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Uncertain Lambert Problem

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This analysis addresses the problem of constructing uncertainties in initial and final satellite orbital state vectors that are dynamically consistent in Lambert's boundary-value problem, given uncertainties in the initial and final position vectors. The structure of the general nonlinear problem is discussed in terms of transformations of the probability density functions of the given positions. For the linear (first-order) variational version of the problem, the covariance matrices of the state variations are presented in explicit terms of the covariance matrix of the given position variations and partitions of the state transition matrix of the nominal orbit. Equivalence is demonstrated between the linear variational results, which involve no iteration, and a weighted batch least-squares differential-correction solution of the orbit determination problem. To illustrate the utility of the working formulas in the linear variational problem, two approaches are taken: 1) an analytic approach to solve directly for the covariance terms, based on linearized dynamics, and 2) a numerical Monte Carlo-based approach that generates a sample covariance. The numerical results for state covariance from the analytic approach agree with results obtained by batch least-squares differential correction, and both agree with results from the Monte Carlo approach to within differences explainable by the limitations of sampling.

Nomenclature

A	=	coefficient matrix in a general linear least-squares estimation problem
D_x	=	region of integration with respect to $\boldsymbol{x}$ coordinates
D_y	=	region of integration with respect to $\boldsymbol{y}$ coordinates
G	=	mapping from initial and final velocities to initial position
$\boldsymbol{g}$	=	continuous one-to-one transformation between vectors of equal dimension
H	=	mapping from initial and final velocities to final position
$\boldsymbol{h}$	=	inverse of $\boldsymbol{g}$
$I_{3\times3}$	=	identity matrix of dimension 3
K_r	=	mapping from initial position and velocity to final position
K_v	=	mapping from initial position and velocity to final velocity
L	=	mapping from initial and final position vectors to initial velocity vector (solution of the deterministic Lambert problem)
M	=	mapping from final position and velocity to initial position
N	=	transformation matrix from initial and final position variations to final position and velocity variations
$p_a(\boldsymbol{a})$	=	probability density function of the random vector $\boldsymbol{a}$
$p_{a,b}(\boldsymbol{a},\boldsymbol{b})$	=	joint probability density function of the random vectors $\boldsymbol{a}$ and $\boldsymbol{b}$
R	=	transformation matrix from initial and final position variations to initial position and velocity variations
$\boldsymbol{r}_1$	=	initial position vector
$\boldsymbol{r}_2$	=	final position vector
S	=	transformation matrix from initial and final position variations to initial and final velocity variations
$\boldsymbol{v}_1$	=	initial velocity vector
$\boldsymbol{v}_2$	=	final velocity vector
W	=	weight matrix in a general linear least-squares problem
$\boldsymbol{x}$, $\boldsymbol{y}$	=	generic vectors used in the rule for transforming probability density functions and in the statement of a general least-squares problem
$\Delta\boldsymbol{a}$	=	first-order variation of the vector $\boldsymbol{a}$
Φ	=	state transition matrix in the linear variational problem
Φ_{rr}	=	partition of the state transition matrix mapping initial position variation to final position variation
Φ_{rv}	=	partition of the state transition matrix mapping initial velocity variation to final position variation
Φ_{vr}	=	partition of the state transition matrix mapping initial position variation to final velocity variation
Φ_{vv}	=	partition of the state transition matrix mapping initial velocity variation to final velocity variation
$0_{3\times3}$	=	zero matrix of dimension 3

Superscript

T	=	matrix transpose operator

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I. Introduction

LAMBERT'S orbital boundary-value problem of determining the trajectory that connects two position vectors at known times has been extensively studied for more than 200 years and continues to motivate new analyses. Battin's modern contributions to the literature

of this subject are especially well known [1]. In the classical statement of the problem, the motion is assumed to be Keplerian. For that case, the calculation of the velocity vector at initial time reduces to the solution of a single transcendental equation involving the given time of flight. Of course, the same boundary-value problem can be posed for perturbed orbital motion. In this case, the usual procedure is to begin with the Keplerian solution as a first approximation and then to differentially correct the initial velocity vector to satisfy the boundary conditions when the full force model is applied.

Whatever formulation is chosen to solve the Lambert problem, the algorithm constructs initial and final orbital states that are dynamically consistent. The present analysis addresses the problem of constructing initial and final state uncertainties that are also dynamically consistent, given a fixed time of flight between uncertain initial and final position vectors. No restriction to Keplerian motion is required: the entire discussion applies equally well to perturbed orbital motion. However, it is assumed throughout that the motion follows from deterministic dynamics.

The problem of providing state uncertainties for Lambert solutions has been neglected. For example, the seminal paper on uncertainty propagation for orbital motion, that by Junkins et al. [2], treats only the initial-value problem, as do virtually all studies that cite this one, up to the present time. Yet, some applications may demand such an addition to the basic Lambert problem. In the field of surveillance tracking of space objects, for example, it is important to have realistic uncertainties for initial orbit solutions to perform reliable data association over long time spans and to initialize Bayesian estimators for orbit refinement. The success of these applications depends on the initial orbit state uncertainties being consistent in the dynamical sense described here.

Among the very few sources that mention the uncertain Lambert problem explicitly is the paper by Armellin et al. [3]. These authors use Taylor differential algebra to derive high-order polynomials that can represent the mapping of error distributions from observations to the initial orbit solutions themselves. Although this formalism is an elegant and efficient way to derive Taylor expansions up to some given order, it is not easy to tell in advance what that order should be in any particular problem, a difficulty common to all Taylor series methods.

In this analysis, the general problem is described, without any Taylor series approximations, in terms of the formal transformation of probability density functions. This approach has been outlined in recent years for the initial-value problem by many authors, including Park and Scheeres [4], Scheeres et al. [5], Fujimoto and Scheeres [6], Fujimoto et al. [7], Weisman et al. [8–10], Majji et al. [11], Weisman and Jah [12], and others. These authors treat the orbital state error propagation as a solution of a Fokker–Planck–Kolmogorov (FPK) equation and take advantage of the special case of "diffusionless" (deterministic) dynamics. In that special case, the FPK equation reduces to a form analogous to the continuity (conservation of mass) equation in fluid mechanics, namely, to Liouville's equation. The orbital state probability density function, which satisfies Liouville's continuity equation, turns out to be an integral invariant of a type first discussed by Poincaré [13]. Then, the time propagation of the probability density function reduces essentially to a transformation of variables, that is, the time evolution of the state itself, with the state variables properly incorporated as arguments in the density function. However, getting to this result does not necessarily require the apparatus of FPK equations and integral invariants, as many of the previous authors, particularly Weisman et al. [8–11], have noted. Here, in this treatment of the boundary-value problem, deterministic dynamics are assumed from the start, so that the necessary transformations of variables are, at least in principle, available from the solution of the ordinary differential equations of motion. The transformation rule for multiple integrals, a standard result in real analysis [14], is then the only mathematical machinery needed to describe the basic structure of the problem. Of course, this conceptual simplification of the problem can still call for difficult solution procedures in practice, but specific solution methods are not discussed in this analysis.

Because the problem is nonlinear, even in its Keplerian version, practically none of the required mappings is available in explicit terms. Hence, Monte Carlo methods may be the preferred way to verify that sufficiently accurate probability density functions for the orbital state have been derived, whatever approximate method the analyst has used to derive them. The discussion of the general nonlinear problem presented here is intended to encourage the straightforward application of Monte Carlo methods, without imposing any further assumptions on the motion or probability density functions.

This analysis does develop a formal solution of the linear variational Lambert problem with uncertain initial and final position variations because, in this approximation, the required mappings can be given explicitly. The solution for the initial and final state covariance matrices is given in explicit terms of covariances of the position variations and partitions of the transition matrix of the nominal orbit. Also noted is the fact that the transformation of the probability density functions can be used in the linear variational problem to accommodate non-Gaussian uncertainties in the given position variations.

To illustrate the results of the linear variational analysis, the Monte Carlo approach is implemented for the case of Gaussian position uncertainty. The approach is simple to implement in this case and does not require a state transition matrix. Numerical comparisons between the explicit analytic approach, the Monte Carlo approach, and numerical orbit determination via batch least-squares differential correction are presented for two representative Lambert-problem examples.

II. General Problem

A solution of Lambert's problem provides the mapping from two position vectors at two times to velocity at the initial time:

$$\boldsymbol{v}_1 = L(\boldsymbol{r}_1, \boldsymbol{r}_2) \tag{1}$$

Given $\boldsymbol{v}_1$, the corresponding value of $\boldsymbol{v}_2$ is obtained by propagating the initial state $(\boldsymbol{r}_1, \boldsymbol{v}_1)$ to the final time via a solution of the initial-value problem:

$$\boldsymbol{r}_2 = K_r(\boldsymbol{r}_1, \boldsymbol{v}_1) \tag{2}$$

$$\boldsymbol{v}_2 = K_v(\boldsymbol{r}_1, \boldsymbol{v}_1) = K_v(\boldsymbol{r}_1, L(\boldsymbol{r}_1, \boldsymbol{r}_2)) \tag{3}$$

Consider the case when the given position vectors are uncertain and seek the corresponding uncertainty in the initial and final orbital states. Uncertainties in the initial or final times are not considered, so that time of flight appears as a fixed parameter in the problem.

With deterministic dynamics and known probability densities for the positions, then, at least in principle, the probability densities for the velocities can be obtained by transformation of variables. The general transformation rule for probability density functions is well known and can be summarized as follows. Given n-dimensional random vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ related by continuous one-to-one transformations $\boldsymbol{y} = \boldsymbol{g}(\boldsymbol{x})$ and $\boldsymbol{x} = \boldsymbol{h}(\boldsymbol{y})$, and given the probability density function $p_{\boldsymbol{x}}(\boldsymbol{x})$, the probability that $\boldsymbol{x}$ lies in some region D_x can be computed either in terms of $\boldsymbol{x}$ or in terms of $\boldsymbol{y}$. A standard theorem on multiple integrals [14] expresses this choice as

$$\int_{D_x} p_x(\boldsymbol{x})\,\mathrm{d}x_1\mathrm{d}x_2\cdots\mathrm{d}x_n = \int_{D_y} \left|\det\left(\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{y}}\right)\right| p_x(\boldsymbol{h}(\boldsymbol{y}))\,\mathrm{d}y_1\mathrm{d}y_2\cdots\mathrm{d}y_n \tag{4}$$

In this equation, the region of integration D_y is the image of D_x under the transformation $\boldsymbol{g}$, that is, $\boldsymbol{h}^{-1}$, and the integral on the right-hand side is the probability that $\boldsymbol{y}$ lies in the region D_y. Hence, the probability density function for $\boldsymbol{y}$ can be identified as the transformed integrand:

$$p_y(\boldsymbol{y}) = \left|\det\left(\frac{\partial \boldsymbol{h}}{\partial \boldsymbol{y}}\right)\right| p_x(\boldsymbol{h}(\boldsymbol{y})) \tag{5}$$

A subtlety of the rule is that the vectors $\boldsymbol{x}$ and $\boldsymbol{y}$ must have the same dimension, so that, for example, marginal (lower-dimensional) probability density functions cannot be obtained merely by a point transformation. Now, because the mapping in question is from six dimensions of position to six dimensions of velocity and is continuous, assume that the following mappings also exist:

$$\boldsymbol{r}_1 = G(\boldsymbol{v}_1, \boldsymbol{v}_2) \tag{6}$$

$$\boldsymbol{r}_2 = H(\boldsymbol{v}_1, \boldsymbol{v}_2) \tag{7}$$

Whether this assumption is always justified is an important question, and the issue is reconsidered at the end of this section. When they exist, the mappings [Eqs. (6) and (7)] together constitute the inverse of the mappings L and K_v in Eqs. (1) and (3). Consequently, if the joint density of the positions is $p_{r_1,r_2}(\boldsymbol{r}_1, \boldsymbol{r}_2)$, then the joint density of the velocities is

$$p_{v_1,v_2}(\boldsymbol{v}_1, \boldsymbol{v}_2) = \left|\det\begin{pmatrix} \frac{\partial G}{\partial \boldsymbol{v}_1} & \frac{\partial G}{\partial \boldsymbol{v}_2} \\ \frac{\partial H}{\partial \boldsymbol{v}_1} & \frac{\partial H}{\partial \boldsymbol{v}_2} \end{pmatrix}\right| p_{r_1,r_2}(G(\boldsymbol{v}_1, \boldsymbol{v}_2), H(\boldsymbol{v}_1, \boldsymbol{v}_2)) \tag{8}$$

If the random position vectors are statistically independent, the joint position density can be factored as $p_{r_1,r_2}(\boldsymbol{r}_1, \boldsymbol{r}_2) = p_{r_1}(G(\boldsymbol{v}_1, \boldsymbol{v}_2))p_{r_2}(H(\boldsymbol{v}_1, \boldsymbol{v}_2))$. However, in general, this form cannot be factored as $p_{v_1,v_2}(\boldsymbol{v}_1, \boldsymbol{v}_2) = p_{v_1}(\boldsymbol{v}_1)p_{v_2}(\boldsymbol{v}_2)$. The velocity uncertainties in the Lambert problem are statistically dependent, even if the given random position vectors are statistically independent. This dependence is required so that the state probability densities at initial and final times, $p_{r_1,v_1}(\boldsymbol{r}_1, \boldsymbol{v}_1)$ and $p_{r_2,v_2}(\boldsymbol{r}_2, \boldsymbol{v}_2)$, correctly represent the given position probability densities. In any case, using Eqs. (1) and (3), the covariance of the velocities is

$$\begin{bmatrix} \mathrm{cov}(\boldsymbol{v}_1, \boldsymbol{v}_1) & \mathrm{cov}(\boldsymbol{v}_1, \boldsymbol{v}_2) \\ \mathrm{cov}(\boldsymbol{v}_2, \boldsymbol{v}_1) & \mathrm{cov}(\boldsymbol{v}_2, \boldsymbol{v}_2) \end{bmatrix} = \begin{bmatrix} \mathrm{cov}(L(\boldsymbol{r}_1, \boldsymbol{r}_2), L(\boldsymbol{r}_1, \boldsymbol{r}_2)) & \mathrm{cov}(L(\boldsymbol{r}_1, \boldsymbol{r}_2), K_v(\boldsymbol{r}_1, L(\boldsymbol{r}_1, \boldsymbol{r}_2))) \\ \mathrm{cov}(K_v(\boldsymbol{r}_1, L(\boldsymbol{r}_1, \boldsymbol{r}_2)), L(\boldsymbol{r}_1, \boldsymbol{r}_2)) & \mathrm{cov}(K_v(\boldsymbol{r}_1, L(\boldsymbol{r}_1, \boldsymbol{r}_2)), K_v(\boldsymbol{r}_1, L(\boldsymbol{r}_1, \boldsymbol{r}_2))) \end{bmatrix} \tag{9}$$

where now the expectations on the right-hand side are computed with respect to the given uncertain positions.

The initial state is a transformation of the given position vectors:

$$\begin{Bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{v}_1 \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{r}_1 \\ L(\boldsymbol{r}_1, \boldsymbol{r}_2) \end{Bmatrix} \tag{10}$$

Then, the covariance of the initial state is given explicitly by

$$\begin{bmatrix} \mathrm{cov}(\boldsymbol{r}_1, \boldsymbol{r}_1) & \mathrm{cov}(\boldsymbol{r}_1, \boldsymbol{v}_1) \\ \mathrm{cov}(\boldsymbol{v}_1, \boldsymbol{r}_1) & \mathrm{cov}(\boldsymbol{v}_1, \boldsymbol{v}_1) \end{bmatrix} = \begin{bmatrix} \mathrm{cov}(\boldsymbol{r}_1, \boldsymbol{r}_1) & \mathrm{cov}(\boldsymbol{r}_1, L(\boldsymbol{r}_1, \boldsymbol{r}_2)) \\ \mathrm{cov}(L(\boldsymbol{r}_1, \boldsymbol{r}_2), \boldsymbol{r}_1) & \mathrm{cov}(L(\boldsymbol{r}_1, \boldsymbol{r}_2), L(\boldsymbol{r}_1, \boldsymbol{r}_2)) \end{bmatrix} \tag{11}$$

Similarly, using Eq. (3), the covariance of the final state is

$$\begin{bmatrix} \mathrm{cov}(\boldsymbol{r}_2, \boldsymbol{r}_2) & \mathrm{cov}(\boldsymbol{r}_2, \boldsymbol{v}_2) \\ \mathrm{cov}(\boldsymbol{v}_2, \boldsymbol{r}_2) & \mathrm{cov}(\boldsymbol{v}_2, \boldsymbol{v}_2) \end{bmatrix} = \begin{bmatrix} \mathrm{cov}(\boldsymbol{r}_2, \boldsymbol{r}_2) & \mathrm{cov}(\boldsymbol{r}_2, K_v(\boldsymbol{r}_1, L(\boldsymbol{r}_1, \boldsymbol{r}_2))) \\ \mathrm{cov}(K_v(\boldsymbol{r}_1, L(\boldsymbol{r}_1, \boldsymbol{r}_2)), \boldsymbol{r}_2) & \mathrm{cov}(K_v(\boldsymbol{r}_1, L(\boldsymbol{r}_1, \boldsymbol{r}_2)), K_v(\boldsymbol{r}_1, L(\boldsymbol{r}_1, \boldsymbol{r}_2))) \end{bmatrix} \tag{12}$$

However, the transformation Eq. (10) can be inverted, using Eq. (2), to give the two positions explicitly in terms of initial state:

$$\begin{Bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{r}_1 \\ K_r(\boldsymbol{r}_1, \boldsymbol{v}_1) \end{Bmatrix} \tag{13}$$

This relation allows the probability density function of the initial state, $p_{r_1,v_1}(\boldsymbol{r}_1, \boldsymbol{v}_1)$, to be expressed in terms of the joint probability density function of the given position vectors, $p_{r_1,r_2}(\boldsymbol{r}_1, \boldsymbol{r}_2)$:

$$p_{r_1,v_1}(\boldsymbol{r}_1, \boldsymbol{v}_1) = \left|\det\left[\frac{\partial}{\partial(\boldsymbol{r}_1, \boldsymbol{v}_1)}\begin{Bmatrix} \boldsymbol{r}_1 \\ K_r(\boldsymbol{r}_1, \boldsymbol{v}_1) \end{Bmatrix}\right]\right| p_{r_1,r_2}(\boldsymbol{r}_1, K_r(\boldsymbol{r}_1, \boldsymbol{v}_1)) \tag{14}$$

As before, in case the positions are statistically independent, the joint position density can be factored as $p_{r_1,r_2}(\boldsymbol{r}_1, K_r(\boldsymbol{r}_1, \boldsymbol{v}_1)) = p_{r_1}(\boldsymbol{r}_1)p_{r_2}(K_r(\boldsymbol{r}_1, \boldsymbol{v}_1))$. However, in general, the state probability density function cannot be otherwise factored: the initial position and velocity must be statistically dependent. Some simplification is possible because of the special form of the determinant:

$$\det\left[\frac{\partial}{\partial(\boldsymbol{r}_1, \boldsymbol{v}_1)}\begin{Bmatrix} \boldsymbol{r}_1 \\ K_r(\boldsymbol{r}_1, \boldsymbol{v}_1) \end{Bmatrix}\right] = \det\begin{bmatrix} I_{3\times 3} & 0_{3\times 3} \\ \frac{\partial K_r}{\partial \boldsymbol{r}_1} & \frac{\partial K_r}{\partial \boldsymbol{v}_1} \end{bmatrix} = \det\left[\frac{\partial K_r}{\partial \boldsymbol{v}_1}\right] \tag{15}$$

This reduction follows from the cofactor expansion of the determinant, taken along the first row, then the second row, and then the third.

Similarly, the probability density function of the final state can be expressed in terms of the joint density of the given positions. The final state is a transformation of the two positions, analogously to Eq. (10). Using Eq. (3), the transformation can be written as

$$\begin{Bmatrix} \boldsymbol{r}_2 \\ \boldsymbol{v}_2 \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{r}_2 \\ K_v(\boldsymbol{r}_1, \boldsymbol{v}_1) \end{Bmatrix} = \begin{Bmatrix} \boldsymbol{r}_2 \\ K_v(\boldsymbol{r}_1, L(\boldsymbol{r}_1, \boldsymbol{r}_2)) \end{Bmatrix} \tag{16}$$

The inverse of this transformation, giving the positions in terms of final state, is well defined, although Eq. (3) has to be solved for $\boldsymbol{r}_1$ in terms of final state:

$$\boldsymbol{v}_2 = K_v(\boldsymbol{r}_1, L(\boldsymbol{r}_1, \boldsymbol{r}_2)) \Rightarrow \boldsymbol{r}_1 = M(\boldsymbol{r}_2, \boldsymbol{v}_2) \tag{17}$$

Then, analogously to Eq. (13), the required transformation is

$$\begin{Bmatrix} \boldsymbol{r}_1 \\ \boldsymbol{r}_2 \end{Bmatrix} = \begin{Bmatrix} M(\boldsymbol{r}_2, \boldsymbol{v}_2) \\ \boldsymbol{r}_2 \end{Bmatrix} \tag{18}$$

The probability density function of the final state is

$$p_{r_2,v_2}(\boldsymbol{r}_2, \boldsymbol{v}_2) = \left|\det\left[\frac{\partial}{\partial(\boldsymbol{r}_2, \boldsymbol{v}_2)}\begin{Bmatrix} M(\boldsymbol{r}_2, \boldsymbol{v}_2) \\ \boldsymbol{r}_2 \end{Bmatrix}\right]\right| p_{r_1,r_2}(M(\boldsymbol{r}_2, \boldsymbol{v}_2), \boldsymbol{r}_2) \tag{19}$$

In case the positions are statistically independent, the joint position density on the right-hand side can be factored as $p_{r_1,r_2}(M(\boldsymbol{r}_2,\boldsymbol{v}_2),\boldsymbol{r}_2) = p_{r_1}(M(\boldsymbol{r}_2,\boldsymbol{v}_2))p_{r_2}(\boldsymbol{r}_2)$. In general, the final position and velocity will be statistically dependent. The special form of the determinant allows some simplification:

$$\det\left[\frac{\partial}{\partial(\boldsymbol{r}_2,\boldsymbol{v}_2)}\begin{Bmatrix} M(\boldsymbol{r}_2,\boldsymbol{v}_2) \\ \boldsymbol{r}_2 \end{Bmatrix}\right] = \det\begin{bmatrix} \frac{\partial M}{\partial \boldsymbol{r}_2} & \frac{\partial M}{\partial \boldsymbol{v}_2} \\ I_{3\times3} & 0_{3\times3} \end{bmatrix} = -\det\left[\frac{\partial M}{\partial \boldsymbol{v}_2}\right] \tag{20}$$

This reduction follows from the cofactor expansion of the determinant, taken along the sixth row, then the fifth row, and then the fourth. The negative sign is a result of the alternating pattern of signs in the cofactor expansion, although only the absolute value of the determinant is needed.

An obvious difficulty is that the nonlinear mappings outlined previously are not available in explicit terms, even for Keplerian motion. At least conceptually, the probability densities of the given positions can be sampled in Monte Carlo fashion to derive the joint velocity probability density, the individual marginal probability densities of $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$, and the probability density functions of the initial and final states. If only the first two moments of the probability density functions are needed, then, under the appropriate assumptions [15–17], the unscented transformation of the position covariances would be sufficient to derive the approximate covariance matrices of initial and final velocities, as well as of the initial and final states, using the mappings L and K_v. The full Monte Carlo sampling would not be needed in that case.

A less obvious difficulty is that the Lambert problem has a special case in which the velocity vectors are indeterminate, namely, when the position vectors are collinear so that the plane of motion is indeterminate. It is well known that, in this case, the radial and transverse components of $\boldsymbol{v}_1$ are still well defined but that the direction of the transverse component is ambiguous [1]. This difficulty is purely geometrical and is inherent in the problem, but it means that some of the previous transformations among the position and velocity vectors may not exist in the collinear case. In the deterministic Lambert problem with exactly collinear positions, one simply supplies the missing plane orientation a priori, essentially reformulating the problem from six dimensions of position and velocity to four. The more serious difficulty is that the deterministic Lambert problem is inherently ill-conditioned in a neighborhood of collinear positions, and the uncertain Lambert problem necessarily has the same ill-conditioning. Moreover, for realistic probability density functions of position, it will not be possible to invoke some kind of prior information to reduce the dimension of the problem. The difficulty will show up as extreme sensitivity of velocity variations with respect to position variations, correctly reflecting the nature of the problem but rendering numerical results suspect.

III. Linear Variational Problem

In the case of linear dynamics describing the first-order variational motion near a nominal orbit, all the required mappings can be obtained explicitly, and so a formal solution of the variational uncertain Lambert problem is possible. Define the nominal orbit by the pair of vectors $(\boldsymbol{r}_1, \boldsymbol{v}_1)$ and begin with the solution of the variational equations in the initial-value form

$$\Delta\boldsymbol{r}_2 = \frac{\partial K_r}{\partial \boldsymbol{r}_1}\Delta\boldsymbol{r}_1 + \frac{\partial K_r}{\partial \boldsymbol{v}_1}\Delta\boldsymbol{v}_1 \tag{21}$$

$$\Delta\boldsymbol{v}_2 = \frac{\partial K_v}{\partial \boldsymbol{r}_1}\Delta\boldsymbol{r}_1 + \frac{\partial K_v}{\partial \boldsymbol{v}_1}\Delta\boldsymbol{v}_1 \tag{22}$$

Assume that the nominal orbit has been obtained as a solution of the Lambert problem and that all partial derivatives are functions of known (deterministic) initial state and time of flight. As before, further assume that the time of flight is fixed. Now consider the variational Lambert problem, which is to obtain the mapping from given values of position variation, $\Delta\boldsymbol{r}_1$ and $\Delta\boldsymbol{r}_2$, to the velocity variation at the initial time, $\Delta\boldsymbol{v}_1$. Based on this mapping, the uncertainties in the variations of the initial and final states can be derived, given uncertainties in the position variations.

To this end, write the two previous equations in more standard notation as

$$\begin{Bmatrix} \Delta\boldsymbol{r}_2 \\ \Delta\boldsymbol{v}_2 \end{Bmatrix} = \begin{bmatrix} \frac{\partial K_r}{\partial \boldsymbol{r}_1} & \frac{\partial K_r}{\partial \boldsymbol{v}_1} \\ \frac{\partial K_v}{\partial \boldsymbol{r}_1} & \frac{\partial K_v}{\partial \boldsymbol{v}_1} \end{bmatrix}\begin{Bmatrix} \Delta\boldsymbol{r}_1 \\ \Delta\boldsymbol{v}_1 \end{Bmatrix} \triangleq \begin{bmatrix} \Phi_{rr} & \Phi_{rv} \\ \Phi_{vr} & \Phi_{vv} \end{bmatrix}\begin{Bmatrix} \Delta\boldsymbol{r}_1 \\ \Delta\boldsymbol{v}_1 \end{Bmatrix} \triangleq \Phi\begin{Bmatrix} \Delta\boldsymbol{r}_1 \\ \Delta\boldsymbol{v}_1 \end{Bmatrix} \tag{23}$$

where Φ is the usual state transition matrix. The solution of the variational Lambert problem presents itself immediately as

$$\Delta\boldsymbol{v}_1 = \Phi_{rv}^{-1}(\Delta\boldsymbol{r}_2 - \Phi_{rr}\Delta\boldsymbol{r}_1) \tag{24}$$

This expression corresponds to the mapping L in Eq. (1) in the general problem.

Equation (24), or its equivalent, appears in many spacecraft guidance and navigation analyses. For example, Battin [18] developed this relation in a slightly different form and discussed its use for differentially correcting circumlunar trajectories. Battin later presented similar expressions in ([1] Chap. 9). Moreover, beginning with the original development of the Clohessy–Wiltshire equations [19], an immense number of papers, such as Mullins [20] for merely one example, have presented boundary-value solutions of the linear equations of relative motion. Most of these treat the linear variational equations for motion near a circular reference orbit. Many different types of state variables have been used, reflecting the diversity of particular problems in satellite orbit control. Yet, a common issue underlies all these analyses: the matrix inverse in Eq. (24) may not exist because that partition of the state transition matrix Φ_{rv}, may become singular. Inevitably, numerical solutions for $\Delta\boldsymbol{v}_1$ become unreliable in some neighborhood of the singular condition, no matter what state variables are chosen. Wen et al. [21] recently analyzed the singularities of this problem in terms of differential orbital elements, interpreting geometrically the various ways that the ill-conditioning appears when these particular elements are used as state variables. These authors considered only the case of a circular nominal orbit, although they recommended that a more comprehensive singularity analysis be done for all types of orbits. In fact, the difficulty that manifests as a singular or ill-conditioned matrix Φ_{rv} stems from the indeterminateness of the general Lambert problem: when the given positions on the nominal orbit are nearly collinear, velocity variations become extremely sensitive to position variations. Hence, this difficulty in the linear variational Lambert problem will occur with any type of nominal orbit.

When the matrix inverse Φ_{rv}^{-1} exists, Eq. (24) allows an analysis of uncertainties in this problem. In particular, the initial state variation can be computed as a linear transformation of the position variations:

$$\begin{Bmatrix} \Delta\boldsymbol{r}_1 \\ \Delta\boldsymbol{v}_1 \end{Bmatrix} = \begin{bmatrix} I_{3\times3} & 0_{3\times3} \\ -\Phi_{rv}^{-1}\Phi_{rr} & \Phi_{rv}^{-1} \end{bmatrix}\begin{Bmatrix} \Delta\boldsymbol{r}_1 \\ \Delta\boldsymbol{r}_2 \end{Bmatrix} \triangleq R\begin{Bmatrix} \Delta\boldsymbol{r}_1 \\ \Delta\boldsymbol{r}_2 \end{Bmatrix} \tag{25}$$

The covariance of the initial state variation follows as a linear (similarity) transformation of the covariance of the position variations:

$$\begin{bmatrix} \mathrm{cov}(\Delta\boldsymbol{r}_1,\Delta\boldsymbol{r}_1) & \mathrm{cov}(\Delta\boldsymbol{r}_1,\Delta\boldsymbol{v}_1) \\ \mathrm{cov}(\Delta\boldsymbol{v}_1,\Delta\boldsymbol{r}_1) & \mathrm{cov}(\Delta\boldsymbol{v}_1,\Delta\boldsymbol{v}_1) \end{bmatrix} = R\begin{bmatrix} \mathrm{cov}(\Delta\boldsymbol{r}_1,\Delta\boldsymbol{r}_1) & \mathrm{cov}(\Delta\boldsymbol{r}_1,\Delta\boldsymbol{r}_2) \\ \mathrm{cov}(\Delta\boldsymbol{r}_2,\Delta\boldsymbol{r}_1) & \mathrm{cov}(\Delta\boldsymbol{r}_2,\Delta\boldsymbol{r}_2) \end{bmatrix}R^T \tag{26}$$

Straightforward algebraic steps produce

$$\begin{aligned}\mathrm{cov}(\Delta\boldsymbol{r}_1,\Delta\boldsymbol{v}_1) &= [\mathrm{cov}(\Delta\boldsymbol{v}_1,\Delta\boldsymbol{r}_1)]^T \\ &= -\mathrm{cov}(\Delta\boldsymbol{r}_1,\Delta\boldsymbol{r}_1)\Phi_{rr}^T\Phi_{rv}^{-T} + \mathrm{cov}(\Delta\boldsymbol{r}_1,\Delta\boldsymbol{r}_2)\Phi_{rv}^{-T}\end{aligned} \tag{27}$$

$$\begin{aligned}\operatorname{cov}(\Delta\boldsymbol{v}_1,\Delta\boldsymbol{v}_1) &= \Phi_{rv}^{-1}\Phi_{rr}\operatorname{cov}(\Delta\boldsymbol{r}_1,\Delta\boldsymbol{r}_1)\Phi_{rr}^T\Phi_{rv}^{-T}\\ &- \Phi_{rv}^{-1}\Phi_{rr}\operatorname{cov}(\Delta\boldsymbol{r}_1,\Delta\boldsymbol{r}_2)\Phi_{rv}^{-T}\\ &- \Phi_{rv}^{-1}\operatorname{cov}(\Delta\boldsymbol{r}_2,\Delta\boldsymbol{r}_1)\Phi_{rr}^T\Phi_{rv}^{-T} + \Phi_{rv}^{-1}\operatorname{cov}(\Delta\boldsymbol{r}_2,\Delta\boldsymbol{r}_2)\Phi_{rv}^{-T}\end{aligned} \tag{28}$$

Similarly, the variations of the final state are a linear transformation of the position variations. Substituting Eq. (25) into Eq. (23) produces

$$\begin{aligned}\begin{Bmatrix}\Delta\boldsymbol{r}_2\\ \Delta\boldsymbol{v}_2\end{Bmatrix} &= \begin{bmatrix}\Phi_{rr} & \Phi_{rv}\\ \Phi_{vr} & \Phi_{vv}\end{bmatrix}\begin{Bmatrix}\Delta\boldsymbol{r}_1\\ \Delta\boldsymbol{v}_1\end{Bmatrix}\\ &= \begin{bmatrix}\Phi_{rr} & \Phi_{rv}\\ \Phi_{vr} & \Phi_{vv}\end{bmatrix}\begin{bmatrix}I_{3\times3} & 0_{3\times3}\\ -\Phi_{rv}^{-1}\Phi_{rr} & \Phi_{rv}^{-1}\end{bmatrix}\begin{Bmatrix}\Delta\boldsymbol{r}_1\\ \Delta\boldsymbol{r}_2\end{Bmatrix}\end{aligned} \tag{29}$$

$$\begin{Bmatrix}\Delta\boldsymbol{r}_2\\ \Delta\boldsymbol{v}_2\end{Bmatrix} = \begin{bmatrix}0_{3\times3} & I_{3\times3}\\ (\Phi_{vr}-\Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr}) & (\Phi_{vv}\Phi_{rv}^{-1})\end{bmatrix}\begin{Bmatrix}\Delta\boldsymbol{r}_1\\ \Delta\boldsymbol{r}_2\end{Bmatrix} \triangleq N\begin{Bmatrix}\Delta\boldsymbol{r}_1\\ \Delta\boldsymbol{r}_2\end{Bmatrix} \tag{30}$$

The covariance of the final state variations is then

$$\begin{bmatrix}\operatorname{cov}(\Delta\boldsymbol{r}_2,\Delta\boldsymbol{r}_2) & \operatorname{cov}(\Delta\boldsymbol{r}_2,\Delta\boldsymbol{v}_2)\\ \operatorname{cov}(\Delta\boldsymbol{v}_2,\Delta\boldsymbol{r}_2) & \operatorname{cov}(\Delta\boldsymbol{v}_2,\Delta\boldsymbol{v}_2)\end{bmatrix} = N\begin{bmatrix}\operatorname{cov}(\Delta\boldsymbol{r}_1,\Delta\boldsymbol{r}_1) & \operatorname{cov}(\Delta\boldsymbol{r}_1,\Delta\boldsymbol{r}_2)\\ \operatorname{cov}(\Delta\boldsymbol{r}_2,\Delta\boldsymbol{r}_1) & \operatorname{cov}(\Delta\boldsymbol{r}_2,\Delta\boldsymbol{r}_2)\end{bmatrix}N^T \tag{31}$$

Straightforward algebraic steps produce

$$\begin{aligned}\operatorname{cov}(\Delta\boldsymbol{r}_2,\Delta\boldsymbol{v}_2) &= [\operatorname{cov}(\Delta\boldsymbol{v}_2,\Delta\boldsymbol{r}_2)]^T\\ &= \operatorname{cov}(\Delta\boldsymbol{r}_2,\Delta\boldsymbol{r}_1)(\Phi_{vr}-\Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr})^T\\ &+ \operatorname{cov}(\Delta\boldsymbol{r}_2,\Delta\boldsymbol{r}_2)\Phi_{rv}^{-T}\Phi_{vv}^T\end{aligned} \tag{32}$$

$$\begin{aligned}\operatorname{cov}(\Delta\boldsymbol{v}_2,\Delta\boldsymbol{v}_2) &= (\Phi_{vr}-\Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr})\operatorname{cov}(\Delta\boldsymbol{r}_1,\Delta\boldsymbol{r}_1)\\ &\times(\Phi_{vr}-\Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr})^T\\ &+ (\Phi_{vr}-\Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr})\operatorname{cov}(\Delta\boldsymbol{r}_1,\Delta\boldsymbol{r}_2)\Phi_{rv}^{-T}\Phi_{vv}^T\\ &+ (\Phi_{vv}\Phi_{rv}^{-1})\operatorname{cov}(\Delta\boldsymbol{r}_2,\Delta\boldsymbol{r}_1)(\Phi_{vr}-\Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr})^T\\ &+ (\Phi_{vv}\Phi_{rv}^{-1})\operatorname{cov}(\Delta\boldsymbol{r}_2,\Delta\boldsymbol{r}_2)\Phi_{rv}^{-T}\Phi_{vv}^T\end{aligned} \tag{33}$$

Although these covariance relations are sufficient to describe Gaussian errors in the variational Lambert problem, the previous transformations of probability densities can be used to describe general errors in this problem. For example, the inverse of the linear transformation Eq. (25) gives the position variations in terms of the initial state variations:

$$\begin{Bmatrix}\Delta\boldsymbol{r}_1\\ \Delta\boldsymbol{r}_2\end{Bmatrix} = \begin{bmatrix}I_{3\times3} & 0_{3\times3}\\ -\Phi_{rv}^{-1}\Phi_{rr} & \Phi_{rv}^{-1}\end{bmatrix}^{-1}\begin{Bmatrix}\Delta\boldsymbol{r}_1\\ \Delta\boldsymbol{v}_1\end{Bmatrix} \tag{34}$$

Using the solution of the variational initial-value problem [Eq. (23)], the matrix inverse is trivial:

$$\begin{Bmatrix}\Delta\boldsymbol{r}_1\\ \Delta\boldsymbol{r}_2\end{Bmatrix} = \begin{bmatrix}I_{3\times3} & 0_{3\times3}\\ \Phi_{rr} & \Phi_{rv}\end{bmatrix}\begin{Bmatrix}\Delta\boldsymbol{r}_1\\ \Delta\boldsymbol{v}_1\end{Bmatrix} \tag{35}$$

Then, analogously with Eq. (14) for initial state density, the joint density of the initial state variations is

$$p_{\Delta r_1,\Delta v_1}(\Delta\boldsymbol{r}_1,\Delta\boldsymbol{v}_1) = \left|\det\begin{bmatrix}I_{3\times3} & 0_{3\times3}\\ \Phi_{rr} & \Phi_{rv}\end{bmatrix}\right| p_{\Delta r_1,\Delta r_2}(\Delta\boldsymbol{r}_1,\Phi_{rr}\Delta\boldsymbol{r}_1+\Phi_{rv}\Delta\boldsymbol{v}_1) \tag{36}$$

If the position variations are statistically independent, their joint density can be factored as $p_{\Delta r_1,\Delta r_2}(\Delta\boldsymbol{r}_1,\Phi_{rr}\Delta\boldsymbol{r}_1+\Phi_{rv}\Delta\boldsymbol{v}_1) = p_{\Delta r_1}(\Delta\boldsymbol{r}_1)p_{\Delta r_2}(\Phi_{rr}\Delta\boldsymbol{r}_1+\Phi_{rv}\Delta\boldsymbol{v}_1)$. However, in general, the initial state variations are statistically dependent. As before, the determinant simplifies to

$$\det\begin{bmatrix}I_{3\times3} & 0_{3\times3}\\ \Phi_{rr} & \Phi_{rv}\end{bmatrix} = \det\Phi_{rv} \tag{37}$$

The probability density function of the final state variation can also be obtained in terms of the probability density of the position variations. The variation of the final state in terms of the position variations is given in Eq. (30). The inverse of that transformation is needed, which gives the position variations in terms of the final state variations:

$$\begin{Bmatrix}\Delta\boldsymbol{r}_1\\ \Delta\boldsymbol{r}_2\end{Bmatrix} = \begin{bmatrix}0_{3\times3} & I_{3\times3}\\ (\Phi_{vr}-\Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr}) & (\Phi_{vv}\Phi_{rv}^{-1})\end{bmatrix}^{-1}\begin{Bmatrix}\Delta\boldsymbol{r}_2\\ \Delta\boldsymbol{v}_2\end{Bmatrix} \tag{38}$$

The matrix inverse is obtained straightforwardly as follows. Solving for $\Delta\boldsymbol{r}_1$ in terms of $\Delta\boldsymbol{r}_2$ and $\Delta\boldsymbol{v}_2$ from the lower partition of Eq. (30) produces

$$\begin{aligned}\Delta\boldsymbol{r}_1 &= -(\Phi_{vr}-\Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr})^{-1}(\Phi_{vv}\Phi_{rv}^{-1})\Delta\boldsymbol{r}_2\\ &+ (\Phi_{vr}-\Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr})^{-1}\Delta\boldsymbol{v}_2\end{aligned} \tag{39}$$

This expression corresponds to the mapping M in Eq. (17) in the previous general problem and allows the matrix inverse in Eq. (38) to be rewritten in explicit terms:

$$\begin{Bmatrix}\Delta\boldsymbol{r}_1\\ \Delta\boldsymbol{r}_2\end{Bmatrix} = \begin{bmatrix}-(\Phi_{vr}-\Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr})^{-1}(\Phi_{vv}\Phi_{rv}^{-1}) & (\Phi_{vr}-\Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr})^{-1}\\ I_{3\times3} & 0_{3\times3}\end{bmatrix} \times\begin{Bmatrix}\Delta\boldsymbol{r}_2\\ \Delta\boldsymbol{v}_2\end{Bmatrix} \tag{40}$$

Then, analogously with Eq. (19) for final state density, the joint density of the final state variations becomes

$$p_{\Delta r_2,\Delta v_2}(\Delta\boldsymbol{r}_2,\Delta\boldsymbol{v}_2) = \left|\det\begin{bmatrix}-(\Phi_{vr}-\Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr})^{-1}(\Phi_{vv}\Phi_{rv}^{-1}) & (\Phi_{vr}-\Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr})^{-1}\\ I_{3\times3} & 0_{3\times3}\end{bmatrix}\right| p_{\Delta r_1,\Delta r_2}(\Delta\boldsymbol{r}_1,\Delta\boldsymbol{r}_2) \tag{41}$$

where now $\Delta\boldsymbol{r}_1$ is given by Eq. (39). If the position variations are statistically independent, then their joint density can be factored as $p_{\Delta r_1,\Delta r_2}(\Delta\boldsymbol{r}_1,\Delta\boldsymbol{r}_2) = p_{\Delta r_1}(\Delta\boldsymbol{r}_1)p_{\Delta r_2}(\Delta\boldsymbol{r}_2)$. However, in general, the final state variations will be statistically dependent. In the same manner as before, the determinant simplifies to the form

$$\begin{aligned}&\det\begin{bmatrix}-(\Phi_{vr}-\Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr})^{-1}(\Phi_{vv}\Phi_{rv}^{-1}) & (\Phi_{vr}-\Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr})^{-1}\\ I_{3\times3} & 0_{3\times3}\end{bmatrix}\\ &= -\det[(\Phi_{vr}-\Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr})^{-1}]\end{aligned} \tag{42}$$

Now the joint probability density function of the velocity variations can be developed in explicit terms of the densities of the position variations. The lower partitions of Eqs. (25) and (30), respectively, give the mapping from position variations to velocity variations:

$$\begin{Bmatrix}\Delta\boldsymbol{v}_1\\ \Delta\boldsymbol{v}_2\end{Bmatrix} = \begin{bmatrix}-(\Phi_{rv}^{-1}\Phi_{rr}) & (\Phi_{rv}^{-1})\\ (\Phi_{vr}-\Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr}) & (\Phi_{vv}\Phi_{rv}^{-1})\end{bmatrix}\begin{Bmatrix}\Delta\boldsymbol{r}_1\\ \Delta\boldsymbol{r}_2\end{Bmatrix} \triangleq S\begin{Bmatrix}\Delta\boldsymbol{r}_1\\ \Delta\boldsymbol{r}_2\end{Bmatrix} \tag{43}$$

This transformation is sufficient to describe Gaussian state errors in the linear variational problem because the covariance matrix of the velocity variations is available explicitly as

$$\begin{bmatrix} \mathrm{cov}(\Delta \boldsymbol{v}_1, \Delta \boldsymbol{v}_1) & \mathrm{cov}(\Delta \boldsymbol{v}_1, \Delta \boldsymbol{v}_2) \\ \mathrm{cov}(\Delta \boldsymbol{v}_2, \Delta \boldsymbol{v}_1) & \mathrm{cov}(\Delta \boldsymbol{v}_2, \Delta \boldsymbol{v}_2) \end{bmatrix} = S \begin{bmatrix} \mathrm{cov}(\Delta \boldsymbol{r}_1, \Delta \boldsymbol{r}_1) & \mathrm{cov}(\Delta \boldsymbol{r}_1, \Delta \boldsymbol{r}_2) \\ \mathrm{cov}(\Delta \boldsymbol{r}_2, \Delta \boldsymbol{r}_1) & \mathrm{cov}(\Delta \boldsymbol{r}_2, \Delta \boldsymbol{r}_2) \end{bmatrix} S^T \tag{44}$$

In fact, the expressions for $\mathrm{cov}(\Delta \boldsymbol{v}_1, \Delta \boldsymbol{v}_1)$ and $\mathrm{cov}(\Delta \boldsymbol{v}_2, \Delta \boldsymbol{v}_2)$ have already been developed in Eqs. (28) and (33). However, now the explicit expression for the cross-covariance can be offered. Straightforward algebraic steps produce

$$\begin{aligned} \mathrm{cov}(\Delta \boldsymbol{v}_1, \Delta \boldsymbol{v}_2) &= [\mathrm{cov}(\Delta \boldsymbol{v}_2, \Delta \boldsymbol{v}_1)]^T \\ &= -(\Phi_{rv}^{-1}\Phi_{rr})\mathrm{cov}(\Delta \boldsymbol{r}_1, \Delta \boldsymbol{r}_1)(\Phi_{vr} - \Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr})^T \\ &- (\Phi_{rv}^{-1}\Phi_{rr})\mathrm{cov}(\Delta \boldsymbol{r}_1, \Delta \boldsymbol{r}_2)(\Phi_{vv}\Phi_{rv}^{-1})^T \\ &+ (\Phi_{rv}^{-1})\mathrm{cov}(\Delta \boldsymbol{r}_2, \Delta \boldsymbol{r}_1)(\Phi_{vr} - \Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr})^T \\ &+ (\Phi_{rv}^{-1})\mathrm{cov}(\Delta \boldsymbol{r}_2, \Delta \boldsymbol{r}_2)(\Phi_{vv}\Phi_{rv}^{-1})^T \end{aligned} \tag{45}$$

Moreover, the apparently complicated inverse matrix of the transformation Eq. (43),

$$\begin{Bmatrix} \Delta \boldsymbol{r}_1 \\ \Delta \boldsymbol{r}_2 \end{Bmatrix} = \begin{bmatrix} -(\Phi_{rv}^{-1}\Phi_{rr}) & (\Phi_{rv}^{-1}) \\ (\Phi_{vr} - \Phi_{vv}\Phi_{rv}^{-1}\Phi_{rr}) & (\Phi_{vv}\Phi_{rv}^{-1}) \end{bmatrix}^{-1} \begin{Bmatrix} \Delta \boldsymbol{v}_1 \\ \Delta \boldsymbol{v}_2 \end{Bmatrix} \triangleq S^{-1} \begin{Bmatrix} \Delta \boldsymbol{v}_1 \\ \Delta \boldsymbol{v}_2 \end{Bmatrix} \tag{46}$$

which is needed for the transformation of probability densities, turns out to be surprisingly simple. The lower partition of the initial-value solution [Eq. (23)] gives the initial position variation in terms of velocity variations:

$$\Delta \boldsymbol{r}_1 = \Phi_{vr}^{-1}(\Delta \boldsymbol{v}_2 - \Phi_{vv}\Delta \boldsymbol{v}_1) \tag{47}$$

Substituting this value into the upper partition of the initial-value solution [Eq. (23)] gives the final position variation in terms of velocity variations:

$$\Delta \boldsymbol{r}_2 = \Phi_{rr}(\Phi_{vr}^{-1}(\Delta \boldsymbol{v}_2 - \Phi_{vv}\Delta \boldsymbol{v}_1)) + \Phi_{rv}\Delta \boldsymbol{v}_1 \tag{48}$$

Then, the desired inverse transformation is

$$\begin{Bmatrix} \Delta \boldsymbol{r}_1 \\ \Delta \boldsymbol{r}_2 \end{Bmatrix} = \begin{bmatrix} -(\Phi_{vr}^{-1}\Phi_{vv}) & (\Phi_{vr}^{-1}) \\ (\Phi_{rv} - \Phi_{rr}\Phi_{vr}^{-1}\Phi_{vv}) & (\Phi_{rr}\Phi_{vr}^{-1}) \end{bmatrix} \begin{Bmatrix} \Delta \boldsymbol{v}_1 \\ \Delta \boldsymbol{v}_2 \end{Bmatrix} = S^{-1} \begin{Bmatrix} \Delta \boldsymbol{v}_1 \\ \Delta \boldsymbol{v}_2 \end{Bmatrix} \tag{49}$$

The matrix inverse S^{-1} is obtained merely by interchanging the subscript labels r and v in the forward transformation matrix S in Eq. (43), a memorable if not intuitively plausible result. This relation corresponds to the mappings G and H cited in Eqs. (6) and (7) for the general nonlinear problem. Similarly, by interchanging symbols for r and v in the solution of the variational Lambert problem Eq. (24), one obtains the mapping from velocity variations to initial position variation in Eq. (47). The simplicity and symmetry of these relations suggest that there must be some correspondingly elegant feature of the general problem that has yet to be elucidated.

Now, analogously to Eq. (8) in the general problem, the joint probability density of the velocity variations can be expressed as

$$p_{\Delta v_1, \Delta v_2}(\Delta \boldsymbol{v}_1, \Delta \boldsymbol{v}_2) = \left| \det \begin{bmatrix} -(\Phi_{vr}^{-1}\Phi_{vv}) & (\Phi_{vr}^{-1}) \\ (\Phi_{rv} - \Phi_{rr}\Phi_{vr}^{-1}\Phi_{vv}) & (\Phi_{rr}\Phi_{vr}^{-1}) \end{bmatrix} \right| p_{\Delta r_1, \Delta r_2}(\Delta \boldsymbol{r}_1, \Delta \boldsymbol{r}_2) \tag{50}$$

where the position variations on the right-hand side are given in terms of the velocity variations by Eq. (49). The velocity variations will, in general, be statistically dependent in the variational Lambert problem even when the position variations are statistically independent. There is no obvious simplification of the determinant, although it is the reciprocal of the determinant of S, the transformation matrix that maps position variations to velocity variations.

Finally, it should be noted that the covariance matrices of the initial and final state variations, as well as the covariance of the initial and final velocity variations, can be derived by using weighted batch least-squares estimates. This approach is equivalent to the previous developments, as shown next. The linear least-squares problem $\min_x(\boldsymbol{y} - A\boldsymbol{x})^T W(\boldsymbol{y} - A\boldsymbol{x})$, where $\boldsymbol{x}$ represents a parameter vector to be estimated and $\boldsymbol{y}$ represents a vector of given values ("measurements"), is well posed if the dimension of $\boldsymbol{y}$ is at least equal to that of $\boldsymbol{x}$, the matrix A has rank at least equal to the dimension of $\boldsymbol{x}$, and the weight matrix W, which has dimension equal to that of $\boldsymbol{y}$, has full rank. It is straightforward to prove that, if $W = \mathrm{cov}(\boldsymbol{y})^{-1}$, then $\mathrm{cov}(\boldsymbol{x}) = (A^T W A)^{-1}$. Now, using this least-squares formulation, consider the given position variations as measurements, $\boldsymbol{y} = \begin{Bmatrix} \Delta \boldsymbol{r}_1 \\ \Delta \boldsymbol{r}_2 \end{Bmatrix}$, and compute the covariances of several different state vectors as if those state vectors were estimated from the measurements. Because states of dimension 6 will be computed in terms of measurements of dimension 6, the matrix A is invertible, so that the state covariance can be computed as $\mathrm{cov}(\boldsymbol{x}) = A^{-1}W^{-1}A^{-T} = A^{-1}\mathrm{cov}(\boldsymbol{y})A^{-T}$.

For the choice $\boldsymbol{x} = \begin{Bmatrix} \Delta \boldsymbol{r}_1 \\ \Delta \boldsymbol{v}_1 \end{Bmatrix}$, the matrix A^{-1}, which maps position variations into initial state variations, is the matrix R given in Eq. (25). The expression $\mathrm{cov}(\boldsymbol{x}) = A^{-1}\mathrm{cov}(\boldsymbol{y})A^{-T}$ matches that for the covariance of the initial state variation given in Eq. (26).

For the choice $\boldsymbol{x} = \begin{Bmatrix} \Delta \boldsymbol{r}_2 \\ \Delta \boldsymbol{v}_2 \end{Bmatrix}$, the matrix A^{-1}, which maps position variations into final state variations, is the matrix N given in Eq. (30). The expression $\mathrm{cov}(\boldsymbol{x}) = A^{-1}\mathrm{cov}(\boldsymbol{y})A^{-T}$ matches that for the covariance of the final state variation given in Eq. (31).

For the choice $\boldsymbol{x} = \begin{Bmatrix} \Delta \boldsymbol{v}_1 \\ \Delta \boldsymbol{v}_2 \end{Bmatrix}$, the matrix A^{-1}, which maps position variations into initial and final velocity variations, is the matrix S given in Eq. (43). The expression $\mathrm{cov}(\boldsymbol{x}) = A^{-1}\mathrm{cov}(\boldsymbol{y})A^{-T}$ matches that for the covariance of the initial and final velocity variations given in Eq. (44). This equivalence means, among other things, that least-squares orbit differential correction software can be used to check numerical results obtained from the covariance expressions presented previously for initial and final state variations.

IV. Monte Carlo Approach

The fundamental concept behind the Monte Carlo approach is the generation of representatively sampled inputs to the system to create representatively sampled outputs of the system. Thus, numerically, the relationship between the input and output distributions is produced using the full system dynamics. This approach is valid for any input or output probability distribution. However, one would like to have representative knowledge of these distributions, and that knowledge has to be ensured by proper sampling. Generally, as the number of samples increases, the output distribution is represented more accurately, in accordance with the law of large numbers.

For the Lambert problem, the inputs are position vectors and their associated probability density functions, $p_{r_1}(\boldsymbol{r}_1)$ and $p_{r_2}(\boldsymbol{r}_2)$. As an aside, in the example problem presented here, the given position vectors are assumed to be statistically independent. This is done mainly for convenience; however, such an assumption is not necessarily unrealistic. Suppose that position measurements were supplied by tracking sensors. If the measured positions have been supplied by different sensors, or if there is a sufficiently long interval of time between the measurements, then statistical independence is a reasonable assumption. On the other hand, if the measurements originate from the same sensor within a very short interval of time, then the assumption of independence would be less reasonable.

In any case, statistical independence is a convenient assumption for illustrating the calculations. Samples from each position probability density function are taken to produce position vectors $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ that are input into a Lambert solver. The Lambert solver returns sample output velocity vectors $\boldsymbol{v}_1$ and $\boldsymbol{v}_2$. Given the family of samples of $\boldsymbol{r}_1$ and $\boldsymbol{v}_1$ as well as $\boldsymbol{r}_2$ and $\boldsymbol{v}_2$, the entire Lambert solution

uncertainty $p_{r_1,v_1}(\boldsymbol{r}_1, \boldsymbol{v}_1)$ and $p_{r_2,v_2}(\boldsymbol{r}_2, \boldsymbol{v}_2)$ can be statistically represented. The state transition matrix is not needed.

Further restricting the problem to independent Gaussian input distributions, via $\text{cov}(\Delta\boldsymbol{r}_1, \Delta\boldsymbol{r}_1)$ and $\text{cov}(\Delta\boldsymbol{r}_2, \Delta\boldsymbol{r}_2)$, makes the implementation and application of the Monte Carlo approach very straightforward. Many scientific programming languages have statistical libraries that allow multivariate random sample generation for normal distributions. For relatively small $\text{cov}(\Delta\boldsymbol{r}_1, \Delta\boldsymbol{r}_1)$ and $\text{cov}(\Delta\boldsymbol{r}_2, \Delta\boldsymbol{r}_2)$, it can be assumed that the total probability distribution function will be Gaussian, and a six-dimensional sample covariance matrix can be created from the samples straightforwardly. Statistical programming libraries that provide for multivariate random sample generation for normal distributions often also include functions for estimating normal distributions from random samples.

For this work, the Monte Carlo approach with Gaussian distributions was programmed in MATLAB using the mvnrnd and cov functions to produce the random $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$ samples and then to create the output six dimensional sample covariance for $\boldsymbol{r}_1$ and $\boldsymbol{v}_1$, respectively. The function cov merely accumulates the sample covariance matrix for a set of vector samples. The function mvnrnd generates vector samples from a specified multivariate Gaussian probability density function. The sampling starts with uniformly distributed pseudorandom numbers on the interval (0,1) for each vector component, followed by transformations involving the mean and covariance to obtain sample vectors having the specified Gaussian distribution. The default (0,1) pseudorandom number generator is a Mersenne Twister algorithm of the type introduced by Matsumoto and Nishimura [22]. This class of algorithms is guaranteed to produce pseudorandom sequences having extremely long period and is considered more than adequate for the work presented here. With these functions available, the primary sampling code is limited to a handful of lines that define the input states and covariance, loop through the number of samples, call the Lambert solver for each sample pair, end the loop, and then compute the sample covariance. Battin's Lambert solver [1] was used to generate the results discussed next.

V. Numerical Results

Two orbit classes were used to compare the analytic and Monte Carlo approaches: one in low Earth orbit (LEO) and one in geosynchronous equatorial orbit (GEO). The LEO case is meant to represent the accuracy one may have from radar tracks, whereas the GEO case is more representative of the initial orbit determination problem with optical observations and hypothesized range information. In both cases, the goals are to demonstrate that the state covariance from the analytic approach agrees with results obtained by batch least-squares differential correction, as expected from the equivalence established earlier, and to investigate the precision of the Monte Carlo approach given sampling limitations.

For the LEO case, an epoch state was chosen and propagated 20 min, about 0.2 revolutions, with two-body dynamics to produce the second state. A state transition matrix was also generated during the propagation. Table 1 contains the two states in inertial coordinates, and Table 2 contains the associated state transition matrix in Cartesian coordinates. This test case is a near-circular orbit with a radius of about 7000 km and 2 deg inclination.

As a basis of comparison, the position components of the state were treated as observations in a weighted, batch least-squares differential correction. The position covariance matrices were assumed to be $0.01 \times I_{3\times3}$ in units of square kilometers, corresponding to a position standard deviation of 100 m in each component. With actual radar data, the uncertainty along the range direction would likely be lower, the perpendicular-to-range components would likely be larger, and the correlation terms would likely not be zero. However, this example is meant simply to demonstrate the method and not address initial orbit determination accuracy. The estimated state and associated partial derivatives were all in Cartesian coordinates. This is a choice of convenience because it does not require any additional transformations between the estimation state and the input position observations. Table 3 contains the covariance matrix that resulted from the differential correction. One can observe the numerical precision limitations in the off-diagonal position covariance.

The same state transition matrix from Table 2 and position covariance matrix described previously were used to evaluate Eqs. (27) and (28). It should be noted that the state transition matrix from Table 2 was used as a matter of convenience and that one can use any method desired for generating the state transition matrix as long as one takes care to ensure that the state and the input position covariance are in a common frame. With the linearized variational equation approach, the $0.01 \times I_{3\times3}$ position covariance matrix used in the differential correction becomes $\text{cov}(\Delta\boldsymbol{r}_1, \Delta\boldsymbol{r}_1)$ and $\text{cov}(\Delta\boldsymbol{r}_2, \Delta\boldsymbol{r}_2)$, whereas $\text{cov}(\Delta\boldsymbol{r}_1, \Delta\boldsymbol{r}_2)$ is set to zero because the states are assumed to be statistically independent. Given $\text{cov}(\Delta\boldsymbol{r}_1, \Delta\boldsymbol{r}_1)$, $\text{cov}(\Delta\boldsymbol{r}_1, \Delta\boldsymbol{v}_1)$, and $\text{cov}(\Delta\boldsymbol{v}_1, \Delta\boldsymbol{v}_1)$, the six-dimensional covariance at the epoch time can be reconstructed. Table 4 contains this covariance for this test case. One can see excellent agreement with Table 3, verifying that Eqs. (27) and (28) are equivalent to a batch least-squares state estimate. Because the position covariance was taken to be exactly the input, no numerical precision issues are evident in those correlation terms.

Next, Monte Carlo simulations were performed for the LEO test case. Position vector samples at the initial and final times were generated from Gaussian error distributions characterized by $\text{cov}(\Delta\boldsymbol{r}_1, \Delta\boldsymbol{r}_1)$ and $\text{cov}(\Delta\boldsymbol{r}_2, \Delta\boldsymbol{r}_2)$ being $0.01 \times I_{3\times3}$. The resulting velocity was calculated through Battin's Lambert solver for each sample, and a sample covariance matrix was calculated from the entire population of position and velocity samples at the epoch time. This was completed for sample sizes of 1000, 10,000, 100,000, and 1,000,000. Tables 5–8 contain the resulting sample covariance. The structure of the results is generally similar to the analytical results. Although there are some numerical differences even with the million-sample case, particularly in the correlation terms of the position components, the differences can be attributed mainly to sampling limitations. For example, an order-of-magnitude estimate based on the law of large numbers is that one can expect about three significant figures of agreement, with 95% confidence, using 10^6 samples in this particular six-dimensional problem. Similarly, achieving four or five

Table 1 LEO test case position and velocity vectors

Parameter	State 1	State 2
Time, s	0	1200
X, km	−2039.8845	−6995.7285
Y, km	6672.88669	−166.39802
Z, km	232.675383	−7.0380479
$\dot{X}$, km/s	−7.236669	0.15969047
$\dot{Y}$, km/s	−2.2063637	−7.5422634
$\dot{Z}$, km/s	−0.0783	−0.2633659

Table 2 LEO test case state transition matrix

X	Y	Z	$\dot{X}$	$\dot{Y}$	$\dot{Z}$
1.4500×10^{0}	-1.4127×10^{0}	-4.9128×10^{-2}	1.5614×10^{3}	-4.8581×10^{2}	-1.6848×10^{1}
-9.8093×10^{-1}	1.7118×10^{0}	5.0193×10^{-2}	-3.9396×10^{2}	1.2686×10^{3}	1.3127×10^{1}
-3.4050×10^{-2}	5.0117×10^{-2}	2.7132×10^{-1}	-1.3640×10^{1}	1.3111×10^{1}	8.9117×10^{2}
1.7062×10^{-3}	-2.8024×10^{-3}	-9.7383×10^{-5}	2.3998×10^{0}	-1.0939×10^{0}	-3.7828×10^{-2}
-1.1850×10^{-3}	3.8290×10^{-4}	4.9504×10^{-5}	-6.6342×10^{-1}	7.6037×10^{-1}	1.6965×10^{-2}
-4.0900×10^{-5}	4.9220×10^{-5}	-1.0389×10^{-3}	-2.2795×10^{-2}	1.6889×10^{-2}	2.7184×10^{-1}

Table 3 LEO Cartesian covariance resulting from differential correction

X	Y	Z	$\dot{X}$	$\dot{Y}$	$\dot{Z}$
1.0000×10^{-2}	-1.0728×10^{-17}	-4.7434×10^{-20}	-7.6155×10^{-6}	5.3656×10^{-6}	1.8657×10^{-7}
-1.0728×10^{-17}	1.0000×10^{-2}	-1.3553×10^{-19}	5.3656×10^{-6}	-1.1824×10^{-5}	-3.0629×10^{-7}
-4.7434×10^{-20}	-1.3553×10^{-19}	1.0000×10^{-2}	1.8657×10^{-7}	-3.0629×10^{-7}	-3.0372×10^{-6}
-7.6155×10^{-6}	5.3656×10^{-6}	1.8657×10^{-7}	1.4447×10^{-8}	-6.5077×10^{-9}	-2.2710×10^{-10}
5.3656×10^{-6}	-1.1824×10^{-5}	-3.0629×10^{-7}	-6.5077×10^{-9}	2.4970×10^{-8}	3.9871×10^{-10}
1.8657×10^{-7}	-3.0629×10^{-7}	-3.0372×10^{-6}	-2.2710×10^{-10}	3.9871×10^{-10}	1.3534×10^{-8}

Table 4 LEO Cartesian covariance resulting from the linear variational formulas

X	Y	Z	$\dot{X}$	$\dot{Y}$	$\dot{Z}$
1.0000×10^{-2}	0.0000×10^{0}	0.0000×10^{0}	-7.6155×10^{-6}	5.3656×10^{-6}	1.8657×10^{-7}
0.0000×10^{0}	1.0000×10^{-2}	0.0000×10^{0}	5.3656×10^{-6}	-1.1824×10^{-5}	-3.0629×10^{-7}
0.0000×10^{0}	0.0000×10^{0}	1.0000×10^{-2}	1.8657×10^{-7}	-3.0629×10^{-7}	-3.0372×10^{-6}
-7.6155×10^{-6}	5.3656×10^{-6}	1.8657×10^{0}	1.4447×10^{-8}	-6.5077×10^{-9}	-2.2710×10^{-10}
5.3656×10^{-6}	-1.1824×10^{-5}	-3.0629×10^{-7}	-6.5077×10^{-9}	2.4970×10^{-8}	3.9871×10^{-10}
1.8657×10^{-7}	-3.0629×10^{-7}	-3.0372×10^{-6}	-2.2710×10^{-10}	3.9871×10^{-10}	1.3534×10^{-8}

Table 5 LEO Cartesian covariance resulting from the 1000 sample Monte Carlo analysis

X	Y	Z	$\dot{X}$	$\dot{Y}$	$\dot{Z}$
9.9195×10^{-3}	-1.6879×10^{-4}	-2.9854×10^{-4}	-7.9153×10^{-6}	5.6088×10^{-6}	2.1506×10^{-7}
-1.6879×10^{-4}	9.2938×10^{-3}	-4.0303×10^{-4}	4.7665×10^{-6}	-1.1144×10^{-5}	1.4903×10^{-7}
-2.9854×10^{-4}	-4.0303×10^{-4}	9.3241×10^{-3}	1.1237×10^{-7}	-1.3992×10^{-7}	-2.8297×10^{-6}
-7.9153×10^{-6}	4.7665×10^{-6}	1.1237×10^{-7}	1.4709×10^{-8}	-5.8149×10^{-9}	-3.5184×10^{-10}
5.6088×10^{-6}	-1.1144×10^{-5}	-1.3992×10^{-7}	-5.8149×10^{-9}	2.4782×10^{-8}	-3.3554×10^{-10}
2.1506×10^{-7}	1.4903×10^{-7}	-2.8297×10^{-6}	-3.5184×10^{-10}	-3.3554×10^{-10}	1.4142×10^{-8}

Table 6 LEO Cartesian covariance resulting from the 10,000 sample Monte Carlo analysis

X	Y	Z	$\dot{X}$	$\dot{Y}$	$\dot{Z}$
1.0305×10^{-2}	-4.8904×10^{-5}	2.1522×10^{-4}	-7.8676×10^{-6}	5.6492×10^{-6}	9.1843×10^{-8}
-4.8904×10^{-5}	9.9712×10^{-2}	-1.2778×10^{-4}	5.4253×10^{-6}	-1.1633×10^{-5}	-3.8221×10^{-7}
2.1522×10^{-4}	-1.2778×10^{-4}	1.0157×10^{-2}	-1.3721×10^{-8}	-1.0029×10^{-7}	-3.1447×10^{-6}
-7.8676×10^{-6}	5.4253×10^{-6}	-1.3721×10^{-8}	1.4486×10^{-8}	-6.7487×10^{-9}	-1.3349×10^{-10}
5.6492×10^{-6}	-1.1633×10^{-5}	-1.0029×10^{-7}	-6.7487×10^{-9}	2.4679×10^{-8}	6.2012×10^{-10}
9.1843×10^{-8}	-3.8221×10^{-7}	-3.1447×10^{-6}	-1.3349×10^{-10}	6.2012×10^{-10}	1.3845×10^{-8}

Table 7 LEO Cartesian covariance resulting from the 100,000 sample Monte Carlo analysis

X	Y	Z	$\dot{X}$	$\dot{Y}$	$\dot{Z}$
9.9527×10^{-3}	5.8547×10^{-5}	4.8305×10^{-5}	-7.5445×10^{-6}	5.2566×10^{-6}	1.9256×10^{-7}
5.8547×10^{-5}	1.0050×10^{-2}	-1.6153×10^{-5}	5.3332×10^{-6}	-1.1867×10^{-5}	-2.7241×10^{-7}
4.8305×10^{-5}	-1.6153×10^{-5}	1.0022×10^{-2}	1.0820×10^{-7}	-3.0531×10^{-7}	-3.0331×10^{-6}
-7.5445×10^{-6}	5.3332×10^{-6}	1.0820×10^{-7}	1.4376×10^{-8}	-6.4198×10^{-9}	-1.5446×10^{-10}
5.2566×10^{-6}	-1.1867×10^{-5}	-3.0531×10^{-7}	-6.4198×10^{-9}	2.4981×10^{-8}	4.3499×10^{-10}
1.9256×10^{-7}	-2.7241×10^{-7}	-3.0331×10^{-6}	-1.5446×10^{-10}	4.3499×10^{-10}	1.3548×10^{-8}

significant figures of agreement could call for as many as 10^8 or 10^{10} samples, respectively. This situation illustrates how Monte Carlo methods might verify expected trends in a qualitative sense, even when sampling limitations preclude a high-confidence quantitative validation.

To verify that the differences between the approaches was, in fact, due to sampling limitations in the Monte Carlo approach and not due to nonlinearities in the analytical and differential correction solutions, the LEO test case was revisited with much smaller input position uncertainties. The same state and state transition matrices from

Table 8 LEO Cartesian covariance resulting from the 1,000,000 sample Monte Carlo analysis

X	Y	Z	$\dot{X}$	$\dot{Y}$	$\dot{Z}$
9.9822×10^{-3}	1.1171×10^{-5}	1.1491×10^{-5}	-7.5890×10^{-6}	5.3414×10^{-6}	1.8108×10^{-7}
1.1171×10^{-5}	1.0001×10^{-2}	-1.6935×10^{-5}	5.3468×10^{-6}	-1.1829×10^{-5}	-3.3937×10^{-7}
1.1491×10^{-5}	-1.6935×10^{-5}	9.9936×10^{-3}	1.6711×10^{-7}	-2.7242×10^{-7}	-3.0458×10^{-6}
-7.5890×10^{-6}	5.3468×10^{-6}	1.6711×10^{-7}	1.4412×10^{-8}	-6.4688×10^{-9}	-2.4737×10^{-10}
5.3414×10^{-6}	-1.1829×10^{-5}	-2.7242×10^{-7}	-6.4688×10^{-9}	2.4976×10^{-8}	4.3208×10^{-10}
1.8108×10^{-7}	-3.3937×10^{-7}	-3.0458×10^{-6}	-2.4737×10^{-10}	4.3208×10^{-10}	1.3536×10^{-8}

Table 9 Small LEO Cartesian covariance resulting from the linear variational formulas

X	Y	Z	$\dot{X}$	$\dot{Y}$	$\dot{Z}$
1.0000×10^{-6}	0.0000×10^{0}	0.0000×10^{0}	-7.6155×10^{-10}	5.3656×10^{-10}	1.8657×10^{-11}
0.0000×10^{0}	1.0000×10^{-6}	0.0000×10^{0}	5.3656×10^{-10}	-1.1824×10^{-09}	-3.0629×10^{-11}
0.0000×10^{0}	0.0000×10^{0}	1.0000×10^{-6}	1.8657×10^{-11}	-3.0629×10^{-11}	-3.0372×10^{-10}
-7.6155×10^{-10}	5.3656×10^{-10}	1.8657×10^{-11}	1.4447×10^{-12}	-6.5077×10^{-13}	-2.2710×10^{-14}
5.3656×10^{-10}	-1.1824×10^{-09}	-3.0629×10^{-11}	-6.5077×10^{-13}	2.4970×10^{-12}	3.9871×10^{-14}
1.8657×10^{-11}	-3.0629×10^{-11}	-3.0372×10^{-10}	-2.2710×10^{-14}	3.9871×10^{-14}	1.3534×10^{-12}

Table 10 Small LEO Cartesian covariance resulting from the 1,000,000 sample Monte Carlo analysis

X	Y	Z	$\dot{X}$	$\dot{Y}$	$\dot{Z}$
9.9937×10^{-7}	-2.0186×10^{-10}	-9.9562×10^{-10}	-7.6148×10^{-10}	5.3685×10^{-10}	1.9594×10^{-11}
-2.0186×10^{-10}	1.0013×10^{-6}	5.8727×10^{-10}	5.3696×10^{-10}	-1.1839×10^{-9}	-3.0233×10^{-11}
-9.9562×10^{-10}	5.8727×10^{-10}	9.9913×10^{-7}	1.9721×10^{-11}	-3.0606×10^{-11}	-3.0319×10^{-10}
-7.6148×10^{-10}	5.3696×10^{-10}	1.9721×10^{-11}	1.4441×10^{-12}	-6.5271×10^{-13}	-2.3848×10^{-14}
5.3685×10^{-10}	-1.1839×10^{-9}	-3.0606×10^{-11}	-6.5271×10^{-13}	2.4976×10^{-12}	3.9811×10^{-14}
1.9594×10^{-11}	-3.0233×10^{-11}	-3.0319×10^{-10}	-2.3848×10^{-14}	3.9811×10^{-14}	1.3535×10^{-12}

Tables 1 and 2 were used, but the differential correction weighting matrix was formed using $\mathrm{cov}(\Delta \boldsymbol{r}_1, \Delta \boldsymbol{r}_1)$ and $\mathrm{cov}(\Delta \boldsymbol{r}_2, \Delta \boldsymbol{r}_2)$ set to $10^{-6} \times I_{3\times3}$ in units of square kilometers, corresponding to a 1 m position uncertainty in each component. Again, the two input positions are 1200 s or roughly 0.2 orbits apart. Table 9 contains the resulting covariance from the linearized variational equations, which is identical to the results from the differential correction, and Table 10 contains the resulting covariance from a 1,000,000 sample Monte Carlo analysis. One can see similar precision differences between the two, as previously observed. Given the 1 m position uncertainties and fairly limited time of flight between the states, the linearized dynamics should be adequate, and the differences are due to sampling limitations in the Monte Carlo approach.

The entire Monte Carlo code executed in 21 s for 100,000 samples on a single processing core. Execution time scaled linearly with the number of samples, and given that processing of each sample is independent, the code would be well suited to parallelization to take advantage of modern multicore processors. Even with parallel processing and software optimization, the Monte Carlo approach is not likely to provide a precise answer in a timely manner. However, the strength of the method is its simplicity to implement and its ability to generate and propagate a particle distribution without any linearization assumptions.

The GEO test case followed a protocol similar to the LEO test case described previously. For this case, an epoch state was chosen and propagated 2 h, about 0.08 revolutions, with two-body dynamics to produce the second state. A state transition matrix was also generated during the propagation. Table 11 contains the two states in inertial coordinates, and Table 12 contains the associated state transition matrix in Cartesian coordinates. This test case is a near-circular orbit with a radius of about 42,164 km and 2 deg inclination.

Table 11 GEO test case position and velocity vectors

Parameter	State 1	State 2
Time, s	−12287.00747	7200
X, km	40193.35817	−30880.86911
Y, km	1401.493154	28562.21819
Z, km	232.675383	992.0445991
$\dot{X}$, km/s	−2.948617500	−2.096520385
$\dot{Y}$, km/s	−0.8989940607	−2.256397418
$\dot{Z}$, km/s	−0.03191220323	−0.07916652167

Results were initially generated for the GEO case with small position uncertainty. A weighted, batch, least-squares differential correction was performed using the input position vectors as observations with the covariance matrices assumed to be $10^{-6} \times I_{3\times3}$ in units of square kilometers, corresponding to a position standard deviation in each component of 1 m. Table 13 contains the covariance matrix that resulted from the differential correction. Similar to the LEO results, one can observe the numerical precision limitations in the off-diagonal position covariance. The $10^{-6} \times I_{3\times3}$ position covariance matrix was then used for $\mathrm{cov}(\Delta \boldsymbol{r}_1, \Delta \boldsymbol{r}_1)$ and $\mathrm{cov}(\Delta \boldsymbol{r}_2, \Delta \boldsymbol{r}_2)$ in the linearized variational equations of Eqs. (27) and (28) along with the state transition matrix from Table 12. Table 14 contains the resulting covariance. One can again see that the results match the results of the differential correction to within expected machine precision. Finally, a 1,000,000 sample Monte Carlo analysis was conducted, and Table 15 contains the resulting sample covariance. Much like the LEO test case, the Monte Carlo approach recovers the covariance to a limited precision, consistent with order of magnitude estimates from sampling theory. It should be noted that many of the zero values displayed in Table 15 are due to limited precision in the program output.

Next, a GEO test case was examined where the initial uncertainties were meant to be representative of track initiation for optical sensors where the range is coarsely hypothesized. In this case, the X and Z standard deviations are 1 km, and the Y standard deviation, close to the range projection, is 100 km. The resulting input position covariances for both times are

$$\mathrm{cov}(\Delta \boldsymbol{r}_1, \Delta \boldsymbol{r}_1) = \mathrm{cov}(\Delta \boldsymbol{r}_2, \Delta \boldsymbol{r}_2) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 10,000 & 0 \\ 0 & 0 & 1 \end{bmatrix} \mathrm{km}^2 \quad (51)$$

The states and state transition matrix remain the same as in Tables 11 and 12, respectively. Ideally, the uncertainty in measurement space

Table 12 GEO test case state transition matrix

X	Y	Z	$\dot{X}$	$\dot{Y}$	$\dot{Z}$
9.5570×10^{-1}	-1.7034×10^{-1}	-5.9328×10^{-3}	7.1630×10^{3}	-4.4707×10^{2}	-1.5562×10^{1}
-1.6448×10^{-1}	1.1975×10^{0}	1.1606×10^{-2}	-4.3997×10^{2}	7.5911×10^{3}	2.5056×10^{1}
-5.7281×10^{-3}	1.1605×10^{-2}	8.6471×10^{-1}	-1.5314×10^{1}	2.5055×10^{1}	6.8723×10^{3}
-1.9929×10^{-6}	-5.4047×10^{-5}	-1.8813×10^{-6}	1.0207×10^{0}	-2.0165×10^{-1}	-7.0146×10^{-3}
-5.0043×10^{-5}	4.8248×10^{-5}	2.9615×10^{-6}	-1.9579×10^{-1}	1.1324×10^{0}	9.3261×10^{-3}
-1.7415×10^{-6}	2.9608×10^{-6}	-3.6702×10^{-5}	-6.8101×10^{-3}	9.3250×10^{-3}	8.6471×10^{-1}

Table 13 Small GEO Cartesian covariance resulting from differential correction

X	Y	Z	$\dot{X}$	$\dot{Y}$	$\dot{Z}$
1.0000×10^{-6}	-4.2485×10^{-23}	2.0680×10^{-24}	-1.3255×10^{-10}	1.3984×10^{-11}	4.8716×10^{-13}
-4.2485×10^{-23}	1.0000×10^{-6}	4.9631×10^{-24}	1.3984×10^{-11}	-1.5693×10^{-10}	-1.0853×10^{-12}
2.0680×10^{-24}	4.9631×10^{-24}	1.0000×10^{-6}	4.8716×10^{-13}	-1.0853×10^{-12}	-1.2582×10^{-10}
-1.3255×10^{-10}	1.3984×10^{-11}	4.8716×10^{-13}	3.7465×10^{-14}	-1.8201×10^{-15}	-6.3482×10^{-17}
1.3984×10^{-11}	-1.5693×10^{-10}	-1.0853×10^{-12}	-1.8201×10^{-15}	4.2371×10^{-14}	1.8722×10^{-16}
4.8716×10^{-13}	-1.0853×10^{-12}	-1.2582×10^{-10}	-6.3482×10^{-17}	1.8722×10^{-16}	3.7007×10^{-14}

Table 14 Small GEO Cartesian covariance resulting from the linear variational formulas

X	Y	Z	$\dot{X}$	$\dot{Y}$	$\dot{Z}$
1.0000×10^{-6}	0.0000×10^{0}	0.0000×10^{0}	-1.3255×10^{-10}	1.3984×10^{-11}	4.8716×10^{-13}
0.0000×10^{0}	1.0000×10^{-6}	0.0000×10^{0}	1.3984×10^{-11}	-1.5693×10^{-10}	-1.0853×10^{-12}
0.0000×10^{0}	0.0000×10^{0}	1.0000×10^{-6}	4.8716×10^{-13}	-1.0853×10^{-12}	-1.2582×10^{-10}
-1.3255×10^{-10}	1.3984×10^{-11}	4.8716×10^{-13}	3.7465×10^{-14}	-1.8201×10^{-15}	-6.3482×10^{-17}
1.3984×10^{-11}	-1.5693×10^{-10}	-1.0853×10^{-12}	-1.8201×10^{-15}	4.2371×10^{-14}	1.8722×10^{-16}
4.8716×10^{-13}	-1.0853×10^{-12}	-1.2582×10^{-10}	-6.3482×10^{-17}	1.8722×10^{-16}	3.7007×10^{-14}

Table 15 Small GEO Cartesian covariance resulting from the 1,000,000 sample Monte Carlo analysis

X	Y	Z	$\dot{X}$	$\dot{Y}$	$\dot{Z}$
9.9937×10^{-7}	-2.0186×10^{-10}	-9.9562×10^{-10}	-1.3255×10^{-10}	1.4078×10^{-11}	6.9600×10^{-13}
-2.0186×10^{-10}	1.0013×10^{-6}	5.8727×10^{-10}	1.3921×10^{-11}	-1.5709×10^{-10}	-1.0800×10^{-12}
-9.9562×10^{-10}	5.8727×10^{-10}	9.9913×10^{-7}	5.5700×10^{-13}	-9.9000×10^{-13}	-1.2567×10^{-10}
-1.3255×10^{-10}	1.3921×10^{-11}	5.5700×10^{-13}	3.7000×10^{-14}	-2.0000×10^{-15}	0.0000×10^{0}
1.4078×10^{-11}	-1.5709×10^{-10}	-9.9000×10^{-13}	-2.0000×10^{-15}	4.2000×10^{-14}	0.0000×10^{0}
6.9600×10^{-13}	-1.0800×10^{-12}	-1.2567×10^{-10}	0.0000×10^{0}	0.0000×10^{0}	3.7000×10^{-14}

Table 16 Large GEO Cartesian covariance resulting from differential correction

X	Y	Z	$\dot{X}$	$\dot{Y}$	$\dot{Z}$
1.0000×10^{0}	2.5947×10^{-13}	3.4694×10^{-17}	-1.3255×10^{-4}	1.3984×10^{-5}	4.8716×10^{-7}
2.5947×10^{-13}	1.0000×10^{4}	-6.7502×10^{-13}	1.3984×10^{-1}	-1.5693×10^{0}	-1.0853×10^{-2}
3.4694×10^{-17}	-6.7502×10^{-13}	1.0000×10^{0}	4.8716×10^{-7}	-1.0853×10^{-6}	-1.2582×10^{-4}
-1.3255×10^{-4}	1.3984×10^{-1}	4.8716×10^{-7}	2.6735×10^{-6}	-1.1037×10^{-5}	-1.9007×10^{-7}
1.3984×10^{-5}	-1.5693×10^{0}	-1.0853×10^{-6}	-1.1037×10^{-5}	4.2108×10^{-4}	1.0903×10^{-6}
4.8716×10^{-7}	-1.0853×10^{-2}	-1.2582×10^{-4}	-1.9007×10^{-7}	1.0903×10^{-6}	5.0935×10^{-8}

Table 17 Large GEO Cartesian covariance resulting from the linear variational formulas

X	Y	Z	$\dot{X}$	$\dot{Y}$	$\dot{Z}$
1.0000×10^{0}	0.0000×10^{0}	0.0000×10^{0}	-1.3255×10^{-4}	1.3984×10^{-5}	4.8716×10^{-7}
0.0000×10^{0}	1.0000×10^{4}	0.0000×10^{0}	1.3984×10^{-1}	-1.5693×10^{0}	-1.0853×10^{-2}
0.0000×10^{0}	0.0000×10^{0}	1.0000×10^{0}	4.8716×10^{-7}	-1.0853×10^{-6}	-1.2582×10^{-4}
-1.3255×10^{-4}	1.3984×10^{-1}	4.8716×10^{-7}	2.6735×10^{-6}	-1.1037×10^{-5}	-1.9007×10^{-7}
1.3984×10^{-5}	-1.5693×10^{0}	-1.0853×10^{-6}	-1.1037×10^{-5}	4.2108×10^{-4}	1.0903×10^{-6}
4.8716×10^{-7}	-1.0853×10^{-2}	-1.2582×10^{-4}	-1.9007×10^{-7}	1.0903×10^{-6}	5.0940×10^{-8}

would be mapped into the Cartesian position frame and would not be the same for the two input position vectors. However, to compare results with the differential correction program, the observation weights have to be consistent, and this simple approach was taken for demonstration purposes. The goal of this exercise is to demonstrate the correctness of the methods and not investigate initial orbit determination accuracy using the methods.

Table 18 Large GEO Cartesian covariance resulting from the 1,000,000 sample Monte Carlo analysis

X	Y	Z	$\dot{X}$	$\dot{Y}$	$\dot{Z}$
1.0015×10^{0}	-3.4574×10^{-2}	-1.2731×10^{-4}	-1.3396×10^{-4}	3.6351×10^{-6}	7.8374×10^{-7}
-3.4574×10^{-2}	9.9898×10^{3}	5.2645×10^{-2}	1.3969×10^{-1}	-1.5679×10^{0}	-1.0846×10^{-2}
-1.2731×10^{-4}	5.2645×10^{-2}	9.9939×10^{-1}	1.0102×10^{-7}	-2.5886×10^{-5}	-1.2569×10^{-4}
-1.3396×10^{-4}	1.3969×10^{-1}	1.0102×10^{-7}	2.6707×10^{-6}	-1.1023×10^{-5}	-1.8994×10^{-7}
3.6351×10^{-6}	-1.5679×10^{0}	-2.5886×10^{-5}	-1.1023×10^{-5}	4.2080×10^{-4}	1.0897×10^{-6}
7.8374×10^{-7}	-1.0846×10^{-2}	-1.2569×10^{-4}	-1.8994×10^{-7}	1.0897×10^{-6}	5.0926×10^{-8}

Covariances were again generated using the weighted, batch, least-squares differential correction, linearized variational equations, and the Monte Carlo approaches described previously. Tables 16 and 17 contain the resulting covariance for the differential correction and the linearized variational equations, respectively. One can observe that there is excellent agreement with differences explained by numerical precision, as with the other cases. Table 18 contains the sample covariance that resulted from a 1,000,000 sample Monte Carlo analysis. A similar level of agreement to the other approaches is seen with differences attributable to sampling limitations. Consistent results and differences were observed over all of the test cases comparing the weighted, batch, least-squares differential correction, linearized variational equations, and the Monte Carlo approaches.

VI. Conclusions

The uncertain Lambert problem with random terminal positions, fixed time of flight, and deterministic dynamics presents a potentially useful extension of the classical two-position-and-time orbital boundary-value problem. The probability density functions of initial and final state, as well as the joint probability density function of the initial and final velocities, can be expressed formally in terms of transformations of the joint probability density function of the given position vectors. Monte Carlo methods are indicated for actually solving the general problem. Explicit expressions for the covariance matrices of the velocities and of the initial and final state can also be obtained, which in some cases might be evaluated with sufficient accuracy by an unscented transformation.

In the linear variational problem, one can obtain the covariance matrices of the initial state variations, the final state variations, and the initial and final velocity variations in explicit terms of the state transition matrix and covariances of the initial and final position variations. These covariance expressions are sufficient to describe Gaussian uncertainties. Numerical results for these expressions agree to expected numerical precision with weighted, batch least-squares differential correction results, illustrating the analytical equivalence established here between these two approaches. The transformation of probability density functions can be applied in the linear variational problem as well as in the general problem, accommodating non-Gaussian position variations. However, the case of non-Gaussian position variations in either problem would typically have to be treated by a Monte Carlo method.

Monte Carlo methods allow for the handling of both non-Gaussian position variations and nonlinear dynamics. However, representative sampling and uncertainty representation can present challenges, illustrated here by example. A simple Monte Carlo framework was presented with results for the linear Gaussian problem. The benefits of the Monte Carlo approach, even for the linear Gaussian problem, are that its implementation is conceptually simple and it does not require a state transition matrix.

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