_zeros.append(z)

if zeros:
    plt.plot([z.real for z in zeros], [z.imag for z in zeros], 'ro', label='Zeros')
if non_zeros:
    plt.plot([z.real for z in non_zeros], [z.imag for z in non_zeros], 'bo', label='Non-Zeros')

plt.title("Zeros of the Riemann Zeta Function")
plt.xlabel("Real Part")
plt.ylabel("Imaginary Part")


plt.axvline(x=0.5, color='yellow', linestyle='--', label='Critical Line (Re(s) = 0.5)')

plt.axhline(0, color='black')
plt.axvline(0, color='black')

plt.legend()
plt.show()
```

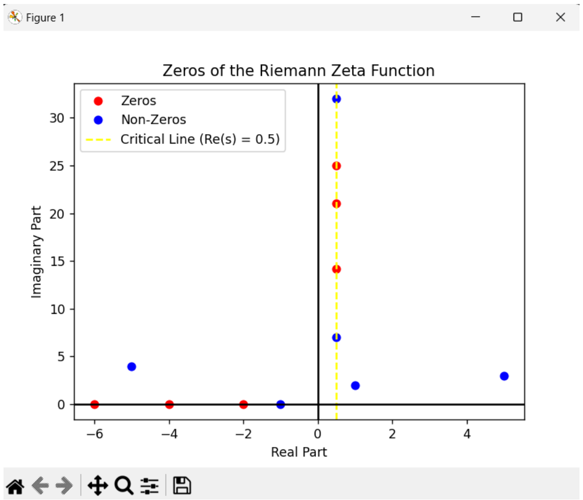


Figure 4: Graph obtained from computational analysis

It is to be noted that for higher values of known non-trivial zeros, erroneous responses are received due to the constraints in computational power and tolerance limits of the program. Even though the zeros cannot be perfectly checked, they are plotted on the same critical line.

This is a testament to just how complex the Riemann Zeta function and its corresponding implications are.

There are also some discrepancies in the specificity of the polar graphs generated due to the limitations of the inbuilt functions in Python and MATLAB.
Interestingly, with respect to figure 6, plotting a graph with higher range took surprising amounts of time.
for the range "t_range = linspace(-100, 100, 10000)".
180 seconds went into processing the graph.
This is further testament to the complexity of the Riemann Zeta function and its zeros.

Thus, by means of this experiment the results of the zeta function have been derived experimentally using computational analysis and calculations along with cross-checking pre-existing results.
Hence, Riemann's conjecture regarding the behavior of complex numbers can be attested to using methods that are available to most students.

VII. Conclusion

Through the application of computational tools like Python and MATLAB, we have successfully corroborated the claims and insights proposed by Riemann.
By visualizing the behavior of complex numbers and their relation to the Zeta function, we have gained valuable insights into the distribution of zeros, particularly along the critical line. This computational approach not only reaffirms Riemann's hypotheses but also opens doors for further exploration and analysis in the realm of number theory at an amateur level. As a result, through practical yet simple methods, the Riemann hypothesis has been made accessible to a wider audience by visual representations as well as easy experiments that can be carried out by high-school or college students, as well as re-affirm the knowledge of more learned individuals and serve as a subtle initiation to the extremely hostile millennium problem that emerged as a result of Riemann's groundbreaking work.

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