Ehrenfest Chains
and Poincaré's Recurrence Theorem

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A MATTER OF CONSISTENCY

Consistency between the 2^{nd} Principle of Thermodynamics and microscopical models:

1. 2^{nd} Principle $\nleftrightarrow$ Classical Mechanics;

We start by considering the stochastic model.

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3. 2^{nd} Principle $\overbrace{\longleftrightarrow}^{?}$ Quantum Mechanics;

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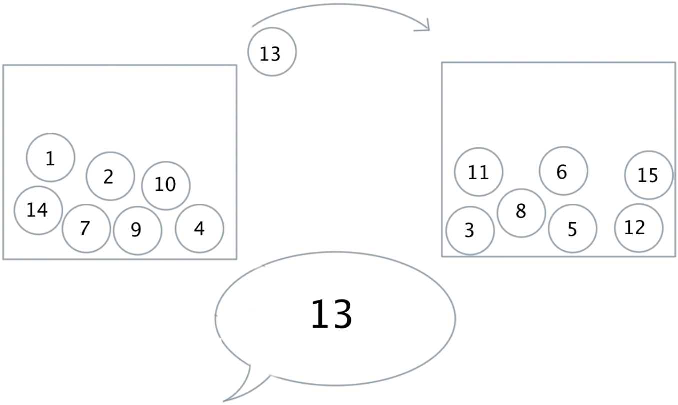
THE MODEL



The model

Features of Ehrenfest Chains:

- Purely stochastic;
- Time is a discrete variable;
- Markov property.

We can analyze the model by means of the theory of Markov Chains!

MARKOV CHAINS

Definition (Markov chain)

A discrete stochastic process $X = \{X_n, n \geq 0\}$ on the state space S is said to be a *Markov chain* if for any sequence $\{x_i\}_i \subset S$

$$P\{X_{n+1} = x_{n+1} \mid X_k = x_k,\ 0 \leq k \leq n\} = P\{X_{n+1} = x_{n+1} \mid X_n = x_n\}$$

Markov Chains

Definition (Communication between states)

We say that a state x *leads* to state y and write $x \longrightarrow y$ if $\exists n \geq 0 \ : \ p^n(x,y) > 0$. If both $x \longrightarrow y$ and $x \longleftarrow y$ hold, we say that x and y *communicate*, and write $x \longleftrightarrow y$.

The relation $\longleftrightarrow$ is an equivalence relation!

Definition (Irreducibility)

An MC is said to be irreducible if its state space is an equivalence class with respect to $\longleftrightarrow$.

MARKOV CHAINS

The following, fundamental result holds:

Theorem

Let $\{X_n\}_n$ be an irreducible MC with finite state space S: then all the states of $\{X_n\}_n$ are recurrent, and the MC is said to be recurrent.

MARKOV CHAINS

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Furthermore, for the EC it is possible to compute a mean recurrence time as

$$\tau(x) = 2^{2N} \frac{x!(2N-x)!}{2N!}$$

For a state with x particles out of all the $2N$ particles involved.

A MATTER OF CONSISTENCY

Consistency between the 2^{nd} Principle of Thermodynamics and microscopical models:

1. 2^{nd} Principle $\not\leftrightarrow$ Classical Mechanics;
2. 2^{nd} Principle $\not\leftrightarrow$ Stochastic model; (!)
3. 2^{nd} Principle $\overbrace{\longleftrightarrow}^{?}$ Quantum Mechanics;

What about QM?

Physical hypotheses

Features of the system:

- N particles with mass m;
- No mutual interactions;
- Enclosure in a finite volume;
- No spin.

Main problem: exchange degeneracy.

MATHEMATICAL ISSUES

Main points we need to discuss:

1. How do we swap functions?
2. What is a symmetrization?

And then we can prove the theorem!

MATHEMATICAL ISSUES

1. SWAPPING FUNCTIONS

We work with:

1. The group of N-permutations: $\mathcal{P}_N$;
2. The Hilbertian tensor product of N copies of the single particle's Hilbert space: $\mathcal{H}^{\otimes N}$.

$\Rightarrow$ There exists a representation of $\mathcal{P}_N$ on $\mathcal{H}^{\otimes N}$!

MATHEMATICAL ISSUES

2. SYMMETRIZING

Symmetrization and anti-symmetrization are indeed projections:

- $\mathcal{A} : \mathcal{H}^{\otimes N} \to \Lambda(\mathcal{H}^{\otimes N}) : \phi \mapsto \frac{1}{N!} \sum_{\sigma \in \mathscr{P}_N} \epsilon_\sigma \sigma^{\otimes} \phi$
- $\mathcal{S} : \mathcal{H}^{\otimes N} \to S(\mathcal{H}^{\otimes N}) : \phi \mapsto \frac{1}{N!} \sum_{\sigma \in \mathscr{P}_N} \sigma^{\otimes} \phi$

Where $\Lambda(\mathcal{H}^{\otimes N})$ and $S(\mathcal{H}^{\otimes N})$ are respectively the set of all anti-symmetric and symmetric tensors of $\mathcal{H}^{\otimes N}$.

THE THEOREM

Theorem (Poincaré's Recurrence Theorem)

Let $\psi(t,x)$ the wavefunction of the considered system. Then $\forall t_0 \in \mathbb{R}, \forall \varepsilon > 0\ \exists\ T = T(\varepsilon) > t_0$ such that

$$||\psi(t_0,\cdot) - \psi(T,\cdot)|| < \varepsilon$$

Central point:

$$\forall \varepsilon > 0,\ \forall x \in \mathbb{R},\ \exists n, m \in \mathbb{N}\ :\ |nx - m| < \varepsilon$$

A paradox?

- Difference between *figures* and *real objects*: we should look for contradiction between theoretical previsions and experimental results;
- Poincaré's theorem and Ehrenfest's model highlight the fact we still haven't found a microscopical theory consistent with the 2^{nd} Law of Thermodynamics;
- Implying such a theory exists...