PAPER AAS 92- 194

Simplifications & corrections

MULTIPLE REVOLUTION SOLUTIONS

TO

LAMBERT'S PROBLEM

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AAS/AIAA Spaceflight Mechanics Meeting

COLORADO SPRINGS, COLORADO FEBRUARY 24-26, 1992

AAS Publications Office, P. O. Box 28130, San Diego, CA 92198

AAS 92-194

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In minimum-fuel impulsive spacecraft trajectories, long-duration coast arcs between thrust impulses can occur. If the coast time is long enough to allow more than one complete revolution of the central body, the solution becomes complicated. Lambert's Problem, which is the determination of the orbit, given the terminal radius vectors and the transfer time, has a multiplicity of solutions. For a transfer time long enough to allow N revolutions of the central body there exist $2N + 1$ trajectories which satisfy the boundary value problem. An algorithm based on the the classical Lagrange formulation for an elliptic orbit is developed and demonstrated which determines all the trajectories.

INTRODUCTION

The orbital boundary-value problem called Lambert's Problem has received considerable attention over the years. It can be thought of as an orbit determination problem and also a problem in spacecraft targeting. In the former interpretation one seeks to determine the conic orbit which connects two radius vector measurements in a known time interval. In the latter interpretation one seeks to determine the velocity vector at a given initial radius vector which will transfer a spacecraft to a given final radius vector in a specified time interval.

Battin[1] provides an analysis of the problem along with historical references to fundamental work by Lagrange (1778) and Gauss (1809) in the solution to Lambert's Problem. Additional research by Lancaster, Blanchard, and Devaney[2], Sun, Vinh, and Chern[3], Battin and Vaughan[4], and Gooding[5] has also been published in recent years.

The solution to Lambert's Problem in the multiple revolution case is discussed in Refs. 1, 3, and 5, using various formulations of the problem, and has also been analyzed by Loechler[6]. The present study of the multiple revolution case is based on Ref. 7 which utilizes the classical Lagrange formulation of Lambert's Problem for an elliptic orbit. The authors feel that in doing so, the problem is easier to understand in terms of the relationship between the transfer time and the semimajor axis of the orbit. The determination of the maximum number of revolutions N for a specified transfer time and all $2N + 1$ of the orbit solutions can be readily obtained.

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LAMBERT'S PROBLEM FORMULATION

The basic orbital geometry is shown in Fig. 1.

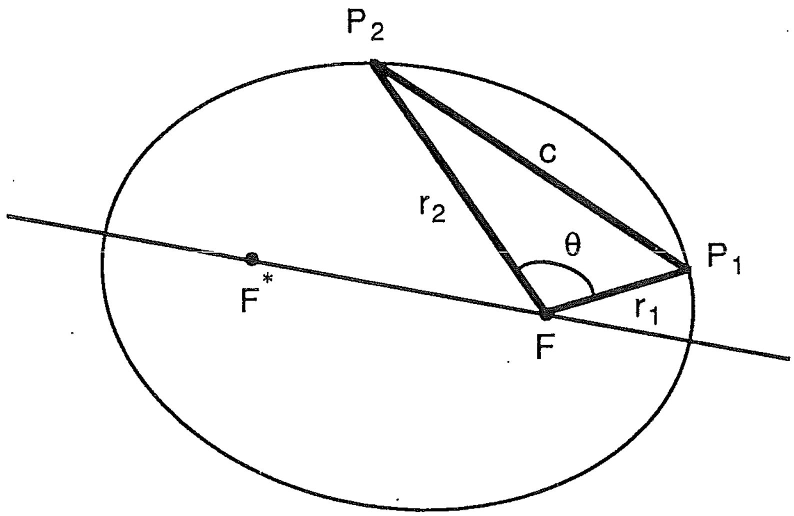


Fig. 1 The transfer orbit geometry

The two terminal radius vectors $\mathbf{r}_1$and $\mathbf{r}_2$ locate the terminal points P_1 and P_2 relative to the focus of the ellipse F at the center of gravitational attraction. The times corresponding to the terminal points are t_1 and t_2. The angle θ between the radius vectors is the *transfer angle*, and the length c is the *chord*. The triangle FP_1P_2 is often referred to as the *space triangle*.

Lagrange's formulation of Lambert's Problem generalized to the multiple revolution case is given in terms of the transfer time $t_f \equiv t_2 - t_1$ as:

$$\sqrt{\mu}\, t_f = a^{3/2}\,[2N\pi + \alpha - \beta - (\sin\alpha - \sin\beta)] \tag{1}$$

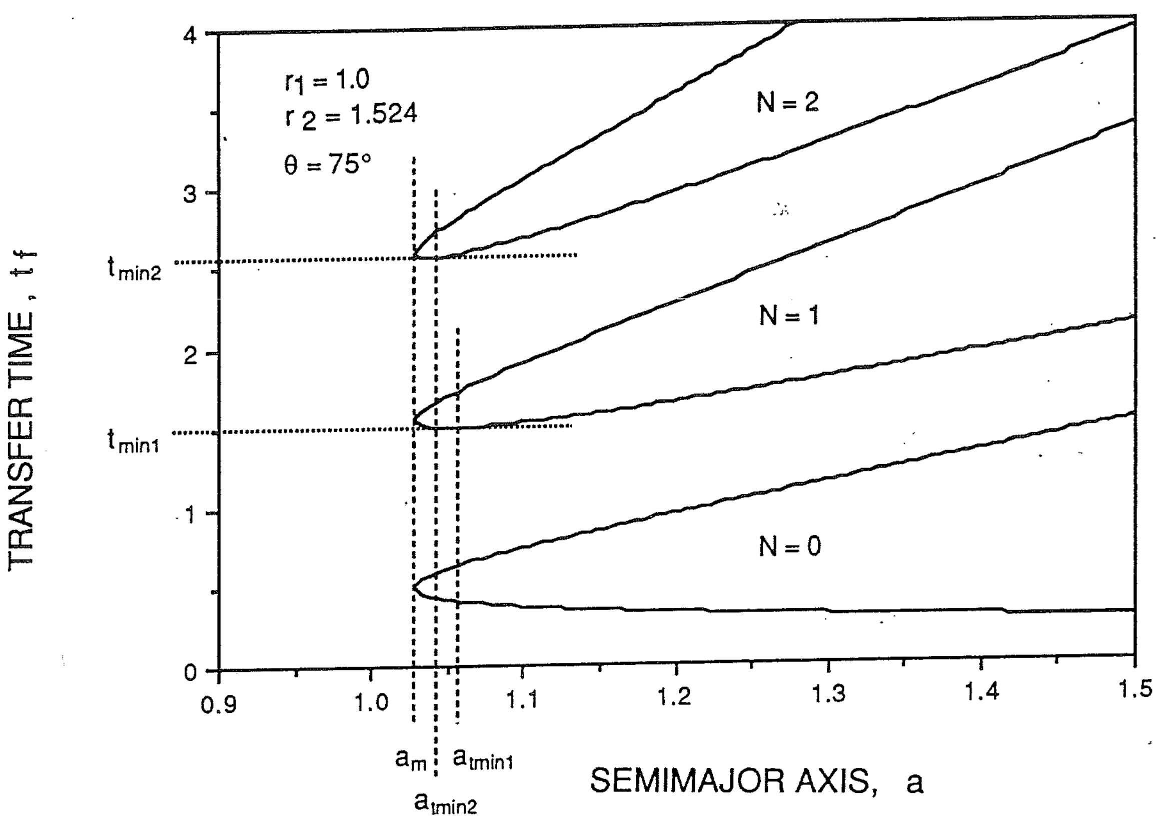
where μ is the gravitational parameter, a is the semimajor axis of the transfer orbit, N is the number of complete revolutions of the focus, and α and β are variables defined as:

$$\sin\frac{\alpha}{2} = \left[\frac{s}{2a}\right]^{1/2} \quad ; \quad \sin\frac{\beta}{2} = \left[\frac{s-c}{2a}\right]^{1/2} \tag{2}$$

where the variable $s \equiv \frac{1}{2}(r_1 + r_2 + c)$ is the *semiperimeter* of the space triangle. A geometric interpretation of the classical variables α and β has been given by Prussing[8,9].

Equation (1) and (2) then represents a statement of Lambert's Theorem, namely that the transfer time t_f depends only on the two geometrical variables $r_1 + r_2$ and c and the semimajor axis of the transfer orbit a.

A graph of transfer time t_f vs. semimajor axis a is shown in Fig. 2 for $N = 0$, 1, and 2 with $r_1 = 1$ and $r_2 = 1.524$ and $\theta = 75°$. In the numerical examples presented, a set of canonical units of length and time are used, for which $\mu = 4\pi^2$. In these units a unit radius circular orbit has a period of one time unit.



s/2 **Fig. 2 Transfer time vs semimajor axis**

Note that in Fig. 2 two things are evident. First, there exists a minimum value of semimajor axis a, denoted by a_m, which is a well-known property of Lambert's Problem[1] : $a_m = ½s$. This value corresponds to the *minimum energy ellipse*. Second, for each value of $N > 0$, there exists a *minimum* value t_{minN} of the transfer time t_f.

The value of the time t_{m_N} corresponding to the minimum energy value a_m on each curve in Fig. 2 (not to be confused with the minimum transfer time t_{minN}) plays a critical role in determining the solutions to Eq. (1). The value of t_{m_N} is obtained by substituting the value of a_m into Eq. (1) and noting that $\alpha_m = \pi$ from Eq. (2):

$$\sqrt{\mu}t_{m_N} = [(2N+1)\pi - \beta_m + \sin\beta_m] \qquad (3)$$

$\left(\frac{s}{2}\right)^{3/2}$

where

$$\sin\frac{\beta_m}{2} = \left[\frac{s-c}{s}\right]^{½} \qquad (4)$$

, denoted by

As seen in Fig. 2 the time t_{m_N} separates the transfer time curve for each value of N into an upper and lower portion. As discussed in Refs. 8 and 9, on the lower portion of the curve the value of α used in Eq. (1) is simply the principal value α_o of the $\sin^{-1}$ function used to solve Eq. (2). However, on the upper portion of the curve, the value of α to be used in Eq. (1) is $2\pi - \alpha_o$. This is the way in which Eq. (1) generates two values of t_f for a given value of a shown in Fig. 2, one on the lower portion of each curve and one on the upper portion.

Whether or not the principal value β_o of the $\sin^{-1}$ function is the correct value for β depends only on the transfer geometry[8,9]. The value of $\beta = \beta_o$ if $0 \le \theta < \pi$ and $\beta = \cancel{2}\pi$ - β_o if $\pi \le \theta < 2\pi$. This applies to the value of β_m in Eq. (3) also.

Regarding the existence of a minimum transfer time, the fact that there is no minimum value of the transfer time for $N = 0$ makes sense, because as the value of a increases for an elliptic orbit, the value of the transfer time approaches a lower bound given by the transfer time t_p on a parabolic transfer orbit. The equation for the parabolic transfer time t_p is

$$\sqrt{\mu}\, t_p = \frac{\sqrt{2}}{3}\left[s^{3/2} - \mathrm{sgn}(\sin\theta)\,(s - c)^{3/2}\right] \tag{5}$$

where the notation sgn denotes the *signum* function, in a term which accounts for whether the transfer angle is less than or greater than π. For the case shown in Fig. 2 t_p = 0.197.

Given a value for the transfer time t_f, denoted by T, the value of the semimajor axis a for the $N = 0$ case is unique. However, for each $N > 0$, there are, in general, two values for a, unless T is equal to the minimum transfer time for a particular value of N, in which case only one value exists for that value of N (See Fig. 2).

Given the transfer time T, one must first calculate the minimum values of the transfer times t_{minN} for each $N > 0$ for the given space triangle and determine the largest allowable value of N, denoted by N_{MAX} by comparing the the value of T with the value of t_{minN} for each N. Once the value of N_{MAX} is known, all the solutions for the values of a corresponding to the specified T must then be determined.

The basic problem then becomes clear. The solution to the multiple revolution Lambert's Problem requires a two-step process: For a specified space triangle and transfer time T:

1. Determine the value of N_{MAX}.

2. Determine the $2N_{MAX} + 1$ values of a for the specified value T. (If $T = t_{minN_{MAX}}$, there are only $2N_{MAX}$ solutions).

DETERMINATION OF THE MINIMUM TRANSFER TIME

The determination of the minimum transfer time t_{minN} for each $N > 0$ follows an analysis of Stern[10]. First, one analytically determines an expression for the slope $\frac{\partial t_f}{\partial a}$. As derived in the Appendix, the resulting equation is:

$$\frac{\partial t_f}{\partial a} = \frac{\tfrac{1}{2}\left[\dfrac{a}{\mu}\right]^{1/2}}{\sin(\alpha - \beta) + (\sin\alpha - \sin\beta)}\, f(a) \tag{6}$$

where the function $f(a)$ is given by:

$$f(a) = [6N\pi + 3(\alpha - \beta) - (\sin\alpha - \sin\beta)]$$

$$\cdot [\sin(\alpha - \beta) + (\sin\alpha - \sin\beta)] - 8[1 - \cos(\alpha - \beta)] \tag{7}$$

The minimum transfer time t_{minN} for each value of $N > 0$ will then occur at a value of a for which $f(a) = 0$.

The determination of the unique zero ~~root~~ of $f(a)$ for each value of $N > 0$ can be readily accomplished using a simple Newton iteration on the value of a. The solution for a is ~~know~~ to be larger than the value a_m shown in Fig. 2. A convenient starting value is therefore $a_o = 1.001\ a_m$ for the iteration algorithm:

known

$$a_{k+1} = a_k - \frac{f(a)}{f'(a)}, \quad k = 0, 1, 2, \cdots \tag{8}$$

The value of the derivative $f'(a)$ needed in Eq. (8) is derived in the Appendix to be

$$f'(a) = \frac{\partial f}{\partial a} = ([6N\pi + 3\xi - \eta][\cos\xi + \cos\alpha]$$

$$+ [3 - \cos\alpha][\sin\xi + \eta] - 8\sin\xi)(-\frac{1}{a}\tan\frac{\alpha}{2})$$

$$+ ([6N\pi + 3\xi - \eta][-\cos\xi - \cos\alpha]$$

$$+ [-3 - \cos\beta][\sin\xi + \eta] + 8\sin\xi)(-\frac{1}{a}\tan\frac{\beta}{2}) \tag{9}$$

where $\xi \equiv \alpha - \beta$ and $\eta \equiv \sin\alpha - \sin\beta$ are defined for convenience as in the Appendix.

DETERMINATION OF a AND e FOR A GIVEN TRANSFER TIME

Once the number of solutions for the semimajor axis a is known, a simple Newton iteration can be used to determine them. In terms of the specified transfer time T, one defines the problem:

$$g(a) = t_f(a) - T = 0 \tag{10}$$

and seeks to determine all the solutions to Eq. (10). Because the function $g(a)$ and the transfer time differ only by an additive constant, their derivatives are equal and the derivative $g'(a)$ needed for the Newton algorithm is simply given by Eq. (6).

As mentioned previously, for each branch of the transfer time curve corresponding to a value of N, the principal value α_o is used for values of $t_f \le t_{m_N}$, which constitutes the lower portion of the curve. On the upper portion $\alpha = 2\pi - \alpha_o$ is the value used. This simplifies the iteration to determine $g(a) = 0$ in the case that $T \ge t_{minN_{MAX}}$. In this case, as shown in Fig. 3 for $N_{MAX} = 2$, the smallest solution for a is a_{U2} on the *upper* portion of the $n = 2$ branch, and the next larger solution for a is a_{L2} on the *lower* portion of the $N = 2$ branch. In these two cases a different value of α is used in Eq. (1) for the same value of N to determine $g(a)$ in Eq. (10).

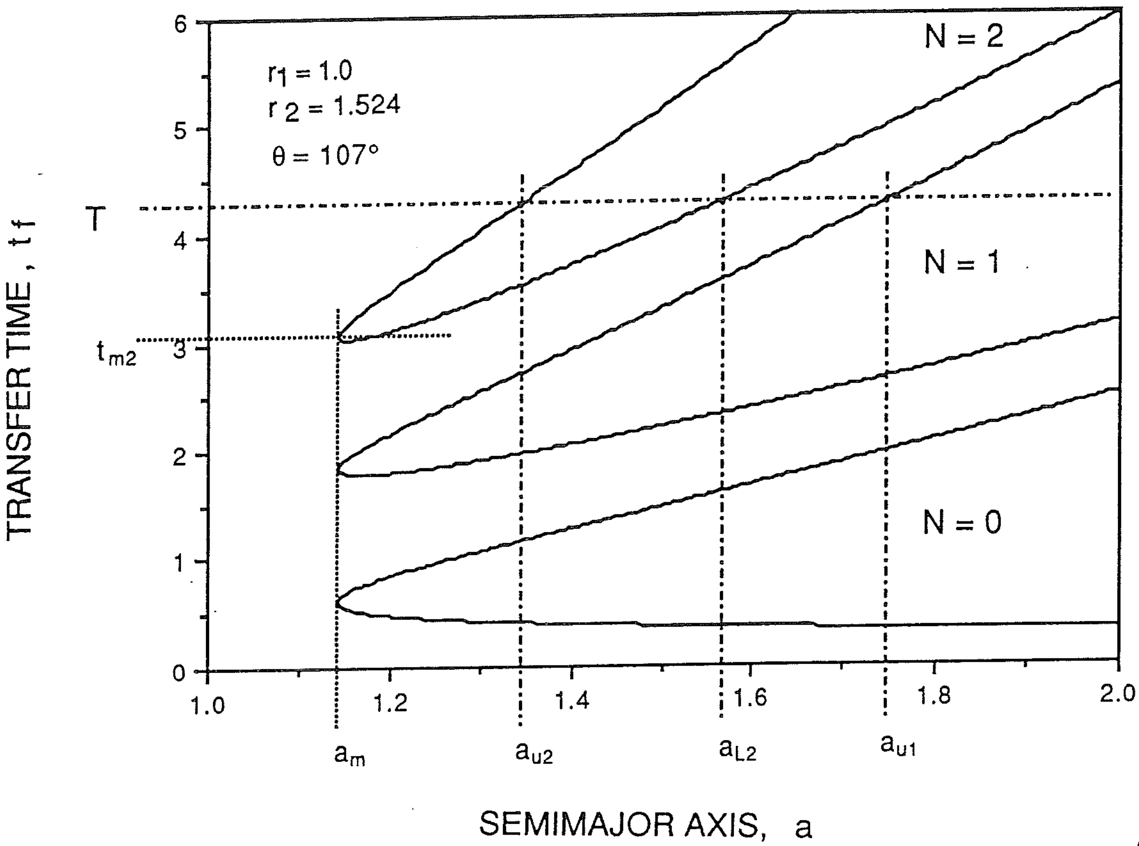


Fig. 3 Solutions for $T \geq t_{minN_{MAX}}$

A more difficult case occurs if $t_{minN_{MAX}} \leq T < t_{m_{NMAX}}$, as shown in Fig. 4. In this case both solutions for a for $N = N_{MAX}$ lie on the lower portion of the curve and $\alpha = \alpha_o$ for both solutions. Knowing that the two solutions lie on opposite sides of a_{tmin} allows one to chose starting values for the Newton iteration to solve Eq. (10). which will converge to each correct solution.

Once each value of semimajor axis a has been determined, it is of interest to also know the value of the eccentricity e in order to completely define the orbit in the orbital plane defined by the terminal radius vectors $\mathbf{r}_1$ and $\mathbf{r}_2$. A convenient formula to determine the value of the eccentricity e is[1]:

$$e = 1 - \frac{4(s - r_1)(s - r_2)}{c^2} \sin^2 \left[\frac{\alpha + \beta}{2} \right] \tag{11}$$

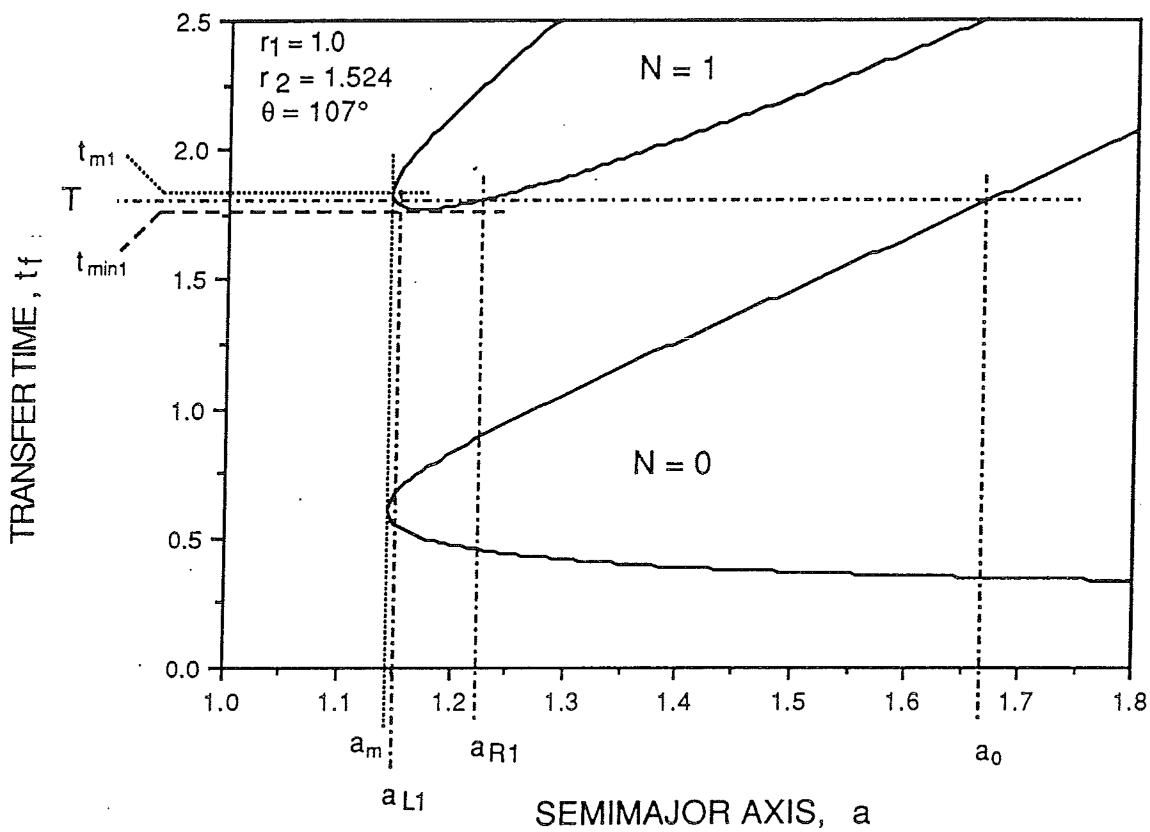


Fig. 4 Solutions for $t_{minN_{MAX}} \le T < t_{m_{NMAX}}$

NUMERICAL EXAMPLES

Two example cases are presented below of solutions to the multiple revolution problem.

CASE 1: A simple test case is presented first, namely a multiple revolution circular orbit. The orbital geometry is specified by $r_1 = r_2 = 1$, $\theta = 90$ °. The chord c is determined to be 1.41421 and the semimajor axis on the minimum energy ellipse a_m is 0.85355. A value of transfer time T is specified as 2.25 time units.

Table 1 shows the numerical results for the variables of interest, indicating that $N_{MAX} = 2$ because $t_{min2} \le T < t_{min3}$.

Table 1

CASE 1 MINIMUM TRANSFER TIMES

N	t_{m_N}	a_m	t_{minN}	a_{minN}
0	0.38172	0.85355		
1	1.17030	0.85355	1.13374	0.87212
2	1.95888	0.85355	1.93736	0.85988
3	2.74746	0.85355	2.73217	0.85674

Table 2 shows the values of semimajor axis and eccentricity for all $2N_{MAX} + 1 = 5$ trajectories, using the notation of Fig. 3.

Table 2

CASE 1 MULTIPLE REVOLUTION RESULTS

N	a_U	e_U	a_L	e_L
0	1.82313	0.89328		
1	1.15950	0.78506	1.61725	0.43672
2	0.90112	0.60260	1.00000	0.00000

The results for CASE 1 illustrate some typical characteristic of multiple revolution solutions. The results in Table 1 are consistent with the trends evident in Fig. 2. As the value of t_{minN} increases with N, the value of a_{minN} decreases toward a_m. In Table 2 the test case verifies that for $T = 2.25$ units, the solution on the lower portion of the $N = N_{MAX} = 2$ curve is indeed a circular orbit as it should be, with 2.25 revolutions of the focus in the specified transfer time. In addition, the general trends shown in Fig. 3 are evident. The value of a_U is less than the value of a_L for a given value of N and the value of a_U decreases with increasing N.

CASE 2: A case with different magnitude terminal radii and a large transfer angle is considered. The orbital geometry is specified by $r_1 = 1$, $r_2 = 2$, $\theta = 240$ °. The chord is determined to be 2.64575 and the semimajor axis on the minimum energy ellipse a_m is 1.41144. A value of transfer time T is specified as 6 time units.

Table 3 shows the numerical results for the variables of interest, indicating that $N_{MAX} = 3$ because $t_{min3} \leq T < t_{min4}$.

Table 3

CASE 2 MINIMUM TRANSFER TIMES

N	t_{m_N}	a_m	t_{minN}	a_{minN}
0	0.83272	1.41144		
1	2.50956	1.41144	2.44318	1.44217
2	4.18641	1.41144	4.15203	1.42191
3	5.86325	1.41144	5.84212	1.41670
4	7.54009	1.41144	7.52625	1.41460

Table 4 shows the values of semimajor axis and eccentricity for all $2N_{MAX} + 1 = 7$ trajectories.

Table 4

CASE 2 MULTIPLE REVOLUTION RESULTS

N	a_U	e_U	a_L	e_L
0	3.44963	0.71553		
1	2.18562	0.54308	3.14374	0.86821
2	1.68185	0.41310	1.96329	0.74877
3	1.41897	0.41256	1.46562	0.54734

The results for CASE 2 are consistent with the trends shown in the figures also. However, note that the solution having the smallest eccentricity for CASE 2 is *not* the solution on the lower portion of the $N = N_{MAX}$ curve, as it is in CASE 1. As seen in Table 4, the solution on the upper portion of the curve has the smallest eccentricity and the smallest semimajor axis.

CONCLUDING REMARKS

A two-step process is developed to determine all the multiple revolution solutions to Lambert's Problem. The classical formulation of Lagrange in terms of the transfer time as a function of semimajor axis of an elliptic orbit is used. This provides a structure for the problem which is helpful in visualizing the number and location of solutions and in choosing starting values for the iterations.

The first step is to determine the maximum number N_{MAX} of complete revolutions possible for the given transfer geometry and the specified transfer time T. This is accomplished by calculating the minimum value of the transfer time for increasing values of N until the value of T is exceeded.

The second step is to determine all $2N_{MAX} + 1$ trajectories which solve the boundary value problem. This is accomplished by iteration on the value of a from a suitably chosen starting value for each solution. A simple Newton iteration algorithm results in successful determination of the solutions. A more sophisticated algorithm, such as the Laguerre method used successfully by Conway[11] on Kepler's Equation, may provide improved performance.

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APPENDIX

In this appendix, Eq. (7) is derived starting from Eq. (1) for the transfer time:

$$\sqrt{\mu}\, t_f = a^{3/2}\,[2N\pi + \alpha - \beta - (\sin\alpha - \sin\beta)] \qquad (1,\ \text{A.1})$$

where the variables α and β are given by Eq. (2):

$$\sin\frac{\alpha}{2} = \left[\frac{s}{2a}\right]^{1/2} \ ; \ \sin\frac{\beta}{2} = \left[\frac{s-c}{2a}\right]^{1/2} \qquad (2,\ \text{A.2})$$

Equation (A.1) can be rewritten using the definitions $\xi \equiv \alpha - \beta$ and $\eta \equiv \sin\alpha - \sin\beta$ as:

$$\sqrt{\mu}\, t_f = a^{3/2}\,[2N\pi + \xi - \eta] \qquad (\text{A.3})$$

Taking the partial derivative with respect to a results in

$$\sqrt{\mu}\,\frac{\partial t_f}{\partial a} = \frac{3}{2}\, a^{1/2}\,[2N\pi + \xi - \eta]$$

$$+\, a^{3/2}\left[\left[\frac{\partial\alpha}{\partial a} - \frac{\partial\beta}{\partial a}\right] - \left[\cos\alpha\,\frac{\partial\alpha}{\partial a} - \cos\beta\,\frac{\partial\beta}{\partial a}\right]\right] \qquad (\text{A.4})$$

$\sin^2\frac{\alpha}{2} = \frac{s}{2a} = \frac{1-\cos}{2}$

Grouping like terms, ?

$$\sqrt{\mu}\,\frac{\partial t_f}{\partial a} = \frac{1}{2}\, a^{1/2}\,[6N\pi + 3\xi - 3\eta +$$

$$\underbrace{2a\,(1-\cos\alpha)}_{2s}\frac{\partial\alpha}{\partial a} - \underbrace{2a\,(1-\cos\beta)}_{2(s-c)}\frac{\partial\beta}{\partial a}] \qquad (\text{A.5})$$

From Stern[10] expressions are derived for:

$$\frac{\partial\alpha}{\partial a} = \frac{1-\cos\alpha}{a\sin\alpha} = -\,\frac{1}{a}\tan\frac{\alpha}{2} \qquad (\text{A.6})$$

$$\frac{\partial\beta}{\partial a} = \frac{1-\cos\beta}{a\sin\beta} = -\,\frac{1}{a}\tan\frac{\beta}{2} \qquad (\text{A.7})$$

Substitution into Eq. (A.5) yields: not in Stern

$$\sqrt{\mu}\,\frac{\partial t_f}{\partial a} = \frac{1}{2}\, a^{1/2}\,[6N\pi + 3\xi - 3\eta$$

$$-\,4\,(\tan\frac{\alpha}{2} - \tan\frac{\beta}{2})] \qquad (\text{A.8})$$

Finally, after some manipulation, the relationships presented as Eqs. (6) and (7) in the text are obtained:

$$\frac{\partial t_f}{\partial a} = \frac{\frac{1}{2}\left[\frac{a}{\mu}\right]^{\frac{1}{2}}}{\sin\xi + \eta} f(a) \qquad \text{(A.9, 6)}$$

where the function $f(a)$ is given by:

$$f(a) = [6N\pi + 3\xi - \eta]$$

$$\cdot [\sin\xi + \eta] - 8[1 - \cos\xi] \qquad \text{(A.10, 7)}$$

The expression for $f'(a)$ needed for the Newton iterative algorithm is derived using:

$$f'(a) = \frac{\partial f}{\partial a} = \frac{\partial f}{\partial \alpha}\frac{\partial \alpha}{\partial a} + \frac{\partial f}{\partial \beta}\frac{\partial \beta}{\partial a} \qquad \text{(A.11)}$$

The required partial derivatives are:

$$\frac{\partial f}{\partial \alpha} = [6N\pi + 3\xi - \eta][\cos\xi + \cos\alpha]$$

$$+ [3 - \cos\alpha][\sin\xi + \eta] - 8\sin\xi \qquad \text{(A.12)}$$

and

$$\frac{\partial f}{\partial \beta} = [6N\pi + 3\xi - \eta][-\cos\xi - \cos\beta]$$

$$+ [-3 + \cos\beta][\sin\xi + \eta] + 8\sin\xi \qquad \text{(A.13)}$$

Combining Eqs. (A.6), (A.7), (A.12) and (A.13) yields the result given in the text as Eq. (8):

$$f'(a) = ([6N\pi + 3\xi - \eta][\cos\xi + \cos\alpha]$$

$$+ [3 - \cos\alpha][\sin\xi + \eta] - 8\sin\xi)(-\frac{1}{a}\tan\frac{\alpha}{2})$$

$$+ ([6N\pi + 3\xi - \eta][-\cos\xi - \cos\alpha]$$

$$+ [-3 - \cos\beta][\sin\xi + \eta] + 8\sin\xi)(-\frac{1}{a}\tan\frac{\beta}{2}) \qquad \text{(A.14, 9)}$$