Lecture 11

Stoke’s Theorem

Objective:

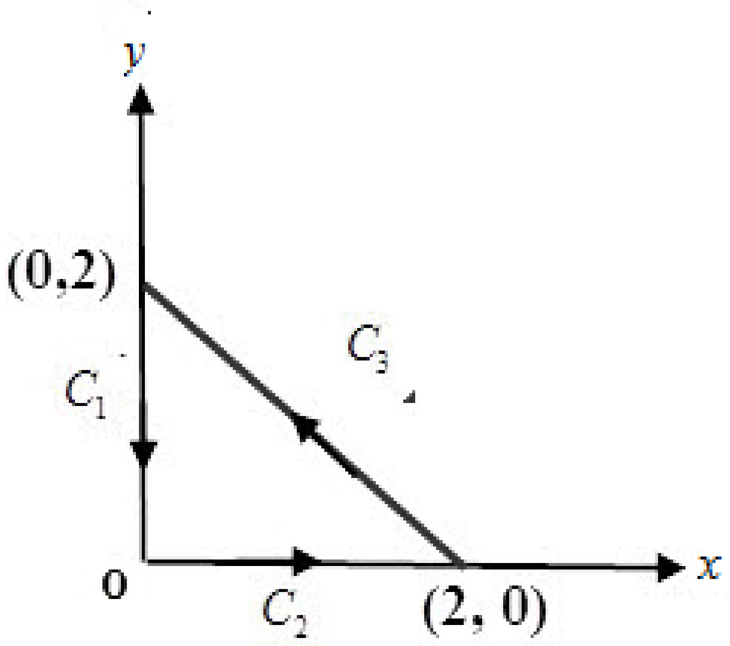
- **To know the statement of Stoke's theorem**
- **To know how to verify Stoke's theorem in a vector field and given contour .**

Stokes's Theorem:

Let S be the open surface (two-sided) and C be the closed boundary of S, the vector field A is continuous on S. Then

$$\int_{s} (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \boldsymbol{ds} = \oint_{c} \boldsymbol{A} \cdot \boldsymbol{dl} \ .$$

Example 1: **Assume that a vector field** $A = \hat{x}(2x^2 + y^2) + \hat{y}(xy - y^2)$**, (a) find** $\oint_c A \cdot dl$ **around the triangular contour, (b) find** $\int_s (\nabla \times A) \cdot ds$ **over triangular arc, (c) verify Stokes's theorem.**



Solution:

(a) $\boldsymbol{dl} = \hat{\mathbf{x}}\, dx + \hat{\mathbf{y}}\, dy \therefore \boldsymbol{A} \cdot \boldsymbol{dl} = \left(2x^2 + y^2\right)dx + \left(xy - y^2\right)dy$

Path c_1; $x = 0, dx = 0, \oint_{c_1} \boldsymbol{A} \cdot \boldsymbol{dl} = -\int_2^0 y^2 dy = \frac{8}{3}$.

Path c_2; $y = 0, dy = 0, \oint_{c_2} \boldsymbol{A} \cdot \boldsymbol{dl} = \int_0^2 2x^2 dx = \frac{16}{3}$.

Path c_3; $y = 2 - x, dy = -dx,$

$$\oint_{c_3} \boldsymbol{A} \cdot \boldsymbol{dl} = \int_2^0 \left[\left\{2x^2 + (2-x)^2\right\} - \left\{x(2-x) - (2-x)^2\right\}\right] dx = -\frac{28}{3}.$$

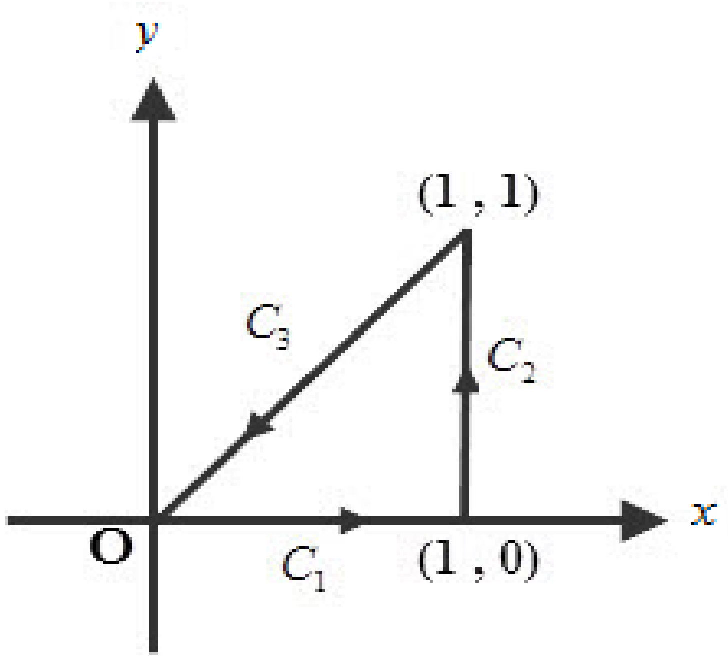
Total, $\oint_c \boldsymbol{A} \cdot \boldsymbol{dl} = -\frac{4}{3}$.

Now, $\boldsymbol{\nabla} \times \boldsymbol{A} = -\hat{\boldsymbol{z}}\, \boldsymbol{y}, \boldsymbol{ds} = \hat{\boldsymbol{z}}\, \boldsymbol{dxdy}$

$$\therefore \int_s (\boldsymbol{\nabla} \times \boldsymbol{A}) \cdot \boldsymbol{ds} = -\int_0^2 \int_0^{2-x} \boldsymbol{ydydx} = -\frac{4}{3}.$$

(c) Stokes's theorem is verified.

Example 2: **Assume that a vector field $A = \hat{x}\, xy - \hat{y}\left(x^2 + 2y^2\right)$, (a) find $\oint_c \ A \cdot dl$ around the triangular contour, (b) find $\int_s \ (\nabla \times A) \cdot ds$ over triangular arc.**



Solution:

(a) $\boldsymbol{dl} = \hat{\mathbf{x}}\, \boldsymbol{dx} + \hat{\mathbf{y}}\, \boldsymbol{dy} \therefore \boldsymbol{A} \cdot \boldsymbol{dl} = \boldsymbol{xy\, dx} - \left(\boldsymbol{x^2 + 2y^2}\right) \boldsymbol{dy}$

Path c_1; $y = 0,\ dy = 0, \oint_{c_1} A \cdot dl = \int_0^1 xy\, dx - \left(x^2 + 2y^2\right) dy = 0\cdot$

Path c_2; $x = 1,\ dx = 0, \oint_{c_2} A \cdot dl = \int_0^1 xy\, dx - \left(x^2 + 2y^2\right) dy$

$$= \int_0^1 -\left(1 + 2y^2\right) dy = -\frac{5}{3}\cdot$$

Path c_3; $y = x,\ \ dy = dx,$

$$\oint_{c_3} A \cdot dl = \int_1^0 xy\, dx - \left(x^2 + 2y^2\right) dy = \int_1^0 y^2\, dy - \left(y^2 + 2y^2\right) dy = \frac{2}{3} \cdot$$

Total, $\oint_c\ A \cdot dl = -\frac{5}{3} + \frac{2}{3} = -1 \cdot$

Now, $\nabla \times A = \nabla \times A = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy & -\left(x^2 + 2y^2\right) & 0 \end{vmatrix} = -3x\, \hat{z}, ds = \hat{z}\, dxdy$

$$\therefore \int_s (\nabla \times A) \cdot ds = -\int_0^1 \int_0^x 3x\, dydx = -1 \cdot$$

Stokes's theorem is verified.

Sample MCQ:

1. Which one indicates the correct statement of Stoke's theorem?

a) $\int_s (\nabla \times A) \cdot ds = \oint_c A \cdot dl$

b) $\int_s (\nabla . A) ds = \oint_c A \cdot dl$

c) $\int_v (\nabla \times A) \cdot dv = \oint_c A \cdot dl$

d) $\int_v (\nabla . A) d\mathrm{v} = \oint_c A \cdot dl$

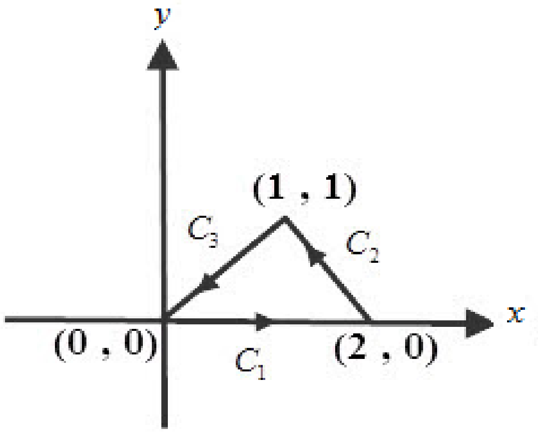
2. Assume that a vector field $A = \hat{\mathrm{x}}\, xy - \hat{\mathrm{y}}(x^2 + 2y^2)$, find $\oint_{c_2} A \cdot dl$ using the following triangular arc.

a) $\frac{11}{3}$

b) 0

c) $-\frac{11}{3}$

d) 5



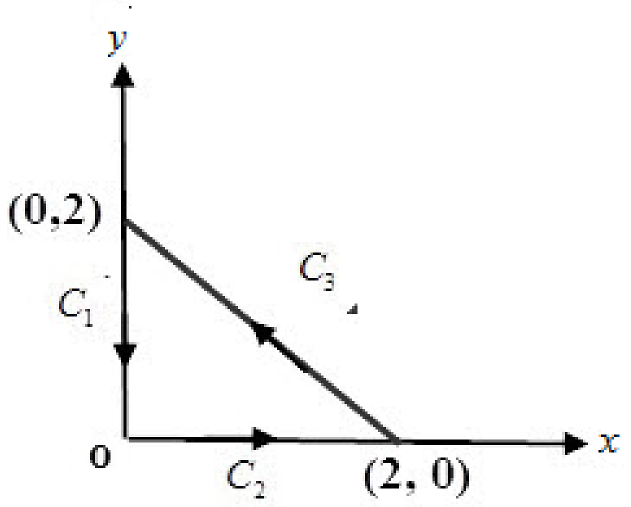
3. Assume that a vector field $A = \hat{x}(2x^2 + y^2) + \hat{y}(xy - y^2)$, find $\oint_{c3} A \cdot dl$ around the given triangular contour

$a)\ 0$

$b) -28/3$

$c)\ 8/3$

$d)\ 16/3$



4. For the same vector field and triangular contour given in question (3), along C_2 $A \cdot dl = ?$

$a)\ A \cdot dl = (2x^2)dx$

$b)\ A \cdot dl = (xy - y^2)dy$

$c)\ A \cdot dl = (2x^2 + y^2)dx$

$d)\ A \cdot dl = (2x^2 + y^2)dx + (xy - y^2)dy$

Outcome:

Clear concept about the Stoke's theorem in triangular contour.