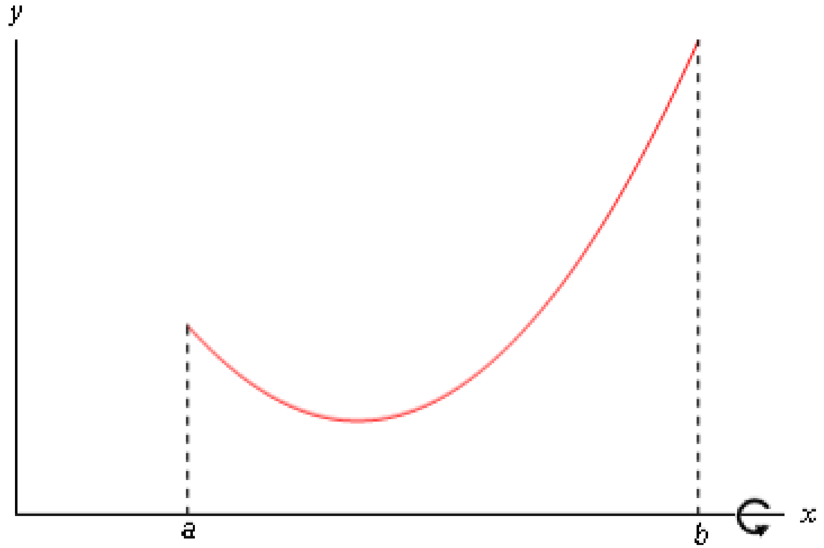
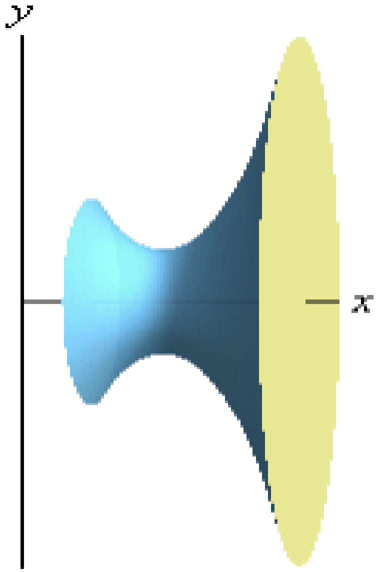
# 2.2 Volumes of Solids of Revolution

## What is Solids of Revolution

If a region is rotated completely (i.e. through $2\pi$ radians) about a straight line, the solid formed is a solid of revolution. Any cross section perpendicular to the axis of rotation is circular.

To get a solid of revolution let's start with a function $y = f(x)$, on an interval $[a, b]$(Left side graph). Let's rotate the curve about $x$ −axis(although it could be any vertical or horizontal axis) so that we get the following(right-side graph) three dimensional region.



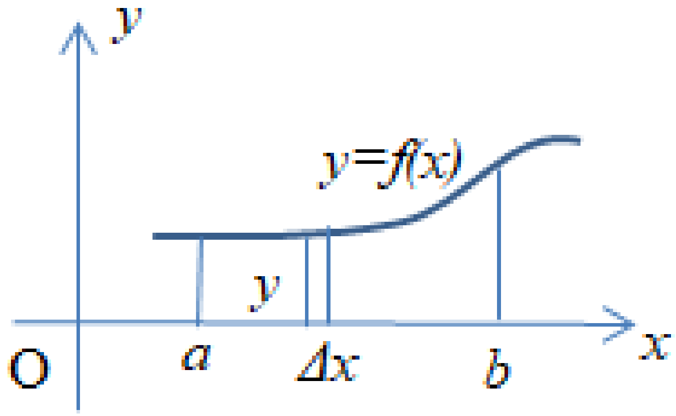




**Now we are going to find the volume of the object**

## Volume of Solids of Revolution

Let us consider a solid generated by revolving about the *x*-axis of a region *R* bounded by a curve $y = f(x)$, the *x*-axis and the lines $x = a, x = b$.

The region *R* can be divided into small strips. When a typical strip of length $y$ and width $\Delta x$ is rotated completely about the *x*-axis, it forms a circular disc.



# Volume of Solids of Revolution

The volume $\Delta V$ of the disc is , $\Delta V \approx \pi y^2 \Delta x$

The volume of the solid can be divided into small discs. Summing all the discs as $\Delta x \to 0$we have the volume of revolution $V_x$, **about the $x$ −axis**

$$V_x = \lim_{\Delta x \to 0} \sum_{x=a}^{x=b} \pi y^2 \Delta x = \int_a^b \pi y^2 dx$$

In the same way, when a region bounded by the curve $x = f(y)$, , the *y*-axis and the lines $y = c, y = d$ is rotated **about the *y*-axis**, the solid formed has volume

$$V_y = \int_c^d \pi x^2 dy$$

This method is often called **method of disks** or the **method of rings**

# Volume of Solids of Revolution

If we have two function $y = f(x)$ and $y = g(x)$ where $f(x) > g(x)$ and bounded by $x = a, x =$ b then volume solid of revolution is **about $\boldsymbol{x}$ −axis** is given by

$$\mathrm{V}_x = \int_a^b \pi\,((f(x))^2 - (g(x))^2)dx$$

$$\mathrm{V}_x = \int_a^b \pi\,((\text{outer radius})^2 - (\text{inner radius})^2)dx$$

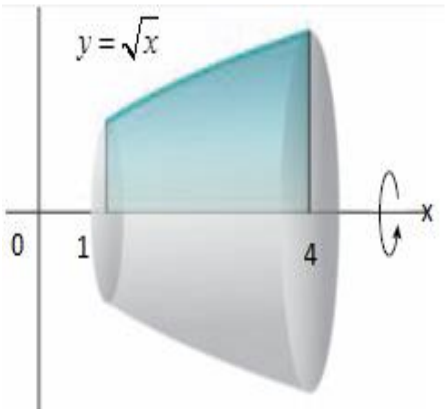
Similarly for the volume of solid of revolution about **y-axis** is,

$$\mathrm{V}_\mathrm{y} = \int_a^b \pi\,((f(y))^2 - (g(y))^2)dy$$

## Example set-2.2.1

**1.** Find the volume of the solid that is obtained when the region under the curve $y = \sqrt{x}$ over the interval $[1,4]$ is revolved about the ***x*-axis**.

**Solution:**

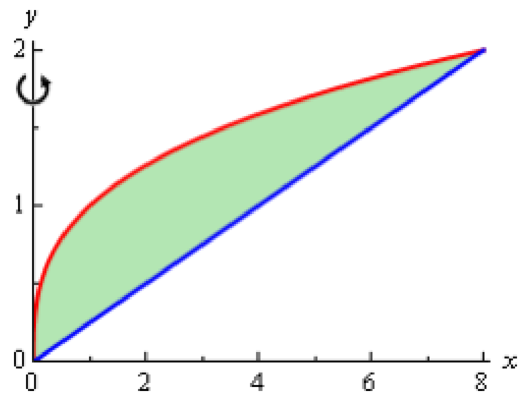


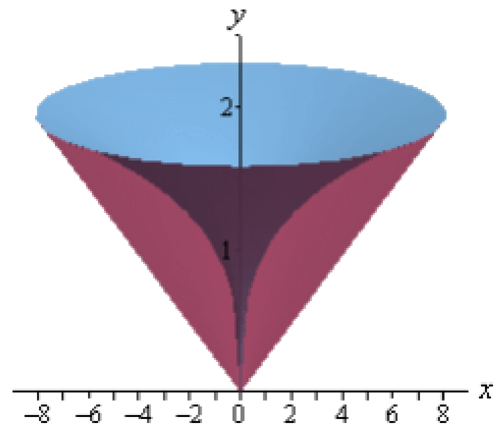

The volume is ,

$$V_x = \int_a^b \pi y^2 dx = \int_a^b \pi (f(x))^2 dx = \int_1^4 \pi(\sqrt{x})^2 dx = \pi \int_1^4 x\, dx = \frac{15\pi}{2}$$

**2.** Find the volume of the solid that is obtained when the region under the curve $y = \sqrt[3]{x}$ and $y = \frac{x}{4}$ that lies in the first quadrant and is revolved about the ***y*-axis**.

**Solution:**




$y = \sqrt[3]{x} \quad \Rightarrow \quad x = y^3$ ........(1)

$y = \dfrac{x}{4} \quad \Rightarrow \quad x = 4y$ ........(2)

$$y^3 = 4y$$

$$y(y^2 - 4) = 0$$

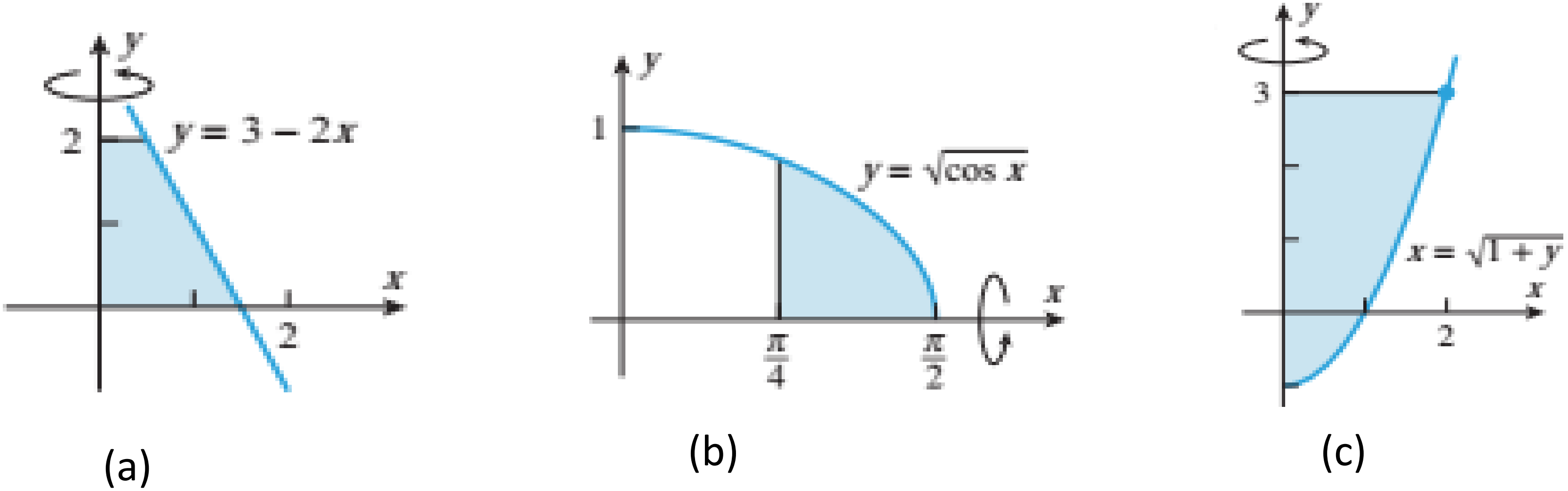
$$\therefore y = 0, 2, -2$$

So, the volume is ,

$$V_y = \int_a^b \pi((right)^2 - (left)^2)dy = \pi \int_0^2 (16y^2 - y^6)dy = \frac{512\pi}{21}$$

# Exercise set-2.2.1

**1.** Find the volume of the solid that results when the shaded region is revolved about the indicated axis:



**2.** Find the volume of the solid when the region enclosed by the given curves is revolved about the ***x*-axis.**

(a) $y = \sqrt{x},\ x = 9.$
(b) $y = x^2, x = 0, x = 2.$
(c) $y = x^2 - 4x + 5, x = 1, x = 4.$
(d) $y = x, y = 1, x = 0.$

**3.** Find the volume of the solid when the region enclosed by the given curves is revolved about the ***y*-axis.**

(a) $y = \sqrt{x}, x = 0, y = 3.$
(b) $x = 1 - y^2,\ x = 0.$
(c) $y = \frac{1}{x}, y = 1, y = 2.$

**4.** **Calculus– James Stewart - 8th edition**

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