The η invariant under cone-edge degeneration

Nelvis Fornasin

Albert-Ludwig-Universität Freiburg

The η invariant under cone-edge degeneration

The η invariant

Let (M^n, g) be a spin manifold. We will consider two operators:

- The Hodge operator $d + d^*$ acting on forms;
- The spin Dirac operator acting on spinors.

Let D be either of these operators[1]. Then

$$\eta(D) := \int_0^\infty \frac{\operatorname{tr} De^{-tD^2}}{\sqrt{\pi t}} dt = \int_0^\infty \frac{\sum_{i=0}^\infty \lambda_i e^{-t\lambda_i^2}}{\sqrt{\pi t}} dt$$

Here:

- e^{-tD^2} is the *heat kernel* of D, acting on the same bundle as D;
- $\{\lambda_i\}_{i\in\mathbb{N}}$ is the set of eigenvalues of D;
- $\operatorname{tr} De^{-tD^2} = \sum_{i=0}^\infty \lambda_i e^{-t\lambda_i^2}$.

[1] Assume that D is (essentially) self adjoint.

The η invariant

Takeaway 1

There exist two equivalent theories for treating the η invariant: microlocal analysis and spectral theory.

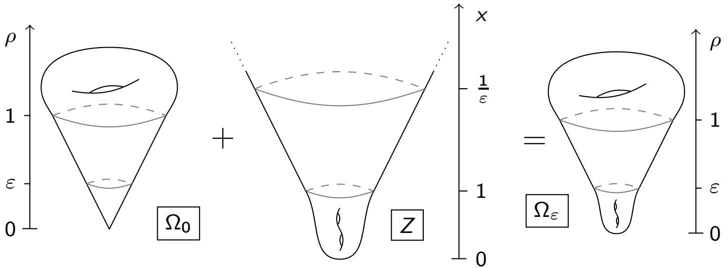
$$\eta(D) := \underbrace{\int_0^\infty \frac{\operatorname{tr} De^{-tD^2}}{\sqrt{\pi t}} dt}_{\text{Microlocal analysis}} = \underbrace{\int_0^\infty \frac{\sum_{i=0}^\infty \lambda_i e^{-t\lambda_i^2}}{\sqrt{\pi t}} dt}_{\text{Spectral theory}} \qquad (1)$$

Both are important for my thesis. Following Sher [S15], I split $\eta(D)$ in two summands and treat each of them in the most convenient framework:

$$\eta(D) = \underbrace{\int_0^1 \frac{\operatorname{tr} De^{-tD^2}}{\sqrt{\pi t}} dt}_{\text{Short time component}} + \underbrace{\int_1^\infty \frac{\sum_{i=0}^\infty \lambda_i e^{-t\lambda_i^2}}{\sqrt{\pi t}} dt}_{\text{Long time component}}$$

Conic degeneration

Use surgery to resolve a conic singularity:

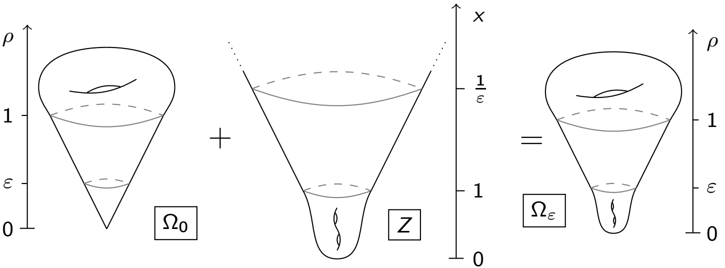


Comments

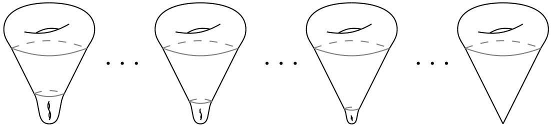
$\{\rho = 0\} \subseteq \Omega_0$ is the *singular set*. In this case $S = \{pt\}$.
Ω_0 has a *conic singularity* at p iff in a neighbourhood of p it is isometric to the cone $(C(Y), d\rho^2 + \rho^2 h)$, where (Y, h) is a smooth Riemannian manifold. (Y, h) is called *link*.

Conic degeneration

Use surgery to resolve a conic singularity:

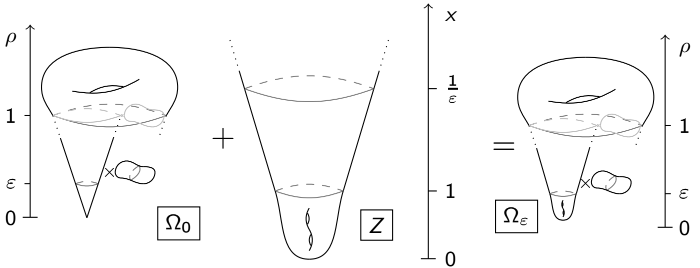


As $\varepsilon \to 0$, Ω_ε degenerates to Ω_0:



Cone-edge degeneration

Use surgery to resolve a cone-edge singularity:

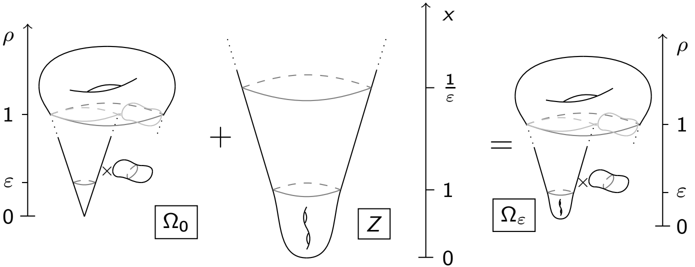


Comments

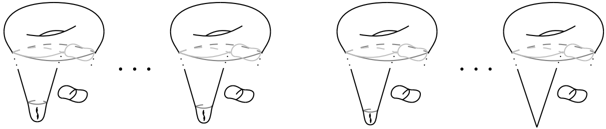
$\{\rho = 0\} \subseteq \Omega_0$ is the *singular set*. In this case $S = (B, k)$.
Ω_0 has a *cone-edge singularity* at B iff in a neighbourhood of B it is isometric to $(C(Y) \times B, d\rho^2 + \rho^2 h + k)$, where (Y, h) and (B, k) are smooth Riemannian manifolds. (B, k) is called *edge*.

Cone-edge degeneration

Use surgery to resolve a cone-edge singularity:



As $\varepsilon \to 0$, Ω_ε degenerates to Ω_0:



Aim of my phd project

I studied the behaviour of the η invariant under cone-edge degeneration. Let D_ε be the Hodge or spin Dirac operator on Ω_ε,

General aim

Compute $\lim_{\varepsilon\to 0} \eta(D_\varepsilon) - \eta(D_0)$ under degeneration.

Takeaway 2

Spectral invariants don't behave well under degeneration. Easy example: topologically $\Omega_\varepsilon = \Omega_1 \ \forall \varepsilon \in (0, 1]$, so for the k^{th} Betti number $b^k(\varepsilon)$ of Ω_ε we have

$$\lim_{\varepsilon\to 0} b^k(\varepsilon) - b^k(0) = b^k(1) - b^k(0) \neq 0 \text{ in general}$$

Characterising this defect is an interesting problem!

First, a definition

Definition (Generalised Witt condition)

Let $\dim Z = n$, D_Y the Hodge/spin Dirac operator of the link Y.

- The Hodge operator is *admissible* if and only if:

$$\begin{cases} \sigma(D_Y^2|_{\Lambda^{n/2-1}}) \cap \{0\} = \sigma(D_Y^2|_{\Lambda^{n/2}}) \cap [0,1] = \emptyset & \text{if } n \text{ is even} \\ \sigma(D_Y^2|_{\Lambda^{(n-1)/2}}) \cap \left[0, \frac{3}{4}\right] = \emptyset & \text{if } n \text{ is odd} \end{cases}$$

- The Dirac operator is *admissible* if and only if:

$$\sigma(D_Y^2) \cap \left[0, \frac{9}{4}\right] = \emptyset$$

Remark

For $Y = S^{n-1}/\Gamma$, $n > 4$, both operators are admissible.

Results

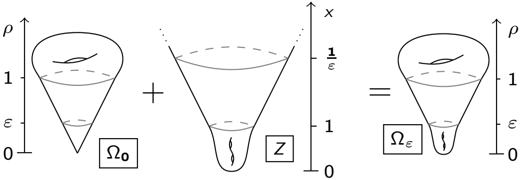
Theorem (N. - Conic degeneration)

Let $\dim Z \geq 3$. *Assume that* D_ε *is admissible, then:*

$$\lim_{\varepsilon \to 0} \eta(D_\varepsilon) = \eta(D_0) + \eta_R(D_Z) \tag{2}$$

where $\eta_R(D_Z)$ *is the rescaled* η *invariant of* D_Z.

The result about conic degeneration is non-trivial not only because it shows that there is additional term appearing in the limit, but also for the sheer fact that the limit exists!



Results

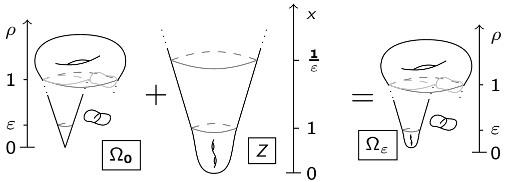
Theorem (N. - Cone-edge degeneration)

Let $\dim Z \geq 3$. *Assume that* D_ε *is admissible, then:*

$$\lim_{\varepsilon \to 0} \eta(D_\varepsilon) = \eta(D_0) + \mathrm{ind}_R(D_Z)\eta(D_B) + \mathrm{ind}(D_B)\eta_R(D_Z) \qquad (3)$$

where $\mathrm{ind}_R(D_Z) = \mathrm{ind}_{L^2}(D_Z)$, *and* $\eta_R(D_Z)$ *is the rescaled* η *invariant.*

The form of the extra term is a generalisation of the product formula for the η invariant. Non-trivial: $\mathrm{ind}_R(D_Z) = \mathrm{ind}_{L^2}(D_Z)$.



The general idea

These theorems are analytical in nature. In the spirit of Index Theory, they are better used when paired with integrality conditions:

Sample input

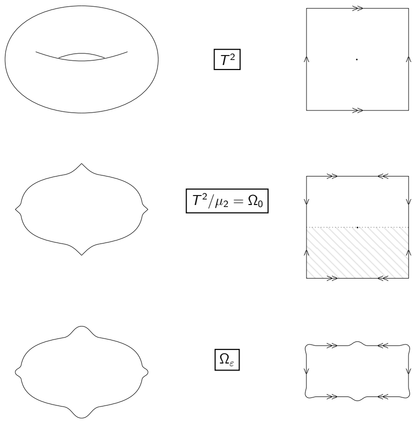
For a given family of Ω_ε, $\eta(D_\varepsilon) \in \mathbb{Z}\ \forall \varepsilon \in (0, 1]$.

This turns the limit expressions into exact formulas, e.g.

$$\exists \varepsilon_0 > 0 : \eta(D_\varepsilon) = \eta(D_0) + \mathrm{ind}_R(D_Z)\eta(D_B) + \mathrm{ind}(D_B)\eta_R(D_Z)\ \ \forall \varepsilon \leq \varepsilon_0$$

Results similar to the Sample Input exist, as we'll see soon; we want to use the equation to compute $\eta(D_\varepsilon)$, so one also needs to find a way to compute the rest of the terms.

Torus orbifolds



- The family of manifolds Ω_ε we'll consider comes from Joyce's generalisation [J96] of the Kummer construction;
- We consider a quotient T^7/Γ, with Γ a given finite group. In this case - unlike in the picture - we get actual singularities;
- More precisely, we get a cone-edge singularity with edge S^1.

Here we use the theorem

This is the integrality result we'll use, stemming from previous work of [CGN15]:

Theorem (Scaduto, '18)

Let $\Omega_0 = T^7/\Gamma$, let Ω_ε be the corresponding degenerating family. Then

$$\bar{\nu}(\Omega_\varepsilon) = 3\eta(B_\varepsilon) - 24\eta(D_\varepsilon) \in \mathbb{Z}\ \forall\varepsilon \in (0,1] \tag{4}$$

where B_ε is the signature operator on Ω_ε, D_ε the spin Dirac operator on Ω_ε.

We obtain an exact expression: $\exists\varepsilon_0 \in (0,1]$ such that

$$\begin{aligned}\bar{\nu}(\Omega_\varepsilon) =& 3(\eta(B_0) + \mathrm{ind}_R(B_Z)\eta(B_{S^1}) + \mathrm{ind}(B_{S^1})\eta_R(B_Z)) \\ &- 24(\eta(D_0) + \mathrm{ind}_R(D_Z)\eta(D_{S^1}) + \mathrm{ind}(D_{S^1})\eta_R(D_Z)) \\ &= 3\eta(B_0) - 24\eta(D_0) = \bar{\nu}(\Omega_0) \quad \forall\varepsilon \le \varepsilon_0\end{aligned} \tag{5}$$

Computation of $\bar{\nu}(\Omega_0)$

The group Γ is the finite group generated by α and β:

$$\begin{cases} \alpha : & [(x_1, \ldots, x_7)] \mapsto [(x_2, x_3, x_7, -x_6, -x_4, x_1, x_5)] \\ \beta : & [(x_1, \ldots, x_7)] \mapsto \left[\left(\frac{1}{2} - x_1, \frac{1}{2} - x_2, -x_3, -x_4, \frac{1}{2} + x_5, \frac{1}{2} + x_6, x_7\right)\right] \end{cases}$$

Both α and β commute with the orientation reversing isometry

$$\iota : [(x_1, \ldots, x_7)] \mapsto [-(x_1, \ldots, x_7)]$$

So ι descends to the quotient, hence $\eta(B_{T^7/\Gamma}) = 0 = \eta(D_{T^7/\Gamma})$. One also shows

$$\eta(B_{T^7/\Gamma}) = \eta(B_0), \quad \eta(D_{T^7/\Gamma}) = \eta(D_0)$$

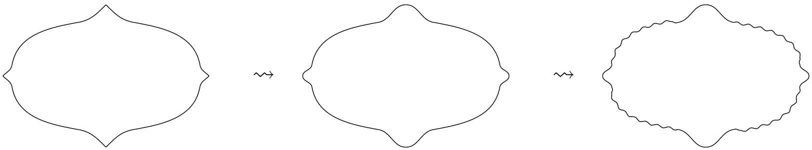
So $0 = \bar{\nu}(\Omega_0) = \bar{\nu}(\Omega_\varepsilon)\ \forall \varepsilon \leq \varepsilon_0$.

Main quest

Option 1: keep following Joyce's construction and compute $\bar{\nu}(\chi)$. So far I have

$$\bar{\nu}(\chi) = 24\left(h(T^7/\Gamma) + h_{L^2}(Z)h(S^1) - 2\,\mathrm{spf}\left((D_{M_s})_{s\in[0,1]}\right) - h(\chi)\right)$$

where $h(\cdot) := \dim\ker(D_\cdot)$, spf denotes spectral flow.



$\boxed{T^7/\Gamma \simeq \Omega_0}$ $\boxed{\Omega_\varepsilon}$ $\boxed{\chi}$

Secondary quests

Option 2: extend the definition of admissible operators to include the case $Y = \mathbb{R}P^3$: this would allow to access more of Joyce's examples. However, I expect the same formula to hold, and formal computations yield $\bar{\nu}(\Omega_\varepsilon) = 0$ for all of these examples as well.

Option 3: we saw $\mathrm{ind}_R(D_Z) = \mathrm{ind}_{L^2}(D_Z)$. From work by Gilles Carron [C01], $\mathrm{ind}_{L^2}(D_Z) = \mathrm{ind}_{APS}(D_{\hat{Z}})$, where $\hat{Z} = \{z \in Z : \rho(z) \leq 1\} \subseteq Z$. It would be interesting to obtain a similar interpretation for the rescaled η invariant $\eta_R(D_Z)$.

Option 4: W.D Gillam and S. Molcho proposed [GM15] a log-geometric reformulation of Melrose's theory, which is the theoretic foundation of this project. Working on this one would arguably lose sight of the trees, but start seeing the woods.

Bibliography

S15 Sher, David A. "Conic degeneration and the determinant of the Laplacian." Journal d'Analyse Mathématique 126.1 (2015): 175-226;

J96 Joyce, Dominic. "Compact Riemannian 7-manifolds with holonomy G2. II." Journal of Differential Geometry 43 (1996), 329?375

CGN15 Crowley, Diarmuid, Sebastian Goette, and Johannes Nordström. "An analytic invariant of G2 manifolds." arXiv preprint arXiv:1505.02734 (2015);

C01 Carron, Gilles. "Théorèmes de l'indice sur les variétés non-compactes." Journal für die Reine und Angewandte Mathematik 541 (2001): 81-116;

GM15 Gillam, William D., and Samouil Molcho. "Log differentiable spaces and manifolds with corners." arXiv preprint arXiv:1507.06752 (2015).

Thank you for your attention!

Q&A