# The Conjugate Unscented Transform

Nagavenkat Adurthi

Department of Mechanical & Aerospace Engineering
University at Buffalo

October 12, 2011

### TO EVALUATE THE INTEGRAL

$$E[f(x)] = \int f(x)N(x,\mu|P)dx$$

**An Experimental Investigation**

- We would like to compute Multi-Dimensional Expectation Integrals.
- Analytical expressions for these multi-dimension integrals exist only for linear systems and only for few moments
- the well celebrated Kalman filter provides the analytical expressions for mean and covariance of linear system subject to Gaussian white noise and Gaussian initial condition errors
- one often do not have direct analytical solution for these integrals and have to approximate integral values by making use of computational methods.
- Monte Carlo (MC) methods, Gaussian Quadrature Rule, Unscented Transformation (UT) and Cubature methods.

- We would like to compute Multi-Dimensional Expectation Integrals.
- Analytical expressions for these multi-dimension integrals exist only for linear systems and only for few moments
- the well celebrated Kalman filter provides the analytical expressions for mean and covariance of linear system subject to Gaussian white noise and Gaussian initial condition errors
- one often do not have direct analytical solution for these integrals and have to approximate integral values by making use of computational methods.
- Monte Carlo (MC) methods, Gaussian Quadrature Rule, Unscented Transformation (UT) and Cubature methods.

- We would like to compute Multi-Dimensional Expectation Integrals.
- Analytical expressions for these multi-dimension integrals exist only for linear systems and only for few moments
- the well celebrated Kalman filter provides the analytical expressions for mean and covariance of linear system subject to Gaussian white noise and Gaussian initial condition errors
- one often do not have direct analytical solution for these integrals and have to approximate integral values by making use of computational methods.
- Monte Carlo (MC) methods, Gaussian Quadrature Rule, Unscented Transformation (UT) and Cubature methods.

- We would like to compute Multi-Dimensional Expectation Integrals.
- Analytical expressions for these multi-dimension integrals exist only for linear systems and only for few moments
- the well celebrated Kalman filter provides the analytical expressions for mean and covariance of linear system subject to Gaussian white noise and Gaussian initial condition errors
- one often do not have direct analytical solution for these integrals and have to approximate integral values by making use of computational methods.
- Monte Carlo (MC) methods, Gaussian Quadrature Rule, Unscented Transformation (UT) and Cubature methods.

- We would like to compute Multi-Dimensional Expectation Integrals.
- Analytical expressions for these multi-dimension integrals exist only for linear systems and only for few moments
- the well celebrated Kalman filter provides the analytical expressions for mean and covariance of linear system subject to Gaussian white noise and Gaussian initial condition errors
- one often do not have direct analytical solution for these integrals and have to approximate integral values by making use of computational methods.
- Monte Carlo (MC) methods, Gaussian Quadrature Rule, Unscented Transformation (UT) and Cubature methods.

- These methods basically differ from each other in the generation of these specific points
- MC methods involves random samples from the specified pdf while Gaussian quadrature scheme involves deterministic points
- Both deterministic Gaussian quadrature and MC methods are very popular but both methods require extensive computational resources and effort
- For one-dimensional integrals, one needs $m$ quadrature points according to the Gaussian quadrature scheme to reproduce the expectation integrals involving $2m-1$ degree polynomial functions.
- for a generic $N$-dimensional system, one needs to take the tensor product of $1-$dimensional $m$ quadrature points and hence one would require a total of $m^N$ quadrature points (also known as cubature points)

- These methods basically differ from each other in the generation of these specific points
- MC methods involves random samples from the specified pdf while Gaussian quadrature scheme involves deterministic points
- Both deterministic Gaussian quadrature and MC methods are very popular but both methods require extensive computational resources and effort
- For one-dimensional integrals, one needs $m$ quadrature points according to the Gaussian quadrature scheme to reproduce the expectation integrals involving $2m-1$ degree polynomial functions.
- for a generic $N$-dimensional system, one needs to take the tensor product of $1-$dimensional $m$ quadrature points and hence one would require a total of $m^N$ quadrature points (also known as cubature points)

- These methods basically differ from each other in the generation of these specific points
- MC methods involves random samples from the specified pdf while Gaussian quadrature scheme involves deterministic points
- Both deterministic Gaussian quadrature and MC methods are very popular but both methods require extensive computational resources and effort
- For one-dimensional integrals, one needs $m$ quadrature points according to the Gaussian quadrature scheme to reproduce the expectation integrals involving $2m-1$ degree polynomial functions.
- for a generic $N$-dimensional system, one needs to take the tensor product of $1-$dimensional $m$ quadrature points and hence one would require a total of $m^N$ quadrature points (also known as cubature points)

- These methods basically differ from each other in the generation of these specific points
- MC methods involves random samples from the specified pdf while Gaussian quadrature scheme involves deterministic points
- Both deterministic Gaussian quadrature and MC methods are very popular but both methods require extensive computational resources and effort
- For one-dimensional integrals, one needs $m$ quadrature points according to the Gaussian quadrature scheme to reproduce the expectation integrals involving $2m-1$ degree polynomial functions.
- for a generic $N$-dimensional system, one needs to take the tensor product of $1-$dimensional $m$ quadrature points and hence one would require a total of $m^N$ quadrature points (also known as cubature points)

- These methods basically differ from each other in the generation of these specific points
- MC methods involves random samples from the specified pdf while Gaussian quadrature scheme involves deterministic points
- Both deterministic Gaussian quadrature and MC methods are very popular but both methods require extensive computational resources and effort
- For one-dimensional integrals, one needs $m$ quadrature points according to the Gaussian quadrature scheme to reproduce the expectation integrals involving $2m-1$ degree polynomial functions.
- for a generic $N$-dimensional system, one needs to take the tensor product of $1-$dimensional $m$ quadrature points and hence one would require a total of $m^N$ quadrature points (also known as cubature points)

- Even for a moderate dimension system involving 6 random variables, the number of points required to evaluate the expectation integral with only 5 points along each direction is $5^6 = 15,625$
- But fortunately the Gaussian quadrature rule is *not minimal* for $N \geq 2$ and there exists cubature rules with reduced number of points
- This forms the basis of our motivation to the work presented in this paper.

- Even for a moderate dimension system involving 6 random variables, the number of points required to evaluate the expectation integral with only 5 points along each direction is $5^6 = 15,625$
- But fortunately the Gaussian quadrature rule is *not minimal* for $N \geq 2$ and there exists cubature rules with reduced number of points
- This forms the basis of our motivation to the work presented in this paper.

- Even for a moderate dimension system involving 6 random variables, the number of points required to evaluate the expectation integral with only 5 points along each direction is $5^6 = 15,625$
- But fortunately the Gaussian quadrature rule is *not minimal* for $N \geq 2$ and there exists cubature rules with reduced number of points
- This forms the basis of our motivation to the work presented in this paper.

Consider a discrete dynamic system with noise

$$x_{k+1} = f(x_k, k) + v_k$$

The PDF of this dynamic system propagates according to the **Chapman Kolmogorov Equation**.

$$P(x_{k+1}) = \int P(x_{k+1}|x_k)P(x_k)dx_k$$

# Motivation - Nonlinear Filtering: From the basics

Consider a discrete dynamic system with noise

$$x_{k+1} = f(x_k, k) + v_k$$

The PDF of this dynamic system propagates according to the **Chapman Kolmogorov Equation**.

$$P(x_{k+1}) = \int P(x_{k+1}|x_k)P(x_k)dx_k$$

Consider a discrete dynamic system with noise

$$x_{k+1} = f(x_k, k) + v_k$$

The PDF of this dynamic system propagates according to the **Chapman Kolmogorov Equation**.

$$P(x_{k+1}) = \int P(x_{k+1}|x_k)P(x_k)dx_k$$

Consider a discrete dynamic system with noise

$$x_{k+1} = f(x_k, k) + \nu_k$$

The PDF of this dynamic system propagates according to the **Chapman Kolmogorov Equation**.

$$P(x_{k+1}) = \int P(x_{k+1}|x_k)P(x_k)dx_k$$

- $P(x_k)$ in the CKE need not be gaussian at all times even though the initial condition was gaussian.
- It would be gaussian at all time only when system is linear and the noise is also gaussian. In this case it is very easy to **solve this equation analytically**.
- Incase the the system in nonlinear and noise is gaussian, the CKE would be

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

- $P(x_k)$ in the CKE need not be gaussian at all times even though the initial condition was gaussian.
- It would be gaussian at all time only when system is linear and the noise is also gaussian. In this case it is very easy to **solve this equation analytically**.
- Incase the the system in nonlinear and noise is gaussian, the CKE would be

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

- $P(x_k)$ in the CKE need not be gaussian at all times even though the initial condition was gaussian.
- It would be gaussian at all time only when system is linear and the noise is also gaussian. In this case it is very easy to **solve this equation analytically**.
- Incase the the system in nonlinear and noise is gaussian, the CKE would be

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

- $P(x_k)$ in the CKE need not be gaussian at all times even though the initial condition was gaussian.
- It would be gaussian at all time only when system is linear and the noise is also gaussian. In this case it is very easy to **solve this equation analytically**.
- Incase the the system in nonlinear and noise is gaussian, the CKE would be

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

- $P(x_k)$ is not always gaussian and there is no analytical solution to this equation.
- Hence the EKF emerged that can provide an **analytical solution** to this CKE by considering the following **two approximations**
- $P(x_k)$ is replaced by an equivalent gaussian PDF that has the same first two moments as the original PDF $P(x_k)$.
- $f(x_k, k)$ is linearized.

- $P(x_k)$ is not always gaussian and there is no analytical solution to this equation.
- Hence the EKF emerged that can provide an **analytical solution** to this CKE by considering the following **two approximations**
- $P(x_k)$ is replaced by an equivalent gaussian PDF that has the same first two moments as the original PDF $P(x_k)$.
- $f(x_k, k)$ is linearized.

- $P(x_k)$ is not always gaussian and there is no analytical solution to this equation.
- Hence the EKF emerged that can provide an **analytical solution** to this CKE by considering the following **two approximations**
- $P(x_k)$ is replaced by an equivalent gaussian PDF that has the same first two moments as the original PDF $P(x_k)$.
- $f(x_k, k)$ is linearized.

- $P(x_k)$ is not always gaussian and there is no analytical solution to this equation.
- Hence the EKF emerged that can provide an **analytical solution** to this CKE by considering the following **two approximations**
- $P(x_k)$ is replaced by an equivalent gaussian PDF that has the same first two moments as the original PDF $P(x_k)$.
- $f(x_k, k)$ is linearized.

- The EKF works well for systems in which the linearized dynamics is a good approximation to the nonlinear system- **Higher order terms in Taylor series are negligible**.
- If the nonlinearity is too strong the EKF would diverge.
- Above that during the linearization process the computation of the jacobian is **computationally expensive**.
- The EKF **disregards the actual state PDF** and propagates only the first two moments of the state PDF. The linearized dynamics are used in the propagation of the first two moments.

- The EKF works well for systems in which the linearized dynamics is a good approximation to the nonlinear system- **Higher order terms in Taylor series are negligible**.
- If the nonlinearity is too strong the EKF would diverge.
- Above that during the linearization process the computation of the jacobian is **computationally expensive**.
- The EKF **disregards the actual state PDF** and propagates only the first two moments of the state PDF. The linearized dynamics are used in the propagation of the first two moments.

- The EKF works well for systems in which the linearized dynamics is a good approximation to the nonlinear system- **Higher order terms in Taylor series are negligible**.
- If the nonlinearity is too strong the EKF would diverge.
- Above that during the linearization process the computation of the jacobian is **computationally expensive**.
- The EKF **disregards the actual state PDF** and propagates only the first two moments of the state PDF. The linearized dynamics are used in the propagation of the first two moments.

- The EKF works well for systems in which the linearized dynamics is a good approximation to the nonlinear system- **Higher order terms in Taylor series are negligible**.
- If the nonlinearity is too strong the EKF would diverge.
- Above that during the linearization process the computation of the jacobian is **computationally expensive**.
- The EKF **disregards the actual state PDF** and propagates only the first two moments of the state PDF. The linearized dynamics are used in the propagation of the first two moments.

# EKF

Propagation of mean

$$\mu_{k+1} = f(\mu_k)$$

Propagation of Covariance

$$P_{k+1} = AP_kA^T + Q_k$$

Where A is the jacobian of the system.

$$A = \frac{\partial f}{\partial x}|_{\mu_k}$$

Propagation of mean

$$\mu_{k+1} = f(\mu_k)$$

Propagation of Covariance

$$P_{k+1} = AP_kA^T + Q_k$$

Where A is the jacobian of the system.

$$A = \frac{\partial f}{\partial x}|_{\mu_k}$$

Propagation of mean

$$\mu_{k+1} = f(\mu_k)$$

Propagation of Covariance

$$P_{k+1} = AP_kA^T + Q_k$$

Where A is the jacobian of the system.

$$A = \frac{\partial f}{\partial x}\Big|_{\mu_k}$$

Propagation of mean

$$\mu_{k+1} = f(\mu_k)$$

Propagation of Covariance

$$P_{k+1} = AP_kA^T + Q_k$$

Where A is the jacobian of the system.

$$A = \frac{\partial f}{\partial x}|_{\mu_k}$$

- The LRKF also uses linearized dynamics but the linearization is not done using the taylor series (hence evaluating jacobian) but by a method called **statistical linearization**.
- Firstly sample points are chosen about the current mean at time $k$ such that the mean of the samples and covariance of the samples **match** the current mean and current covariance.

- The LRKF also uses linearized dynamics but the linearization is not done using the taylor series (hence evaluating jacobian) but by a method called **statistical linearization**.
- Firstly sample points are chosen about the current mean at time $k$ such that the mean of the samples and covariance of the samples **match** the current mean and current covariance.

- Each point is propagated using the **nonlinear dynamics** of the system to time step $k+1$.
- Now between the current sample points at time $k$ and the corresponding propagated points at time $k+1$ a **linear model is fit** which gives rise to the linearized dynamics of the system at time $k$.
- Now this linearized dynamics is used to compute the mean and covariance at time step $k+1$.

- Each point is propagated using the **nonlinear dynamics** of the system to time step $k+1$.
- Now between the current sample points at time $k$ and the corresponding propagated points at time $k+1$ a **linear model is fit** which gives rise to the linearized dynamics of the system at time $k$.
- Now this linearized dynamics is used to compute the mean and covariance at time step $k+1$.

# LRKF

- Each point is propagated using the **nonlinear dynamics** of the system to time step $k+1$.
- Now between the current sample points at time $k$ and the corresponding propagated points at time $k+1$ a **linear model is fit** which gives rise to the linearized dynamics of the system at time $k$.
- Now this linearized dynamics is used to compute the mean and covariance at time step $k+1$.

# LRKF

The sample points at time $k$ are chosen such that

$$\mu_k = \frac{1}{n}\sum_{i=1}^{N} X^i$$

$$P_k = \frac{1}{n}\sum_{i=1}^{N} (X^i - \mu_k)(X^i - \mu_k)^T$$

And now each sample point is individually propagated using nonlinear dynamics

$$Y^i = f(X^i)$$

The sample points at time $k$ are chosen such that

$$\mu_k = \frac{1}{n}\sum_{i=1}^{N} X^i$$

$$P_k = \frac{1}{n}\sum_{i=1}^{N} (X^i - \mu_k)(X^i - \mu_k)^T$$

And now each sample point is individually propagated using nonlinear dynamics

$$Y^i = f(X^i)$$

The sample points at time $k$ are chosen such that

$$\mu_k = \frac{1}{n}\sum_{i=1}^{N} X^i$$

$$P_k = \frac{1}{n}\sum_{i=1}^{N}(X^i - \mu_k)(X^i - \mu_k)^T$$

And now each sample point is individually propagated using nonlinear dynamics

$$Y^i = f(X^i)$$

The sample points at time $k$ are chosen such that

$$\mu_k = \frac{1}{n}\sum_{i=1}^{N} X^i$$

$$P_k = \frac{1}{n}\sum_{i=1}^{N} (X^i - \mu_k)(X^i - \mu_k)^T$$

And now each sample point is individually propagated using nonlinear dynamics

$$Y^i = f(X^i)$$

The sample points at time $k$ are chosen such that

$$\mu_k = \frac{1}{n}\sum_{i=1}^{N} X^i$$

$$P_k = \frac{1}{n}\sum_{i=1}^{N} (X^i - \mu_k)(X^i - \mu_k)^T$$

And now each sample point is individually propagated using nonlinear dynamics

$$Y^i = f(X^i)$$

Now trying to fit a linear model between the points $(Y^i, X^i)$

$$Y = AX + B$$

This is standard linear fit procedure by minimizing the least square error

$$e_i = Y^i - AX^i - B$$
$$E = (e_i)^T (e_i)$$

The mean and covariance at time $k+1$ can be found from the Kalman filter propagation equations

$$\mu_{k+1} = A\mu_k$$
$$P_{k+1} = AP_kA^T + Q_k$$

Now trying to fit a linear model between the points $(Y^i, X^i)$

$$Y = AX + B$$

This is standard linear fit procedure by minimizing the least square error

$$e_i = Y^i - AX^i - B$$
$$E = (e_i)^T (e_i)$$

The mean and covariance at time $k+1$ can be found from the Kalman filter propagation equations

$$\mu_{k+1} = A\mu_k$$
$$P_{k+1} = AP_kA^T + Q_k$$

Now trying to fit a linear model between the points $(Y^i, X^i)$

$$Y = AX + B$$

This is standard linear fit procedure by minimizing the least square error

$$\begin{aligned} e_i &= Y^i - AX^i - B \\ E &= (e_i)^T (e_i) \end{aligned}$$

The mean and covariance at time $k+1$ can be found from the Kalman filter propagation equations

$$\begin{aligned} \mu_{k+1} &= A\mu_k \\ P_{k+1} &= AP_kA^T + Q_k \end{aligned}$$

Now trying to fit a linear model between the points $(Y^i, X^i)$

$$Y = AX + B$$

This is standard linear fit procedure by minimizing the least square error

$$e_i = Y^i - AX^i - B$$
$$E = (e_i)^T (e_i)$$

The mean and covariance at time $k+1$ can be found from the Kalman filter propagation equations

$$\mu_{k+1} = A\mu_k$$
$$P_{k+1} = AP_kA^T + Q_k$$

Now trying to fit a linear model between the points $(Y^i, X^i)$

$$Y = AX + B$$

This is standard linear fit procedure by minimizing the least square error

$$e_i = Y^i - AX^i - B$$
$$E = (e_i)^T (e_i)$$

The mean and covariance at time $k+1$ can be found from the Kalman filter propagation equations

$$\mu_{k+1} = A\mu_k$$
$$P_{k+1} = AP_kA^T + Q_k$$

# LRKF

Now trying to fit a linear model between the points $(Y^i, X^i)$

$$Y = AX + B$$

This is standard linear fit procedure by minimizing the least square error

$$\begin{aligned} e_i &= Y^i - AX^i - B \\ E &= (e_i)^T (e_i) \end{aligned}$$

The mean and covariance at time $k+1$ can be found from the Kalman filter propagation equations

$$\begin{aligned} \mu_{k+1} &= A\mu_k \\ P_{k+1} &= AP_kA^T + Q_k \end{aligned}$$

Now trying to fit a linear model between the points $(Y^i, X^i)$

$$Y = AX + B$$

This is standard linear fit procedure by minimizing the least square error

$$e_i = Y^i - AX^i - B$$
$$E = (e_i)^T (e_i)$$

The mean and covariance at time $k+1$ can be found from the Kalman filter propagation equations

$$\mu_{k+1} = A\mu_k$$
$$P_{k+1} = AP_kA^T + Q_k$$

Now trying to fit a linear model between the points $(Y^i, X^i)$

$$Y = AX + B$$

This is standard linear fit procedure by minimizing the least square error

$$e_i = Y^i - AX^i - B$$
$$E = (e_i)^T (e_i)$$

The mean and covariance at time $k+1$ can be found from the Kalman filter propagation equations

$$\mu_{k+1} = A\mu_k$$
$$P_{k+1} = AP_kA^T + Q_k$$

- The UKF works in the same ways as the LRKF but the samples are chosen in a **determined** way such that they always match the mean and covariance or higher moments at the current step.
- The points are propagated using the **nonlinear dynamics**. The mean and covariance at time step $k+1$ are calculated from these propagated points.
- The **first approximation** the UKF does to the CKE is that it replaces the current state PDF with a gaussian PDF with same first two moments.

- The UKF works in the same ways as the LRKF but the samples are chosen in a **determined** way such that they always match the mean and covariance or higher moments at the current step.
- The points are propagated using the **nonlinear dynamics**. The mean and covariance at time step $k+1$ are calculated from these propagated points.
- The **first approximation** the UKF does to the CKE is that it replaces the current state PDF with a gaussian PDF with same first two moments.

- The UKF works in the same ways as the LRKF but the samples are chosen in a **determined** way such that they always match the mean and covariance or higher moments at the current step.
- The points are propagated using the **nonlinear dynamics**. The mean and covariance at time step $k+1$ are calculated from these propagated points.
- The **first approximation** the UKF does to the CKE is that it replaces the current state PDF with a gaussian PDF with same first two moments.

# UKF

- There is **no linearizations involved**, instead the integral is itself evaluated approximately using quadrature points of the gaussian weighting function-**second approximation**.
- The UKF like the EKF only keeps track of the first two moments of the state PDF.
- We can still improve the **second approximation** by better evaluating the integrals.

# UKF

- There is **no linearizations involved**, instead the integral is itself evaluated approximately using quadrature points of the gaussian weighting function-**second approximation**.
- The UKF like the EKF only keeps track of the first two moments of the state PDF.
- We can still improve the **second approximation** by better evaluating the integrals.

- There is **no linearizations involved**, instead the integral is itself evaluated approximately using quadrature points of the gaussian weighting function-**second approximation**.
- The UKF like the EKF only keeps track of the first two moments of the state PDF.
- We can still improve the **second approximation** by better evaluating the integrals.

One could look at the UKF in the following perspective, starting from CKE

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

Replacing the state PDF with gaussian pdf with equivvalent mean $\mu_k$ and covariance $P_k$.

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)N(x_k)dx_k$$

Calculating the mean of $P(x_{k+1}$ by integrating on both side wrt $x_{k+1}$.

$$\begin{aligned}\mu_{k+1} &= \int \int N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k \\ &= \int f(x_k)N(x_k)dx_k\end{aligned}$$

# UKF

One could look at the UKF in the following perspective, starting from CKE

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

Replacing the state PDF with gaussian pdf with equivvalent mean $\mu_k$ and covariance $P_k$.

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)N(x_k)dx_k$$

Calculating the mean of $P(x_{k+1}$ by integrating on both side wrt $x_{k+1}$.

$$\begin{aligned}\mu_{k+1} &= \int\int N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k \\ &= \int f(x_k)N(x_k)dx_k\end{aligned}$$

# UKF

One could look at the UKF in the following perspective, starting from CKE

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

Replacing the state PDF with gaussian pdf with equivvalent mean $\mu_k$ and covariance $P_k$.

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)N(x_k)dx_k$$

Calculating the mean of $P(x_{k+1}$ by integrating on both side wrt $x_{k+1}$.

$$\begin{aligned}\mu_{k+1} &= \int\int N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k \\ &= \int f(x_k)N(x_k)dx_k\end{aligned}$$

# UKF

One could look at the UKF in the following perspective, starting from CKE

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

Replacing the state PDF with gaussian pdf with equivvalent mean $\mu_k$ and covariance $P_k$.

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)N(x_k)dx_k$$

Calculating the mean of $P(x_{k+1}$ by integrating on both side wrt $x_{k+1}$.

$$\begin{aligned}
\mu_{k+1} &= \int\int N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k \\
&= \int f(x_k)N(x_k)dx_k
\end{aligned}$$

# UKF

One could look at the UKF in the following perspective, starting from CKE

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

Replacing the state PDF with gaussian pdf with equivvalent mean $\mu_k$ and covariance $P_k$.

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)N(x_k)dx_k$$

Calculating the mean of $P(x_{k+1}$ by integrating on both side wrt $x_{k+1}$.

$$\begin{aligned}\mu_{k+1} &= \int\int N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k \\ &= \int f(x_k)N(x_k)dx_k\end{aligned}$$

# UKF

One could look at the UKF in the following perspective, starting from CKE

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

Replacing the state PDF with gaussian pdf with equivvalent mean $\mu_k$ and covariance $P_k$.

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)N(x_k)dx_k$$

Calculating the mean of $P(x_{k+1}$ by integrating on both side wrt $x_{k+1}$.

$$\begin{aligned}\mu_{k+1} &= \int\int N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k \\ &= \int f(x_k)N(x_k)dx_k\end{aligned}$$

One could look at the UKF in the following perspective, starting from CKE

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

Replacing the state PDF with gaussian pdf with equivvalent mean $\mu_k$ and covariance $P_k$.

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)N(x_k)dx_k$$

Calculating the mean of $P(x_{k+1}$ by integrating on both side wrt $x_{k+1}$.

$$\begin{aligned}\mu_{k+1} &= \int\int N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k \\ &= \int f(x_k)N(x_k)dx_k\end{aligned}$$

The covariance can be calculated from the second raw moment

$$
\begin{aligned}
E[x_{k+1}x_{k+1}^T] &= \int\int x_{k+1}x_{k+1}^T N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k \\
&= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k
\end{aligned}
$$

By parallel axis theorem for moments the covariance is calculated as

$$
P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T
$$

- Thus by evaluating two integrals we get the mean and covariance.
- The second raw moment and the mean need to be evaluated accurate enough or else the parallel axis theorem for moments might render the **covariance to be positive semi definite**.

The covariance can be calculated from the second raw moment

$$E[x_{k+1}x_{k+1}^T] = \int\int x_{k+1}x_{k+1}^T N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k$$
$$= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k$$

By parallel axis theorem for moments the covariance is calculated as

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

- Thus by evaluating two integrals we get the mean and covariance.
- The second raw moment and the mean need to be evaluated accurate enough or else the parallel axis theorem for moments might render the **covariance to be positive semi definite**.

# UKF

The covariance can be calculated from the second raw moment

$$
\begin{aligned}
E[x_{k+1}x_{k+1}^T] &= \int\int x_{k+1}x_{k+1}^T N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k \\
&= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k
\end{aligned}
$$

By parallel axis theorem for moments the covariance is calculated as

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

- Thus by evaluating two integrals we get the mean and covariance.
- The second raw moment and the mean need to be evaluated accurate enough or else the parallel axis theorem for moments might render the **covariance to be positive semi definite**.

# UKF

The covariance can be calculated from the second raw moment

$$
\begin{aligned}
E[x_{k+1}x_{k+1}^T] &= \int\int x_{k+1}x_{k+1}^T N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k \\
&= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k
\end{aligned}
$$

By parallel axis theorem for moments the covariance is calculated as

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

- Thus by evaluating two integrals we get the mean and covariance.
- The second raw moment and the mean need to be evaluated accurate enough or else the parallel axis theorem for moments might render the **covariance to be positive semi definite**.

The covariance can be calculated from the second raw moment

$$E[x_{k+1}x_{k+1}^T] = \int\int x_{k+1}x_{k+1}^T N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k$$
$$= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k$$

By parallel axis theorem for moments the covariance is calculated as

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

- Thus by evaluating two integrals we get the mean and covariance.
- The second raw moment and the mean need to be evaluated accurate enough or else the parallel axis theorem for moments might render the **covariance to be positive semi definite**.

The covariance can be calculated from the second raw moment

$$E[x_{k+1}x_{k+1}^T] = \int\int x_{k+1}x_{k+1}^T N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k$$
$$= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k$$

By parallel axis theorem for moments the covariance is calculated as

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

- Thus by evaluating two integrals we get the mean and covariance.
- The second raw moment and the mean need to be evaluated accurate enough or else the parallel axis theorem for moments might render the **covariance to be positive semi definite**.

The covariance can be calculated from the second raw moment

$$E[x_{k+1}x_{k+1}^T] = \int\int x_{k+1}x_{k+1}^T N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k$$
$$= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k$$

By parallel axis theorem for moments the covariance is calculated as

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

- Thus by evaluating two integrals we get the mean and covariance.
- The second raw moment and the mean need to be evaluated accurate enough or else the parallel axis theorem for moments might render the **covariance to be positive semi definite**.

In terms of gaussian quadrature, the integrals would be

$$\begin{aligned}
\mu_{k+1} &= \int f(x_k)N(x_k)dx_k \\
&= \sum_{i=1}^{N} w^i f(x_k^i) \\
E[x_{k+1}x_{k+1}^T] &= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k \\
&= \sum_{i=1}^{N} w^i f(x_k^i)f(x_k^i)^T + Q_k \sum_{i=1}^{N} w^i \\
&= \sum_{i=1}^{N} w^i f(x_k^i)f(x_k^i)^T + Q_k
\end{aligned}$$

In terms of gaussian quadrature, the integrals would be

$$\begin{aligned}
\mu_{k+1} &= \int f(x_k) N(x_k) dx_k \\
&= \sum_{i=1}^{N} w^i f(x_k^i) \\
E[x_{k+1} x_{k+1}^T] &= \int (f(x_k) f(x_k)^T + Q_k) N(x_k) dx_k \\
&= \sum_{i=1}^{N} w^i f(x_k^i) f(x_k^i)^T + Q_k \sum_{i=1}^{N} w^i \\
&= \sum_{i=1}^{N} w^i f(x_k^i) f(x_k^i)^T + Q_k
\end{aligned}$$

In terms of gaussian quadrature, the integrals would be

$$\begin{aligned}
\mu_{k+1} &= \int f(x_k)N(x_k)dx_k \\
&= \sum_{i=1}^{N} w^i f(x_k^i) \\
E\left[x_{k+1}x_{k+1}^T\right] &= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k \\
&= \sum_{i=1}^{N} w^i f(x_k^i)f(x_k^i)^T + Q_k \sum_{i=1}^{N} w^i \\
&= \sum_{i=1}^{N} w^i f(x_k^i)f(x_k^i)^T + Q_k
\end{aligned}$$

In terms of gaussian quadrature, the integrals would be

$$
\begin{aligned}
\mu_{k+1} &= \int f(x_k)N(x_k)dx_k \\
&= \sum_{i=1}^{N} w^i f(x_k^i) \\
E[x_{k+1}x_{k+1}^T] &= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k \\
&= \sum_{i=1}^{N} w^i f(x_k^i)f(x_k^i)^T + Q_k \sum_{i=1}^{N} w^i \\
&= \sum_{i=1}^{N} w^i f(x_k^i)f(x_k^i)^T + Q_k
\end{aligned}
$$

# UKF

In terms of gaussian quadrature, the integrals would be

$$
\begin{aligned}
\mu_{k+1} &= \int f(x_k) N(x_k) dx_k \\
&= \sum_{i=1}^{N} w^i f(x_k^i) \\
E[x_{k+1} x_{k+1}^T] &= \int (f(x_k) f(x_k)^T + Q_k) N(x_k) dx_k \\
&= \sum_{i=1}^{N} w^i f(x_k^i) f(x_k^i)^T + Q_k \sum_{i=1}^{N} w^i \\
&= \sum_{i=1}^{N} w^i f(x_k^i) f(x_k^i)^T + Q_k
\end{aligned}
$$

In terms of gaussian quadrature, the integrals would be

$$\begin{aligned}
\mu_{k+1} &= \int f(x_k)N(x_k)dx_k \\
&= \sum_{i=1}^{N} w^i f(x_k^i) \\
E[x_{k+1}x_{k+1}^T] &= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k \\
&= \sum_{i=1}^{N} w^i f(x_k^i)f(x_k^i)^T + Q_k \sum_{i=1}^{N} w^i \\
&= \sum_{i=1}^{N} w^i f(x_k^i)f(x_k^i)^T + Q_k
\end{aligned}$$

The covariance is

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

After some simplification we could write it in a more general way as

$$P_k = \sum_{i=1}^{N} w^i [f(x_k^i) - \mu_{k+1}][f(x_k^i) - \mu_{k+1}]^T + Q_k$$

Thus the UKF boils down to just evaluating the integrals involving gaussian kernel using the quadrature points.

The covariance is

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

After some simplification we could write it in a more general way as

$$P_k = \sum_{i=1}^{N} w^i [f(x_k^i) - \mu_{k+1}][f(x_k^i) - \mu_{k+1}]^T + Q_k$$

Thus the UKF boils down to just evaluating the integrals involving gaussian kernel using the quadrature points.

The covariance is

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

After some simplification we could write it in a more general way as

$$P_k = \sum_{i=1}^{N} w^i [f(x_k^i) - \mu_{k+1}][f(x_k^i) - \mu_{k+1}]^T + Q_k$$

Thus the UKF boils down to just evaluating the integrals involving gaussian kernel using the quadrature points.

The covariance is

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

After some simplification we could write it in a more general way as

$$P_k = \sum_{i=1}^{N} w^i [f(x_k^i) - \mu_{k+1}][f(x_k^i) - \mu_{k+1}]^T + Q_k$$

Thus the UKF boils down to just evaluating the integrals involving gaussian kernel using the quadrature points.

The covariance is

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

After some simplification we could write it in a more general way as

$$P_k = \sum_{i=1}^{N} w^i [f(x_k^i) - \mu_{k+1}][f(x_k^i) - \mu_{k+1}]^T + Q_k$$

Thus the UKF boils down to just evaluating the integrals involving gaussian kernel using the quadrature points.

We try to propose a **new method of Gaussian Cubature** to evaluate the integrals and hence may be a potential application as a **new filter**. Later we show that the UKF and CKF are inline with the present construction. Basic Philosophy of CUT:

- The basic philosophy in this analysis is "‘**To evaluate the integral involving gaussian weight function with as few points as possible**".
- Any Gaussian Quadrature rule has to capture the moments of the continuous PDF
- For example consider the Gauss-Hermite quadrature, as we increase the number of quadrature points more moments are captured and hence higher degree polynomials can be integrated.

We try to propose a **new method of Gaussian Cubature** to evaluate the integrals and hence may be a potential application as a **new filter**. Later we show that the UKF and CKF are inline with the present construction. Basic Philosophy of CUT:

- The basic philosophy in this analysis is "'**To evaluate the integral involving gaussian weight function with as few points as possible**".
- Any Gaussian Quadrature rule has to capture the moments of the continuous PDF
- For example consider the Gauss-Hermite quadrature, as we increase the number of quadrature points more moments are captured and hence higher degree polynomials can be integrated.

We try to propose a **new method of Gaussian Cubature** to evaluate the integrals and hence may be a potential application as a **new filter**. Later we show that the UKF and CKF are inline with the present construction. Basic Philosophy of CUT:

- The basic philosophy in this analysis is "‘**To evaluate the integral involving gaussian weight function with as few points as possible**".
- Any Gaussian Quadrature rule has to capture the moments of the continuous PDF
- For example consider the Gauss-Hermite quadrature, as we increase the number of quadrature points more moments are captured and hence higher degree polynomials can be integrated.

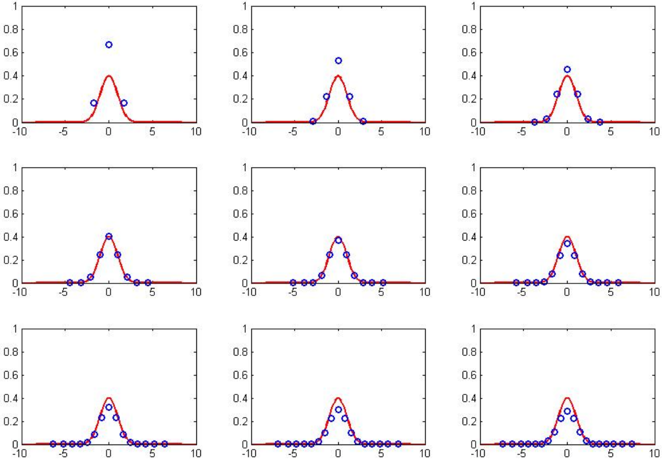



FIGURE: Guass hermite quadrature for 1D

- In general for a N-Dimensional system, to integrate a polynomial of degree$2m+1$ **we need a total of** $(m+1)^N$ **quadrature points**.
- This is a **very big number** for higher dimensional system and **this is the basis of our motivation to develop a method with reduced number of points**.
- Ideally one would like to capture all the **infinite** moments of the PDF.
- In practice this is difficult to achieve or might be computationally expensive. Thus often only the lower order moments are captured.This highly limits the type of funtions that can be integrated with good numerical accuracy.

The 1D Gauss Hermite rule can be extended to any dimension for an i.i.d set of random variable $(x_1, x_2, ..x_N)^T$.

$$E[f(x_1, x_2, ...x_N)]$$

$$= \int \int ... \int f(x_1, x_2, ...x_N) N(x_1, x_2, ...x_N, 0|I) dx_1 dx_2 ... dx_N$$
$$= \int \int ... [\int f(x_1, x_2, ...x_N) N(x_1, 0|1) dx_1] N(x_2, 0|1) dx_2 N(x_3, 0|1)$$
$$... N(x_N, 0|1) dx_N$$

$$= \sum_{i_N=1}^{n} w_N^{i_N} \sum_{i_{N-1}=1}^{n} w_{N-1}^{i_{N-1}} ... \sum_{i_2=1}^{n} w_2^{i_2} \sum_{i_1=1}^{n} w_1^{i_1} f(x_1^{i_1}, x_2^{i_2}, ... x_N^{i_N})$$

The 1D Gauss Hermite rule can be extended to any dimension for an i.i.d set of random variable $(x_1, x_2, ..x_N)^T$.

$$E[f(x_1, x_2, ...x_N)]$$

$$= \int\int ... \int f(x_1, x_2, ...x_N) N(x_1, x_2, ...x_N, 0|I) dx_1 dx_2 ... dx_N$$
$$= \int\int ...[\int f(x_1, x_2, ...x_N) N(x_1, 0|1) dx_1] N(x_2, 0|1) dx_2 N(x_3, 0|1)$$
$$...N(x_N, 0|1) dx_N$$

$$= \sum_{i_N=1}^{n} w_N^{i_N} \sum_{i_{N-1}=1}^{n} w_{N-1}^{i_{N-1}} ... \sum_{i_2=1}^{n} w_2^{i_2} \sum_{i_1=1}^{n} w_1^{i_1} f(x_1^{i_1}, x_2^{i_2}, ...x_N^{i_N})$$

The 1D Gauss Hermite rule can be extended to any dimension for an i.i.d set of random variable $(x_1, x_2, ..x_N)^T$.

$$E[f(x_1, x_2, ...x_N)]$$

$$= \int\int ... \int f(x_1, x_2, ...x_N) N(x_1, x_2, ...x_N, 0|I) dx_1 dx_2 ... dx_N$$
$$= \int\int ...[\int f(x_1, x_2, ...x_N) N(x_1, 0|1) dx_1] N(x_2, 0|1) dx_2 N(x_3, 0|1)$$
$$...N(x_N, 0|1) dx_N$$

$$= \sum_{i_N=1}^{n} w_N^{i_N} \sum_{i_{N-1}=1}^{n} w_{N-1}^{i_{N-1}} ... \sum_{i_2=1}^{n} w_2^{i_2} \sum_{i_1=1}^{n} w_1^{i_1} f(x_1^{i_1}, x_2^{i_2}, ...x_N^{i_N})$$

The 1D Gauss Hermite rule can be extended to any dimension for an i.i.d set of random variable $(x_1, x_2, ..x_N)^T$.

$$E[f(x_1, x_2, ...x_N)]$$

$$= \int\int ... \int f(x_1, x_2, ...x_N) N(x_1, x_2, ...x_N, 0|I) dx_1 dx_2 ... dx_N$$
$$= \int\int ...[\int f(x_1, x_2, ...x_N) N(x_1, 0|1) dx_1] N(x_2, 0|1) dx_2 N(x_3, 0|1)$$
$$...N(x_N, 0|1) dx_N$$

$$= \sum_{i_N=1}^{n} w_N^{i_N} \sum_{i_{N-1}=1}^{n} w_{N-1}^{i_{N-1}} ... \sum_{i_2=1}^{n} w_2^{i_2} \sum_{i_1=1}^{n} w_1^{i_1} f(x_1^{i_1}, x_2^{i_2}, ...x_N^{i_N})$$

The 1D Gauss Hermite rule can be extended to any dimension for an i.i.d set of random variable $(x_1, x_2, ..x_N)^T$.

$$E[f(x_1, x_2, ...x_N)]$$

$$= \int\int ... \int f(x_1, x_2, ...x_N) N(x_1, x_2, ...x_N, 0|I) dx_1 dx_2 ... dx_N$$

$$= \int\int ...[\int f(x_1, x_2, ...x_N) N(x_1, 0|1) dx_1] N(x_2, 0|1) dx_2 N(x_3, 0|1)$$

$$...N(x_N, 0|1) dx_N$$

$$= \sum_{i_N=1}^{n} w_N^{i_N} \sum_{i_{N-1}=1}^{n} w_{N-1}^{i_{N-1}} ... \sum_{i_2=1}^{n} w_2^{i_2} \sum_{i_1=1}^{n} w_1^{i_1} f(x_1^{i_1}, x_2^{i_2}, ...x_N^{i_N})$$

The 1D Gauss Hermite rule can be extended to any dimension for an i.i.d set of random variable $(x_1, x_2, ..x_N)^T$.

$$E[f(x_1, x_2, \ldots x_N)]$$

$$= \int \int \ldots \int f(x_1, x_2, \ldots x_N) N(x_1, x_2, \ldots x_N, 0|I) dx_1 dx_2 \ldots dx_N$$
$$= \int \int \ldots [\int f(x_1, x_2, \ldots x_N) N(x_1, 0|1) dx_1] N(x_2, 0|1) dx_2 N(x_3, 0|1)$$
$$\ldots N(x_N, 0|1) dx_N$$

$$= \sum_{i_N=1}^{n} w_N^{i_N} \sum_{i_{N-1}=1}^{n} w_{N-1}^{i_{N-1}} \ldots \sum_{i_2=1}^{n} w_2^{i_2} \sum_{i_1=1}^{n} w_1^{i_1} f(x_1^{i_1}, x_2^{i_2}, \ldots x_N^{i_N})$$

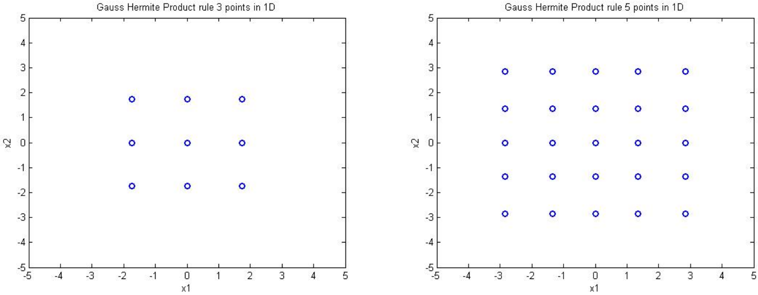



FIGURE: Guass hermite product rule for 2D

# Transformation of a Normal PDF with Arbitrary mean and Covariance into a Normal PDF with Zero Mean and Identity Covariance

- Any jointly distributed Normal PDF can be transformed into a Gaussian PDF of zero mean and identity covariance of same dimension.
- This way one could find the cubature points in the i.i.d space and then transform each point into the original jointly distributed space.

$$
\begin{aligned}
(x-\mu)^T P^{-1}(x-\mu) &= (x-\mu)^T U^T \Sigma U (x-\mu) \\
&= (x-\mu)^T U^T \sqrt{\Sigma}\sqrt{\Sigma} U (x-\mu)
\end{aligned}
$$

$$
y = \sqrt{\Sigma} U (x-\mu)
$$

$$
(x-\mu)^T P^{-1}(x-\mu) = y^T y
$$

# Transformation of a Normal PDF with Arbitrary mean and Covariance into a Normal PDF with Zero Mean and Identity Covariance

- Any jointly distributed Normal PDF can be transformed into a Gaussian PDF of zero mean and identity covariance of same dimension.
- This way one could find the cubature points in the i.i.d space and then transform each point into the original jointly distributed space.

$$
\begin{aligned}
(x-\mu)^T P^{-1}(x-\mu) &= (x-\mu)^T U^T \Sigma U (x-\mu) \\
&= (x-\mu)^T U^T \sqrt{\Sigma}\sqrt{\Sigma} U (x-\mu)
\end{aligned}
$$

$$y = \sqrt{\Sigma} U (x-\mu)$$

$$(x-\mu)^T P^{-1}(x-\mu) = y^T y$$

# Transformation of a Normal PDF with Arbitrary mean and Covariance into a Normal PDF with Zero Mean and Identity Covariance

- Any jointly distributed Normal PDF can be transformed into a Gaussian PDF of zero mean and identity covariance of same dimension.
- This way one could find the cubature points in the i.i.d space and then transform each point into the original jointly distributed space.

$$
\begin{aligned}
(x-\mu)^T P^{-1}(x-\mu) &= (x-\mu)^T U^T \Sigma U (x-\mu) \\
&= (x-\mu)^T U^T \sqrt{\Sigma}\sqrt{\Sigma} U (x-\mu)
\end{aligned}
$$

$$y = \sqrt{\Sigma} U (x-\mu)$$

$$(x-\mu)^T P^{-1}(x-\mu) = y^T y$$

# Transformation of a Normal PDF with Arbitrary mean and Covariance into a Normal PDF with Zero Mean and Identity Covariance

- Any jointly distributed Normal PDF can be transformed into a Gaussian PDF of zero mean and identity covariance of same dimension.
- This way one could find the cubature points in the i.i.d space and then transform each point into the original jointly distributed space.

$$
\begin{aligned}
(x-\mu)^T P^{-1}(x-\mu) &= (x-\mu)^T U^T \Sigma U (x-\mu) \\
&= (x-\mu)^T U^T \sqrt{\Sigma}\sqrt{\Sigma} U (x-\mu)
\end{aligned}
$$

$$y = \sqrt{\Sigma} U (x-\mu)$$

$$(x-\mu)^T P^{-1}(x-\mu) = y^T y$$

# Transformation of a Normal PDF with Arbitrary mean and Covariance into a Normal PDF with Zero Mean and Identity Covariance

- Any jointly distributed Normal PDF can be transformed into a Gaussian PDF of zero mean and identity covariance of same dimension.
- This way one could find the cubature points in the i.i.d space and then transform each point into the original jointly distributed space.

$$
\begin{aligned}
(x-\mu)^T P^{-1}(x-\mu) &= (x-\mu)^T U^T \Sigma U (x-\mu) \\
&= (x-\mu)^T U^T \sqrt{\Sigma}\sqrt{\Sigma} U (x-\mu)
\end{aligned}
$$

$$y = \sqrt{\Sigma} U (x-\mu)$$

$$(x-\mu)^T P^{-1}(x-\mu) = y^T y$$

# Transformation of a Normal PDF with Arbitrary mean and Covariance into a Normal PDF with Zero Mean and Identity Covariance

- Any jointly distributed Normal PDF can be transformed into a Gaussian PDF of zero mean and identity covariance of same dimension.
- This way one could find the cubature points in the i.i.d space and then transform each point into the original jointly distributed space.

$$
\begin{aligned}
(x-\mu)^T P^{-1}(x-\mu) &= (x-\mu)^T U^T \Sigma U (x-\mu) \\
&= (x-\mu)^T U^T \sqrt{\Sigma}\sqrt{\Sigma} U (x-\mu)
\end{aligned}
$$

$$
y = \sqrt{\Sigma} U (x-\mu)
$$

$$
(x-\mu)^T P^{-1}(x-\mu) = y^T y
$$

# Transformation of a Normal PDF with Arbitrary mean and Covariance into a Normal PDF with Zero Mean and Identity Covariance

- Any jointly distributed Normal PDF can be transformed into a Gaussian PDF of zero mean and identity covariance of same dimension.
- This way one could find the cubature points in the i.i.d space and then transform each point into the original jointly distributed space.

$$
\begin{aligned}
(x-\mu)^T P^{-1}(x-\mu) &= (x-\mu)^T U^T \Sigma U(x-\mu) \\
&= (x-\mu)^T U^T \sqrt{\Sigma}\sqrt{\Sigma} U(x-\mu)
\end{aligned}
$$

$$
\begin{aligned}
y &= \sqrt{\Sigma} U(x-\mu) \\
(x-\mu)^T P^{-1}(x-\mu) &= y^T y
\end{aligned}
$$

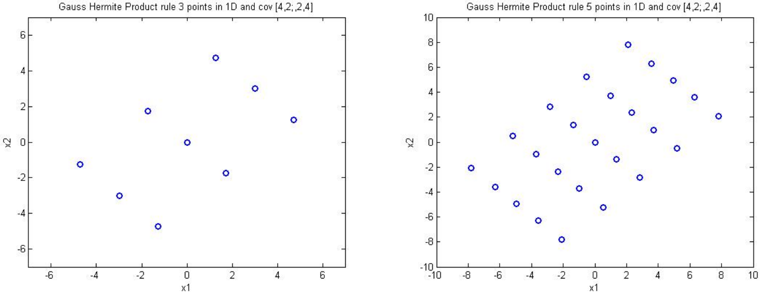



FIGURE: Guass hermite product rule for 2D for Cov $[4,2;2,4]$

**Expectation Integral**

- Let us consider the problem of computing expected value of a function $f(x)$ with respect to a Gaussian density function.
- In addition consider the mean and covarince of the Gaussian function to be zero and unity, respectively.

$$E[f(x)] = \int f(x)N(x,0|1)dx \tag{1}$$

**Expectation Integral**

- Let us consider the problem of computing expected value of a function $f(x)$ with respect to a Gaussian density function.
- In addition consider the mean and covarince of the Gaussian function to be zero and unity, respectively.

$$E[f(x)] = \int f(x)N(x,0|1)dx \tag{1}$$

**Expectation Integral**

- Let us consider the problem of computing expected value of a function $f(x)$ with respect to a Gaussian density function.
- In addition consider the mean and covarince of the Gaussian function to be zero and unity, respectively.

$$E[f(x)] = \int f(x)N(x,0|1)dx \tag{1}$$

**Expectation Integral**

- Let us consider the problem of computing expected value of a function $f(x)$ with respect to a Gaussian density function.
- In addition consider the mean and covarince of the Gaussian function to be zero and unity, respectively.

$$E[f(x)] = \int f(x)N(x,0|1)dx \tag{1}$$

$$E[f(x)] = \int f(x)N(x,0|1)dx \tag{2}$$

Taking the Taylor series expansion of $f(x)$ about the mean

$$E[f(x)] = f(0) + \nabla f(0)E[x] + \frac{1}{2!}\nabla^2 f(0)E[x^2] + \frac{1}{3!}\nabla^3 f(0)E[x^3] + \frac{1}{4!}\nabla^4 f(0)E[x^4]\ldots \tag{3}$$

- Notice that the problem of evaluating the expected value of nonlinear function $f(x)$ has reduced to computing higher order moments of $x$.
- Thus by increasing the number of terms in the Taylor series expansion, one can obtain more accurate value of the expectation integral.

$$E[f(x)] = \int f(x)N(x,0|1)dx \tag{2}$$

Taking the Taylor series expansion of $f(x)$ about the mean

$$E[f(x)] = f(0) + \nabla f(0)E[x] + \frac{1}{2!}\nabla^2 f(0)E[x^2] + \frac{1}{3!}\nabla^3 f(0)E[x^3] + \frac{1}{4!}\nabla^4 f(0)E[x^4]\ldots \tag{3}$$

- Notice that the problem of evaluating the expected value of nonlinear function $f(x)$ has reduced to computing higher order moments of $x$.
- Thus by increasing the number of terms in the Taylor series expansion, one can obtain more accurate value of the expectation integral.

$$E[f(x)] = \int f(x) N(x, 0|1) dx \tag{2}$$

Taking the Taylor series expansion of $f(x)$ about the mean

$$E[f(x)] = f(0) + \nabla f(0) E[x] + \frac{1}{2!}\nabla^2 f(0) E[x^2] + \frac{1}{3!}\nabla^3 f(0) E[x^3] + \frac{1}{4!}\nabla^4 f(0) E[x^4] \ldots \tag{3}$$

- Notice that the problem of evaluating the expected value of nonlinear function $f(x)$ has reduced to computing higher order moments of $x$.
- Thus by increasing the number of terms in the Taylor series expansion, one can obtain more accurate value of the expectation integral.

$$E[f(x)] = \int f(x)N(x,0|1)dx \tag{2}$$

Taking the Taylor series expansion of $f(x)$ about the mean

$$\begin{aligned} E[f(x)] &= f(0) + \nabla f(0)E[x] + \frac{1}{2!}\nabla^2 f(0)E[x^2] \\ &+ \frac{1}{3!}\nabla^3 f(0)E[x^3] + \frac{1}{4!}\nabla^4 f(0)E[x^4]\ldots \end{aligned} \tag{3}$$

- Notice that the problem of evaluating the expected value of nonlinear function $f(x)$ has reduced to computing higher order moments of $x$.
- Thus by increasing the number of terms in the Taylor series expansion, one can obtain more accurate value of the expectation integral.

- Now consider the discrete approximation of the expectation integral as a weighted average of $f(x)$ evaluated at each quadrature/sigma point set
- Let the sigma point set be $(x_1, x_2, ..., x_n)$ with corresponding weights $(w_1, w_2, ...w_3)$:

$$E[f(x)] \approx \sum_{i=1}^{n} w_i f(x_i) \tag{4}$$

- Now consider the discrete approximation of the expectation integral as a weighted average of $f(x)$ evaluated at each quadrature/sigma point set
- Let the sigma point set be $(x_1, x_2, ..., x_n)$ with corresponding weights $(w_1, w_2, ...w_3)$:

$$E[f(x)] \approx \sum_{i=1}^{n} w_i f(x_i) \tag{4}$$

- Now consider the discrete approximation of the expectation integral as a weighted average of $f(x)$ evaluated at each quadrature/sigma point set
- Let the sigma point set be $(x_1, x_2, ..., x_n)$ with corresponding weights $(w_1, w_2, ...w_3)$:

$$E[f(x)] \approx \sum_{i=1}^{n} w_i f(x_i) \tag{4}$$

- Now consider the discrete approximation of the expectation integral as a weighted average of $f(x)$ evaluated at each quadrature/sigma point set
- Let the sigma point set be $(x_1, x_2, ..., x_n)$ with corresponding weights $(w_1, w_2, ...w_3)$:

$$E[f(x)] \approx \sum_{i=1}^{n} w_i f(x_i) \tag{4}$$

- Now consider the discrete approximation of the expectation integral as a weighted average of $f(x)$ evaluated at each quadrature/sigma point set
- Let the sigma point set be $(x_1, x_2, ..., x_n)$ with corresponding weights $(w_1, w_2, ...w_3)$:

$$E[f(x)] \approx \sum_{i=1}^{n} w_i f(x_i) \tag{4}$$

- Now consider the discrete approximation of the expectation integral as a weighted average of $f(x)$ evaluated at each quadrature/sigma point set
- Let the sigma point set be $(x_1, x_2, ..., x_n)$ with corresponding weights $(w_1, w_2, ...w_3)$:

$$E[f(x)] \approx \sum_{i=1}^{n} w_i f(x_i) \tag{4}$$

Now, substitution of the Taylor series expansion about the mean for $f(x_i)$

$$E[f(x)] = f(0)(\sum_{i=1}^{n} w_i) + \frac{1}{2!}\nabla^2 f(0)(\sum_{i=1}^{n} w_i x_i^2)$$
$$\frac{1}{4!}\nabla^4 f(0)(\sum_{i=1}^{n} w_i x_i^4) + \dots \quad (5)$$

Now, substitution of the Taylor series expansion about the mean for $f(x_i)$

$$E[f(x)] = f(0)(\sum_{i=1}^{n} w_i) + \frac{1}{2!}\nabla^2 f(0)(\sum_{i=1}^{n} w_i x_i^2)$$
$$\frac{1}{4!}\nabla^4 f(0)(\sum_{i=1}^{n} w_i x_i^4) + \ldots \tag{5}$$

Now, substitution of the Taylor series expansion about the mean for $f(x_i)$

$$E[f(x)] = f(0)(\sum_{i=1}^{n} w_i) + \frac{1}{2!}\nabla^2 f(0)(\sum_{i=1}^{n} w_i x_i^2)$$
$$\frac{1}{4!}\nabla^4 f(0)(\sum_{i=1}^{n} w_i x_i^4) + \ldots \tag{5}$$

Comparing the coefficients of $f(0)$ and derivatives of $f(x)$ evaluated at 0 in the two equations

$$E[f(x)] = f(0) + \nabla f(0)E[x] + \frac{1}{2!}\nabla^2 f(0)E[x^2]$$
$$\frac{1}{3!}\nabla^3 f(0)E[x^3] + \frac{1}{4!}\nabla^4 f(0)E[x^4]... \tag{6}$$
$$E[f(x)] = f(0)(\sum_{i=1}^{n} w_i) + \frac{1}{2!}\nabla^2 f(0)(\sum_{i=1}^{n} w_i x_i^2)$$
$$\frac{1}{4!}\nabla^4 f(0)(\sum_{i=1}^{n} w_i x_i^4) + ... \tag{7}$$

*moment constraint equations:*

$$\sum_{i=1}^{n} w_i x_i^m = E[x^m], \quad m = 0, 1, 2, \cdots \tag{8}$$

Comparing the coefficients of $f(0)$ and derivatives of $f(x)$ evaluated at 0 in the two equations

$$E[f(x)] = f(0) + \nabla f(0)E[x] + \frac{1}{2!}\nabla^2 f(0)E[x^2]$$
$$\frac{1}{3!}\nabla^3 f(0)E[x^3] + \frac{1}{4!}\nabla^4 f(0)E[x^4]... \quad (6)$$
$$E[f(x)] = f(0)(\sum_{i=1}^{n} w_i) + \frac{1}{2!}\nabla^2 f(0)(\sum_{i=1}^{n} w_i x_i^2)$$
$$\frac{1}{4!}\nabla^4 f(0)(\sum_{i=1}^{n} w_i x_i^4) + ... \quad (7)$$

*moment constraint equations:*

$$\sum_{i=1}^{n} w_i x_i^m = E[x^m], \quad m = 0, 1, 2, \cdots \quad (8)$$

Comparing the coefficients of $f(0)$ and derivatives of $f(x)$ evaluated at 0 in the two equations

$$E[f(x)] = f(0) + \nabla f(0)E[x] + \frac{1}{2!}\nabla^2 f(0)E[x^2] \\ \frac{1}{3!}\nabla^3 f(0)E[x^3] + \frac{1}{4!}\nabla^4 f(0)E[x^4]... \tag{6}$$

$$E[f(x)] = f(0)(\sum_{i=1}^{n} w_i) + \frac{1}{2!}\nabla^2 f(0)(\sum_{i=1}^{n} w_i x_i^2) \\ \frac{1}{4!}\nabla^4 f(0)(\sum_{i=1}^{n} w_i x_i^4) + ... \tag{7}$$

*moment constraint equations:*

$$\sum_{i=1}^{n} w_i x_i^m = E[x^m], \quad m = 0, 1, 2, \cdots \tag{8}$$

Comparing the coefficients of $f(0)$ and derivatives of $f(x)$ evaluated at 0 in the two equations

$$E[f(x)] = f(0) + \nabla f(0)E[x] + \frac{1}{2!}\nabla^2 f(0)E[x^2]$$
$$\frac{1}{3!}\nabla^3 f(0)E[x^3] + \frac{1}{4!}\nabla^4 f(0)E[x^4]... \tag{6}$$
$$E[f(x)] = f(0)(\sum_{i=1}^{n} w_i) + \frac{1}{2!}\nabla^2 f(0)(\sum_{i=1}^{n} w_i x_i^2)$$
$$\frac{1}{4!}\nabla^4 f(0)(\sum_{i=1}^{n} w_i x_i^4) + ... \tag{7}$$

*moment constraint equations:*

$$\sum_{i=1}^{n} w_i x_i^m = E[x^m], \quad m = 0, 1, 2, \cdots \tag{8}$$

Comparing the coefficients of $f(0)$ and derivatives of $f(x)$ evaluated at 0 in the two equations

$$E[f(x)] = f(0) + \nabla f(0)E[x] + \frac{1}{2!}\nabla^2 f(0)E[x^2]$$
$$\frac{1}{3!}\nabla^3 f(0)E[x^3] + \frac{1}{4!}\nabla^4 f(0)E[x^4]... \tag{6}$$
$$E[f(x)] = f(0)(\sum_{i=1}^{n} w_i) + \frac{1}{2!}\nabla^2 f(0)(\sum_{i=1}^{n} w_i x_i^2)$$
$$\frac{1}{4!}\nabla^4 f(0)(\sum_{i=1}^{n} w_i x_i^4) + ... \tag{7}$$

*moment constraint equations:*

$$\sum_{i=1}^{n} w_i x_i^m = E[x^m], \quad m = 0, 1, 2, \cdots \tag{8}$$

- Notice that these constraint equations conveys that the sigma points $x_i$ should satisfy the infinite moment equations in the domain of the pdf.
- Thus, if $f(x)$ is a polynomial function of degree $d$ then sigma points should be chosen to reproduce first $d$ moments of the pdf exactly to guarantee the exact evaluation of the expectation integral.
- For non-polynomial functions, one has to judiciously choose the number of moment equations which should be satisfied.

- Notice that these constraint equations conveys that the sigma points $x_i$ should satisfy the infinite moment equations in the domain of the pdf.
- Thus, if $f(x)$ is a polynomial function of degree $d$ then sigma points should be chosen to reproduce first $d$ moments of the pdf exactly to guarantee the exact evaluation of the expectation integral.
- For non-polynomial functions, one has to judiciously choose the number of moment equations which should be satisfied.

- Notice that these constraint equations conveys that the sigma points $x_i$ should satisfy the infinite moment equations in the domain of the pdf.
- Thus, if $f(x)$ is a polynomial function of degree $d$ then sigma points should be chosen to reproduce first $d$ moments of the pdf exactly to guarantee the exact evaluation of the expectation integral.
- For non-polynomial functions, one has to judiciously choose the number of moment equations which should be satisfied.

To evaluate the cubature points and weights of any PDF, the following procedure is followed

- Evaluate the**moments** of the continuous PDF
- Assume the**positions and weights** of each cubature point as variables
- Equate the expressions of moment equations for the discrete system to the known moments of the continuous PDF
- Solve these set of **nonlinear moment constraint equations**

To evaluate the cubature points and weights of any PDF, the following procedure is followed

- Evaluate the**moments** of the continuous PDF
- Assume the**positions and weights** of each cubature point as variables
- Equate the expressions of moment equations for the discrete system to the known moments of the continuous PDF
- Solve these set of **nonlinear moment constraint equations**

To evaluate the cubature points and weights of any PDF, the following procedure is followed

- Evaluate the**moments** of the continuous PDF
- Assume the**positions and weights** of each cubature point as variables
- Equate the expressions of moment equations for the discrete system to the known moments of the continuous PDF
- Solve these set of **nonlinear moment constraint equations**

To evaluate the cubature points and weights of any PDF, the following procedure is followed

- Evaluate the**moments** of the continuous PDF
- Assume the**positions and weights** of each cubature point as variables
- Equate the expressions of moment equations for the discrete system to the known moments of the continuous PDF
- Solve these set of **nonlinear moment constraint equations**

The equivalent discrete moments are written interms of the assumed variables- **positions and weights**

TABLE: Equivalent Moments for 2D till order 4

| Continuous | Discrete | Continuous | Dicrete |
|---|---|---|---|
| $E[x]$ | $\sum_{i=1}^{n} w_i x_i$ | $E[y]$ | $\sum_{i=1}^{n} w_i y_i$ |
| $E[x^2]$ | $\sum_{i=1}^{n} w_i x_i^2$ | $E[xy]$ | $\sum_{i=1}^{n} w_i x_i y_i$ |
| $E[y^2]$ | $\sum_{i=1}^{n} w_i y_i^2$ | $E[x^4]$ | $\sum_{i=1}^{n} w_i x_i^4$ |
| $E[y^4]$ | $\sum_{i=1}^{n} w_i y_i^4$ | $E[x^3 y]$ | $\sum_{i=1}^{n} w_i x_i^3 y_i$ |
| $E[y^3 x]$ | $\sum_{i=1}^{n} w_i y_i^3 x_i$ | $E[x^2 y^2]$ | $\sum_{i=1}^{n} w_i x_i^2 y_i^2$ |

Table 1 gives the moment constraint equations that have to be solved for the **weights** $w_i$ **and points** $x_i$.

The equivalent discrete moments are written interms of the assumed variables- **positions and weights**

TABLE: Equivalent Moments for 2D till order 4

| Continuous | Discrete | Continuous | Dicrete |
|---|---|---|---|
| $E[x]$ | $\sum_{i=1}^{n} w_i x_i$ | $E[y]$ | $\sum_{i=1}^{n} w_i y_i$ |
| $E[x^2]$ | $\sum_{i=1}^{n} w_i x_i^2$ | $E[xy]$ | $\sum_{i=1}^{n} w_i x_i y_i$ |
| $E[y^2]$ | $\sum_{i=1}^{n} w_i y_i^2$ | $E[x^4]$ | $\sum_{i=1}^{n} w_i x_i^4$ |
| $E[y^4]$ | $\sum_{i=1}^{n} w_i y_i^4$ | $E[x^3 y]$ | $\sum_{i=1}^{n} w_i x_i^3 y_i$ |
| $E[y^3 x]$ | $\sum_{i=1}^{n} w_i y_i^3 x_i$ | $E[x^2 y^2]$ | $\sum_{i=1}^{n} w_i x_i^2 y_i^2$ |

Table 1 gives the moment constraint equations that have to be solved for the **weights** $w_i$ **and points** $x_i$**.**

The equivalent discrete moments are written interms of the assumed variables- **positions and weights**

TABLE: Equivalent Moments for 2D till order 4

| Continuous | Discrete | Continuous | Dicrete |
|---|---|---|---|
| $E[x]$ | $\sum_{i=1}^{n} w_i x_i$ | $E[y]$ | $\sum_{i=1}^{n} w_i y_i$ |
| $E[x^2]$ | $\sum_{i=1}^{n} w_i x_i^2$ | $E[xy]$ | $\sum_{i=1}^{n} w_i x_i y_i$ |
| $E[y^2]$ | $\sum_{i=1}^{n} w_i y_i^2$ | $E[x^4]$ | $\sum_{i=1}^{n} w_i x_i^4$ |
| $E[y^4]$ | $\sum_{i=1}^{n} w_i y_i^4$ | $E[x^3 y]$ | $\sum_{i=1}^{n} w_i x_i^3 y_i$ |
| $E[y^3 x]$ | $\sum_{i=1}^{n} w_i y_i^3 x_i$ | $E[x^2 y^2]$ | $\sum_{i=1}^{n} w_i x_i^2 y_i^2$ |

Table 1 gives the moment constraint equations that have to be solved for the **weights** $w_i$ **and points** $x_i$**.**

# Moments of Continuous Normal PDF

- The higher order moments of a Normal PDF can be calculated from **just the mean and Covariance.**
- Particularly the Higher order raw moments just need the covariance matrix.

The moments are given by the following theorem

**Isserilis Theorem**

$$E[x_1 x_2 x_3 \ldots . x_{2n}] = \sum \prod E[x_i x_j]$$

- The higher order moments of a Normal PDF can be calculated from **just the mean and Covariance.**
- Particularly the Higher order raw moments just need the covariance matrix.

The moments are given by the following theorem

**ISSERILIS THEOREM**

$$E[x_1 x_2 x_3 \ldots . x_{2n}] = \sum \prod E[x_i x_j]$$

- The higher order moments of a Normal PDF can be calculated from **just the mean and Covariance.**
- Particularly the Higher order raw moments just need the covariance matrix.

The moments are given by the following theorem

**ISSERILIS THEOREM**

$$E[x_1 x_2 x_3 \ldots . x_{2n}] = \sum \prod E[x_i x_j]$$

- The higher order moments of a Normal PDF can be calculated from **just the mean and Covariance.**
- Particularly the Higher order raw moments just need the covariance matrix.

The moments are given by the following theorem

**ISSERILIS THEOREM**

$$E[x_1 x_2 x_3 \ldots . x_{2n}] = \sum \prod E[x_i x_j]$$

- The higher order moments of a Normal PDF can be calculated from **just the mean and Covariance.**
- Particularly the Higher order raw moments just need the covariance matrix.

The moments are given by the following theorem

**Isserilis Theorem**

$$E[x_1 x_2 x_3 \ldots . x_{2n}] = \sum \prod E[x_i x_j]$$

for example the **fourth moment** is

$$E[x_1x_2x_3x_4] = E[x_1x_2]E[x_3x_4] + E[x_1x_3]E[x_4x_5] + E[x_1x_4]E[x_2x_3]$$

And the **sixth moment** is

$$E[x_1x_2x_3x_4x_5x_6] =$$

$$\begin{aligned}
=&E[x_1x_2]E[x_3x_4]E[x_5x_6] + E[x_1x_2]E[x_3x_5]E[x_4x_6]\\
&+E[x_1x_2]E[x_3x_6]E[x_4x_5] + E[x_1x_3]E[x_2x_4]E[x_5x_6]\\
&+E[x_1x_3]E[x_2x_5]E[x_4x_6] + E[x_1x_3]E[x_2x_6]E[x_4x_5]\\
&+E[x_1x_4]E[x_2x_3]E[x_5x_6] + E[x_1x_4]E[x_2x_5]E[x_3x_6]\\
&+E[x_1x_4]E[x_2x_6]E[x_3x_5] + E[x_1x_5]E[x_2x_3]E[x_4x_6]\\
&+E[x_1x_5]E[x_2x_4]E[x_3x_6] + E[x_1x_5]E[x_2x_6]E[x_3x_4]\\
&+E[x_1x_6]E[x_2x_3]E[x_4x_5] + E[x_1x_6]E[x_2x_4]E[x_3x_5]\\
&+E[x_1x_6]E[x_2x_5]E[x_3x_4]
\end{aligned}$$

for example the **fourth moment** is

$$E[x_1x_2x_3x_4] = E[x_1x_2]E[x_3x_4] + E[x_1x_3]E[x_4x_5] + E[x_1x_4]E[x_2x_3]$$

And the **sixth moment** is

$$E[x_1x_2x_3x_4x_5x_6] =$$

$$\begin{aligned}
=&E[x_1x_2]E[x_3x_4]E[x_5x_6] + E[x_1x_2]E[x_3x_5]E[x_4x_6] \\
&+ E[x_1x_2]E[x_3x_6]E[x_4x_5] + E[x_1x_3]E[x_2x_4]E[x_5x_6] \\
&+ E[x_1x_3]E[x_2x_5]E[x_4x_6] + E[x_1x_3]E[x_2x_6]E[x_4x_5] \\
&+ E[x_1x_4]E[x_2x_3]E[x_5x_6] + E[x_1x_4]E[x_2x_5]E[x_3x_6] \\
&+ E[x_1x_4]E[x_2x_6]E[x_3x_5] + E[x_1x_5]E[x_2x_3]E[x_4x_6] \\
&+ E[x_1x_5]E[x_2x_4]E[x_3x_6] + E[x_1x_5]E[x_2x_6]E[x_3x_4] \\
&+ E[x_1x_6]E[x_2x_3]E[x_4x_5] + E[x_1x_6]E[x_2x_4]E[x_3x_5] \\
&+ E[x_1x_6]E[x_2x_5]E[x_3x_4]
\end{aligned}$$

for example the **fourth moment** is

$$E[x_1x_2x_3x_4] = E[x_1x_2]E[x_3x_4] + E[x_1x_3]E[x_4x_5] + E[x_1x_4]E[x_2x_3]$$

And the **sixth moment** is

$$E[x_1x_2x_3x_4x_5x_6] =$$

$$\begin{aligned}
=&E[x_1x_2]E[x_3x_4]E[x_5x_6] + E[x_1x_2]E[x_3x_5]E[x_4x_6]\\
&+E[x_1x_2]E[x_3x_6]E[x_4x_5] + E[x_1x_3]E[x_2x_4]E[x_5x_6]\\
&+E[x_1x_3]E[x_2x_5]E[x_4x_6] + E[x_1x_3]E[x_2x_6]E[x_4x_5]\\
&+E[x_1x_4]E[x_2x_3]E[x_5x_6] + E[x_1x_4]E[x_2x_5]E[x_3x_6]\\
&+E[x_1x_4]E[x_2x_6]E[x_3x_5] + E[x_1x_5]E[x_2x_3]E[x_4x_6]\\
&+E[x_1x_5]E[x_2x_4]E[x_3x_6] + E[x_1x_5]E[x_2x_6]E[x_3x_4]\\
&+E[x_1x_6]E[x_2x_3]E[x_4x_5] + E[x_1x_6]E[x_2x_4]E[x_3x_5]\\
&+E[x_1x_6]E[x_2x_5]E[x_3x_4]
\end{aligned}$$

As the Normal PDF is symmetric, we would like to **exploit this symmetry** in finding the cubature points. The points we like to seek have the following properties

- **Fully symmetric** :- thus satisfying all odd order moments at no further cost
- Weights **sum up** to 1
- All weights are to be **positive**
- As the points are symmetric, each point has an equivalent point located on the other side of the mean at the same distance.
- These two points together lie on a line passing through the mean. We later name this line as a particular **'axis'.**
- And these two points also have the same weight to balance each other

As the Normal PDF is symmetric, we would like to **exploit this symmetry** in finding the cubature points. The points we like to seek have the following properties

- **Fully symmetric** :- thus satisfying all odd order moments at no further cost
- Weights **sum up** to 1
- All weights are to be **positive**
- As the points are symmetric, each point has an equivalent point located on the other side of the mean at the same distance.
- These two points together lie on a line passing through the mean. We later name this line as a particular **'axis'.**
- And these two points also have the same weight to balance each other

As the Normal PDF is symmetric, we would like to **exploit this symmetry** in finding the cubature points. The points we like to seek have the following properties

- **Fully symmetric** :- thus satisfying all odd order moments at no further cost
- Weights **sum up** to 1
- All weights are to be **positive**
- As the points are symmetric, each point has an equivalent point located on the other side of the mean at the same distance.
- These two points together lie on a line passing through the mean. We later name this line as a particular **'axis'.**
- And these two points also have the same weight to balance each other

As the Normal PDF is symmetric, we would like to **exploit this symmetry** in finding the cubature points. The points we like to seek have the following properties

- **Fully symmetric** :- thus satisfying all odd order moments at no further cost
- Weights **sum up** to 1
- All weights are to be **positive**
- As the points are symmetric, each point has an equivalent point located on the other side of the mean at the same distance.
- These two points together lie on a line passing through the mean. We later name this line as a particular **'axis'.**
- And these two points also have the same weight to balance each other

As the Normal PDF is symmetric, we would like to **exploit this symmetry** in finding the cubature points. The points we like to seek have the following properties

- **Fully symmetric** :- thus satisfying all odd order moments at no further cost
- Weights **sum up** to 1
- All weights are to be **positive**
- As the points are symmetric, each point has an equivalent point located on the other side of the mean at the same distance.
- These two points together lie on a line passing through the mean. We later name this line as a particular **'axis'.**
- And these two points also have the same weight to balance each other

As the Normal PDF is symmetric, we would like to **exploit this symmetry** in finding the cubature points. The points we like to seek have the following properties

- **Fully symmetric** :- thus satisfying all odd order moments at no further cost
- Weights **sum up** to 1
- All weights are to be **positive**
- As the points are symmetric, each point has an equivalent point located on the other side of the mean at the same distance.
- These two points together lie on a line passing through the mean. We later name this line as a particular **'axis'.**
- And these two points also have the same weight to balance each other

To proceed with the method we describe, the following **terminology or definitions** that might be handy

### GENERATOR SET

- This definition has been used quite a lot in literature to describe a set containing all permutations, including sign permutations of the elements in a set $\{u_1, u_2, u_3, ..., u_r, 0, 0, 0...., 0\}$, where $u_1, u_2, ...$ are real numbers.
- for example the generator set $(1,1,0)$ is equivalent to $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(-1,1,0)$, $(1,-1,0)$, $(-1,0,1)$, $(1,0,-1)$, $(0,-1,1)$, $(0,1,-1)$, $(0,-1,-1)$, $(-1,-1,0)$, $(-1,0,-1)$

We introduce some new terminology with respect to the Identity Covariance Normal PDF of $N^{th}$-Dimension with zero mean, that will just aid our intuition

To proceed with the method we describe, the following **terminology or definitions** that might be handy

### GENERATOR SET

- This definition has been used quite a lot in literature to describe a set containing all permutations, including sign permutations of the elements in a set $\{u_1, u_2, u_3, ..., u_r, 0, 0, 0...., 0\}$, where $u_1, u_2, ...$ are real numbers.
- for example the generator set $(1,1,0)$ is equivalent to $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(-1,1,0)$, $(1,-1,0)$, $(-1,0,1)$, $(1,0,-1)$, $(0,-1,1)$, $(0,1,-1)$, $(0,-1,-1)$, $(-1,-1,0)$, $(-1,0,-1)$

We introduce some new terminology with respect to the Identity Covariance Normal PDF of $N^{th}$-Dimension with zero mean, that will just aid our intuition

To proceed with the method we describe, the following **terminology or definitions** that might be handy

### GENERATOR SET

- This definition has been used quite a lot in literature to describe a set containing all permutations, including sign permutations of the elements in a set $\{u_1, u_2, u_3, ..., u_r, 0, 0, 0...., 0\}$, where $u_1, u_2, ...$ are real numbers.
- for example the generator set $(1,1,0)$ is equivalent to $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(-1,1,0)$, $(1,-1,0)$, $(-1,0,1)$, $(1,0,-1)$, $(0,-1,1)$, $(0,1,-1)$, $(0,-1,-1)$, $(-1,-1,0)$, $(-1,0,-1)$

We introduce some new terminology with respect to the Identity Covariance Normal PDF of $N^{th}$-Dimension with zero mean, that will just aid our intuition

To proceed with the method we describe, the following **terminology or definitions** that might be handy

### GENERATOR SET

- This definition has been used quite a lot in literature to describe a set containing all permutations, including sign permutations of the elements in a set $\{u_1, u_2, u_3, ..., u_r, 0, 0, 0...., 0\}$, where $u_1, u_2, ...$ are real numbers.
- for example the generator set $(1,1,0)$ is equivalent to $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(-1,1,0)$, $(1,-1,0)$, $(-1,0,1)$, $(1,0,-1)$, $(0,-1,1)$, $(0,1,-1)$, $(0,-1,-1)$, $(-1,-1,0)$, $(-1,0,-1)$

We introduce some new terminology with respect to the Identity Covariance Normal PDF of $N^{th}$-Dimension with zero mean, that will just aid our intuition

To proceed with the method we describe, the following **terminology or definitions** that might be handy

### GENERATOR SET

- This definition has been used quite a lot in literature to describe a set containing all permutations, including sign permutations of the elements in a set $\{u_1, u_2, u_3, ..., u_r, 0, 0, 0...., 0\}$, where $u_1, u_2, ...$ are real numbers.
- for example the generator set $(1,1,0)$ is equivalent to $(1,1,0)$, $(1,0,1)$, $(0,1,1)$, $(-1,1,0)$, $(1,-1,0)$, $(-1,0,1)$, $(1,0,-1)$, $(0,-1,1)$, $(0,1,-1)$, $(0,-1,-1)$, $(-1,-1,0)$, $(-1,0,-1)$

We introduce some new terminology with respect to the Identity Covariance Normal PDF of $N^{th}$-Dimension with zero mean, that will just aid our intuition

- Once the appropriate axis are chosen, the points are constrained to be on these axis such that a symmetric set of points that have the same weight are at equal distance from the origin.
- The distance variables are $r_1, r_2 \ldots$ measured from the origin to the appropriate set of points.
- The points at distance $r_1$ have weight $w_1$ and so on.

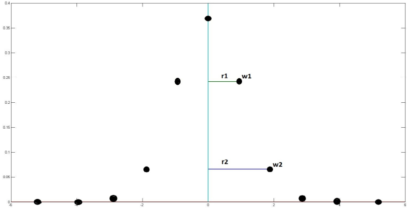



FIGURE: distances and weights

- Once the appropriate axis are chosen, the points are constrained to be on these axis such that a symmetric set of points that have the same weight are at equal distance from the origin.
- The distance variables are $r_1, r_2 ...$ measured from the origin to the appropriate set of points.
- The points at distance $r_1$ have weight $w_1$ and so on.

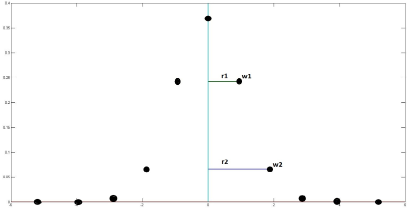



FIGURE: distances and weights

- Once the appropriate axis are chosen, the points are constrained to be on these axis such that a symmetric set of points that have the same weight are at equal distance from the origin.
- The distance variables are $r_1$, $r_2$... measured from the origin to the appropriate set of points.
- The points at distance $r_1$ have weight $w_1$ and so on.

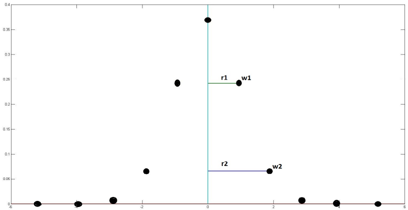



FIGURE: distances and weights

- Once the appropriate axis are chosen, the points are constrained to be on these axis such that a symmetric set of points that have the same weight are at equal distance from the origin.
- The distance variables are $r_1, r_2 \ldots$ measured from the origin to the appropriate set of points.
- The points at distance $r_1$ have weight $w_1$ and so on.

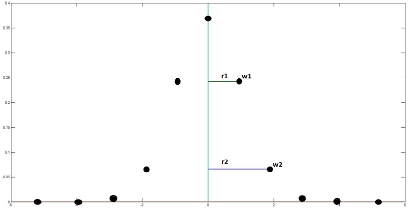



FIGURE: distances and weights

- Once the appropriate axis are chosen, the points are constrained to be on these axis such that a symmetric set of points that have the same weight are at equal distance from the origin.
- The distance variables are $r_1$, $r_2$... measured from the origin to the appropriate set of points.
- The points at distance $r_1$ have weight $w_1$ and so on.

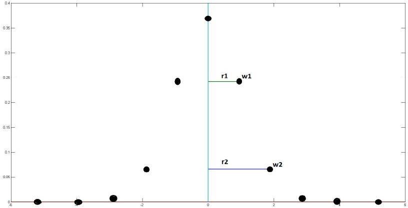


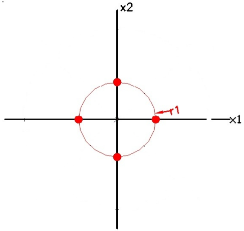

FIGURE: distances and weights

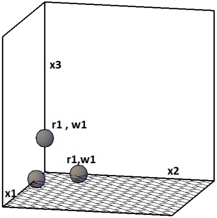
## Principal axis

The orthogonal axis in cartesian space intersecting at the origin. These are the axis corresponding to each column of the identity covariance matrix. Thus there are $N$ principal axis or $2N$ distinct points on the principal axis for N-Dimensional system. We list the points on principal axis as

$$\sigma_i \in \{\pm\sqrt{P}_j | \{j\} \in \{1,2,...,N\}\} \quad (9)$$

$$i = 1,2,3,...,2N. \quad (10)$$



For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$E[X_i^2] = 1$$
$$E[X_iX_j] = 0$$
$$E[X_i^4] = 3$$
$$E[X_i^3X_j] = 0$$
$$E[X_i^2X_j^2] = 1$$
$$E[X_i^2X_jX_k] = 0$$
$$E[X_iX_jX_kX_l] = 0$$

Where $\{i, j, k, l\} \in \{1, 2, 3, ..., N\}$ & $i \neq j \neq k \neq l$.

# Cubature points constrained to the principal axis to satisfy the $2^{nd}$ order moments

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$
\begin{aligned}
E[X_i^2] &= 1 \\
E[X_i X_j] &= 0 \\
E[X_i^4] &= 3 \\
E[X_i^3 X_j] &= 0 \\
E[X_i^2 X_j^2] &= 1 \\
E[X_i^2 X_j X_k] &= 0 \\
E[X_i X_j X_k X_l] &= 0
\end{aligned}
$$

Where $\{i, j, k, l\} \in \{1, 2, 3, ..., N\}$ & $i \neq j \neq k \neq l$.

# Cubature points constrained to the principal axis to satisfy the $2^{nd}$ order moments

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$
\begin{aligned}
E[X_i^2] &= 1 \\
E[X_i X_j] &= 0 \\
E[X_i^4] &= 3 \\
E[X_i^3 X_j] &= 0 \\
E[X_i^2 X_j^2] &= 1 \\
E[X_i^2 X_j X_k] &= 0 \\
E[X_i X_j X_k X_l] &= 0
\end{aligned}
$$

Where $\{i, j, k, l\} \in \{1, 2, 3, ..., N\}$ & $i \neq j \neq k \neq l$.

# CUBATURE POINTS CONSTRAINED TO THE PRINCIPAL AXIS TO SATISFY THE $2^{nd}$ ORDER MOMENTS

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$
\begin{aligned}
E[X_i^2] &= 1 \\
E[X_i X_j] &= 0 \\
E[X_i^4] &= 3 \\
E[X_i^3 X_j] &= 0 \\
E[X_i^2 X_j^2] &= 1 \\
E[X_i^2 X_j X_k] &= 0 \\
E[X_i X_j X_k X_l] &= 0
\end{aligned}
$$

Where $\{i, j, k, l\} \in \{1, 2, 3, ..., N\}$ & $i \neq j \neq k \neq l$.

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$
\begin{aligned}
E[X_i^2] &= 1 \\
E[X_i X_j] &= 0 \\
E[X_i^4] &= 3 \\
E[X_i^3 X_j] &= 0 \\
E[X_i^2 X_j^2] &= 1 \\
E[X_i^2 X_j X_k] &= 0 \\
E[X_i X_j X_k X_l] &= 0
\end{aligned}
$$

Where $\{i, j, k, l\} \in \{1, 2, 3, ..., N\}$ & $i \neq j \neq k \neq l$.

# Cubature points constrained to the principal axis to satisfy the $2^{nd}$ order moments

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$
\begin{aligned}
E[X_i^2] &= 1 \\
E[X_i X_j] &= 0 \\
E[X_i^4] &= 3 \\
E[X_i^3 X_j] &= 0 \\
E[X_i^2 X_j^2] &= 1 \\
E[X_i^2 X_j X_k] &= 0 \\
E[X_i X_j X_k X_l] &= 0
\end{aligned}
$$

Where $\{i, j, k, l\} \in \{1, 2, 3, ..., N\}$ & $i \neq j \neq k \neq l$.

# CUBATURE POINTS CONSTRAINED TO THE PRINCIPAL AXIS TO SATISFY THE $2^{nd}$ ORDER MOMENTS

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$
\begin{aligned}
E[X_i^2] &= 1 \\
E[X_i X_j] &= 0 \\
E[X_i^4] &= 3 \\
E[X_i^3 X_j] &= 0 \\
E[X_i^2 X_j^2] &= 1 \\
E[X_i^2 X_j X_k] &= 0 \\
E[X_i X_j X_k X_l] &= 0
\end{aligned}
$$

Where $\{i, j, k, l\} \in \{1, 2, 3, ..., N\}$ & $i \neq j \neq k \neq l$.

# Cubature points constrained to the principal axis to satisfy the $2^{nd}$ order moments

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$E[X_i^2] = 1$$
$$E[X_iX_j] = 0$$
$$E[X_i^4] = 3$$
$$E[X_i^3X_j] = 0$$
$$E[X_i^2X_j^2] = 1$$
$$E[X_i^2X_jX_k] = 0$$
$$E[X_iX_jX_kX_l] = 0$$

Where $\{i, j, k, l\} \in \{1, 2, 3, ..., N\}$ & $i \neq j \neq k \neq l$.

## Cubature points constrained to the principal axis to satisfy the $2^{nd}$ order moments

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$
\begin{aligned}
E[X_i^2] &= 1 \\
E[X_i X_j] &= 0 \\
E[X_i^4] &= 3 \\
E[X_i^3 X_j] &= 0 \\
E[X_i^2 X_j^2] &= 1 \\
E[X_i^2 X_j X_k] &= 0 \\
E[X_i X_j X_k X_l] &= 0
\end{aligned}
$$

Where $\{i, j, k, l\} \in \{1, 2, 3, ..., N\}$ & $i \neq j \neq k \neq l$.

# Cubature points constrained to the principal axis to satisfy the $2^{nd}$ order moments

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$
\begin{aligned}
E[X_i^2] &= 1 \\
E[X_i X_j] &= 0 \\
E[X_i^4] &= 3 \\
E[X_i^3 X_j] &= 0 \\
E[X_i^2 X_j^2] &= 1 \\
E[X_i^2 X_j X_k] &= 0 \\
E[X_i X_j X_k X_l] &= 0
\end{aligned}
$$

Where $\{i, j, k, l\} \in \{1, 2, 3, ..., N\}$ & $i \neq j \neq k \neq l$.

- Consider a fully symmetric set of cubature points that lie on the principal axis at a distance of $r_1$ from the origin and have weight of $w_1$.
- The moments that have any odd powers for the random varable in the set of moments is are already satisfied due to**symmetry of cubature points.**
- Only the moments with **all even powers** have to be satisfied

$$E[X_i^2] = 1$$
$$E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1$$

- Consider a fully symmetric set of cubature points that lie on the principal axis at a distance of $r_1$ from the origin and have weight of $w_1$.
- The moments that have any odd powers for the random varable in the set of moments is are already satisfied due to**symmetry of cubature points.**
- Only the moments with **all even powers** have to be satisfied

$$E[X_i^2] = 1$$
$$E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1$$

- Consider a fully symmetric set of cubature points that lie on the principal axis at a distance of $r_1$ from the origin and have weight of $w_1$.
- The moments that have any odd powers for the random varable in the set of moments is are already satisfied due to**symmetry of cubature points.**
- Only the moments with **all even powers** have to be satisfied

$$E[X_i^2] = 1$$
$$E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1$$

- Consider a fully symmetric set of cubature points that lie on the principal axis at a distance of $r_1$ from the origin and have weight of $w_1$.
- The moments that have any odd powers for the random varable in the set of moments is are already satisfied due to**symmetry of cubature points.**
- Only the moments with **all even powers** have to be satisfied

$$E[X_i^2] = 1$$
$$E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1$$

- Consider a fully symmetric set of cubature points that lie on the principal axis at a distance of $r_1$ from the origin and have weight of $w_1$.
- The moments that have any odd powers for the random varable in the set of moments is are already satisfied due to**symmetry of cubature points.**
- Only the moments with **all even powers** have to be satisfied

$$E[X_i^2] = 1$$
$$E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1$$

- Consider a fully symmetric set of cubature points that lie on the principal axis at a distance of $r_1$ from the origin and have weight of $w_1$.
- The moments that have any odd powers for the random varable in the set of moments is are already satisfied due to**symmetry of cubature points.**
- Only the moments with **all even powers** have to be satisfied

$$E[X_i^2] = 1$$
$$E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1$$

- Consider a fully symmetric set of cubature points that lie on the principal axis at a distance of $r_1$ from the origin and have weight of $w_1$.
- The moments that have any odd powers for the random varable in the set of moments is are already satisfied due to**symmetry of cubature points.**
- Only the moments with **all even powers** have to be satisfied

$$
\begin{aligned}
E[X_i^2] &= 1 \\
E[X_i^4] &= 3 \\
E[X_i^2 X_j^2] &= 1
\end{aligned}
$$

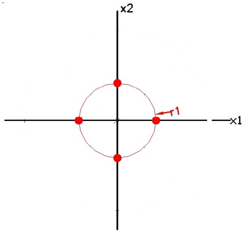



FIGURE: distances and weights

For example consider the 2D system. The points on these axis when enumerated as a list is $(r1,0)$, $(0,r1)$, $(-r1,0)$, $(0,-r1)$. The equivalent discrete moments are

$$2r_1^2 w_1 = 1$$
$$2r_1^4 w_1 = 3$$
$$0 \neq 1$$

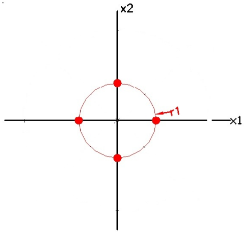



FIGURE: distances and weights

For example consider the 2D system. The points on these axis when enumerated as a list is $(r1,0)$, $(0,r1)$, $(-r1,0)$, $(0,-r1)$. The equivalent discrete moments are

$$2r_1^2 w_1 = 1$$

$$2r_1^4 w_1 = 3$$

$$0 \neq 1$$

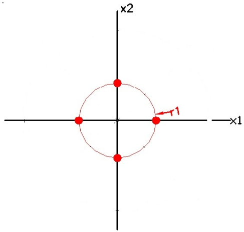



FIGURE: distances and weights

For example consider the 2D system. The points on these axis when enumerated as a list is $(r1,0)$, $(0,r1)$, $(-r1,0)$, $(0,-r1)$. The equivalent discrete moments are

$$2r_1^2 w_1 = 1$$
$$2r_1^4 w_1 = 3$$
$$0 \neq 1$$

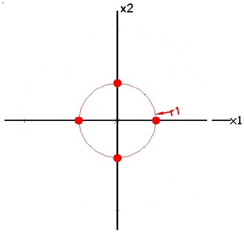



FIGURE: distances and weights

For example consider the 2D system. The points on these axis when enumerated as a list is $(r1,0)$, $(0,r1)$, $(-r1,0)$, $(0,-r1)$. The equivalent discrete moments are

$$2r_1^2 w_1 = 1$$
$$2r_1^4 w_1 = 3$$
$$0 \neq 1$$

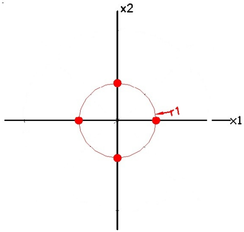



FIGURE: distances and weights

For example consider the 2D system. The points on these axis when enumerated as a list is $(r1,0)$, $(0,r1)$, $(-r1,0)$, $(0,-r1)$. The equivalent discrete moments are

$$2r_1^2 w_1 = 1$$
$$2r_1^4 w_1 = 3$$
$$0 \neq 1$$

This set of equations are infact the same for any dimension.The central weight is calculated as

$$w_0 = 1 - 2Nw_1 \tag{11}$$

One of the $4^{th}$ order cross moment **cannot be satisfied** by selecting such points on the principal axis.

In general **no cross moment** of any order can be satisfied by the points just on the **principal axis.**

$$E[X_i^2 X_j^2] \neq 1 \tag{12}$$

The 4th moment is also evaluted here just to compare different filters such as UKF and CKF in terms of the **4th order moment error**.

This set of equations are infact the same for any dimension.The central weight is calculated as

$$w_0 = 1 - 2Nw_1 \tag{11}$$

One of the $4^{th}$ order cross moment **cannot be satisfied** by selecting such points on the principal axis.
In general **no cross moment** of any order can be satisfied by the points just on the **principal axis.**

$$E[X_i^2 X_j^2] \neq 1 \tag{12}$$

The 4th moment is also evaluted here just to compare different filters such as UKF and CKF in terms of the **4th order moment error**.

This set of equations are infact the same for any dimension.The central weight is calculated as

$$w_0 = 1 - 2Nw_1 \tag{11}$$

One of the $4^{th}$ order cross moment **cannot be satisfied** by selecting such points on the principal axis.

In general **no cross moment** of any order can be satisfied by the points just on the **principal axis.**

$$E[X_i^2 X_j^2] \neq 1 \tag{12}$$

The 4th moment is also evaluted here just to compare different filters such as UKF and CKF in terms of the **4th order moment error**.

This set of equations are infact the same for any dimension.The central weight is calculated as

$$w_0 = 1 - 2Nw_1 \tag{11}$$

One of the $4^{th}$ order cross moment **cannot be satisfied** by selecting such points on the principal axis.

In general **no cross moment** of any order can be satisfied by the points just on the **principal axis.**

$$E[X_i^2 X_j^2] \neq 1 \tag{12}$$

The 4th moment is also evaluted here just to compare different filters such as UKF and CKF in terms of the **4th order moment error**.

This set of equations are infact the same for any dimension.The central weight is calculated as

$$w_0 = 1 - 2Nw_1 \tag{11}$$

One of the $4^{th}$ order cross moment **cannot be satisfied** by selecting such points on the principal axis.
In general **no cross moment** of any order can be satisfied by the points just on the **principal axis.**

$$E[X_i^2 X_j^2] \neq 1 \tag{12}$$

The 4th moment is also evaluted here just to compare different filters such as UKF and CKF in terms of the **4th order moment error**.

This set of equations are infact the same for any dimension.The central weight is calculated as

$$w_0 = 1 - 2Nw_1 \tag{11}$$

One of the $4^{th}$ order cross moment **cannot be satisfied** by selecting such points on the principal axis.
In general **no cross moment** of any order can be satisfied by the points just on the **principal axis.**

$$E[X_i^2 X_j^2] \neq 1 \tag{12}$$

The 4th moment is also evaluted here just to compare different filters such as UKF and CKF in terms of the **4th order moment error**.

This set of equations are infact the same for any dimension.The central weight is calculated as

$$w_0 = 1 - 2Nw_1 \tag{11}$$

One of the $4^{th}$ order cross moment **cannot be satisfied** by selecting such points on the principal axis.
In general **no cross moment** of any order can be satisfied by the points just on the **principal axis.**

$$E[X_i^2 X_j^2] \neq 1 \tag{12}$$

The 4th moment is also evaluted here just to compare different filters such as UKF and CKF in terms of the **4th order moment error**.

- The $2N+1$ sigma points for the unscented Transform are chosen such that 1 point is the origin and $2N$ points of equal weight are constrained to lie on the principal axis.
- The distance of each point on the principal axis is chosen such that the second order moments are satisfied.
- A tuning parameter $\kappa$ is introduced such that one of the 4th moment can be tuned.
- This works well for systems till dimension 3 after which the central weight becomes negative.
- The present framework is shown to **be inline** with the $2n+1$ Unscented sigma points by working our way backwards.

# THE $2N+1$ UNSCENTED TRANSFORM METHOD

- The $2N+1$ sigma points for the unscented Transform are chosen such that 1 point is the origin and $2N$ points of equal weight are constrained to lie on the principal axis.
- The distance of each point on the principal axis is chosen such that the second order moments are satisfied.
- A tuning parameter $\kappa$ is introduced such that one of the 4th moment can be tuned.
- This works well for systems till dimension 3 after which the central weight becomes negative.
- The present framework is shown to **be inline** with the $2n+1$ Unscented sigma points by working our way backwards.

# The $2N+1$ Unscented Transform Method

- The $2N+1$ sigma points for the unscented Transform are chosen such that 1 point is the origin and $2N$ points of equal weight are constrained to lie on the principal axis.
- The distance of each point on the principal axis is chosen such that the second order moments are satisfied.
- A tuning parameter $\kappa$ is introduced such that one of the 4th moment can be tuned.
- This works well for systems till dimension 3 after which the central weight becomes negative.
- The present framework is shown to **be inline** with the $2n+1$ Unscented sigma points by working our way backwards.

# THE $2N+1$ UNSCENTED TRANSFORM METHOD

- The $2N+1$ sigma points for the unscented Transform are chosen such that 1 point is the origin and $2N$ points of equal weight are constrained to lie on the principal axis.
- The distance of each point on the principal axis is chosen such that the second order moments are satisfied.
- A tuning parameter $\kappa$ is introduced such that one of the 4th moment can be tuned.
- This works well for systems till dimension 3 after which the central weight becomes negative.
- The present framework is shown to **be inline** with the $2n+1$ Unscented sigma points by working our way backwards.

- The $2N+1$ sigma points for the unscented Transform are chosen such that 1 point is the origin and $2N$ points of equal weight are constrained to lie on the principal axis.
- The distance of each point on the principal axis is chosen such that the second order moments are satisfied.
- A tuning parameter $\kappa$ is introduced such that one of the 4th moment can be tuned.
- This works well for systems till dimension 3 after which the central weight becomes negative.
- The present framework is shown to **be inline** with the $2n+1$ Unscented sigma points by working our way backwards.

The suggested points by the authors of UT are

$$x_0 = \mu \qquad w_0 = \frac{\kappa}{(N+\kappa)}$$

$$x_i = \mu + (sqrt(N+\kappa)P)_i \qquad w_i = \frac{1}{2(N+\kappa)}$$

$$x_{i+N} = \mu - (sqrt(N+\kappa)P)_i \qquad w_{i+N} = \frac{1}{2(N+\kappa)}$$

An **elegant selection for** $r_1$ is chosen and then rest of the variables are solved in terms of this selction

$$r_1 = \sqrt{N+\kappa}$$

$$w_1 = \frac{1}{2(N+\kappa)}$$

$$w_0 = 1 - 2Nw_1 = \frac{\kappa}{(N+\kappa)}$$

$$2r_1^4 w_1 \equiv N + \kappa = 3$$

The suggested points by the authors of UT are

$$x_0 = \mu \qquad w_0 = \frac{\kappa}{(N+\kappa)}$$

$$x_i = \mu + (sqrt(N+\kappa)P)_i \qquad w_i = \frac{1}{2(N+\kappa)}$$

$$x_{i+N} = \mu - (sqrt(N+\kappa)P)_i \qquad w_{i+N} = \frac{1}{2(N+\kappa)}$$

An **elegant selection for** $r_1$ is chosen and then rest of the variables are solved in terms of this selction

$$r_1 = \sqrt{N+\kappa}$$

$$w_1 = \frac{1}{2(N+\kappa)}$$

$$w_0 = 1 - 2Nw_1 = \frac{\kappa}{(N+\kappa)}$$

$$2r_1^4 w_1 \equiv N+\kappa = 3$$

The suggested points by the authors of UT are

$$x_0 = \mu \qquad w_0 = \frac{\kappa}{(N+\kappa)}$$

$$x_i = \mu + (sqrt(N+\kappa)P)_i \qquad w_i = \frac{1}{2(N+\kappa)}$$

$$x_{i+N} = \mu - (sqrt(N+\kappa)P)_i \qquad w_{i+N} = \frac{1}{2(N+\kappa)}$$

An **elegant selection for** $r_1$ is chosen and then rest of the variables are solved in terms of this selction

$$r_1 = \sqrt{N+\kappa}$$

$$w_1 = \frac{1}{2(N+\kappa)}$$

$$w_0 = 1 - 2Nw_1 = \frac{\kappa}{(N+\kappa)}$$

$$2r_1^4 w_1 \equiv N+\kappa = 3$$

The suggested points by the authors of UT are

$$x_0 = \mu \qquad w_0 = \frac{\kappa}{(N+\kappa)}$$

$$x_i = \mu + (sqrt(N+\kappa)P)_i \qquad w_i = \frac{1}{2(N+\kappa)}$$

$$x_{i+N} = \mu - (sqrt(N+\kappa)P)_i \qquad w_{i+N} = \frac{1}{2(N+\kappa)}$$

An **elegant selection for** $r_1$ is chosen and then rest of the variables are solved in terms of this selction

$$r_1 = \sqrt{N+\kappa}$$

$$w_1 = \frac{1}{2(N+\kappa)}$$

$$w_0 = 1 - 2Nw_1 = \frac{\kappa}{(N+\kappa)}$$

$$2r_1^4 w_1 \equiv N+\kappa = 3$$

The suggested points by the authors of UT are

$$x_0 = \mu \qquad w_0 = \frac{\kappa}{(N+\kappa)}$$

$$x_i = \mu + (sqrt(N+\kappa)P)_i \qquad w_i = \frac{1}{2(N+\kappa)}$$

$$x_{i+N} = \mu - (sqrt(N+\kappa)P)_i \qquad w_{i+N} = \frac{1}{2(N+\kappa)}$$

An **elegant selection for** $r_1$ is chosen and then rest of the variables are solved in terms of this selction

$$r_1 = \sqrt{N+\kappa}$$

$$w_1 = \frac{1}{2(N+\kappa)}$$

$$w_0 = 1 - 2Nw_1 = \frac{\kappa}{(N+\kappa)}$$

$$2r_1^4 w_1 \equiv N+\kappa = 3$$

The suggested points by the authors of UT are

$$x_0 = \mu \qquad w_0 = \frac{\kappa}{(N+\kappa)}$$

$$x_i = \mu + (sqrt(N+\kappa)P)_i \qquad w_i = \frac{1}{2(N+\kappa)}$$

$$x_{i+N} = \mu - (sqrt(N+\kappa)P)_i \qquad w_{i+N} = \frac{1}{2(N+\kappa)}$$

An **elegant selection for** $r_1$ is chosen and then rest of the variables are solved in terms of this selction

$$r_1 = \sqrt{N+\kappa}$$

$$w_1 = \frac{1}{2(N+\kappa)}$$

$$w_0 = 1 - 2Nw_1 = \frac{\kappa}{(N+\kappa)}$$

$$2r_1^4 w_1 \equiv N+\kappa = 3$$

The suggested points by the authors of UT are

$$x_0 = \mu \qquad w_0 = \frac{\kappa}{(N+\kappa)}$$

$$x_i = \mu + (sqrt(N+\kappa)P)_i \qquad w_i = \frac{1}{2(N+\kappa)}$$

$$x_{i+N} = \mu - (sqrt(N+\kappa)P)_i \qquad w_{i+N} = \frac{1}{2(N+\kappa)}$$

An **elegant selection for** $r_1$ is chosen and then rest of the variables are solved in terms of this selction

$$r_1 = \sqrt{N+\kappa}$$

$$w_1 = \frac{1}{2(N+\kappa)}$$

$$w_0 = 1 - 2Nw_1 = \frac{\kappa}{(N+\kappa)}$$

$$2r_1^4 w_1 \equiv N + \kappa = 3$$

The suggested points by the authors of UT are

$$x_0 = \mu \qquad w_0 = \frac{\kappa}{(N+\kappa)}$$

$$x_i = \mu + (sqrt(N+\kappa)P)_i \qquad w_i = \frac{1}{2(N+\kappa)}$$

$$x_{i+N} = \mu - (sqrt(N+\kappa)P)_i \qquad w_{i+N} = \frac{1}{2(N+\kappa)}$$

An **elegant selection for** $r_1$ is chosen and then rest of the variables are solved in terms of this selction

$$r_1 = \sqrt{N+\kappa}$$

$$w_1 = \frac{1}{2(N+\kappa)}$$

$$w_0 = 1 - 2Nw_1 = \frac{\kappa}{(N+\kappa)}$$

$$2r_1^4 w_1 \equiv N+\kappa = 3$$

The suggested points by the authors of UT are

$$x_0 = \mu \qquad w_0 = \frac{\kappa}{(N+\kappa)}$$

$$x_i = \mu + (sqrt(N+\kappa)P)_i \qquad w_i = \frac{1}{2(N+\kappa)}$$

$$x_{i+N} = \mu - (sqrt(N+\kappa)P)_i \qquad w_{i+N} = \frac{1}{2(N+\kappa)}$$

An **elegant selection for** $r_1$ is chosen and then rest of the variables are solved in terms of this selction

$$r_1 = \sqrt{N+\kappa}$$

$$w_1 = \frac{1}{2(N+\kappa)}$$

$$w_0 = 1 - 2Nw_1 = \frac{\kappa}{(N+\kappa)}$$

$$2r_1^4 w_1 \equiv N+\kappa = 3$$

- The authors of CKF provide a very mathematically rigourous and elegant way of of finding the second moment equivalent $2N$ cubature points that can again accurately integrate polynomials of degree 3 or less.
- Spherical-radial coordinates are employed rather than cartesian coordinates to find these $2N$ symmetric cubature points on the principal axis.
- Contrary to the $2N+1$ Unscented transform sigma points, they only have $2N$ cubature points on the principal axis and no point on the mean.

- The authors of CKF provide a very mathematically rigourous and elegant way of of finding the second moment equivalent 2$N$ cubature points that can again accurately integrate polynomials of degree 3 or less.
- Spherical-radial coordinates are employed rather than cartesian coordinates to find these 2$N$ symmetric cubature points on the principal axis.
- Contrary to the $2N+1$ Unscented transform sigma points, they only have 2$N$ cubature points on the principal axis and no point on the mean.

- The authors of CKF provide a very mathematically rigourous and elegant way of of finding the second moment equivalent $2N$ cubature points that can again accurately integrate polynomials of degree 3 or less.
- Spherical-radial coordinates are employed rather than cartesian coordinates to find these $2N$ symmetric cubature points on the principal axis.
- Contrary to the $2N+1$ Unscented transform sigma points, they only have $2N$ cubature points on the principal axis and no point on the mean.

- This is equivalent to **setting** $\kappa = 0$ in the Unscented transform sigma points thus making the central weight $w_0$ 0.
- An alternate derivation to the one given in their paper is provided which is less rigourous but **more intuitive** using the present frame work.

As there is no point at the mean, the central weight can be taken 0.

$$\begin{aligned} 0 &= 1 - 2Nw_1 \\ w_1 &= \frac{1}{2N} \\ r_1 &= \sqrt{N} \end{aligned}$$

- This is equivalent to **setting** $\kappa = 0$ in the Unscented transform sigma points thus making the central weight $w_0$ 0.
- An alternate derivation to the one given in their paper is provided which is less rigourous but **more intuitive** using the present frame work.

As there is no point at the mean, the central weight can be taken 0.

$$
\begin{aligned}
0 &= 1 - 2Nw_1 \\
w_1 &= \frac{1}{2N} \\
r_1 &= \sqrt{N}
\end{aligned}
$$

- This is equivalent to **setting** $\kappa = 0$ in the Unscented transform sigma points thus making the central weight $w_0$ 0.
- An alternate derivation to the one given in their paper is provided which is less rigourous but **more intuitive** using the present frame work.

As there is no point at the mean, the central weight can be taken 0.

$$0 = 1 - 2Nw_1$$

$$w_1 = \frac{1}{2N}$$

$$r_1 = \sqrt{N}$$

- This is equivalent to **setting** $\kappa = 0$ in the Unscented transform sigma points thus making the central weight $w_0$ 0.
- An alternate derivation to the one given in their paper is provided which is less rigourous but **more intuitive** using the present frame work.

As there is no point at the mean, the central weight can be taken 0.

$$
\begin{aligned}
0 &= 1 - 2Nw_1 \\
w_1 &= \frac{1}{2N} \\
r_1 &= \sqrt{N}
\end{aligned}
$$

- This is equivalent to **setting** $\kappa = 0$ in the Unscented transform sigma points thus making the central weight $w_0$ 0.
- An alternate derivation to the one given in their paper is provided which is less rigourous but **more intuitive** using the present frame work.

As there is no point at the mean, the central weight can be taken 0.

$$
\begin{aligned}
0 &= 1 - 2Nw_1 \\
w_1 &= \frac{1}{2N} \\
r_1 &= \sqrt{N}
\end{aligned}
$$

- This is equivalent to **setting** $\kappa = 0$ in the Unscented transform sigma points thus making the central weight $w_0$ 0.
- An alternate derivation to the one given in their paper is provided which is less rigourous but **more intuitive** using the present frame work.

As there is no point at the mean, the central weight can be taken 0.

$$0 = 1 - 2Nw_1$$

$$w_1 = \frac{1}{2N}$$

$$r_1 = \sqrt{N}$$

- This is equivalent to **setting** $\kappa = 0$ in the Unscented transform sigma points thus making the central weight $w_0$ 0.
- An alternate derivation to the one given in their paper is provided which is less rigourous but **more intuitive** using the present frame work.

As there is no point at the mean, the central weight can be taken 0.

$$
\begin{aligned}
0 &= 1 - 2Nw_1 \\
w_1 &= \frac{1}{2N} \\
r_1 &= \sqrt{N}
\end{aligned}
$$

# Comparison of UT and CKF cubature points using the error in the 4th moment

- The authors of CKF claim that their method is superior to UKF. They outrightly criticize that UT has no mathematical basis.
- Infact the UT can **capture one of the 4th order moment exactly** but the CKF has an error in all the fourth order moments.
- This is also illustrated by examples later on in the results section.

Absolute error in fourth moment for UT

$$|3-(N+\kappa)| \tag{13}$$

Absolute error in fourth moment for CKF

$$|2N^2\frac{1}{2N}-3| \equiv |N-3| \tag{14}$$

# Comparison of UT and CKF cubature points using the error in the 4th moment

- The authors of CKF claim that their method is superior to UKF. They outrightly criticize that UT has no mathematical basis.
- Infact the UT can **capture one of the 4th order moment exactly** but the CKF has an error in all the fourth order moments.
- This is also illustrated by examples later on in the results section.

Absolute error in fourth moment for UT

$$|3-(N+\kappa)| \tag{13}$$

Absolute error in fourth moment for CKF

$$|2N^2\frac{1}{2N}-3| \equiv |N-3| \tag{14}$$

# Comparison of UT and CKF cubature points using the error in the 4th moment

- The authors of CKF claim that their method is superior to UKF. They outrightly criticize that UT has no mathematical basis.
- Infact the UT can **capture one of the 4th order moment exactly** but the CKF has an error in all the fourth order moments.
- This is also illustrated by examples later on in the results section.

Absolute error in fourth moment for UT

$$|3-(N+\kappa)| \tag{13}$$

Absolute error in fourth moment for CKF

$$|2N^2\frac{1}{2N}-3| \equiv |N-3| \tag{14}$$

## Comparison of UT and CKF cubature points using the error in the 4th moment

- The authors of CKF claim that their method is superior to UKF. They outrightly criticize that UT has no mathematical basis.
- Infact the UT can **capture one of the 4th order moment exactly** but the CKF has an error in all the fourth order moments.
- This is also illustrated by examples later on in the results section.

Absolute error in fourth moment for UT

$$|3-(N+\kappa)| \tag{13}$$

Absolute error in fourth moment for CKF

$$|2N^2\frac{1}{2N}-3| \equiv |N-3| \tag{14}$$

- The authors of CKF claim that their method is superior to UKF. They outrightly criticize that UT has no mathematical basis.
- Infact the UT can **capture one of the 4th order moment exactly** but the CKF has an error in all the fourth order moments.
- This is also illustrated by examples later on in the results section.

Absolute error in fourth moment for UT

$$|3-(N+\kappa)| \tag{13}$$

Absolute error in fourth moment for CKF

$$|2N^2\frac{1}{2N}-3| \equiv |N-3| \tag{14}$$

# Comparison of UT and CKF cubature points using the error in the 4th moment

- The authors of CKF claim that their method is superior to UKF. They outrightly criticize that UT has no mathematical basis.
- Infact the UT can **capture one of the 4th order moment exactly** but the CKF has an error in all the fourth order moments.
- This is also illustrated by examples later on in the results section.

Absolute error in fourth moment for UT

$$|3-(N+\kappa)| \tag{13}$$

Absolute error in fourth moment for CKF

$$|2N^2\frac{1}{2N}-3| \equiv |N-3| \tag{14}$$

- The authors of CKF claim that their method is superior to UKF. They outrightly criticize that UT has no mathematical basis.
- Infact the UT can **capture one of the 4th order moment exactly** but the CKF has an error in all the fourth order moments.
- This is also illustrated by examples later on in the results section.

Absolute error in fourth moment for UT

$$|3-(N+\kappa)| \tag{13}$$

Absolute error in fourth moment for CKF

$$|2N^2\frac{1}{2N}-3| \equiv |N-3| \tag{14}$$

# Comparison of UT and CKF cubature points using the error in the 4th moment

- The authors of CKF claim that their method is superior to UKF. They outrightly criticize that UT has no mathematical basis.
- Infact the UT can **capture one of the 4th order moment exactly** but the CKF has an error in all the fourth order moments.
- This is also illustrated by examples later on in the results section.

Absolute error in fourth moment for UT

$$|3-(N+\kappa)| \tag{13}$$

Absolute error in fourth moment for CKF

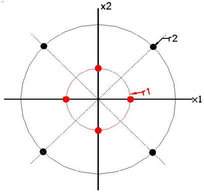
$$|2N^2\frac{1}{2N}-3| \equiv |N-3| \tag{14}$$

## $M^{th}$-Conjugate axis

For a N-Dimensional system where $M \leq N$,the axis that are constructed from all the combinations of the set of principle axis taken M at a time. For example, the $N^{th}$-Conjugate set of axis for N-Dimensional system have $2^N$ distinct points or $2^{N-1}$ axis and the $2^{nd}$-Conjugate set of axis for N-Dimensional system has $2N(N-1)$ distinct points or $N(N-1)$ axis. We label the set of $M^{th}$ conjugate axis as $c^M$, where the points are listed as $c_i^M$

$$c_i^M \in \{\pm\sigma_{n_1} \pm \sigma_{n_2} \pm \ldots \pm \sigma_{n_M} | \{n_1, n_2, \ldots, n_M\} \subset \{1, 2, \ldots, N\}\}$$

$$i = 1, 2, \ldots, 2^M \binom{N}{M}$$



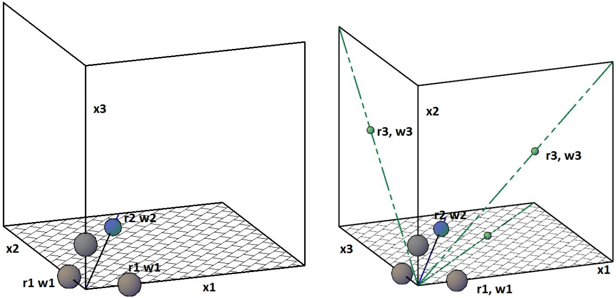




FIGURE: distances and weights

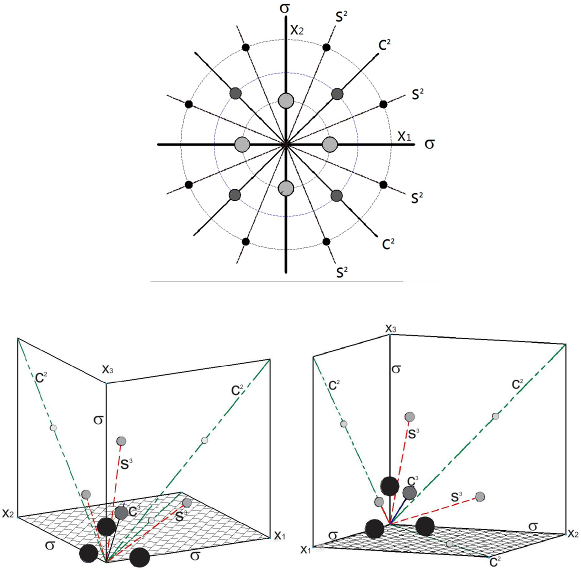



FIGURE: Symmetric set of points and axis 2D and 3D space

- The authors of UT suggest a method to capture the fourth order moments
- But this method suffers the same problem of having a **negative weight above dimension 4**.
- The authors suggest to constrain 1 set of points on the principal axis and another set on the $2^{nd}$-conjugate axis
- A formal derivation in terms of the present framework is shown

$$
\begin{aligned}
2r_1^2 w_1 + 4(N-1)r_2^2 w_2 &= 1 \\
2r_1^4 w_1 + 4(N-1)r_2^4 w_2 &= 3 \\
4r_2^4 w_1 &= 1 \\
1 - 2Nw_1 - 2N(N-1)w_2 &= w_0
\end{aligned}
$$

- The authors of UT suggest a method to capture the fourth order moments
- But this method suffers the same problem of having a **negative weight above dimension 4**.
- The authors suggest to constrain 1 set of points on the principal axis and another set on the $2^{nd}$-conjugate axis
- A formal derivation in terms of the present framework is shown

$$
\begin{aligned}
2r_1^2 w_1 + 4(N-1) r_2^2 w_2 &= 1 \\
2r_1^4 w_1 + 4(N-1) r_2^4 w_2 &= 3 \\
4r_2^4 w_1 &= 1 \\
1 - 2Nw_1 - 2N(N-1) w_2 &= w_0
\end{aligned}
$$

- The authors of UT suggest a method to capture the fourth order moments
- But this method suffers the same problem of having a **negative weight above dimension 4**.
- The authors suggest to constrain 1 set of points on the principal axis and another set on the $2^{nd}$-conjugate axis
- A formal derivation in terms of the present framework is shown

$$
\begin{aligned}
2r_1^2 w_1 + 4(N-1)r_2^2 w_2 &= 1 \\
2r_1^4 w_1 + 4(N-1)r_2^4 w_2 &= 3 \\
4r_2^4 w_1 &= 1 \\
1 - 2Nw_1 - 2N(N-1)w_2 &= w_0
\end{aligned}
$$

- The authors of UT suggest a method to capture the fourth order moments
- But this method suffers the same problem of having a **negative weight above dimension 4**.
- The authors suggest to constrain 1 set of points on the principal axis and another set on the $2^{nd}$-conjugate axis
- A formal derivation in terms of the present framework is shown

$$\begin{aligned}
2r_1^2 w_1 + 4(N-1)r_2^2 w_2 &= 1 \\
2r_1^4 w_1 + 4(N-1)r_2^4 w_2 &= 3 \\
4r_2^4 w_1 &= 1 \\
1 - 2Nw_1 - 2N(N-1)w_2 &= w_0
\end{aligned}$$

- The authors of UT suggest a method to capture the fourth order moments
- But this method suffers the same problem of having a **negative weight above dimension 4**.
- The authors suggest to constrain 1 set of points on the principal axis and another set on the $2^{nd}$-conjugate axis
- A formal derivation in terms of the present framework is shown

$$
\begin{aligned}
2r_1^2 w_1 + 4(N-1) r_2^2 w_2 &= 1 \\
2r_1^4 w_1 + 4(N-1) r_2^4 w_2 &= 3 \\
4r_2^4 w_1 &= 1 \\
1 - 2Nw_1 - 2N(N-1) w_2 &= w_0
\end{aligned}
$$

- The authors of UT suggest a method to capture the fourth order moments
- But this method suffers the same problem of having a **negative weight above dimension 4**.
- The authors suggest to constrain 1 set of points on the principal axis and another set on the $2^{nd}$-conjugate axis
- A formal derivation in terms of the present framework is shown

$$2r_1^2 w_1 + 4(N-1) r_2^2 w_2 = 1$$
$$2r_1^4 w_1 + 4(N-1) r_2^4 w_2 = 3$$
$$4r_2^4 w_1 = 1$$
$$1 - 2N w_1 - 2N(N-1) w_2 = w_0$$

- The authors of UT suggest a method to capture the fourth order moments
- But this method suffers the same problem of having a **negative weight above dimension 4**.
- The authors suggest to constrain 1 set of points on the principal axis and another set on the $2^{nd}$-conjugate axis
- A formal derivation in terms of the present framework is shown

$$2r_1^2 w_1 + 4(N-1)r_2^2 w_2 = 1$$
$$2r_1^4 w_1 + 4(N-1)r_2^4 w_2 = 3$$
$$4r_2^4 w_1 = 1$$
$$1 - 2Nw_1 - 2N(N-1)w_2 = w_0$$

- The authors of UT suggest a method to capture the fourth order moments
- But this method suffers the same problem of having a **negative weight above dimension 4**.
- The authors suggest to constrain 1 set of points on the principal axis and another set on the $2^{nd}$-conjugate axis
- A formal derivation in terms of the present framework is shown

$$
\begin{aligned}
2r_1^2w_1 + 4(N-1)r_2^2w_2 &= 1 \\
2r_1^4w_1 + 4(N-1)r_2^4w_2 &= 3 \\
4r_2^4w_1 &= 1 \\
1 - 2Nw_1 - 2N(N-1)w_2 &= w_0
\end{aligned}
$$

- The authors of UT suggest a method to capture the fourth order moments
- But this method suffers the same problem of having a **negative weight above dimension 4**.
- The authors suggest to constrain 1 set of points on the principal axis and another set on the $2^{nd}$-conjugate axis
- A formal derivation in terms of the present framework is shown

$$
\begin{aligned}
2r_1^2 w_1 + 4(N-1) r_2^2 w_2 &= 1 \\
2r_1^4 w_1 + 4(N-1) r_2^4 w_2 &= 3 \\
4r_2^4 w_1 &= 1 \\
1 - 2N w_1 - 2N(N-1) w_2 &= w_0
\end{aligned}
$$

The simplified set of equations are got by first solving for $w_1$ and $w_2$ and then substituting them in the first equation

$$w_2 = \frac{1}{4r_2^4}$$
$$w_1 = \frac{4-N}{2r_1^4}$$

Substituting these weights into the first constraint equation we have only one cosntraint equation

$$r_1^2 r_2^2 = r_1^2(N-1) + r_2^2(4-N)$$

Even though we solve for $r_1$ and $r_2$ we always have a negative/zero weight for $N > 3$

The simplified set of equations are got by first solving for $w_1$ and $w_2$ and then substituting them in the first equation

$$w_2 = \frac{1}{4r_2^4}$$
$$w_1 = \frac{4-N}{2r_1^4}$$

Substituting these weights into the first constraint equation we have only one cosntraint equation

$$r_1^2 r_2^2 = r_1^2(N-1) + r_2^2(4-N)$$

Even though we solve for $r_1$ and $r_2$ we always have a negative/zero weight for $N > 3$

The simplified set of equations are got by first solving for $w_1$ and $w_2$ and then substituting them in the first equation

$$w_2 = \frac{1}{4r_2^4}$$
$$w_1 = \frac{4-N}{2r_1^4}$$

Substituting these weights into the first constraint equation we have only one cosntraint equation

$$r_1^2 r_2^2 = r_1^2(N-1) + r_2^2(4-N)$$

Even though we solve for $r_1$ and $r_2$ we always have a negative/zero weight for $N > 3$

The simplified set of equations are got by first solving for $w_1$ and $w_2$ and then substituting them in the first equation

$$w_2 = \frac{1}{4r_2^4}$$
$$w_1 = \frac{4-N}{2r_1^4}$$

Substituting these weights into the first constraint equation we have only one cosntraint equation

$$r_1^2 r_2^2 = r_1^2(N-1) + r_2^2(4-N)$$

Even though we solve for $r_1$ and $r_2$ we always have a negative/zero weight for $N > 3$

The simplified set of equations are got by first solving for $w_1$ and $w_2$ and then substituting them in the first equation

$$w_2 = \frac{1}{4r_2^4}$$
$$w_1 = \frac{4-N}{2r_1^4}$$

Substituting these weights into the first constraint equation we have only one cosntraint equation

$$r_1^2 r_2^2 = r_1^2(N-1) + r_2^2(4-N)$$

Even though we solve for $r_1$ and $r_2$ we always have a negative/zero weight for $N > 3$

The simplified set of equations are got by first solving for $w_1$ and $w_2$ and then substituting them in the first equation

$$w_2 = \frac{1}{4r_2^4}$$
$$w_1 = \frac{4-N}{2r_1^4}$$

Substituting these weights into the first constraint equation we have only one cosntraint equation

$$r_1^2 r_2^2 = r_1^2(N-1) + r_2^2(4-N)$$

Even though we solve for $r_1$ and $r_2$ we always have a negative/zero weight for $N > 3$

# The Conjugate Unscented Transform Method to capture all the 4th moment

- The first set of points are chosen on the principal axis at a distance of $r_1$ and weight $w_1$ each.
- The second set of points are chosen on the Nth-Conjugate axis at a distance of $r_2$ and each have a weight of $w_2$.

Forming the moment constraint equations along with the central weight

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 &= 3 \\
2^N r_2^4 w_2 &= 1 \\
1 - 2N w_1 - 2^N w_2 &= w_0
\end{aligned}
$$

- The first set of points are chosen on the principal axis at a distance of $r_1$ and weight $w_1$ each.
- The second set of points are chosen on the Nth-Conjugate axis at a distance of $r_2$ and each have a weight of $w_2$.

Forming the moment constraint equations along with the central weight

$$\begin{aligned} 2r_1^2 w_1 + 2^N r_2^2 w_2 &= 1 \\ 2r_1^4 w_1 + 2^N r_2^4 w_2 &= 3 \\ 2^N r_2^4 w_2 &= 1 \\ 1 - 2N w_1 - 2^N w_2 &= w_0 \end{aligned}$$

# The Conjugate Unscented Transform Method to capture all the 4th moment

- The first set of points are chosen on the principal axis at a distance of $r_1$ and weight $w_1$ each.
- The second set of points are chosen on the Nth-Conjugate axis at a distance of $r_2$ and each have a weight of $w_2$.

Forming the moment constraint equations along with the central weight

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 &= 3 \\
2^N r_2^4 w_2 &= 1 \\
1 - 2N w_1 - 2^N w_2 &= w_0
\end{aligned}
$$

# The Conjugate Unscented Transform Method to capture all the 4th moment

- The first set of points are chosen on the principal axis at a distance of $r_1$ and weight $w_1$ each.
- The second set of points are chosen on the Nth-Conjugate axis at a distance of $r_2$ and each have a weight of $w_2$.

Forming the moment constraint equations along with the central weight

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 &= 3 \\
2^N r_2^4 w_2 &= 1 \\
1 - 2N w_1 - 2^N w_2 &= w_0
\end{aligned}
$$

# The Conjugate Unscented Transform Method to capture all the 4th moment

- The first set of points are chosen on the principal axis at a distance of $r_1$ and weight $w_1$ each.
- The second set of points are chosen on the Nth-Conjugate axis at a distance of $r_2$ and each have a weight of $w_2$.

Forming the moment constraint equations along with the central weight

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 &= 3 \\
2^N r_2^4 w_2 &= 1 \\
1 - 2N w_1 - 2^N w_2 &= w_0
\end{aligned}
$$

- The first set of points are chosen on the principal axis at a distance of $r_1$ and weight $w_1$ each.
- The second set of points are chosen on the Nth-Conjugate axis at a distance of $r_2$ and each have a weight of $w_2$.

Forming the moment constraint equations along with the central weight

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 &= 3 \\
2^N r_2^4 w_2 &= 1 \\
1 - 2N w_1 - 2^N w_2 &= w_0
\end{aligned}
$$

- The first set of points are chosen on the principal axis at a distance of $r_1$ and weight $w_1$ each.
- The second set of points are chosen on the Nth-Conjugate axis at a distance of $r_2$ and each have a weight of $w_2$.

Forming the moment constraint equations along with the central weight

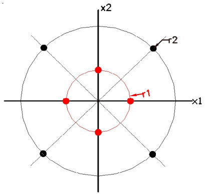
$$\begin{aligned} 2r_1^2 w_1 + 2^N r_2^2 w_2 &= 1 \\ 2r_1^4 w_1 + 2^N r_2^4 w_2 &= 3 \\ 2^N r_2^4 w_2 &= 1 \\ 1 - 2N w_1 - 2^N w_2 &= w_0 \end{aligned}$$

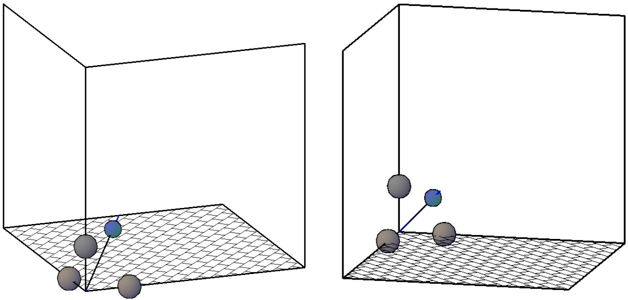
- The first set of points are chosen on the principal axis at a distance of $r_1$ and weight $w_1$ each.
- The second set of points are chosen on the Nth-Conjugate axis at a distance of $r_2$ and each have a weight of $w_2$.

Forming the moment constraint equations along with the central weight

$$\begin{aligned} 2r_1^2 w_1 + 2^N r_2^2 w_2 &= 1 \\ 2r_1^4 w_1 + 2^N r_2^4 w_2 &= 3 \\ 2^N r_2^4 w_2 &= 1 \\ 1 - 2N w_1 - 2^N w_2 &= w_0 \end{aligned}$$

# CUT4



These set of equations can be **solved analytically** in the following 2 schemes because there is no constraint on $w_0$ except that it should be positive

**Solution 1**

The $w_0$ is solved for by minimizing the square of error in sixth moment

$$Min \quad (2r_1^6 w_1 + 2^N r_2^6 w_2 - 15)^2$$

This is applied for dimensions 1 and 2

TABLE: Optimization Solution for $N = 1$ and $N = 2$

| Variable | $N = 1$ | $N = 2$ |
|---|---|---|
| $r_1$ | 1.4861736616297834 | 2.6060099476935847 |
| $r_2$ | 3.2530871022700643 | 1.190556300661233 |
| $w_0$ | 0.5811010092660772 | 0.41553535186548973 |
| $w_1$ | 0.20498484723245053 | 0.021681819434216532 |
| $w_2$ | 0.0044646481345109 | 0.12443434259941118 |

These set of equations can be **solved analytically** in the following 2 schemes because there is no constraint on $w_0$ except that it should be positive

**Solution 1**

The $w_0$ is solved for by minimizing the square of error in sixth moment

$$Min \quad (2r_1^6 w_1 + 2^N r_2^6 w_2 - 15)^2$$

This is applied for dimensions 1 and 2

TABLE: Optimization Solution for $N = 1$ and $N = 2$

| Variable | $N = 1$ | $N = 2$ |
|---|---|---|
| $r_1$ | 1.48617366162978 34 | 2.6060099476935847 |
| $r_2$ | 3.2530871022700643 | 1.190556300661233 |
| $w_0$ | 0.5811010092660772 | 0.41553535186548973 |
| $w_1$ | 0.20498484723245053 | 0.021681819434216532 |
| $w_2$ | 0.00446464813451093 | 0.12443434259941118 |

These set of equations can be **solved analytically** in the following 2 schemes because there is no constraint on $w_0$ except that it should be positive

**Solution 1**

The $w_0$ is solved for by minimizing the square of error in sixth moment

$$Min \quad (2r_1^6 w_1 + 2^N r_2^6 w_2 - 15)^2$$

This is applied for dimensions 1 and 2

TABLE: Optimization Solution for $N = 1$ and $N = 2$

| Variable | $N = 1$ | $N = 2$ |
|---|---|---|
| $r_1$ | 1.4861736616297834 | 2.6060099476935847 |
| $r_2$ | 3.2530871022700643 | 1.190556300661233 |
| $w_0$ | 0.5811010092660772 | 0.41553535186548973 |
| $w_1$ | 0.20498484723245053 | 0.021681819434216532 |
| $w_2$ | 0.0044646481345109 3 | 0.12443434259941118 |

These set of equations can be **solved analytically** in the following 2 schemes because there is no constraint on $w_0$ except that it should be positive

**Solution 1**

The $w_0$ is solved for by minimizing the square of error in sixth moment

$$Min \quad (2r_1^6 w_1 + 2^N r_2^6 w_2 - 15)^2$$

This is applied for dimensions 1 and 2

TABLE: Optimization Solution for $N = 1$ and $N = 2$

| Variable | $N = 1$ | $N = 2$ |
|---|---|---|
| $r_1$ | 1.4861736616297834 | 2.6060099476935847 |
| $r_2$ | 3.2530871022700643 | 1.190556300661233 |
| $w_0$ | 0.5811010092660772 | 0.41553535186548973 |
| $w_1$ | 0.20498484723245053 | 0.021681819434216532 |
| $w_2$ | 0.0044464813451093 | 0.1244343259941118 |

These set of equations can be **solved analytically** in the following 2 schemes because there is no constraint on $w_0$ except that it should be positive

**Solution 1**

The $w_0$ is solved for by minimizing the square of error in sixth moment

$$Min \quad (2r_1^6 w_1 + 2^N r_2^6 w_2 - 15)^2$$

This is applied for dimensions 1 and 2

TABLE: Optimization Solution for $N = 1$ and $N = 2$

| Variable | $N = 1$ | $N = 2$ |
|---|---|---|
| $r_1$ | 1.4861736616297834 | 2.6060099476935847 |
| $r_2$ | 3.2530871022700643 | 1.190556300661233 |
| $w_0$ | 0.5811010092660772 | 0.41553535186548973 |
| $w_1$ | 0.20498484723245053 | 0.021681819434216532 |
| $w_2$ | 0.0044646481345093 | 0.12443434259941118 |

These set of equations can be **solved analytically** in the following 2 schemes because there is no constraint on $w_0$ except that it should be positive

**Solution 1**

The $w_0$ is solved for by minimizing the square of error in sixth moment

$$Min \quad (2r_1^6 w_1 + 2^N r_2^6 w_2 - 15)^2$$

This is applied for dimensions 1 and 2

TABLE: Optimization Solution for $N = 1$ and $N = 2$

| Variable | $N = 1$ | $N = 2$ |
|---|---|---|
| $r_1$ | 1.4861736616297834 | 2.6060099476935847 |
| $r_2$ | 3.2530871022700643 | 1.190556300661233 |
| $w_0$ | 0.5811010092660772 | 0.41553535186548973 |
| $w_1$ | 0.20498484723245053 | 0.021681819434216532 |
| $w_2$ | 0.0044646481345109 | 0.12443434259941118 |

**Solution 2** For dimension greater than 2, the central weight is eliminated thus **reducing 1 point**. For $N > 2$

$$w_0 = 0$$

$$r1 = \sqrt{\frac{N+2}{2}}$$

$$r2 = \sqrt{\frac{N+2}{N-2}}$$

$$w_1 = \frac{1}{r_1^4} = \frac{4}{(N+2)^2}$$

$$w_2 = \frac{1}{2^N r_2^4} = \frac{(N-2)^2}{2^N (N+2)^2}$$

**Solution 2** For dimension greater than 2, the central weight is eliminated thus **reducing 1 point**. For $N > 2$

$$w_0 = 0$$

$$r1 = \sqrt{\frac{N+2}{2}}$$

$$r2 = \sqrt{\frac{N+2}{N-2}}$$

$$w_1 = \frac{1}{r_1^4} = \frac{4}{(N+2)^2}$$

$$w_2 = \frac{1}{2^N r_2^4} = \frac{(N-2)^2}{2^N (N+2)^2}$$

**Solution 2** For dimension greater than 2, the central weight is eliminated thus **reducing 1 point**. For $N > 2$

$$w_0 = 0$$

$$r1 = \sqrt{\frac{N+2}{2}}$$

$$r2 = \sqrt{\frac{N+2}{N-2}}$$

$$w_1 = \frac{1}{r_1^4} = \frac{4}{(N+2)^2}$$

$$w_2 = \frac{1}{2^N r_2^4} = \frac{(N-2)^2}{2^N (N+2)^2}$$

**Solution 2** For dimension greater than 2, the central weight is eliminated thus **reducing 1 point**. For $N > 2$

$$w_0 = 0$$

$$r1 = \sqrt{\frac{N+2}{2}}$$

$$r2 = \sqrt{\frac{N+2}{N-2}}$$

$$w_1 = \frac{1}{r_1^4} = \frac{4}{(N+2)^2}$$

$$w_2 = \frac{1}{2^N r_2^4} = \frac{(N-2)^2}{2^N (N+2)^2}$$

**Solution 2** For dimension greater than 2, the central weight is eliminated thus **reducing 1 point**. For $N > 2$

$$w_0 = 0$$

$$r1 = \sqrt{\frac{N+2}{2}}$$

$$r2 = \sqrt{\frac{N+2}{N-2}}$$

$$w_1 = \frac{1}{r_1^4} = \frac{4}{(N+2)^2}$$

$$w_2 = \frac{1}{2^N r_2^4} = \frac{(N-2)^2}{2^N(N+2)^2}$$

**Solution 2** For dimension greater than 2, the central weight is eliminated thus **reducing 1 point**. For $N > 2$

$$w_0 = 0$$

$$r1 = \sqrt{\frac{N+2}{2}}$$

$$r2 = \sqrt{\frac{N+2}{N-2}}$$

$$w_1 = \frac{1}{r_1^4} = \frac{4}{(N+2)^2}$$

$$w_2 = \frac{1}{2^N r_2^4} = \frac{(N-2)^2}{2^N (N+2)^2}$$

**Solution 2** For dimension greater than 2, the central weight is eliminated thus **reducing 1 point**. For $N > 2$

$$w_0 = 0$$

$$r1 = \sqrt{\frac{N+2}{2}}$$

$$r2 = \sqrt{\frac{N+2}{N-2}}$$

$$w_1 = \frac{1}{r_1^4} = \frac{4}{(N+2)^2}$$

$$w_2 = \frac{1}{2^N r_2^4} = \frac{(N-2)^2}{2^N (N+2)^2}$$

**Solution 2** For dimension greater than 2, the central weight is eliminated thus **reducing 1 point**. For $N > 2$

$$
\begin{aligned}
w_0 &= 0 \\
r1 &= \sqrt{\frac{N+2}{2}} \\
r2 &= \sqrt{\frac{N+2}{N-2}} \\
w_1 = \frac{1}{r_1^4} &= \frac{4}{(N+2)^2} \\
w_2 = \frac{1}{2^N r_2^4} &= \frac{(N-2)^2}{2^N (N+2)^2}
\end{aligned}
$$

TABLE: Sigma Points for CUT4

| | Position | Weights |
|---|---|---|
| $1 \leq i \leq 2N$ | $X_i = r_1 \sigma_i$ | $W_i = w_1$ |
| $1 \leq i \leq 2^N$ | $X_{i+2N} = r_2 c_i^N$ | $W_{i+2N} = w_2$ |
| Central weight | $X_0 = \mathbf{0}$ | $W_0 = w_0$ |
| $n = 2N + 2^N \ (+1)$ | | |

### IN SUMMARY

- Solve the distance and weight variables
- Generate the principle axis and Conjugate axis
- Generate the sigma points
- Transform the points to match the required Mean and Covariance.

TABLE: Sigma Points for CUT4

| | Position | Weights |
|---|---|---|
| $1 \leq i \leq 2N$ | $X_i = r_1 \sigma_i$ | $W_i = w_1$ |
| $1 \leq i \leq 2^N$ | $X_{i+2N} = r_2 c_i^N$ | $W_{i+2N} = w_2$ |
| Central weight | $X_0 = \mathbf{0}$ | $W_0 = w_0$ |
| $n = 2N + 2^N\ (+1)$ | | |

### IN SUMMARY

- Solve the distance and weight variables
- Generate the principle axis and Conjugate axis
- Generate the sigma points
- Transform the points to match the required Mean and Covariance.

TABLE: Sigma Points for CUT4

| | Position | Weights |
|---|---|---|
| $1 \leq i \leq 2N$ | $X_i = r_1 \sigma_i$ | $W_i = w_1$ |
| $1 \leq i \leq 2^N$ | $X_{i+2N} = r_2 c_i^N$ | $W_{i+2N} = w_2$ |
| Central weight | $X_0 = \mathbf{0}$ | $W_0 = w_0$ |
| $n = 2N + 2^N\ (+1)$ | | |

### IN SUMMARY

- Solve the distance and weight variables
- Generate the principle axis and Conjugate axis
- Generate the sigma points
- Transform the points to match the required Mean and Covariance.

TABLE: Sigma Points for CUT4

| | Position | Weights |
|---|---|---|
| $1 \le i \le 2N$ | $X_i = r_1 \sigma_i$ | $W_i = w_1$ |
| $1 \le i \le 2^N$ | $X_{i+2N} = r_2 c_i^N$ | $W_{i+2N} = w_2$ |
| Central weight | $X_0 = \mathbf{0}$ | $W_0 = w_0$ |
| $n = 2N + 2^N \; (+1)$ | | |

### IN SUMMARY

- Solve the distance and weight variables
- Generate the principle axis and Conjugate axis
- Generate the sigma points
- Transform the points to match the required Mean and Covariance.

**Results of integration compared to Gauss Hermite integration for 3D system**

The total number of cubature points involved in this method to capture all the moments till 4th order is$2N+2^N$

$$P = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 9 & 1 \\ 1 & 1 & 16 \end{bmatrix}$$

$$F = x_1^4 + x_2^4 + x_3^4 + x_1^3 x_2 + x_1^2 x_2^2 + \\ + x_3^2 x_2^2 + x_1^2 x_3^2 + x_1^3 x_3 + x_2^3 x_3 + x_3^3 x_2$$

| No. of pts | $2^3 = 8$ | $3^3 = 27$ | $4^3 = 64$ | Analytical |
|---|---|---|---|---|
| GH | 1056.95571 | 1797.99999 | 1798.00 | 1798.00 |
| % error wrt Truth | 41.2149211 | 4.299e-013 | 3.0350e-013 | 0 |

| | No. of pts | Integration result | % error |
|---|---|---|---|
| CUT4 | 15 | 1797.999999 | 6.3552080e-010 |

| No. of pts | $2^3 = 8$ | $3^3 = 27$ | $4^3 = 64$ | Analytical |
|---|---|---|---|---|
| GH | 1056.95571 | 1797.99999 | 1798.00 | 1798.00 |
| % error wrt Truth | 41.2149211 | 4.299e-013 | 3.0350e-013 | 0 |

| | No. of pts | Integration result | % error |
|---|---|---|---|
| CUT4 | 15 | 1797.999999 | 6.3552080e-010 |

**Results of integration compared to Gauss Hermite integration for 8D system**
The total number of cubature points involved in this method to capture all the moments till 4th order is $2N + 2^N$

$$P = 10I_{8x8}$$

$$F = x_1^4 + x_8^2 * x_2^2 + x_3^2 + x_4^2 * x_5^2$$

| No. of pts | $2^8 = 8$ | $3^8 = 6561$ | $4^8 = 65536$ | Analytical |
|---|---|---|---|---|
| GH | 256 | 614 | 614 | 614 |
| % error | 32.573 | 5.18441605e-013 | 6.591614e-012 | 0 |

| | No. of pts | Integration result | % error |
|---|---|---|---|
| CUT4 | 273 | 614 | 1.0368832104e-012 |

| No. of pts | $2^8 = 8$ | $3^8 = 6561$ | $4^8 = 65536$ | Analytical |
|---|---|---|---|---|
| GH | 256 | 614 | 614 | 614 |
| % error | 32.573 | 5.18441605e-013 | 6.591614e-012 | 0 |

| | No. of pts | Integration result | % error |
|---|---|---|---|
| CUT4 | 273 | 614 | 1.0368832104e-012 |

$$P_1 = \begin{bmatrix} 114.2595 & 90.1397 & 8.9751 \\ 90.1397 & 92.2504 & 29.1237 \\ 8.9751 & 29.1237 & 84.0908 \end{bmatrix} \quad (15)$$

$$P_2 = 100I_{(10x10)} \quad (16)$$

$$f(X) = (\sqrt{1+X^TX})^4 \quad (17)$$

TABLE: CUT4

| method | $n_1$ | $\mu_1$ % error | $n_2$ | $\mu_2$ % error |
|---|---|---|---|---|
| GH2 | 8 | 52.36 | 1024 | 16.39 |
| GH3 | 27 | $4.89e-014$ | 59049 | $7.23e-012$ |
| **CUT4** | 14 | 0 | 1044 | $6.72e-012$ |

**Cubature points that are $6^{th}$ moment equivalent**

- We **attempt** to capture all the $6^{th}$ order moments till 6D
- In addition to the axis chosen for the fourth moment one additional set of axis is considered
- 1 point on the mean, first set of points on the principal axis at distance $r_1$ and each weight $w_1$.
- Second set on the Nth-Conjugate axis with distance $r_2$ and weight $w_2$.
- Third set of axis are chosen on the 2nd conjugate axis with points at distance $r_3$ and weight $w_3$

**Cubature points that are $6^{th}$ moment equivalent**

- We **attempt** to capture all the $6^{th}$ order moments till 6D
- In addition to the axis chosen for the fourth moment one additional set of axis is considered
- 1 point on the mean, first set of points on the principal axis at distance $r_1$ and each weight $w_1$.
- Second set on the Nth-Conjugate axis with distance $r_2$ and weight $w_2$.
- Third set of axis are chosen on the 2nd conjugate axis with points at distance $r_3$ and weight $w_3$

**Cubature points that are $6^{th}$ moment equivalent**

- We **attempt** to capture all the $6^{th}$ order moments till 6D
- In addition to the axis chosen for the fourth moment one additional set of axis is considered
- 1 point on the mean, first set of points on the principal axis at distance $r_1$ and each weight $w_1$.
- Second set on the Nth-Conjugate axis with distance $r_2$ and weight $w_2$.
- Third set of axis are chosen on the 2nd conjugate axis with points at distance $r_3$ and weight $w_3$

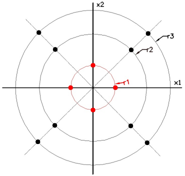
# 6TH ORDER CONJUGATE UNSCENTED TRANSFORM - CUT6

**Cubature points that are $6^{th}$ moment equivalent**

- We **attempt** to capture all the $6^{th}$ order moments till 6D
- In addition to the axis chosen for the fourth moment one additional set of axis is considered
- 1 point on the mean, first set of points on the principal axis at distance $r_1$ and each weight $w_1$.
- Second set on the Nth-Conjugate axis with distance $r_2$ and weight $w_2$.
- Third set of axis are chosen on the 2nd conjugate axis with points at distance $r_3$ and weight $w_3$

**Cubature points that are $6^{th}$ moment equivalent**

- We **attempt** to capture all the $6^{th}$ order moments till 6D
- In addition to the axis chosen for the fourth moment one additional set of axis is considered
- 1 point on the mean, first set of points on the principal axis at distance $r_1$ and each weight $w_1$.
- Second set on the Nth-Conjugate axis with distance $r_2$ and weight $w_2$.
- Third set of axis are chosen on the 2nd conjugate axis with points at distance $r_3$ and weight $w_3$



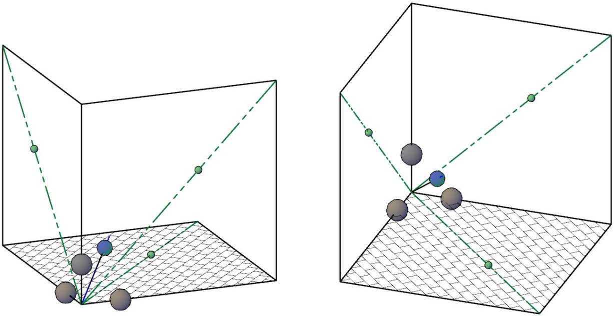



FIGURE: distances and weights

# MOMENTS TILL 6TH ORDER

$$E[X_i^2] = 1 \qquad E[X_iX_j] = 0$$
$$E[X_i^4] = 3 \qquad E[X_i^3X_j] = 0$$
$$E[X_i^2X_j^2] = 1 \qquad E[X_i^2X_jX_k] = 0$$
$$E[X_iX_jX_kX_l] = 0 \qquad E[X_i^6] = 15$$
$$E[X_i^5X_j] = 0 \qquad E[X_i^4X_j^2] = 3$$
$$E[X_i^4X_jX_k] = 0 \qquad E[X_i^3X_j^3] = 0$$
$$E[X_i^3X_j^2X_k] = 0 \qquad E[X_i^3X_j^2] = 0$$
$$E[X_i^2X_j^2X_k^2] = 1 \qquad E[X_iX_jX_kX_lX_mX_n] = 0$$

# Moments till 6th order

$$E[X_i^2] = 1 \qquad E[X_iX_j] = 0$$
$$E[X_i^4] = 3 \qquad E[X_i^3X_j] = 0$$
$$E[X_i^2X_j^2] = 1 \qquad E[X_i^2X_jX_k] = 0$$
$$E[X_iX_jX_kX_l] = 0 \qquad E[X_i^6] = 15$$
$$E[X_i^5X_j] = 0 \qquad E[X_i^4X_j^2] = 3$$
$$E[X_i^4X_jX_k] = 0 \qquad E[X_i^3X_j^3] = 0$$
$$E[X_i^3X_j^2X_k] = 0 \qquad E[X_i^3X_j^2] = 0$$
$$E[X_i^2X_j^2X_k^2] = 1 \qquad E[X_iX_jX_kX_lX_mX_n] = 0$$

$$E[X_i^2] = 1$$
$$E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1$$
$$E[X_i X_j X_k X_l] = 0$$
$$E[X_i^5 X_j] = 0$$
$$E[X_i^4 X_j X_k] = 0$$
$$E[X_i^3 X_j^2 X_k] = 0$$
$$E[X_i^2 X_j^2 X_k^2] = 1$$

$$E[X_i X_j] = 0$$
$$E[X_i^3 X_j] = 0$$
$$E[X_i^2 X_j X_k] = 0$$
$$E[X_i^6] = 15$$
$$E[X_i^4 X_j^2] = 3$$
$$E[X_i^3 X_j^3] = 0$$
$$E[X_i^3 X_j^2] = 0$$
$$E[X_i X_j X_k X_l X_m X_n] = 0$$

# Moments till 6th order

$$E[X_i^2] = 1 \qquad E[X_iX_j] = 0$$

$$E[X_i^4] = 3 \qquad E[X_i^3X_j] = 0$$

$$E[X_i^2X_j^2] = 1 \qquad E[X_i^2X_jX_k] = 0$$

$$E[X_iX_jX_kX_l] = 0 \qquad E[X_i^6] = 15$$

$$E[X_i^5X_j] = 0 \qquad E[X_i^4X_j^2] = 3$$

$$E[X_i^4X_jX_k] = 0 \qquad E[X_i^3X_j^3] = 0$$

$$E[X_i^3X_j^2X_k] = 0 \qquad E[X_i^3X_j^2] = 0$$

$$E[X_i^2X_j^2X_k^2] = 1 \qquad E[X_iX_jX_kX_lX_mX_n] = 0$$

# Moments till 6th order

$$E[X_i^2] = 1 \qquad E[X_iX_j] = 0$$
$$E[X_i^4] = 3 \qquad E[X_i^3X_j] = 0$$
$$E[X_i^2X_j^2] = 1 \qquad E[X_i^2X_jX_k] = 0$$
$$E[X_iX_jX_kX_l] = 0 \qquad E[X_i^6] = 15$$
$$E[X_i^5X_j] = 0 \qquad E[X_i^4X_j^2] = 3$$
$$E[X_i^4X_jX_k] = 0 \qquad E[X_i^3X_j^3] = 0$$
$$E[X_i^3X_j^2X_k] = 0 \qquad E[X_i^3X_j^2] = 0$$
$$E[X_i^2X_j^2X_k^2] = 1 \qquad E[X_iX_jX_kX_lX_mX_n] = 0$$

$$E[X_i^2] = 1 \qquad E[X_iX_j] = 0$$

$$E[X_i^4] = 3 \qquad E[X_i^3X_j] = 0$$

$$E[X_i^2X_j^2] = 1 \qquad E[X_i^2X_jX_k] = 0$$

$$E[X_iX_jX_kX_l] = 0 \qquad E[X_i^6] = 15$$

$$E[X_i^5X_j] = 0 \qquad E[X_i^4X_j^2] = 3$$

$$E[X_i^4X_jX_k] = 0 \qquad E[X_i^3X_j^3] = 0$$

$$E[X_i^3X_j^2X_k] = 0 \qquad E[X_i^3X_j^2] = 0$$

$$E[X_i^2X_j^2X_k^2] = 1 \qquad E[X_iX_jX_kX_lX_mX_n] = 0$$

$$E[X_i^2] = 1 \qquad E[X_iX_j] = 0$$
$$E[X_i^4] = 3 \qquad E[X_i^3X_j] = 0$$
$$E[X_i^2X_j^2] = 1 \qquad E[X_i^2X_jX_k] = 0$$
$$E[X_iX_jX_kX_l] = 0 \qquad E[X_i^6] = 15$$
$$E[X_i^5X_j] = 0 \qquad E[X_i^4X_j^2] = 3$$
$$E[X_i^4X_jX_k] = 0 \qquad E[X_i^3X_j^3] = 0$$
$$E[X_i^3X_j^2X_k] = 0 \qquad E[X_i^3X_j^2] = 0$$
$$E[X_i^2X_j^2X_k^2] = 1 \qquad E[X_iX_jX_kX_lX_mX_n] = 0$$

$$E[X_i^2] = 1 \qquad E[X_iX_j] = 0$$
$$E[X_i^4] = 3 \qquad E[X_i^3X_j] = 0$$
$$E[X_i^2X_j^2] = 1 \qquad E[X_i^2X_jX_k] = 0$$
$$E[X_iX_jX_kX_l] = 0 \qquad E[X_i^6] = 15$$
$$E[X_i^5X_j] = 0 \qquad E[X_i^4X_j^2] = 3$$
$$E[X_i^4X_jX_k] = 0 \qquad E[X_i^3X_j^3] = 0$$
$$E[X_i^3X_j^2X_k] = 0 \qquad E[X_i^3X_j^2] = 0$$
$$E[X_i^2X_j^2X_k^2] = 1 \qquad E[X_iX_jX_kX_lX_mX_n] = 0$$

$$E[X_i^2] = 1 \qquad E[X_iX_j] = 0$$
$$E[X_i^4] = 3 \qquad E[X_i^3X_j] = 0$$
$$E[X_i^2X_j^2] = 1 \qquad E[X_i^2X_jX_k] = 0$$
$$E[X_iX_jX_kX_l] = 0 \qquad E[X_i^6] = 15$$
$$E[X_i^5X_j] = 0 \qquad E[X_i^4X_j^2] = 3$$
$$E[X_i^4X_jX_k] = 0 \qquad E[X_i^3X_j^3] = 0$$
$$E[X_i^3X_j^2X_k] = 0 \qquad E[X_i^3X_j^2] = 0$$
$$E[X_i^2X_j^2X_k^2] = 1 \qquad E[X_iX_jX_kX_lX_mX_n] = 0$$

# MOMENT CONSTRAINT EQUATIONS TILL ORDER 6

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 + 4(N-1) r_3^2 w_3 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 + 4(N-1) r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 4 r_3^4 &= 1 \\
2r_1^6 w_1 + 2^N r_2^6 w_2 + 4(N-1) r_3^6 w_3 &= 15 \\
2^N r_2^6 w_2 + 4 r_3^6 &= 3 \\
2^N r_2^6 w_2 &= 1
\end{aligned}
$$

$$2r_1^2w_1 + 2^N r_2^2 w_2 + 4(N-1)r_3^2 w_3 = 1$$

$$2r_1^4w_1 + 2^N r_2^4 w_2 + 4(N-1)r_3^4 w_3 = 3$$

$$2^N r_2^4 w_2 + 4r_3^4 = 1$$

$$2r_1^6w_1 + 2^N r_2^6 w_2 + 4(N-1)r_3^6 w_3 = 15$$

$$2^N r_2^6 w_2 + 4r_3^6 = 3$$

$$2^N r_2^6 w_2 = 1$$

# Moment constraint equations till order 6

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 + 4(N-1) r_3^2 w_3 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 + 4(N-1) r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 4 r_3^4 &= 1 \\
2r_1^6 w_1 + 2^N r_2^6 w_2 + 4(N-1) r_3^6 w_3 &= 15 \\
2^N r_2^6 w_2 + 4 r_3^6 &= 3 \\
2^N r_2^6 w_2 &= 1
\end{aligned}
$$

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 + 4(N-1) r_3^2 w_3 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 + 4(N-1) r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 4r_3^4 &= 1 \\
2r_1^6 w_1 + 2^N r_2^6 w_2 + 4(N-1) r_3^6 w_3 &= 15 \\
2^N r_2^6 w_2 + 4r_3^6 &= 3 \\
2^N r_2^6 w_2 &= 1
\end{aligned}
$$

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 + 4(N-1) r_3^2 w_3 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 + 4(N-1) r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 4 r_3^4 &= 1 \\
2r_1^6 w_1 + 2^N r_2^6 w_2 + 4(N-1) r_3^6 w_3 &= 15 \\
2^N r_2^6 w_2 + 4 r_3^6 &= 3 \\
2^N r_2^6 w_2 &= 1
\end{aligned}
$$

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 + 4(N-1) r_3^2 w_3 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 + 4(N-1) r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 4 r_3^4 &= 1 \\
2r_1^6 w_1 + 2^N r_2^6 w_2 + 4(N-1) r_3^6 w_3 &= 15 \\
2^N r_2^6 w_2 + 4 r_3^6 &= 3 \\
2^N r_2^6 w_2 &= 1
\end{aligned}
$$

# Moment constraint equations till order 6

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 + 4(N-1) r_3^2 w_3 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 + 4(N-1) r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 4 r_3^4 &= 1 \\
2r_1^6 w_1 + 2^N r_2^6 w_2 + 4(N-1) r_3^6 w_3 &= 15 \\
2^N r_2^6 w_2 + 4 r_3^6 &= 3 \\
2^N r_2^6 w_2 &= 1
\end{aligned}
$$

Procedure followed

- These nonlinear equations are in a way linear with respect to the weights
- The last 3 equations can be solved for $w_1$, $w_2$ and $w_3$ symbolically in terms of the other variables $r_1$, $r_2$ and $r_3$
- Substituting these into the first 3 equations the overall order of the system is reduced.

$$w_1 = \frac{8-N}{r_1^6}$$

$$w_2 = \frac{1}{2^N r_2^6}$$

$$w_3 = \frac{1}{2r_3^6}$$

Procedure followed

- These nonlinear equations are in a way linear with respect to the weights
- The last 3 equations can be solved for $w_1$, $w_2$ and $w_3$ symbolically in terms of the other variables $r_1$, $r_2$ and $r_3$
- Substituting these into the first 3 equations the overall order of the system is reduced.

$$w_1 = \frac{8-N}{r_1^6}$$

$$w_2 = \frac{1}{2^N r_2^6}$$

$$w_3 = \frac{1}{2r_3^6}$$

Procedure followed

- These nonlinear equations are in a way linear with respect to the weights
- The last 3 equations can be solved for $w_1$, $w_2$ and $w_3$ symbolically in terms of the other variables $r_1$, $r_2$ and $r_3$
- Substituting these into the first 3 equations the overall order of the system is reduced.

$$w_1 = \frac{8-N}{r_1^6}$$
$$w_2 = \frac{1}{2^N r_2^6}$$
$$w_3 = \frac{1}{2 r_3^6}$$

Procedure followed

- These nonlinear equations are in a way linear with respect to the weights
- The last 3 equations can be solved for $w_1$, $w_2$ and $w_3$ symbolically in terms of the other variables $r_1$, $r_2$ and $r_3$
- Substituting these into the first 3 equations the overall order of the system is reduced.

$$w_1 = \frac{8-N}{r_1^6}$$
$$w_2 = \frac{1}{2^N r_2^6}$$
$$w_3 = \frac{1}{2r_3^6}$$

Procedure followed

- These nonlinear equations are in a way linear with respect to the weights
- The last 3 equations can be solved for $w_1$, $w_2$ and $w_3$ symbolically in terms of the other variables $r_1$, $r_2$ and $r_3$
- Substituting these into the first 3 equations the overall order of the system is reduced.

$$
\begin{aligned}
w_1 &= \frac{8-N}{r_1^6} \\
w_2 &= \frac{1}{2^N r_2^6} \\
w_3 &= \frac{1}{2r_3^6}
\end{aligned}
$$

Procedure followed

- These nonlinear equations are in a way linear with respect to the weights
- The last 3 equations can be solved for $w_1$, $w_2$ and $w_3$ symbolically in terms of the other variables $r_1$, $r_2$ and $r_3$
- Substituting these into the first 3 equations the overall order of the system is reduced.

$$w_1 = \frac{8-N}{r_1^6}$$

$$w_2 = \frac{1}{2^N r_2^6}$$

$$w_3 = \frac{1}{2r_3^6}$$

Procedure followed

- These nonlinear equations are in a way linear with respect to the weights
- The last 3 equations can be solved for $w_1$, $w_2$ and $w_3$ symbolically in terms of the other variables $r_1$, $r_2$ and $r_3$
- Substituting these into the first 3 equations the overall order of the system is reduced.

$$w_1 = \frac{8-N}{r_1^6}$$

$$w_2 = \frac{1}{2^N r_2^6}$$

$$w_3 = \frac{1}{2r_3^6}$$

Procedure followed

- These nonlinear equations are in a way linear with respect to the weights
- The last 3 equations can be solved for $w_1$, $w_2$ and $w_3$ symbolically in terms of the other variables $r_1$, $r_2$ and $r_3$
- Substituting these into the first 3 equations the overall order of the system is reduced.

$$w_1 = \frac{8-N}{r_1^6}$$

$$w_2 = \frac{1}{2^N r_2^6}$$

$$w_3 = \frac{1}{2r_3^6}$$

# In Summary CUT6, $(N \leq 6)$

Table: Sigma Points for CUT6, $(N \leq 6)$

| | Position | Weights |
|---|---|---|
| $1 \leq i \leq 2N$ | $X_i = r_1 \sigma_i$ | $W_i = w_1$ |
| $1 \leq i \leq 2^N$ | $X_{i+2N} = r_2 c_i^N$ | $W_{i+2N} = w_2$ |
| $1 \leq i \leq 2N(N-1)$ | $X_{i+2N+2^N} = r_3 c_i^2$ | $W_{i+2N+2^N} = w_3$ |
| Central weight | $X_0 = \mathbf{0}$ | $W_0 = w_0$ |
| $n = 2N^2 + 2^N + 1$ | | |

## In Summary for $(N \leq 6)$

- Solve the distance and weight variables
- Generate the principle axis and Conjugate axis
- Generate the sigma points
- Transform the points to match the required Mean and Covariance.

# In Summary CUT6, $(N \leq 6)$

Table: Sigma Points for CUT6, $(N \leq 6)$

| | Position | Weights |
|---|---|---|
| $1 \leq i \leq 2N$ | $X_i = r_1 \sigma_i$ | $W_i = w_1$ |
| $1 \leq i \leq 2^N$ | $X_{i+2N} = r_2 c_i^N$ | $W_{i+2N} = w_2$ |
| $1 \leq i \leq 2N(N-1)$ | $X_{i+2N+2^N} = r_3 c_i^2$ | $W_{i+2N+2^N} = w_3$ |
| Central weight | $X_0 = \mathbf{0}$ | $W_0 = w_0$ |
| $n = 2N^2 + 2^N + 1$ | | |

## In Summary for $(N \leq 6)$

- Solve the distance and weight variables
- Generate the principle axis and Conjugate axis
- Generate the sigma points
- Transform the points to match the required Mean and Covariance.

# In Summary CUT6, $(N \leq 6)$

Table: Sigma Points for CUT6, $(N \leq 6)$

| | Position | Weights |
|---|---|---|
| $1 \leq i \leq 2N$ | $X_i = r_1 \sigma_i$ | $W_i = w_1$ |
| $1 \leq i \leq 2^N$ | $X_{i+2N} = r_2 c_i^N$ | $W_{i+2N} = w_2$ |
| $1 \leq i \leq 2N(N-1)$ | $X_{i+2N+2^N} = r_3 c_i^2$ | $W_{i+2N+2^N} = w_3$ |
| Central weight | $X_0 = \mathbf{0}$ | $W_0 = w_0$ |
| $n = 2N^2 + 2^N + 1$ | | |

## In Summary for $(N \leq 6)$

- Solve the distance and weight variables
- Generate the principle axis and Conjugate axis
- Generate the sigma points
- Transform the points to match the required Mean and Covariance.

Table: Sigma Points for CUT6, $(N \leq 6)$

| | Position | Weights |
|---|---|---|
| $1 \leq i \leq 2N$ | $X_i = r_1 \sigma_i$ | $W_i = w_1$ |
| $1 \leq i \leq 2^N$ | $X_{i+2N} = r_2 c_i^N$ | $W_{i+2N} = w_2$ |
| $1 \leq i \leq 2N(N-1)$ | $X_{i+2N+2^N} = r_3 c_i^2$ | $W_{i+2N+2^N} = w_3$ |
| Central weight | $X_0 = \mathbf{0}$ | $W_0 = w_0$ |
| $n = 2N^2 + 2^N + 1$ | | |

### In Summary for $(N \leq 6)$

- Solve the distance and weight variables
- Generate the principle axis and Conjugate axis
- Generate the sigma points
- Transform the points to match the required Mean and Covariance.

- The previous method of CUT6 is valid only till dimension 6.
- After dimension 6 we have negative weights.
- The problem of negative weight can be avoided by choosing $3^{rd}$-conjugate axis instead of $2^{nd}$-conjugate axis.
- But again this proposed approach is valid only for $7 \le N \le 9$.

- The previous method of CUT6 is valid only till dimension 6.
- After dimension 6 we have negative weights.
- The problem of negative weight can be avoided by choosing $3^{rd}$-conjugate axis instead of $2^{nd}$-conjugate axis.
- But again this proposed approach is valid only for $7 \leq N \leq 9$.

# Sigma Points for CUT6, $(7 \leq N \leq 9)$

- The previous method of CUT6 is valid only till dimension 6.
- After dimension 6 we have negative weights.
- The problem of negative weight can be avoided by choosing $3^{rd}$-conjugate axis instead of $2^{nd}$-conjugate axis.
- But again this proposed approach is valid only for $7 \leq N \leq 9$.

- The previous method of CUT6 is valid only till dimension 6.
- After dimension 6 we have negative weights.
- The problem of negative weight can be avoided by choosing $3^{rd}$-conjugate axis instead of $2^{nd}$-conjugate axis.
- But again this proposed approach is valid only for $7 \leq N \leq 9$.

The set of $6^{th}$ order moment constraint equations in terms of these new sigma points are given as:

$$
\begin{aligned}
2r_1^2w_1 + 2^N r_2^2 w_2 + 4(N-1)(N-2)r_3^2 w_3 &= 1 \\
2r_1^4w_1 + 2^N r_2^4 w_2 + 4(N-1)(N-2)r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 8(n-2)r_3^4 w_3 &= 1 \\
2r_1^6w_1 + 2^N r_2^6 w_2 + 4(N-1)r_3^6 w_3 &= 15 \\
2^N r_2^6 w_2 + 8(n-2)r_3^6 w_3 &= 3 \\
2^N r_2^6 w_2 + 8r_3^6 w_3 &= 1 \\
1 - 2Nw_1 - 2^N w_2 - (4n(n-1)(n-2)/3)w_3 &= w_0
\end{aligned}
$$

# Sigma Points for CUT6, $(7 \leq N \leq 9)$

The set of $6^{th}$ order moment constraint equations in terms of these new sigma points are given as:

$$
\begin{aligned}
2r_1^2w_1 + 2^N r_2^2 w_2 + 4(N-1)(N-2)r_3^2 w_3 &= 1 \\
2r_1^4w_1 + 2^N r_2^4 w_2 + 4(N-1)(N-2)r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 8(n-2)r_3^4 w_3 &= 1 \\
2r_1^6w_1 + 2^N r_2^6 w_2 + 4(N-1)r_3^6 w_3 &= 15 \\
2^N r_2^6 w_2 + 8(n-2)r_3^6 w_3 &= 3 \\
2^N r_2^6 w_2 + 8r_3^6 w_3 &= 1 \\
1 - 2Nw_1 - 2^N w_2 - (4n(n-1)(n-2)/3)w_3 &= w_0
\end{aligned}
$$

The set of $6^{th}$ order moment constraint equations in terms of these new sigma points are given as:

$$
\begin{aligned}
2r_1^2w_1 + 2^N r_2^2 w_2 + 4(N-1)(N-2)r_3^2 w_3 &= 1 \\
2r_1^4w_1 + 2^N r_2^4 w_2 + 4(N-1)(N-2)r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 8(n-2)r_3^4 w_3 &= 1 \\
2r_1^6w_1 + 2^N r_2^6 w_2 + 4(N-1)r_3^6 w_3 &= 15 \\
2^N r_2^6 w_2 + 8(n-2)r_3^6 w_3 &= 3 \\
2^N r_2^6 w_2 + 8r_3^6 w_3 &= 1 \\
1 - 2Nw_1 - 2^N w_2 - (4n(n-1)(n-2)/3)w_3 &= w_0
\end{aligned}
$$

The set of $6^{th}$ order moment constraint equations in terms of these new sigma points are given as:

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 + 4(N-1)(N-2) r_3^2 w_3 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 + 4(N-1)(N-2) r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 8(n-2) r_3^4 w_3 &= 1 \\
2r_1^6 w_1 + 2^N r_2^6 w_2 + 4(N-1) r_3^6 w_3 &= 15 \\
2^N r_2^6 w_2 + 8(n-2) r_3^6 w_3 &= 3 \\
2^N r_2^6 w_2 + 8 r_3^6 w_3 &= 1 \\
1 - 2N w_1 - 2^N w_2 - (4n(n-1)(n-2)/3) w_3 &= w_0
\end{aligned}
$$

The set of $6^{th}$ order moment constraint equations in terms of these new sigma points are given as:

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 + 4(N-1)(N-2) r_3^2 w_3 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 + 4(N-1)(N-2) r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 8(n-2) r_3^4 w_3 &= 1 \\
2r_1^6 w_1 + 2^N r_2^6 w_2 + 4(N-1) r_3^6 w_3 &= 15 \\
2^N r_2^6 w_2 + 8(n-2) r_3^6 w_3 &= 3 \\
2^N r_2^6 w_2 + 8 r_3^6 w_3 &= 1 \\
1 - 2N w_1 - 2^N w_2 - (4n(n-1)(n-2)/3) w_3 &= w_0
\end{aligned}
$$

The set of $6^{th}$ order moment constraint equations in terms of these new sigma points are given as:

$$
\begin{aligned}
2r_1^2w_1 + 2^N r_2^2 w_2 + 4(N-1)(N-2)r_3^2 w_3 &= 1 \\
2r_1^4w_1 + 2^N r_2^4 w_2 + 4(N-1)(N-2)r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 8(n-2)r_3^4 w_3 &= 1 \\
2r_1^6w_1 + 2^N r_2^6 w_2 + 4(N-1)r_3^6 w_3 &= 15 \\
2^N r_2^6 w_2 + 8(n-2)r_3^6 w_3 &= 3 \\
2^N r_2^6 w_2 + 8r_3^6 w_3 &= 1 \\
1 - 2Nw_1 - 2^N w_2 - (4n(n-1)(n-2)/3)w_3 &= w_0
\end{aligned}
$$

The set of $6^{th}$ order moment constraint equations in terms of these new sigma points are given as:

$$
\begin{aligned}
2r_1^2w_1 + 2^N r_2^2 w_2 + 4(N-1)(N-2)r_3^2 w_3 &= 1 \\
2r_1^4w_1 + 2^N r_2^4 w_2 + 4(N-1)(N-2)r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 8(n-2)r_3^4 w_3 &= 1 \\
2r_1^6w_1 + 2^N r_2^6 w_2 + 4(N-1)r_3^6 w_3 &= 15 \\
2^N r_2^6 w_2 + 8(n-2)r_3^6 w_3 &= 3 \\
2^N r_2^6 w_2 + 8r_3^6 w_3 &= 1 \\
1 - 2Nw_1 - 2^N w_2 - (4n(n-1)(n-2)/3)w_3 &= w_0
\end{aligned}
$$

The set of $6^{th}$ order moment constraint equations in terms of these new sigma points are given as:

$$
\begin{aligned}
2r_1^2w_1 + 2^N r_2^2 w_2 + 4(N-1)(N-2)r_3^2 w_3 &= 1 \\
2r_1^4w_1 + 2^N r_2^4 w_2 + 4(N-1)(N-2)r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 8(n-2)r_3^4 w_3 &= 1 \\
2r_1^6w_1 + 2^N r_2^6 w_2 + 4(N-1)r_3^6 w_3 &= 15 \\
2^N r_2^6 w_2 + 8(n-2)r_3^6 w_3 &= 3 \\
2^N r_2^6 w_2 + 8r_3^6 w_3 &= 1 \\
1 - 2Nw_1 - 2^N w_2 - (4n(n-1)(n-2)/3)w_3 &= w_0
\end{aligned}
$$

The set of $6^{th}$ order moment constraint equations in terms of these new sigma points are given as:

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 + 4(N-1)(N-2) r_3^2 w_3 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 + 4(N-1)(N-2) r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 8(n-2) r_3^4 w_3 &= 1 \\
2r_1^6 w_1 + 2^N r_2^6 w_2 + 4(N-1) r_3^6 w_3 &= 15 \\
2^N r_2^6 w_2 + 8(n-2) r_3^6 w_3 &= 3 \\
2^N r_2^6 w_2 + 8 r_3^6 w_3 &= 1 \\
1 - 2N w_1 - 2^N w_2 - (4n(n-1)(n-2)/3) w_3 &= w_0
\end{aligned}
$$

**In Summary**

TABLE: Sigma Points for CUT6, $(7 \le N \le 9)$

| | Position | Weights |
|---|---|---|
| $1 \le i \le 2N$ | $X_i = r_1 \sigma_i$ | $W_i = w_1$ |
| $1 \le i \le 2^N$ | $X_{i+2N} = r_2 c_i^N$ | $W_{i+2N} = w_2$ |
| $1 \le i \le 4n(n-1)(n-2)/3$ | $X_{i+2N+2^N} = r_3 c_i^3$ | $W_{i+2N+2^N} = w_3$ |
| Central weight | $X_0 = \mathbf{0}$ | $W_0 = w_0$ |
| $n = 2N + 2^N + 4n(n-1)(n-2)/3 + 1$ | | |

**Results of integration compared to Gauss Hermite integration for 4D system**

The total number of cubature points involved in this method to capture all the moments till 6th order is $2N^2 + 2^N + 1$

$$P = \begin{bmatrix} 4 & 1 & 2 & 1 \\ 1 & 9 & 2 & 3 \\ 2 & 2 & 16 & 4 \\ 1 & 3 & 4 & 25 \end{bmatrix}$$

$$F = x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_1^2 x_2^2 x_3^2 + \\ + x_1^3 x_3 + x_1^2 x_2^4 + x_3^4 x_2^2 + x_1^2 x_3^4 + x_1^3 x_3 + x_2^3 x_3 + x_3^3 x_2$$

| No. of pts | $2^4 = 16$ | $3^4 = 81$ | $4^4 = 256$ | $5^4 = 64$ | Analytical |
|---|---|---|---|---|---|
| GH | 84462.6354 | 293311.8446 | 375417.9999 | 375417.9999 | 375417.9999 |
| % error | 77.5017 | 21.8705 | 9.0702e-012 | 8.9152e-012 | 0 |

| | No. of pts | Integration result | % error |
|---|---|---|---|
| CUT6 | 49 | 3.7541800e+005 | 8.062475e-013 |

| No. of pts | $2^4 = 16$ | $3^4 = 81$ | $4^4 = 256$ | $5^4 = 64$ | Analytical |
|---|---|---|---|---|---|
| GH | 84462.6354 | 293311.8446 | 375417.9999 | 375417.9999 | 375417.9999 |
| % error | 77.5017 | 21.8705 | 9.0702e-012 | 8.9152e-012 | 0 |

| | No. of pts | Integration result | % error |
|---|---|---|---|
| CUT6 | 49 | 3.7541800e+005 | 8.062475e-013 |

**Results of integration compared to Gauss Hermite integration for 6D system**

The total number of cubature points involved in this method to capture all the moments till 6th order is $2N^2 + 2^N + 1$

$$P = 10I_{6x6} \tag{18}$$

$$F = x_1^6 + x_1^2 x_2^2 x_3^2 + x_1^4 x_6^2 + x_5^4 + x_3^2 x_4^2$$

# Numerical Results of CUT6

| No. of pts | $2^6 = 64$ | $3^6 = 729$ | $4^6 = 4096$ | $5^6 = 15625$ | Ar |
|---|---|---|---|---|---|
| GH | 6.035000e+003 | 1.943500e+004 | 2.543500e+004 | 2.543500e+004 | 2. |
| % error | 76.2728 | 23.589 | 1.5633e-011 | 1.58906e-011 | 0 |

| | No. of pts | Integration result | % error |
|---|---|---|---|
| CUT6 | 137 | 2.543500e+004 | 3.4583111e-008 |

| No. of pts | $2^6 = 64$ | $3^6 = 729$ | $4^6 = 4096$ | $5^6 = 15625$ | Ar |
|---|---|---|---|---|---|
| GH | 6.035000e+003 | 1.943500e+004 | 2.543500e+004 | 2.543500e+004 | 2. |
| % error | 76.2728 | 23.589 | 1.5633e-011 | 1.58906e-011 | 0 |

| | No. of pts | Integration result | % error |
|---|---|---|---|
| CUT6 | 137 | 2.543500e+004 | 3.4583111e-008 |

$$P_3 = 100 I_{(4x4)}$$
$$P_4 = 100 I_{(9x9)}$$

Table: CUT6

| method | $n_3$ | $\mu_3$ % error | $n_4$ | $\mu_4$ % error |
|---|---|---|---|---|
| GH3 | 81 | 12.45 | 19683 | 4.18 |
| GH4 | 256 | $2.31e-013$ | 262144 | $1.37e-009$ |
| **CUT6** | 49 | $6.49e-013$ | 1203 | $6.26e-009$ |

# Cubature points to capture ALL the 8th order moments

- All the axis used before were not able to capture all the 8th order moments
- Only points that lie in the full N-D space enter the 8th order moments.
- I was only able to capture the moments till 6th dimension.
- Due to the complex nature of the equation I was not able to find a trend in the set of equations to generalize it.
- I have only done this case by case till 6th-Dimension.
- But in all cases I have selected the **SAME** set of axis

# Cubature points to capture ALL the 8th order moments

- All the axis used before were not able to capture all the 8th order moments
- Only points that lie in the full N-D space enter the 8th order moments.
- I was only able to capture the moments till 6th dimension.
- Due to the complex nature of the equation I was not able to find a trend in the set of equations to generalize it.
- I have only done this case by case till 6th-Dimension.
- But in all cases I have selected the **SAME** set of axis

# CUBATURE POINTS TO CAPTURE ALL THE 8TH ORDER MOMENTS

- All the axis used before were not able to capture all the 8th order moments
- Only points that lie in the full N-D space enter the 8th order moments.
- I was only able to capture the moments till 6th dimension.
- Due to the complex nature of the equation I was not able to find a trend in the set of equations to generalize it.
- I have only done this case by case till 6th-Dimension.
- But in all cases I have selected the **SAME** set of axis

# Cubature points to capture ALL the 8th order moments

- All the axis used before were not able to capture all the 8th order moments
- Only points that lie in the full N-D space enter the 8th order moments.
- I was only able to capture the moments till 6th dimension.
- Due to the complex nature of the equation I was not able to find a trend in the set of equations to generalize it.
- I have only done this case by case till 6th-Dimension.
- But in all cases I have selected the **SAME** set of axis

# Cubature points to capture ALL the 8th order moments

- All the axis used before were not able to capture all the 8th order moments
- Only points that lie in the full N-D space enter the 8th order moments.
- I was only able to capture the moments till 6th dimension.
- Due to the complex nature of the equation I was not able to find a trend in the set of equations to generalize it.
- I have only done this case by case till 6th-Dimension.
- But in all cases I have selected the **SAME** set of axis

# CUBATURE POINTS TO CAPTURE ALL THE 8TH ORDER MOMENTS

- All the axis used before were not able to capture all the 8th order moments
- Only points that lie in the full N-D space enter the 8th order moments.
- I was only able to capture the moments till 6th dimension.
- Due to the complex nature of the equation I was not able to find a trend in the set of equations to generalize it.
- I have only done this case by case till 6th-Dimension.
- But in all cases I have selected the **SAME** set of axis

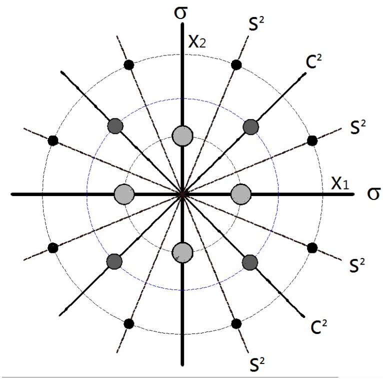
## $N^{th}$-Scaled Conjugate axis

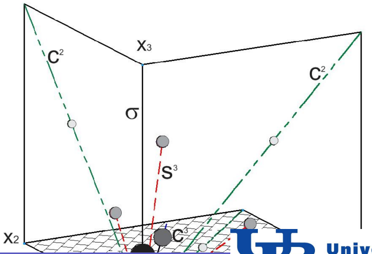
For a N-Dimensional system ,the axis that are constructed from all the combinations of the set of principle axis such that in every combination exactly one principle axis is scaled by a scaling parameter 'h' .The $N^{th}$-Scaled Conjugate set of axis for N-Dimensional system has $N2^N$ distinct points or $N2^{N-1}$ axis. We label the set of $N^{th}$-Scaled conjugate axis as $s^N(h)$, where the points are listed as $s_i^N(h)$

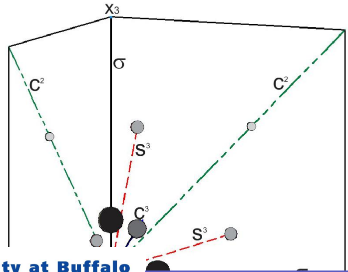
$$s_i^N(h) \in \{(\pm h\sigma_{n_1} \pm \sigma_{n_2} \pm ... \pm \sigma_{n_N})|(n_1,n_2,...,n_N) \in (1,2,...,N) \tag{19}$$

$$\&n_1 \neq n_2 \neq ... \neq n_N\} \tag{20}$$

$$i = 1,2,3,...,N2^N \tag{21}$$


σ
X2
S²
C²
S²
X1
σ
S²
C²
S²


C²
X3
C²
σ
S³
C³
X2


X3
σ
C²
C²
S³
C³
S³

- 1 point on the mean
- A set of points on the principal axis
- A set of points on the Nth-conjugate axis
- A set of points on the 2nd-conjugate axis
- A set of points on the Nth- Scaled conjugate axis
- The scaling parameter h has to appropriately chosen

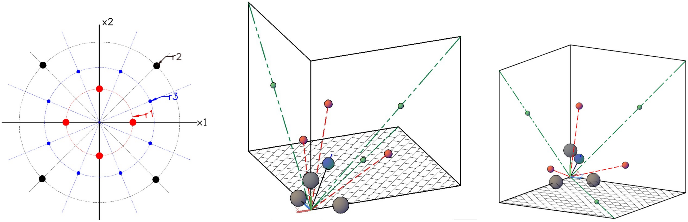



FIGURE: distances and weights

- 1 point on the mean
- A set of points on the principal axis
- A set of points on the Nth-conjugate axis
- A set of points on the 2nd-conjugate axis
- A set of points on the Nth- Scaled conjugate axis
- The scaling parameter h has to appropriately chosen

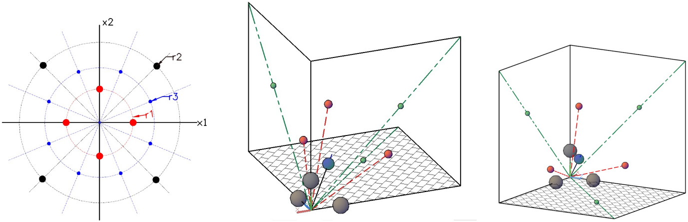



FIGURE: distances and weights

- 1 point on the mean
- A set of points on the principal axis
- A set of points on the Nth-conjugate axis
- A set of points on the 2nd-conjugate axis
- A set of points on the Nth- Scaled conjugate axis
- The scaling parameter h has to appropriately chosen

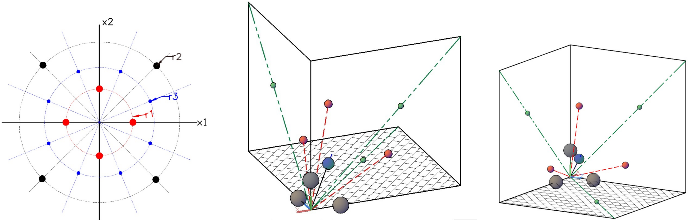



FIGURE: distances and weights

- 1 point on the mean
- A set of points on the principal axis
- A set of points on the Nth-conjugate axis
- A set of points on the 2nd-conjugate axis
- A set of points on the Nth- Scaled conjugate axis
- The scaling parameter h has to appropriately chosen

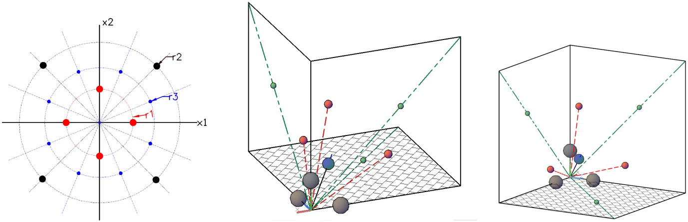



FIGURE: distances and weights

- 1 point on the mean
- A set of points on the principal axis
- A set of points on the Nth-conjugate axis
- A set of points on the 2nd-conjugate axis
- A set of points on the Nth- Scaled conjugate axis
- The scaling parameter h has to appropriately chosen

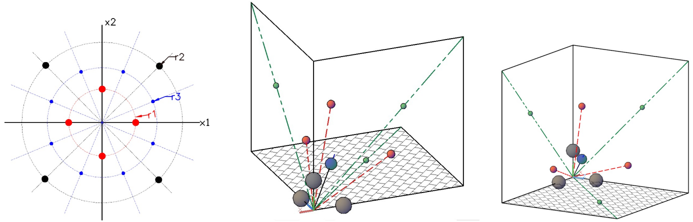



FIGURE: distances and weights

- 1 point on the mean
- A set of points on the principal axis
- A set of points on the Nth-conjugate axis
- A set of points on the 2nd-conjugate axis
- A set of points on the Nth- Scaled conjugate axis
- The scaling parameter h has to appropriately chosen

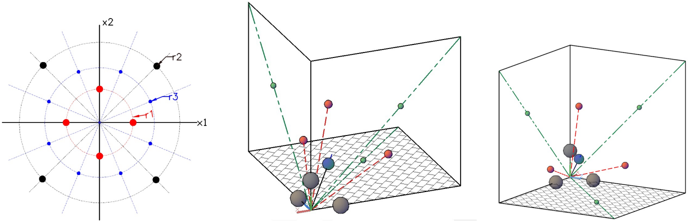



FIGURE: distances and weights

- 1 point on the mean
- A set of points on the principal axis
- A set of points on the Nth-conjugate axis
- A set of points on the 2nd-conjugate axis
- A set of points on the Nth- Scaled conjugate axis
- The scaling parameter h has to appropriately chosen

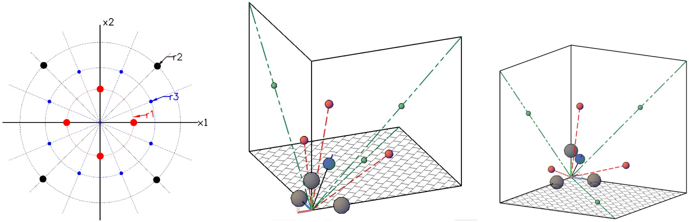



FIGURE: distances and weights

- 1 point on the mean
- A set of points on the principal axis
- A set of points on the Nth-conjugate axis
- A set of points on the 2nd-conjugate axis
- A set of points on the Nth- Scaled conjugate axis
- The scaling parameter h has to appropriately chosen

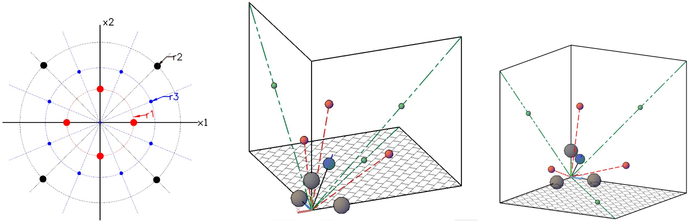



FIGURE: distances and weights

Only the even moments are shown that are to be satisfied. There are **11 non-zero moments for any dimension** in all

$$E[X_i^2] = 1 \qquad E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1 \qquad E[X_i^6] = 15$$
$$E[X_i^4 X_j^2] = 3 \qquad E[X_i^2 X_j^2 X_k^2] = 1$$
$$E[X_i^8] = 105 \qquad E[X_i^6 X_j^2] = 15$$
$$E[X_i^4 X_j^4] = 9 \qquad E[X_i^4 X_j^2 X_k^2] = 3$$
$$E[X_i^2 X_j^2 X_k^2 X_l^2] = 1$$

Only the even moments are shown that are to be satisfied. There are **11 non-zero moments for any dimension** in all

$$E[X_i^2] = 1 \qquad E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1 \qquad E[X_i^6] = 15$$
$$E[X_i^4 X_j^2] = 3 \qquad E[X_i^2 X_j^2 X_k^2] = 1$$
$$E[X_i^8] = 105 \qquad E[X_i^6 X_j^2] = 15$$
$$E[X_i^4 X_j^4] = 9 \qquad E[X_i^4 X_j^2 X_k^2] = 3$$
$$E[X_i^2 X_j^2 X_k^2 X_l^2] = 1$$

Only the even moments are shown that are to be satisfied. There are **11 non-zero moments for any dimension** in all

$$E[X_i^2] = 1 \qquad E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1 \qquad E[X_i^6] = 15$$
$$E[X_i^4 X_j^2] = 3 \qquad E[X_i^2 X_j^2 X_k^2] = 1$$
$$E[X_i^8] = 105 \qquad E[X_i^6 X_j^2] = 15$$
$$E[X_i^4 X_j^4] = 9 \qquad E[X_i^4 X_j^2 X_k^2] = 3$$
$$E[X_i^2 X_j^2 X_k^2 X_l^2] = 1$$

Only the even moments are shown that are to be satisfied. There are **11 non-zero moments for any dimension** in all

$$E[X_i^2] = 1 \qquad E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1 \qquad E[X_i^6] = 15$$
$$E[X_i^4 X_j^2] = 3 \qquad E[X_i^2 X_j^2 X_k^2] = 1$$
$$E[X_i^8] = 105 \qquad E[X_i^6 X_j^2] = 15$$
$$E[X_i^4 X_j^4] = 9 \qquad E[X_i^4 X_j^2 X_k^2] = 3$$
$$E[X_i^2 X_j^2 X_k^2 X_l^2] = 1$$

Only the even moments are shown that are to be satisfied. There are **11 non-zero moments for any dimension** in all

$$E[X_i^2] = 1 \qquad E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1 \qquad E[X_i^6] = 15$$
$$E[X_i^4 X_j^2] = 3 \qquad E[X_i^2 X_j^2 X_k^2] = 1$$
$$E[X_i^8] = 105 \qquad E[X_i^6 X_j^2] = 15$$
$$E[X_i^4 X_j^4] = 9 \qquad E[X_i^4 X_j^2 X_k^2] = 3$$
$$E[X_i^2 X_j^2 X_k^2 X_l^2] = 1$$

Only the even moments are shown that are to be satisfied. There are **11 non-zero moments for any dimension** in all

$$E[X_i^2] = 1 \qquad E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1 \qquad E[X_i^6] = 15$$
$$E[X_i^4 X_j^2] = 3 \qquad E[X_i^2 X_j^2 X_k^2] = 1$$
$$E[X_i^8] = 105 \qquad E[X_i^6 X_j^2] = 15$$
$$E[X_i^4 X_j^4] = 9 \qquad E[X_i^4 X_j^2 X_k^2] = 3$$
$$E[X_i^2 X_j^2 X_k^2 X_l^2] = 1$$

Only the even moments are shown that are to be satisfied. There are **11 non-zero moments for any dimension** in all

$$E[X_i^2] = 1 \qquad E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1 \qquad E[X_i^6] = 15$$
$$E[X_i^4 X_j^2] = 3 \qquad E[X_i^2 X_j^2 X_k^2] = 1$$
$$E[X_i^8] = 105 \qquad E[X_i^6 X_j^2] = 15$$
$$E[X_i^4 X_j^4] = 9 \qquad E[X_i^4 X_j^2 X_k^2] = 3$$
$$E[X_i^2 X_j^2 X_k^2 X_l^2] = 1$$

# Moment constraint equations till 8th moment for 4D system

$$2r_1^2w_1+16r_2^2w_2+12r_3^2w_3+16r_4^2w_4+24r_5^2w_5+48r_6^2w_6+16h^2r_6^2w_6=1$$
$$2r_1^4w_1+16r_2^4w_2+12r_3^4w_3+16r_4^4w_4+24r_5^4w_5+48r_6^4w_6+16h^4r_6^4w_6=3$$
$$16r_2^4w_2+4r_3^4w_3+16r_4^4w_4+16r_5^4w_5+32r_6^4w_6+32h^2r_6^4w_6=1$$
$$2r_1^6w_1+16r_2^6w_2+12r_3^6w_3+16r_4^6w_4+24r_5^6w_5+48r_6^6w_6+16h^6r_6^6w_6=15$$
$$16r_2^6w_2+4r_3^6w_3+16r_4^6w_4+16r_5^6w_5+32r_6^6w_6+16h^2r_6^6w_6+16h^4r_6^6w_6=3$$
$$16r_2^6w_2+16r_4^6w_4+8r_5^6w_5+16r_6^6w_6+48h^2r_6^6w_6=1$$
$$2r_1^8w_1+16r_2^8w_2+12r_3^8w_3+16r_4^8w_4+24r_5^8w_5+48r_6^8w_6+16h^8r_6^8w_6=105$$
$$16r_2^8w_2+4r_3^8w_3+16r_4^8w_4+16r_5^8w_5+32r_6^8w_6+16h^2r_6^8w_6+16h^6r_6^8w_6=15$$
$$16r_2^8w_2+4r_3^8w_3+16r_4^8w_4+16r_5^8w_5+32r_6^8w_6+32h^4r_6^8w_6=9$$
$$16r_2^8w_2+16r_4^8w_4+8r_5^8w_5+16r_6^8w_6+32h^2r_6^8w_6+16h^4r_6^8w_6=3$$
$$16r_2^8w_2+16r_4^8w_4+64h^2r_6^8w_6=1$$

# Moment constraint equations till 8th moment for 5D system

$$2r_1^2w_1+32r_2^2w_2+16r_3^2w_3+32r_4^2w_4+48r_5^2w_5+128r_6^2w_6+32h^2r_6^2w_6=1$$
$$2r_1^4w_1+32r_2^4w_2+16r_3^4w_3+32r_4^4w_4+48r_5^4w_5+128r_6^4w_6+32h^4r_6^4w_6=3$$
$$32r_2^4w_2+4r_3^4w_3+32r_4^4w_4+24r_5^4w_5+96r_6^4w_6+64h^2r_6^4w_6=1$$
$$2r_1^6w_1+32r_2^6w_2+16r_3^6w_3+32r_4^6w_4+48r_5^6w_5+128r_6^6w_6+32h^6r_6^6w_6=15$$
$$32r_2^6w_2+4r_3^6w_3+32r_4^6w_4+24r_5^6w_5+96r_6^6w_6+32h^2r_6^6w_6+32h^4r_6^6w_6=3$$
$$32r_2^6w_2+32r_4^6w_4+8r_5^6w_5+64r_6^6w_6+96h^2r_6^6w_6=1$$
$$2r_1^8w_1+32r_2^8w_2+16r_3^8w_3+32r_4^8w_4+48r_5^8w_5+128r_6^8w_6+32h^8r_6^8w_6=105$$
$$32r_2^8w_2+4r_3^8w_3+32r_4^8w_4+24r_5^8w_5+96r_6^8w_6+32h^2r_6^8w_6+32h^6r_6^8w_6=15$$
$$32r_2^8w_2+4r_3^8w_3+32r_4^8w_4+24r_5^8w_5+96r_6^8w_6+64h^4r_6^8w_6=9$$
$$32r_2^8w_2+32r_4^8w_4+8r_5^8w_5+64r_6^8w_6+64h^2r_6^8w_6+32h^4r_6^8w_6=3$$
$$32r_2^8w_2+32r_4^8w_4+32r_6^8w_6+128h^2r_6^8w_6=1$$

$$
\begin{aligned}
h &= 3\\
r_5 &= 2\\
r_1 &= 2.314370817280744 7\\
r_2 &= 0.8390942773980102\\
r_3 &= 1.830752125326649 4\\
r_4 &= 1.397039743064495 9\\
r_6 &= 1.113478632736702 1\\
w_1 &= 0.0105290342215466 07\\
w_2 &= 0.0151440196395375 72\\
w_3 &= 0.0052828996967816 825\\
w_4 &= 0.0010671298950159 158\\
w_5 &= 0.0006510416666666 666\\
w_6 &= 0.0001377601759207 4394
\end{aligned} \tag{22}
$$

TABLE: Cut8 : for $2 \leq N \leq 6$,

| Variable | 2D | 3D | 4D | 5D |
|---|---|---|---|---|
| $r_1$ | 2.068136061121187 | 2.255137265545780 | 2.201709071472343 | 2.314370817280745 |
| $r_2$ | 0.8491938499087475 | 0.7174531274600530 | 0.7941993714175681 | 0.8390942773980102 |
| $r_3$ | 1.138654980847415 | 1.843019437068797 | 1.872574360506295 | 1.830752125326649 |
| $r_4$ | 1.861619935018895 | 1.558481032725744 | 1.329116430064565 | 1.397039743064496 |
| $r_5$ | – | – | 2 | 2 |
| $r_6$ | – | 1.305561500466050 | 1.125865581272049 | 1.113478632736702 |
| $w_1$ | 0.04382264267013926 | 0.024631993437193266 | 0.01811008737283111 | 0.010529034221546607 |
| $w_2$ | 0.1405096621714662 | 0.08151009408908164 | 0.032063273384586845 | 0.015144019639537572 |
| $w_3$ | 0.0009215768861610588 | 0.009767235524166815 | 0.006614353755080834 | 0.0052828996967816825 |
| $w_4$ | 0.01240953967762697 | 0.00577248937435553 | 0.003489906522946932 | 0.0010671298950159158 |
| $w_5$ | – | – | 0.0006510416666666666 | 0.0006510416666666666 |
| $w_6$ | – | 0.000279472936899139 | 0.00025218336987488566 | 0.00013776017592074394 |
| $h$ | 3 | 2.74 | 3 | 3 |

**Results of integration compared to Gauss Hermite integration for 5D system**

For a N-D system the total number of points required to capture the 8th moment by this scheme are
$1+2N^2+\frac{4N(N-1)(N-2)}{3}+(N+2)2^N$.
The covariance of the gaussian Kernel

$$P = 1000I_{5x5} \tag{23}$$

$$X = [x_1, x_2, x_3, x_4, x_5] \tag{24}$$

$$\begin{aligned} F(X) = {} & x_1^8+x_2^8+x_3^6+x_4^6+x_1^4x_5^4+x_2^4x_3^4+x_4^2+x_1^2x_2^2x_3^2x_4^2+ \\ & +x_1^3x_3+x_1^4x_2^2x_5^2+x_3^4x_2^2+x_1^2x_3^4+x_1^3x_3+x_2^3x_3+x_3^3x_2 \end{aligned} \tag{25}$$

Evaluating the integral

$$f = \int F(X)N(X,0|P)dX \tag{26}$$

# Numerical Results for CUT8

| No. of pts | $2^5 = 32$ | $3^5 = 243$ | $4^5 = 1024$ | $5^5 = 3125$ | 1 |
|---|---|---|---|---|---|
| GH | 6.0962801e+012 | 7.6616634e+013 | 1.84964e+014 | 2.329646e+014 | 2 |
| % error | 97.38317 | 67.11233 | 20.6039857 | 5.184531e-011 | 0 |

| | No. of pts | Integration result | % error wrt Truth |
|---|---|---|---|
| CUT8 | 355 | 2.3296463e+014 | 5.1509964e-011 |

| No. of pts | $2^5 = 32$ | $3^5 = 243$ | $4^5 = 1024$ | $5^5 = 3125$ | 1 |
|---|---|---|---|---|---|
| GH | 6.0962801e+012 | 7.6616634e+013 | 1.84964e+014 | 2.329646e+014 | 2 |
| % error | 97.38317 | 67.11233 | 20.6039857 | 5.184531e-011 | 0 |

| | No. of pts | Integration result | % error wrt Truth |
|---|---|---|---|
| CUT8 | 355 | 2.3296463e+014 | 5.1509964e-011 |

$$P_5 = 100 I_{(5x5)}$$

$$P_6 = 100 I_{(6x6)}$$

Table: CUT8

| method | $n_5$ | $\mu_5$ % error | $n_6$ | $\mu_6$ % error |
|---|---|---|---|---|
| GH4 | 1024 | 3.45 | 4096 | 2.49 |
| GH5 | 3125 | $4.73e-012$ | 15625 | $9.29e-012$ |
| **CUT8** | 355 | $7.52e-012$ | 745 | $6.63e-012$ |

**Polar to Cartesian coordinates transformation**

- A radar has error in its radial and angular measurements
- We would like to see how this error is transformed into cartesian coordinates
- We would like to compute the expected value in the cartesian coordinates

In effect we are trying to evaluate the integral

$$
\begin{aligned}
E[(x,y)^T] &= E[(rcos(\theta), rsin(\theta))^T] \\
&= \int\int (rcos(\theta), rsin(\theta))^T N((r,\theta),(\mu_r,\mu_\theta|P))drd\theta
\end{aligned}
$$

**Polar to Cartesian coordinates transformation**

- A radar has error in its radial and angular measurements
- We would like to see how this error is transformed into cartesian coordinates
- We would like to compute the expected value in the cartesian coordinates

In effect we are trying to evaluate the integral

$$
\begin{aligned}
E[(x,y)^T] &= E[(rcos(\theta), rsin(\theta))^T] \\
&= \int\int (rcos(\theta), rsin(\theta))^T N((r,\theta),(\mu_r,\mu_\theta|P))drd\theta
\end{aligned}
$$

**Polar to Cartesian coordinates transformation**

- A radar has error in its radial and angular measurements
- We would like to see how this error is transformed into cartesian coordinates
- We would like to compute the expected value in the cartesian coordinates

In effect we are trying to evaluate the integral

$$
\begin{aligned}
E[(x,y)^T] &= E[(rcos(\theta), rsin(\theta))^T] \\
&= \int\int (rcos(\theta), rsin(\theta))^T N((r,\theta),(\mu_r,\mu_\theta|P))drd\theta
\end{aligned}
$$

**Polar to Cartesian coordinates transformation**

- A radar has error in its radial and angular measurements
- We would like to see how this error is transformed into cartesian coordinates
- We would like to compute the expected value in the cartesian coordinates

In effect we are trying to evaluate the integral

$$E[(x,y)^T] = E[(rcos(\theta), rsin(\theta))^T]$$
$$= \int\int (rcos(\theta), rsin(\theta))^T N((r,\theta),(\mu_r,\mu_\theta|P))drd\theta$$

**Polar to Cartesian coordinates transformation**

- A radar has error in its radial and angular measurements
- We would like to see how this error is transformed into cartesian coordinates
- We would like to compute the expected value in the cartesian coordinates

In effect we are trying to evaluate the integral

$$\begin{aligned} E[(x,y)^T] &= E[(rcos(\theta), rsin(\theta))^T] \\ &= \int\int (rcos(\theta), rsin(\theta))^T N((r,\theta), (\mu_r, \mu_\theta | P)) dr d\theta \end{aligned}$$

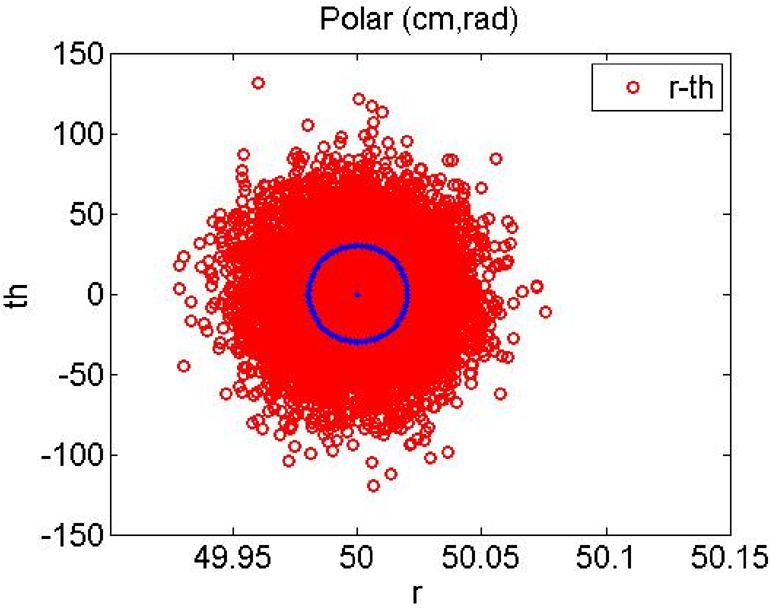



FIGURE: 2D and 3D Cubature points to satisfy 6th order moments

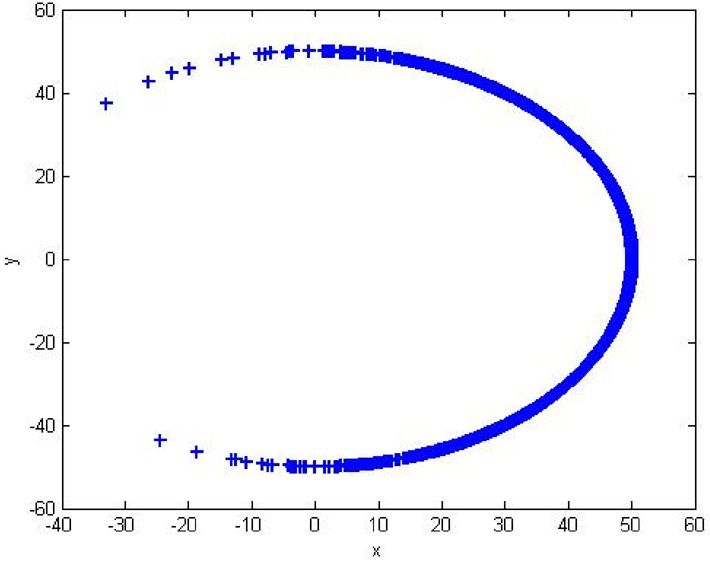



FIGURE: 2D and 3D Cubature points to satisfy 6th order moments

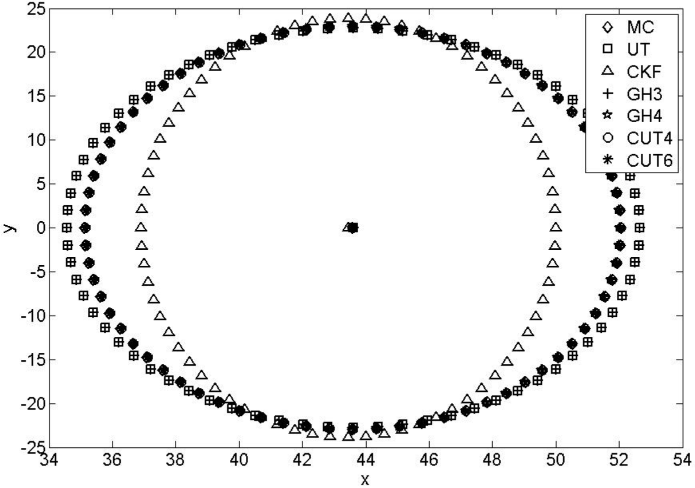



FIGURE: 2D and 3D Cubature points to satisfy 6th order moments

**Normal Distribution** The integral being evaluated is

$$E[N(x,\mu|P)] = \int N(x,\mu|P)N(x,\mu|P)dx$$

The parameters used are

$$\mu = 0$$
$$P = I$$

# TRUE VALUE OF INTEGRAL

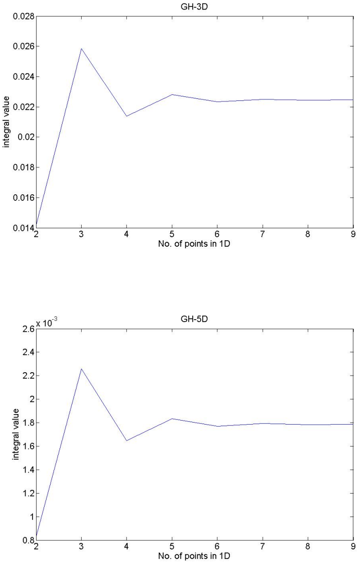


TABLE: Results of the integration interms of % rel. error

| Dimension | CKF | UT | CUT4 | CUT6 | CUT8 |
|---|---|---|---|---|---|
| 3 | 36.902 | 36.902 | 27.042 | 13.273 | 1.5688 |
| 4 | 45.880 | 45.880 | 46.571 | 23.276 | 2.0279 |
| 5 | 53.581 | 53.581 | 73.768 | 28.056 | 6.8219 |
| 6 | 60.186 | 60.186 | 110.90 | 13.393 | 15.793 |

TABLE: Number of points in each method

| Dim | GH-5 | GH-6 | GH-7 | CKF | UT | CUT4 | CUT6 | CUT8 |
|---|---|---|---|---|---|---|---|---|
| 3 | 125 | 216 | 343 | 6 | 7 | 15 | 27 | 59 |
| 4 | 625 | 1296 | 2401 | 8 | 9 | 25 | 49 | 161 |
| 5 | 3125 | 7776 | 16807 | 10 | 11 | 43 | 83 | 355 |
| 6 | 15625 | 46656 | 117649 | 12 | 13 | 77 | 137 | 745 |

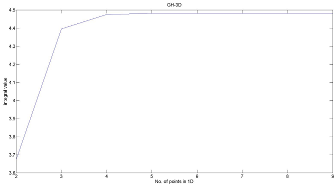
**Normal Distribution** The integral being evaluated is

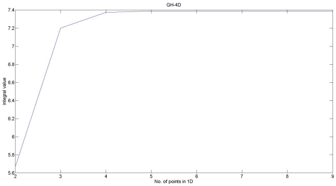
$$E[N(x,\mu|P)] = \int exp(-\sum_{i=1}^{N} x_i) N(x,\mu|P) dx$$

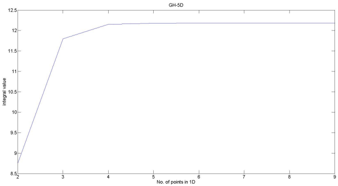
The parameters used are

$$\mu = 0$$
$$P = I$$

# TRUE VALUE OF INTEGRAL



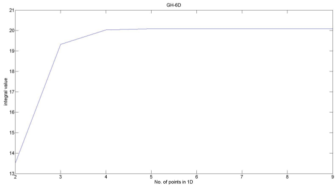



TABLE: Results of the integration interms of % rel. error

| Dimension | CKF | UT | CUT4 | CUT6 | CUT8 |
|---|---|---|---|---|---|
| 3 | 34.9670 | 34.9670 | 5.6510 | 1.5589 | 0.0813 |
| 4 | 49.0842 | 49.0842 | 7.7049 | 2.0998 | 0.6467 |
| 5 | 61.1601 | 61.1601 | 9.7754 | 7.0631 | 1.4656 |
| 6 | 70.9523 | 70.9523 | 12.0018 | 11.6941 | 1.2176 |

TABLE: Number of points in each method

| Dim | GH-4 | GH-5 | GH-6 | CKF | UT | CUT4 | CUT6 | CUT8 |
|---|---|---|---|---|---|---|---|---|
| 3 | 64 | 125 | 216 | 6 | 7 | 15 | 27 | 59 |
| 4 | 256 | 625 | 1296 | 8 | 9 | 25 | 49 | 161 |
| 5 | 1024 | 3125 | 7776 | 10 | 11 | 43 | 83 | 355 |
| 6 | 4096 | 15625 | 46656 | 12 | 13 | 77 | 137 | 745 |

$$f(\mathbf{X}) = (\sqrt{1+\mathbf{X}^T\mathbf{X}})^{\alpha}, \mathbf{X} \in \mathbb{R}^N$$

- This benchmark problem was introduced by CKF.
- It has been discussed that computing the expectation of $f(\mathbf{X})$ for negative values for $\alpha$ is a challenging task.
- Negative values lead to a delta-sequence functions, which is a very notorious function when using quadrature rules.
- For simulation purposes, $\alpha = -3$ and the covariance of the zero mean Gaussian pdf is assumed to be 0.1 times the identity matrix. T
- The covariance is intentionally scaled down by 0.1 to make the integral value converge with reasonable number of points.

$f(\mathbf{X}) = (\sqrt{1+\mathbf{X}^T\mathbf{X}})^{\alpha}, \mathbf{X} \in \mathbb{R}^N$

- This benchmark problem was introduced by CKF.
- It has been discussed that computing the expectation of $f(\mathbf{X})$ for negative values for $\alpha$ is a challenging task.
- Negative values lead to a delta-sequence functions, which is a very notorious function when using quadrature rules.
- For simulation purposes, $\alpha = -3$ and the covariance of the zero mean Gaussian pdf is assumed to be 0.1 times the identity matrix. T
- The covariance is intentionally scaled down by 0.1 to make the integral value converge with reasonable number of points.

$f(\mathbf{X}) = (\sqrt{1+\mathbf{X}^T\mathbf{X}})^\alpha, \mathbf{X} \in \mathbb{R}^N$

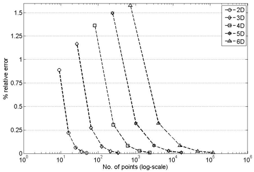
- This benchmark problem was introduced by CKF.
- It has been discussed that computing the expectation of $f(\mathbf{X})$ for negative values for $\alpha$ is a challenging task.
- Negative values lead to a delta-sequence functions, which is a very notorious function when using quadrature rules.
- For simulation purposes, $\alpha = -3$ and the covariance of the zero mean Gaussian pdf is assumed to be 0.1 times the identity matrix. T
- The covariance is intentionally scaled down by 0.1 to make the integral value converge with reasonable number of points.

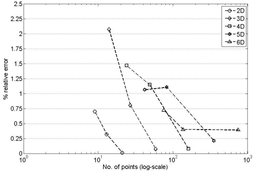
$f(\mathbf{X}) = (\sqrt{1+\mathbf{X}^T\mathbf{X}})^{\alpha}, \mathbf{X} \in \mathbb{R}^N$

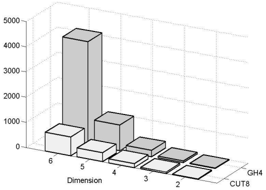
- This benchmark problem was introduced by CKF.
- It has been discussed that computing the expectation of $f(\mathbf{X})$ for negative values for $\alpha$ is a challenging task.
- Negative values lead to a delta-sequence functions, which is a very notorious function when using quadrature rules.
- For simulation purposes, $\alpha = -3$ and the covariance of the zero mean Gaussian pdf is assumed to be 0.1 times the identity matrix. T
- The covariance is intentionally scaled down by 0.1 to make the integral value converge with reasonable number of points.

$f(\mathbf{X}) = (\sqrt{1+\mathbf{X}^T\mathbf{X}})^\alpha, \mathbf{X} \in \mathbb{R}^N$

- This benchmark problem was introduced by CKF.
- It has been discussed that computing the expectation of $f(\mathbf{X})$ for negative values for $\alpha$ is a challenging task.
- Negative values lead to a delta-sequence functions, which is a very notorious function when using quadrature rules.
- For simulation purposes, $\alpha = -3$ and the covariance of the zero mean Gaussian pdf is assumed to be 0.1 times the identity matrix. T
- The covariance is intentionally scaled down by 0.1 to make the integral value converge with reasonable number of points.




Therefore to achieve 0.5% accuracy, we need atleast GH4 for Gauss Hemrite quadrature rule and atleast CUT8. There is a significant difference in the number of points to achieve this order of accuracy.

# Objective 2

**To evaluate the Uniform Expectation integral**

$$E[f(x)] = \int_{-1}^{1} f(x)dx$$

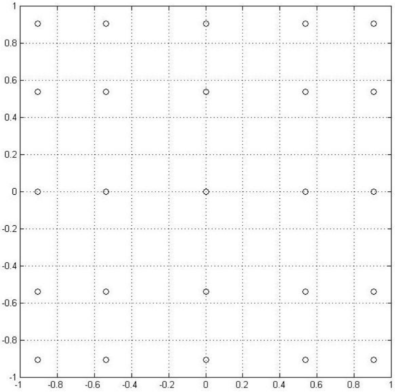
The domain of integration is a N-Dimensional Hypercube with each side of length 2 units.

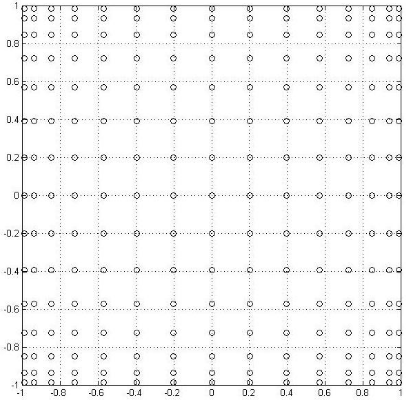
- The unifrom Distribution is also a centrally symmetric function
- The same axis of symmetry can be used
- But there is an additional constraint invloved: ***All the cubature points must lie within the domain of the pdf i.e. within a hypercube***
- Because of this additional constraint, the procedure is even more difficult

- The unifrom Distribution is also a centrally symmetric function
- The same axis of symmetry can be used
- But there is an additional constraint invloved: ***All the cubature points must lie within the domain of the pdf i.e. within a hypercube***
- Because of this additional constraint, the procedure is even more difficult

- The unifrom Distribution is also a centrally symmetric function
- The same axis of symmetry can be used
- But there is an additional constraint invloved: ***All the cubature points must lie within the domain of the pdf i.e. within a hypercube***
- Because of this additional constraint, the procedure is even more difficult

- The unifrom Distribution is also a centrally symmetric function
- The same axis of symmetry can be used
- But there is an additional constraint invloved: ***All the cubature points must lie within the domain of the pdf i.e. within a hypercube***
- Because of this additional constraint, the procedure is even more difficult


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1


1
0.8
0.6
0.4
0.2
0
-0.2
-0.4
-0.6
-0.8
-1
-1
-0.8
-0.6
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1

We are looking for cubature points of uniform pdf with the following properties

- The sigma points have to satisfy the **moments of the uniform pdf**.
- All the weights to be **positive**.
- All the weights **sum upto 1**.
- All sigma points lie **within the hypercube**.
- All weights and sigma points have to be **real**.

We are looking for cubature points of uniform pdf with the following properties

- The sigma points have to satisfy the **moments of the uniform pdf**.
- All the weights to be **positive**.
- All the weights **sum upto 1**.
- All sigma points lie **within the hypercube**.
- All weights and sigma points have to be **real**.

We are looking for cubature points of uniform pdf with the following properties

- The sigma points have to satisfy the **moments of the uniform pdf**.
- All the weights to be **positive**.
- All the weights **sum upto 1**.
- All sigma points lie **within the hypercube**.
- All weights and sigma points have to be **real**.

We are looking for cubature points of uniform pdf with the following properties

- The sigma points have to satisfy the **moments of the uniform pdf**.
- All the weights to be **positive**.
- All the weights **sum upto 1**.
- All sigma points lie **within the hypercube**.
- All weights and sigma points have to be **real**.

We are looking for cubature points of uniform pdf with the following properties

- The sigma points have to satisfy the **moments of the uniform pdf**.
- All the weights to be **positive**.
- All the weights **sum upto 1**.
- All sigma points lie **within the hypercube**.
- All weights and sigma points have to be **real**.

For a 1-D unifrom pdf between -1 and 1, the moments are given by

$$\mu_n = E[x^n] = \frac{(2)^n + (-2)^n}{2^{(n+1)}(n+1)} \tag{27}$$

All odd moments are zero

For N-Dimensional moments, we can consider an i.i.d. space such that

$$E[x_1^{n_1} x_2^{n_2} \cdots x_m^{n_m}] = \mu_{n_1} \mu_{n_2} \cdots \mu_{n_m} \tag{28}$$

For dimension $2 \leq N \leq 5$

- Choose principal axis and Nth-Conjugate axis
- Works only till 5-D after which some points go outside the cube
- This is also documented by Stroud in his paper. His work is mostly on symmetrical spherical regions.

Example integrating a 4th degree polynomial in 5D

| | No. of pts | Integration result |
|---|---|---|
| CUT4-Uniform | 42 | 7.5556 |
| GL3 | 243 | 7.5556 |

For dimension $N = 4$

- Choose principal axis ,2nd Conjugate Axis and Nth-Conjugate axis

Example integrating a 6th degree polynomial in 4D

| | No. of pts | Integration result |
|---|---|---|
| CUT6-Uniform | 57 | 15.2603 |
| GL4 | 256 | 15.2603 |

# Understanding the structure of Gauss Legendre Product rule

For a 4D system the GL5 has the structure

Table: GL5 Structure

| | | | |
|---|---|---|---|
| 1 | 1 | 1 | 1 |
| 0.5942 | 1 | 1 | 1 |
| 0 | 1 | 1 | 1 |
| 0.5942 | 0.5942 | 1 | 1 |
| 0 | 0 | 1 | 1 |
| 0.5942 | 0.5942 | 0.5942 | 1 |
| 0 | 0.5942 | 0.5942 | 1 |
| 0 | 0 | 0.5942 | 1 |
| 0 | 0 | 0 | 1 |
| 0 | 0 | 0 | 0 |

Table: The generator set

| | | | | |
|---|---|---|---|---|
| 1 | 1 | 1 | 1 | $r_1$ |
| h | 1 | 1 | 1 | $r_2$ |
| 0 | 1 | 1 | 1 | $r_3$ |
| p | q | 1 | 1 | $r_4$ |
| 0 | 0 | 1 | 1 | $r_5$ |
| r | s | t | 1 | $r_6$ |
| 0 | k | m | 1 | $r_7$ |
| 0 | 0 | a | 1 | $r_8$ |
| 0 | 0 | 0 | 1 | $r_9$ |
| 0 | 0 | 0 | 0 | $r_{10}$ |

## COST FUNCTION

How do we frame a cost function such that we have minimum points as posible while satisfying the moment equations.

## MOMENT COSNTRAINTS

- The constraints have polynomial type nonlinearity
- The degree of the polynomials can be reduced by some substitutions
- Gradient based methods of optimization might fail.
- Polynomial solver might help ( *Currently working on it*)
- With polynomials we have the issue of complex roots

## Cost Function

How do we frame a cost function such that we have minimum points as posible while satisfying the moment equations.

## Moment cosntraints

- The constraints have polynomial type nonlinearity
- The degree of the polynomials can be reduced by some substitutions
- Gradient based methods of optimization might fail.
- Polynomial solver might help ( *Currently working on it*)
- With polynomials we have the issue of complex roots

## Cost Function

How do we frame a cost function such that we have minimum points as posible while satisfying the moment equations.

## Moment cosntraints

- The constraints have polynomial type nonlinearity
- The degree of the polynomials can be reduced by some substitutions
- Gradient based methods of optimization might fail.
- Polynomial solver might help ( *Currently working on it*)
- With polynomials we have the issue of complex roots

### Cost Function

How do we frame a cost function such that we have minimum points as posible while satisfying the moment equations.

### Moment cosntraints

- The constraints have polynomial type nonlinearity
- The degree of the polynomials can be reduced by some substitutions
- Gradient based methods of optimization might fail.
- Polynomial solver might help ( *Currently working on it*)
- With polynomials we have the issue of complex roots

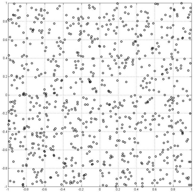
## COST FUNCTION

How do we frame a cost function such that we have minimum points as posible while satisfying the moment equations.

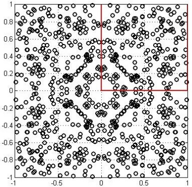
## MOMENT COSNTRAINTS

- The constraints have polynomial type nonlinearity
- The degree of the polynomials can be reduced by some substitutions
- Gradient based methods of optimization might fail.
- Polynomial solver might help ( *Currently working on it*)
- With polynomials we have the issue of complex roots

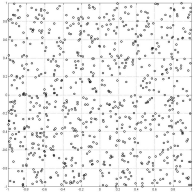
- Consider a 2 Dimensional square (-1,-1) to (1,1)
- Take a sample from the red square (0,0) to (1,1)
- make this sample **fully symmetric**

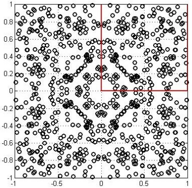


| Moment | conv-rand-samp | Full-sym-rand-samp | True |
|---|---|---|---|
| $E[x_1^2]$ | 0.3407 | 0.3207 | 0.3333 |
| $E[x_1 x_2]$ | 0.0120 | 0 | 0 |
| $E[x_2^2]$ | 0.3328 | 0.3207 | 0.3333 |
| $E[x_1^4]$ | 0.2073 | 0.1830 | 0.2 |
| $E[x_1^3 x_2]$ | 0.0052 | 0 | 0 |
| $E[x_1^2 x_2^2]$ | 0.1139 | 0.1082 | 0.1111 |
| $E[x_1^1 x_2^3]$ | 0.0026 | 0 | 0 |
| $E[x_2^4]$ | 0.1997 | 0.1830 | 0.2 |
| Tot. samp | 800 | 800 | – |



- Consider a 2 Dimensional square (-1,-1) to (1,1)
- Take a sample from the red square (0,0) to (1,1)
- make this sample **fully symmetric**

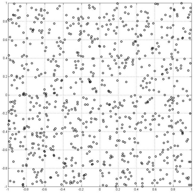


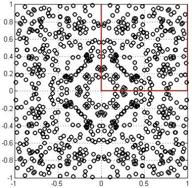


| Moment | conv-rand-samp | Full-sym-rand-samp | True |
|---|---|---|---|
| $E[x_1^2]$ | 0.3407 | 0.3207 | 0.3333 |
| $E[x_1x_2]$ | 0.0120 | 0 | 0 |
| $E[x_2^2]$ | 0.3328 | 0.3207 | 0.3333 |
| $E[x_1^4]$ | 0.2073 | 0.1830 | 0.2 |
| $E[x_1^3x_2]$ | 0.0052 | 0 | 0 |
| $E[x_1^2x_2^2]$ | 0.1139 | 0.1082 | 0.1111 |
| $E[x_1^1x_2^3]$ | 0.0026 | 0 | 0 |
| $E[x_2^4]$ | 0.1997 | 0.1830 | 0.2 |
| Tot. samp | 800 | 800 | – |

# Random Idea 1: A Fully symmetric Sample

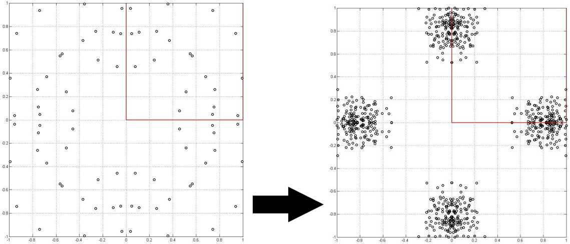
- Consider a 2 Dimensional square (-1,-1) to (1,1)
- Take a sample from the red square (0,0) to (1,1)
- make this sample **fully symmetric**






| Moment | conv-rand-samp | Full-sym-rand-samp | True |
|---|---|---|---|
| $E[x_1^2]$ | 0.3407 | 0.3207 | 0.3333 |
| $E[x_1x_2]$ | 0.0120 | 0 | 0 |
| $E[x_2^2]$ | 0.3328 | 0.3207 | 0.3333 |
| $E[x_1^4]$ | 0.2073 | 0.1830 | 0.2 |
| $E[x_1^3x_2]$ | 0.0052 | 0 | 0 |
| $E[x_1^2x_2^2]$ | 0.1139 | 0.1082 | 0.1111 |
| $E[x_1^1x_2^3]$ | 0.0026 | 0 | 0 |
| $E[x_2^4]$ | 0.1997 | 0.1830 | 0.2 |
| Tot. samp | 800 | 800 | – |

Can we have an iterative process such that we can sample and then resample again with respect to the constraints.... **the samples might converge to the quadrature points**. Can we build such an Algorithm to ***ESTIMATE the cubature points***



The samples in the second figure have the same 2nd moment as that of uniform pdf. I have created the samples from gaussian pdf centered about the exact cubature point.

- Is there a better way to solve the fully nonlinear system of moment equations.
- How to find the minimal number of cubature points -or- to be optimistic how to prove that the cubature points by this method is minimal.
- Is it really advantageous to develop higher order methods from a filtering point of view. If the first approximation is dominantly wrong does it help in using higher order cubature points

- Is there a better way to solve the fully nonlinear system of moment equations.
- How to find the minimal number of cubature points -or- to be optimistic how to prove that the cubature points by this method is minimal.
- Is it really advantageous to develop higher order methods from a filtering point of view. If the first approximation is dominantly wrong does it help in using higher order cubature points

- Is there a better way to solve the fully nonlinear system of moment equations.
- How to find the minimal number of cubature points -or- to be optimistic how to prove that the cubature points by this method is minimal.
- Is it really advantageous to develop higher order methods from a filtering point of view. If the first approximation is dominantly wrong does it help in using higher order cubature points

- How to identify the non-polynomial type of functions that can be integrated by these methods accurately, How do we develop Error estimates. For example for 1D Gauss Hermite quadrature the error estimate is $E = \frac{n!\sqrt{\pi}}{2^n(2n)!} f^{2n}(\xi)$
- How do we generalize this method or how do we find a mathematically rigorous theory/algorithm to generate cubature points for any moment and any dimension.
- Can we find the cubature points for any other PDF in the same manner.

- How to identify the non-polynomial type of functions that can be integrated by these methods accurately, How do we develop Error estimates. For example for 1D Gauss Hermite quadrature the error estimate is $E = \frac{n!\sqrt{\pi}}{2^n(2n)!} f^{2n}(\xi)$
- How do we generalize this method or how do we find a mathematically rigorous theory/algorithm to generate cubature points for any moment and any dimension.
- Can we find the cubature points for any other PDF in the same manner.

- How to identify the non-polynomial type of functions that can be integrated by these methods accurately, How do we develop Error estimates. For example for 1D Gauss Hermite quadrature the error estimate is $E = \frac{n!\sqrt{\pi}}{2^n(2n)!} f^{2n}(\xi)$
- How do we generalize this method or how do we find a mathematically rigorous theory/algorithm to generate cubature points for any moment and any dimension.
- Can we find the cubature points for any other PDF in the same manner.