Lecture 9

Divergence theorem

Objective

- To discuss about Divergence theorem
- To discuss application of Divergence theorem

Gauss's divergence theorem

Statement:

The surface integral of the normal component of a vector function **A** taken around a closed surface S is equal to the integral of the divergence of **A** taken over the volume V enclosed by the surface S.

Mathematically, $$\int_V \nabla \cdot \mathbf{A}\, dv = \oint_S \mathbf{A} \cdot d\mathbf{s}$$

Gauss's divergence theorem

Example:

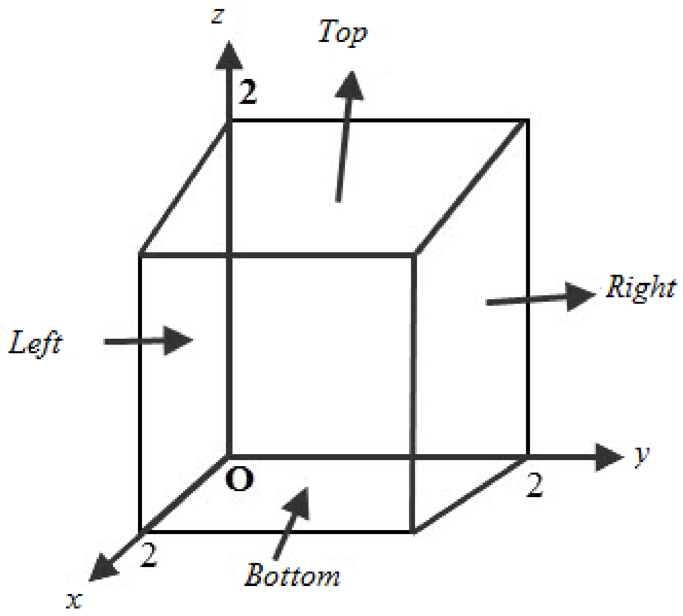
For the vector field $\mathbf{A} = \hat{x}\, xz - \hat{y}\, yz^2 - \hat{z}\, xy$, verify the divergence theorem by computing

(a) the total outward flux flowing through the surface of a cube centered at the origin and with sides equal to 2 units each and parallel to the Cartesian axes,

(b) the integral of $\nabla \cdot \mathbf{A}$over the cube's volume.

Gauss’s divergence theorem

Solution:



Step 1: Evaluate the surface integral over the six faces.

i. Front face: $x = 2, d\mathbf{s} = \hat{x}\, dydz \therefore \int_{\substack{\text{front}\\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = \int_0^2 \int_0^2 xzdydz = 8.$

Gauss's divergence theorem

Solution:

ii. Back face: $x = 0, d\mathbf{s} = -\hat{\mathrm{x}}\, dydz \therefore \int_{\substack{\text{back} \\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = 0.$

iii. Right face: $y = 2, d\mathbf{s} = \hat{\mathrm{y}}\, dxdz \therefore \int_{\substack{\text{right} \\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = \int_0^2 \int_0^2 -yz^2 dxdz = -\frac{32}{3}.$

iv. Left face: $y = 0, d\mathbf{s} = -\hat{\mathrm{y}}\, dxdz \therefore \int_{\substack{\text{left} \\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = 0.$

v. Top face: $z = 2, d\mathbf{s} = \hat{\mathrm{z}}\, dxdy \therefore \int_{\substack{\text{top} \\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = \int_0^2 \int_0^2 -xydydx = -4.$

vi. Bottom face: $z = 0, d\mathbf{s} = -\hat{\mathrm{z}}\, dxdy \therefore \int_{\substack{\text{bottom} \\ \text{face}}} \mathbf{A} \cdot d\mathbf{s} = \int_0^2 \int_0^2 xydydx = 4.$

Gauss's divergence theorem

Solution:

Step 2: Adding the above six values.

Thus, we have $\oint_S \mathbf{A} \cdot d\mathbf{s} = 8 + 0 - \frac{32}{3} + 0 - 4 + 4 = -\frac{8}{3} \cdot$

Step 3: Find the divergence of **A**.

$\nabla \cdot \mathbf{A} = \frac{\partial}{\partial x}(xz) + \frac{\partial}{\partial y}(-yz^2) + \frac{\partial}{\partial z}(-xy) = z - z^2.$

Hence $\int_V \nabla \cdot \mathbf{A}\, dv = \int_0^2 \int_0^2 \int_0^2 (z - z^2) dx\, dy\, dz = -\frac{8}{3}$

Step 4: Check whether $\int_V \nabla \cdot \mathbf{A}\, dv = \oint_S \mathbf{A} \cdot d\mathbf{s}$.

Since, the result of step 2 and step 3 are equal.

Therefore, The divergence theorem is therefore verified.

Sample Exercise

1. For the vector field $\mathbf{A} = \hat{\mathrm{x}}\, xy + \hat{\mathrm{y}}\, y^2 z + \hat{\mathrm{z}}\, xz$, verify the divergence theorem by computing

(a) the total outward flux flowing through the surface of a cube centered at the origin and with sides equal to 2 units each and parallel to the Cartesian axes,

(b) the integral of $\nabla \cdot \mathbf{A}$ over the cube's volume.

2. Given $A = \hat{\mathrm{x}}\; x^2 + \hat{\mathrm{y}}\; xy + \hat{\mathrm{z}}\; yz$,verify the divergence theorem over a cube one unit on each side. The cube is situated in the first octant of the Cartesian coordinate system with one corner at the origin.

Sample MCQ

- Which of the following is the mathematical definition of Divergence theorem?

a) $\int_V \nabla \cdot \mathbf{A}\, dv = \oint_s \mathbf{A} \cdot d\mathbf{s}$ b) $\int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \oint_c \mathbf{A} \cdot d\mathbf{l}$. c) both d) none

Outcome

After this lecture student will

- know the basic idea of Divergence theorem
- be able to solve problem using Divergence theorem

Next class

- Divergence theorem in Cylindrical and Spherical coordinate