Equivariant Cohomology and Localization

Nelvis Fornasin

Ludwig-Maximilians-Universität München

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Overview

1. Introduction
   - The idea
   - Equivariant cohomology

2. Localization
   - The localization theorem
   - Symplectic geometry
   - Generalized flag manifolds

Outline of the problem

Main idea

Use symmetries of the space to simplify computations.

How?

Symmetries are encoded in the geometry of the manifold.

Algebra

Extract the desired information with cohomology.

Two approaches

Hypotheses

Compact, connected Lie group G acting on a compact manifold M.

Borel (1959)

Consider the universal bundle

$$G \to EG \to BG$$

Define equivariant cohomology:

$$H^*_G(M) = H^*((EG \times M)/G)$$

Cartan (1950)

Consider the Cartan complex

$$C_G(M) = (S(\mathfrak{g}^*) \otimes \Omega(M))^G$$

Define equivariant cohomology:

$$H^*_G(M) = H^*(C_G(M))$$

Cartan's model (I)

Start with polynomials over $\mathfrak{g}$, valued in $\Omega(M)$:

$$(S(\mathfrak{g}^*) \otimes \Omega(M))^k = \oplus_{2i+j=k} S(\mathfrak{g}^*)^i \otimes \Omega(M)^j$$

To define a differential, consider the *fundamental vector field* related to $X \in \mathfrak{g}$:

$$\underline{X}_x = \frac{d}{dt}_0 e^{tX} \cdot x$$

Then for $p \in S(\mathfrak{g}^*) \otimes \Omega(M)$:

$$(d_G p)(X) = d(p(X)) - \iota_{\underline{X}}(p(X))$$

And $d_G^2(p)(X) = -L_{\underline{X}}(p(X))$. This is general not zero!

Cartan's model (II)

Restrict to $G-$invariant polynomials $(S(\mathfrak{g}^*)\otimes\Omega(M))^G$:

$$p(\mathrm{Ad}_{g^{-1}}X) = g^* p(X)$$

set $g = e^{tX}$ and derive in $t=0$:

$$0 = p(\mathrm{ad}_X(X)) = \frac{d}{dt}_0 (e^{tX})^* p(X) = L_{\underline{X}} p(X)$$

for these polynomials $d_G^2 = 0$!

$C_G(M) = ((S(\mathfrak{g}^*)\otimes\Omega(M))^G, d_G)$ is the Cartan complex of M.

Remarks

Cartan's model

$$\begin{cases} p(\mathrm{Ad}_{g^{-1}} X) = g^* p(X) \\ d_G p(X) = d(p(X)) - \iota_{\underline{X}}(p(X)) \end{cases}$$

- Suppose G acts trivially on M. We obtain $(S(\mathfrak{g}^*) \otimes \Omega(M))^G \simeq S(\mathfrak{g}^*)^G \otimes \Omega(M)$, $d_G = d$, then

$$H_G^*(M) = (S(\mathfrak{g}^*))^G \otimes H^*(M)$$

- Suppose G acts trivially on itself. We obtain $(S(\mathfrak{g}^*) \otimes \Omega(M))^G \simeq S(\mathfrak{g}^*) \otimes \Omega(M)^G$.

Set $M = *$, $G = T^l$:

$$H_T^*(*) = \mathcal{C}[u_1, \ldots, u_l]$$

Statement

Assume T^l acts on an orientable M, with a discrete set of fixed points $p_1, \ldots, p_k$. Then $\dim M = 2n$, and for an $[\omega] \in H_T^{2l}(M)$

$$\omega = 1 \otimes \omega_{2n} + p_1 \otimes \omega_{2n-2} + \cdots + p_n \otimes \omega_0$$

denote by α_j^i, $j = 1, \ldots, n$ the weights of the isotropy representation of T at p^i. Then

Theorem (Localization theorem)

$$\int_M \omega_{2n} = (2\pi)^n \sum_{i=1}^{k} \frac{\omega_0(p_i)}{\Pi_j \alpha_j^i}$$

How to use it

Suppose we want to compute $\int_M \nu$.
$\Rightarrow$ Find an equivariant extension

$$\omega = 1 \otimes \nu + p_1 \otimes \omega_{2n-2} + \cdots + p_n \otimes \omega_0$$

$\Rightarrow$ Express the integral as the sum of fixed point data

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$$(2\pi)^n \sum_{i=1}^{k} \frac{\omega_0(p_i)}{\prod_j \alpha_j^i}$$

Does such an extension always exist? How can we find it?

Equivariant symplectic forms

Symplectic action of T on (M, ω). We want to extend ω to

$$\omega_{eq} = 1 \otimes \omega - p_1 \otimes \phi = \omega - \varphi$$

imposing $d_G \omega_{eq} = 0$ yields $\iota_X \omega = d\varphi_X$.
$\Rightarrow$ This is the definition of a *comoment map* for the action:

$$\tau : \mathfrak{g} \to C^\infty(M) : \; X \mapsto \varphi_X$$

$\Rightarrow$ One shows:

Theorem

There is a $1:1$ correspondence between equivariant extensions of ω and comoment maps for the action.

The stationary phase approximation (I)

Consider an S^1-action. Then $\omega_{eq} = \omega - u\phi$, and

$$\int_M \frac{\omega^n}{n!} e^{-u\phi} = \int_M e^{\omega_{eq}} = (2\pi)^n \sum_{i=1}^{k} \frac{e^{-u\phi(p_i)}}{\prod_j \alpha_j^i}$$

Figure: Aequorea victoria

This formula has a very intuitive interpretation as the so-called stationary phase approximation: suppose that the light emitted from the A.v. travels in spherical waves.

The stationary phase approximation (II)

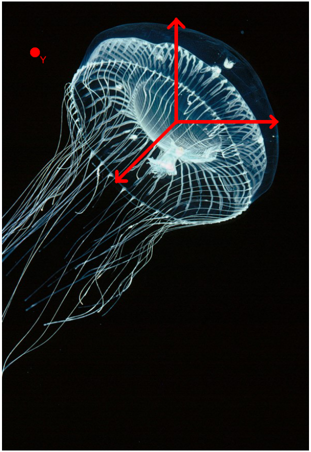

Figure: Aequorea victoria

The intensity in y can be computed as

$$I(y) = \int_{\mathbb{R}^3} \nu(x) \frac{e^{ik||x-y||}}{||x-y||} d^3x$$
$$= \int_{\mathbb{R}^3} a(x) e^{ik\phi(x)} d^3x$$

where $\nu(x)$ is the density of the A.v.
k is "large" $\Rightarrow$ Main contributions in the stationary points of ϕ.

Fixed points

Definition

Generalized flag manifolds can be equivalently defined as:

- Quotients $G/C(S)$, where $C(S)$ is the centralizer of a torus S in G;
- Orbits of the adjoint action of G on $\mathfrak{g}$.

Let us take the first point of view. We have the following

Theorem

The fixed point action of a maximal torus $T \subset G$ on $G/C(S)$ has finitely many fixed points, corresponding to representatives of $W(G)/W(C(S))$.

The moment map

Suppose that the group G is semisimple. This has two consequences:

- Given a symplectic action of G, there exists a unique moment map;
- The adjoint representation is isomorphic to the coadjoint representation.

One proves that there exists a canonical symplectic structure, with moment map of the T-action

$$\varphi : M \hookrightarrow \mathfrak{g}^* \simeq \mathfrak{t}^*$$

Conclusions

- Localization is a powerful machinery, which condensates information from the whole space to a set of points;
- This advantages are however only apparent, unless we can elaborate the data on points;
- This is the case for GFM. We saw how to derive the position of points and the moment map for the canonical symplectic form;
- The product of weights $\Pi_j \alpha_j^i$ carries local information about the moment map, as can be seen from the stationary phase approximation: not easy to compute.