Chapter-2

Applications of Definite Integrals

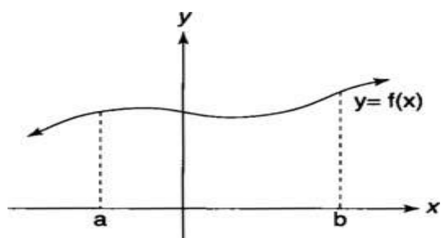
2.1 Area of Regions Between Two Graphs

Definite integrals could be used to determine the area of the region between the graph of a function and the *x*-axis or the *y*-axis.

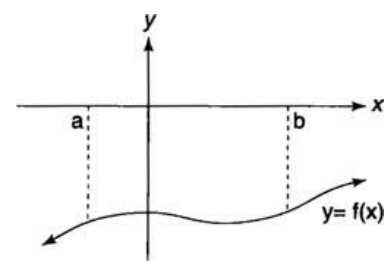
Recall that:

If $f(x) \geq 0$ or $f(x) \leq 0$ for $a \leq x \leq b$ then the area of the region bounded by the curve $y = f(x)$, the x −axis and the line $x = a$ and $x = b$ is

$$A = \int_a^b |f(x)|dx$$



$f(x) \geq 0$



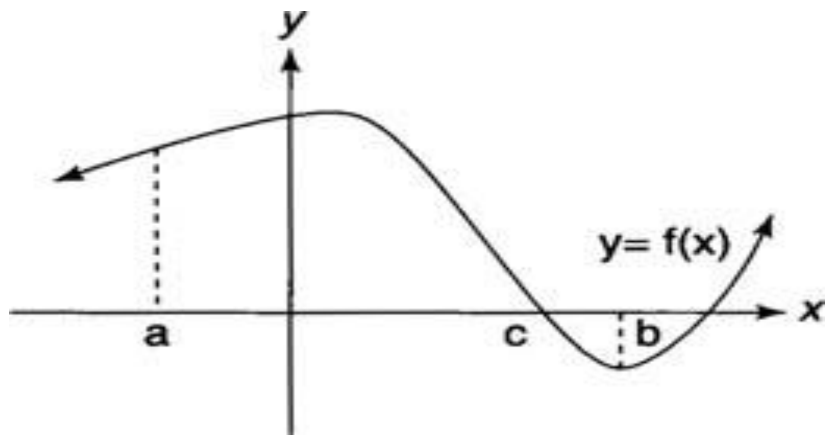
$f(x) \leq 0$

Similarly.
If g$(y) \geq 0$ or g$(y) \leq 0$ for c$\leq x \leq d$ then the area of the region bounded by the curve $x = f(y)$, the y −axis and the line $y = c$ and $y = d$ is

$$A = \int_c^d |g(y)|dy$$

Also,

If $f(x) \geq 0$ on $[a, c]$ and $f(x) \leq 0$ on $[c, b]$, then the area A of the region bounded by the graph of $f(x)$, the x −axis, and the lines $x = a$ and $x = b$ would be determined by the following definite integrals:

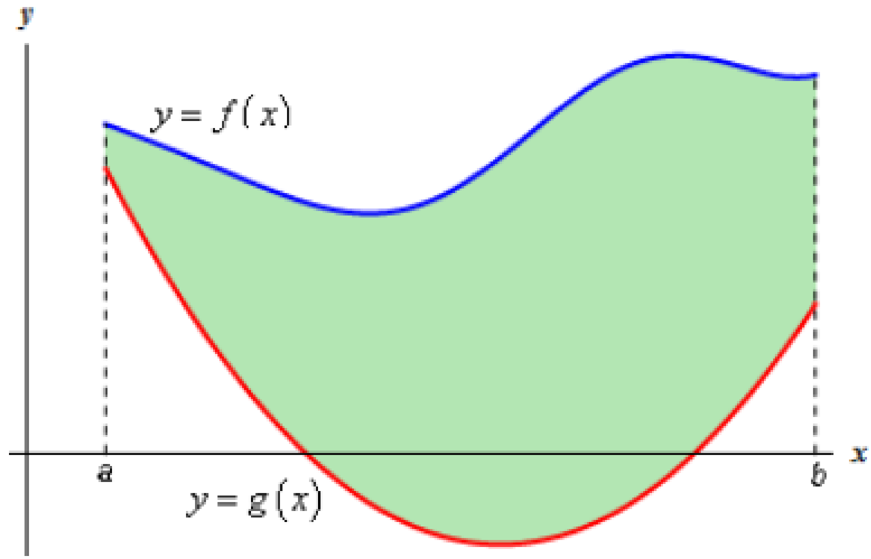


$$\mathrm{A} = \int_a^b |f(x)| dx \quad = \int_a^c f(x) dx - \int_c^b f(x) dx$$

Area Between Two Curves

First Case:

In the first case we want to determine the area between $y = f(x)$ and $y = g(x)$ on the interval $[a,b]$. We are also going to assume that $f(x) \geq g(x)$. Take a look at the following sketch to get an idea of what we're initially going to look at.



So the Area is,

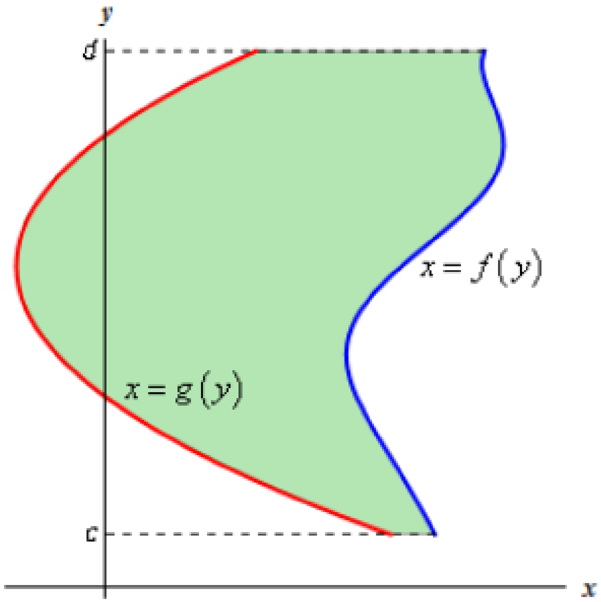
$$A = \int_a^b (f(x) - g(x))dx$$

In other words,

$$A = \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx, \qquad a \leq x \leq b$$

Second Case:

The second case is almost identical to the first case. Here we are going to determine the area between $x = f(y)$ and $x = g(y)$ on the interval $[c, d]$ with $f(y) \geq g(y)$.



So the Area is,

$$A = \int_c^d (f(y) - g(y))dy$$

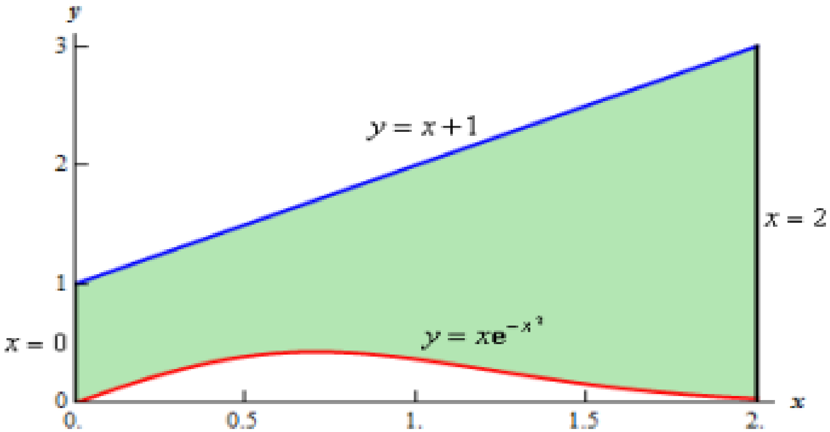
In other words,

$$A = \int_c^d \begin{pmatrix} \text{right} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{left} \\ \text{function} \end{pmatrix} dy, \qquad c \leq y \leq d$$

Example set-2.1

1. Write down the area in integral form and hence evaluate it

(a)



Solution:

Here, $y = x + 1$(Upper function)

$y = xe^{-x^2}$(Lower function)

$x = 0$ and $x = 2$

So the area is $A = \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx$

$= \int_0^2 x + 1 - xe^{-x^2}\, dx$

$= \int_0^2 x\, dx + \int_0^2 1\, dx - \int_0^2 x e^{-x^2} dx$

$= \left[\frac{x^2}{2}\right]_0^2 + [x]_0^2 - \int_0^4 x e^{-u} \frac{du}{2x}$

$= \frac{1}{2} \cdot 2^2 + 2 - \int_0^4 \frac{e^{-u}}{2}\, du$

$= 2 + 2 + \frac{1}{2}\left[e^{-u}\right]_0^4$

$= 4 + \frac{1}{2}\left[e^{-4} - e^0\right]$

$= 4 + \frac{1}{2}(0.0183 - 1)$

$= 3.509$

set,

$u = x^2$

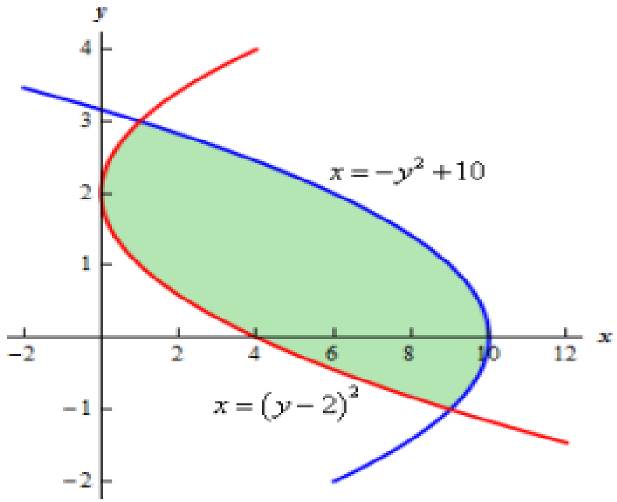
$du = 2x\,dx$

$dx = \frac{du}{2x}$

x	0	2
u	0	4

Example set-2.1

(b)



Solution:

Here, $x = -y^2 + 10$(Right function)

$x = (y-2)^2$ (Left function)

$y = -1$ and $y = 3$

So the area is, $A = \int_{-1}^{3} (-y^2 + 10 - (y-2)^2)dy$

$= \int_{-1}^{3} (-y^2 + 10 - y^2 + 4y - 4))dy$

$= \int_{-1}^{3} (-2y^2 + 4y + 6)dy$

$= \left[-2.\frac{y^3}{3} + 4 \cdot \frac{y^2}{2} + 6y\right]_{-1}^{3}$

$= \left[-\frac{2}{3} y^3 + 2y^2 + 6y\right]_{-1}^{3}$

$= \left(-\frac{2}{3} \cdot 3^3 + 2 . 3^2 + 6 . 3\right) - \left(-\frac{2}{3}(-1)^3 + 2(-1)^2 + 6(-1)\right)$

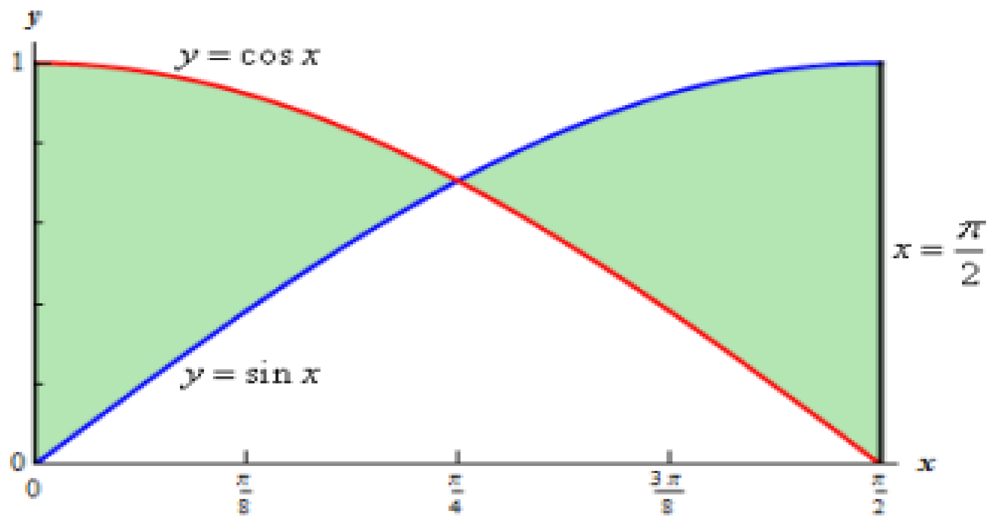
$= (-18 + 18 + 18) - \left(\frac{2}{3} + 2 - 6\right)$

$= 18 - 0{\cdot}66 - 2 + 6$

$= 21{\cdot}34$

Example set-2.1

(c)



Solution: The area is,

$$A = \int_0^{\frac{\pi}{4}} \cos x - \sin x \, dx + \int_{\pi/4}^{\pi/2} \sin x - \cos x \, dx$$

$$= (\sin x + \cos x)\Big|_0^{\frac{\pi}{4}} + (-\cos x - \sin x)\Big|_{\pi/4}^{\pi/2}$$

$$= \sqrt{2} - 1 + \left(\sqrt{2} - 1\right)$$

$$= 2\sqrt{2} - 2 = 0.828427$$

Example set-2.1

2. Sketch the region enclosed by $y = 9 - x^2$ and the x −axis. Hence find its area.

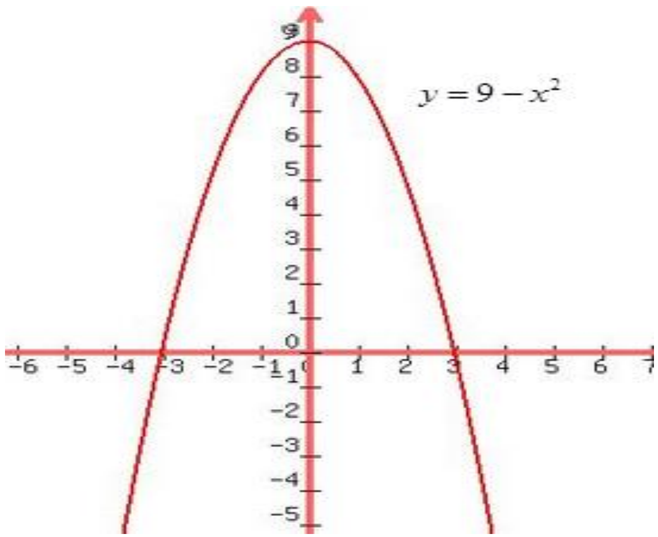
Solution: The region is shown in the Figure given below,

$y = 9 - x^2$......(1), $y = 0$ ... ... (2)(x −axis)

So, from eq (1) & (2):

$$9 - x^2 = 0$$

$$x^2 = 9 \quad \therefore x = \pm 3$$



So the area is,

$$A = \int_{-3}^{3} (9 - x^2 - 0)dx$$

$$= 2\int_{0}^{3} (9 - x^2)dx$$

$$= 36$$

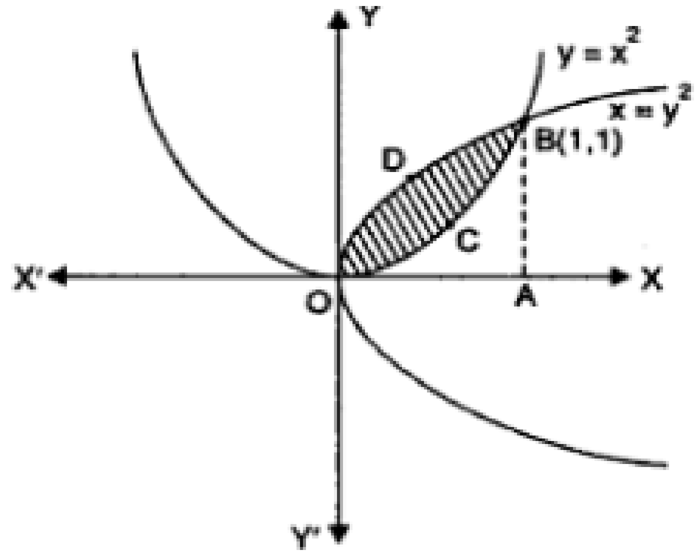
Example set-2.1

3. Sketch the region enclosed by the parabolas $y = x^2$ and $x = y^2$. Hence find its area.

Solution: The region is shown in the Figure given below,

$$y = x^2 \text{......}(1),\ x = y^2 \ \ \ldots \ldots (2)$$

$$y = \sqrt{x} \text{......}(3)$$



So, from eq (1) & (3):

$$x^2 = \sqrt{x}$$
$$x^4 = x$$
$$x(x^3 - 1) = 0 \qquad \therefore x = 0,1$$

So the area is,

$$= \int_0^1 (\sqrt{x} - x^2)dx$$

$$= \left(\frac{2}{3}x^{\frac{3}{2}} - \frac{1}{3}x^3\right)\Bigg|_0^1$$

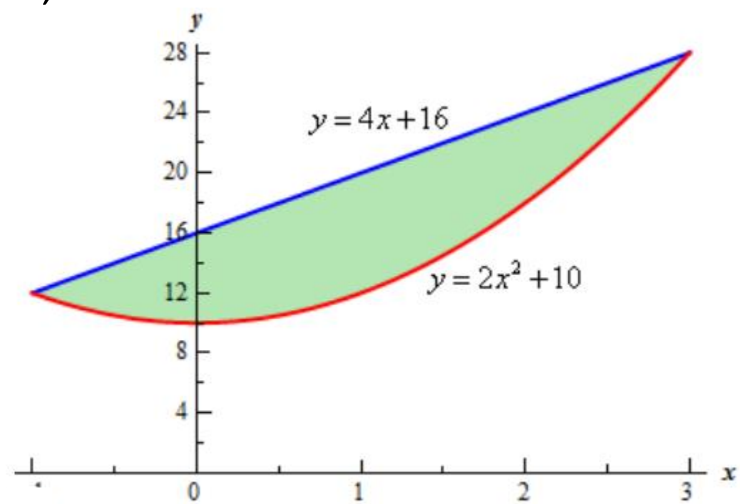
$$= \frac{1}{3}$$

Example set-2.1

4. Sketch the region enclosed by the parabolas $y = 2x^2 + 10$ and $y = 4x + 16$. Hence find its area.

Solution: The region is shown in the Figure given below,

$y = 2x^2 + 10$……(1) , $y = 4x + 16$……(2)



So, from eq (1) & (2):

$$2x^2 + 10 = 4x + 16$$
$$2x^2 - 4x - 6 = 0$$
$$2(x+1)(x-3) = 0$$
$$\therefore x = -1,3$$

So the area is,

$$A = \int_a^b \begin{pmatrix} \text{upper} \\ \text{function} \end{pmatrix} - \begin{pmatrix} \text{lower} \\ \text{function} \end{pmatrix} dx$$

$$= \int_{-1}^{3} 4x + 16 - (2x^2 + 10)\, dx$$

$$= \int_{-1}^{3} -2x^2 + 4x + 6\, dx$$

$$= \left(-\frac{2}{3}x^3 + 2x^2 + 6x \right) \Bigg|_{-1}^{3}$$

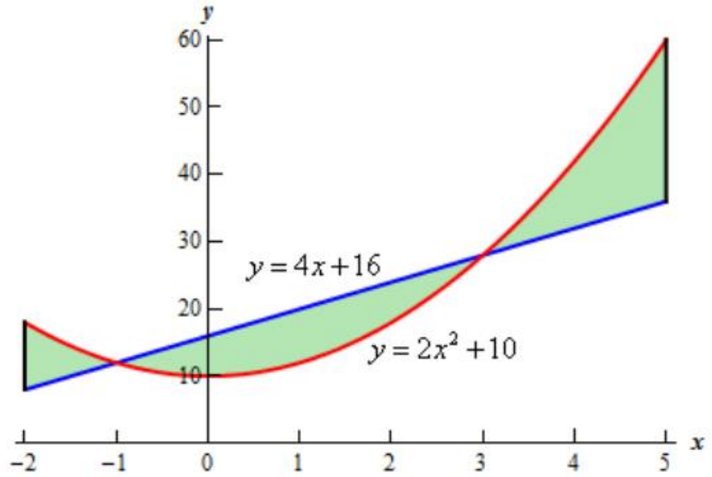
$$= \frac{64}{3}$$

Example set-2.1

5. Sketch the region enclosed by the parabolas $y = 2x^2 + 10$ and $y = 4x + 16$, $x = -2$ and $x = 5$ Hence find its area.

Solution: The region is shown in the Figure given below,

$y = 2x^2 + 10$……(1) , $y = 4x + 16$……(2)



So, from eq (1) & (2):

$$2x^2 + 10 = 4x + 16$$

$$2x^2 - 4x - 6 = 0$$

$$2(x+1)(x-3) = 0$$

$$\therefore x = -1, 3$$

So the area is,

$$A = \int_{-2}^{-1} 2x^2 + 10 - (4x+16)\,dx + \int_{-1}^{3} 4x + 16 - (2x^2+10)\,dx + \int_{3}^{5} 2x^2 + 10 - (4x+16)\,dx$$

$$= \int_{-2}^{-1} 2x^2 - 4x - 6\,dx + \int_{-1}^{3} -2x^2 + 4x + 6\,dx + \int_{3}^{5} 2x^2 - 4x - 6\,dx$$

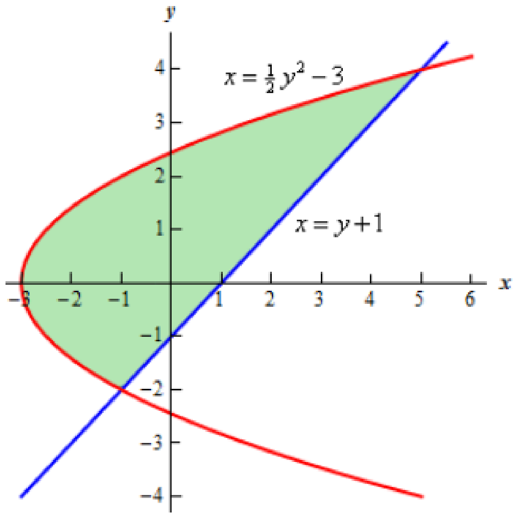
$$= \left(\frac{2}{3}x^3 - 2x^2 - 6x\right)\Bigg|_{-2}^{-1} + \left(-\frac{2}{3}x^3 + 2x^2 + 6x\right)\Bigg|_{-1}^{3} + \left(\frac{2}{3}x^3 - 2x^2 - 6x\right)\Bigg|_{3}^{5}$$

$$= \frac{14}{3} + \frac{64}{3} + \frac{64}{3}$$

$$= \frac{142}{3}$$

Example set-2.1

6. Determine the area enclosed by $x = \frac{1}{2}y^2 - 3$ and $y = x - 1$



Solution:

$x = \frac{1}{2}y^2 - 3$ ......(1) and $y = x - 1$......(2)

$x = y + 1$......(3)

So, from eq (1) & (3):

$x = \frac{1}{2}y^2 - 3$ and $x = y + 1$

$$y + 1 = \frac{1}{2}y^2 - 3$$

$$2y + 2 = y^2 - 6$$

$$0 = y^2 - 2y - 8$$

$$0 = (y-4)(y+2)$$

$$\therefore y = -2, 4$$

So the area is,

$$A = \int_c^d \begin{pmatrix}\text{right}\\ \text{function}\end{pmatrix} - \begin{pmatrix}\text{left}\\ \text{function}\end{pmatrix} dy$$

$$= \int_{-2}^{4} (y+1) - \left(\frac{1}{2}y^2 - 3\right) dy$$

$$= \int_{-2}^{4} -\frac{1}{2}y^2 + y + 4\,dy$$

$$= \left(-\frac{1}{6}y^3 + \frac{1}{2}y^2 + 4y\right)\Bigg|_{-2}^{4}$$

$$= 18$$

Exercise set-2.1

1. Sketch the region enclosed by the following curves and then find its area.

(a) $y = f(x) = x,\ 1 \le x \le 3$ and the x-axis.

(b) $y = f(x) = x^3,\ 1 \le x \le 2$ and the x-axis.

(c) $y = f(x) = x^2 + x + 4,\ 1 \le x \le 3.$

(d) $y = f(x) = \sin x\,, 0 \le x \le \frac{3\pi}{2}$ and the x-axis.

(e) $y = x^2 + 2$, the x-axis and the lines $x = 1$ and $x = 2$.

(f) $y = x^2 - 4$ and the x-axis.

(g) $x = 1 - y^2$ and the y-axis.

(h) $y = f(x) = x(1 - x)(2 - x)$ and the x-axis.

2. Sketch the region enclosed by the following curves and then find its area.

(a) $y = x^2$ and $y = x$

(b) $y = x(x-3)$ and the ordinates $x = 0, x = 5$

(c) $y = x^2$ and $y = 2 - x, x = 0, x \geq 0$

(d) $y = 3x - x^2$ and $y = x$

(e) $x = y^2$ and $y = x - 2$

(g) $y^2 = 4x + 4$ and $4x - y = 16$

3. **Calculus– James Stewart - 8th edition**

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