Answers to questions in
Lab 1: Filtering operations

Name: Ait lahmouch Nadir **Program:** Master in Machine Learning

Instructions: Complete the lab according to the instructions in the notes and respond to the questions stated below. Keep the answers short and focus on what is essential. Illustrate with figures only when explicitly requested.

Good luck!

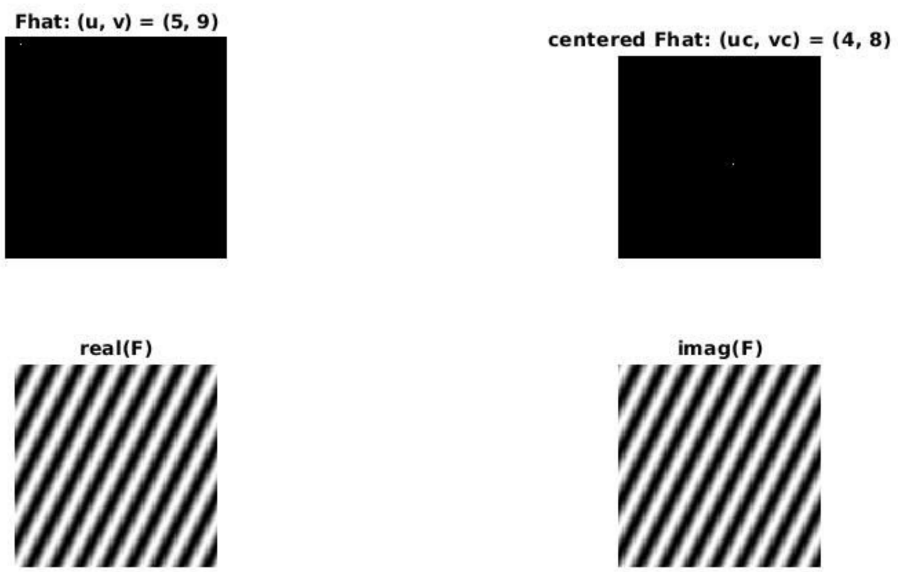
Question 1: Repeat this exercise with the coordinates p and q set to (5, 9), (9, 5), (17, 9), (17, 121), (5, 1) and (125, 1) respectively. What do you observe?

Answers:

The direction and the length of the sine wave depends on (p, q). But the magnitude remains the same. Also, there is a phase shift between the wavelength of the real and imaginary parts.

Question 2: Explain how a position (p, q) in the Fourier domain will be projected as a sine wave in the spatial domain. Illustrate with a Matlab figure.

Answers:



We have :

$$F(x) = \frac{1}{N} \sum_{u \in [0..N-1]2} \widehat{F}(u)\, exp(\frac{j2\pi u^T x}{N})$$

$$F(x_1, x_2) = \frac{1}{N}\, exp(\frac{j2\pi(px_1 + qx_2)}{N})$$

because $\widehat{F}(u) = 1$ when $u = (p, q)$
and $\widehat{F}(u) = 0$ when $u \neq (p, q)$

$$\Rightarrow \quad F(x_1, x_2) = \frac{1}{N}\, cos(\frac{2\pi(px_1 + qx_2)}{N}) + j\, \frac{1}{N}\, sin(\frac{2\pi(px_1 + qx_2)}{N})$$

We can see in the equation above that the real and imaginary parts of the projection are sine waves, therefore the projection is a sine wave too. We can see that in the figure above.

Question 3: How large is the amplitude? Write down the expression derived from Equation (4) in the notes. Complement the code (variable amplitude) accordingly.

Answers:

We already derivated the equation in the question above, therefore we have after derivating the equation (4):

$$F(x_1, x_2) = \frac{1}{N}\, exp(\frac{j2\pi(px_1 + qx_2)}{N})$$

And we know that the amplitude is the modulus of F(x), therefore:

$$\left|F(x_1, x_2)\right| = \frac{1}{N}$$

In our case, $N = 128$:

$$A = \frac{1}{128}$$

P.S: in Matlab, the normalizing factor is $\frac{1}{N^2}$, and that's why $A = \frac{1}{128^2}$.

Question 4: How does the direction and length of the sine wave depend on p and q? Write down the explicit expression that can be found in the lecture notes. Complement the code (variable wavelength) accordingly.

Answers:

The expression of the wavelength is as follows:

$$\lambda = \frac{2\pi}{|\omega|} = \frac{2\pi}{\sqrt{\omega_1^2 + \omega_2^2}}$$

$$\text{where} \quad \omega_1 = \frac{2\pi u_c}{N} \text{ and } \omega_2 = \frac{2\pi v_c}{N}$$

((u_c, v_c) are the new coord. of u after the shift.

$$\lambda = \frac{2\pi}{\sqrt{(\frac{2\pi u_c}{N})^2 + (\frac{2\pi v_c}{N})^2}}$$

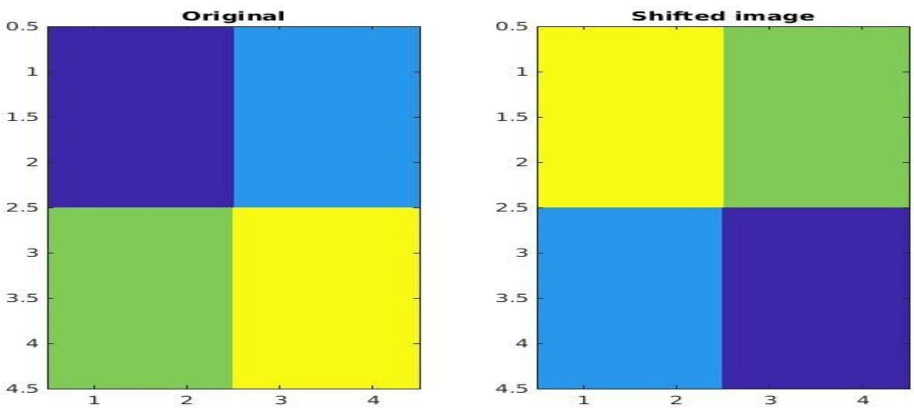
$$\lambda = \frac{N^2}{\sqrt{u_c^2 + v_c^2}}$$

The vector $(u_c,\ v_c)$ is giving the direction of the sine wave.

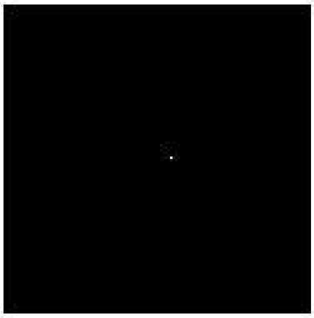
Question 5: What happens when we pass the point in the center and either p or q exceeds half the image size? Explain and illustrate graphically with Matlab!

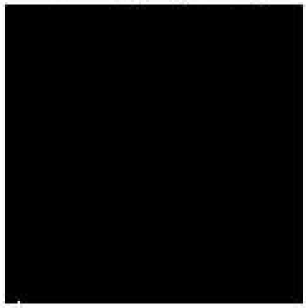
Answers:
The function fftshift places the smallest frequency at the origin (0,0) (center of the image).
The result of this code is :

```
pic = [1 1 2 2; 1 1 2 2; 3 3 4 4; 3 3 4 4]
subplot(1,2,1)
imagesc(pic)
title("Original")
subplot(1,2,2)
imagesc(fftshift(pic))
title("Shifted image")
```



Therefore, if we pass a point $(p,\ q)$ which one of the coordinates or both exceeds half the size of the image, the new corresponding point $(p_c,\ q_c)$ changes region in the new image.
And here's an example:





Question 6: What is the purpose of the instructions following the question *What is done by these instructions?* in the code?

Answers:

The *fftshift* function places the smallest frequency in the middle. So the purpose of this instruction is to place (u, v) in the new map. And it does that by computing new coordinates $(u_c, v_c) \in [\frac{-N}{2}, \frac{N}{2}]$.

Question 7: Why are these Fourier spectra concentrated to the borders of the images? Can you give a mathematical interpretation? Hint: think of the frequencies in the source image and consider the resulting image as a Fourier transform applied to a 2D function. It might be easier to analyze each dimension separately!

Answers:

Let's find a mathematical interpretation of why the Fourier spectra are concentrated in the margin. We will do the the calculation using the image F.
Let's derivate the 2D Fourier transform:

$$\widehat{F}(p, q) = \frac{1}{128} \sum_{p,q=0}^{128} F(x_1, x_2)\, exp(\frac{j2\pi(px_1 + qx_2)}{128})$$

We can separate the two dimensions:

$$\widehat{F}(p, q) = \frac{1}{128} \sum_{x_1=56}^{71} exp(\frac{j2\pi px_1}{128}) \sum_{x_2=0}^{128} exp(\frac{j2\pi qx_2}{128})$$

$F(x_1, x_2) = 1$ only when $x_1 \in [56, 71]$

Also, if $q \neq 0$ *then* $\sum_{x_2=0}^{128} exp(\frac{j2\pi qx_2}{128}) = 0 \Rightarrow \widehat{F}(p, q) = 0$ *when* $q \neq 0$.

And therefore, the fourier spectra is concentrated in the borders of the pic (the first column)

The same analysis can be done to the rest of the pics.

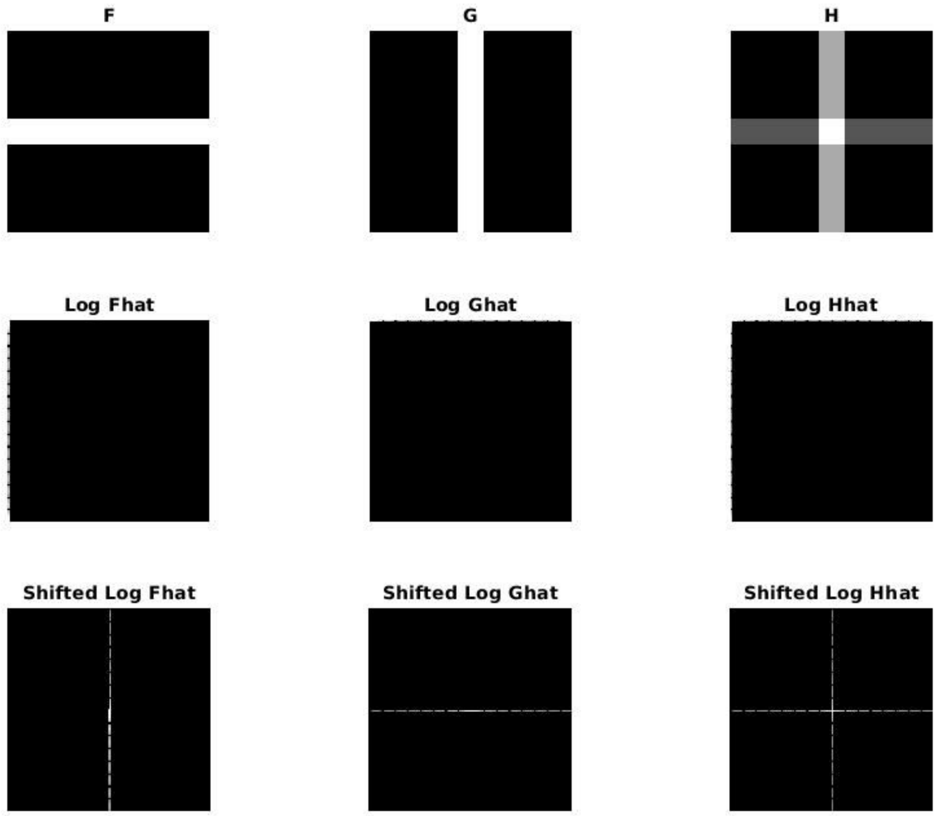
Question 8: Why is the logarithm function applied?

Answers:

The logarithm is applied to exponentially reduce the Fourier spectrum. The dynamic range of the Fourier coefficients is too large to be displayed on the screen, therefore all other values appear as black. Using the logarithm makes the details visible.

Question 9: What conclusions can be drawn regarding linearity? From your observations can you derive a mathematical expression in the general case?

Answers:



We can see in the figures that the fourier transform of image H is a linear combination of the fourier transforms of F and G. It follows this property:

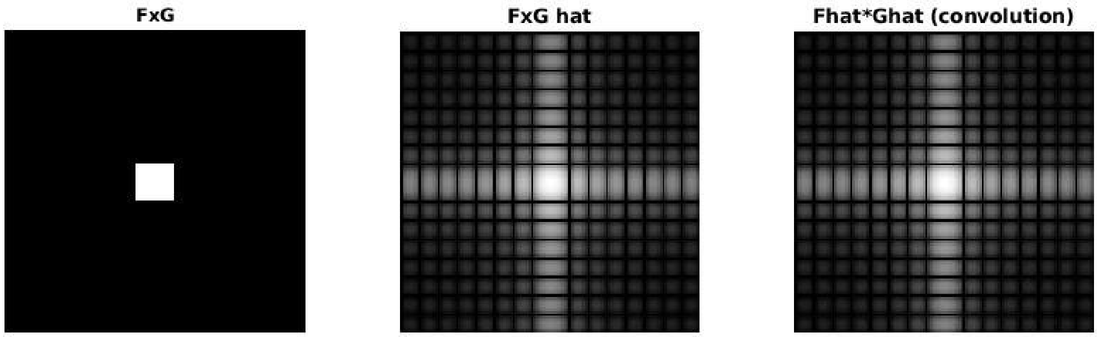
$$\mathcal{F}\,(\mathrm{F} + 2\mathrm{G}\,) = \ \mathcal{F}\,(\mathrm{F}) + 2\,(\mathrm{G})$$

So we can say that the fourier transform is linear, and follow this property:

$$\mathcal{F}\,(a\mathrm{F} + b\mathrm{G}\,) = \ a\,\mathcal{F}\,(\mathrm{F}) + b\,\mathcal{F}\,(\mathrm{G})$$

Question 10: Are there any other ways to compute the last image? Remember what multiplication in the Fourier domain equals in the spatial domain! Perform these alternative computations in practice.

Answers:



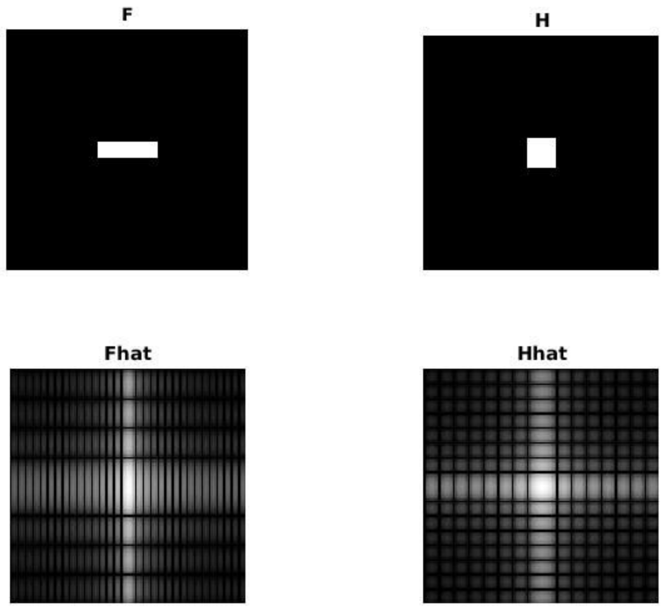
As shown in the figures, multiplication in the spatial domain is the same as convolution in the Fourier domain. And the opposite is valid too. And here's the property.

$$F\,(fg) = F\,(f) * F\,(g)$$

$$F\,(f * g) = F\,(f)\;F\,(g)$$

Question 11: What conclusions can be drawn from comparing the results with those in the previous exercise? See how the source images have changed and analyze the effects of scaling.

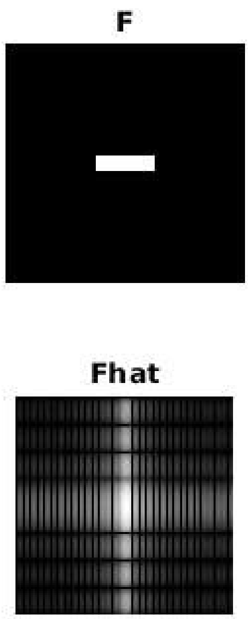
Answers:

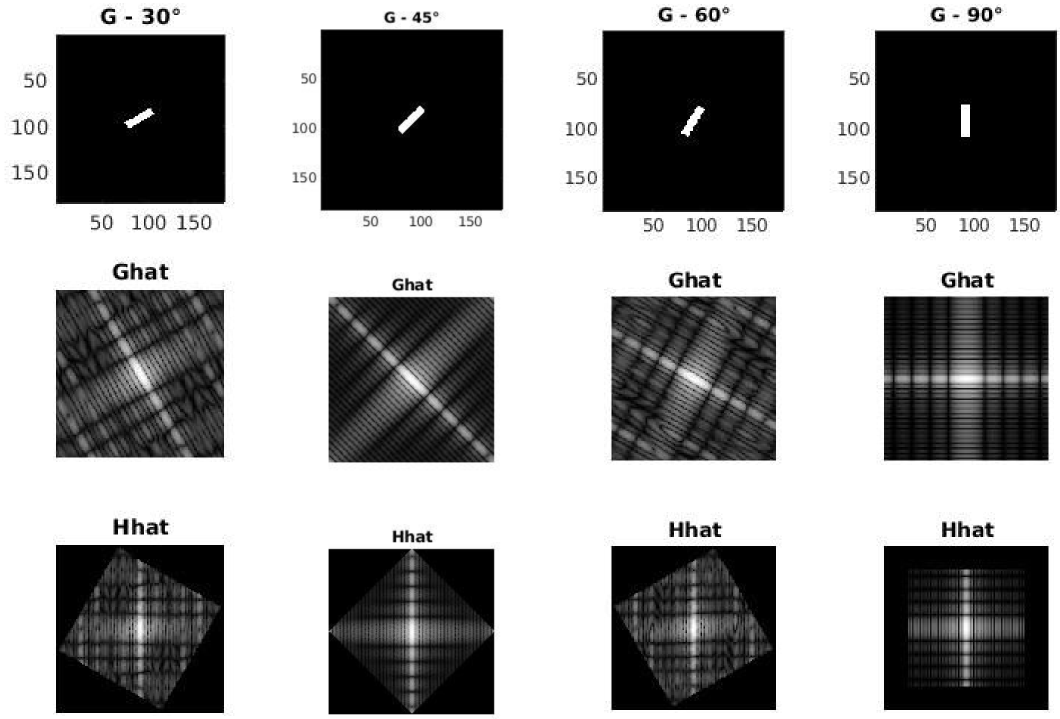


As shown in the figures, scaling in the spatial domain corresponds to scaling in the fourier domain. Compression of an axis in the fourier domain corresponds to expansion of the same axis in the spatial domain.

Question 12: What can be said about possible similarities and differences? Hint: think of the frequencies and how they are affected by the rotation.

Answers:

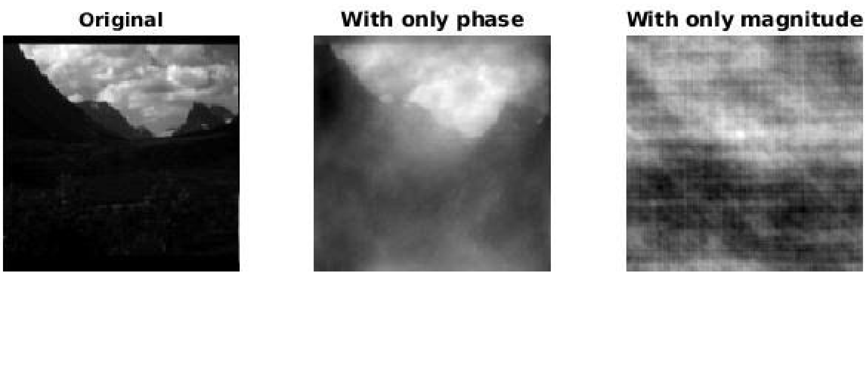


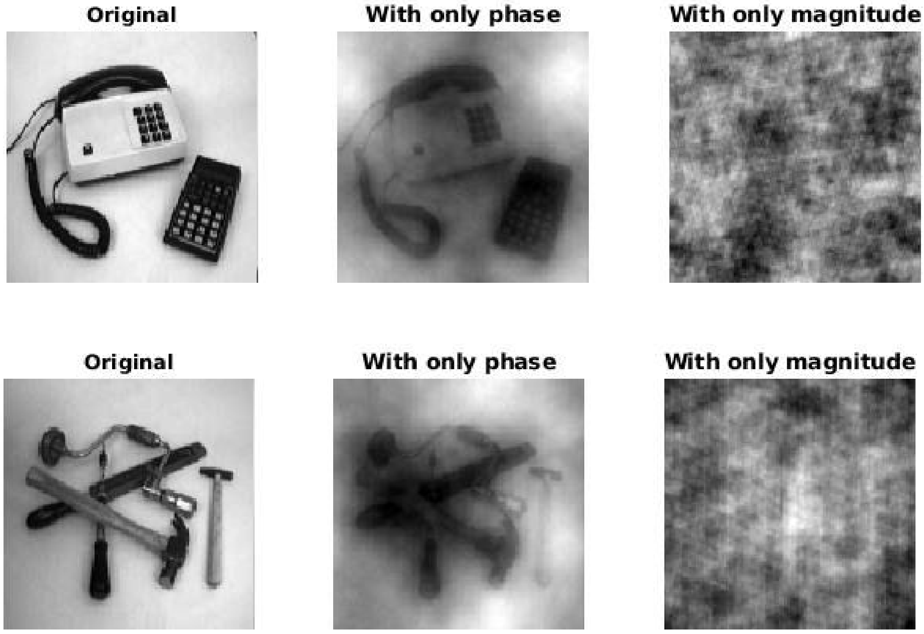


We can conclude from the figures above that a rotation in the spatial domain leads to a rotation with the same angle in the fourier domain. The frequencies don't change from image to image, they're just rotated. However, we can notice some distortions in the fourier domain for the angles 30° and 60°, but i guess this is normal because the corresponding edges in the rotated pics show some irregularities.

Question 13: What information is contained in the phase and in the magnitude of the Fourier transform?

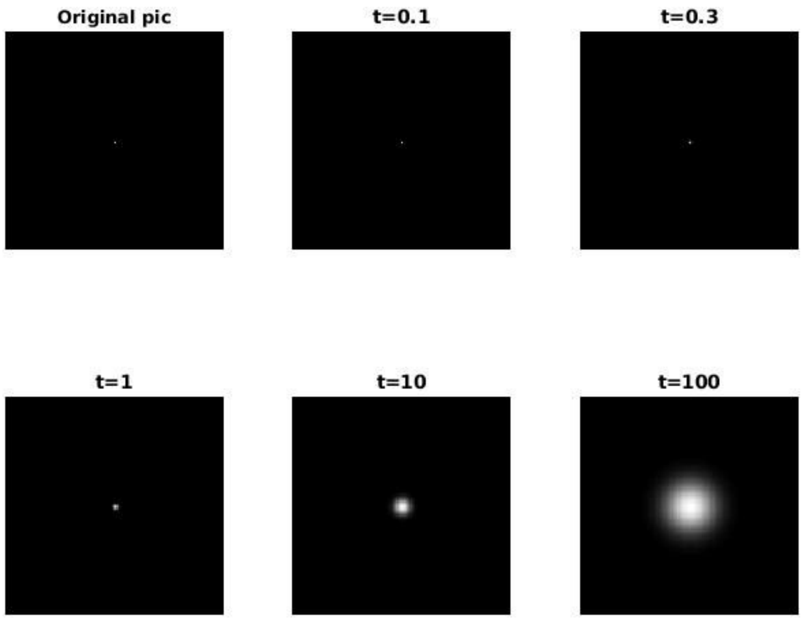
Answers:
From our observations on the figures (see below), we see that the phase is responsible for the edges of the pic as after only keeping it, the image becomes noisy or cloudy but the edges are still there.
However, with only the magnitude, we can't reconstitute the pic, and with randomization, the pic becomes all noise, so we can convince ourselves that magnitude is responsible for the intensity of the colors in an image.





Question 14: Show the impulse response and variance for the above-mentioned t-values. What are the variances of your discretized Gaussian kernel for t = 0.1, 0.3, 1.0, 10.0 and 100.0?

Answers:



The first 6 figures above show the impulse response of the discretized Gaussian kernel for the given t-values. The sharp point in the middle in the beginning seems to spread and get blurry each time we increase the value of t. The variances for all the t-values are given below.

for t = 0.1 : Variance = $\begin{pmatrix} 0.0133 & 0 \\ 0 & 0.0133 \end{pmatrix}$

for t = 0.3 : Variance = $\begin{pmatrix} 0.2811 & 0 \\ 0 & 0.2811 \end{pmatrix}$

for t = 1 : Variance = $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

for t = 10 : Variance $\begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}$ =

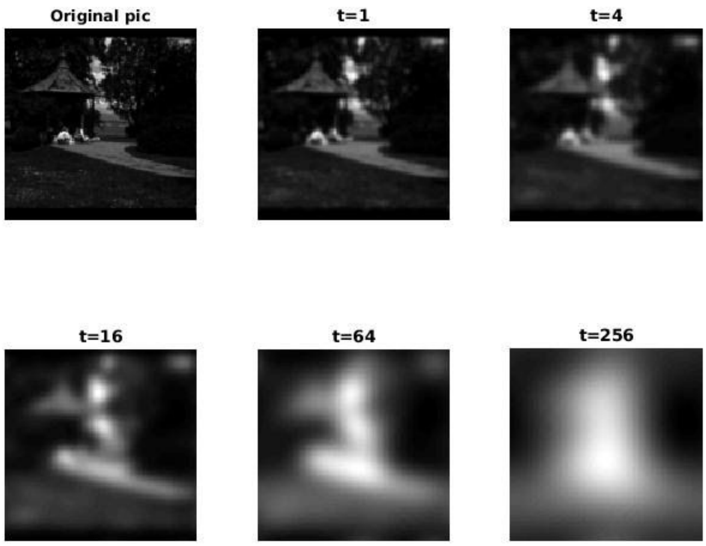
for t = 100 : Variance $\begin{pmatrix} 100 & 0 \\ 0 & 100 \end{pmatrix}$ =

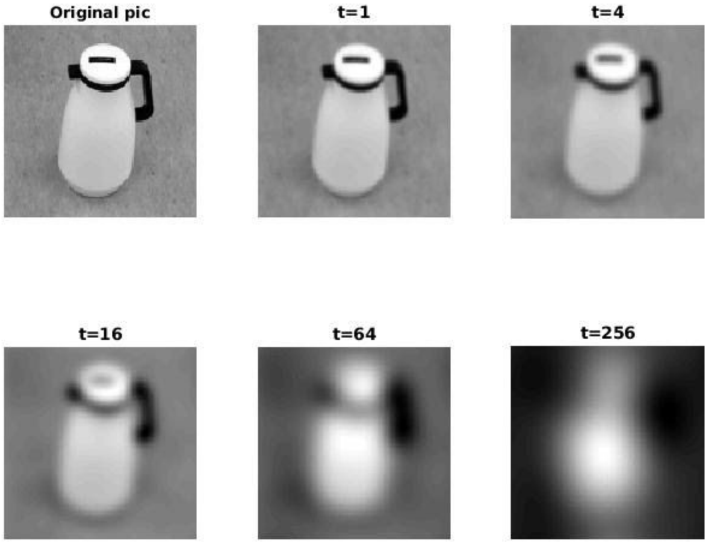
Question 15: Are the results different from or similar to the estimated variance? How does the result correspond to the ideal continuous case? Lead: think of the relation between spatial and Fourier domains for different values of t.

Answers:
We can see in the previous question that for t >= 1, the variances correspond exactly to the variances in the ideal continuous case (t times the identity matrix). But for t < 1, it's pretty close to it, and i think the difference comes from the discretization that we did to the Gaussian convolution by using a sampled version of the Gaussian function.

Question 16: Convolve a couple of images with Gaussian functions of different variances (like t = 1.0, 4.0, 16.0, 64.0 and 256.0) and present your results. What effects can you observe?

Answers:

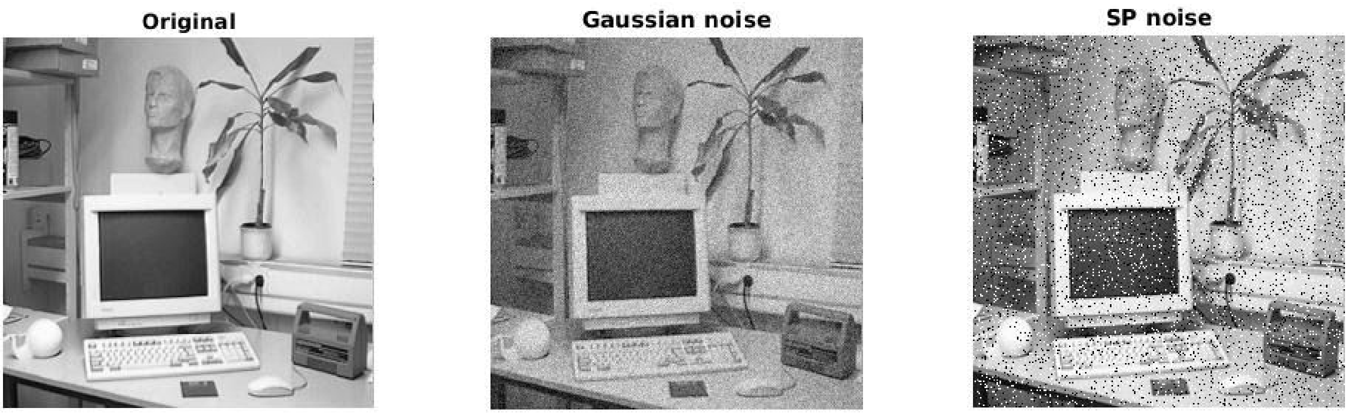


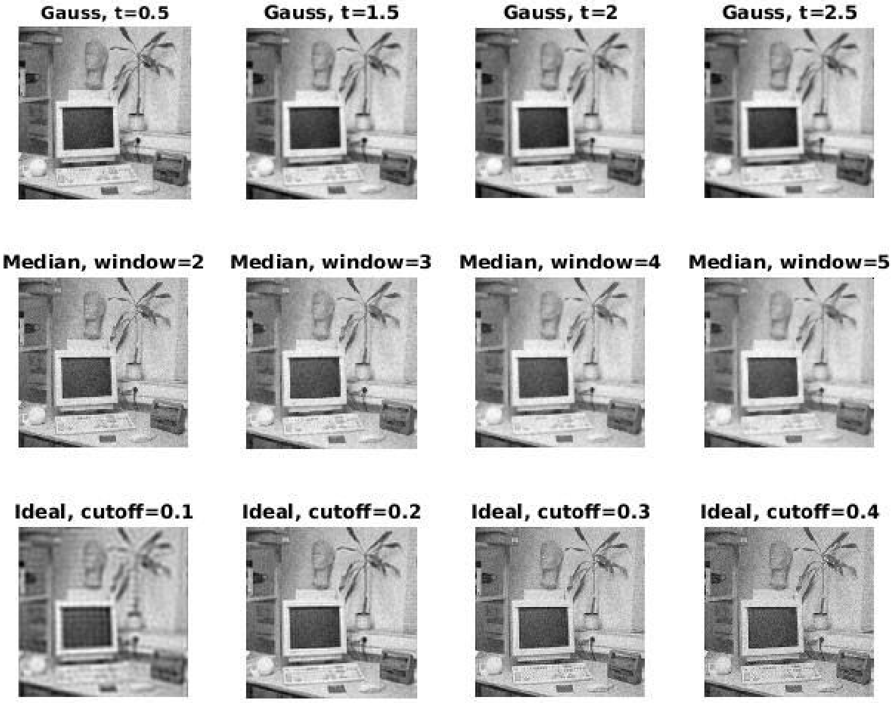


The figures show that the higher is the value of t the more blurry the image gets. And this is normal as when t increases, the variance increases and more high frequencies are getting dropped.

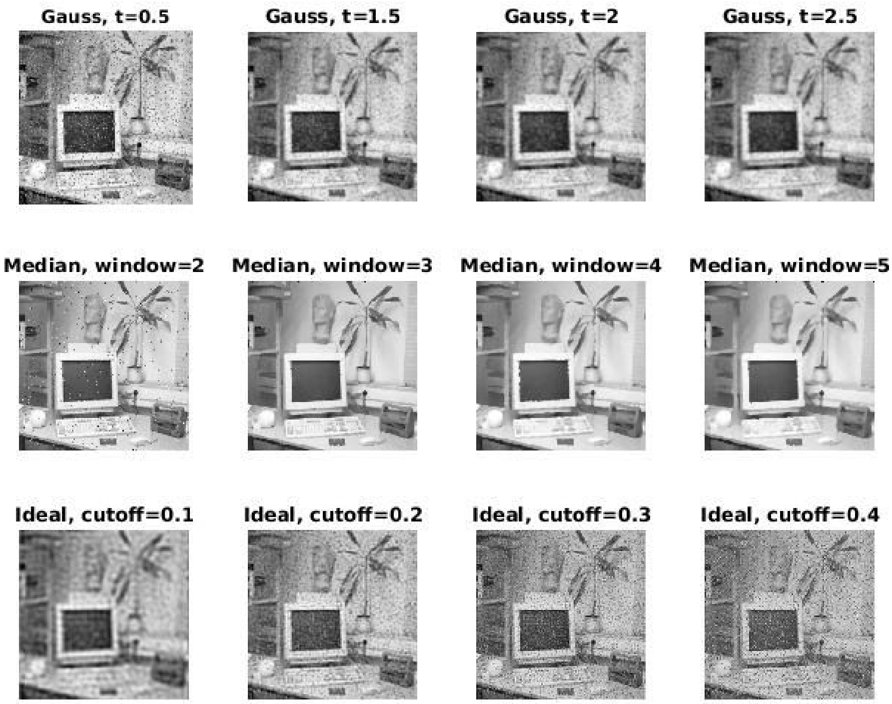
Question 17: What are the positive and negative effects for each type of filter? Describe what you observe and name the effects that you recognize. How do the results depend on the filter parameters? Illustrate with Matlab figure(s).

Answers:
All the figures below show the effect of the three filters on the images with noise.





Smoothing on Gaussian noise



Smoothing on SP noise

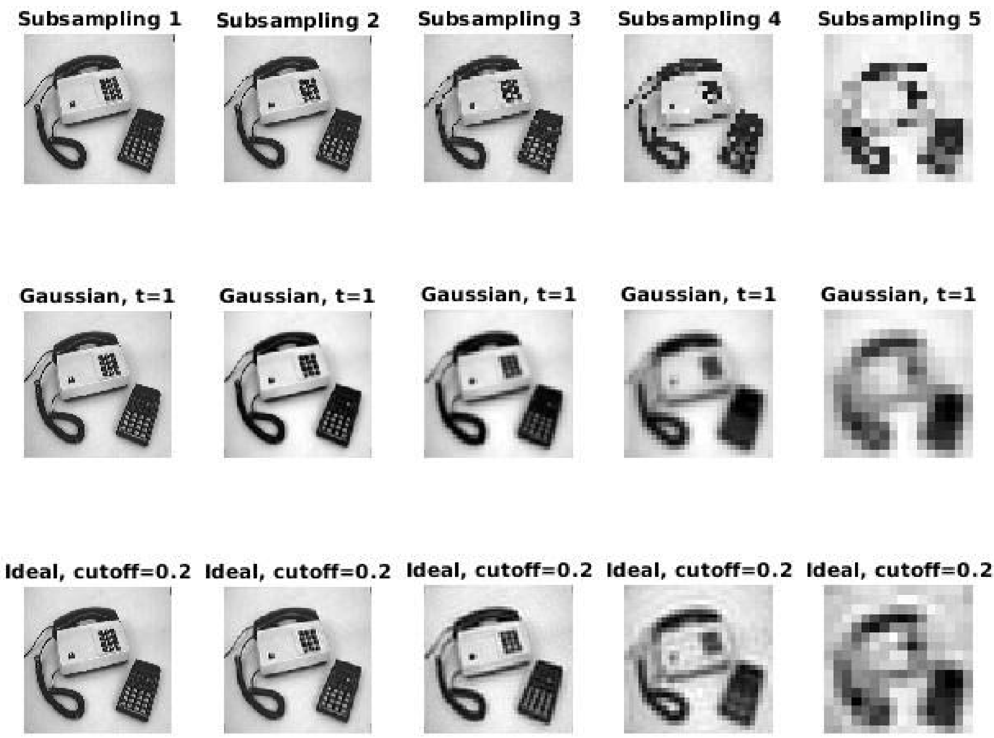
Filters / Effects	Positive effects	Negative effects	Observations
Gaussian smoothing	Good blurring, good effect on Gaussian noise.	Bad effect on the SP noise.	Image more blurred as we increase *t*.
Median filtering	Good effect on SP noise (removes the bad pixels), preserves edges.	Loss of visible details, becomes like a painting.	Less information and colors as we increase the size of the window.
Ideal low-pass filter	Remove frequencies over the cut-off. Smoothing of Gaussian noise.	Bad effect on SP noise, does not remove it.	Imag goes smoother when cut-off decreases.

Question 18: What conclusions can you draw from comparing the results of the respective methods?

Answers:
From our observations, we would choose the Gaussian filter to reduce Gaussian noise, and Median filter to reduce SP noise.

Question 19: What effects do you observe when subsampling the original image and the smoothed variants? Illustrate both filters with the best results found for iteration i = 4.

Answers:



We chose t = 1 and c = 0.2 from the observation above as they gave the best results in previous questions.
We can see that performing smoothing before subsampling is a good idea as the images are better after a few iterations. Indeed, this smoothing reduces aliasing artifacts in the first image so when we subsample, the image is clearer.

Question 20: What conclusions can you draw regarding the effects of smoothing when combined with subsampling? Hint: think in terms of frequencies and side effects.

Answers:
Let R_{min} be the minimum sampling rate to recreate the exact same signal from the subsampling and F_{max} the highest frequency in an image. We know from the Sampling theorem, we should have :

$$R_{min} >= 2 F_{max}$$

Therefore, as we know that smoothing reduces the highest frequencies in the image, it will decrease the possible value of R_{min} . And then, combining both of them will give a clearer subsampling.
