## The equivalence between the isomorphism between two objects, the isomorphism of the pushforward, and the isomorphism of the pushback

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This short project is dedicated to proving an important theorem. In summary, it states that we can learn a lot about an object X by considering it's relationships to other objects.

In this document we will denote the morphisms between objects X and Y in a category C by C(X,Y). It is often written as  $\operatorname{Hom}_C(X,Y)$  or  $\operatorname{Hom}(X,Y)$  if the category is clear from context. We will write maps on the right.

**Theorem.** Let X and Y be objects from a category C. Then the following are equivalent:

- 1. The morphism  $f: X \to Y$  is an isomorphism;
- 2. For any object Z, the pushforward  $f_{\star}: C(Z,X) \to C(Z,Y)$  is an isomorphism (of sets);
- 3. For any object Z, the pullback  $f^*: C(Y,Z) \to C(X,Z)$  is an isomorphism (of sets).

*Proof.* We will prove  $1 \Leftrightarrow 3$ , then  $1 \Leftrightarrow 2$ , and finally  $2 \Rightarrow 3$ . For convenience, we have included relevant commutative diagrams in Figure 1.

 $(1 \Leftrightarrow 3)$  First, assume that f is an isomorphism. In other words, the inverse  $f^{-1}: Y \to X$  exists. We want to show that the pullback  $f^*$  is an isomorphism of sets. We will do this by constructing a map and showing it is the inverse of  $f^*$ . Consider  $(f^{-1})^*$ , the pullback of  $f^{-1}$ . Let  $g_1 \in C(Y, Z)$ . Then

$$g_1 f^*(f^{-1})^* = f^{-1} f g_1 = g_1.$$

Similarly, let  $g_2 \in C(X, Z)$ . Then

$$g_2(f^{-1})^* f^* = f f^{-1} g_2 = g_2.$$

Hence  $(f^{-1})^* = (f^*)^{-1}$ , and  $f^*$  is an isomorphism.

Next, assume that  $f^*$  is an isomorphism for all Z. We want to show that f is an isomorphism. Take Z = X, so the pullback is a map between C(Y, X) and C(X, X). Since f is surjective, we can find a  $h \in C(Y, X)$  such that  $hf^* = fh = \mathrm{id}_X$ . This seems like a good candidate for our inverse for f.

Now consider Z = Y, so the pullback is a map between C(Y,Y) and C(X,Y). Since

$$\mathrm{id}_Y f^\star = f \mathrm{id}_Y = f$$

and

$$(hf)f^* = f(hf) = (fh)f = \mathrm{id}_X f = f,$$

we can deduce that  $hf = id_Y$  as  $f^*$  is injective. Therefore h is an inverse for f, and f is an isomorphism.

 $(1 \Leftrightarrow 2)$  The reasoning to this is very similar to above, so we omit it.

 $(2 \Rightarrow 3)$  Assume  $f_*$  is an isomorphism. From above, we know that f is an isomorphism, so  $f^{-1}$  exists. Consider the pullback  $(f^{-1})^*: C(X,Z) \to C(Y,Z)$ . Let  $g_1 \in C(Y,Z)$ . Then

$$g_1 f^*(f^{-1})^* = g_1 f f^{-1} = g.$$

Similarly, let  $g_2 \in C(X, Z)$ . Then

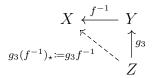
$$g_2(f^{-1})^* f^* = g_2 f^{-1} f = g.$$

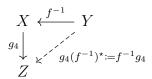
As  $g_1$  and  $g_2$  were arbitary, we have shown that  $(f^{-1})^* = (f^*)^{-1}$ , and therefore  $f^*$  is an isomorphism.

 $X \xrightarrow{f} Y$   $g_2 f_* := f g_2 \xrightarrow{g} Z$ 

(a) The pushforward  $f_{\star}$ .

(b) The pullback  $f^*$ .





(c) The pushforward  $(f^{-1})_{\star}$ .

(d) The pullback  $(f^{-1})^*$ .

Figure 1: Commutative diagrams of pushforwards and pullbacks.