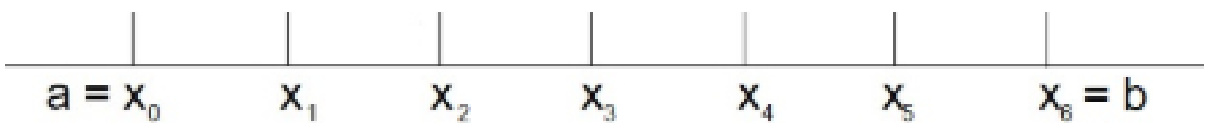
1.2 Definite Integrals

1.2.1 Riemann and Trapezoidal Sum

Consider a function $f(x)$ which is defined (i.e. bounded) over the closed interval$[a, b]$.

Consider a partition *P* of $[a, b]$ into n subintervals by the points

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b$$

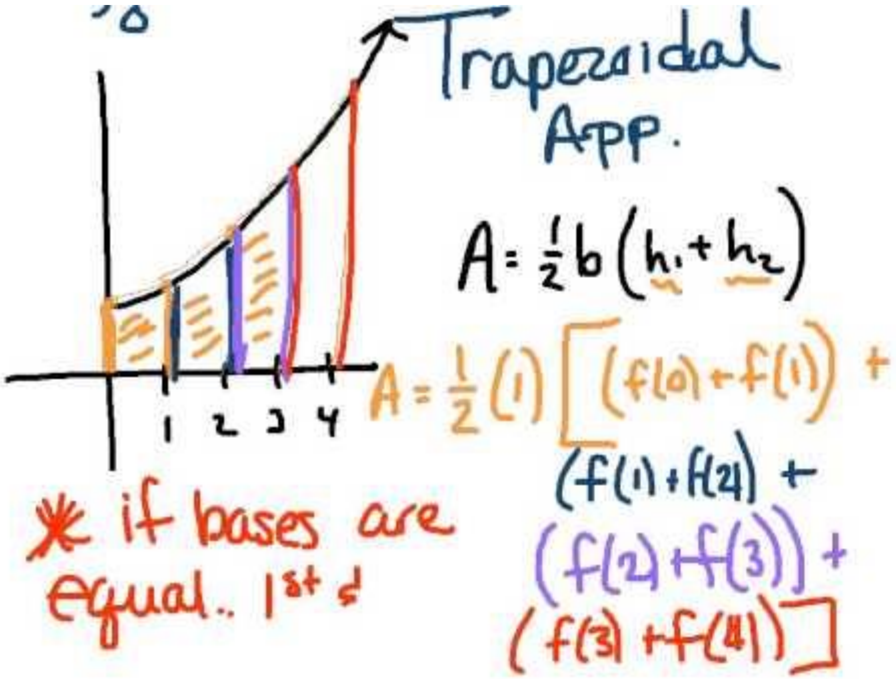


This partition corresponds to the subintervals

$$[x_0, x_1], [x_1, x_2], [x_2, x_3], \cdots, [x_{n-1}, x_n]$$

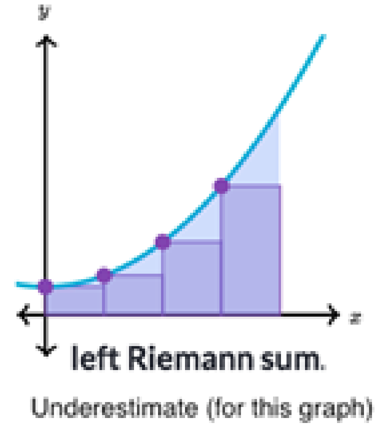
1.2.1 **Riemann and Trapezoidal Sum**

A **Riemann sum** is an approximation of the area under a curved by dividing it into multiple simple shapes (like rectangles or trapezoids). Riemann sums use rectangles to approximate the area under a curve. Another useful integration rule is the **Trapezoidal Rule**.

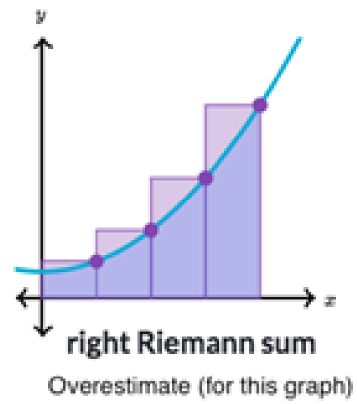


Left, middle and right Riemann sums

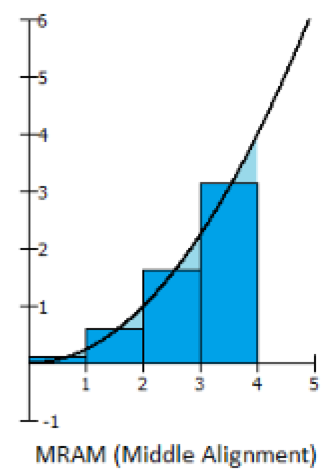
To make a Riemann sum, we must choose how we are going to make our rectangles. One possible choice is to make our rectangles touch the curve with their to-left corners. This is called a **left Riemann sum.**



Another choice is to make our rectangles touch the curve with their top-right corners. This is a **right Riemann sum.**



In a **middle Riemann sum,** the height of each rectangle is equal to the value of the function at the **middle point** of its base.



In each $[x_{r-1},\ x_r]$ choose any point c_r such that $x_{r-1} \le c_r \le x_r$. Then the sum

$$S_n = \sum_{r=1}^{n} f(c_r)(x_r - x_{r-1})$$

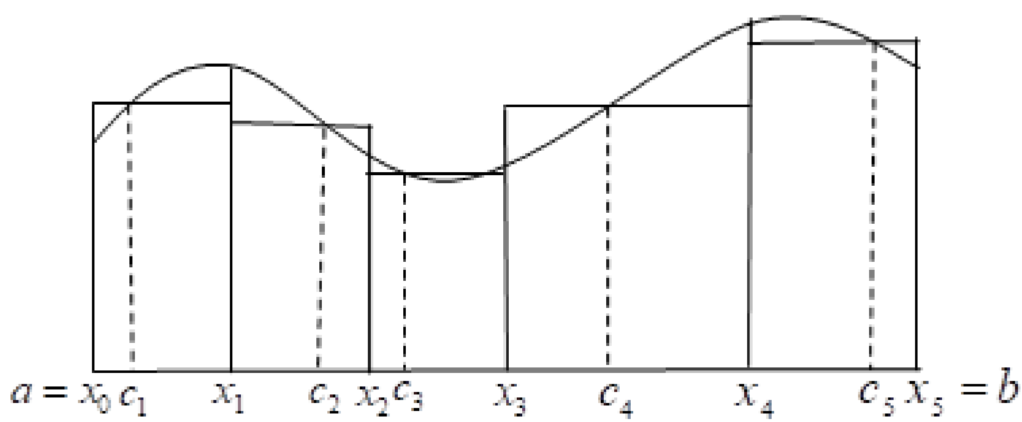
$$= \sum_{r=1}^{n} f(c_r)\Delta x_r\,, \qquad\qquad \Delta x_r = x_r - x_{r-1}$$

is called a **Riemann sum** for the function $f(x)$ on $[a, b]$.

Suppose $f(x) \geq 0$, on $[a, b]$. Then the Riemann sum

$$S_n = \sum_{r=1}^{n} f(c_r)\Delta x_r$$

is the sum of the areas of the *n* rectangles shown below, and thus represents an approximation to the area under the graph on $[a, b]$. Figure below illustrates the case where *n* = 5.



Different choice of the nodal points c_r give different values of the Riemann sums.

Commonly used Riemann sums are :

left Riemann sum ($c_r = x_{r-1}$),

right Riemann sum($c_r = x_r$) and

middle Riemann sum($c_r = \frac{x_{r-1}+x_r}{2}$).

If we use

$$f(c_r) = \frac{f(x_{r-1}) + f(x_r)}{2},$$

average of the heights at end points of the subinterval, it is called the **Trapezoidal Riemann sum**.

Summary of Riemann Sum:

Let a function $f(x)$ is defined in the closed interval $[a, b]$.

In evaluation of Riemann sums we commonly use equal subintervals. Dividing $[a, b]$ into n equal sub-intervals of the length

$$\Delta x = \frac{b-a}{n}$$

Riemann sum of $f(x)$ over the interval $[a, b]$ is

$$S_n = \sum_{r=1}^{n} f(c_r)\,\Delta x = \Delta x \sum_{r=1}^{n} f(c_r)$$

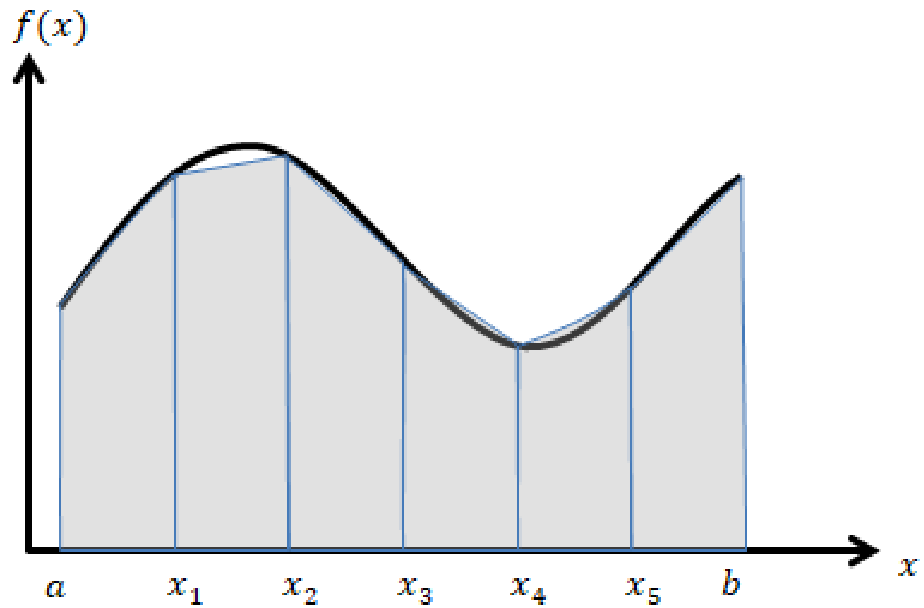
For all r if	The sum S_n will be called
$c_r = x_{r-1}$	left Riemann sum
$c_r = x_r$	right Riemann sum
$c_r = (x_r + x_{r-1})/2$	middle Riemann sum
$f(c_r) = \dfrac{f(x_{r-1}) + f(x_r)}{2}$	Trapezoidal Riemann sum

1.2.2 **Numerical Integration (The Trapezoidal Rule)**

First we subdivide the interval $[a, b]$ into n subintervals of width $\Delta x = \frac{b-a}{n}$.

Then on each interval we will approximate the function by a straight line joining the function values at either endpoint on the interval.

The following figure illustrates the case for $n = 6$.



Each of these shaded objects is a trapezoid (hence the rule's name) and as we can see some of them do a very good approximation to the actual area under the curve.

The area of the trapezoid in the interval $[x_r, x_{r+1}]$ is given by,

$$A_r = \left(f(x_r) + f(x_{r+1})\right) \times \frac{\Delta x}{2}.$$

Then sum of the area of the n trapeziums (e.g. 6 in the above figure) will approximate the area under the curve and is given by,

$$\int_a^b f(x)dx$$

$$\approx \left(f(x_0) + f(x_1)\right) \times \frac{\Delta x}{2} + \left(f(x_1) + f(x_2)\right) \times \frac{\Delta x}{2} + \cdots + \left(f(x_{n-1}) + f(x_n)\right)$$

$$\times \frac{\Delta x}{2}$$

$$\approx \frac{\Delta x}{2}\left(f(x_0) + 2f(x_1) + 2f(x_2) + \cdots + 2f(x_{n-1}) + f(x_n)\right)$$

Which is known as the composite Trapezoidal rule.

Example set 1.2.1

Example 1:

Find the area under the curve $f(x) = x^4 - 3x^2 + 3$ by using different Riemann sum over the interval [0,1.6] using 8 subintervals.

Solution:

The following table shows the estimated area, using different Riemann sum, under the curve $f(x) = x^4 - 3x^2 + 3$ over the interval $[0,1.6]$using 8 equal subintervals .

Here, $a = 0$ and $b = 1.6$ and $n = 8$

Then the length of each subintervals is,

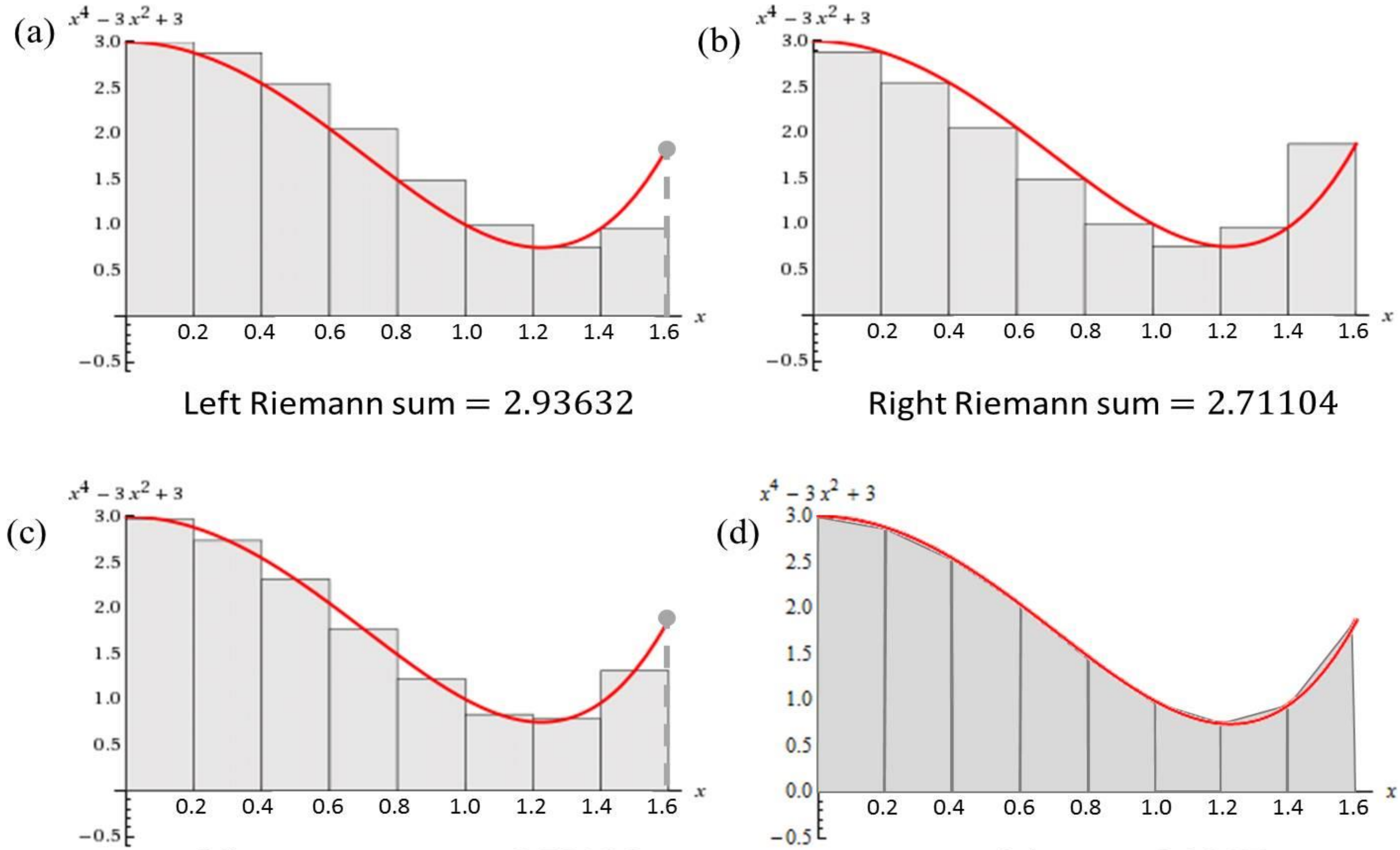
$$\Delta x = \frac{b-a}{n} = \frac{1.6}{8} = 0.2$$

Riemann sum $[x_{r-1}, x_r]$	left Riemann sum		right Riemann sum		middle Riemann sum	
	c_r	$f(c_r)$	c_r	$f(c_r)$	c_r	$f(c_r)$
$[0.0, 0.2]$	0.0	3	0.2	2.8816	0.1	2.9701
$[0.2, 0.4]$	0.2	2.8816	0.4	2.5456	0.3	2.7381
$[0.4, 0.6]$	0.4	2.5456	0.6	2.0496	0.5	2.3125
$[0.6, 0.8]$	0.6	2.0496	0.8	1.4896	0.7	1.7701
$[0.8, 1.0]$	0.8	1.4896	1.0	1.0000	0.9	1.2261
$[1.0, 1.2]$	1.0	1.0000	1.2	0.7536	1.1	0.8341
$[1.2, 1.4]$	1.2	0.7536	1.4	0.9616	1.3	0.7861
$[1.4, 1.6]$	1.4	0.9616	1.6	1.8736	1.5	1.3125
$\sum f(c_r)$	14.6816		13.5552		13.9496	
$\Delta x * \sum f(c_r)$	2.9363		2.7110		2.7899	

The Trapezoidal Riemann sum is,

$$S_n = [3 + 2 \times (2.8816 + 2.5456 + 2.0496 + 1.4896 + 1.0000 + 0.7536 + 0.9616) + 1.8736] \times \frac{0.2}{2}$$
$$= 2.8237.$$

The following figures show the geometrical interpretation of the above Riemann sums,



Left Riemann sum $= 2.93632$

Right Riemann sum $= 2.71104$

Middle Riemann sum$= 2.78992$

Trapezoidal sum $= 2.8237$

Note that the exact value of the area is 2.80115 which is calculated using the integration will be considered later.

Example 2:

Use the Trapezoidal rule with $n = 5$ to approximate the integral $\int_{0.5}^{1} \sqrt{1 + e^{x^2}}\ dx$ to 3 decimal places.

Solution:

Here $a = 0.5$, $b = 1$ and $n = 5$. So $\Delta x = \frac{1-0.5}{5} = 0.1$ and $f(x) = \sqrt{1 + e^{x^2}}$.

Hence,

x	0.5	0.6	0.7	0.8	0.9	1
f(x)	1.5113	1.5599	1.6224	1.7019	1.8022	1.9283

Using the Trapezoidal rule, we have

$$\int_{0.5}^{1} \sqrt{1 + e^{x^2}}\ dx \approx \frac{0.1}{2}[1.5113 + 2(1.5599) + 2(1.6224)$$

$$+2(1.7019) + 2(1.8022) + 1.9283]$$

$$=0.8406 \approx 0.841$$

Example 3:

Evaluate $\int_0^{\pi} \sqrt{(3+\cos x)}\ dx$. to three decimal places using Trapezoidal rule with four subintervals.[Note that in calculating the values of cos *x* use radian mode]

Solution:

Here $a = 0$, $b = \pi$ and $n = 4$. So,$\Delta x = \frac{\pi}{4}$ and $f(x) = \sqrt{3+\cos x}$.

Hence,

x	0	π/4	π/2	3π/4	π
f(x)	2.0000	1.9254	1.7321	1.5142	1.4142

Using Trapezoidal rule we have

$$\int_0^{\pi} \sqrt{(3+\cos x)}\ dx$$

$$\approx \frac{1}{2}.\frac{\pi}{4}[2 + 2 * (1.9254 + 1.7321 + 1.5142) + 1.4142]$$

$$\approx \frac{\pi}{8} \times 13.7576 \ \approx 5.403$$

Exercise set 1.2.1

1. Estimate the value the following integrals to 3 decimal places using '*n*' subintervals of equal length using **(i) left Riemann sum, (ii) right Riemann sum, (iii) middle Riemann sum and (iv) Trapezoidal rule.**

(a) $\int_0^2 e^{-3x}dx$ (n = 4), (b) $\int_1^7 \frac{1}{\sqrt{x^3+1}}\,dx$ (n = 6), (c) $\int_3^5 \frac{1}{1-\ln x}\,dx$ (n = 4),

(d) $\int_0^1 \sin(x)\cos(x^2)\,dx$ (n = 4), (e) $\int_0^1 \sin(x^2)\,dx$ (n = 5).

2. **Calculus– James Stewart - 8th edition**

P- 388 Ex # 1, 3, 7, 9
P- 524 Ex # 7, 8, 9 (n= 4, 6) LRS, RRS, MRS, TR