Lecture-12

Stokes Theorem in Cylindrical and Spherical Coordinate

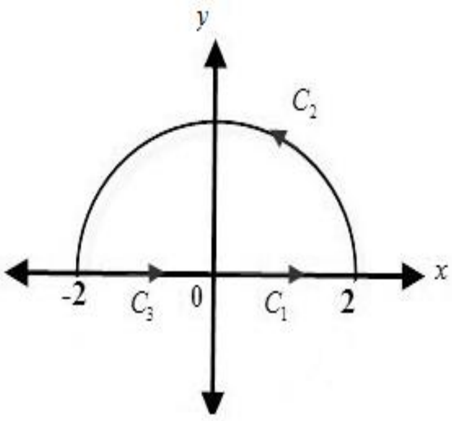
Objective:

- Discuss about Stokes theorem
- verification of Stokes theorem on a closed boundary of an open surface.

Verification of Stokes theorem

Problem: Assume that a vector field $\mathbf{A} = \hat{\mathrm{r}}\, r\cos\phi + \widehat{\phi}\sin\phi$, (a) find $\oint_c \mathbf{A}\cdot d\mathbf{l}$ over the semicircular contour, and (b) find $\int_s (\nabla\times\mathbf{A})\cdot d\mathbf{s}$ over the surface of the semicircle.

$d\mathbf{l} = \hat{\mathrm{r}}\, dr + \widehat{\phi}\, rd\phi \therefore \mathbf{A}\cdot d\mathbf{l} = r\cos\phi\, dr + \sin\phi\, r\, d\phi$



Path c_1: $\phi = 0,\ d\phi = 0,\ \oint_{c_1} \mathbf{A}\cdot d\mathbf{l} = \int_0^2 r\cos\phi\, dr = 2$

Path c_2: $r = 2,\ dr = 0,\ \oint_{c_2} \mathbf{A}\cdot d\mathbf{l} = \int_0^\pi \sin\phi\, r\, d\phi = 4.$

Path c_3: $\phi = \pi,\ d\phi = 0,$

$\oint_{c_3} \mathbf{A}\cdot d\mathbf{l} = \int_2^0 r\cos\varphi\, dr = -\int_2^0 r\, dr = 2\cdot$

Total, $\oint_c \mathbf{A}\cdot d\mathbf{l} = 2 + 4 + 2 = 8\cdot$

Verification of Stokes theorem

Now, $\boldsymbol{\nabla} \times \mathbf{A} = \boldsymbol{\nabla} \times \mathbf{A} = \frac{1}{r}\begin{vmatrix} \hat{\mathrm{r}} & r\widehat{\phi} & \hat{\mathrm{z}} \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ r\cos\phi & r\sin\phi & 0 \end{vmatrix} = \hat{\mathrm{z}}\frac{1}{r}(\sin\phi + r\sin\phi), d\boldsymbol{s} =$ $\hat{\mathrm{z}}\, r\; dr d\phi$

$\therefore \int_s \; (\boldsymbol{\nabla} \times \mathbf{A}) \cdot d\mathbf{s} = \int_0^{\pi} \int_0^{2} (\sin\phi + r\sin\phi)\; dr d\phi = 8 \cdot$

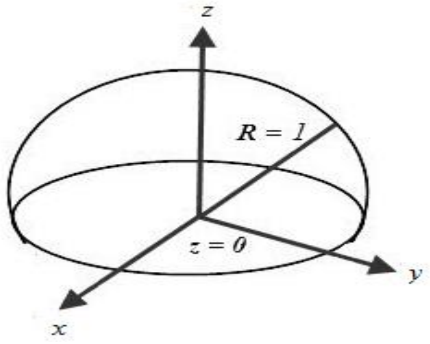
Stokes theorem is verified.

Verification of Stokes theorem

Problem: Verify Stokes's theorem for the vector field, $\mathbf{A} = \widehat{\mathrm{R}}\cos\theta + \widehat{\phi}\sin\theta$ by evaluating it on the hemisphere of unit radius.

$$\nabla \times \mathbf{A} = \frac{\mathbf{1}}{R^2 \sin\theta} \begin{vmatrix} \widehat{\mathrm{R}} & R\,\hat{\theta} & R\sin\theta\,\widehat{\phi} \\ \dfrac{\partial}{\partial R} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial \phi} \\ \cos\theta & R\cdot 0 & R\sin\theta\sin\theta \end{vmatrix}$$

$$= \frac{\widehat{\mathrm{R}}\,2\mathrm{R}\sin\theta\cos\theta - \hat{\theta}\,R\sin^2\theta + \widehat{\phi}\,R\sin^2\theta}{R^2\sin\theta}.$$



and $d\mathbf{s} = \widehat{\mathrm{R}}(Rd\theta)(R\sin\theta\, d\phi)$

$(\nabla \times \mathbf{A}).\mathrm{ds} = 2\mathrm{R}\sin\theta\cos\theta\; d\theta\; d\phi, \quad (R = 1).$

Verification of Stokes theorem

$$\int_S (\nabla \times \mathbf{A}) \cdot d\mathbf{s} = \int_{\theta=0}^{\pi/2} \int_{\varphi=0}^{2\pi} 2 \sin\theta \cos\theta \, d\phi d\theta$$

$$= 4\pi \int_0^{\pi/2} \sin\theta \cos\theta \, d\theta = 4\pi \int_0^1 u \, du = 2\pi.$$

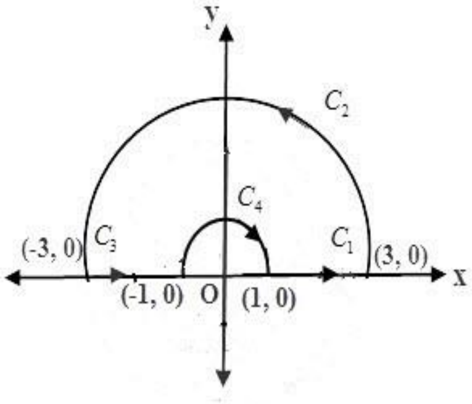
Again, $d\mathbf{l} = \widehat{\phi} \, R \sin\theta \, d\phi \therefore \mathbf{A} \cdot d\mathbf{l} = R \sin^2\theta \, d\phi = d\phi, (R = 1 \,\&\, \theta = \pi/2).$

$$\oint_c \mathbf{A} \cdot d\mathbf{l} = \int_0^{2\pi} d\phi = 2\pi.$$

$\therefore$ Stokes's theorem is verified.

Sample Question

Assume that a vector field, $\mathbf{A} = \hat{\mathrm{r}}\, r \sin\phi + \hat{\phi} \cos\phi$, (a) find $\oint_c \mathbf{A} \cdot d\mathbf{l}$ over the semicircular contour shown below, and (b) find $\int_s (\nabla \times \mathbf{A}) \cdot d\mathbf{s}$ over the surface of the semicircles.



Sample MCQ

- Which of the following is the mathematical definition for Stokes theorem

a) $\int_s (\nabla \times A) \cdot ds = \oint_c A \cdot dl$.

b) $\int_s (\nabla . A) \cdot ds = \oint_c A \cdot dl$.

c) $\int_s (\nabla \times A) \cdot ds = \oint_c A \times dl$.

d) *none*

Outcome

After this lecture students

- will be able to verify Stokes theorem