Lecture 6

Coordinate Systems

Objective:

- To know the relationship between three coordinate systems
- To know how to transfer point & vector from one coordinate to another.

Relation between the coordinate systems:

Transformation	Coordinate Variables	Unit Vectors	Vector Components
Cartesian to cylindrical	$r = \sqrt[+]{x^2 + y^2}$ $\phi = \tan^{-1}(y/x)$ $z = z$	$\hat{\mathbf{r}} = \hat{\mathbf{x}} \cos\phi + \hat{\mathbf{y}} \sin\phi$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_r = A_x \cos\phi + A_y \sin\phi$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$ $A_z = A_z$
Cylindrical to Cartesian	$x = r \cos\phi$ $y = r \sin\phi$ $z = z$	$\hat{\mathbf{x}} = \hat{\mathbf{r}} \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{r}} \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{z}}$	$A_x = A_r \cos\phi - A_\phi \sin\phi$ $A_y = A_r \sin\phi + A_\phi \cos\phi$ $A_z = A_z$
Cartesian to spherical	$R = \sqrt[+]{x^2 + y^2 + z^2}$ $\theta = \tan^{-1}[\sqrt[+]{x^2 + y^2}/z]$ $\phi = \tan^{-1}(y/x)$	$\hat{\mathbf{R}} = \hat{\mathbf{x}} \sin\theta \cos\phi$ $\quad + \hat{\mathbf{y}} \sin\theta \sin\phi + \hat{\mathbf{z}} \cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{x}} \cos\theta \cos\phi$ $\quad + \hat{\mathbf{y}} \cos\theta \sin\phi - \hat{\mathbf{z}} \sin\theta$ $\hat{\boldsymbol{\phi}} = -\hat{\mathbf{x}} \sin\phi + \hat{\mathbf{y}} \cos\phi$	$A_R = A_x \sin\theta \cos\phi$ $\quad + A_y \sin\theta \sin\phi + A_z \cos\theta$ $A_\theta = A_x \cos\theta \cos\phi$ $\quad + A_y \cos\theta \sin\phi - A_z \sin\theta$ $A_\phi = -A_x \sin\phi + A_y \cos\phi$
Spherical to Cartesian	$x = R \sin\theta \cos\phi$ $y = R \sin\theta \sin\phi$ $z = R \cos\theta$	$\hat{\mathbf{x}} = \hat{\mathbf{R}} \sin\theta \cos\phi$ $\quad + \hat{\boldsymbol{\theta}} \cos\theta \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi$ $\hat{\mathbf{y}} = \hat{\mathbf{R}} \sin\theta \sin\phi$ $\quad + \hat{\boldsymbol{\theta}} \cos\theta \sin\phi + \hat{\boldsymbol{\phi}} \cos\phi$ $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$	$A_x = A_R \sin\theta \cos\phi$ $\quad + A_\theta \cos\theta \cos\phi - A_\phi \sin\phi$ $A_y = A_R \sin\theta \sin\phi$ $\quad + A_\theta \cos\theta \sin\phi + A_\phi \cos\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$
Cylindrical to spherical	$R = \sqrt[+]{r^2 + z^2}$ $\theta = \tan^{-1}(r/z)$ $\phi = \phi$	$\hat{\mathbf{R}} = \hat{\mathbf{r}} \sin\theta + \hat{\mathbf{z}} \cos\theta$ $\hat{\boldsymbol{\theta}} = \hat{\mathbf{r}} \cos\theta - \hat{\mathbf{z}} \sin\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$	$A_R = A_r \sin\theta + A_z \cos\theta$ $A_\theta = A_r \cos\theta - A_z \sin\theta$ $A_\phi = A_\phi$
Spherical to cylindrical	$r = R \sin\theta$ $\phi = \phi$ $z = R \cos\theta$	$\hat{\mathbf{r}} = \hat{\mathbf{R}} \sin\theta + \hat{\boldsymbol{\theta}} \cos\theta$ $\hat{\boldsymbol{\phi}} = \hat{\boldsymbol{\phi}}$ $\hat{\mathbf{z}} = \hat{\mathbf{R}} \cos\theta - \hat{\boldsymbol{\theta}} \sin\theta$	$A_r = A_R \sin\theta + A_\theta \cos\theta$ $A_\phi = A_\phi$ $A_z = A_R \cos\theta - A_\theta \sin\theta$

Transformation (Cartesian coordinates to cylindrical and vice versa):

		A_r	A_ϕ	A_z
		$\hat{r}$	$\hat{\phi}$	$\hat{z}$
A_x	$\hat{x}$	$\cos\phi$	$-\sin\phi$	0
A_y	$\hat{y}$	$\sin\phi$	$\cos\phi$	0
A_z	$\hat{z}$	0	0	1

Transformation (Spherical coordinates to cylindrical coordinates and vice versa):

		A_R	A_θ	A_ϕ
		$\hat{R}$	$\hat{\theta}$	$\hat{\phi}$
A_r	$\hat{r}$	$\sin\theta$	$\cos\theta$	0
A_z	$\hat{z}$	$\cos\theta$	$-\sin\theta$	0
A_ϕ	$\hat{\phi}$	0	0	1

Transformation (spherical coordinates to Cartesian coordinates and vice versa):

		A_R	A_θ	A_ϕ
		$\hat{R}$	$\hat{\theta}$	$\hat{\phi}$
A_x	$\hat{x}$	$\sin\theta\cos\phi$	$\cos\theta\cos\phi$	$-\sin\phi$
A_y	$\hat{y}$	$\sin\theta\sin\phi$	$\cos\theta\ \sin\phi$	$\cos\phi$
A_z	$\hat{z}$	$\cos\theta$	$-\sin\theta$	0

Vector Transformation:

Example 1: **Transform vector $\vec{A} = \hat{x}(x+y) + \hat{y}(y-x) + \hat{z}z$ to cylindrical coordinates.**

Solution: We know, cylindrical coordinate $\vec{A} = \hat{r}A_r + \hat{\phi}A_\phi + \hat{z}A_z$

$$A_r = A_x \cos\phi + A_y \sin\phi = (x+y)\cos\phi + (y-x)\sin\phi$$

$$= (r\cos\phi + r\sin\phi)\cos\phi + (r\sin\phi - r\cos\phi)\sin\phi$$

$$= r\cos^2\phi + r\sin\phi\cos\phi + r\sin^2\phi - r\sin\phi\cos\phi$$

$$= r\left(\cos^2\phi + \sin^2\phi\right) = r$$

$$A_\phi = -A_x \sin\phi + A_y \cos\phi$$

$$= -(x + y)\sin\phi + (y - x)\cos\phi$$

$$= -(r\cos\phi + r\sin\phi)\sin\phi + (r\sin\phi - r\cos\phi)\cos\phi$$

$$= -r\cos\phi\sin\phi - r\sin^2\phi + r\cos\phi\sin\phi - r\cos^2\phi$$

$$= -r\left(\sin^2\phi + \cos^2\phi\right)$$

$$= -r$$

$$A_z = z$$

$$\therefore \vec{A} = \hat{r}r - \hat{\phi}r + \hat{z}z$$

Example 2: Express vector $\vec{A} = \hat{r}\, r - \hat{\phi}\, \mathrm{r} + \hat{z}\, z$ in Cartesian coordinate.

Solution: We know in Cartesian coordinate $\vec{A} = \hat{x}A_x + \hat{y}A_y + \hat{z}A_z$

$$A_x = A_r \cos\varphi - A_\varphi \sin\varphi$$

$$= r\cos\varphi + \mathrm{r}\sin\varphi$$

$$= x + y$$

$$A_y = A_r \sin\varphi + A_\varphi \cos\varphi$$

$$= r\sin\varphi - r\cos\varphi$$

$$= y - x$$

$$A_z = z$$

Hence, $\therefore \vec{A} = \hat{x}\,(x + y) + \hat{y}\,(y - x\,) + \hat{z}\,z$

Some Related Exercise:

Transform the following vectors into cylindrical coordinates at the indicated points:

1) $A = \hat{\mathbf{x}}(x + y) + \hat{\mathbf{y}}(x + y) + \hat{\mathbf{z}}\,\mathbf{z}\ \mathbf{at}\ p = (4, 0, -4)$

2) $B = \hat{\mathbf{R}} \sin\theta + \hat{\boldsymbol{\theta}} \cos\theta + \hat{\boldsymbol{\phi}} \cos^2\phi\ \ at\ p = \left(2, \frac{\pi}{2}, \frac{\pi}{4}\right)$

Transform the following vectors into spherical coordinates at the indicated points:

1) $A = \hat{\mathbf{x}}y - \hat{\mathbf{y}}x\ at\ p = (1, -1, 0)$

2) $B = \hat{\mathbf{z}} \sin\phi\ \ at\ p = \left(2, \frac{\pi}{4}, 2\right)$

Transform the following vectors into cartesian coordinates at the indicated points:

1) $A = -\hat{\boldsymbol{\phi}} R \sin\theta\ at\ p = (1, \frac{\pi}{2}, 0)$

2) $B = -\hat{\mathbf{r}} \cos\phi - \hat{\boldsymbol{\phi}} \sin\phi + \hat{\mathbf{z}}\,\mathbf{z}\ at\ p = (2, \frac{2\pi}{3}, 2\sqrt{3})$

Sample MCQ:

1. **Which one is the differential volume in cylindrical coordinate?**

   a) $\boldsymbol{dv\, rdr\, d\varphi\, dz}$

   b) $\boldsymbol{dv\, \varphi dr\, d\varphi\, dz}$

   c) $\boldsymbol{dv\, zdr\, d\varphi\, dz}$

2. Which one is the correct vector expression of $\vec{A} = \hat{x}(x+y) + \hat{y}(y-x) + \hat{z}z$ in spherical coordinate?

a) $\vec{A} = \hat{R} - \hat{\phi}\, R \sin\theta$

b) $\vec{A} = \hat{R}\, R - \hat{\phi}\, R \sin\theta$

c) $\vec{A} = \hat{R}\, R - \hat{\phi}\, \sin\theta$

Outcome:

- **Clear concept about three coordinate systems and the relationship between them.**
- **Point & vector can be easily transformed from one coordinate to another.**