Lecture 8

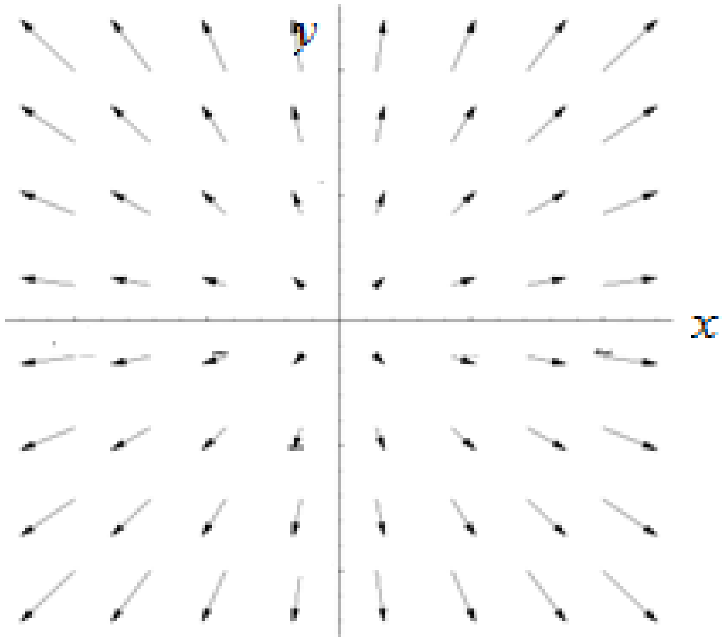
Divergence and Curl

Objective

- To discuss about divergence
- To discuss about curl

The divergence of a vector function

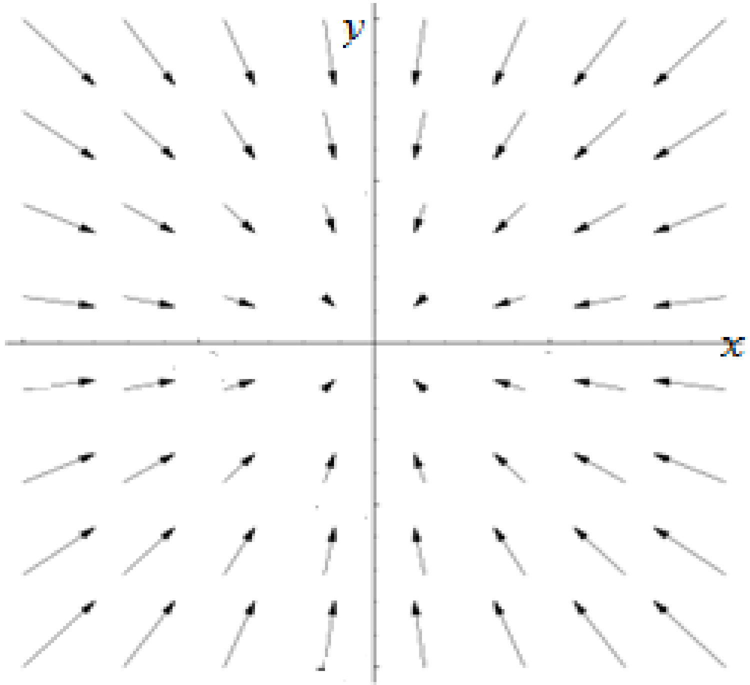
The divergence of a vector field is relatively easy to understand intuitively. Imagine that the vector field **A** pictured below gives the velocity of some fluid flow. It appears that the fluid is exploding outward from the origin.



$$\nabla \cdot \mathbf{A} > 0$$

The divergence of a vector function

In contrast, the below vector field represents fluid flowing so that it compresses as it moves toward the origin. Since this compression of fluid is the opposite of expansion, the divergence of this vector field is negative.

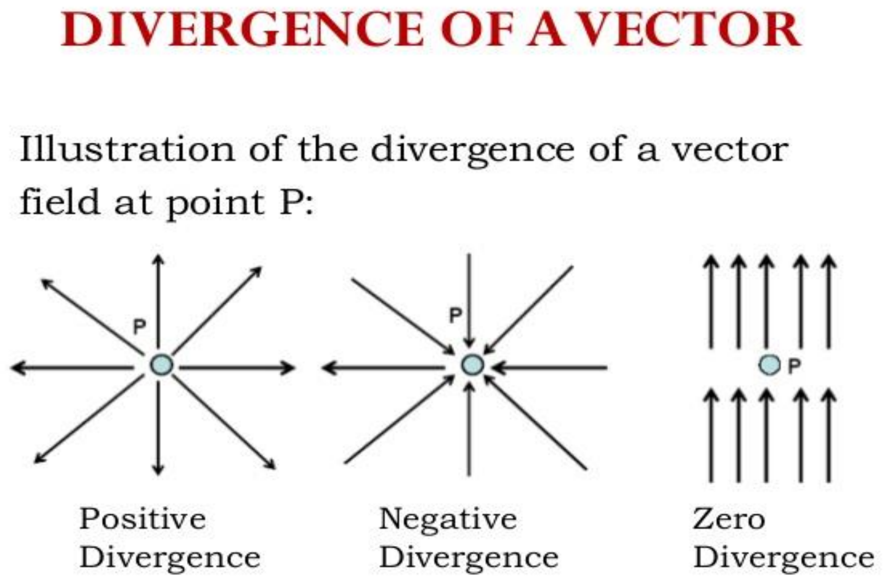


$$\nabla \cdot \mathbf{A} < 0$$

The divergence of a vector function

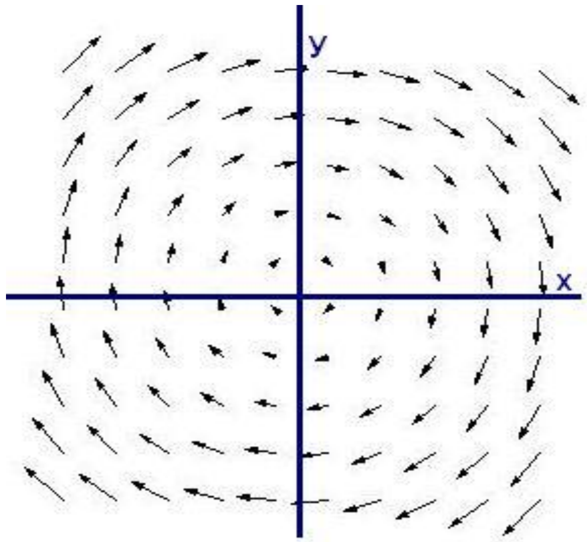
Lastly, a **solenoidal vector field** (also known as an **incompressible vector field**, a **divergence-free vector field**) is a vector field $\boldsymbol{A}$ with divergence zero at all points in the field. That is

$$\boldsymbol{\nabla} \cdot \mathbf{A} = \mathbf{0}$$



The curl of a vector function

The curl of a vector field captures the idea of how a flow may rotate.



If curl$\mathbf{A} = 0$ then $\mathbf{A}$ is called **conservative or irrotational**.

Formula for divergence and curl

Let, $\mathbf{A} = \hat{\mathrm{u}}_1 A_1(u_1, u_2, u_3) + \hat{\mathrm{u}}_2 A_2(u_1, u_2, u_3) + \hat{\mathrm{u}}_3 A_3(u_1, u_2, u_3)$ is a vector.

The divergence and the curl is given by

$$\mathrm{div}\mathbf{A} = \boldsymbol{\nabla} \cdot \mathbf{A} = \frac{1}{h_1 h_2 h_3}\left[\frac{\partial}{\partial u_1}(A_1 h_2 h_3) + \frac{\partial}{\partial u_2}(h_1 A_2 h_3) + \frac{\partial}{\partial u_3}(h_1 h_2 A_3)\right] \cdot$$

$$\mathrm{curl}\mathbf{A} = \boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{h_1 h_2 h_3}\begin{vmatrix} \hat{\mathrm{u}}_1 h_1 & \hat{\mathrm{u}}_2 h_2 & \hat{u}_3 h_3 \\ \dfrac{\partial}{\partial u_1} & \dfrac{\partial}{\partial u_2} & \dfrac{\partial}{\partial u_3} \\ h_1 A_1 & h_2 A_2 & h_3 A_3 \end{vmatrix}$$

Example of divergence and curl

Example: Determine divergence and curl of the vector $\mathbf{A} = \hat{\mathrm{x}}x^2 + \hat{\mathrm{y}}\, 2xy$.

Also check whether the vector fields solenoidal, conservative or both.

Solution:

$$\mathrm{div}\mathbf{A} = \nabla \cdot \mathbf{A} = \left[\frac{\partial}{\partial x}(x^2) + \frac{\partial}{\partial y}(2xy)\right] = 2x + 2x = 4x.$$

$$\mathrm{curl}\mathbf{A} = \nabla \times \mathbf{A} = \begin{vmatrix} \hat{\mathrm{x}} & \hat{\mathrm{y}} & \hat{\mathrm{z}} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2 & 2xy & 0 \end{vmatrix} = \hat{\mathrm{z}}\, 2y.$$

$\therefore$ **A** is not solenoidal or conservative.

Example of divergence and curl

Example: Determine divergence and curl of the vector $\mathbf{A} = \hat{\mathrm{r}}\frac{\sin\phi}{r^2} + \hat{\phi}\frac{\cos\phi}{r^2}$

Also check whether the vector fields solenoidal, conservative or both.

Solution:

$$\mathrm{div}\mathbf{A} = \nabla\cdot\mathbf{A} = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(A_1h_2h_3) + \frac{\partial}{\partial u_2}(h_1A_2h_3) + \frac{\partial}{\partial u_3}(h_1h_2A_3)\right]$$

$$= \frac{1}{r}\left[\frac{\partial}{\partial r}\left(\frac{\sin\phi}{r^2}\,r\right) + \frac{\partial}{\partial\phi}\left(\frac{\cos\phi}{r^2}\right) + \frac{\partial}{\partial z}(0\cdot r)\right] = -\frac{\sin\phi}{r^3} - \frac{\sin\phi}{r^3} = -\frac{2\sin\phi}{r^3}.$$

$$\mathrm{curl}\mathbf{A} = \nabla\times\mathbf{A} = \frac{1}{h_1h_2h_3}\begin{vmatrix} \hat{\mathrm{u}}_1h_1 & \hat{\mathrm{u}}_2h_2 & \hat{\mathrm{u}}_3h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix} = \frac{1}{r}\begin{vmatrix} \hat{\mathrm{r}} & \hat{\phi}\,\mathrm{r} & \hat{\mathrm{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial\phi} & \frac{\partial}{\partial z} \\ \frac{\sin\phi}{r^2} & \frac{\cos\phi}{r^2}r & 0 \end{vmatrix} = -\hat{\mathrm{z}}\frac{2\cos\phi}{r^2}.$$

$\therefore$ **A** is not solenoidal or conservative.

Example of divergence and curl

Example: Determine divergence and curl of the vector $\mathbf{A} = \hat{\mathrm{R}}(Re^{-R})$

Also check whether the vector fields solenoidal, conservative or both.

Solution:

$$\mathrm{div}\mathbf{A} = \boldsymbol{\nabla}\cdot\mathbf{A} = \frac{1}{R^2\sin\theta}\left[\frac{\partial}{\partial R}(Re^{-R}\cdot R^2\sin\theta) + \frac{\partial}{\partial \theta}(0\cdot R\sin\theta) + \frac{\partial}{\partial \phi}(0\cdot R)\right]$$

$$= e^{-R}(3-R).$$

$$\boldsymbol{\nabla}\times\mathbf{A} = \frac{1}{R^2\sin\theta}\begin{vmatrix} \hat{\mathrm{R}} & \hat{\theta}\,R & \hat{\phi}\,R\sin\theta \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ Re^{-R} & R\cdot 0 & R\sin\theta\cdot 0 \end{vmatrix} = 0.$$

$\therefore$ **A** is conservative but not solenoidal.

Example of divergence and curl

Example: Test whether $\mathbf{A} = \hat{\mathrm{x}}(y^2 \cos x + z^3) + \hat{\mathrm{y}}(2y \sin x - y) + \hat{\mathrm{z}}(3xz^2 + 2)$ is a conservative force field. If conservative, find the scalar potential T such that $\mathbf{A} = \boldsymbol{\nabla} T$. Hence find the work done in moving an object in this field from $(0, 1, -1)$ to $\left(\frac{\pi}{2}, -1, 2\right)$.

Solution: We know for conservative force field curl$\mathbf{A} = \boldsymbol{\nabla} \times \mathbf{A} = 0$

$$\boldsymbol{\nabla} \times \mathbf{A} = \begin{vmatrix} \hat{\mathrm{x}} & \hat{\mathrm{y}} & \hat{\mathrm{z}} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ y^2 \cos x + z^3 & 2y \sin x - y & 3xz^2 + 2 \end{vmatrix} = 0$$

Hence $\mathbf{A}$ is conservative force field.

Example of divergence and curl (continued)

Let $T(x, y, z)$be a scalar potential of **A,** i.e.$\mathbf{A} = \boldsymbol{\nabla} T \cdot$

$$\therefore\ T(x, y, z) = \int \mathbf{A} \cdot d\mathbf{l}$$

$$= \int [\hat{\mathrm{x}}(y^2 \cos x + z^3) + \hat{\mathrm{y}}(2y \sin x - y\,) + \hat{\mathrm{z}}(3xz^2 + 2)].\,[\hat{\mathrm{x}}\,dx + \hat{\mathrm{y}}\,dy + \hat{\mathrm{z}}dz]$$

$$= \int [(y^2 \cos x + z^3)dx + \ (2y \sin x - y\,)dy + (3xz^2 + 2)dz]$$

$$= \int d\left(xz^3 + y^2 \sin x + 2z - \frac{y^2}{2}\right) = xz^3 + y^2 \sin x + 2z - \frac{y^2}{2} + c, \quad c \text{ is a constant.}$$

Now, work done, $= \int_{(0,1,-1)}^{\left(\frac{\pi}{2},-1,2\right)} \mathbf{A} \cdot d\mathbf{l} = \left[xz^3 + y^2 \sin x + 2z - \frac{y^2}{2}\right]_{(0,1,-1)}^{\left(\frac{\pi}{2},-1,2\right)} = 4\pi + 7 \cdot$

Sample Exercise

1. Determine divergence and curl. Also check each of the following vector fields solenoidal, conservative or both.

(a) $\mathbf{A} = \hat{\mathrm{x}}\ zy^3 + \hat{\mathrm{y}}\ 2y\sin(xy) + \hat{\mathrm{z}}\ 3x^2 \ln z$

(b) $\mathbf{A} = \hat{\mathrm{r}}\frac{\sin\phi}{r} + \hat{\phi}\ \frac{\cos\phi}{r^2}$

(c) $\mathbf{A} = \hat{\mathrm{R}}\cos\theta + \hat{\theta}(R - \sin\theta)$

2. (a) Test whether $\mathbf{A} = \hat{\mathrm{x}}\ (\sin y + 1) + \hat{\mathrm{y}}\ (2yz + x\cos y\ ) + \hat{\mathrm{z}}\ (y^2 - 3)$ is a conservative force field. If conservative, find the scalar potential T such that $\mathbf{A} = \boldsymbol{\nabla}\ \mathrm{T}$. Hence find the work done in moving an object in this field from $(1, -1, 5)$ to $\left(2, \frac{\pi}{2}, 1\right)$ ·

(b) Test whether $\mathbf{A} = \hat{\mathrm{r}}\ (2rz - \cos\phi) + \hat{\phi}\ (\sin\phi) + \hat{\mathrm{z}}\ (r^2)$is a conservative force field. If conservative, find the scalar potential T such that $\mathbf{A} = \boldsymbol{\nabla}\ \mathrm{T}$. Hence find the work done in moving an object in this field from $(1, 0, -1)$ to $(1, \pi, 0)$ ·

Sample MCQ

- Which of the following is true for the vector $= \hat{\mathrm{x}}\, y^2 + \hat{\mathrm{y}}\, 2xy$?

a) The vector is solenoidal b) The vector is conservative c) None d) Both

- Which of the following is the condition for a vector field $\mathbf{A}$ to be solenoidal ?

a) $\mathbf{\nabla} \times \mathbf{A} = 0$ b) $\mathbf{\nabla} \times \mathbf{A} > 0$ c) $\mathbf{\nabla} . \mathbf{A} < 0$ d) $\mathbf{\nabla} . \mathbf{A} = 0$

- Which of the following is the condition for a vector field $\mathbf{A}$ to be conservative ?

a) $\mathbf{\nabla} \times \mathbf{A} = 0$ b) $\mathbf{\nabla} \times \mathbf{A} > 0$ c) $\mathbf{\nabla} . \mathbf{A} < 0$ d) $\mathbf{\nabla} . \mathbf{A} = 0$

- Which of the following is true for the vector $= \hat{\mathrm{x}}\, x^2 - \hat{\mathrm{y}}\, 4xy + \hat{\mathrm{z}}\, 2xz$?

a) The vector is solenoidal b) The vector is conservative c) both d) none

Outcome

After this lecture student will know

- How to calculate divergence and curl
- How to identify whether a vector is solenoidal or not
- How to identify whether a vector field is conservative or not

Next class

- Gauss Divergence theorem