Chapter-3

Improper Integrals

Gamma Function

and

Beta Function

3.1 Improper Integrals

An improper integral is an extended concept of a definite integral that has infinite limits on one or both ends of the interval and/or an integrand that becomes infinite at one or more points within the interval of integration .

Improper integral is called convergent if the limit of the integral exists with finite value and divergent if the limit of the integral does not exist or has infinite value.

Example 3.1.1

$$\int_0^1 x dx = \left[\frac{x^2}{2}\right]_0^1 = \left[\frac{1}{2} - 0\right] = \frac{1}{2}$$

Definite Integral

Improper integral with **infinite** limit

Example 3.1.2

$$\int_1^\infty \frac{dx}{x^3} = \lim_{b\to\infty} \int_1^b \frac{dx}{x^3} = \lim_{b\to\infty} \left[-\frac{1}{2x^2}\right]_1^b$$
$$= \lim_{b\to\infty} \left[-\frac{1}{2b^2} + \frac{1}{2}\right] = \frac{1}{2}$$

Convergent

Example 3.1.3

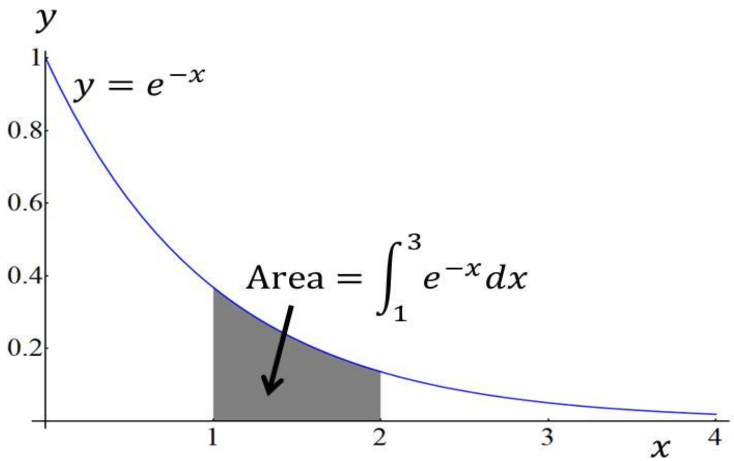
$$\int_{-\infty}^0 \frac{xdx}{1+x^2} = \lim_{a\to-\infty} \int_a^0 \frac{xdx}{1+x^2} = \lim_{a\to-\infty} \left[\frac{1}{2}\ln(x^2+1)\right]_a^0 \Rightarrow \quad \left[\because \int \frac{f'(x)}{f(x)} = ln(f(x)) + c\right]$$
$$= \lim_{a\to-\infty} \left[-\frac{1}{2}\ln(a^2+1)\right] \to -\infty$$

Divergent

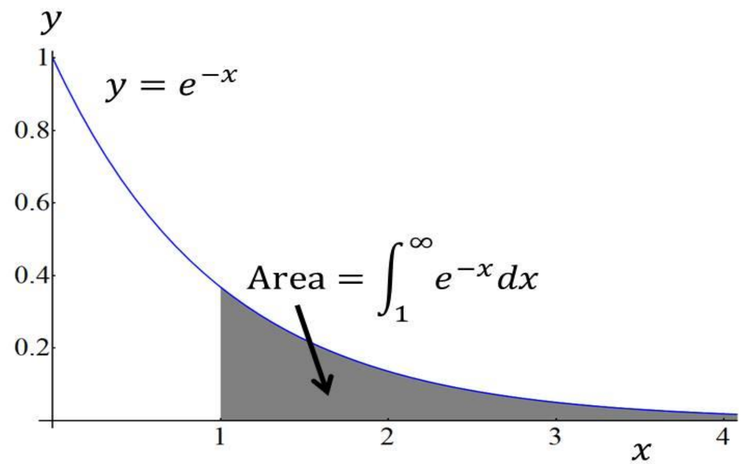
Example 3.1.4

$$\int_{-\infty}^{\infty} \frac{dx}{1+(x-1)^2} = \int_{-\infty}^{1} \frac{dx}{1+(x-1)^2} + \int_{1}^{\infty} \frac{dx}{1+(x-1)^2}$$

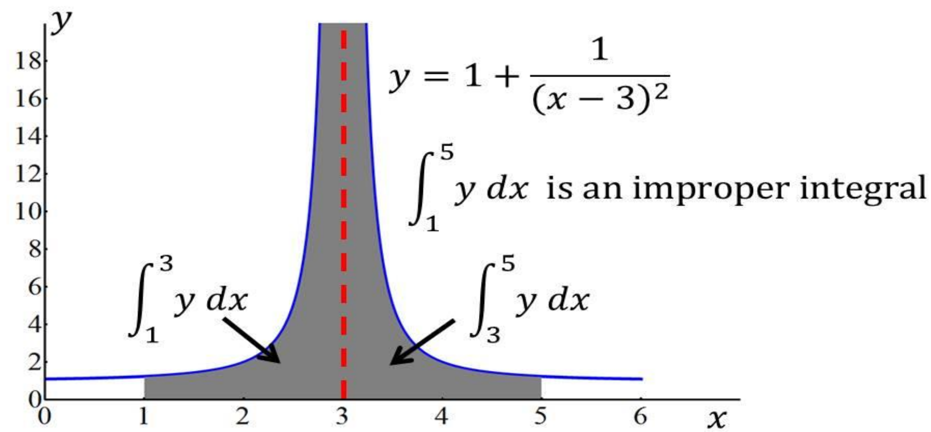
Geometrical concept of the definite integral and improper integral.



Definite Integral



Improper Integral



3.2 The Gamma function

The gamma function is denoted by $\Gamma(n)$ is defined by

$$\Gamma(n) = \int_0^\infty x^{n-1} e^{-x} dx$$

where, for convergence of the integral, $n > 0$.

Some useful formula

1. $\Gamma(n+1) = n\Gamma(n)$ or $\Gamma(n) = (n-1)\Gamma(n-1)$, when n is fraction

2. $\Gamma(n+1) = n!$ or $\Gamma(n) = (n-1)!$, when n is an integer

3. $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$

4. $\Gamma(n) = \dfrac{\Gamma(n+1)}{n}$, when n is negative

Example 3.2.1

$$\Gamma\left(\frac{5}{2}\right) = \Gamma\left(\frac{3}{2}+1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}\Gamma\left(\frac{1}{2}+1\right) = \frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}.\frac{1}{2}\sqrt{\pi}$$

OR

$$\Gamma\left(\frac{5}{2}\right) = \left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}.\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right) = \frac{3}{2}.\frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{3}{2}.\frac{1}{2}\sqrt{\pi}$$

Example 3.2.2

$$\Gamma(-\frac{5}{2}) = \frac{\Gamma(-\frac{5}{2}+1)}{(-\frac{5}{2})} = \frac{\Gamma(-\frac{3}{2})}{(-\frac{5}{2})} = \frac{\Gamma(-\frac{3}{2}+1)}{(-\frac{5}{2})(-\frac{3}{2})} = \frac{\Gamma(-\frac{1}{2})}{(-\frac{5}{2})(-\frac{3}{2})} = \frac{\Gamma(-\frac{1}{2}+1)}{(-\frac{5}{2})(-\frac{3}{2})(-\frac{1}{2})} = \frac{\Gamma(\frac{1}{2})}{(-\frac{5}{2})(-\frac{3}{2})(-\frac{1}{2})} = -\frac{8}{15}\sqrt{\pi}.$$

Example 3.2.3

$$\Gamma(4) = (4-1)! = 3! = 6 \text{ or}$$
$$\Gamma(4) = \Gamma(3+1) = 3! = 6$$

Class practice

1. $\Gamma\left(-\frac{1}{2}\right) = ?$
2. $\frac{\Gamma\left(\frac{5}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = ?$
3. $\frac{6\Gamma\left(\frac{8}{3}\right)}{5\Gamma\left(\frac{2}{3}\right)} = ?$
4. $\Gamma(6) = ?$

Home work

3.2.1

1. $\Gamma\left(\frac{7}{2}\right)$
2. $\frac{\Gamma\left(\frac{9}{2}\right)}{\Gamma\left(\frac{1}{2}\right)}$
3. $\frac{2\Gamma\left(\frac{10}{3}\right)}{3\Gamma\left(\frac{1}{3}\right)}$
4. $\Gamma(10)$
5. $\Gamma\left(-\frac{2}{7}\right)$

Example 3.2.4: Solve the integral $\int_0^{\infty} x^5 e^{-x} dx$

Solution:

$$\int_0^{\infty} x^5 e^{-x} dx = \int_0^{\infty} x^{5+1-1} e^{-x} dx = \int_0^{\infty} x^{6-1} e^{-x} dx$$

$$= \Gamma(6) \qquad \because \Gamma(n) = \int_0^{\infty} t^{n-1} e^{-t} dt$$

$$= 5!$$
$$= 120$$

Example 3.2.5: Solve the integral $\int_0^\infty x^{\frac{3}{2}} e^{-x} dx$

Solution:

$$\int_0^\infty x^{\frac{3}{2}} e^{-x} dx = \int_0^\infty x^{\frac{3}{2}+1-1} e^{-x} dx = \int_0^\infty x^{\frac{5}{2}-1} e^{-x} dx$$

$$= \Gamma\left(\frac{5}{2}\right) \qquad \because \Gamma(n) = \int_0^\infty t^{n-1} e^{-t} dt$$

$$= \left(\frac{5}{2} - 1\right)\Gamma\left(\frac{5}{2} - 1\right) = \frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{3}{2}.\left(\frac{3}{2} - 1\right)\Gamma\left(\frac{3}{2} - 1\right) = \frac{3}{2}.\frac{1}{2}.\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{3}{2}.\frac{1}{2}.\sqrt{\pi} = \frac{3}{4}.\sqrt{\pi}$$

Example 3.2.6: Consider the integral $\int_0^\infty \frac{e^{-4x}}{x^{1/2}}\,dx$.

Solution:

$$\int_0^\infty \frac{e^{-4x}}{x^{1/2}}\,dx = \int_0^\infty e^{-4x} x^{-\frac{1}{2}}\,dx$$

Let, $u = 4x \quad i.e. \;\; x = u/4 \;\; \therefore \, dx = \frac{du}{4}$

Changing Limit

x	*u*
0	0
∞	∞

The integral becomes

$$\int_0^\infty e^{-u}\left(\frac{u}{4}\right)^{-\frac{1}{2}} . \frac{du}{4}$$

$$= \frac{1}{2}\int_0^\infty u^{-1/2} e^{-u}\,du$$

$$= \frac{1}{2}\Gamma\left(\frac{1}{2}\right) \qquad \because \Gamma(n) = \int_0^\infty t^{n-1}e^{-t}dt$$

$$= \frac{1}{2}\sqrt{\pi}\,.$$

Home work
3.2.2

(a) $\int_0^\infty x^6\, e^{-x} dx$ (b) $\int_0^\infty \sqrt{x} e^{-3x} dx$ (c) $\int_0^\infty x^4 e^{-x^2} dx$

(d) $\int_0^\infty x^5 e^{-2x^2}\, dx$ (e) $\int_0^\infty \sqrt{y}\, e^{-y^2}\, dy$

3.3 The Beta function

The beta function is denoted by $\mathrm{B}(m,n)$ is defined by

$$\mathrm{B}(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\,dx$$

where, for convergence of the integral, $m > 0$, $n > 0$.

Relation between the Gamma- and Beta Functions

$$\mathrm{B}(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Gamma- and Beta Functions could be used to solve the following particular integral

$$\int_0^{\pi/2} \sin^p x \cos^q x \;\; dx = \frac{1}{2}\mathrm{B}\left(\frac{p+1}{2}, \frac{q+1}{2}\right) == \frac{\Gamma\left(\frac{p+1}{2}\right)\,\Gamma\left(\frac{q+1}{2}\right)}{2\,\Gamma\left(\frac{p+q+2}{2}\right)}$$

Example 3.3.1: Evaluate $B(3,2)$

Solution:

$$\mathrm{B}(3,2) = \frac{\Gamma(3)\Gamma(2)}{\Gamma(5)} = \frac{2!\,1!}{4!} = \frac{1}{12}$$

Example 3.3.2: Evaluate $B\left(\frac{3}{2},\frac{1}{2}\right)$

Solution:

$$\mathrm{B}\left(\frac{3}{2},\frac{1}{2}\right) = \frac{\Gamma\left(\frac{3}{2}\right)\Gamma\left(\frac{1}{2}\right)}{\Gamma(2)} = \frac{\left(\frac{3}{2}-1\right)\Gamma\left(\frac{3}{2}-1\right)\sqrt{\pi}}{1!} = \frac{\frac{1}{2}\Gamma\left(\frac{1}{2}\right)\sqrt{\pi}}{1!} = \frac{1}{2}.\sqrt{\pi}.\sqrt{\pi} = \frac{\pi}{2}$$

Example 3.3.4: Solve the integral $\int_0^1 t^4(1-t)^3 dt$

Solution:

$$\int_0^\infty t^4(1-t)^3 dt = \int_0^\infty t^{4+1-1}(1-t)^{3+1-1} dt$$

$$= \int_0^\infty t^{5-1}(1-t)^{4-1} dt$$

$$= B(5,4)$$

$$\mathrm{B}(m,n) = \int_0^1 x^{m-1}(1-x)^{n-1}\, dx$$

$$= \frac{\Gamma(5)\ \Gamma(4)}{\Gamma(9)} = \frac{4!\,3!}{8!} = \frac{1}{280}$$

$$\mathrm{B}(m,n) = \frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}$$

Example 3.3.5: Solve the integral $\int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{1}{2}}dx$

Solution:

$$\int_0^1 x^{\frac{3}{2}}(1-x)^{\frac{1}{2}}dx = \int_0^1 x^{\frac{3}{2}+1-1}(1-x)^{\frac{1}{2}+1-1}dx$$

$$= \int_0^1 x^{\frac{5}{2}-1}(1-x)^{\frac{3}{2}-1}dx$$

$$= B\left(\frac{5}{2},\frac{3}{2}\right)$$

$$= \frac{\Gamma\left(\frac{5}{2}\right)\ \Gamma\left(\frac{3}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{3}{2}\right)} = \frac{\Gamma\left(\frac{5}{2}\right)\ \Gamma\left(\frac{3}{2}\right)}{\Gamma(4)} = \frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}.\frac{1}{2}\cdot\sqrt{\pi}}{3!} = \frac{\pi}{16}$$

Example 3.3.6 Solve the integral $\int_0^2 \frac{x^2}{\sqrt{2x - x^2}} dx$

Solution:

$$I = \int_0^2 \frac{x^2}{\sqrt{2x - x^2}} dx$$

$$= \int_0^2 \frac{x^2}{\sqrt{2x\left(1 - \frac{x}{2}\right)}} dx$$

$$= \frac{1}{\sqrt{2}} \int_0^2 x^2 . x^{-\frac{1}{2}} \left(1 - \frac{x}{2}\right)^{-\frac{1}{2}} dx$$

$$= \frac{1}{\sqrt{2}} \int_0^2 x^{\frac{3}{2}} \left(1 - \frac{x}{2}\right)^{-\frac{1}{2}} dx$$

Let, $u = \frac{x}{2}$ $i.e.$ $x = 2u$ $\therefore dx = 2du$

Changing Limit

x	u
0	0
2	1

$$I = \frac{1}{\sqrt{2}} \int_0^1 (2u)^{\frac{3}{2}} (1-u)^{-\frac{1}{2}} 2du$$

$$= 2.\, 2^{-\frac{1}{2}}.\, 2^{\frac{3}{2}} \int_0^1 u^{\frac{3}{2}+1-1} (1-u)^{-\frac{1}{2}+1-1} du$$

$$= 4 \int_0^1 u^{\frac{5}{2}-1} (1-u)^{\frac{1}{2}-1} du$$

$$= 4B\left(\frac{5}{2}, \frac{1}{2}\right)$$

$$= 4. \frac{\Gamma\left(\frac{5}{2}\right)\ \Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{5}{2}+\frac{1}{2}\right)} = 4. \frac{\Gamma\left(\frac{5}{2}\right)\ \Gamma\left(\frac{1}{2}\right)}{\Gamma(3)} = 4. \frac{\frac{3}{2} \cdot \frac{1}{2} \cdot \sqrt{\pi}.\sqrt{\pi}}{2!} = 4. \frac{3\pi}{8} = \frac{3\pi}{2}$$

Home work
3.3

(a) $\int_0^1 x^4(1-x)^3\, dx$ (b) $\int_0^4 \frac{x^2}{\sqrt{4-x}}\, d$ (c) $\int_0^1 y^4\sqrt{1-y^2}\, dy$

(d) $B\left(\frac{7}{2},1\right)$ $(e)\ B(10{,}11)$