The Conjugate Unscented Transform - An Approach to Evaluate Multi-Dimensional Stochastic Integrals

Nagavenkat Adurthi, Graduate Student, nagavenk@buffalo.edu
Puneet Singla, Assistant Professor, psingla@buffalo.edu
Tarunraj Singh, Professor, tsingh@buffalo.edu

University at Buffalo, State University of New York, Amherst, NY 14260-4400

Abstract— **This paper presents few methods to evaluate fully symmetric sigma points with positive weights for the Unscented Transform that can capture higher order moments of the normal probability density function. This work can be considered as a potential extension to the $2n+1$ Unscented Transform rule for nonlinear filtering or it can be used as an efficient Gaussian Cubature rule with reduced number of points compared to the Gauss Hermite product rule. Mainly three sets of sigma points are explored, firstly the set of sigma points that are 4th degree exact for any dimension. Secondly, the set of sigma points of that are 6th degree exact for dimensions $2 \leq N \leq 9$ and thirdly the set of sigma points that are 8th degree exact for dimensions $2 \leq N \leq 6$. Each method is compared to the Gauss Hermite product rule which is known to be exact for polynomial functions. The results are very encouraging for the fact that the same order of accuracy can be achieved with a small fraction of the number of points used by Gauss Hermite product rule. Finally results for a few non-polynomial type functions are shown that would motivate one to develop even higher order cubature methods with reduced number of points.**

I. INTRODUCTION

Numerous fields of science and engineering present the problem of uncertainty characterization and propagation through nonlinear dynamic systems with stochastic excitation and uncertain initial conditions. One may be interested in predicting the probability of collision of asteroid with Earth, diffusion of toxic materials through atmosphere, control of movement and planning of actions of autonomous systems, the optimization of financial investment policies, active control of structural vibrations induced by earthquakes or wind loads, or simply the computation of the prediction step in the design of a Bayes filter. All these applications require the study of the relevant stochastic dynamic system and involve computing multi-dimensional expected value integrals involving appropriate probability density function (pdf). Analytical expressions for these multi-dimension integrals exist only for linear systems and only for few moments. For example, the well celebrated Kalman filter provides the analytical expressions for mean and covariance of linear system subject to Gaussian white noise and Gaussian initial condition errors. However, one often do not have direct analytical solution for these integrals and have to approximate integral values by making use of computational methods.

Several computation techniques exist in the literature to approximate the expectation integral values, the most popular being Monte Carlo (MC) methods, Gaussian Quadrature Rule, Unscented Transformation and Cubature methods. All these methods involve an approximation of the expectation integral as a weighted sum of integrand values at specified points within the domain of integration. These methods basically differ from each other in the generation of specific points also known as sample points. For example, MC methods involves random samples from the specified pdf while Gaussian quadrature scheme involves deterministic sample points carefully chosen to reproduce exactly the integrals involving polynomial functions. Both deterministic Gaussian quadrature and MC methods are very popular but both methods require extensive computational resources and effort, and becomes increasingly infeasible for computing expectation integrals in high-dimension space [1].

For one-dimensional integrals, one needs m quadrature points according to the Gaussian quadrature scheme to reproduce the expectation integrals involving $2m-1$ degree polynomial functions. However, for a generic N-dimensional system, one needs to take the tensor product of $1-$dimensional m quadrature points and hence one would require a total of m^N quadrature points (also known as cubature points for $N \geq 2$) [1]. Even for a moderate dimension system involving 6 random variables, the number of points required to evaluate the expectation integral with only 5 points along each direction is $5^6 = 15,625$. This is a huge number of points that might be computationally expensive to use especially when the evaluation of function at each cubature point itself can be an expensive procedure, e.g., one may need to solve a system of partial differential equations to compute the function of interest. But fortunately the Gaussian quadrature rule is *not minimal* for $N \geq 2$ and there exists cubature rules with reduced number of points [1]. This forms the basis of our motivation to the work presented in this paper.

An extensive amount of work has been done in this field to develop cubature/quadrature rules which can exactly reproduce integrals involving polynomials of degree less than or equal to '$d = 2m-1$' with fewer points, i.e., less than m^N. A good description of non-product Gaussian cubature rules, particularly for symmetric density functions, that are second, third and fifth degree exact in any dimension is provided in Ref. [2]. In general to develop a cubature rule of degree 'd' that is applicable to any dimension is still an open problem. A cubature rule of degree 2 with $N+1$ points and a cubature method of degree 3 with $2N$ points for a general centrally symmetric weight function (such as the normal and uniform pdf) in N-dimensions is developed in Ref. [3]. A fully symmetric integration rules with minimal points for 2-dimension are developed that are exact for degree $9-15$ in Ref. [4]. A 19 point cubature rule is developed, for symmetric regions in 2-dimensions that are exact for degree 9 or less in Ref. [5]. For a 2-dimesnional integral, with symmetric weight function a 12 point cubature rule for degree 7 is developed in [6]. It is even claimed that this is the minimal number of points required and there exist many such 12 point cubature rule for degree 7 in 2-dimensional systems. The highly acclaimed Unscented Transform (UT) with only $2N+1$ points is a degree 3 cubature method, where these cubature points are now called *sigma points* [7]. Similar to the UT method, a more recent development is the Cubature Kalman filter which is again exact to degree 3 but uses only $2N$ points [8].

Most of the non-product cubature rule possess certain similarities, they exploit the symmetry of the weight function, assume a structure for the cubature points, solve a system of nonlinear equations. Hence these methods also suffer some similar drawbacks such as inconsistency of the set of non-linear equations, the presence of negative weights/complex roots and the inability to provide a generalized solution that works for any dimension. The present work heads in a similar manner and tries to overcome some of these difficulties.

The primary objective of this paper can be stated as "To find a fully symmetric sigma/cubature point set with all positive weights and reduced number of points that is *equivalent* to the set of cubature points of Gaussian quadrature product rule of *same order*". By *equivalent to same order* mean that for a polynomial of order $2m-1$ in generic N-dimensions, both the new reduced sigma point set from the proposed method known as Conjugate Unscented Transform method ('CUT') and the m^N quadrature points from the Gaussian quadrature product rule result in same order of relative percentage error. The organization of paper is as follows: firstly, we review and analytically compare few popular cubature rules that are widely used in nonlinear filtering, particularly the Unscented Transfrom and Cubature Kalman Filter. A detailed study of how these methods tend to pick the sigma/cubature points provides a valuable insight upon which our development of the framework for the Cunjugate Unscented Transfrom is based. Secondly, we describe this framework/methodology to evaluate the sigma point sets for the three methods we intend to propose namely CUT4, CUT6 and CU8. Finally, numerical experiment results are presented to show the efficacy of the proposed methods.

II. Expectation Integral and Cubature Methods

Let us consider the problem of computing expected value of a function $f(x)$ with respect of a Gaussian density function with mean μ and covariance P:

$$E[f(x)] = \int f(x)N(x,\mu|P)dx \tag{1}$$

Taking the Taylor series expansion of $f(x)$ about μ and substituting in the aforementioned expression leads to

$$E[f(x)] = f(\mu) + \nabla f(\mu)E[\delta x] + \frac{1}{2!}\nabla^2 f(\mu)E[\delta x^2] + \frac{1}{3!}\nabla^3 f(\mu)E[\delta x^3] + \frac{1}{4!}\nabla^4 f(\mu)E[\delta x^4]... \tag{2}$$

where $\delta x = (x-\mu)$. Notice that the problem of evaluating the expected value of nonlinear function $f(x)$ has reduced to computing higher order moments of δx. Thus by increasing the number of terms in the Taylor series expansion, one can obtain more accurate value of the expectation integral of (1).

A. Sigma Point Set to capture moments of Normal PDF

Without loss of generality consider a normal PDF with 0 mean and identity covariance of appropriate dimension. Under this assumption all the odd order central and raw moments of the normal PDF are 0. For a sigma point set $(x_1, x_2, ..., x_n)$ with corresponding weights $(w_1, w_2, ...w_3)$, the expected value is approximated by a weighted sum as in (3).

$$E[f(x)] = \sum_{i=1}^{n} w_i f(x_i) \tag{3}$$

Consider the Taylor series approximation of the function value at each point about the mean as in (4)

$$f(x_i) = f(0) + \frac{1}{2!}\nabla^2 f(0)x_i^2 + \frac{1}{4!}\nabla^4 f(0)x_i^4 + \frac{1}{6!}\nabla^6 f(0)x_i^6 ... \tag{4}$$

which on substituting into (3) results in (5) that is in terms of the sigma points and associated weights.

$$E[f(x)] = f(0)(\sum_{i=1}^{n} w_i) + \frac{1}{2!}\nabla^2 f(0)(\sum_{i=1}^{n} w_i x_i^2) + \frac{1}{4!}\nabla^4 f(0)(\sum_{i=1}^{n} w_i x_i^4) + \frac{1}{6!}\nabla^6 f(0)(\sum_{i=1}^{n} w_i x_i^6) \tag{5}$$

(6)

Upon comparing the coefficients of (5) to (2), the resulting set of equations are called the moment constraint eqautions (7).

$$\sum_{i=1}^{n} w_i x_i^m = E[\delta x^m], \quad m = 0, 1, 2, \cdots \tag{7}$$

When more sigma points are evaluated such that they can satisfy the higher order moment constraint equations (7), the integral value evaluated by (3) tends to converge to the true value of the integral. Though only a 1-Dimensional case is shown the moment constraint equations can also be extended to N-Dimensions, such as in [7]. As mentioned in the objective, we intend to seek a set of *fully symmetric* sigma points, as these sigma points have the advantage of satisfying all odd order moment constraint equations. Thus a sigma point set or a cubature rule that can satisfy all the moment constraint equations till the even order 'm' can be used to evaluate the expectation integral of polynomials that have monomial terms with degree $d \leq m$ or any odd degree.

III. Sigma points that are 2^{nd} moment equivalent

In this section we review the methods of UT and CKF. Both these methods have certain similarities specifically in the way they choose the structure of the sigma points. The sigma points that are 2^{nd} moment equivalent can completely satisfy the moment constraint equations till 2^{nd} order. The two evident similarities are that they constrain the sigma points to lie on specific axis and they are 2^{nd} moment equivalent. To proceed with a general analysis the following definition is introduced with respect to a zero mean and identity covariance matrix Normal PDF

Principal axis: It is defined as the orthogonal axis in cartesian space intersecting at the origin. These are the axis corresponding to each column of the identity covariance matrix. Thus there are N principal axis or $2N$ distinct points on the principal axis for N-Dimensional system. Let the set $D = \{1,2,3,...,N\}$ We list the points on the principal axis as

$$\sigma_i \in \{\pm(I)_j | j \in D\} \quad i = 1,2,3,...,2N. \tag{8}$$

Each point on the principle axis is at a distance $|\sigma_i|$ or in this case at 1 unit distance from the origin, where $|x| = \sqrt{x_1^2 + x_2^2 \cdots x_N^2}$. The principle axis labeled as σ and the corresponding points are shown for a 2D case in Fig(1(a)) while for a 3D case only one of the quadrant is shown in two perspective views as Fig(1(b)) and Fig(1(c)). All the 8 quadrants in the 3D case are symmetrical.

1) Sigma points constrained to the principal axis: For an i.i.d random variables $(X_1, X_2, ..., X_N)$ the N-Dimensional normal PDF has Identity covariance and zero mean. The first four moments are

$$\begin{aligned} E[X_i^2] &= 1, \quad & E[X_iX_j] &= 0, \quad & E[X_i^4] &= 3 \\ E[X_i^3X_j] &= 0, \quad & E[X_i^2X_j^2] &= 1, \quad & E[X_i^2X_jX_k] &= 0 \\ E[X_iX_jX_kX_l] &= 0, \end{aligned} \tag{9}$$

Where $\{i,j,k,l\} \subset \{1,2,3,...,N\}$. Consider a fully symmetric set of sigma points that lie on the principal axis such that the distance of each point from the origin r_1 and each has weight of w_1. Additionally consider a point at the origin with weight w_0. The moments that have any odd powers for the random variable in the set of moments in (9) are already satisfied due to symmetry of $2N+1$ sigma points. The non-zero/even moment constraint equations till order 4, interms of the variables r_1 and w_1 are

$$E[X_i^2] \equiv 2r_1^2 w_1 = 1 \tag{10}$$

$$E[X_i^4] \equiv 2r_1^4 w_1 = 3 \tag{11}$$

$$E[X_i^2X_j^2] \equiv 0 \neq 1 \tag{12}$$

Notice that the central weight at the origin does not show up in any moment constraint equation. As all the N-random variables X_i and the sigma points are symmetric, we just need to form the moment constraint equations for any one set of random variables. The 4^{th} order cross moment $E[X_i^2X_j^2]$ in (12) cannot be satisfied by the sigma points that are constrained to be only on the principal axis. Infact no cross moment of any order can be satisfied by points just on the principal axis.The central weight is calculated as

$$w_0 = 1 - 2Nw_1 \tag{13}$$

In the following sub sections, the derivation of the sigma points of UT and the cubature points of CKF are illustrated with respect to the analysis presented in this section.

2) $2N+1$ sigma points of the Unscented Transform-UKF: The $2N+1$ sigma points for the Unscented Transform in [7] are chosen such that 1 point is the origin and $2N$ points of each weight w_1 are constrained to lie on the principal axis at distance scaled by r_1. The distance of each point on the principal axis is chosen such that the second order moments are satisfied. A tuning parameter κ is introduced such that one of the 4th moment can be tuned. This works well for systems till dimension 3 after which the central weight becomes negative. The present framework is shown to be inline with the $2N+1$ Unscented sigma points by working our way backwards. The suggested points are

$$X_0 = \mu \qquad W_0 = \kappa/(N+\kappa) \tag{14}$$

$$X_i = \mu + (\sqrt{(N+\kappa)P})_i \qquad W_i = 1/[2(N+\kappa)] \tag{15}$$

$$X_{i+N} = \mu - (\sqrt{(N+\kappa)P})_i \qquad W_{i+N} = 1/[2(N+\kappa)] \tag{16}$$

With $\mu = \vec{0}_{(N\times 1)}$ and $P = I_{(N\times N)}$, by an elegant selection of $r_1 = \sqrt{N+\kappa}$ one could solve for w_1 from (10), w_0 from (13) and κ from (11) as

$$w_1 = 1/2(N+\kappa) \tag{17}$$

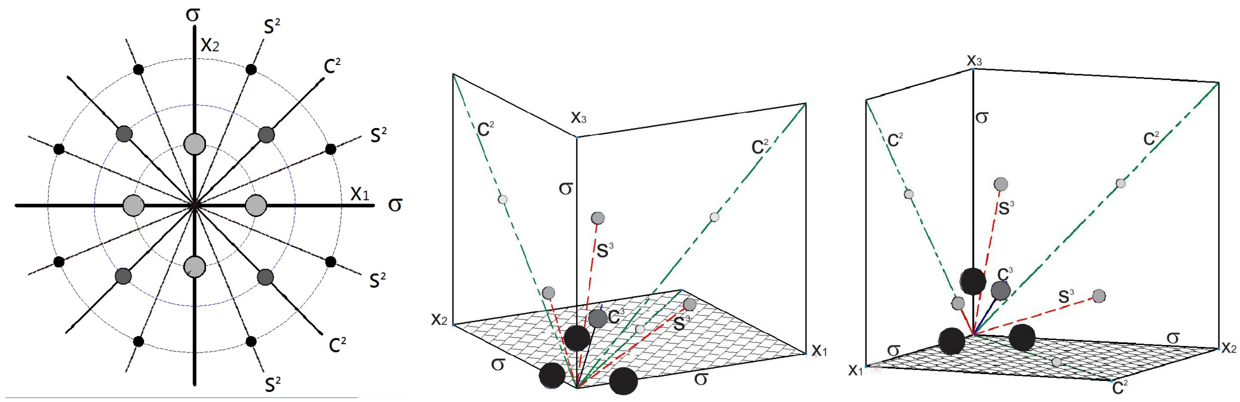
$$w_0 = 1 - 2Nw_1 = \kappa/(N+\kappa) \tag{18}$$

$$N + \kappa = 3 \tag{19}$$

Thus under a normal distribution assumption the tuning parameter $\kappa = 3 - N$. When the dimension of system is greater than 3, $\kappa < 0$ and the central weight becomes negative leading to some numerical issues. Thus the fully symmetric $2N+1$ Unscented Transform sigma points are second moment equivalent and can integrate polynomials of degree 3 or less accurately.

3) $2n$ Cubature points-CKF: The authors of [8] provide a mathematically rigourous and elegant way of finding the second moment equivalent $2N$ cubature points that can again accurately integrate polynomials of degree 3 or less. Spherical-radial coordinates are employed to find these $2N$ symmetric cubature points on the principal axis. Contrary to the $2N+1$ Unscented transform sigma points, they only have $2N$ cubature points on the principal axis and no point on the mean. This is equivalent to setting $\kappa = 0$ in the Unscented transform sigma points thus making the central weight $w_0 = 0$. An alternate derivation, using the present framework, to the one given in [8] is provided which is less rigourous but more intuitive. As the central weight w_0 is 0, we get w_1 from (13), r_1 from (10) as

$$w_1 = 1/(2N) \quad \text{and} \quad r_1 = \sqrt{N} \tag{20}$$



(a) 2D axis (b) 3D 1^{st} Quadrant-view 1 (c) 3D 1^{st} Quadrant-view 2

Fig. 1. Symmetric set of points and axis 2D and 3D space

The absolute error in 4^{th} moment equation (11) is

$$|2N^2\frac{1}{2N}-3| \equiv |N-3| \tag{21}$$

For systems till dimension 3, the Unscented Transform points can better match one of the 4^{th} order moment than the CKF-Cubature points. After dimension 3 both the method have error in the 4th order moments as they were not designed to capture all the 4th order moments.

IV. Methodology

For a general N-dimensional system, the set of nonlinear moment constraint equations are very tedious to solve. The methods of UT and CKF provide an insight that "the points can be constrained to some carefully selected axis" and then all that remains is to solve for the distances of these points from the origin and their corresponding weights. The set of moment constraint equations along with these additional constraints makes the system easier to solve. Based on the drawbacks of UT and CKF and the idea of constraining points to selected axis, a common framework is followed throughout the paper can be summarized as

1) As we know that the Gaussian PDF is symmetric, we exploit this by constraining the sigma points to various symmetric axis. We later define some of these axis as and when they are required.
2) Sigma points on the same set of symmetric axis are equidistant from the mean and have equal weight. For the i^{th} set of axis , the distance scaling variables are labeled r_i and weight variables are labeled w_i.
3) We find all the moments of the continuous multidimensional normal PDF. The moments of a normal PDF are given by the Isserlis theorem which is formally restated in [9]. By which, given any covariance matrix 'P' we can find all higher order moments just from the entries of P.
4) We enumerate the list of points and then form the moment constraint equations till required order in terms of the variable r_i and w_i. For example the moment constraint equations till order 4 for a 2D system are shown in Table I. As the points (x_i, y_i) are constrained to a line, they can be replaced in terms of a single variable namely the distance r_i from the origin.
5) These nonlinear set of equations are solved for r_i and w_i, that result in the required sigma point set.

TABLE I
Moment Constraint equations for 2D till order 4

Continuous	Discrete	Continuous	Dicrete
$E[x]$	$\sum_{i=1}^{n} w_i x_i$	$E[y]$	$\sum_{i=1}^{n} w_i y_i$
$E[x^2]$	$\sum_{i=1}^{n} w_i x_i^2$	$E[xy]$	$\sum_{i=1}^{n} w_i x_i y_i$
$E[y^2]$	$\sum_{i=1}^{n} w_i y_i^2$	$E[x^4]$	$\sum_{i=1}^{n} w_i x_i^4$
$E[y^4]$	$\sum_{i=1}^{n} w_i y_i^4$	$E[x^3 y]$	$\sum_{i=1}^{n} w_i x_i^3 y_i$
$E[y^3 x]$	$\sum_{i=1}^{n} w_i y_i^3 x_i$	$E[x^2 y^2]$	$\sum_{i=1}^{n} w_i x_i^2 y_i^2$

In the following sections we derive the sigma point sets this framework to satisfy the moment constraint equations of order 4, 6 and 8 respectively. As we go to higher order moment constraint equations we introduce new axis that aid in the derivation of these sigma point sets.

V. Sigma points that are 4th moment equivalent

Sigma points that are 4^{th} moment equivalent can completely satisfy the moment constraint equations till 4^{th} order. The following definition is introduced

M^{th}-Conjugate axis: For a N-Dimensional system where $M \leq N$,the axis that are constructed from all the combinations, including the sign permutations, of the set of principal axis taken M at a time. For example, the N^{th}-Conjugate set of axis for N-Dimensional system have 2^N distinct points or 2^{N-1} axis and the 2^{nd}-Conjugate set of axis for N-Dimensional system has $2N(N-1)$ distinct points or $N(N-1)$ axis. We label the set of M^{th} conjugate axis as c^M, where the points are listed as c_i^M, where $i = 1, 2, ..., 2^M \binom{N}{M}$

$$c_i^M \in \{(\pm\sigma_{n_1} \pm \sigma_{n_2} \pm ... \pm \sigma_{n_M}) | \{n_1, n_2, ..., n_M\} \subset D\} \tag{22}$$

For zero mean and identity covariance, each point on the M^{th}-Conjugate axis is at a distance of $|c_i^M| = \sqrt{M}$ from the origin. The set of conjugate axis labeled as c^M are shown for 2D in Fig(1(a)) and for a 3D case in Fig(1(b)) and

Fig(1(c)), which are just 2 different perspective views of the first quadrant.

A. Sigma points constrained to principal axis and 2^{nd}- Conjugate axis

In appendex IV of [10], the authors provides a method to calculate the cubature points that can capture all the fourth order moments. But this method also suffers from the presence of a negative/zero weight for dimensions higher than or equal to 4.The reason can be shown analytically. Let the points be chosen such that 1 point of weight w_0 lies on the origin, $2N$ points of weight w_1 each lie on the principal axis at a distance of r_1 and $2N(N-1)$ points of weight w_2 each lie on the 2^{nd}- Conjugate axis at a distance $r_2|c_i^2|$. Following the similar framework, the moment constraint equations till order 4 are

$$2r_1^2 w_1 + 4(N-1) r_2^2 w_2 = 1 \quad (23)$$
$$2r_1^4 w_1 + 4(N-1) r_2^4 w_2 = 3 \quad (24)$$
$$4r_2^4 w_2 = 1 \quad (25)$$
$$1 - 2Nw_1 - 2N(N-1)w_2 = w_0 \quad (26)$$

The main variables to solve for are r_1, w_1, r_2, w_2 from (23)-(25) and then w_0 is found from (26). Thus there are 4 main variables and 3 equations indicating that there needs to be some kind of optimization procedure. The authors of [10] chooses to minimize the error in one of the 6th order moment. Before the optimization is framed, one can simplify the constraint equations as

$$w_2 = \frac{1}{4r_2^4}, w_1 = \frac{4-N}{2r_1^4}, \quad r_1^2 r_2^2 = r_1^2(N-1) + r_2^2(4-N) \quad (27)$$

There is only one constraint in (27) to solve interms of the variables r_1 and r_2. Even though it can be solved by minimizing the 6th moment error one would still result in a negative/zero weight above dimension 3. In the next section we describe the 4^{th} moment equivalent sigma points of Conjugate Unscented Transform that is able to overcome the problem of negative weight by choosing a different set of axis.

B. 4^{th} order Conjugate Unscented Transfrom: CUT4

In this section we solve for sigma points that are 4th order equivalent, fully symmetric and has all the weights to be positive. A similar work can be found in formula (IV) of [11], where the cubature rule is developed for uniform distribution. The points are selected such that $2N$ points of weight w_1 each lie on the principal axis at a distance of r_1 and 2^N points of weight w_2 lie on the N^{th}- Conjugate axis at a distance of $r_2\sqrt{N}$. Once the points are enumerated as in Table (III) and substituted into the moment constraint equations till order 4, the set of equations (28)-(30) are to be solved. This general set of equations were formed by first forming them separately for dimension 3,4 and 5 and then finding a general set of equations. This set of generalized equations were numerically tested to be exact till dimension 10. Though it is also shown to be analytically correct for any dimension.

TABLE II

CUT4: OPTIMIZED SOLUTION FOR $N = 1$ AND $N = 2$

Variable	$N = 1$	$N = 2$
r_1	1.4861736616297834	2.6060099476935847
r_2	3.2530871022700643	1.190556300661233
w_0	0.5811010092660772	0.41553535186548973
w_1	0.20498484723245053	0.021681819434216532
w_2	0.00446464813451093	0.12443434259941118
No. of points	5	9

TABLE III

SIGMA POINTS FOR CUT4

	Position	Weights
$1 \le i \le 2N$	$X_i = r_1 \sigma_i$	$W_i = w_1$
$1 \le i \le 2^N$	$X_{i+2N} = r_2 c_i^N$	$W_{i+2N} = w_2$
Central weight	$X_0 = \mathbf{0}$	$W_0 = w_0$
$n = 2N + 2^N\,(+1)$		

$$2r_1^2 w_1 + 2^N r_2^2 w_2 = 1 \quad (28)$$
$$2r_1^4 w_1 + 2^N r_2^4 w_2 = 3 \quad (29)$$
$$2^N r_2^4 w_2 = 1 \quad (30)$$

And the central weight at the mean is given by

$$1 - 2Nw_1 - 2^N w_2 = w_0 \quad (31)$$

The central weight can be assigned a value such that the square of the 6th moment constraint error $(2r_1^6 w_1 + 2^N r_2^6 w_2 - 15)^2$ is minimized subjected to the constraint of (28)-(30) or even to be 0 thus reducing one point. The solution of the former scheme is shown in Table (II) for dimensions 1 and 2 while the latter scheme is shown for dimensions greater than 2 .

In summary for $N > 2$

$$r_1 = \sqrt{\frac{N+2}{2}}, \qquad r_2 = \sqrt{\frac{N+2}{N-2}} \quad (32)$$
$$w_1 = \frac{1}{r_1^4} = \frac{4}{(N+2)^2}, \quad w_2 = \frac{1}{2^N r_2^4} = \frac{(N-2)^2}{2^N (N+2)^2} \quad (33)$$

C. 6^{th} order Conjugate Unscented Transform: CUT6

This section describes a procedure to solve all the moment constraint equations till 6^{th} order for dimensions $N \le 9$. Firstly the moments till 6^{th} order for a Normal PDF are evaluated, then the moment constraint equations are formed by selecting the appropriate axis. The points on the principal axis have each a weight of w_1 and are at a distance r_1 from the origin. The points on the N^{th}-Conjugate axis have each a weight of w_2 and are located at a distance of $r_2\sqrt{N}$. The third set of points have weight w_3 and are located at a distance of $r_3\sqrt{2}$ along the 2^{nd}-Conjugate axis. The points

TABLE IV
SIGMA POINTS FOR CUT6, ($N \leq 6$)

	Position	Weights
$1 \leq i \leq 2N$	$X_i = r_1\sigma_i$	$W_i = w_1$
$1 \leq i \leq 2^N$	$X_{i+2N} = r_2 c_i^N$	$W_{i+2N} = w_2$
$1 \leq i \leq 2N(N-1)$	$X_{i+2N+2^N} = r_3 c_i^2$	$W_{i+2N+2^N} = w_3$
Central weight	$X_0 = \mathbf{0}$	$W_0 = w_0$
$n = 2N^2 + 2^N + 1$		

TABLE V
SIGMA POINTS FOR CUT6, ($7 \leq N \leq 9$)

	Position	Weights
$1 \leq i \leq 2N$	$X_i = r_1\sigma_i$	$W_i = w_1$
$1 \leq i \leq 2^N$	$X_{i+2N} = r_2 c_i^N$	$W_{i+2N} = w_2$
$1 \leq i \leq 4n(n-1)(n-2)/3$	$X_{i+2N+2^N} = r_3 c_i^3$	$W_{i+2N+2^N} = w_3$
Central weight	$X_0 = \mathbf{0}$	$W_0 = w_0$
$n = 2N + 2^N + 4n(n-1)(n-2)/3 + 1$		

are enumerated as in Table (IV) and the moment constraint equations are formed using the moments for the continuous normal PDF in (34)

$$\begin{aligned} E[X_i^2] &= 1, & E[X_i^4] &= 3, & E[X_i^2 X_j^2] &= 1 \\ E[X_i^6] &= 15 & E[X_i^4 X_j^2] &= 3 & E[X_i^2 X_j^2 X_k^2] &= 1 \end{aligned} \tag{34}$$

As the points chosen are fully symmetric only the moment with all even powers are to be satisfied. The general set of moment constraint equations are
For $N < 7$

$$2r_1^2 w_1 + 2^N r_2^2 w_2 + 4(N-1) r_3^2 w_3 = 1 \tag{35}$$
$$2r_1^4 w_1 + 2^N r_2^4 w_2 + 4(N-1) r_3^4 w_3 = 3 \tag{36}$$
$$2^N r_2^4 w_2 + 4 r_3^4 w_3 = 1 \tag{37}$$
$$2r_1^6 w_1 + 2^N r_2^6 w_2 + 4(N-1) r_3^6 w_3 = 15 \tag{38}$$
$$2^N r_2^6 w_2 + 4 r_3^6 w_3 = 3 \tag{39}$$
$$2^N r_2^6 w_2 = 1 \tag{40}$$
$$1 - 2Nw_1 - 2^N w_2 - 2N(N-1) w_3 = w_0 \tag{41}$$

Solving the 3 equations (38)-(40) for the weights analytically ,

$$w_1 = \frac{8-n}{r_1^6}, \quad w_2 = \frac{1}{2^n r_2^6}, \quad w_3 = \frac{1}{2r_3^6} \tag{42}$$

Once the weights are symbolically solved, they can be substituted into (35)-(37), then there are only 3 polynomial equations in terms of r_1,r_2 and r_3. This reduced system of equations is much easier to solve than the original system of equations. Above dimension 6 the central weight becomes negative and above 8 one weight starts to become negative, hence this procedure is valid for $N \leq 6$. A very similar method having the same drawback is presented in [12], for which the author derives the cubature rule for degree 7 and for specific dimensions $N = 3,4,6,7$. The problem of negative weight can be solved by choosing another set of axis as, For $7 \leq N \leq 9$, choose 3^{rd}-Conjugate axis instead of 2^{nd}-conjugate axis. The set of 6th moment constraint equations in terms of the points, as enumerated in Table (V), constrained to principal axis (r_1,w_1), N^{th}- conjugate axis (r_2,w_2) and 3^{rd}-conjugate axis (r_3,w_3) are For $7 \leq N \leq 9$

$$2r_1^2 w_1 + 2^N r_2^2 w_2 + 4(N-1)(N-2) r_3^2 w_3 = 1 \tag{43}$$
$$2r_1^4 w_1 + 2^N r_2^4 w_2 + 4(N-1)(N-2) r_3^4 w_3 = 3 \tag{44}$$
$$2^N r_2^4 w_2 + 8(n-2) r_3^4 w_3 = 1 \tag{45}$$
$$2r_1^6 w_1 + 2^N r_2^6 w_2 + 4(N-1) r_3^6 w_3 = 15 \tag{46}$$
$$2^N r_2^6 w_2 + 8(n-2) r_3^6 w_3 = 3 \tag{47}$$
$$2^N r_2^6 w_2 + 8 r_3^6 w_3 = 1 \tag{48}$$
$$1 - 2Nw_1 - 2^N w_2 - (4n(n-1)(n-2)/3) w_3 = w_0 \tag{49}$$

Exploiting the linearity in the weights, one can again reduce the overall order and number of variables in the system. This set of equations (43)-(48) are preferred till dimension 9 only as above dimension 9 the central weight becomes negative. But if one allows the presence of negative weight the equations can be solved untill dimension 13 after which some roots become complex.

D. 8^{th} order Conjugate Unscented Transform: CUT8

This section describes an *attempt* to solve all the moment constraint equations till order 8 by choosing appropriate axis. We describe the analysis in this section to be an attempt as we were able to 'exactly' solve the 8th order moment constraint equations till only dimension 6, above which the weight becomes negative. We provide a procedure that is done separately for each dimension. The procedure presented might apparently look laborious or in a way de-motivating but the *striking* results do infact justify our *endeavour*. A new set of axis is defined for a N-D system with zero mean and Identity covariance as follows
N^{th}-Scaled Conjugate axis: For a N-Dimensional system ,the axis that are constructed from all the combinations,including sign permutations, of the set of principal axis such that in every combination exactly one principal axis is scaled by a scaling parameter 'h' .The N^{th}-Scaled Conjugate set of axis for N-Dimensional system has $N2^N$ distinct points or $N2^{N-1}$ axis. We label the set of N^{th}-Scaled conjugate axis as $s^N(h)$, and the points are listed as $s_i^N(h)$ where $i = 1,2,3,...,N2^N$

$$s_i^N(h) \in \{(\pm h\sigma_{n_1} \pm \sigma_{n_2} \pm ... \pm \sigma_{n_N}) | \{n_1, n_2, ..., n_N\} \subset D\} \tag{50}$$

The scaled conjugate axis labeled as $s^N(h)$ are shown for a 2D case in Fig(1(a)) where the scaling paramter h is 2.414

while for a 3D case that is shown in Fig(1(b)) and Fig(1(c)) the scaling parameter h is taken as 2.74. The procedure followed is described for a general dimension of $N \leq 6$.

1) The first set of points are on the principal axis such that $X_i^{(1)} = r_1 \sigma_i$ and $W_i^{(1)} = w_1$.
2) The second set of points are on the N^{th}-Conjugate axis such that $X_i^{(2)} = r_2 c_i^N$ and $W_i^{(2)} = w_2$.
3) The third set of points are on the 2^{nd}-Conjugate axis such that $X_i^{(3)} = r_3 c_i^2$ and $W_i^{(3)} = w_3$.
4) The fourth set of points are again on the N^{th}-Conjugate axis such that $X_i^{(4)} = r_4 c_i^N$ and $W_i^{(4)} = w_4$.
5) The fifth set of points are on the 3^{rd}-Conjugate axis such that $X_i^{(5)} = r_5 c_i^3$ and $W_i^{(5)} = w_5$.
6) The sixth set of points are on the N^{th}-Scaled Conjugate axis such that $X_i^{(6)} = r_6 s_i^N(h)$ and $W_i^{(6)} = w_6$.
7) The scaling parameter 'h' has to be appropriate chosen
8) Enumerate the points and form the moment constraint equations till order 8 using the moments in (52).

The sigma point set is

$$\begin{aligned} &\{X^{(1)}, X^{(2)}, X^{(3)}, X^{(4)}, X^{(5)}, X^{(6)}, X_0\} \\ &\{W^{(1)}, W^{(2)}, W^{(3)}, W^{(4)}, W^{(5)}, W^{(6)}, W_0\} \end{aligned} \tag{51}$$

The even/non-zero moments till order 8 are

$$\begin{aligned} E[X_i^2] &= 1, & E[X_i^4] &= 3, & E[X_i^2 X_j^2] &= 1, \\ E[X_i^6] &= 15, & E[X_i^4 X_j^2] &= 3, & E[X_i^8] &= 105, \\ E[X_i^6 X_j^2] &= 15, & E[X_i^4 X_j^4] &= 9, & E[X_i^4 X_j^2 X_k^2] &= 3, \\ E[X_i^2 X_j^2 X_k^2] &= 1, & E[X_i^2 X_j^2 X_k^2 X_l^2] &= 1, \end{aligned} \tag{52}$$

A particular case of 5D system is developed as an example. The set of moment equations for 5D system till order 8 are derived and solved for the corresponding sigma points. The points are enumerated in the procedure described and once substituted into the appropriate moment constraint equations result in exactly 11 moment constraint equations as in (53).

$$\begin{aligned} 2r_1^2 w_1 + 32 r_2^2 w_2 + 16 r_3^2 w_3 + 32 r_4^2 w_4 + 48 r_5^2 w_5 + 128 r_6^2 w_6 + 32 h^2 r_6^2 w_6 &= 1 \\ 2r_1^4 w_1 + 32 r_2^4 w_2 + 16 r_3^4 w_3 + 32 r_4^4 w_4 + 48 r_5^4 w_5 + 128 r_6^4 w_6 + 32 h^4 r_6^4 w_6 &= 3 \\ 32 r_2^4 w_2 + 4 r_3^4 w_3 + 32 r_4^4 w_4 + 24 r_5^4 w_5 + 96 r_6^4 w_6 + 64 h^2 r_6^4 w_6 &= 1 \\ 2r_1^6 w_1 + 32 r_2^6 w_2 + 16 r_3^6 w_3 + 32 r_4^6 w_4 + 48 r_5^6 w_5 + 128 r_6^6 w_6 + 32 h^6 r_6^6 w_6 &= 15 \\ 32 r_2^6 w_2 + 4 r_3^6 w_3 + 32 r_4^6 w_4 + 24 r_5^6 w_5 + 96 r_6^6 w_6 + 32 h^2 r_6^6 w_6 + 32 h^4 r_6^6 w_6 &= 3 \\ 32 r_2^6 w_2 + 32 r_4^6 w_4 + 8 r_5^6 w_5 + 64 r_6^6 w_6 + 96 h^2 r_6^6 w_6 &= 1 \\ 2r_1^8 w_1 + 32 r_2^8 w_2 + 16 r_3^8 w_3 + 32 r_4^8 w_4 + 48 r_5^8 w_5 + 128 r_6^8 w_6 + 32 h^8 r_6^8 w_6 &= 105 \\ 32 r_2^8 w_2 + 4 r_3^8 w_3 + 32 r_4^8 w_4 + 24 r_5^8 w_5 + 96 r_6^8 w_6 + 32 h^2 r_6^8 w_6 + 32 h^6 r_6^8 w_6 &= 15 \\ 32 r_2^8 w_2 + 4 r_3^8 w_3 + 32 r_4^8 w_4 + 24 r_5^8 w_5 + 96 r_6^8 w_6 + 64 h^4 r_6^8 w_6 &= 9 \\ 32 r_2^8 w_2 + 32 r_4^8 w_4 + 8 r_5^8 w_5 + 64 r_6^8 w_6 + 64 h^2 r_6^8 w_6 + 32 h^4 r_6^8 w_6 &= 3 \\ 32 r_2^8 w_2 + 32 r_4^8 w_4 + 32 r_6^8 w_6 + 128 h^2 r_6^8 w_6 &= 1 \end{aligned} \tag{53}$$

These set of 11 equations that are formed from points as described in the procedure tend to work very well till dimension 6 after which some weights become negative. Though this is done separately for each dimension it is a one-time procedure, as the results for each dimension till 6 can be evaluated and the points can be stored. As the sigma points are evaluated for an i.i.d set of normal random variables, each sigma point can undergo an affine transformation based on the mean and covariance matrix of the arbitrary Normal PDF. A detailed description of such transformation is given in chapter one of [2] and proposition 4.2 of [8].These 11 moment constraint equations can be solved in a similar manner to the previous sections. Firstly by analytically solving for the 6 weights from the last 6 equations, which is easily achieved with any symbolic computation environment and then substituting them into the first 5 equations leaves one with only 5 equations of reduced order and 6 variables r_1, r_2, r_3, r_4, r_5 and r_6. One could minimize the error in 10th moment or as our objective is to solve these set of equations one could assume a value for a variable and solve all the other 5 variables. Table (VI) is computed by assuming a value for r_5 and solving the 5 equations in 5 variables. In 2D case, there are only 8 moment constraint equations and hence the sigma points $(X^{(5)}, X^{(6)})$ in (51) are unnecessary, in effect there are only 8 variables r_1, r_2, r_3, r_4 and w_1, w_2, w_3, w_4 to solve. In the 3D case, there are 10 moments constraint equations, hence the sigma points $(X^{(5)})$ in (51) is dropped and the 10 variables to solve for are r_1, r_2, r_3, r_4, r_6 and w_1, w_2, w_3, w_4, w_6. The solutions for all dimensions $2 \leq N \leq 6$ are shown in Table (VI). The solutions have been verified till dimension 6 and can be directly used to generate the sigma points as in (51). One has to be careful as while rounding off the solution to even the 4th decimal place could result in significant error.

VI. Results

This section illustrates few examples using the methods mentioned in this paper.

A. Polynomial functions

All the methods are compared against the Gauss Hermite Product rule for multidimensional integral as it is known to exactly integrate polynomials.In general for a polynomial of degree $2m-1$ the Gauss Hermite Product rule for N^{th}-Dimensional integral requires m^N cubature points. In every case the integral that is being evaluated in the domian $(-\infty, \infty)$ is of the form

$$\mu = E[f(\mathbf{X})] = \int f(\mathbf{X}) N(\mathbf{X}, 0|P) d\mathbf{X} \tag{54}$$

where $\mathbf{X} = [X_1, X_2, ..., X_N]^T$ and $f(X)$ is a polynomial function such that $f: R^N \to R$ i.e. f is real valued. The normal weighting function has mean $\mathbf{0}$ and covariance matrix denoted as P, appropriately defined in each case. A sample function is taken from [8] as

$$f(X) = (\sqrt{1 + X^T X})^p \tag{55}$$

where the value of p is varied in each case to get various degree polynomials. The notations GH2,GH3,GH4$\cdots$ stand for the Gauss Hermite Product rule with 2,3,4$\cdots$ points in 1 dimension, thereby $2^N, 3^N, 4^N \cdots$ total points. In each case the relative error in the integral value, the corresponding covariance matrix of the weight function used and the total number of points are represented by ε_i, P_i and n_i respectively.

TABLE VI

SOLUTIONS FOR $2 \leq N \leq 6$, 8^{th} MOMENT CONSTRAINT EQUATIONS, CUT8

Variable	2D	3D	4D	5D	6D
r_1	2.068136061121187	2.255137265545780	2.201709071472343	2.314370817280745	2.449489742783178
r_2	0.8491938499087475	0.7174531274600530	0.7941993714175681	0.8390942773980102	0.8938246941221211
r_3	1.138654980847415	1.843019437068797	1.872574360506295	1.830752125326649	1.732050807568877
r_4	1.861619935018895	1.558481032725744	1.329116430064565	1.397039743064496	1.531963037906212
r_5	–	–	2	2	2
r_6	–	1.305561500466050	1.125865581272049	1.113478632736702	1.095445115010332
w_1	0.04382264267013926	0.024631993437193266	0.01811008737283111	0.010529034221546607	0.006172839506172839
w_2	0.1405096621714662	0.08151009408908164	0.032063273384586845	0.015144019639537572	0.006913443044833937
w_3	0.0009215768861610588	0.009767235524166815	0.006614353755080834	0.0052828996967816825	0.004115226337448559
w_4	0.01240953967762697	0.00577248937435553	0.003489906522946932	0.0010671298950159158	0.0002183265828666806
w_5	–	–	0.0006510416666666666	0.0006510416666666666	0.0006510416666666666
w_6	–	0.000279472936899139	0.00025218336987488566	0.00013776017592074394	0.00007849171328446504
h	3	2.74	3	3	3

TABLE VII

GH VS CUT4: % REL. ERROR AND NO. OF POINTS

method	n_1	ε_1 % error	n_2	ε_2 % error
GH2	8	52.36	1024	16.39
GH3	27	$4.89e-014$	59049	$7.23e-012$
CUT4	14	0	1044	$6.72e-012$

TABLE VIII

GH VS CUT6: % REL. ERROR AND NO. OF POINTS

method	n_3	ε_3 % error	n_4	ε_4 % error
GH3	81	12.45	19683	4.18
GH4	256	$2.31e-013$	262144	$1.37e-009$
CUT6	49	$6.49e-013$	1203	$6.26e-009$

TABLE IX

GH VS CUT8: % REL. ERROR AND NO. OF POINTS

method	n_5	ε_5 % error	n_6	ε_6 % error
GH4	1024	3.45	4096	2.49
GH5	3125	$4.73e-012$	15625	$9.29e-012$
CUT8	355	$7.52e-012$	745	$6.63e-012$

1) Polynomials of degree 5 or less: Taking $p = 4$ in (55) gives a polynomial of degree 4. Table (VII) compares the integral value evaluate by 4th moment equivalent sigma points (CUT4) and the Gauss Hermite product rule. The Gauss Hermite product rule would need atleast 3^N for a N-Dimensional system. ε_1 and ε_2 are calculated using a covariance matrices P_1 (randomly generated) and P_2 (identity matrix) respectively. The relative error is calculated using GH5. The number of points required by the CUT4 method is significantly less than that required Gauss Hermite rule while having the same order of accuracy. This difference in number of points becomes extremely contrasting for higher dimensions such as $N = 10$ thus highlighting the advantage of CUT4.

$$P_1 = \begin{bmatrix} 114.2595 & 90.1397 & 8.9751 \\ 90.1397 & 92.2504 & 29.1237 \\ 8.9751 & 29.1237 & 84.0908 \end{bmatrix}, P_2 = 100 I_{(10\times 10)}$$

2) Polynomials of degree 7 or less: Taking $p = 6$ in (55) gives a polynomial of degree 6. Table (VIII) compares the integral value evaluate by 6th moment equivalent sigma points (CUT6) and the Gauss Hermite product rule. The Gauss Hermite product rule would need atleast 4^N for a N-Dimensional system. ε_3 and ε_4 are calculated using a covariance matrices P_3 and P_4 respectively. The relative error is calculated using GH7. Hence with the sheer reduction in number of points used by CUT6 one could capture the integral with same order of accuracy as that of Gauss Hermite . Where $P_3 = 100 I_{(4\times 4)}$ and $P_4 = 100 I_{(9\times 9)}$

3) Polynomials of degree 9 or less: Taking $p = 8$ in (55) gives a polynomial of order 8. Table (IX) compares the integral value evaluate by 8th moment equivalent sigma points (CUT8) and the Gauss Hermite product rule. The Gauss Hermite product rule would need atleast 5^N. ε_5 and ε_6 are calculated using a covariance matrices P_5 and P_6 respectively. The relative error is calculated using GH9. CUT8 is able to match GH5 to same order of accuracy with fewer number of points. Where $P_5 = 100 I_{(5\times 5)}$, $P_6 = 100 I_{(6\times 6)}$

B. Non-polynomial functions

1) Polar to Cartesian coordinates: The example on polar to cartesian coordinates transformation from [10] is one way to graphically illustrulate the advantage in capturing higher order moments. The example is re-simulated in Fig(2), using the parameters described in [10] as $\mu_r = 1$, $\mu_\theta = 90$, $\sigma_r = 0.02^2$, $\sigma_\theta = 15^2$. The mean and 1-σ contours are plotted for each method. All the methods,except CKF, seem to do equally well compared to the Monte Carlo simulations using 3×10^6 samples in 2D. CKF predicts the mean correctly but covariance is slightly under estimated due to error in all the 4th moments. The Unscented Transform can do better than CKF as having $\kappa = 1$ captures one 4th order moment. For the same example when the parameters are changed as $\mu_r = 50$, $\mu_\theta = 0$, $\sigma_r = 0.02^2$, $\sigma_\theta = 30^2$, though not a realistic choice, the higher moment methods tend to do better as seen in Fig(3). The CKF slightly under estimates the covariance while the UT and GH3 slightly over estimates the covariance. CUT4 in 2D can additionally capture one of the 6th order moment and hence does better than GH3.

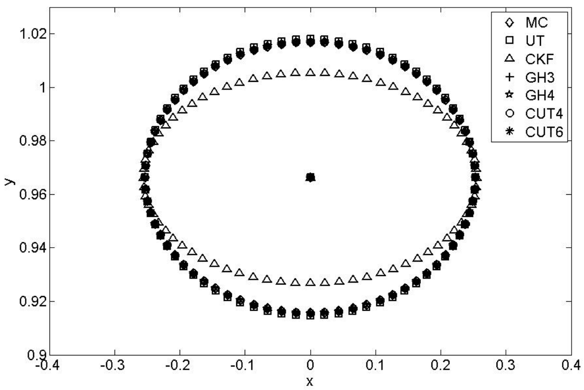


Fig. 2. Polar to Cartesian, $\mu_r = 1, \mu_\theta = 90, \sigma_r = 0.02^2, \sigma_\theta = 15^2$

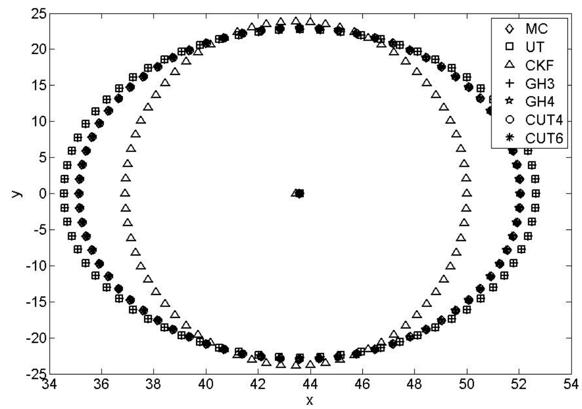


Fig. 3. Polar to Cartesian, $\mu_r = 50, \mu_\theta = 0, \sigma_r = 0.02^2, \sigma_\theta = 30^2$

2) $p = -3$: Having p in (55) to be negative makes the function to behave as a delta function and hence difficult to integrate using quadrature points. This may be because most of the quadrature points fall in the region where the function value is near to zero and hence does not help in the integral value. The convergence of the integral is highly effected by increasing the covariance of the normal PDF. This is characterized by the oscillatory type behaviour of the integral. Even taking a large number of points does not always guarantee the convergence. But by decreasing the magnitude of covariance, most of the quadrature points tend to fall within the region where the function value is significant. Under this condition one can consider the integral value to converge. This example has been shown just to motivate the fact that 'as one can capture more moments the integral value tends to converge, this statement is in agreement with equation (2). The covariance is taken as $P = 0.1I$ where I is the identity matrix of appropriate dimension. The covariance is intentionally scaled down by 0.1 to make the integral converge. In Fig(4), the Gauss Hermite product rule is used to compare the convergence of the integral in 2D,3D,4D,5D and 6D(belonging to each curve in Fig(4)) as the number of points are increased. The points are increased from 3 to 7 in single dimension resulting in GH3,GH4,GH5,GH6 and GH7 for each curve. In Fig(5), the CUT4,CUT6 and CUT8 is used in 2D,3D,4D,5D and 6D to compare the performance of the methods in evaluating the same integral. From Fig(5), it can be seen that CUT8 can achieve a relative error just below 0.5% in all dimensions considered. To achieve the same accuracy of 0.5%, the Gauss Hermite Product rule would need atleast 4 points(GH4) in a single dimension, thus 4^N in total. In Fig(6), the number of points required by GH4 and CUT8 are compared for each dimension.

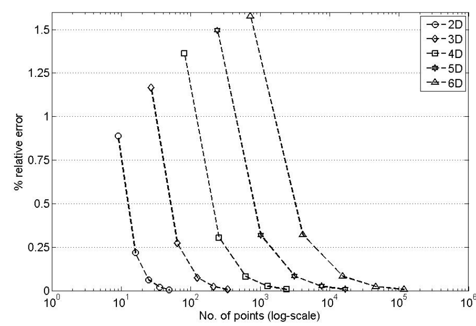


Fig. 4. Rel. error for Gauss Hermite Product rule in each Dim.

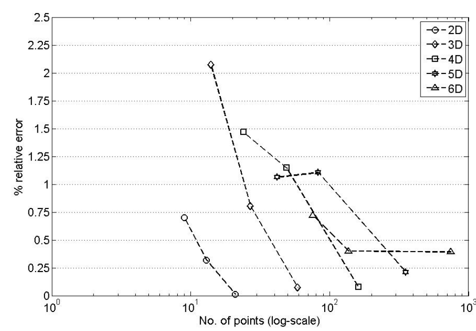


Fig. 5. Rel. error for CUT4,CUT6 and CUT8 in each Dim.

VII. CONCLUSIONS

It has been customary to use the Guass-Hermite product rule to evaluate multi-dimensional integrals involving the normal probability density function. But the Gauss Hermite product rule not only involves a lot of computational cost but even computational time. Especially for applications in online or real-time filtering, one would prefer a cubature rule with as minimal points as possible without the compromise in accuracy. This has been the prime focus of this paper. Thus we have been able to develop a fully symmetric set

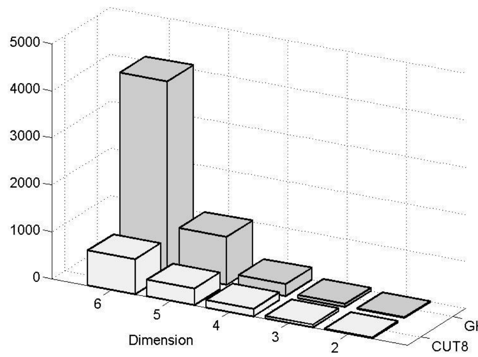


Fig. 6. GH vs CUT8: Points required to achieve 0.5% Rel. error

of sigma points that are 4th order, 6th order and 8th order equivalent. Developing higher order cubature rules for higher dimensions with minimal points still remains a challenge. When integrating polynomial functions, each sigma point set can be considered a direct replacement for the equivalent Gauss Hermite product rule of same order.

VIII. ACKNOWLEDGMENTS

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