Lecture 5

Coordinate Systems

Objective:

- To know the relationship between three coordinate systems
- To know how to transfer point & vector from one coordinate to another.

Cartesian coordinates:

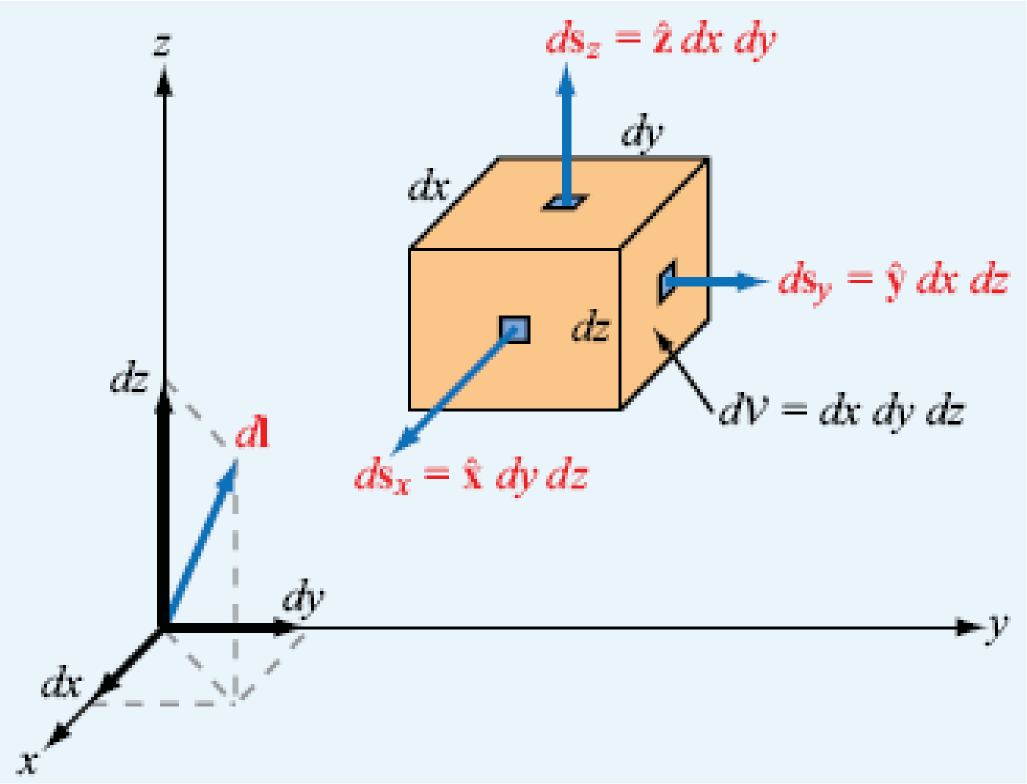


Figure: Differential length, area, and volume in Cartesian coordinates.

Cylindrical coordinates:

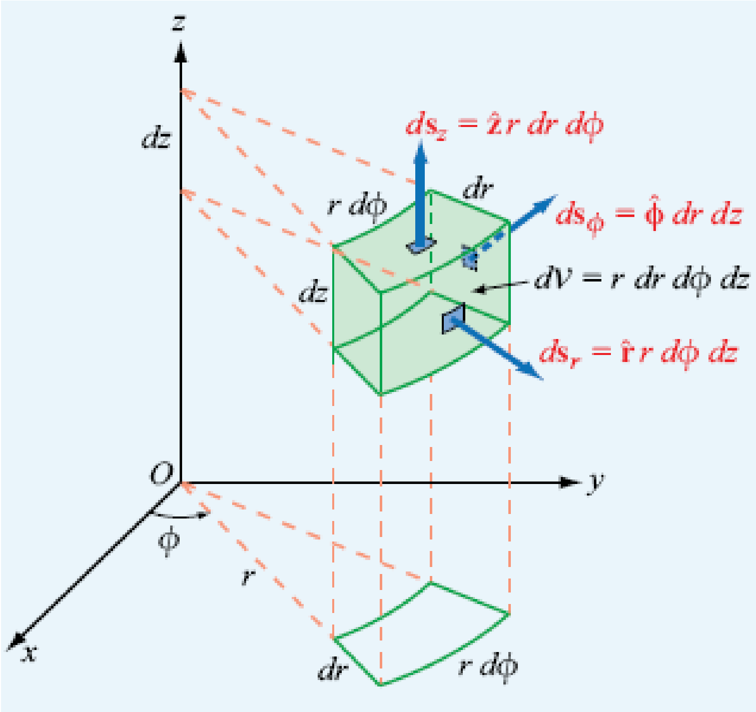


Figure: Differential length, area, and volume in cylindrical coordinates.

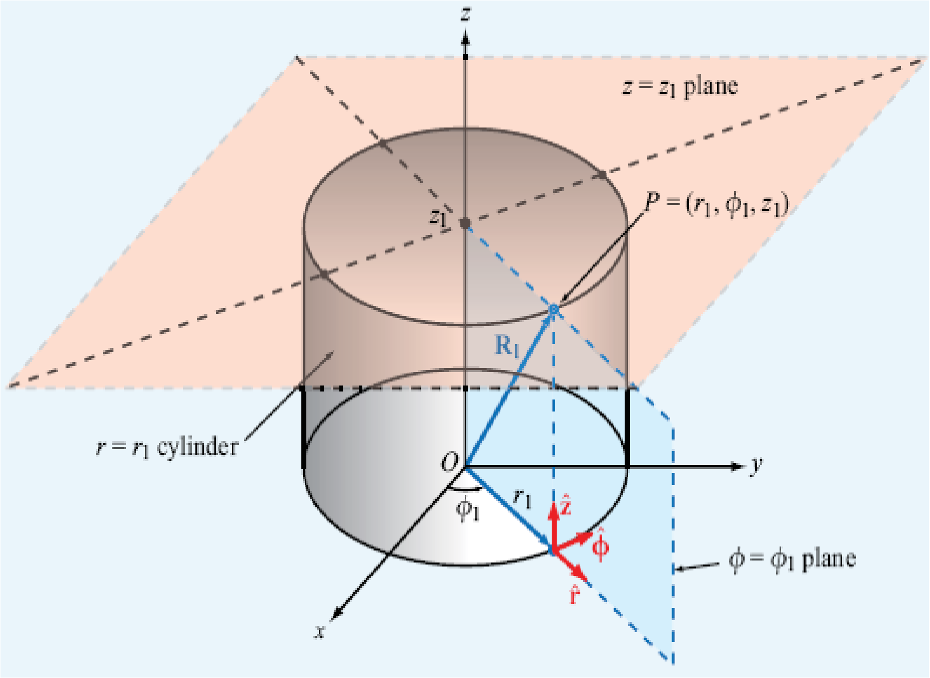
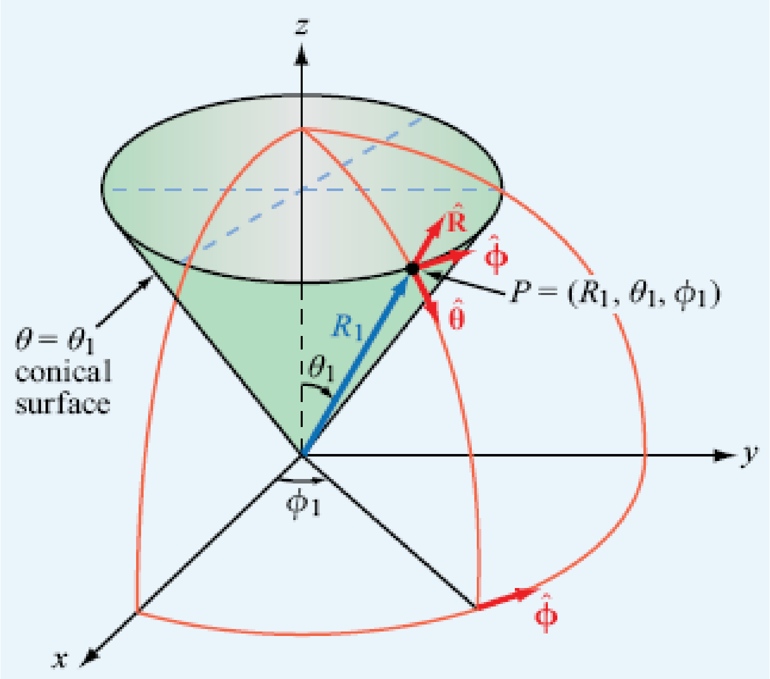


Figure: Point $P(r_1, \phi_1, z_1)$ in cylindrical coordinates; r_1 is the radial distance from the origin in the xy plane, ϕ_1 is the angle in xy plane measured from the x axis toward the y axis, and z_1 is the vertical distance from the xy plane.

Spherical coordinates:



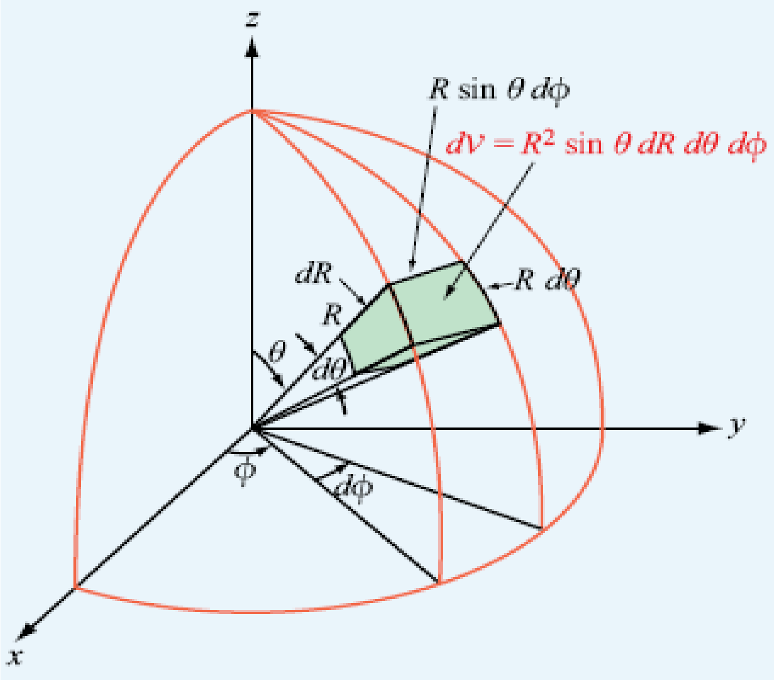


Figure: point $P(R_1, \theta_1, \phi_1)$ in spherical coordinates.

Figure: Differential volume in spherical coordinates.

Relation between the coordinate systems:

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x \cos\phi + A_y \sin\phi$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r\cos\phi$ $y = r\sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}} \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}} \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r \cos\phi - A_\phi \sin\phi$ $A_y = A_r \sin\phi + A_\phi \cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2 + y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}} \sin\theta\cos\phi + \hat{\mathbf{y}} \sin\theta\sin\phi + \hat{\mathbf{z}}\cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}} \cos\theta\cos\phi + \hat{\mathbf{y}} \cos\theta\sin\phi - \hat{\mathbf{z}}\sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$	$A_R = A_x \sin\theta\cos\phi + A_y \sin\theta\sin\phi + A_z \cos\theta$ $A_\theta = A_x \cos\theta\cos\phi + A_y \cos\theta\sin\phi - A_z \sin\theta$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$
Spherical to Cartesian	$x = R\sin\theta\cos\phi$ $y = R\sin\theta\sin\phi$ $z = R\cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}} \sin\theta\cos\phi + \hat{\boldsymbol{\theta}} \cos\theta\cos\phi - \hat{\boldsymbol{\phi}} \sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}} \sin\theta\sin\phi + \hat{\boldsymbol{\theta}} \cos\theta\sin\phi + \hat{\boldsymbol{\phi}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$	$A_x = A_R \sin\theta\cos\phi + A_\theta \cos\theta\cos\phi - A_\phi \sin\phi$ $A_y = A_R \sin\theta\sin\phi + A_\theta \cos\theta\sin\phi + A_\phi \cos\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2 + z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}} \sin\theta + \hat{\mathbf{z}} \cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}} \cos\theta - \hat{\mathbf{z}} \sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r \sin\theta + A_z \cos\theta$ $A_\theta = A_r \cos\theta - A_z \sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R\sin\theta$ $\phi = \phi$ $z = R\cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}} \sin\theta + \hat{\boldsymbol{\theta}} \cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$	$A_r = A_R \sin\theta + A_\theta \cos\theta$ $A_\phi = A_\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$

Point Transformation:

Example 1: Transform $\left(\sqrt{2}, \frac{3\pi}{4}, 3\right)$ from cylindrical coordinates to Cartesian coordinates.

Solution: $x = r\cos\phi = \sqrt{2}\cos\frac{3\pi}{4} = -1,$

$$y = r\sin\phi = \sqrt{2}\sin\frac{3\pi}{4} = 1,$$

$$z = z = 3$$

So, Cartesian point $(x, y, z) = (-1,1,3)$

Example 2: **Transform $\left(1, 0, \sqrt{3}\right)$ from Cartesian coordinate to spherical coordinate.**

Solution: $\boldsymbol{R = \sqrt{x^2 + y^2 + z^2} = \sqrt{1^2 + 0^2 + (\sqrt{3})^2} = \sqrt{4} = 2}$

$$\boldsymbol{\theta = tan^{-1}\left(\frac{\sqrt{x^2 + y^2}}{z}\right) = tan^{-1}\left(\frac{\sqrt{1^2 + 0^2}}{(\sqrt{3})}\right) = \frac{\pi}{6}}$$

$$\boldsymbol{\phi = tan^{-1}\frac{y}{x} = tan^{-1}\frac{0}{1} = 0}$$

So, the spherical point $\boldsymbol{(R, \theta, \phi) = \left(2, \frac{\pi}{6}, 0\right)}$

Some Related Exercise:

Convert the coordinates of the following points from Cartesian to cylindrical and spherical coordinates:

a) $p_1 = \left(-1, \sqrt{3}, -2\sqrt{3}\right)$

b) $p_2 = (4, 0, -4)$

c) $p_3 = (\sqrt{8}, -\sqrt{8}, 4)$

Convert the coordinates of the following points from cylindrical to Cartesian and spherical coordinates:

a) $p_1 = (2, \frac{2\pi}{3}, 2\sqrt{3})$

b) $p_2 = (\sqrt{3}, 0, -1)$

c) $p_3 = (4\sqrt{3}, \pi, -4)$

Sample MCQ:

1. Which of the following point in cylindrical coordinate is equivalent to the point $(-1, 1, 3)$ in Cartesian coordinate?

a) $\left(\sqrt{2}, \frac{3\pi}{4}, 3\right)$

b) $\left(\sqrt{2}, \frac{\pi}{4}, 3\right)$

c) $\left(\sqrt{2}, -\frac{3\pi}{4}, 3\right)$

d) $\left(\sqrt{2}, -\frac{\pi}{4}, 3\right)$

2. Which of the following point in Cartesian coordinate is equivalent to the point $\left(2, \frac{\pi}{6}, 0\right)$ in cylindrical coordinate?

a) $(1, 0, 0)$

b) $(-1, 0, \sqrt{3})$

c) $(1, 0, -\sqrt{3})$

d) $(1, 0, \sqrt{3})$

3. Which of the following could be true for the point $(2, 0, -2)$?

a) **it is a point in cylindrical coordinate**
b) **it is a point in Cartesian coordinate**
c) **both of them**
d) **none of them**

Outcome:

- Clear concept about three coordinate systems and the relationship between them.
- Point & vector can be easily transformed from one coordinate to another.