CHAPTER 5

LAMBERT'S PROBLEM

5.1 INTRODUCTION

A fundamental problem in astrodynamics is the transfer of a spacecraft from one point in space to another. An example application is spacecraft targeting, in which the final point (the "target") is a planet or space station moving in a known orbit. In this situation, one might want the spacecraft to either *intercept* the target (match position only) or *rendezvous* with the target (match both position and velocity).

The initial point for an orbital rendezvous or interception is typically the location of the spacecraft in its orbit at the initial time. However, in other applications, such as ascent trajectories from the surface of the moon, the initial point can be at rest on the surface. Common to all orbit transfer applications is the determination of two-body orbits that connect specified initial and final points.

5.2 TRANSFER ORBITS BETWEEN SPECIFIED POINTS

As shown in Fig. 5.1, consider points P_1 and P_2 described by radius vectors $\mathbf{r}_1$ and $\mathbf{r}_2$ relative to the focus F at the center of attraction. The end points P_1 and P_2 are separated by the transfer angle θ and the chord c . The triangle FP_1P_2 is sometimes referred to as the *space triangle* for the transfer.

First, let us investigate the possible transfer orbits between the specified endpoints P_1 and P_2. For the case of elliptic transfer orbits, this can be accomplished using the simple geometric property shown in Fig. 5.2 A similar

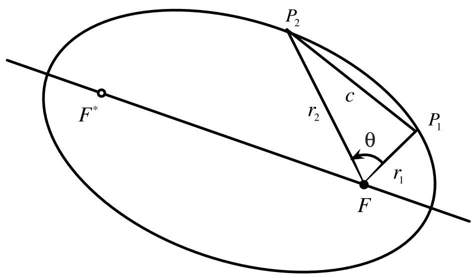


FIGURE 5.1 Transfer Orbit Geometry

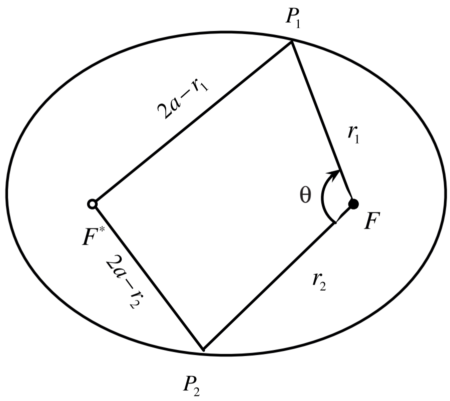


FIGURE 5.2 A Geometric Property of Ellipses

analysis can be done for parabolic and hyperbolic transfer orbits, but is not presented here.

The geometric property used is that the sum of the distances from any point on the ellipse to the focus and the vacant focus is a constant having value $2a$. This is the familiar property by which one can draw an ellipse by anchoring a piece of string at two points with thumbtacks, draw the string taut with the point of a pencil, and trace out an ellipse. In this mechanical device the thumbtacks locate the focus and vacant focus, and the string length is $2a$. Thus in Fig. 5.2

$$P_1F + P_1F^* = 2a \tag{5.1}$$

and

$$P_2F + P_2F^* = 2a$$

or

$$P_1F^* = 2a - r_1 \tag{5.2}$$

and

$$P_2F^* = 2a - r_2$$

For the remainder of the discussion, let us assume that $r_2 \geq r_1$, which implies no loss of generality, since the transfer orbit can be traversed in either direction. Because gravity is a conservative (nondissipative) force, one can determine the orbit that solves the boundary value problem in the reverse direction (starting at the final point P_2 and ending at the initial point P_1) by simply letting time run backward on the original orbit from P_2 to P_1. This represents a valid forward time solution with the original velocity vector replaced by its negative.

For a given space triangle and specified value of semimajor axis a, Fig. 5.3 shows that a vacant focus is located at the intersection of two circles centered at P_1 and P_2 having respective radii $2a - r_1$ and $2a - r_2$ [Eq. (5.2)].

As shown in Fig. 5.3, the circular arcs for a given value of $a = a_k$ intersect at two points labeled F_k^* and $\tilde{F}_k^*$ that are equidistant from the chord c. This means that for the value of a depicted, there are two elliptic transfer orbits between P_1 and P_2. As we will see, these two transfer orbits for the same value of a have different eccentricities and transfer times, but they have the same total energy.

From Fig. 5.3 it is evident that the distance FF^* is less than the distance $F\tilde{F}^*$. Because the distance from the focus of an ellipse to the vacant focus is $2ae$ (Sec. 1.5), this implies that ellipse with vacant focus at F^* has the smaller eccentricity: $e < \tilde{e}$. Figure 5.4. shows the two elliptic transfer orbits for the

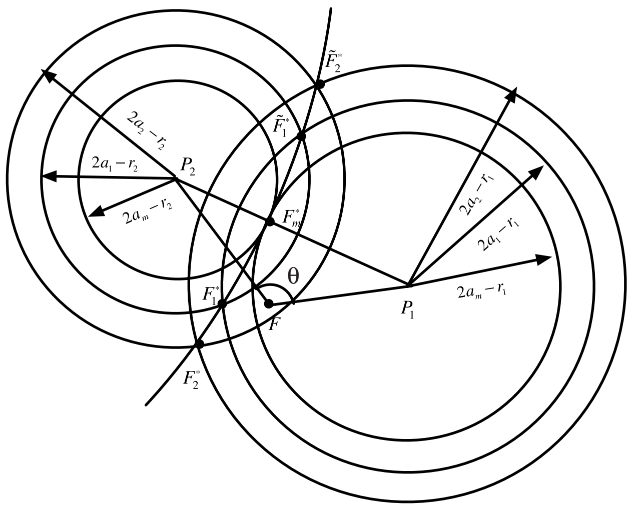


FIGURE 5.3 Vacant Focus Locations

case $r_2 = 1.524r_1$ (earth to Mars) with $\theta = 107°$ and a specified semimajor axis value of $a = 1.36r_1$. The numerical values of the two eccentricities are $e = 0.2768$ and $\tilde{e} = 0.6789$.

Returning to Fig. 5.3 two other aspects of the problem are evident from the geometry. First, as the value of a is varied, the vacant foci describe a locus formed by the intersections of the circles of varying radii centered at P_1 and P_2. This *locus of the focus* has the property that at any point on it the *difference* in the distances to the fixed points P_1 and P_2 is a constant, equal to $r_2 - r_1$. This implies that the locus itself (the solid line in Fig. 5.3) is a hyperbola with foci at P_1 and P_2!.

The second aspect of the problem that is evident from Fig. 5.3 is that as the value of a is decreased from the values shown, the two vacant foci approach the point F_m^* on the chord between P_1 and P_2. For all values of a less than this value there is *no intersection* of the circles centered at P_1 and P_2, which implies that no elliptic transfer connecting P_1 and P_2 exists for values of a less than a certain minimum value. This minimum value is denoted by a_m, and its value is easily calculated from the geometry of the point F_m^* in Fig. 5.3:

$$(2\,a_m - r_2) + (2\,a_m - r_1) = c \tag{5.3}$$

or

$$a_m = \frac{s}{2} \tag{5.4}$$

where

$$s \equiv \frac{r_1 + r_2 + c}{2} \tag{5.5}$$

Note that s is the *semiperimeter* of the space triangle $F P_1 P_2$.
Figure 5.4 shows two elliptic orbits value of a for an Earth–Mars Transfer. These transfer have:

$$\begin{aligned}
\mathbf{r}_2 &= 1.524\, r_1 \\
.26 = e &< e^* = .68 \\
\theta &= 107° \\
a &= 1.36\, r_1 \\
a_m &= 1.14\, r_1
\end{aligned}$$

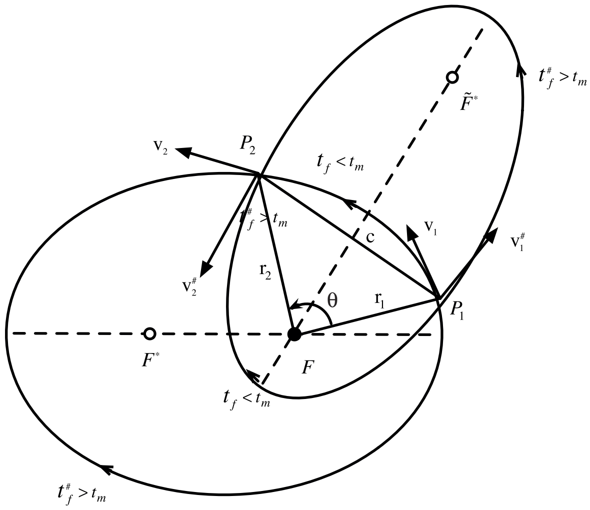


FIGURE 5.4 Two Elliptic Transfer Orbits with the Same Value of a for an Earth–Mars Transfer

In addition to describing the value a_m geometrically,one can also interpret it dynamically by recalling that the value of a for a conic orbit is a measure of its total energy (Sec. 1.7). Thus the ellipse having semimajor axis a_m is the *minimum-energy* ellipse that connects the specified endpoints P_1 and P_2. Orbits having a value of a less than a_m simply do not have enough energy at point P_1 to reach point P_2. Some days are like that.

One other interesting geometric property of the elliptic transfer orbits between P_1 and P_2 concerns the eccentricities of these orbits. As will be shown, the locus of the eccentricity vectors is a straight line that is normal to the chord, as shown by Battin, Fill, and Shepperd in [5.1].

To demonstrate this, one uses the basic polar equation for a conic section,

$$r = \frac{p}{1 + e\cos f} \tag{5.6}$$

to write

$$\mathbf{e} \cdot \mathbf{r}_1 = p - r_1; \qquad \mathbf{e} \cdot \mathbf{r}_2 = p - r_2 \tag{5.7}$$

Subtracting and dividing by the chord c yields

$$-\mathbf{e} \cdot \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{c} = \frac{r_2 - r_1}{c} \tag{5.8}$$

Because $(\mathbf{r}_2 - \mathbf{r}_1)/c$ is a unit vector along the chord directed from P_1 to P_2, Eq. (5.8) implies that the eccentricity vectors for all the transfer orbits have a constant projection along the chord direction. This, in turn, implies that the locus of the eccentricity vectors is a straight line normal to the chord as shown in Fig. 5.5.

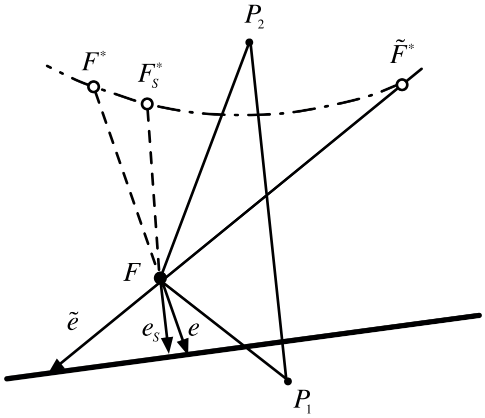


FIGURE 5.5 Locus of eccentricity vectors

Also evident from Fig. 5.5 is the fact that there is a transfer ellipse of *minimum eccentricity* e_s whose value is simply [5.3]

$$e_s = \frac{r_2 - r_1}{c} \tag{5.9}$$

which is, interestingly, the reciprocal of the eccentricity of the hyperbolic locus of the vacant focus. This minimum eccentricity ellipse is also termed the *fundamental ellipse* because the point P_1 has the same relationship to the occupied focus F as P_2 does to the vacant focus F^*, due to the fact that the major axis of the ellipse is parallel to the chord.

5.3 LAMBERT'S THEOREM

A primary concern in orbit transfer is the transfer time, defined as the time required to travel from point P_1 to point P_2. In the spacecraft targeting example mentioned earlier, the spacecraft is at point P_1 in its orbit at a time t_1 and the target will be at point P_2 in its orbit at a later time t_2. The transfer time is then $t_2 - t_1$ and the crucial issue is the determination of the transfer orbit that connects the specified endpoints in the given transfer time. Because of the work of Johann Heinrich Lambert (1728–1779), this is often called *Lambert's problem*.

The theorem which bears his name is due to his conjecture in 1761, based on geometric reasoning, that the time required to traverse an elliptic arc between specified endpoints *depends only on the semimajor axis of the ellipse, and on two geometric properties of the space triangle, namely the chord length and the sum of the radii from the focus to points* P_1 *and* P_2:

$$t_2 - t_1 = F(a, c, r_1 + r_2) \tag{5.10}$$

This result was actually first obtained analytically by Lagrange in 1778, one year before Lambert's death.

This is the third instance encountered of an important property of elliptic orbits that depends on the semimajor axis but is independent of the eccentricity. The other two are the period of an elliptic orbit and the total energy of the orbit (i.e., the velocity as a function of the radius given by the vis-viva equation).

In order to derive an equation describing the transfer time, we make use of our previously obtained equation relating time and position, namely, Kepler's equation. The essential difference here is that Lambert's problem is an orbital *boundary value problem*, whereas Kepler's equation describes an *initial value problem*. If one lets E_2 and E_1 denote the (unknown) values of eccentric anomaly at times t_2 and t_1 on the transfer ellipse, Kepler's equation yields:

$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}[E_2 - E_1 - e\,(\sin E_2 - \sin E_1)] \tag{5.11}$$

This form is not very convenient for the boundary value problem because both a and e of the transfer orbit are unknown (to be determined) and because, according to Lambert's theorem, the transfer time does not actually depend on the value of e. In order to get the transfer time equation into a more convenient form, define

$$E_P = 1/2\,(E_2 + E_1)$$

and

$$E_M = 1/2\,(E_2 - E_1) > 0 \tag{5.12}$$

Using the fact that $r = a(1 - e\cos E)$,

$$r_1 + r_2 = a\,[2 - e\,(\cos\,E_1 + \cos E_2)] \tag{5.13}$$

$$= 2_a\,[1 - e\,\cos\,E_p \cos E_M] \tag{5.14}$$

In terms of cartesian coordinates with origin at the geometric center of the ellipse with the x-axis along the major axis (see Fig. 2.1):

$$x = a\cos E \tag{5.15}$$

$$y = b\sin E; \qquad b = a(1 - e^2)^{1/2} \tag{5.16}$$

and the chord distance can be obtained from

$$\begin{aligned} c^2 &= (x_2 - x_1)^2 + (y_2 - y_1)^2 \\ &= a^2[(\cos E_2 - \cos E_1)^2 + (1 - e^2)(\sin E_2 - \sin E_1)^2] \\ &= 4a^2 \sin^2 E_M (1 - e^2 \cos^2 E_P) \end{aligned} \tag{5.17}$$

The temptation is irresistible to let

$$\cos\xi = e\cos E_P \tag{5.18}$$

which is allowable only because the numerical value of e does not exceed unity. This leads to a perfect square on the right-hand side of Eq. (5.17), resulting in

$$c = 2a \sin E_M \sin \xi \tag{5.19}$$

Equation (5.14) can then be rewritten as

$$r_1 + r_2 = 2a\,(1 - \cos E_M \cos \xi) \tag{5.20}$$

Finally, let

$$\alpha = \xi + E_M \tag{5.21a}$$

$$\beta = \xi - E_M \tag{5.21b}$$

Note that $\alpha - \beta = E_2 - E_1$.
Now one can combine Eqs. (5.19), (5.20), and (5.21) to write:

$$r_1 + r_2 + c = 2a\,(1 - \cos\alpha) = 4a \sin^2(\alpha/2) \tag{5.22}$$

$$r_1 + r_2 - c = 2a\,(1 - \cos\beta) = 4a \sin^2(\beta/2) \tag{5.23}$$

Equation (5.11) for the transfer time becomes

$$\sqrt{\mu}\,(t_2 - t_1) = 2a^{3/2}(E_M - \cos\xi \sin E_M) \tag{5.24}$$

whence our final result is

$$\sqrt{\mu}\,(t_2 - t_1) = a^{3/2}[\alpha - \beta - (\sin\alpha - \sin\beta)] \tag{5.25}$$

To summarize, Eq. (5.25), sometimes called *Lambert's equation*, describes the elliptic transfer time $t_2 - t_1$ for the case of less than one complete revolution $0 \le \theta < 2\pi$, where the variables α and β are determined from Eqs. (5.22) and (5.23) as

$$\sin\left[\frac{\alpha}{2}\right] = \left[\frac{s}{2a}\right]^{1/2} \tag{5.26}$$

$$\sin\left[\frac{\beta}{2}\right] = \left[\frac{s-c}{2a}\right]^{1/2} \tag{5.27}$$

As defined before, s is the semiperimeter of the space triangle, equal to $1/2(r_1 + r_2 + c)$. Note that Lambert's theorem as stated in Eq. (5.10) has been proved, since the angles α and β depend only on a, c and $r_1 + r_2$.

5.4 PROPERTIES OF THE SOLUTIONS TO LAMBERT'S EQUATION

Equation (5.25) must provide the solutions for the transfer times on all elliptic arcs of less than one revolution that connect points P_1 and P_2 for a give space triangle and a specified value for a. Returning to Fig. 5.4, it can be seen that for

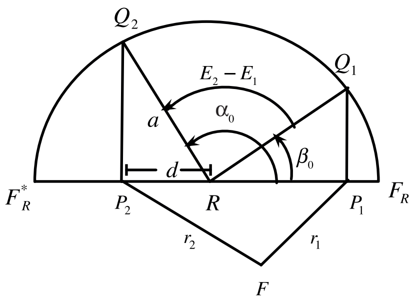


FIGURE 5.6 Interpretation of Angles α_0 and β_0

$a > a_m$ there are four such arcs. (Recall that the direction of traversal on these arcs is irrelevant to the analysis because time-reversed solutions are valid as forward time solutions.) Two of these arcs have a transfer angle $\theta < \pi$ as shown; the other two correspond to a transfer angle greater than π, and are formed by the remaining portions of the ellipses containing the smaller transfer angle arcs.

These four solutions for the transfer time correspond to quadrant ambiguities associated with the angles α and β. The principal values of the inverse sine function used to solve Eqs. (5.26) and (5.27) yield angles α_0 and β_0 characterized by $0 \leq \beta_0 \leq \alpha_0 \leq \pi$. In order to determine which one of the four arcs corresponds to these principal values and what quadrant corrections are needed for the other arcs, it is convenient to utilize the geometric interpretation of the angles α and β derived by Prussing [5.2].

The derivation of the geometric interpretation is based on two properties of elliptic motion: (1) the transfer time must satisfy Kepler's equation, and (2) the shape of the transfer orbit can be altered by moving the focus F and the vacant focus F^* without altering the transfer time or the angles α and β as *long as* $r_1 + r_2$, c, and *a remain unchanged*. Using this property the focus and vacant focus can be moved to the locations F_R and F_R^* shown in Fig. 5.6, which define a *rectilinear elliptic orbit* ($e = 1, p = 0$) between points P_1 and P_2. This rectilinear orbit has the same values of $r_1 + r_2$, c, and a and hence the same transfer time, α and β as the original orbit.

Kepler's equation for the transfer time between two points in an elliptic orbit whose locations are specified by the values of eccentric anomaly E is

$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}[E_2 - E_1 - e\,(\sin E_2 - \sin E_1)] \tag{5.28}$$

By comparing Eq. (5.28) with (5.25), one can interpret the angles α and β as *the values of eccentric anomaly on the rectilinear ellipse* between P_1 *and* P_2 *having the same value of a, c, and* $r_1 + r_2$.

The geometric interpretation of α and β and the quadrants for these angles then follows the classical geometric interpretation of eccentric anomaly encountered previously (Sec. 2.2). As shown in Fig. 5.6, one constructs the auxiliary circle of radius a centered at the center R of the rectilinear ellipse, located a distance $d = s - a$ from P_2. Points Q_1 and Q_2 are the intersections

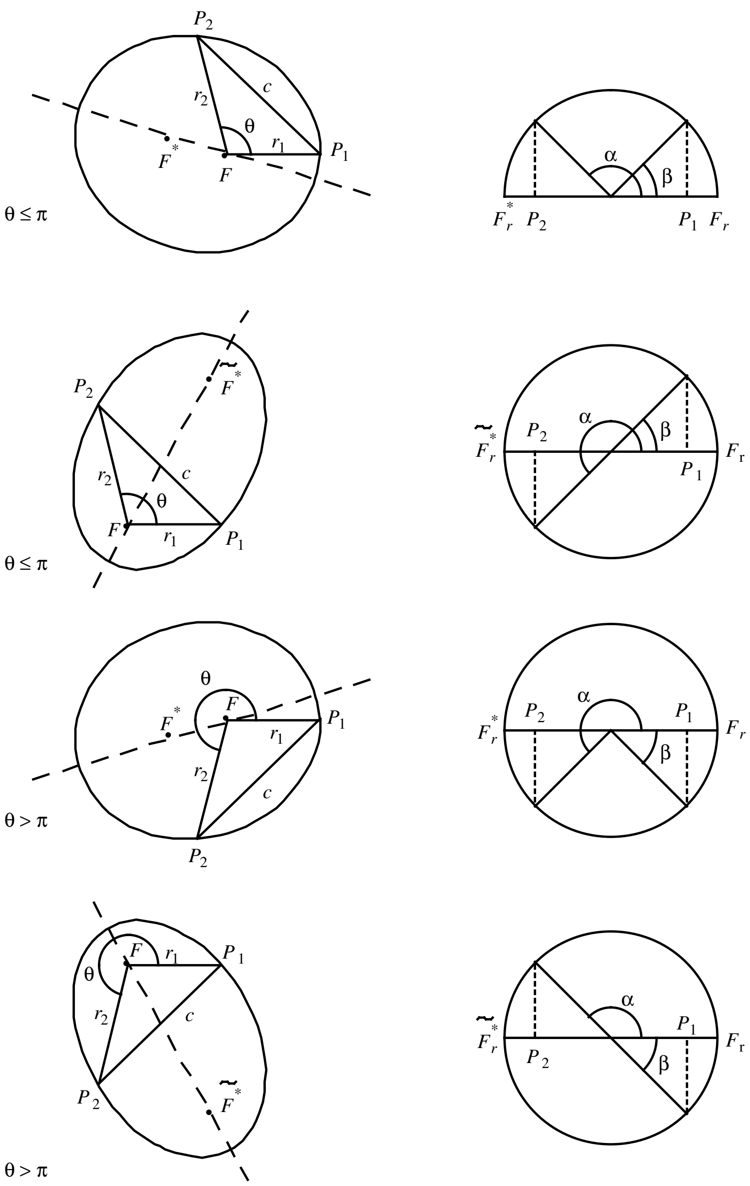


FIGURE 5.7 The Four Elliptic Arcs (Same Value of a)

of lines normal to the chord through P_1 and P_2 with the auxiliary circle. The principal value angles α_0 and β_0 are shown in Fig. 5.6. Also shown is the fact that the *difference* $\alpha - \beta$ (regardless of quadrant) is equal to the difference in the values of eccentric anomaly on the *original* elliptic path between points P_1 and P_2 [see Eq. (5.21)].

Figure 5.7 shows separately the four elliptic arcs originally shown in Fig. [5.4], along with the corresponding geometric interpretation of the angles

α and β Battin [5.3]. The top two figures depict the case $\theta \leq \pi$ and are characterized by $\beta = \beta_0 \leq \pi$. The top figure corresponds to the shorter transfer time for the given transfer angle for which $\alpha = \alpha_0 \leq \pi$. The second figure corresponds to the longer transfer time, for which $\alpha = 2\pi - \alpha_0 \geq \pi$.

For a given transfer angle the behavior of the transfer time $t_2 - t_1$ as a function of semimajor axis is shown in Fig. 5.8. For $a > a_m$ the two elliptic arcs are characterized by one having a transfer time *defined as* t_F, which is less than t_m, the time on the minimum energy arc. The other has a transfer time *defined as* $t_F^{\#}$, which is greater than t_m. The value of t_m is easily determined from Eq. (5.25) using the fact that $a_m = s/2$ [Eq. (5.4)]. Equations (5.26) and (5.27) then tell us that

$$\alpha_m = \pi, \qquad \sin(\beta_m/2) = \left[\frac{s-c}{s}\right]^{1/2} \tag{5.29}$$

and the time is given by

$$\sqrt{\mu}\, t_m = \left[\frac{s^3}{8}\right]^{1/2} (\pi - \beta_m + \sin\beta_m) \tag{5.30}$$

where for $0 \leq \theta \leq \pi, \beta_m = \beta_{m_\circ}$ and for $\pi \leq \theta \leq 2\pi, \beta_m = -\beta_{m_\circ}$. The limiting value of transfer time along the lower part of the curve of Fig. 5.8 is the *parabolic transfer time* t_p, which is approached asymptotically as $a \to \infty$. This will be discussed in more detail shortly.

Returning to Fig. 5.7, the lower two figures correspond to $\theta \geq \pi$ for which $\beta = -\beta_0$ with $\alpha = \alpha_0$ on the shorter transfer time arc and $\alpha = 2\pi - \alpha_0$ on the longer time arc.

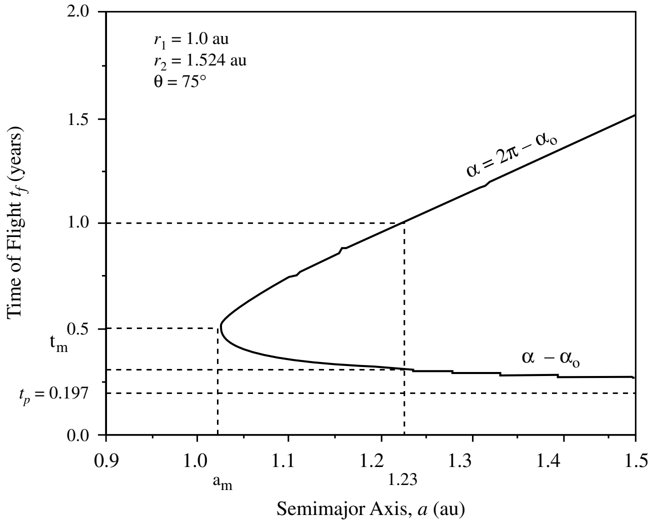


FIGURE 5.8 Transfer Time vs. Semimajor Axis

To summarize, a single equation (5.25) can be used for all elliptic arcs with $0 \leq \theta < 2\pi$, with the values of α and β determined from the principal values $0 \leq \beta_0 \leq \alpha_0 \leq \pi$ as follows:

$$0 \leq \theta < \pi, \qquad \beta = \beta_0 \tag{5.31a}$$

$$\pi \leq \theta < 2\pi, \qquad \beta = -\beta_0 \tag{5.31b}$$

$$t_2 - t_1 = t_F \leq t_m, \qquad \alpha = \alpha_0 \tag{5.32a}$$

$$t_2 - t_1 = t_F^{\#} > t_m, \qquad \alpha = 2\pi - \alpha_0 \tag{5.32b}$$

Note that for the special case $\theta = \pi, s = c$ and $\beta_0 = 0$; therefore, β is continuous at $\theta = \pi$. Also, for $t_2 - t_1 = t_m$, as mentioned in Eq. (5.29) $\alpha_0 = \pi$, and α is continuous at the transfer time t_m.

The equation describing the parabolic transfer time between specified endpoints is called *Euler's equation* (not that Euler needs another equation named after him) because this special case was published by Euler in 1743, almost 20 years before Lambert's more general result. It can be obtained by carefully taking the limit of the Lambert equation for an elliptic orbit as $a \to \infty$ (Prob. 5.3). The result can be compactly written using the *signum function*, sgn, defined by

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{for } x > 0 \\ -1 & \text{for } x < 0 \end{cases} \tag{5.33}$$

Euler's equation is then

$$\sqrt{\mu}(t_2 - t_1) = \sqrt{\mu}\, t_p = \frac{\sqrt{2}}{3}\left[s^{3/2} - \ \mathrm{sgn}\ (\sin\theta)(s-c)^{3/2}\right] \tag{5.34}$$

where the term sgn $(\sin\theta)$ automatically accounts for the required sign change going from transfer angles less than π to greater than π. This equation yields a value of $t_p = 0.197$ years for the earth-Mars transfer geometry in Fig. 5.8. The value of t_p is important, since for an elliptic transfer to exist between specified endpoints, the transfer time must be greater than t_p.

A universal formulation of Lambert's equation exists using the special functions S and C described in Chap. 2. The Battin–Vaughan algorithm [5.4] and the Gooding procedure [5.5] are excellent methods for the iterative solution to Lambert's equation for the transfer orbit given the transfer time, and are based on universal variables. A simple Newton iteration on the semimajor axis a (applicable to elliptic orbits only) is given in Sec. (5.7). This application is discussed in more detail in Sec. 5.6, and 5.7. A complete analysis of Lambert's problem in universal variables is given in [5.3].

▶ EXAMPLE 5.1

Consider the example transfer depicted in Fig. 5.8, with $r_1 = 1$ au, $r_2 = 1.524$ au, and $\theta = 75°$. The terminal radii represent an earth–Mars transfer orbit. For this geometry the chord $c = 1.592$ au, and the semiperimeter $s = 2.058$ au.

From Eq. (5.4) the value of a_m is determined to be 1.03 au, as shown in Fig. 5.8. From Eqs. (5.29) and (5.30) this yields a value of time on the minimum energy ellipse of $t_m = 3.117$ canonical time units ($\mu = 1$), which corresponds to 181 days.

For a transfer time $t_2 - t_1$ equal to 115 days = 1.978 time units, Eq. (5.25) can be solved (by numerical iteration) for the value of a. Because the value of θ is less than 180°, β is simply β_0, as indicated in Eq. (5.31a). Also, because the specified transfer time is less than t_m, the solution for a lies on the lower portion of the curve in Fig. 5.8, and the specified time is denoted by t_F. The value of α that applies in this case is simply α_0 as indicated in Eq. (5.32a). The value of a obtained is 1.232 au, which corresponds to $\alpha_0 = 2.305 = 132.1°$ and $\beta_o = 0.900 = 51.6°$. Thus the unique value of a has been determined for the specified transfer time.

The other value of transfer time for this same value of a lies on the upper portion of the curve in Fig. 5.8, corresponding to the transfer time $t_F^{\#}$. To determine this value, $\alpha = 2\pi - \alpha_0$ in Eq. (5.25), as indicated in Eq. (5.32b). The value obtained is $t_F^{\#} = 6.279$ time units = 365 days.

5.5 THE TERMINAL VELOCITY VECTORS

The terminal velocity vectors $\mathbf{v}_1$ at $\mathbf{r}_1$ and $\mathbf{v}_2$ at $\mathbf{r}_2$ can be conveniently expressed in terms of a set of skewed unit vectors, one along the *local* radius vector and the other along the chord. Specifically, let

$$\begin{aligned} \mathbf{u}_1 &\equiv \frac{\mathbf{r}_1}{r_1} \\ \mathbf{u}_2 &\equiv \frac{\mathbf{r}_2}{r_2} \\ \mathbf{u}_c &\equiv \frac{(\mathbf{r}_2 - \mathbf{r}_1)}{c} \end{aligned} \tag{5.35}$$

as shown in Fig. 5.9.

It can be shown that the velocity vector $\mathbf{v}_1$ can be expressed as

$$\mathbf{v}_1 = (B + A)\mathbf{u}_c + (B - A)\mathbf{u}_1 \tag{5.36}$$

where

$$A = \left[\frac{\mu}{4a}\right]^{1/2} \cot\left(\frac{\alpha}{2}\right) \tag{5.37a}$$

$$B = \left[\frac{\mu}{4a}\right]^{1/2} \cot\left(\frac{\beta}{2}\right) \tag{5.37b}$$

with the values of α and β being determined from Eqs. (5.26) and (5.27) with quadrant modifications given by Eqs. (5.31) and (5.32).

Equation (5.36) can be used to determine the terminal velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ for *all* the transfer ellipses having a given value of semimajor axis a.

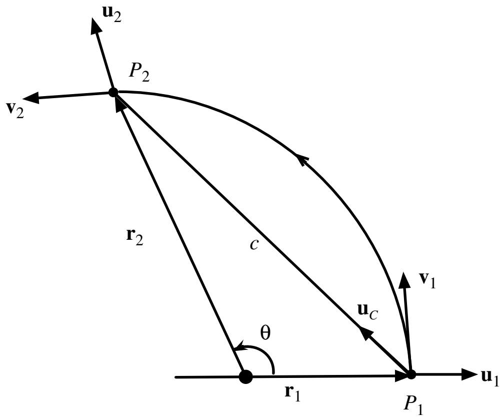


FIGURE 5.9 Unit Vector Definitions

For specified values of the transfer angle $\theta \le \pi$ and semimajor axis a, the principal values α_0 and β_0 in Eq. (5.37) yield the correct components of $\mathbf{v}_1$ for the case $t_2 - t_1 = t_F < t_m$. The initial velocity $\mathbf{v}_1^{\#}$ on the other transfer ellipse having the same value of a, but with $t_2 - t_1 = t_F^{\#} > t_m$ is obtained by using $\alpha = 2\pi - \alpha_0$ in Eq. (5.37). This has the effect of merely changing the algebraic sign of the coefficient A in Eq. (5.36). This change in sign is equivalent to *interchanging the chordal and radial components* of the velocity vector in Eq. (5.36). Thus, the components of $\mathbf{v}_1^{\#}$ are easily obtained from the components of $\mathbf{v}_1$, and, because the value for a is the same for both, $|\mathbf{v}_1| = |\mathbf{v}_1^{\#}|$ (see Fig. 5.4).

The components the final velocity $\mathbf{v}_2$ at $\mathbf{r}_2$ can also be obtained using Eq. (5.36) by considering a transfer backward in time from P_2 to P_1. In this context the velocity vector $-\mathbf{v}_2$ is the "initial" velocity, the chordal unit vector toward the final point is $-\mathbf{u}_c$, and the "initial" radial unit vector is $\mathbf{u}_2$. Equation (5.36) then becomes:

$$-\mathbf{v}_2 = (B + A)(-\mathbf{u}_c) + (B - A)\mathbf{u}_2 \tag{5.38}$$

or

$$\mathbf{v}_2 = (B + A)\mathbf{u}_c - (B - A)\mathbf{u}_2 \tag{5.39}$$

Comparing Eqs. (5.36) and (5.39), one can see that *the chordal components of the terminal velocities* $\mathbf{v}_1$ *and* $\mathbf{v}_2$ *are equal, whereas the local radial components are the negatives of each other.*

Finally, the terminal velocities on the transfer ellipse for which $\pi < \theta < 2\pi$ are obtained by changing the sign on β, as in Eq. (5.31). This has the effect of changing the sign of the coefficient B in Eq. (5.36), which is equivalent to replacing the chordal component by the negative of the local radial components, and vice versa. This makes sense, because the velocity vectors for

$\theta > \pi$ and transfer time t_F are simply the negatives of the velocity vectors for $\theta < \pi$ and transfer time $t_F^{\#}$ (see Fig. 5.4).

In the typical application, for which the transfer angle θ and the transfer time $t_2 - t_1$ are specified, the transfer ellipse and the corresponding terminal velocity vectors are unique. The appropriate values of α and β must be used to obtain the correct orbit and the terminal velocities.

It should be noted that for $\theta = \pi$ Eq. (5.36) is indeterminant because the chordal unit vector and the local radial unit vectors become parallel. In this case, an alternate form of the velocity vector equation must be used.

▶ EXAMPLE 5.2

A spacecraft is to travel from earth to Venus. The transfer angle (the true anomaly difference through which the spacecraft actually travels) is to be 135°.

a) The semimajor axis of the transfer ellipse is to be 1.1 au. Is this a feasible semimajor axis for an elliptic transfer for this case?

Working in normalized units, $r_1 = 1$ au, $r_2 = .723$, $\theta = 135°$, thus $c = 1.595$ and $s = 1.659$. Therefore $a_m = s/2 = 0.830$ au. A transfer with $a = 1.1$ au is thus larger than the minimum possible semimajor axis and is thus feasible.

b) The particular transfer desired is to have a time of flight $t_f > t_m$. Find the time of flight (in days).

For $t_f > t_m$ and $\theta < \pi$, we must have $\alpha = 2\pi - \alpha_0$ and $\beta = \beta_0$. Then

$$\sin\frac{\alpha_0}{2} = \sqrt{\frac{s}{2a}} = 0.868 \rightarrow \alpha_0 = 2.104$$

$$\sin\frac{\beta_0}{2} = \sqrt{\frac{s-c}{2a}} = .171 \rightarrow \beta_0 = .343$$

So $\alpha = 2\pi - \alpha_0 = 4.179$ and $\beta = \beta_0 = 0.343$. Then

$$t_f = \frac{a^{3/2}}{\sqrt{\mu}}[\alpha - \beta - (\sin\alpha - \sin\beta)] = 5.807 \text{ tu} = 337.6 \text{ days}$$

c) What is the required speed (not velocity) of the spacecraft at departure on the transfer ellipse?

$$v = \sqrt{\mu\left(\frac{2}{r} - \frac{1}{a}\right)} = \sqrt{1\left[\frac{2}{1} - \frac{1}{1.1}\right]} = 1.045 \text{ au/tu } = 31.11 \text{ km/sec.}$$

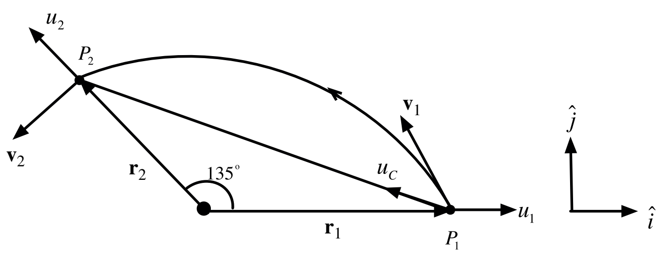
d) What impulse (km/s) is required of the spacecraft at departure?

$$r_1 = 1\hat{\imath}, \qquad r_2 = 0.723[\cos 135°\hat{\imath} + \sin 135°\hat{\jmath}] = -0.511\hat{\imath} + 0.511\hat{\jmath}$$

So $\mathbf{u}_c = \dfrac{\mathbf{r}_2 - \mathbf{r}_1}{c} = -0.947\,\hat{\imath} + 0.321\,\hat{\jmath}$
The factors A and B of Eqs. (5.37) are

$$A = \sqrt{\frac{\mu}{4a}}\cot\left[\frac{\alpha}{2}\right] = -0.272, \qquad B = \sqrt{\frac{\mu}{4a}}\cot\left[\frac{\beta}{2}\right] = 2.752$$

Then $\mathbf{v}_1 = (B + A)\mathbf{u}_c + (B - A)\mathbf{u}_1 = 0.675\,\hat{\imath} + 0.795\hat{\jmath}$ in canonical units. But $\mathbf{v}_{\text{earth}} = 1\hat{\jmath}$, so $\Delta\mathbf{v}_1 = \mathbf{v}_1 - \mathbf{v}_{\text{earth}} = 0.675\hat{\imath} - 0.205\hat{\jmath}$, so $|\,\Delta\mathbf{v}_1\,| = 0.706 = 21.01$ km/s.



EXAMPLE 5.3

The velocity vectors corresponding to the transfer shown in Fig. 5.8 utilize the unit vectors

$$\mathbf{u}_1^T = [1\ 0\ 0], \quad \mathbf{u}_2^T = [0.2588\ 0.9659\ 0], \quad \text{and} \quad \mathbf{u}_c^T = [-0.3804\ 0.9248\ 0]$$

which yields, using Eqs. (5.36) and (5.39):

$$\mathbf{v}_1^T = [0.3015\ 1.0476\ 0]; \ \mathbf{v}_2^T = [-0.6205\ 0.3401\ 0]$$

While not a direct result from the Lambert problem, it can be noted that the universal variable x at point P_1 is, of course, 0, and the value at point P_2 is determined from the universal Kepler's Eq. (2.39) to be $x = 1.5600$.

5.6 APPLICATIONS OF LAMBERT'S EQUATION

The typical application of Lambert's Eq.(5.25) is to determine the orbit and the terminal velocity vectors for specified $\mathbf{r}_1, \mathbf{r}_2$, and $t_2 - t_1$. The procedure is as follows:

1. Calculate the parabolic transfer time t_p using Eq. (5.34). If the specified $t_2 - t_1 > t_p$, an elliptic transfer orbit exists. Otherwise the orbit must be parabolic or hyperbolic.
2. Calculate t_m, using Eq. (5.30) and note whether $t_2 - t_1$ is greater than or less than t_m. This determines the correct value of α in Eq. (5.32). The

correct value of β is determined by the value of the transfer angle θ through Eq. (5.31).

3. Iteratively solve Lambert's Eq. (5.25) for the unique value of semimajor axis a. Standard iteration algorithms can be used, but the powerful universal algorithms developed by Battin and Vaughan [5.4] and Gooding [5.5] are recommended.
4. Once the value of a is determined, the terminal velocity vectors can be determined using Eq. (5.36), as discussed in Sec 5.5.

The value of the eccentricity of the transfer orbit is best determined by first obtaining the parameter:

$$p = \frac{4a(s-r_1)(s-r_2)}{c^2}\sin^2\left[\frac{\alpha+\beta}{2}\right] \tag{5.40}$$

where α and β are determined as before by Eqs. (5.31) and (5.32). Then e can be determined from a and p using $p = a(1-e^2)$.

5.7 MULTIPLE-REVOLUTION LAMBERT SOLUTIONS

One further aspect of Lambert's problem is a long-duration arc for which the transfer time is long enough to allow one or more complete revolutions of the focus, in which case $\theta \geq 2\pi$. In this case the transfer orbit is nonunique. One can always find a sufficiently large value of a so that $\theta < 2\pi$ for the given transfer time. But, as will be shown in this section, there are a total of $2N+1$ distinct solutions if the transfer time is long enough to allow N complete revolutions of the focus, for which $2N\pi \leq \theta < 2(N+1)\pi$. The transfer time for $N > 0$ is related to the transfer time for $N = 0$ [Eq. (5.25)] by simply adding the term NT, where T is the period of the elliptic transfer orbit given by Eq. (1.41):

$$\sqrt{\mu}(t_2 - t_1) = a^{3/2}\left[2N\pi + \alpha - \beta - (\sin\alpha - \sin\beta)\right] \tag{5.41}$$

The analysis of the multirevolution Lambert problem presented here is from Ref. [5.6]. A graph of transfer time $t_2 - t_1$ in years vs. semimajor axis a in au is shown in Fig. 5.10 for $N = 0, 1, 2$. This figure is a generalization of Fig. 5.8 (with a different transfer angle) to display multirevolution solutions. Note that for each value of $N > 0$ there exists a *minimum* value t_{minN} of the the transfer time. The fact that there is no minimum value of the transfer time for $N = 0$ makes sense, because as the value of a increases for an elliptic orbit, the value of the transfer time approaches a lower bound given by the transfer time t_p on a parabolic transfer orbit.

The value of the time t_{m_N} corresponding to the minimum energy value a_m on each curve in Fig. 5.10 (not to be confused with the minimum transfer time t_{min_N}) plays a critical role in determining the solutions to Eq. (5.41). The value of t_{m_N} is obtained by substituting $a_m = s/2$ into Eq. (5.41) and noting

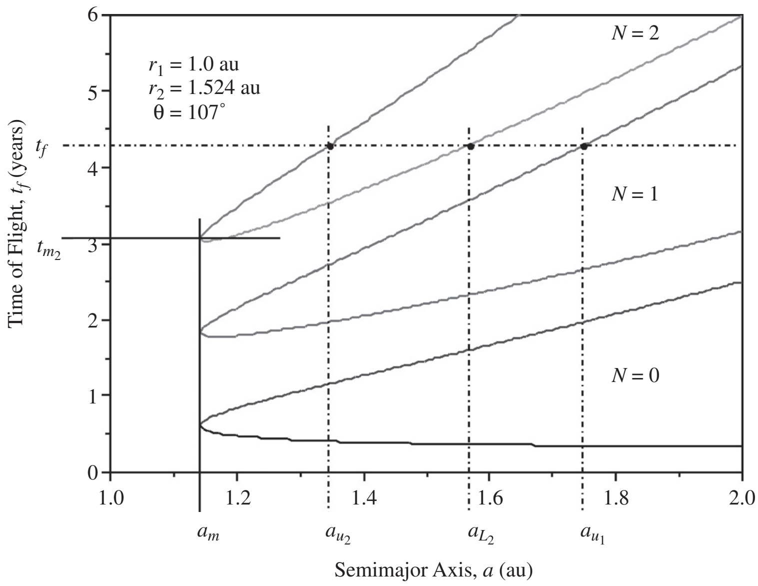


FIGURE 5.10 Transfer Time vs. Semimajor Axis for Multirevolution Cases

that $\alpha_m = \pi$,

$$\sqrt{\mu}t_{m_N} = \left(\frac{s}{2}\right)^{3/2}[(2N+1)\pi - \beta_m + \sin\beta_m] \tag{5.42}$$

The time t_{m_N} separates the transfer-time curve for each value of N into an upper and lower portion, as in Fig. 5.8.

Let Δt denote the specified value of the transfer time $t_2 - t_1$. The value of the semimajor axis a for the $N = 0$ case is unique, as shown in Fig. 5.8. However, for each $N > 0$, there are, in general, two values for a, unless Δt is equal to the minimum transfer time for a particular value of N, in which case only one value exists (see Fig. 5 .l0).

For a specified transfer time Δt, one must first calculate the minimum values of the transfer time t_{min_N} for each $N > 0$ for the given space triangle and determine the largest allowable value of N, denoted by N_{MAX} by comparing the value of Δt with the value of t_{minN} for each N. Once the value of N_{MAX} is known, all the solutions for the values of a corresponding to the specified Δt can then be determined.

The basic problem then becomes clear. The solution to the multiple-revolution Lambert's Problem requires a two-step process: For a specified space triangle and transfer time Δt:

1. Determine the value of N_{MAX}.
2. Determine the $2N_{MAX} + 1$ values of a for the specified value Δt. (If $\Delta t = t_{minN_{MAX}}$, there are only $2N_{MAX}$ solutions because the two solutions for N_{MAX} are equal.)

The minimum transfer time t_{minN} for each $N > 0$ can be determined as follows. First, one analytically determines an expression for the slope $\partial(t_2 - t_1)/\partial a$.

$$\frac{\partial(t_2 - t_1)}{\partial a} = \frac{\frac{1}{2}\left(\frac{a}{\mu}\right)^{\frac{1}{2}}}{\sin(\alpha - \beta) + (\sin\alpha - \sin\beta)} f(a) \tag{5.43}$$

where the function $f(a)$ is given by

$$\begin{aligned} f(a) = {} & [6N\pi + 3(\alpha - \beta) - (\sin\alpha - \sin\beta)].[\sin(\alpha - \beta) + (\sin\alpha - \sin\beta)] \\ & - 8[1 - \cos(\alpha - \beta)] \end{aligned} \tag{5.44}$$

The minimum transfer time t_{minN} for each value of $N > 0$ will then occur at a value of a for which $f(a) = 0$. Note that the derivative shown in Eq. (5.43) can also be used in a Newton iteration on Eq. (5.25) for which $N = 0$.

The determination of the unique zero of $f(a)$ for each value of $N > 0$ can be accomplished using a Newton iteration on the value of a. The solution for a will be larger than the value a_m, so a convenient starting value for the iteration is 1.001 a_m.

The value of the derivative $f'(a)$ needed in the Newton iteration is given by

$$\begin{aligned} f'(a) = \frac{\partial f}{\partial a} = {} & ([6N\pi + 3\xi - \eta][\cos\xi + \cos\alpha] \\ & + [3 - \cos\alpha][\sin\xi + \eta] - 8\sin\xi)\left(-\frac{1}{a}\tan\frac{\alpha}{2}\right) \\ & + ([6N\pi + 3\xi - \eta][-\cos\xi - \cos\alpha] \\ & + [-3 - \cos\beta][\sin\xi + \eta] + 8\sin\xi)\left(-\frac{1}{a}\tan\frac{\beta}{2}\right) \end{aligned} \tag{5.45}$$

where $\xi \equiv \alpha - \beta$ and $\eta \equiv \sin\alpha - \sin\beta$.

Once the number of solutions for the semimajor axis a is determined, a Newton iteration can be used to determine their values. In terms of the specified transfer time Δt, one uses Eq. (5.41) to define the problem,

$$g(a) = (t_2 - t_1)(a) - \Delta t = 0 \tag{5.46}$$

and seeks to determine all the values of a that satisfy Eq. (5.46). Because the function $g(a)$ and the transfer time differ only by an additive constant, their derivatives are equal, and the derivative $g'(a)$ needed for the Newton algorithm is given by Eq. (5.43).

As indicated in Eqs. (5.32a) and (5.32b), for each branch of the transfer-time curve corresponding to a value of N, the principal value α_0 is used if $\Delta t \leq t_{m_N}$, corresponding to the lower portion of the curve. If $\Delta t > t_{m_N}$ then $\alpha = 2\pi - \alpha_0$ is the value used, corresponding to the upper portion of the curve. This simplifies the iteration to determine $g(a) = 0$ in the case that

$\Delta t \geq t_{minN_{MAX}}$. In this case, as shown in Fig. 5.10 for $N_{MAX} = 2$, the smallest solution for a is a_{U2} on the *upper* portion of the $N = 2$ branch, and the next larger solution for a is a_{L2} on the lower portion of the $N = 2$ branch. In these two cases the two different values of a are used in Eq. (5.41) for the same value of N to determine $g(a)$ in Eq. (5.46).

A more difficult case occurs if $t_{minN_{MAX}} \Delta t \leq t_{m_{N_{MAX}}}$. In this case both solutions for a for $N = N_{MAX}$ lie on the lower portion of the curve, and $\alpha = \alpha_0$ for both solutions. But these two solutions lie on opposite sides of a_{tmin}, so one can choose appropriate different starting values for the iterative solutions to Eq. (5.46).

▶ EXAMPLE 5.4

The orbital geometry is specified by $r_1 = 1$ au, $r_2 = 2$ au, $\theta = 240°$. The chord is determined to be $c = 2.64575$ au, and the semimajor axis on the minimum energy ellipse is $a_m = 1.41144$ au. The specified transfer time is $\Delta t = 6$ years. Table 5.1 shows the minimum transfer times for different values of N, indicating that $N_{MAX} = 3$ because $t_{min3} \leq \Delta t < t_{min4}$.

As indicated in Table 5.1 and shown in Fig. 5.10, t_{minN} is slightly less than t_{m_N} and a_{minN} is slightly greater that a_m for each value of N.

Table 5.2 shows the values of the semimajor axis and eccentricity for all $2N_{MAX} + 1 = 7$ transfers. The value of the eccentricity is calculated using Eq. (5.40) along with $p = a(1 - e^2)$. a_U and a_L are, respectively, the values on the upper and lower portions of the curves, as shown in Fig. 5.10.

As shown in Table 5.2 and Fig. 5.l0, a_U is less than a_L for each value of N. Also, the values of a decrease as N increases.

TABLE 5.1 MINIMUM TRANSFER TIMES

N	t_{m_N}	a_m	t_{minN}	a_{minN}
0	0.83272	1.41144		
1	2.50956	1.41144	2.44318	1.44217
2	4.18641	1.41144	4.15203	1.42191
3	5.86325	1.41144	5.84212	1.41670
4	7.54009	1.41144	7.52625	1.41460

TABLE 5.2 MULTIPLE-REVOLUTION TRANSFERS

N	a_U	e_U	a_L	e_L
0	3.44963	0.71553		
1	2.18562	0.54308	3.14374	0.86821
2	1.68185	0.41310	1.96329	0.74877
3	1.41897	0.41256	1.46562	0.54734

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Problems

5.1
a) For a given space triangle, determine expressions for the terminal velocity vectors $\mathbf{v}_{1_m}$ and $\mathbf{v}_{2_m}$ on the minimum energy orbit between P_1 and P_2 in terms of the unit vectors $\mathbf{u}_c, \mathbf{u}_1$, and $\mathbf{u}_2$.
b) Interpret the directions of these velocity vectors geometrically in terms of the unit vector directions.

5.2 Consider the earth and Jupiter to be in coplanar circular orbits of radii 1 au and 5.2 au, respectively.

a) Considering the transfer angle θ as a variable, determine the range of values of a_m for all the possible earth-Jupiter transfer ellipses.
b) For $\theta = 150°$ and $a = 5$ au, calculate the values of a_m (in au), $t_m, t_F, t_F^\#$ and t_p (in years).
c) Calculate $\mathbf{v}_1$, and $\mathbf{v}_1^\#$ (in EMOS) for the two transfer ellipses of (b).
d) Calculate the magnitudes of $\mathbf{v}_1$, and $\mathbf{v}_1^\#$.
e) Calculate p and $\tilde{p}$ (in au) along with e and $\tilde{e}$.
f) for the two ellipses, perform the graphical construction for α and β described in the text.

5.3* Determine the expression (5.34) for t_p, the transfer time on a parabolic orbit between points P_1 and P_2. Start with Eq. (5.25) for an elliptic orbit, proceed to the limit as $a \to \infty$, Be sure to account for the two cases $\theta \le \pi$ and $\theta > \pi$.

5.4 Calculate the sum of t_F for $\theta < \pi$ and $t_F^\#$ for $\theta > \pi$ for a given elliptic orbit, and interpret your result using Fig. 5.4.

5.5 Show that $p_m = 2(s - r_1)(s - r_2)/c = r_1 r_2 (1 - \cos\theta)/c$.

5.6
a) Specialize the expressions for α, β, and $t_2 - t_1$ to the case of a *circular* arc of radius r_c and transfer angle θ.

b) Generalize the results to N complete revolutions of the focus.
c) Perform the graphical construction for the angles α and β.

5.7 For the case $r_1 = r_2 \equiv r_0$ and arbitrary transfer angle θ,

a) Construct the locus of the focus.
b) For a value of a equal to r_o determine the values of e and $\tilde{e}$ and the corresponding values of p and $\tilde{p}$.

5.8 Determine the terminal velocity vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ [Eqs. (5.36) and (5.39)] for a parabolic orbit. Accomplish this by evaluating the variables A and B in Eqs. (5.37a) and (5.37b) in the limit as $a \to \infty$.

5.9 There are 4 possible elliptic trajectories connecting the two points shown in Fig. 5.4. Two have transfer angles of 107° two have transfer angles of 253°; all have the same semimajor axis of 1.36 au.

a) Find the times of flight (in days), t_m, on the elliptic trajectory (not shown) connecting the points and having minimum semimajor axis a_m.
b) Find the times of flight (in days) for each of the 4 trajectories shown in Fig. 5.4. Associate each flight time with the corresponding α and β expressed as functions of α_0 and β_0, e.g., $\alpha = 2\pi - \alpha_0, \beta = \beta_0$.
c) For the case $\theta = 107°, t_F > t_m$, construct a figure, such as that of Fig. 5.6, using the "auxiliary circle," given only $r_1 + r_2, a$, and c, i.e., without *a priori* knowledge of α and β. Measure α and β with a protractor, and compare to the exact values that were found in the course of solving part (b).