Lecture-3

Cayley-Hamilton theorem

Objective:

- Discussion about Cayley-Hamilton theorem
- Verification of Cayley-Hamilton theorem
- Finding inverse of a square matrix

Methodology

- We will find inverse using Cayley-Hamilton theorem

Cayley-Hamilton theorem

Every square matrix satisfies its characteristic equation

$$i.e\ A^n + a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots \ldots\ldots + +a_1A + a_0I = 0$$

Example: Verify the Cayley-Hamilton theorem for the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

- **Characteristic matrix:** $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$
- **Characteristic polynomial:** $|A - \lambda I| = \lambda^3 - \lambda^2 - 15\lambda - 15 = 0$

To verify the theorem we have to show tha $A^3 - A^2 - 15A - 15I = 0$

- $A^2 = \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix}$ and $A^3 = \begin{bmatrix} 44 & 33 & 53 \\ 33 & 6 & 21 \\ 53 & 21 & 41 \end{bmatrix}$

Verification

$$\therefore A^3 - A^2 - 15A - 15I$$

$$= \begin{bmatrix} 44 & 33 & 53 \\ 33 & 6 & 21 \\ 53 & 21 & 41 \end{bmatrix} - \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix} - 15\begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} - 15\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = 0$$

$$\therefore A^3 - A^2 - 15A - 15I = 0$$

Hence Cayley-Hamilton theorem is verified.

Finding Inverse

Example: Using Cayley-Hamilton theorem find the inverse of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix}$

Solution:

characteristic matrix: $A - \lambda I = \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 & 3 \\ 2 & -1-\lambda & 1 \\ 3 & 1 & 1-\lambda \end{bmatrix}$

characteristic equation: $|A - \lambda I| = 0 \implies \lambda^3 - \lambda^2 - 15\lambda - 15 = 0$

According to the Cayley –Hamilton theorem $\quad A^3 - A^2 - 15A - 15I = 0$

$\implies A^2 - A - 15I - 15A^{-1} = 0 \quad \implies A^{-1} = \frac{1}{15}[A^2 - A - 15I]$

$A^2 = \begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix}$

$\therefore A^{-1} = \frac{1}{15}\left[\begin{bmatrix} 14 & 3 & 8 \\ 3 & 6 & 6 \\ 8 & 6 & 11 \end{bmatrix} - \begin{bmatrix} 1 & 2 & 3 \\ 2 & -1 & 1 \\ 3 & 1 & 1 \end{bmatrix} - 15\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}\right] = \frac{1}{15}\begin{bmatrix} -2 & 1 & 5 \\ 1 & -8 & 5 \\ 5 & 5 & -5 \end{bmatrix}.$

Sample Question

State the Cayley-Hamilton theorem. Hence find the inverse of the following matrices using Cayley –Hamilton theorem and verify your result.

$a)\quad A=\begin{bmatrix}2&0&1\\-2&3&4\\-5&5&6\end{bmatrix}$ b) $\quad A=\begin{bmatrix}1&2&3\\2&5&3\\1&0&8\end{bmatrix}$ c) $\quad A=\begin{bmatrix}1&-1&1\\2&-1&0\\1&-1&0\end{bmatrix}$

Outcome

After this lecture students

- Will be able to verify Cayley-Hamilton theorem
- Will be able to find the inverse of a square matrix using Cayley-Hamilton theorem

Next class

- Vector Space
- Linear combination