INTEGRAL CALCULUS
AND
ORDINARY DIFFERENTIAL EQUATIOSNS

MULTIPLE INTEGRATION

Multiple Integration

- **Multiple Integration:** The integrals of functions of more than one variable are known as **multiple integrals** and are evaluated by a process involving iterated integrals.
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- **Partial Integration:** The process in which the integration is performed with respect to one variable treating the other variable(s) as constant is called partial integration.
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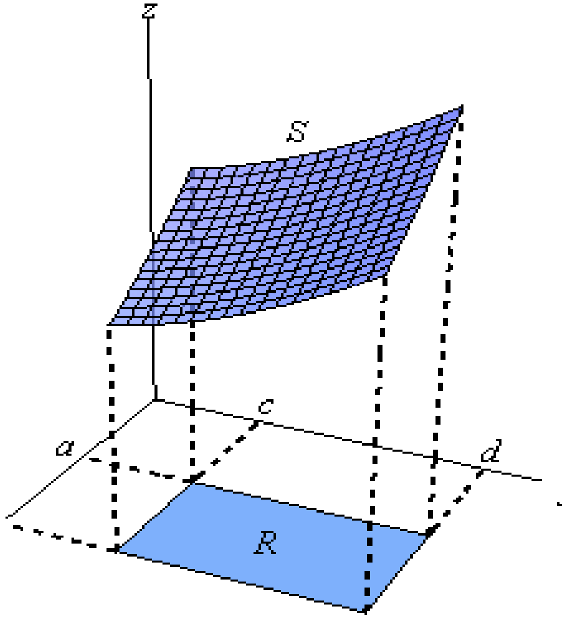
- **Iterated Integral:** A definite integral which is evaluated stage by stage using partial integration is called an iterated (successive or repeated) integral.

- **Double Integrals:** The double integral may be defined geometrically in much the same way as the definite Riemann integral.

- **Double Integrals over the rectangular region:**

  If R is the region defined by $R = \{(x, y) | a \leq x \leq b, c \leq y \leq d\}$, then

$$\iint_R f(x,y)dA = \int_a^b \int_c^d f(x,y)dydx = \int_c^d \int_a^b f(x,y)dxdy,$$ where $s = f(x,y)$.



Double Integrals over the rectangular region:

Example: Evaluate the iterated integral $\int_0^1 \int_1^2 (x^2 + xy)dxdy$

Solution: $\int_0^1 \int_1^2 (x^2 + xy)dxdy = \int_0^1 \left[\int_1^2 (x^2 + xy)dx\right] dy$

$$= \int_0^1 \left[\frac{x^3}{3} + \frac{x^2 y}{2}\right]_{x=1}^{x=2} dy$$

$$= \int_0^1 \left[\frac{8-1}{3} + \frac{(4-1)y}{2}\right] dy$$

$$= \int_0^1 \left[\frac{7}{3} + \frac{3y}{2}\right] dy .$$

$$= \left[\frac{7}{3} y + \frac{3}{2}\frac{y^2}{2}\right]_{y=0}^{y=1}$$

$$= \frac{37}{12}$$

Class Practice:

Evaluate the followings:

1. $\int_0^1 \int_0^2 (x+2)\,dydx$
2. $\int_2^4 \int_0^3 (x+y)\,dxdy$
3. $\int_0^1 \int_x^y xy\;dydx$
4. $\int_0^1 \int_{y^2}^y (x^2y+xy^2)\,dxdy$
5. $\int_1^2 \int_1^y (\frac{1}{x}+\frac{1}{y})\,dxdy$
6. $\int_0^1 \int_0^{\sqrt{x}} ye^{x^2}\;dy\,dx$
7. $\int_0^{\sqrt{\frac{\pi}{2}}} \int_0^{x^2} x\cos y\,dy\,dx$
8. $\int_0^1 \int_0^{x^2} (x^2+y)\,dydx$
9. $\int_0^{\pi/2} \int_0^2 r\sqrt{4-r^2}\;dr\,d\theta$
10. $\int_0^1 \int_{-x}^x (x^2-y^2)\,dy\,dx$
11. $\int_0^{\frac{\pi}{2}} \int_0^{\sin\theta} r\cos\theta\,drd\theta$

Home Work

Iterated Integral (P-993) Example # 4, 5, 6

Page – 999 Ex # 15 – 21, 27, 28, 29, 34

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Double Integrals over the non-rectangular region:

(a) If R is the region defined by $R = \{(x, y) | a \leq x \leq b, f_1(x) \leq y \leq f_2(x)\}$, then

$$\iint_R f(x, y)dA = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y)dydx\,.$$

(b) If R is the region defined by $R = \{(x, y) | c \leq y \leq d, g_1(y) \leq x \leq g_2(y)\}$, then

$$\iint_R f(x, y)dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y)dxdy\,.$$

Double Integrals over the non-rectangular region:

Example: Evaluate $\iint_R y^2 x \, dA$ over the rectangle

$$R = \{(x, y) | -3 \le x \le 2, 0 \le y \le 1\}.$$

Solution: $\iint_R y^2 x \, dA$

$$= \int_{-3}^{2} \int_{0}^{1} y^2 x dy dx \, .$$

$$= \int_{-3}^{2} x \left[\frac{y^3}{3}\right]_0^1 dx$$

$$= \frac{1}{3} \int_{-3}^{2} x \, dx$$

$$= \frac{1}{3} \left[\frac{x^2}{2}\right]_{-3}^{2}$$

$$= -\frac{5}{6}$$

Double Integrals over the non-rectangular region:

Example: Evaluate $\iint_R xy\,dA$ over the region

$$R = \{(x, y): \frac{1}{2}x \le y \le \sqrt{x}, 2 \le x \le 4\}.$$

Solution: $\iint_R x\,y dA = \int_2^4 \int_{\frac{x}{2}}^{\sqrt{x}} xydydx$

$$= \int_2^4 x\left[\frac{y^2}{2}\right]_{\frac{x}{2}}^{\sqrt{x}} dx$$

$$= \frac{1}{2}\int_2^4 x[x - \frac{x^2}{4}]\,dx$$

$$= \frac{1}{2}\int_2^4 [x^2 - \frac{x^3}{4}]dx]$$

$$= \frac{1}{2}\left[\frac{x^3}{3} - \frac{x^4}{16}\right]_2^4 = \frac{11}{6}$$

Class Practice:

Evaluate the following:

1. $\iint_R (x^2 + xy^3)\, dA$ over the rectangle $R = \{(x, y) : 0 \le x \le 1, 1 \le y \le 2\}$.
2. $\iint_R (xy - y^2) dA$ where R is rectangle whose vertices are $(-1,0), (0,0), (0,1), \text{and } (-1,1)$.
3. $\iint_R (2x + y)\, dA$ over the rectangle $R = \{(x, y) | 3 \le x \le 5, 1 \le y \le 2\}$.
4. $\iint_R (x^2 + y^2)\, dA$ where R is rectangle whose vertices are $(0,1), (1,1), (1,2)$and$(0,2)$.
5. $\iint_R (x + y)\, dA$, where R is the region bounded by $y = 1, y = x^2$ *and* $x \ge 0$.
6. $\iint_R x\, dA$ over the triangular region R enclosed by the lines $x + 2y = 2, x = 0$ and $y = 0$.

Home Work

Double Integral over general regions (P- 1001) Example # 1, 3

Page- 1008 Ex # 1-4, 7, 8, 9, 17, 18

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Application of Double Integrals:

Area: Plane area of a closed bounded region *R* is $A = \iint_R dA$.

Example: Using double integrals, find the finite area bounded by the following curves $y = -x^2$and$y = x - 2$.

Solution: $A = \iint_R 1\, dA$

$$= \int_{x=-2}^{x=1} \int_{y=x-2}^{-x^2} 1 dydx$$

$$= \int_{x=-2}^{x=1} [y]_{x-2}^{-x^2}\, dx$$

$$= \int_{x=-2}^{x=1} [-x^2 - x + 2] dx$$

$$= \left[-\frac{x^3}{3} - \frac{x^2}{2} + 2x\right]_{-2}^{1}$$

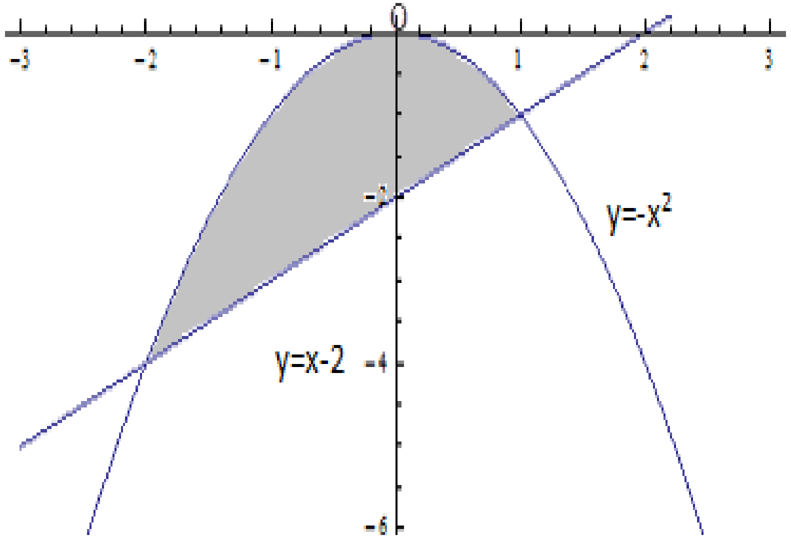
$$= \frac{11}{6}.$$

$-x^2 = x - 2$

$x^2 + x - 2 = 0$

$(x + 2)(x - 1) = 0$

$x = -2, 1$



Application of Double Integrals:

Area: Plane area of a closed bounded region R is $A = \iint_R dA$.

Example: Using double integrals, find the finite area bounded by the following curves $y = x^2$ and $y = x + 6$.

$x^2 = x + 6$
$x^2 - x - 6 = 0$
$(x - 3)(x + 2) = 0$
$x = -2, 3$

Solution: $A = \iint_R 1\, dA$

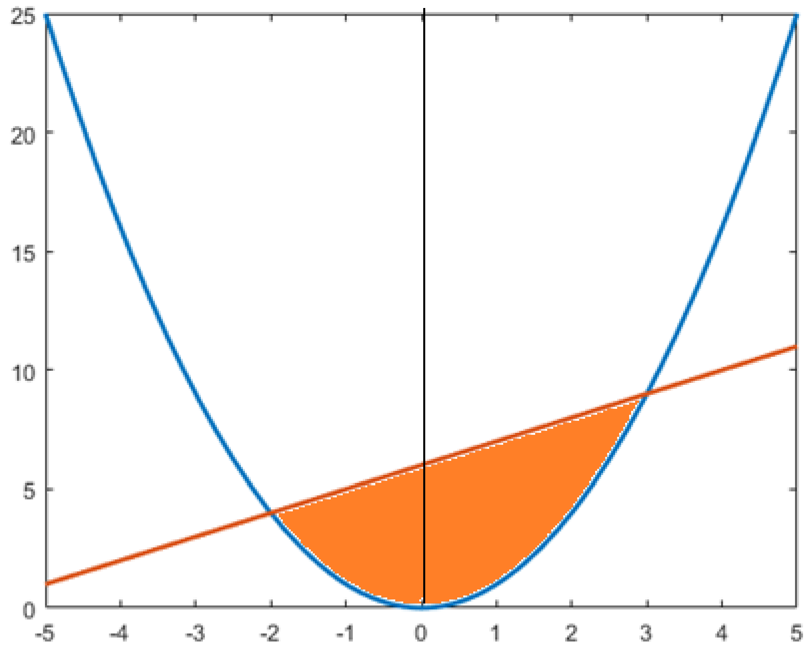
$$= \int_{x=-2}^{x=3} \int_{y=x^2}^{y=x+6} 1 dy dx$$

$$= \int_{x=-2}^{x=3} [y]_{x^2}^{x+6}\, dx$$

$$= \int_{x=-2}^{x=3} [x + 6 - x^2] dx$$

$$= \left[\frac{x^2}{2} + 6x - \frac{x^3}{3}\right]_{-2}^{3}$$

$$= \frac{125}{6}.$$



Class Practice:

Sketch the region and **using** double integrals, find the finite area bounded by the following curve (s).

1. $\boldsymbol{y = 2x - x^2}$ and *x*-axis
2. $\boldsymbol{x^2 = 4y, 8y = x^2 + 16}$
3. $\boldsymbol{y = -x, x = 0,\ y = 2}$

Iterated Integrals in Three Variables:

Example: Evaluate $\int_{-1}^{2}\int_{0}^{3}\int_{0}^{2} xy^2z^3 dzdydx$

Solution: $\int_{-1}^{2}\int_{0}^{3}\int_{0}^{2} xy^2z^3 dzdydx$

$$= \int_{-1}^{2}\int_{0}^{3} xy^2 \left[\frac{z^4}{4}\right]_0^2 dydx$$

$$= 4\int_{-1}^{2}\int_{0}^{3} xy^2\, dydx$$

$$= 4\int_{-1}^{2} x \left[\frac{y^3}{3}\right]_0^3 dx$$

$$= 36\int_{-1}^{2} x\, dx$$

$$= 36\left[\frac{x^2}{2}\right]_{-1}^2$$

$$= 54$$

Iterated Integrals in Three Variables:

Example: Evaluate $\int_0^3 \int_0^1 \int_{-1}^1 (x^2 + yz) dzdydx$

Solution: $\int_0^3 \int_0^1 \int_{-1}^1 (x^2 + yz) dzdydx$

$$= \int_0^3 \int_0^1 \left(x^2 z + y\frac{z^2}{2}\right)_{z=-1}^{z=1} dy\, dx$$

$$= \int_0^3 \int_0^1 2x^2\, dy\, dx$$

$$= \int_0^2 2x^2 [y]_0^1\, dx$$

$$= \int_0^3 2x^2 \cdot 1 dx$$

$$= 2\left[\frac{x^3}{3}\right]_0^3$$

$$= 18$$

Class Practice:

Evaluate the following iterated integral

1. $\int_0^2 \int_{-3}^0 \int_{-1}^1 (x^2 + yz)dzdydx$
2. $\int_1^2 \int_0^1 \int_{-1}^1 (x^2 + y^2 + z^2)dxdydz$
3. $\int_0^1 \int_0^{y^2} \int_0^{x+y} x \ dzdxdy$
4. $\int_0^1 \int_0^x \int_0^{x-y} xdzdydx$
5. $\int_0^2 \int_{-1}^{y^2} \int_{-1}^z yzdxdzdy$
6. $\int_0^{2\pi} \int_0^2 \int_0^{4-r^2} zr \ dzdrd\theta$
7. $\int_0^{2\pi} \int_0^\pi \int_0^a r^3 \sin\theta \ drd\theta d\varphi,$

Home Work

Triple Integral (P-1030) Example # 1

Page- 1037 Ex # 3-7

Mass and center of mass

The co-ordinates $(\overline{x}, \overline{y})$of the center of mass of a lamina occupying the region *D* and having density function $\rho(x, y)$ are

$$\overline{x} = \frac{1}{m}\iint_D x\rho(x, y)\, dA$$

$$\overline{y} = \frac{1}{m}\iint_D y\rho(x, y)\, dA$$

Where the mass m is given by

$$m = \iint_D \rho(x, y)\, dA$$

Mass and center of mass

Example: Find the mass and center of mass of the lamina that occupies the region D and has the given density function ρ. Where $\boldsymbol{D = \{(x, y)|0 \leq x \leq 1, 0 \leq y \leq 2\}}$ **and** $\boldsymbol{\rho(x, y) = x + y}$.

Solution: $\boldsymbol{m = \int_0^2 \int_0^1 (x + y) dx\, dy}$

$$\boldsymbol{= \int_0^2 \left(\frac{x^2}{2} + xy\right)\Big|_0^1 dy}$$

$$\boldsymbol{= \int_0^2 \left(\frac{1}{2} + y\right) dy}$$

$$\boldsymbol{= \left(\frac{1}{2} y + \frac{y^2}{2}\right)\Big|_0^2}$$

$$\boldsymbol{= 3}$$

Mass and center of mass

$$\overline{x} = \frac{1}{m}\iint_D \ x\rho(x, y)\, dA$$

$$\overline{x} = \frac{1}{3}\int_0^2 \int_0^1 x(x + y)dx\, dy$$

$$= \frac{1}{3}\int_0^2 \int_0^1 \left(x^2 + xy\right)dx\, dy$$

$$= \frac{1}{3}\int_0^2 \left(\frac{x^3}{3} + \frac{x^2}{2}y\right)\Bigg|_0^1 dy$$

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$$= \frac{1}{3}\int_0^2 \left(\frac{1}{3} + \frac{1}{2}y\right) dy$$

$$= \frac{1}{3}\left(\frac{1}{3}y + \frac{1}{2}\frac{y^2}{2}\right)\Bigg|_0^2$$

$$= \frac{5}{9}$$

Mass and center of mass

$$\bar{y} = \frac{1}{m}\iint_D \ y\rho(x,y)\,dA$$

$$\bar{y} = \frac{1}{3}\int_0^2\int_0^1 y(x+y)dx\,dy$$

$$= \frac{1}{3}\int_0^2\int_0^1 (xy+y^2)dx\,dy$$

$$= \frac{1}{3}\int_0^2 \left(\frac{x^2}{2}y + xy^2\right)\Big|_0^1 dy$$

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$$= \frac{1}{3}\int_0^2 \left(\frac{1}{2}y + y^2\right) dy$$

$$= \frac{1}{3}\left(\frac{1}{2}\frac{y^2}{2} + \frac{y^3}{3}\right)\Big|_0^2$$

$$= \frac{11}{9}$$

Mass and center of mass

Class practice:

1. Find the mass and center of mass of the lamina that occupies the region *D* and has the given density function ρ. Where $\boldsymbol{D = \{(x, y) | 0 \le x \le 1, 0 \le y \le 2\}}$ **and** $\boldsymbol{\rho(x, y) = y^2}$.
2. Find the mass and center of mass of the lamina that occupies the region *D* and has the given density function ρ. Where $\boldsymbol{D = \{(x, y) | 0 \le x \le 1, 0 \le y \le 1\}}$ **and** $\boldsymbol{\rho(x, y) = 2x}$.

Home Work

Page-1017, Example # 2

Page- 1024 Ex # 3-10