Lesson plan 16

Damped Simple Harmonic Motion

- A pendulum will swing only briefly underwater, because the water exerts on the pendulum a drag force that quickly eliminates the motion.
- A pendulum swinging in air does better, but still the motion dies out eventually, because the air exerts a drag force on the pendulum (and friction acts at its support point), transferring energy from the pendulum's motion.
- **When the motion of an oscillator is reduced by an external force, the oscillator and its motion are said to be damped.**

- An idealized example of a damped oscillator is shown in Fig. 15-16, where a block with mass m oscillates vertically on a spring with spring constant k.
- From the block, a rod extends to a vane (both assumed massless) that is submerged in a liquid.
- As the vane moves up and down, the liquid exerts an inhibiting drag force on it and thus on the entire oscillating system.

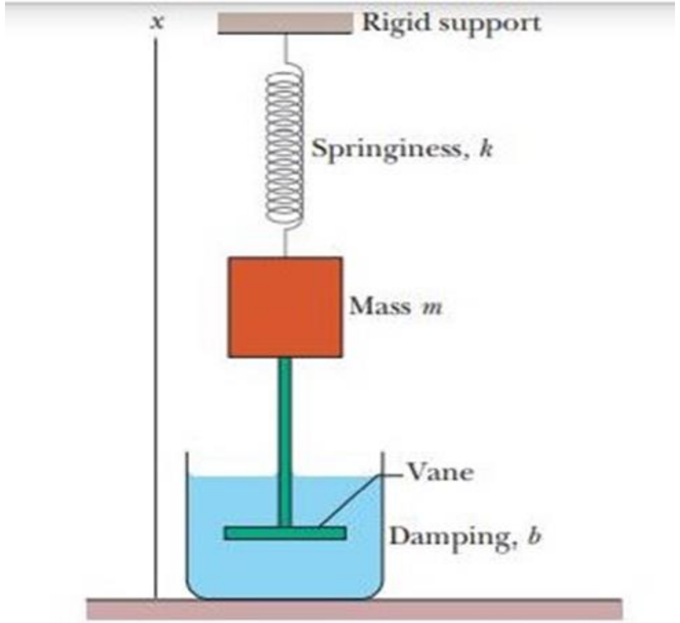

Figure 15-16 An idealized damped simple harmonic oscillator. A vane immersed in a liquid exerts a damping force on the block as the block oscillates parallel to the x axis.

- With time, the mechanical energy of the block–spring system decreases, as energy is transferred to thermal energy of the liquid and vane.

- Let us assume the liquid exerts a damping force $\overrightarrow{F_d}$ that is proportional to the velocity $\vec{v}$ of the vane and block (an assumption that is accurate if the vane moves slowly).
- Then, for force and velocity components along the x axis in Fig. 15-16, we have

$$\boldsymbol{F_d = -bv} \quad \ldots \ldots \ldots \ldots \ldots (1)$$

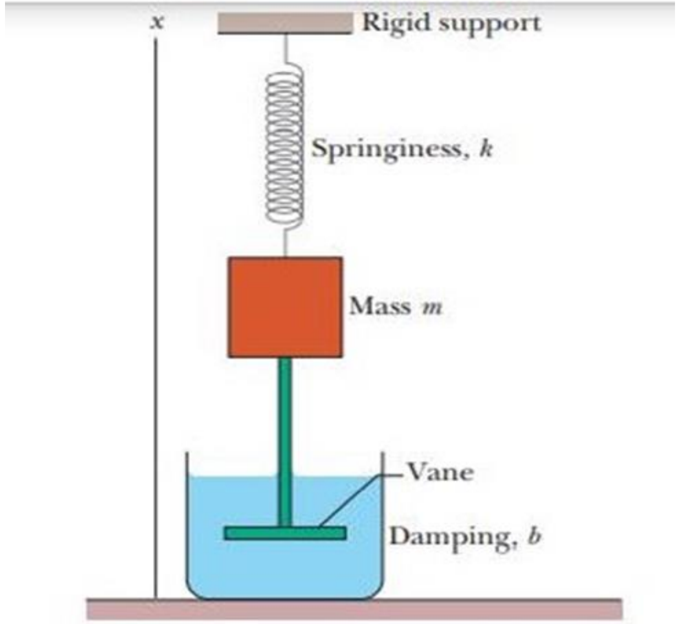

Figure 15-16 An idealized damped simple harmonic oscillator. A vane immersed in a liquid exerts a damping force on the block as the block oscillates parallel to the x axis.

- where b is a damping constant that depends on the characteristics of both the vane and the liquid and has the SI unit of kilogram per second.
- The minus sign indicates that $\overrightarrow{\boldsymbol{F_d}}$ opposes the motion.

Damped Oscillations:

- The force on the block from the spring is $\boldsymbol{F_s = -kx}$.

- Let us assume that the gravitational force on the block is negligible relative to $\mathbf{F_d}$ and $\mathbf{F_s}$.

- Then we can write Newton's second law for components along the x axis ($\boldsymbol{F_{net,x} = ma_x}$) as

$$-bv - kx = ma \quad \ldots\ldots\ldots\ldots\ldots (2)$$

- Substituting for $\boldsymbol{v} = \frac{dx}{dt}$ and for $\boldsymbol{a} = \frac{d^2x}{dt^2}$ and rearranging give us the differential equation

$$\mathbf{m}\frac{\mathbf{d^2x}}{\mathbf{dt^2}} + \mathbf{b}\frac{\mathbf{dx}}{\mathbf{dt}} + \mathbf{kx} = \mathbf{0} \quad \ldots\ldots\ldots\ldots (3)$$

- The solution of this equation is

$$x(t) = x_m e^{-bt/2m} \cos(\omega' t + \varphi)$$

where $\boldsymbol{x_m}\, \boldsymbol{e^{-bt/2m}}$ is the amplitude and $\boldsymbol{\omega'}$ is the angular frequency of the damped oscillator .

- This angular frequency is given by

$$\omega' = \sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}$$

- If ***b = 0*** (there is no damping), then above equation reduces to ($\omega = \sqrt{\frac{k}{m}}$) for the angular frequency of an undamped oscillator, and displacement equation reduces to equation for the displacement of an undamped oscillator.

- If the damping constant is small but not zero (so that b $\ll \sqrt{k/m}$), then $\omega \approx \omega'$

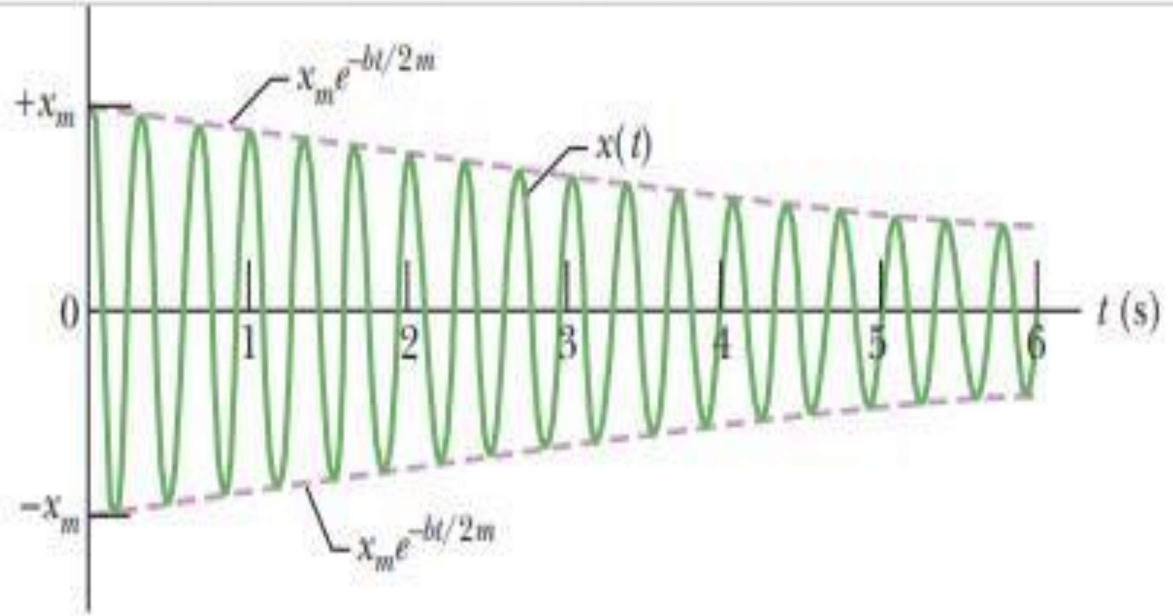

Figure 15-17 The displacement function $x(t)$ for the damped oscillator of Fig. 15-16. The amplitude, which is $x_m\, e^{-bt/2m}$, decreases exponentially with time.

Problem 58: *For the damped oscillator system shown in Fig. 15-16, with m = 250 g, k = 85 N/m, and b = 70 g/s, T= 0.34 sec, what is the ratio of the oscillation amplitude at the end of 20 cycles to the initial oscillation amplitude?*

Solution: The displacement of the damped oscillation is

$$x(t) = x_m e^{-\frac{bt}{2m}} \cos(\omega' t + \varphi)$$

Initially at **t = 0**, $x(t) = x_m \cos\varphi$ .... *(1)*

The displacement after 20 cycle at t = 20T is

$$x'(t) = x_m e^{-\frac{10bT}{m}} \cos(20\omega' T + \varphi)$$

$$= x_m e^{-\frac{10bT}{m}} \cos(40\pi + \varphi) \quad ; [\omega' = \frac{2\pi}{T}]$$

$$\rightarrow x'(t) = x_m e^{-\frac{10bT}{m}} \cos\varphi$$

So

$$\frac{x'(t)}{x(t)} = e^{-\frac{10\, b\, T}{m}}$$

Substituting m = 0.25 kg, b = 0.07 kg/s, T = 0.34 sec

$$\frac{x'(t)}{x(t)} = \quad 0.39$$

Problem 60: *The suspension system of a 2000 kg automobile "sags"10 cm when the chassis is placed on it. Also, the oscillation amplitude decreases by 50% each cycle. Estimate the values of (a) the spring constant k and (b) the damping constant b for the spring and shock absorber system of one wheel, assuming each wheel supports 500 kg.*

Solution: **Given. $x = 10$ cm $= 0.10$ m**

(a) Hooke's law, $\mathbf{F = kx}$

$\rightarrow$ $\mathbf{mg = kx}$

One quarter of the vehicle mass, $m = 2000/4 = 500$ kg

$\rightarrow$ $\mathbf{k = mg/x = 500(9.8)/0.10 = 49000\ N/m}$

[Here displacement downward and spring force upward]

For one cycle, t = T s

$$(\boldsymbol{b})\ \boldsymbol{x_m e^{-\frac{bT}{2m}} = 50\%\ x_m}$$

$$\rightarrow\ \boldsymbol{e^{-\frac{bT}{2m}} = \frac{50}{100} = 0.50}$$

$$\rightarrow\ \boldsymbol{ln\ [e^{-\frac{bT}{2m}}] = ln\ (0.50)} \quad \rightarrow -\boldsymbol{\frac{bT}{2m}} = -\ \mathbf{0.69} \quad \rightarrow \boldsymbol{\frac{bT}{2m}} = \mathbf{0.69}$$

Time period of a damped SHM, $T = \frac{2\pi}{\omega'}$

$$\rightarrow T = \frac{2\pi}{\sqrt{\frac{k}{m} - \frac{b^2}{4m^2}}}$$

$$\rightarrow\ T^2 = \frac{4\pi^2}{\frac{k}{m} - \frac{b^2}{4m^2}} = \frac{4\pi^2}{\frac{4mk - b^2}{4m^2}} = \frac{16\pi^2 m^2}{4mk\ -\ b^2}$$

$$\mathbf{Here,}\quad T^2 = \frac{16\pi^2 m^2}{4mk - b^2} \;;\quad \text{and} \quad \frac{\boldsymbol{bT}}{\boldsymbol{2m}} = \mathbf{0.69}$$

Now,

$$\frac{\boldsymbol{b^2 T^2}}{\boldsymbol{4m^2}} = \mathbf{0.48} \rightarrow \frac{\boldsymbol{b^2}}{\boldsymbol{4m^2}}\left(\frac{16\pi^2 m^2}{4mk - b^2}\right) = \mathbf{0.48} \rightarrow \boldsymbol{b^2}\left(\frac{4\pi^2}{4mk - b^2}\right) = \mathbf{0.48}$$

$$\rightarrow 4\pi^2 b^2 = 0.48(4mk - b^2)$$

$$\rightarrow 4\pi^2 b^2 = 1.92mk - 0.48b^2$$

$$\rightarrow 4\pi^2 b^2 + 0.48b^2 = 1.92mk$$

$$\rightarrow b^2 (4\pi^2 + 0.48) = 1.92(500)(49000)$$

$$\rightarrow 39.96b^2 = 47040000$$

$$\rightarrow b^2 = 1177177.18$$

$$\rightarrow b = \sqrt{1177177.18}$$

$$\rightarrow b = 1084.98 \text{ kg/s}$$