Project Numerical Analysis Report - Bending of Bernoulli Beams

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1 Introduction

The aim of this project is to create and implement an algorithm which solves the elastic Bernoulli beam problem using Finite Element Method. The main task is divided into three sub-problems, that is, the simulation of:

- a static beam
- a dynamic beam
- and a structure of beams

The solutions of these problems were implemented consecutively, with each problem building on the prior solutions.

In the static beam problem the goal is to simulate the bending of an elastic Bernoulli beam given a load distribution and a set of boundary conditions (supports). In the dynamic beam problem the load and boundary conditions are time dependent which in turn causes for a time dependent curvature of the beam. The structure of beams problem is a composition of several dynamic beams conected as a structure.

There are three deliverables in this project. The first is this report of the project. The second is the MatLab code, the algorithm that solves the Bernoulli Beam problem. And finally there is also a presentation where much of the content of this report will be presented.

2 Euler-Bernoulli Beams

Euler-Bernoulli beam theory is used to calculate deflections of elastic beams subject to a load. As such, it is a simplification of the linear theory of elasticity. The theory is valid for relatively small deflections and relatively thin beams. For larger deflections and thicker beams the error in computation may render the result invalid. Furthermore the beams are assumed to be only under lateral load. There exists an extended version of this theory that takes axial forces and the extension/compression of the beam into consideration.

It is also worthwhile to point out that in most of the literature the elastic beam is referred to as an "Euler-Bernoulli beam". However it is sometimes referred to simply as a "Bernoulli Beam". In this report we may refer to an Euler-Bernoulli beam simply as a Bernoulli beam.

2.1 Static Bending Equation

The equation for the deflection of a laterally loaded beam has the following form:

$$(EIw'')''(x) = q(x), \quad x \in [0, L], \quad (q \in V, \ (EIw'') \in V) \tag{1}$$

where: $w(x)$ is the displacement of the beam compared to the neutral axis at x, $E = E(x)$ is the Young's modulus (a measure of stiffness of elastic isotropic material), $q(x)$ is the load at x and $I = I(x)$ is the moment of inertia of the cross section of the beam, i.e.:

$$I(x) = \int_{\text{cross section at x}} z^2 \, dy \, dz \tag{2}$$

Additionally, the bending moment at the point x is given by following formula:

$$M^x(w) = EIw''(x) \tag{3}$$

and the shear force at x by:

$$Q^x(w) = -(EIw'')'(x) \tag{4}$$

The general solution of the static bending equation is as follows:

$$w = \left(\frac{q^{[2]} + p_1}{EI} \right)^{[2]} + p_2 \tag{5}$$

where p_1, p_2 are arbitrary polynomials of degree ≤ 1. In order to specify a particular solution four boundary conditions are necessary (there are two free parameters in p_1 and two in p_2). (For different Boundary Conditions see Section 3)

2.2 Dynamic Bending Equation

In the dynamic case the deflection of the beam as well as all forces and boundary conditions depend on time. When we say the boundary conditions depend on time we mean the values in the boundary condition may change but not the type of condition. The bending equation for the dynamic case has an additional term corresponding to inertia, which assures that the beam is not only moved by forces currently acting on it but also accounts for momentum. The dynamic bending equation for Bernoulli beam is given by formula:

$$\mu\,\ddot{w} + (EIw'')'' = q, \tag{6}$$

where $\mu = \mu(x)$ is the mass density and the $\ddot{w}$ denotes second derivative of w with respect to time t.

2.3 Axially and Laterally Loaded Beams

In the structural problems we have to consider both axial and lateral loads. The beams in the structure can be placed at different angles but the general analysis of this case can be simplified to axially and laterally loaded beams. The bending equation is of the same form as for a dynamic single beam:

$$\mu\,\ddot{w} + (EIw'')'' = q \tag{7}$$

The extension/compression equation is of the form:

$$\mu\,\ddot{v} + (EAv')' = q \tag{8}$$

where $A = A(x)$ is the area of the beam, $w(x)$ is the normal component of the deflection at x and $v(x)$ is its axial deflection. Those two equations create a system which describes the problem of laterally and axially loaded beams:

$$\begin{cases} \mu\,\ddot{w} + (EIw'')'' = q \\ \mu\,\ddot{v} - (EAv')' = q \end{cases} \tag{9}$$

In this case there are 6 free parameters for the solution. So the boundary conditions must contain two additional values, as compared to the lateral case, corresponding to the constraints in axial direction.

3 Various Boundary Conditions

The boundary conditions in the Bernoulli beam problem symbolize different types of physical supports that can be applied to the ends of the beam. Obviously they have significant effect on the solution of the problem. Each end of the beam must have have the right combination of boundary conditions. The following are the four characteristic values at an end:

- height
- slope
- shear force
- moment of force

For the boundary conditions at each end two of those values must be given and two left unknown. More specifically, of the two pairs: height/shear force and slope/moment of force, one value of each pair must be given and the other unknown. What follows are some common boundary conditions.

3.1 Clamped

A common support for a beam, is for an end to be ”clamped”. The clamped end of the beam is fixed in a way that does not allow translation or rotation. The height and the slope of a beam at this point are given and are fixed. The shear force and moment of force are the reactions which are unknown. The role of reaction forces and moments is to prevent the translation and rotation. An example of this type of support is a beam bolted to the wall.

3.2 Pinned

Another common type of support is for an end of a beam to be ”pinned”. Here the end of the beam is fixed in a way that prevents only translation but allows the beam to rotate freely. In this type support height and arbitrary moment of force are given but reaction shear force and slope are unknown. An example of this type of support is the attached end of a pendulum.

3.3 Free

It is possible to have no support at one end of a beam. In this case height and slope are unknown, while arbitrary shear force and moment of force need to be given.

3.4 Others

Aside from the above mentioned boundary conditions one might have in the Bernoulli beam problem, there exist many others. In the scope of the current project, and hence also this report, we consider only the ones mentioned here.

3.5 Different combinations of boundary conditions

A single beam must be supported on each end in one of the ways listed above. It should be pointed out here that one should combine these different types of supports wisely. While it would make sense to combine two clamped supports, two pinned supports, or even clamped and free supports, it does not make sense to combine pinned and free. Although such a case is hypothetically possible, the problem then becomes a pendulum problem and not a Bernoulli beam problem. An even worse combination would be to combine a free support with a free support. Again, although such a case is hypothetically possible, for example on the international space station, such a problem is no longer a Bernoulli beam problem, does not have a unique solution, and is therefore not well posed. These last two combinations are also not solvable numerically as the extended matrix (explained later) is singular.

3.6 Boundary conditions for structural problems

In the single beam problems it is assumed that no axial forces are acting on the beam and the length of the beam is not changing. For structural problems axial forces have to be taken into consideration.

A second boundary condition that must be considered as well, is the type of connections between beams in the structure. That is, beams can either be connected such that the angle between them is fixed, or a ”stiff” connection, or they are connected such that the joint allows the angle to change.

4 Finite Elements Method

As seen in section (2) the Bernoulli Beam equation is a differential equation and will thus require a particular numerical solver. The Finite Element Method (FEM) is today a widely used method to solve exactly such problems. This is also the method that we will use to solve the Bernoulli beam problem. The basic approach of FEM is to approximate the solution in a finite function space, which is defined via considering the domain of the problem as many connected sub domains called Finite Elements.

In using FEM one rewrites the problem in its weak form and then solves the "weak" problem, for example, via the Galerkin Method. The weak formulation is found in the following way. Consider a boundary value problem

$$\begin{aligned} -u''(x) &= f(x) \text{ in } (a,b) \\ u(a) &= u(b) = 0 \end{aligned}$$

Then for any smooth function v the following is true:

$$\begin{aligned} -\int_a^b u''(x)dx &= \int_a^b f(x)dx \\ \Leftrightarrow -\int_a^b u''(x)v(x)dx &= \int_a^b f(x)v(x)dx \end{aligned}$$

Apply Integration by Parts to the term on the right-hand-side and we get the following

$$\int_a^b u'(x)v'(x)dx = \int_a^b f(x)v(x)dx + u'(x)v(x)|_a^b \tag{10}$$

4.1 Galerkin Method

The Galerkin method solves the weak formulation (10) in a specific function space. Given an n-dimensional function space V_h with a set of basis functions $\{\phi_i(x)|i = 1,\ldots,n\}$, we seek to construct an approximate solution $U(x)$ in the form

$$U(x) = \sum_{i=1}^{n} w_i\phi_i(x)$$

Here $w_i, i = 1,...,n$ are weighting factors to be determined such that for each basis function $\phi_j(x), j = 1,\ldots,n$, equation (10) is satisfied.

4.2 Basis Functions

In this project, two types of basis functions are used. To solve the bending equation, the function space is a space of piecewise cubic polynomials as described below.

First of all, the interval [0, L] is evenly discretized into n grid points $0 = x_1, x_2, \ldots, x_n = L$. Basis functions $\phi_1, \phi_2, \ldots, \phi_{2n-1}, \phi_{2n} \in V_h$ are defined as follows. For $i = 2,\ldots,n-1$,

$$\phi_{2i-1}(x) = \begin{cases} \bar{\phi}_3\left(\frac{x-x_{i-1}}{h}\right) & x \in [x_{i-1}, x_i] \\ \bar{\phi}_1\left(\frac{x-x_i}{h}\right) & x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise.} \end{cases} \qquad \phi_{2i}(x) = \begin{cases} h\,\bar{\phi}_4\left(\frac{x-x_{i-1}}{h}\right) & x \in [x_{i-1}, x_i] \\ h\,\bar{\phi}_2\left(\frac{x-x_i}{h}\right) & x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise.} \end{cases}$$

Additionally:

$$\phi_1(x) = \begin{cases} \bar{\phi}_1\left(\frac{x}{h}\right) & x \in [0,h] \\ 0 & \text{otherwise.} \end{cases} \qquad \phi_2(x) = \begin{cases} h\,\bar{\phi}_2\left(\frac{x}{h}\right) & x \in [0,h] \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\phi_{2n-1}(x) = \begin{cases} \bar{\phi}_3\left(\frac{x-x_{n-1}}{h}\right) & x \in [x_{n-1}, L] \\ 0 & \text{otherwise.} \end{cases} \qquad \phi_{2n}(x) = \begin{cases} h\,\bar{\phi}_4\left(\frac{x-x_{n-1}}{h}\right) & x \in [x_{n-1}, L] \\ 0 & \text{otherwise,} \end{cases}$$

where

$$\begin{aligned} \bar{\phi}_1(\xi) &= 1 - 3\xi^2 + 2\xi^3, & \bar{\phi}_3(\xi) &= 3\xi^2 - 2\xi^3, \\ \bar{\phi}_2(\xi) &= \xi(\xi - 1)^2, & \bar{\phi}_4(\xi) &= \xi^2(\xi - 1). \end{aligned}$$

Each function $u \in V_h$ can be uniquely written as

$$u = \sum_{k=1}^{2n} u_k \phi_k = \sum_{i=1}^{n} (u_{2i-1}\, \phi_{2i-1} + u_{2i}\, \phi_{2i})$$

with $u_{2i-1} := u(x_i)$ and $u_{2i} := u'(x_i)$.

As to the extension/compression equation (axial loads), a space of piecewise linear functions is used. The basis functions $\psi_1, \ldots, \psi_n$ are defined as follows. For $i = 2, \ldots, n-1$,

$$\psi_i(x) = \begin{cases} \frac{x - x_{i-1}}{h}, & x \in [x_{i-1}, x_i] \\ \frac{x_{i+1} - x}{h} & x \in [x_i, x_{i+1}] \\ 0 & \text{otherwise.} \end{cases}$$

Additionally:

$$\psi_1(x) = \begin{cases} \frac{h - x}{h} & x \in [0, h] \\ 0 & \text{otherwise.} \end{cases}$$

and

$$\psi_n(x) = \begin{cases} \frac{x - x_{n-1}}{h} & x \in [x_{n-1}, L] \\ 0 & \text{otherwise,} \end{cases}$$

5 Implementation

Here then is how it all comes together into a working solution. One starts by obtaining the weak formulations. Next using the Galerkin Ansatz one can formulate the mass and stiffness matrices. After that the boundary conditions are applied to get a uniquely defined system. And finally, in solving the dynamic case the Newark method is used.

5.1 Weak formulations

In this section the problem is reformulated such that we can find a solution subject to a selected set of test functions, or test vectors. In the Bernoulli Beam problem the weak solution is a very good approximation of the actual solution.

5.1.1 Static

The bending of a static beam is described by Equation 1. Let $w \in V$ and and $Q_0, Q_L, M_0, M_L \in \mathbb{R}$. Q_0' and Q_L are the shear forces on the boundaries, and M_0 and M_L are the bending moments. Then the following statements are equivalent.

(a) w satisfies the bending equation (1) and the boundary conditions

$$Q^0(w) = Q_0, \quad Q^L(w) = Q_L, \quad M^0(w) = M_0, \quad M^L(w) = M_L. \tag{11}$$

(b) For all $\psi \in V$,

$$\int_0^L EIw''\psi'' = \int_0^L q\psi + b(\psi), \tag{12}$$

where

$$b(\psi) = Q_L\psi(L) - Q_0\psi(0) + M_L\psi'(L) - M_0\psi'(0).$$

For the proof, see Chapter 4 of the handout *Bending of Bernoulli beams and FEM* by Michael Karow.

5.1.2 Dynamic

The same procedure can by applied to the time dependent bending equation (6) seen in chapter 2.2. This yields the following weak formulation:

$$\int_0^L \ddot{w}\psi + \int_0^L EIw''\psi'' = \int_0^L q(\cdot,t)\psi + b(\psi,t), \tag{13}$$

where

$$b(\psi,t) = Q_L(t)\psi(L) - Q_0(t)\psi(0) + M_L(t)\psi'(L) - M_0(t)\psi'(0)$$

and $\psi \in V$ ($\psi = \psi(x)$ does not depend on time).

5.2 Stiffness and Mass Matrices

Now the mass and stiffness matrices are derived by using the Galerkin method as described in Section 4.1.

5.2.1 Stiffness Matrix

Insert the Ansatz in the weak formulation (12) and let $\psi = \phi_j, j = 1,\dots,N$. This yields the equations

$$\int_0^L EIw_h''\phi_j'' = \int_0^L q\phi_j + b(\phi_j), \qquad j = 1,\dots,N. \tag{14}$$

Observe that

$$\int_0^L EIw_h''\phi_j'' = \sum_{k=1}^N \left(\int_0^L EI\phi_k''\phi_j''\right) w_k.$$

Hence, (14) is equivalent to

$$\sum_{k=1}^N \left(\int_0^L EI\phi_k''\phi_j''\right) w_k = \int_0^L q\phi_j \;+ Q_L\phi_j(L) - Q_0\phi_j(0) + M_L\phi_j'(L) - M_0\phi_j'(0), \qquad j = 1,\dots,N.$$

These are N linear equations for the N unknowns $w_1,\dots,w_N$. The equations can be written in matrix-vector-form as follows.

$$\boldsymbol{S}\underline{\boldsymbol{w}} = \underline{\boldsymbol{q}} + Q_L\,\underline{\boldsymbol{e}}_L - Q_0\,\underline{\boldsymbol{e}}_0 + M_L\,\underline{\boldsymbol{d}}_L - M_0\,\underline{\boldsymbol{d}}_0, \tag{15}$$

where

$$\underline{\boldsymbol{w}} = \begin{bmatrix} w_1 \\ \vdots \\ w_N \end{bmatrix}, \qquad \underline{\boldsymbol{q}} = \begin{bmatrix} \int q\phi_1 \\ \vdots \\ \int q\phi_N \end{bmatrix}, \qquad \underline{\boldsymbol{e}}_x = \begin{bmatrix} \phi_1(x) \\ \vdots \\ \phi_N(x) \end{bmatrix}, \qquad \underline{\boldsymbol{d}}_x = \begin{bmatrix} \phi_1'(x) \\ \vdots \\ \phi_N'(x) \end{bmatrix}, \tag{16}$$

and

$$\boldsymbol{S} = \begin{bmatrix} s_{11} & \dots & s_{1N} \\ \vdots & & \vdots \\ s_{N1} & \dots & s_{NN} \end{bmatrix}, \qquad s_{jk} = \int_0^L EI\phi_k''\phi_j''.$$

The symmetric matrix $S \in \mathbb{R}^{N\times N}$ is called the stiffness matrix. Using the basis functions proposed in chapter 4.1 yields the following local stiffness matrix:

$$\boldsymbol{S} = \frac{1}{h}\begin{bmatrix} 12 & 6h & -12 & 6h \\ 6h & 4h^2 & -6h & 2h^2 \\ -12 & -6h & 12 & -6h \\ 6h & 2h^2 & -6h & 4h^2 \end{bmatrix}$$

5.2.2 Mass Matrix

Inserting in (13) for w the (time depentent) Galerkin Ansatz

$$w_h(x,t)\sum_{k=1}^{N} w_k(t)\phi_k(x)$$

and for ψ the basis functions $\phi_j, \quad j=1,\dots,N$ we obtain the ordinary differential equation of second order

$$\boldsymbol{M}\ddot{\underline{\boldsymbol{w}}}(t)+\boldsymbol{S}\underline{\boldsymbol{w}}(t)=\underline{\boldsymbol{q}}(t)+Q_L(t)\,\underline{\boldsymbol{e}}_L-Q_0(t)\,\underline{\boldsymbol{e}}_0+M_L(t)\,\underline{\boldsymbol{d}}_L-M_0(t)\,\underline{\boldsymbol{d}}_0,$$

where $\underline{\boldsymbol{w}}(t), \underline{\boldsymbol{e}}_x, \underline{\boldsymbol{d}}_x$ and $\underline{\boldsymbol{q}}$ are defined as in (16) and

$$\boldsymbol{M}=\begin{bmatrix} m_{11} & \dots & m_{1N} \\ \vdots & & \vdots \\ m_{N1} & \dots & m_{NN}\end{bmatrix}, \qquad m_{jk}=\int_0^L \mu\phi_k\phi_j$$

is the mass matrix. Using the basis functions proposed in chapter 4.2 yields the following local mass matrix:

$$\boldsymbol{M}=\begin{bmatrix} \frac{13}{35} & \frac{11}{210}h & \frac{9}{70} & \frac{-13}{420}h \\ \frac{11}{210}h & \frac{1}{105}h^2 & \frac{13}{420}h & \frac{-1}{140}h^2 \\ \frac{9}{70} & \frac{13}{420}h & \frac{13}{35} & \frac{-11}{210}h \\ \frac{-13}{420}h & \frac{-1}{140}h^2 & \frac{-11}{210}h & \frac{1}{105}h^2 \end{bmatrix}$$

5.3 Implementation of Boundary Conditions

The only thing left to do now in order to get a solvable system is to incorporate the boundary conditions as discussed in section 3

5.3.1 Static

In order to get a unique solution of the beam equation one has to add at least two Dirichlet boundary conditions. For a left sided clamped beam (cantilever) these conditions are $w(0)=a$, $w'(L)=b$, where $a,b\in\mathbb{R}$ are given. The Galerkin approximation should satisfy these conditions as well, that is.

$$a_x=w_h(x)=\underline{\boldsymbol{e}}_x^\top\underline{\boldsymbol{w}}, \qquad b_x=w_h'(x)=\underline{\boldsymbol{d}}_x^\top\underline{\boldsymbol{w}}$$

These equations combined with (15) are equivalent to

$$\begin{bmatrix} \boldsymbol{S} & \underline{\boldsymbol{e}}_0 & \underline{\boldsymbol{d}}_0 \\ \underline{\boldsymbol{e}}_0^\top & 0 & 0 \\ \underline{\boldsymbol{d}}_0^\top & 0 & 0 \end{bmatrix}\begin{bmatrix}\underline{\boldsymbol{w}} \\ Q_0 \\ M_0\end{bmatrix}=\begin{bmatrix}\underline{\boldsymbol{q}}+Q_L\,\underline{\boldsymbol{e}}_L+M_L\,\underline{\boldsymbol{d}}_L \\ a_0 \\ b_0\end{bmatrix}. \tag{17}$$

If the quantities on the right hand side of this linear equation are given then $\underline{\boldsymbol{w}}$, Q_0 and M_0 can be computed.

If the beam is supported (but not not clamped) at both ends, then the corresponding equation is

$$\begin{bmatrix} \boldsymbol{S} & \underline{\boldsymbol{e}}_0 & -\underline{\boldsymbol{e}}_L \\ \underline{\boldsymbol{e}}_0^\top & 0 & 0 \\ -\underline{\boldsymbol{e}}_L^\top & 0 & 0 \end{bmatrix}\begin{bmatrix}\underline{\boldsymbol{w}} \\ Q_0 \\ Q_L\end{bmatrix}=\begin{bmatrix}\underline{\boldsymbol{q}}-M_0\,\underline{\boldsymbol{d}}_0+M_L\,\underline{\boldsymbol{d}}_L \\ a_0 \\ -a_L\end{bmatrix} \tag{18}$$

If the beam is supported on one end and clamped at the other end, then the corresponding equation is

$$\begin{bmatrix} \boldsymbol{S} & \underline{\boldsymbol{e}}_0 & -\underline{\boldsymbol{e}}_L & \underline{\boldsymbol{d}}_0 \\ \underline{\boldsymbol{e}}_0^\top & 0 & 0 & 0 \\ \underline{\boldsymbol{e}}_L^\top & 0 & 0 & 0 \\ \underline{\boldsymbol{d}}_0^\top & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{\boldsymbol{w}} \\ -Q_0 \\ -Q_L \\ -M_0 \end{bmatrix} = \begin{bmatrix} \underline{\boldsymbol{q}} + M_L\, \underline{\boldsymbol{d}}_L \\ a_0 \\ a_L \\ b_0 \end{bmatrix} \tag{19}$$

The equations (17),(18) and (19) are of the form

$$\underbrace{\begin{bmatrix} \boldsymbol{S} & \boldsymbol{C} \\ \boldsymbol{C}^\top & 0 \end{bmatrix}}_{\boldsymbol{S_e}} \begin{bmatrix} \underline{\boldsymbol{x}} \\ \underline{\boldsymbol{\mu}} \end{bmatrix} = \begin{bmatrix} \underline{\boldsymbol{f}} \\ \underline{\boldsymbol{a}} \end{bmatrix} \tag{20}$$

where $\boldsymbol{S}$ is symmetric and positive semidefinite. The matrix $\boldsymbol{S_e}$ is called the extended stiffness matrix. The matrix $\boldsymbol{C}$ must be such that $\boldsymbol{S_e}$ is nonsingular. Otherwise (20) does not have a unique solution.

5.3.2 Dynamic

The formulation of the linear system for the dynamic problem is very similar to that of the static problem. The things that change are of course that the curve now depends on time and that momentum must be taken into account. The new term you see below is what changes practically in the linear system to account for momentum. One more thing to point out here is that the system now has some differential equations and some algebraic equations which makes this system what is called a system of Differential Algebraic Equations or a (DAE).

$$\begin{bmatrix} \boldsymbol{M} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \ddot{\underline{\boldsymbol{w}}} \\ \ddot{Q}_0 \\ \ddot{M}_0 \end{bmatrix} + \begin{bmatrix} \boldsymbol{S} & \underline{\boldsymbol{e}}_0 & \underline{\boldsymbol{d}}_0 \\ \underline{\boldsymbol{e}}_0^\top & 0 & 0 \\ \underline{\boldsymbol{d}}_0^\top & 0 & 0 \end{bmatrix} \begin{bmatrix} \underline{\boldsymbol{w}} \\ Q_0 \\ M_0 \end{bmatrix} = \begin{bmatrix} \underline{\boldsymbol{q}} + Q_L\, \underline{\boldsymbol{e}}_L + M_L\, \underline{\boldsymbol{d}}_L \\ a \\ b \end{bmatrix}.$$

This DAE is of the form

$$\underbrace{\begin{bmatrix} \boldsymbol{M} & 0 \\ 0 & 0 \end{bmatrix}}_{\boldsymbol{M_e}} \begin{bmatrix} \ddot{\underline{\boldsymbol{x}}} \\ \ddot{\underline{\boldsymbol{\mu}}} \end{bmatrix} + \underbrace{\begin{bmatrix} \boldsymbol{S} & \boldsymbol{C} \\ \boldsymbol{C}^\top & 0 \end{bmatrix}}_{\boldsymbol{S_e}} \begin{bmatrix} \underline{\boldsymbol{x}} \\ \underline{\boldsymbol{\mu}} \end{bmatrix} = \begin{bmatrix} \underline{\boldsymbol{f}} \\ \underline{\boldsymbol{a}} \end{bmatrix} \tag{21}$$

with extended mass matrix $\boldsymbol{M_e}$ and extended stiffness matrix $\boldsymbol{S_e}$ (Note: $\underline{\boldsymbol{\mu}}$ is the vector of constraint forces and moments and not a vector of mass densities.)

Implementing various boundary conditions in the dynamic case is the same as in the static case by updating the extended stiffness matrix $\boldsymbol{S_e}$

5.4 Newmark Method

Although the formulation of the linear system for the dynamic problem is similar to that of the static problem, finding the time dependent solution is not quite as straight forward. There is however a well established method to solve this problem called the Newmark method. This is the method explained here.

Let $x : [t_1, t_e] \to \mathbb{R}^n$ be a solution of the equation

$$f(\ddot{u}(t), \dot{u}(t), u(t), t) = 0.$$

Furthermore, let $t_1 < t_2 < \ldots < t_{n-1} < t_n = t_e$ be a partition of the interval $[t_1, t_e]$. We want to generate approximations u_j, $\dot{u}_j$, $\ddot{u}_j$ for $u(t_j)$, $\dot{u}(t_j)$, $\ddot{u}(t_j)$ which satisfy

$$u_1 = u(t_1), \quad \dot{u}_1 = \dot{u}(t_1), \quad \ddot{u}_1 = \ddot{u}(t_1). \tag{22}$$

and

$$f(\ddot{u}_{j+1}, \dot{u}_{j+1}, u_{j+1}, t_{j+1}) = 0, \qquad j = 1, \ldots, n-1 \tag{23}$$

For the derivation of an algorithm we consider Taylor-expansions with remainder:

$$u(t_{j+1}) = u(t_j) + \dot{u}(t_j)\, h_j + \frac{\ddot{u}(\tau_1)}{2}\, h_j^2 \qquad \tau_1 \in [t_j, t_{j+1}], \tag{24}$$

$$\dot{u}(t_{j+1}) = \dot{u}(t_j) + \ddot{u}(\tau_2)\, h_j, \qquad \tau_2 \in [t_j, t_{j+1}], \tag{25}$$

where

$$h_j = t_{j+1} - t_j$$

is the stepsize. In these formulation we replace $u(t_{j+1}), u(t_j), \dot{u}(t_j), \dot{u}(t_{j+1})$ by their approximations. The unknown quantities $\ddot{u}(\tau_1)$ und $\ddot{u}(\tau_2)$ are replaced by weighted means of $\ddot{u}_j$ and $\ddot{u}_{j+1}$. This yields the formulas

$$u_{j+1} = u_j + \dot{u}_j\, h_j + ((\frac{1}{2} - \beta)\, \ddot{u}_j + \beta\, \ddot{u}_{j+1})\, h_j^2, \tag{26}$$

$$\dot{u}_{j+1} = \dot{u}_j + ((1-\gamma)\, \ddot{u}_j + \gamma\, \ddot{u}_{j+1})\, h_j. \tag{27}$$

Here, $\beta \in [0, \frac{1}{2}]$, $\gamma \in [0, 1]$ are fixed values. The Equations (22), (23), (26) und (27) define the Newmark-method. For practical reasons we collect the j-dependent parts of the right hand sides of (26) and (27):

$$u_j^* = u_j + \dot{u}_j\, h_j + (\frac{1}{2} - \beta)\, \ddot{u}_j\, h_j^2, \tag{28}$$

$$\dot{u}_j^* = \dot{u}_j + (1-\gamma)\, \ddot{u}_j\, h_j. \tag{29}$$

With these quantities the equation (26) and (27) can be written as

$$u_{j+1} = u_j^* + \beta\, \ddot{u}_{j+1}\, h_j^2, \qquad \dot{u}_{j+1} = \dot{u}_j^* + \gamma\, \ddot{u}_{j+1}\, h_j. \tag{30}$$

This combined with (23) yields

$$f(\, \ddot{u}_{j+1},\ \dot{u}_j^* + \gamma\ \ddot{u}_{j+1}\, h_j,\ u_j^* + \beta\, \ddot{u}_{j+1}\, h_j^2,\ t_{j+1}\,) = 0. \tag{31}$$

Now, the Newmark algorithm is the following:

- Let $u_1 = u(t_1), \quad \dot{u}_1 = \dot{u}(t_1), \quad \ddot{u}_1 = \ddot{u}(t_1)$.
- For $j = 1$ to $n-1$:
  (a) Compute u_j^* and $\dot{u}_j^*$ using (28) and (29).
  (b) Compute the solution $\ddot{u}_{j+1}$ of (31).
  (c) Compute u_{j+1} and $\dot{u}_{j+1}$ using (30).

5.5 A Structure of Beams

In a structure of beams, the displacement of each beam involves two directions, described by Equation (9).

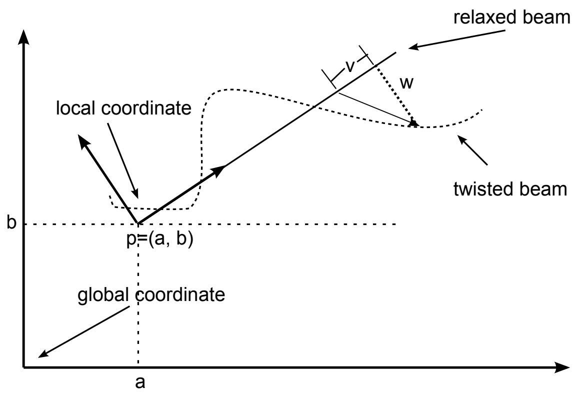

Figure 1: The relation between local and global coordinate systems

5.5.1 Local and Global Coordinate Systems

The coordinates $v(x,t)$ and $w(x,t)$ used in Equation (9) are local coordinates. The corresponding global coordinates $\beta_g(x,t)$ are achieved by rotation and translation:

$$\boldsymbol{\beta(x,t)} = \begin{bmatrix} x+v(x,t) \\ w(x,t) \end{bmatrix} = \begin{bmatrix} x \\ 0 \end{bmatrix} + \begin{bmatrix} v(x,t) \\ w(x,t) \end{bmatrix}$$

$$\boldsymbol{\beta_g(x,t)} = p + \boldsymbol{D_\phi}\beta(x,t) = \begin{bmatrix} a+\cos(\phi)\,(x+v(x,t)) - \sin(\phi)\,w(x,t) \\ b+\sin(\phi)\,(x+v(x,t)) + \cos(\phi)\,w(x,t) \end{bmatrix}$$

with $\boldsymbol{D_\phi}$ being the rotation matrix

$$\boldsymbol{D_\phi} = \begin{bmatrix} \cos(\phi) & -\sin(\phi) \\ \sin(\phi) & \cos(\phi) \end{bmatrix}$$

5.5.2 Stiffness and Mass Matrices

The weak formulation for the bending equation is shown in the last section, thus so are the corresponding stiff and mass matrices. Following the same step, we can obtain the weak formulation of the extension/compression equation

$$\int_0^L \mu\ddot{v}(x,t)\psi(x)dx + \int_0^L EAv'(x,t)\psi'(x)dx = \int_0^L f(x,t)\psi(x)dx + EAv'(L,t)\psi(L) - EAv'(0,t)\psi(0)$$

Let $b_L(t) = EAv'(L,t)$ and $b_0(t) = EAv'(0,t)$, and solve the weak formulation in the function space mentioned in Section 4.2

$$\sum_{i=1}^{n}(\int_0^L \mu\psi_i\psi_j)\ddot{v}_i + \sum_{i=1}^{n}(\int_0^L EA\psi_i'\psi_j')v_i = \int_0^L f\psi_j + b_L\psi(L) - b_0\psi(0)$$

with $v_1,\ldots,v_n$ to be determined and $v = \sum_{i=1}^{n} v_i\psi_i$. The matrix form of the above equation is as follows

$$M\underline{\boldsymbol{\ddot{v}}} + S\underline{\boldsymbol{v}} = \underline{\boldsymbol{f}} + b_L\underline{\boldsymbol{e_L}} - b_0\underline{\boldsymbol{e_0}}$$

where $\boldsymbol{S}$ and $\boldsymbol{M}$ are the stiffness matrix and mass matrix respectively, and

$$S_ij = \int_0^L EA\phi_i'\phi_j', \qquad M_ij = \int_0^L \mu\psi_i\psi_j$$

and

$$\underline{\boldsymbol{v}} = \begin{bmatrix} v_1 \\ \vdots \\ v_N \end{bmatrix}, \qquad \underline{\boldsymbol{f}} = \begin{bmatrix} \int f\psi_1 \\ \vdots \\ \int f\psi_N \end{bmatrix}, \qquad \underline{\boldsymbol{e}}_x = \begin{bmatrix} \psi_1(x) \\ \vdots \\ \psi_N(x) \end{bmatrix}$$

Now consider a structure comprised of N beams, labeled as $Beam_1, \ldots, Beam_N$, with $\boldsymbol{v_1} = \begin{bmatrix} v_{1,1}, \ldots, v_{1,n} \end{bmatrix}^T, \ldots, \boldsymbol{v_N} = \begin{bmatrix} v_{N,1}, \ldots, v_{N,n} \end{bmatrix}^T$ and $\boldsymbol{w_1} = \begin{bmatrix} w_{1,1}, \ldots, w_{1,2n} \end{bmatrix}^T, \ldots, \boldsymbol{w_N} = \begin{bmatrix} w_{N,1}, \ldots, w_{N,2n} \end{bmatrix}^T$ to characterize their displacement in the local coordinate systems. These vectors are time dependent and solutions of $2N$ independent linear ODEs. Merging these ODEs into a large linear system, we get

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t) + \boldsymbol{S}\boldsymbol{u}(t) = \boldsymbol{f}(t)$$

where

$$\boldsymbol{u(t)} = \begin{bmatrix} v_{1,1}(t) \\ \vdots \\ v_{1,n}(t) \\ w_{1,1}(t) \\ \vdots \\ w_{1,2n}(t) \\ \vdots \\ v_{N,1}(t) \\ \vdots \\ v_{N,n}(t) \\ w_{N,1}(t) \\ \vdots \\ w_{N,2n}(t) \end{bmatrix}$$

and $\boldsymbol{S}$, $\boldsymbol{M}$ the assembled stiffness and mass matrices

$$\boldsymbol{M} = \begin{bmatrix} M_L^{(1)} & & & & & \\ & M_T^{(1)} & & & & \\ & & M_L^{(2)} & & & \\ & & & M_T^{(2)} & & \\ & & & & \ddots & \\ & & & & & M_T^{(m)} \end{bmatrix}, \quad \boldsymbol{S} = \begin{bmatrix} S_L^{(1)} & & & & & \\ & S_T^{(1)} & & & & \\ & & S_L^{(2)} & & & \\ & & & S_T^{(2)} & & \\ & & & & \ddots & \\ & & & & & S_T^{(m)} \end{bmatrix}$$

5.5.3 Boundary Conditions and Extended Matrices

A variety of boundary conditions must be applied to the ends of beams. Some conditions are included in the right hand side of the ODE, while others must be provided as linear constraints.

Of the latter are the positions and slopes of endpoints, and the connection of end points of different beams. These constraints can be written in the form

$$\boldsymbol{c}^T \boldsymbol{u} = \boldsymbol{q}$$

Therefore we have the extended system

$$\begin{bmatrix} \boldsymbol{M} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{0} \end{bmatrix} \ddot{\boldsymbol{u}} + \begin{bmatrix} \boldsymbol{S} & \boldsymbol{c} \\ \boldsymbol{c}^T & \boldsymbol{0} \end{bmatrix} \begin{bmatrix} \boldsymbol{u} \\ * \end{bmatrix} = \begin{bmatrix} \boldsymbol{F} \\ \boldsymbol{q} \end{bmatrix}$$

Here are some examples

1. If the end of $Beam_1$ is clamped at the origin $x = 0$ with a slope of 0, then the constraints are as follows

$$v_1(0,t) = 0, \qquad w_1(0,t) = 0, \qquad w_1'(0,t) = 0$$

   or in the vector form

$$v_{1,1}(t) = 0, \qquad w_{1,1} = 0, \qquad w_{1,2} = 0$$

   The corresponding constraint can be expressed as

$$\underbrace{\begin{bmatrix} 1 & 0 & \dots & 0 & 0 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & 0 & 1 & 0 & \dots & \dots & 0 \\ 0 & 0 & \dots & 0 & 0 & 1 & \dots & \dots & 0 \end{bmatrix}}_{\boldsymbol{c}^T} u(t) = 0$$

2. If the end of $Beam_1$ $(x = L)$ is allowed to only move vertically the constraint is as follows

$$\cos(\phi_1)\, v_1(L_1,t) - \sin(\phi_1)\, w_1(L_1,t) = 0$$

   or in vector form

$$\cos(\phi_1) v_{1,n}(t) - \sin(\phi_1) w_{1,2n-1} = 0$$

   and is represented in the linear system as

$$\begin{bmatrix} 0 & \dots & 0 & \cos(\phi_1) & 0 & \dots & 0 & -\sin(\phi_1) \dots & 0 \end{bmatrix} u(t) = 0$$

3. If the end of $Beam_1$ at $x_1 = L$ is connected to the end of $Beam_2$ at $x_2 = 0$ then the following must hold

$$\underbrace{\begin{bmatrix} \cos(\phi_1) & -\sin(\phi_1) \\ \sin(\phi_1) & \cos(\phi_1) \end{bmatrix}}_{D_{\phi_1}} \begin{bmatrix} v_1(L_1,t) \\ w_1(L_1,t) \end{bmatrix} = \underbrace{\begin{bmatrix} \cos(\phi_2) & -\sin(\phi_2) \\ \sin(\phi_2) & \cos(\phi_2) \end{bmatrix}}_{D_{\phi_2}} \begin{bmatrix} v_1(0,t) \\ w_1(0,t) \end{bmatrix}.$$

   i.e.

$$\begin{array}{rcl}\cos(\phi_1)\, v_1(L_1,t) - \sin(\phi_1)\, w_1(L_1,t) - \cos(\phi_2)\, v_2(0,t) + \sin(\phi_2)\, w_2(0,t) & = & 0 \\ \sin(\phi_1)\, v_1(L_1,t) + \cos(\phi_1)\, w_1(L_1,t) - \sin(\phi_2)\, v_2(0,t) - \cos(\phi_2)\, w_2(0,t) & = & 0.\end{array}$$

or in the vector form

$$\begin{array}{rcl}\cos(\phi_1)\, v_{1,n}(t) - \sin(\phi_1)\, w_{1,2n-1}(t) - \cos(\phi_2)\, v_{2,1}(t) + \sin(\phi_2)\, w_{2,1}(t) & = & 0 \\ \sin(\phi_1)\, v_{1,n}(t) + \cos(\phi_1)\, w_{1,2n-1}(t) - \sin(\phi_2)\, v_{2,1}(t) - \cos(\phi_2)\, w_{2,1}(t) & = & 0.\end{array}$$

and is represented in the linear system as

$$\begin{bmatrix} 0\ldots0 & \cos(\phi_1) & 0\ldots0 & -\sin(\phi_1) & 0\ldots0 & -\cos(\phi_2) & 0\ldots0 & \sin(\phi_2) & 0\ldots0 \\ 0\ldots0 & \sin(\phi_1) & 0\ldots0 & \cos(\phi_1) & 0\ldots0 & -\sin(\phi_2) & 0\ldots0 & -\cos(\phi_2) & 0\ldots0 \end{bmatrix} u(t) = 0$$

If, in addition, the connection between these beams is ridgid we also need the following constraint

$$w_1'(L_1,t) - w_2'(0,t) = 0$$

or

$$w_{1,2n}(t) - w_{2,2}(t) = 0$$

6 Results

In the previous sections the theory and implementation of the theory was presented. What follows are the results, computations, error analysis and selected outputs of of this project. For the static case we show some sample problems with a computed solution and compare the results with the exact solution. Then in the dynamic case the total energy of the system is computed which should be constant for an isolated system. And for the structural case deflections of our computation are compared with those computed by use of an external software.

6.1 Static

In the static problem a sample of four solutions with different boundary conditions and types of loads are presented. For each of them the numerical solution is compared with the exact solution[1] and the error, calculated as absolute value of the difference between the exact and the numerical solution at a given point, is calculated and shown.

In the plots below, the subplot on the left shows the deflection of the beam with the same scale on both axes. Values of the deflections, forces, and moments at both ends of the beam are also shown. The subplot on the right shows the error of the approximation.

Unless specified otherwise, the modulus of elasticity E, length of the beam l and moment of inertia I are assumed to be to 1 in the rest of this report. It is also assumed that all the values are in the basic SI units. The number of grid points used for calculations is 2^7 for all the problems.

[1]Source of the exact solutions: http://www.engineersedge.com/beam_calc_menu.shtml

The first problem is the cantilever beam, a beam with one end clamped and the other free. The beam is loaded with a uniformly distributed force of 5N. The exact solution for this problem is given by the formula:

$$y(x) = \frac{Wx^2}{24EIl}(2l^2 + (2l - x)^2) \tag{32}$$

where W is the total value of distributed load. The results for this problem are presented in Fig. 2. The reaction force and the moment at the support have the values we would expect (sum of forces and moments acting on the beam is equal to zero). The error of computation is relatively small and it increases with the distance from the support as would be expected.

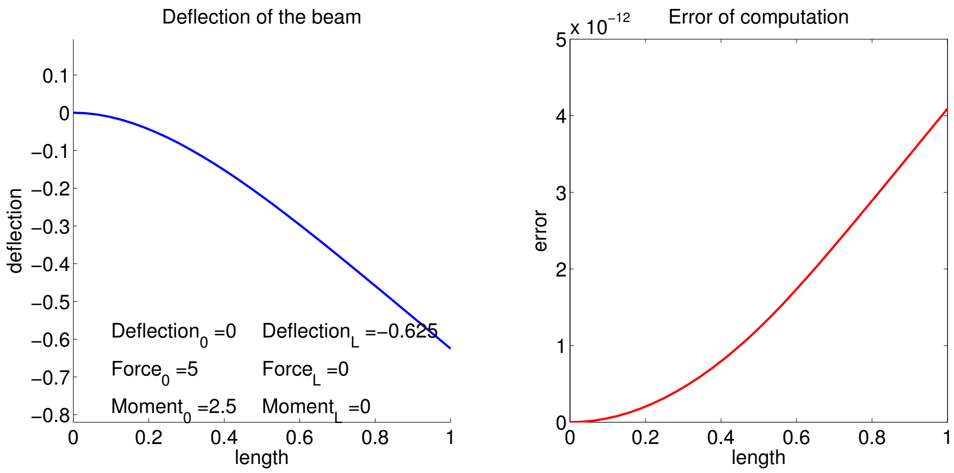

Figure 2: Results for cantilever beam with uniformly distributed load of 5N

The second problem is a beam that is clamped at both ends. The beam is uniformly loaded with a total force of 120N. The exact solution for this problem is given by the formula:

$$y(x) = \frac{Wx^2}{24EIl}(l - x)^2 \tag{33}$$

where W is the total load. The results for this problem are presented in Fig. 3. The reaction forces and the moments at the supports have correct values (sum of forces and moments acting on the beam is equal to zero). The error of computation is relatively small. The significant decrease of the value of error around $x = 0.6$ is caused by a change of sign of the difference between exact and numerical solution.

The third problem is a beam that is clamped at one end and pinned at the other. The beam is loaded at the center by a force of 20N. The exact solution for this problem is given by the formula:

$$y(x) = \begin{cases} \frac{Wx^2}{96EI}(9l - 11x) & x \in [0, l/2] \\ \frac{W(l-x)^2}{96EI}(3l^2 - 5(l - x)^2) & x \in (l/2, l] \end{cases} \tag{34}$$

where W is the total value of distributed load. The results for this problem are presented on the Fig. 4. The reaction forces and the moments at the supports have correct values (sum of forces and moments acting on the beam is equal to zero). The error of computation is acceptable but significantly larger than in previous cases.

The fourth problem is the simply supported beam, a beam pinned at both ends. The beam is uniformly loaded with total force of 20N. The exact solution for this problem is given by the formula:

$$y(x) = \frac{Wx(l - x)}{24EIl}(l^2 + x(l - x)) \tag{35}$$

where W is the total value of distributed load. The results for this problem are presented on the Fig. 5. The reaction forces at the supports have correct values (sum of forces and moments

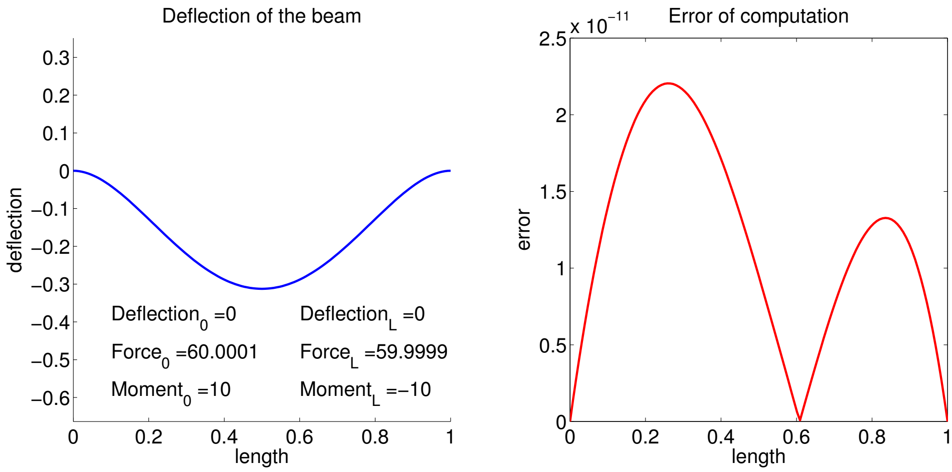

Figure 3: Results for clamped - clamped beam with uniformly distributed load of 120N

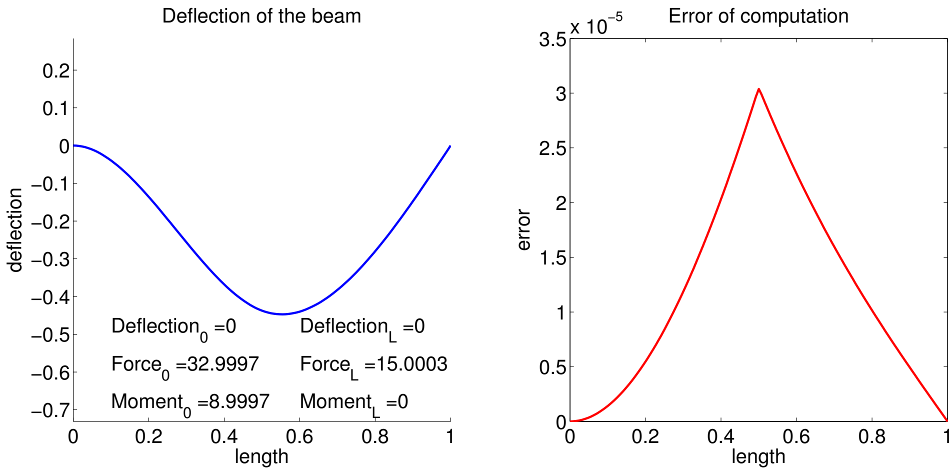

Figure 4: Results for clamped - pinned beam with load of 48N at the center

acting on the beam is equal to zero). The error of computation is relatively small. The significant decrease of the value of error around $x = 0.4$ is caused by a change of sign of the difference between exact and numerical solution.

6.2 Dynamic

In the dynamic case we selected two problems with different types of boundary conditions and loads. In both problems the beam was firstly deflected by a force, as in static case, which was released after $t = 0$ the beam then freely vibrates without any damping. The total energy of the system was calculated in every time step. The freely vibrating beam is an isolated system in which the total energy should be constant. If this requirement is met then the numerical solution is probably a good one.

In this system there are two types of energy: potential energy of deflection and kinetic energy. Potential energy is given by formula 36 and the kinetic energy by 37. The total energy is a sum of those two energies.

$$U = \int_0^L \frac{1}{2EI} M^2 dx = \int_0^L \frac{EI}{2} \left(\frac{\partial^2 w}{\partial x^2} \right)^2 dx \tag{36}$$

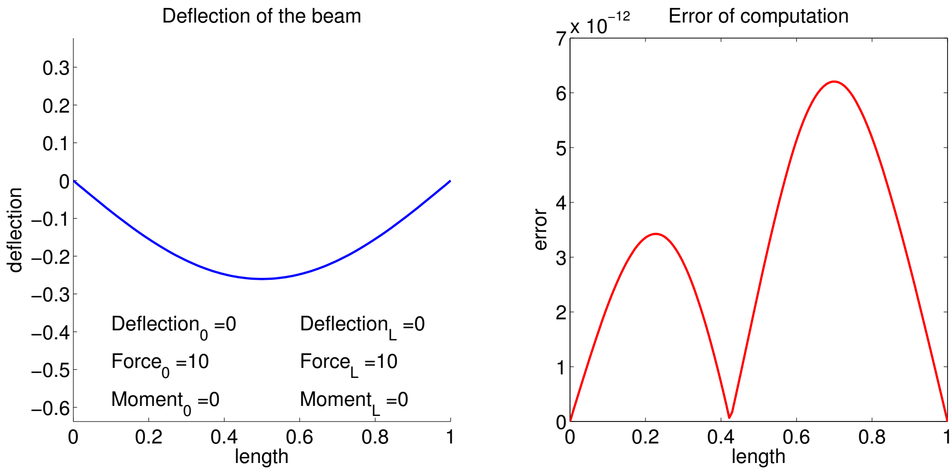

Figure 5: Results for simply supported beam with uniformly distributed load of 20N

$$T = \int_0^L \frac{1}{2}\mu \left(\frac{\partial w}{\partial t}\right)^2 dx \tag{37}$$

To calculate the energies in the system the time derivative and second spatial derivative of the deflection function are used. For the energy analysis those were calculated using backward difference applied to the deflection function (for the time derivative) and central difference applied to the slope function (for the second spatial derivative). Because this approach is an approximation it significantly reduces the accuracy of the calculated energy but the obtained results are still useful for the analysis of the stability of the total energy in the system.

The number of grid points used for calculations for problems throughout the rest of this report are 2^5.

The first problem is a cantilever beam deflected by a uniform load of 5N acting upwards. The load is released after $t = 0$. The shape of the beam at given time points is shown in the Fig. 6. The potential, kinetic and total energy as a function of time are plotted in the Fig. 7. The shapes of the functions of kinetic energy and potential energy are correct. At $t = 0$ all energy is in form of potential energy because the beam is at the furthest deflection. At the point around $t = 0.5$ the potential energy is equal to zero because there is no deflection and whole energy is in form of kinetic energy (the velocity of the beam is at its maximum). It can be easily seen that the total energy of the system is fluctuating when it should be constant. This is caused by numerical errors of the approximation of the derivatives as mentioned above. The total energy of the system for longer time interval is shown in the Fig. 8. The figure shows that despite the fluctuations the total energy seems to be stable even for a longer interval.

The second problem is a beam with both ends pinned and deflected by uniform load of 400N acting downwards on the left half of the beam and 400N acting upwards on the right half of the beam. This system is later referred as pinned-pinned beam. The load is released at $t = 0$. The shape of the beam at given time points is shown in Fig. 9. The potential, kinetic and total energy as a function of time are plotted in the Fig. 10. The shapes of kinetic and potential energy are correct, similar to the previous case. Also as in the previous case the total energy fluctuates but remains stable over a longer time frame as seen in Fig. 11.

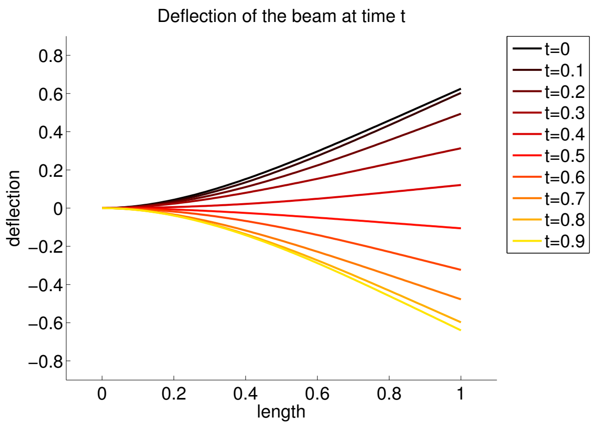

Figure 6: Vibrating cantilever beam

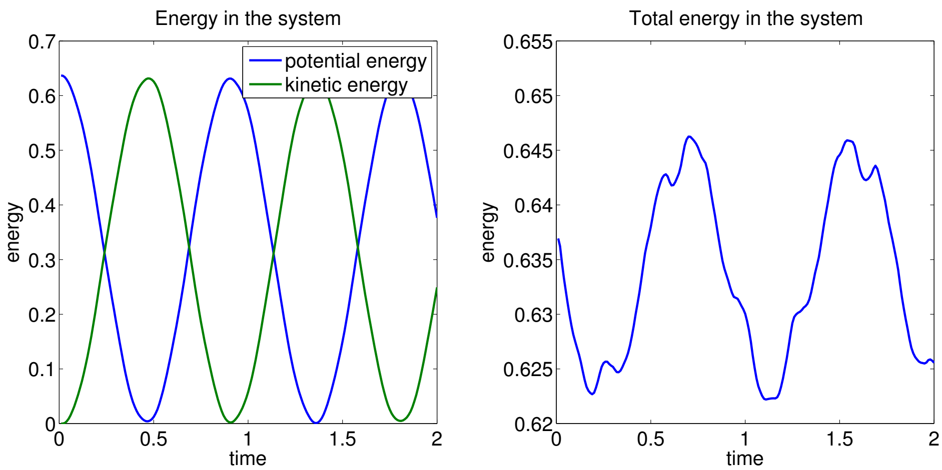

Figure 7: Energy of vibrating cantilever beam

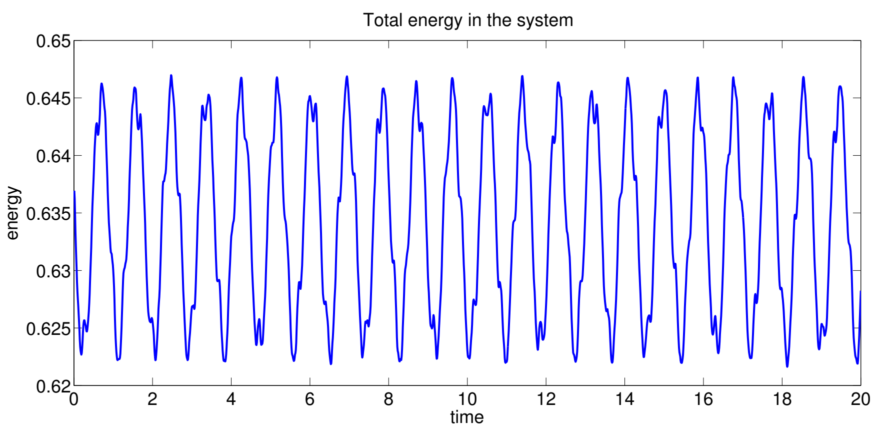

Figure 8: Stability of the energy in the system of vibrating cantilever beam

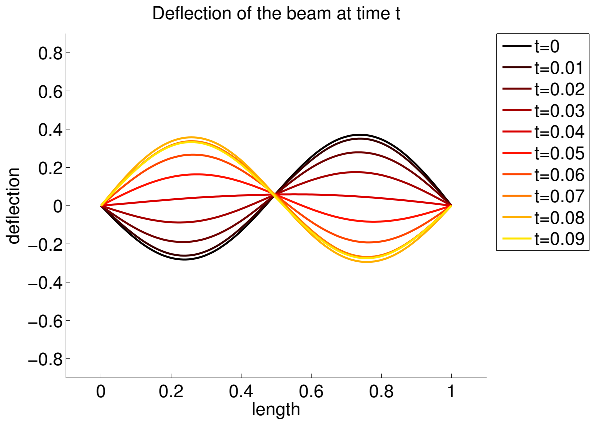

Figure 9: Vibrating pinned-pinned beam

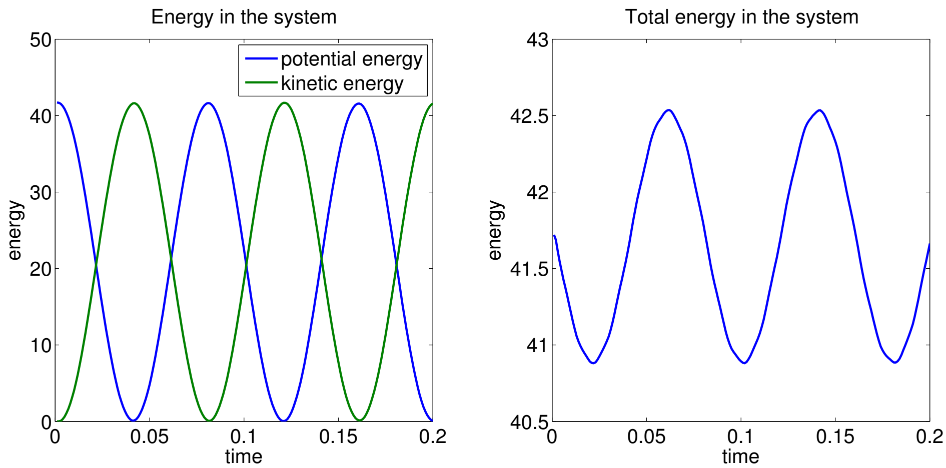

Figure 10: Energy of vibrating pinned-pinned beam

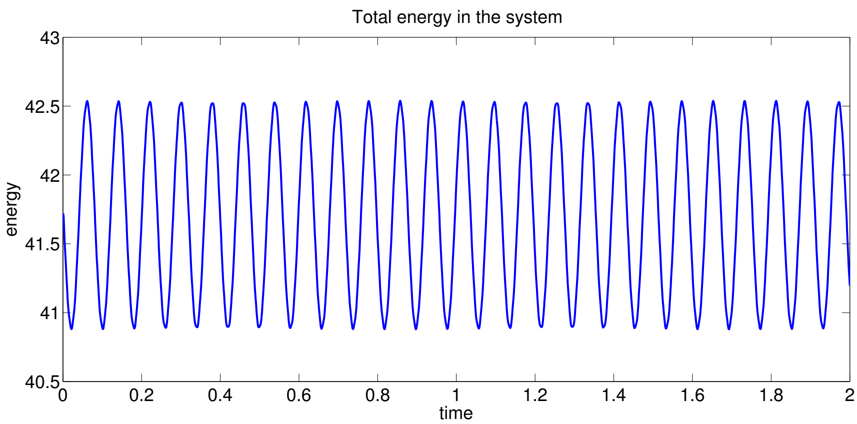

Figure 11: Stability of energy of vibrating pinned-pinned beam

6.3 Structure of Beams

The structural problems can be simulated either in static or dynamic case. Results for two structures in static case are presented in this chapter, and compared with results obtained from the free online software: Sopromat[2]. The software calculates deflections of the nodes (joints of beams) of arbitrary constructions. The deflections of joints are shown on output figures from Matlab code as well. Please note that the deflections are eqidistant despite the depictions of the displacement not being of equal scale in the two different softwares.

The first structure, referred to later as tower, is shown in Fig. 12. The structure is loaded with uniformly distributed load of 1N acting on the central beam and directed to the right. The left bottom joint is clamped to the ground and the right bottom joint is pinned to the ground. The rest of joints are free. All the joints between the beams are "welded" (they do not allow rotation). When comparing our results (see Fig. 13) with the results from Sopromat (see Fig. 14) the simulated deflections are almost identical.

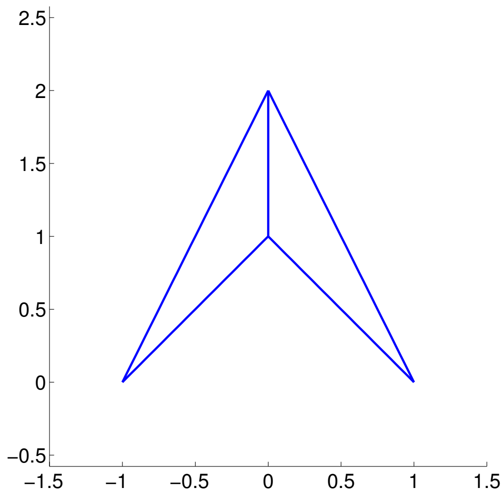

Figure 12: The unloaded tower structure

The second structure, referred to later as the bridge, is shown in the Fig. 15. The structure is loaded with a uniformly distributed load of 1N acting on the central bottom beam and directed downwards. The left joint is clamped to the ground and the right joint is pinned to the ground. The rest of joints are free. All the joints between the beams are "welded" (they do not allow rotation). Our results are shown on the Fig. 16. The values of deflections of joints can be compared with the results from Sopromat (see Fig. 17). The values are almost identical.

[2] Adres: http://en.sopromat.org/2009/#

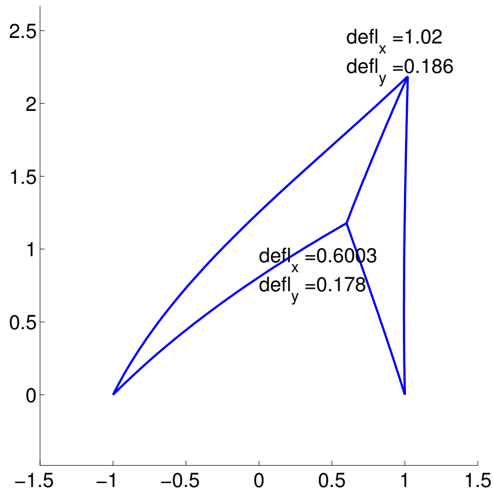

Figure 13: The tower structure loaded with the distributed load of 1N applied on the central beam and directed to the right

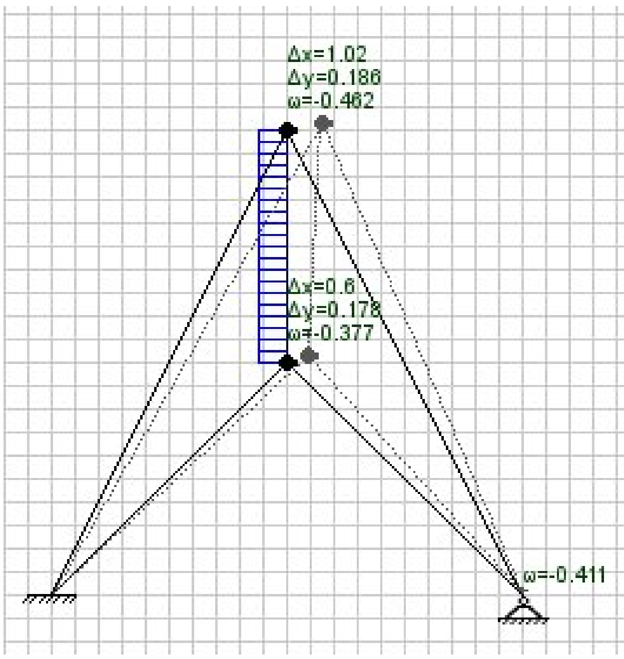

Figure 14: The solution from Sopromat software for the tower structure loaded with the distributed load of 1N applied on the central beam and directed to the right

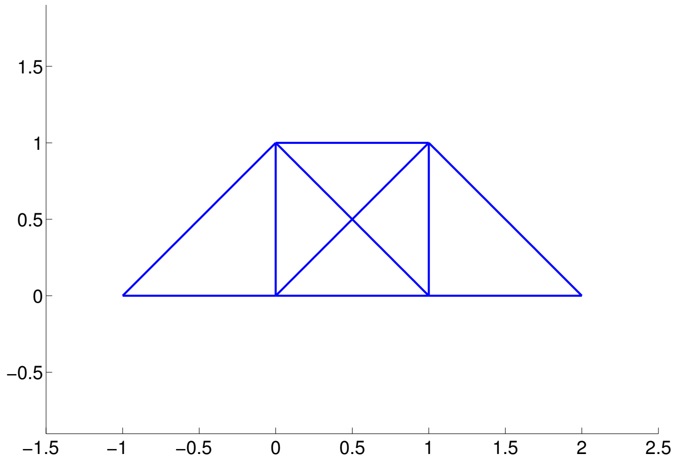

Figure 15: The unloaded bridge structure

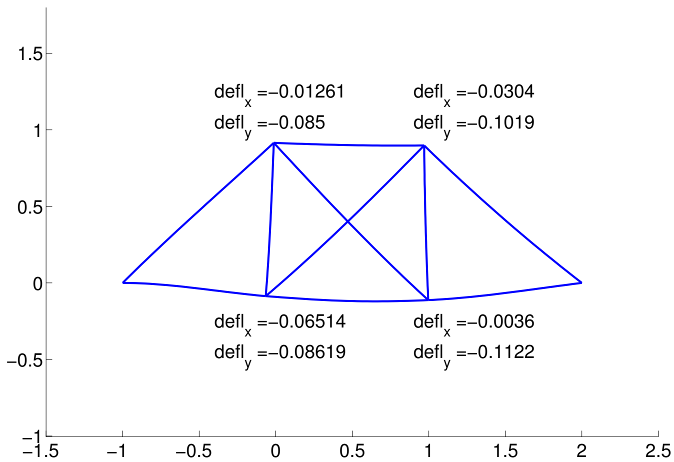

Figure 16: The bridge structure loaded with the distributed load of 1N applied on the central bottom beam and directed downwards

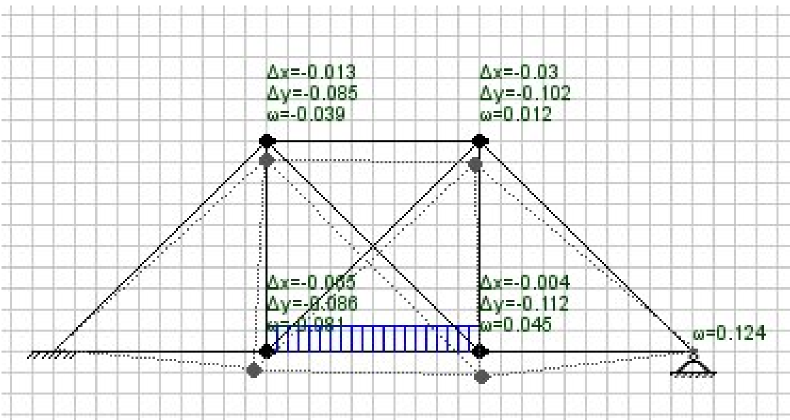

Figure 17: The solution from Sopromat software for the bridge structure loaded with the distributed load of 1N applied on the central bottom beam and directed downwards

7 Conclusions

The aim of this project was to create and implement an algorithm which solves the elastic Bernoulli beam problem using Finite Element Method. The problem was divided into three phases: solving the problem of single beam under static load, solving the problem of single beam under dynamic load and solving the problem of structure consisting many beams.

As demonstrated in Section 6 the program created during the project is able to solve all those types of problems including different boundary conditions and different distributions of loads. And it is able to give the output of numerical computation in numerical and graphical form.

The results obtained with use of this program were compared with exact solutions (for the static case), analyzed in terms of stability of energy in the system (for the dynamic case) and compared with external software (for the structural case). Those tests show that the results are of very good quality.

Although all goals of this project were achieved there are possible further improvements. Using another type of basis functions would make it possible to analyze which type of basis functions fits the best for the Bernoulli beam problem. Additional error analysis in terms of the changing step sizes would be a good field for discussion of convergence and stability. And lastly, creating a Graphical User Interface would make using the program much easier and more intuitive.