Lesson 8
Contour integration

- Last lecture we talked about expansion in Taylor series:

$$f(z) = \sum_{k=0}^{\infty} \hat{f}_k z^k$$

- We also discussed analytic functions
- This lecture we will discuss contour integration
- This will show that analyticity in the unit circle is equivalent to having a Taylor series
- We will also look into Laurent series

$$f(z) = \sum_{k=-\infty}^{\infty} \hat{f}_k z^k$$

Contour integration

- A *curve* in the complex plane is defined by a map from an interval $I = (a,b)$:

$$\Gamma = M(I) \quad \text{where} \quad M : I \to \mathbb{C}$$

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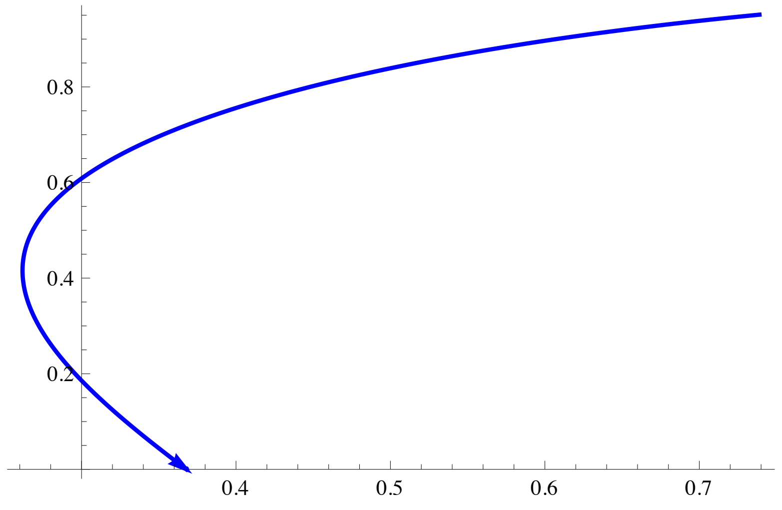

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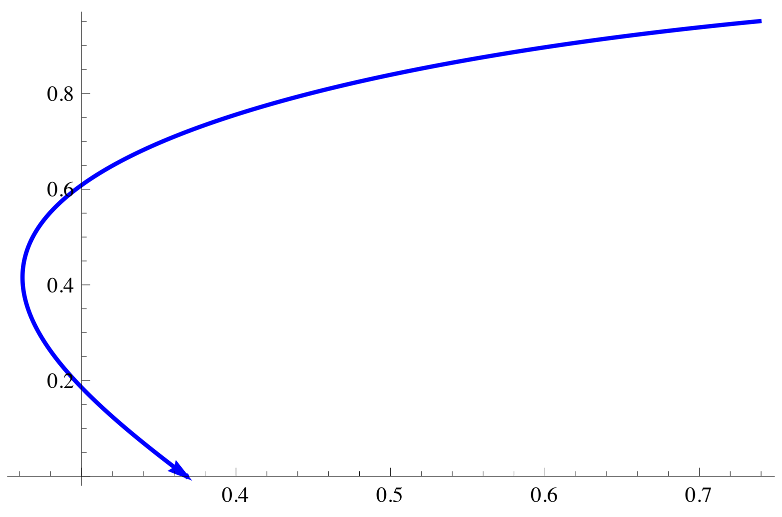
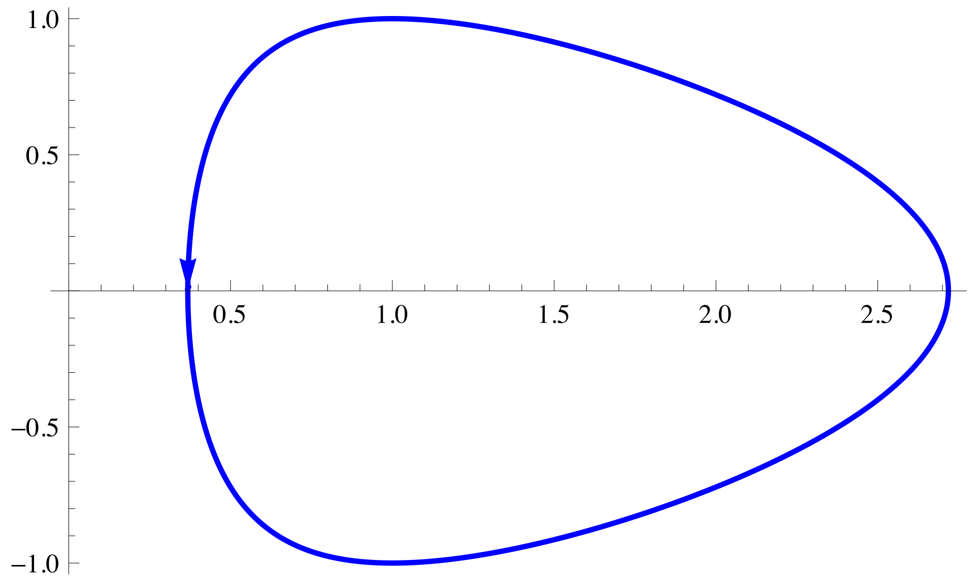
$\Gamma =$ 

$\Gamma =$ 

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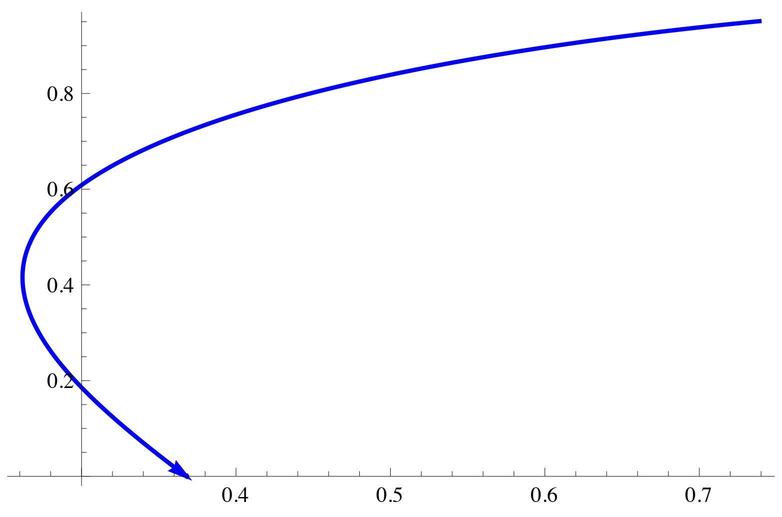
$\Gamma =$ 

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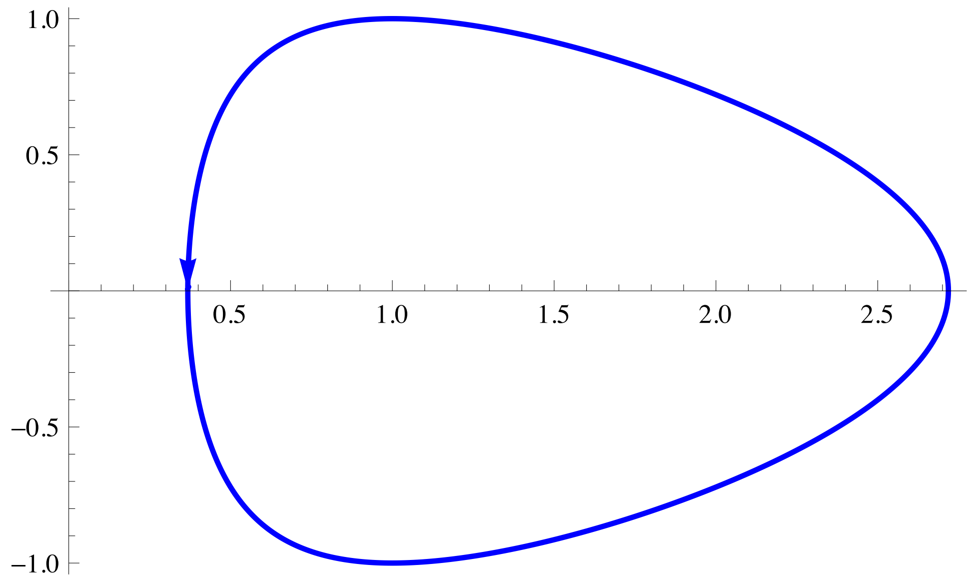
$\Gamma =$ 

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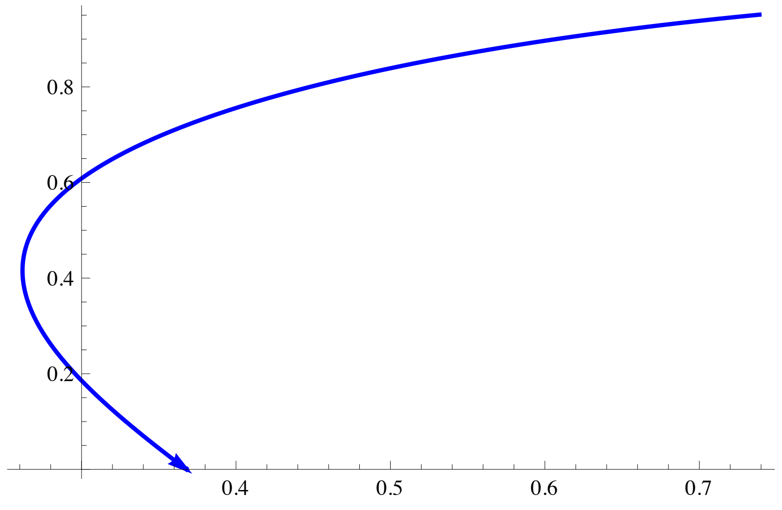
$\Gamma =$ 

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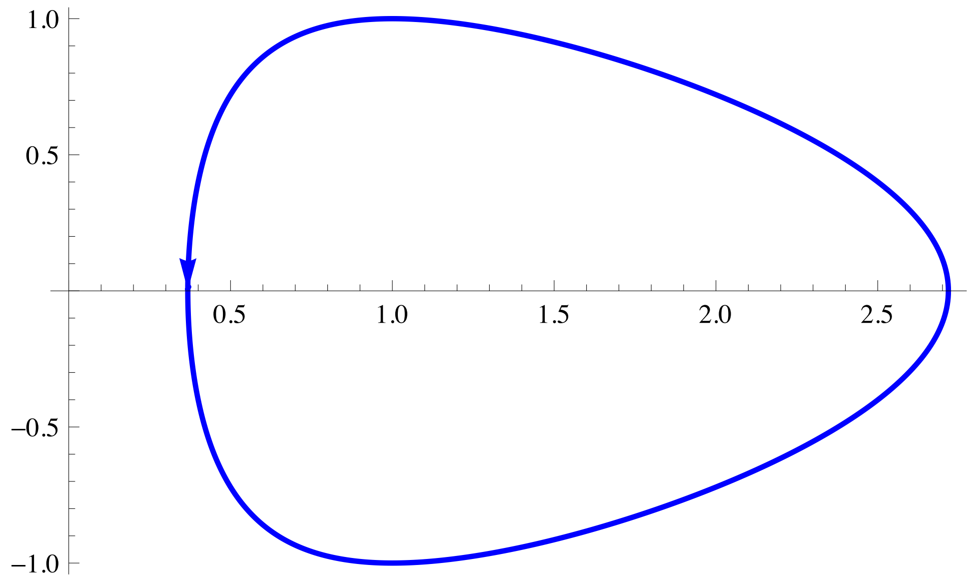
$\Gamma =$ 

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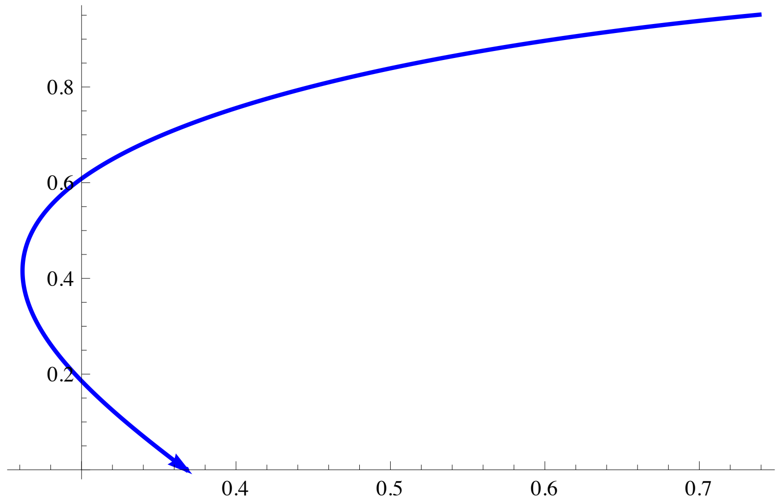
$\Gamma =$ 

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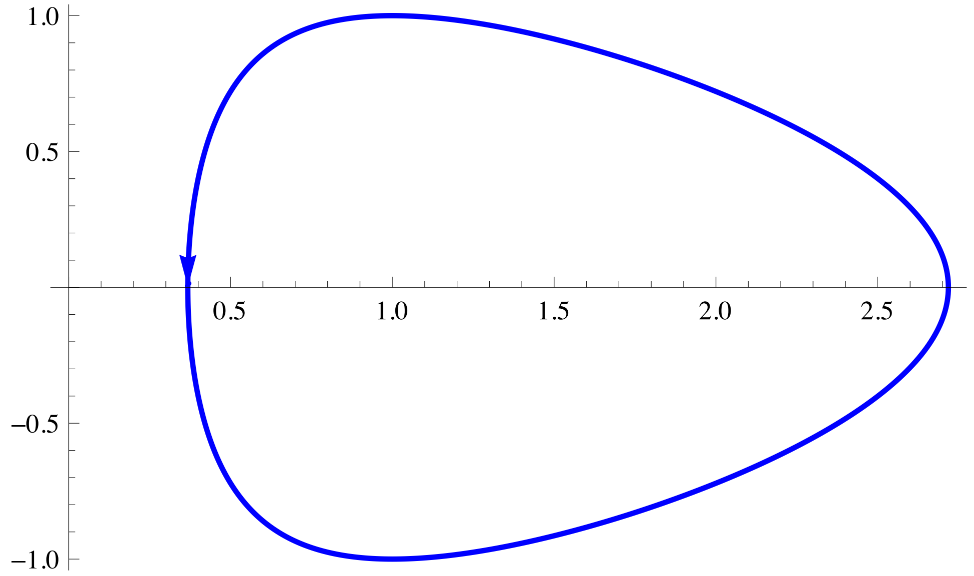
$\Gamma =$ 

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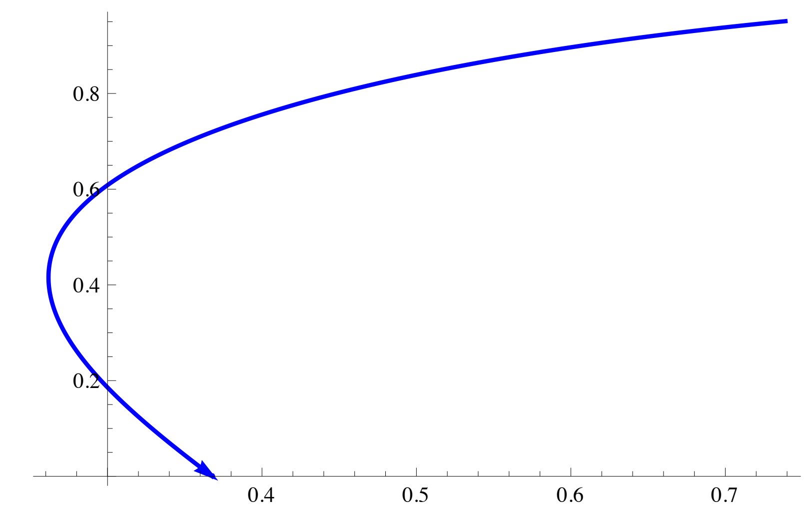
$\Gamma =$ 

- A *Jordon curve* is a curve which is
  - Closed: it forms a loop $M(\mathrm{a}) = M(\mathrm{b})$
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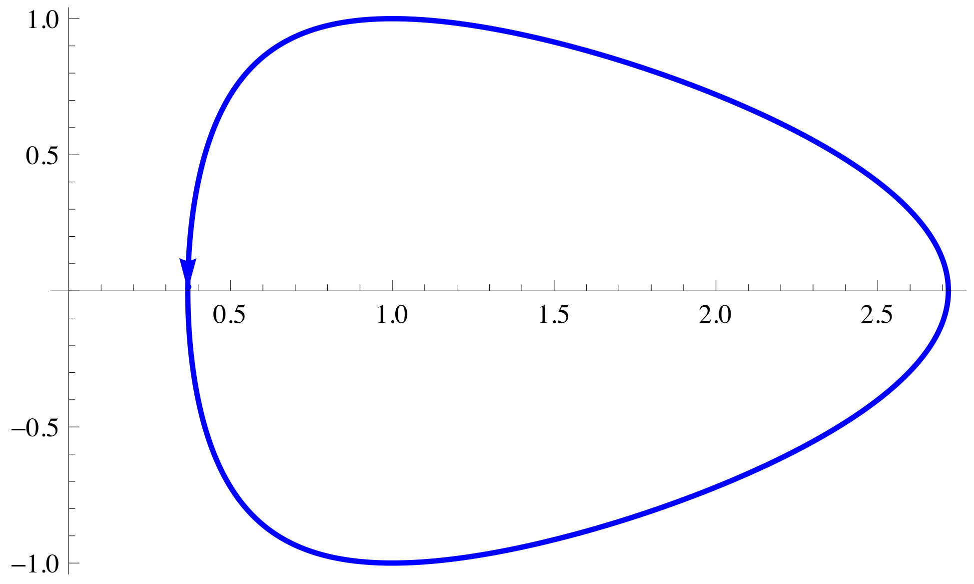
  - Oriented: it has a left and right

$\Gamma =$ 

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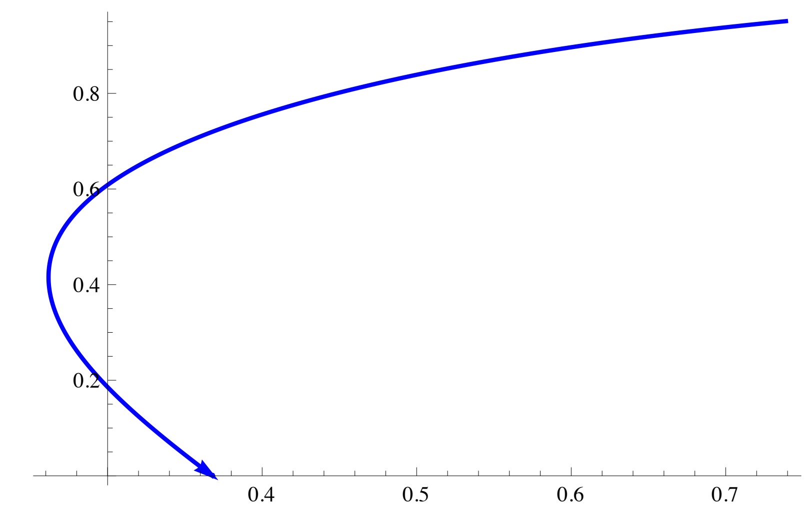

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  - Closed: it forms a loop $M(\mathrm{a}) = M(\mathrm{b})$
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  - Oriented: it has a left and right
  - Sufficiently smooth: M is continuously differentiable

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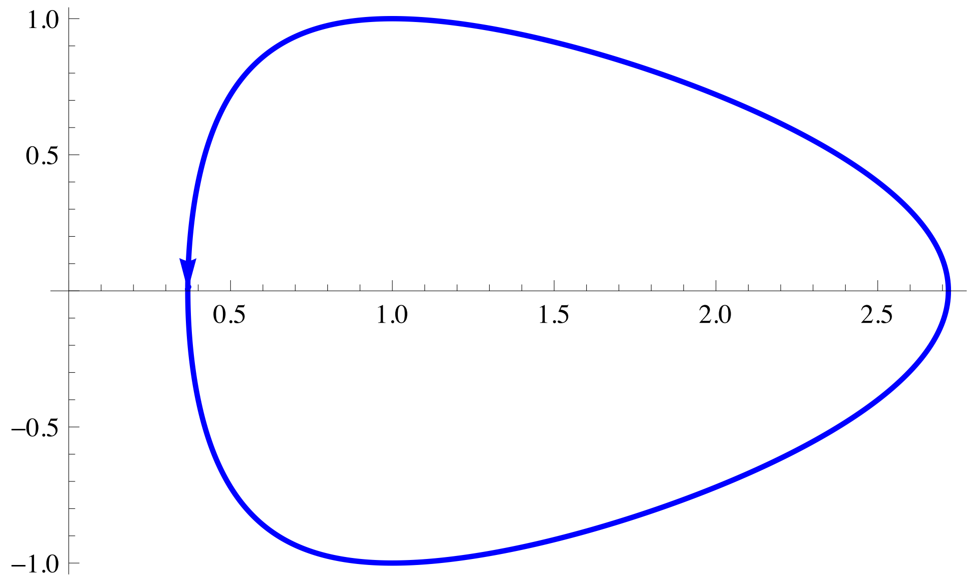
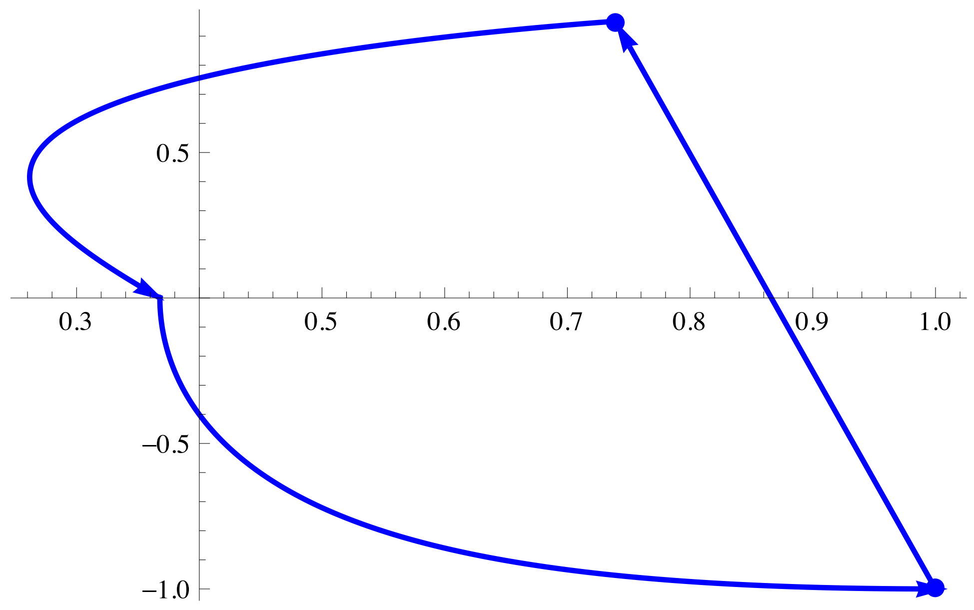


- A *contour* in the complex plane is defined by a finite number of smooth curves
- A *Jordon contour* is a contour which is
  - Closed
  - Simply connected
  - Oriented
  - Sufficiently smooth: the contour is continuous and piecewise differentiable

Example

$\Gamma =$ 

- Let $M \in \mathrm{C}^1[I]$ define a curve $\Gamma = M(I)$
- Contour integration over the curve Γ is defined by

$$\int_\Gamma f(z)\,\mathrm{d}z = \int_a^b f(M(x))M'(x)\,\mathrm{d}x$$

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$$\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_\ell = M_1(I) \cup \cdots \cup M_\ell(I)$$

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- When Γ is a Jordan contour, we denote

$$\oint_\Gamma f(z)\,\mathrm{d}z = \int_\Gamma f(z)\,\mathrm{d}z$$

- Now consider the curve Γ as lying in $\mathbb{R}^2$: let $\Gamma = M(\mathbb{T})$ where

$$M(\theta) = (x(\theta), y(\theta))$$

for $x : \mathbb{T} \to \mathbb{R}$ and $y : \mathbb{T} \to \mathbb{R}$ so that $M :\to \mathbb{R}^2$

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- Recall: a *line integral* in $\mathbb{R}^2$ over the curve Γ is defined by

$$\oint_\Gamma (u(x,y)\ \mathrm{d}x + v(x,y)\,\mathrm{d}y) = \int_{-\pi}^{\pi} (u(M(\theta))x'(\theta) + v(M(\theta))y'(\theta)\,\mathrm{d}\theta$$

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- We can thus recast complex contour integration as line integration for $f(z) = u(z) + \mathrm{i}v(z)$:

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$$\oint_\Gamma f(z)\,\mathrm{d}z = \int_{-\pi}^{\pi} f(M(\theta))M'(\theta)\,\mathrm{d}\theta$$
$$= \int_{-\pi}^{\pi} \left[u(M(\theta))x'(\theta) - v(M(\theta))y'(\theta) + \mathrm{i}\,(u(M(\theta))y'(\theta) + v(M(\theta))x'(\theta))\right]\,\mathrm{d}\theta$$

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$$\begin{aligned}\oint_\Gamma f(z)\, \mathrm{d}z &= \int_{-\pi}^{\pi} f(M(\theta))M'(\theta)\, \mathrm{d}\theta \\ &= \int_{-\pi}^{\pi} [u(M(\theta))x'(\theta) - v(M(\theta))y'(\theta) + \mathrm{i}\,(u(M(\theta))y'(\theta) + v(M(\theta))x'(\theta))]\; \mathrm{d}\theta \\ &= \oint_\Gamma (u\, \mathrm{d}x - v\, \mathrm{d}y) + \mathrm{i} \oint_\Gamma (v\, \mathrm{d}x + u\, \mathrm{d}y)\end{aligned}$$

Theorem: (*Green's theorem*)

Suppose that u, v and their partial derivatives are continuous throughout a simply connected domain D, whose boundary is Γ, oriented in the positive way (the left-hand side of Γ is inside D). Then

$$\oint_\Gamma (u\,\mathrm{d}x + v\,\mathrm{d}y) = \iint_D \left(\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right)\,\mathrm{d}x\,\mathrm{d}y$$

Theorem: if f is analytic in a domain D and $\Gamma \subset D$ is a Jordan contour oriented so that the interior of the contour lies in D, then

$$\oint_\Gamma f(z)\,\mathrm{d}z = 0$$

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- Restrict D to be inside Γ

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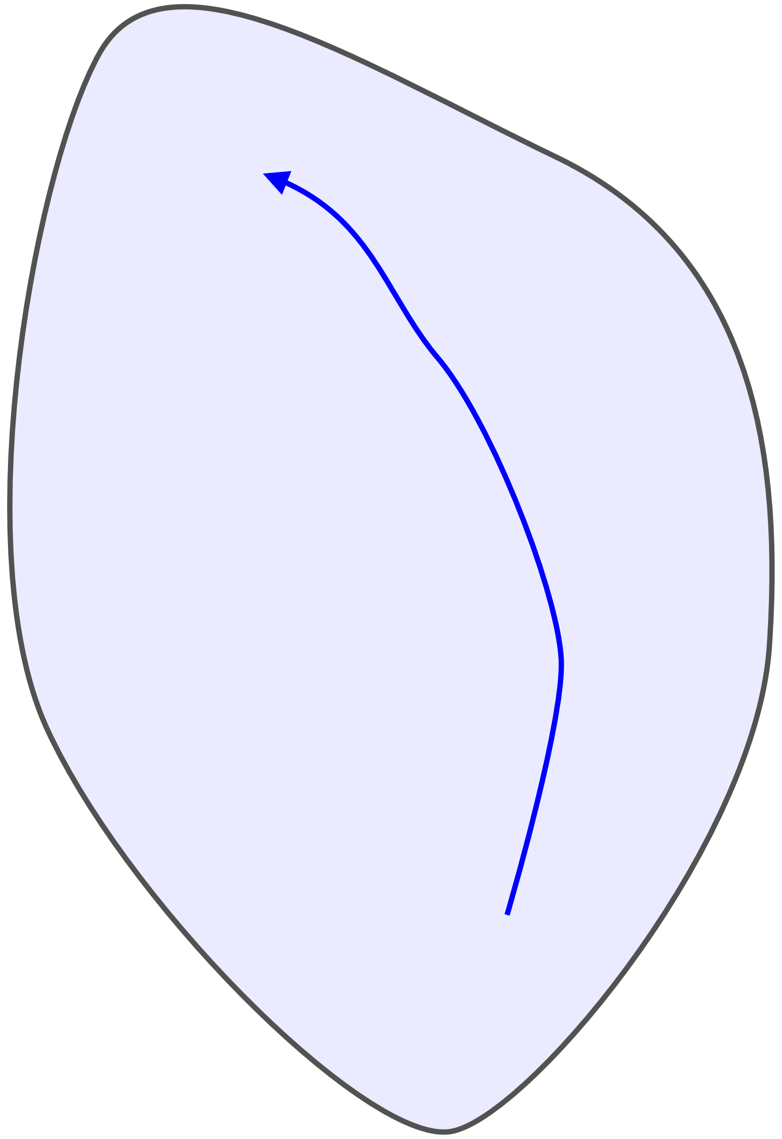
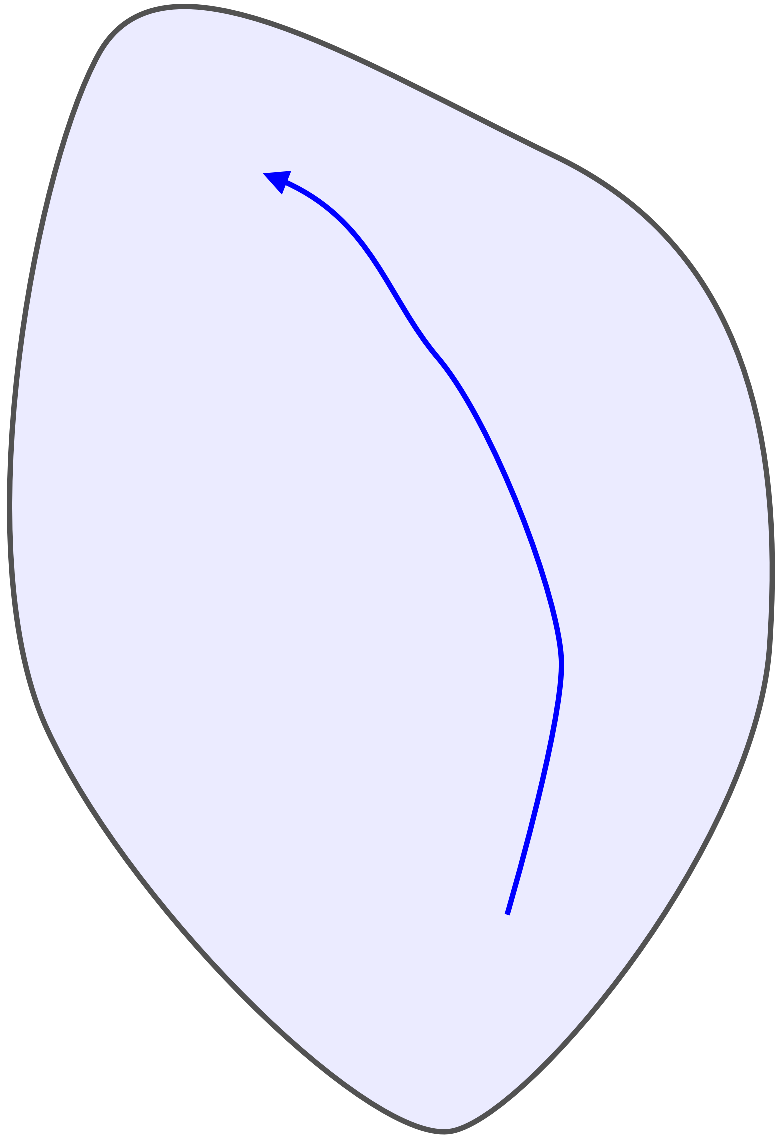
- These integrals are zero because the Cauchy–Riemann conditions are satisfied

Contour deformation

$$\int_\Gamma f \, \mathrm{d}z$$



$$\int_\Gamma f \, \mathrm{d}z \qquad \oint_C f \, \mathrm{d}z$$



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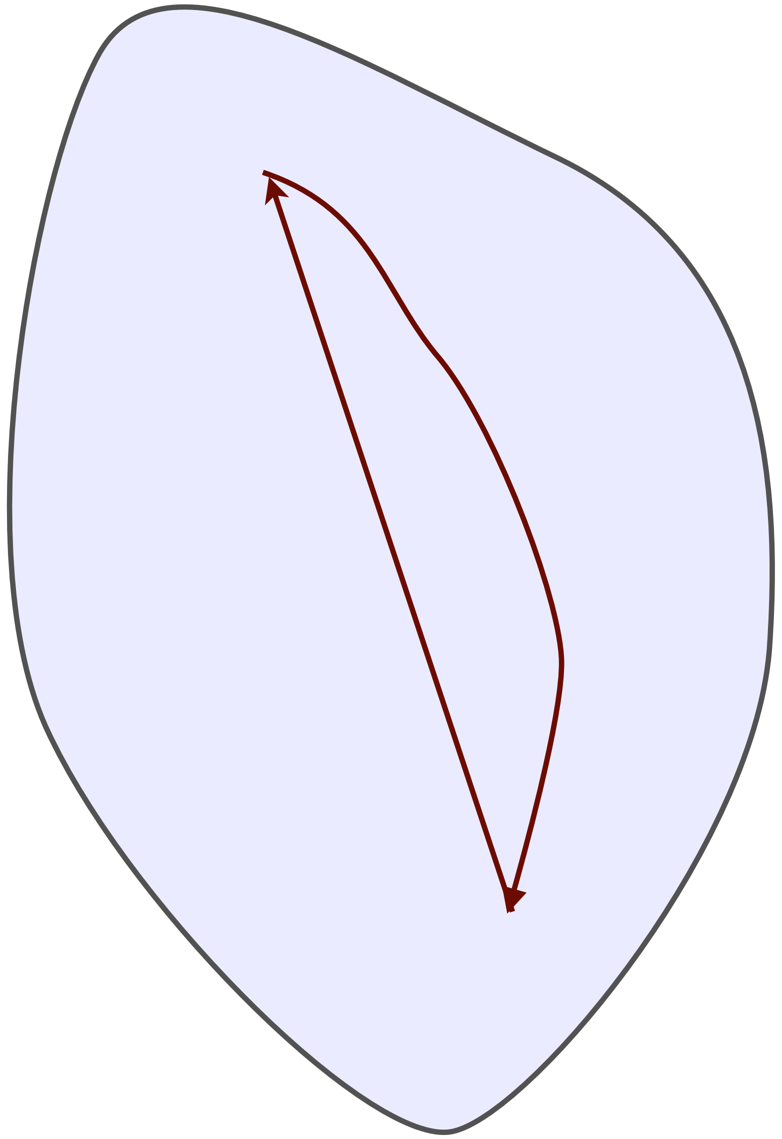

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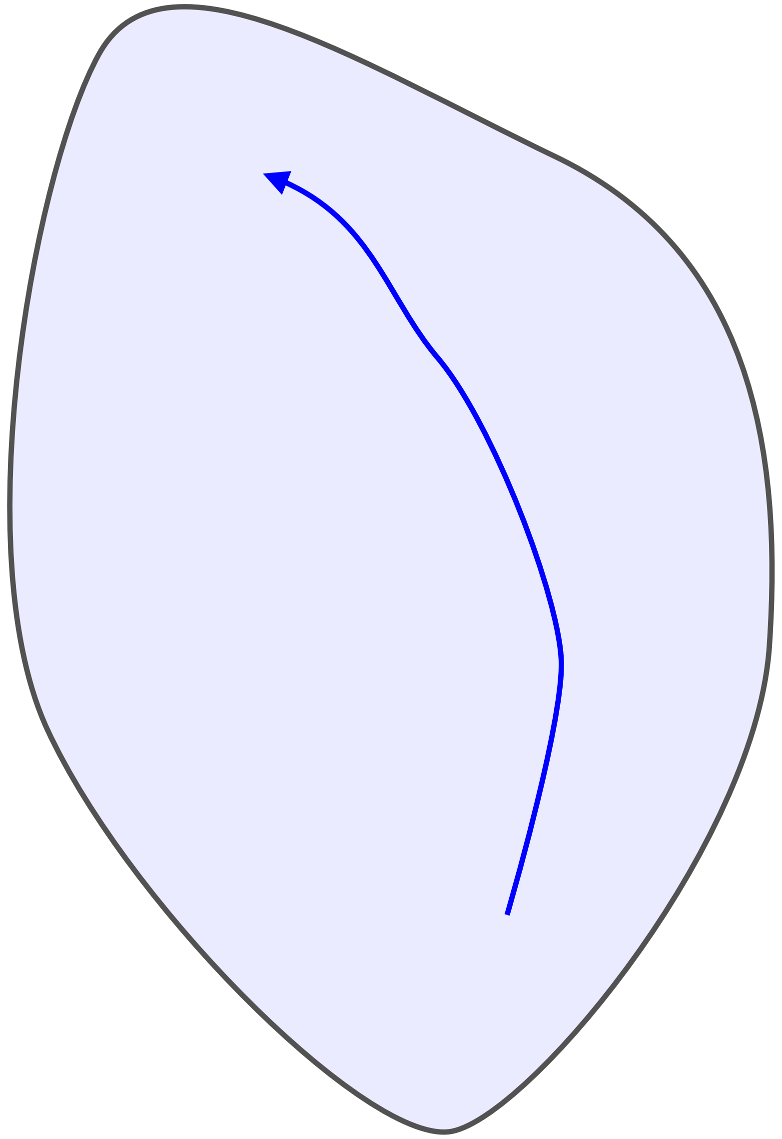

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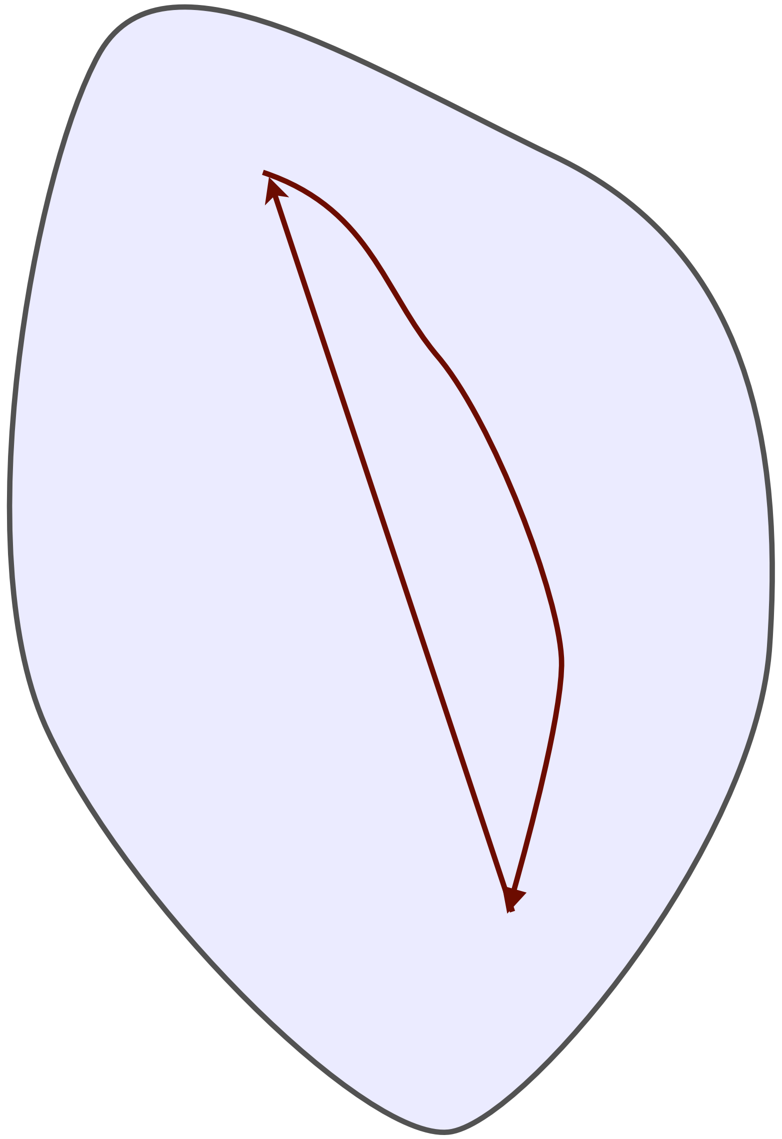

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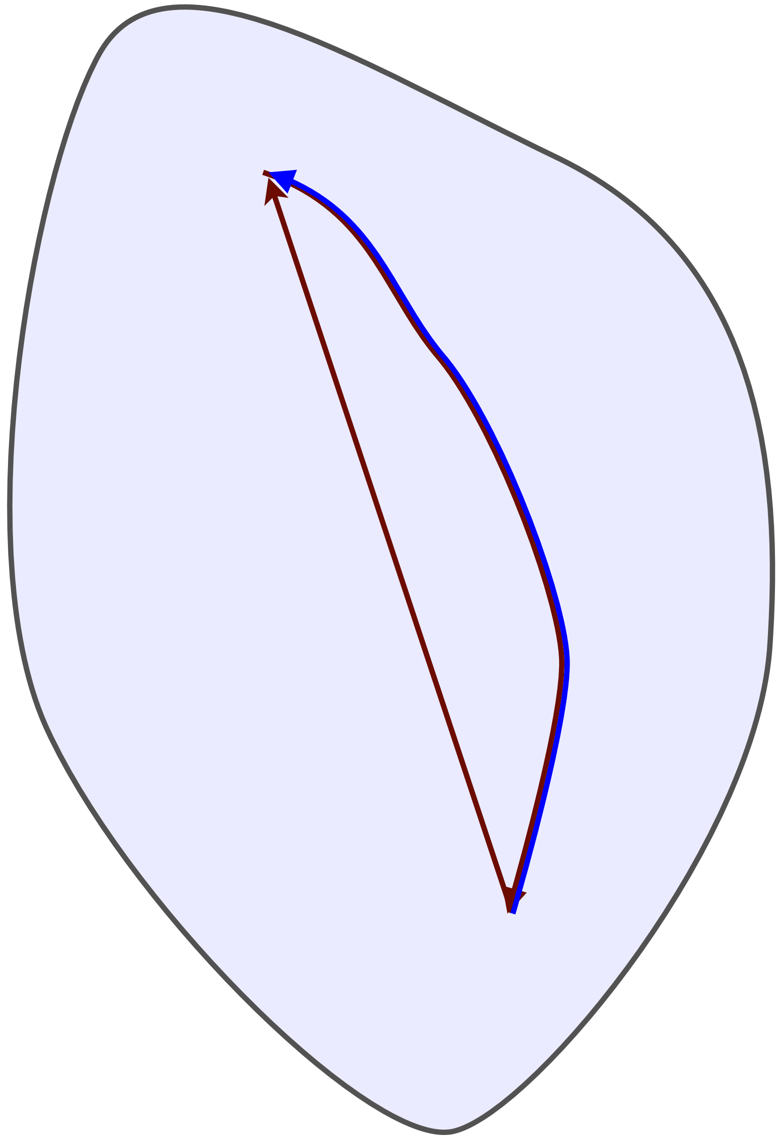
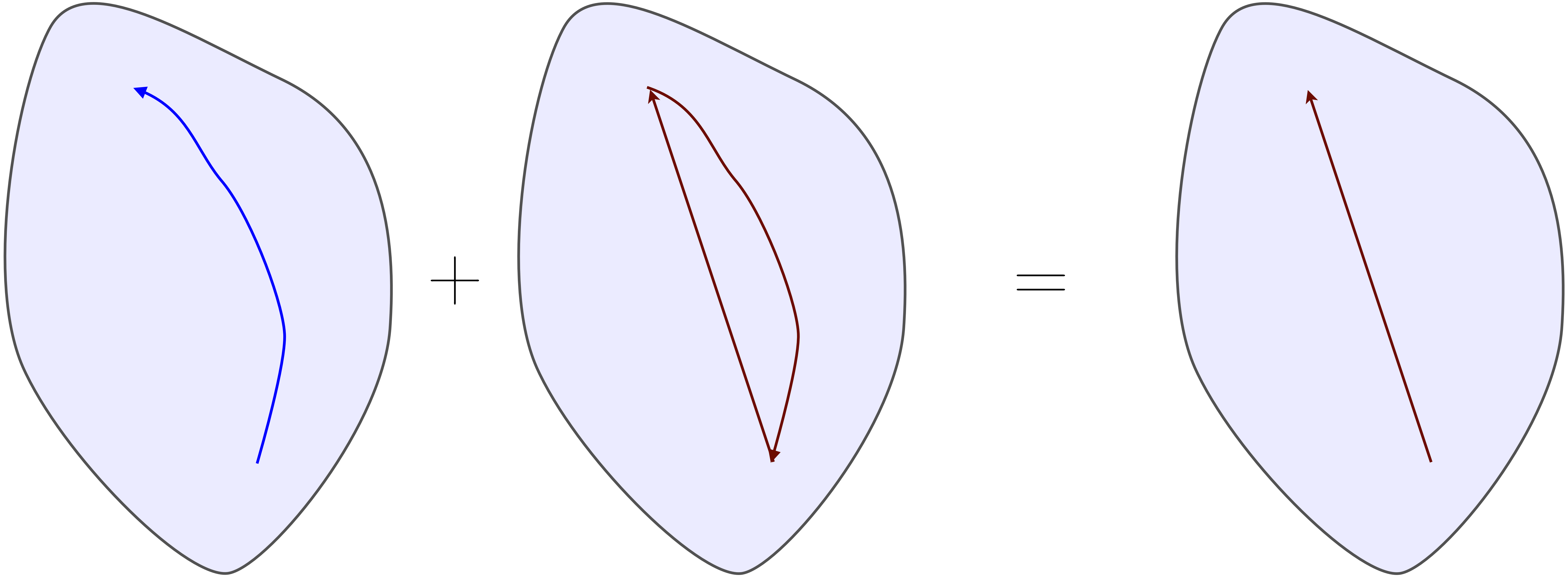


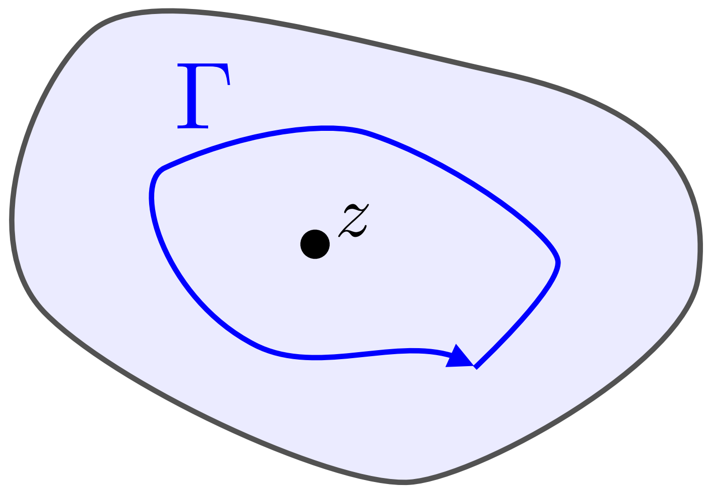
$$\int_\Gamma f\,\mathrm{d}z \quad + \quad 0 = \oint_C f\,\mathrm{d}z \quad = \quad \int_L f\,\mathrm{d}z$$



Theorem (*Cauchy's integral formula*): If $f(z)$ is analytic in a simply connected domain D and $\Gamma \subset D$ is a Jordan contour, then

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_\Gamma \frac{f(t)}{t-z}\,\mathrm{d}t$$

for all z in the interior of Γ



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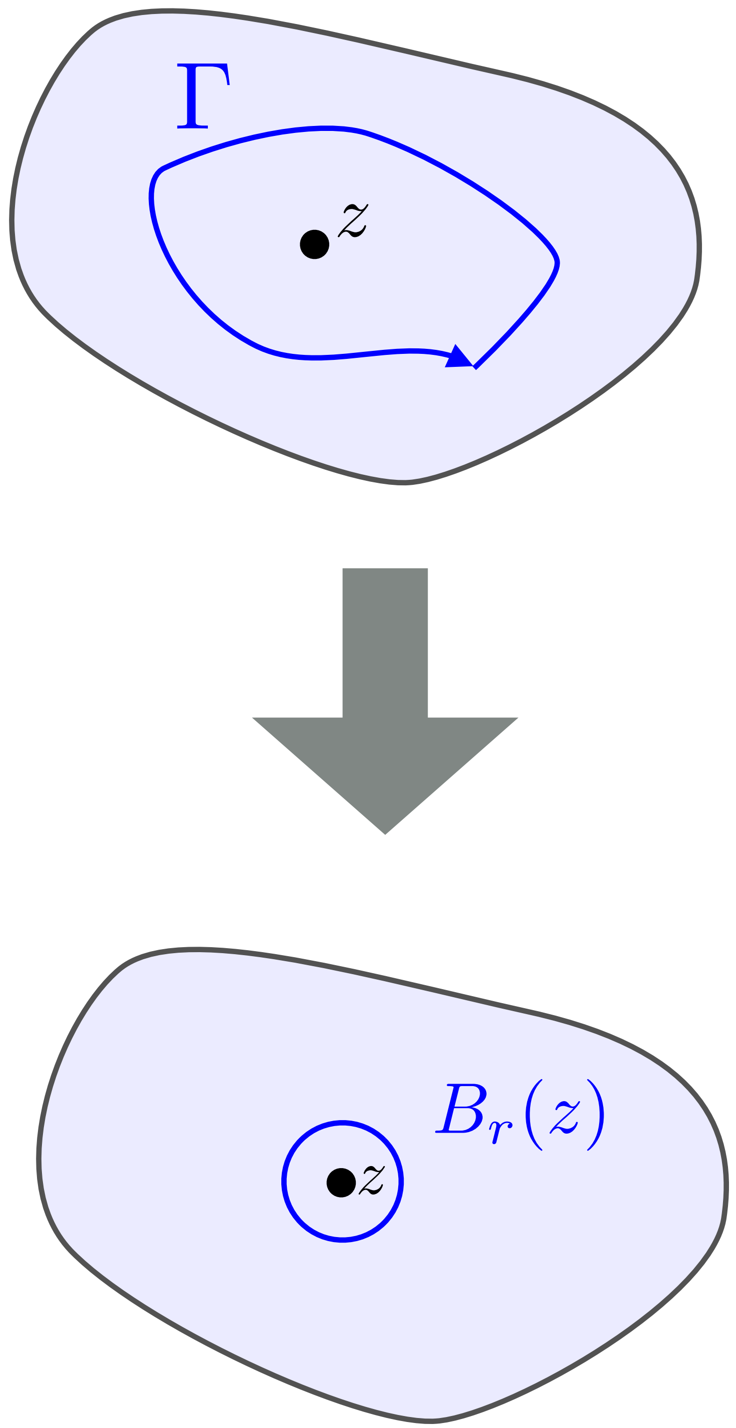

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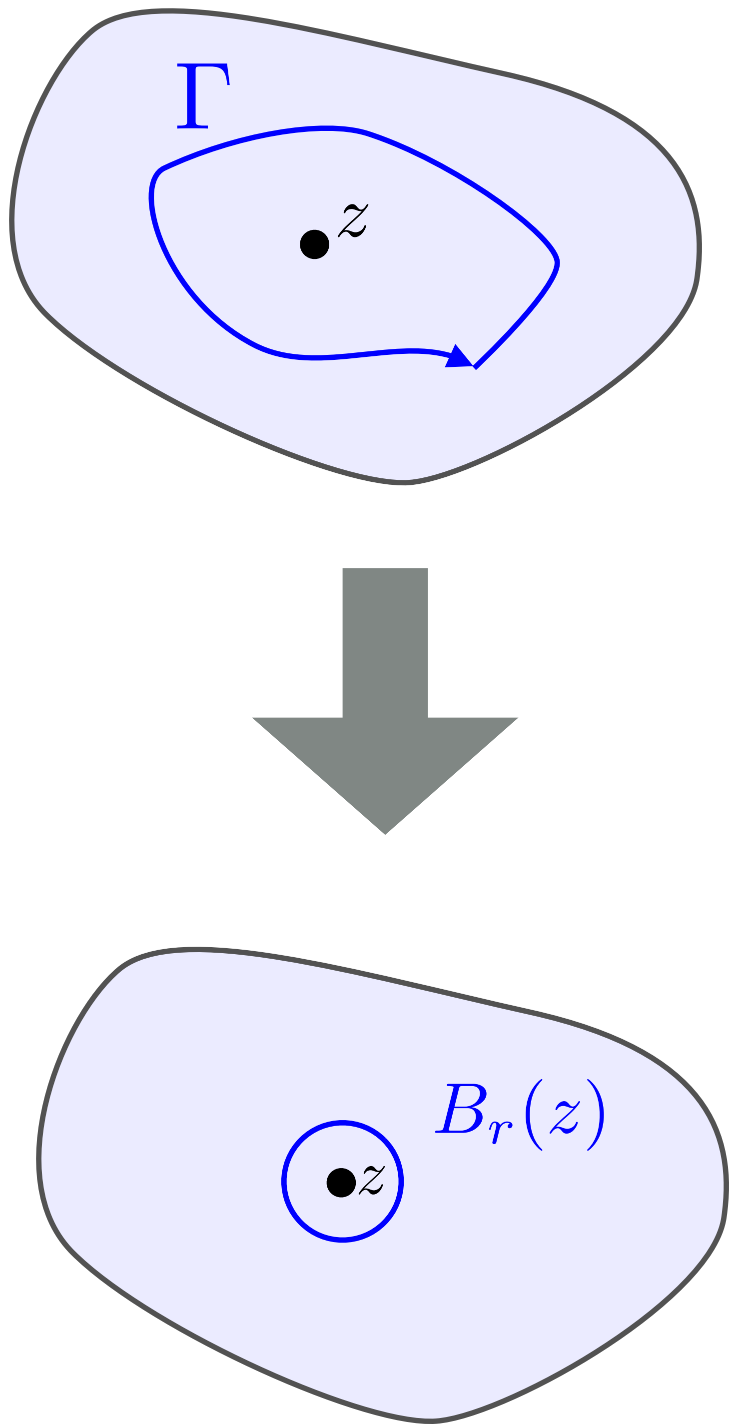

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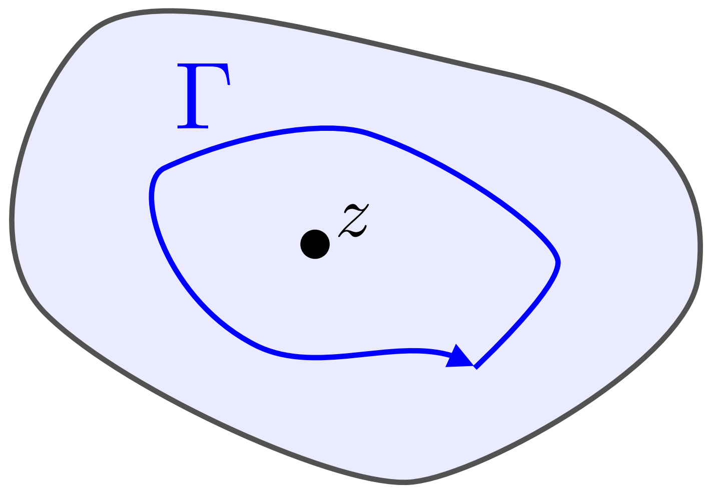

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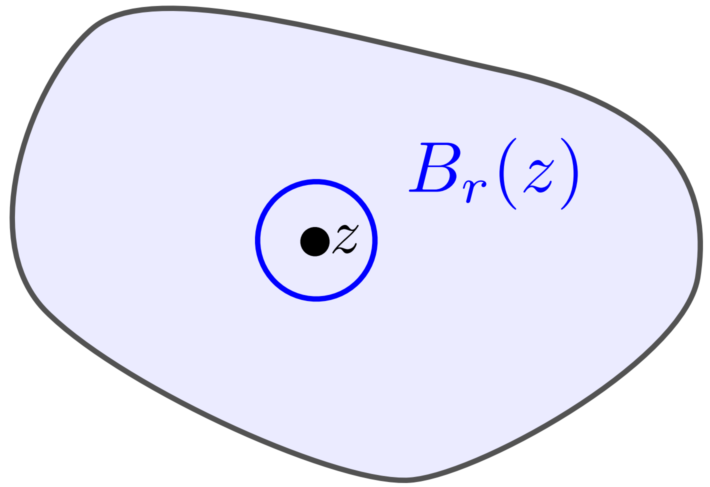
We find that

$$\frac{1}{2\pi \mathrm{i}} \oint_{B_r(z)} \frac{1}{t - z} \, \mathrm{d}t \overset{(w = t - z)}{=} \frac{1}{2\pi \mathrm{i}} \oint_{B_r(0)} \frac{1}{w} \, \mathrm{d}w$$



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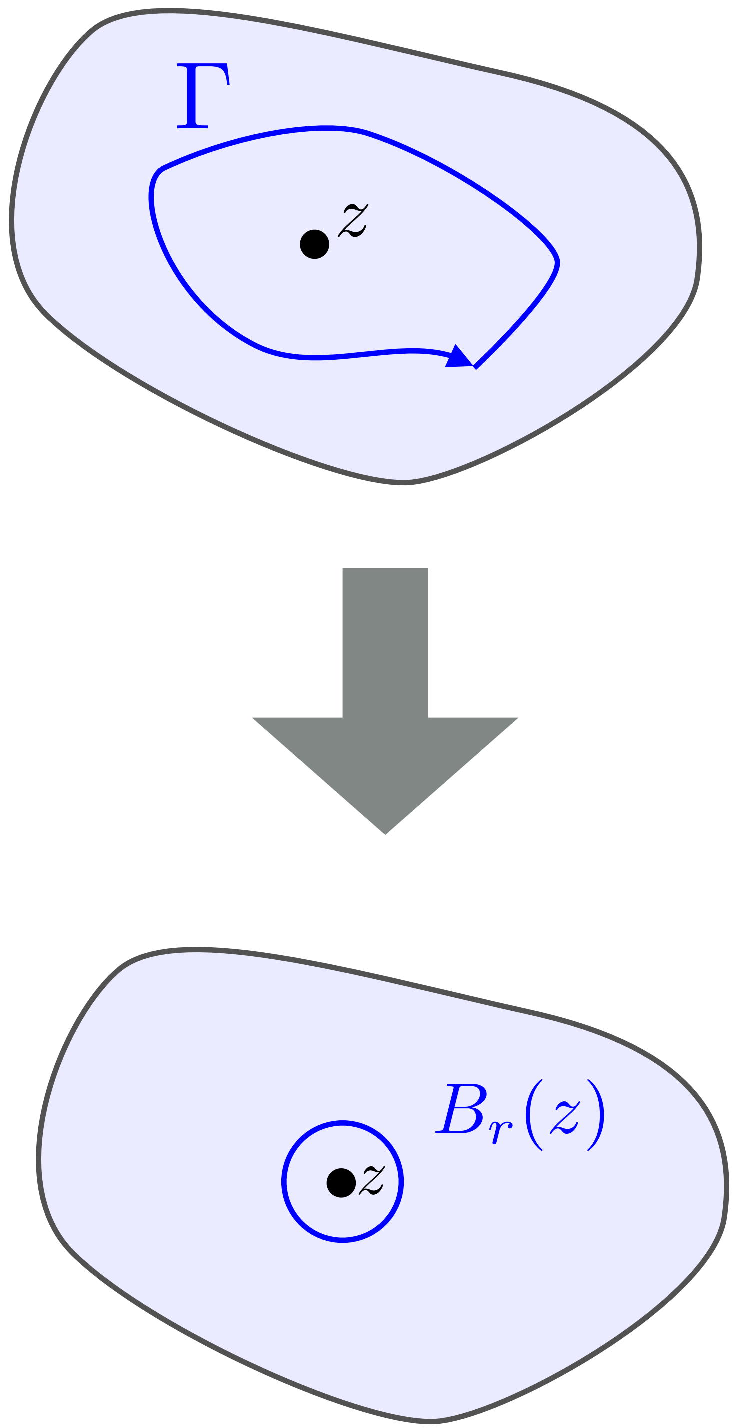

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- Because f is continuous at z, the second integral vanishes

Corollary: If $f(z)$ is analytic, then it is infinitely differentiable. In particular, if it is analytic in a simply connected domain D and $\Gamma \subset D$ is a Jordan contour, then

$$f^{(\lambda)}(z) = \frac{\lambda!}{2\pi \mathrm{i}} \oint_\Gamma \frac{f(t)}{(t-z)^{\lambda+1}} \, \mathrm{d}t$$

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Proof:

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- Therefore, we can interchange derivatives and integrals:

$$f'(z) = \frac{\mathrm{d}}{\mathrm{d}z} \frac{1}{2\pi \mathrm{i}} \int_\Gamma \frac{f(t)}{t-z} \,\mathrm{d}t = \frac{1}{2\pi \mathrm{i}} \int_\Gamma f(t) \frac{\mathrm{d}}{dz} \frac{1}{t-z} \,\mathrm{d}t$$

Corollary: If $f(z)$ is analytic, then it is infinitely differentiable. In particular, if it is analytic in a simply connected domain D and $\Gamma \subset D$ is a Jordan contour, then

$$f^{(\lambda)}(z) = \frac{\lambda!}{2\pi \mathrm{i}} \oint_\Gamma \frac{f(t)}{(t-z)^{\lambda+1}} \, \mathrm{d}t$$

for all z in the interior of Γ

Proof:

- $$\frac{f(t)}{t-z}$$

  is continuously differentiable on Γ

- Therefore, we can interchange derivatives and integrals:

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- The corollary follows by induction

Remark:

- We have assumed in the Cauchy–Riemann conditions that, for $f(x+\mathrm{i}y) = u(x,y) + \mathrm{i}v(x,y)$,

$$u_x, u_y, v_x, v_y$$

  are continuous

- This condition can be omitted
- All the theorems on analytic functions, including the preceding theorem, can be adapted to not assume that they are continuous
- The (adapted) preceding theorem then shows that f is in fact infinitely differentiable
- Therefore, the partial derivates u_x, u_y, v_x, v_y are continuous

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Useful properties of analytic functions

Lemma: Suppose f is analytic in the disk $z + R\mathbb{D}$ and $|f|$ is bounded by M in $z + R\mathbb{U}$. Then

$$\left|f^{(\lambda)}(z)\right| \leq \frac{\lambda! M}{R^\lambda}$$

Lemma: Suppose f is analytic in the disk $z + R\mathbb{D}$ and $|f|$ is bounded by M in $z + R\mathbb{U}$. Then

$$\left|f^{(\lambda)}(z)\right| \leq \frac{\lambda! M}{R^\lambda}$$

Proof:

$$\left|f^{(\lambda)}(z)\right| = \frac{\lambda!}{2\pi} \left| \oint_{z+R\mathbb{U}} \frac{f(t)}{(t-z)^{\lambda+1}} \, \mathrm{d}t \right|$$

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$$\begin{aligned}\left|f^{(\lambda)}(z)\right| &= \frac{\lambda!}{2\pi}\left|\oint_{z+R\mathbb{U}} \frac{f(t)}{(t-z)^{\lambda+1}}\,\mathrm{d}t\right| \\ &\le \frac{\lambda!}{2\pi}\oint_{z+R\mathbb{U}} |f(t)|\left|\frac{1}{(t-z)^{\lambda+1}}\right|\,|\,\mathrm{d}t|\end{aligned}$$

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\left|f^{(\lambda)}(z)\right| &= \frac{\lambda!}{2\pi}\left|\oint_{z+R\mathbb{U}} \frac{f(t)}{(t-z)^{\lambda+1}}\,\mathrm{d}t\right| \\
&\leq \frac{\lambda!}{2\pi}\oint_{z+R\mathbb{U}} |f(t)|\left|\frac{1}{(t-z)^{\lambda+1}}\right|\,|\,\mathrm{d}t| \\
&\leq \frac{\lambda! M}{2\pi R^{\lambda+1}}\oint_{R\mathbb{U}} |\,\mathrm{d}t| = \frac{\lambda! M}{2\pi R^{\lambda+1}}\int_{-\pi}^{\pi}\left|R\mathrm{e}^{\mathrm{i}\theta}\right|\,\mathrm{d}\theta
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&= \frac{\lambda! M}{R^\lambda}
\end{aligned}$$

Theorem (Liouville): Suppose f is analytic everywhere in $\mathbb{C}$ and bounded. Then f is constant.

Proof:

- Let $f(z) \leq M$ everywhere
- For every R we have
$$|f'(z)| \leq \frac{M}{R}$$
- Letting $R \to \infty$ shows that $f'(z) = 0$

Theorem (Maximum principles):

(i): Suppose f is analytic in an open set D. Then $|f(z)|$ cannot have a maximum in D unless $f(z)$ is constant.

(ii): If D is bounded and $|f(z)|$ is continuous on the boundary ∂D, then $|f(z)|$ takes it maximum on ∂D

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- Then we have a contradiction:

$$|f(z_0)| = \left|\frac{1}{2\pi \mathrm{i}} \oint_{z_0+R\mathbb{U}} \frac{f(t)}{t-z_0} \,\mathrm{d}t\right| = \frac{1}{2\pi}\int_0^{2\pi} \left|f(z_0 + r\mathrm{e}^{\mathrm{i}\theta})\right| \,\mathrm{d}\theta$$

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- Part (ii) follows since a continuous function on a bounded domain must achieve its maximum

Examples

- $|\mathrm{e}^z| = \left|\mathrm{e}^x \mathrm{e}^{\mathrm{i}y}\right| = \mathrm{e}^x$ is maximized at the furtherest right point in D
- $\left|\frac{1}{z}\right| = \frac{1}{|z|}$ is maximized at $z = 0$ since it is not analytic there. Otherwise it is maximizes at the smallest point in D

Taylor series revisited

Theorem (Liouville): Suppose f is analytic in the closed unit disk $\mathbb{D}$ and is *sufficiently smooth* on the unit circle $\mathbb{U}$. Then f has a Taylor series at zero that converges inside $\mathbb{D}$.

Proof:

- The Taylor series at zero is

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

- Each of the coefficients is precisely the Fourier coefficients:

$$\frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi \mathrm{i}} \oint_{\mathbb{U}} \frac{f(t)}{t^{k+1}} \, \mathrm{d}t = \hat{f}_k$$

- The smoothness of f on $\mathbb{U}$ tells us that coefficients decay, and hence the Taylor series has radius of convergence at least one

Theorem: Suppose $f(z)$ is analytic in a disk of radius $\rho > 1$. Then its Taylor series decays like

$$\hat{f}_k = \mathcal{O}(\rho^{-k})$$

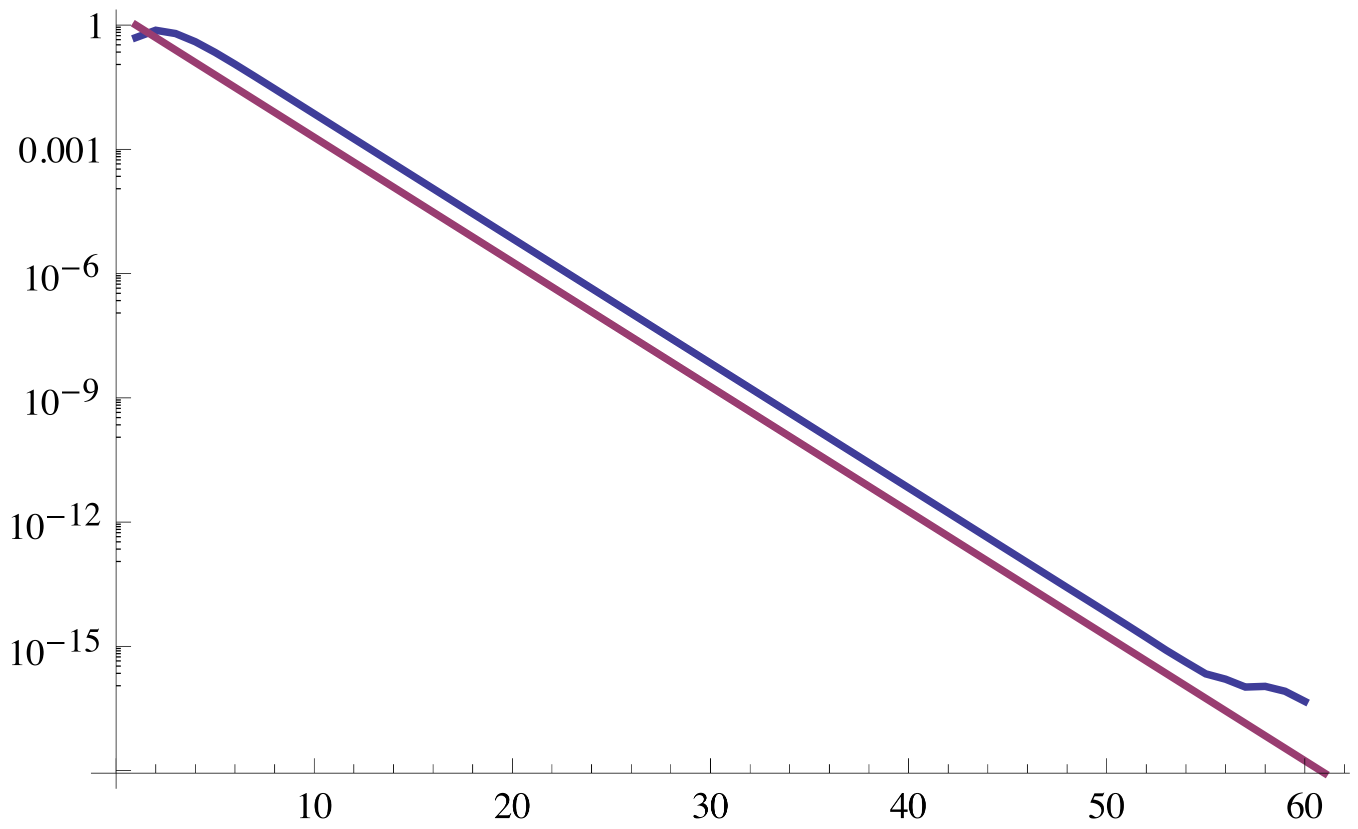
Proof:

- Let M denote the maximum of f in $\rho\mathbb{D}$
- We have

$$\left|\hat{f}_k\right| \leq \left|\frac{f^{(k)}(0)}{k!}\right| \leq \frac{M}{\rho^k}$$

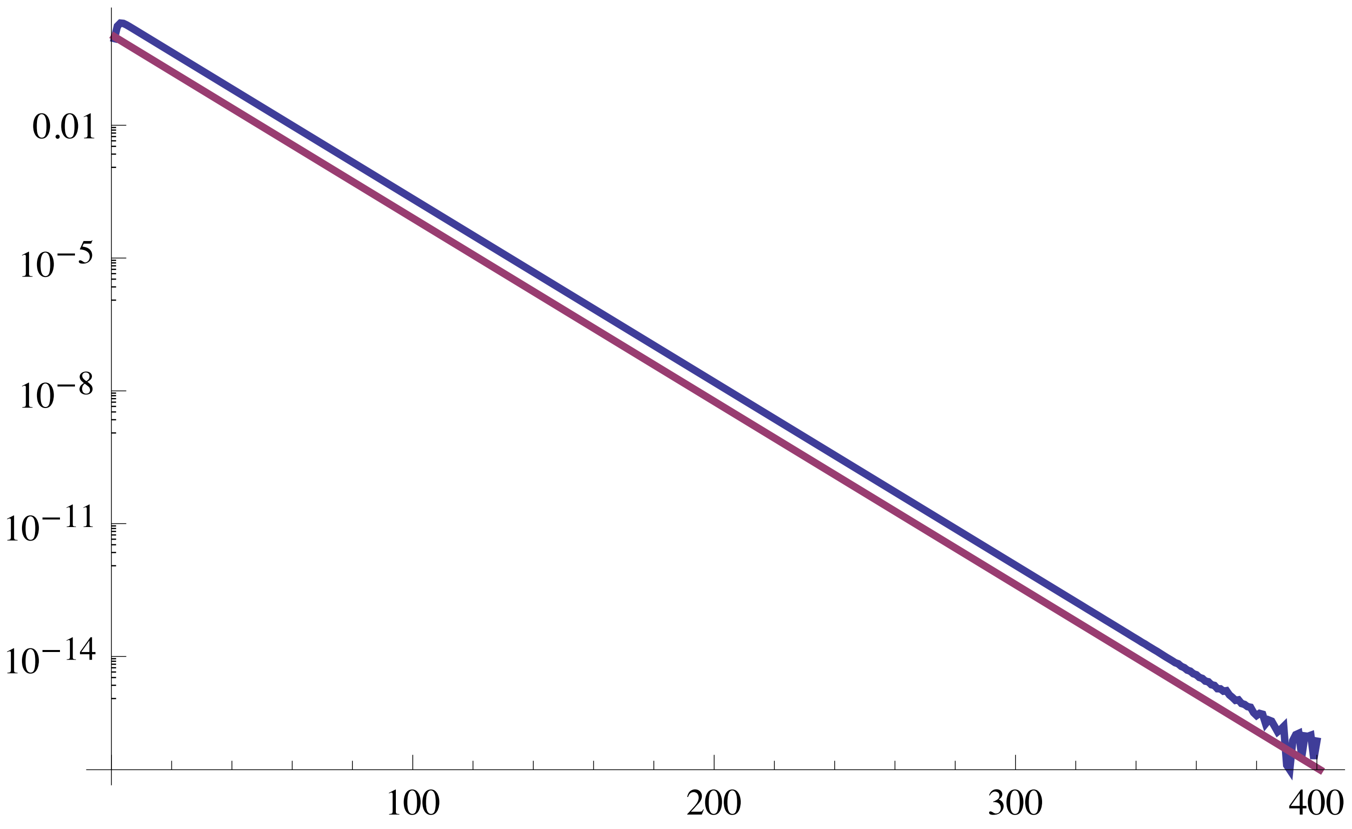
$$\frac{e^z}{z-2}$$

$$\frac{e^z}{z-1.1}$$



Taylor coefficients

2^{-k}



Taylor coefficients

1.1^{-k}

Theorem: The error in the approximate Taylor series is maximized on the unit circle.

Proof: The error term

$$f(z) - \sum_{k=0}^{n} \left\langle \mathrm{e}^{\mathrm{i}k\theta}, f \right\rangle_m z^k$$

is analytic, hence its maximum is on the boundary

Analyticity at infinity

- We have defined analyticity everywhere except at ∞
- We call $f(z)$ analytic at infinity if

$$f\left(\frac{1}{z}\right)$$

  is analytic at zero

  - Examples: $1, \frac{1}{z}, \frac{1}{z^2}, \frac{z+1}{z-1}, \mathrm{e}^{1/z}$
- We thus say f is analytic in a domain D containing ∞ if it is analytic at every finite point of D and analytic at ∞
- An important example is analyticity outside the unit circle

- If f is analytic everywhere outside the unit disk including at infinity, $f(1/z)$ is analytic inside the unit disk

- Thus we can expand it in a Taylor series

$$f(1/z) = \sum_{k=0}^{\infty} c_k z^k$$

- In other words,

$$f(z) = \sum_{k=-\infty}^{0} c_{-k} z^k$$

which converges outside the unit circle

- Thus $c_{-k} = \hat{f}_k$.

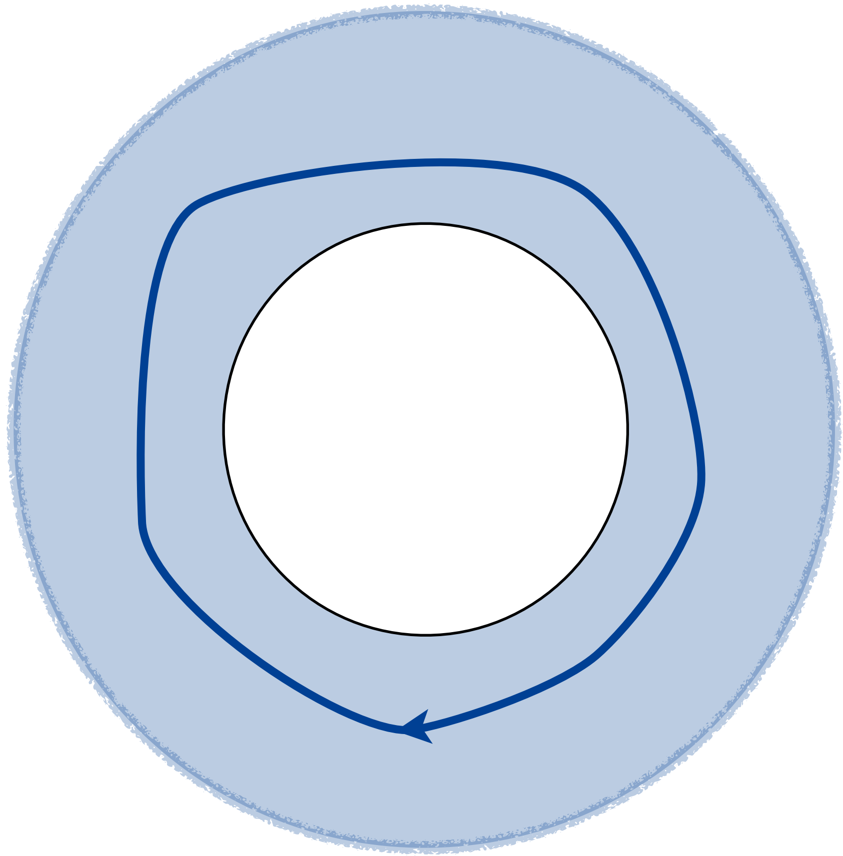
Theorem: The only functions that are analytic everywhere in $\mathbb{C}$, including at ∞, are constant.

Proof:

- Because $f(z)$ is analytic inside the unit circle, $\hat{f}_k = 0$ for $k > 0$
- Because $f(z)$ is analytic outside the unit circle, $\hat{f}_k = 0$ for $k < 0$
- Thus $f(z) = \hat{f}_0$

Lemma: Suppose f is analytic in a domain D containing infinity so that $1/D$ is simply connected, and Γ is a curve in D. If $f(z) = \mathcal{O}(z^{-2})$ then

$$\oint_\Gamma f(z)\, \mathrm{d}z = 0$$



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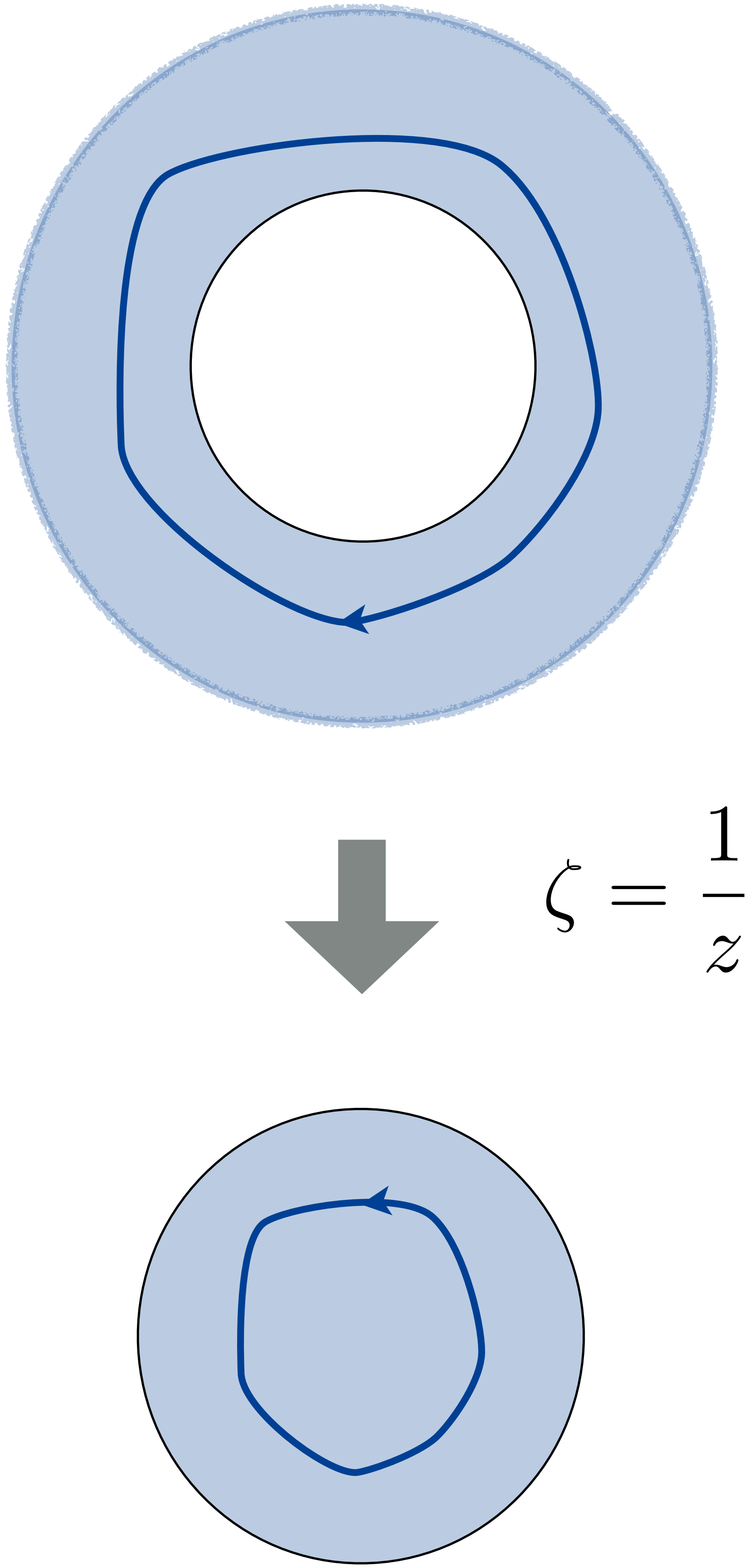
Proof:

- Note that

$$\oint_\Gamma f(z)\,\mathrm{d}z = \oint_{1/\Gamma} \frac{f(1/\zeta)}{\zeta^2}\,\mathrm{d}\zeta$$



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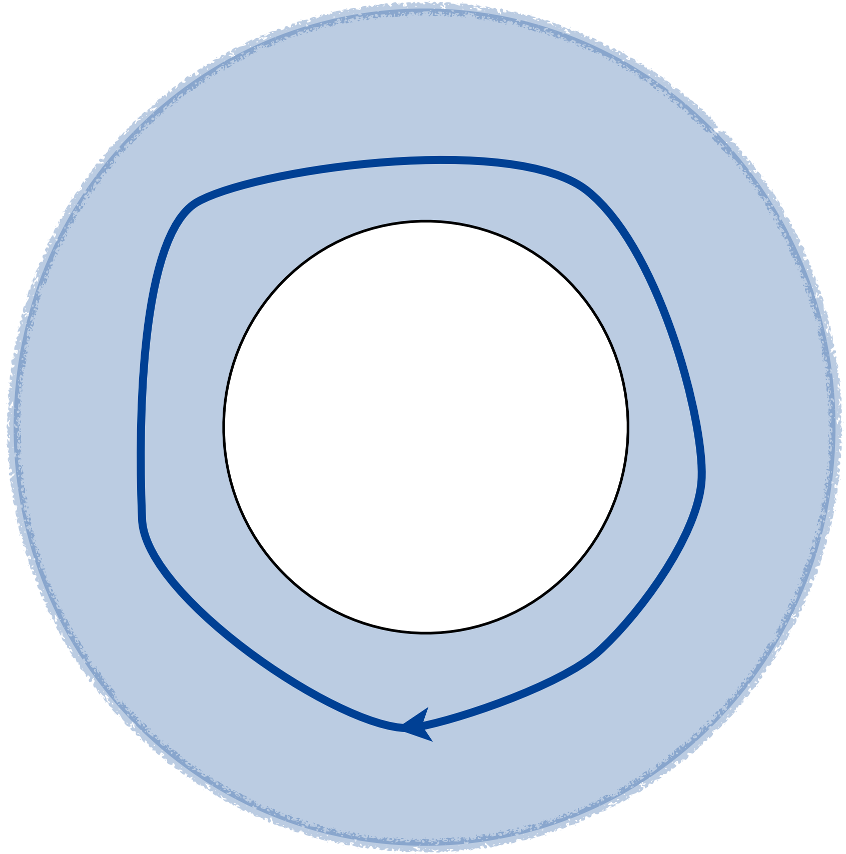
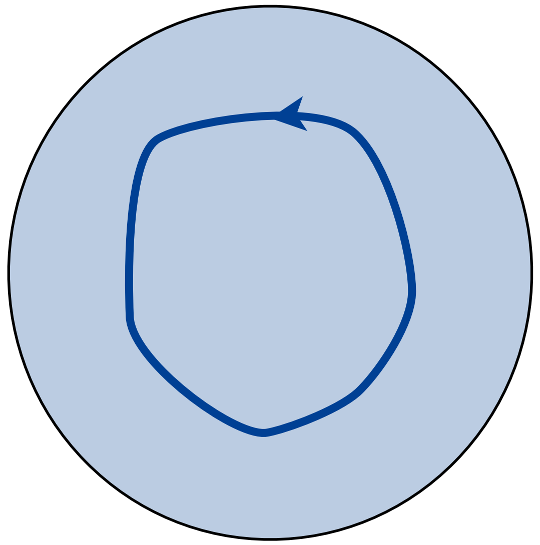
- We know that $f(1/\zeta) = \mathcal{O}\left(\zeta^2\right)$
- Therefore

$$\frac{f(1/\zeta)}{\zeta^2} = \sum_{k=0}^{\infty} \hat{f}_{-k}\zeta^{k-2}$$

  converges



$$\zeta = \frac{1}{z}$$



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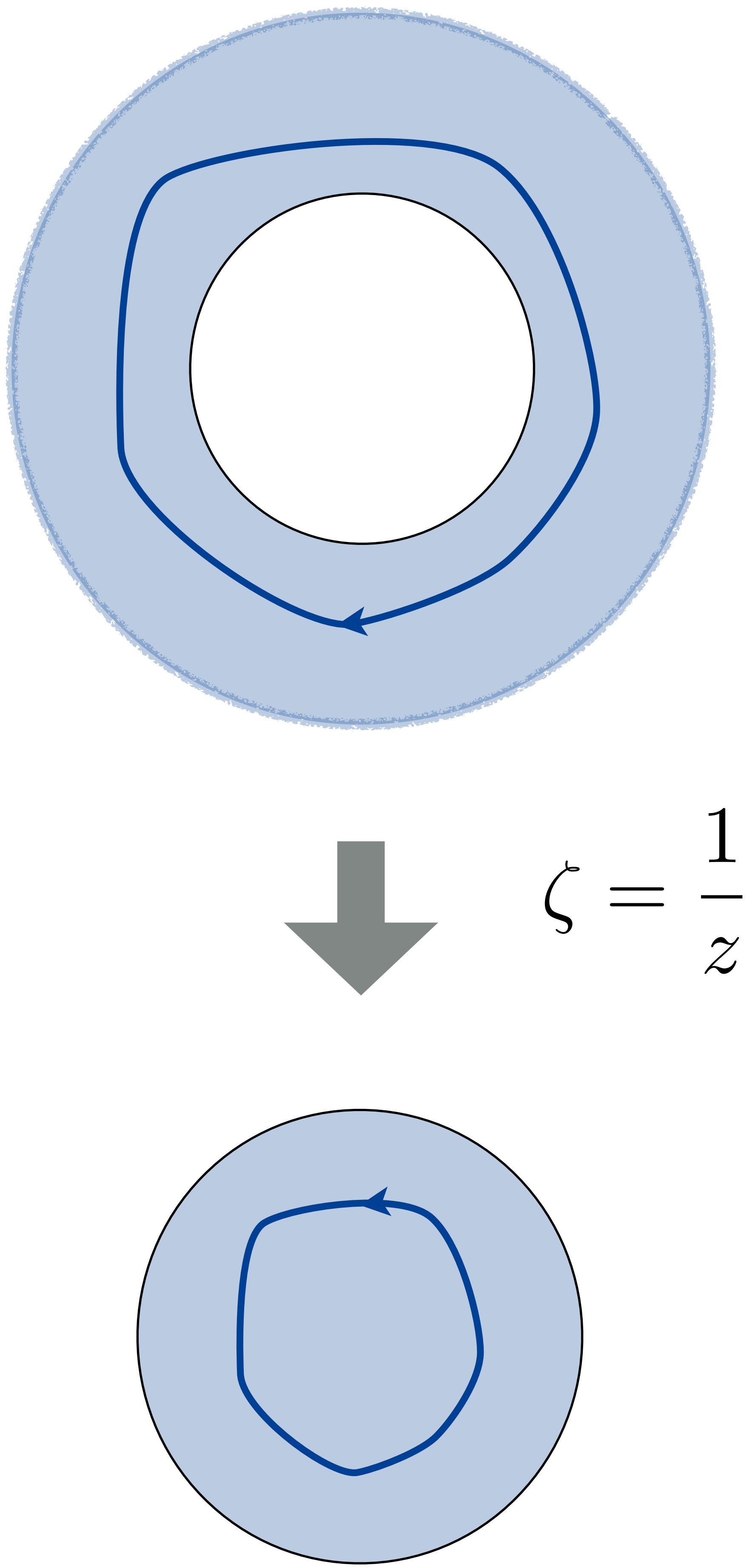
- Therefore

$$\frac{f(1/\zeta)}{\zeta^2} = \sum_{k=0}^{\infty} \hat{f}_{-k}\zeta^{k-2}$$

  converges

- In other words, it is analytic in the unit disk, and

$$\oint_\Gamma \frac{f(1/\zeta)}{\zeta^2}\,\mathrm{d}\zeta = 0$$



Corollary:

$$\oint_{\mathbb{U}} \frac{t^k}{t - z} \, \mathrm{d}t = 0$$

if $k \geq 0$ and z is outside the unit circle or $k < 0$ and z is inside the unit circle

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- For $k \geq 0$ and z outside the unit circle,

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is analytic inside the unit circle, hence the integral is zero

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Proof:

- For $k \geq 0$ and z outside the unit circle,

  $$\frac{t^k}{t - z}$$

  is analytic inside the unit circle, hence the integral is zero

- For $k < 0$ and z inside the unit circle,

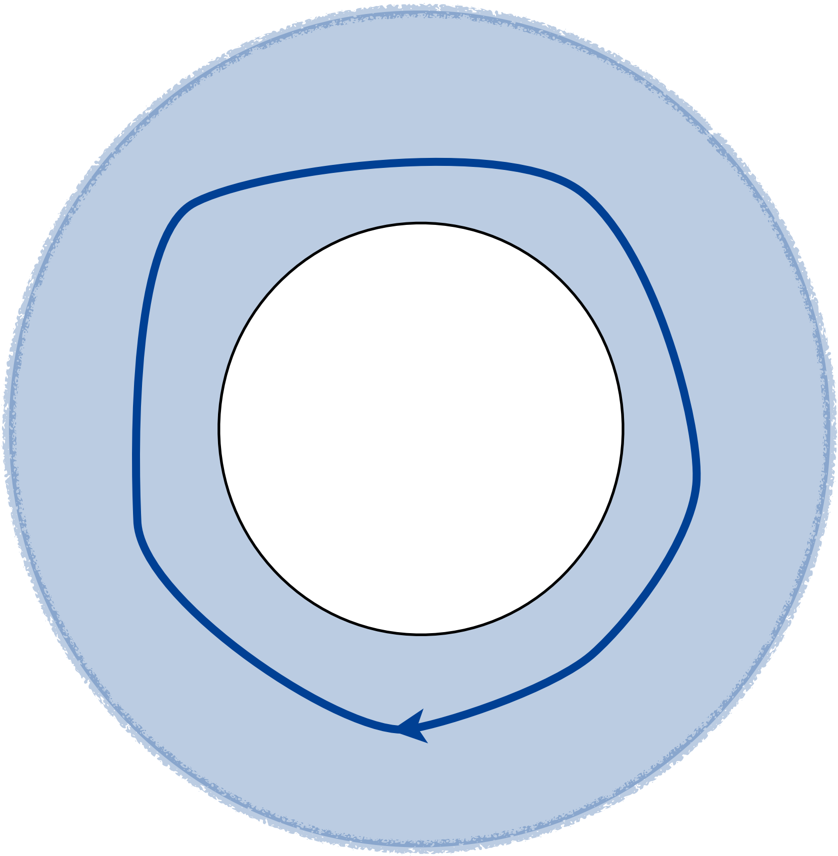
  $$\frac{t^k}{t - z}$$

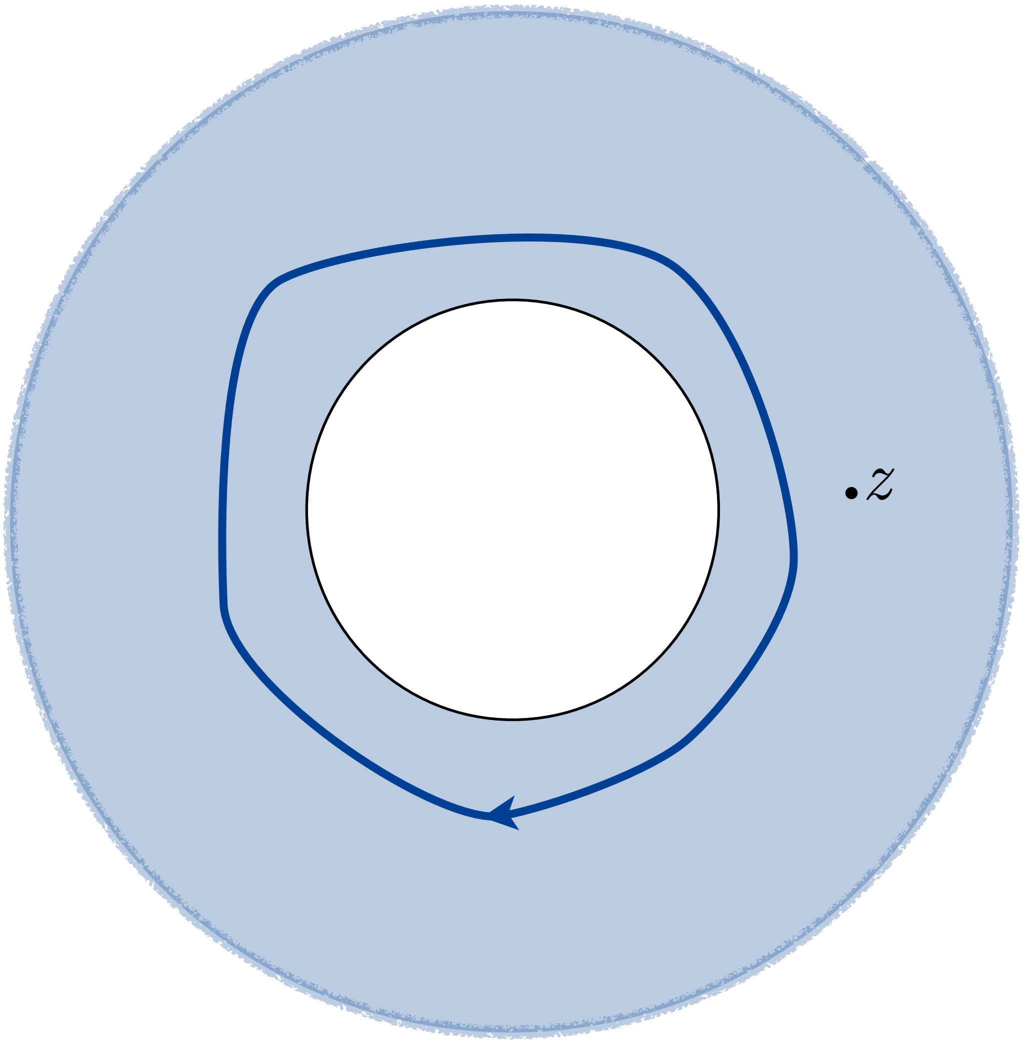
  is analytic outside the unit circle and behaves like $\mathcal{O}\big(t^{k-1}\big) = \mathcal{O}\big(t^{-2}\big)$, hence the integral is also zero

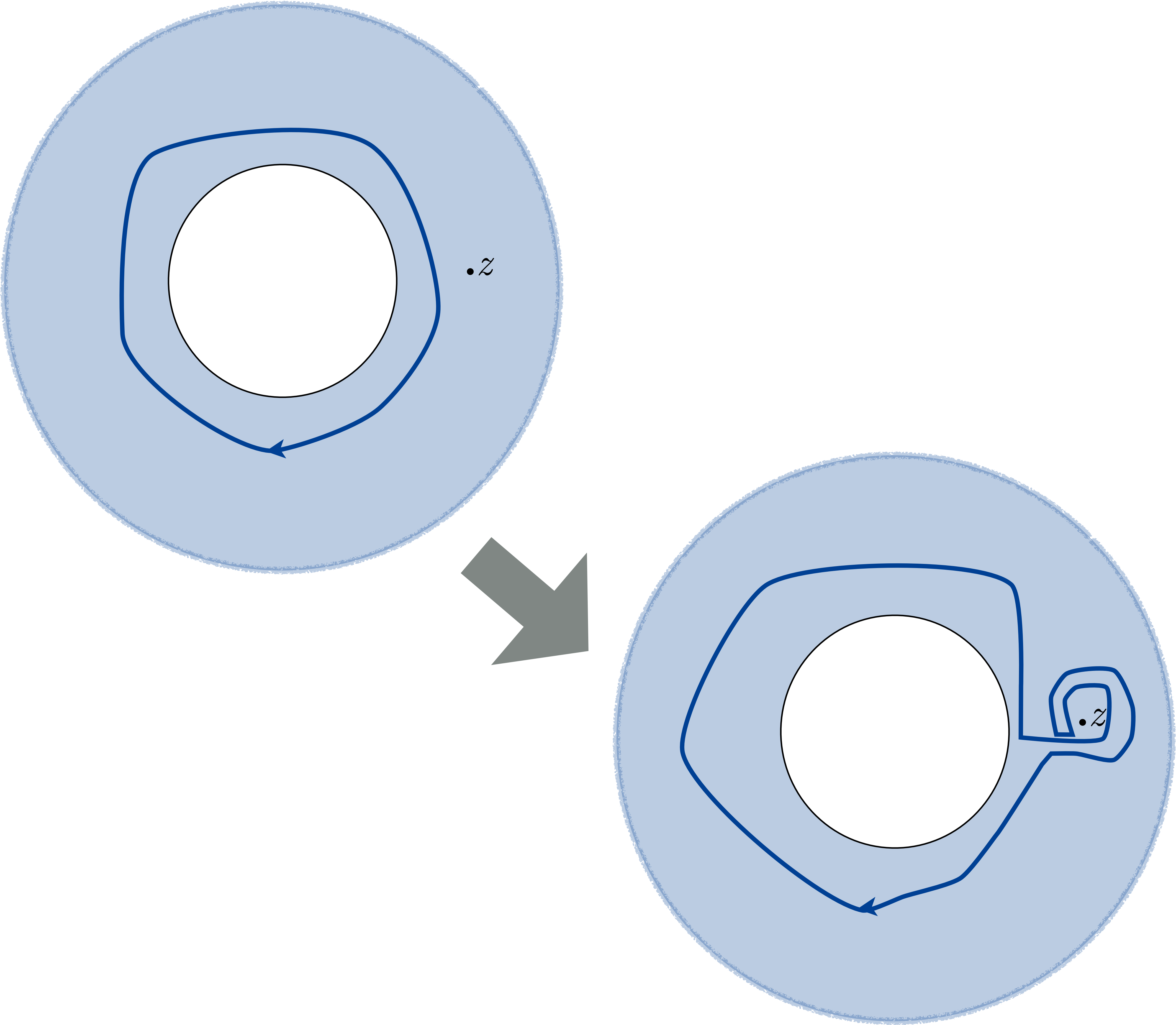
Lemma: Suppose f is analytic in a domain D containing infinity so that $1/D$ is simply connected, and Γ is a curve in D. If $f(z) = \mathcal{O}\big(z^{-2}\big)$ then

$$f(z) = \frac{1}{2\pi\mathrm{i}} \oint_\Gamma \frac{f(t)}{t - z} \, \mathrm{d}t$$

when Γ is oriented with the interior of the contour inside D




z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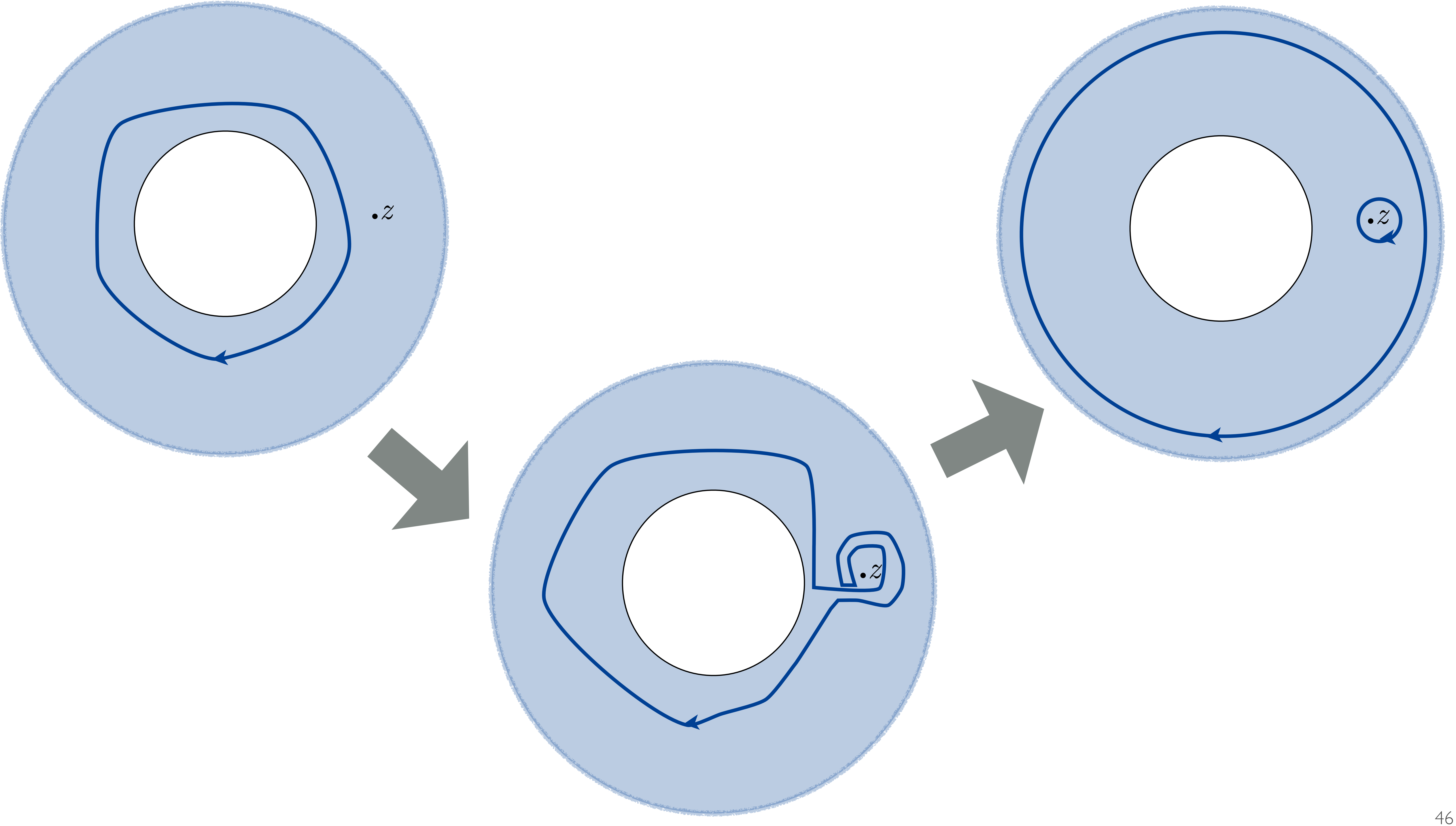
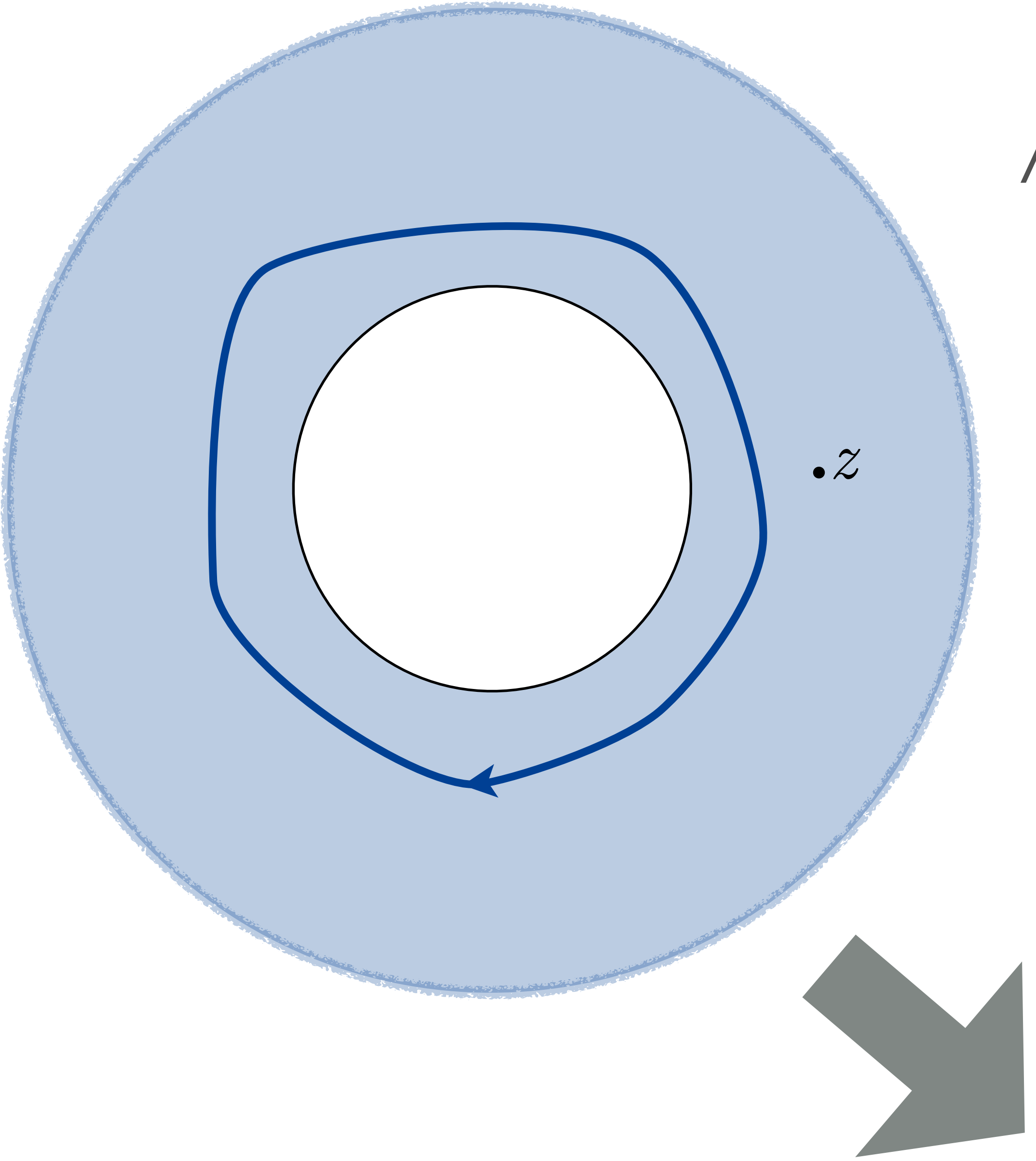
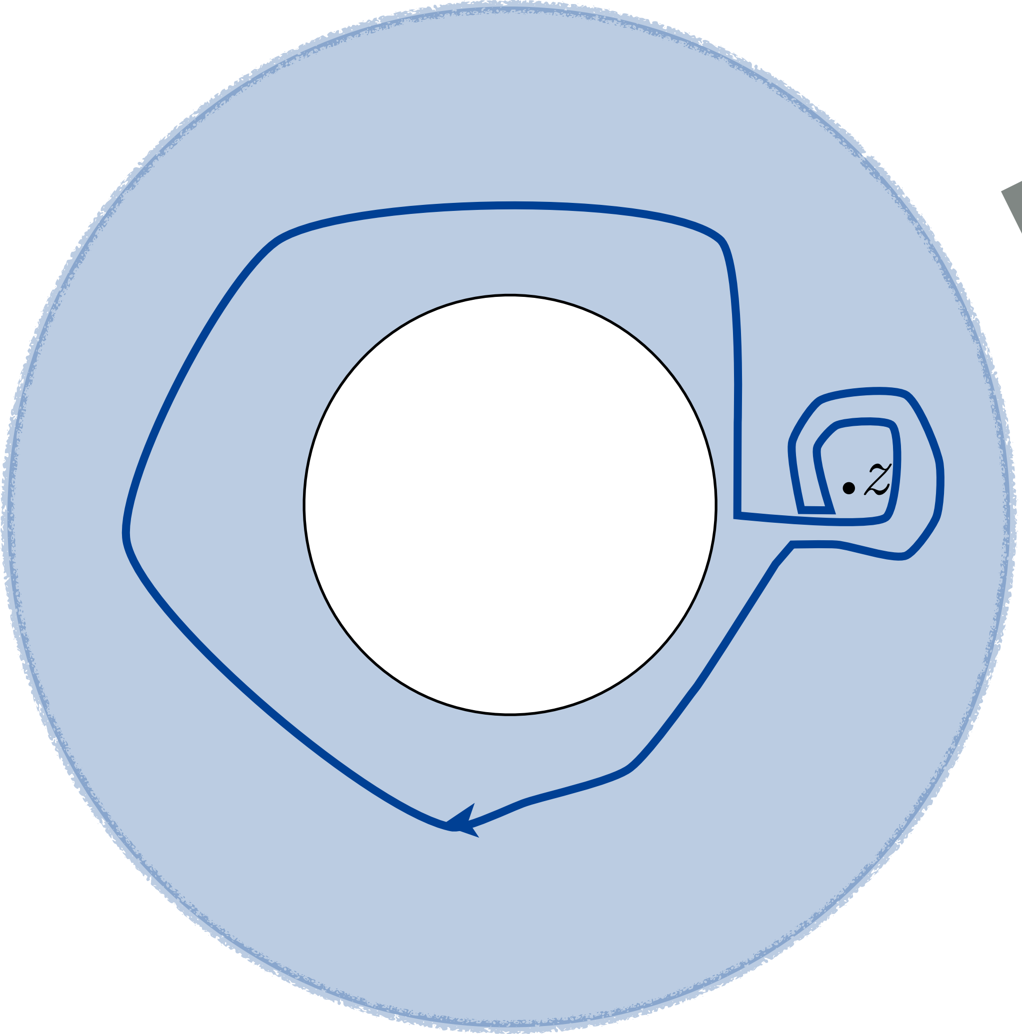


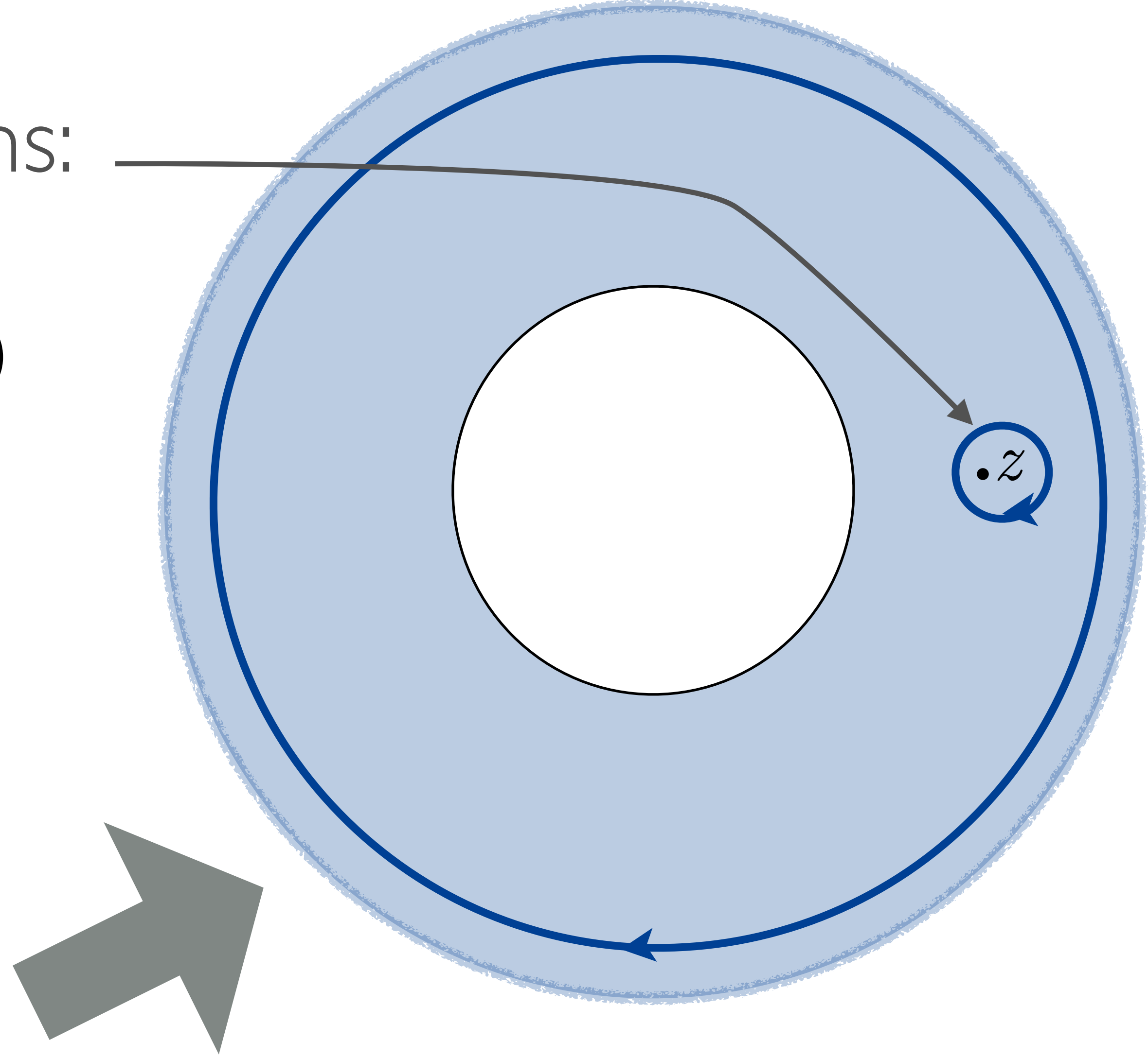
z
z
z

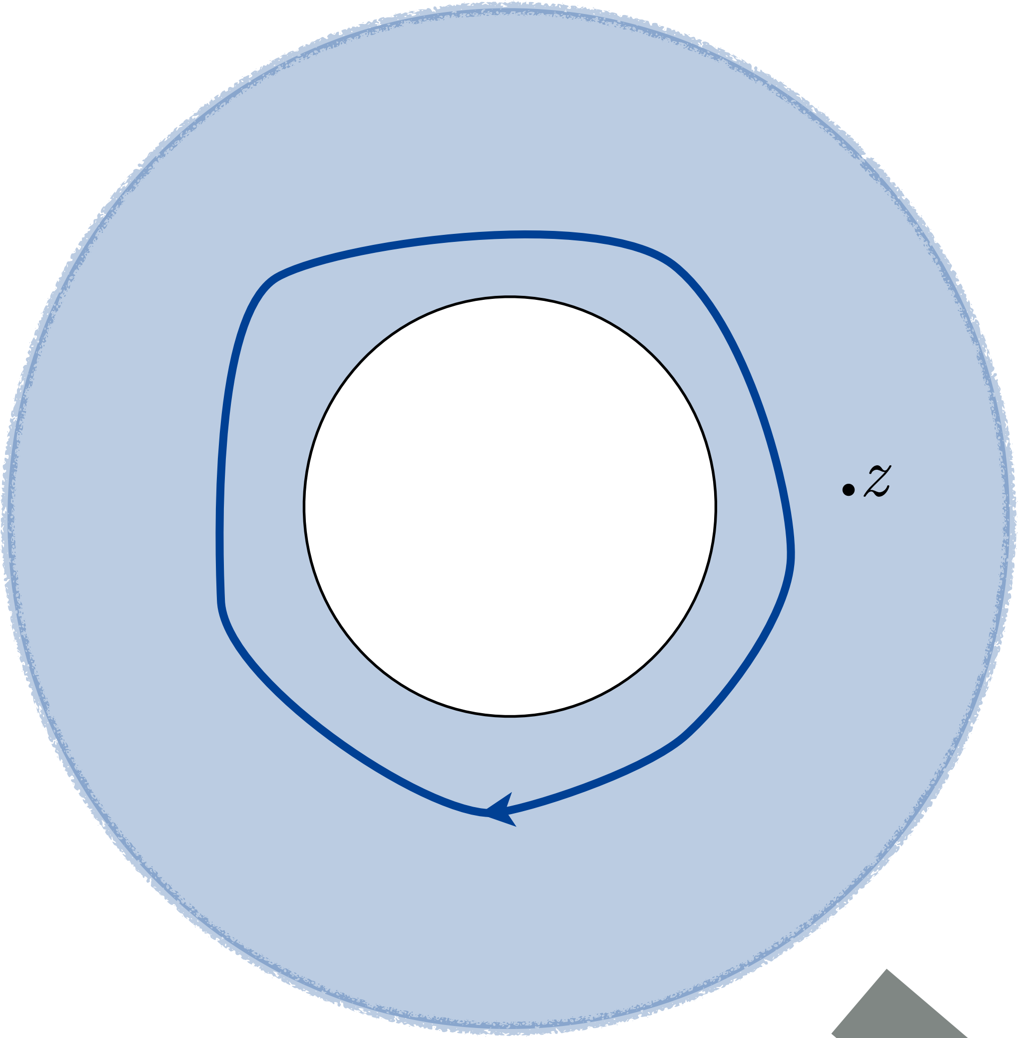
After switching orientations:

$$-\frac{1}{2\pi \mathrm{i}}\oint_{B_r(z)} \frac{f(t)}{t-z}\,\mathrm{d}t = -f(z)$$



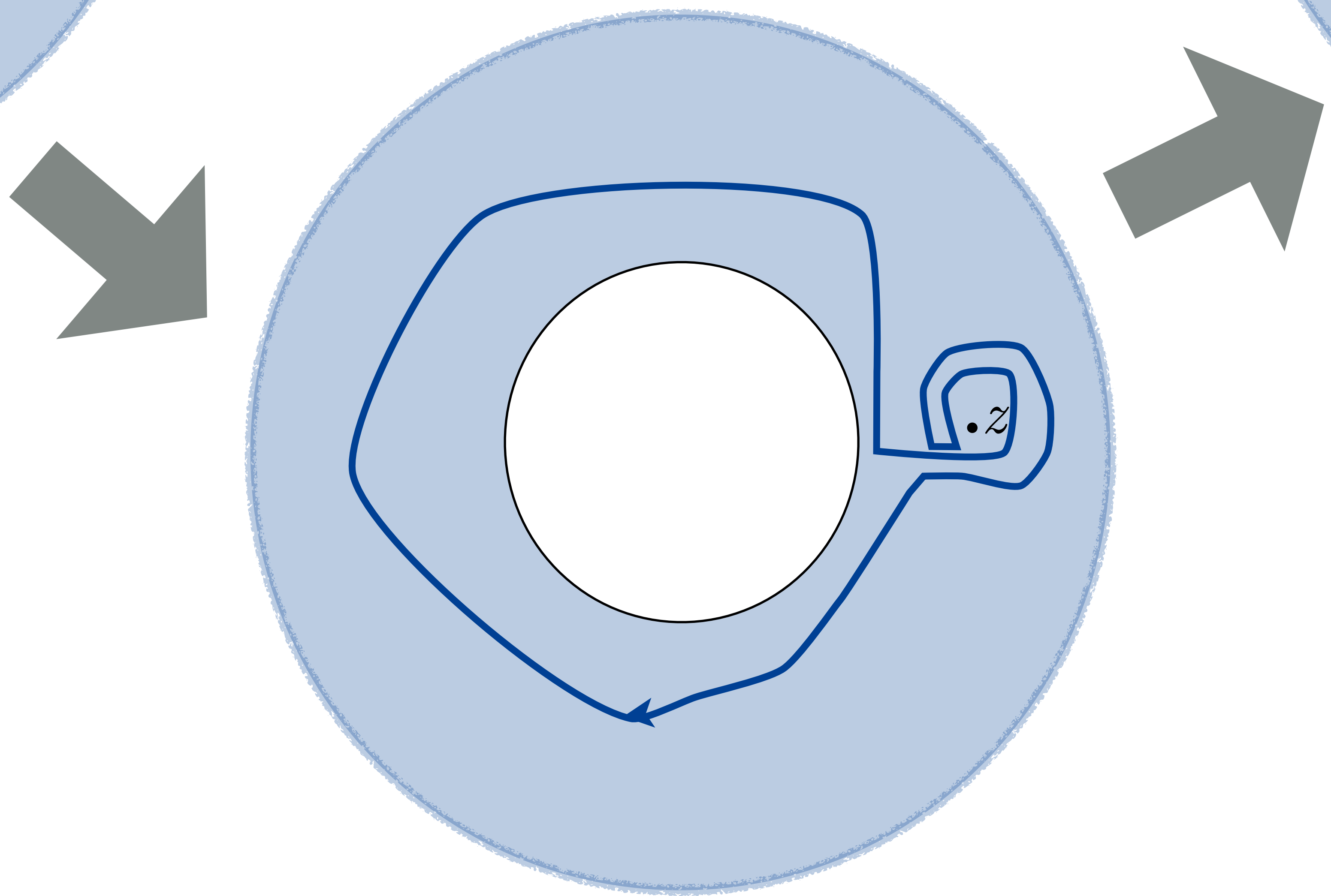


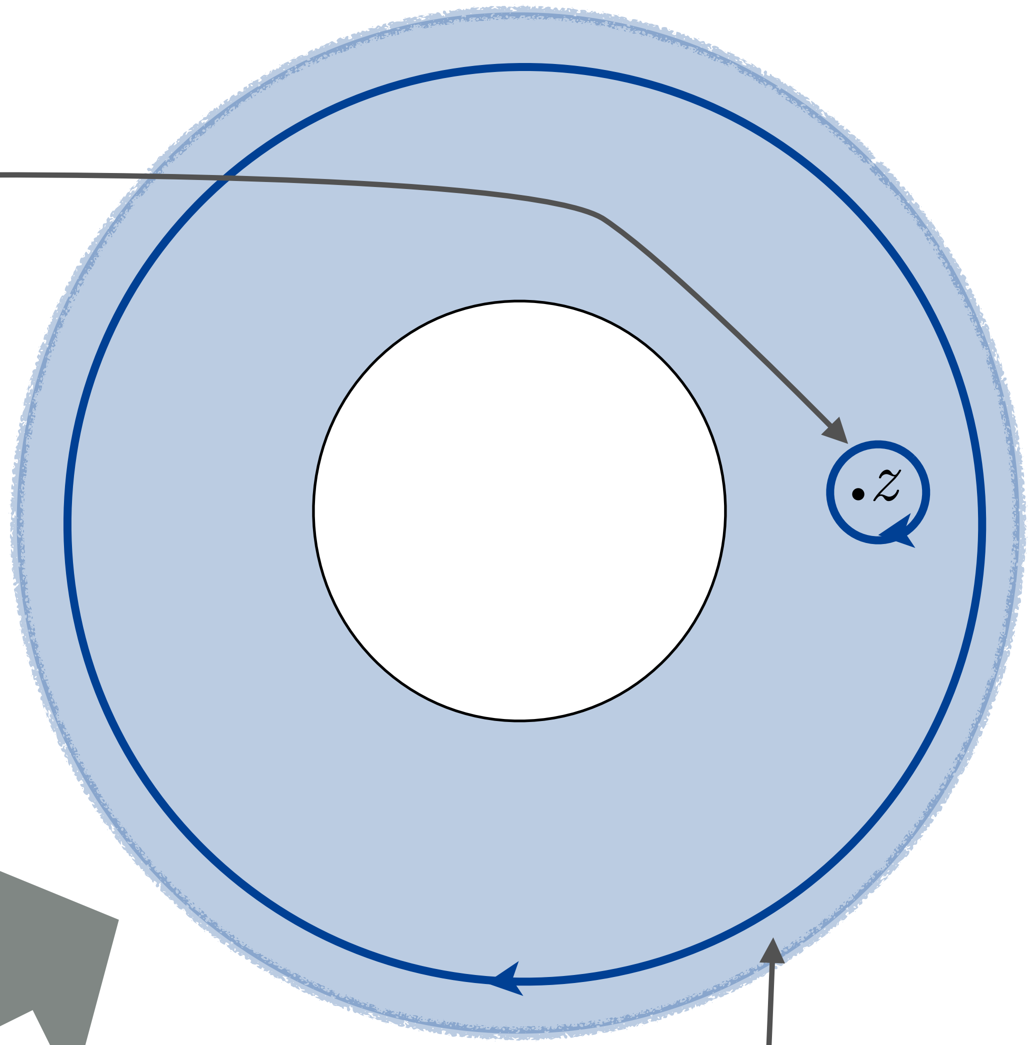




After switching orientations:

$$-\frac{1}{2\pi \mathrm{i}} \oint_{B_r(z)} \frac{f(t)}{t-z}\,\mathrm{d}t = -f(z)$$





Zero due to analyticity

Laurent series

- Suppose f is smooth on the unit circle but neither analytic everywhere inside or outside the unit disk
  - Examples: $\mathrm{e}^{1/z+z}$ and $\frac{z+2}{z}$
- Then we know for sure that it cannot be represented by a Taylor series
- However, we can still expand in Fourier series

$$f(\mathrm{e}^{\mathrm{i}\theta}) = \sum_{k=-\infty}^{\infty} \hat{f}_k \mathrm{e}^{\mathrm{i}k\theta}$$

- This leads to *Laurent series* on the unit circle

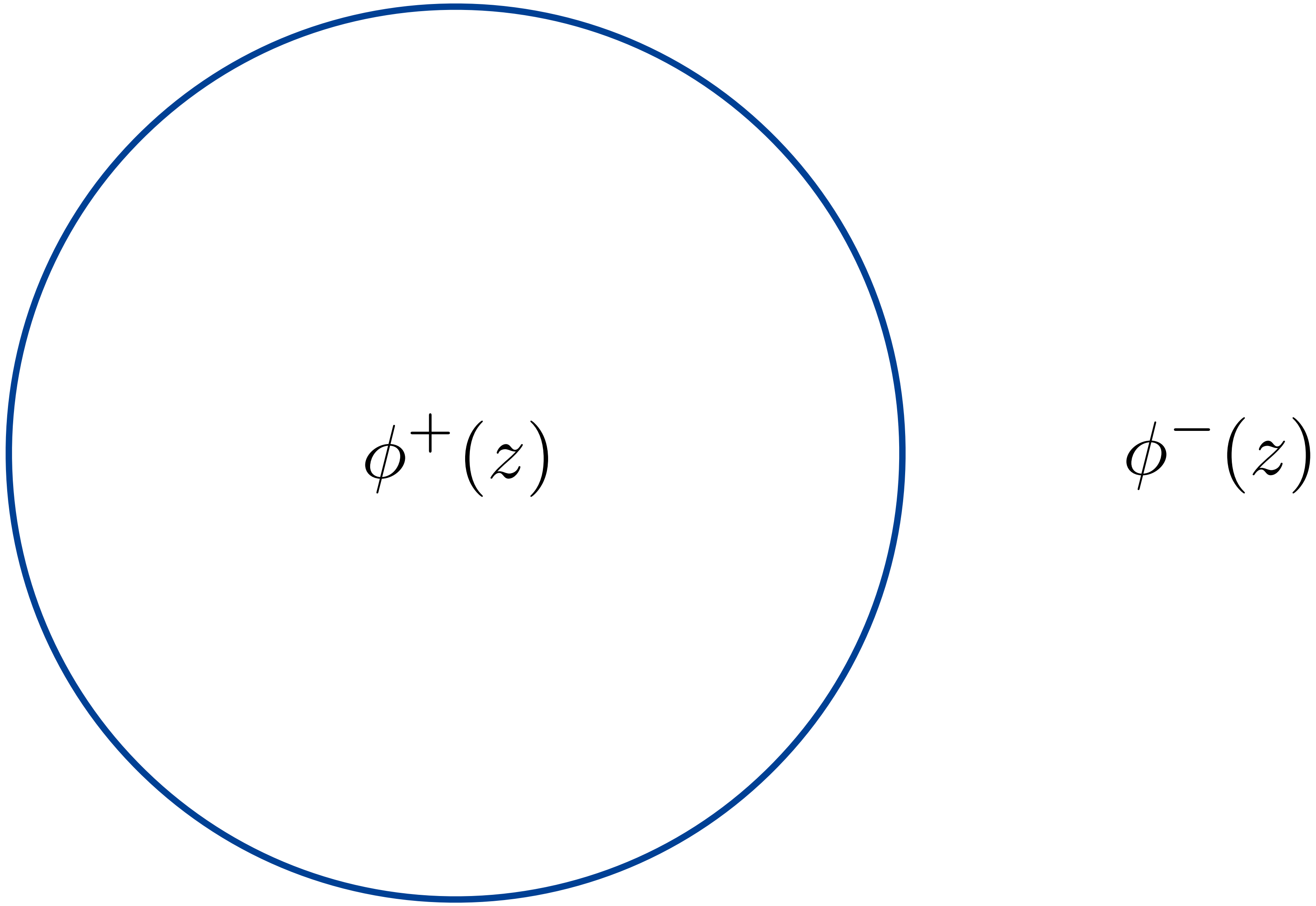
$$f(z) = \sum_{k=-\infty}^{\infty} \hat{f}_k z^k$$

- We will see that this can be used to decompose f into a function ϕ^+ analytic inside the unit circle and ϕ^- analytic outside the unit circle such that

$$\phi^+(z) + \phi^-(z) = f(z)$$

for z on the unit circle

$$\phi^+(z) + \phi^-(z) = f(z)$$



$$\phi^+(z) + \phi^-(z) = f(z)$$

$$\phi^+(z) = \sum_{k=0}^{\infty} \hat{f}_k z^k$$

$$\phi^-(z) = \sum_{k=-\infty}^{-1} \hat{f}_k z^k$$

$$\phi^+(z) + \phi^-(z) = f(z)$$

We now show

$$\phi^+(z) = \frac{1}{2\pi\mathrm{i}} \oint \frac{f(t)}{t-z}\,\mathrm{d}t \qquad \phi^-(z) = -\frac{1}{2\pi\mathrm{i}} \oint \frac{f(t)}{t-z}\,\mathrm{d}t$$

Theorem: For f sufficiently smooth,

$$\phi^+(z) = \frac{1}{2\pi \mathrm{i}} \oint \frac{f(t)}{t - z} \, \mathrm{d}t$$

and

$$\phi^-(z) = -\frac{1}{2\pi \mathrm{i}} \oint \frac{f(t)}{t - z} \, \mathrm{d}t$$

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Proof:

- For $|z| < 0$ we have

$$\frac{1}{2\pi \mathrm{i}} \oint \frac{f(t)}{t-z}\,\mathrm{d}t = \frac{1}{2\pi \mathrm{i}} \sum_{k=-\infty}^{\infty} \hat{f}_k \oint \frac{t^k}{t-z}\,\mathrm{d}t$$

Theorem: For f sufficiently smooth,

$$\phi^+(z) = \frac{1}{2\pi\mathrm{i}} \oint \frac{f(t)}{t-z}\,\mathrm{d}t$$

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Proof:

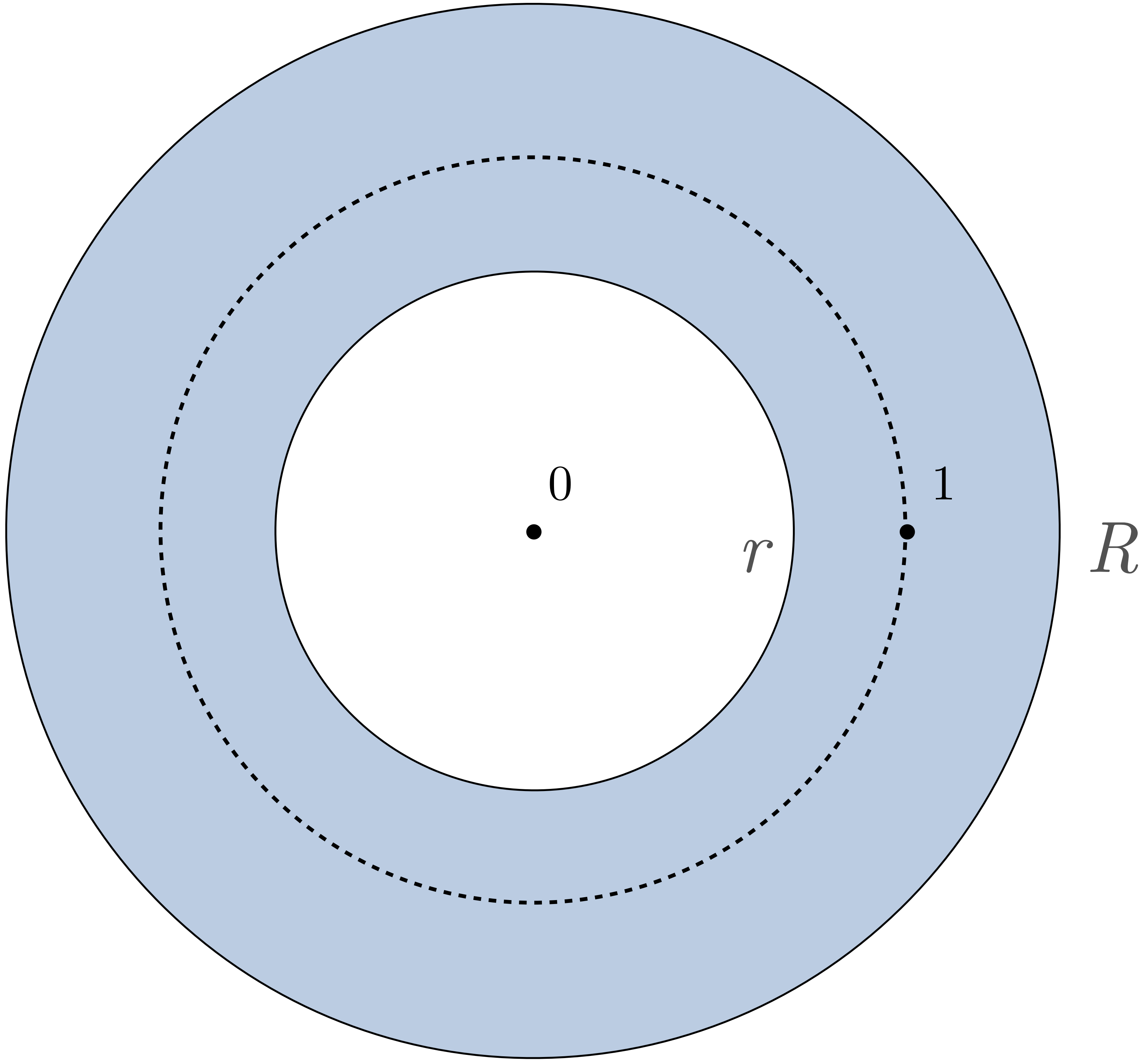
- For $|z| < 0$ we have

$$\begin{aligned}\frac{1}{2\pi\mathrm{i}} \oint \frac{f(t)}{t-z}\,\mathrm{d}t &= \frac{1}{2\pi\mathrm{i}} \sum_{k=-\infty}^{\infty} \hat{f}_k \oint \frac{t^k}{t-z}\,\mathrm{d}t \\ &= \frac{1}{2\pi\mathrm{i}} \sum_{k=0}^{\infty} \hat{f}_k \oint \frac{t^k}{t-z}\,\mathrm{d}t = \frac{1}{2\pi\mathrm{i}} \oint \frac{\phi^+(z)}{t-z}\,\mathrm{d}t \\ &= \phi^+(z)\end{aligned}$$

- Similar logic shows the formula for $\phi^-(z)$.

Analyticity in an ellipse

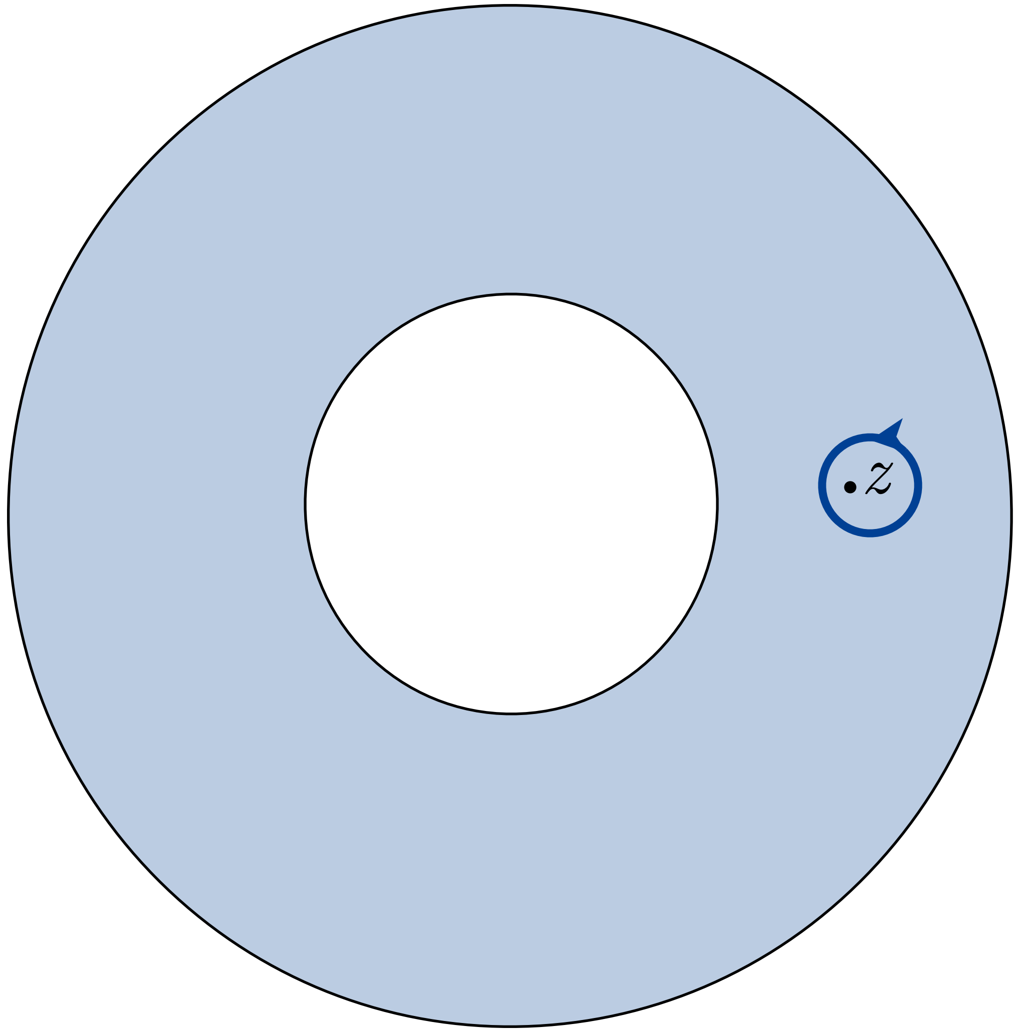
- Suppose f is analytic in an annulus A



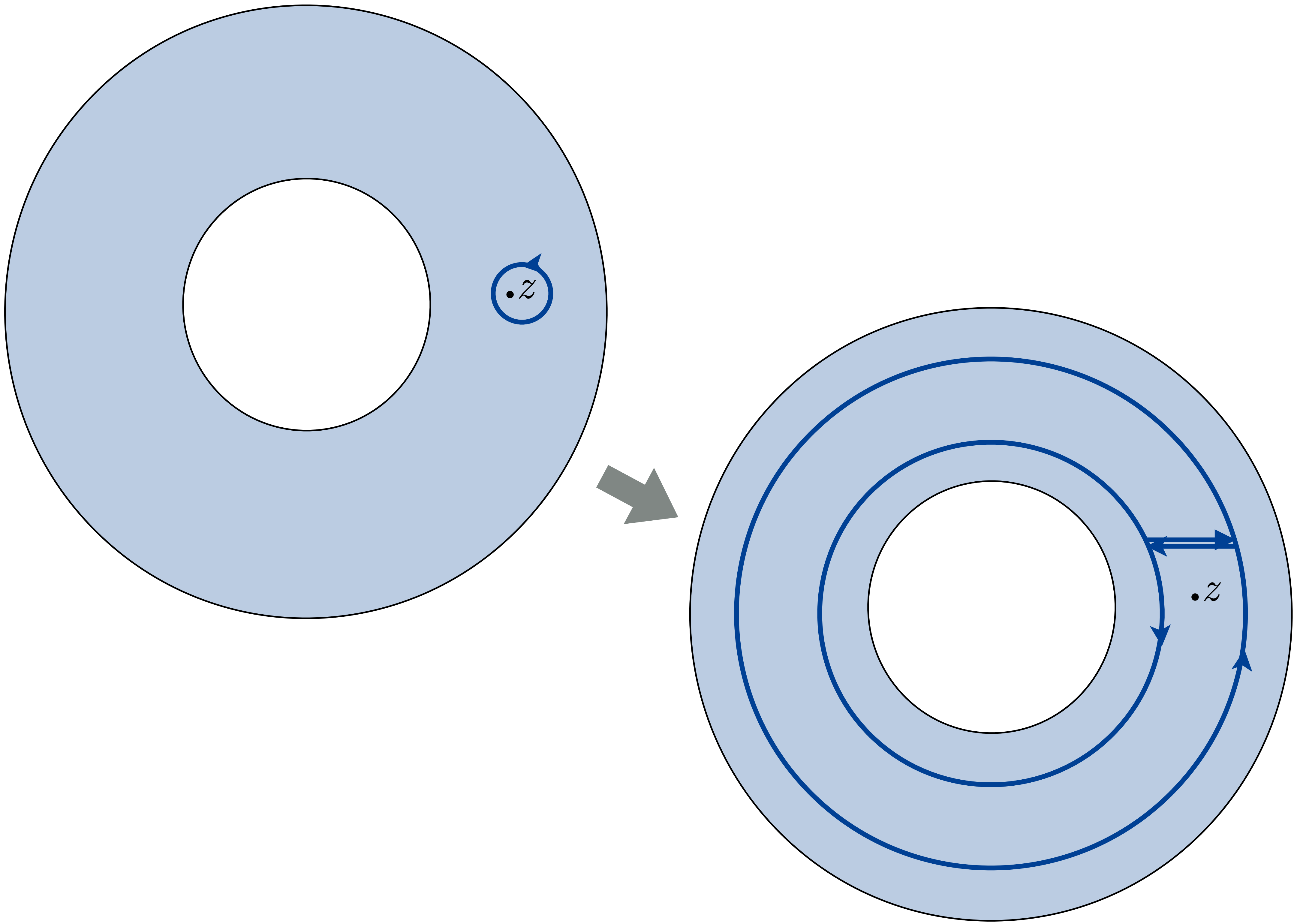
- We will show for all z in A

$$f(z) = \phi^+(z) + \phi^-(z)$$
$$= \sum_{k=-\infty}^{\infty} \hat{f}_k z^k$$

$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{B_r(z)} \frac{f(t)}{t - z} \, \mathrm{d}t$$

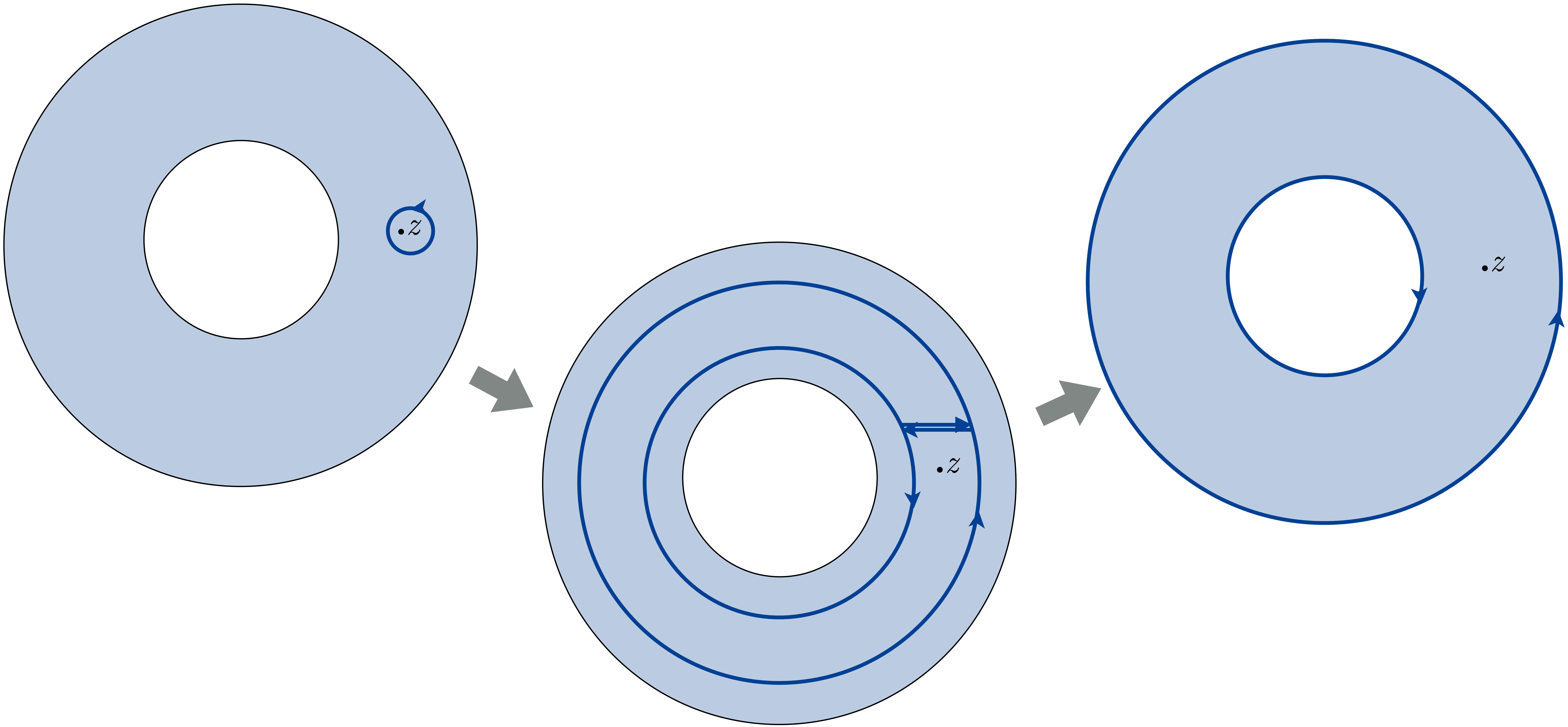


$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{B_r(z)} \frac{f(t)}{t-z}\,\mathrm{d}t$$



$$f(z) = \frac{1}{2\pi \mathrm{i}} \oint_{B_r(z)} \frac{f(t)}{t-z}\,\mathrm{d}t$$

$$= \frac{1}{2\pi \mathrm{i}} \oint_{R\mathbb{U}} \frac{f(t)}{t-z}\,\mathrm{d}t - \frac{1}{2\pi \mathrm{i}} \oint_{r\mathbb{U}} \frac{f(t)}{t-z}\,\mathrm{d}t$$



Theorem:

$$\left|\hat{f}_k\right| = \mathcal{O}\left(R^{-k}\right)$$

for $k > 0$ and

$$\left|\hat{f}_k\right| = \mathcal{O}\left(r^{-k}\right)$$

for $k < 0$

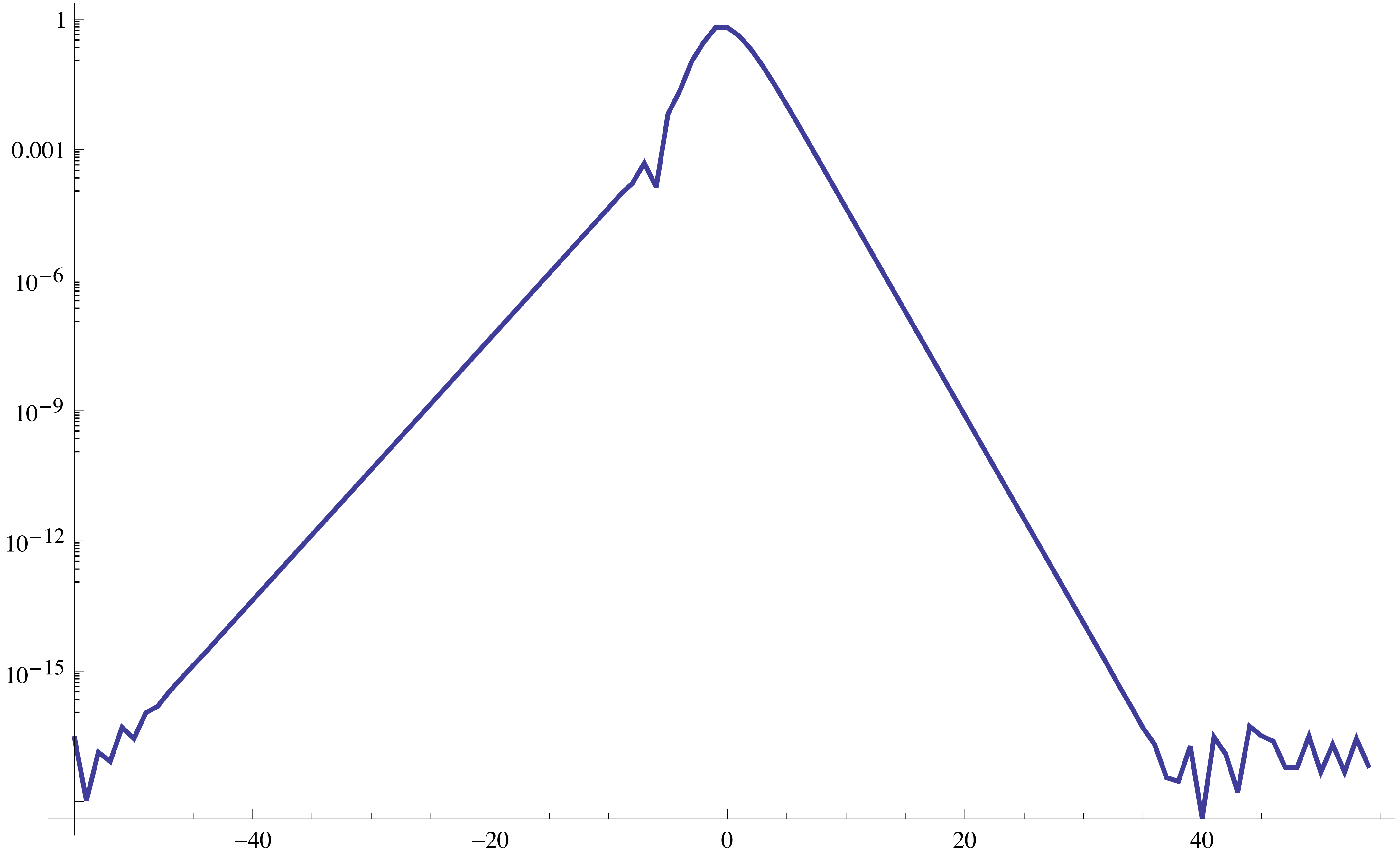
Proof: Let M denote the maximum of f. We have

$$\left|\hat{f}_k\right| = \left|\frac{1}{2\pi\mathrm{i}}\int_{\mathbb{U}} f(z)z^{-k-1}\,\mathrm{d}z\right| = \frac{1}{2\pi}\left|\int_{R\mathbb{U}} f(z)z^{-k-1}\,\mathrm{d}z\right| \leq MR^{-k}$$

Similarly,

$$\left|\hat{f}_k\right| = \frac{1}{2\pi}\left|\int_{r\mathbb{U}} f(z)z^{-k-1}\,\mathrm{d}z\right| \leq Mr^{-k}$$

Computed Laurent coefficients for $f(z) = \dfrac{e^{z+1/z}}{(z+1/2)(z-3)}$



- We analytically continue ϕ^+ by defining it as

$$\phi^+(z) = \frac{1}{2\pi\mathrm{i}} \oint_{R\mathbb{U}} \frac{f(t)}{t - z} \,\mathrm{d}t$$

- By deforming back to the unit circle, this equals $\phi^+(z)$ inside the unit circle

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- Similarly, we obtain

$$\phi^-(z) = -\frac{1}{2\pi\mathrm{i}} \oint_{r\mathbb{U}} \frac{f(t)}{t - z} \, \mathrm{d}t$$

- Thus for $z \in A$

$$\phi^+(z) + \phi^-(z) = f(z)$$

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- Thus for $z \in A$

$$\phi^+(z) + \phi^-(z) = f(z)$$

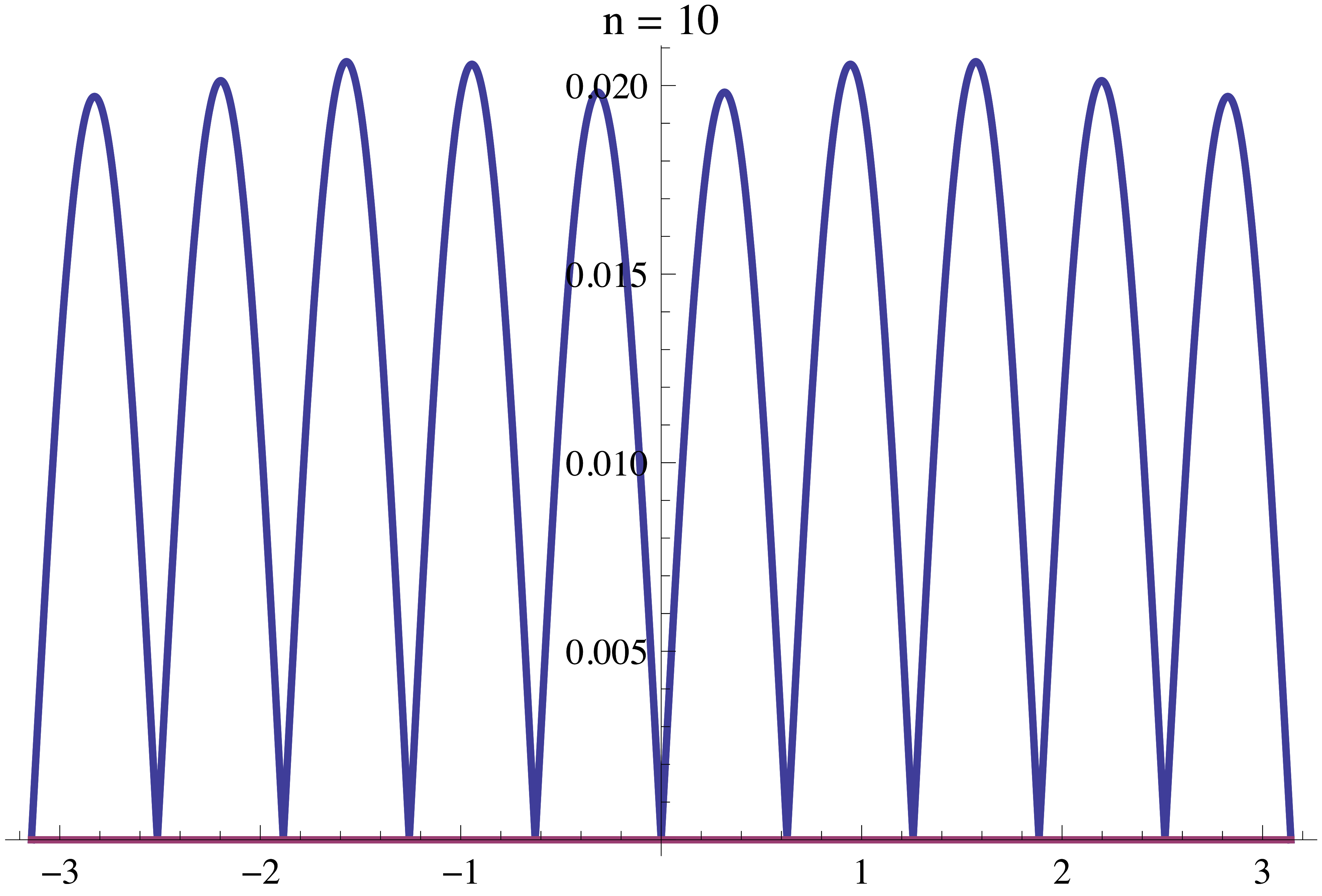
- Furthermore, the radius of convergence of ϕ^+ is R (Why?), hence for $z \in A$

$$\phi^+(z) = \sum_{k=0}^{\infty} \hat{f}_k z^k$$

- Similarly,

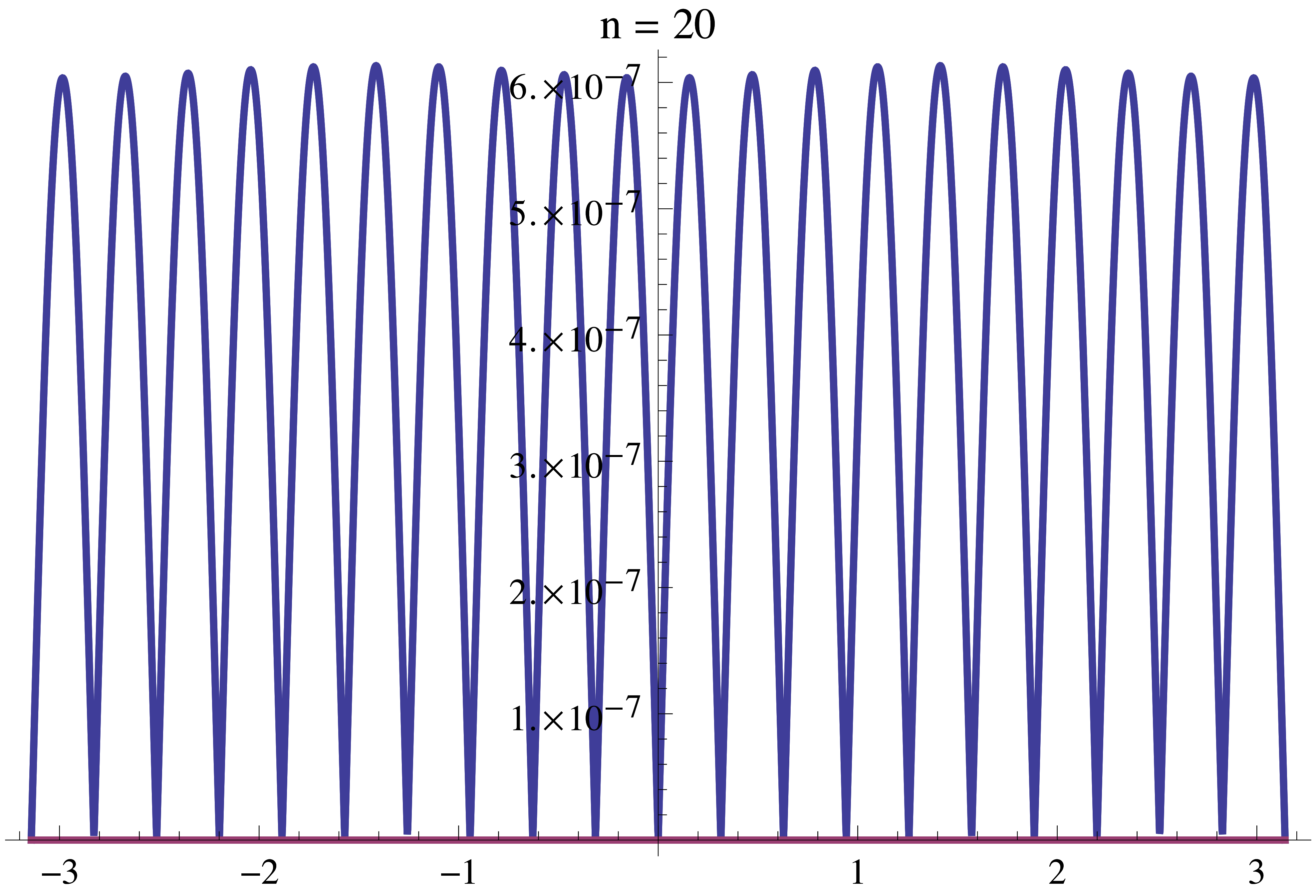
$$\phi^-(z) = \sum_{k=-\infty}^{-1} \hat{f}_k z^k$$

Error in approximating $f(z) = \mathrm{e}^{z+1/z}$ by approximate Fourier series



$z \in \mathbb{U}$

Error in approximating $f(z) = \mathrm{e}^{z+1/z}$ by approximate Fourier series



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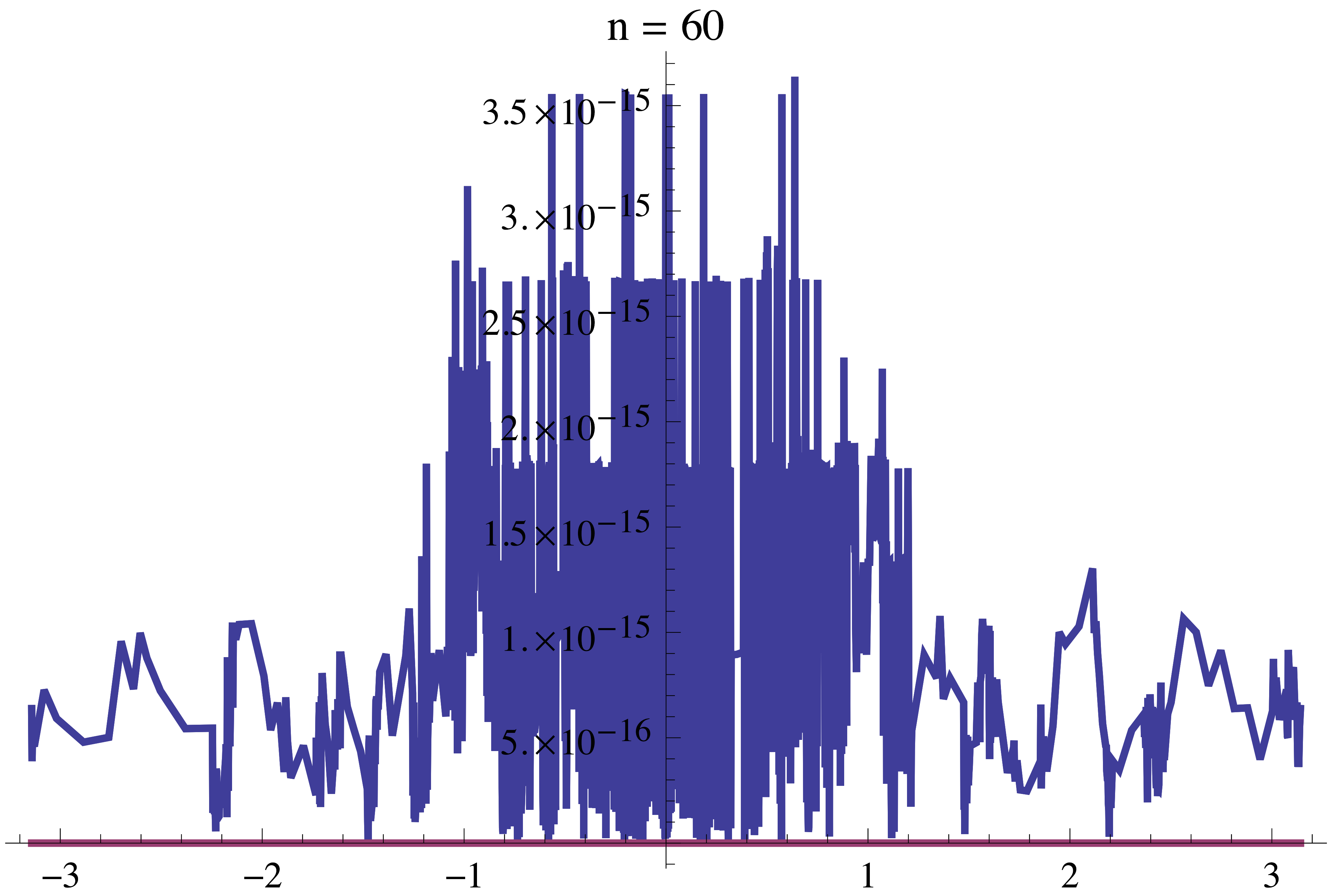

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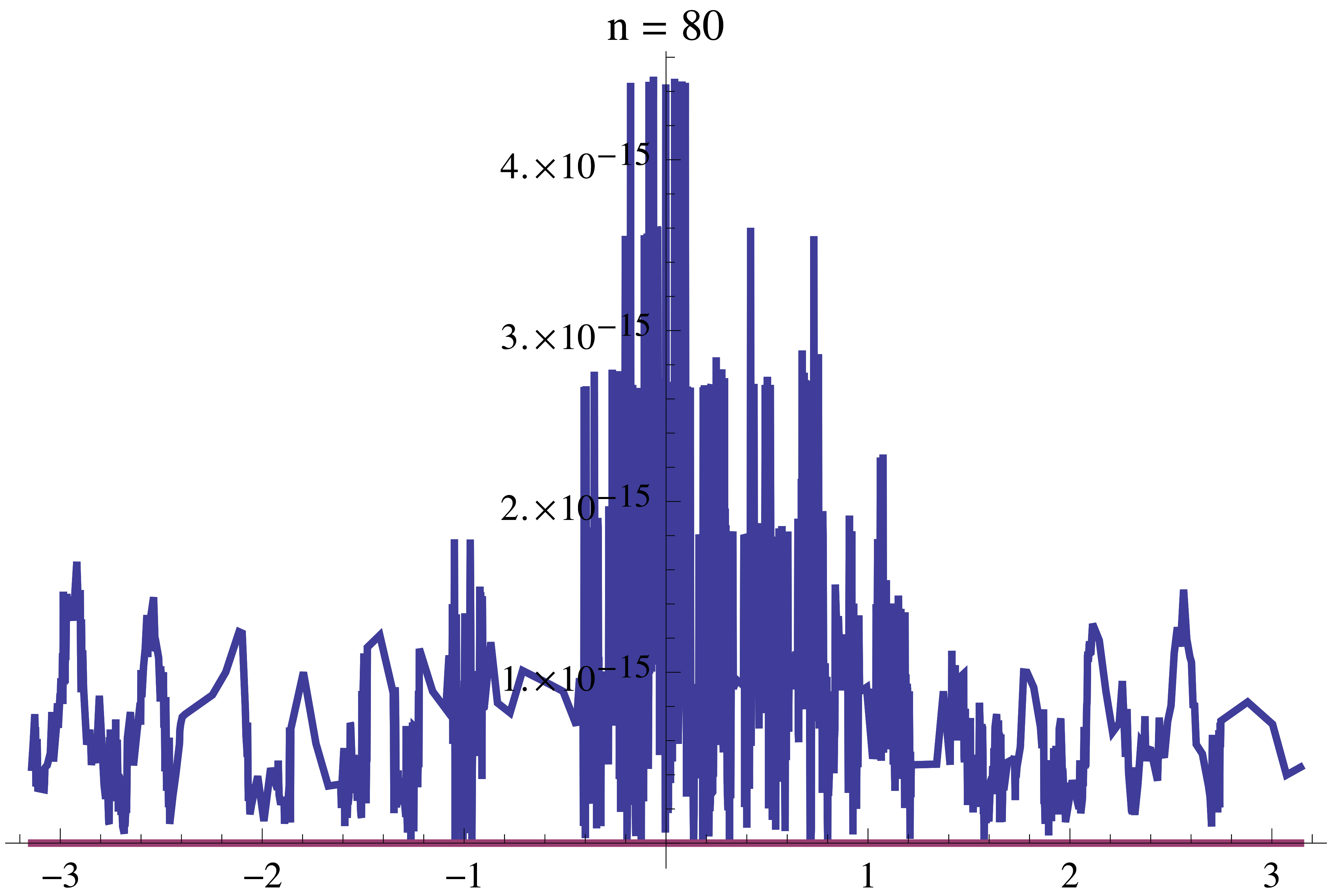

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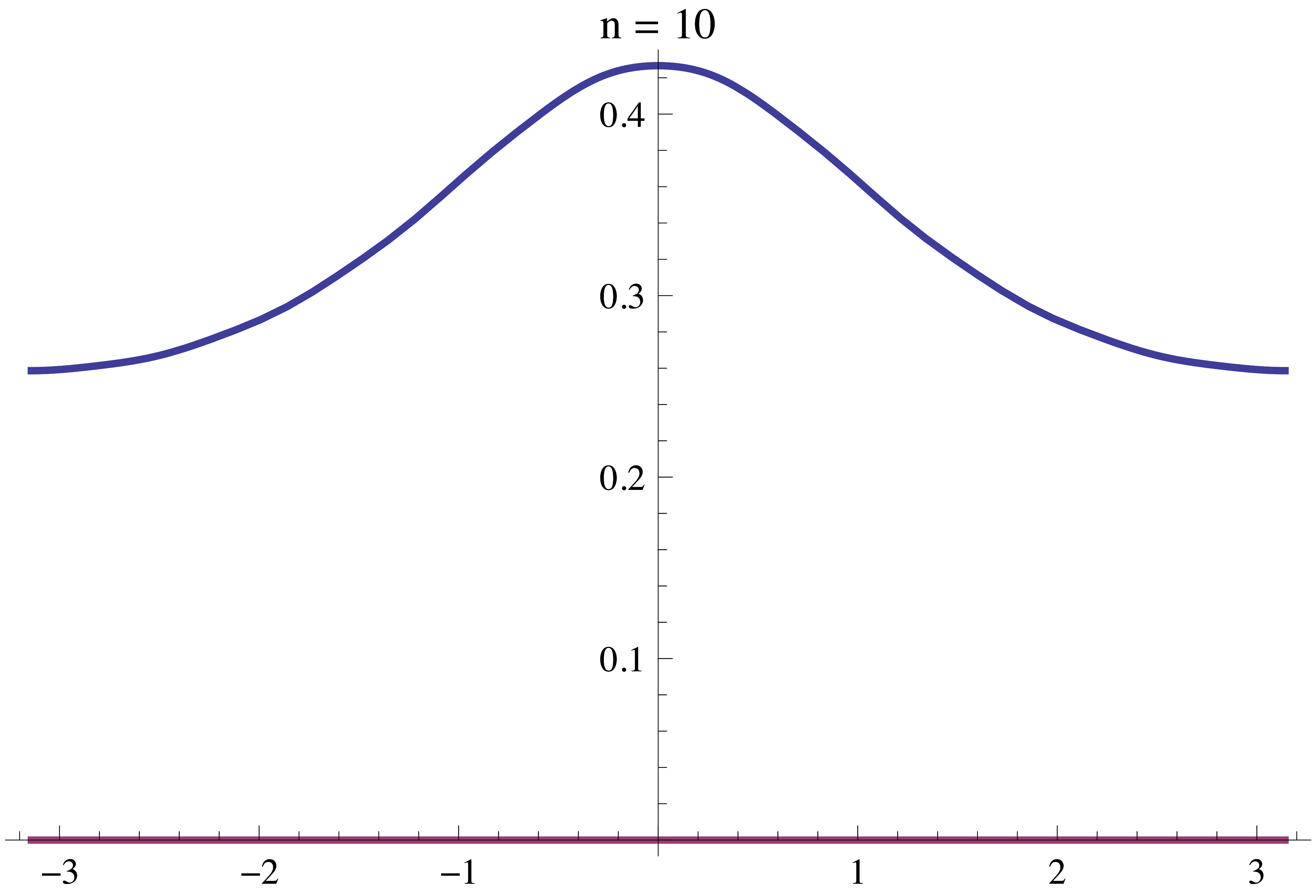
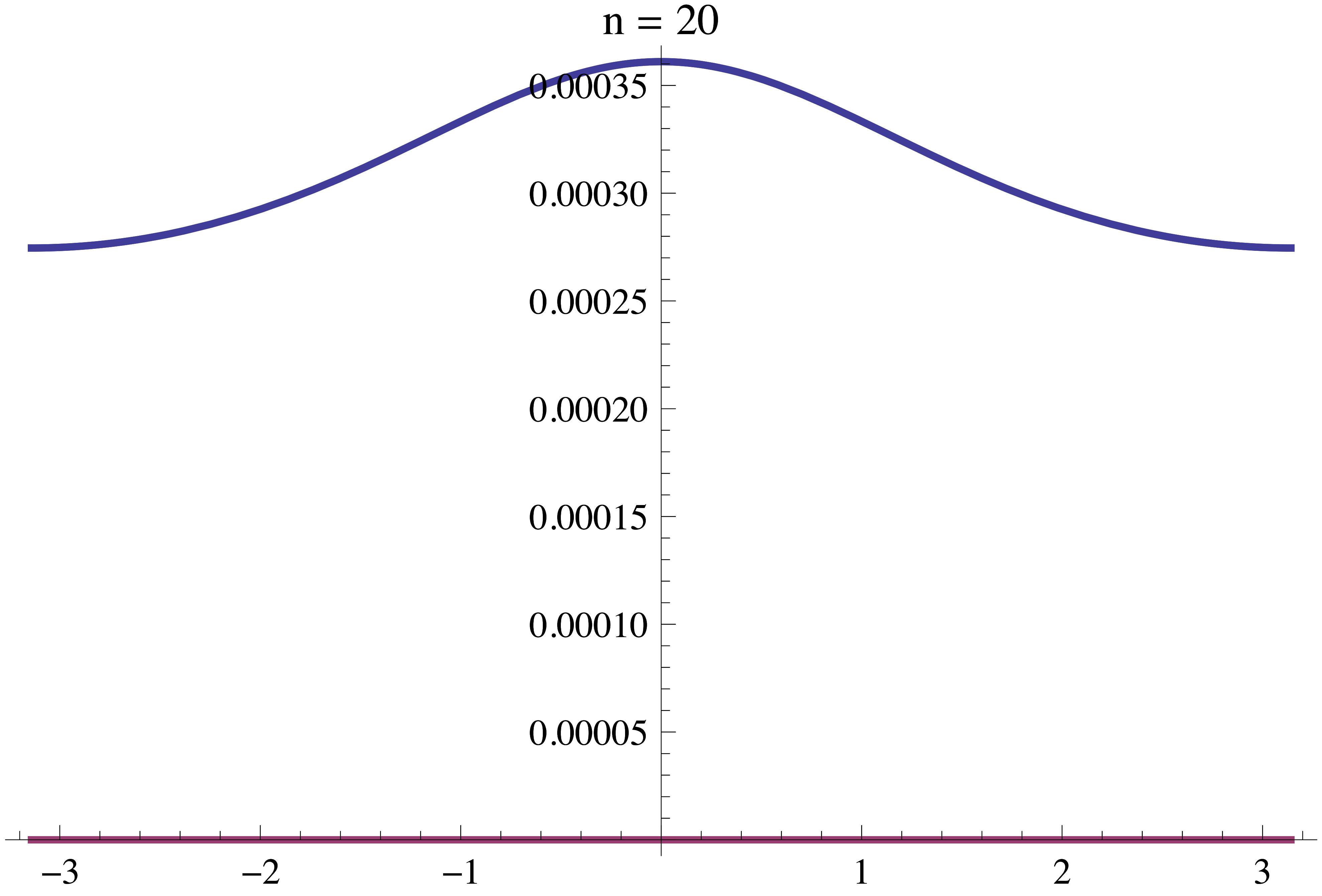


$z \in 2\mathbb{U}$

Error in approximating $f(z) = \mathrm{e}^{z+1/z}$ by approximate Fourier series



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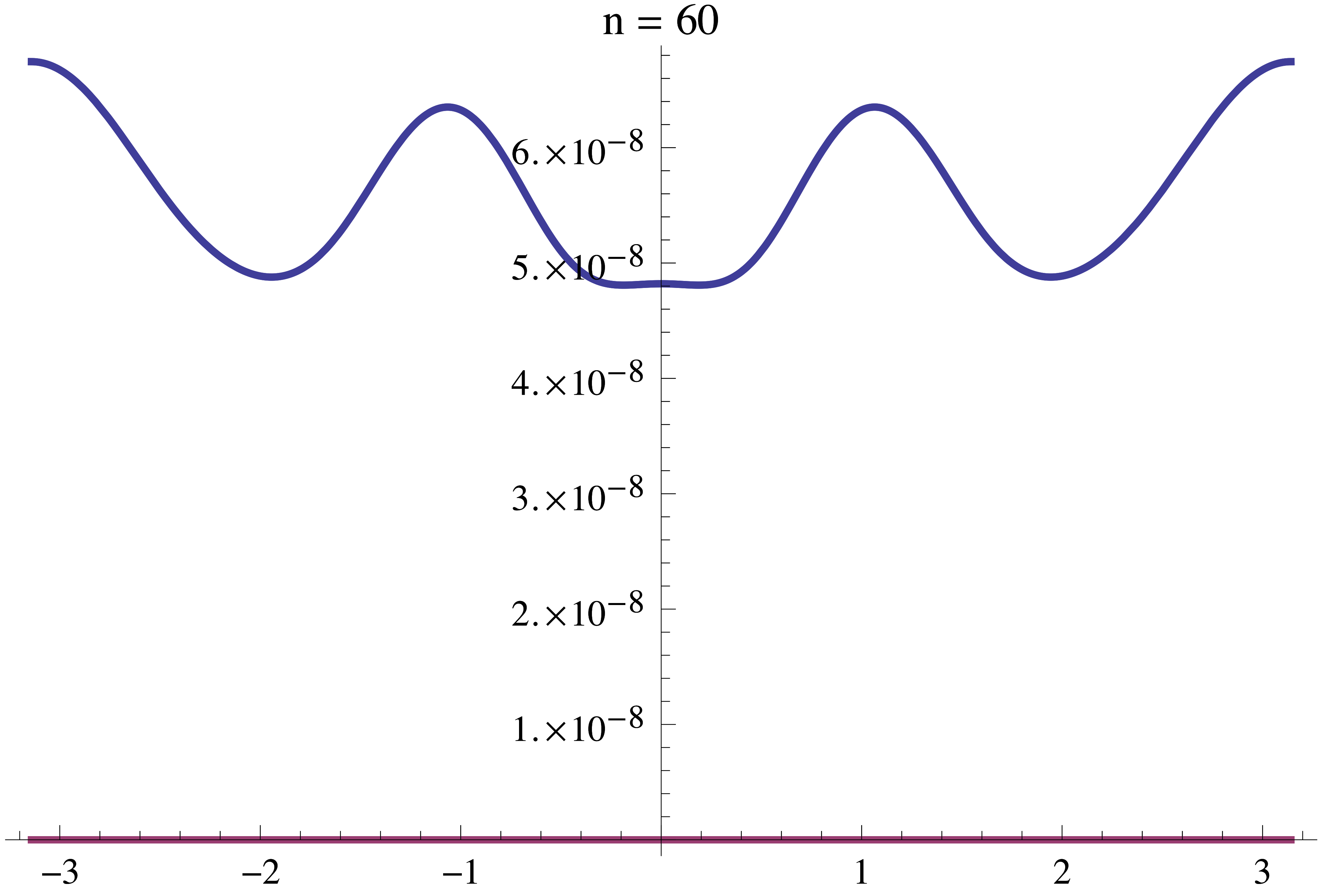

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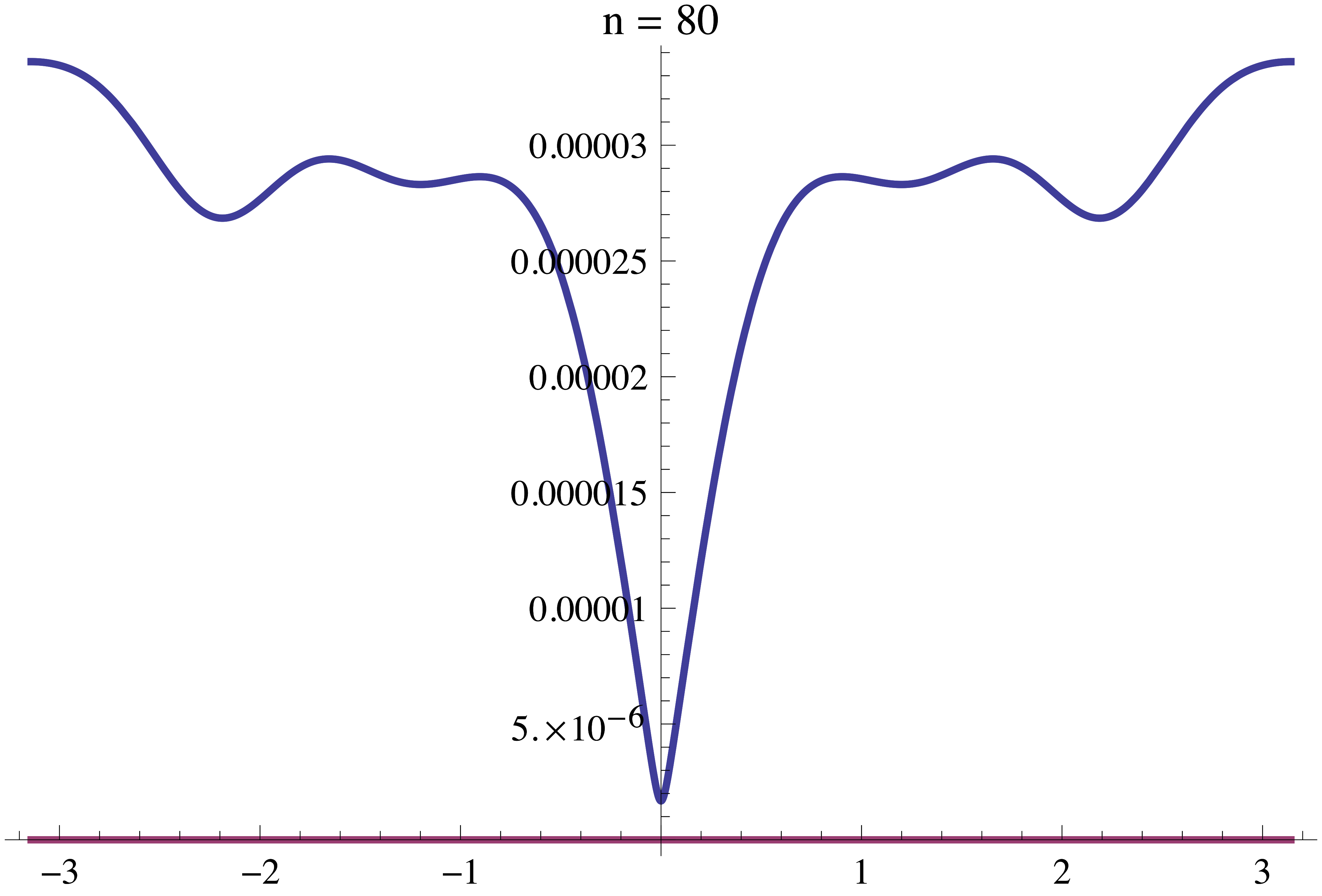

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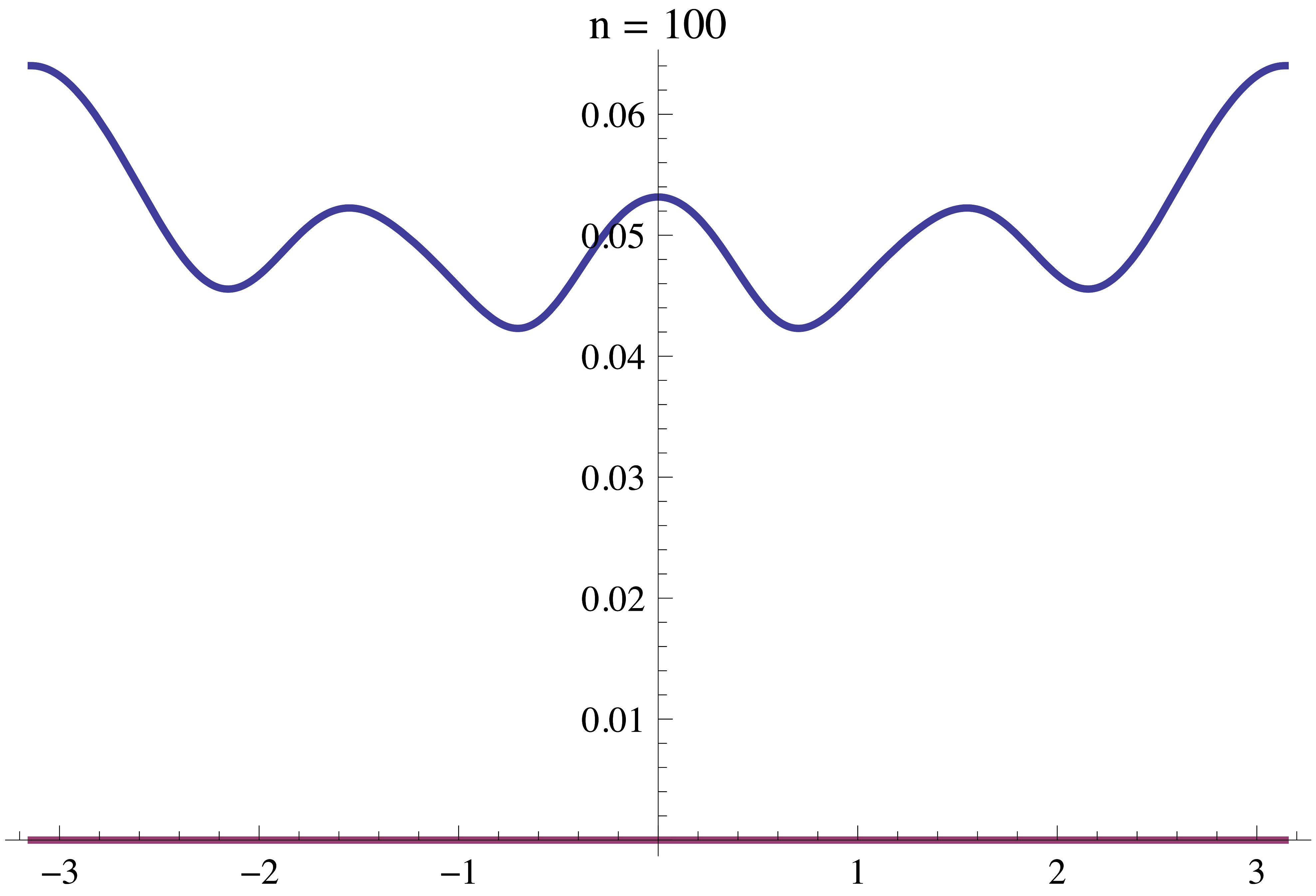

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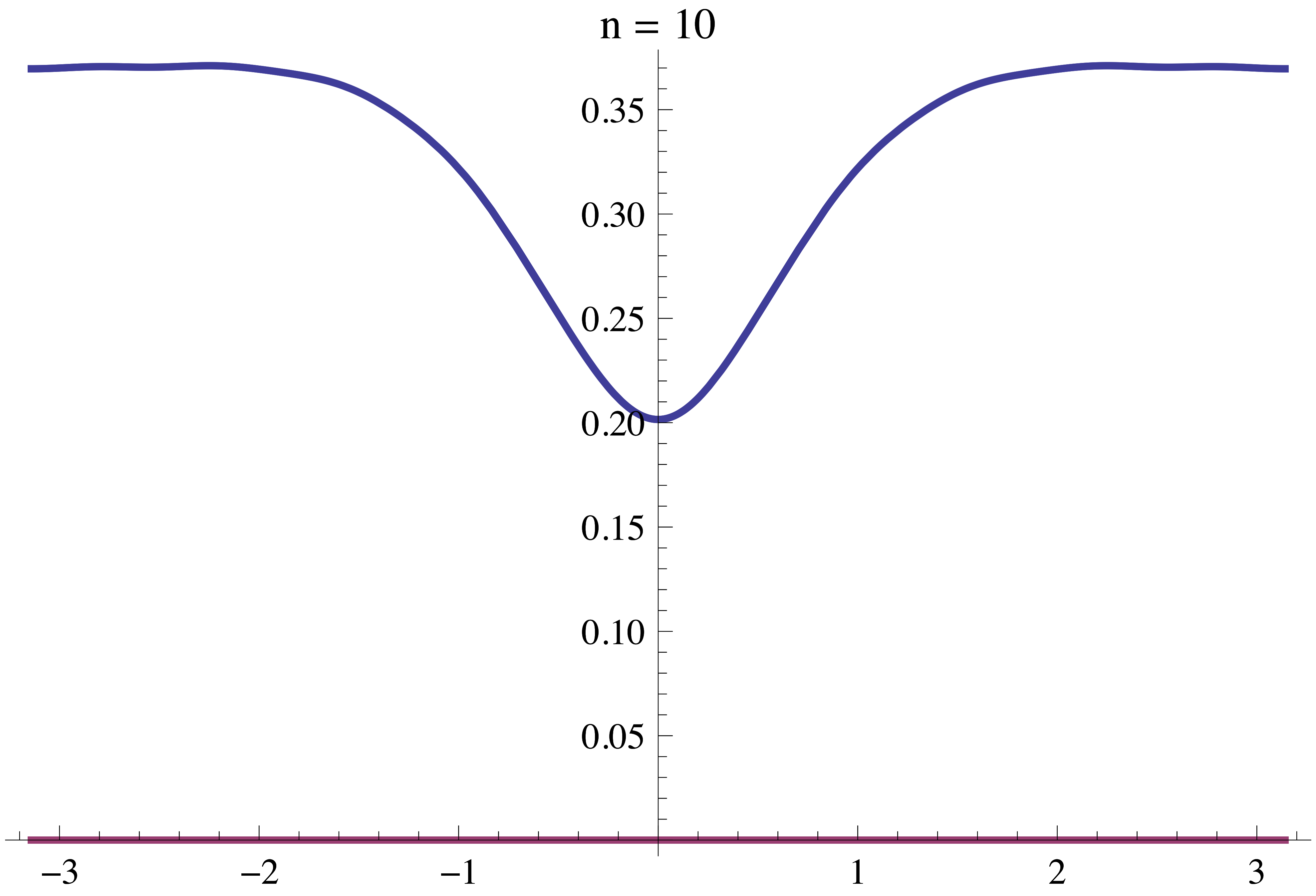
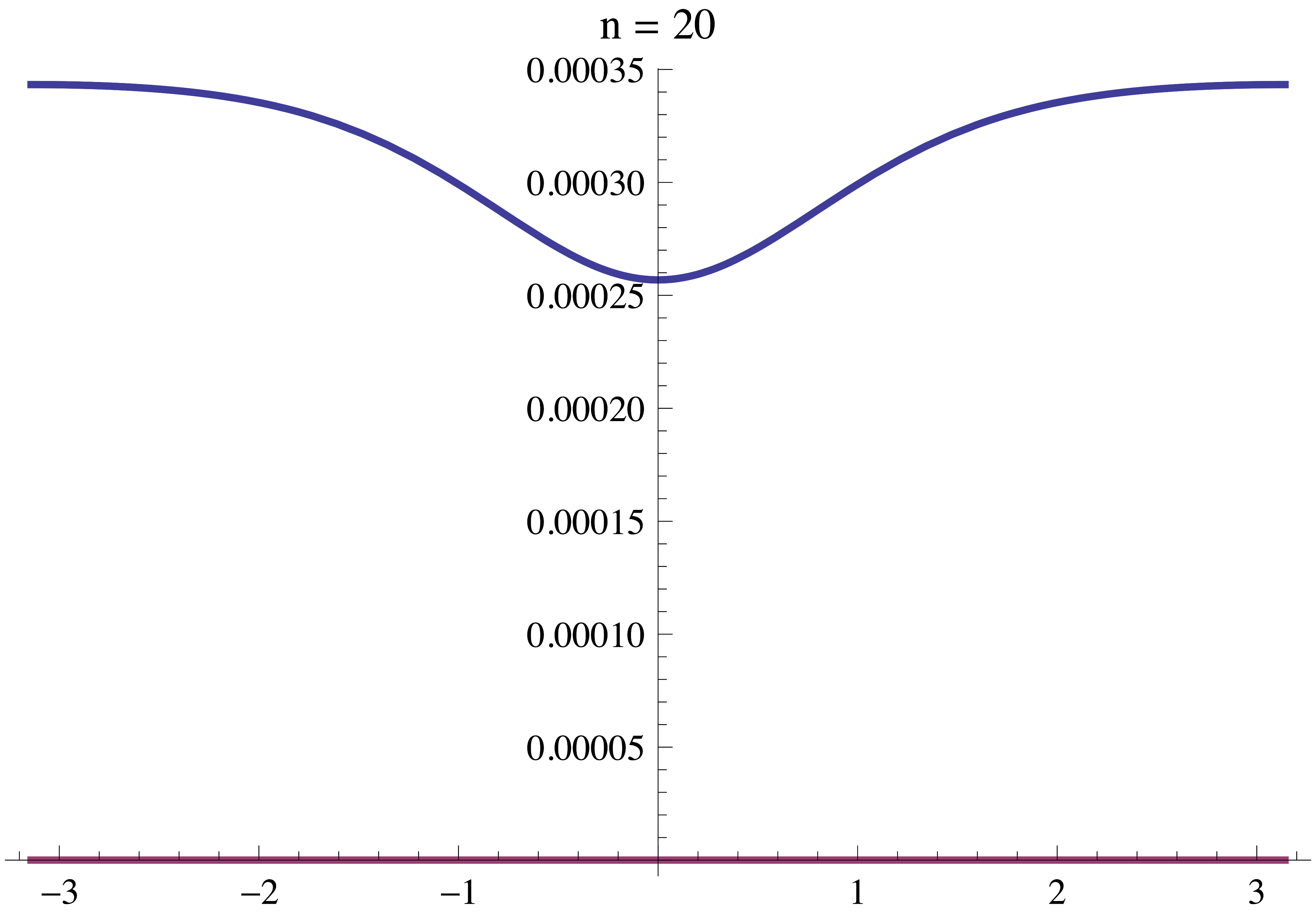


$$z \in \frac{\mathbb{U}}{2}$$

Error in approximating $f(z) = \mathrm{e}^{z+1/z}$ by approximate Fourier series



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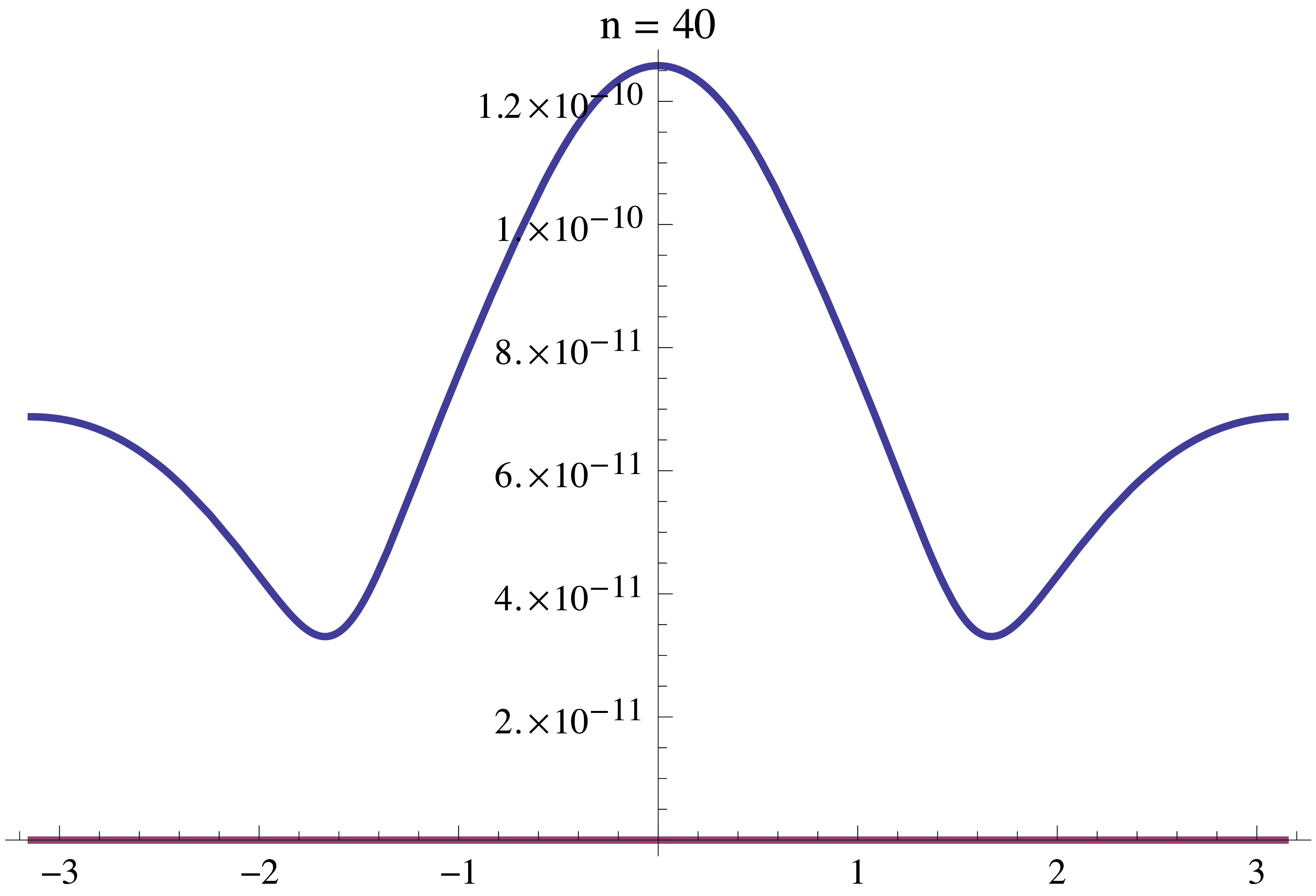
Error in approximating $f(z) = e^{z+1/z}$ by approximate Fourier series



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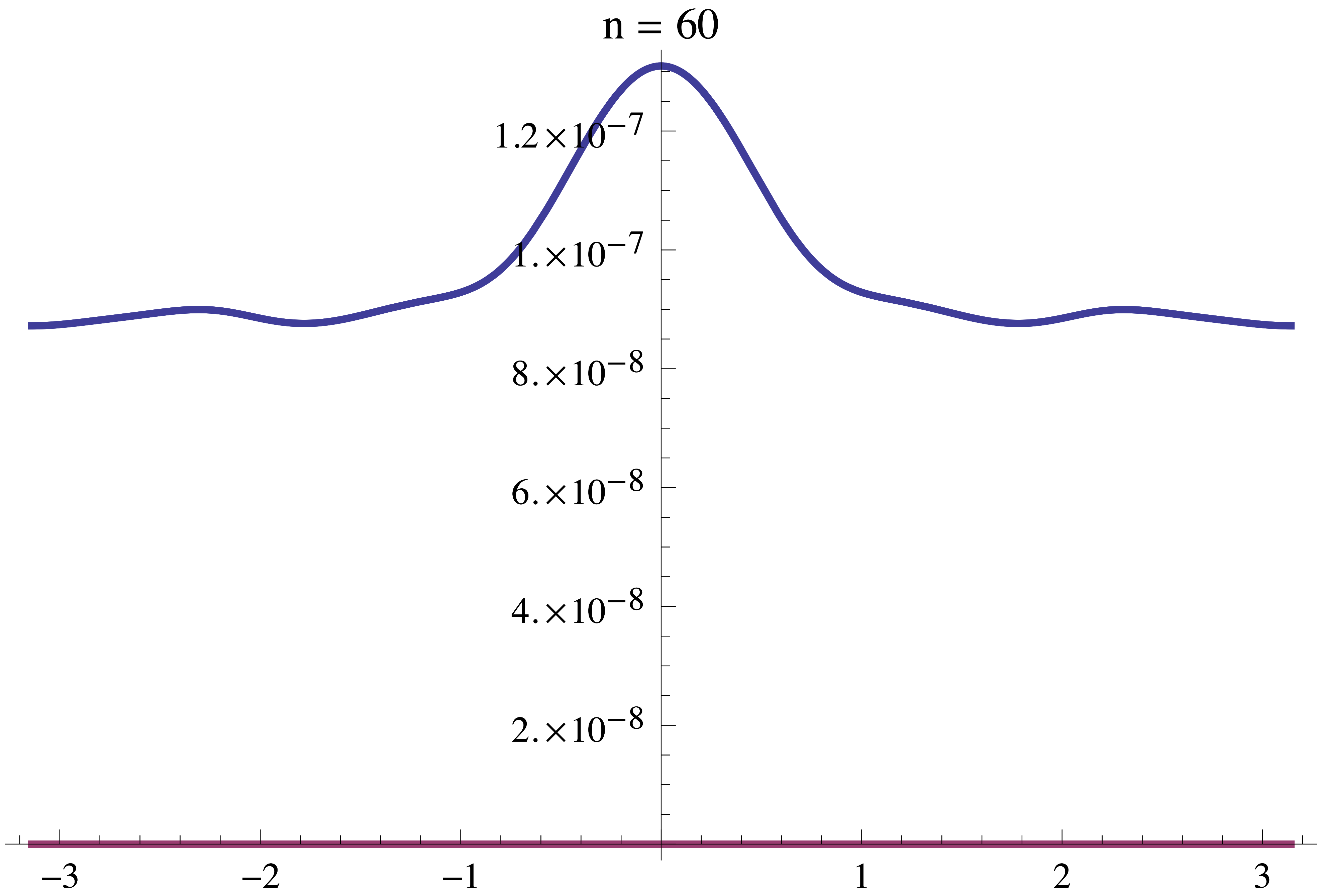

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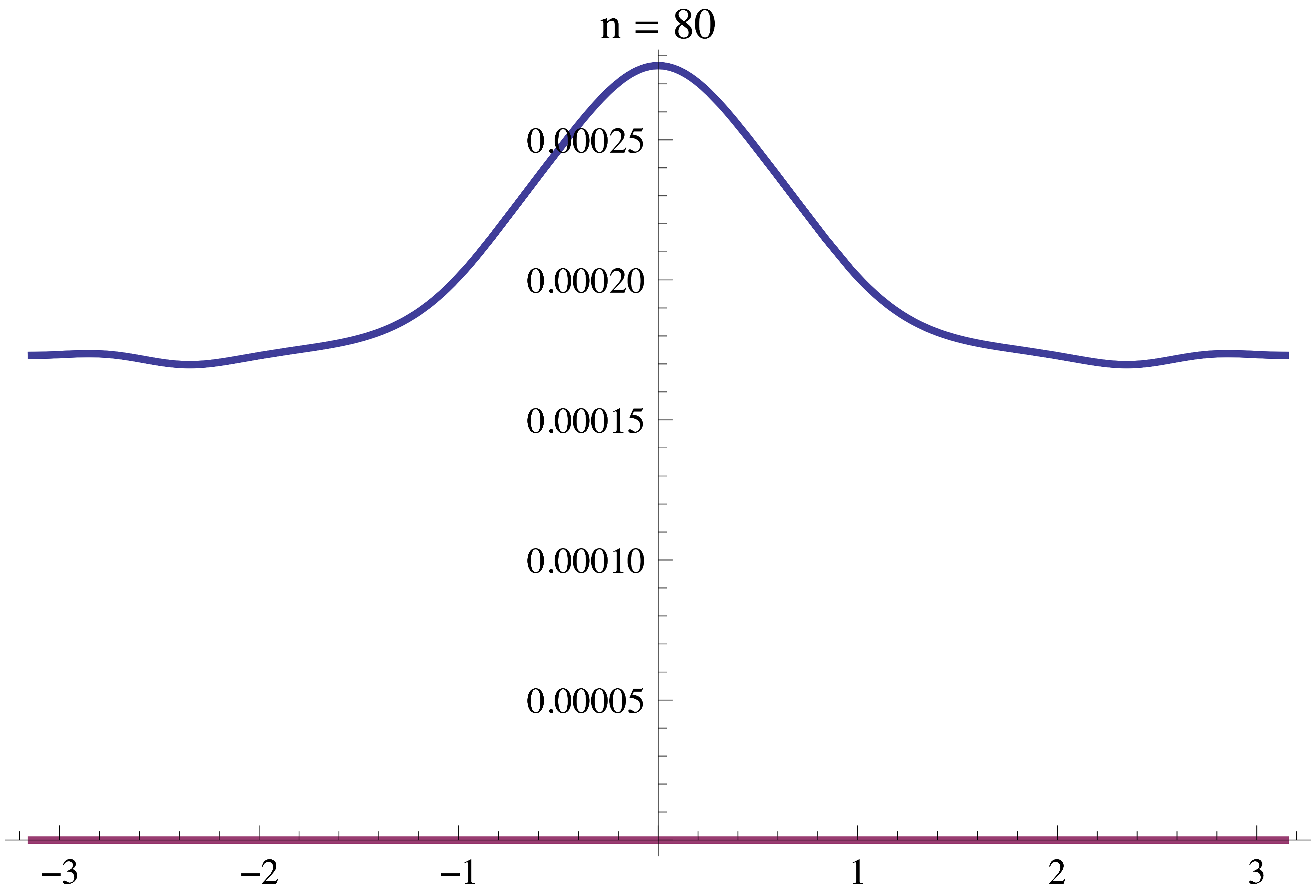

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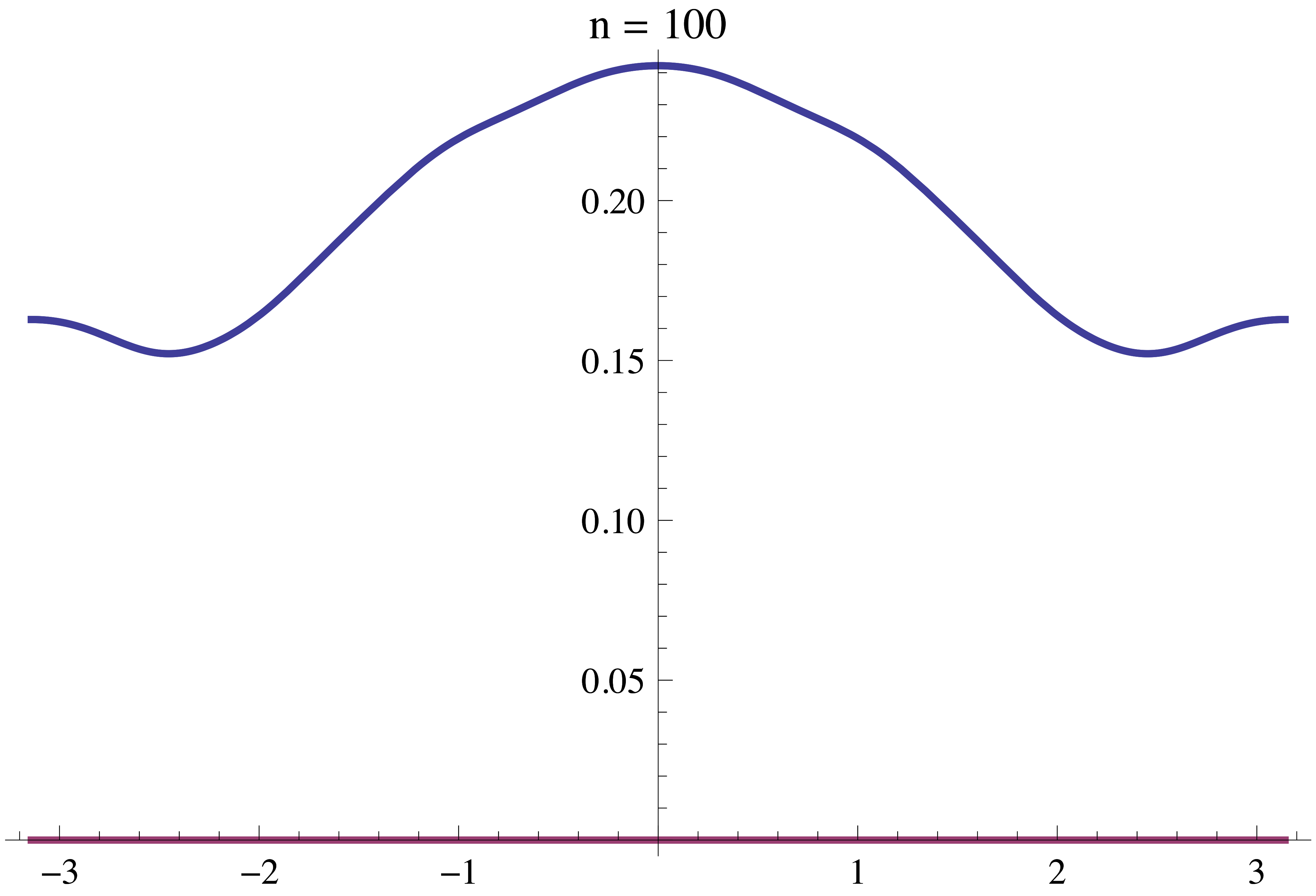

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