Lecture 4

Vector Space

Objective:

- **To have knowledge on Vector space and subspace,**
- **To find linear combination of vectors,**
- **To check the linear dependency and independency of vectors.**

Vectors in $\mathbb{R}^3$:

The set of all ordered triplets of real numbers is called three-dimensional vector space and is denoted by $\mathbb{R}^3$

$$\mathbb{R}^3 = \{(x, y, z): x, y, z \in \mathbb{R}\}, \text{ where } \mathbb{R}^3 = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$$

Vectors in $\mathbb{R}^n$:

If n is a positive integer then the set of all ordered *n* triplets of real numbers is called vector n-space and is denoted by $\mathbb{R}^n$ and if $u \in \mathbb{R}^n$; $u = (u_1, u_2, \ldots \ldots, u_n)$, then u is called a n-dimensional vector in $\mathbb{R}^n$. A particular *n* triplets in $\mathbb{R}^n$ is called co-ordinates of point.

Vector Spaces:

Let F be a field of scalars and V be a non-empty set of vectors. If V contains the following rules of vector addition and scalar multiplication, then V is called vector space over F.

In vector addition:

- $\boldsymbol{A_1 : u, v \in V \Rightarrow (u + v) \in V}$
- $\boldsymbol{A_2 : u, v, w \in V \Rightarrow (u + v) + w = u + (v + w)}$
- $\boldsymbol{A_3 : u, 0 \in V \Rightarrow (u + 0) = (0 + u) = u,}$ $\quad$ **here 0 is the zero vector**
- $\boldsymbol{A_4 : u, v \in V \Rightarrow u + v = v + u}$
- $\boldsymbol{A_5 : u \in V \Rightarrow -u \in V \Rightarrow u + (-u) = (-u) + u = 0}$

In scalar multiplication:

- $\boldsymbol{M_1: a \in F \; and \; u \in V \Rightarrow au \in V}$
- $\boldsymbol{M_2: a \in F \; and \; u, v \in V \Rightarrow a(u+v) = au + av}$
- $\boldsymbol{M_3: a, b \in F \; and \; u \in V \Rightarrow (a+b)u = au + bu}$
- $\boldsymbol{M_4: a, b \in F \; and \; u \in V \Rightarrow (ab)u = a(bu)}$
- $\boldsymbol{M_3: 1 \in F \;\; \Rightarrow 1.u = u.1 = u; \quad u \in F}$

Subspace:

Let W be a non empty subset of a vector space V over the field F. We call W a subspace of V if and only if W is a vector space over the field F under the laws of vector addition and scalar multiplication defined on V, or equivalently, W is a subspace of V wherever $w_1, w_2 \in W$ and $\alpha, \beta \in F$ implies that $\alpha w_1 + \beta w_2 \in W$.

Linear combination:

Let $\boldsymbol{V}(F)$ be a vector space, where $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \dots \boldsymbol{v}_n \in \boldsymbol{V}$ and $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n \in F$. If $\alpha_1\boldsymbol{v}_1 + \alpha_2\boldsymbol{v}_2 + \alpha_3\boldsymbol{v}_3 + \cdots + \alpha_n\boldsymbol{v}_n = \boldsymbol{u} \in \boldsymbol{V}$, then $\boldsymbol{u}$ is called linear combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3 \dots \boldsymbol{v}_n$.

Example: Write the vector $\boldsymbol{u} = (1, -2, 5)$ as a linear combination of the vectors

$\boldsymbol{u_1} = (1,1,1)$, $\boldsymbol{u_2} = (1,2,3)$ and $\boldsymbol{u_3} = (2,-1,1)$.

Solution: Let $x_1\,\boldsymbol{u_1} + x_2\,\boldsymbol{u_2} + x_3\,\boldsymbol{u_3} = \boldsymbol{u}$; where x_1, x_2, x_3 are scalar

$$x_1(1,1,1) + x_2(1,2,3) + x_3(2,-1,1) = (1,-2,5)$$

$$(x_1, x_1, x_1) + (x_2, 2x_2, 3x_2) + (2x_3, -x_3, x_3) = (1,-2,5)$$

$$(x_1 + x_2 + 2x_3, x_1 + 2x_2 - x_3, x_1 + 3x_2 + x_3) = (1,-2,5)$$

Equating corresponding components

$$x_1 + x_2 + 2x_3 = 1$$

$$x_1 + 2x_2 - x_3 = -2$$

$$x_1 + 3x_2 + x_3 = 5$$

Solving using elementary row operation,

Augmented matrix: $\left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 1 & 2 & -1 & -2 \\ 1 & 3 & 1 & 5 \end{array}\right] \xrightarrow[r_3 - r_1]{r_2 - r_1} \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 2 & -1 & 4 \end{array}\right] r_3 - 2r_2 \left[\begin{array}{ccc|c} 1 & 1 & 2 & 1 \\ 0 & 1 & -3 & -3 \\ 0 & 0 & 5 & 10 \end{array}\right]$

Now $5x_3 = 10$, x_3=2

$$x_2 - 3x_3 = -3, x_2 = 3$$

$$x_1 + x_2 + 2x_3 = 1, x_1 = -6$$

we get $x_1 = -6, x_2 = 3$ **and** $x_3 = 2$.

$$\therefore -6\, u_1 + 3u_2 + 2u_3 = u$$

Hence u **is a linear combination of the vectors** u_1, u_2 **and** u_3.

Linear dependency of vectors:

Let $V(F)$ be a vector space, where $v_1, v_2, v_3 \dots v_n \in V$ and $\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n \in F$. If $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n = 0$, and at least one of the element of the set $\{\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n\}$ is not zero, then the vectors $v_1, v_2, v_3 \dots v_n$ are linearly dependent.

Linear independency of vectors:

Let $V(F)$ be a vector space and $v_1, v_2, v_3, \dots v_n \in V$ and $\alpha_1, \alpha_2, \alpha_3, \dots \alpha_n \in F$. If $\alpha_1 v_1 + \alpha_2 v_2 + \alpha_3 v_3 + \cdots + \alpha_n v_n = 0$, and all of the elements of the set $\{\alpha_1, \alpha_2, \alpha_3 \dots \alpha_n\}$ are zero , then the vectors $v_1, v_2, v_3 \dots v_n$ are linearly independent.

Example: Test whether the vectors $\boldsymbol{u_1} = (1,0,1)$, $\boldsymbol{u_2} = (-3,2,6)$ & $\boldsymbol{u_3} = (4,5,-2)$ are linearly dependent or independent.

Step 1: Let $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0$; where $\alpha_1, \alpha_2, \alpha_3$ are scalars.

$$\alpha_1(1,0,1) + \alpha_2(-3,2,6) + \alpha_3(4,5,-2) = (0,0,0)$$

$$(\alpha_1 - 3\alpha_2 + 4\alpha_3, 0 + 2\alpha_2 + 5\alpha_3, \alpha_1 + 6\alpha_2 - 2\alpha_3) = (0,0,0$$

Step 2: We can write from above equation

$$\alpha_1 - 3\alpha_2 + 4\alpha_3 = 0$$

$$0 + 2\alpha_2 + 5\alpha_3 = 0$$

$$\alpha_1 + 6\alpha_2 - 2\alpha_3 = 0$$

Step 3: Solving the above linear system, we get

$$\alpha_1 = 0, \quad \alpha_2 = 0 \; and \; \alpha_3 = 0.$$

Hence the vectors $u_1, u_2 \, and \, u_3$ are linearly independent.

Some Related Exercise:

Write the vector u as a linear combination of the vectors u_1, u_2 and u_3, where

- $u = (5, 9, 5)$, $u_1 = (1, -1, 3)$, $u_2 = (2, 1, 4)$, $u_3 = (3, 2, 5)$.
- $u = (6, 20, 2)$, $u_1 = (1, 2, 3)$, $u_2 = (1, 3, -2)$, $u_3 = (1, 4, 1)$.
- $u = (4, 2, 1, 0)$, $u_1 = (3, 1, 0, 1)$, $u_2 = (1, 2, 3, 1)$, $u_3 = (0, 3, 6, 6)$.

Test whether the following vectors are linearly independent or dependent.

- $u_1 = (2, -1, 4)$, $u_2 = (3, 6, 2)$, $u_3 = (2, 10, -4)$
- $u_1 = (3, 0, 1, -1)$, $u_2 = (2, -1, 0, 1)$, $u_3 = (1, 1, 1, -2)$
- $u_1 = (1, -1, 2)$, $u_2 = (3, -5, 1)$, $u_3 = (2, 7, 8)$, $u_4 = (-1, 1, 1)$
- $u_1 = (2, 1, 2)$, $u_2 = (0, 1, -1)$, $u_3 = (4, 3, 3)$.

Sample MCQ:

1. Let we have a set of vectors $\{u, u_1, u_2, u_3\}$ and let $u = \alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3$.

If $u = (1, -2, 5), u_1 = (1, 1, 1), u_2 = (1, 2, 3)$ and $u_3 = (2, -1, 1)$ which of the following will be the value of $\alpha_1, \alpha_2, \alpha_3$?

a) $\{6, -3, 2\}$
b) $\{0, 4, -1\}$
c) $\{5, -3, 1\}$
d) $\{-6, 3, 2\}$

2. Consider the set of vectors $\{u_1, u_2, u_3\}$ and let $\alpha_1 u_1 + \alpha_2 u_2 + \alpha_3 u_3 = 0$. For which of the following values of $\alpha_1, \alpha_2, \alpha_3$ the vectors will be linearly independent?

a) $\{1, 0, 0\}$
b) $\{0, 0, 0\}$
c) $\{0, 1, 0\}$
d) $\{0, 0, 1\}$

Outcome:

- Students got the knowledge on Vector space and subspace,
- They can easily find linear combination of vectors,
- It will be easy for the students to check the linear dependency and independency of vectors.