Certain Thoughts on Uncertainty Analysis for Dynamical Systems

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Probabilistic Analysis of Volcanic Hazards: Current Methodologies and Vision for Future Efforts

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- Stochastic System: $\dot{\mathbf{x}} = \mathbf{A}(\boldsymbol{\Theta})\mathbf{x} + \mathbf{B}(\boldsymbol{\Theta})\mathbf{u} + \mathbf{G}(\boldsymbol{\Theta})\boldsymbol{\eta}, \ \mathbf{x}(t_0) = \boldsymbol{\mu}_0.$
  - Source of Uncertainties: system parameters, initial conditions, input to the system, modeling error.
- Robust modeling of the propagation of these uncertainties is important to accurately quantify the uncertainty in the solution at any future time.

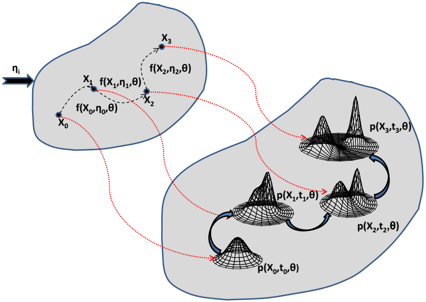

FIGURE: State and pdf transition.

LINEAR SYSTEMS WITH KNOWN PARAMETER: $\dot{\mathbf{x}} = \mathbf{A}(\Theta_0)\mathbf{x} + \mathbf{B}(\Theta_0)\mathbf{u} + \mathbf{G}(\Theta_0)\eta$

- State pdf is Gaussian assuming initial condition and input uncertainty to be Gaussian in nature.
- Drawback: appending unknown parameters as additional state variables lead to non-linear system with non-gaussian pdfs.

LINEAR SYSTEMS WITH UNKNOWN PARAMETER: $\dot{\mathbf{x}} = \mathbf{A}(\Theta)\mathbf{x} + \mathbf{B}(\Theta)\mathbf{u} + \mathbf{G}(\Theta)\eta$

- Generalized Polynomial Chaos (gPC): propagate time-invariant parametric uncertainty through an otherwise deterministic system of equations.
- Drawback: Using gPC series expansion for the time-varying stochastic forcing terms is computationally expensive.

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- Generalized Polynomial Chaos (gPC): propagate time-invariant parametric uncertainty through an otherwise deterministic system of equations.
- Drawback: Using gPC series expansion for the time-varying stochastic forcing terms is computationally expensive.

- Main challenge: characterizing the uncertainty in the system states due to *both parametric and temporal stochastic uncertainties* simultaneously
  - Approximate Solution to exact problem: Multiple-model estimation method, Monte Carlo (MC) methods.
  - Exact solution to approximate problem: Gaussian closure, Equivalent Linearization, and Stochastic Averaging.
- Main Objective: *develop analytical means to accurately characterize the state pdf of a linear system subject to initial condition uncertainty, white noise excitation and possibly non-Gaussian parametric uncertainty.*
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METHOD 1: CONDITIONING FIRST ON UNCERTAIN PARAMETERS

- The conditional state pdf $p(\mathbf{x}|\boldsymbol{\Theta})$ is a normal distribution with mean $\boldsymbol{\mu}$ and covariance $\boldsymbol{\Sigma}$, i.e. $p(\mathbf{x}|\boldsymbol{\Theta}) = \mathcal{N}(\mathbf{x};\boldsymbol{\mu},\boldsymbol{\Sigma})$.

$$\dot{\boldsymbol{\mu}} = \mathbf{A}(\boldsymbol{\Theta})\boldsymbol{\mu} + \mathbf{B}\mathbf{u} \quad (1)$$

$$\dot{\boldsymbol{\Sigma}} = \mathbf{A}(\boldsymbol{\Theta})\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\mathbf{A}^T(\boldsymbol{\Theta}) + \mathbf{G}(\boldsymbol{\Theta})\mathbf{Q}\mathbf{G}^T(\boldsymbol{\Theta}) \quad (2)$$

- These conditional moment propagation equations are exact and depend only on the initial moments and the model parameters.
- The complete distribution of the original state vector $\mathbf{x}$ is:

$$p(t,\mathbf{x}) = \int_{\boldsymbol{\Omega}} p(t,\mathbf{x}|\boldsymbol{\Theta}(\boldsymbol{\xi}))p(\boldsymbol{\xi})d\boldsymbol{\xi} \quad (3)$$

$$= \int_{\boldsymbol{\Omega}} \mathcal{N}(t,\mathbf{x};\boldsymbol{\mu}(t,\boldsymbol{\Theta}),\boldsymbol{\Sigma}(t,\boldsymbol{\Theta}))p(\boldsymbol{\xi})d\boldsymbol{\xi} \quad (4)$$

POLYNOMIAL CHAOS

- Polynomial chaos is a term originated by *Norbert Wiener in* 1938, to describe the members of the span of Hermite polynomial functionals of a Gaussian process.
- According to the Cameron-Martin Theorem, the Fourier-Hermite polynomial chaos expansion converges, in the L^2 sense, to *any arbitrary process with finite variance* (which applies to most physical processes).
- This has been generalized by Xiu et al. (2002) to efficiently use the orthogonal polynomials from the Askey-scheme to model various probability distributions.

Generalized Polynomial Chaos: Basic Goal

- Approximate the stochastic system states in terms of a finite-dimensional series expansion in the infinite-dimensional stochastic space.
- The unknown coefficients are determined by minimizing an appropriate norm of the residual.
- The basis can be chosen for a given pdf, to represent the random variable with the fewest number of terms.
  - For example, Hermite polynomials for Gaussian r.v. and Legendre polynomials for uniform r.v.
- The completeness of the space allows for the accurate representation of any random variable, with a given probability density function (pdf), by a *suitable projection on the space spanned by a carefully selected basis.*

METHOD 1: $p(t,\mathbf{x}) = \int\limits_{\boldsymbol{\Omega}} \mathcal{N}(t,\mathbf{x};\boldsymbol{\mu}(t,\boldsymbol{\Theta}),\boldsymbol{\Sigma}(t,\boldsymbol{\Theta}))p(\boldsymbol{\xi})d\boldsymbol{\xi}$

- $p(\mathbf{x}|\boldsymbol{\Theta}) = \mathcal{N}(\mathbf{x};\boldsymbol{\mu},\boldsymbol{\Sigma})$

$$\dot{\boldsymbol{\mu}} = \mathbf{A}(\boldsymbol{\Theta})\boldsymbol{\mu} + \mathbf{B}\mathbf{u} \tag{5}$$

$$\dot{\boldsymbol{\Sigma}} = \mathbf{A}(\boldsymbol{\Theta})\boldsymbol{\Sigma} + \boldsymbol{\Sigma}\mathbf{A}^T(\boldsymbol{\Theta}) + \mathbf{G}(\boldsymbol{\Theta})\mathbf{Q}\mathbf{G}^T(\boldsymbol{\Theta}) \tag{6}$$

- Combining the state mean and covariance terms into a new state vector $\mathbf{z} \in \mathbb{R}^N$ where $N = n(n+3)/2$ leads to

$$\dot{\mathbf{z}} = \mathbf{L}(\boldsymbol{\Theta})\mathbf{z} + \mathbf{M}(\boldsymbol{\Theta})\mathbf{w} \tag{7}$$

Method 1: $\dot{\mathbf{z}} = \mathbf{L}(\boldsymbol{\Theta})\mathbf{z} + \mathbf{M}(\boldsymbol{\Theta})\mathbf{w}$

- According to gPC

$$z_i(t,\boldsymbol{\xi}) = \sum_{r=0}^{P} z_{i_r}(t)\phi_r(\boldsymbol{\xi}) = \mathbf{z}_i^T(t)\boldsymbol{\Phi}(\boldsymbol{\xi}) \tag{8}$$

$$\boldsymbol{\Theta}_j(\boldsymbol{\xi}) = \sum_{r=0}^{P} \theta_{j_r}\phi_r(\boldsymbol{\xi}) = \theta_j^T\boldsymbol{\Phi}(\boldsymbol{\xi}) \tag{9}$$

$$L_{ij}(\boldsymbol{\Theta}) = \sum_{r=0}^{P} L_{ij_r}\phi_r(\boldsymbol{\xi}) = \mathbf{l}_{ij}^T\boldsymbol{\Phi}(\boldsymbol{\xi}) \tag{10}$$

$$M_{ij}(\boldsymbol{\Theta}) = \sum_{r=0}^{P} M_{ij_r}\phi_r(\boldsymbol{\xi}) = \mathbf{m}_{ij}^T\boldsymbol{\Phi}(\boldsymbol{\xi}) \tag{11}$$

$$L_{ij_r} = \frac{\langle L_{ij}(\boldsymbol{\Theta}(\boldsymbol{\xi})),\phi_r(\boldsymbol{\xi})\rangle}{\langle \phi_r(\boldsymbol{\xi}),\phi_r(\boldsymbol{\xi})\rangle} \quad M_{ij_r} = \frac{\langle M_{ij}(\boldsymbol{\Theta}(\boldsymbol{\xi})),\phi_r(\boldsymbol{\xi})\rangle}{\langle \phi_r(\boldsymbol{\xi}),\phi_r(\boldsymbol{\xi})\rangle}$$

where $\langle u(\boldsymbol{\xi}), v(\boldsymbol{\xi})\rangle = \int_{\Omega} u(\boldsymbol{\xi})v(\boldsymbol{\xi})p(\boldsymbol{\xi})d\boldsymbol{\xi}$

METHOD 1: $\dot{\mathbf{z}} = \mathbf{L}(\boldsymbol{\Theta})\mathbf{z} + \mathbf{M}(\boldsymbol{\Theta})\mathbf{w}$

- Substitution of the approximate expressions for $\mathbf{x}$ and $\boldsymbol{\Theta}$ in moment equations leads to:

$$e_i(\boldsymbol{\xi}) = \dot{\mathbf{z}}_i^T \Phi(\boldsymbol{\xi}) - \sum_{j=1}^{N} \mathbf{l}_{ij}^T \Phi(\boldsymbol{\xi}) \mathbf{z}_j^T(t) \Phi(\boldsymbol{\xi}) - \sum_{j=1}^{q} \mathbf{m}_{ij}^T \Phi(\boldsymbol{\xi}) w_j(t)$$

- Galerkin projection leads to following deterministic equations:

$$\langle e_i(\boldsymbol{\xi}), \phi_r(\boldsymbol{\xi}) \rangle = 0 \text{ for } i = 1, \ldots, N \text{ and } r = 0, \ldots, P$$

$$\dot{\mathbf{c}} = \mathbf{A}_p \mathbf{c} + \mathbf{B}_p \mathbf{w} \tag{12}$$

where, $\mathbf{c}(t) = [\mathbf{z}_1^T(t), \mathbf{z}_2^T(t), \ldots, \mathbf{z}_N^T(t)]^T \in \mathbb{R}^{N(P+1)}$ is a vector of the gPC coefficients.

METHOD 1: $p(t,\mathbf{x}) = \int_{\boldsymbol{\Omega}} \mathcal{N}(t,\mathbf{x};\boldsymbol{\mu}(t,\boldsymbol{\Theta}),\boldsymbol{\Sigma}(t,\boldsymbol{\Theta}))p(\boldsymbol{\xi})d\boldsymbol{\xi}$

- The pdf of state vector $\mathbf{x}$ can be computed as follows:

$$p(\mathbf{x}) = \int_{\boldsymbol{\Omega}} \mathcal{N}\left(t,\mathbf{x};\sum_{r=0}^{P} z_{i_r}(t)\phi_r(\boldsymbol{\xi})\right)p(\boldsymbol{\xi})d\boldsymbol{\xi} \tag{13}$$

- Quadrature integration scheme can be used to evaluate the aforementioned integrals.
  - Polynomial Chaos Quadrature (PCQ) [Dalbey et al., 2008]
- The first two moments of the actual state vector $\mathbf{x}$ can be estimated analytically, as follows:

$$\mathbf{E}[x_i(t)] = \int_{\boldsymbol{\Omega}} \mathbf{E}[\mathcal{N}(t,\mathbf{x};\sum_{r=0}^{P} z_{i_r}(t)\phi_r(\boldsymbol{\xi}))]p(\boldsymbol{\xi})d\boldsymbol{\xi} = \mu_{i_0}(t) \tag{14}$$

$$\mathbf{E}[x_i^2(t)] = \int_{\boldsymbol{\Omega}} \mathbf{E}[x_i^2(t)|\boldsymbol{\xi}]p(\boldsymbol{\xi})d\boldsymbol{\xi} = \sum_{r=0}^{P} \mu_{i_r}^2(t)\langle\phi_r^2\rangle + \Sigma_{ii_0}(t) \tag{15}$$

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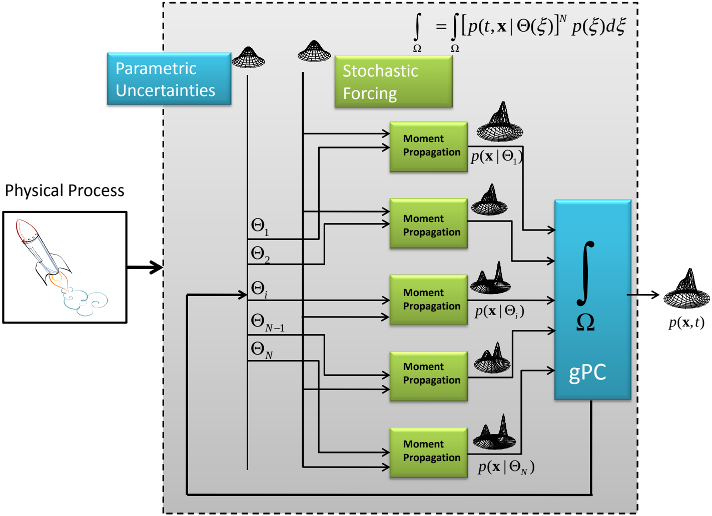
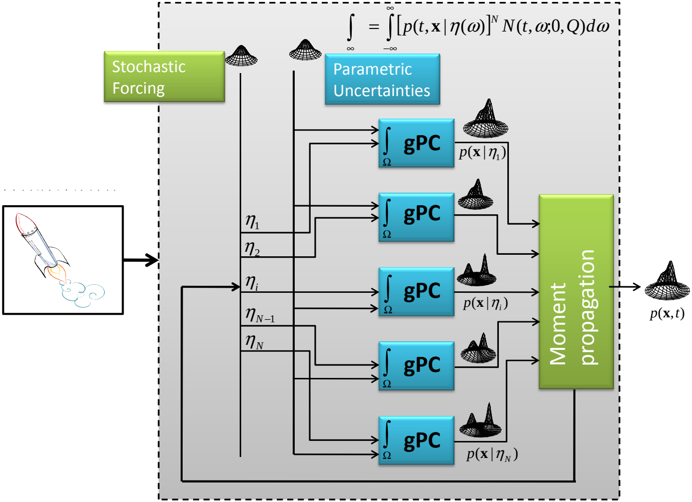


(a) Method 1: Conditioning first on uncertain parameters [Konda et al., 2011]



(b) Method 2: Conditioning first on Gaussian stochastic forcing [Konda et al., 2011]

- One can compute the sensitivity of mean and covariance of conditional pdf $p(t,\mathbf{x}|\boldsymbol{\Theta}) = \mathcal{N}(t,\mathbf{x};\mathbf{z}(\boldsymbol{\xi}))$ with respect to unknown parameter vector $\boldsymbol{\Theta}(\boldsymbol{\xi})$.
- Computational Cost:
    - Method 1: $n(n+3)(P+1)/2$ simultaneous deterministic ODEs.
    - Method 2: $n(P+1)\left[n(P+1)+3\right]/2$ simultaneous deterministic ODEs.
    - SMC: nN differential equations, N being number of Monte Carlo samples.

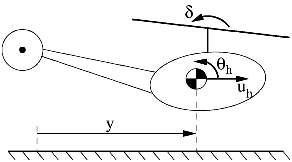
- $\dot{\mathbf{x}} = A\mathbf{x} + B\mathbf{u} + B_w\mathbf{u}_w$, $\mathbf{x} = \{u_h, q_h, \theta_h, y\}^T$

$$A = \begin{bmatrix} p_1 & p_2 & -g & 0 \\ p_3 & p_4 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} p_5 \\ p_6 \\ 0 \\ 0 \end{bmatrix}, \quad B_w = \begin{bmatrix} -p_1 \\ -p_3 \\ 0 \\ 0 \end{bmatrix}$$

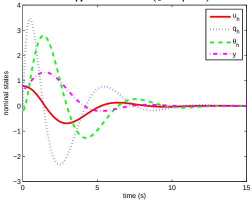
- Four aerodynamic parameters are assumed to uniformly distributed with following bounds:

$$\mathbf{p}_{lb} = [-0.049, 0.001, 0.126, -3.354]^T, \quad \mathbf{p}_{ub} = [-0.003, 0.025, 2.394, -0.177]^T$$

- Modeling errors due to unsteady flow, u_w are assumed to be zero mean Gaussian white noise with a variance of $\sigma_w^2 = 18(ft/s)^2$.

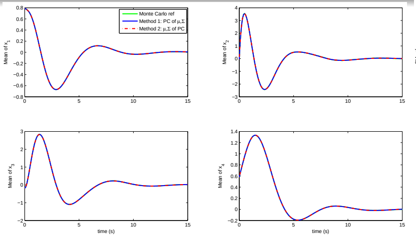


(c) Hovering helicopter

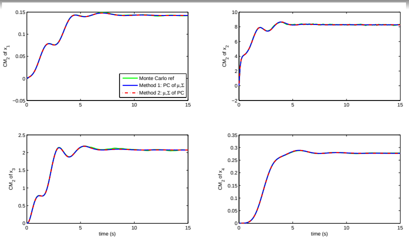


(d) Nominal states

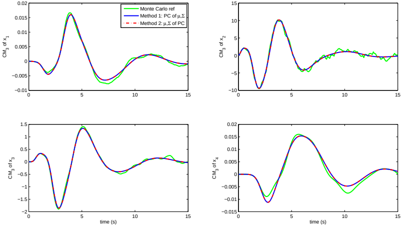
Numerical Experiment
Hovering Helicopter model



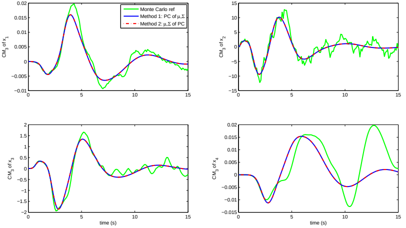
(e) Propagation of mean



(f) Propagation of variance



(g) Propagation of 3rd central moment



(h) 3rd central moment with 10000 Monte Carlo runs

Figure: Propagation of moments of the states [Konda et al., 2011]

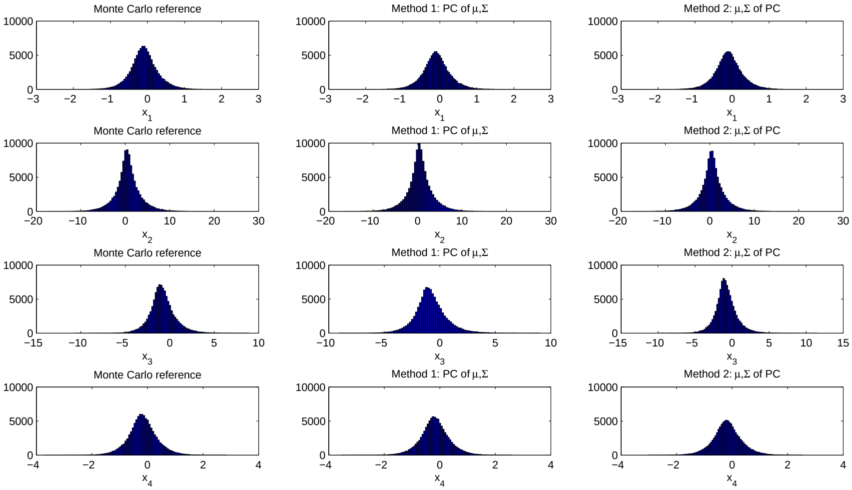


FIGURE: Histograms of the states [Konda et al., 2011]

- The `BENT` integral eruption column model was used to produce eruption column parameters (mass loading, column height, grain size distribution) given *a specific atmospheric sounding and source conditions.*
  - `BENT` takes into consideration atmospheric (wind) conditions as given by atmospheric sounding data.
  - Plume rise height is given as *a function of volcanic source and environmental conditions.*
- The `PUFF` Lagrangian model was used to propagate ash parcels in *a given wind field (NCEP Reanalysis).*
  - `PUFF` takes into account dry deposition as well as dispersion and advection.
- Polynomial chaos quadrature (PCQ) was used to select sample points and weights in the uncertain input space of vent radius, vent velocity, mean particle size and particle size variance.

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TABLE: Eruption source parameters based on observations of Eyjafjallajökull volcano and information from other similar eruptions.

Parameter	Value range	PDF	Comment
Vent radius, b_0, m	65-150	Uniform	Measured from radar image of summit vents
Vent velocity, w_0, m/s	Range: 45-124	Uniform	M. Ripepe, Geneva, Switzerland, 2010, presentation
Mean grain size, Md_φ	2 boxcars: 1.5-2 and 3-5	Multi-Modal Uniform	Woods and Bursik (1991), Table 1, vulcanian and phreatoplinian. A. Hoskuldsson, Iceland meeting 2010, presentation
σ_φ	1.9 ± 0.6	Uniform	Woods and Bursik (1991), Table 1, vulcanian and phreatoplinian.

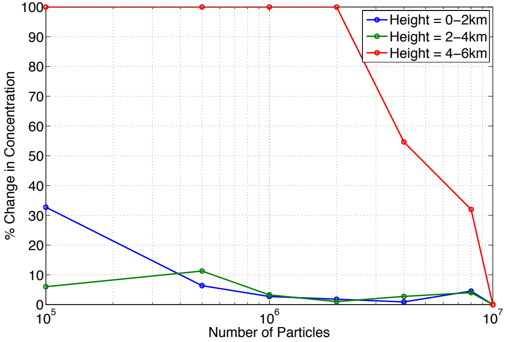
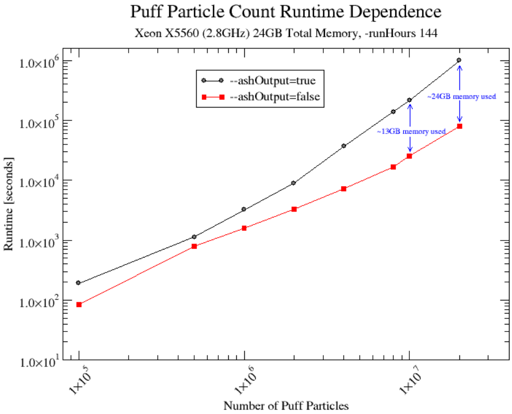


FIGURE: Concentration (52N 13.5E) vs. Number of `PUFF` Particles



nAsh	10^5	5×10^5	10^6	2×10^6	4×10^6	8×10^6	10^7
Conc.	7.40×10^{-5}	1.17×10^{-4}	1.07×10^{-4}	1.12×10^{-4}	1.09×10^{-4}	1.15×10^{-4}	1.10×10^{-4}

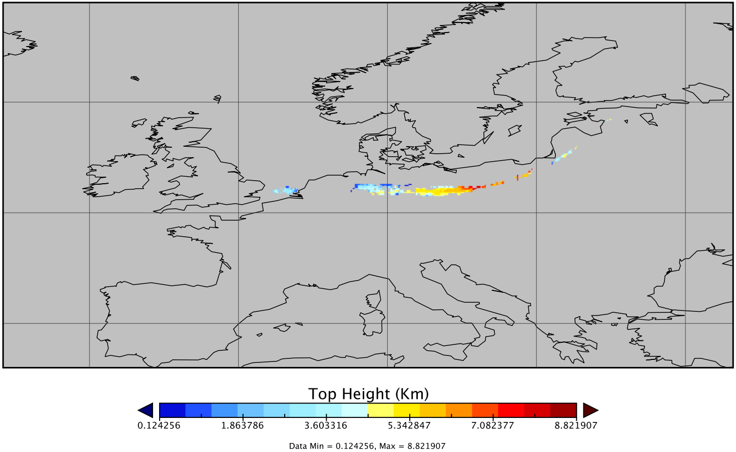


Figure: SEVIRI Data: Ash Top Height (16th April, 2010)

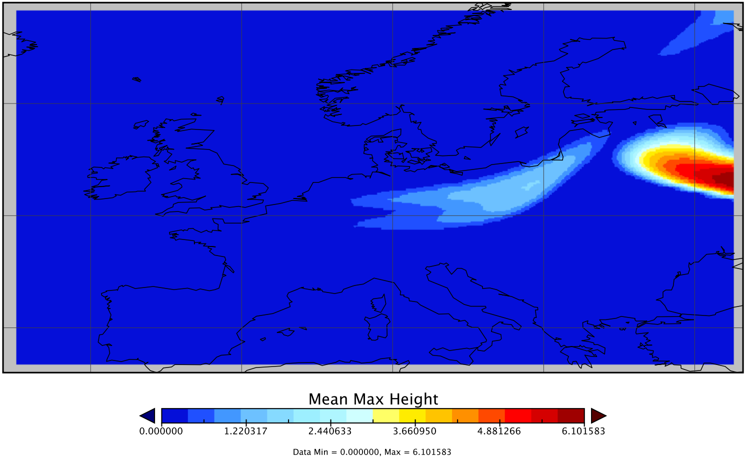


FIGURE: PCQ Runs: Ash Top Height (Mean) (16th April, 2010)

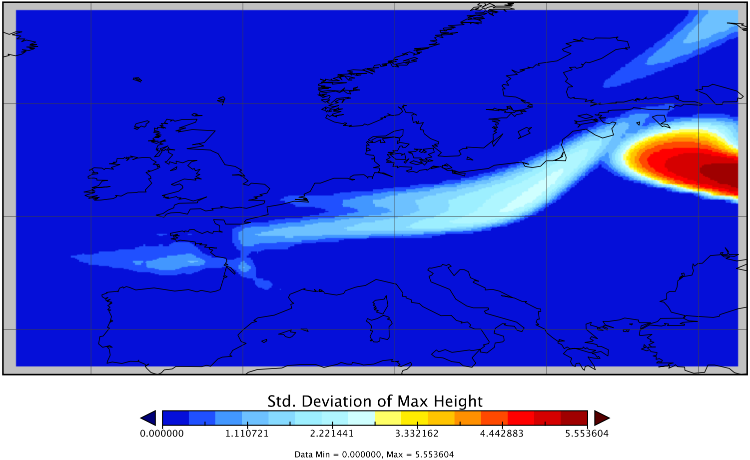


Figure: PCQ Runs: Ash Top Height (Std. Dev.) (16th April, 2010)

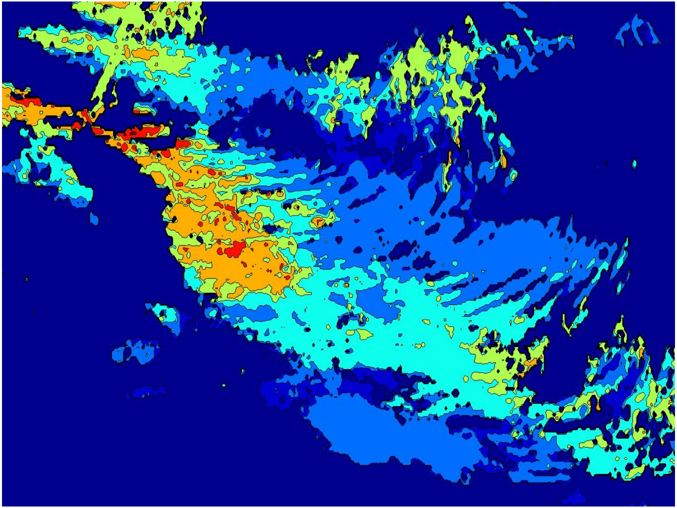
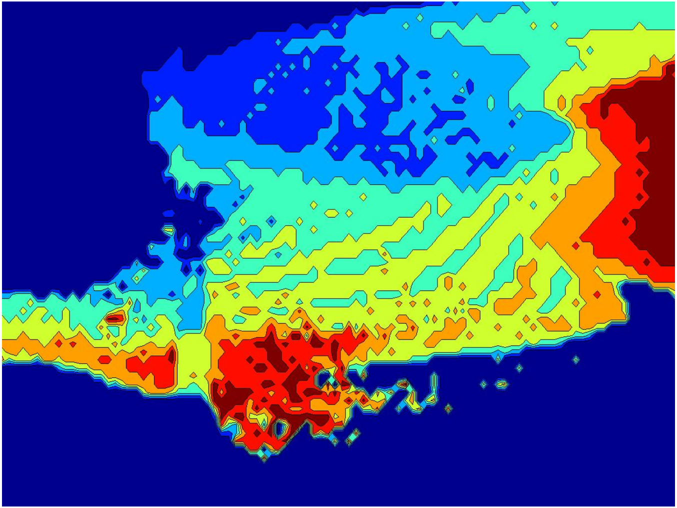


FIGURE: SEVIRI Data: Ash Top Height (15th &16th April, 2010)



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- Efficient hybrid Bayesian approach is developed for the accurate determination of uncertainty propagation in linear dynamic models with
  - *Parametric, initial condition uncertainties, and driven by additive white Gaussian noise (AWGN) process.*
  - Can be extended for nonlinear systems using *Kolmogorov equation.*
- The uncertainty due to stochastic forcing is propagated using mean and covariance propagation equations and that due to *uncertain model parameters using polynomial chaos.*
  - The moment propagation equations are exact only for white Gaussian stochastic forcing in linear dynamic models, the polynomial chaos approach can be used for any probability distribution of model parameters.
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- We were able to predict ash footprint with good confidence even though source uncertainty is large.
  - *False positives are large.*
  - Predicted average mass loading is of *similar magnitude* assuming a 1-km thick down wind plume as suggested by CALIPSO data.
  - Convergence in concentration value is slow.
- The PCQ approach uses `BENT` and `PUFF` as black-box models
  - Any other models for ash dispersion can be included in the uncertainty analysis.
- Effect of uncertainty in wind data still needs to be analyzed.
  - Uncertainty Marriage
- Question arise: *Can we estimate the distribution of source parameters using Satellite imagery or other sensory data?*
  - Likelihood functions: validating the sensor data.
  - *uniqueness of the parameters.*

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Concluding Remarks

What needs to be done

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