PROBLEM 01:

a/ Total possible outcome of three dices $= 6^3 = 216$

Combinations for the sum of three dices is 09
$= (6,1,2), (6,2,1), (5,1,3), (5,2,2), (5,3,1), (4,1,4), (4,2,3), (4,3,2),$
$(4,4,1), (3,1,5), (3,2,4), (3,3,3), (3,4,2), (3,5,1), (2,1,6), (2,2,5), (2,3,4),$
$(2,4,3), (2,5,2), (2,6,1), (1,2,6), (1,3,5), (1,4,4), (1,5,3), (1,6,2)$

Total 25 combination. So, $P(Sum=9) = 25/216$

Combinations for the sum of three dices is 10
$= (6,1,3), (6,2,2), (6,3,1), (5,1,4), (5,2,3), (5,3,2), (5,4,1), (4,1,5), (4,2,4),$
$(4,3,3), (4,4,2), (4,5,1), (3,1,6), (3,2,5), (3,3,4), (3,4,3), (3,5,2), (3,6,1),$
$(2,2,6), (2,3,5), (2,4,4), (2,5,3), (2,6,2), (1,3,6), (1,4,5), (1,5,4), (1,6,3)$

Total 27 Combinations. So, $P(Sum=10) = 27/216$

b/

Probability of at least one six out of 4 dice throws —

Probability of not getting a six in a single dice throw $= 5/6$
Probability of not getting a six on any four dice throws $= (5/6)^4$

So, $P(\text{at least one six}) = 1 - (5/6)^4 = 0.5177$

Probability of at least one time two sixes when throwing two dices 24 times —

Probability of getting one six in a single dice = 1/6
Probability of getting two six on a roll of two dice $= \frac{1}{6} \times \frac{1}{6} = 1/36$

We can use binomial distribution. The random variable X represents the number of sixes in 24 dice rolls. We have to find $P(X \geq 2)$, which is the probability of getting at least two sixes

Here, Total number of trials, $n = 24$

Probability of getting two six on a roll of two dice, $p = 1/36$

$$X \sim \text{Binomial}(24, 1/36)$$

$$f(x) = \binom{n}{x} p^x (1-p)^{n-x}, \quad x = 0, 1, \ldots n, \quad \Bigg\| \quad \binom{n}{x} = \frac{n!}{x!\,(n-x)!}$$

Now, $P(X \geq 2) = 1 - \left(P(X=0) + P(X=1)\right)$

Probability of getting zero six in two dice in 24 rolls —

$$P(X=0) = \binom{24}{0} \left(\frac{1}{36}\right)^0 \left(\frac{35}{36}\right)^{24} = 1 \times 1 \times \left(\frac{35}{36}\right)^{24} = 0.5085$$

Probability of getting one six in two dice in 24 rolls —

$$P(X=1) = \binom{24}{1} \left(\frac{1}{36}\right)^1 \left(\frac{35}{36}\right)^{23} = \frac{24!}{1! \cdot 23!} \times \frac{1}{36} \times \left(\frac{35}{36}\right)^{23}$$

$$= \frac{24}{36} \times \left(\frac{35}{36}\right)^{23} = 0.3488$$

So, $P(X \geq 2) = 1 - \left(P(X=0) + P(X=1)\right) = 1 - 0.5085 - 0.3488 = 0.1427$

Thus, it is more likely to get at least one six out of 4 dice throws c

At least one 6 in 6 dice throws, $X \sim Binomial(6, 1/6)$

$$P(X \geq 1) = 1 - P(X=0)$$

$$P(X=0) = \binom{6}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^6 = \left(\frac{5}{6}\right)^6$$

$$\text{So, } P(X \geq 1) = 1 - \left(\frac{5}{6}\right)^6 = 0.6651$$

At least two 6 in 12 dice throws, $X \sim Binomial(12, 1/6)$

$$P(X \geq 2) = 1 - \left(P(X=0) + P(X=1)\right)$$

$$P(X=0) = \binom{12}{0} \cdot \left(\frac{1}{6}\right)^0 \cdot \left(\frac{5}{6}\right)^{12} = \left(\frac{5}{6}\right)^{12}$$

$$P(X=1) = \binom{12}{1} \cdot \left(\frac{1}{6}\right)^1 \cdot \left(\frac{5}{6}\right)^{11} = 2 \times \left(\frac{5}{6}\right)^{11}$$

$$\text{So, } P(X \geq 2) = 1 - \left(\frac{5}{6}\right)^{12} - \left(2 \times \left(\frac{5}{6}\right)^{11}\right) = 0.6187$$

At least three 6 in 18 dice throws, $X \sim Binomial(18, 1/6)$

$$P(X \geq 3) = 1 - \left(P(X=0) + P(X=1) + P(X=2)\right)$$

$$P(X=0) = \binom{18}{0}\left(\frac{1}{6}\right)^0\left(\frac{5}{6}\right)^{18} = \left(\frac{5}{6}\right)^{18}$$

$$P(X=1) = \binom{18}{1}\left(\frac{1}{6}\right)^1\left(\frac{5}{6}\right)^{17} = 3 \cdot \left(\frac{5}{6}\right)^{17}$$

$$P(X=2) = \binom{18}{2}\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^{16} = \frac{18 \times 17}{2 \times 36} \times \left(\frac{5}{6}\right)^{16}$$

$$\text{So, } P(X \geq 3) = 1 - \left(\frac{5}{6}\right)^{18} - \left(3 \times \left(\frac{5}{6}\right)^{17}\right) - \left(\frac{17}{4} \times \left(\frac{5}{6}\right)^{16}\right)$$

$$= 0.5973$$

d. Here, we use negative binomial distribution.

$$f(x) = \binom{x-1}{k-1} p^k (1-p)^{x-k}, \quad x = k, k+1, \ldots$$

Number of trials, $x = 8$

Number of success (three sixes) $= 3$

Probability of success (rolling a six) $= \frac{1}{6}$

So, $P(X=8) = \binom{7}{2} \left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right)^5 = \frac{7!}{2!\,5!} \cdot \left(\frac{1}{6}\right)^3 \cdot \left(\frac{5}{6}\right)^5 = 0.039$

PROBLEM 02:

Given CDF, $F_a(t) = e^{-e^{-(t-a)}}$, $t \in \mathbb{R}$

a/

To verify that a given function is a proper CDF, it must satisfy following conditions —

01. As $t \to -\infty$, $F_a(t) \to 0$ & as $t \to +\infty$, $F_a(t) \to 1$
02. A CDF must be a non-decreasing function, meaning it never decreases as t increases
03. The CDF must satisfy $0 \le F_a(t) \le 1$ for all t.

Verify Condition 01 —

$$F_a(t) = e^{-e^{-(t-a)}} = e^{-e^{a-t}}$$

As $t \to -\infty$, $e^{a-t} \to +\infty$ [As e^{a-t} becomes $e^{+\infty}$]

So, $e^{-e^{a-t}} \to e^{-\infty}$ which is 0.

Thus, as $t \to -\infty$, $F_a(t) \to 0$

Again, As $t \to +\infty$, $e^{a-t} \to 0$ [As e^{a-t} becomes $e^{-\infty}$]

So, $e^{-e^{a-t}} \to e^0$ which is 1.

Thus, as $t \to +\infty$, $F_a(t) \to 1$

Verify Condition 02 —

To check if the function is non-decreasing, we check if the derivative of $F_a(t)$ is non-negative.

$$\frac{d}{dt} F_a(t) = \frac{d}{dt} e^{-e^{-(t-a)}}$$

$$= e^{-e^{-(t-a)}} \cdot \frac{d}{dt}\left(-e^{-(t-a)}\right)$$

$$= e^{-e^{-(t-a)}} \cdot (-e^{-(t-a)}) \cdot \frac{d}{dt}(-(t-a))$$

$$= e^{-e^{-(t-a)}} \cdot (-e^{-(t-a)}) \cdot (-1)$$

$$= e^{-e^{-(t-a)}} \cdot e^{-(t-a)}$$

Since, both $e^{-e^{-(t-a)}}$ and $e^{-(t-a)}$ are positive for all t, the derivative is positive.

Thus, $F_a(t)$ is a non decreasing function.

Verify Condition 03 —

Since, $F_a(t)$ is non-decreasing and approaches 0 and 1 at the extremes, it must lie between 0 and 1 for all t.

As all the condition verifies, $F_a(t)$ is a PROPER CDF.

b/

To calculate the mean $E(T)$, we need to use the probability density function (PDF) $f_a(t)$, which is the derivative of the CDF $F_a(t)$ with respect to t.

So, $f_a(t) = \frac{d}{dt} F_a(t) = e^{-e^{-(t-a)}} \cdot e^{-(t-a)}$

Now, the mean $E(T)$ of a continuous random variable is given by

$$E(T) = \int_{-\infty}^{\infty} t \cdot f_a(t) \cdot dt$$

$$= \int_{-\infty}^{\infty} t \cdot e^{-(t-a)} \cdot e^{-e^{-(t-a)}} \cdot dt$$

Let, $s = e^{-(t-a)}$

$ds = -e^{-(t-a)} dt$

$dt = \frac{-ds}{s}$

Now, As $t \to -\infty$, $s \to \infty$

As $t \to \infty$, $s \to 0$

Again, $s = e^{-(t-a)}$

$\log s = \log(e^{-(t-a)})$

$\Rightarrow \log s = -(t-a)$

$\Rightarrow t = a - \log s$

So, $E(T) = \int_{\infty}^{0} (a - \log s) \cdot s \cdot e^{-s} \cdot \frac{-ds}{s}$

$= \int_{\infty}^{0} -\left((a-\log s) \cdot e^{-s} \cdot ds\right)$

$= \int_{0}^{\infty} (a - \log s) e^{-s} ds = \int_{0}^{\infty} (ae^{-s} - \log s \cdot e^{-s}) ds$

Breaking the integral into two parts

$E(T) = a \int_{0}^{\infty} e^{-s} ds - \int_{0}^{\infty} \log s \cdot e^{-s} ds$

$= a\left[-e^{-s}\right]_{0}^{\infty} + \gamma \quad \left[\text{As } \gamma = -\int_{0}^{\infty} \log(x) e^{-x} dx\right]$

$= a(-e^{-\infty} + e^{-0}) + \gamma$

$= \underline{\underline{a + \gamma}}$

c

$$f_a(t) = e^{-e^{-(t-a)}}, \quad t \in \mathbb{R}$$

Given, $a = 5$

$$P(T \le 4) = f_a(4) = e^{-e^{-(4-5)}} = e^{-e} = 0.0659$$

$$P(T > 9) = 1 - P(T \le 9) = 1 - f_a(9) = 1 - e^{-e^{-(9-5)}} = 1 - e^{-e^{-4}} = 1 - 0.9818 = 0.0182$$

$$P(5 < T \le 6) = P(T \le 6) - P(T \le 5) = f_a(6) - f_a(5) = e^{-e^{-(6-5)}} - e^{-e^{-(5-5)}} = e^{-e^{-1}} - e^{-e^{-0}} = 0.3243$$

d

Let, $f_a(t) = U$, where $U \in (0,1)$ is the uniform random variable.

$$U = e^{-e^{-(t-a)}}$$

$$\Rightarrow \log U = \log\left(e^{-e^{-(t-a)}}\right)$$

$$\Rightarrow \log U = -e^{-(t-a)}$$

$$\Rightarrow -\log U = e^{-(t-a)}$$

$$\Rightarrow \log(-\log U) = \log e^{-(t-a)}$$

$$\Rightarrow \log(-\log U) = -t + a$$

$$\Rightarrow t = a - \log(-\log U)$$

$$\Rightarrow t = a - \log(\log U^{-1}) \quad [\text{As } \log U^{-1} = -\log U]$$

Thus, the inverse CDF is $f_a^{-1}(U) = a - \log(\log U^{-1})$

PROBLEM 03:

Given, $f_\lambda(x) = \frac{1}{2\lambda} \exp\left(-\frac{|x|}{\lambda}\right), x \in \mathbb{R}, \lambda > 0$

where $|x|$ is the absolute value of x with $E(|x|) = \lambda$

a/

The likelihood function $L(\lambda)$ is the product of the individual densities for the observations ——

$$L(\lambda | x_1, x_2, \cdots x_n) = \prod_{i=1}^{n} f_\lambda(x_i)$$

$$= \prod_{i=1}^{n} \frac{1}{2\lambda} \exp\left(-\frac{|x_i|}{\lambda}\right)$$

$$= \left(\frac{1}{2\lambda}\right)^n \prod_{i=1}^{n} \exp\left(-\frac{|x_i|}{\lambda}\right)$$

$$= \left(\frac{1}{2\lambda}\right)^n \exp\left(\frac{-1}{\lambda} \sum_{i=1}^{n} |x_i|\right)$$

$$\log L(\lambda | x_1, x_2, \cdots x_n) = \log\left(\left(\frac{1}{2\lambda}\right)^n \exp\left(\frac{-1}{\lambda} \sum_{i=1}^{n} |x_i|\right)\right)$$

$$= \log\left(\frac{1}{2\lambda}\right)^n + \log\left(\exp\left(\frac{-1}{\lambda} \sum_{i=1}^{n} |x_i|\right)\right) \quad [\log(ab) = \log a + \log b]$$

$$= n \log \frac{1}{2\lambda} + \left(\frac{-1}{\lambda} \sum_{i=1}^{n} |x_i|\right) \quad [\log a^n = n \log a ; \log(\exp(a)) = a]$$

$$= -n \log 2\lambda - \frac{1}{\lambda} \sum_{i=1}^{n} |x_i| \quad [\log \frac{a}{b} = -\log \frac{b}{a}]$$

To find the MLE of λ, we take the derivative of the log likelihood function with respect to λ and set it equal to zero.

$$\frac{d}{d\lambda} \log L(\lambda | x_1, x_2, \cdots, x_n) = \frac{d}{d\lambda}\left(-n \log 2\lambda - \frac{1}{\lambda} \sum_{i=1}^{n} |x_i|\right)$$

$$= \frac{d}{d\lambda}(-n\log 2\lambda) - \frac{d}{d\lambda}\left(\frac{1}{\lambda}\sum_{i=1}^{n}|x_i|\right)$$

$$= \frac{-n}{2\lambda}\frac{d}{d\lambda}(2\lambda) - \left(-\frac{1}{\lambda^2}\sum_{i=1}^{n}|x_i|\right) \quad \left[\frac{d}{dx}\log x = \frac{1}{x}\cdot\frac{d}{dx}x\right]$$

$$= \frac{-n}{\lambda} + \frac{1}{\lambda^2}\sum_{i=1}^{n}|x_i|$$

Set the derivative equal to zero,

$$\frac{-n}{\lambda} + \frac{1}{\lambda^2}\sum_{i=1}^{n}|x_i| = 0$$

$$\Rightarrow -n\lambda + \sum_{i=1}^{n}|x_i| = 0 \quad [\text{multiply by } \lambda^2 \text{ on both side}]$$

$$\Rightarrow \lambda = \frac{1}{n}\sum_{i=1}^{n}|x_i|$$

So, the maximum likelihood estimator (MLE) of λ is

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n}|x_i|$$

Now, an estimator $\hat{\lambda}$ is unbiased if $E(\hat{\lambda}) = \lambda$

Given, $E(|X|) = \lambda$

$$\text{So, } E(\hat{\lambda}) = E\left(\frac{1}{n}\sum_{i=1}^{n}|x_i|\right)$$

$$= \frac{1}{n}\sum_{i=1}^{n}E(|x_i|)$$

$$= \frac{1}{n}\sum_{i=1}^{n}\lambda = \frac{1}{n}\cdot n\lambda = \lambda$$

Thus, the estimator $\hat{\lambda}$ is an unbiased estimator of λ