Chapter 6

Ordinary Differential Equation

DE: A differential equation (DE) is an equation that involves differentials or derivatives.

Definition :

A differential equation is an equation containing an unknown function and its derivatives.

Examples:.

1. $$\frac{dy}{dx} = 2x + 3$$

2. $$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + ay = 0$$

3. $$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$$

y is dependent variable and ***x*** is independent variable, and these are ordinary differential equations

Order of Differential Equation

The order of the differential equation is order of the highest derivative in the differential equation.

Differential Equation	ORDER
$\frac{dy}{dx} = 2x + 3$	1
$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 9y = 0$	2
$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$	3

Degree of Differential Equation

The **degree** of a differential equation is **power of the highest order derivative term** in the differential equation.

Differential Equation	Degree
$\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + ay = 0$	1
$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + 6y = 3$	1
$\left(\frac{d^2y}{dx^2}\right)^3 + \left(\frac{dy}{dx}\right)^5 + 3 = 0$	3

Initial Value Problem (IVP)

If a differential equation is required to satisfy conditions on the dependent variable and its derivatives specified at one value of the independent variable, these conditions are called initial conditions and the problem is called initial value problem (IVP).

Example: A stone is dropped from the top of a tower of height *h* under gravity can be expressed as

$\frac{d^2y}{dt^2} = g$ (neglecting friction)

with initial conditions when $t = 0, y = h, \frac{dy}{dt} = 0$

FIRST ORDER DIFFERENTIAL EQUATIONS

An ordinary differential equation of order one can be written as

$$\frac{dy}{dx} = f(x, y)$$

Methods of Solution

Separation of Variables

If it is possible to rearrange the terms of the DE in two groups each containing only one variable, the equation is said to be separable. A separable equation is of the form

$$\frac{dy}{dx} = f(x)g(y)$$

Example 6.1 : Solve the following differential equation $xy\frac{dy}{dx} = (1+x^2)(1+y^2)$

Solution: The equation is separable and can be written as

$$\frac{y}{1+y^2}dy = \frac{(1+x^2)}{x}dx = \left(\frac{1}{x}+x\right)dx$$

Integrating, we get $\frac{1}{2}\ln(1+y^2) = \ln|x| + \frac{x^2}{2} + \frac{A}{2}$, *A* is a constant of integration

Simplifying, $\ln(1+y^2) = \ln(x^2) + x^2 + A$

Example 6.2: Solve the following IVP

$$\frac{dy}{dt} = e^{-y}(2t-4), y(0) = 0$$

Solution: The equation is separable and can be written as

$$e^{y} dy = (2t-4)dt$$

Integrating, we get $e^{y} = t^2 - 4t + c$

Using initial condition, we have $c = 1$

Therefore, $e^{y} = t^2 - 4t + 1 \quad \Rightarrow y(t) = \ln(t^2 - 4t + 1)$

Example 6.3: Solve the following initial value problem

$$\frac{dy}{dx} = \frac{1+y^2}{1+x^2}, \quad y(0) = 1$$

Solution: The equation is separable and can be written as

$$\frac{dy}{1+y^2} = \frac{dx}{1+x^2}$$

Integrating, we get $\tan^{-1} y = \tan^{-1} x + c$

Using initial condition, we have $c = \frac{\pi}{4}$

Therefore, $\tan^{-1} y = \tan^{-1} x + \frac{\pi}{4}$.

Class Practice: Find the solution of the following DEs using separation of variables

1. $\frac{dy}{dt} = (t-1),\ y(0) = 2$

2. $\frac{dy}{dt} = \frac{y}{1+y^2}$

3. $\frac{dy}{dt} = \frac{t}{y^2},\ y(1) = 1$

4. $xy^3 dx + e^{x^2}\, dy = 0$

First Order Linear Differential Equations (General Form):

$$\boldsymbol{\frac{dy}{dt} + p(t)y = q(t)} \text{......}(1)$$

Working rule:

- At first find the integrating factor(IF) by using the following formula

$$\mathrm{IF} = \mu(t) = e^{\int p(t)dt}$$

- On integration we get the general solution as

$$y = \frac{1}{\mu(t)} \int [q(t).\mu(t)]dt + c$$

.

Example 6.4: Find the general solution of $y' + y \tan x = \sin 2x$

Solution: Here $p(x) = \tan x$.

Now, $\mu(t) = e^{\int \tan x \, dx} = e^{\ln \sec x} = \sec x$.

On integration we get the general solution as

$$y = \frac{1}{\sec x} \int [(2 \sin x \cos x) \sec x] dt$$

$$y = \frac{1}{\sec x} (-2 \cos x) + C$$

$$y = -2 \cos^2 x + C$$

Example 6.5: Find the solution of the IVP $\frac{dy}{dt} + \frac{y}{t} = e^t, y(0) = 0$.

Solution: Now, $\mu(t) = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$

On integration we get the general solution as,

$$y = \frac{1}{t} \int (e^t . t) dt$$

$$\mathrm{y} = \frac{1}{t}(t\, e^t - e^t) + c \text{ [using integration by parts]}$$

Using initial condition, we have $c = 1$

Therefore, the solution is $y = \frac{1}{t}(t\, e^t - e^t + 1)$

Class Practice: Find the solution of the following first order linear DEs

1. $\frac{dy}{dt} + \frac{2}{t}\, y = t - 1$

2. $\frac{dy}{dt} - \frac{2y}{t} = 2t^2,\ y(-2) = 4$

3. $\frac{dy}{dt} = -2ty + 4\, e^{-t^2}, y(0) = 3$

4. $\frac{dy}{dt} = -\frac{y}{1+t} + t^2$