Lecture-2

Eigenvalues & Eigenvectors

Objective:

- How to solve system of differential equation

Methodology:

- We will solve system of differential equation with the help eigenvalue and eigenvector

Solving system of differential equation

Example: Solve the differential equation

$$\begin{cases} \dot{x_1}(t) = -1.5x_1(t) + 0.5x_2(t) \\ \dot{x_2}(t) = x_1(t) - x_2(t) \end{cases}$$

with $x_1(0) = 5, x_2(0) = 4$ where $\dot{x_1}(t) = \frac{dx_1}{dt}$ and $\dot{x_2}(t) = \frac{dx_2}{dt}$.

Solution: Let, $X(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}$ and $\dot{X}(t) = \begin{pmatrix} \dot{x_1}(t) \\ \dot{x_2}(t) \end{pmatrix}$

So, $X(0) = \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = \begin{pmatrix} 5 \\ 4 \end{pmatrix}$.

$$A = \begin{pmatrix} -1.5 & 0.5 \\ 1 & -1 \end{pmatrix}$$

Characteristic polynomial $|A - \lambda I| = \begin{vmatrix} -1.5 - \lambda & 0.5 \\ 1 & -1 - \lambda \end{vmatrix}$

- **Characteristic equation:** $(\lambda + 1.5)(\lambda + 1) - 0.5 = 0$
- **Eigenvalues:** $\lambda = -0.5, -2$

Finding eigenvectors

For eigenvectors we need to solve $\begin{pmatrix} -1.5-\lambda & 0.5 \\ 1 & -1-\lambda \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0$

- When $\lambda = -0.5$, then

$$\begin{pmatrix} -1 & 0.5 \\ 1 & -0.5 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Longrightarrow \begin{cases} -v_1 + 0.5v_2 = 0 \\ v_1 - 0.5v_2 = 0 \end{cases}$$

Solving the above system, we get $-v_1 + \frac{1}{2}v_2 = 0.$

- v_2 free variable. Let $v_2 = 2a$ then $v_1 = a$
- eigenvector corresponding to $\lambda = -0.5$ is $\boldsymbol{V}_1 = \begin{pmatrix} a \\ 2a \end{pmatrix}$

Finding eigenvectors(continued)

- When $\lambda = -2$, then

$$\begin{pmatrix} 0.5 & 0.5 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = 0 \Longrightarrow \begin{cases} 0.5v_1 + 0.5v_2 = 0 \\ \quad v_1 \quad + \quad v_2 = 0 \end{cases}$$

Solving the above system, we get $0.5v_1 + 0.5v_2 = 0$

- v_2 free variable. Let $v_2 = b$ then $v_1 = -b$
- eigenvector corresponding to $\lambda = -2$ is $\boldsymbol{V}_2 = \begin{pmatrix} -b \\ b \end{pmatrix}$

Finding solutions of the system

The solution of the system of differential equation can be written as,

$$X(t) = C_1 V_1 e^{\lambda_1 t} + C_2 V_2 e^{\lambda_2 t} \Longrightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = C_1 \begin{pmatrix} a \\ 2a \end{pmatrix} e^{-0.5t} + C_2 \begin{pmatrix} -b \\ b \end{pmatrix} e^{-2t}$$

$$\Longrightarrow \begin{pmatrix} x_1(0) \\ x_2(0) \end{pmatrix} = C_1 \begin{pmatrix} a \\ 2a \end{pmatrix} + C_2 \begin{pmatrix} -b \\ b \end{pmatrix} \Longrightarrow \begin{pmatrix} 5 \\ 4 \end{pmatrix} = C_1 \begin{pmatrix} a \\ 2a \end{pmatrix} + C_2 \begin{pmatrix} -b \\ b \end{pmatrix}$$

We can write $\begin{cases} aC_1 - bC_2 = 5 \\ 2aC_1 + bC_2 = 4 \end{cases}$ Solution: $C_1 = 3/a$ and $C_2 = -2/b$.

Therefore $\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \frac{3}{a} \begin{pmatrix} a \\ 2a \end{pmatrix} e^{-0.5t} - \frac{2}{b} \begin{pmatrix} -b \\ b \end{pmatrix} e^{-2t}$

$$\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \end{pmatrix} e^{-0.5t} - 2 \begin{pmatrix} -1 \\ 1 \end{pmatrix} e^{-2t} \Rightarrow \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} = \begin{pmatrix} 3e^{-0.5t} \\ 6e^{-0.5t} \end{pmatrix} - \begin{pmatrix} -2e^{-2t} \\ 2e^{-2t} \end{pmatrix}$$

$$\therefore x_1(t) = 3e^{-0.5t} + 2e^{-2t} \qquad x_2(t) = 6e^{-0.5t} - 2e^{-2t}$$

Sample Question

Solve the following system of differential equations using eigenvalue and eigenvector

a) $\begin{cases} \dot{x_1}(t) = x_1(t) + 2x_2(t) \\ \dot{x_2}(t) = 3x_1(t) + 2x_2(t) \end{cases}$ with $x_1(0) = 0, x_2(0) = -4.$

b) $\begin{cases} \dot{x_1}(t) = -5x_1(t) + x_2(t) \\ \dot{x_2}(t) = 4x_1(t) - 2x_2(t) \end{cases}$ with $x_1(0) = 1, x_2(0) = 2.$

Outcome

After this lecture students

- Will know how to solve system of differential equation using eigenvalues and eigenvectors.

Next class

- Cayley-Hamilton theorem