An enumerator E starts with a blank input on its work tape. If the enumerator doesn't halt, it may print an infinite list of strings. The language enumerated by E is the collection of all the strings that it eventually prints out. Moreover, E may generate the strings of the language in any order, possibly with repetitions. Now we are ready to develop the connection between enumerators and Turing-recognizable languages.

THEOREM 3.21

A language is Turing-recognizable if and only if some enumerator enumerates it.

PROOF First we show that if we have an enumerator E that enumerates a language A, a TM M recognizes A. The TM M works in the following way.

```
M = "On input $w$:
    1. Run $E$. Every time that $E$ outputs a string, compare it with $w$.
    2. If $w$ ever appears in the output of $E$, accept."
```

Clearly, M accepts those strings that appear on E's list.

Now we do the other direction. If TM M recognizes a language A, we can construct the following enumerator E for A. Say that $s_1, s_2, s_3, \ldots$ is a list of all possible strings in Σ^*.

```
E = "Ignore the input.
    1. Repeat the following for $i = 1, 2, 3, \ldots$.
    2.     Run $M$ for $i$ steps on each input, $s_1, s_2, \ldots, s_i$.
    3.     If any computations accept, print out the corresponding $s_j$."
```

If M accepts a particular string s, eventually it will appear on the list generated by E. In fact, it will appear on the list infinitely many times because M runs from the beginning on each string for each repetition of step 1. This procedure gives the effect of running M in parallel on all possible input strings.

EQUIVALENCE WITH OTHER MODELS

So far we have presented several variants of the Turing machine model and have shown them to be equivalent in power. Many other models of general purpose computation have been proposed. Some of these models are very much like Turing machines, but others are quite different. All share the essential feature of Turing machines—namely, unrestricted access to unlimited memory—distinguishing them from weaker models such as finite automata and pushdown automata. Remarkably, *all* models with that feature turn out to be equivalent in power, so long as they satisfy reasonable requirements.[3]

[3] For example, one requirement is the ability to perform only a finite amount of work in a single step.

To understand this phenomenon, consider the analogous situation for programming languages. Many, such as Pascal and LISP, look quite different from one another in style and structure. Can some algorithm be programmed in one of them and not the others? Of course not—we can compile LISP into Pascal and Pascal into LISP, which means that the two languages describe *exactly* the same class of algorithms. So do all other reasonable programming languages. The widespread equivalence of computational models holds for precisely the same reason. Any two computational models that satisfy certain reasonable requirements can simulate one another and hence are equivalent in power.

This equivalence phenomenon has an important philosophical corollary. Even though we can imagine many different computational models, the class of algorithms that they describe remains the same. Whereas each individual computational model has a certain arbitrariness to its definition, the underlying class of algorithms that it describes is natural because the other models arrive at the same, unique class. This phenomenon has had profound implications for mathematics, as we show in the next section.

3.3

THE DEFINITION OF ALGORITHM

Informally speaking, an *algorithm* is a collection of simple instructions for carrying out some task. Commonplace in everyday life, algorithms sometimes are called *procedures* or *recipes*. Algorithms also play an important role in mathematics. Ancient mathematical literature contains descriptions of algorithms for a variety of tasks, such as finding prime numbers and greatest common divisors. In contemporary mathematics, algorithms abound.

Even though algorithms have had a long history in mathematics, the notion of algorithm itself was not defined precisely until the twentieth century. Before that, mathematicians had an intuitive notion of what algorithms were, and relied upon that notion when using and describing them. But that intuitive notion was insufficient for gaining a deeper understanding of algorithms. The following story relates how the precise definition of algorithm was crucial to one important mathematical problem.

HILBERT'S PROBLEMS

In 1900, mathematician David Hilbert delivered a now-famous address at the International Congress of Mathematicians in Paris. In his lecture, he identified 23 mathematical problems and posed them as a challenge for the coming century. The tenth problem on his list concerned algorithms.

Before describing that problem, let's briefly discuss polynomials. A ***polynomial*** is a sum of terms, where each ***term*** is a product of certain variables and a constant, called a ***coefficient***. For example,

$$6 \cdot x \cdot x \cdot x \cdot y \cdot z \cdot z = 6x^3yz^2$$

is a term with coefficient 6, and

$$6x^3yz^2 + 3xy^2 - x^3 - 10$$

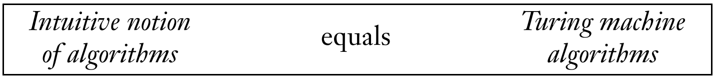
is a polynomial with four terms, over the variables x, y, and z. For this discussion, we consider only coefficients that are integers. A ***root*** of a polynomial is an assignment of values to its variables so that the value of the polynomial is 0. This polynomial has a root at $x = 5$, $y = 3$, and $z = 0$. This root is an ***integral root*** because all the variables are assigned integer values. Some polynomials have an integral root and some do not.

Hilbert's tenth problem was to devise an algorithm that tests whether a polynomial has an integral root. He did not use the term *algorithm* but rather "a process according to which it can be determined by a finite number of operations."[4] Interestingly, in the way he phrased this problem, Hilbert explicitly asked that an algorithm be "devised." Thus he apparently assumed that such an algorithm must exist—someone need only find it.

As we now know, no algorithm exists for this task; it is algorithmically unsolvable. For mathematicians of that period to come to this conclusion with their intuitive concept of algorithm would have been virtually impossible. The intuitive concept may have been adequate for giving algorithms for certain tasks, but it was useless for showing that no algorithm exists for a particular task. Proving that an algorithm does not exist requires having a clear definition of algorithm. Progress on the tenth problem had to wait for that definition.

The definition came in the 1936 papers of Alonzo Church and Alan Turing. Church used a notational system called the λ-calculus to define algorithms. Turing did it with his "machines." These two definitions were shown to be equivalent. This connection between the informal notion of algorithm and the precise definition has come to be called the ***Church–Turing thesis***.

The Church–Turing thesis provides the definition of algorithm necessary to resolve Hilbert's tenth problem. In 1970, Yuri Matijasevič, building on the work of Martin Davis, Hilary Putnam, and Julia Robinson, showed that no algorithm exists for testing whether a polynomial has integral roots. In Chapter 4 we develop the techniques that form the basis for proving that this and other problems are algorithmically unsolvable.



Intuitive notion of algorithms equals *Turing machine algorithms*

FIGURE 3.22
The Church–Turing thesis

[4] Translated from the original German.

Let's phrase Hilbert's tenth problem in our terminology. Doing so helps to introduce some themes that we explore in Chapters 4 and 5. Let

$$D = \{p|\, p \text{ is a polynomial with an integral root}\}.$$

Hilbert's tenth problem asks in essence whether the set D is decidable. The answer is negative. In contrast, we can show that D is Turing-recognizable. Before doing so, let's consider a simpler problem. It is an analog of Hilbert's tenth problem for polynomials that have only a single variable, such as $4x^3 - 2x^2 + x - 7$. Let

$$D_1 = \{p|\, p \text{ is a polynomial over } x \text{ with an integral root}\}.$$

Here is a TM M_1 that recognizes D_1:

```
M_1 = "On input ⟨p⟩: where p is a polynomial over the variable x.
    1. Evaluate p with x set successively to the values 0, 1, −1, 2, −2, 3,
       −3, . . . . If at any point the polynomial evaluates to 0, accept."
```

If p has an integral root, M_1 eventually will find it and accept. If p does not have an integral root, M_1 will run forever. For the multivariable case, we can present a similar TM M that recognizes D. Here, M goes through all possible settings of its variables to integral values.

Both M_1 and M are recognizers but not deciders. We can convert M_1 to be a decider for D_1 because we can calculate bounds within which the roots of a single variable polynomial must lie and restrict the search to these bounds. In Problem 3.21 you are asked to show that the roots of such a polynomial must lie between the values

$$\pm\, k\, \frac{c_{\max}}{c_1},$$

where k is the number of terms in the polynomial, $c_{\max}$ is the coefficient with the largest absolute value, and c_1 is the coefficient of the highest order term. If a root is not found within these bounds, the machine *rejects*. Matijasevič's theorem shows that calculating such bounds for multivariable polynomials is impossible.

TERMINOLOGY FOR DESCRIBING TURING MACHINES

We have come to a turning point in the study of the theory of computation. We continue to speak of Turing machines, but our real focus from now on is on algorithms. That is, the Turing machine merely serves as a precise model for the definition of algorithm. We skip over the extensive theory of Turing machines themselves and do not spend much time on the low-level programming of Turing machines. We need only to be comfortable enough with Turing machines to believe that they capture all algorithms.

With that in mind, let's standardize the way we describe Turing machine algorithms. Initially, we ask: What is the right level of detail to give when describing such algorithms? Students commonly ask this question, especially when preparing solutions to exercises and problems. Let's entertain three possibilities. The first is the *formal description* that spells out in full the Turing machine's states, transition function, and so on. It is the lowest, most detailed level of description. The second is a higher level of description, called the *implementation description*, in which we use English prose to describe the way that the Turing machine moves its head and the way that it stores data on its tape. At this level we do not give details of states or transition function. The third is the *high-level description*, wherein we use English prose to describe an algorithm, ignoring the implementation details. At this level we do not need to mention how the machine manages its tape or head.

In this chapter, we have given formal and implementation-level descriptions of various examples of Turing machines. Practicing with lower level Turing machine descriptions helps you understand Turing machines and gain confidence in using them. Once you feel confident, high-level descriptions are sufficient.

We now set up a format and notation for describing Turing machines. The input to a Turing machine is always a string. If we want to provide an object other than a string as input, we must first represent that object as a string. Strings can easily represent polynomials, graphs, grammars, automata, and any combination of those objects. A Turing machine may be programmed to decode the representation so that it can be interpreted in the way we intend. Our notation for the encoding of an object O into its representation as a string is $\langle O \rangle$. If we have several objects $O_1, O_2, \ldots, O_k$, we denote their encoding into a single string $\langle O_1, O_2, \ldots, O_k \rangle$. The encoding itself can be done in many reasonable ways. It doesn't matter which one we pick because a Turing machine can always translate one such encoding into another.

In our format, we describe Turing machine algorithms with an indented segment of text within quotes. We break the algorithm into stages, each usually involving many individual steps of the Turing machine's computation. We indicate the block structure of the algorithm with further indentation. The first line of the algorithm describes the input to the machine. If the input description is simply w, the input is taken to be a string. If the input description is the encoding of an object as in $\langle A \rangle$, the Turing machine first implicitly tests whether the input properly encodes an object of the desired form and rejects it if it doesn't.

EXAMPLE 3.23

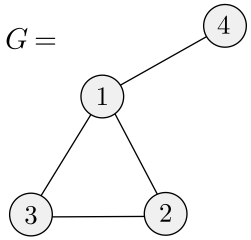
Let A be the language consisting of all strings representing undirected graphs that are connected. Recall that a graph is ***connected*** if every node can be reached from every other node by traveling along the edges of the graph. We write

$$A = \{\langle G \rangle |\ G \text{ is a connected undirected graph}\}.$$

The following is a high-level description of a TM M that decides A.

```
M = "On input ⟨G⟩, the encoding of a graph G:
    1. Select the first node of G and mark it.
    2. Repeat the following stage until no new nodes are marked:
    3.     For each node in G, mark it if it is attached by an edge to a
           node that is already marked.
    4. Scan all the nodes of G to determine whether they all are
       marked. If they are, accept; otherwise, reject."
```

For additional practice, let's examine some implementation-level details of Turing machine M. Usually we won't give this level of detail in the future and you won't need to either, unless specifically requested to do so in an exercise. First, we must understand how $\langle G\rangle$ encodes the graph G as a string. Consider an encoding that is a list of the nodes of G followed by a list of the edges of G. Each node is a decimal number, and each edge is the pair of decimal numbers that represent the nodes at the two endpoints of the edge. The following figure depicts such a graph and its encoding.



$\langle G\rangle =$

`(1,2,3,4)((1,2),(2,3),(3,1),(1,4))`

FIGURE 3.24
A graph G and its encoding $\langle G\rangle$

When M receives the input $\langle G\rangle$, it first checks to determine whether the input is the proper encoding of some graph. To do so, M scans the tape to be sure that there are two lists and that they are in the proper form. The first list should be a list of distinct decimal numbers, and the second should be a list of pairs of decimal numbers. Then M checks several things. First, the node list should contain no repetitions; and second, every node appearing on the edge list should also appear on the node list. For the first, we can use the procedure given in Example 3.12 for TM M_4 that checks element distinctness. A similar method works for the second check. If the input passes these checks, it is the encoding of some graph G. This verification completes the input check, and M goes on to stage 1.

For stage 1, M marks the first node with a dot on the leftmost digit.

For stage 2, M scans the list of nodes to find an undotted node n_1 and flags it by marking it differently—say, by underlining the first symbol. Then M scans the list again to find a dotted node n_2 and underlines it, too.

Now M scans the list of edges. For each edge, M tests whether the two underlined nodes n_1 and n_2 are the ones appearing in that edge. If they are, M dots n_1, removes the underlines, and goes on from the beginning of stage 2. If they aren't, M checks the next edge on the list. If there are no more edges, $\{n_1, n_2\}$ is not an edge of G. Then M moves the underline on n_2 to the next dotted node and now calls this node n_2. It repeats the steps in this paragraph to check, as before, whether the new pair $\{n_1, n_2\}$ is an edge. If there are no more dotted nodes, n_1 is not attached to any dotted nodes. Then M sets the underlines so that n_1 is the next undotted node and n_2 is the first dotted node and repeats the steps in this paragraph. If there are no more undotted nodes, M has not been able to find any new nodes to dot, so it moves on to stage 4.

For stage 4, M scans the list of nodes to determine whether all are dotted. If they are, it enters the accept state; otherwise, it enters the reject state. This completes the description of TM M. ■

EXERCISES

3.1 This exercise concerns TM M_2, whose description and state diagram appear in Example 3.7. In each of the parts, give the sequence of configurations that M_2 enters when started on the indicated input string.

a. 0.

^{A}**b.** 00.

c. 000.

d. 000000.

3.2 This exercise concerns TM M_1, whose description and state diagram appear in Example 3.9. In each of the parts, give the sequence of configurations that M_1 enters when started on the indicated input string.

^{A}**a.** 11.

b. 1#1.

c. 1##1.

d. 10#11.

e. 10#10.

^{A}**3.3** Modify the proof of Theorem 3.16 to obtain Corollary 3.19, showing that a language is decidable iff some nondeterministic Turing machine decides it. (You may assume the following theorem about trees. If every node in a tree has finitely many children and every branch of the tree has finitely many nodes, the tree itself has finitely many nodes.)

3.4 Give a formal definition of an enumerator. Consider it to be a type of two-tape Turing machine that uses its second tape as the printer. Include a definition of the enumerated language.