Lecture-1

Eigenvalues & Eigenvectors

Objective:

- Discussion about eigenvalues & eigenvectors
- How to find eigenvalues & eigenvectors of a matrix

What are eigenvalues and eigenvectors?

Let $A = (a_{ij})_{n\times n}$ is a square matrix. A non-zero vector $\boldsymbol{V}$ in $\mathbb{R}^n$ is called an **eigenvector** of A if $A\boldsymbol{V}$ is a scalar multiple of $\boldsymbol{V}$; that is $A\boldsymbol{V} = \lambda\boldsymbol{V}$ for some scalar λ. The scalar λ is called an **eigenvalue** of A and $\boldsymbol{V}$ is called the eigenvector of A corresponding to λ.

- **Characteristic matrix:** $A - \lambda I$
- **Characteristic polynomial:** $|A - \lambda I|$
- **Characteristic equation:** $|A - \lambda I| = 0$
- The roots of the characteristic equation $|A - \lambda I| = 0$ are called **characteristic roots or eigenvalues** of matrix A.

Finding eigenvalues

Example: Find the eigenvalues and eigenvectors of the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & 5 & 2 \end{bmatrix}$

characteristic matrix: $A - \lambda I = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & 5 & 2 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1-\lambda & 2 & -1 \\ 0 & -2-\lambda & 0 \\ 0 & 5 & 2-\lambda \end{bmatrix}$

characteristic polynomial : $|A - \lambda I| = \begin{vmatrix} 1-\lambda & 2 & -1 \\ 0 & -2-\lambda & 0 \\ 0 & 5 & 2-\lambda \end{vmatrix}$

characteristic equation: $|A - \lambda I| = 0 \Rightarrow (\lambda - 1)(\lambda + 2)(\lambda - 2) = 0 \Rightarrow \lambda = 1, -2, 2$

So, the characteristic roots or the Eigenvalues of A is $\lambda = 1, -2, 2$

Eigenvectors: $(A - \lambda I)\boldsymbol{V} = 0$ So, $\begin{bmatrix} 1-\lambda & 2 & -1 \\ 0 & -2-\lambda & 0 \\ 0 & 5 & 2-\lambda \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0$

Finding eigenvectors

- When $\lambda = 1$

$$\begin{bmatrix} 0 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 5 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Longrightarrow \begin{cases} 0 \cdot v_1 + 2 \cdot v_2 - 1 \cdot v_3 = 0 \\ 0 \cdot v_1 - 3 \cdot v_2 + 0 \cdot v_3 = 0 \\ 0 \cdot v_1 + 5 \cdot v_2 + 1 \cdot v_3 = 0 \end{cases}$$

Solving we get $v_2 = v_3 = 0$. Hence v_1 is a free variable.

Let, $v_1 = a$, where a is any real number.

the eigenvector of A corresponding to the eigenvalue $\lambda = 1$ are the non-zero vectors of the form

$$\boldsymbol{V} = \begin{bmatrix} a \\ 0 \\ 0 \end{bmatrix}$$

Finding eigenvectors(continued)

- When $\lambda = -2$, then

$$\begin{bmatrix} 3 & 2 & -1 \\ 0 & 0 & 0 \\ 0 & 5 & 4 \end{bmatrix}\begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Longrightarrow \begin{cases} 3 \cdot v_1 + 2 \cdot v_2 - 1 \cdot v_3 = 0 \\ 0 \cdot v_1 - 0 \cdot v_2 + 0 \cdot v_3 = 0 \\ 0 \cdot v_1 + 5 \cdot v_2 + 4 \cdot v_3 = 0 \end{cases}$$

This system has one free variable. Let $v_3 = b$.

$\therefore v_2 = -\frac{4b}{5}$ and $\therefore v_1 = -\frac{13b}{15}$

Therefore, $\boldsymbol{V} = \begin{bmatrix} -\frac{13b}{15} \\ -\frac{4b}{5} \\ b \end{bmatrix}$.

Finding eigenvectors(continued)

When $\lambda = 2$,

$$\begin{bmatrix} -1 & 2 & -1 \\ 0 & -4 & 0 \\ 0 & 5 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = 0 \Longrightarrow \begin{cases} -1 \cdot v_1 + 2 \cdot v_2 - 1 \cdot v_3 = 0 \\ 0 \cdot v_1 - 4 \cdot v_2 + 0 \cdot v_3 = 0 \\ 0 \cdot v_1 + 5 \cdot v_2 + 0 \cdot v_3 = 0 \end{cases}$$

Hence, $v_2 = 0$ and v_3 is free variable. Let $v_3 = c$ then we have $v_1 = -c$

So for $\lambda = -2$ the eigenvector is $V = \begin{bmatrix} -c \\ 0 \\ c \end{bmatrix}$

Sample Question

Find the eigenvalues and eigenvectors of the following matrices

a) $\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ b) $\begin{bmatrix} -1 & 0 \\ 3 & 2 \end{bmatrix}$ c) $\begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & -5 & 2 \end{bmatrix}$ d) $\begin{bmatrix} 2 & 3 & 3 \\ 1 & 3 & 2 \\ -1 & -4 & -3 \end{bmatrix}$

Sample MCQ

Consider the matrix $A = \begin{bmatrix} 1 & 2 & -1 \\ 0 & -2 & 0 \\ 0 & 5 & 2 \end{bmatrix}$.

1) Which of the following is the characteristic polynomial of A ?

$a)(\lambda + 1)(\lambda + 2)(\lambda - 2)$ b) $(\lambda - 1)(\lambda - 3)(\lambda - 2)$

c) $(\lambda - 1)(\lambda + 2)(\lambda - 2)$ d) $(\lambda + 5)(\lambda + 2)(\lambda - 2)$

2) Which of the following is the eigenvalue of A ?

a) 2 b) -3 c) 4 d) -5

3) Which of the following can't be the eigenvector of A ?

a) $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ b) $\begin{pmatrix} 13 \\ 12 \\ -15 \end{pmatrix}$ c) $\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$ d) $\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}$.

Outcome

After this lecture students

- Will know about eigenvalues and eigenvectors
- Will be able to calculate eigenvalues
- Will be able to calculate eigenvectors

Next class

- Application of eigenvalue and eigenvector to Solve system of differential equation