1. $\Gamma\left(-\frac{1}{2}\right) = \dfrac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -2\,\Gamma\frac{1}{2} = -2\sqrt{\pi}$

2. $\dfrac{\Gamma(5/2)}{\Gamma\left(\frac{1}{2}\right)} = \dfrac{\left(\frac{5}{2}-1\right)\Gamma\left(\frac{5}{2}-1\right)}{\sqrt{\pi}} = \dfrac{\frac{3}{2}\,\Gamma\frac{3}{2}}{\sqrt{\pi}} = \dfrac{3}{4}$

3. $\dfrac{6\,\Gamma\left(\frac{8}{3}\right)}{5\,\Gamma\left(\frac{2}{3}\right)} = \dfrac{6.\left(\frac{8}{3}-1\right)\Gamma\left(\frac{8}{3}-1\right)}{5\,\Gamma\left(\frac{2}{3}\right)} = \dfrac{6.\,\frac{5}{3}\,\Gamma\left(\frac{5}{3}\right)}{5\,\Gamma\left(\frac{2}{3}\right)}$

$= \dfrac{2.5\left(\frac{5}{3}-1\right)\Gamma\left(\frac{5}{3}-1\right)}{5\,\Gamma\left(\frac{2}{3}\right)} = \dfrac{2.\,\frac{2}{3}\,\Gamma\left(\frac{2}{3}\right)}{\Gamma\left(\frac{2}{3}\right)} = \dfrac{4}{3}$

4. $\Gamma 6 = (6-1)! = 5! = \cancel{2}\,5\times4\times3\times2\times1 = 120$

1. $$\Gamma\left(\frac{7}{2}\right) = \Gamma\left(\frac{5}{2}+1\right) = \frac{5}{2}\Gamma\left(\frac{5}{2}\right) = \frac{5}{2}\Gamma\left(\frac{3}{2}+1\right)$$

$$= \frac{5}{2}\cdot\frac{3}{2}\Gamma\left(\frac{3}{2}\right) = \frac{15}{4}\Gamma\left(\frac{1}{2}+1\right) = \frac{15}{4}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$$

$$= \frac{15}{8}\Gamma\frac{1}{2} = \frac{15}{8}\sqrt{\pi}$$

2. $$\frac{\Gamma\left(\frac{9}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\frac{7}{2}\Gamma\left(\frac{7}{2}\right)}{\Gamma\left(\frac{1}{2}\right)} = \frac{\frac{7}{2}\cdot\frac{5}{2}\Gamma\frac{5}{2}}{\Gamma\frac{1}{2}} = \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\Gamma\frac{3}{2}}{\Gamma\frac{1}{2}}$$

$$= \frac{\frac{7}{2}\cdot\frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma\left(\frac{1}{2}\right)}{\Gamma\frac{1}{2}} = \frac{105}{16}$$

(3) $$\frac{2\Gamma\left(\frac{10}{3}\right)}{3\Gamma\left(\frac{1}{3}\right)} = \frac{2\Gamma\left(\frac{7}{3}+1\right)}{3\Gamma\left(\frac{1}{3}\right)} = \frac{2\cdot\frac{7}{3}\Gamma\left(\frac{7}{3}\right)}{3\Gamma\left(\frac{1}{3}\right)}$$

$$= \frac{2\cdot\frac{7}{3}\Gamma\left(\frac{4}{3}+1\right)}{3\Gamma\left(\frac{1}{3}\right)} = \frac{2\cdot\frac{7}{3}\cdot\frac{4}{3}\Gamma\left(\frac{4}{3}\right)}{3\Gamma\left(\frac{1}{3}\right)}$$

$$= \frac{2\cdot\frac{7}{3}\cdot\frac{4}{3}\cdot\frac{4}{3}\Gamma\left(\frac{1}{3}+1\right)}{3\Gamma\left(\frac{1}{3}\right)} = \frac{2\cdot\frac{7}{3}\cdot\frac{4}{3}\cdot\frac{1}{3}\Gamma\left(\frac{1}{3}\right)}{3\Gamma\left(\frac{1}{3}\right)}$$

$$= 2\cdot\frac{7}{3}\cdot\frac{4}{3}\cdot\frac{1}{3}\cdot\frac{1}{3} = \frac{56}{81}$$

(4) $$\Gamma(10) = \Gamma(9+1) = 9! = 362880$$

(5) $$\Gamma\left(-\frac{7}{2}\right) = \frac{\Gamma\left(-\frac{7}{2}+1\right)}{-\frac{7}{2}} = -\frac{2}{7}\cdot\Gamma\left(-\frac{5}{2}\right)$$

$$= -\frac{2}{7}\cdot\frac{\Gamma\left(-\frac{5}{2}+1\right)}{-\frac{5}{2}} = -\frac{2}{7}\cdot\left(-\frac{2}{5}\right)\Gamma\left(-\frac{3}{2}\right)$$

$$= \frac{4}{35}\frac{\Gamma\left(-\frac{3}{2}+1\right)}{-\frac{3}{2}} = \frac{4}{35}\left(-\frac{2}{3}\right)\Gamma\left(-\frac{1}{2}\right)$$

$$= -\frac{8}{105}\frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}} = -\frac{8}{105}\times\left(-\frac{2}{1}\right)\Gamma\left(\frac{1}{2}\right)$$

$$= \cancel{\frac{16}{105}}\ \frac{16}{105}\sqrt{\pi}.$$

(a) $\int_0^\infty x^6 e^{-x} dx = \int_0^\infty x^{7-1} e^{-x} dx = \Gamma 7 = 6! = 720$

(b) $\int_0^\infty \sqrt{x} e^{-3x} dx$

let, $u = 3x$

$x = \frac{u}{3}$

$dx = \frac{du}{3}$

x	0	∞
u	0	∞

$= \int_0^\infty \left(\frac{u}{3}\right)^{1/2} e^{-u} \frac{du}{3}$

$= \frac{1}{3}\left(\frac{1}{3}\right)^{1/2} \int_0^\infty u^{1/2} e^{-u} du$

$= \frac{1}{3\sqrt{3}} \int_0^\infty u^{3/2-1} e^{-u} du$

$= \frac{1}{3\sqrt{3}} \Gamma\left(\frac{3}{2}\right) = \frac{1}{3\sqrt{3}} \cdot \frac{1}{2} \Gamma\left(\frac{1}{2}\right)$

$= \frac{1}{6\sqrt{3}} \sqrt{\pi}$

(c) $\int_0^\infty x^4 e^{-x^2}\,dx$

let, $u = x^2$

$du = 2x\,dx$

$dx = \frac{du}{2x}$

$$\begin{array}{c|c|c} x & 0 & \infty \\ \hline u & 0 & \infty \end{array}$$

$= \int_0^\infty u^2 e^{-u} \cdot \frac{du}{2x}$

$= \frac{1}{2}\int_0^\infty u^2 e^{-u} \frac{1}{\sqrt{u}} \cancel{x}\, du$

$= \frac{1}{2}\int_0^\infty u^2 e^{-u} \cdot u^{-1/2}\, du$

$= \frac{1}{2}\int_0^\infty u^{3/2} e^{-u}\, du$

$= \frac{1}{2}\int_0^\infty u^{5/2-1} e^{-u}\, du$

$= \frac{1}{2}\Gamma(5/2) = \frac{1}{2}\,\frac{3}{2}\Gamma(\frac{3}{2}) = \frac{1}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}\Gamma(\frac{1}{2})$

$= \frac{3}{8}\sqrt{\pi}$

(d) $\int_0^\infty x^5 e^{-2x}\,dx$

let, $u = 2x^2$

$du = 4x\,dx$

$dx = \frac{du}{4x}$

$$\begin{array}{c|c|c} x & 0 & \infty \\ \hline u & 0 & \infty \end{array}$$

$= \int_0^\infty x^5 e^{-u} \frac{du}{4x}$

$= \frac{1}{4}\int_0^\infty x^4 e^{-u}\, du$

$= \frac{1}{4}\int_0^\infty (\frac{u}{2})^2 e^{-u}\, du$

$= \frac{1}{16}\int_0^\infty u^2 e^{-u}\, du$

$= \frac{1}{16}\int_0^\infty u^{3-1} e^{-u}\, du$

$= \frac{1}{16}\Gamma(3) = \frac{1}{16}\,2! = \frac{1}{8}.$

(e) $\int_0^\infty \sqrt{y}\, e^{-y^2} dy$

let, $u = y^2$

$du = 2y\,dy$

$dy = \frac{du}{2y}$

x	0	∞
u	0	∞

$= \int_0^\infty y^{1/2} e^{-u} \frac{du}{2y}$

$= \frac{1}{2} \int_0^\infty y^{-1/2} e^{-u} du$

$= \frac{1}{2} \int_0^\infty (\sqrt{u})^{-1/2} e^{-u} du$

$= \frac{1}{2} \int_0^\infty u^{-1/4} e^{-u} du$

$= \frac{1}{2} \int_0^\infty u^{3/4 - 1} e^{-u} du$

$= \frac{1}{2} \Gamma(3/4) = \frac{1}{2} \frac{1}{4} \Gamma(1/4)$

$= \frac{1}{8} \Gamma\left(\frac{1}{4}\right)$

(a) $\int_0^1 x^4(1-x^3)dx$

$= \int_0^1 x^{5-1}(1-x)^{4-1} dx$

$= B(5,4)$

$= \frac{\Gamma 5\, \Gamma 4}{\Gamma 5+4} = \frac{4! \times 3!}{8!} = \frac{4\times3\times2\times3\times2}{8\times7\times6\times5\times4\times3\times2}$

$= \frac{1}{280}$

(b) let, $I = \int_0^4 \frac{x^2}{\sqrt{4-x}} dx$

$= \int_0^4 \frac{x^2}{\sqrt{4(1-\frac{x}{4})}} dx = \frac{1}{\sqrt{4}} \int_0^4 \frac{x^2}{}$

$\therefore I = \frac{1}{\sqrt{4}} \int_0^4 x^2 (1-\frac{x}{4})^{-1/2} dx$

Putting $\frac{x}{4} = u$

$x = 4u \quad \therefore dx = 4du \quad \begin{array}{c|c|c} x & 0 & 4 \\ \hline u & 0 & 1 \end{array}$

So $I = \frac{1}{\sqrt{4}} \int_0^1 (4u)^2 (1-u)^{-1/2} 4\, du$

$= \frac{1}{\sqrt{4}} \cdot 4^2 \cdot 4 \int_0^1 u^2 (1-u)^{-1/2} du$

$= \frac{1}{\sqrt{4}} \cdot 4^2 \cdot \sqrt{4} \cdot \sqrt{4} \int_0^1 u^{3-1} (1-u)^{\frac{1}{2}-1} du$

$= 16\sqrt{4}\, B(3, \frac{1}{2})$

$= 16\sqrt{4} \cdot \frac{\Gamma 3\, \Gamma \frac{1}{2}}{\Gamma 3+\frac{1}{2}} = 16\sqrt{4} \frac{2!\sqrt{\pi}}{\Gamma \frac{7}{2}}$

$= 16\sqrt{4} \cdot \frac{2\sqrt{\pi}}{\frac{1}{2} \cdot \Gamma \frac{1}{2}} = 16\sqrt{4} \frac{2\sqrt{\pi}}{\frac{1}{2}\sqrt{\pi}} = 128$

(c) $\int_0^1 y^4\sqrt{1-y^2}\,dy$

let, $y^2 = u$

$\therefore du = 2y\,dy \quad \therefore dy = \dfrac{du}{2y}$

y	0	1
u	0	1

$$= \int_0^1 y^4 \sqrt{1-u}\cdot\frac{du}{2y}$$

$$= \frac{1}{2}\int_0^1 y^3\sqrt{1-u}\,du$$

$$= \frac{1}{2}\int_0^1 (y^2)^{3/2}\sqrt{1-u}\,du$$

$$= \frac{1}{2}\int_0^1 u^{3/2}(1-u)^{1/2}\,du$$

$$= \frac{1}{2}\int_0^1 u^{\frac{5}{2}-1}(1-u)^{\frac{3}{2}-1}\,du$$

$$= \frac{1}{2}B\left(\frac{5}{2},\frac{3}{2}\right) = \frac{1}{2}\,\frac{\Gamma\frac{5}{2}\,\Gamma\frac{3}{2}}{\Gamma\left(\frac{5}{2}+\frac{3}{2}\right)}$$

$$= \frac{1}{2}\,\frac{\frac{3}{2}\cdot\frac{1}{2}\cdot\sqrt{\pi}\cdot\frac{1}{2}\sqrt{\pi}}{\Gamma 4} = \frac{1}{2}\,\frac{\frac{3}{8}\sqrt{\pi}}{3!}$$

$$= \frac{\sqrt{\pi}}{32}$$

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(d) $B\left(\frac{7}{2}, 1\right)$

$$= \frac{\Gamma\frac{7}{2}\,\Gamma 1}{\Gamma\left(\frac{7}{2}+1\right)} = \frac{\Gamma\frac{7}{2}\,\Gamma 1}{\Gamma\frac{9}{2}} = \frac{\Gamma\frac{7}{2}\cdot 1}{\frac{7}{2}\,\Gamma\frac{7}{2}} = \frac{2}{7}$$

(e) $B(10, 11)$

$$= \frac{\Gamma 10\,\Gamma 11}{\Gamma(10+11)} = \frac{9!\ 10!}{\Gamma 21} = \frac{9!,\ 10!}{21!}$$

$$= \frac{9\times 8\times 7\times 6\times 5\times 4\times 3\times 2\times 10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2}{21\times 20\times 19\times 18\times 17\times 16\times 15\times 14\times 13\times 12\times 11\times 10\times 9\times 8\times 7\times 6\times 5\times 4\times 3\times 2}$$

$$= 5.41\times 10^{-7}$$