INTEGRAL CALCULUS
AND
ORDINARY DIFFERENTIAL EQUATIOSNS

METHODS OF INTEGRATION

Integration by Parts

$$\int u \cdot v\, dx = u \int v\, dx - \int \left(\frac{du}{dx} \int v\, dx \right) dx$$

Example: Evaluate $\int x^2 e^{3x} dx$

Here $u = x^2, v = e^{3x}$

$$\int x^2 e^{3x} dx = x^2 \int e^{3x} dx - \int \left(\frac{d}{dx} x^2 \int e^{3x} dx \right) dx$$
$$= x^2 \frac{1}{3} e^{3x} - \int \left(2x \frac{1}{3} e^{3x} \right) dx$$
$$= \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \int x \cdot\ e^{3x}\, dx$$
$$= \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \left[x \int e^{3x} dx - \int \left(\frac{d}{dx} x\ \int e^{3x} dx \right) dx \right]$$
$$= \frac{1}{3} x^2 e^{3x} - \frac{2}{3} \left[x \frac{1}{3} e^{3x} - \int 1 \cdot \frac{1}{3} e^{3x}\, dx \right]$$
$$= \frac{1}{3} x^2 e^{3x} - \frac{2}{9} x\, e^{3x} + \frac{2}{9} \int e^{3x}\, dx$$
$$= \frac{1}{3} x^2 e^{3x} - \frac{2}{9} x\, e^{3x} + \frac{2}{9} \cdot \frac{1}{3} e^{3x}$$
$$= \frac{1}{3} x^2 e^{3x} - \frac{2}{9} x\, e^{3x} + \frac{2}{27} e^{3x}$$

ILATE

I=Inverse
L=Log
A=Algebra
T=Trigonometry
E=e

$$\int u \cdot v \, dx = u \int v \, dx - \int \left(\frac{du}{dx} \int v \, dx \right) dx$$

Example: Evaluate $\int_1^2 x^2 \ln x \, dx$

$$\int x^2 \ln x \, dx = \ln x \int x^2 \, dx - \int \left[\frac{d}{dx} \ln x \int x^2 \, dx \right] dx$$

$$= \ln x \frac{x^3}{3} - \int \left[\frac{1}{x} \frac{x^3}{3} \right] dx$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{3} \int x^2 \, dx$$

$$= \frac{1}{3} x^3 \ln x - \frac{1}{3} \frac{x^3}{3}$$

$$\therefore \int_1^2 x^2 \ln x \, dx = \left[\frac{1}{3} x^3 \ln x - \frac{1}{9} x^3 \right]_1^2$$

$$= \left[\left(\frac{1}{3} 2^3 \ln 2 - \frac{1}{9} 2^3 \right) - \left(\frac{1}{3} 1^3 \ln 1 - \frac{1}{9} 1^3 \right) \right]$$

$$= \frac{8}{3} \ln 2 - \frac{8}{9} - 0 + \frac{1}{9}$$

$$= \frac{8}{3} \ln 2 - \frac{7}{9}$$

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Class Practice:

Evaluate

1. $\int x^2 \sin 2x \, dx$
2. $\int x \sin(2x + 1) \, dx$
3. $\int_0^{\pi} (2x^2 + 1) \cos 2x \, dx$

Home Work

Integration by parts (P-472) Example # 1, 2, 3, 4, 5

Page – 476 Ex # 3, 5, 6, 8, 17

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Integrals of the form $\int \sin Ax \cos Bx \, dx$, $\int \cos Ax \cos Bx \, dx$, $\int \sin Ax \sin Bx \, dx$

Necessary Trigonometric Formulas

$$sin\, A\, cos\, B = \frac{1}{2}[sin(A+B) + sin(A-B)]$$

$$cos\, A\, cos\, B = \frac{1}{2}[cos(A+B) + cos(A-B)]$$

$$sin\, A\, sin\, B = \frac{1}{2}[cos(A-B) - cos(A+B)]$$

Integrals of the form $\int \sin Ax \cos Bx \, dx$, $\int \cos Ax \cos Bx \, dx$, $\int \sin Ax \sin Bx \, dx$

Example: Evaluate $\int sin\, 7\, x\, cos\, 3\, x\, dx$

$$\int sin\, 7\, x\, cos\, 3\, x\, dx = \frac{1}{2}\int [sin\, 1\, 0x + sin\, 4\, x]dx$$

$$= -\frac{1}{20} cos\, 1\, 0x - \frac{1}{8} cos\, 4\, x + C$$

Class Practice:

Evaluate the following:

1. $\int \sin 4x \cos 4x \, dx$
2. $\int \sin 3x \sin 2x \, dx$
3. $\int_0^{\pi/6} cos\, 4\, x\, sin\, 2\, xdx$
4. $\int_0^{\pi/4} \cos 4x \cos x \, dx$

Home Work

Page 485. Ex: 41, 42

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Integration of irrational functions using trigonometric substitution

Expression in the integrand	Substitution
$\sqrt{a^2 - x^2}$	$x = a\sin\theta$
$\sqrt{a^2 + x^2}$	$x = a\tan\theta$
$\sqrt{x^2 - a^2}$	$x = a\sec\theta$

Example: Evaluate $\int \frac{dx}{x^2\sqrt{4-x^2}}$

Let, $x = 2\,sin\,\theta\,, dx = 2\,cos\,\theta\,d\theta$

So, $\int \frac{2\,cos\,\theta d\theta}{4\sin^2\theta\,\sqrt{4-4\sin^2\theta}} = \int \frac{2\cos\ \theta}{4\sin^2\theta\sqrt{4(1-\sin^2\theta)}}\,d\theta$

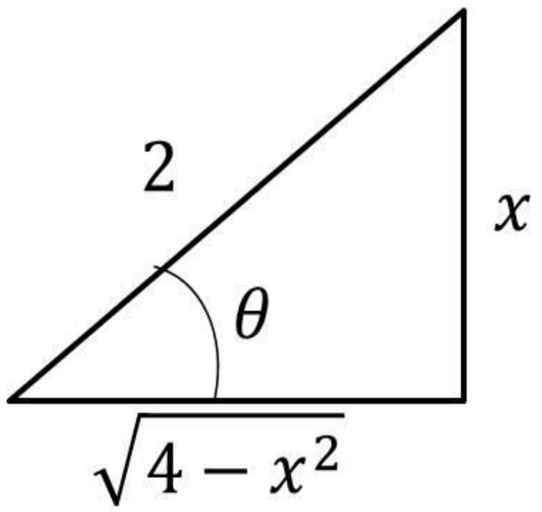
$$= \int \frac{2\cos\theta}{4\sin^2\theta\;2\sqrt{\cos^2\theta}}\,d\theta$$

$$= \frac{1}{4}\int \frac{d\theta}{\sin^2\theta}$$

$$= \frac{1}{4}\int \csc^2\theta\,d\theta$$

$$= -\frac{1}{4}\cot\theta + C$$

$$= -\frac{1}{4}\frac{\sqrt{4-x^2}}{x} + C.$$



Class Practice:

Evaluate the following:

1. $\int \frac{1}{\sqrt{4-x^2}}\,dx$

2. $\int_0^1 x\sqrt{1-x^2}\,dx$

3. $\int \frac{\sqrt{x^2-4}}{x}\,dx$

Home Work

Trigonometric Substitution (P-486) Example # 1, 6, 7

P – 491 Ex # 1-6, 10-14, 21-24

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Integration of the form $\int sin^m x \, cos^n x \, dx$

Strategy for Evaluating $\int \sin^m x \cos^n x \, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x \, dx = \int \sin^m x \, (\cos^2 x)^k \cos x \, dx$$

$$= \int \sin^m x \, (1 - \sin^2 x)^k \cos x \, dx$$

Then substitute $u = \sin x$.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x \, dx = \int (\sin^2 x)^k \cos^n x \, \sin x \, dx$$

$$= \int (1 - \cos^2 x)^k \cos^n x \, \sin x \, dx$$

Then substitute $u = \cos x$. [Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2} \sin 2x$$

Integration of the form $\int sin^m x \, cos^n x \, dx$

Evaluate $\int sin^4 x \, cos^5 x \, dx$

Solution: $\int sin^4 x \, cos^5 x \, dx = \int sin^4 x \, cos^4 x \, cos \, x \, dx$

$$= \int sin^4 x \, (cos^2 x)^2 \, cos \, x \, dx$$

$$= \int sin^4 x \, (1 - sin^2 x)^2 \, cos \, x \, dx$$

Let,$u = sin \, x \, , \frac{du}{dx} = cos \, x \, , du = cos \, x \, dx$

$$\int sin^4 x \, (1 - sin^2 x)^2 \, cos \, x \, dx = \int u^4 (1 - u^2)^2 \, du$$

$$= \int \, (u^4 - 2u^6 + u^8) du$$

$$= \frac{1}{5} u^5 - \frac{2}{7} u^7 + \frac{1}{9} u^9 + C$$

$$= \frac{1}{5} sin^5 x - \frac{2}{7} sin^5 x + \frac{1}{9} sin^5 x + C$$

Class practice

1. Evaluate $\int sin^2 x \, cos^2 x \, dx$
2. Evaluate $\int_0^{\pi/2} sin^3 x \, cos^2 x \, dx$
3. Evaluate $\int sin^5 x \, cos^3 x \, dx$
4. Evaluate $\int_0^{\pi/6} sin^2 3x \, cos^3 3x \, dx$

Home Work

Page 484. Ex: 1, 2, 11, 17

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