MAT 3103: Computational Statistics and Probability

Chapter 3: Probability

Probability:

A probability is a number that reflects the chance or likelihood that a particular event will occur. Probabilities can be expressed as proportions that range from 0 to 1, and they can also be expressed as percentages ranging from 0% to 100%. A probability of 0 indicates that there is no chance that a particular event will occur, whereas a probability of 1 indicates that an event is certain to occur. The probability that the sun will rise from the north tomorrow is 0, whereas the probability that an individual currently alive will die one day is 1. A probability of 0.45 (45%) indicates that there are 45 chances out of 100 for the event to occur.

Application of probability for engineering:

Application areas: Modeling of text and web data, network traffic modeling, probabilistic analysis of algorithms and graphs, reliability modeling, simulation algorithms, data mining, and speech recognition.

The mathematical methods that we will use to analyze these applications will include basic principles of probability such as Bayes rule, conditional probability, random variables, expectation, and Markov chains. These are only the few examples you can actually develop logical thinking for series and sequence programming. We use probability to measure the success or failure of something. Same goes for the use of probability in programming. Programmers use probability to measure the success of the program before running it.

In modern computer science, software engineering, and other fields, the need arises to make decisions under uncertainty. Presenting probability and statistical methods, simulation techniques, and modeling tools, Probability and Statistics for Computer Scientists helps students solve problems and make optimal decisions in uncertain conditions, select stochastic models, compute probabilities and forecasts, and evaluate performance of computer systems and networks. If you combine probability and statistics with a computer science curriculum, you get a data science curriculum! If you have interest in becoming a data scientist (or any related profession), then that should be motivation for you.

Probability Theory is one of the most important courses in Electrical Engineering. In most Universities you are taught an Introduction to Probability Theory and Statistics and then in other classes you are learning about more specific areas depending on the needs of each class. For instance, if you are going to choose a) Control Systems, b) Signals, c) Information Theory, d) Communications, then you are certainly going to need probabilities in most of your classes. But that is not the case if you are going to choose Energy or Motors.

Probability is particularly relevant in the area of quality engineering and being able to design reliable circuits where you have to take into account the tolerance of various components, and the predicted lifetimes. What is the mean-time-to-failure (MTTF) of your system? Where can you get by with a 20% tolerance capacitor instead of a more expensive 5% one? Communications theory and error correction are all about probability and statistics, where the maximum amount of information is jammed into the smallest amount of bandwidth, while dealing with possibly very noisy channels.

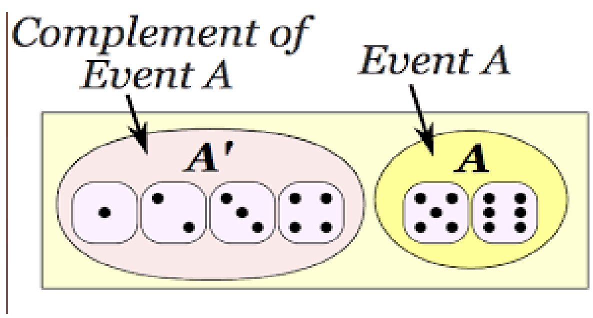
Some relevant terms which are needed to understand probability:

- **Experiment:** The work or activity that generates the results to be studied.
- **Random experiment:** It is an experiment whose outcomes cannot be predicted with certainty in advance, and these outcomes depend on chance.
- **Outcome:** The result of an experiment.
- **Sample space:** Set of all possible outcomes. It is usually denoted by the symbol S.

The above-mentioned terms can be better understood from the following set of examples:

Random experiment	**Outcome**	**Sample space**
Tossing a coin (once)	Head (H), tail (T)	S = {H, T}
Tossing a coin (twice)	H, T	S = {HH, HT, TH, TT}

- **Event:** Any subset of a sample space is an event. It is denoted by capital letters, e.g., A/B/C. In an experiment of rolling a die, an event can be of getting any of the numbers from 1 to 6 on its uppermost face. E.g., A = getting number 4 when a die is rolled. A is an event.
- **Mutually exclusive events:** The events are said to be mutually exclusive when they do not occur simultaneously. If a student is in class, he/she cannot be at shopping mall in the same time. If a ball is white, it cannot be red.
- **Equally likely events:** Events are said to be equally likely, when there is equal chance of occurring. In rolling a die, all six faces are equally likely to come.
- **Exhaustive outcomes:** All possible outcomes of a random experiment are exhaustive outcomes. In the sample space (S) given above the outcomes HH, HT, TH and TT are exhaustive outcomes.
- **Favorable outcomes:** Number of outcomes in favor of an event is known as favorable outcomes. It is denoted by m ($\leq$ n).
- **Complementary events:** The complement of an event A, denoted A' or A^c, is the event not A. If the probability of an event, A, is P(A), then the probability that the event would not occur (also called the complementary event) is 1 – P(A).

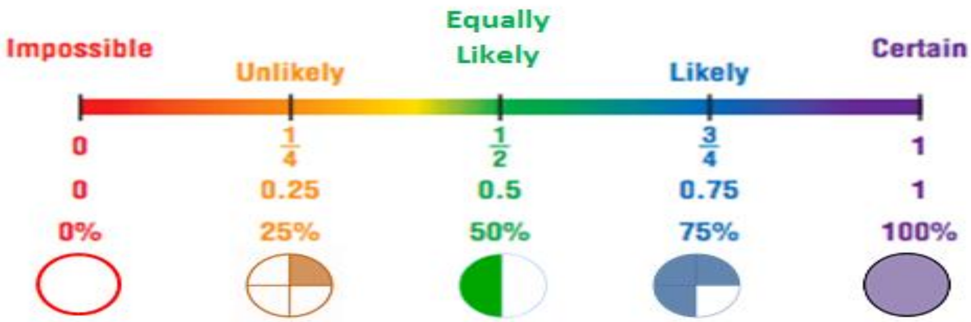


- **Independent events:** Two or more events are said to be independent when the occurrence of one trial does not affect the other. If a coin is tossed one by one, then in a trial the head or tail may come which never describes anything what event will come in second trial. So, the second trial is completely independent to that of the first trial.

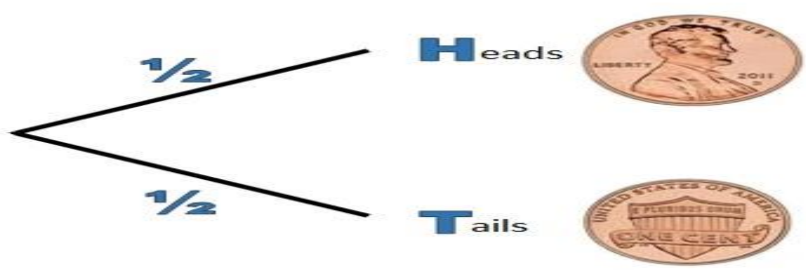
Probability: If a random experiment shows n exhaustive, mutually exclusive and equally likely outcomes and if m ($\leq$ n) outcomes are in favor of an event A, then the probability of an event A is measured by:

$$P(A) = \frac{\text{Number of favourable outcomes}}{\text{Total number of possible outcomes}} = \frac{n(A)}{n(S)} = \frac{m}{n}.$$

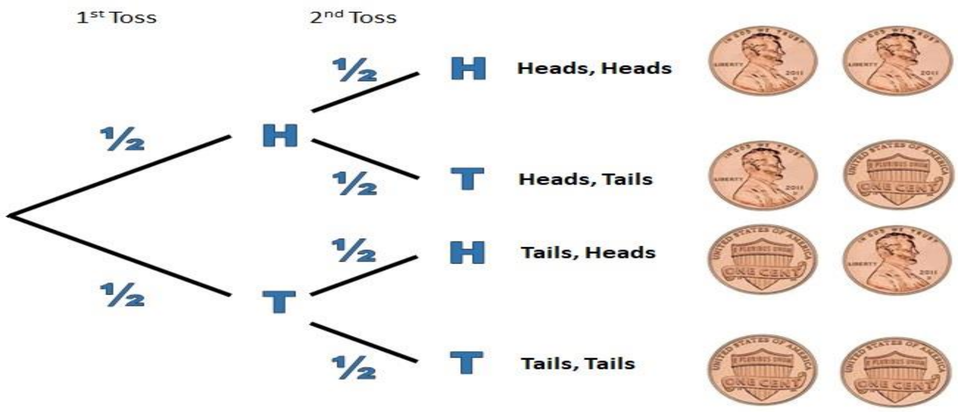
Interpretation rule of probability:



Tree diagram: A diagram that represents probabilities. If a fair coin is tossed once, then



If a fair coin is tossed twice, then



Conditional probability: The conditional probability of an event A in relationship to an event B is the probability that event A occurs given that event B has already occurred, denoted as P(A|B), read as the probability of A given B. Mathematically, $P(A|B) = \frac{P(A \cap B)}{P(B)}$.

Additive law of probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$.

Multiplicative law of probability: P(AB) = P(A) P(B), where A and B are independent events.

With replacement: Selecting with replacement is exactly what it sounds like – we are picking something out of a bag (bucket, drawer, group, etc.), putting it back in there (i.e., the replacement), and picking another one of those somethings out.

Without replacement: In some experiments, the sample space may change for the different events. For example, a marble may be taken from a bag with 20 marbles and then a second marble is taken without replacing the first marble. The sample space for the second event is then 19 marbles instead of 20 marbles. This is called probability without replacement.

Example 3.1: A fair coin is tossed once. What is the probability that a head will be shown?
Solution: A fair coin means, the probability of observing head (H) equal to the probability of observing tail (T). A coin is tossed once, so the sample space is S = {H; T}. Let us now define the event that head will be shown. Thus, the event set A = {H} and

$$P(\mathrm{A}) = \frac{m}{n} = \frac{1}{2}.$$

Example 3.2: Two fair coins are tossed once. What is the probability that (a) both coins will show head (b) at least one coin will show tail (c) at most (or at best) one coin will show head, and (d) none of the coins will show head?
Solution: Sample space for two coins is S = {HH; HT; TH; TT} and the total cases, n = 4.

(a) Both the coin show heads (H), that is, the event set A = {HH}. Thus m = 1 and $P(\mathrm{A}) = \frac{m}{n} = \frac{1}{4}$.

(b) At least one coin will show tail (T), that is, the event set B = {HT; TH; TT} and m = 3. Thus

$$P(\mathrm{B}) = \frac{m}{n} = \frac{3}{4}.$$

(c) At best one coin will show head, that is, C = {HT; TH; TT} and $P(\mathrm{C}) = \frac{m}{n} = \frac{3}{4}$.

(d) None of the coins will show head, that is, the event set D = {TT} and $P(\mathrm{D}) = \frac{m}{n} = \frac{1}{4}$.

Example 3.3: Three fair coins are tossed once. What is the probability that (a) at least two coins will show head (b) first or third coin will show head (c) third coin shows head given the first coin shows head?

Solution: Sample space for three coins is

	Second coin				
First coin	H				T
H	TT				HT
T	TH				TT
Third coin	First two coins				Sample space
	HH	HT	TH	TT	S = {HHH; HHT; HTH; HTT; THH; THT; TTH; TTT}
H	HHH	HHT	HTH	HTT	
T	THH	THT	TTH	TTT	

The all possible outcomes, n = 8.

(a) Let A be an event that at least two coins will show head. So, A = {HHH; HHT; HTH; THH},

$$P(\mathrm{A}) = \frac{m}{n} = \frac{4}{8} = \frac{1}{2}.$$

Now, the first coin will show head, that is, B = {HHH; HHT; HTH; HTT} and $P(\mathrm{B}) = \frac{4}{8}$,

the third coin will show head, that is, C = {TTH; THH; HTH; HHH} and $P(\mathrm{C}) = \frac{4}{8}$,

$$\text{So, } P(\mathrm{B} \cap \mathrm{C}) = \frac{2}{8}.$$

(b) $P(\mathrm{B} \cup \mathrm{C}) = P(\mathrm{B}) + P(\mathrm{C}) - P(\mathrm{B} \cap \mathrm{C}) = \frac{4}{8} + \frac{4}{8} - \frac{2}{8} = \frac{6}{8}$.

(c) $P(B|C) = \frac{\mathrm{P(B \cap C)}}{\mathrm{P(C)}} = \frac{\frac{2}{8}}{\frac{4}{8}} = \frac{2}{4}$.

Problem 3.4: A die is rolled. Find the probability that an even number is obtained.

Solution: Let us first write the sample space: S = {1, 2, 3, 4, 5, 6}.

Let, E = an even number is obtained. So, E = {2, 4, 6}.

Then, $P(\mathrm{E}) = \frac{\mathrm{n(E)}}{\mathrm{n(S)}} = \frac{m}{n} = \frac{3}{6}$.

Problem 3.5: Two bits are produced one by one using an electronic device. The device is such that it produces Fine (F) bit 50% times and Noisy (N) bits 50% times. Find the probability that (a) both bits are fine, (b) one bit is fine, and (c) at least one bit is fine.

Solution: Let us first construct the sample space, S.

<table>
<tr><td rowspan="2">First bit</td><td colspan="2">Second bit</td><td>Sample space</td></tr>
<tr><td>F</td><td>N</td><td rowspan="3">S = {FF, FN, NF, NN}</td></tr>
<tr><td>F</td><td>FF</td><td>FN</td></tr>
<tr><td>N</td><td>NF</td><td>NN</td></tr>
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(a) Let A = both bits are fine = {FF}. Then, $P(\text{A}) = \frac{\text{n(A)}}{\text{n(S)}} = \frac{1}{4}$.

(b) Let B = one is fine bit = {FN, NF}. Then, $P(\text{B}) = \frac{\text{n(B)}}{\text{n(S)}} = \frac{2}{4}$.

(c) Let C = at least one bit is fine = {FF, FN, NF}. Then, $P(\text{C}) = \frac{\text{n(C)}}{\text{n(S)}} = \frac{3}{4}$.

Problem 3.6: There are 100 vehicles at a car park. 60 of them are cars, 30 are vans and the remaining are lorries. If every vehicle is equally likely to leave, find the probability of: (a) van leaving first, and (b) lorry leaving first.

Solution: (a) Let A be the event of a van leaving first. Then, $P(\text{A}) = \frac{\text{n(A)}}{\text{n(S)}} = \frac{30}{100}$.

(b) Let B be the event of a lorry leaving first. Then, $P(\text{B}) = \frac{\text{n(B)}}{\text{n(S)}} = \frac{10}{100}$.

Problem 3.7: In a box there are 30 bulbs. The bulbs are identified by identity number 1 to 30. One bulb is selected randomly. Find the probability that the selected bulb has the identity number (a) either multiple of 3 or 5, and (b) even under the condition that it is multiple of 3.

Solution: (a) Let A = multiple of 3 = {3, 6, 9, 12, 15, 18, 21, 24, 27, 30}. So, $P(\text{A}) = \frac{\text{n(A)}}{\text{n(S)}} = \frac{10}{30}$.

B = multiple of 5 = {5, 10, 15, 20, 25, 30}. Then, $P(\text{B}) = \frac{\text{n(B)}}{\text{n(S)}} = \frac{6}{30}$.

Now, A∩B = {15, 30}. Then, $P(\text{A}\cap\text{B}) = \frac{\text{n(A}\cap\text{B)}}{\text{n(S)}} = \frac{2}{30}$.

So, $P(\text{A}\cup\text{B}) = P(\text{A}) + P(\text{B}) - P(\text{A}\cap\text{B}) = \frac{10}{30} + \frac{6}{30} - \frac{2}{30} = \frac{14}{30} = \frac{7}{15}$.

(b) Let, D = even no. = {2, 4, 6, 8, 10, 12, 14, 16, 18, 20, 22, 24, 26, 28, 30}.

Now, D∩A = {6, 12, 18, 24, 30}. Then, $P(\text{D}\cap\text{A}) = \frac{\text{n(D}\cap\text{A)}}{\text{n(S)}} = \frac{5}{30}$.

So, $P\,(\text{D}|\text{A}) = \frac{P(\text{D}\cap\text{A})}{P(\text{A})} = \frac{5/30}{10/30} = \frac{5}{10}$.

Problem 3.8: The probability that Lima passes mathematics is $\frac{7}{8}$and the probability that she passes statistics is $\frac{3}{4}$. If the probability of passing both the course is $\frac{3}{5}$, what is the probability that Lima will pass at least one of the courses?

Solution: Let, A be the event of passing mathematics and B be the event of passing statistics.

So, $P(\text{A}\cup\text{B}) = P(\text{A}) + P(\text{B}) - P(\text{A}\cap\text{B}) = \frac{7}{8} + \frac{3}{4} - \frac{3}{5} = \frac{31}{40}$.

Problem 3.9: Three consecutive phone calls are monitored. The calls may be either Voice call (V) or Data call (D), where P(V) = P(D) = 50%. Voice call means someone is speaking and Data call means it carries a signal. Find the probability that out of the three calls, there will be (a) no data calls, (b) at least one voice call, and (c) at best one voice call.

Solution: Let us first construct the sample space, S.

First call	Second call	
	V	D
V	VV	VD
D	DV	DD

Third call	First two calls				Sample space
	VV	VD	DV	DD	
V	VVV	VVD	VDV	VDD	S = {VVV, VVD, VDD, VDV, DDV, DVV, DVD, DDD}
D	DVV	DVD	DDV	DDD	

(a) P (no data calls) = P(VVV) = $\frac{1}{8}$.

(b) P (at least one voice call) = P (VVV, VVD, VDD, VDV, DDV, DVV, DVD) = $\frac{7}{8}$.

(c) P (at best one voice call) = P (VDD, DDV, DVD, DDD) = $\frac{4}{8}$.

Problem 3.10: Three consecutive phone calls are monitored. The calls may be either Voice call (V) or Data call (D), if P(V) = 10% and P(D) = 90%, Find the probability that out of the three calls, there will be (a) no data calls, (b) at least one voice call, and (c) at best one voice call.

Solution: Given, P(V) = 10% = 0.1 and P(D) = 90% = 0.9.

(a) P (no data calls) = P(VVV) = P(V) P(V) P(V) = (0.1) (0.1) (0.1) = 0.001.

(b) P (at least one voice call) = 1 - P (no voice call) = 1 - P (DDD)

= 1 - P(D) P(D) P(D) = 1 - (0.9) (0.9) (0.9) = 1 – 0.729 = 0.271.

(c) P (at best one voice call) = P (VDD, DDV, DVD, DDD)

= P(VDD) + P(DDV) + P(DVD) + P(DDD)

= P(V) P(D) P(D) + P(D) P(D) P(V) + P(D) P(V) P(D) + P(D) P(D) P(D)

= (0.1) (0.9) (0.9) + (0.9) (0.9) (0.1) + (0.9) (0.1) (0.9) + (0.9) (0.9) (0.9)

= 0.972.

Problem 3.11: Eighty-five per cent e-mails sent from a cyber cafe reach to the destination properly. Once 3 mails are checked randomly, Find the probability that **(**a**)** all 3 reach properly, **(**b**)** two reach properly, **(**c**)** at least one reaches properly and **(**d**)** at best two reach properly.

Solution: Let, R = reach properly the e-mail and N = not reach properly the e-mail.

Sent of 3 e-mails can occur in n = 2^3 = 8 ways. The sample space is-

S = {RRR, RRN, RNR, RNN, NRR, NRN, NNR, NNN}

Given, P(R) = 0.85 and P(N) = $-1$0.85 = 0.15. So, R and N are not equally likely.

a) Let, A: all 3 reach properly; where, A = {RRR}

P(A) = P(RRR) = 0.85×0.85×0.85 = 0.6141.

b) Let, B: two reach properly; where, B = {RRN, RNR, NRR}

P(B) = P(RRN)+ P(RNR)+ P(NRR)

= (0.85×0.85×0.15) + (0.85×0.15×0.85) + (0.15×0.85×0.85) = 0.3251.

c) Let, C: at least one reaches properly;

where, C = {RRR, RRN, RNR, RNN, NRR, NRN, NNR} $\bar{C}$ = {NNN}

P(C) =1$-P$ ($\bar{C}$) = 1$-$ P(NNN) = 1$-$ (0.15×0.15×0.15) = 0.9966.

d) Let, D: at best two reach properly;

where, D = {RRN, RNR, RNN, NRR, NRN, NNR, NNN}, $\bar{D}$ = {RRR}

P(D) =1$-$ P ($\bar{D}$) = 1$-P$(RRR) = 1$-$ (0.85×0.85×0.85) = 0.3859.

Problem 3.12: There are 7 Vivo and 5 LG mobile sets in a box. Two sets are drawn at random. Find the probability that (a) both are Vivo sets, and (b) one set is Vivo, other is LG.

Solution: There are total 7+5=12 sets. Two sets can be selected in 12_{c_2} ways.

(a) Let A = both sets are Vivo. A can occur in 7_{c_2} ways. So, $P(\mathrm{A}) = \frac{\mathrm{n(A)}}{\mathrm{n(S)}} = \frac{7_{c_2}}{12_{c_2}}$.

(b) Let B = one set is Vivo, other is LG. B can occur in $7_{c_1} \times 5_{c_1}$ ways. So, $P(\mathrm{B}) = \frac{\mathrm{n(B)}}{\mathrm{n(S)}} = \frac{7_{c_1} \text{x}\, 5_{c_1}}{12_{c_2}}$.

Problem 3.13: There are 7 Vivo and 5 LG mobile sets in a box. Two sets are drawn one by one with replacement. Find the probability that (a) both are Vivo sets, and (b) one set is Vivo, other is LG.

Solution: Let us first construct the sample space, S.

First draw	Second draw		Sample space
	Vivo (V)	LG (L)	S = {VV, VL, LV, LL}
Vivo (V)	VV	VL	
LG (L)	LV	LL	

(a) P (both sets are Vivo) = P(VV) = P(V) × P(V) = $\frac{7}{12} \text{ x } \frac{7}{12} = \frac{49}{144}$.

(b) P (one set is Vivo, other is LG) = P (VL ∪ LV) = P(VL) + P(LV)

= P(V) × P(L) + P(L) × P(V) = $\frac{7}{12} \times \frac{5}{12} + \frac{5}{12} \times \frac{7}{12} = \frac{35}{144} + \frac{35}{144} = \frac{70}{144}$.

Problem 3.14: Find the probabilities specified in Problem 7 if sets are drawn without replacement.

Solution: (a) P (both sets are Vivo) = P(VV) = P(V) × P(V) = $\frac{7}{12} \times \frac{6}{11} = \frac{42}{132}$.

(b) P (one set is Vivo, other is LG) = P (VL ∪ LV) = P(VL) + P(LV)

= P(V) × P(L) + P(L) × P(V) = $\frac{7}{12} \times \frac{5}{11} + \frac{5}{12} \times \frac{7}{11} = \frac{35}{132} + \frac{35}{132} = \frac{70}{132}$.

Problem 3.15: In a box there are four balls numbered as 1, 2, 3, and 4. Two balls are drawn one by one with replacement. Find the probability that (a) sum of the numbers is 5 or first drawn ball has the number 3. (b) sum of the numbers is 5 given that second drawn ball bears the number 3.

Solution: Let us first construct the sample space.

First draw	Second draw			
	1	2	3	4
1	(1, 1)	(1, 2)	(1, 3)	(1, 4)
2	(2, 1)	(2, 2)	(2, 3)	(2, 4)
3	(3, 1)	(3, 2)	(3, 3)	(3, 4)
4	(4, 1)	(4, 2)	(4, 3)	(4, 4)

Let A = sum of the numbers is 5 = {(1, 4), (2, 3), (3, 2), (4, 1)}. Then, $P(\text{A}) = \frac{\text{n(A)}}{\text{n(S)}} = \frac{4}{16}$.

B = first ball has the number 3 = {(3, 1), (3, 2), (3, 3), (3, 4)}. Then, $P(\text{B}) = \frac{\text{n(B)}}{\text{n(S)}} = \frac{4}{16}$.

C = second ball has the number 3 = {(1, 3), (2, 3), (3, 3), (4, 3)}. Then, $P(\text{C}) = \frac{\text{n(C)}}{\text{n(S)}} = \frac{4}{16}$.

Now, A∩B = {(3, 2)}. So, $P(\text{A}\cap\text{B}) = \frac{\text{n(A}\cap\text{B)}}{\text{n(S)}} = \frac{1}{16}$.

Again, A∩C = {(2, 3)}. So, $P(\text{A}\cap\text{C}) = \frac{\text{n(A}\cap\text{C)}}{\text{n(S)}} = \frac{1}{16}$.

(a) $P(\text{A}\cup\text{B}) = P(\text{A}) + P(\text{B}) - P(\text{A}\cap\text{B}) = \frac{4}{16} + \frac{4}{16} - \frac{1}{16} = \frac{7}{16}$.

(b) $P\,(\text{A}|\text{C}) = \frac{P(\text{A}\cap\text{C})}{P(\text{C})} = \frac{\frac{1}{16}}{\frac{4}{16}} = \frac{1}{4}$.

Problem 3.16: Find the probabilities in Problem 3.15 if balls are drawn without replacement.

Solution: In case of without replacement, outcomes as (1, 1), (2, 2), (3, 3), and (4, 4) will not be counted. Hence, our total number of possible outcomes will be 16 – 4 = 12. Then,

A = {(1, 4), (2, 3), (3, 2), (4, 1)}. Then, $P(\text{A}) = \frac{\text{n(A)}}{\text{n(S)}} = \frac{4}{12}$.

B = {(3, 1), (3, 2), (3, 4)}. Then, $P(\text{B}) = \frac{\text{n(B)}}{\text{n(S)}} = \frac{3}{12}$.

C = {(1, 3), (2, 3), (4, 3)}. Then, $P(\text{C}) = \frac{\text{n(C)}}{\text{n(S)}} = \frac{3}{12}$.

Now, A∩B = {(3, 2)}. So, $P(\text{A}\cap\text{B}) = \frac{\text{n(A}\cap\text{B)}}{\text{n(S)}} = \frac{1}{12}$.

Again, A∩C = {(2, 3)}. So, $P(\text{A}\cap\text{C}) = \frac{\text{n(A}\cap\text{C)}}{\text{n(S)}} = \frac{1}{12}$.

(a) $P(\text{A}\cup\text{B}) = P(\text{A}) + P(\text{B}) - P(\text{A}\cap\text{B}) = \frac{4}{12} + \frac{3}{12} - \frac{1}{12} = \frac{6}{12}$.

(b) $P\,(\text{A}|\text{C}) = \frac{P(\text{A}\cap\text{C})}{P(\text{C})} = \frac{\frac{1}{12}}{\frac{3}{12}} = \frac{1}{3}$.

Problem 3.17: Suppose there are 30 students, out of which 12 are from EEE department and 18 are from CSE department. The CGPA of 8 EEE and 12 CSE students are found to be good. One student is selected randomly. Find the probability that the selected student is (a) from EEE dept. given that his CGPA is not good, and (b) from CSE dept. or his CGPA is good.

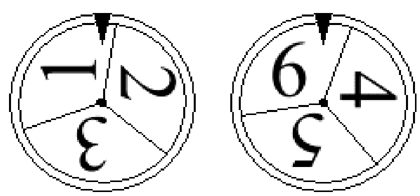
Solution: Let us first construct the sample space, S.

Department	CGPA		Total
	Good (G)	Not good ($\overline{\mathrm{G}}$)	
EEE (E)	8	4	12
CSE (C)	12	6	18
Total	20	10	30

(a) $P\,(\mathrm{E}\,|\,\overline{\mathrm{G}}) = \frac{P(\mathrm{E}\cap\overline{\mathrm{G}})}{P(\overline{\mathrm{G}})} = \frac{4/30}{10/30} = \frac{4}{10}$.

(b) $P(\mathrm{C}\cup\mathrm{G}) = P(\mathrm{C}) + P(\mathrm{G}) - P(\mathrm{C}\cap\mathrm{G}) = \frac{18}{30} + \frac{20}{30} - \frac{12}{30} = \frac{26}{30}$.

Problem 3.18: Julia spins 2 spinners. One is labeled as 1, 2 and 3. The other is labeled 4, 5 and 6.

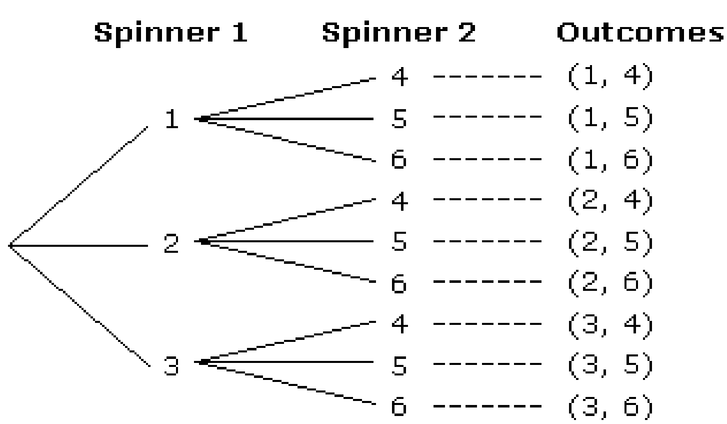


a) Draw a tree diagram for the experiment.

b) What is the probability that the spinners stop at “3” and “4”?

c) Find the probability that the spinners do not stop at “3” and “4”.

Solution: (a) Let us first draw the tree diagram.

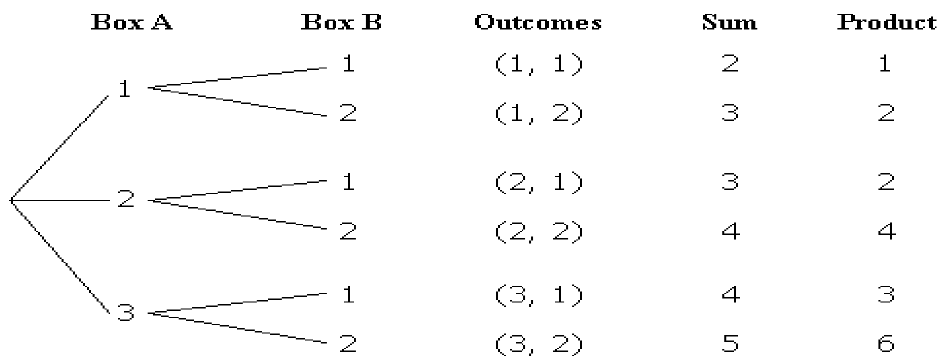


(b) P (the spinners stop at "3" and "4") = P (3, 4) = $\frac{1}{9}$.

(c) P (the spinners do not stop at "3" and "4") = 1 - P (3, 4) = 1 - $\frac{1}{9}$ = $\frac{8}{9}$.

Problem 3.19: Box A contains three cards numbered as 1, 2 and 3. Box B contains 2 cards numbered as 1 and 2. One card is drawn randomly from each box. Draw a tree diagram to list all the possible outcomes. Find the probability that (a) sum of the numbers is 4, and (b) sum is equal to the product.

Solution: Let us first draw the tree diagram.



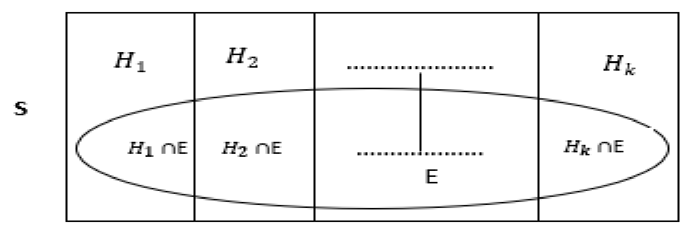
(a) P (sum of the numbers is 4) = $\frac{2}{6}$.

(b) P (sum is equal to the Product) = $\frac{1}{6}$.

Bayes' theorem and data science:

Are you planning to become a data scientist? If yes, you must know Bayes' Theorem. No data scientist can work without a complete understanding of conditional probability and Bayesian inference. So, today, we will discuss the same with the help of examples and applications. Bayes' Theorem is most widely used in Machine Learning as a classifier that makes use of Naive Bayes' Classifier. It has also emerged as an advanced algorithm for the development of Bayesian Neural Networks. The applications of Bayes' Theorem are everywhere in the field of Data Science.

Bayes' theorem: Let, S is the sample space having n equally likely outcomes. With some of outcomes let us define an event E. With some of outcomes of E we can define, separate mutually exclusive events $H_1, H_2, \ldots, H_k$. Then-



We have, $E = H_1 \cap E + H_2 \cap E + \ldots + H_k \cap E$

$\therefore P(E) = P(H_1 \cap E) + P(H_2 \cap E) + \ldots + P(H_k \cap E)$

Again, $P(E/H_i) = \frac{P(H_i \cap E)}{P(H_i)}$; $i = 1, 2, \ldots\ldots, k$ $\quad \therefore P(H_i \cap E) = P(H_i)\, P(E/H_i)$

Now, Bayes' theorem states that, $P(H_i/E) = \frac{P(H_i \cap E)}{P(E)} = \frac{P(H_i)\, P(E/H_i)}{\sum P(H_i)\, P(E/H_i)}$; $i = 1, 2, \ldots\ldots, k$

$P(H_i/E)$- This is the posterior probability. Posteriori basically means deriving theory out of given

evidence. It denotes the conditional probability of H (hypothesis), given the evidence E.

$P(E/H_i)$- It is the conditional probability of the occurrence of the evidence, given the hypothesis.

P(H)- It is the prior probability which is without the involvement of the data or the evidence.
P(E)- This is the probability of the occurrence of evidence regardless of the hypothesis.

Problem 3.20: You are planning a picnic today, but the morning is cloudy. Oh no! 50% of all rainy days start off cloudy! But cloudy mornings are common (40% of days start cloudy). This is a dry month (only 3 of 30 days tend to be rainy, or 10%). What is the chance of rain during the day?

Solution: $P\,(\text{Rain}|\text{Cloud}) = \frac{\text{P(Cloud}|\text{Rain) P(Rain)}}{\text{P(Cloud)}} = \frac{0.50 \text{ x } 0.10}{0.40} = 0.125.$

=> **A 12.5% chance of rain. Not too bad, let's have a picnic!!!!!!!!!!!!!!!!!!** <=

Problem 3.21: In a box there are 70% mathematics books and 30% electrical engineering books. Among mathematics books 40% are foreign books and among electrical engineering books 50% are foreign books. A foreign book is selected. What is the probability that the selected one is an electrical engineering book?

Solution: Let, E = Foreign book, H_1= Mathematics book, H_2= Electrical engineering book.

$P(H_1) = 0.70$, $P(E|H_1) = 0.40$. So, $P(E|H_1)\,P(H_1) = 0.40 \times 0.70 = 0.28$

$P(H_2) = 0.30$, $P(E|H_2) = 0.50$. So, $P(E|H_2)\,P(H_2) = 0.50 \times 0.30 = 0.15$

$\sum P(E|H_i)\,P(H_i) = P(E|H_1)\,P(H_1) + P(E|H_2)\,P(H_2) = 0.28 + 0.15 = 0.43$

So, $P(H_2|E) = \frac{P(E|H_2)P(H_2)}{\sum P(E|H_i)P(H_i)} = \frac{0.15}{0.43} = \frac{15}{43}.$

Exercise 3

3.1. Tickets are numbered as 1 to 20, mixed up and then one is drawn randomly. Find the probability that the ticket drawn has a number which is a multiple of 3 or 5.

3.2. A bag contains 2 red, 3 green and 2 blue balls. Two balls are drawn at random. What is the probability that none of the balls drawn is blue?

3.3. There are 8 red, 7 blue and 6 green balls in a box. One ball is picked up randomly. What is the probability that it is neither red nor green?

3.4. What is the probability of getting a sum 9 from two throws of a die?

3.5. Three unbiased coins are tossed. What is the probability of getting at most two heads?

3.6. Two die are thrown at once. Find the probability of getting two numbers whose product is even.

3.7. There are 15 boys and 10 girls in a class. Three students are selected at random. Find the probability that 1 girl and 2 boys are selected.

3.8. In a lottery, there are 10 prizes and 25 blanks. A lottery is drawn at random. What is the probability of getting a prize?

3.9. A bag contains 4 white, 5 red and 6 blue balls. Three balls are drawn at random from the bag. Find the probability that all of them are red.

3.10. In an office there are 5 computers identified by serial number 1, 2, 3, 4 and 5. Two computers are selected by two persons who work at (i)different working hours, (ii) same working hour. Find the probability that sum of the numbers of the selected computers is (a) 8 or first selected computer has the number 3, and (b) 6 given that second selected computer has the number 4.

3.11. There are 5 electronic engineers and 6 computers engineers in a mobile operator's office. A committee of 4 is to be formed to perform a duty. Find the probability that the committee will consist of (a) all electronic engineers, and (b) 2 electronic engineers and 2 computer engineers.

3.12. In a packet there are 7 Samsung and 5 Nokia mobile phone sets. Two sets are drawn one after another with replacement. Find the probability that (a) both are Samsung sets, and (b) one set is Samsung and another one is Nokia.

3.13. Find the probabilities specified in problem 3.12 if sets are drawn without replacement.

3.14. 75% signals sent from a server reach to its goal properly. Once 3 signals are checked randomly. Find the probability that out of 3, (a) at least 1 reaches properly, and (b) at best 2 reach properly.

3.15. There are 50 computers in an office. Of them, 20 are ACER and 30 are Dell. The computers are investigated and found that 12 ACER and 25 Dell computers are good. One computer is selected at random. Find the probability that the selected computer is (a) either Dell or good, and (b) ACER given that it is good.

3.16. In a communication system, signals are sent, where some of the signals are Faded (F) and some reached to the destination Properly (P). If power of the signals is not appropriate, then 50% signals are faded. An inspection team observed 3 consecutive signals. Find the probability that (a) at least one signal is faded, and (b) at best one signal is faded.

3.17. Find the probabilities specified in problem 3.16 if 10% signals are faded.

3.18. Find the probability of a number that is odd or less than 5 when a fair die is rolled.

3.19. While watching a game of Champions League football in a cafe, you observe someone who is clearly supporting Manchester United in the game. What is the probability that they were actually born within 25 miles of Manchester? Assume that the probability that a randomly selected person in a typical local bar environment is born within 25 miles of Manchester is 1/20, the chance that a person born within 25 miles of Manchester actually supports United is 7/10, and the probability that a person not born within 25 miles of Manchester supports United with probability 1/10.

3.20. Two dies are thrown at once. Find the probability of getting two numbers whose product is even.

Sample MCQs

1. In a company there are six robots numbered 1, 2, 3, 4, 5, and 6 to serve the employees. Two robots are selected one by one without replacement, Find the probability that both robots are of same number.

a) $\frac{1}{36}$ b) $\frac{0}{30}$ c) $\frac{5}{30}$ d) $\frac{1}{5}$

2. Two digits are randomly from the digits 1, 2, 3, 4, 5. Find the probability that sum of the digits will be even.

a) $\frac{1}{10}$ b) $\frac{3}{10}$ c) $\frac{4}{20}$ d) $\frac{1}{5}$

3. A group of students are trained to write a program. They are successful in 80% cases. They are asked to write three programs. Find the probability that at least one reaches properly.

a) 0.9661 b) 0.9920 c) 0.0008 d) None of them

4. Out of 20 Robots 12 are installed by Company A and 8 are installed by Company B. Eight Robots of A and 6 installations of B doing well. One installation is chosen at random to observe its performance. Find the probability that the selected Robot is of Company A under the condition that its performance is good.

a) $\frac{4}{7}$ b) $\frac{4}{10}$ c) $\frac{6}{20}$ d) $\frac{2}{6}$

5. In an office there are 15 Philips and 5 Samsung computers. If 5 computers are selected at random, what is the probability that 2 are Philips and 3 are Samsung?

a) 0.2681 b) 0.0930 c) 0.0677 d) None of them

6. Signals are sent from Station - 1 and Station - 2. Fifty signals from Station- 1 and 30 signals from Station - 2 are sent. It is known that 20% sent from Station - 1 and 30% sent from Station -

2 do not reach properly. One day on random investigation it is found that one signal is not reached properly. Find the probability that the signal is sent from Station - 2.

a) $\frac{4}{7}$ b) $\frac{9}{10}$ c) $\frac{3}{20}$ d) $\frac{9}{19}$

7. Two unbiased dice are thrown once. Find the probability that sum of the upper faces of the dice is 8 or first selected ball bears number 3.

a) $\frac{1}{36}$ b) $\frac{11}{36}$ c) $\frac{5}{30}$ d) $\frac{1}{5}$

8. In a packet there are 6 green circuits and 4 precise circuits. Two circuits are drawn one by one with replacement. Find the probability that one circuit is green and another one is precise.

a) 0.0681 b) 0.3930 c) 0.5277 d) 0.4800

9. A coin is biased so that a head is twice as likely to occur as a tail. If the coin is tossed twice, what is the probability of getting exactly 2 heads?

a) 0.4444 b) 0.3930 c) 0.5277 d) 0.4800

Reference Book: *Statistics and Probability for Engineering Applications*, D. J. Decoursey, Elsevier science, 2003.