The Conjugate Unscented Transform

Nagavenkat Adurthi

Department of Mechanical & Aerospace Engineering
University at Buffalo

August 24, 2011

From the basics

From the basics

Consider a discrete dynamic system with noise

$$x_{k+1} = f(x_k, k) + \nu_k$$

From the basics

Consider a discrete dynamic system with noise

$$x_{k+1} = f(x_k, k) + \nu_k$$

The PDF of this dynamic system propagates according to the **Chapman Kolmogorov Equation.**

From the basics

Consider a discrete dynamic system with noise

$$x_{k+1} = f(x_k, k) + \nu_k$$

The PDF of this dynamic system propagates according to the **Chapman Kolmogorov Equation.**

$$P(x_{k+1}) = \int P(x_{k+1}|x_k)P(x_k)dx_k$$

- $P(x_k)$ in the CKE need not be gaussian at all times even though the initial condition was gaussian.

- $P(x_k)$ in the CKE need not be gaussian at all times even though the initial condition was gaussian.
- It would be gaussian at all time only when system is linear and the noise is also gaussian. In this case it is very easy to **solve this equation analytically**.

- $P(x_k)$ in the CKE need not be gaussian at all times even though the initial condition was gaussian.
- It would be gaussian at all time only when system is linear and the noise is also gaussian. In this case it is very easy to **solve this equation analytically**.
- Incase the the system in nonlinear and noise is gaussian, the CKE would be

- $P(x_k)$ in the CKE need not be gaussian at all times even though the initial condition was gaussian.
- It would be gaussian at all time only when system is linear and the noise is also gaussian. In this case it is very easy to **solve this equation analytically**.
- Incase the the system in nonlinear and noise is gaussian, the CKE would be

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

Extended Kalman Filter

- $P(x_k)$ is not always gaussian and there is no analytical solution to this equation.

Extended Kalman Filter

- $P(x_k)$ is not always gaussian and there is no analytical solution to this equation.
- Hence the EKF emerged that can provide an **analytical solution** to this CKE by considering the following **two approximations**

Extended Kalman Filter

- $P(x_k)$ is not always gaussian and there is no analytical solution to this equation.
- Hence the EKF emerged that can provide an **analytical solution** to this CKE by considering the following **two approximations**
- $P(x_k)$ is replaced by an equivalent gaussian PDF that has the same first two moments as the original PDF $P(x_k)$.

Extended Kalman Filter

- $P(x_k)$ is not always gaussian and there is no analytical solution to this equation.
- Hence the EKF emerged that can provide an **analytical solution** to this CKE by considering the following **two approximations**
- $P(x_k)$ is replaced by an equivalent gaussian PDF that has the same first two moments as the original PDF $P(x_k)$.
- $f(x_k, k)$ is linearized.

- The EKF works well for systems in which the linearized dynamics is a good approximation to the nonlinear system- **Higher order terms in Taylor series are negligible.**

- The EKF works well for systems in which the linearized dynamics is a good approximation to the nonlinear system- **Higher order terms in Taylor series are negligible.**
- If the nonlinearity is too strong the EKF would diverge.

- The EKF works well for systems in which the linearized dynamics is a good approximation to the nonlinear system- **Higher order terms in Taylor series are negligible.**
- If the nonlinearity is too strong the EKF would diverge.
- Above that during the linearization process the computation of the jacobian is **computationally expensive.**

- The EKF works well for systems in which the linearized dynamics is a good approximation to the nonlinear system- **Higher order terms in Taylor series are negligible.**
- If the nonlinearity is too strong the EKF would diverge.
- Above that during the linearization process the computation of the jacobian is **computationally expensive.**
- The EKF **disregards the actual state PDF** and propagates only the first two moments of the state PDF. The linearized dynamics are used in the propagation of the first two moments.

EKF

EKF

Propagation of mean

$$\mu_{k+1} = f(\mu_k)$$

EKF

Propagation of mean

$$\mu_{k+1} = f(\mu_k)$$

Propagation of Covariance

$$P_{k+1} = AP_kA^T + Q_k$$

EKF

Propagation of mean

$$\mu_{k+1} = f(\mu_k)$$

Propagation of Covariance

$$P_{k+1} = AP_kA^T + Q_k$$

Where A is the jacobian of the system.

$$A = \frac{\partial f}{\partial x}|_{\mu_k}$$

Linear Regression Kalman Filter

- The LRKF also uses linearized dynamics but the linearization is not done using the taylor series (hence evaluating jacobian) but by a method called **statistical linearization.**

Linear Regression Kalman Filter

- The LRKF also uses linearized dynamics but the linearization is not done using the taylor series (hence evaluating jacobian) but by a method called **statistical linearization.**
- Firstly sample points are chosen about the current mean at time k such that the mean of the samples and covariance of the samples **match** the current mean and current covariance.

LRKF

- Each point is propagated using the **nonlinear dynamics** of the system to time step $k+1$.

LRKF

- Each point is propagated using the **nonlinear dynamics** of the system to time step $k+1$.
- Now between the current sample points at time k and the corresponding propagated points at time $k+1$ a **linear model is fit** which gives rise to the linearized dynamics of the system at time k.

LRKF

- Each point is propagated using the **nonlinear dynamics** of the system to time step $k + 1$.
- Now between the current sample points at time k and the corresponding propagated points at time $k + 1$ a **linear model is fit** which gives rise to the linearized dynamics of the system at time k.
- Now this linearized dynamics is used to compute the mean and covariance at time step $k + 1$.

LRKF

The sample points at time k are chosen such that

LRKF

The sample points at time k are chosen such that

$$\mu_k = \frac{1}{n} \sum_{i=1}^{N} X^i$$

LRKF

The sample points at time k are chosen such that

$$\mu_k = \frac{1}{n}\sum_{i=1}^{N} X^i$$

$$P_k = \frac{1}{n}\sum_{i=1}^{N}(X^i - \mu_k)(X^i - \mu_k)^T$$

LRKF

The sample points at time k are chosen such that

$$\mu_k = \frac{1}{n} \sum_{i=1}^{N} X^i$$

$$P_k = \frac{1}{n} \sum_{i=1}^{N} (X^i - \mu_k)(X^i - \mu_k)^T$$

And now each sample point is individually propagated using nonlinear dynamics

LRKF

The sample points at time k are chosen such that

$$\mu_k = \frac{1}{n}\sum_{i=1}^{N} X^i$$

$$P_k = \frac{1}{n}\sum_{i=1}^{N}(X^i - \mu_k)(X^i - \mu_k)^T$$

And now each sample point is individually propagated using nonlinear dynamics

$$Y^i = f(X^i)$$

LRKF

Now trying to fit a linear model between the points (Y^i, X^i)

LRKF

Now trying to fit a linear model between the points (Y^i, X^i)

$$Y = AX + B$$

LRKF

Now trying to fit a linear model between the points (Y^i, X^i)

$$Y = AX + B$$

This is standard linear fit procedure by minimizing the least square error

LRKF

Now trying to fit a linear model between the points (Y^i, X^i)

$$Y = AX + B$$

This is standard linear fit procedure by minimizing the least square error

$$e_i = Y^i - AX^i - B$$

LRKF

Now trying to fit a linear model between the points (Y^i, X^i)

$$Y = AX + B$$

This is standard linear fit procedure by minimizing the least square error

$$e_i = Y^i - AX^i - B$$
$$E = (e_i)^T(e_i)$$

LRKF

Now trying to fit a linear model between the points (Y^i, X^i)

$$Y = AX + B$$

This is standard linear fit procedure by minimizing the least square error

$$e_i = Y^i - AX^i - B$$
$$E = (e_i)^T(e_i)$$

The mean and covariance at time $k+1$ can be found from the Kalman filter propagation equations

LRKF

Now trying to fit a linear model between the points (Y^i, X^i)

$$Y = AX + B$$

This is standard linear fit procedure by minimizing the least square error

$$e_i = Y^i - AX^i - B$$
$$E = (e_i)^T (e_i)$$

The mean and covariance at time $k+1$ can be found from the Kalman filter propagation equations

$$\mu_{k+1} = A\mu_k$$

LRKF

Now trying to fit a linear model between the points (Y^i, X^i)

$$Y = AX + B$$

This is standard linear fit procedure by minimizing the least square error

$$e_i = Y^i - AX^i - B$$
$$E = (e_i)^T (e_i)$$

The mean and covariance at time $k+1$ can be found from the Kalman filter propagation equations

$$\mu_{k+1} = A\mu_k$$
$$P_{k+1} = AP_kA^T + Q_k$$

The Unscented Kalman Filter-UKF

- The UKF works in the same ways as the LRKF but the samples are chosen in a **determined** way such that they always match the mean and covariance or higher moments at the current step.

The Unscented Kalman Filter-UKF

- The UKF works in the same ways as the LRKF but the samples are chosen in a **determined** way such that they always match the mean and covariance or higher moments at the current step.
- The points are propagated using the **nonlinear dynamics**. The mean and covariance at time step $k+1$ are calculated from these propagated points.

The Unscented Kalman Filter-UKF

- The UKF works in the same ways as the LRKF but the samples are chosen in a **determined** way such that they always match the mean and covariance or higher moments at the current step.
- The points are propagated using the **nonlinear dynamics**. The mean and covariance at time step $k+1$ are calculated from these propagated points.
- The **first approximation** the UKF does to the CKE is that it replaces the current state PDF with a gaussian PDF with same first two moments.

UKF

- There is **no linearizations involved**, instead the integral is itself evaluated approximately using quadrature points of the gaussian weighting function-**second approximation.**

UKF

- There is **no linearizations involved**, instead the integral is itself evaluated approximately using quadrature points of the gaussian weighting function-**second approximation.**
- The UKF like the EKF only keeps track of the first two moments of the state PDF.

UKF

- There is **no linearizations involved**, instead the integral is itself evaluated approximately using quadrature points of the gaussian weighting function-**second approximation.**
- The UKF like the EKF only keeps track of the first two moments of the state PDF.
- We can still improve the **second approximation** by better evaluating the integrals.

UKF

One could look at the UKF in the following perspective, starting from CKE

UKF

One could look at the UKF in the following perspective, starting from CKE

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

UKF

One could look at the UKF in the following perspective, starting from CKE

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

Replacing the state PDF with gaussian pdf with equivvalent mean μ_k and covariance P_k.

UKF

One could look at the UKF in the following perspective, starting from CKE

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

Replacing the state PDF with gaussian pdf with equivvalent mean μ_k and covariance P_k.

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)N(x_k)dx_k$$

UKF

One could look at the UKF in the following perspective, starting from CKE

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

Replacing the state PDF with gaussian pdf with equivvalent mean μ_k and covariance P_k.

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)N(x_k)dx_k$$

Calculating the mean of $P(x_{k+1}$ by integrating on both side wrt x_{k+1}.

UKF

One could look at the UKF in the following perspective, starting from CKE

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

Replacing the state PDF with gaussian pdf with equivvalent mean μ_k and covariance P_k.

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)N(x_k)dx_k$$

Calculating the mean of $P(x_{k+1}$ by integrating on both side wrt x_{k+1}.

$$\mu_{k+1} = \int \int N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k$$

UKF

One could look at the UKF in the following perspective, starting from CKE

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)P(x_k)dx_k$$

Replacing the state PDF with gaussian pdf with equivvalent mean μ_k and covariance P_k.

$$P(x_{k+1}) = \int N(x_{k+1}|x_k)N(x_k)dx_k$$

Calculating the mean of $P(x_{k+1}$ by integrating on both side wrt x_{k+1}.

$$\begin{aligned}\mu_{k+1} &= \int \int N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k \\ &= \int f(x_k)N(x_k)dx_k\end{aligned}$$

UKF

The covariance can be calculated from the second raw moment

UKF

The covariance can be calculated from the second raw moment

$$E[x_{k+1}x_{k+1}^T] = \int\int x_{k+1}x_{k+1}^T N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k$$

UKF

The covariance can be calculated from the second raw moment

$$
\begin{aligned}
E[x_{k+1}x_{k+1}^T] &= \int \int x_{k+1}x_{k+1}^T N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k \\
&= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k
\end{aligned}
$$

UKF

The covariance can be calculated from the second raw moment

$$
\begin{aligned}
E[x_{k+1}x_{k+1}^T] &= \int\int x_{k+1}x_{k+1}^T N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k \\
&= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k
\end{aligned}
$$

By parallel axis theorem for moments the covariance is calculated as

UKF

The covariance can be calculated from the second raw moment

$$E[x_{k+1}x_{k+1}^T] = \int\int x_{k+1}x_{k+1}^T N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k$$
$$= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k$$

By parallel axis theorem for moments the covariance is calculated as

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

UKF

The covariance can be calculated from the second raw moment

$$E[x_{k+1}x_{k+1}^T] = \int\int x_{k+1}x_{k+1}^T N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k$$
$$= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k$$

By parallel axis theorem for moments the covariance is calculated as

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

- Thus by evaluating two integrals we get the mean and covariance.

UKF

The covariance can be calculated from the second raw moment

$$E[x_{k+1}x_{k+1}^T] = \int\int x_{k+1}x_{k+1}^T N(x_{k+1} - f(x_k)|Q_k)dx_{k+1}N(x_k)dx_k$$
$$= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k$$

By parallel axis theorem for moments the covariance is calculated as

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

- Thus by evaluating two integrals we get the mean and covariance.
- The second raw moment and the mean need to be evaluated accurate enough or else the parallel axis theorem for moments might render the **covariance to be positive semi definite**.

UKF

In terms of gaussian quadrature, the integrals would be

UKF

In terms of gaussian quadrature, the integrals would be

$$\mu_{k+1} = \int f(x_k)N(x_k)dx_k$$

UKF

In terms of gaussian quadrature, the integrals would be

$$\mu_{k+1} = \int f(x_k) N(x_k) dx_k$$
$$= \sum_{i=1}^{N} w^i f(x_k^i)$$

UKF

In terms of gaussian quadrature, the integrals would be

$$\mu_{k+1} = \int f(x_k)N(x_k)dx_k$$
$$= \sum_{i=1}^{N} w^i f(x_k^i)$$
$$E[x_{k+1}x_{k+1}^T] = \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k$$

UKF

In terms of gaussian quadrature, the integrals would be

$$\begin{aligned}
\mu_{k+1} &= \int f(x_k)N(x_k)dx_k \\
&= \sum_{i=1}^{N} w^i f(x_k^i) \\
E[x_{k+1}x_{k+1}^T] &= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k \\
&= \sum_{i=1}^{N} w^i f(x_k^i)f(x_k^i)^T + Q_k \sum_{i=1}^{N} w^i
\end{aligned}$$

UKF

In terms of gaussian quadrature, the integrals would be

$$\begin{aligned}
\mu_{k+1} &= \int f(x_k)N(x_k)dx_k \\
&= \sum_{i=1}^{N} w^i f(x_k^i) \\
E[x_{k+1}x_{k+1}^T] &= \int (f(x_k)f(x_k)^T + Q_k)N(x_k)dx_k \\
&= \sum_{i=1}^{N} w^i f(x_k^i)f(x_k^i)^T + Q_k \sum_{i=1}^{N} w^i \\
&= \sum_{i=1}^{N} w^i f(x_k^i)f(x_k^i)^T + Q_k
\end{aligned}$$

UKF

The covariance is

UKF

The covariance is

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

UKF

The covariance is

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

After some simplification we could write it in a more general way as

UKF

The covariance is

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

After some simplification we could write it in a more general way as

$$P_k = \sum_{i=1}^{N} w^i[f(x_k^i) - \mu_{k+1}][f(x_k^i) - \mu_{k+1}]^T + Q_k$$

UKF

The covariance is

$$P_k = E[x_{k+1}x_{k+1}^T] - \mu_{k+1}\mu_{k+1}^T$$

After some simplification we could write it in a more general way as

$$P_k = \sum_{i=1}^{N} w^i [f(x_k^i) - \mu_{k+1}][f(x_k^i) - \mu_{k+1}]^T + Q_k$$

Thus the UKF boils down to just evaluating the integrals involving gaussian kernel using the quadrature points.

Objective

To evaluate the integral

$$E[f(x)] = \int f(x) N(x, \mu | P) dx$$

The Conjugate Unscented Transform- CUT

We try to propose a **new method of Gaussian Cubature** to evaluate the integrals and hence may be a potential application as a **new filter.** Later we show that the UKF and CKF are inline with the present construction. Basic Philosophy of CUT:

- The basic philosophy in this analysis is "‘**To evaluate the integral involving gaussian weight function with as few points as possible**".

The Conjugate Unscented Transform- CUT

We try to propose a **new method of Gaussian Cubature** to evaluate the integrals and hence may be a potential application as a **new filter.** Later we show that the UKF and CKF are inline with the present construction. Basic Philosophy of CUT:

- The basic philosophy in this analysis is "‘**To evaluate the integral involving gaussian weight function with as few points as possible**".
- Any Gaussian Quadrature rule has to capture the moments of the continuous PDF

The Conjugate Unscented Transform- CUT

We try to propose a **new method of Gaussian Cubature** to evaluate the integrals and hence may be a potential application as a **new filter.** Later we show that the UKF and CKF are inline with the present construction. Basic Philosophy of CUT:

- The basic philosophy in this analysis is "“**To evaluate the integral involving gaussian weight function with as few points as possible**".
- Any Gaussian Quadrature rule has to capture the moments of the continuous PDF
- For example consider the Gauss-Hermite quadrature, as we increase the number of quadrature points more moments are captured and hence higher degree polynomials can be integrated.

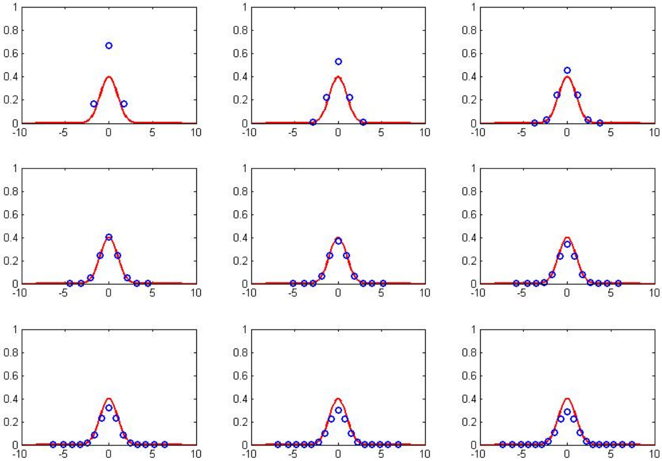

Figure: Guass hermite quadrature for 1D

- In general for a N-Dimensional system, to integrate a polynomial of degree$2m+1$ **we need a total of** $(m+1)^N$ **quadrature points**.
- This is a **very big number** for higher dimensional system and **this is the basis of our motivation to develop a method with reduced number of points**.
- Ideally one would like to capture all the **infinite** moments of the PDF.
- In practice this is difficult to achieve or might be computationally expensive. Thus often only the lower order moments are captured.This highly limits the type of funtions that can be integrated with good numerical accuracy.

Gauss Hermite Product rule

Gauss Hermite Product rule

The 1D Gauss Hermite rule can be extended to any dimension for an i.i.d set of random variable $(x_1, x_2, ..x_N)^T$.

Gauss Hermite Product rule

The 1D Gauss Hermite rule can be extended to any dimension for an i.i.d set of random variable $(x_1, x_2, ..x_N)^T$.

$$E[f(x_1, x_2, ...x_N)]$$

Gauss Hermite Product rule

The 1D Gauss Hermite rule can be extended to any dimension for an i.i.d set of random variable $(x_1, x_2, ..x_N)^T$.

$$E[f(x_1, x_2, ...x_N)]$$

$$= \int \int ... \int f(x_1, x_2, ...x_N) N(x_1, x_2, ...x_N, 0|I) dx_1 dx_2 ... dx_N$$

Gauss Hermite Product rule

The 1D Gauss Hermite rule can be extended to any dimension for an i.i.d set of random variable $(x_1, x_2, ..x_N)^T$.

$$E[f(x_1, x_2, ...x_N)]$$

$$= \int \int ... \int f(x_1, x_2, ...x_N) N(x_1, x_2, ...x_N, 0|I) dx_1 dx_2 ... dx_N$$
$$= \int \int ... [\int f(x_1, x_2, ...x_N) N(x_1, 0|1) dx_1] N(x_2, 0|1) dx_2 N(x_3, 0|1)$$
$$... N(x_N, 0|1) dx_N$$

Gauss Hermite Product rule

The 1D Gauss Hermite rule can be extended to any dimension for an i.i.d set of random variable $(x_1, x_2, ..x_N)^T$.

$$E[f(x_1, x_2, ...x_N)]$$

$$= \int \int ... \int f(x_1, x_2, ...x_N) N(x_1, x_2, ...x_N, 0|I) dx_1 dx_2 ... dx_N$$

$$= \int \int ...[\int f(x_1, x_2, ...x_N) N(x_1, 0|1) dx_1] N(x_2, 0|1) dx_2 N(x_3, 0|1)$$

$$...N(x_N, 0|1) dx_N$$

$$= \sum_{i_N=1}^{n} w_N^{i_N} \sum_{i_{N-1}=1}^{n} w_{N-1}^{i_{N-1}} ... \sum_{i_2=1}^{n} w_2^{i_2} \sum_{i_1=1}^{n} w_1^{i_1} f(x_1^{i_1}, x_2^{i_2}, ...x_N^{i_N})$$

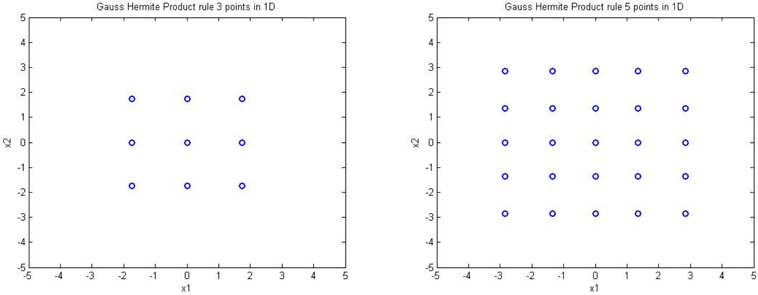

Figure: Guass hermite product rule for 2D

Transformation of a Normal PDF with Arbitrary mean and Covariance into a Normal PDF with Zero Mean and Identity Covariance

- Any jointly distributed Normal PDF can be transformed into a Gaussian PDF of zero mean and identity covariance of same dimension.

Transformation of a Normal PDF with Arbitrary mean and Covariance into a Normal PDF with Zero Mean and Identity Covariance

- Any jointly distributed Normal PDF can be transformed into a Gaussian PDF of zero mean and identity covariance of same dimension.
- This way one could find the cubature points in the i.i.d space and then transform each point into the original jointly distributed space.

Transformation of a Normal PDF with Arbitrary mean and Covariance into a Normal PDF with Zero Mean and Identity Covariance

- Any jointly distributed Normal PDF can be transformed into a Gaussian PDF of zero mean and identity covariance of same dimension.
- This way one could find the cubature points in the i.i.d space and then transform each point into the original jointly distributed space.

Transformation of a Normal PDF with Arbitrary mean and Covariance into a Normal PDF with Zero Mean and Identity Covariance

- Any jointly distributed Normal PDF can be transformed into a Gaussian PDF of zero mean and identity covariance of same dimension.
- This way one could find the cubature points in the i.i.d space and then transform each point into the original jointly distributed space.

$$(x-\mu)^T P^{-1}(x-\mu) = (x-\mu)^T U^T \Sigma U (x-\mu)$$

Transformation of a Normal PDF with Arbitrary mean and Covariance into a Normal PDF with Zero Mean and Identity Covariance

- Any jointly distributed Normal PDF can be transformed into a Gaussian PDF of zero mean and identity covariance of same dimension.
- This way one could find the cubature points in the i.i.d space and then transform each point into the original jointly distributed space.

$$
\begin{aligned}
(x-\mu)^T P^{-1}(x-\mu) &= (x-\mu)^T U^T \Sigma U (x-\mu) \\
&= (x-\mu)^T U^T \sqrt{\Sigma}\sqrt{\Sigma} U (x-\mu)
\end{aligned}
$$

Transformation of a Normal PDF with Arbitrary mean and Covariance into a Normal PDF with Zero Mean and Identity Covariance

- Any jointly distributed Normal PDF can be transformed into a Gaussian PDF of zero mean and identity covariance of same dimension.
- This way one could find the cubature points in the i.i.d space and then transform each point into the original jointly distributed space.

$$\begin{aligned}(x-\mu)^T P^{-1}(x-\mu) &= (x-\mu)^T U^T \Sigma U(x-\mu) \\ &= (x-\mu)^T U^T \sqrt{\Sigma}\sqrt{\Sigma} U(x-\mu)\end{aligned}$$

$$y = \sqrt{\Sigma} U(x-\mu)$$

Transformation of a Normal PDF with Arbitrary mean and Covariance into a Normal PDF with Zero Mean and Identity Covariance

- Any jointly distributed Normal PDF can be transformed into a Gaussian PDF of zero mean and identity covariance of same dimension.
- This way one could find the cubature points in the i.i.d space and then transform each point into the original jointly distributed space.

$$
\begin{aligned}
(x-\mu)^T P^{-1}(x-\mu) &= (x-\mu)^T U^T \Sigma U(x-\mu) \\
&= (x-\mu)^T U^T \sqrt{\Sigma}\sqrt{\Sigma} U(x-\mu)
\end{aligned}
$$

$$
\begin{aligned}
y &= \sqrt{\Sigma} U(x-\mu) \\
(x-\mu)^T P^{-1}(x-\mu) &= y^T y
\end{aligned}
$$

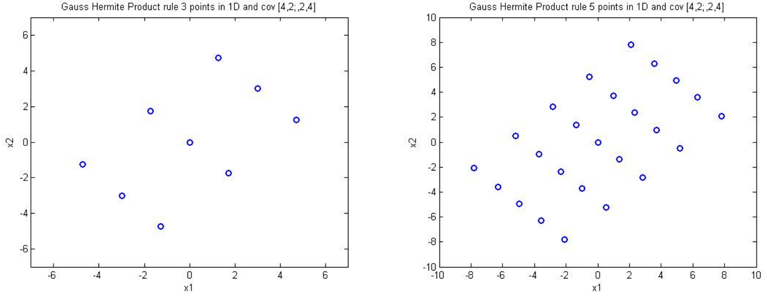

Figure: Guass hermite product rule for 2D for Cov $[4,2;2,4]$

Evaluating the Cubature points

To evalute the cubature points and weights of any PDF, the following procedure is followed

- Evaluate the**moments** of the continuous PDF

Evaluating the Cubature points

To evalute the cubature points and weights of any PDF, the following procedure is followed

- Evaluate the**moments** of the continuous PDF
- Assume the**positions and weights** of each cubature point as variables

Evaluating the Cubature points

To evalute the cubature points and weights of any PDF, the following procedure is followed

- Evaluate the**moments** of the continuous PDF
- Assume the**positions and weights** of each cubature point as variables
- Equate the expressions of moment equations for the discrete system to the known moments of the continuous PDF

Evaluating the Cubature points

To evalute the cubature points and weights of any PDF, the following procedure is followed

- Evaluate the**moments** of the continuous PDF
- Assume the**positions and weights** of each cubature point as variables
- Equate the expressions of moment equations for the discrete system to the known moments of the continuous PDF
- Solve these set of **nonlinear moment constraint equations**

Moment constraint equations

The equivalent discrete moments are written interms of the assumed variables- **positions and weights**

Moment constraint equations

The equivalent discrete moments are written interms of the assumed variables- **positions and weights**

Table: Equivalent Moments for 2D till order 4

Continuous	Discrete	Continuous	Dicrete
$E[x]$	$\sum_{i=1}^{n} w_i x_i$	$E[y]$	$\sum_{i=1}^{n} w_i y_i$
$E[x^2]$	$\sum_{i=1}^{n} w_i x_i^2$	$E[xy]$	$\sum_{i=1}^{n} w_i x_i y_i$
$E[y^2]$	$\sum_{i=1}^{n} w_i y_i^2$	$E[x^4]$	$\sum_{i=1}^{n} w_i x_i^2$
$E[y^4]$	$\sum_{i=1}^{n} w_i y_i^2$	$E[x^3 y]$	$\sum_{i=1}^{n} w_i x_i^3 y_i$
$E[y^3 x]$	$\sum_{i=1}^{n} w_i y_i^3 x_i$	$E[x^2 y^2]$	$\sum_{i=1}^{n} w_i x_i^2 y_i^2$

Moment constraint equations

The equivalent discrete moments are written interms of the assumed variables- **positions and weights**

Table: Equivalent Moments for 2D till order 4

Continuous	Discrete	Continuous	Dicrete
$E[x]$	$\sum_{i=1}^{n} w_i x_i$	$E[y]$	$\sum_{i=1}^{n} w_i y_i$
$E[x^2]$	$\sum_{i=1}^{n} w_i x_i^2$	$E[xy]$	$\sum_{i=1}^{n} w_i x_i y_i$
$E[y^2]$	$\sum_{i=1}^{n} w_i y_i^2$	$E[x^4]$	$\sum_{i=1}^{n} w_i x_i^2$
$E[y^4]$	$\sum_{i=1}^{n} w_i y_i^2$	$E[x^3 y]$	$\sum_{i=1}^{n} w_i x_i^3 y_i$
$E[y^3 x]$	$\sum_{i=1}^{n} w_i y_i^3 x_i$	$E[x^2 y^2]$	$\sum_{i=1}^{n} w_i x_i^2 y_i^2$

Table 1 gives the moment constraint equations that have to be solved for the **weights** w_i **and points** x_i**.**

Moments of Continuous Normal PDF

- The higher order moments of a Normal PDF can be calculated from **just the mean and Covariance.**

Moments of Continuous Normal PDF

- The higher order moments of a Normal PDF can be calculated from **just the mean and Covariance.**
- Particularly the Higher order raw moments just need the covariance matrix.

Moments of Continuous Normal PDF

- The higher order moments of a Normal PDF can be calculated from **just the mean and Covariance.**
- Particularly the Higher order raw moments just need the covariance matrix.

Moments of Continuous Normal PDF

- The higher order moments of a Normal PDF can be calculated from **just the mean and Covariance.**
- Particularly the Higher order raw moments just need the covariance matrix.

The moments are given by the following theorem

Moments of Continuous Normal PDF

- The higher order moments of a Normal PDF can be calculated from **just the mean and Covariance.**
- Particularly the Higher order raw moments just need the covariance matrix.

The moments are given by the following theorem

Isserilis Theorem

$$E[x_1 x_2 x_3 \ldots . x_{2n}] = \sum \prod E[x_i x_j]$$

for example the **fourth moment** is

for example the **fourth moment** is

$$E[x_1 x_2 x_3 x_4] = E[x_1 x_2]E[x_3 x_4] + E[x_1 x_3]E[x_4 x_5] + E[x_1 x_4]E[x_2 x_3]$$

for example the **fourth moment** is

$$E[x_1x_2x_3x_4] = E[x_1x_2]E[x_3x_4] + E[x_1x_3]E[x_4x_5] + E[x_1x_4]E[x_2x_3]$$

And the **sixth moment** is

$$E[x_1x_2x_3x_4x_5x_6] =$$

$$\begin{aligned}
=&E[x_1x_2]E[x_3x_4]E[x_5x_6] + E[x_1x_2]E[x_3x_5]E[x_4x_6] \\
&+ E[x_1x_2]E[x_3x_6]E[x_4x_5] + E[x_1x_3]E[x_2x_4]E[x_5x_6] \\
&+ E[x_1x_3]E[x_2x_5]E[x_4x_6] + E[x_1x_3]E[x_2x_6]E[x_4x_5] \\
&+ E[x_1x_4]E[x_2x_3]E[x_5x_6] + E[x_1x_4]E[x_2x_5]E[x_3x_6] \\
&+ E[x_1x_4]E[x_2x_6]E[x_3x_5] + E[x_1x_5]E[x_2x_3]E[x_4x_6] \\
&+ E[x_1x_5]E[x_2x_4]E[x_3x_6] + E[x_1x_5]E[x_2x_6]E[x_3x_4] \\
&+ E[x_1x_6]E[x_2x_3]E[x_4x_5] + E[x_1x_6]E[x_2x_4]E[x_3x_5] \\
&+ E[x_1x_6]E[x_2x_5]E[x_3x_4]
\end{aligned}$$

Normal Distribution

As the Normal PDF is symmetric, we would like to **exploit this symmetry** in finding the cubature points. The points we like to seek have the following properties

- **Fully symmetric** :- thus satisfying all odd order moments at no further cost

Normal Distribution

As the Normal PDF is symmetric, we would like to **exploit this symmetry** in finding the cubature points. The points we like to seek have the following properties

- **Fully symmetric** :- thus satisfying all odd order moments at no further cost
- Weights **sum up** to 1

Normal Distribution

As the Normal PDF is symmetric, we would like to **exploit this symmetry** in finding the cubature points. The points we like to seek have the following properties

- **Fully symmetric** :- thus satisfying all odd order moments at no further cost
- Weights **sum up** to 1
- All weights are to be **positive**

Normal Distribution

As the Normal PDF is symmetric, we would like to **exploit this symmetry** in finding the cubature points. The points we like to seek have the following properties

- **Fully symmetric** :- thus satisfying all odd order moments at no further cost
- Weights **sum up** to 1
- All weights are to be **positive**
- As the points are symmetric, each point has an equivalent point located on the other side of the mean at the same distance.

Normal Distribution

As the Normal PDF is symmetric, we would like to **exploit this symmetry** in finding the cubature points. The points we like to seek have the following properties

- **Fully symmetric** :- thus satisfying all odd order moments at no further cost
- Weights **sum up** to 1
- All weights are to be **positive**
- As the points are symmetric, each point has an equivalent point located on the other side of the mean at the same distance.
- These two points together lie on a line passing through the mean. We later name this line as a particular '**axis**'.

Normal Distribution

As the Normal PDF is symmetric, we would like to **exploit this symmetry** in finding the cubature points. The points we like to seek have the following properties

- **Fully symmetric** :- thus satisfying all odd order moments at no further cost
- Weights **sum up** to 1
- All weights are to be **positive**
- As the points are symmetric, each point has an equivalent point located on the other side of the mean at the same distance.
- These two points together lie on a line passing through the mean. We later name this line as a particular **'axis'.**
- And these two points also have the same weight to balance each other

Terminology

Generator set

- This definition has been used quite a lot in literature to describe a set containing all permutations, including sign permutations of the elements in a set $\{u_1, u_2, u_3, ..., u_r, 0, 0, 0...., 0\}$, where $u_1, u_2, ...$ are real numbers.

Terminology

To proceed with the method we describe, the following **terminology or definitions** that might be handy

Generator set

- This definition has been used quite a lot in literature to describe a set containing all permutations, including sign permutations of the elements in a set $\{u_1, u_2, u_3, ..., u_r, 0, 0, 0...., 0\}$, where $u_1, u_2, ...$ are real numbers.
- for example the generator set $(1, 1, 0)$ is equivalent to $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, $(-1, 1, 0)$, $(1, -1, 0)$, $(-1, 0, 1)$, $(1, 0, -1)$, $(0, -1, 1)$, $(0, 1, -1)$, $(0, -1, -1)$, $(-1, -1, 0)$, $(-1, 0, -1)$

Terminology

To proceed with the method we describe, the following **terminology or definitions** that might be handy

Generator set

- This definition has been used quite a lot in literature to describe a set containing all permutations, including sign permutations of the elements in a set $\{u_1, u_2, u_3, ..., u_r, 0, 0, 0...., 0\}$, where $u_1, u_2, ...$ are real numbers.
- for example the generator set $(1, 1, 0)$ is equivalent to $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, $(-1, 1, 0)$, $(1, -1, 0)$, $(-1, 0, 1)$, $(1, 0, -1)$, $(0, -1, 1)$, $(0, 1, -1)$, $(0, -1, -1)$, $(-1, -1, 0)$, $(-1, 0, -1)$

Terminology

To proceed with the method we describe, the following **terminology or definitions** that might be handy

Generator set

- This definition has been used quite a lot in literature to describe a set containing all permutations, including sign permutations of the elements in a set $\{u_1, u_2, u_3, ..., u_r, 0, 0, 0...., 0\}$, where $u_1, u_2, ...$ are real numbers.
- for example the generator set $(1, 1, 0)$ is equivalent to $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, $(-1, 1, 0)$, $(1, -1, 0)$, $(-1, 0, 1)$, $(1, 0, -1)$, $(0, -1, 1)$, $(0, 1, -1)$, $(0, -1, -1)$, $(-1, -1, 0)$, $(-1, 0, -1)$

Terminology

To proceed with the method we describe, the following **terminology or definitions** that might be handy

Generator set

- This definition has been used quite a lot in literature to describe a set containing all permutations, including sign permutations of the elements in a set $\{u_1, u_2, u_3, ..., u_r, 0, 0, 0...., 0\}$, where $u_1, u_2, ...$ are real numbers.
- for example the generator set $(1, 1, 0)$ is equivalent to $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, $(-1, 1, 0)$, $(1, -1, 0)$, $(-1, 0, 1)$, $(1, 0, -1)$, $(0, -1, 1)$, $(0, 1, -1)$, $(0, -1, -1)$, $(-1, -1, 0)$, $(-1, 0, -1)$

We introduce some new terminology with respect to the Identity Covariance Normal PDF of N^{th}-Dimension with zero mean, that will just aid our intuition

The variables used

- Once the appropriate axis are chosen, the points are constrained to be on these axis such that a symmetric set of points that have the same weight are at equal distance from the origin.

The variables used

- Once the appropriate axis are chosen, the points are constrained to be on these axis such that a symmetric set of points that have the same weight are at equal distance from the origin.
- The distance variables are r_1, r_2... measured from the origin to the appropriate set of points.

The variables used

- Once the appropriate axis are chosen, the points are constrained to be on these axis such that a symmetric set of points that have the same weight are at equal distance from the origin.
- The distance variables are r_1, r_2... measured from the origin to the appropriate set of points.
- The points at distance r_1 have weight w_1 and so on.

The variables used

- Once the appropriate axis are chosen, the points are constrained to be on these axis such that a symmetric set of points that have the same weight are at equal distance from the origin.
- The distance variables are r_1, r_2... measured from the the origin to the appropriate set of points.
- The points at distance r_1 have weight w_1 and so on.

The variables used

- Once the appropriate axis are chosen, the points are constrained to be on these axis such that a symmetric set of points that have the same weight are at equal distance from the origin.
- The distance variables are r_1, r_2... measured from the the origin to the appropriate set of points.
- The points at distance r_1 have weight w_1 and so on.

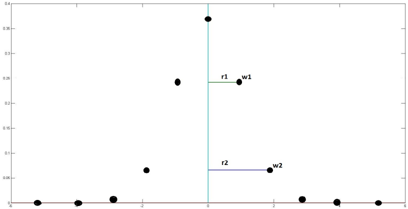

Figure: distances and weights

Principle axis

The axis formed from the generator set $\{1, 0, 0, ..., 0\}$. For example consider the 3D case for which the generator set is $\{(1, 0, 0), (0, 1, 0), (0, 0, 1), (-1, 0, 0), (0, -1, 0), (0, 0, -1)\}$. The equivalent principle axis are just the orthogonal axis in 3D space $(1, 0, 0)$, $(0, 1, 0)$ and $(0, 0, 1)$. There are $2N$ points and N principle/orthogonal axis

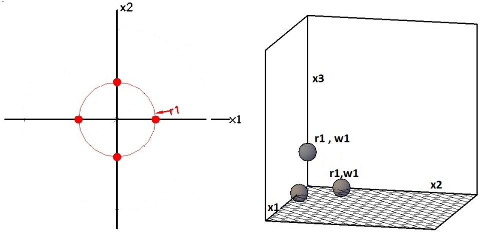

Figure: distances and weights

M^{th}-Conjugate axis

- For a Normal PDF of Nth Dimension, the axis that are formed from the generator set $\{1, 1, 1, ..., 1\}$.

M^{th}-Conjugate axis

- For a Normal PDF of Nth Dimension, the axis that are formed from the generator set $\{1, 1, 1, ..., 1\}$.
- For a 3D system the 3^{rd}-conjugate axis are the lines that pass through the origin and the points from the generator set $(1, 1, 1)$, $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$, $(-1, -1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, -1)$.

M^{th}-Conjugate axis

- For a Normal PDF of Nth Dimension, the axis that are formed from the generator set $\{1, 1, 1, ..., 1\}$.
- For a 3D system the 3^{rd}-conjugate axis are the lines that pass through the origin and the points from the generator set $(1, 1, 1)$, $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$, $(-1, -1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, -1)$.
- There are 2^N points $2^{(N-1)}$ axis.

M^{th}-Conjugate axis

- For a Normal PDF of Nth Dimension, the axis that are formed from the generator set $\{1, 1, 1, ..., 1\}$.
- For a 3D system the 3^{rd}-conjugate axis are the lines that pass through the origin and the points from the generator set $(1, 1, 1)$, $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$, $(-1, -1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, -1)$.
- There are 2^N points $2^{(N-1)}$ axis.
- The 2^{nd}- conjugate axis for a 3D system are the axis formed from the generator set $\{1, 1, 0\}$

M^{th}-Conjugate axis

- For a Normal PDF of Nth Dimension, the axis that are formed from the generator set $\{1, 1, 1, ..., 1\}$.
- For a 3D system the 3^{rd}-conjugate axis are the lines that pass through the origin and the points from the generator set $(1, 1, 1)$, $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$, $(-1, -1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, -1)$.
- There are 2^N points $2^{(N-1)}$ axis.
- The 2^{nd}- conjugate axis for a 3D system are the axis formed from the generator set $\{1, 1, 0\}$
- The set of points $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, $(-1, 1, 0)$, $(1, -1, 0)$, $(-1, 0, 1)$, $(1, 0, -1)$, $(0, -1, 1)$, $(0, 1, -1)$, $(0, -1, -1)$, $(-1, -1, 0)$, $(-1, 0, -1)$

M^{th}-Conjugate axis

- For a Normal PDF of Nth Dimension, the axis that are formed from the generator set $\{1, 1, 1, ..., 1\}$.
- For a 3D system the 3^{rd}-conjugate axis are the lines that pass through the origin and the points from the generator set $(1, 1, 1)$, $(-1, 1, 1)$, $(1, -1, 1)$, $(1, 1, -1)$, $(-1, -1, 1)$, $(1, -1, -1)$, $(-1, 1, -1)$, $(-1, -1, -1)$.
- There are 2^N points $2^{(N-1)}$ axis.
- The 2^{nd}- conjugate axis for a 3D system are the axis formed from the generator set $\{1, 1, 0\}$
- The set of points $(1, 1, 0)$, $(1, 0, 1)$, $(0, 1, 1)$, $(-1, 1, 0)$, $(1, -1, 0)$, $(-1, 0, 1)$, $(1, 0, -1)$, $(0, -1, 1)$, $(0, 1, -1)$, $(0, -1, -1)$, $(-1, -1, 0)$, $(-1, 0, -1)$
- There are $2N(N-1)$ points and $N(N-1)$ axis

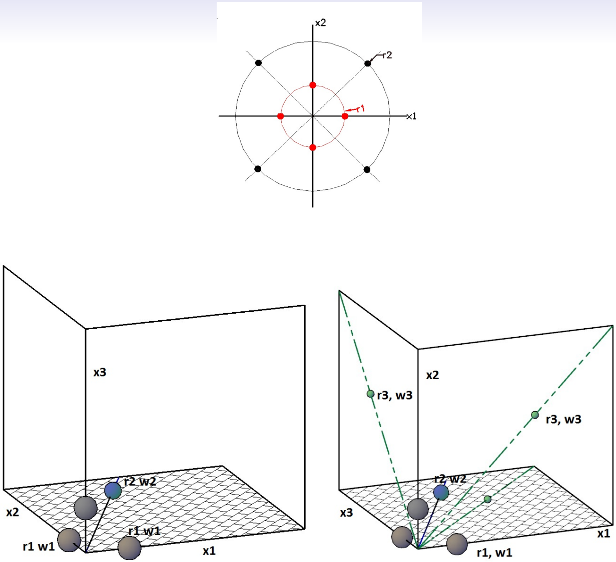

Figure: distances and weights

N^{th}- Scaled Conjugate axis

- For a Normal PDF of Nth Dimension, the axis that are formed from the generator set $\{h, 1, 1, ..., 1\}$.

N^{th}- Scaled Conjugate axis

- For a Normal PDF of Nth Dimension, the axis that are formed from the generator set $\{h, 1, 1, ..., 1\}$.
- For a 3D system the 3^{rd}-Scaled conjugate axis are the lines that pass through the origin and the points from the generator set $(h, 1, 1)$, $(-h, 1, 1)$, $(h, -1, 1)$, $(h, 1, -1)$, $(-h, -1, 1)$, $(h, -1, -1)$, $(-h, 1, -1)$, $(-h, -1, -1)$, $(1, h, 1)$, $(-1, h, 1)$, $(1, -h, 1)$, $(1, h, -1)$, $(-1, -h, 1)$, $(1, -h, -1)$, $(-1, h, -1)$, $(-1, -h, -1)$, $(h, 1, 1)$, $(-h, 1, 1)$, $(1, -1, h)$, $(1, 1, -h)$, $(-1, -1, h)$, $(1, -1, -h)$, $(-1, 1, -h)$, $(-1, -1, -h)$.

N^{th}- Scaled Conjugate axis

- For a Normal PDF of Nth Dimension, the axis that are formed from the generator set $\{h, 1, 1, ..., 1\}$.
- For a 3D system the 3^{rd}-Scaled conjugate axis are the lines that pass through the origin and the points from the generator set $(h, 1, 1)$, $(-h, 1, 1)$, $(h, -1, 1)$, $(h, 1, -1)$, $(-h, -1, 1)$, $(h, -1, -1)$, $(-h, 1, -1)$, $(-h, -1, -1)$, $(1, h, 1)$, $(-1, h, 1)$, $(1, -h, 1)$, $(1, h, -1)$, $(-1, -h, 1)$, $(1, -h, -1)$, $(-1, h, -1)$, $(-1, -h, -1)$, $(h, 1, 1)$, $(-h, 1, 1)$, $(1, -1, h)$, $(1, 1, -h)$, $(-1, -1, h)$, $(1, -1, -h)$, $(-1, 1, -h)$, $(-1, -1, -h)$.
- There are $N2^N$ points and $N2^{(N-1)}$ axis.

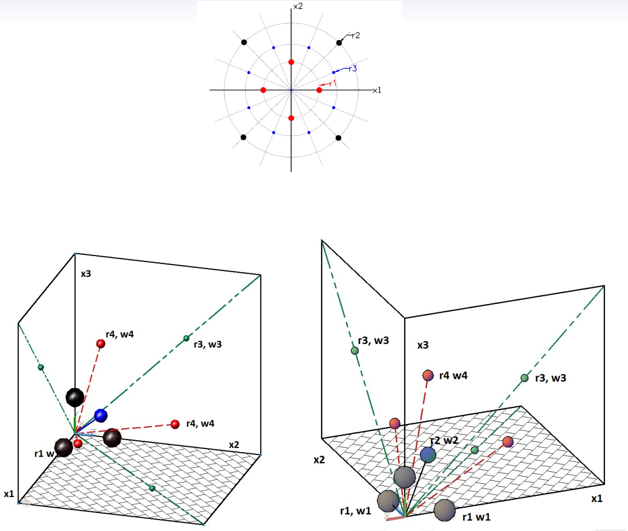

Figure: distances and weights

Cubature points constrained to the principle axis to satisfy the 2^{nd} order moments

Cubature points constrained to the principle axis to satisfy the 2^{nd} order moments

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

Cubature points constrained to the principle axis to satisfy the 2^{nd} order moments

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$E[X_i^2] = 1$$

Cubature points constrained to the principle axis to satisfy the 2^{nd} order moments

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$E[X_i^2] = 1$$
$$E[X_i X_j] = 0$$

Cubature points constrained to the principle axis to satisfy the 2^{nd} order moments

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$E[X_i^2] = 1$$
$$E[X_iX_j] = 0$$
$$E[X_i^4] = 3$$

Cubature points constrained to the principle axis to satisfy the 2^{nd} order moments

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$E[X_i^2] = 1$$
$$E[X_i X_j] = 0$$
$$E[X_i^4] = 3$$
$$E[X_i^3 X_j] = 0$$

Cubature points constrained to the principle axis to satisfy the 2^{nd} order moments

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$
\begin{aligned}
E[X_i^2] &= 1 \\
E[X_i X_j] &= 0 \\
E[X_i^4] &= 3 \\
E[X_i^3 X_j] &= 0 \\
E[X_i^2 X_j^2] &= 1
\end{aligned}
$$

Cubature points constrained to the principle axis to satisfy the 2^{nd} order moments

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$
\begin{aligned}
E[X_i^2] &= 1 \\
E[X_i X_j] &= 0 \\
E[X_i^4] &= 3 \\
E[X_i^3 X_j] &= 0 \\
E[X_i^2 X_j^2] &= 1 \\
E[X_i^2 X_j X_k] &= 0
\end{aligned}
$$

Cubature points constrained to the principle axis to satisfy the 2^{nd} order moments

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$
\begin{aligned}
E[X_i^2] &= 1 \\
E[X_i X_j] &= 0 \\
E[X_i^4] &= 3 \\
E[X_i^3 X_j] &= 0 \\
E[X_i^2 X_j^2] &= 1 \\
E[X_i^2 X_j X_k] &= 0 \\
E[X_i X_j X_k X_l] &= 0
\end{aligned}
$$

Cubature points constrained to the principle axis to satisfy the 2^{nd} order moments

For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The **first four moments** of this continuous PDF are

$$
\begin{aligned}
E[X_i^2] &= 1 \\
E[X_i X_j] &= 0 \\
E[X_i^4] &= 3 \\
E[X_i^3 X_j] &= 0 \\
E[X_i^2 X_j^2] &= 1 \\
E[X_i^2 X_j X_k] &= 0 \\
E[X_i X_j X_k X_l] &= 0
\end{aligned}
$$

Where $\{i, j, k, l\} \in \{1, 2, 3, ..., N\}$ & $i \neq j \neq k \neq l$.

- Consider a fully symmetric set of cubature points that lie on the principle axis at a distance of r_1 from the origin and have weight of w_1.

- Consider a fully symmetric set of cubature points that lie on the principle axis at a distance of r_1 from the origin and have weight of w_1.
- The moments that have any odd powers for the random varable in the set of moments is are already satisfied due to**symmetry of cubature points.**

- Consider a fully symmetric set of cubature points that lie on the principle axis at a distance of r_1 from the origin and have weight of w_1.
- The moments that have any odd powers for the random varable in the set of moments is are already satisfied due to**symmetry of cubature points.**
- Only the moments with **all even powers** have to be satisfied

- Consider a fully symmetric set of cubature points that lie on the principle axis at a distance of r_1 from the origin and have weight of w_1.
- The moments that have any odd powers for the random varable in the set of moments is are already satisfied due to**symmetry of cubature points.**
- Only the moments with **all even powers** have to be satisfied

- Consider a fully symmetric set of cubature points that lie on the principle axis at a distance of r_1 from the origin and have weight of w_1.
- The moments that have any odd powers for the random varable in the set of moments is are already satisfied due to**symmetry of cubature points.**
- Only the moments with **all even powers** have to be satisfied

$$E[X_i^2] = 1$$

- Consider a fully symmetric set of cubature points that lie on the principle axis at a distance of r_1 from the origin and have weight of w_1.
- The moments that have any odd powers for the random varable in the set of moments is are already satisfied due to**symmetry of cubature points.**
- Only the moments with **all even powers** have to be satisfied

$$E[X_i^2] = 1$$
$$E[X_i^4] = 3$$

- Consider a fully symmetric set of cubature points that lie on the principle axis at a distance of r_1 from the origin and have weight of w_1.
- The moments that have any odd powers for the random varable in the set of moments is are already satisfied due to**symmetry of cubature points.**
- Only the moments with **all even powers** have to be satisfied

$$
\begin{aligned}
E[X_i^2] &= 1 \\
E[X_i^4] &= 3 \\
E[X_i^2 X_j^2] &= 1
\end{aligned}
$$

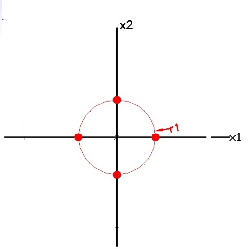

Figure: distances and weights

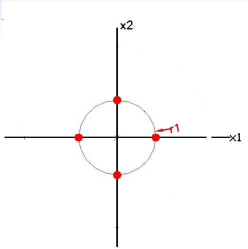

Figure: distances and weights

For example consider the 2D system. The points on these axis when enumerated as a list is $(r1, 0)$, $(0, r1)$, $(-r1, 0)$, $(0, -r1)$. The equivalent discrete moments are

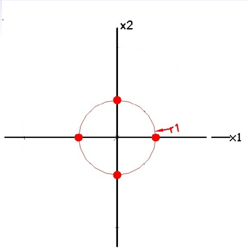

Figure: distances and weights

For example consider the 2D system. The points on these axis when enumerated as a list is $(r1, 0)$, $(0, r1)$, $(-r1, 0)$, $(0, -r1)$. The equivalent discrete moments are

$$2r_1^2 w_1 = 1$$

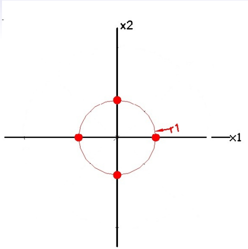

Figure: distances and weights

For example consider the 2D system. The points on these axis when enumerated as a list is $(r1, 0)$, $(0, r1)$, $(-r1, 0)$, $(0, -r1)$. The equivalent discrete moments are

$$
\begin{aligned}
2r_1^2 w_1 &= 1 \\
2r_1^4 w_1 &= 3
\end{aligned}
$$

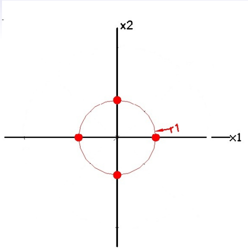

Figure: distances and weights

For example consider the 2D system. The points on these axis when enumerated as a list is $(r1, 0)$, $(0, r1)$, $(-r1, 0)$, $(0, -r1)$. The equivalent discrete moments are

$$2r_1^2 w_1 = 1$$
$$2r_1^4 w_1 = 3$$
$$0 \neq 1$$

(1)

(2)

This set of equations are infact the same for any dimension.The central weight is calculated as

(1)

(2)

This set of equations are infact the same for any dimension.The central weight is calculated as

$$w_0 = 1 - 2Nw_1 \tag{1}$$

(2)

This set of equations are infact the same for any dimension.The central weight is calculated as

$$w_0 = 1 - 2Nw_1 \tag{1}$$

One of the 4^{th} order cross moment **cannot be satisfied** by selecting such points on the principle axis.

$$\tag{2}$$

This set of equations are infact the same for any dimension.The central weight is calculated as

$$w_0 = 1 - 2Nw_1 \tag{1}$$

One of the 4^{th} order cross moment **cannot be satisfied** by selecting such points on the principle axis.
In general **no cross moment** of any order can be satisfied by the points just on the **principle axis.**

$$\tag{2}$$

This set of equations are infact the same for any dimension.The central weight is calculated as

$$w_0 = 1 - 2Nw_1 \tag{1}$$

One of the 4^{th} order cross moment **cannot be satisfied** by selecting such points on the principle axis.
In general **no cross moment** of any order can be satisfied by the points just on the **principle axis.**

$$E[X_i^2 X_j^2] \neq 1 \tag{2}$$

This set of equations are infact the same for any dimension.The central weight is calculated as

$$w_0 = 1 - 2Nw_1 \tag{1}$$

One of the 4^{th} order cross moment **cannot be satisfied** by selecting such points on the principle axis.
In general **no cross moment** of any order can be satisfied by the points just on the **principle axis.**

$$E[X_i^2 X_j^2] \neq 1 \tag{2}$$

The 4th moment is also evaluted here just to compare different filters such as UKF and CKF in terms of the **4th order moment error**.

The $2N+1$ Unscented Transform Method

- The $2N+1$ sigma points for the unscented Transform are chosen such that 1 point is the origin and $2N$ points of equal weight are constrained to lie on the principle axis.

The $2N + 1$ Unscented Transform Method

- The $2N + 1$ sigma points for the unscented Transform are chosen such that 1 point is the origin and $2N$ points of equal weight are constrained to lie on the principle axis.
- The distance of each point on the principle axis is chosen such that the second order moments are satisfied.

The $2N + 1$ Unscented Transform Method

- The $2N + 1$ sigma points for the unscented Transform are chosen such that 1 point is the origin and $2N$ points of equal weight are constrained to lie on the principle axis.
- The distance of each point on the principle axis is chosen such that the second order moments are satisfied.
- A tuning parameter κ is introduced such that one of the 4th moment can be tuned.

The $2N + 1$ Unscented Transform Method

- The $2N + 1$ sigma points for the unscented Transform are chosen such that 1 point is the origin and $2N$ points of equal weight are constrained to lie on the principle axis.
- The distance of each point on the principle axis is chosen such that the second order moments are satisfied.
- A tuning parameter κ is introduced such that one of the 4th moment can be tuned.
- This works well for systems till dimension 3 after which the central weight becomes negative.

The $2N + 1$ Unscented Transform Method

- The $2N + 1$ sigma points for the unscented Transform are chosen such that 1 point is the origin and $2N$ points of equal weight are constrained to lie on the principle axis.
- The distance of each point on the principle axis is chosen such that the second order moments are satisfied.
- A tuning parameter κ is introduced such that one of the 4th moment can be tuned.
- This works well for systems till dimension 3 after which the central weight becomes negative.
- The present framework is shown to **be inline** with the $2n + 1$ Unscented sigma points by working our way backwards.

The suggested points by the authors of UT paper are

The suggested points by the authors of UT paper are

$$x_0 = \mu \qquad\qquad w_0 = \frac{\kappa}{(N + \kappa)}$$

The suggested points by the authors of UT paper are

$$x_0 = \mu \qquad w_0 = \frac{\kappa}{(N + \kappa)}$$

$$x_i = \mu + (sqrt(N + \kappa)P)_i \qquad w_i = \frac{1}{2(N + \kappa)}$$

The suggested points by the authors of UT paper are

$$x_0 = \mu \qquad w_0 = \frac{\kappa}{(N+\kappa)}$$

$$x_i = \mu + (sqrt(N+\kappa)P)_i \qquad w_i = \frac{1}{2(N+\kappa)}$$

$$x_{i+N} = \mu - (sqrt(N+\kappa)P)_i \qquad w_{i+N} = \frac{1}{2(N+\kappa)}$$

The suggested points by the authors of UT paper are

$$x_0 = \mu \qquad w_0 = \frac{\kappa}{(N+\kappa)}$$

$$x_i = \mu + (sqrt(N+\kappa)P)_i \qquad w_i = \frac{1}{2(N+\kappa)}$$

$$x_{i+N} = \mu - (sqrt(N+\kappa)P)_i \qquad w_{i+N} = \frac{1}{2(N+\kappa)}$$

An **elegant selection for** r_1 is chosen and then rest of the variables are solved in terms of this selction

The suggested points by the authors of UT paper are

$$x_0 = \mu \qquad w_0 = \frac{\kappa}{(N+\kappa)}$$

$$x_i = \mu + (sqrt(N+\kappa)P)_i \qquad w_i = \frac{1}{2(N+\kappa)}$$

$$x_{i+N} = \mu - (sqrt(N+\kappa)P)_i \qquad w_{i+N} = \frac{1}{2(N+\kappa)}$$

An **elegant selection for** r_1 is chosen and then rest of the variables are solved in terms of this selction

$$r_1 = \sqrt{N+\kappa}$$

The suggested points by the authors of UT paper are

$$x_0 = \mu \qquad w_0 = \frac{\kappa}{(N+\kappa)}$$

$$x_i = \mu + (sqrt(N+\kappa)P)_i \qquad w_i = \frac{1}{2(N+\kappa)}$$

$$x_{i+N} = \mu - (sqrt(N+\kappa)P)_i \qquad w_{i+N} = \frac{1}{2(N+\kappa)}$$

An **elegant selection for** r_1 is chosen and then rest of the variables are solved in terms of this selction

$$r_1 = \sqrt{N+\kappa}$$

$$w_1 = \frac{1}{2(N+\kappa)}$$

The suggested points by the authors of UT paper are

$$x_0 = \mu \qquad w_0 = \frac{\kappa}{(N+\kappa)}$$

$$x_i = \mu + (sqrt(N+\kappa)P)_i \qquad w_i = \frac{1}{2(N+\kappa)}$$

$$x_{i+N} = \mu - (sqrt(N+\kappa)P)_i \qquad w_{i+N} = \frac{1}{2(N+\kappa)}$$

An **elegant selection for** r_1 is chosen and then rest of the variables are solved in terms of this selction

$$r_1 = \sqrt{N+\kappa}$$

$$w_1 = \frac{1}{2(N+\kappa)}$$

$$w_0 = 1 - 2Nw_1 = \frac{\kappa}{(N+\kappa)}$$

The suggested points by the authors of UT paper are

$$x_0 = \mu \qquad w_0 = \frac{\kappa}{(N+\kappa)}$$

$$x_i = \mu + (sqrt(N+\kappa)P)_i \qquad w_i = \frac{1}{2(N+\kappa)}$$

$$x_{i+N} = \mu - (sqrt(N+\kappa)P)_i \qquad w_{i+N} = \frac{1}{2(N+\kappa)}$$

An **elegant selection for** r_1 is chosen and then rest of the variables are solved in terms of this selction

$$r_1 = \sqrt{N+\kappa}$$

$$w_1 = \frac{1}{2(N+\kappa)}$$

$$w_0 = 1 - 2Nw_1 = \frac{\kappa}{(N+\kappa)}$$

$$2r_1^4 w_1 \equiv N + \kappa = 3$$

The $2N$ Cubature Kalman Filter

- The authors of CKF provide a very mathematically rigourous and elegant way of of finding the second moment equivalent $2N$ cubature points that can again accurately integrate polynomials of degree 3 or less.

The $2N$ Cubature Kalman Filter

- The authors of CKF provide a very mathematically rigourous and elegant way of of finding the second moment equivalent $2N$ cubature points that can again accurately integrate polynomials of degree 3 or less.
- Spherical-radial coordinates are employed rather than cartesian coordinates to find these $2N$ symmetric cubature points on the principle axis.

The $2N$ Cubature Kalman Filter

- The authors of CKF provide a very mathematically rigourous and elegant way of of finding the second moment equivalent $2N$ cubature points that can again accurately integrate polynomials of degree 3 or less.
- Spherical-radial coordinates are employed rather than cartesian coordinates to find these $2N$ symmetric cubature points on the principle axis.
- Contrary to the $2N + 1$ Unscented transform sigma points, they only have $2N$ cubature points on the principle axis and no point on the mean.

- This is equivalent to **setting** $\kappa = 0$ in the Unscented transform sigma points thus making the central weight w_0 0.

- This is equivalent to **setting** $\kappa = 0$ in the Unscented transform sigma points thus making the central weight w_0 0.
- An alternate derivation to the one given in their paper is provided which is less rigourous but **more intuitive** using the present frame work.

- This is equivalent to **setting** $\kappa = 0$ in the Unscented transform sigma points thus making the central weight w_0 0.
- An alternate derivation to the one given in their paper is provided which is less rigourous but **more intuitive** using the present frame work.

- This is equivalent to **setting** $\kappa = 0$ in the Unscented transform sigma points thus making the central weight w_0 0.
- An alternate derivation to the one given in their paper is provided which is less rigourous but **more intuitive** using the present frame work.

As there is no point at the mean, the central weight can be taken 0.

- This is equivalent to **setting** $\kappa = 0$ in the Unscented transform sigma points thus making the central weight w_0 0.
- An alternate derivation to the one given in their paper is provided which is less rigourous but **more intuitive** using the present frame work.

As there is no point at the mean, the central weight can be taken 0.

$$0 = 1 - 2Nw_1$$

- This is equivalent to **setting** $\kappa = 0$ in the Unscented transform sigma points thus making the central weight w_0 0.
- An alternate derivation to the one given in their paper is provided which is less rigourous but **more intuitive** using the present frame work.

As there is no point at the mean, the central weight can be taken 0.

$$0 = 1 - 2Nw_1$$
$$w_1 = \frac{1}{2N}$$

- This is equivalent to **setting** $\kappa = 0$ in the Unscented transform sigma points thus making the central weight w_0 0.
- An alternate derivation to the one given in their paper is provided which is less rigourous but **more intuitive** using the present frame work.

As there is no point at the mean, the central weight can be taken 0.

$$
\begin{aligned}
0 &= 1 - 2Nw_1 \\
w_1 &= \frac{1}{2N} \\
r_1 &= \sqrt{N}
\end{aligned}
$$

Comparison of UT and CKF cubature points using the error in the 4th moment

- The authors of CKF claim that their method is superior to UKF. They outrightly criticize that UT has no mathematical basis.

$$ \tag{3} $$

$$ \tag{4} $$

Comparison of UT and CKF cubature points using the error in the 4th moment

- The authors of CKF claim that their method is superior to UKF. They outrightly criticize that UT has no mathematical basis.
- Infact the UT can **capture one of the 4th order moment exactly** but the CKF has an error in all the fourth order moments.

$$(3)$$

$$(4)$$

Comparison of UT and CKF cubature points using the error in the 4th moment

- The authors of CKF claim that their method is superior to UKF. They outrightly criticize that UT has no mathematical basis.
- Infact the UT can **capture one of the 4th order moment exactly** but the CKF has an error in all the fourth order moments.
- This is also illustrated by examples later on in the results section.

$$ \tag{3} $$

$$ \tag{4} $$

Comparison of UT and CKF cubature points using the error in the 4th moment

- The authors of CKF claim that their method is superior to UKF. They outrightly criticize that UT has no mathematical basis.
- Infact the UT can **capture one of the 4th order moment exactly** but the CKF has an error in all the fourth order moments.
- This is also illustrated by examples later on in the results section.

$$ \tag{3} $$

$$ \tag{4} $$

Comparison of UT and CKF cubature points using the error in the 4th moment

- The authors of CKF claim that their method is superior to UKF. They outrightly criticize that UT has no mathematical basis.
- Infact the UT can **capture one of the 4th order moment exactly** but the CKF has an error in all the fourth order moments.
- This is also illustrated by examples later on in the results section.

Absolute error in fourth moment for UT

(3)

(4)

Comparison of UT and CKF cubature points using the error in the 4th moment

- The authors of CKF claim that their method is superior to UKF. They outrightly criticize that UT has no mathematical basis.
- Infact the UT can **capture one of the 4th order moment exactly** but the CKF has an error in all the fourth order moments.
- This is also illustrated by examples later on in the results section.

Absolute error in fourth moment for UT

$$|3 - (N + \kappa)| \tag{3}$$

$$\tag{4}$$

Comparison of UT and CKF cubature points using the error in the 4th moment

- The authors of CKF claim that their method is superior to UKF. They outrightly criticize that UT has no mathematical basis.
- Infact the UT can **capture one of the 4th order moment exactly** but the CKF has an error in all the fourth order moments.
- This is also illustrated by examples later on in the results section.

Absolute error in fourth moment for UT

$$|3 - (N + \kappa)| \tag{3}$$

Absolute error in fourth moment for CKF

$$\tag{4}$$

Comparison of UT and CKF cubature points using the error in the 4th moment

- The authors of CKF claim that their method is superior to UKF. They outrightly criticize that UT has no mathematical basis.
- Infact the UT can **capture one of the 4th order moment exactly** but the CKF has an error in all the fourth order moments.
- This is also illustrated by examples later on in the results section.

Absolute error in fourth moment for UT

$$|3-(N+\kappa)| \tag{3}$$

Absolute error in fourth moment for CKF

$$|2N^2\frac{1}{2N}-3| \equiv |N-3| \tag{4}$$

Cubature points that are 4th order equivalent

- The authors of UT suggest a method to capture the fourth order moments

Cubature points that are 4th order equivalent

- The authors of UT suggest a method to capture the fourth order moments
- But this method suffers the same problem of having a **negative weight above dimension 4.**

Cubature points that are 4th order equivalent

- The authors of UT suggest a method to capture the fourth order moments
- But this method suffers the same problem of having a **negative weight above dimension 4.**
- The authors suggest to constrain 1 set of points on the principle axis and another set on the 2^{nd}-conjugate axis

Cubature points that are 4th order equivalent

- The authors of UT suggest a method to capture the fourth order moments
- But this method suffers the same problem of having a **negative weight above dimension 4.**
- The authors suggest to constrain 1 set of points on the principle axis and another set on the 2^{nd}-conjugate axis
- A formal derivation in terms of the present framework is shown

Cubature points that are 4th order equivalent

- The authors of UT suggest a method to capture the fourth order moments
- But this method suffers the same problem of having a **negative weight above dimension 4.**
- The authors suggest to constrain 1 set of points on the principle axis and another set on the 2^{nd}-conjugate axis
- A formal derivation in terms of the present framework is shown

Cubature points that are 4th order equivalent

- The authors of UT suggest a method to capture the fourth order moments
- But this method suffers the same problem of having a **negative weight above dimension 4.**
- The authors suggest to constrain 1 set of points on the principle axis and another set on the 2^{nd}-conjugate axis
- A formal derivation in terms of the present framework is shown

$$2r_1^2 w_1 + 4(N-1) r_2^2 w_2 = 1$$

Cubature points that are 4th order equivalent

- The authors of UT suggest a method to capture the fourth order moments
- But this method suffers the same problem of having a **negative weight above dimension 4.**
- The authors suggest to constrain 1 set of points on the principle axis and another set on the 2^{nd}-conjugate axis
- A formal derivation in terms of the present framework is shown

$$2r_1^2 w_1 + 4(N-1) r_2^2 w_2 = 1$$
$$2r_1^4 w_1 + 4(N-1) r_2^4 w_2 = 3$$

Cubature points that are 4th order equivalent

- The authors of UT suggest a method to capture the fourth order moments
- But this method suffers the same problem of having a **negative weight above dimension 4.**
- The authors suggest to constrain 1 set of points on the principle axis and another set on the 2^{nd}-conjugate axis
- A formal derivation in terms of the present framework is shown

$$
\begin{aligned}
2r_1^2 w_1 + 4(N-1)r_2^2 w_2 &= 1 \\
2r_1^4 w_1 + 4(N-1)r_2^4 w_2 &= 3 \\
4r_2^4 w_1 &= 1
\end{aligned}
$$

Cubature points that are 4th order equivalent

- The authors of UT suggest a method to capture the fourth order moments
- But this method suffers the same problem of having a **negative weight above dimension 4.**
- The authors suggest to constrain 1 set of points on the principle axis and another set on the 2^{nd}-conjugate axis
- A formal derivation in terms of the present framework is shown

$$
\begin{aligned}
2r_1^2 w_1 + 4(N-1)r_2^2 w_2 &= 1 \\
2r_1^4 w_1 + 4(N-1)r_2^4 w_2 &= 3 \\
4r_2^4 w_1 &= 1 \\
1 - 2Nw_1 - 2N(N-1)w_2 &= w_0
\end{aligned}
$$

The simplified set of equations are got by first solving for w_1 and w_2 and then substituting them in the first equation

The simplified set of equations are got by first solving for w_1 and w_2 and then substituting them in the first equation

$$w_2 = \frac{1}{4r_2^4}$$

The simplified set of equations are got by first solving for w_1 and w_2 and then substituting them in the first equation

$$w_2 = \frac{1}{4r_2^4}$$

$$w_1 = \frac{4 - N}{2r_1^4}$$

The simplified set of equations are got by first solving for w_1 and w_2 and then substituting them in the first equation

$$w_2 = \frac{1}{4r_2^4}$$
$$w_1 = \frac{4-N}{2r_1^4}$$

Substituting these weights into the first constraint equation we have only one cosntraint equation

The simplified set of equations are got by first solving for w_1 and w_2 and then substituting them in the first equation

$$w_2 = \frac{1}{4r_2^4}$$
$$w_1 = \frac{4 - N}{2r_1^4}$$

Substituting these weights into the first constraint equation we have only one cosntraint equation

$$r_1^2 r_2^2 = r_1^2(N - 1) + r_2^2(4 - N)$$

The simplified set of equations are got by first solving for w_1 and w_2 and then substituting them in the first equation

$$w_2 = \frac{1}{4r_2^4}$$
$$w_1 = \frac{4-N}{2r_1^4}$$

Substituting these weights into the first constraint equation we have only one cosntraint equation

$$r_1^2 r_2^2 = r_1^2(N-1) + r_2^2(4-N)$$

Even though we solve for r_1 and r_2 we always have a negative/zero weight for $N > 3$

The Conjugate Unscented Transform Method to capture all the 4th moment

- The first set of points are chosen on the principle axis at a distance of r_1 and weight w_1 each.

The Conjugate Unscented Transform Method to capture all the 4th moment

- The first set of points are chosen on the principle axis at a distance of r_1 and weight w_1 each.
- The second set of points are chosen on the Nth-Conjugate axis at a distance of r_2 and each have a weight of w_2.

The Conjugate Unscented Transform Method to capture all the 4th moment

- The first set of points are chosen on the principle axis at a distance of r_1 and weight w_1 each.
- The second set of points are chosen on the Nth-Conjugate axis at a distance of r_2 and each have a weight of w_2.

The Conjugate Unscented Transform Method to capture all the 4th moment

- The first set of points are chosen on the principle axis at a distance of r_1 and weight w_1 each.
- The second set of points are chosen on the Nth-Conjugate axis at a distance of r_2 and each have a weight of w_2.

Forming the moment constraint equations along with the central weight

The Conjugate Unscented Transform Method to capture all the 4th moment

- The first set of points are chosen on the principle axis at a distance of r_1 and weight w_1 each.
- The second set of points are chosen on the Nth-Conjugate axis at a distance of r_2 and each have a weight of w_2.

Forming the moment constraint equations along with the central weight

$$2r_1^2 w_1 + 2^N r_2^2 w_2 = 1$$

The Conjugate Unscented Transform Method to capture all the 4th moment

- The first set of points are chosen on the principle axis at a distance of r_1 and weight w_1 each.
- The second set of points are chosen on the Nth-Conjugate axis at a distance of r_2 and each have a weight of w_2.

Forming the moment constraint equations along with the central weight

$$2r_1^2 w_1 + 2^N r_2^2 w_2 = 1$$
$$2r_1^4 w_1 + 2^N r_2^4 w_2 = 3$$

The Conjugate Unscented Transform Method to capture all the 4th moment

- The first set of points are chosen on the principle axis at a distance of r_1 and weight w_1 each.
- The second set of points are chosen on the Nth-Conjugate axis at a distance of r_2 and each have a weight of w_2.

Forming the moment constraint equations along with the central weight

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 &= 3 \\
2^N r_2^4 w_2 &= 1
\end{aligned}
$$

The Conjugate Unscented Transform Method to capture all the 4th moment

- The first set of points are chosen on the principle axis at a distance of r_1 and weight w_1 each.
- The second set of points are chosen on the Nth-Conjugate axis at a distance of r_2 and each have a weight of w_2.

Forming the moment constraint equations along with the central weight

$$\begin{aligned} 2r_1^2 w_1 + 2^N r_2^2 w_2 &= 1 \\ 2r_1^4 w_1 + 2^N r_2^4 w_2 &= 3 \\ 2^N r_2^4 w_2 &= 1 \\ 1 - 2Nw_1 - 2^N w_2 &= w_0 \end{aligned}$$

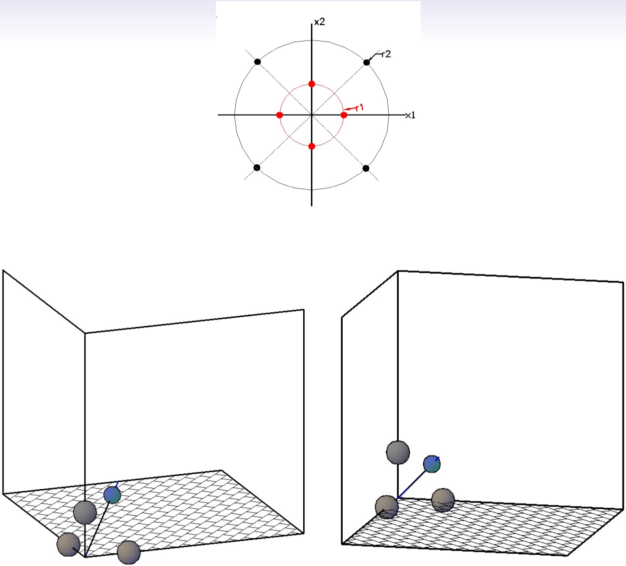

Figure: distances and weights

These set of equations can be **solved analytically** in the following 2 schemes because there is no constraint on w_0 except that it should be positive

These set of equations can be **solved analytically** in the following 2 schemes because there is no constraint on w_0 except that it should be positive

Solution 1

These set of equations can be **solved analytically** in the following 2 schemes because there is no constraint on w_0 except that it should be positive

Solution 1

The w_0 is solved for by minimizing the square of error in sixth moment

These set of equations can be **solved analytically** in the following 2 schemes because there is no constraint on w_0 except that it should be positive

Solution 1

The w_0 is solved for by minimizing the square of error in sixth moment

$$Min \quad (2r_1^6 w_1 + 2^N r_2^6 w_2 - 15)^2$$

These set of equations can be **solved analytically** in the following 2 schemes because there is no constraint on w_0 except that it should be positive

Solution 1

The w_0 is solved for by minimizing the square of error in sixth moment

$$Min \quad (2r_1^6 w_1 + 2^N r_2^6 w_2 - 15)^2$$

This is applied for dimensions 1 and 2

These set of equations can be **solved analytically** in the following 2 schemes because there is no constraint on w_0 except that it should be positive

Solution 1

The w_0 is solved for by minimizing the square of error in sixth moment

$$Min \quad (2r_1^6 w_1 + 2^N r_2^6 w_2 - 15)^2$$

This is applied for dimensions 1 and 2

Table: Optimization Solution for $N = 1$ and $N = 2$

Variable	$N = 1$	$N = 2$
r_1	1.4861736616297834	2.6060099476935847
r_2	3.2530871022700643	1.190556300661233
w_0	0.5811010092660772	0.41553535186548973
w_1	0.20498484723245053	0.021681819434216532
w_2	0.0044646481345109 3	0.12443434259941118

Solution 2

Solution 2

For dimension greater than 2, the central weight is eliminated thus **reducing 1 point**.

Solution 2

For dimension greater than 2, the central weight is eliminated thus **reducing 1 point**. For $N > 2$

Solution 2

For dimension greater than 2, the central weight is eliminated thus **reducing 1 point**. For $N > 2$

$$w_0 = 0$$

Solution 2

For dimension greater than 2, the central weight is eliminated thus **reducing 1 point**. For $N > 2$

$$w_0 = 0$$

$$r1 = \sqrt{\frac{N+2}{2}}$$

Solution 2

For dimension greater than 2, the central weight is eliminated thus **reducing 1 point.** For $N > 2$

$$
\begin{aligned}
w_0 &= 0 \\
r1 &= \sqrt{\frac{N+2}{2}} \\
r2 &= \sqrt{\frac{N+2}{N-2}}
\end{aligned}
$$

Solution 2

For dimension greater than 2, the central weight is eliminated thus **reducing 1 point.** For $N > 2$

$$
\begin{aligned}
w_0 &= 0 \\
r1 &= \sqrt{\frac{N+2}{2}} \\
r2 &= \sqrt{\frac{N+2}{N-2}} \\
w_1 = \frac{1}{r_1^4} &= \frac{4}{(N+2)^2}
\end{aligned}
$$

Solution 2

For dimension greater than 2, the central weight is eliminated thus **reducing 1 point.** For $N > 2$

$$
\begin{aligned}
w_0 &= 0 \\
r1 &= \sqrt{\frac{N+2}{2}} \\
r2 &= \sqrt{\frac{N+2}{N-2}} \\
w_1 = \frac{1}{r_1^4} &= \frac{4}{(N+2)^2} \\
w_2 = \frac{1}{2^N r_2^4} &= \frac{(N-2)^2}{2^N (N+2)^2}
\end{aligned}
$$

Results of integration compared to Gauss Hermite integration for 3D system

The total number of cubature points involved in this method to capture all the moments till 4th order is$2N + 2^N$

$$P = \begin{bmatrix} 4 & 2 & 1 \\ 2 & 9 & 1 \\ 1 & 1 & 16 \end{bmatrix}$$

$$\begin{aligned} F = {} & x_1^4 + x_2^4 + x_3^4 + x_1^3 x_2 + x_1^2 x_2^2 + \\ & + x_3^2 x_2^2 + x_1^2 x_3^2 + x_1^3 x_3 + x_2^3 x_3 + x_3^3 x_2 \end{aligned}$$

No. of pts	$2^3 = 8$	$3^3 = 27$	$4^3 = 64$	Analytical
GH	1056.95571	1797.99999	1798.00	1798.00
% error wrt Truth	41.2149211	4.299e-013	3.0350e-013	0

No. of pts	$2^3 = 8$	$3^3 = 27$	$4^3 = 64$	Analytical
GH	1056.95571	1797.99999	1798.00	1798.00
% error wrt Truth	41.2149211	4.299e-013	3.0350e-013	0

	No. of pts	Integration result	% error
CUT4	15	1797.999999	6.3552080e-010

Results of integration compared to Gauss Hermite integration for 8D system

The total number of cubature points involved in this method to capture all the moments till 4th order is $2N + 2^N$

$$P = 10 I_{8x8} \tag{5}$$

$$F = x_1^4 + x_8^2 * x_2^2 + x_3^2 + x_4^2 * x_5^2$$

No. of pts	$2^8 = 8$	$3^8 = 6561$	$4^8 = 65536$	Analytical
GH	256	614	614	614
% error	32.573	5.18441605e-013	6.591614e-012	0

No. of pts	$2^8 = 8$	$3^8 = 6561$	$4^8 = 65536$	Analytical
GH	256	614	614	614
% error	32.573	5.18441605e-013	6.591614e-012	0

	No. of pts	Integration result	% error
CUT4	273	614	1.0368832104e-012

Cubature points that are 6^{th} order equivalent

- We **attempt** to capture all the 6^{th} order moments till 6D

Cubature points that are 6^{th} order equivalent

- We **attempt** to capture all the 6^{th} order moments till 6D
- In addition to the axis chosen for the fourth moment one additional set of axis is considered

Cubature points that are 6^{th} order equivalent

- We **attempt** to capture all the 6^{th} order moments till 6D
- In addition to the axis chosen for the fourth moment one additional set of axis is considered
- 1 point on the mean, first set of points on the principle axis at distance r_1 and each weight w_1.

Cubature points that are 6^{th} order equivalent

- We **attempt** to capture all the 6^{th} order moments till 6D
- In addition to the axis chosen for the fourth moment one additional set of axis is considered
- 1 point on the mean, first set of points on the principle axis at distance r_1 and each weight w_1.
- Second set on the Nth-Conjugate axis with distance r_2 and weight w_2.

Cubature points that are 6^{th} order equivalent

- We **attempt** to capture all the 6^{th} order moments till 6D
- In addition to the axis chosen for the fourth moment one additional set of axis is considered
- 1 point on the mean, first set of points on the principle axis at distance r_1 and each weight w_1.
- Second set on the Nth-Conjugate axis with distance r_2 and weight w_2.
- Third set of axis are chosen on the 2nd conjugate axis with points at distance r_3 and weight w_3

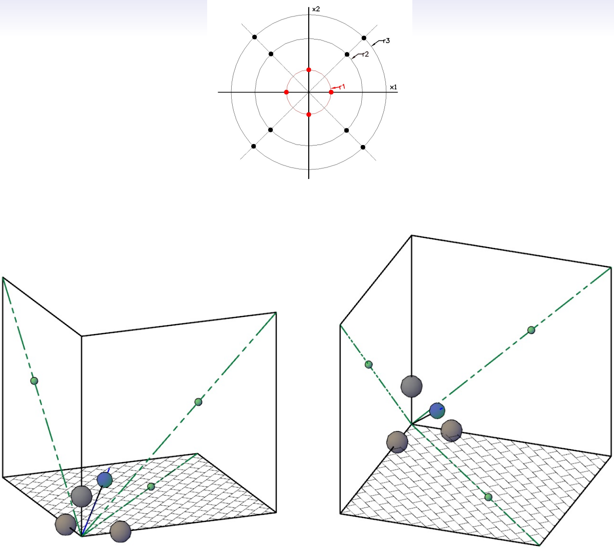

Figure: distances and weights

Moments till 6th order

Moments till 6th order

$$E[X_i^2] = 1 \qquad\qquad E[X_i X_j] = 0$$

Moments till 6th order

$$E[X_i^2] = 1 \qquad E[X_i X_j] = 0$$
$$E[X_i^4] = 3 \qquad E[X_i^3 X_j] = 0$$

Moments till 6th order

$$E[X_i^2] = 1 \qquad E[X_i X_j] = 0$$
$$E[X_i^4] = 3 \qquad E[X_i^3 X_j] = 0$$
$$E[X_i^2 X_j^2] = 1 \qquad E[X_i^2 X_j X_k] = 0$$

Moments till 6th order

$$E[X_i^2] = 1 \qquad E[X_i X_j] = 0$$
$$E[X_i^4] = 3 \qquad E[X_i^3 X_j] = 0$$
$$E[X_i^2 X_j^2] = 1 \qquad E[X_i^2 X_j X_k] = 0$$
$$E[X_i X_j X_k X_l] = 0 \qquad E[X_i^6] = 15$$

Moments till 6th order

$$E[X_i^2] = 1 \qquad E[X_iX_j] = 0$$

$$E[X_i^4] = 3 \qquad E[X_i^3X_j] = 0$$

$$E[X_i^2X_j^2] = 1 \qquad E[X_i^2X_jX_k] = 0$$

$$E[X_iX_jX_kX_l] = 0 \qquad E[X_i^6] = 15$$

$$E[X_i^5X_j] = 0 \qquad E[X_i^4X_j^2] = 3$$

Moments till 6th order

$$E[X_i^2] = 1$$
$$E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1$$
$$E[X_i X_j X_k X_l] = 0$$
$$E[X_i^5 X_j] = 0$$
$$E[X_i^4 X_j X_k] = 0$$

$$E[X_i X_j] = 0$$
$$E[X_i^3 X_j] = 0$$
$$E[X_i^2 X_j X_k] = 0$$
$$E[X_i^6] = 15$$
$$E[X_i^4 X_j^2] = 3$$
$$E[X_i^3 X_j^3] = 0$$

Moments till 6th order

$$
\begin{aligned}
E[X_i^2] &= 1 \\
E[X_i^4] &= 3 \\
E[X_i^2 X_j^2] &= 1 \\
E[X_i X_j X_k X_l] &= 0 \\
E[X_i^5 X_j] &= 0 \\
E[X_i^4 X_j X_k] &= 0 \\
E[X_i^3 X_j^2 X_k] &= 0
\end{aligned}
\qquad
\begin{aligned}
E[X_i X_j] &= 0 \\
E[X_i^3 X_j] &= 0 \\
E[X_i^2 X_j X_k] &= 0 \\
E[X_i^6] &= 15 \\
E[X_i^4 X_j^2] &= 3 \\
E[X_i^3 X_j^3] &= 0 \\
E[X_i^3 X_j^2] &= 0
\end{aligned}
$$

Moments till 6th order

$$E[X_i^2] = 1 \qquad E[X_i X_j] = 0$$
$$E[X_i^4] = 3 \qquad E[X_i^3 X_j] = 0$$
$$E[X_i^2 X_j^2] = 1 \qquad E[X_i^2 X_j X_k] = 0$$
$$E[X_i X_j X_k X_l] = 0 \qquad E[X_i^6] = 15$$
$$E[X_i^5 X_j] = 0 \qquad E[X_i^4 X_j^2] = 3$$
$$E[X_i^4 X_j X_k] = 0 \qquad E[X_i^3 X_j^3] = 0$$
$$E[X_i^3 X_j^2 X_k] = 0 \qquad E[X_i^3 X_j^2] = 0$$
$$E[X_i^2 X_j^2 X_k^2] = 1 \qquad E[X_i X_j X_k X_l X_m X_n] = 0$$

Moment constraint equations till order 6

Moment constraint equations till order 6

$$2r_1^2 w_1 + 2^N r_2^2 w_2 + 4(N-1) r_3^2 w_3 = 1$$

Moment constraint equations till order 6

$$2r_1^2 w_1 + 2^N r_2^2 w_2 + 4(N-1) r_3^2 w_3 = 1$$
$$2r_1^4 w_1 + 2^N r_2^4 w_2 + 4(N-1) r_3^4 w_3 = 3$$

Moment constraint equations till order 6

$$\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 + 4(N-1) r_3^2 w_3 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 + 4(N-1) r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 4 r_3^4 &= 1
\end{aligned}$$

Moment constraint equations till order 6

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 + 4(N-1) r_3^2 w_3 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 + 4(N-1) r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 4 r_3^4 &= 1 \\
2r_1^6 w_1 + 2^N r_2^6 w_2 + 4(N-1) r_3^6 w_3 &= 15
\end{aligned}
$$

Moment constraint equations till order 6

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 + 4(N-1) r_3^2 w_3 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 + 4(N-1) r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 4 r_3^4 &= 1 \\
2r_1^6 w_1 + 2^N r_2^6 w_2 + 4(N-1) r_3^6 w_3 &= 15 \\
2^N r_2^6 w_2 + 4 r_3^6 &= 3
\end{aligned}
$$

Moment constraint equations till order 6

$$
\begin{aligned}
2r_1^2 w_1 + 2^N r_2^2 w_2 + 4(N-1) r_3^2 w_3 &= 1 \\
2r_1^4 w_1 + 2^N r_2^4 w_2 + 4(N-1) r_3^4 w_3 &= 3 \\
2^N r_2^4 w_2 + 4 r_3^4 &= 1 \\
2r_1^6 w_1 + 2^N r_2^6 w_2 + 4(N-1) r_3^6 w_3 &= 15 \\
2^N r_2^6 w_2 + 4 r_3^6 &= 3 \\
2^N r_2^6 w_2 &= 1
\end{aligned}
$$

Solution to the system

- These nonlinear equations are in a way linear with respect to the weights

Solution to the system

Procedure followed

- These nonlinear equations are in a way linear with respect to the weights
- The last 3 equations can be solved for w_1, w_2 and w_3 symbolically in terms of the other variables r_1, r_2 and r_3

Solution to the system

Procedure followed

- These nonlinear equations are in a way linear with respect to the weights
- The last 3 equations can be solved for w_1, w_2 and w_3 symbolically in terms of the other variables r_1, r_2 and r_3
- Substituting these into the first 3 equations the overall order of the system is reduced.

Solution to the system

Procedure followed

- These nonlinear equations are in a way linear with respect to the weights
- The last 3 equations can be solved for w_1, w_2 and w_3 symbolically in terms of the other variables r_1, r_2 and r_3
- Substituting these into the first 3 equations the overall order of the system is reduced.

Solution to the system

Procedure followed

- These nonlinear equations are in a way linear with respect to the weights
- The last 3 equations can be solved for w_1, w_2 and w_3 symbolically in terms of the other variables r_1, r_2 and r_3
- Substituting these into the first 3 equations the overall order of the system is reduced.

Solution to the system

Procedure followed

- These nonlinear equations are in a way linear with respect to the weights
- The last 3 equations can be solved for w_1, w_2 and w_3 symbolically in terms of the other variables r_1, r_2 and r_3
- Substituting these into the first 3 equations the overall order of the system is reduced.

$$w_1 = \frac{8 - N}{r_1^6}$$

Solution to the system

Procedure followed

- These nonlinear equations are in a way linear with respect to the weights
- The last 3 equations can be solved for w_1, w_2 and w_3 symbolically in terms of the other variables r_1, r_2 and r_3
- Substituting these into the first 3 equations the overall order of the system is reduced.

$$w_1 = \frac{8 - N}{r_1^6}$$

$$w_2 = \frac{1}{2^N r_2^6}$$

Solution to the system

Procedure followed

- These nonlinear equations are in a way linear with respect to the weights
- The last 3 equations can be solved for w_1, w_2 and w_3 symbolically in terms of the other variables r_1, r_2 and r_3
- Substituting these into the first 3 equations the overall order of the system is reduced.

$$w_1 = \frac{8 - N}{r_1^6}$$

$$w_2 = \frac{1}{2^N r_2^6}$$

$$w_3 = \frac{1}{2 r_3^6}$$

Results of integration compared to Gauss Hermite integration for 4D system

The total number of cubature points involved in this method to capture all the moments till 6th order is $2N^2 + 2^N + 1$

$$P = \begin{bmatrix} 4 & 1 & 2 & 1 \\ 1 & 9 & 2 & 3 \\ 2 & 2 & 16 & 4 \\ 1 & 3 & 4 & 25 \end{bmatrix}$$

$$F = x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_1^4 + x_2^4 + x_3^4 + x_4^4 + x_1^2 x_2^2 x_3^2 + \\ + x_1^3 x_3 + x_1^2 x_2^4 + x_3^4 x_2^2 + x_1^2 x_3^4 + x_1^3 x_3 + x_2^3 x_3 + x_3^3 x_2$$

No. of pts	$2^4 = 16$	$3^4 = 81$	$4^4 = 256$	$5^4 = 64$	Analytical
GH	84462.6354	293311.8446	375417.9999	375417.9999	375417.9999
% error	77.5017	21.8705	9.0702e-012	8.9152e-012	0

No. of pts	$2^4 = 16$	$3^4 = 81$	$4^4 = 256$	$5^4 = 64$	Analytical
GH	84462.6354	293311.8446	375417.9999	375417.9999	375417.9999
% error	77.5017	21.8705	9.0702e-012	8.9152e-012	0

	No. of pts	Integration result	% error
CUT6	49	3.7541800e+005	8.062475e-013

Results of integration compared to Gauss Hermite integration for 6D system

The total number of cubature points involved in this method to capture all the moments till 6th order is $2N^2 + 2^N + 1$

$$P = 10I_{6x6} \tag{6}$$

$$F = x_1^6 + x_1^2 x_2^2 x_3^2 + x_1^4 x_6^2 + x_5^4 + x_3^2 x_4^2$$

No. of pts	$2^6 = 64$	$3^6 = 729$	$4^6 = 4096$	$5^6 = 15625$	Analytical
GH	6.035000e+003	1.943500e+004	2.543500e+004	2.543500e+004	2.5434999e+004
% error	76.2728	23.589	1.5633e-011	1.58906e-011	0

No. of pts	$2^6 = 64$	$3^6 = 729$	$4^6 = 4096$	$5^6 = 15625$	Analytical
GH	6.035000e+003	1.943500e+004	2.543500e+004	2.543500e+004	2.5434999e+004
% error	76.2728	23.589	1.5633e-011	1.58906e-011	0

	No. of pts	Integration result	% error
CUT6	137	2.543500e+004	3.4583111e-008

Cubature points to capture ALL the 8th order moments

- All the axis used before were not able to capture all the 8th order moments

Cubature points to capture ALL the 8th order moments

- All the axis used before were not able to capture all the 8th order moments
- Only points that lie in the full N-D space enter the 8th order moments.

Cubature points to capture ALL the 8th order moments

- All the axis used before were not able to capture all the 8th order moments
- Only points that lie in the full N-D space enter the 8th order moments.
- I was only able to capture the moments till 6th dimension.

Cubature points to capture ALL the 8th order moments

- All the axis used before were not able to capture all the 8th order moments
- Only points that lie in the full N-D space enter the 8th order moments.
- I was only able to capture the moments till 6th dimension.
- Due to the complex nature of the equation I was not able to find a trend in the set of equations to generalize it.

Cubature points to capture ALL the 8th order moments

- All the axis used before were not able to capture all the 8th order moments
- Only points that lie in the full N-D space enter the 8th order moments.
- I was only able to capture the moments till 6th dimension.
- Due to the complex nature of the equation I was not able to find a trend in the set of equations to generalize it.
- I have only done this case by case till 6th-DImension.

Cubature points to capture ALL the 8th order moments

- All the axis used before were not able to capture all the 8th order moments
- Only points that lie in the full N-D space enter the 8th order moments.
- I was only able to capture the moments till 6th dimension.
- Due to the complex nature of the equation I was not able to find a trend in the set of equations to generalize it.
- I have only done this case by case till 6th-DImension.
- But in all cases I have selected the **SAME** set of axis

- 1 point on the mean

- 1 point on the mean
- A set of points on the principle axis

- 1 point on the mean
- A set of points on the principle axis
- A set of points on the Nth-conjugate axis

- 1 point on the mean
- A set of points on the principle axis
- A set of points on the Nth-conjugate axis
- A set of points on the 2nd-conjugate axis

- 1 point on the mean
- A set of points on the principle axis
- A set of points on the Nth-conjugate axis
- A set of points on the 2nd-conjugate axis
- A set of points on the Nth- Scaled conjugate axis

- 1 point on the mean
- A set of points on the principle axis
- A set of points on the Nth-conjugate axis
- A set of points on the 2nd-conjugate axis
- A set of points on the Nth- Scaled conjugate axis
- The scaling parameter h has to appropriately chosen

- 1 point on the mean
- A set of points on the principle axis
- A set of points on the Nth-conjugate axis
- A set of points on the 2nd-conjugate axis
- A set of points on the Nth- Scaled conjugate axis
- The scaling parameter h has to appropriately chosen

- 1 point on the mean
- A set of points on the principle axis
- A set of points on the Nth-conjugate axis
- A set of points on the 2nd-conjugate axis
- A set of points on the Nth- Scaled conjugate axis
- The scaling parameter h has to appropriately chosen

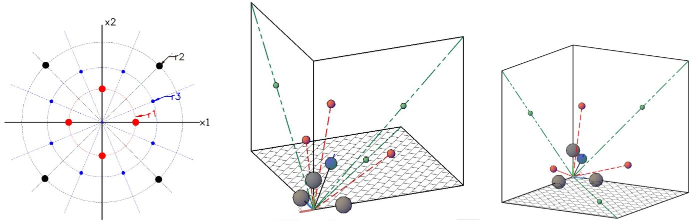

Figure: distances and weights

Moment constraint equations till 8th moment

Only the even moments are shown that are to be satisfied.
There are **11 non-zero moments for any dimension** in all

Moment constraint equations till 8th moment

Only the even moments are shown that are to be satisfied.
There are **11 non-zero moments for any dimension** in all

$$E[X_i^2] = 1 \qquad E[X_i^4] = 3$$

Moment constraint equations till 8th moment

Only the even moments are shown that are to be satisfied.
There are **11 non-zero moments for any dimension** in all

$$E[X_i^2] = 1 \qquad E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1 \qquad E[X_i^6] = 15$$

Moment constraint equations till 8th moment

Only the even moments are shown that are to be satisfied.
There are **11 non-zero moments for any dimension** in all

$$E[X_i^2] = 1 \qquad E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1 \qquad E[X_i^6] = 15$$
$$E[X_i^4 X_j^2] = 3 \qquad E[X_i^2 X_j^2 X_k^2] = 1$$

Moment constraint equations till 8th moment

Only the even moments are shown that are to be satisfied.
There are **11 non-zero moments for any dimension** in all

$$E[X_i^2] = 1 \qquad E[X_i^4] = 3$$
$$E[X_i^2 X_j^2] = 1 \qquad E[X_i^6] = 15$$
$$E[X_i^4 X_j^2] = 3 \qquad E[X_i^2 X_j^2 X_k^2] = 1$$
$$E[X_i^8] = 105 \qquad E[X_i^6 X_j^2] = 15$$

Moment constraint equations till 8th moment

Only the even moments are shown that are to be satisfied.
There are **11 non-zero moments for any dimension** in all

$$E[X_i^2] = 1 \qquad E[X_i^4] = 3$$

$$E[X_i^2 X_j^2] = 1 \qquad E[X_i^6] = 15$$

$$E[X_i^4 X_j^2] = 3 \qquad E[X_i^2 X_j^2 X_k^2] = 1$$

$$E[X_i^8] = 105 \qquad E[X_i^6 X_j^2] = 15$$

$$E[X_i^4 X_j^4] = 9 \qquad E[X_i^4 X_j^2 X_k^2] = 3$$

Moment constraint equations till 8th moment

Only the even moments are shown that are to be satisfied.
There are **11 non-zero moments for any dimension** in all

$$E[X_i^2] = 1$$

$$E[X_i^4] = 3$$

$$E[X_i^2 X_j^2] = 1$$

$$E[X_i^6] = 15$$

$$E[X_i^4 X_j^2] = 3$$

$$E[X_i^2 X_j^2 X_k^2] = 1$$

$$E[X_i^8] = 105$$

$$E[X_i^6 X_j^2] = 15$$

$$E[X_i^4 X_j^4] = 9$$

$$E[X_i^4 X_j^2 X_k^2] = 3$$

$$E[X_i^2 X_j^2 X_k^2 X_l^2] = 1$$

Moment constraint equations till 8th moment for 4D system

$$
\begin{aligned}
2r_1^2w_1 + 16r_2^2w_2 + 12r_3^2w_3 + 16r_4^2w_4 + 24r_5^2w_5 + 48r_6^2w_6 + 16h^2r_6^2w_6 &= 1\\
2r_1^4w_1 + 16r_2^4w_2 + 12r_3^4w_3 + 16r_4^4w_4 + 24r_5^4w_5 + 48r_6^4w_6 + 16h^4r_6^4w_6 &= 3\\
16r_2^4w_2 + 4r_3^4w_3 + 16r_4^4w_4 + 16r_5^4w_5 + 32r_6^4w_6 + 32h^2r_6^4w_6 &= 1\\
2r_1^6w_1 + 16r_2^6w_2 + 12r_3^6w_3 + 16r_4^6w_4 + 24r_5^6w_5 + 48r_6^6w_6 + 16h^6r_6^6w_6 &= 15\\
16r_2^6w_2 + 4r_3^6w_3 + 16r_4^6w_4 + 16r_5^6w_5 + 32r_6^6w_6 + 16h^2r_6^6w_6 + 16h^4r_6^6w_6 &= 3\\
16r_2^6w_2 + 16r_4^6w_4 + 8r_5^6w_5 + 16r_6^6w_6 + 48h^2r_6^6w_6 &= 1\\
2r_1^8w_1 + 16r_2^8w_2 + 12r_3^8w_3 + 16r_4^8w_4 + 24r_5^8w_5 + 48r_6^8w_6 + 16h^8r_6^8w_6 &= 105\\
16r_2^8w_2 + 4r_3^8w_3 + 16r_4^8w_4 + 16r_5^8w_5 + 32r_6^8w_6 + 16h^2r_6^8w_6 + 16h^6r_6^8w_6 &= 15\\
16r_2^8w_2 + 4r_3^8w_3 + 16r_4^8w_4 + 16r_5^8w_5 + 32r_6^8w_6 + 32h^4r_6^8w_6 &= 9\\
16r_2^8w_2 + 16r_4^8w_4 + 8r_5^8w_5 + 16r_6^8w_6 + 32h^2r_6^8w_6 + 16h^4r_6^8w_6 &= 3\\
16r_2^8w_2 + 16r_4^8w_4 + 64h^2r_6^8w_6 &= 1
\end{aligned}
$$

Moment constraint equations till 8th moment for 5D system

$$2r_1^2 w_1 + 32r_2^2 w_2 + 16r_3^2 w_3 + 32r_4^2 w_4 + 48r_5^2 w_5 + 128r_6^2 w_6 + 32h^2 r_6^2 w_6 = 1$$
$$2r_1^4 w_1 + 32r_2^4 w_2 + 16r_3^4 w_3 + 32r_4^4 w_4 + 48r_5^4 w_5 + 128r_6^4 w_6 + 32h^4 r_6^4 w_6 = 3$$
$$32r_2^4 w_2 + 4r_3^4 w_3 + 32r_4^4 w_4 + 24r_5^4 w_5 + 96r_6^4 w_6 + 64h^2 r_6^4 w_6 = 1$$
$$2r_1^6 w_1 + 32r_2^6 w_2 + 16r_3^6 w_3 + 32r_4^6 w_4 + 48r_5^6 w_5 + 128r_6^6 w_6 + 32h^6 r_6^6 w_6 = 15$$
$$32r_2^6 w_2 + 4r_3^6 w_3 + 32r_4^6 w_4 + 24r_5^6 w_5 + 96r_6^6 w_6 + 32h^2 r_6^6 w_6 + 32h^4 r_6^6 w_6 = 3$$
$$32r_2^6 w_2 + 32r_4^6 w_4 + 8r_5^6 w_5 + 64r_6^6 w_6 + 96h^2 r_6^6 w_6 = 1$$
$$2r_1^8 w_1 + 32r_2^8 w_2 + 16r_3^8 w_3 + 32r_4^8 w_4 + 48r_5^8 w_5 + 128r_6^8 w_6 + 32h^8 r_6^8 w_6 = 105$$
$$32r_2^8 w_2 + 4r_3^8 w_3 + 32r_4^8 w_4 + 24r_5^8 w_5 + 96r_6^8 w_6 + 32h^2 r_6^8 w_6 + 32h^6 r_6^8 w_6 = 15$$
$$32r_2^8 w_2 + 4r_3^8 w_3 + 32r_4^8 w_4 + 24r_5^8 w_5 + 96r_6^8 w_6 + 64h^4 r_6^8 w_6 = 9$$
$$32r_2^8 w_2 + 32r_4^8 w_4 + 8r_5^8 w_5 + 64r_6^8 w_6 + 64h^2 r_6^8 w_6 + 32h^4 r_6^8 w_6 = 3$$
$$32r_2^8 w_2 + 32r_4^8 w_4 + 32r_6^8 w_6 + 128h^2 r_6^8 w_6 = 1$$

Solution for 5D system

$$
\begin{aligned}
h &= 3 \\
r_5 &= 2 \\
r_1 &= 2.31437081728074447 \\
r_2 &= 0.83909427739801 02 \\
r_3 &= 1.8307521253266494 \\
r_4 &= 1.3970397430644959 \\
r_6 &= 1.1134786327367021 \\
w_1 &= 0.010529034221546607 \\
w_2 &= 0.015144019639537572 \\
w_3 &= 0.0052828996967816825 \\
w_4 &= 0.0010671298950159158 \\
w_5 &= 0.00065104166666666666 \\
w_6 &= 0.00013776017592074394
\end{aligned} \tag{7}
$$

Results of integration compared to Gauss Hermite integration for 5D system

For a N-D system the total number of points required to capture the 8th moment by this scheme are
$1 + 2N^2 + \frac{4N(N-1)(N-2)}{3} + (N+2)2^N$.
The covariance of the gaussian Kernel

$$P = 1000 I_{5x5} \tag{8}$$

$$X = [x_1, x_2, x_3, x_4, x_5] \tag{9}$$

$$F(X) = x_1^8 + x_2^8 + x_3^6 + x_4^6 + x_1^4 x_5^4 + x_2^4 x_3^4 + x_4^2 + x_1^2 x_2^2 x_3^2 x_4^2 + \\ + x_1^3 x_3 + x_1^4 x_2^2 x_5^2 + x_3^4 x_2^2 + x_1^2 x_3^4 + x_1^3 x_3 + x_2^3 x_3 + x_3^3 x_2 \tag{10}$$

Evaluating the integral

$$f = \int F(X) N(X, 0|P) dX \tag{11}$$

No. of pts	$2^5 = 32$	$3^5 = 243$	$4^5 = 1024$	$5^5 = 3125$	15^5 =759375(Truth)
GH	6.0962801e+012	7.6616634e+013	1.84964e+014	2.329646e+014	2.3296463e+014
% error	97.38317	67.11233	20.6039857	5.184531e-011	0

No. of pts	$2^5 = 32$	$3^5 = 243$	$4^5 = 1024$	$5^5 = 3125$	15^5 =759375(Truth)
GH	6.0962801e+012	7.6616634e+013	1.84964e+014	2.329646e+014	2.3296463e+014
% error	97.38317	67.11233	20.6039857	5.184531e-011	0

	No. of pts	Integration result	% error wrt Truth
NM	355	2.3296463e+014	5.1509964e-011

Results for Non- Polynomial Nonlinearity

Polar to Cartesian coordinates transformation

- A radar has error in its radial and angular measurements

Results for Non- Polynomial Nonlinearity

Polar to Cartesian coordinates transformation

- A radar has error in its radial and angular measurements
- We would like to see how this error is transformed into cartesian coordinates

Results for Non- Polynomial Nonlinearity

Polar to Cartesian coordinates transformation

- A radar has error in its radial and angular measurements
- We would like to see how this error is transformed into cartesian coordinates
- We would like to compute the expected value in the cartesian coordinates

Results for Non- Polynomial Nonlinearity

Polar to Cartesian coordinates transformation

- A radar has error in its radial and angular measurements
- We would like to see how this error is transformed into cartesian coordinates
- We would like to compute the expected value in the cartesian coordinates

Results for Non- Polynomial Nonlinearity

Polar to Cartesian coordinates transformation

- A radar has error in its radial and angular measurements
- We would like to see how this error is transformed into cartesian coordinates
- We would like to compute the expected value in the cartesian coordinates

In effect we are trying to evaluate the integral

$$\begin{aligned} E[(x,y)^T] &= E[(rcos(\theta), rsin(\theta))^T] \\ &= \int\int (rcos(\theta), rsin(\theta))^T N((r,\theta),(\mu_r,\mu_\theta|P))drd\theta \end{aligned}$$

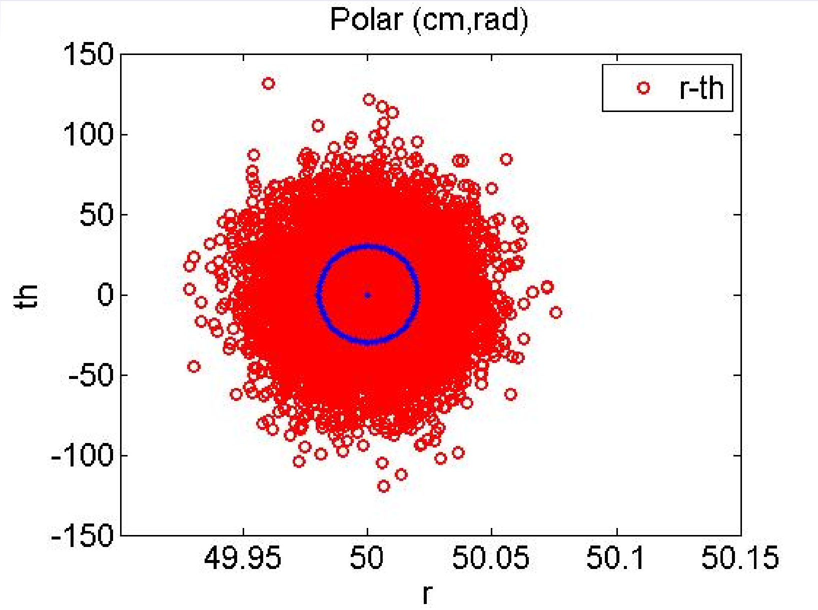

Figure: 2D and 3D Cubature points to satisfy 6th order moments

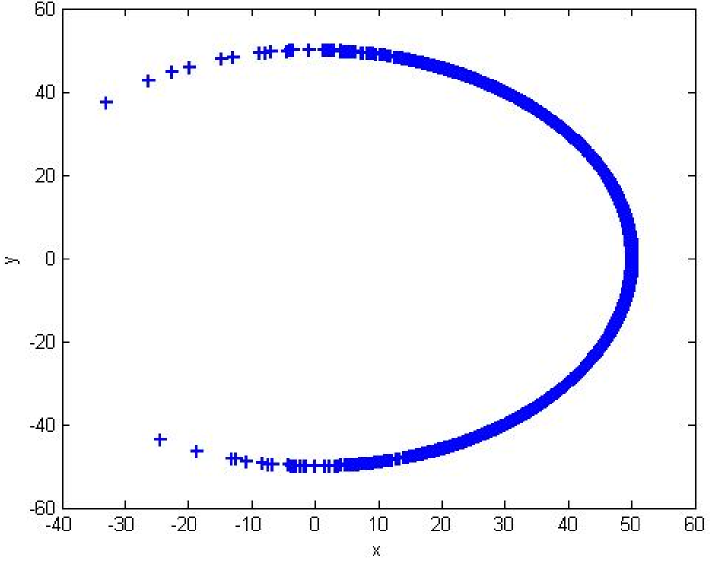

Figure: 2D and 3D Cubature points to satisfy 6th order moments

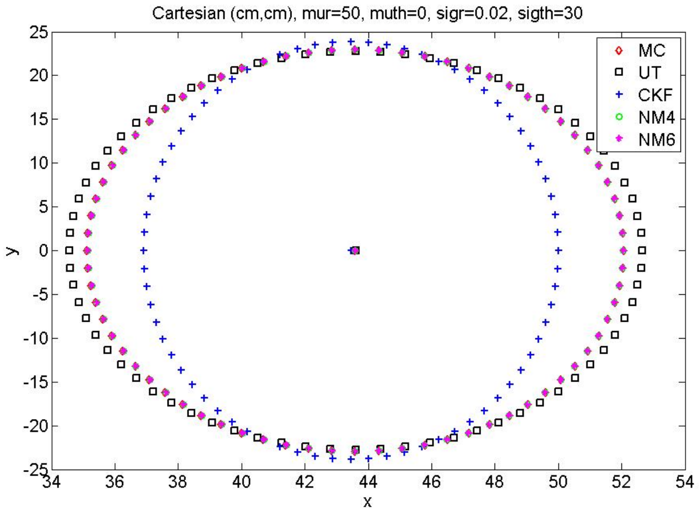

Figure: 2D and 3D Cubature points to satisfy 6th order moments

Expected value of Normal PDF

Normal Distribution The integral being evaluated is

$$E[N(x, \mu|P)] = \int N(x, \mu|P) N(x, \mu|P) dx$$

The parameters used are

$$\mu = 0$$
$$P = I$$

True value of Integral

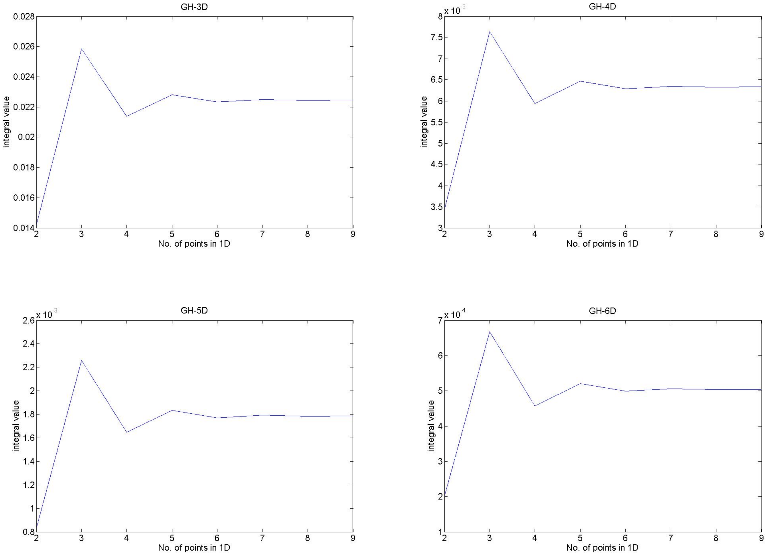

Table: Results of the integration interms of % rel. error

Dimension	CKF	UT	CUT4	CUT6	CUT8
3	36.902	36.902	27.042	13.273	1.5688
4	45.880	45.880	46.571	23.276	2.0279
5	53.581	53.581	73.768	28.056	6.8219
6	60.186	60.186	110.90	13.393	15.793

Table: Number of points in each method

Dim	GH-5	GH-6	GH-7	CKF	UT	CUT4	CUT6	CUT8
3	125	216	343	6	7	15	27	59
4	625	1296	2401	8	9	25	49	161
5	3125	7776	16807	10	11	43	83	355
6	15625	46656	117649	12	13	77	137	745

Expected value of Exponential Function

Normal Distribution The integral being evaluated is

$$E[N(x,\mu|P)] = \int exp(-\sum_{i=1}^{N} x_i)N(x,\mu|P)dx$$

The parameters used are

$$\mu = 0$$
$$P = I$$

True value of Integral

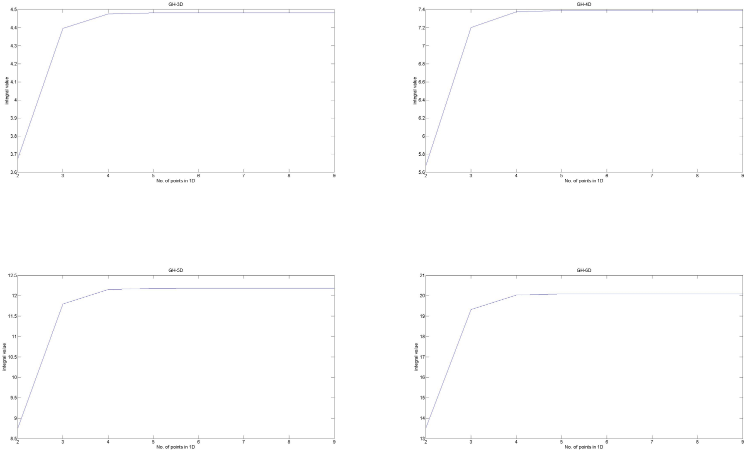

Table: Results of the integration interms of % rel. error

Dimension	CKF	UT	CUT4	CUT6	CUT8
3	34.9670	34.9670	5.6510	1.5589	0.0813
4	49.0842	49.0842	7.7049	2.0998	0.6467
5	61.1601	61.1601	9.7754	7.0631	1.4656
6	70.9523	70.9523	12.0018	11.6941	1.2176

Table: Number of points in each method

Dim	GH-4	GH-5	GH-6	CKF	UT	CUT4	CUT6	CUT8
3	64	125	216	6	7	15	27	59
4	256	625	1296	8	9	25	49	161
5	1024	3125	7776	10	11	43	83	355
6	4096	15625	46656	12	13	77	137	745

Discussion

- Is there a better way to solve the fully nonlinear system of moment equations.

Discussion

- Is there a better way to solve the fully nonlinear system of moment equations.
- How to find the minimal number of cubature points -or- to be optimistic how to prove that the cubature points by this method is minimal.

Discussion

- Is there a better way to solve the fully nonlinear system of moment equations.
- How to find the minimal number of cubature points -or- to be optimistic how to prove that the cubature points by this method is minimal.
- Is it really advantageous to develop higher order methods from a filtering point of view. If the first approximation is dominantly wrong does it help in using higher order cubature points

Discussion

- How to identify the non-polynomial type of functions that can be integrated by these methods accurately .- How do we develop Error estimates. For example for 1D Gauss Hermite quadrature the error estimate is $E = \frac{n!\sqrt{\pi}}{2^n(2n)!} f^{2n}(\xi)$

Discussion

- How to identify the non-polynomial type of functions that can be integrated by these methods accurately .- How do we develop Error estimates. For example for 1D Gauss Hermite quadrature the error estimate is $E = \frac{n!\sqrt{\pi}}{2^n(2n)!} f^{2n}(\xi)$
- How do we generalize this method or how do we find a mathematically rigorous theory/algorithm to generate cubature points for any moment and any dimension.

Discussion

- How to identify the non-polynomial type of functions that can be integrated by these methods accurately .- How do we develop Error estimates. For example for 1D Gauss Hermite quadrature the error estimate is $E = \frac{n!\sqrt{\pi}}{2^n(2n)!} f^{2n}(\xi)$
- How do we generalize this method or how do we find a mathematically rigorous theory/algorithm to generate cubature points for any moment and any dimension.
- Can we find the cubature points for any other PDF in the same manner.