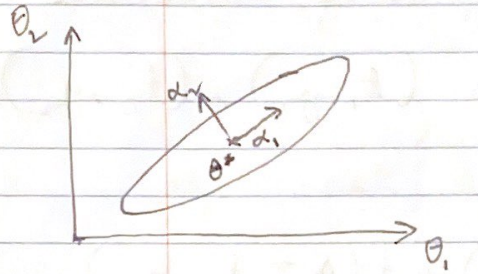
April 13, 2023

H at $\vec{\theta} = \vec{\theta}^*$

$$C(\vec{\theta}) \approx C(\vec{\theta}^*) + \frac{1}{2}(\vec{\theta} - \vec{\theta}^*)^T H (\vec{\theta} - \vec{\theta}^*)$$



Let $\vec{\alpha}_1$ and $\vec{\alpha}_2$ be eigenvectors of $H|_{\vec{\theta} = \vec{\theta}^*}$ with eigenvalues λ_1 and λ_2

and let $|\vec{\alpha}_1| = |\vec{\alpha}_2| = 1$

Along the direction of $\hat{\theta}_1$ (that is, $\vec{\theta} - \vec{\theta}^* = \hat{\theta}_1$)

$\hat{\alpha}_1 = \vec{\alpha}_1$

$\hat{\alpha}_2 = \vec{\alpha}_2$

Write $\hat{\theta}_1$ in $(\hat{\alpha}_1, \hat{\alpha}_2)$ basis

$$C(\vec{\theta}) \approx C(\vec{\theta}^*) + \frac{1}{2}\hat{\theta}_1^T H \hat{\theta}_1$$

$$= C(\vec{\theta}^*) + \frac{1}{2}(k_{11}\hat{\alpha}_1 + k_{21}\hat{\alpha}_2)^T H (k_{11}\hat{\alpha}_1 + k_{21}\hat{\alpha}_2)$$

$$= C(\vec{\theta}^*) + \frac{1}{2}(k_{11}\hat{\alpha}_1 + k_{21}\hat{\alpha}_2)^T (k_{11}\lambda_1\hat{\alpha}_1 + k_{21}\lambda_2\hat{\alpha}_2)$$

$$= C(\vec{\theta}^*) + \frac{1}{2}\Big[k_{11}^2\lambda_1\hat{\alpha}_1^T\hat{\alpha}_1 + k_{21}^2\lambda_2\hat{\alpha}_2^T\hat{\alpha}_2 + k_{11}k_{21}\lambda_2\hat{\alpha}_1^T\hat{\alpha}_2 + k_{11}k_{21}\lambda_1\hat{\alpha}_2^T\alpha_1\Big]$$

(*) H is real and symmetric $\rightarrow$ can be decomposed into a set of real eigenvalues and an orthogonal basis of eigenvectors.

$$C(\vec{\theta}) = C(\vec{\theta}^*) + \frac{1}{2}\Big[\underbrace{k_{11}^2\lambda_1 + k_{21}^2\lambda_2}\Big]$$

This is ≥ 0, and will be our weight for $\hat{\theta}_1$

Similarly for the direction of $\hat{\theta}_2$:

$$C(\vec{\theta}) = C(\vec{\theta}^*) + \frac{1}{2}\Big[\underbrace{k_{12}^2\lambda_1 + k_{22}^2\lambda_2}\Big]$$

where $\hat{\theta}_2 = k_{12}\hat{\alpha}_1 + k_{22}\hat{\alpha}_2$ $\quad \hookrightarrow$ weight for $\hat{\theta}_2$

k_{11} is the first component of $\hat{\alpha}_1$
k_{21} is the " " " $\hat{\alpha}_2$

$$K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} = [\hat{\alpha}_1 \ \hat{\alpha}_2]^T$$

$$[\hat{\alpha}_1 \ \hat{\alpha}_2] = K^T = \begin{pmatrix} k_{11} & k_{21} \\ k_{12} & k_{22} \end{pmatrix}, \quad \vec{\lambda} = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix}$$

$$W = \begin{pmatrix} k_{11}^2 & k_{21}^2 \\ k_{12}^2 & k_{22}^2 \end{pmatrix} \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

where w_1 is the weight of the time point for parameter 1, θ_1

and w_2 is the weight of the time point for parameter 2, θ_2

$$\therefore \quad W = [\hat{\alpha}_1^2 \ \hat{\alpha}_2^2] \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix}$$

$\hat{\alpha}_1$ and $\hat{\alpha}_2$ are 2D vectors.