Chapter 7

Higher Order Linear Differential Equations

The general *n*th order linear differential equations with constant coefficients is

$$a_0 \frac{d^n y}{dx^n} + a_1 \frac{d^{n-1} y}{dx^{n-1}} + \cdot \cdot \cdot + a_{n-1} \frac{dy}{dx} + a_n y = f(x)$$

where $a_{0,}\ a_1, \cdot \cdot \cdot, a_n$ are constants.

In terms of differential operator $D \equiv \dfrac{d}{dx}$, it can be written as

$$(a_0 D^n + a_1 D^{n-1} + \cdot \cdot \cdot + a_{n-1} D + a_n) y = f(x)$$

or, in symbolic form, as $L(D)y = f(x)$

Where $L(D) \equiv a_0 D^n + a_1 D^{n-1} + \cdot \cdot \cdot + a_{n-1} D + a_n$ is a polynomial in *D* of degree *n*.

The equation is said to be homogeneous when $f(x) = 0$ and to be non-homogeneous when $f(x) \neq 0$.

We shall now concentrate on the solution of second order LDEs. The general form of the equation is

$$a_0 \frac{d^2 y}{dx^2} + a_1 \frac{dy}{dx} + a_2 y = 0$$

where a_0, a_1, a_2 are constants.

A trial solution $y = e^{mx}$ leads to the auxiliary equation (AE)

$$a_0 m^2 + a_1 m + a_2 = 0.$$

Suppose the roots of the auxiliary equation are m_1 and m_2.

The general solution, depending on the nature of the roots, may be expressed in any one of the following forms:

(a) If $m_1 \neq m_2$ and both real, the solution is $y = Ae^{m_1 x} + Be^{m_2 x}$.

(a) $(D^2 - 3D + 2)y = 0$

$m^2 - 3m + 2 = 0$

$\Rightarrow (m-1)(m-2) = 0$

$\therefore m = 1, 2$

$y = c_1 e^x + c_2 e^{2x}$

(b) If $m_1 = m_2 = \alpha$ (say),

the two solutions are not independent and becomes $y = (A + Bx)e^{\alpha x}$

(b) $(D^2 - 4D + 4)y = 0$

$m^2 - 4m + 4 = 0$

$\Rightarrow (m-2)^2 = 0$

$\therefore m = 2, 2$

$y = (c_1 + c_2 x)e^{2x}$

(c) If m_1 and m_2 are complex, say $\alpha \pm i\beta$,

the corresponding general solution is $y = e^{\alpha x}(A\cos\beta x + B\sin\beta x)$

(c) $(D^2 - D + 4)y = 0$

$m^2 - m + 4 = 0$

$\Rightarrow m = \frac{+1 \pm \sqrt{1-16}}{2}$ $\left[\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}\right]$

$= \frac{1}{2} \pm \frac{\sqrt{-15}}{2}$

$\therefore m = \frac{1}{2} \pm \frac{\sqrt{15}}{2} i$

Comparing with $m = \alpha \pm i\beta$

$y = \left(c_1 \cos \frac{\sqrt{15}}{2}x + c_2 \sin \frac{\sqrt{15}}{2}x\right)e^{\frac{1}{2}x}$

The above rules may be extended for higher order equations.

Let we are given the auxiliary roots of certain homogeneous DES, write down the general solution based on the previous discussions,

- If $m = 2, -3, 5$ then $y = c_1 e^{2x} + c_2 e^{-3x} + c_3 e^{5x}$
- If $m = 3, 3$ then $y = (c_1 + c_2 x) e^{3x}$
- If $m = 2 \pm 3i$ then $y = e^{2x}(A \cos 3x + B \sin 3x)$

Example 7.1 Find the general solution to $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0$.

Solution: Let $D \equiv \frac{d}{dt}$, then the given equation can be written as

$$(D^2 + 3D + 2)y = 0 \text{.....(1)}$$

Let $y = e^{mt}$ is a trial solution of eqn(1).

Therefore the auxiliary equation is

$$m^2 + 3m + 2 = 0$$

Solving, $m = -1, -2$ (the roots are real & unequal)

Thus the general solution is

$$y = c_1 e^{-t} + c_2 e^{-2t}$$

Example 7.2 Find the general solution to $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$

Solution: Let $D \equiv \frac{d}{dx}$, then the given equation can be written as

$$(D^2 + 6D + 9)y = 0\text{.....(1)}$$

Let $y = e^{mx}$ is a trial solution of eqn(1).

Therefore the auxiliary equation is

$$m^2 + 6m + 9 = 0$$

Solving, $m = -3, -3$ (the roots are real & equal)

Thus the general solution is

$$y = (c_1 + c_2 x)e^{-3x}$$

Example 7.3 Find the solution of the initial value problem

$$\frac{d^2y}{dt^2} + 2\frac{dy}{dt} + 2y = 0, y(0) = 0, y'(0) = 0.$$

Solution: Let $D \equiv \frac{d}{dt}$, then the given equation can be written as

$$(D^2 + 2D + 2)y = 0\text{.....(1)}$$

Let $y = e^{mt}$ is a trial solution of eqn(1).

Therefore the auxiliary equation is

$$m^2 + 2m + 2 = 0$$

Solving, $m = -1 \pm i$ (the roots are complex) [using formula]

Thus the general solution is [with α= - 1, β = 1]

$$y = e^{-t}(A\cos t + B\sin t)$$

Now we are going to apply the initial conditions.

$$y = e^{-t}(A\cos t + B\sin t)\text{......(2)}$$

$$y' = -e^{-t}(A\cos t + B\sin t) + e^{-t}(-A\sin t + B\cos t)\text{......(3)}$$

Using $y(0) = 0$, (2) becomes

$0 = 1(A + 0) \Rightarrow A = 0$

Similarly using $y'(0) = 0$, (3) gives

$$B = 1$$

Therefore eq. (1) becomes

$$y = e^{-t}\sin t$$

Class Practice

Find the general solution of the following differential equations:

1. $\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = 0.$
2. $\frac{d^2y}{dt^2} + 5\frac{dy}{dt} + 6y = 0.$
3. $\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + y = 0.$
4. $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = 0, y(0) = 2, y\,'(0) = -3.$

Non-homogeneous LDEs

The solution of the non-homogeneous linear differential equation

$$L(D)y = f(x)$$

is of the form

$$y = y_c + y_p$$

where y_c is the general solution of $L(D)y = 0$ and y_p is a particular solution of $L(D)y = f(x)$.

The particular solution y_p is called a particular integral (PI) and the general solution y_c of the related homogeneous equation is the **complementary function** (CF).

The method of undetermined Coefficient

	Corresponding RHS $f(x)$	Assumed form of y_p
1.	polynomial of degree n ($n = 0,1,2,3 \cdots$)	polynomial of degree n
2.	$e^{\alpha x}, \alpha \neq 0$ (exponential)	$Ae^{\alpha x}$
3.	sin αx / cos αx , $\alpha \neq 0$(sine/cosine)	$Asin\alpha x + B\ cos\alpha x$

Note: If the trial PS solutions duplicate terms found in y_c, then multiply the trial solution by x repeatedly until it doesn't. The final trial PS solution is the modified expression.

Example 7.4 Find the general solution of $\frac{d^2y}{dt^2} + 4y = 3t + 2$

Solution: Let $D \equiv \frac{d}{dt}$, then the given equation can be written as

$(D^2 + 4)y = 3t + 2$ ……(1)

Consider, $(D^2 + 4)y = 0$…….(2)

Let $y = e^{mt}$ is a trial solution of eqn(2).

Therefore the auxiliary equation is $m^2 + 4 = 0, \Rightarrow \mathrm{m} = \pm 2i$

The complimentary function is $y_c = \mathrm{A}\cos 2t + B\sin 2t$

Again let $y_p = a_0 + a_1 t$

$Dy_p = a_1$

$D^2 y_p = 0$

Subtitling the values of y_p and its derivative into the given differential equation (1), we have

$$4\,a_0 + 4\,a_1 t = 2 + 3t$$

Now, equating like terms, we have,

$$4\,a_0 = 2$$

$$4\,a_1 = 3$$

Solving $a_0 = \frac{1}{2}$ and $a_1 = \frac{3}{4}$

The general solution is

$$y = y_c + y_p$$

$$= \mathrm{A}\cos 2t + B\sin 2t + \frac{1}{2} + \frac{3}{4}\,t$$

Example 7.5 Find the general solution of $\frac{d^2y}{dt^2} - \frac{dy}{dt} - 6y = e^{4t}$

Solution: Let $D \equiv \frac{d}{dt}$, then the given equation can be written as

$$(D^2 - D - 6)y = e^{4t} \text{ … … } (1)$$

Consider, $(D^2 - D - 6)y = 0$…….(2)

Let $y = e^{mt}$ is a trial solution of eqn(2).

Therefore the auxiliary equation is $m^2 - m - 6 = 0$

Solving $m = -2, 3$

The complimentary function is $y_c = c_1e^{-2t} + c_2e^{3t}$

let $y_p = Ae^{4t}$

$$Dy_p = 4Ae^{4t}$$

$$D^2y_p = 16Ae^{4t}$$

(1) implies $(D^2 - D - 6)y_p = e^{4t}$

$$16Ae^{4t} - 4Ae^{4t} - 6Ae^{4t} = e^{4t}$$

$\Rightarrow \quad 6Ae^{4t} = e^{4t}$

Comparing $6A = 1 \Rightarrow A = \frac{1}{6}$

Therefore $y = c_1e^{-2t} + c_2e^{3t} + \frac{1}{6}\ \mathrm{e}^{4\mathrm{t}}$

Example 7.6 Find the general solution of $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = \cos t$

Solution: Let $D \equiv \frac{d}{dt}$, then the given equation can be written as

$$(D^2 + 3D + 2)y = \cos t \dots \dots (1)$$

Consider, $(D^2 + 3D + 2)y = 0$.......(2)

Let $y = e^{mt}$ is a trial solution of eqn(2).

Therefore the auxiliary equation is $m^2 + 3m + 2 = 0$

Solving $m = -2, -1$

The complimentary function is $y_c = c_1 e^{-2t} + c_2 e^{-t}$

let $y_p = A\cos t + B\sin t$

$$Dy_p = -A\sin t + B\cos t$$

$$D^2 y_p = -A\cos t - B\sin t$$

(1) implies

$$-A\cos t - B\sin t + 3(-A\sin t + B\cos t) + 2(A\cos t + B\sin t) = \cos t$$

Simplifying and equating both sides

$$A + 3B = 1$$

$$-3A + B = 0$$

Solving $A = \frac{1}{10}, \ \ B = \frac{3}{10}$

Therefore the general solution is

$$y = y_c + y_p = c_1 e^{-2t} + c_2 e^{-t} + \frac{1}{10}\cos t + \frac{3}{10}\sin t$$

Example 7.7 Solve $\frac{d^2y}{dt^2} + 7\frac{dy}{dt} + 10y = e^{-2t}$

Solution: Let $D \equiv \frac{d}{dt}$, then the given equation can be written as

$$(D^2 + 7D + 10)y = e^{-2t} \dots\dots (1)$$

Consider, $(D^2 + 7D + 10)y = 0$.......(2)

Let $y = e^{mt}$ is a trial solution of eqn(2).

Therefore the auxiliary equation is $m^2 + 7m + 10 = 0$

Solving $m = -5, -2$

The complimentary function is $y_c = c_1e^{-5t} + c_2e^{-2t}$

let $y_p = Ae^{-2t}$

Note that some terms of y_c match those of y_p. Multiplication once by t is necessary to eliminate duplicates. Then the final form of the particular solution is

$$y_p = A\,t\,e^{-2t}$$

Therefore proceeding in similar way , we'll have

$$3Ae^{-2t} = e^{-2t}$$
$$\Rightarrow 3A = 1$$
$$\Rightarrow A = \frac{1}{3}$$

Therefore the general solution is

$$y = y_c + y_p = c_1 e^{-5t} + c_2 e^{-2t} + \frac{1}{3}\, t\, e^{-2t}$$

Class practice

Find the solution of the following differential equations

1. $\frac{d^2y}{dt^2} + 3\frac{dy}{dt} + 2y = t^2.$
2. $\frac{d^2y}{dt^2} + 4y = t - \frac{t^2}{20}.$
3. $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2\,e^{-3t}.$
4. $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = \ e^{-t}, y(0) = y'(0) = 0.$
5. $\frac{d^2y}{dt^2} + 9y = e^{-t}.$
6. $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = \cos t.$
7. $\frac{d^2y}{dt^2} + 6\frac{dy}{dt} + 8y = 2\cos 3t\,, y(0) = y'(0) = 0.$