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# UNIT 1 SETS, RELATIONS AND FUNCTIONS

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## 1.0 INTRODUCTION

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In common parlance, we find people using the words given in the title of this unit. Do they have the same meaning in mathematics? You'll find this out by studying this unit. You will also see how basic the concept of 'set' and 'function' or to any area of mathematics and subjects depend on mathematics.

In this unit we will begin by introducing you to various kinds of sets. You will also study operations like, 'union' and 'intersection'. While doing so you will see in what way Venn diagrams are a useful tool for understanding and working with sets.

Next we will discuss what a relation is, and expose you to some important types of relations. You will come across while studying banking, engineering, information technology and computer science, of course mathematics. As you will see in your study of computer science, an extensive use of functions is made in problem-solving.

Finally, we lead you detailed discussion of functions. Over here we particularly focus on various points of functions and fundamental operations on functions.

## 1.1 OBJECTIVES

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After studying this unit, you should be able to:

- explain what a set, a relation or a function is
  - give examples and non-examples of sets, relations and functions
  - perform different operations on sets
  - establish relationships between operations on sets and those on statements in logic
  - use Venn diagrams
  - explain the difference between a relation and a function.
  - describe different types of relations and functions.
  - define and perform the four basic operations on functions
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## 1.2 INTRODUCING SETS

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In our daily life we encounter collections, like the collection of coins of various countries, a collection of good students in a class, a collection of faculty members of IGNOU, etc. In the first of these examples, it is easy for anybody anywhere to tell

whether an object belongs to this collection or not. If we take the collection of coins of a country, then a coin will be in the collection if it is a coin of that country, not otherwise. The criterion for being a member of the collection is objective and clear. However, if we take the collection of all good students, it is very difficult to say whether a person belongs to this collection or not because the characteristic *good* is not very clearly defined. In this case the collection is not ‘well-defined’, while the previous collection is ‘well-defined’. Similarly, the collection of all the IGNOU students is well-defined.

**Definition:** A **set** is a well-defined collection. The objects belonging to a set are called **elements** or **members** of that set.

We write the elements of a set within curly brackets. For instance, consider the set A of stationary items used by Nazia. We write this as

$$A = \{\text{pen, pencil, eraser, sharpener, paper}\}$$

Another example is the set

$$B = \{\text{Lucknow, Patna, Bhopal, Itanagar, Shillong}\}$$

of the capitals of 5 states of India.

Note that A and B are well-defined collections. However, the collection of short people is not well-defined, and therefore, it is not a set.

Also note that **the elements of a set don’t have to appear ‘similar’**. For example, **{pen, Lucknow, 4} is a set** consisting of 3 clearly defined elements.

As you have seen, we usually, denote sets by capital letters of the English alphabet. We usually denote the elements by small letters a, b, x, y .... If x is an element of a set A, we write this as  $x \in A$  (read as ‘x belongs to A’). If x is not an element of A, we write this as  $x \notin A$  (read as ‘x does not belong to A’).

There are three ways of representing sets: ‘Set-builder form’, ‘Tabular form’ and the pictorial representation through Venn diagrams.

In the ‘**Set-builder form**’, or ‘**property method**’ of representation of sets, we write between brackets { } a variable x, which stands for each of the elements of the set which have the properties p(x), and separate x and p(x) by a symbol ‘:’ or ‘|’ (read as ‘such that’). So the set looks like  $\{x: p(x)\}$  or  $\{x | p(x)\}$ .

For instance, the set  $\{x | x \text{ is a white flower}\}$  is the set of all white flowers, or  $\{x: x \text{ is a natural number and } 2 < x < 11\}$  is the set of natural numbers lying between 2 and 11.

In ‘**Tabular form**’, or the ‘**listing method**’, the elements of a set are listed one by one within the brackets { }, each separated from the other by a comma, as in the examples A and B given above.

The accepted convention for writing a set by the listing method is that elements will not be repeated. For example, in the set  $A = \{4, 2, 8, 2, 6\}$ , 2 is repeated, which is not necessary. So we will write  $A = \{4, 2, 8, 6\}$ .

We shall introduce you to Venn diagrams a little later. For now, let us consider a few more sets.

**Definition:** A set with no element is called the **empty (or null, or void) set**, and is denoted by  $\phi$  or  $\{\}$ .

For example  $A = \{x: x \text{ is an integer between 13 and 17 which is divisible by 6}\}$ , has no element, i.e., A is the **empty set**.

**Definition:** A set having a finite number of elements is called a **finite set**.

For example,  $\{1,2,4,6\}$  is a finite set because it has four elements,  $\phi$ , the null set, is also a finite set because it has zero number of elements; the set of stars in the sky is also a finite set.

**Definition:** A set having infinitely many elements is called an **infinite set**.

For example, the set  $\mathbf{N}$  of natural numbers is infinite. Similarly,  $\mathbf{Q}$ ,  $\mathbf{Z}$ ,  $\mathbf{R}$  and  $\mathbf{C}$ , the set of rational numbers and complex numbers, respectively, are infinite set.

$B$  = The set of all strengthness in a given plane.

Now try the following exercises.

- 
- E1) How would you represent the set of all students who have offered the IGNOU course?
- E2) Explain, with reason, whether or not
- i) the collection of all good teachers is a set
  - ii) the set of points on a line is finite.
- E3) Represent the set of all integers by the listing method.
- 

When we deal with several sets, we need to understand the nature of the elements of those sets, whether the elements of two given sets have some elements in common or not, and so on. These questions involve concepts, which we now define.

**Definition:** A set  $A$  is said to be a **subset** of a set  $B$  if each element of  $A$  is also an element of  $B$ . In this case  $B$  is called a **superset** of  $A$ . If  $A$  is a subset of  $B$ , we represent this by  $A \subseteq B$ .

As a statement in logic we represent this situation as,

$$A \subseteq B \Leftrightarrow [x \in A \Rightarrow x \in B]$$

' $B$  contains  $A$ ' or ' $B$  is a superset of  $A$ ' is represented by  $B \supseteq A$ .

If  $A$  is not a subset of  $B$ , we represent this by  $A \not\subseteq B$ .

For example, if  $A = \{4,5,6\}$  and  $B = \{4,5,7,8,6\}$ , then  $A \subseteq B$ . But if  $C = \{3,4\}$  then  $C \not\subseteq B$ .

**Remark:** If  $A \subseteq B$  and  $B \subseteq C$ , then  $A \subseteq C$ .

**Definition:** Two sets  $A$  and  $B$  are **equal** if every element of  $A$  belongs to  $B$  and every element of  $B$  belongs to  $A$ . We represent this by  $A = B$ .

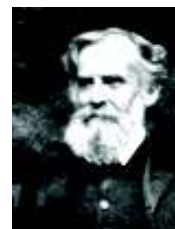
For example, if  $A = \{1,2,3\}$ ,  $B = \{2,3,1\}$ , then  $A \subseteq B$  and  $B \subseteq A$ , so that  $A = B$ .

**Definition:** A set  $A$  is said to be a **proper subset** of a set  $B$  if  $A$  is a subset of  $B$  and  $A$  and  $B$  are not equal. We represent this by  $A \subset B$ .

For example, if  $A = \{4,5,6\}$  and  $B = \{4,5,7,8,6\}$ , then  $A \subset B$ ; and if  $A = \{\text{Java, C, C++}, \text{Cobol}\}$  and  $B = \{\text{Java, C++}\}$ , then  $A \supset B$ .

**Note:** A set can have many subsets and many supersets. For example  $A = \{1,2,3,4,5\}$ ,  $B = \{2,3,4,5,6,7\}$ , and  $C = \{2,3\}$ , then for  $C$ ,  $A$  and  $B$  can be used as supersets. Similarly, if  $X = \{\text{Ram, Rani, Sita, Gita}\}$ ,  $Y = \{\text{Rani}\}$ , and  $Z = \{\text{Sita}\}$ , then  $Y$  and  $Z$  both are subsets of  $X$ .

**Definition:** The **power set** of a set  $A$  is the set of all the subsets of  $A$ , and is denoted by  $P(A)$ .



(1834 -1923)  
Fig 1: John Venn

Mathematically,  $P(A) = \{x : x \subseteq A\}$ .

Note that  $\phi \in P(A)$  and  $A \in P(A)$  for all sets  $A$ .

For example, if  $A = \{1\}$ , then  $P(A) = \{\phi, \{1\}\}$  and

if  $A = \{1, 2\}$ , then  $P(A) = \{\phi, \{1\}, \{2\}, \{1, 2\}\}$

Similarly, if  $A = \{1, 2, 3\}$ , then  $P(A) = \{\phi, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$ .

**Definition:** Any set which is a **superset** of all the sets **under consideration** is known as the **universal set**. This is usually denoted by  $\Omega$ ,  $S$  or  $U$ .

For example, if  $A = \{1, 2, 3\}$ ,  $B = \{3, 4, 6, 9\}$  and  $C = \{0, 1\}$ , then we can take  $U = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$  or  $U = \mathbb{N}$ , or  $U = \mathbb{Z}$  as the universal set.

Note that the universal set can be chosen arbitrarily for a given problem. But once chosen, it is fixed for the discussion of that problem.

**Theorem 1:** If  $A$  is a set with  $n$  elements, then  $|P(A)| = 2^n$ .

**Proof:** We shall prove this by mathematical induction.

For this, we first check if it is true for  $n = 1$ . Then assuming that it is true for  $n = m$ , we prove it for the case  $n = m + 1$ . It will, then, follow that the result will be true  $\forall n \in \mathbb{N}$ .

**Step I:** If  $A = 1$ , then  $P(A) = 2 = 2^1$ .

**Step II:** Assume that the theorem holds for all sets  $A$  of cardinality  $k$ , i.e. if  $|A| = k$ , then  $A$  has  $2^k$  subsets.

**Step III:** Now consider any set  $A = \{x_1, x_2, x_3, \dots, x_k, x_{k+1}\}$ , with  $k+1$  elements. Consider its subset  $B = \{x_1, x_2, x_3, \dots, x_k\}$ . Now  $B$  has  $2^k$  subsets, each being a subset of  $A$ . Now, take any such subset  $\{x_{i_1}, x_{i_2}, \dots, x_{i_r}\}$  of  $B$ . Then  $\{x_{i_1}, x_{i_2}, \dots, x_{i_r}, x_{k+1}\}$  is a subset of  $A$  that is not a subset of  $B$ . So, for each of the  $2^k$  subsets of  $B$ , we attach  $x_{k+1}$  to it to get  $2^k$  more subsets of  $A$ .

You can see that this covers all the subsets of  $A$ .

So the number of subsets of  $A = 2^k + 2^k = 2 \cdot 2^k = 2^{k+1}$ .  
Hence the theorem.

Now try these exercises.

- 
- E4) Give two proper subsets and two supersets of the set of vowels of the English alphabet.
- E5) Find the power set of the set  $A = \{a, e, i, o, u\}$ .
- E6) For which set  $A$ , is  $P(A) = 1$ ?
- E7) If  $A \subseteq B$ , is  $P(A) \subseteq P(B)$ ? Why?
- E8)  $P(A) = P(B) \Rightarrow A = B$ . True or false? Why?
- 

Let us conclude this section with the **pictorial** representation of sets. You know that the pictorial representation of any object helps in understanding the object. This is why a pictorial representation of sets, known as a **Venn diagram**, helps in understanding and dealing with sets.

The English priest and logician John Venn invented the Venn diagram. Through Venn diagrams we can easily visualize the abstract concept of a set and operations on sets. In this diagram, the universal set is usually represented by a rectangle and its subsets are shown as circles or other closed geometrical figures inside this rectangle.

For example,  $A = \{\text{Lucknow, Patna, Bhopal, Itanagar, Shillong}\}$  can be represented using a Venn diagram as in Fig. 2. Here  $U$  could be any superset of  $A$ .

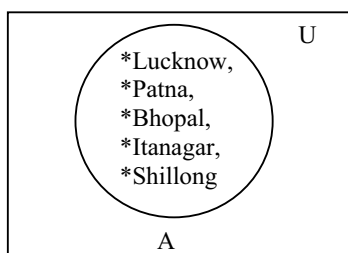


Fig. 2: A Venn diagram

Now that you are familiar with basic definitions related to sets, let us discuss some basic operations that can be performed on sets. This is when we shall appeal to Venn diagrams very often, as you will see.

## 1.3 OPERATIONS ON SETS

Let us now study sets obtained by applying operations on sets. We will cover four operations here, namely, union of sets, intersection of sets, complement of sets and symmetric difference. While studying them you will see how useful a Venn diagram can be for proving results related to these operations. In this section we will also look at some rules that are common to operations on sets and operations on statements, which you studied in Block 1.

### 1.3.1 Basic Operations

In this sub-section we shall define each of the operations one by one.

**Definition:** The **union** of two sets  $A$  and  $B$  is the set of all those elements which are either in  $A$  or in  $B$  or in both  $A$  and  $B$ . This set is denoted by  $A \cup B$ , and read as ‘ $A$  union  $B$ ’.

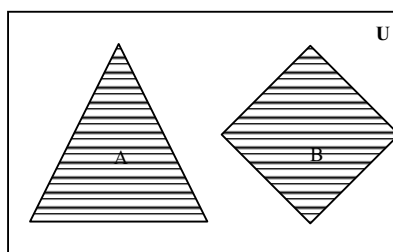
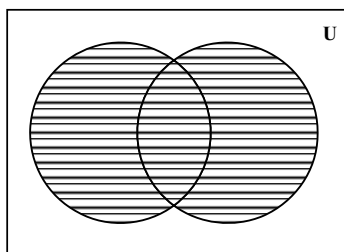
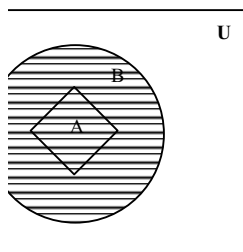
Symbolically,  $A \cup B = \{x : x \in A \text{ or } x \in B\}$

For example, if  $A = \{x : x \text{ is a stamp}\}$  and  $B = \{4, 5\}$ , then

$A \cup B = \{x : x \text{ is a stamp or a natural number lying between 3 and 6}\}$ .

And  $A = \{\text{Ram, Mohan, Ravi}\}$  and  $B = \{\text{Ravi, Rita, Neetu}\}$ , then  $A \cup B = \{\text{Ram, Mohan, Ravi, Rita, Neetu}\}$ .

If  $A \subseteq B$ , then  $A \cup B = B$ , and vice versa. This can be shown immediately using a Venn diagram, as in Fig.3.(a), where  $A$  is shown as the square contained in the circle representing  $B$ . In Fig.3(b),  $A \cup B$  is shown when  $A$  and  $B$  have some elements in common, and in Fig.3(c), we depict  $A \cup B$  when  $A$  and  $B$  have no element in common.



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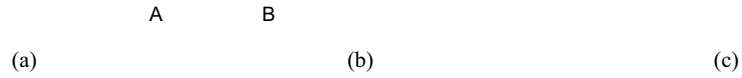


Fig. 3: Venn diagram for union

**Definition:** The **intersection** of sets A and B is the set of all the elements which are common to both A and B. This set is denoted by  $A \cap B$ , and read as ‘A intersection B’.

Symbolically,  $A \cap B = \{x : x \in A \text{ and } x \in B\}$ ;  
 For example  $A = \{1,2,3\}$  and  $B = \{2,1,5,6\}$ , then  $A \cap B = \{1,2\}$ .  
 Again if  $A = \{1\}$  and  $B = \{5\}$  then  $A \cap B = \{\}$  or  $\phi$ .

**Remark:** For any two sets A and B,  $A \cap B \subseteq A \subseteq A \cup B$  and  $A \cap B \subseteq B \subseteq A \cup B$ .

What is  $A \cap B$  if  $A \subseteq B$ ? Do you agree that it is A? Let us use a Venn diagram to check this (see Fig.4(a)). If A and B have some elements in common, then the Venn diagram for  $A \cap B$  looks like Fig 4.(b), and if A and B have no element in common, then the Venn diagram will be as in Fig.4(c).

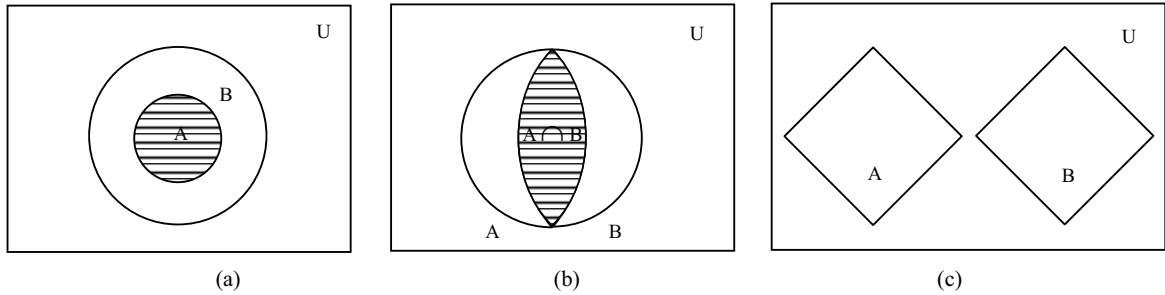


Fig. 4: Venn diagram for intersection of sets

**Definition:** The **difference of two sets** A and B is the set of all those elements of A which are not elements of B. Sometimes, we call this set the **relative component** of B in A. It is denoted by  $A \sim B$  or  $A \setminus B$ , and is read as ‘A complement B’.

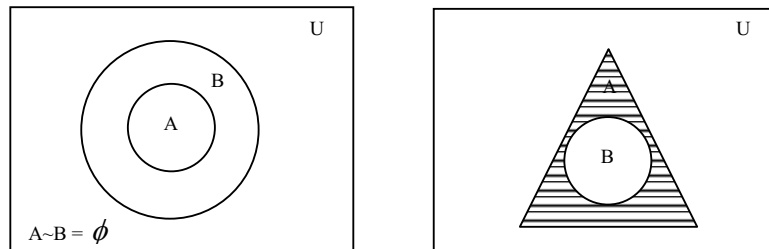
Symbolically,  $A \sim B = \{x : x \in A \text{ and } x \notin B\}$  and  
 $B \sim A = \{x : x \in B \text{ and } x \notin A\}$

For example, if  $A = \{4,5,6,7,8,9\}$  and  $B = \{3,5,2,7\}$ , then  $A \sim B = \{4,6,8,9\}$  and  $B \sim A = \{3,2\}$ . From this example it is clear that  $A \sim B \neq B \sim A$ . In fact, this is usually the case. So, **the difference of sets is not a commutative operation**.

In Fig.5(a),  $A \subseteq B$ , so that  $A \sim B = \phi$ .

In Fig.5(b) we show  $A \sim B$  when  $A \supseteq B$ , and in Fig.5(c) we show  $A \sim B$  when neither  $A \subseteq B$  nor  $B \subseteq A$ .

In Fig. 5(d), we show  $A \sim B$  when A and B are disjoint.



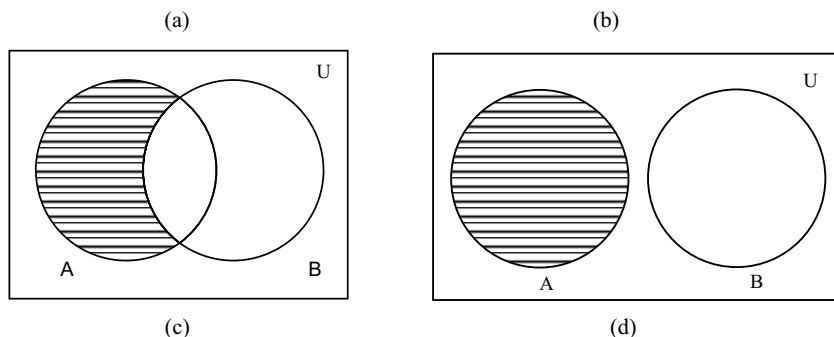


Fig. 5:  $A \sim B$  in different situation is the shaded portion.

There is one particular 'difference' that shows up very often, which we now define.

**Definition:** The **complement of a set**  $A$ , is the set  $U \setminus A$ , and is denoted by  $A'$  or  $A^c$ . For example,  $U = \{\text{Physics, Chemistry, Mathematics}\}$  and  $A = \{\text{Mathematics}\}$ , then the complement of  $A$  is  $A' = \{\text{Physics, Chemistry}\}$ .

The Venn diagram showing the complement of  $A$  is the set of those elements of the universal set  $U$  which are outside  $A$  (see Fig.6).

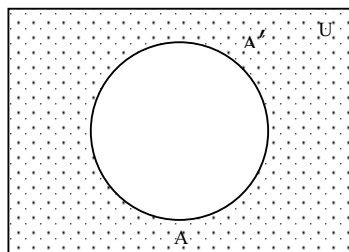


Fig. 6: Venn diagram for  $A'$ .

**Definition:** The **symmetric difference** of two sets  $A$  and  $B$  is the set of all those elements which are in  $A$  or in  $B$ , but not in both. It is denoted by  $A \Delta B$ .

i.e.,  $A \Delta B = (A \sim B) \cup (B \sim A)$ .

Note that  $A \Delta B = B \Delta A$ , i.e. the symmetric difference is commutative.

For example  $A = \{1,2,3,4,5\}$  and  $B = \{3,5,6,7\}$ , then  $A \sim B = \{1,2,4\}$ , and  $B \sim A = \{6,7\}$   
 $\therefore A \Delta B = (A \sim B) \cup (B \sim A) = \{1,2,4,6,7\}$

Now you may try these exercises.

E 9) Make a Venn diagram for  $A \Delta B$  for each of the situations i)  $A \subseteq B$ , ii)  $A \cap B \neq \emptyset$ , iii)  $B \subseteq A$  and  $A \cap B \neq \emptyset$ ; iv)  $A \cap B = \emptyset$ .

E10) Let  $A = \{\text{Math, Physics, Science}\}$ ,  $B = \{\text{Computer, Math, Chemistry}\}$ ,  $C = \{\text{Math}\}$ . Find  $A \cup (B \cap C)$ .

E11) If  $A = \{1,2,3,4,5,6\}$ ,  $B = \{4,5,6,7,8,9\}$ , find i)  $A \sim B$ , ii)  $B \sim A$ , iii)  $A \Delta B$ .

E12) For which sets  $A$  and  $B$  would  $A \sim B = B \sim A$ ?

E13) Write a program in C to perform E 10.

E14) Under what conditions can  $A \cap B = A \cup B$ ?

While discussing these operations, you may be wondering that they seem to satisfy properties very similar to those of propositional logic covered in Block 1 of this course. You are right! Let us look at these aspects now.

### 1.3.2 Properties Common to Logic and Sets

Before looking into the properties we shall first present a very useful principle to you. This will allow you to see how one property can be proved in several situations simultaneously.

**Duality Principle:** The ‘duality principle’ is very convenient for dealing with theorems about sets. Basically if any theorem is given to you, by applying the duality principle you can get another theorem (the dual of the previous one). In any statement involving the union and intersection of sets, we can get from one of the statements to the other by interchanging  $\cap$  with  $\cup$  and  $\phi$  with  $U$ .

For example, the dual of  $A \cup (B \cap C)$  is  $A \cap (B \cup C)$  and the dual of  $U \cup \phi = U$  is  $U \cap \phi = \phi$ . So, for example what is true for  $A \cup (B \cap C)$  will be true for  $A \cap (B \cup C)$  too. This is why if the first property in each of the pairs below is proved the second one follows immediately.

For any universal set  $U$  and subsets  $A$ ,  $B$  and  $C$  of  $U$ , **the following properties hold.**

**i) Associative properties:**

Union:  $A \cup (B \cap C) = (A \cup B) \cap C$

Intersection:  $A \cap (B \cup C) = (A \cap B) \cup C$

**ii) Commutative properties:**

Union:  $A \cup B = B \cup A$

Intersection:  $A \cap B = B \cap A$ .

**iii) Identity:**

Union:  $A \cup \phi = A$

Intersection:  $A \cap U = A$ .

**iv) Complement:**

Union:  $A \cup A' = U$

Intersection:  $A \cap A' = \phi$

**v) Distributive properties:**

Union:  $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

Intersection:  $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

**De Morgan’s Laws:**

For any two sets  $A$  and  $B$  the following laws known as De Morgan’s laws, hold

1.  $(A \cup B)' = A' \cap B'$ , and
2.  $(A \cap B)' = A' \cup B'$

De Morgan’s laws can also be expressed as

1.  $A \sim (B \cup C) = (A \sim B) \cap (A \sim C)$
2.  $A \sim (B \cap C) = (A \sim B) \cup (A \sim C)$

Each of the properties above corresponds to a related property for mathematical statements in logic (which we have covered in Unit 2 and Unit 3 of Block 1 of this course).

Now try these exercises.



Fig. 7: Augustus De Morgan (1806–1871)

Fig. 7: Augustus De Morgan (1806–1871)



E15) Find the dual of

i)  $A \cap (B \cap C) = (A \cap B) \cap C$ , and ii)  $(A \cup B) \cap (A \cup C)$ .

E16) Draw a Venn diagram to represent  $A \cup (B \cap C)$ .

E17) Check whether  $(A \cup B) \cap C = A \cup (B \cap C)$  or not using a Venn diagram.

Let us now focus on subsets of a particular kind of product of sets.

## 1.4 RELATIONS

Sometimes we need to establish relations between two or more sets. For example, a software development company has a set of specialists in different technology domains, or a company gets some projects to develop. Here the company needs to establish a relation between professionals and the project in which they will participate. To solve this type of problem the following concepts are required.

### 1.4.1 Cartesian product

Very often we deal with several sets at a time, and we need to study their combined action. For instance, combinations of a set of teachers and a set of students. In such a situation we can take a product of these sets to handle them simultaneously. To understand this product let us first consider the following definitions.

**Definition:** An **ordered pair**, usually denoted by  $(x,y)$ , is a pair of elements  $x$  and  $y$  of some sets. This is ordered in the sense that  $(x,y) \neq (y,x)$  whenever  $x \neq y$ , that is, the order of placing of the element in the pair matters.

Any two ordered pairs  $(x,y)$  and  $(a,b)$  are equal iff  $x = a$  and  $y = b$ .

For example if,  $A = \{a,b,c\}$  and  $B = \{x,y,z\}$ , then

$$A \times B = \{a,b,c\} \times \{x,y,z\} = \{(a,x), (a,y), (a,z), (b,x), (b,y), (b,z), (c,x), (c,y), (c,z)\}, \text{ and}$$

$$B \times A = \{x,y,z\} \times \{a,b,c\} = \{(x,a), (x,b), (x,c), (y,a), (y,b), (y,c), (z,a), (z,b), (z,c)\}.$$

Now let us think about how  $B \times A$  can be represented geometrically? For instance what is the geometric view of  $\{2\} \times \mathbf{R}$ ? This is the line  $x=2$  given in Fig.8(a).

Now, after seeing geometric representation of  $\{2\} \times \mathbf{R}$ , can you tell what  $\{1,3\} \times \{2,3\} = \{(1,2), (3,2), (1,3), (3,3)\}$  looks like? You will get four points in the first quadrant, as shown in Fig.8(b).



Fig. 8(a):  $\{2\} \times \mathbf{R}$ , i.e.,  $x=2$ . Is or numbers is commutative. Is

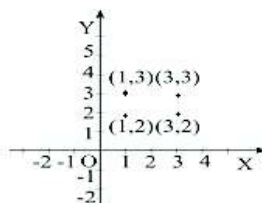


Fig.8(b):  $\{1,3\} \times \{2,3\}$

multiplication the Cartesian

product of sets also commutative? For instance, is  $\{1\} \times \{2\} = \{2\} \times \{1\}$ ? No, because  $(1,2) \neq (2,1)$ . So,  $A \times B \neq B \times A$  usually.

We can extend the definition of  $A \times B$  to define the Cartesian product of  $n$  sets  $A_1, A_2, \dots, A_n$  as follows.

$$A_1 \times A_2 \times A_3 \times \dots \times A_n = \{(x_1, x_2, x_3, \dots, x_n)\} : x_1 \in A_1, \wedge x_2 \in A_2 \wedge x_3 \in A_3 \wedge \dots, \wedge x_n \in A_n\}.$$

iff is short for 'if and only if'.

The element  $(x_1, x_2, \dots, x_n)$  is called an **n-tuple**. For instance, the 3-tuple  $(1, 1, 3) \in \{1\} \times \{1, 2\} \times \{2, 3\}$ .

Now you may try some exercises.

---

E18) If  $X = \{a, b, c\}$  and  $Y = \{1, 2, 3\}$ , find

i)  $X \times X$ , ii)  $X \times Y$ , and iii)  $X \times \phi$ .

E19) Under what conditions on  $A$  and  $B$  is  $A \times B = B \times A$ ?

E20) Give the geometric representation of  $\mathbf{R} \times \{2\}$ .

---

With what you studied in this sub-section, you now have the background to discuss relations.

## 1.4.2 Relations and their Types

We often speak of relations which hold between two or more objects, e.g., discrete mathematics is one of the courses in the IGNOU MCA I<sup>st</sup> semester, Nehru wrote Freedom of India, Chennai is the capital of Tamil Nadu. These are the relations in everyday situations. In these examples some sort of connections between pairs of objects are shown, and hence they express a relation between the pairs of objects.

**Definition:** A relation between two sets  $A$  and  $B$  is a subset of  $A \times B$ . Any subset of  $A \times A$  is a relation on the set  $A$ .

For instance, if  $A = \{1, 2, 3\}$  and  $B = \{p, q\}$ , then the subset  $\{(1, p), (2, q), (2, p)\}$  is a relation on  $A \times B$ . And  $\{(1, 1), (2, 3)\}$  is a relation on  $A$ .

Also,  $R = \{(x, y) \in \mathbf{N} \times \mathbf{N} : x > y\}$  is a relation on  $\mathbf{N}$ , the set of natural numbers, since  $R \subseteq \mathbf{N} \times \mathbf{N}$ .

If  $R \subseteq A \times B$ , we write  $x R_y$  if and only if  $(x, y) \in R$  ( $x R_y$  is read as 'x is related to y').

**Theorem 2:** The total number of distinct relations between a finite set  $A$  and a finite set  $B$  is  $2^{mn}$ , where  $m$  and  $n$  are the number of elements in  $A$  and  $B$ , respectively.

For example,  $R_1 = \mathbf{N} \times L$ , where  $L$  is set of straight lines, in this relation we can give different ordering of the straight lines.

If the relation  $R_2 = \{1, 2, 3\} \times \{l_1, l_2\}$ , then line  $l_1$  and  $l_2$  can get three different ordering.

**Proof:** The number of elements of  $A \times B$  is  $mn$ . Therefore, the number of elements of the power set of  $A \times B$  is  $2^{mn}$  (See Theorem 1). Thus,  $A \times B$  has  $2^{mn}$  different subsets. Now every subset of  $A \times B$  is a relation from  $A$  to  $B$ , by definition. Hence the number of different relations from  $A$  to  $B$  is  $2^{mn}$ .

As you have seen, any and every subset of  $A \times A$  is a relation on  $A$ . However, some relations have special properties. Let us consider these types one by one.

## 1.4.3 Properties of Relations

**Reflexive Relations:** A relation  $R$  on a set  $A$  is called a **reflexive relation** if  $(a, a) \in R \forall a \in A$ .

In other words,  $R$  is reflexive if every element in  $A$  is related to itself.

Thus,  $R$  is **not reflexive** if there is at least one element  $a \in A$  such that  $(a, a) \notin R$ .

For example, if  $A = \{1, 2, 3, 4\}$ , then the relation  $R_1 = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$  in  $A$  is reflexive because for  $x \in A, (x, x) \in R_1$ . However,  $R_2 = \{(1, 1), (2, 1), (4, 4)\}$  is not reflexive since  $2 \in A$ , but  $(2, 2) \notin R_2$ .

**Symmetric Relations:** A relation  $R$  on a set  $A$  is called a **symmetric relation** if  $(a, b) \in R \Rightarrow (b, a) \in R$ . Thus,  $R$  is symmetric if  $bRa$  holds whenever  $aRb$  holds.

A relation  $R$  in a set  $A$  is **not symmetric** if there exist two distinct elements  $a, b \in A$ , such that  $aRb$ , but not  $bRa$ .

For example, if  $L$  is the set of all straight lines in a plane, then the relation  $R$  in  $L$ , defined by ' $x$  is parallel to  $y$ ', is symmetric, since if a straight line  $a$  is parallel to a straight line  $b$ , then  $b$  is also parallel to  $a$ . Thus,  $(a, b) \in R \Rightarrow (b, a) \in R$ .

However, if  $R$  is the relation on  $\mathbb{N}$  defined by ' $xRy$  iff  $x - y > 0$ ', then  $R$  is not symmetric, since,  $4 - 2 > 0$  but  $2 - 4 \leq 0$ . Thus,  $(4, 2) \in R$  but  $(2, 4) \notin R$ .

**Transitive Relations:** A relation  $R$  on a set  $A$  is called a **transitive relation** if whenever  $(a, b) \in R$  and  $(b, c) \in R$ , then  $(a, c) \in R$  for  $a, b, c \in A$ .  
Thus,  $[(a, b) \in R, (b, c) \in R \Rightarrow (a, c) \in R], \forall a, b, c \in A \Rightarrow R$  is transitive.

A relation  $R$  in a set  $A$  is **not transitive** if there exist elements  $a, b, c \in A$ , not necessarily distinct, such that  $(a, b) \in R, (b, c) \in R$  but  $(a, c) \notin R$ .

For example, if  $L$  is the set of all straight lines in a plane and  $R$  is the relation on  $L$  defined by ' $x$  is parallel to  $y$ ' then if  $a$  is parallel to  $b$  and  $b$  is parallel to  $c$ , then  $a$  is parallel to  $c$ . Hence  $R$  is transitive. However, the relation ' $xSy$ ' on  $L$  defined by ' $x$  intersects  $y$ ' is not transitive.

Also, the relation  $R$  on  $A$ , the set of all Indians, defined by ' $xRy$  iff  $x$  loves  $y$ ', is not a transitive relation.

**Equivalence Relations:** A relation  $R$  on a set  $A$  is called an **equivalence relation** if and only if

- (i)  $R$  is reflexive, i.e., for all  $a \in A, (a, a) \in R$ ,
- (ii)  $R$  is symmetric, i.e.,  $(a, b) \in R \Rightarrow (b, a) \in R$ , for all  $a, b \in A$ , and
- (iii)  $R$  is transitive, i.e.,  $(a, b) \in R$  and  $(b, c) \in R \Rightarrow (a, c) \in R$ , for all  $a, b, c \in A$ .

One of the most trivial examples of an equivalence relation is that of '**equality**'.  
For any elements  $a, b, c$  in a set  $A$ ,

- (i)  $a = a$ , i.e., reflexivity
- (ii)  $a = b \Rightarrow b = a$ , i.e., symmetricity
- (iii)  $a = b$  and  $b = c \Rightarrow a = c$ , i.e., transitivity.

Now let us see if ' $xRy$  iff ' $x \leq y$ ' gives an equivalence relation on  $\mathbb{R}$ .

- (i)  $x \leq x$ , i.e.,  $(x, x) \in R$ , i.e.,  $R$  is reflexive.
- (ii) However,  $2 \leq 3$  but  $3 \not\leq 2$ . So,  $R$  is not symmetric.

Thus,  $R$  is not an equivalence relation.

Now you may try these exercises.

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E 21) Let  $A$  be the set of all people on Earth. A relation  $R$  is defined on the set  $A$  by ' $aRb$  if and only if  $a$  loves  $b$ ' for  $a, b \in A$ .

Examine if R is i) reflexive, ii) symmetric, iii) transitive.

Now we shall study a particular kind of relation, which is very useful in mathematics, as well as in computer science, as you will soon see.

## 1.5 FUNCTIONS

A function is a special kind of relation. If we take the example of the set A of students of IGNOU, and the set B of their enrolment numbers. Now consider  $R = \{(a,b) \in A \times B \mid b \text{ is enrolment number of } a\}$ , this is a relation between A and B. It is a 'special' relation, 'special' because to each  $a \in A \exists ! b$  such that  $aRb$ . We call such a relation a function from A to B.

Let us define this term formally.

**Definition:** A function from a non-empty set A to a non-empty set B is a subset R of  $A \times B$  such that for each  $a \in A \exists$  a unique  $b \in B$  such that  $(a,b) \in R$ . So, this relation satisfies the following two conditions:

- (i) for each  $a \in A$ , there is some  $b \in B$  such that  $(a,b) \in R$
- (ii) if  $(a,b) \in R$  and  $(a,b') \in R$  then  $b = b'$ .

We usually present functions as a rule associating elements of one set with another. So, let us present the definition again, with this view.

**Definition:** Let A and B be non-empty sets. A **function** (or a **mapping**) f from A to B is a rule that assigns to each element x in A **exactly one** element y in B. We write this as  $f: A \rightarrow B$ , read it as 'f is a function from A to B'.

Note that

- (i) to each  $a \in A$ , f assigns an element of B; and
- (ii) to each  $a \in A$ , f assigns only **one element** of B.

So, for example, suppose  $A = \{1,2,3\}$ ,  $B = \{1,4,9,11\}$  and f assigns to each member in A its square values. Then f is a function from A to B. But if  $A = \{1,2,3,4\}$ ,  $B = \{1,4,9,10\}$  and f is the same rule, then f is not a function from A to B since no member of B is assigned to the element 4 in A.

Note that the former example,  $11 \in B$ , but there is no element in A which is assigned to 11. This does not matter. It is not necessary that every element of B be related to some element of A.

Functions are not restricted to sets of numbers only. For instance, let A be the set of mothers and B be the set of human beings. Then the rule that assigns to every mother her eldest child is a **function**. But the rule that assigns to each mother her children is **not a function** because it does not relate a unique element of B to each element of A. Now, given a function, we have certain sets and terms that are associated with it. Let us give them some names.

**Definitions:** Let f be a function from A to B. The set A is called the **domain** of the function f and B is called the **co-domain** of f. The set  $\{f(x) \mid x \in A\}$  is called the **range** of f, and is also denoted by  $f(A)$ .

Given an element  $x \in A$ , the unique element of B to which the function f associates, it is denoted by  $f(x)$  and is called the **f-image** (or **image**) of x or the value of the function f for x. We also say that f **maps** x to  $f(x)$ . The element x is referred to as the **pre-image** of  $f(x)$ .

For example, if  $A = \{1,2,3,4\}$ ,  $B = \{1,8,27,64,125\}$ , and the rule  $f$  assigns to each member in  $A$  its cube, then  $f$  is a function from  $A$  to  $B$ . The domain of  $f$  is  $A$ , its codomain is  $B$  and its range is  $\{1,8,27,64\}$ .

Can you tell what will be the domain and codomain for rule  $f : f(x) = \frac{x}{1-x}$  ?

You can see that  $1-x = 0$ , if  $x = 1$ , in this case  $f(x)$  will be undefined.  
Domain of  $f$  can be taken as  $\mathbb{R} \setminus \{1\}$  and codomain can be  $\mathbb{R}$ .

**Remark:** Each element of  $A$  has a **unique image**, and each element of  $B$  need not appear as the image of an element in  $A$ . Further, more than one element of  $A$  can have the same image in  $B$ .

Let us look at some examples of functions, and non-functions now.

i) If  $b$  is a fixed element of  $B$ , then  $f : A \rightarrow B : f(x) = b \quad \forall x \in A$  is called a **constant function**.

Note that if  $b=0$ , then  $f$  is called the **zero map**, and is denoted by  $0$ .

ii)  $f : A \rightarrow A : f(x) = x \quad \forall x \in A$  is called the **identity function**, and is denoted by  $I$ .

iii) Consider  $A = \{1,2,3,4\}$ ,  $B = \{1,4,5\}$  and the rule  $f$  which associates  $1 \rightarrow 1, 2 \rightarrow 4, 3 \rightarrow 5, 4 \rightarrow 5$ . Then  $f$  is a function from  $A$  to  $B$ .

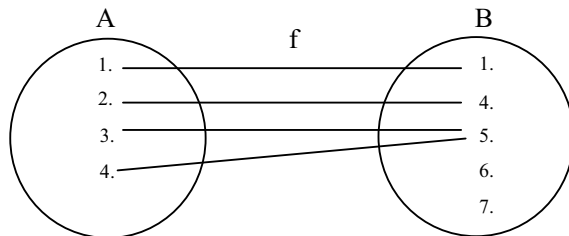


Fig.11: The rule  $f$  is a function

iv) The function  $f$  from  $\mathbb{R}$  to  $\mathbb{Z}$ , defined by the rule that  $f$  maps any real number  $x$  to the greatest integer less than or equal to  $x$ , is known as the **greatest integer function** or the **floor function**. We denote this function's action by  $f(x) = [x]$ , where  $[x]$  is the greatest integer  $\leq x$ .

For example, if  $x = 0.6$  then  $f(x) = [x] = 0$ , if  $x = 2.3$  then  $f(x) = [x] = 2$ , and if  $x = -5$ , then  $[x] = -5$ .

v) Function  $f : \mathbb{R} \rightarrow \mathbb{R} : f(x) = |x|$  is known as the modulus (or absolute value) function, where  $|x|$  is the absolute value of  $x$ .

For example, if  $x = 10$  then  $f(x) = |x| = 10$  and if  $x = -10$ , then  $f(x) = |x| = 10$ .

vi) Now take,  $A = \{a,b,c\}$  and  $B = \{1,2,3,4,5\}$ . Consider the rule  $f$  which associates  $a \rightarrow 1, a \rightarrow 3, b \rightarrow 2, c \rightarrow 3$ . This is not a function from  $A$  to  $B$  because, elements 1 and 3  $\in B$  are assigned to the same element  $a \in A$ .

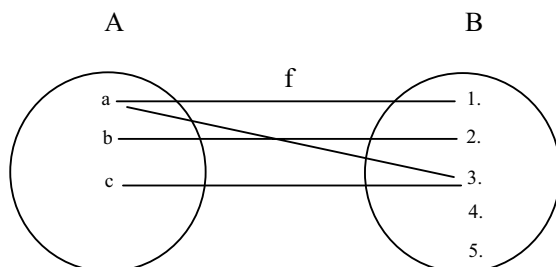


Fig.9: The rule  $f$  is not a function

vii) Consider  $A = \{1, 2, 3\}$ ,  $B = \{1, 4, 5, 6, 7\}$  and the rule  $f$  which associates  $1 \rightarrow 1$ ,  $2 \rightarrow 1$ ,  $2 \rightarrow 4$ . Here  $f$  is not a function from  $A$  to  $B$  since no member of  $B$  is assigned to the element  $3 \in A$ .

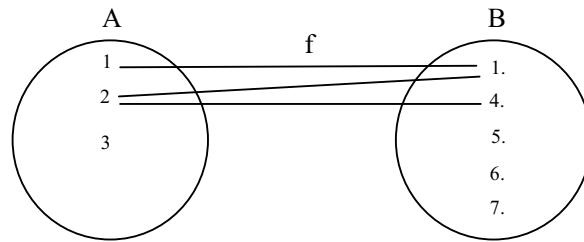


Fig.10: The rule is not a function

Now you may try these exercises.

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E22) Let  $A = \{a, b, c, d\}$ ,  $B = \{1, 2, 3\}$  and  $R = \{(a, 2), (b, 1), (c, 2), (d, 1)\}$ . Is  $R$  a function? Why.

E23) Every function is a relation. Is every relation a function? Why?

E24) Consider the following pseudocode.

```

1. read(n)
2. while n > 1 do
3.   begin
4.     if n is even then n := n div 2
5.     else n := 2n + 1;
6.   end

```

Write a function of  $n$  that describes the operations performed.

---

E25) If  $A = \{1, 2, 3, 4\}$ ,  $B = \{2, 3, 4, 5, 6, 7\}$  and the rule  $f$  assigned to each member in  $A$  is  $f(x) = x + 1$ , then find the domain and range of  $f$ .

---

Now let us discuss some types of functions.

### 1.5.1 Types of Functions

Here we shall look at different types of mappings.

‘Surjective’ comes from the French word ‘sur’, meaning ‘on top of’.

**Onto Mapping:** A mapping  $f: A \rightarrow B$  is said to be an **onto** (or **surjective**) **mapping** if  $f(A) = B$ , that is, the range and co-domain coincide. In this case we say that  **$f$  maps  $A$  onto  $B$** .

For example,  $f: \mathbf{Z} \rightarrow \mathbf{Z} : f(x) = x + 1$ ,  $x \in \mathbf{Z}$ , then every element  $y$  in the co-domain  $\mathbf{Z}$  has a pre-image  $y - 1$  in the domain  $\mathbf{Z}$ . Therefore,  $f(\mathbf{Z}) = \mathbf{Z}$ , and  $f$  is an onto mapping.

**Injective Mapping:** A mapping  $f: A \rightarrow B$  is said to be **injective** (or **one-one**) if the images of distinct elements of  $A$  under  $f$  are distinct, i.e., if  $x_1 \neq x_2$  in  $A$ , then  $f(x_1) \neq f(x_2)$  in  $B$ . This is briefly denoted by saying  $f$  is **1-1**.

For example  $f: \mathbf{R} \rightarrow \mathbf{R}$  be defined by  $f(x) = 2x + 1$ ,  $x \in \mathbf{R}$ , then for  $x_1, x_2 \in \mathbf{R}$  ( $x_1 \neq x_2$ ) we have  $f(x_1) \neq f(x_2)$ . So,  $f$  is **1-1**.

**Bijjective Mapping:** A mapping  $f: A \rightarrow B$  is said to be **bijjective** (or **one-one onto**), if  $f$  is both injective and surjective, i.e., one-one as well as onto.

For example,  $f: \mathbf{Z} \rightarrow \mathbf{Z} : f(x) = x + 2$ ,  $x \in \mathbf{Z}$  is both injective and surjective. So,  $f$  is bijective.

There is a particular kind of bijective function that we use very often. Let us define this.

**Definition:** A bijective mapping  $f : A \rightarrow A$  is said to be a **permutation** on the set  $A$ . Let  $A = \{a_1, a_2, \dots, a_n\}$ , and  $f$  be a bijection from  $A$  onto  $A$  that maps  $a_i$  to  $f(a_i)$ , then we write  $f$  as

$$f = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ f(a_1) & f(a_2) & \dots & f(a_n) \end{pmatrix}. \text{ So, the identity mapping } I = \begin{pmatrix} a_1 & a_2 & \dots & a_n \\ a_1 & a_2 & \dots & a_n \end{pmatrix}.$$

Now, associated with a bijective function, we get another function very naturally, which we now define.

**Definition:** Let  $f : A \rightarrow B$  be a bijective mapping. Then the mapping  $g : B \rightarrow A$  which associates to each element  $b \in B$  the unique element  $a \in A$ , such that  $f(a) = b$ , is called the inverse mapping of the mapping  $f : A \rightarrow B$ . We denote this function  $g$  by  $f^{-1}$ .

Note that a function  $f$  is invertible iff  $f^{-1}$  exists iff  $f$  is bijective.

Hence, if  $f : A \rightarrow B$  is a one-one onto mapping, then  $f^{-1} : B \rightarrow A$  exists, and is also 1-to-1.

Note the inverse of the permutation  $f = \begin{pmatrix} a_1 & a_2 \dots a_n \\ b_1 & b_2 \dots b_n \end{pmatrix}$  is the permutation

$$\begin{pmatrix} b_1 & b_2 \dots b_n \\ a_1 & a_2 \dots a_n \end{pmatrix}.$$

For example,  $A = \mathbf{R} \setminus \{3\}$  and  $B = \mathbf{R} \setminus \{1\}$ , and the function  $f : A \rightarrow B$  is defined by

$$f(x) = \frac{x-2}{x-3}.$$

We can see that  $f$  is a one-to-one function.  
 $\therefore f$  inverse exists.

To get  $f^{-1}(x)$  the following steps are required;

1. Replace  $f(x)$  by  $y$  in the equation describing the function. You will get

$$y = \frac{x-2}{x-3}.$$

2. Interchange  $x$  and  $y$ . In other words, replace every  $x$  by  $y$ , and vice versa. You

$$\text{will get } x = \frac{y-2}{y-3}.$$

3. Solve for  $y$ .

4. Replace  $y$  by  $f^{-1}(x)$ .

$$\text{By applying these steps we get } f^{-1}(x) = \frac{3x-2}{x-1}.$$

Now try these exercises.

E26) Explain why  $f : \mathbf{Z} \rightarrow \mathbf{Z} : f(x) = x^2$  is onto? Domain and range of  $f$  is  $\mathbf{Z}$ .

E27) Which of the following kind of function would you use to provide photo identity numbers? Why?

- i) Constant function, ii) one-to-one function, and iii) identity function.

E28) Find  $f$  inverse of rule  $f : f(x) = x^3 - 3$ .

Now we can see how different operations like addition, subtraction, multiplication and division can be applied on functions.

### 1.5.2 Operations on Functions

If given whose domains ranges are subsets of the **real numbers**, we define the function  $f+g$  by  $(f+g)(x)$  to be the function whose value at  $x$  is the sum of  $f(x)$  and  $g(x)$ . Symbolically,

$(f+g)(x) = f(x) + g(x)$ . This is called pointwise addition of  $f$  and  $g$ .

The domain of  $f+g$  is the **intersection** of the domains of  $f$  and  $g$  since to compute  $(f+g)(x)$  it is necessary and sufficient to compute both  $f(x)$  and  $g(x)$ .

Other operations on functions are defined similarly:

- $(fg)(x) = f(x)g(x)$  (pointwise multiplication)
- $f^p(x) = (f(x))^p$  for any real exponent  $p$  with the domain of  $f^p$  consisting of those points for which the  $p$ -th power of  $f(x)$  makes sense.
- $(f/g)(x) = f(x)/g(x)$ , for  $g(x) \neq 0$  (pointwise multiplication)

For example, if  $f(x) = 3 \sin(x)$  and  $g(x) = x^2$ , then

$$(f+g)(x) = 3 \sin(x) + x^2$$

$$(fg)(x) = 3 \sin(x) \cdot x^2$$

$$(f-g)(x) = 3 \sin(x) - x^2$$

$$(f/g)(x) = 3 \sin(x) / x^2$$

The domains of both  $f$  and  $g$  are all **real numbers**, but the domain of  $f/g$  is  $\{x \mid x \neq 0\}$ .

Now let us consider two functions  $f$  and  $g$  from  $A = \{1,2\}$  to  $B = \{1,2,3,4\}$ , where  $f = \{(1,1), (2,4)\}$ . Let  $g$  be defined by the rule  $g(x) = x^2$  where the domain of  $g$  is the set  $\{1,2\}$ . Here both have the same domain. Since  $f$  and  $g$  assign the same image to each element in the domain, they have the same effect throughout. This is why we treat them as the same, or equal.

**Definition:** If  $f$  and  $g$  are two functions defined on the same domain  $A$  and if  $f(a) = g(a)$  for every  $a \in A$ , then the functions  $f$  and  $g$  are **equal**, i.e.,  $f = g$ .

For example  $f(x) = x^2 + 5$ , where  $x$  is a real number, and  $g(x) = x^2 + 5$ , where  $x$  is a complex number. Then the function  $f$  is not equal to the function  $g$  since they have different domains although  $f(x) = x^2 + 5 = g(x) \forall x \in \mathbf{R}$ . By this example we can conclude that even if  $f(a) = g(a)$ ,  $f$  and  $g$  may not be the same.

So far, the operations you have seen are the same as those for member systems. However, there is yet another operation on functions which we now define.

**Definition:** Let  $f$  and  $g$  be the operation of combining two functions by applying them one after the other. That is, the composition of  $f(x)$  and  $g(x)$ , denoted by,  $f \circ g$ .

For example, consider  $f: \mathbf{R} \rightarrow \mathbf{R}: f(x) = (x^3 + 2x)^3$ . We can write it as the composition of  $g$  and  $h$ , where the value of  $f(x)$  can be obtained by first calculating  $x^3 + 2x$  and then taking its third power. We can write  $g$  for first or inside function  $g(x) = x^3 + 2x$ . We write  $h$  for the second function:  $h(x) = x^3$ . The use of the variable  $x$  is irrelevant, we could as well write  $h(y) = y^3$  for  $y \in \mathbf{R}$ . We can see that  $goh(x) = g(x^3 + 2x) = (x^3 + 2x)^3 = f(x)$ .

In general  $(f \circ g) \neq (g \circ f)$ .

For example, if,  $f(x) = x^2$  and  $g(x) = x+1$ , then  $(f \circ g)(x) = (x+1)^2$  and  $(g \circ f)(x) = x^2 + 1$ .

Here we can see that  $f \circ g \neq g \circ f$ .

Let us see another example, where  $f(x) = x^2$ ,  $g(x) = x+1$ ,  $h(x) = x^3$



Then,  $f \circ (g \circ h)(x) = (x^3 + 1)^2$  and  $(f \circ g) \circ h(x) = (x^3 + 1)^2$ . Here we can see  $f \circ (g \circ h) = (f \circ g) \circ h$ .

Now let us see how you can get **product** of two permutations  $f$  and  $g$  of the same set,

Let  $f = \begin{pmatrix} a_1 & a_2 \dots a_n \\ f(a_1) & f(a_2) \dots f(a_n) \end{pmatrix}$  and  $g = \begin{pmatrix} a_1 & a_2 \dots a_n \\ g(a_1) & g(a_2) \dots g(a_n) \end{pmatrix}$ . Then  $fg = \begin{pmatrix} a_1 & a_2 \dots a_n \\ f[g(a_1)] & f[g(a_2)] \dots f[g(a_n)] \end{pmatrix}$  is itself a permutation.

For example if,  $f = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$ ,  $g = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}$  then

$$fg = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} \text{ and } gf = \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}$$

Note that  $f \circ g \neq g \circ f$ . Thus the **multiplication of permutations is not commutative** in general.

However, the multiplication of permutations is associative. For example, if  $f =$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 4 & 2 \end{pmatrix}, g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 1 & 3 \end{pmatrix} \text{ be the permutations on}$$

$A = \{1, 2, 3, 4\}$ , then

$$fg = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}, gf = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix}, gh = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{pmatrix},$$

$$f(gh) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}, (fg)h = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}.$$

Here we can see the multiplication of permutation is commutative.

Now try these exercises.

E 29) Let  $f(x) = 1/x$  and  $g(x) = x^3 + 2$ . Find the following functions, where  $x \in \mathbf{R}$ .

- i)  $(f + g)(x)$
- ii)  $(f - g)(x)$
- iii)  $(fg)(x)$
- iv)  $(f/g)(x)$

E30) Let  $f(x) = \sqrt{x+1} \quad \forall x \geq -1$  and  $g(x) = x^3 \quad \forall x \in \mathbf{R}$ . Define the following functions. Also give their domains.

- i)  $(f + g)$
- ii)  $(f - g)$
- iii)  $(fg)$
- iv)  $(f/g)$
- v)  $(f \circ g)$

With this we have come to the end of this unit. Let us now summaries what we have covered in this unit.

## 1.6 SUMMARY

In this unit we have covered the following points:

1. We introduced basic concepts related to sets and different ways of representing them.
2. We worked at different operations on sets and there Venn diagram representations.
3. We explored some properties common to operations on sets and logical statements.
4. In the process we also documented the duality principle.
5. We defined relations as a Cartesian product of sets and looked at several examples and type of relations.
6. We defined a function as a particular kind of relation. Then we studied different types of functions as well as basic operations on functions. In the process we considered permutations and their product.

## 1.7 SOLUTIONS / ANSWERS

E1)  $A = \{x : x \text{ is a student of IGNOU.}\}$

- E2) i) The collection of all good teachers is **not** a set because this collection is not well-defined. The characteristic 'good' cannot be measured objectively.  
 ii) The set of points on a line is **not** finite because infinitely many points make a straight line.

E3)  $\{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$ .

E4) Set of vowels of English alphabet  $V = \{a, e, i, o, u\}$ . Two subsets of set  $V$  are  $V_1 = \{a, e\}$ , and  $V_2 = \{i, o\}$ . Two supersets of  $V$  are  $V_3 = \{a, b, c, \dots, z\}$ , and  $V_4 = \{a, c, d, e, i, o, u, \dots, z\}$ .

E5) Powerset of  $A = \{a, e, i, o, u\}$  is

$\{\phi, \{a\}, \{e\}, \{i\}, \{o\}, \{u\}, \{a, e\}, \{a, i\}, \{a, o\}, \{a, u\}, \{e, i\}, \{e, o\}, \{e, u\}, \{i, o\}, \{i, u\}, \{o, u\}, \{a, e, i\}, \{a, i, o\}, \{a, o, u\}, \{e, i, o\}, \{i, o, u\}, \{i, o, u\}, \{a, e, i, o\}, \{e, i, o, u\}, \{a, e, i, o, u\}$   
 .

E6) For empty set  $A = \{\}$  or  $\phi$ ,  $P(A) = 1$

E7) If  $A \subseteq B$ , then  $P(A) \subseteq P(B)$  because every subset of  $A$  is a subset of  $B$ .

E8) If  $P(A) = P(B)$  then  $A \in P(A) = P(B) = A \subseteq B$ . Similarly,  $B \subseteq A$ . Therefore  $A = B$ .

E9)

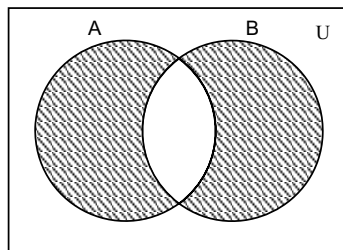


Fig.12: The Shaded portion is  $A \Delta B$

You can try them for the other situations. We are showing in Fig. 12 for the second situation.

E10)  $A \cup (B \cap C) = \{\text{Math, Physics, Science}\} = A$ .

E11) i)  $A \sim B = \{1,2,3\}$

ii)  $B \sim A = \{7,8,9\}$

iii)  $A \Delta B = \{1,2,3,7,8,9\}$

E12) Only if A and B are  $\phi$ .

E13) Write separate functions to find  $A \sim B$ ,  $B \sim A$  and  $A \Delta B$  with passing sets A and B as argument, return the resultant set.

E14)  $A \cap B$  can be equal to  $A \cup B$  if either  $A \subseteq B$  or  $B \subseteq A$ .

E15) i) Dual of  $A \cap (B \cap C) = (A \cap B) \cap C$  is  $A \cup (B \cup C) = (A \cup B) \cup C$ .

ii) Dual of  $(A \cup B) \cap (A \cup C)$  is  $(A \cap B) \cup (A \cap C)$ .

E16)

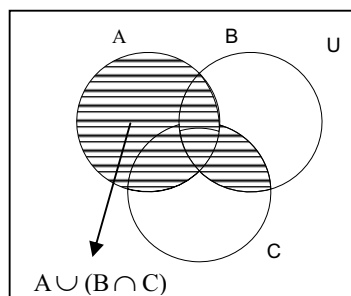


Fig.13: The lined portion represents  $A \cup (B \cap C)$

E17)

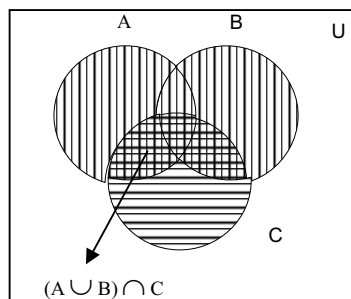


Fig.14(a)

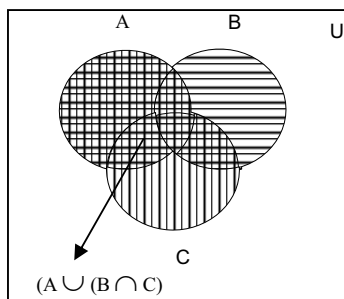


Fig.14(b)

Shaded area in Fig.14 (a) and Fig.14(b) are not same so  $(A \cup B) \cap C$  is not equal to  $A \cup (B \cap C)$ .

E18) i)  $X \times X = \{(a,a), (a,b), (a,c), (b,b), (b,c), (c,c)\}$ .

ii)  $X \times Y = \{(a,1), (b,1), (c,1), (a,2), (b,2), (c,2), (a,3), (b,3), (c,3)\}$ .

iii)  $X \times \phi = \phi$ .

E19)  $A \times B = B \times A$  iff  $A = B$ .

E20) The geometric diagram for  $R \times \{2\}$

will be the line parallel to Y axis.

See Fig.15.

Fig.15:  $y=2$

E21) i) For  $a \in A$ ,  $aRa$  is reflexive because every one loves herself or himself.

ii)  $R$  is not symmetric because if  $a$  loves  $b$  then  $b$  need not love  $a$ , i.e.,  $aRb$  does not always imply  $bRa$ . Thus  $R$  is not symmetric.

iii)  $R$  is not transitive, because if  $a$  loves  $b$  and  $b$  loves  $c$  then  $a$  need not love  $c$ ; i.e., if  $aRb$  and  $bRc$ ,  $aRc$  need not be. Thus,  $R$  is not transitive.

Hence,  $R$  is reflexive but is neither symmetric nor transitive.

E22)  $R$  is a function because each element of  $A$  is assigned to a unique element of  $B$ .

E23) Not every relation is a function. For example, this relation does not satisfy the property that,

a) Each element of  $A$  must have assigned one element in  $B$ .

b) If  $a \in A$  is assigned  $b \in B$  and  $a \in A$  is assigned  $b' \in B$  then  $b = b'$ .

That is why relations those who don't satisfy above properties are not a function

E24) We can see that the code has no effect on the value of  $n \leq 0$ . In the While loop, the value of  $n$  is halved whenever it is even. If  $n$  becomes odd before reaching 1, the second part of the while loop is invoked, and  $n$  remains odd and increases forever.

This shows that  $f: \mathbb{N} \rightarrow \mathbb{N}$  is the function defined by  $f(n) = \begin{cases} 0 & \text{if } n = 0. \\ 1, & \text{if } n \text{ is a power of } 2, \\ \text{undefined} & \text{otherwise} \end{cases}$

E25) The domain of  $f$  is  $\{1, 2, 3, 4\}$  and range of  $f$  is  $\{2, 3, 4, 5\}$ .

E26) Function  $f(x) = x^2$  is one-to-one because for every value of  $x$ ,  $x^2$  will be a number that is different for different  $x$ . Hence,  $f(x) = x^2$  is one-one mapping.

E27) One-to-one function will be used for providing identity card number, because each person must have unique identity numbers.

E28) Step 1:  $y = x^3 - 3$

Step 2:  $x = y^3 - 3$

Step 3:  $y = \sqrt[3]{x + 3}$

Step 4:  $f^{-1}(x) = \sqrt[3]{x + 3}$ .

E29) i)  $(f+g)(x) = \frac{1}{x} + x^3 + 2$

ii)  $(f-g)(x) = \frac{1}{x} - (x^3 + 2)$

iii)  $(f.g)(x) = \left(\frac{1}{x}\right)(x^3 + 2)$

$$\text{iv) } (f/g)(x) = \frac{1}{x(x^3 + 2)} \quad \forall x \in \mathbb{R}.$$

$$\text{E30) i) } (f+g)(x) = \sqrt{x+1} + x^3 \quad \forall x \geq -1$$

$$\text{ii) } (f-g)(x) = \sqrt{x+1} - x^3 \quad \forall x \geq -1$$

$$\text{iii) } (f \cdot g)(x) = \sqrt{x+1} \cdot x^3 \quad \forall x \geq -1$$

$$\text{iv) } (f/g)(x) = \sqrt{x+1} / x^3 \quad \forall x \geq -1, x \neq 0$$

$$\text{v) } (f \circ g)(x) = f(g(x)) = \sqrt{x^3 + 1} \quad \forall x \geq -1.$$

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## UNIT 3 SOME MORE COUNTING PRINCIPLES

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### 3.0 INTRODUCTION

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In this unit, we continue our discussion of the previous unit on combinatorial techniques. We particularly focus on two principles of counting – the pigeonhole principle and the principle of inclusion-exclusion.

In Sec. 3.2 you will see how obvious the pigeonhole principle is. Its proof is very simple, and amazingly, it has several useful applications. We shall also include some of these in this section.

In Sec. 3.3, we focus on the principle (or formula) of inclusion-exclusion. As you will see, this principle tells us how many elements do not fit into any of  $n$  categories. We prove this result and also give a generalisation. Following this, in Sec. 3.4 we give several important applications of inclusion-exclusion.

We shall continue our discussion on combinatorial techniques in the next unit.

### 3.1 OBJECTIVES

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After studying this unit, you should be able to:

- prove the pigeonhole principle, and state the generalised pigeonhole principle;
- identify situations in which these principles apply, and solve related problems;
- prove the principle of inclusion-exclusion;
- apply inclusion-exclusion for counting the number of surjective functions, derangements and for finding discrete probability.

### 3.2 PIGEONHOLE PRINCIPLE

---

Let us start with considering a situation where we have 10 boxes and 11 objects to be placed in them. Wouldn't you agree that regardless of the way the objects are placed in the two boxes at least one box will have more than one object in it? On the face of it, this seems obvious. This is actually an application of the pigeonhole principle, which we now state.

**Theorem 1 (The Pigeonhole Principle):** Let there be  $n$  boxes and  $(n+1)$  objects. Then, under any assignment of objects to the boxes, there will always be a box with more than one object in it.

This can be reworded as: if  $m$  pigeons occupy  $n$  pigeonholes, where  $m > n$ , then there is at least one pigeonhole with two or more pigeons in it.



**Fig. 1** The pigeonhole principle

**Proof:** Let us label the  $n$  pigeonholes  $1, 2, \dots, n$ , and the  $m$  pigeons  $p_1, p_2, \dots, p_m$ . Now, beginning with  $p_1$ , we assign one each of these pigeons the holes numbered  $1, \dots, n$ , respectively. Under this assignment, each hole has one pigeon, but there are still  $(m-n)$  pigeons left. So, in whichever way we place these pigeons, at least one hole will have more than one pigeon in it. This completes the proof!

This result appears very trivial, but has many applications. For example, using it you can show that:

- if 8 people are picked in any way from a group, at least 2 of them will have been born on the same weekday.
- in any group of 13 people, at least two are born in the same month.

Let us consider some examples of its application, in detail.

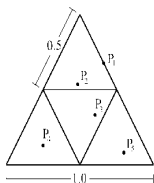
**Example 1:** Assuming that friendship is mutual, show that in any group of people we can always find two persons with the same number of friends in the group.

**Solution:** If there are  $n$  persons in the group, then let the number of friends the  $i$ th person has be  $f(i)$ ,  $i = 1, \dots, n$ . Clearly,  $f(i)$  can take values only between 0 and  $(n-1)$ .

If some  $f(i)$  is 0, it means that the  $i$ th person does not have any friends in the group. In this case, no person can be friends with all the other  $(n-1)$  people. So, no  $f(i)$  can be  $(n-1)$ . So, only one of the values 0 or  $(n-1)$  can be present among the  $f(i)$ 's. So, the  $n$   $f(i)$ 's can take only  $(n-1)$  distinct values. Therefore, by the pigeonhole principle, two  $f(i)$ 's must be equal. Then the corresponding  $i$ 's have the same number of friends in the group.

\* \* \*

**Example 2:** Suppose 5 points are chosen at random within or on the boundary of an equilateral triangle of side 1 metre. Show that we can find two points at a distance of at most  $\frac{1}{2}$  metre.



**Fig. 2**

**Solution:** Divide the triangle into four equilateral triangles of side  $\frac{1}{2}$  m by joining the midpoints of the sides by three line segments (see Fig. 2). These four triangles may now be considered as boxes and the five points as objects. By the pigeonhole principle, at least one of these smaller triangles will have two points in or on it. Clearly, the distance between these two points is at most  $\frac{1}{2}$  metre.

\* \* \*

**Example 3:** Given any ten different positive integers less than 107, show that there will be two disjoint subsets with the same sum.

**Solution:** The highest numbers we could be given would be 97, 98, ..., 106, which add up to 1015. So, consider pigeonholes marked 0, 1, 2, ..., 1015. The set of 10 positive integers have  $2^{10} = 1024$  subsets. Put a subset in the pigeonhole marked with the sum of the numbers in the set. The 1024 subsets have to be put in 1016 pigeonholes. So, some pigeonhole will have more than one subset with the same sum.

Now, note that two subsets that we get with the same sum, may not be disjoint. But, by dropping the common elements in them, we are left with disjoint subsets with the same sum.

\* \* \*

Here are some related exercises for you to do.

- 
- E1) If 10 points are chosen in an equilateral triangle of side 3 cms., show that we can find two points at a distance of at most 1 cm.
- E2) On 11 occasions a pair of persons from a group of 5 was called for a function. Show that some pair of persons must have attended the function at least twice.
- E3) Four persons were found in a queue, independently, on 25 occasions. Show that at least on two occasions they must have been in the queue in the same order.
- 

As you know, **mathematics develops through a process of generalisation**. You know that the principle is valid for  $n+1$  objects and  $n$  boxes. It is natural to ask: what if we have, say,  $4n+1$  objects and 4 boxes? Can we prove a similar principle? In fact, we can, as given below.

**Theorem 2 (The Generalized Pigeonhole Principle):** If  $nm + 1$  objects are distributed among  $m$  boxes, then at least one box will contain more than  $n$  objects.

This can be reworded as: Let  $k$  and  $n$  be positive integers. If  $k$  balls are put into  $n$  boxes, then some box contains at least  $\lceil k/n \rceil + 1$  balls, where  $\lceil x \rceil$  denotes the greatest integer less than  $x$ .

**Proof:** We prove this by contradiction (see Unit 2, Block 1). Suppose all the  $m$  boxes have at most  $n$  objects in them. Then the total number of objects is at most  $nm$ , a contradiction. Hence, the theorem.

Applying this result, we see, for example, that suppose 479 students are enrolled in the course Discrete Mathematics, consisting of 6 units. Then, at least  $\lceil \frac{479}{6} \rceil + 1 = 80$  students are studying the same unit at a given point of time.

Let us consider a few more examples of the application of this principle.

**Example 4:** Show that in any group of 30 people, we can always find 5 people who were born on the same day of the week.

**Solution:** 30 people can be assigned to 7 days of the week. Then at least  $\lceil \frac{30}{7} \rceil + 1 = 5$  of them must have been born on the same day.

\* \* \*

**Example 5:** 20 cards, numbered from 1 to 20, are placed face down on a table. 12 cards are selected one at a time and turned over. If two of the cards add up to 21, the player loses. Is it possible to win this game?

**Solution:** The pairs that can add up to 21 are (1, 20), (2, 19), ..., (10, 11). So, there are 10 such pairs. In turning 12 cards, at least one of these pairs will be included. Therefore, the player will lose.

\* \* \*

**Example 6:** Show that every sequence of  $n^2 + 1$  distinct integers includes either an increasing subsequence of  $n + 1$  numbers or a decreasing subsequence of  $n + 1$  numbers.

**Solution:** Let the sequence be  $a_1, a_2, \dots, a_{n^2+1}$ . Suppose there is no increasing subsequence of  $n + 1$  numbers. For each of these  $a_k$ s, let  $s(k)$  be the length of the longest increasing subsequence beginning at  $a_k$ . Since all  $n^2+1$  of the  $s(k)$ 's are



between 1 and  $n$ , at least  $\left\lceil \frac{n^2 + 1}{n} \right\rceil + 1 = n + 1$  of these numbers are the same. (The  $s(k)$ 's are the objects and the numbers from 1 to  $n$  are the boxes.)

Now, if  $i < j$  and  $s(i) = s(j)$ , then  $a_i > a_j$ . Otherwise  $a_i$  followed by the longest increasing subsequence starting at  $a_j$  would be an increasing subsequence of length  $s(j) + 1$  starting at  $a_i$ . This is a contradiction, since  $s(i) = s(j)$ .

Therefore, all the  $n + 1$  integers  $a_k$ , for which  $s(k) = m$ , must form a decreasing subsequence of length at least  $n + 1$ .

\* \* \*

**Example 7:** Take  $n$  integers, not necessarily distinct. Show that the sum of some of these numbers is a multiple of  $n$ .

**Solution:** Let  $S(m)$  be the sum of the first  $m$  of these numbers. If for some  $r$  and  $m$ ,  $r < m$ ,  $S(m) - S(r)$  is divisible by  $n$ , then  $a_{r+1} + a_{r+2} + \dots + a_m$  is a multiple of  $n$ . This also means that  $S(r)$  and  $S(m)$  leave the same remainder when divided by  $n$ . So, if we cannot find such pairs  $m$  and  $r$ , then it means that the  $n$  numbers  $S(1), S(2), \dots, S(n)$  leave different remainders when divided by  $n$ . But there are only  $n$  possible remainders, viz.,  $0, 1, 2, \dots, (n - 1)$ . So, one of these numbers must leave a remainder of 0. This means that one of the  $S(i)$  is divisible by  $n$ . This completes the proof.

In fact, in this example we have proved that one of the sums of consecutive terms is divisible by  $n$ .

\* \* \*

You may like to try some exercises now.

- 
- E4) If any set of 11 integers is chosen from  $1, \dots, 20$ , show that we can find among them two numbers such that one divides the other.
- E5) If 100 balls are placed in 15 boxes, show that two of the boxes must have the same number of balls.
- E6) If  $a_1, a_2, \dots, a_n$  is a permutation of  $1, 2, \dots, n$  and  $n$  is odd, show that the product  $(a_1 - 1)(a_2 - 2) \dots (a_n - n)$  must be even.
- 

There are several corollaries to Theorem 2. We shall present one of them here.

**Theorem 3:** If a finite set  $S$  is partitioned into  $s$  subsets, then at least one of the subsets has  $\frac{|S|}{k}$  or more elements.

**Proof:** Let  $A_1, \dots, A_k$  be a partition of  $S$ . (This means that  $A_i \cap A_j = \emptyset$  for  $i \neq j$  and  $S = A_1 \cup A_2 \cup \dots \cup A_k$ .) Then the average value of  $|A_i|$  is  $\frac{1}{k} [|A_1| + \dots + |A_k|] = \frac{|S|}{k}$ .

So the largest  $A_i$  has at least  $\frac{|S|}{k}$  elements.

A consequence of this result is the following theorem.

**Theorem 4:** Consider a function  $f: S \rightarrow T$ , where  $S$  and  $T$  are finite sets satisfying  $|S| > r|T|$ . Then at least one of the sets  $f^{-1}(t)$ ,  $t \in T$ , has more than  $r$  elements. ( $f^{-1}(t)$  denotes the inverse image of the set  $\{t\}$ , i.e.,  $f^{-1}(t) = \{x \in S : f(x) = t\}$ .)

**Proof:** The family  $\{f^{-1}(t): t \in T\}$  partitions  $S$  into  $k$  sets with  $k \leq |T|$ . By Theorem 3, some set in this family, say  $f^{-1}(t')$ , has at least  $\frac{|S|}{k}$  members. Since  $\frac{|S|}{k} \geq \frac{|S|}{T} > r$  by our hypothesis,  $f^{-1}(t')$  has more than  $r$  elements.

**Corollary:** If  $f: S \rightarrow T$  and  $|S| > |T|$ , then  **$f$  is not injective**.

**Proof:** Putting  $r = 1$  in Theorem 4, we see that at least one of the sets  $f^{-1}(t)$  has more than one element.

We conclude this section with some more extensions of the pigeonhole principle.

**Theorem 5:** Suppose we put an infinity of objects in a finite number of boxes. Then at least one box must have an infinity of objects.

**Proof:** If every box contains only a finite number of objects, then the total number of objects must be finite. Hence the theorem.

**Theorem 6 (A generalisation of Theorem 3):** Let  $A_1, A_2, \dots, A_k$  be subsets of a finite set  $S$  such that each element of  $S$  is in at least  $t$  of the sets  $A_i$ . Then the average number of elements in the  $A_i$ s is at least  $t \cdot \frac{|S|}{k}$ . (Note that, in this statement, the sets  $A_i$  may overlap.)

We leave the proof to you to do, and give you some related exercises now.

- 
- E7) Every positive integer is given one of the seven colours in VIBGYOR. Show that at least one of the colours must have been used infinitely many times.
- E8) Let  $A$  be a fixed 10-element subset of  $\{1, 2, \dots, 50\}$ . Show that  $A$  possesses two different 5-element subsets, the sum of whose elements are equal.
- E9) The positive integers are grouped into 100 sets. Show that at least one of the sets has an infinity of even numbers. Is it necessary that at least one set should have an infinity of even numbers and an infinity of odd numbers?
- 

Let us now consider another very important counting principle.

### 3.3 INCLUSION-EXCLUSION PRINCIPLE

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Let us begin with considering the following situation: In a sports club with 54 members, 34 play tennis, 22 play golf, and 10 play both. There are 11 members who play handball, of whom 6 play tennis also, 4 play golf also, and 2 play both tennis and golf. How many play none of the three sports?

To answer this, let  $S$  represent the set of all members of the club. Let  $T$  represent the set of tennis playing members,  $G$  represent the set of golf playing members, and  $H$  represent the set of handball playing members. Let us represent the number of elements in  $A$  by  $|A|$ . Consider the number  $|S| - |T| - |G| - |H|$ . Is this the answer to the problem? No, because those who are in  $T$  as well as  $G$  have been subtracted twice. To compensate for this double subtraction, we may now consider the number  $|S| - |T| - |G| - |H| + |T \cap G| + |G \cap H| + |H \cap T|$ . Is this the answer? No, because those playing all the three games have been subtracted thrice and then added thrice. But those members have to be totally excluded also. Hence, we now consider the number

$|S| - |T| - |G| - |H| + |T \cap G| + |G \cap H| + |H \cap T| - |T \cap G \cap H|$ . This is the correct answer. This reduces to  $54 - 34 - 22 - 11 + 10 + 6 + 4 - 2 = 5$ .

To solve this problem we have made inclusions and exclusions alternately to arrive at the correct answer. This is a simple case of **the principle of inclusion and exclusion**. It is also known as the **sieve principle** because we subject the objects to sieves of a progressively finer mesh to arrive at a certain grading.

Let us state and prove this principle now.

$A^c$ , or  $\bar{A}$ , denotes the complement of the set  $A$

**Theorem 7 (The inclusion-exclusion formula):** Let  $A_1, A_2, \dots, A_n$  be  $n$  sets in a universal set  $U$  consisting of  $N$  elements. Let  $S_k$  denote the sum of the sizes of all the sets formed by intersecting  $k$  of the  $A_i$ s at a time. Then the number of elements in none of the sets  $A_1, A_2, \dots, A_n$  is given by

$$\left| \bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_n \right| = N - S_1 + S_2 - S_3 + \dots + (-1)^k S_k + \dots + (-1)^n S_n.$$

RHS is short for 'right-hand side'.

**Proof:** The proof is on the same lines of the counting argument given in the 'sports club' example at the beginning of this section. If an element is in none of the  $A_i$ s, then it should be counted only once, as part of ' $N$ ' in the RHS of the formula above. It is not counted in any of the  $S_k$ s since it is in none of the  $A_i$ s.

Next, an element in exactly one  $A_i$ , say  $A_r$ , is counted once in  $N$ , and once in  $S_1$ , and in none of the other  $S_k$ s. So the net count is  $1 - 1 = 0$ .

Finally, take an element  $x$  in exactly  $m$  of the  $A_i$ s. This is counted once in  $N$ ,  $m$  times in  $S_1$ ,  $C(m, 2)$  times in  $S_2$  (since  $x$  is in  $C(m, 2)$  intersections  $A_i \cap A_j$ ), ...,  $C(m, k)$  times in  $S_k$  for  $k \leq m$ .  $x$  is not counted in any  $S_k$  for  $k > m$ . So the net count of  $x$  in the RHS of the formula is

$$1 - C(m, 1) + C(m, 2) - \dots + (-1)^k C(m, k) + \dots + (-1)^m C(m, m) = 0, \text{ by Identity 2 in Sec. 2.5.}$$

So the only elements that have a net count of 1 in the RHS are those in  $\bigcap_{i=1}^n \bar{A}_i$ . The rest have a net count of 0. Hence the formula.

From this result, we immediately get the following one.

**Corollary:** Given the situation of Theorem 7,

$$\left| A_1 \cup A_2 \cup \dots \cup A_n \right| = S_1 - S_2 + \dots + (-1)^{k-1} S_k + \dots + (-1)^{n-1} S_n.$$

Why don't you try and prove this result? (see E 10.)

What the inclusion-exclusion principle tells us is that to calculate the size of  $A_1 \cup A_2 \cup \dots \cup A_n$ , calculate the size of all possible intersections of the sets  $A_1, A_2, \dots, A_n$ . Add the results obtained by intersecting an odd number of the sets, and then subtract the results obtained by intersecting an even number of the sets. Therefore, this principle is ideally suited to situations in which

- we just want the size of  $A_1 \cup A_2 \cup \dots \cup A_n$ , not a listing of its elements, and
- multiple intersections are fairly easy to count.

Now let us consider some examples in which Theorem 7 is applied.

**Example 8:** How many ways are there to distribute  $r$  distinct objects into five (distinct) boxes with

- at least one empty box?
- no empty box ( $r \geq 5$ )?

**Solution:** Let  $U$  be all possible distributions of  $r$  distinct objects into five boxes. Let  $A_i$  denote the set of possible distributions with the  $i$ th box being empty.

i) Then the required number of distributions with at least one empty box is

$|A_1 \cup A_2 \cup \dots \cup A_5|$ . We have  $N = 5^r$ . Also,  $|A_i| = (5-1)^r$ , the number of distributions in which the objects are put into one of the remaining four boxes. Similarly,  $|A_i \cap A_j| = (5-2)^r$ , and so forth. Thus, by the corollary above, we have

$$\begin{aligned} |A_1 \cup \dots \cup A_5| &= S_1 - S_2 + S_3 - S_4 + S_5 \\ &= C(5,1)4^r - C(5,2)3^r + C(5,3)2^r - C(5,4)1^r + 0 \end{aligned}$$

ii)  $|\bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_5| = 5^r - C(5,1)4^r + C(5,2)3^r - C(5,3)2^r + C(5,4)1^r$ , by

Theorem 7.

\* \* \*

**Example 9:** How many solutions are there to the equation  $x + y + z + w = 20$ , where  $x, y, z, w$  are positive integers such that  $x \leq 6, y \leq 7, z \leq 8, w \leq 9$ ?

**Solution:** To use inclusion-exclusion, we let the objects be the solutions (in positive integers) of the given equation. A solution is in  $A_1$  if  $x > 6$ , in  $A_2$  if  $y > 7$ , in  $A_3$  if  $z > 8$ , and in  $A_4$  if  $w > 9$ . Then what we need is  $|\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4|$ .

Now, to find the total number of **positive** solutions to the given equation, we rewrite it as  $x_1 + y_1 + z_1 + w_1 = 16$ , where  $x_1 = x+1, y_1 = y+1, z_1 = z+1, w_1 = w+1$ . Any non-negative solution of this equation will be a positive solution of the given equation. So, the number of positive solutions is

$$\begin{aligned} N &= C(16+4-1, 16) \text{ (see Example 11 of Unit 2)} \\ &= C(19, 3). \end{aligned}$$

$$\text{Similarly, } |A_1| = C(13, 3), |A_2| = C(12, 3), |A_3| = C(11, 3),$$

$|A_4| = C(10, 3), |A_1 \cap A_2| = C(6, 3), |A_2 \cap A_3| = C(5, 3)$ , and so on. **Note** that for a solution to be in 3 or more  $A_i$ s, the sum of the respective variables would exceed 20, which is not possible. By inclusion-exclusion, we obtain

$$\begin{aligned} |\bar{A}_1 \cap \bar{A}_2 \cap \bar{A}_3 \cap \bar{A}_4| &= C(19, 3) - C(13, 3) - C(12, 3) - C(11, 3) - C(10, 3) \\ &\quad + C(6, 3) + C(5, 3) + C(4, 3) + C(4, 3) + C(3, 3) = 217. \end{aligned}$$

\* \* \*

Now you may try the following exercises.

---

E10) Prove the corollary to Theorem 7.

E11) How many numbers from 0 to 999 are not divisible by either 5 or 7?

---

Let us now consider applications of the inclusion-exclusion principle to some specific problem types.

## 3.4 APPLICATIONS OF INCLUSION-EXCLUSION

In this section we shall consider three broad kinds of applications — for counting the number of surjective functions, finding probability and finding the number of derangements.

### 3.4.1 Application to Surjective Functions

Let us first recall that a function  $f : S \rightarrow T$  is called **surjective** (or **onto**) if  $f(S) = T$ , that is, if for every  $t \in T \exists s \in S$  such that  $f(s) = t$ . Now let us prove a very useful result regarding the number of such functions.

**Theorem 8:** The number of functions from an  $m$ -element set **onto** a  $k$ -element set is

$$\sum_{i=0}^k (-1)^i C(k, i)(k-i)^m, \text{ where } 1 \leq k \leq m.$$

**Proof:** We will use the inclusion-exclusion principle to prove this. For this, we define the objects to be all the functions (not just the onto functions) from  $M$ , an  $m$ -element set, to  $K$ , a  $k$ -element set. For these objects, we will define  $A_i$  to be the set of all  $f : M \rightarrow K$  for which the  $i$ th element of  $K$  is not in  $f(M)$ . Then what we want is

$$\left| \bar{A}_1 \cap \dots \cap \bar{A}_k \right|.$$

Now, the total number of functions from  $M$  to  $K$  is  $k^m$ . Also, the number of mappings that exclude a specific set of  $i$  elements in  $K$  is  $(k-i)^m$ , and there are  $C(k, i)$  such sets. Therefore,  $|A_i| = (k-1)^m$ ,  $|A_i \cap A_j| = (k-2)^m$ , and so on.

Now, applying Theorem 7, we get

$$\left| \bar{A}_1 \cap \dots \cap \bar{A}_k \right| = k^m - C(k, 1)(k-1)^m + C(k, 2)(k-2)^m - \dots + (-1)^{k-1} C(k, k-1)1^m$$

Hence the result.

For example, the number of functions from a five-element set onto a three-element set are  $\sum_{i=0}^3 (-1)^i C(3, i)(3-i)^5$  for  $m = 5$  and  $k = 3$ , that is,  $3^5 - 3 \cdot 2^5 + 3 \cdot 1^5 = 150$ .

Why don't you try some exercises now?

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E12) Eight people enter an elevator. At each of four floors it stops, and at least one person leaves the elevator. After four floors the elevator is empty. In how many ways can this happen?

E13) How many six-digit numbers contain exactly three different digits?

---

Now we look at another application.

### 3.4.2 Application to Probability

An important application of the principle of inclusion-exclusion is used in probability. We have the following theorem.

**Theorem 9:** Suppose  $A_1, A_2, \dots, A_n$  are  $n$  events in a probability space. Then

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = \sum_{r=1}^n (-1)^{r+1} \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} P(A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_r})$$

**Proof:** Let us begin by observing that  $A_1 \cup A_2 \cup \dots \cup A_n$  means that at least one of the events  $A_1, A_2, \dots, A_n$  occurs. Now, let the  $i$ th property be that an outcome belongs to the event  $A_i$ . By De Morgan's law,  $\bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_n$  is the complement of  $A_1 \cup A_2 \cup \dots \cup A_n$ . But the principle of inclusion-exclusion gives

$$\left| \bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_n \right| = N - \sum_{r=1}^n (-1)^{r+1} \sum_{1 \leq i_1 < i_2 < \dots < i_r \leq n} \left| A_{i_1} \cap A_{i_2} \cap \dots \cap A_{i_r} \right|, \text{ where } N \text{ is}$$

the total number of outcomes.

Now, we divide throughout by  $N$  and note that

$$P(A_1 \cup A_2 \cup \dots \cup A_n) = 1 - P(\bar{A}_1 \cap \bar{A}_2 \cap \dots \cap \bar{A}_n), \text{ to get the result.}$$

Let us consider an example of the use of this result.

**Example 12:** Find the probability of a student in a college studying Japanese, given the following data:

All students have to study at least one language out of Hindi, Spanish and Japanese. 65 study Hindi, 45 study Spanish and 42 study Japanese. Further, 20 study Hindi and Spanish, 25 study Hindi and Japanese, 15 study Spanish and Japanese, and 8 study all 3 languages.

**Solution:** The total number of students is  $|H \cup S \cup J|$ , where  $H$ ,  $S$  and  $J$  denote the number of students studying Hindi, Spanish and Japanese, respectively. By the inclusion-exclusion principle,

$$\begin{aligned} |H \cup S \cup J| &= |H| + |S| + |J| - |H \cap S| - |H \cap J| - |S \cap J| + |H \cap S \cap J| \\ &= 65 + 45 + 42 - 20 - 25 - 15 + 8 = 100 \end{aligned}$$

Therefore, the required probability is  $\frac{|J|}{100} = 0.42$ .

\* \* \*

You could do the following exercises now.

---

E14) What is the probability that a 13-card hand has at least one card in each suit?

E15) What is the probability that a number between 1 and 10,000 is divisible by neither 2, 3, 5 nor 7?

---

Let us now come to the use of inclusion-exclusion for counting the number of a particular kind of permutation.

### 3.4.3 Application to Derangements

As you know, a permutation of a set is an arrangement of the elements of a set.

So, for example, a rearrangement  $1 \rightarrow 1, 2 \rightarrow 2, 4 \rightarrow 3, 3 \rightarrow 4$  is a permutation of the 4-element set  $\{1, 2, 3, 4\}$ . In this permutation, the position of the elements 1 and 2 are **fixed**, but the positions of 3 and 4 have been interchanged. Now consider the permutation  $1 \rightarrow 4, 2 \rightarrow 1, 3 \rightarrow 2, 4 \rightarrow 3$ , of  $\{1, 2, 3, 4\}$ , in which the position of

**every element** has been changed. This is an example of a derangement, a term we shall now define.

**Definition:** A **derangement** of a set  $S$  is a permutation of the elements of  $S$  which does not fix any element of  $S$ , i.e., it is a rearrangement of the elements of  $S$  in which the position of every element is altered.

So, if we treat a permutation as a 1-to-1 function from  $S$  to  $S$ , then a derangement is a function  $f: S \rightarrow S$  such that  $f(s) \neq s \forall s \in S$ .

We have the following theorem regarding the number of derangements.

**Theorem 10:** The number of derangements of an  $n$ -element set is  $D_n = n! \sum_{i=0}^n \frac{(-1)^i}{i!}$ .

**Proof:** Let  $A_i$  be the set of all permutations of the  $n$ -element set that fix  $i \forall i = 1, \dots, n$ . Then

$$\begin{aligned} D_n &= \left| \bigcap_{i=1}^n \bar{A}_i \right| = n! - \sum_i |A_i| + \sum_{i < j} |A_i \cap A_j| - \dots + (-1)^n |A_1 \cap \dots \cap A_n| \\ &= n! - C(n, 1)(n-1)! + C(n, 2)(n-2)! - \dots + (-1)^n C(n, n)0! \\ &= n! \left( 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!} \right), \text{ which is the expression we wanted.} \end{aligned}$$

**Remark:** The expression  $\left( 1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!} \right)$  is the beginning of the expansion for  $e^{-1}$ . Even for moderately large values of  $n$ ,  $D_n$  is very close to  $n!e^{-1} = 0.36788 n!$ .

As an extension of Theorem 10, we have the following results.

**Theorem 11:** For a set of  $n$  objects, the number of permutations in which

(i) only  $r$  of these  $n$  objects are deranged is

$$n! - C(r, 1)(n-1)! + C(r, 2)(n-2)! - \dots + (-1)^r C(r, r)(n-r)!;$$

(ii) exactly  $r$  elements are fixed is  $C(n, r) D_{n-r}$ .

We will not prove these formulae here, but shall consider some examples of their applications.

**Example 12:** Let  $n$  books be distributed to  $n$  children. The books are returned and distributed to the children again later on. In how many ways can the books be distributed so that no child will get the same book twice?

**Solution:** The required number is  $(n!)^2 e^{-1}$ , since corresponding to each first distribution, there are  $(n!)e^{-1}$  ways of distributing again.

\* \* \*

**Example 13:** Suppose 10 people have exactly the same briefcases, which they leave at a counter. The cases are handed back to the people randomly. What is the probability that no one gets the right case?

**Solution:** The number of possibilities favourable to the event is  $D_{10}$ . The total number of possibilities is  $10!$ . Thus, the probability that none will get the right briefcase is  $D_{10}/10! = 0.36788$ .

\* \* \*

**Note** that, since  $D_n \approx n!e^{-1}$ , the possibility in all such examples is essentially  $e^{-1}$ , which is independent of  $n$ .

You may now try the following exercises.

- 
- E16) Each of the  $n$  guests at a party puts on a coat when s/he leaves. None of them gets the correct coat. In how many ways can this happen? In how many ways can just one of the guests get the right coat?
- E17) In how many ways can the integers  $1, 2, 3, \dots, 9$  be permuted such that no odd integer will be in its natural position.
- E18) Find the number of permutations in which exactly four of the nine integers  $1, 2, \dots, 9$  are fixed.
- 

With this we come to the end of this unit. In the next unit we shall continue our discussion on ‘counting’ from a slightly different perspective. Let us now summarise what we have covered in this unit.

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## 3.6 SUMMARY

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In this unit, you have studied the following points.

1. The pigeonhole principle, stated in several forms, its proof, and its applications.
  2. The generalized pigeonhole principle, its proof, and applications.
  3. The inclusion-exclusion principle, and its proof.
  4. Finding the number of surjective functions, the discrete probability and the number of derangements, by using the inclusion-exclusion principle.
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## 3.7 SOLUTIONS /ANSWERS

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- E1) By drawing lines parallel to the sides and through the points trisecting each side, we can divide the equilateral triangle into 9 equilateral triangles of side 1 cm. Thus, if 10 points are chosen, at least two of them must lie in one of the 9 triangles.
- E2) 5 persons can be paired in  $C(5, 2) = 10$  ways. Hence, if pairs are invited 11 times, at least one pair must have been invited twice or more times, by the pigeonhole principle.
- E3) Four persons can be arranged in a line in  $4! = 24$  ways. Hence, if we consider 25 occasions, at least on two occasions the same ordering in the queue must have been found, by the pigeonhole principle.
- E4) Consider the following grouping of numbers:
- $\{1, 2, 4, 8, 16\}, \{3, 9, 18\}, \{5, 15\}, \{6, 12\}, \{7, 14\}, \{10, 20\}, \{11\}, \{13\}, \{17\}, \{19\}.$

There are 10 groupings, exhausting all the 20 integers from 1 to 20. If 11 numbers are chosen it is impossible to select at most one from each group. So two numbers have to be selected from some group. Obviously one of them will divide the other.



- E5) Suppose  $x_1, x_2, \dots, x_{15}$  are the number of balls in the 15 boxes, listed in increasing order, assuming that all these numbers are different. Then, clearly,  $x_i \geq i - 1$  for  $i = 1, 2, \dots, 15$ . But then,  $\sum_{i=1}^{15} x_i \geq 14 \cdot 15 / 2 = 105$ .

But the total number of balls is only 100, a contradiction. Thus, the  $x_i$ s cannot all be different.

- E6) In the sequence  $a_1, a_2, \dots, a_n$ , there are  $(n+1)/2$  odd numbers and  $(n-1)/2$  even numbers because  $n$  is odd. Hence, it is impossible to pair all the  $a_i$ s with numbers from  $1, 2, \dots, n$  with opposite parity (evenness and oddness). Hence, in at least one pair  $(i, a_i)$ , both the numbers will be of the same parity. This means that the factor  $(a_i - i)$  will be even, and hence the product will be even.
- E7) Consider the seven colours as containers, and the integers getting the respective colour as their contents. Then we have a distribution of an infinite number of objects among 7 containers. Hence, by Theorem 5, at least one container must have an infinity of objects, that is, the colour of that container must have been used an infinite number of times.

- E8) Let  $H$  be the family of 5-element subsets  $B$  of  $A$ . For each  $B$  in  $H$ , let  $f(B)$  be the sum of the numbers in  $B$ . Obviously, we must have

$$f(B) \geq 1 + 2 + 3 + 4 + 5 = 15, \text{ and } f(B) \leq 46 + 47 + 48 + 49 + 50 = 240.$$

Hence,  $f: H \rightarrow T$  where  $T = \{15, 16, \dots, 240\}$ .

Since  $|T| = 226$  and  $|H| = C(10, 5) = 252$ , by Theorem 4,  $H$  contains different sets with the same image under  $f$ , that is different sets, the sums of whose elements are equal.

- E9) The 100 collections can be considered as containers. There are an infinity of even numbers. When these even numbers are distributed into 100 containers, at least one container must have an infinity of them, by Theorem 5.

- E10) The inclusion-exclusion formula can be rewritten as

$$\left| \bar{A}_1 \cap \dots \cap \bar{A}_n \right| = N - (S_1 - S_2 + \dots + (-1)^{n-1} S_n).$$

$$\text{Also, we know that } \left| \bar{A}_1 \cap \dots \cap \bar{A}_n \right| = N - |A_1 \cup \dots \cup A_n|.$$

Hence the result.

- E11) Let the objects be the integers  $0, 1, \dots, 999$ . Let  $A_1$  be the set of numbers divisible by 5, and  $A_2$  the set of numbers divisible by 7. Now,  $N = 1000$ ,  $|A_1| = 200$ ,  $|A_2| = 143$  and  $|A_1 \cap A_2| = 29$ . So, by Theorem 7, the answer is  $1000 - 200 - 143 + 29 = 686$ .

- E12) The answer to this problem is clearly the number of functions from an 8-element set (the set of people) onto a set of 4-elements (the set of floors). This number is

$$\sum_{i=0}^4 C(4, i)(4-i)^8 = 4^8 - 4 \cdot 3^8 + 6 \cdot 2^8 - 4 \cdot 1^8.$$

- E13) We can choose three digits in  $C(10, 3) = 120$  ways. The number of 6-digit numbers, using all the three digits, is the same as the number of functions from a 6-set onto a 3-set. This number is

$$3^6 - 3 \cdot 2^6 + 3 \cdot 1^6 = 540.$$

Hence, the answer is  $120 \cdot 540 = 64800$ . This will include numbers starting with 0 also.

- E14) The total number of ways in which 13 cards can be chosen from a deck of 52 cards is  $C(52, 13)$ .

If  $A_i$  is a choice of cards, none of which are from the  $i$ th suit, for  $i = 1, 2, 3, 4$ , then  $|A_i| = C(39, 13)$ ,  $|A_i \cap A_j| = C(26, 13)$ , and  $C(A_i \cap A_j \cap A_k) = C(13, 13)$ .

$$\text{So, } |\cap \bar{A}_i| = C(52, 13) - 4C(39, 13) + C(4, 2)C(26, 13) - C(4, 3)C(13, 13)$$

$$\text{Hence, the required probability is } \frac{|\cap \bar{A}_i|}{C(52, 13)}.$$

- E15) If A, B, C, D are the integers divisible by 2, 3, 5, 7, respectively, then

$$\begin{aligned} |\bar{A} \cap \dots \cap \bar{D}| &= 10,000 - \left[ \frac{10000}{2} \right] - \left[ \frac{10000}{3} \right] - \left[ \frac{10000}{5} \right] - \left[ \frac{10000}{7} \right] \\ &+ \left[ \frac{10000}{6} \right] + \left[ \frac{10000}{15} \right] + \left[ \frac{10000}{35} \right] + \left[ \frac{10000}{14} \right] + \left[ \frac{10000}{21} \right] + \left[ \frac{10000}{10} \right] \\ &- \left[ \frac{10000}{30} \right] - \left[ \frac{10000}{42} \right] - \left[ \frac{10000}{105} \right] - \left[ \frac{10000}{70} \right] + \left[ \frac{10000}{210} \right] \\ &= 2285, \text{ where } [x] \text{ denotes the greatest integer } \leq x. \end{aligned}$$

$$\text{Hence, the required probability is } \frac{2285}{10000} = 0.23.$$

- E16) If  $A_r$  is the event that the  $r$ th person gets the right coat, then by Theorem 7,

$$\begin{aligned} |\cap \bar{A}_i| &= n! - \sum_r |A_r| + \sum_{r,s} |A_r \cap A_s| - \dots \\ &= n! - n(n-1)! + C(n, 2)(n-2)! - C(n, 3)(n-3)! + \dots \\ &= C(n, 2)(n-2)! - C(n, 3)(n-3)! + \dots \\ &= n! \left( \sum_{r=0}^n (-1)^r \frac{1}{r!} \right) \end{aligned}$$

The number of ways in which only one person receives the correct coat is the

sum of all possible intersections of  $(n-1) \bar{A}_i$  s. This is

$$n! n(n-1)! \left( \sum_{r=0}^{n-1} (-1)^r \frac{1}{r!} \right) = n! \left( \sum_{r=0}^{n-1} (-1)^r \frac{1}{r!} \right).$$

- E17) 1, 3, 5, 7, 9 are the odd integers.

By Theorem 11(i), the required number of ways is

$$9! - C(5, 1)8! + C(5, 2)7! - C(5, 3)6! + C(5, 4)5! - C(5, 5)4!$$

- E18) By Theorem 11(ii), the required number of permutations is

$$C(9, 4)D_{9-4} = C(9, 4)D_5.$$



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## UNIT 4 PARTITIONS AND DISTRIBUTIONS

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### 4.0 INTRODUCTION

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In the last two units we have exposed you to a variety of combinatorial techniques. In this unit we look at a few more ways of counting arrangements of objects when order matters, and when it doesn't.

In Sec. 4.2, we focus on the ways in which a natural number can be written as a sum of natural numbers. In the process you will be introduced to a useful 'recurrence relation'.

We link this, in Sec. 4.3, with the different ways in which  $n$  objects can be distributed among  $m$  containers. As you will see, there are four broad possible kinds of distributions. In each case, we consider ways of counting all the distributions. In the process you will also be introduced to Stirling numbers.

With this unit we come to the end of our discussion on counting techniques. Some of the problems you have studied here will be looked at from different approaches in our later course MCS-033.

You should attempt the assignment based on the course after studying this unit, and this block.

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### 4.1 OBJECTIVES

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After going through this unit, you should be able to:

- define an integer partition, and count the number of partitions of an integer;
- count the number of ways of distributing distinguishable and indistinguishable objects, respectively, into distinguishable containers;
- count the number of ways of distributing distinguishable and indistinguishable objects, respectively, into indistinguishable containers.

---

### 4.2 INTEGER PARTITIONS

---

Suppose a detergent manufacturer wants to promote her product by giving a gift token with 100 bars out of the whole stock. The lucky persons among her customers will get the gift. Some of them may buy more than one bar at a time with the hope of getting gifts. In how many ways can the 100 gift tokens get distributed? One possible way is that all the 100 bars with gifts are bought by 100 different customers. We can indicate this situation by  $100 = \underbrace{1 + 1 + \dots + 1}_{100 \text{ times}}$ . Another possibility is that somebody buys 2 bars,

somebody else buys 3 bars, and the remaining 95 bars are distributed amongst 95 different people. We are not interested in the order in which the bars are bought. For example, here we are not interested in whether the person who bought 2 bars bought them before the person who bought the 3 bars. So, we can indicate this situation by  $100 = \underbrace{1 + 1 + \dots + 1}_{95 \text{ times}} + 2 + 3$ . More generally, we can indicate each way of distributing

the 100 bars with gifts by  $100 = p_1 + p_2 + p_2 + \dots + p_k$ , where the  $p_i$  are natural numbers, and  $p_1 \leq p_2 \leq \dots \leq p_k$ . Each way of writing 100 in this form is called an **integer partition** of 100. More generally, we have the following definition.

**Definition:** Any representation of  $n \in \mathbf{N}$  as a sum of positive integers in non-increasing order is called a **partition** (or **integer partition**) of  $n$ . Each such partition can be written in the form  $n = p_1 + p_2 + \dots + p_k$ , where  $p_1 \leq p_2 \leq \dots \leq p_k$ .

Here,  $p_1, p_2, \dots, p_k$  are called the **parts** of the partition, and the **number of parts** of the partition is  $k$ .

While we chose 100 in the example above, it is really a huge number in the context of integer partitions. Let us consider a smaller number, say 5. How many partitions of 5 can you think of? There are 7 altogether, namely, 5, 1+4, 2+3, 1+1+3, 1+2+2, 1+1+1+2 and 1+1+1+1+1.

In books, you will often come across the notation  $p(n)$  for the number of partitions of  $n$ .

If we represent the number of partitions of the integer  $n$  by  $P_n$ , we have shown that  $P_5 = 7$ . These partitions have 1, 2, 2, 3, 3, 4 and 5 parts, respectively.

If we represent the number of partitions of  $n$  with exactly  $k$  parts by  $P_n^k$ , then we have  $P_5^1 = 1, P_5^2 = 2, P_5^3 = 2, P_5^4 = 1, P_5^5 = 1$ .

To check your understanding of the material so far, try the following exercises.

- 
- E1) Write down all the partitions of 7. Also find  $P_7^4$  and  $P_7^5$ .
- E2) Let us consider the situation of the detergent manufacturer again. Suppose she only wants to distribute 10 gift tokens in 5 specific sales districts, where the sales are low. What is the number of ways of doing this?
- 

You may wonder if you've found all the partitions in E2. One way to check is by finding out the required number in terms of partitions of smaller numbers, which may be easier to find. One such relation between partitions of  $n$  and  $n+1, n+2$ , etc. is given in the following theorem.

**Theorem 1:**  $P_n^1 + P_n^2 + \dots + P_n^k = P_{n+k}^k, P_n^1 = P_n^n = 1$ , for  $1 \leq k \leq n$ , that is, the number of **partitions of  $n$  with at most  $k$  parts** is the same as the number of **partitions of  $n+k$  with exactly  $k$  parts**.

Before we begin the proof of this theorem, let us consider an example. Let us take  $n = 4, k = 3$ . According to Theorem 1, we must have  $P_4^1 + P_4^2 + P_4^3 = P_7^3$ . Note that  $P_4^1 + P_4^2 + P_4^3$  is the total number of partitions of 4 with 1, 2 or 3 parts, i.e., the number of partitions with **at most 3 parts**. There is one partition of 4 with one part,  $4 = 4$ . Let us write this as a 3-tuple,  $(4, 0, 0)$ , adding two more zeroes since we are considering partitions with at most 3 parts. If we add 1 to all the entries of this 3-tuple, we get  $(4+1, 0+1, 0+1) = (5, 1, 1)$  and  $1+1+5$  is a partition of 7 with three parts. Similarly, consider the partition  $4 = 1+3$  of 4 into two parts. Again, we can write this as  $(1, 3, 0)$ .

Now, if we add 1 to each of the entries, we get (2, 4, 1) and  $1+2+4$  is a partition of 7 into three parts. Conversely, if we take the partition  $7 = 1 + 3 + 3$  of 7 into three parts, write it as (1, 3, 3) and **subtract** 1 from all the entries, we get (0, 2, 2) which corresponds to the partition  $4 = 2+2$  of 4 into 2 parts. In this way, we can match every partition of 4 with **at most 3** parts with a partition of 7 with **exactly 3** parts, and vice versa. This is the basic idea behind our proof of Theorem 1, which we now give.

**Proof of Theorem 1:** The cases  $P_n^1 = 1 = P_n^n$  follow from the definition.

We will prove the general formula now. Let  $M$  be the set of partitions of  $n$  having  $k$  or less parts. We can consider each partition belonging to  $M$  as a  $k$ -tuple after adding as many zeroes as necessary. Define the mapping

$$(p_1, p_2, \dots, p_m, \underbrace{0, 0, \dots, 0}_{(k-m) \text{ times}}) \mapsto (p_1+1, p_2+1, \dots, p_m+1, \underbrace{1, 1, \dots, 1}_{(k-m) \text{ times}}), m \leq k$$

from  $M$  into the set  $M'$  of partitions of  $n+k$  into exactly  $k$  parts. This mapping is bijective, since

- i) two distinct  $k$ -tuples in  $M$  are mapped onto two distinct  $k$ -tuples in  $M'$ ;
- ii) every  $k$ -tuple in  $M'$  is the image of a  $k$ -tuple of  $M$ . This is because, if  $(p_1, p_2, \dots, p_k)$  is a partition of  $n+k$  with  $k$  parts, then it is the image of  $(p_1 - 1, p_2 - 1, \dots, p_k - 1)$  under the mapping above.

Therefore,  $|M| = P_n^1 + \dots + P_n^k = |M'| = P_{n+k}^k$ , and the theorem is proved.

**Note** that  $P_n^k = 0$  if  $n < k$ , since there is no partition of  $n$  with  $k$  parts if  $n < k$ . Also,  $P_n^n = P_n^1 = 1$ .

The formula in Theorem 1 is an identity which allows us to find  $P_n^r$  from values of  $P_m^k$ , where  $m < n$ ,  $k \leq r$ . This is why it is also called a **recurrence relation** for  $P_n^k$ .

Theorem 1 is every useful. For instance, to verify your count in E2, you can use it because  $P_{10}^5 = P_5^1 + P_5^2 + \dots + P_5^5 = 7$ .

From Theorem 1, the  $P_n^k$ s may be calculated recursively as shown in Table 1.

**Table 1 :  $P_n^k$  for  $1 \leq n, k \leq 6$**

$\begin{smallmatrix} k \\ \backslash \\ n \end{smallmatrix}$	1	2	3	4	5	6
1	1	0	0	0	0	0
2	1	1	0	0	0	0
3	1	1	1	0	0	0
4	1	2	1	1	0	0
5	1	2	2	1	1	0
6	1	3	3	2	1	1

In this table, the second entry in the row corresponding to  $n = 4$  is  $P_4^2$ . By Theorem 1,  $P_4^2 = P_2^1 + P_2^2$ , which is the sum of the entries in the row corresponding to  $n = 2$ .

Similarly,  $P_6^3$  is the sum of the entries in the row corresponding to  $n = 3$ .

Now, here is an exercise for you.

---

E3) Use Table 1 to find the values of  $P_7^k, 1 \leq k \leq 6$ .

---

The partition of a number  $n$  into  $k$  parts also tells us how  $n$  objects can be distributed among  $k$  boxes. We will now consider all possibilities of such distributions.

---

## 4.3 DISTRIBUTIONS

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By a distribution we mean a way of placing several objects into a number of containers. For example, consider the distribution of 6 balls among 3 boxes. We may have all 6 balls of different shapes, sizes and colours, i.e., they are all distinguishable. Or, all the balls could be exactly the same, i.e., they are all indistinguishable.

Similarly, all 3 boxes may look different, or all 3 could be exactly the same. So, we see that there are 4 possibilities here.

In fact, we have the following possibilities for any set of  $n$  objects and  $m$  boxes.

**Case 1:** The objects are distinguishable, and so are the boxes;

**Case 2:** The objects are distinguishable and the boxes are indistinguishable;

**Case 3:** The objects are indistinguishable and the boxes are distinguishable;

**Case 4:** The objects are indistinguishable, and so are the boxes.

You may be surprised to know that in each of the cases the number of such distributions is different. In fact, the distribution problem is to count all possible distributions in any of these situations, or in a combination of these cases.

A general guideline for modelling a ‘distribution problem’ is that a distribution of distinct objects corresponds to an arrangement, and a distribution of identical objects corresponds to a selection. Let us consider examples of each of the four cases given above.

- (a) There are twenty students and four colleges. In how many ways can the students be accommodated in the four colleges?

In this example the students, as well as the colleges, are clearly distinguishable. This comes under Case (1).

- (b) Suppose we want to group 100 students into 10 groups of 10 each for the purpose of a medical examination. In how many ways can this be done?

Here the groups are indistinguishable, though the students in them are distinguishable. Hence, this falls under Case (2).

- (c) An employer wants to distribute 100 one-rupee notes among 6 employees. What is the number of ways of doing this?

Though the one-rupee notes can be distinguished by their distinct numbers, we don’t consider them to be distinguishable as far as their use is concerned. The employees, of course, are distinguishable. Hence, this is an example of Case (3).

- (d) There are 1000 one-rupee notes. In how many ways can they be bundled into 20 bundles?

As before, the rupee notes are treated as indistinguishable. Clearly, the bundles are, by themselves, not distinguishable. Only the quantity in each may vary. Hence, this falls under Case (4).

Let us consider each case in some detail now.

### 4.3.1 Distinguishable Objects into Distinguishable Containers

Let us consider the example (A) above. Since any number of students can be put in a college, there are  $20 \times 20 \times 20 \times 20$  possibilities, by the multiplication principle.

More generally, suppose we are distributing  $n$  objects into  $m$  containers, both being distinguishable. Then the total number of such distributions is  $n^m$ .

Let us look at an example.

**Example 1:** Show that the number of words of length  $n$  on an alphabet of  $m$  letters is  $m^n$ .

**Solution:** The  $m$  letters of the alphabet can be used any number of times in a word of  $n$  letters. The word can be considered as  $n$  ordered boxes, each holding a letter from the alphabet. The boxes become distinguishable because they are 'ordered'. The letters of the alphabet are clearly distinguishable. So, the number of ways of doing this is  $m^n$ .

\* \* \*

Several people are confused while solving the problem above. They tend to take the  $m$  letters as the containers instead! Let's consider another example.

**Example 2:** Suppose we have a set  $S$  with  $n$  objects. An  $m$ -sample from this set  $S$  is an ordered arrangement of  $m$  letters taken from  $S$ , with replacement at every draw, in  $m$  draws. Find the number of  $m$ -samples from an  $n$ -element set.

**Solution:** Every  $m$ -sample is a word of length  $m$  from the 'alphabet'  $S$  containing  $n$  letters. Hence, the required number is  $n^m$ .

\* \* \*

Now here are some exercises for you to solve.

- 
- E4) Find the number of three-letter words that can be formed using the letters of the English alphabet. How many of them end in  $x$ ? How many of them have a vowel in the middle position?
- E5) How many five-digit numbers are even? How many five-digit numbers are composed of only odd digits?
- E6) There are 4 women and 5 men. A committee of three, a president, a vice-president, and a secretary, has to be formed from them. In how many ways can this be done if
- the vice-president should be a woman?
  - exactly one out of the vice-president and the secretary should be a woman?
  - there is at least one woman in the committee?
- 

Now suppose, we want to find the number of distributions of  $n$  distinguishable objects into  $m$  distinguishable containers, with **the extra condition** that no container should



contain more than one object. It is clear that this requires  $m \geq n$ . Then we can get all these arrangements by first choosing  $n$  containers to contain exactly one object, and then permuting the  $n$  objects among the chosen containers. This can be done in  $C(m, n) \cdot n! = P(m, n)$  ways. So, we have proved the following result.

**Theorem 2:** The number of ways of distributing  $n$  distinguishable objects into  $m$  distinguishable containers such that no container contains more than one object is  $P(m, n)$ .

For example, the cardinality of the set of 5-digit numbers with all digits being distinct odd numbers is  $P(5, 5)$ . This is because the possible digits are 1, 3, 5, 7, 9.

Why don't you try an exercise now?

- 
- E7) Find the number of  $m$ -letter words with distinct letters, all taken from an alphabet with  $n$  letters, where  $n \geq m$ . Is this different from the number of injective mappings from an  $m$ -element set into an  $n$ -element set, where  $n \geq m$ ? Give reasons for your answer.
- 

Let us now consider the second type of distribution.

### 4.3.2 Distinguishable Objects into Indistinguishable Containers

Here we shall find the number of ways of distributing  $n$  distinguishable objects into  $m$  indistinguishable containers. For this, we first find the number when exactly  $k$  of the containers are occupied. This brings us to Stirling numbers of the second kind, named after James Stirling (1692-1770).

Suppose  $n \geq m$ . The number of distributions of  $n$  distinguishable objects into  $m$  indistinguishable containers **such that no container is empty** is represented by  $S_n^m$ . This number is called the Stirling number of the second kind. As you can see, this is also the number of partitions of a set of  $n$  objects into  $m$  classes.

**Definition:** For natural numbers  $n$  and  $m$ , the **Stirling number of the second kind**,  $S_n^m$ , is the number of partitions of an  $n$ -element set into exactly  $m$  parts.

Note that:

- i)  $S_n^m = 0$  if  $n < m$ , for, if the number of containers exceeds the number of objects, then it is impossible to have all the containers non-empty.
- ii)  $S_n^n = 1$ , since there is only one way of putting  $n$  distinguishable objects in  $n$  indistinguishable boxes so that no box is empty.
- iii)  $S_n^1 = 1$ .

Now, we shall use the inclusion exclusion principle to find the value of  $S_n^m$ .

**Theorem 3:**  $S_n^m = \frac{1}{m!} \sum_{k=0}^m (-1)^k C(m, m-k)(m-k)^n$ ,  $n \geq m$ .

**Proof:** If the  $m$  classes are distinguishable, the number of partitions is the same as the number of functions from an  $n$ -element set onto an  $m$ -element set. As the classes are distinguishable here, we have to divide this number by  $m!$ . The result follows from Theorem 8, Unit 3.

For example, to obtain the Stirling number,  $S_3^3$ , we know that the number of functions from a 5-element set onto a three-element set is 150. So, by Theorem 3,  $S_3^3 = 150/3! = 25$ .

**Remark:** You may be wondering how we have jumped straightaway to the Stirling numbers of the second kind. What about the first kind? We won't be using them in any way here. However, for the sake of completeness, we define **Stirling numbers of the first kind,  $s(n, k)$** , as follows.

For a positive integer  $n$ , and  $0 \leq k \leq n$ ,  $s(n, k)$  is the coefficient of  $x^k$  in the expansion of the multinomial  $x(x-1)(x-2)\dots(x-n+1)$ .

Getting back to  $S_n^m$ , you may feel that the formula in Theorem 3 is a little cumbersome. Sometimes, the following recurrence relation for  $S_n^m$  may be more useful.

**Theorem 4:** If  $1 < m \leq n$ , then  $S_{n+1}^m = S_n^{m-1} + mS_n^m$ .

**Proof:** Let us take  $n+1$  objects, mark one of them, and consider the distribution of these  $n+1$  objects into  $m$  indistinguishable containers. Then we have 2 situations.

**Case (1)** (The marked object is placed in one container without any other objects.): In this case, the remaining  $n$  objects can be placed in  $(m-1)$  containers in  $S_n^{m-1}$  ways.

**Case (2)** (The marked object is placed with at least one more object in a container.): In this case, we can first distribute the  $n$  unmarked objects into  $m$  containers, and then put the marked objects  $m$  to one of these  $m$  containers. So, the number of such partitions is  $mS_n^m$ .

Therefore, by the addition principle, we get  $S_{n+1}^m = S_n^{m-1} + mS_n^m$ .

There is a generalisation of Theorem 4 that is of independent interest, which we now state.

**Theorem 5:**  $S_{n+1}^m = \sum_{k=0}^n C(n, k) S_k^{m-1}$

**Proof:** Let us mark one object in a set of  $(n+1)$  objects. Suppose the marked object is present in a box with  $(n-k+1)$  elements, where  $m-1 \leq k \leq n$ . Then we can choose  $n-k$  more objects to go with the marked object in  $C(n, n-k)$  ways. The remaining  $k$  objects can be distributed into  $(m-1)^n$  boxes in  $S_k^{m-1}$  ways. So the number of ways of distributing the  $n-k$  objects is  $C(n, n-k) S_k^{m-1}$ . The result now follows from the addition principle by allowing  $k$  to vary from 0 to  $n$ .

Let us see some examples of the use of these recurrences.

**Example 3:** Calculate  $S_3^2$  and  $S_4^2$ .

**Solution:** Using Theorem 4, we get  $S_3^2 = S_2^1 + 2 \times S_2^2 = 1 + 2 \times 1 = 3$ , and  $S_4^2 = S_3^1 + 2S_3^2 = 1 + 2 \times 3 = 7$ .

\* \* \*

Now let us find what we had started with in this sub-section.

**Theorem 6:** The number of ways of distributing  $n$  distinguishable objects into  $m$  indistinguishable containers is  $S_n^1 + S_n^2 + \dots + S_n^m$ , where  $n \geq m$ . (Note that here we do not insist that no container is empty.)

**Proof:** When we distribute  $n$  distinguishable objects into  $m$  indistinguishable containers there are  $m$  cases. Case  $(k)$  is that exactly  $k$  containers are non-empty. Here  $k$  varies from 1 to  $m$ . The number of distributions in Case  $(k)$  is  $S_n^k$ . The result now follows from the addition principle.

Let us consider an example.

**Example 4:** In how many ways can 20 students be grouped into 3 groups?

**Solution:** Theorem 6 says that this can be done in  $S_{20}^1 + S_{20}^2 + S_{20}^3$  ways.

Now, using Theorem 3, we get this number to be

$$1 + \frac{1}{2} \sum_{k=0}^2 (-1)^k C(2, 2-k)(2-k)^{20} + \frac{1}{6} \sum_{k=0}^3 (-1)^k C(3, 3-k)(3-k)^{20} \\ = 581,130,734.$$

\* \* \*

Try some exercises now.

- 
- E8) Find the number of surjective functions from an  $n$ -element set onto an  $m$ -element set.
- E9) Find the number of ways of placing  $n$  people in  $n - 1$  rooms, no room being empty.
- 

Let us now consider the third possibility for distributing objects into containers.

### 4.3.3 Indistinguishable Objects into Distinguishable Containers

Suppose there are  $n$  indistinguishable objects and  $m$  distinguishable containers. As the objects are indistinguishable, the distributions depend only on the number of objects in each container. As the containers are distinguishable, they can be assumed to be arranged in a line. Hence, the number of distributions is the number of ways of writing the number  $n$  as the sum  $x_1 + x_2 + \dots + x_m$ , where the  $x_i$ 's are non-negative integers.

We have covered this situation in Theorem 5 of Unit 2. Over there we have shown that **the number of distributions of  $n$  indistinguishable objects into  $m$  distinguishable containers is  $C(m+n-1, n)$** . In particular, the number of non-negative integral solutions of the equation  $x_1 + x_2 + \dots + x_m = n$  is  $C(m+n-1, n)$ .

Incidentally, we note that the number of distributions of  $n$  indistinguishable objects into  $m$  distinguishable containers **with at most one object per container** is  $C(m, n)$ .

Let us consider an example.

**Example 5:** How many distinct solutions are there of  $x + y + z + w = 10$

- in non-negative integers?
- in positive integers?

**Solution:**

- i) From the result quoted above, the answer is  $C(4 + 10 - 1, 10) = 286$ .
- ii) We want  $x, y, z, w$  to be positive. Hence, we can write them respectively as  $X+1, Y+1, Z+1, W+1$ , where  $X, Y, Z, W$  are non-negative. Hence we want the number of non-negative solutions of the equation  $X+1+Y+1+Z+1+W+1=10$ , i.e.,  $X+Y+Z+W=6$ . The answer, now, is  $C(4 + 6 - 1, 6) = 84$ .  
Try some exercises now.

- E10) Show that the number of positive solutions of the equation  $x_1+x_2+\dots+x_n=m$  is  $C(m-1, m-n)$ .
- E11) In how many ways can an employer distribute 100 one-rupee notes among 6 employees so that each gets at least one note?

Let us now consider the fourth case.

### 4.3.4 Indistinguishable Objects into Indistinguishable Containers

Suppose there are  $n$  indistinguishable objects and  $m$  indistinguishable containers. Any distribution is determined purely by an **unordered**  $m$ -tuple of non-negative integers with sum  $n$ . This is equivalent to the number of increasing sequences of length  $m$  of non-negative integers with sum  $n$ . But this is precisely the number of partitions of the integer  $n$  with at most  $m$  parts, viz.,  $P_n^1 + P_n^2 + \dots + P_n^m = P_{n+m}^m$ , from Theorem 1 of this unit.

Let us consider an example of this case.

**Example 6:** In how many ways can 20 identical books be placed in 4 identical boxes?

**Solution:** The answer is  $P_{20}^1 + P_{20}^2 + P_{20}^3 + P_{20}^4 = P_{24}^4$

Why don't you try some exercises now?

- E12) In how many ways can 1000 one-rupee notes be bundled into a maximum of 20 bundles?
- E13) A car manufacture has 5 service centres in a city. 10 identical cars were served in these centres for a particular mechanical defect. In how many ways could the cars have been distributed at the various centres?

With this we have come to the end of this unit. Let us take a quick look at what we have studied in this unit.

## 4.4 SUMMARY

1. A partition of  $n \in \mathbb{N}$  into  $k$  parts is  $x_1+x_2+\dots+x_k=n$ , where  $x_1 \leq x_2 \leq \dots \leq x_k$ .  $P_n$  is the set of all partitions of  $n$ , and  $P_n^k$  is the set of all partitions into exactly  $k$  parts.
2. The proof and applications of the recurrence relation,  
 $P_n^1 + P_n^2 + \dots + P_n^k = P_{n+k}^k, P_n^1 = P_n^n = 1, 1 \leq k \leq n$ .
3. The number of ways of distributing  $n$  objects into  $m$  containers is :

- i)  $n^m$ , if the objects and containers are distinguishable.
- ii)  $\sum_{i=1}^m S_n^i$ , if the objects are distinguishable but the containers are not.  
(Here  $S_j^i$  is a Stirling number of the second kind).
- iii)  $C(m+n-1, n)$  if the objects are not distinguishable but the containers are distinguishable.
- iv)  $P_{n+m}^m$ , if neither the objects nor the containers are distinguishable.

Further, in (i) above, if there is an extra requirement that each container contain at most one object, then the number of distributions is  $P(m, n)$ . Again, in (iii) above, with the same extra requirement, the number of distributions is  $C(m, n)$ .

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## 4.5 SOLUTIONS /ANSWERS

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- E1) In the table below we give all possible partitions of 7.

Table 2

Number of parts	Partitions
1	7
2	1+6, 2+5, 3+4
3	1+1+5, 1+2+4, 1+3+3, 2+2+3
4	1+1+1+4, 1+1+2+3, 1+2+2+2
5	1+1+1+1+3, 1+1+1+2+2
6	1+1+1+1+1+2
7	1+1+1+1+1+1+1

From the table, we see that  $P_7^4 = 3$ ,  $P_7^5 = 2$ .

- E2) The required number is  $P_{10}^5 = 7$ .

- E3)  $P_7^1 = 1 = P_7^7$ .

$$P_7^2 = P_5^1 + P_5^2 = 1 + 2 = 3, \text{ from Table 1.}$$

$$P_7^3 = P_4^1 + P_4^2 + P_4^3 = 1 + 2 + 1 = 4, \text{ from Table 1.}$$

Similarly,  $P_7^4 = P_3^1 + P_3^2 + P_3^3 + P_3^4 = 3$ ,  $P_7^5 = P_2^1 + P_2^2 = 2$  and  $P_7^6 = P_1^1 = 1$ .

- E4) The 26 letters are distinguishable objects. We have to fill then in three distinguishable containers, viz., the first, second, and third positions of a three-lettered word. The solution is  $26^3$ .

If the last letter is to be x, the number is only  $26^2 \times 1$ .

If the middle letter is a vowel, then by the multiplication principle, the answer is  $26 \times 5 \times 26$ .

- E5) The total number of even numbers is  $9 \times 10 \times 10 \times 10 \times 5 = 45,000$ , since the last digit can only be 0, 2, 4, 6 or 8.

The number of 5-digit numbers composed of only odd digits (i.e., 1, 3, 5, 7, 9) is clearly  $5^5 = 3125$ .

E6) i) We can choose a woman for vice-president in 4 ways. To fill the remaining 2 positions we can select 2 from the remaining 8 persons in  $8 \times 7 = 56$  ways. Hence, the required number is  $4 \times 56 = 224$ .

ii) If the vice-president is a woman (chosen in 4 ways), others can be selected in  $5 \times 4 = 20$  ways. Similarly, if the woman is a secretary, the others can be chosen in 20 ways. Hence, by the addition and multiplication principles, the answer is  $20 \times 4 + 20 \times 4 = 160$ .

iii) Without any restriction, three can be selected in  $9 \times 8 \times 7 = 504$  ways. If no woman is to be selected, then it can be done in  $5 \times 4 \times 3 = 60$  ways. What we need is the complement of this. Thus, the required answer is  $504 - 60 = 444$ .

E7) If the alphabet has  $n$  letters, the  $m$ -letter words with distinct letters can be formed in  $n(n-1)(n-2)\dots(n-m+1) = P(n, m)$  ways.

Now, in an injective mapping, images of distinct elements should be distinct (see Unit 1). There are  $n$  possible images for the first element of the  $m$ -set,  $n-1$  possible images for the second, and so on. Hence, the number of such mappings is also  $P(n, m)$ .

E8) Suppose  $N = \{1, 2, \dots, n\}$  and  $M = \{1, 2, \dots, m\}$ . If  $f$  is an onto function from  $N$  to  $M$ , then the inverse images,  $f^{-1}(1), f^{-1}(2), \dots, f^{-1}(k)$  constitute a partition of  $N$  into  $m$  classes. The number of ways in which this can be done is  $S_n^m$ , where the order of partition is immaterial. But, in functions, the order cannot be ignored. So, the distribution can be done in  $m! \cdot S_n^m$  ways.

E9) This is  $S_n^{n-1}$ . This can be done by putting one person each in  $n-2$  rooms and 2 persons in 1 room. This can be done in  $C(n, 2)$  ways. So  $S_n^{n-1} = C(n, 2)$ .

E10) If a positive solution is  $x_1, x_2, \dots, x_n$ , then it can be written as  $X_1+1, X_2+1, \dots, X_n+1$ , where the  $X_i$ 's are non-negative. Thus, the required number is the number of non-negative solutions of  $X_1+X_2+\dots+X_n+n = m$ , which is  $C(n+m-n-1, m-n) = C(m-1, m-n)$ .

E11) This is the number of positive solutions of  $x_1+\dots+x_6 = 100$ . So, the required number is  $C(100-1, 100-6) = C(99, 94) = 71,523,144$ .

E12)  $P_{1000}^1 + P_{1000}^{20} + P_{1020}^{20}$ .

Had the requirement been that there be exactly 20 bundles, then the number would have been  $P_{1000}^{20}$ .

E13)  $P_{10}^1 + P_{10}^2 + P_{10}^3 + P_{10}^4 + P_{10}^5 = P_{15}^5$ .

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# UNIT 1 PROPOSITIONAL CALCULUS

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## Structure

- 1.0 Introduction
- 1.1 Objectives
- 1.2 Propositions
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  - 1.3.1 Disjunction
  - 1.3.2 Conjunction
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  - 1.3.4 Conditional Connectives
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## 1.0 INTRODUCTION

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According to the theory of evolution, human beings have evolved from the lower species over many millennia. The chief asset that made humans “superior” to their ancestors was the ability to reason. How well this ability has been used for scientific and technological development is common knowledge. But no systematic study of logical reasoning seems to have been done for a long time. The first such study that has been found is by Greek philosopher Aristotle (384-322 BC). In a modified form, this type of logic seems to have been taught through the Middle Ages.

Then came a major development in the study of logic, its formalisation in terms of mathematics. It was mainly Leibniz (1646-1716) and George Boole (1815-1864) who seriously studied and developed this theory, called **symbolic logic**. It is the basics of this theory that we aim to introduce you to in this unit and the next one.

In the introduction to the block you have read about what symbolic logic is. Using it we can formalise our arguments and logical reasoning in a manner that can easily show if the reasoning is valid, or is a fallacy. How we symbolise the reasoning is what is presented in this unit.

More precisely, in Section 1.2 (i.e., Sec. 1.2, in brief) we talk about what kind of sentences are acceptable in mathematical logic. We call such sentences statements or propositions. You will also see that a statement can either be true or false. Accordingly, as you will see, we will give the statement a truth value T or F.

In Sec. 1.3 we begin our study of the logical relationship between propositions. This is called propositional calculus. In this we look at some ways of connecting simple propositions to obtain more complex ones. To do so, we use logical connectives like “and” and “or”. We also introduce you to other connectives like “not”, “implies” and “implies and is implied by”. At the same time we construct tables that allow us to find the truth values of the compound statement that we get.

In Sec. 1.4 we consider the conditions under which two statements are “the same”. In such a situation we can safely replace one by the other.

And finally, in Sec 1.5, we talk about some common terminology and notation which is useful for quantifying the objects we are dealing with in a statement.

It is important for you to study this unit carefully, because the other units in this block are based on it. Please be sure to do the exercises as you come to them. Only then will you be able to achieve the following objectives.

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## 1.1 OBJECTIVES

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After reading this unit, you should be able to:

- distinguish between propositions and non-propositions;
- construct the truth table of any compound proposition;
- identify and use logically equivalent statements;
- identify and use logical quantifiers.

Let us now begin our discussion on mathematical logic.

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## 1.2 PROPOSITIONS

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Consider the sentence ‘In 2003, the President of India was a woman’. When you read this declarative sentence, you can immediately decide whether it is true or false. And so can anyone else. Also, it wouldn’t happen that some people say that the statement is true and some others say that it is false. Everybody would have the same answer. So this sentence is either **universally true** or **universally false**.

Similarly, ‘An elephant weighs more than a human being.’ Is a declarative sentence which is either true or false, but not both. In mathematical logic we call such sentences **statements or propositions**.

On the other hand, consider the declarative sentence ‘Women are more intelligent than men’. Some people may think it is true while others may disagree. So, it is neither universally true nor universally false. Such a sentence is not acceptable as a statement or proposition in mathematical logic.

Note that a **proposition should be either uniformly true or uniformly false**. For example, ‘An egg has protein in it.’, and ‘The Prime Minister of India has to be a man.’ are both propositions, the first one true and the second one false.

Would you say that the following are propositions?

‘Watch the film.  
‘How wonderful!’  
‘What did you say?’

Actually, none of them are declarative sentences. (The first one is an order, the second an exclamation and the third is a question.) And therefore, none of them are propositions.

Now for some mathematical propositions! You must have studied and created many of them while doing mathematics. Some examples are

Two plus two equals four.  
Two plus two equals five.  
 $x + y > 0$  for  $x > 0$  and  $y > 0$ .  
A set with  $n$  elements has  $2^n$  subsets.

Of these statements, three are true and one false (which one?).

Now consider the algebraic sentence ‘ $x + y > 0$ ’. Is this a proposition? Are we in a position to determine whether it is true or false? Not unless we know the values that  $x$  and  $y$  can take. For example, it is false for  $x = 1, y = -2$  and true if  $x = 1, y = 0$ . Therefore, ‘ $x + y > 0$ ’ is not a proposition, while ‘ $x + y > 0$  for  $x > 0, y > 0$ ’ is a proposition.



Why don't you try this short exercise now?

- E1) Which of the following sentences are statements? What are the reasons for your answer?
- i) The sun rises in the West.
  - ii) How far is Delhi from here?
  - iii) Smoking is injurious to health.
  - iv) There is no rain without clouds.
  - v) What is a beautiful day!
  - vi) She is an engineering graduates.
  - vii)  $2^n + n$  is an even number for infinitely many  $n$ .
  - viii)  $x + y = y + x$  for all  $x, y \in \mathbf{R}$ .
  - ix) Mathematics is fun.
  - x)  $2^n = n^2$ .

Usually, when dealing with propositions, we shall denote them by lower case letters like  $p, q$ , etc. So, for example, we may denote

'Ice is always cold.' by  $p$ , or

' $\cos^2 \theta + \sin^2 \theta = 1$  for  $\theta \in [0, 2\pi]$ ' by  $q$ .

We shall sometimes show this by saying

$p$ : Ice is always cold., or

$q$ :  $\cos^2 \theta + \sin^2 \theta = 1$  for  $\theta \in [0, 2\pi]$ .

Now, given a proposition, we know that it is either true or false, but not both. If it is **true**, we will allot it the **truth value T**. If it is **false**, its **truth value will be F**. So, for example, the truth value of

'Ice melts at  $30^\circ \text{C}$ .' is F, while that of ' $x^2 \geq 0$  for  $x \in \mathbf{R}$ ' is T.

Here are some exercises for you now.

- E2) Give the truth values of the propositions in E1.
- E3) Give two propositions each, the truth values of which are T and F, respectively. Also give two examples of sentences that are not propositions.

Let us now look at ways of connecting simple propositions to obtain compound statements.

## 1.3 LOGICAL CONNECTIVES

When you're talking to someone, do you use very simple sentences only? Don't you use more complicated ones which are joined by words like 'and', 'or', etc? In the same way, most statements in mathematical logic are combinations of simpler statements joined by words and phrases like 'and', 'or', 'if ... then', 'if and only if', etc. These words and phrases are called **logical connectives**. There are 6 such connectives, which we shall discuss one by one.

### 1.3.1 Disjunction

Consider the sentence 'Alice or the mouse went to the market.'. This can be written as 'Alice went to the market or the mouse went to the market.' So, this statement is actually made up of two simple statements connected by 'or'. We have a term for such a compound statement.

**Definition:** The **disjunction** of two propositions  $p$  and  $q$  is the compound statement

Sometimes, as in the context of logic circuits (See unit 3), we will use 1 instead of T and 0 instead of F.

**p or q**, denoted by  $p \vee q$ .

For example, 'Zarina has written a book or Singh has written a book.' Is the disjunction of p and q, where

p : Zarina has written a book, and

q : Singh has written a book.

Similarly, if p denotes ' $2 > 0$ ' and q denotes ' $2 < 5$ ', then  $p \vee q$  denotes the statement '2 is greater than 0 or 2 is less than 5.'

Let us now look at how the truth value of  $p \vee q$  depends upon the truth values of p and q. For doing so, let us look at the example of Zarina and Singh, given above. If even one of them has written a book, then the compound statement  $p \vee q$  is true. Also, if both have written books, the compound statement  $p \vee q$  is again true. Thus, if the truth value of even one out of p and q is T, then that of ' $p \vee q$ ' is T. Otherwise, the truth value of  $p \vee q$  is F. This holds for any pair of propositions p and q. To see the relation between the truth values of p, q and  $p \vee q$  easily, we put this in the form of a table (Table 1), which we call a **truth table**.

**Table 1: Truth table for disjunction**

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

How do we form this table? We consider the truth values that p can take – T or F. Now, when p is true, q can be true or false. Similarly, when p is false q can be true or false. In this way there are 4 possibilities for the compound proposition  $p \vee q$ . Given any of these possibilities, we can find the truth value of  $p \vee q$ . For instance, consider the third possibility, i.e., p is false and q is true. Then, by definition,  $p \vee q$  is true. In the same way, you can check that the other rows are consistent.

Let us consider an example.

**Example 1:** Obtain the truth value of the disjunction of 'The earth is flat' and ' $3 + 5 = 2$ '.

**Solution:** Let p denote 'The earth is flat,' and q denote ' $3 + 5 = 2$ '. Then we know that the truth values of both p and q are F. Therefore, the truth value of  $p \vee q$  is F.

\*\*\*

Try an exercise now.

---

E4) Write down the disjunction of the following propositions, and give its truth value.

- i)  $2 + 3 = 7$ ,
  - ii) Radha is an engineer.
- 

We also use the term 'inclusive or' for the connective we have just discussed. This is because  $p \vee q$  is true even when both p and q are true. But, what happens when we want to ensure that only one of them should be true? Then we have the following connective.

**Definition:** The **exclusive disjunction** of two propositions p and q is the statement '**Either p is true or q is true, but both are not true.**'. Either p is true or q is true, but both are not true.'. We denote this by  $p \oplus q$ .

So, for example, if  $p$  is ' $2 + 3 = 5$ ' and  $q$  the statement given in E4(ii), then  $p \oplus q$  is the statement 'Either  $2 + 3 = 5$  or Radha is an engineer'. This will be true only if Radha is not an engineer.

In general, how is the truth value of  $p \oplus q$  related to the truth values of  $p$  and  $q$ ? This is what the following exercise is about.

---

E5) Write down the truth table for  $\oplus$ . Remember that  $p \oplus q$  is not true if both  $p$  and  $q$  are true.

---

Now let us look at the logical analogue of the coordinating conjunction 'and'.

### 1.3.2 Conjunction

As in ordinary language, we use 'and' to combine simple propositions to make compound ones. For instance, ' $1 + 4 \neq 5$  and Prof. Rao teaches Chemistry.' is formed by joining ' $1 + 4 \neq 5$ ' and 'Prof. Rao teaches Chemistry' by 'and'. Let us define the formal terminology for such a compound statement.

**Definition:** We call the compound statement ' $p$  and  $q$ ' the **conjunction** of the statements  $p$  and  $q$ . We denote this by  $p \wedge q$ .

For instance, ' $3 + 1 \neq 7 \wedge 2 > 0$ ' is the conjunction of ' $3 + 1 \neq 7$ ' and ' $2 > 0$ '. Similarly, ' $2 + 1 = 3 \wedge 3 = 5$ ' is the conjunction of ' $2 + 1 = 3$ ' and ' $3 = 5$ '.

Now, when would  $p \wedge q$  be true? Do you agree that this could happen only when both  $p$  and  $q$  are true, and not otherwise? For instance, ' $2 + 1 = 3 \wedge 3 = 5$ ' is not true because ' $3 = 5$ ' is false.

So, the truth table for conjunction would be as in Table 2.

**Table 2: Truth table for conjunction**

P	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

To see how we can use the truth table above, consider an example.

**Example 2:** Obtain the truth value of the conjunction of ' $2 \div 5 = 1$ ' and 'Padma is in Bangalore.'.

**Solution:** Let  $p : 2 \div 5 = 1$ , and  
 $q$ : Padma is in Bangalore.

Then the truth value of  $p$  is F. Therefore, from Table 3 you will find that the truth value of  $p \wedge q$  is F.

\*\*\*

Why don't you try an exercise now?

---

E6) Give the set of those real numbers  $x$  for which the truth value of  $p \wedge q$  is T, where  $p : x > -2$ , and  $q : x + 3 \neq 7$

---

If you look at Tables 1 and 2, do you see a relationship between the truth values in their last columns? You would be able to formalize this relationship after studying the next connective.

### 1.3.3 Negation

You must have come across young children who, when asked to do something, go ahead and do exactly the opposite. Or, when asked if they would like to eat, say rice and curry, will say ‘No’, the ‘negation’ of yes! Now, if  $p$  denotes the statement ‘I will eat rice.’, how can we denote ‘I will not eat rice.’? Let us define the connective that will help us do so.

**Definition:** The **negation** of a proposition  $p$  is ‘**not  $p$** ’, denoted by  $\sim p$ .

For example, if  $p$  is ‘Dolly is at the study center.’, then  $\sim p$  is ‘Dolly is not at the study center’. Similarly, if  $p$  is ‘No person can live without oxygen.’,  $\sim p$  is ‘At least one person can live without oxygen.’

Now, regarding the truth value of  $\sim p$ , you would agree that it would be T if that of  $p$  is F, and vice versa. Keeping this in mind you can try the following exercises.

- 
- E7) Write down  $\sim p$ , where  $p$  is
- i)  $0 - 5 \neq 5$
  - ii)  $n > 2$  for every  $n \in \mathbf{N}$ .
  - iii) Most Indian children study till class 5.

- E8) Write down the truth table of negation.
- 

Let us now discuss the conditional connectives, representing ‘If ..., then ...’ and ‘if and only if’.

### 1.3.4 Conditional Connectives

Consider the proposition ‘If Ayesha gets 75% or more in the examination, then she will get an A grade for the course.’. We can write this statement as ‘If  $p$ , and  $q$ ’, where

- $p$ : Ayesha gets 75% or more in the examination, and
- $q$ : Ayesha will get an A grade for the course.

This compound statement is an example of the implication of  $q$  by  $p$ .

**Definition:** Given any two propositions  $p$  and  $q$ , we denote the statement ‘**If  $p$ , then  $q$** ’ by  $p \rightarrow q$ . We also read this as ‘ $p$  **implies**  $q$ ’. or ‘ $p$  is sufficient for  $q$ ’, or ‘ $p$  **only if**  $q$ ’. We also call  $p$  the **hypothesis** and  $q$  the conclusion. Further, a statement of the form  $p \rightarrow q$  is called a **conditional statement** or a **conditional proposition**.

So, for example, in the conditional proposition ‘If  $m$  is in  $\mathbf{Z}$ , then  $m$  belongs to  $\mathbf{Q}$ .’ the hypothesis is ‘ $m \in \mathbf{Z}$ ’ and the conclusion is ‘ $m \in \mathbf{Q}$ ’.

Mathematically, we can write this statement as

$$m \in \mathbf{Z} \rightarrow m \in \mathbf{Q}.$$

Let us analyse the statement  $p \rightarrow q$  for its truth value. Do you agree with the truth table we’ve given below (Table 3)? You may like to check it out while keeping an example from your surroundings in mind.

**Table 3: Truth table for implication**

$p$	$q$	$p \rightarrow q$
T	T	T
T	F	F
F	T	T
F	F	T

You may wonder about the third row in Table 3. But, consider the example ' $3 < 0 \rightarrow 5 > 0$ '. Here the conclusion is true regardless of what the hypothesis is. And therefore, the conditional statement remains true. In such a situation we say that the **conclusion is vacuously true**.

Why don't you try this exercise now?

- 
- E9) Write down the proposition corresponding to  $p \rightarrow q$ , and determine the values of  $x$  for which it is false, where  
 $p : x + y = xy$  where  $x, y \in \mathbf{R}$   
 $q : x \leq 0$  for every  $x \in \mathbf{Z}$ .
- 

Now, consider the implication 'If Jahanara goes to Baroda, then she doesn't participate in the conference at Delhi.'. What would its converse be? To find it, the following definition may be useful.

**Definition:** The **converse** of  $p \rightarrow q$  is  $q \rightarrow p$ . In this case we also say 'p is necessary for q', or 'p if q'.

So, in the example above, the converse of the statement would be 'If Jahanara doesn't participate in the conference at Delhi, then she goes to Baroda.'. This means that Jahanara's non-participation in the conference at Delhi is necessary for her going to Baroda.

Now, what happens when we combine an implication and its converse?

To show ' $p \rightarrow q$  and  $q \rightarrow p$ ', we introduce a shorter notation.

**Definition:** Let  $p$  and  $q$  be two propositions. The compound statement  $(p \rightarrow q) \wedge (q \rightarrow p)$  is the **biconditional** of  $p$  and  $q$ . We denote it by  $p \leftrightarrow q$ , and read it as 'p if and only q'.

We usually shorten 'if and only if' to **iff**.

We also say that 'p **implies and is implied by** q'. or 'p is **necessary and sufficient for** q'.

For example, 'Sudha will gain weight if and only if she eats regularly.' Means that 'Sudha will gain weight if she eats regularly **and** Sudha will eat regularly if she gains weight.'

One point that may come to your mind here is whether there's any difference in the two statements  $p \leftrightarrow q$  and  $q \leftrightarrow p$ . When you study Sec. 1.4 you will realize why they are inter-changeable.

Let us now consider the truth table of the biconditional, i.e., of the **two-way** implication.

To obtain its truth values, we need to use Tables 2 and 3, as you will see in Table 4. This is because, to find the value of  $(p \rightarrow q) \wedge (q \rightarrow p)$  we need to know the values of each of the simpler statements involved.

**Table 4: Truth table for two-way implication.**

p	q	$p \rightarrow q$	$q \rightarrow p$	$p \leftrightarrow q$
T	T	T	T	T
T	F	F	T	F
F	T	T	F	F
F	F	T	T	T

The two connectives  $\rightarrow$  and  $\leftrightarrow$  are called **conditional connectives**.

As you can see from the last column of the table (and from your own experience),  $p \leftrightarrow q$  is true only when both  $p$  and  $q$  are true or both  $p$  and  $q$  are false. In other words,  $p \leftrightarrow q$  is true only when  $p$  and  $q$  have the same truth values. Thus, for example, 'Parimala is in America iff  $2 + 3 = 5$ ' is true only if 'Parimala is in America,' is true.

Here are some related exercises.

---

E10) For each of the following compound statements, first identify the simple propositions  $p$ ,  $q$ ,  $r$ , etc., that are combined to make it. Then write it in symbols, using the connectives, and give its truth value.

- i) If triangle ABC is equilateral, then it is isosceles.
- ii)  $a$  and  $b$  are integers if and only if  $ab$  is a rational number.
- iii) If Raza has five glasses of water and Sudha has four cups of tea, then Shyam will not pass the math examination.
- iv) Mariam is in Class 1 or in Class 2.

E11) Write down two propositions  $p$  and  $q$  for which  $q \rightarrow p$  is true but  $p \leftrightarrow q$  is false.

---

Now, how would you determine the truth value of a proposition which has more than one connective in it? For instance, does  $\sim p \vee q$  mean  $(\sim p) \vee q$  or  $\sim (p \vee q)$ ? We discuss some rules for this below.

### 1.3.5 Precedence Rule

While dealing with operations on numbers, you would have realized the need for applying the BODMAS rule. According to this rule, when calculating the value of an arithmetic expression, we first calculate the value of the Bracketed portion, then apply **Of, Division, Multiplication, Addition and Subtraction, in this order**. While calculating the truth value of compound propositions involving more than one connective, we have a similar **convention** which tells us which connective to apply first.

Why do we need such a convention? Suppose we didn't have an order of preference, and want to find the truth of, say  $\sim p \vee q$ . Some of us may consider **the value of  $(\sim p) \vee q$ , and some may consider  $\sim (p \vee q)$ . The truth values can be different in these cases. For instance, if  $p$  and  $q$  are both true, then  $(\sim p) \vee q$  is true, but  $\sim (p \vee q)$  is false. So, for the purpose of unambiguity, we agree to such an order or rule. Let us see what it is.**

**The rule of precedence:** The order of preference in which the connectives are applied in a formula of propositions that has no brackets is

- i)  $\sim$
- ii)  $\wedge$
- iii)  $\vee$  and  $\oplus$
- iv)  $\rightarrow$  and  $\leftrightarrow$

Note that the 'inclusive or' and 'exclusive or' are both third in the order of preference. However, if both these appear in a statement, we first apply the left most one. So, for instance, in  $p \vee q \oplus \sim p$ , we first apply  $\vee$  and then  $\oplus$ . The same applies to the 'implication' and the 'biconditional', which are both fourth in the order of preference.

To clearly understand how this rule works, let us consider an example.

**Example 3:** Write down the truth table of  $p \rightarrow q \wedge \sim r \leftrightarrow r \oplus q$

**Solution:** We want to find the required truth value when we are given the truth values of  $p$ ,  $q$  and  $r$ . According to the rule of precedence given above, we need to first find the truth value of  $\sim r$ , then that of  $(q \wedge \sim r)$ , then that of  $(r \oplus q)$ , and then that of  $p \rightarrow (q \wedge \sim r)$ , and finally the truth value of  $[p \rightarrow (q \wedge \sim r)] \leftrightarrow r \oplus q$ .

So, for instance, suppose  $p$  and  $q$  are true, and  $r$  is false. Then  $\sim r$  will have value T,  $q \wedge \sim r$  will be T,  $r \oplus q$  will be T,  $p \rightarrow (q \wedge \sim r)$  will be T, and hence,  $p \rightarrow q \wedge \sim r \leftrightarrow r \oplus q$  will be T.

You can check that the rest of the values are as given in Table 5. Note that we have 8 possibilities ( $=2^3$ ) because there are 3 simple propositions involved here.

Table 5: Truth table for  $p \rightarrow q \wedge \sim r \leftrightarrow r \oplus q$

p	q	r	$\sim r$	$q \wedge \sim r$	$r \oplus q$	$p \rightarrow q \wedge \sim r$	$p \rightarrow q \wedge \sim r \leftrightarrow r \oplus q$
T	T	T	F	F	F	F	T
T	T	F	T	T	T	T	T
T	F	T	F	F	T	F	F
T	F	F	T	F	F	F	T
F	T	T	F	F	F	T	F
F	T	F	T	T	T	T	T
F	F	T	F	F	T	T	T
F	F	F	T	F	F	T	F

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You may now like to try some exercises on the same lines.

- 
- E12) In Example 3, how will the truth values of the compound statement change if you first apply  $\leftrightarrow$  and then  $\rightarrow$ ?
- E13) In Example 3, if we replace  $\oplus$  by  $\wedge$ , what is the new truth table?
- E14) From the truth table of  $p \wedge q \vee \sim r$  and  $(p \wedge q) \vee (\sim r)$  and see where they differ.
- E15) How would you bracket the following formulae to correctly interpret them?  
[For instance,  $p \vee \sim q \wedge r$  would be bracketed as  $p \vee ((\sim q) \wedge r)$ .]
- $p \vee q$ ,
  - $\sim q \rightarrow \sim p$ ,
  - $p \rightarrow q \leftrightarrow \sim p \vee q$ ,
  - $p \oplus q \wedge r \rightarrow \sim p \vee q \leftrightarrow p \wedge r$ .
- 

So far we have considered different ways of making new statements from old ones. But, are all these new ones distinct? Or are some of them the same? And “same” in what way? This is what we shall now consider.

## 1.4 LOGICAL EQUIVALENCE

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‘Then you should say what you mean’, the March Hare went on. ‘I do,’ Alice hastily replied, ‘at least ... at least I mean what I say – that’s the same thing you know.’ ‘Not the same thing a bit!’ said the Hatter. ‘Why you might just as well say that “I see what I eat” is the same thing as “I eat what I see”!’

-from ‘Alice in Wonderland’  
by Lewis Carroll

In Mathematics, as in ordinary language, there can be several ways of saying the same thing. In this section we shall discuss what this means in the context of logical statements.

Consider the statements ‘If Lala is rich, then he must own a car.’ and ‘if Lala doesn’t own a car, then he is not rich.’ Do these statements mean the same thing? If we write the first one as  $p \rightarrow q$ , then the second one will be  $(\sim q) \rightarrow (\sim p)$ . How do the truth values of both these statements compare?

We find out in the following table.

Table 6

p	q	$\sim p$	$\sim q$	$p \rightarrow q$	$\sim q \rightarrow \sim p$
T	T	F	F	T	T
T	F	F	T	F	F
F	T	T	F	T	T
F	F	T	T	T	T

Consider the last two columns of Table 6. You will find that ‘ $p \rightarrow q$ ’ and ‘ $\sim q \rightarrow \sim p$ ’ have the same truth value for every choice of truth values of p and q. When this happens, we call them equivalent statements.

**Definition:** We call two propositions r and s **logically equivalent** provided they have the same truth value for every choice of truth values of simple propositions involved in them. We denote this fact by  $r \equiv s$ .

So, from Table 6 we find that  $(p \rightarrow q) \equiv (\sim q \rightarrow \sim p)$ .

You can also check that  $(p \leftrightarrow q) \equiv (q \leftrightarrow p)$  for any pair of propositions p and q.

As another example, consider the following equivalence that is often used in mathematics. You could also apply it to obtain statements equivalent to ‘Neither a borrower, nor a lender be.’!

**Example 4:** For any two propositions p and q, show that  $\sim(p \vee q) \equiv \sim p \wedge \sim q$ .

**Solution:** Consider the following truth table.

Table 7

p	q	$\sim p$	$\sim q$	$p \vee q$	$\sim(p \vee q)$	$\sim p \wedge \sim q$
T	T	F	F	T	F	F
T	F	F	T	T	F	F
F	T	T	F	T	F	F
F	F	T	T	F	T	T

You can see that the last two columns of Table 7 are identical. Thus, the truth values of  $\sim(p \vee q)$  and  $\sim p \wedge \sim q$  agree for every choice of truth values of p and q. Therefore,  $\sim(p \vee q) \equiv \sim p \wedge \sim q$ .

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The equivalence you have just seen is one of **De Morgan’s laws**. You might have already come across these laws in your previous studies of basic Mathematics.

The other law due to De Morgan is similar :  $\sim(p \wedge q) \equiv \sim p \vee \sim q$ .

In fact, there are several such laws about equivalent propositions. Some of them are the following, where, as usual, p, q and r denote propositions.



Fig. 1: Augustus De Morgan (1806-1871) was born in Madurai



- a) **Double negation law** :  $\sim(\sim p) \equiv p$   
 b) **Idempotent laws**:  $p \wedge p \equiv p$ ,  
 $p \vee p \equiv p$   
 c) **Commutativity**:  $p \vee q \equiv q \vee p$   
 $p \wedge q \equiv q \wedge p$   
 d) **Associativity**:  $(p \vee q) \vee r \equiv p \vee (q \vee r)$   
 $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$   
 e) **Distributivity**:  $\vee(q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$   
 $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

We ask you to prove these laws now.

- 
- E16) Show that the laws given in (a)-(e) above hold true.  
 E17) Prove that the relation of 'logical equivalence' is an equivalence relation.  
 E18) Check whether  $(\sim p \vee q)$  and  $(p \rightarrow q)$  are logically equivalent.
- 

The laws given above and the equivalence you have checked in E18 are commonly used, and therefore, **useful to remember**. You will also be applying them in Unit 3 of this Block in the context of switching circuits.

Let us now consider some propositional formulae which are always true or always false. Take, for instance, the statement 'If Bano is sleeping and Pappu likes ice-cream, then Beno is sleeping'. You can draw up the truth table of this compound proposition and see that it is always true. This leads us to the following definition.

**Definition:** A compound proposition that is true for all possible truth values of the simple propositions involved in it is called a **tautology**. Similarly, a proposition that is false for all possible truth values of the simple propositions that constitute it is called a **contradiction**.

Let us look at some example of such propositions.

**Example 5:** Verify that  $p \wedge q \wedge \sim p$  is a contradiction and  $p \rightarrow q \leftrightarrow \sim p \vee q$  is a tautology.

**Solution:** Let us simultaneously draw up the truth tables of these two propositions below.

**Table 8**

p	q	$\sim p$	$p \wedge q$	$p \wedge q \wedge \sim p$	$p \rightarrow q$	$\sim p \vee q$	$p \rightarrow q \leftrightarrow \sim p \vee q$
T	T	F	T	F	T	T	T
T	F	F	F	F	F	F	T
F	T	T	F	F	T	T	T
F	F	T	F	F	T	T	T

Looking at the fifth column of the table, you can see that  $p \wedge q \wedge \sim p$  is a contradiction. This should not be surprising since  $p \wedge q \wedge \sim p \equiv (p \wedge \sim p) \wedge q$  (check this by using the various laws given above).

And what does the last column of the table show? Precisely that  $p \rightarrow q \leftrightarrow \sim p \vee q$  is a tautology.

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Why don't you try an exercise now?

- 
- E19) Let  $T$  denote a tautology (i.e., a statement whose truth value is always T) and  $F$  a contradiction. Then, for any statement  $p$ , show that

- i)  $p \vee T \equiv T$
- ii)  $p \wedge T \equiv p$
- iii)  $p \vee F \equiv p$
- iv)  $p \wedge F \equiv F$

Another way of proving that a proposition is a tautology is to use the properties of logical equivalence. Let us look at the following example.

**Example 6:** Show that  $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$  is a tautology.

**Complementation law:**  
 $q \wedge \sim q$  is a contradiction.

**Solution:**  $[(p \rightarrow q) \wedge \sim q] \rightarrow \sim p$   
 $\equiv [(\sim p \vee q) \wedge \sim q] \rightarrow \sim p$ , using E18, and symmetricity of  $\equiv$ .  
 $\equiv [(\sim p \wedge \sim q) \vee (q \wedge \sim q)] \rightarrow \sim p$ , by De Morgan's laws.  
 $\equiv [(\sim p \wedge \sim q) \vee F] \rightarrow \sim p$ , since  $q \wedge \sim q$  is always false.  
 $\equiv (\sim p \wedge \sim q) \rightarrow \sim p$ , using E18.

Which is tautology.

And therefore the proposition we started with is a tautology.

\*\*\*

The laws of logical equivalence can also be used to prove some other logical equivalences, without using truth tables. Let us consider an example.

**Example 7:** Show that  $(p \rightarrow \sim q) \wedge (p \rightarrow \sim r) \equiv \sim [p \wedge (q \vee r)]$ .

**Solution:** We shall start with the statement on the left hand side of the equivalence that we have to prove. Then, we shall apply the laws we have listed above, or the equivalence in E 18, to obtain logically equivalent statements. We shall continue this process till we obtain the statement on the right hand side of the equivalence given above. Now

$$\begin{aligned}
 & (p \rightarrow \sim q) \wedge (p \rightarrow \sim r) \\
 & \equiv (\sim p \vee q) \wedge (\sim p \vee \sim r), \text{ by E18} \\
 & \equiv \sim p \vee (\sim q \wedge \sim r), \text{ by distributivity} \\
 & \equiv \sim p \vee [\sim (q \vee r)], \text{ by De Morgan's laws} \\
 & \equiv \sim [p \wedge (q \vee r)], \text{ by De Morgan's laws}
 \end{aligned}$$

So we have proved the equivalence that we wanted to.

\*\*\*

You may now like to try the following exercises on the same lines.

E20) Use the laws given in this section to show that

$$\sim (\sim p \wedge q) \wedge (p \vee q) \equiv p.$$

E21) Write down the statement 'If it is raining and if rain implies that no one can go to see a film, then no one can go to see a film.' As a compound proposition. Show that this proposition is a tautology, by using the properties of logical equivalence.

E22) Give an example, with justification, of a compound proposition that is neither a tautology nor a contradiction.

Let us now consider proposition-valued functions.

## 1.5 LOGICAL QUANTIFIERS

In Sec. 1.2, you read that a sentence like ‘She has gone to Patna.’ Is not a proposition, unless who ‘she’ is clearly specified.

Similarly, ‘ $x > 5$ ’ is not a proposition unless we know the values of  $x$  that we are considering. Such sentences are examples of ‘propositional functions’.

**Definition:** A **propositional function**, or a **predicate**, in a variable  $x$  is a sentence  $p(x)$  involving  $x$  that becomes a proposition when we give  $x$  a definite value from the set of values it can take. We usually denote such functions by  $p(x)$ ,  $q(x)$ , etc. The set of values  $x$  can take is called the universe of discourse.

So, if  $p(x)$  is ‘ $x > 5$ ’, then  $p(x)$  is not a proposition. But when we give  $x$  particular values, say  $x = 6$  or  $x = 0$ , then we get propositions. Here,  $p(6)$  is a true proposition and  $p(0)$  is a false proposition.

Similarly, if  $q(x)$  is ‘ $x$  has gone to Patna.’, then replacing  $x$  by ‘Taj Mahal’ gives us a false proposition.

**Note** that a predicate is usually not a proposition. But, of course, every proposition is a propositional function in the same way that every real number is a real-valued function, namely, the constant function.

Now, can all sentences be written in symbolic form by using only the logical connectives? What about sentences like ‘ $x$  is prime and  $x + 1$  is prime for some  $x$ .’? How would you symbolize the phrase ‘for some  $x$ ’, which we can rephrase as ‘there exists an  $x$ ’? You must have come across this term often while studying mathematics.

**We use the symbol ‘ $\exists$ ’ to denote this quantifier, ‘there exists’.** The way we use it is, for instance, to rewrite ‘There is at least one child in the class.’ as ‘ $(\exists x \text{ in } U)p(x)$ ’, where  $p(x)$  is the sentence ‘ $x$  is in the class.’ and  $U$  is the set of all children.

$\exists$  is called the **existential quantifier**.

Now suppose we take the negative of the proposition we have just stated. Wouldn’t it be ‘There is no child in the class.’? We could symbolize this as ‘for all  $x$  in  $U$ ,  $q(x)$ ’ where  $x$  ranges over all children and  $q(x)$  denotes the sentence ‘ $x$  is not in the class.’, i.e.,  $q(x) \equiv \sim p(x)$ .

We have a **mathematical symbol for the quantifier ‘for all’, which is ‘ $\forall$ ’.** So the proposition above can be written as

‘ $(\forall x \in U)q(x)$ ’, or ‘ $q(x)$ ,  $\forall x \in U$ ’.

$\forall$  is called the **universal quantifier**.

An example of the use of the existential quantifier is the true statement.

$(\exists x \in \mathbf{R})(x + 1 > 0)$ , which is read as ‘There exists an  $x$  in  $\mathbf{R}$  for which  $x + 1 > 0$ .’

Another example is the false statement

$(\exists x \in \mathbf{N})(x - \frac{1}{2} = 0)$ , which is read as ‘There exists an  $x$  in  $\mathbf{N}$  for which  $x - \frac{1}{2} = 0$ .’

An example of the use of the universal quantifier is  $(\forall x \notin \mathbf{N})(x^2 > x)$ , which is read as ‘for every  $x$  not in  $\mathbf{N}$ ,  $x^2 > x$ .’ Of course, this is a false statement, because there is at least one  $x \notin \mathbf{N}$ ,  $x \in \mathbf{R}$ , for which it is false.

We often use both quantifiers together, as in the statement called **Bertrand’s postulate**:

$(\forall n \in \mathbf{N} \setminus \{1\})(\exists x \in \mathbf{N})(x \text{ is a prime number and } n < x < 2n)$ .

In words, this is ‘for every integer  $n > 1$  there is a prime number lying strictly between  $n$  and  $2n$ .’

As you have already read in the example of a child in the class,  
 $(\forall x \in U)p(x)$  is logically equivalent to  $\sim (\exists x \in U) (\sim p(x))$ . Therefore,  
 $\sim (\forall x \in U)p(x) \equiv \sim \sim (\exists x \in U) (\sim p(x)) \equiv (\exists x \in U) (\sim p(x))$ .

This is one of the rules for negation that relate  $\forall$  and  $\exists$ . The two rules are

$$\begin{aligned}\sim (\forall x \in U)p(x) &\equiv (\exists x \in U) (\sim p(x)), \text{ and} \\ \sim (\exists x \in U)p(x) &\equiv (\forall x \in U) (\sim p(x))\end{aligned}$$

Where  $U$  is the set of values that  $x$  can take.

Now, consider the proposition

‘There is a criminal who has committed every crime.’

We could write this in symbols as

$$(\exists c \in A) (\forall x \in B) (c \text{ has committed } x)$$

Where, of course,  $A$  is the set of criminals and  $B$  is the set of crimes (determined by law).

What would its negation be? It would be

$$\sim (\exists c \in A) (\forall x \in B) (c \text{ has committed } x)$$

Where, of course,  $A$  is the set of criminals and  $B$  is the set of crimes (determined by law).

What would its negation be? It would be

$$\begin{aligned}\sim (\exists c \in A) (\forall x \in B) (c \text{ has committed } x) \\ \equiv (\forall c \in A) [\sim (\forall x \in B) (c \text{ has committed } x)] \\ \equiv (\forall c \in A) (\exists x \in B) (c \text{ has not committed } x).\end{aligned}$$

We can interpret this as ‘For every criminal, there is a crime that this person has not committed.’

These are only some examples in which the quantifiers occur singly, or together.

Sometimes you may come across situations (as in E23) where you would use  $\exists$  or  $\forall$  twice or more in a statement. It is in situations like this or worse [say,  $(\forall x_1 \in U_1) (\exists x_2 \in U_2) (\exists x_3 \in U_2) (\exists x_3 \in U_3) (\forall x_4 \in U_4) \dots (\exists x_n \in U_n)p$ ]

where our rule for negation comes in useful. In fact, applying it, in a trice we can say that the negation of this seemingly complicated example is

$$(\exists x_1 \in U_1) (\forall x_2 \in U_2) (\forall x_3 \in U_3) (\exists x_4 \in U_4) \dots (\forall x_n \in U_n) (\sim p).$$

Why don’t you try some exercise now?

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E23) How would you present the following propositions and their negations using logical quantifiers? Also interpret the negations in words.

- i) The politician can fool all the people all the time.
- ii) Every real number is the square of some real number.
- iii) There is lawyer who never tell lies.

E24) Write down suitable mathematical statements that can be represented by the following symbolic propositions. Also write down their negations. What is the truth value of your propositions?

- i)  $(\forall x) (\exists y)p$
  - ii)  $(\exists x) (\exists y) (\forall z)p$ .
- 

A predicate can be a function in two **or more** variables.

And finally, let us look at a very useful quantifier, which is very closely linked to  $\exists$ . You would need it for writing, for example, 'There is one and only one key that fits the desk's lock.' In symbols. The symbol is  $\exists!$  X which stands for '**there is one and only one x**' (which is the same as '**there is a unique x**' or '**there is exactly one x**').

So, the statement above would be  $(\exists! X \in A) (x \text{ fits the desk's lock})$ , where A is the set of keys.

For other examples, try and recall the statements of uniqueness in the mathematics that you've studied so far. What about 'There is a unique circle that passes through three non-collinear points in a plane.'? How would you represent this in symbols? If x denotes a circle, and y denotes a set of 3 non-collinear points in a plane, then the proposition is

$$(\forall y \in P) (\exists! X \in C) (x \text{ passes through } y).$$

Here C denotes the set of circles, and P the set of sets of 3 non-collinear points. And now, some short exercises for you!

E25) Which of the following propositions are true (where x, y are in R)?

- i)  $(x \geq 0) \rightarrow (\exists y) (y^2 = x)$
- ii)  $(\forall x) (\exists! y) (y^2 = x^3)$
- iii)  $(\exists x) (\exists! y) (xy = 0)$

Before ending the unit, let us take quick look at what we have covered in it.

## 1.6 SUMMARY

In this unit, we have considered the following points.

1. What a mathematically acceptable statement (or proposition) is.
2. The definition and use of logical connectives:
 

Give propositions p and q,

  - i) their disjunction is 'p and q', denoted by  $p \vee q$ ;
  - ii) their exclusive disjunction is 'either p or q', denoted by  $p \oplus q$ ;
  - iii) their conjunction is 'p and q', denoted by  $p \wedge q$ ;
  - iv) the negation of p is 'not p', denoted by  $\sim p$ ;
  - v) 'if p, then q' is denoted by  $p \rightarrow q$ ;
  - vi) 'p if and only if q' is denoted by  $p \leftrightarrow q$ ;
3. The truth tables corresponding to the 6 logical connectives.
4. Rule of precedence : In any compound statement involving more than one connective, we first apply ' $\sim$ ', then ' $\wedge$ ', then ' $\vee$ ' and ' $\oplus$ ', and last of all ' $\rightarrow$ ' and ' $\leftrightarrow$ '.
5. The meaning and use of logical equivalence, denoted by ' $\equiv$ '.
6. The following laws about equivalent propositions:
  - i) **De Morgan's laws:**  $\sim (p \wedge q) \equiv \sim p \vee \sim q$   
 $\sim (p \vee q) \equiv \sim p \wedge \sim q$
  - ii) **Double negation law:**  $\sim (\sim p) \equiv p$
  - iii) **Idempotent laws:**  $p \wedge p \equiv p$ ,  
 $p \vee p \equiv p$
  - iv) **Commutativity:**  $p \vee q \equiv q \vee p$   
 $p \wedge q \equiv q \wedge p$
  - v) **Associativity:**  $(p \vee q) \vee r \equiv p \vee (q \vee r)$   
 $(p \wedge q) \wedge r \equiv p \wedge (q \wedge r)$
  - vi) **Distributivity:**  $p \vee (q \wedge r) \equiv (p \vee q) \wedge (p \vee r)$   
 $p \wedge (q \vee r) \equiv (p \wedge q) \vee (p \wedge r)$

- vii)  $(\sim p \vee q) \equiv p \rightarrow q$  (ref. E18).
7. Logical quantifiers: ‘For every’ denoted by ‘ $\forall$ ’, ‘there exist’ denoted by ‘ $\exists$ ’, and ‘there is one and only one’ denoted by ‘ $\exists!$ ’.
8. The rule of negation related to the quantifiers:  
 $\sim (\forall x \in U)p(x) \equiv (\exists x \in U)(\sim p(x))$   
 $\sim (\exists x \in U)p(x) \equiv (\forall x \in U)(\sim p(x))$

Now we have come to the end of this unit. You should have tried all the exercises as you came to them. You may like to check your solutions with the ones we have given below.

## 1.7 SOLUTIONS/ ANSWERS

- E1) (i), (iii), (iv), (vii), (viii) are statements because each of them is universally true or universally false.  
 (ii) is a question.  
 (v) is an exclamation.  
 The truth or falsity of (vi) depends upon who ‘she’ is.  
 (ix) is a subjective sentence.  
 (x) will only be a statement if the value(s)  $n$  takes is/are given.  
 Therefore, (ii), (v), (vi), (ix) and (x) are not statements.
- E2) The truth value of (i) is F, and of all the others is T.
- E3) The disjunction is  
 ‘ $2+3 = 7$  or Radha is an engineer.’.  
 Since ‘ $2+3 = 7$ ’ is always false, the truth value of this disjunction depends on the truth value of ‘Radha is an engineer.’. If this is T, then we use the third row of Table 1 to get the required truth value as T. If Radha is not an engineer, then we get the required truth value as F.

Table 9: Truth table for ‘exclusive or’

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

- E4)  $p$  will be a true proposition for  $x \in ]-2, \infty[$  and  $x \neq 4$ , i.e., for  $x \in ]-2, 4[ \cup ]4, \infty[$ .
- E5) i)  $0 - 5 = 5$   
 ii) ‘ $n$  is not greater than 2 for every  $n \in \mathbf{N}$ .’, or ‘There is at least one  $n \in \mathbf{N}$  for which  $n \leq 2$ .’  
 iii) There are some Indian children who do not study till Class 5.

- E6) Table 10: Truth table for negation

p	$\sim p$
T	F
F	T

- E7)  $p \rightarrow q$  is the statement ‘If  $x + y = xy$  for  $x, y \in \mathbf{R}$ , then  $x \neq 0$  for every  $\in \mathbf{Z}$ ’.

In this case,  $q$  is false. Therefore, the conditional statement will be true if  $p$  is false also, and it will be false for those values of  $x$  and  $y$  that make  $p$  true.

So,  $p \rightarrow q$  is false for all those real numbers  $x$  of the form  $\frac{y}{y-1}$ , where

$y \in \mathbf{R} \setminus \{1\}$ . This is because if  $x = \frac{y}{y-1}$  for some  $y \in \mathbf{R} \setminus \{1\}$ , then  $x + y = xy$ ,

i.e.,  $p$  will be true.

- E8) i)  $p \rightarrow q$ , where  $p$  :  $\Delta ABC$  is isosceles. If  $q$  is true, then  $p \rightarrow q$  is true. If  $q$  is false, then  $p \rightarrow q$  is true only when  $p$  is false. So, if  $\Delta ABC$  is an isosceles triangle, the given statement is always true. Also, if  $\Delta ABC$  is not isosceles, then it can't be equilateral either. So the given statement is again true.

ii)  $p$  :  $a$  is an integer.

$q$  :  $b$  is an integer.

$r$  :  $ab$  is a rational number

The given statement is  $(p \wedge q) \leftrightarrow r$ .

Now, if  $p$  is true and  $q$  is true, then  $r$  is still true.

So,  $(p \wedge q) \leftrightarrow r$  will be true if  $p \wedge q$  is true, or when  $p \wedge q$  is false and  $r$  is false.

In all the other cases  $(p \wedge q) \leftrightarrow r$  will be false.

iii)  $p$  : Raza has 5 glasses of water.

$q$  : Sudha has 4 cups of tea.

$r$  : Shyam will pass the math exam.

The given statement is  $(p \wedge q) \rightarrow \sim r$ .

This is true when  $\sim r$  is true, or when  $r$  is true and  $p \wedge q$  is false.

In all the other cases it is false.

iv)  $p$  : Mariam is in Class 1.

$q$  : Mariam is in Class 2.

The given statement is  $p \oplus q$ .

This is true only when  $p$  is true or when  $q$  is true.

- E9) There are infinitely many such examples. You need to give one in which  $p$  is true but  $q$  is false.

- E10) Obtain the truth table. The last column will now have entries TTFTTTT.

- E11) According to the rule of precedence, given the truth values of  $p, q, r$  you should first find those of  $\sim r$ , then of  $q \wedge \sim r$ , and  $r \wedge q$ , and  $p \rightarrow q \wedge \sim r$ , and finally of  $(p \rightarrow q \wedge \sim r) \leftrightarrow r \wedge q$ .

Referring to Table 5, the values in the sixth and eighth columns will be replaced by

$r \wedge q$	$p \rightarrow q \wedge \sim r \leftrightarrow r \wedge q$
T	F
F	F
F	T
F	T
T	T
F	F
F	F
F	F

E12) They should both be the same, viz.,

p	q	r	$\sim r$	$p \wedge q$	$(p \wedge q) \vee (\sim r)$
T	T	T	F	T	T
T	T	F	T	T	T
T	F	T	F	F	F
T	F	F	T	F	T
F	T	T	F	F	F
F	T	F	T	F	T
F	F	T	F	F	F
F	F	F	T	F	T

- E13) i)  $(\sim p) \vee q$   
 ii)  $(\sim q) \rightarrow (\sim p)$   
 iii)  $(p \rightarrow q) \leftrightarrow [(\sim p) \vee q]$   
 iv)  $[(p \oplus (q \wedge r)) \rightarrow [(\sim p) \vee q]] \leftrightarrow (p \wedge r)$

E14) a)

p	$\sim p$	$\sim(\sim p)$
T	F	T
F	T	F

The first and third columns prove the double negation law.

b)

p	q	$p \vee q$	$q \vee p$
T	T	T	T
T	F	T	T
F	T	T	T
F	F	F	F

The third and fourth columns prove the commutativity of  $\vee$ .

E15) For any three propositions p, q, r:

- i)  $p \equiv p$  is trivially true.  
 ii) if  $p \equiv q$ , then  $q \equiv p$  (if p has the same truth value as q for all choices of truth values of p and q, then clearly q has the same truth values as p in all the cases.  
 iii) if  $p \equiv q$  and  $q \equiv r$ , then  $p \equiv r$  (reason as in (ii) above).

Thus,  $\equiv$  is reflexive, symmetric and transitive.

E16)

p	q	$\sim p$	$\sim p \vee q$	$p \rightarrow q$
T	T	F	T	T
T	F	F	F	F
F	T	T	T	T
F	F	T	T	T

The last two columns show that  $[(\sim p) \vee q] \equiv (p \rightarrow q)$ .

E17) i)

p	$\mathcal{T}$	$p \vee \mathcal{T}$
T	T	T
F	T	T

The second and third columns of this table show that  $p \vee \mathcal{T} = \mathcal{T}$ .



ii)

p	$\mathcal{F}$	$p \wedge \mathcal{F}$
T	F	F
F	F	F

The second and third columns of this table show that  $p \wedge \mathcal{F} = F$ .  
You can similarly check (ii) and (iii).

- E18)  $\sim (\sim p \wedge q) \wedge (p \vee q)$   
 $\equiv (\sim(\sim p) \vee \sim q) \wedge (p \wedge q)$ , by De Morgan's laws.  
 $\equiv (p \vee \sim q) \wedge (p \vee q)$ , by the double negation law.  
 $\equiv p \vee (\sim q \wedge q)$ , by distributivity  
 $\equiv p \vee \mathcal{F}$ , where  $\mathcal{F}$  denotes a contradiction  
 $\equiv p$ , using E 19.

- E19) p: It is raining.  
q: Nobody can go to see a film.  
Then the given proposition is  
 $[p \wedge (p \rightarrow q)] \rightarrow q$   
 $\equiv p \wedge (\sim p \vee q) \rightarrow q$ , since  $(p \rightarrow q) \equiv (\sim p \vee q)$   
 $\equiv (p \wedge \sim p) \vee (p \wedge q) \rightarrow q$ , by De Morgan's law  
 $\equiv \mathcal{F} \vee (p \wedge q) \rightarrow q$ , since  $p \wedge \sim p$  is a contradiction  
 $\equiv (\mathcal{F} \vee p) \wedge (F \vee q) \rightarrow q$ , by De Morgan's law  
 $\equiv p \wedge q \rightarrow q$ , since  $\mathcal{F} \vee p \equiv p$ .  
which is a tautology.

- E20) There are infinitely many examples. One such is:  
'If Venkat is on leave, then Shabnam will work on the computer'. This is of the form  $p \rightarrow q$ . Its truth values will be T or F, depending on those of p and q.

- E21) i)  $(\forall t \in [0, \infty]) (\forall x \in H)p(x, t)$  is the given statement where  $p(x, t)$  is the predicate 'The politician can fool x at time t second.', and H is the set of human beings.  
Its negation is  $(\exists t \in [0, \infty]) (\exists x \in H) (\sim p(x, t))$ , i.e., there is somebody who is not fooled by the politician at least for one moment.

- ii) The given statement is  
 $(\forall x \in \mathbf{R}) (\exists y \in \mathbf{R}) (x = y^2)$ . Its negation is  
 $(\exists x \in \mathbf{R}) (\forall y \in \mathbf{R}) (x \neq y^2)$ , i.e.,  
there is a real number which is not the square of any real number.

- iii) The given statement is  
 $(\exists x \in L) (\forall t \in [0, \infty]) p(x, t)$ , where L is the set of lawyers and  $p(x, t) : x$  does not lie at time t. The negation is  
 $(\forall x \in L) (\exists t \in [0, \infty]) (\sim p)$ , i.e., every lawyer tells a lie at some time.

- E22) i) For example,  
 $(\forall x \in \mathbf{N}) (\exists y \in \mathbf{Z}) (\frac{x}{y} \in \mathbf{Q})$  is a true statement. Its negation is  
 $\exists x \in \mathbf{N} (\forall y \in \mathbf{Z}) (\frac{x}{y} \notin \mathbf{Q})$   
You can try (ii) similarly.

- E23) (i), (iii) are true.  
(ii) is false (e.g., for  $x = -1$  there is no y such that  $y^2 = x^3$ ).

## Elementary Logic

(iv) is equivalent to  $(\forall x \in \mathbf{R}) [\sim (\exists! y \in \mathbf{R}) (x + y = 0)]$ , i.e., for every  $x$  there is no unique  $y$  such that  $x + y = 0$ . This is clearly false, because for every  $x$  **there is** a unique  $y (= -x)$  such that  $x + y = 0$ .



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## UNIT 2 COMBINATORICS — AN INTRODUCTION

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### 2.0 INTRODUCTION

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Let us start with thinking about how to assess the efficiency of a computer programme. For this we would need to estimate the number of times each procedure is called during the execution of the programme. How would we do this? The theory of combinatorics helps us in this matter, as you will see while studying this unit.

Combinatorics deals with counting the number of ways in which objects can be arranged according to some pattern (listing). Mostly, it deals with a finite number of objects and a finite number of ways of arranging them. Sometimes an infinite number of objects and infinite number of ways in which they can be arranged are also considered. However, in this unit and block, we shall restrict our discussion to a finite number of objects.

We start our discussion in Sec. 2.2, with two counting principles. These principles help us in counting the number of ways in which a task can be done when it consists of several subtasks, and there are many possible ways of doing the subtasks.

In Sec. 2.3 we look at arrangements of objects in which the order matters. Such arrangements are called permutations. Here we look at various linear and circular permutations, and how to count their number in a given situation.

In Sec. 2.4, we consider arrangements of objects in which the order does not matter. Such arrangements are called combinations. We will consider situations that require us to count combinations. You will see that most of these situations require us to apply the multiplication principle also.

In the next section, Sec. 2.5, we consider binomial and multinomial coefficients. We see how they are related to the objects studied in Sec. 2.4.

Finally, in Sec. 2.6, we consider the applications of what we have presented in the rest of the unit, for finding the probability of the occurrence of an event. As you will see, this application is natural, since we use similar counting arguments for obtaining discrete probabilities. This discussion will be useful for you, for instance, in coding theory as well as in designing **reliable** computer systems.

We continue our study of combinatorics in the next unit. We also have a section of miscellaneous exercises at the end of the block of which several are based on this unit. Doing these exercises, and every exercise given in the unit, will help you achieve the following objectives of this unit.

---

## 2.1 OBJECTIVES

---

After going through this unit, you should be able to:

- explain the multiplication and addition principles, and apply them;
- differentiate between situations involving permutations and those involving combinations;
- perform calculations involving permutations and combinations;
- prove and use formulae involving binomial and multinomial coefficients;
- apply the concepts presented so far for calculating combinatorial probabilities.

---

## 2.2 MULTIPLICATION AND ADDITION PRINCIPLES

---

Let us start with considering the following situation: Suppose a shop sells six styles of pants. Each style is available in 8 lengths, six waist sizes, and four colours. How many different kinds of pants does the shop need to stock?

There are 6 possible types of pants; then for each type, there are 8 possible length sizes; for each of these, there are 6 possible waist sizes; and each of these is available in 4 different colours. So, if you sit down to count all the possibilities, you will find a huge number, and may even miss some out! However, if you apply the multiplication principle, you will have the answer in a jiffy!

So, what is the multiplication principle? There are various ways of explaining this principle. One way is the following:

Suppose that a task/procedure consists of a sequence of subtasks or steps, say, Subtask 1, Subtask 2, ..., Subtask  $k$ . Furthermore, suppose that Subtask 1 can be performed in  $n_1$  ways, Subtask 2 can be performed in  $n_2$  ways after Subtask 1 has been performed, Subtask 3 can be performed in  $n_3$  ways after Subtask 1 and Subtask 2 have been performed, and so on. Then **the multiplication principle** says that the number of ways in which the whole task can be performed is  $n_1 \cdot n_2 \cdot \dots \cdot n_k$ .

Let us consider this principle in the context of boxes and objects filling them. Suppose there are  $m$  boxes. Suppose the first box can be filled up in  $k(1)$  ways. For every way of filling the first box, suppose there are  $k(2)$  ways of filling the second box. Then the two boxes can be filled up in  $k(1) \cdot k(2)$  ways. In general, if for every way of filling the first  $(r - 1)$  boxes, the  $r$ th box can be filled up in  $k(r)$  ways, for  $r = 2, 3, \dots, m$ , then the total number of ways of filling all the boxes is  $k(1) \cdot k(2) \cdot \dots \cdot k(m)$ .

So let us see how the multiplication principle can be applied to the situation above (the shop selling pants). Here  $k(1) = 6$ ,  $k(2) = 8$ ,  $k(3) = 6$  and  $k(4) = 4$ . So, the different kinds of pants are  $6 \times 8 \times 6 \times 4 = 1152$  in number.

Let's consider one more example.

**Example 1:** Suppose we want to choose two persons from a party consisting of 35 members as president and vice-president. In how many ways can this be done?

**Solution:** Here, Subtask 1 is 'choosing a president'. This can be done in 35 ways. Subtask 2 is 'choosing a vice-president'. For each choice of president, we can choose the vice-president in 34 ways. Therefore, the total number of ways in which Subtasks 1 and 2 can be done is  $35 \times 34 = 1190$ .

\* \* \*

There is another fundamental principle called the **addition principle**. This is applied in situations like the following one:

Suppose that a task consists of performing exactly one subtask from among a collection of disjoint (mutually exclusive) subtasks, say, Subtask 1, Subtask 2, ..., Subtask  $k$ . (i.e., the task is performed if **either** Subtask 1 is performed, **or** Subtask 2, ..., or Subtask  $k$  is performed.) Further, suppose that Subtask  $i$  can be performed in  $n_i$  ways,  $i = 1, 2, \dots, k$ . Then, the number of ways in which the task can be performed is the sum  $n_1 + n_2 + \dots + n_k$ .

Let us consider an example of its application.

**Example 2:** There are three political parties,  $P_1$ ,  $P_2$  and  $P_3$ . The party  $P_1$  has 4 members,  $P_2$  has 5 members and  $P_3$  has 6 members in an assembly. Suppose we want to select two persons, both from the same party, to become president and vice-president. In how many ways can this be done?

**Solution:** From  $P_1$ , we can do the task in  $4 \times 3 = 12$  ways, using the multiplication principle. From  $P_2$ , it can be done in  $5 \times 4 = 20$  ways. From  $P_3$  it can be done in  $6 \times 5 = 30$  ways. So, by the addition principle, the number of ways of doing the task is  $12 + 20 + 30 = 62$ .

\* \* \*

Though both these principles seem simple, quite a number of combinatorial enumerations can be done with them. For instance, what we see from Example 2 is that the addition principle helps us to count all possible arrangements grouped into mutually exclusive and exhaustive classes.

Why don't you try a few exercises that involve the use of these principles now?

- 
- E1) Give a situation related to computing in which the addition principle is used, and one in which the multiplication principle is used.
  - E2) Find the number of words of length 4, meaningful or not, made with the letters  $a, b, \dots, j$ .
  - E3) If  $n$  couples are at a dance, in how many ways can the men and women be paired for a single dance?
  - E4) How many integers between 100 and 999 consist of distinct even digits?
  - E5) Consider all the numbers between 100 and 999 that have distinct digits. How many of them are odd?
- 

Let us now consider certain arrangements of objects, in which the order in which they are arranged matters.

## 2.3 PERMUTATIONS

---

Suppose we have 15 books that we want to arrange on a shelf. How many ways are there of doing it? Using the multiplication principle, you would say —

$$15 \times 14 \times 13 \times \dots \times 2 \times 1 = 15!$$

Each of these arrangements of the books is a permutation of the books. Let us define this term formally.

**Definition:** An arrangement of a set of  $n$  objects **in a given order** is called a **permutation** of the objects (taken altogether at a time).

$n!$  denotes ' **$n$  factorial**', which means  $n \times (n-1) \times \dots \times 2 \times 1$  for any  $n \in \mathbb{N}$ .)

An **ordered** arrangement of the  $n$  objects, taking  $r$  at a time, (where  $r \leq n$ ) is called a **permutation of the  $n$  objects taking  $r$  at a time**. The total number of such permutations is denoted by  $P(n,r)$ .

As an example, let us consider picking out books, three at a time, from the shelf of 15 books. The first book can be chosen in 15 ways, the next in 14 ways, and the third in 13 ways. So the multiplication principle tells us that the total number of permutations of the 15 books taken 3 at a time is  $P(15,3) = 15 \times 14 \times 13$ .

Other notations used for  $P(n,r)$  are  ${}^n P_r$ ,  $P_r^n$ ,  ${}_n P_r$ .

Again, consider the permutations of a,b,c,d, taken 2 at a time. These are ab, ba, ac, ca, ad, da, bc, cb, bd, db, cd, dc. (Note that ab and ba are considered different even though they consist of the same two objects.) Or, we can argue combinatorically as above: The first letter can be chosen in 4 ways, and then the next letter can be chosen. We can list out all the cases in 3 ways. So, the total number of permutations are  $P(4,2) = 4 \times 3 = 12$ .

Now, is there a formula for finding the value of  $P(n,r)$ ? This is what the following theorem tells us.

**Theorem 1:** The number of permutations of  $n$  objects, taken  $r$  at a time, where  $0 \leq r \leq n$ , is given by  $P(n,r) = \frac{n!}{(n-r)!}$

Consider  $r$  boxes arranged in a line. Choose one object out of  $n$  and place it in the first box. This can be done in  $n$  ways. Then from the remaining  $(n-1)$  objects choose one and place it in the second box. The first two boxes can be filled in  $n(n-1)$  ways. We continue this operation till the  $r$ th box is filled. So, by the multiplication principle, the total number of ways of doing this is  $n(n-1)(n-2) \dots (n-r+1)$ .

$$\begin{aligned} P(n,r) &= n(n-1) \dots (n-r+1) \\ &= n(n-1) \dots (n-r+1)(n-r)(n-r-1) \dots 3.2.1 \\ &= (n-r) \dots (n-r-1) \dots 3.2.1 \\ &= n!/(n-r)! \end{aligned}$$

We define  $0! = 1$

**Proof:** In particular, Theorem 1 tells us that the number of permutation of  $n$  objects, taken all at a time, is given by

$$P(n,n) = n!$$

$$\text{and } P(n, 0) = 1 \quad \forall n \in \mathbb{N}.$$

So, for example, by Theorem 1 we can find

$$P(6,4) = 6.5.4.3 = 6!/(6-4)! \text{ And } P(6,0) = 1.$$

Why don't you try some exercises now?

---

E6) If  $m$  and  $n$  are positive integers, show that  $(m+n)! \geq m! + n!$ .

E7) How many 3-digit numbers can be formed from the 6 digits 2,3,5,7,8,9 if repetitions are not allowed? How many of these numbers are less than 400? How many are even?

E8) How many ways are there to rank  $n$  candidates for the job of chief engineer? In how many rankings will Ms. Sheela be in the second place.

---

In defining the concept of permutation we assumed that the objects were distinguishable. What does this mean, and what happens if we remove this assumption? Let's see.

### 2.3.1 Permutation of Objects Not Necessarily Distinct

We have shown that there are  $P(n,r)$  ways to choose  $r$  objects from a set of  $n$  distinct objects and arrange them in linear order. Here we consider the same problem with the relaxed condition that some of the objects in the collection may not be distinguishable.

For example, we consider permutations of the letters of the word DISTINCT. Here there are 8 letters of which 2 are I, 2 are T, and three are 4 other different letters. To count the permutations in such a situation, we have the following result.

**Theorem 2:** Suppose there are  $n$  objects classified into  $k$  distinct types, with  $m_1$  identical objects of the first type,  $m_2$  identical objects of the second type, ..., and  $m_k$  identical objects of the  $k$ th type, where  $m_1 + m_2 + \dots + m_k = n$ . Then the number of distinct arrangements of these  $n$  objects, denoted by  $P(n; m_1, m_2, \dots, m_k)$  is  $\frac{n!}{m_1! m_2! \dots m_k!}$ .

**Proof:** Let  $x$  be the number of such permutations. If the objects of Type  $i$  are considered distinct, then they can be arranged amongst themselves in  $m_i!$  ways, where  $i = 1, 2, \dots, k$ . Therefore, by the multiplication principle, the total number of permutations of these  $n$  distinct objects, taken all at a time, is  $x m_1! m_2! \dots m_k!$ .

But this is precisely  $n!$  when there are  $n$  distinct objects.

Hence,  $x m_1! m_2! \dots m_k! = n!$ , that is,  $x = n! / m_1! m_2! \dots m_k!$

So for example, this result tells us that the number of distinct 8 letter words, not necessarily meaningful, that we can make from the letter of the word "DISTINCT" is  $\frac{8!}{2!2!1!1!1!1!} = 14$ .

Here are some related exercises.

- 
- E9) How many permutations are there of the letters, taken all at a time, of the words  
(i) ASSESSES, (ii) PATTIVEERANPATTI?
- E10) How many licence plates can be made if each should have 3 letters of the English alphabet with no letter repeated? What will be the answer if the letters can be repeated?
- 

So far, we have considered permutations of objects as linear arrangements of objects; this means that we visualize arrangements of objects in a **line**. But there is a variant in which the objects are arranged along the circumference of a circle. Let us consider that now.

### 2.3.2 Circular Permutation

Consider an arrangement of 4 objects, a,b,c,d as in Fig. 1. We observe the objects in the clockwise direction. On the circumference there is no preferred origin, and hence the permutations abcd, bcda, cdab, dabc will look exactly alike. So, each linear



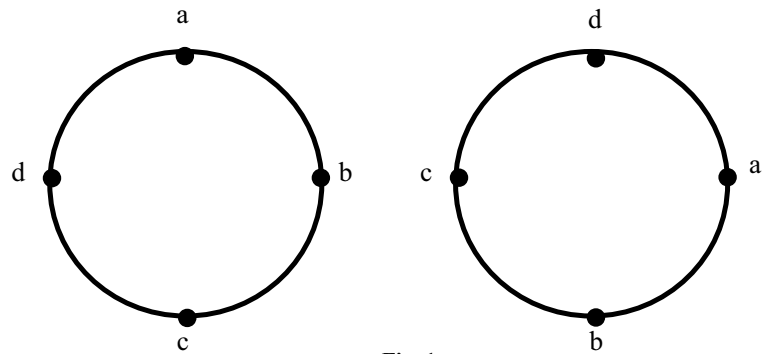


Fig. 1

permutation, when treated as a circular permutation, is repeated 4 times. Similarly, if  $n$  objects are placed in a circular arrangement, each linear arrangement is repeated  $n$  times. So, if we consider all the  $n!$  permutations of  $n$  things, each circular permutation will be indistinguishable from the  $(n-1)$  others obtained by the process of rotating the objects in the same order. So the number of distinct circular permutations will be  $n!/n = (n-1)!$ . Thus, we have shown that **the number of circular permutations of  $n$  things, taken all at a time, is  $(n-1)!$ .**

Let us consider some examples.

**Example 3:** In how many distinct ways is it possible to seat eight persons at a round table?

**Solution:** Clearly we need the number of circular permutations of 8 things. Hence the answer is  $7! = 5040$ .

\* \* \*

**Example 4:** In the preceding question, what would be the answer if a certain pair among the eight persons

- (i) must not sit in adjacent seats?
- (ii) must sit in adjacent seats

**Solution:** To answer (i), let us first solve (ii) from  $7!$  we have to subtract the number of cases in which the pair of persons sit together. If we consider the pair as forming one unit, then we have the circular permutations of 7 objects, which is  $(7-1)!$  (Note that this is the answer for (ii).) But even as a unit they can be arranged in two ways. Hence the required answer is  $2(6!)$ . Now to answer (i), we must subtract these possibilities from the total number of ways of seating all the people. This is  $7! - 2(6!) = 3600$ .

\* \* \*

**Example 5:** Suppose there are five married couples and they (10 people) are made to sit about a round table so that neither two men nor two women sit together. Find the number of such circular arrangements.

**Solution:** Five females can be made to sit about a round table in  $(5-1)! = 4!$  ways. One male can be seated in between two females. There are five positions, and hence they can be made to sit in  $5!$  ways. By the multiplication principle, the total number of ways of such seating arrangements is  $4! \times 5! = 2880$ .

\* \* \*

**Example 6:** Consider seven people seated about a round table. How many circular arrangements are possible if at least one of them will not have the same neighbours in any two arrangements?

**Solution:** The two distinct arrangements in Fig. 2 show that each has the same neighbours.

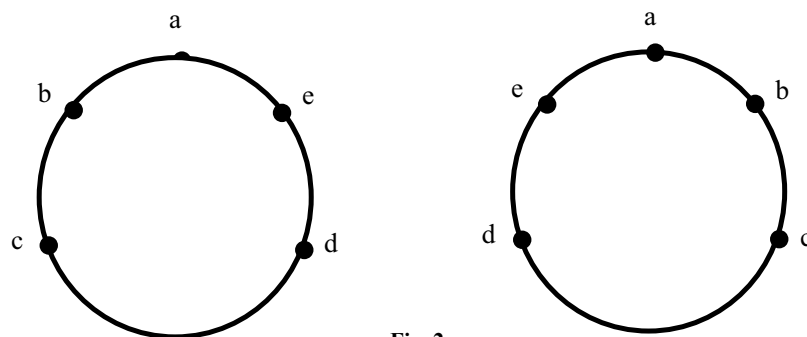


Fig. 2

Hence, the total number of circular arrangements  $= (7-1)! \times \frac{1}{2} = 360$ .

\* \* \*

You may try the following exercise.

---

E11) If there are 7 men and 5 women, how many circular arrangements are possible in which women do not sit adjacent to each other?

---

Permutations apply to ordered arrangement of objects. What happens if order does not matter? Let's see.

---

## 2.4 COMBINATIONS

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Let's begin by considering a situation where we want to choose a committee of 3 faculty members from a group of seven faculty members. In how many distinct ways can this be done? Here order doesn't matter, because choosing  $F_1, F_2, F_3$  is the same as choosing  $F_2, F_1, F_3$ , and so on. (Here  $F_i$  denotes the  $i$ th faculty member.) So, for every choice of members, to avoid repetition, we have to divide by  $3!$ . Thus, the

number would be  $\frac{7 \times 6 \times 5}{3!} = \frac{7!}{3!4!}$ .

More generally, suppose there are  $n$  distinct objects and we want to select  $r$  objects, where  $r \leq n$ , where the order of **the objects in the selection does not matter**. This is called a **combination** of  $n$  things taken  $r$  at a time. The number of ways of doing this is represented by  ${}_nC_r$ ,  ${}^nC_r$ ,  $C_r^n$ ,  $\binom{n}{r}$  or  $C(n, r)$ . We will use the notation  $C(n, r)$ , in conformity with the notation  $P(n, r)$  for permutations. We read  $C(n, r)$  as 'n choose r' to emphasize the fact that only **choice** is involved but **not ordering**.

In the example that we started the section with, you saw that the number of combinations was  $7!/3!4!$ , i.e.,  $\frac{P(7,3)}{3!}$ . In fact, this relationship between  $C(n, r)$  and  $P(n, r)$  is true in general. We have the following result.

**Theorem 3:** The number of combinations of  $n$  objects, taken  $r$  at a time, where  $0 \leq r \leq n$  is given by

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)!r!}.$$

**Proof:**  $C(n, r)$  counts the number of ways of choosing  $r$  out of  $n$  distinct objects without regard to the order. Any one of these choices is simply a subset of  $r$  objects of the set of  $n$  objects we have. Such a set can be ordered in  $r!$  ways. Thus, to each combination, there corresponds  $r!$  permutations. Hence there are  $r!$  times as many permutations as there are combinations. Hence, by the multiplication principle, we get

$$P(n, r) = r! C(n, r)$$

$$\text{Therefore, } C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)!r!}.$$

Using Theorem 3, we can very quickly find out, for instance, how many ways there are of choosing 2 rooms out of 20 rooms offered. This is  $C(20, 2) = \frac{20!}{18!2!} = 190$ .

Now, to find  $C(20, 2)$ , I took a short cut. I cancelled  $18!$  from the number and denominator. In practice, I only needed to calculate  $\frac{20 \times 19}{2 \times 1}$ . This practice is useful,

in general, i.e., we use the identity  $C(n, r) = \frac{n(n-1)\dots r \text{ factors}}{r(r-1)\dots r \text{ factors}}$  for calculations. In

fact, sometimes  $r$  is much larger than  $n-r$ , in which case we cancel  $r!$ . This is also what the following result suggests.

**Theorem 4:**  $C(n, r) = C(n, n-r)$ , for  $0 \leq r \leq n$ ,  $n \in \mathbf{N}$ .

**Proof 1:** For every choice of  $r$  things from  $n$  things, there uniquely corresponds a choice of  $n-r$  things from those  $n$  objects, which are the unchosen objects. This one-to-one correspondence shows that these numbers must be the same. This proves the theorem.

$$\text{Proof 2: } C(n, r) = \frac{n!}{(n-r)!r!} = \frac{n!}{(n-r)!(n-n+r)!} = C(n, n-r).$$

Because of these two theorems we have, for instance,

$$C(n, n) = C(n, 0) = P(n, 0) = 1. \quad C(n, 1) = C(n, n-1) = P(n, 1) = n.$$

The numbers  $C(n, r)$  are also called the binomial coefficients as they occur as the coefficients of  $x^r$  in the expansion of  $(1+x)^n$  in ascending powers of  $x$ , as you will see in Sec. 1.5. At this stage, let us consider some examples involving  $C(n, r)$ .

**Example 7:** Evaluate  $C(6, 2)$ ,  $C(7, 4)$  and  $C(9, 3)$ .

$$\text{Solution: } C(6, 2) = \frac{6 \cdot 5}{2 \cdot 1} = 15, \quad C(7, 4) = \frac{7 \cdot 6 \cdot 5}{3 \cdot 2 \cdot 1} = 35, \quad \text{and } C(9, 3) = \frac{9 \cdot 8 \cdot 7}{3 \cdot 2 \cdot 1} = 84.$$

**Example 8:** Find the number of distinct sets of 5 cards that can be dealt from a deck of 52 cards.

**Solution:** The order in which the cards are dealt is not important. So, the required

$$\text{number is } C(52, 5) = \frac{52!}{5!47!} = \frac{52 \times 51 \times 50 \times 49 \times 48}{5 \times 4 \times 3 \times 2 \times 1} = 2,598,960.$$

**Example 9:** Suppose a valid computer password consists of 8 characters, the first of which is the digit 1, 3 or 5. The rest of the 7 characters are either English alphabets or a digit. How many different passwords are possible?

**Solution:** Firstly, the initial character can be chosen in  $C(3, 1)$  ways. Now, there are 26 alphabets and 10 digits to choose the rest of the characters from, and repetition is allowed. So, the total number of possibilities for these characters is  $(26+10)^7$ .

Therefore, by the multiplication principle, the number of passwords possible are  $C(3, 1) \cdot 36^7$ .

Here are some exercises now.

- 
- E12) At a certain office, a committee consisting of one male and one female worker is to be constituted from among 12 men and 15 women workers. In how many distinct ways can this be done?
- E13) In how many ways can a prize winner choose any 3 CDs from the ‘Ten Best’ list?
- E14) How many different 7-person committees can be formed, each containing 3 women and 4 men, from a set of 20 women and 30 men?
- 

So far we have been considering combinations of distinct objects. Let us now look at combinations in which repetitions are allowed. We start with considering the following situations.

Suppose five friends stop at a sweet shop where each of them has one of the following: a samosa, a dosa, and a vada. The order of consumption does not matter. How many different purchases are possible?

Let  $s$ ,  $t$ , and  $d$  represent samosa, dosa, vada, respectively. In the following table we have listed some possible ways of purchasing these. For instance, the second row represents the possibility that all 5 friends order only dosas.

s	d	v
x	x	xxx
xxx	xxxx	xx

These orders can also be represented by  $x$ 's and  $|$ 's. For instance, the first row can be written as  $x|x|xxx$ . So, any order will consist of five  $x$ 's and two  $|$ 's.

Conversely, any sequence of five  $x$ 's and two  $|$ 's represents an order. So, there is a 1-to-1 correspondence between the orders placed and sequences of five  $x$ 's and two  $|$ 's. But the number of such sequences is just the number of distinct ways of placing 2  $|$ 's in 7 possible places. This is  $C(7,2)$ .

More generally, if we wish to select with repetition,  $r$  out of  $n$  distinct objects, we are considering all arrangements of  $r$  of one kind (say  $x$ 's) and  $n - 1$  of the other kind (say  $|$ 's) (because  $(n - 1)$   $|$ 's are needed to separate  $n$  types).

The following result gives us the total number of such possibilities.

**Theorem 5:** Let  $n$  and  $r$  be natural numbers. Then the number of solutions in natural numbers, to the equation  $x_1 + x_2 + \dots + x_n = r$ , is  $C(n + r - 1, r)$ . Equivalently, the

number of ways to choose  $r$  objects from a collection of  $n$  objects, with repetition allowed, is  $C(n + r - 1, r)$ .

**Proof:** Any string will consist of  $r$  objects and  $n - 1$  bars, to denote the  $n$  different categories in which these objects can fall. So, it will be a string of length  $n + r - 1$ , containing exactly  $r$  stars and  $n - 1$  bars. The total number of such strings is the number of ways we can position  $(n - 1)$  bars in  $r$  different places. This is  $C(n + r - 1, r)$ .

Now we demonstrate how such strings correspond to solution of the equation  $x_1 + \dots + x_n = r$ .

$n - 1$  bars in the string divide the string into  $n$  substrings of stars. The number of stars in these  $n$  substrings are the values of  $x_1, x_2, \dots, x_n$ . Since there are  $r$  stars altogether, the sum is  $r$ . Therefore, is a one-to-one correspondence between the strings and the solutions, and the theorem is proved.

Let us consider examples of the use of this result.

**Example 10:** In how many ways can a prize winner choose three books from a list of 10 best sellers, if repeats are allowed?

**Solution:** Here, note that a person can choose all three books to be the same title. Applying Theorem 5, the solution is  $C(10 + 3 - 1, 3) = C(12, 3) = 220$ .

\* \* \*

**Example 11:** Determine the number of integer solutions to the equation  $x_1 + x_2 + x_3 + x_4 = 7$ , where  $x_i \geq 0$  for all  $i = 1, 2, 3, 4$ .

**Solution:** The solution of the equation corresponds to a selection, with repetition, of size 7 from a collection of size 4. Hence, there are  $C(4 + 7 - 1, 7) = 120$  solutions. ( $n = 4, r = 7$  in Theorem 5.)

\* \* \*

So, from this sub-section, we see the equivalence of the following:

- (a) The number of integer solutions of the equation  $x_1 + x_2 + \dots + x_n = r, x_i \geq 0, 1 \leq i \leq n$ .
- (b) The number of selections, with repetition, of size  $r$  from a collection of size  $n$ .
- (c) The number of ways  $r$  identical objects can be distributed among  $n$  distinct containers.

Why don't you try some exercises now?

---

E15) A student in a college hostel is allowed four fruits per day. There are 6 different types of fruits from which she can choose what she wants. For how many days can a student make a different selection?

E16) An urn contains 15 balls, 8 of which are red and 7 are black. In how many ways can:

- i) 5 balls be chosen so that all 5 are red?
  - ii) 7 balls be chosen so that at least 5 are red?
-

In this section we have considered choosing  $r$  objects, with repetition, out of  $n$  objects, regardless of order. What happens when order comes into the picture? Let's consider an example.

**Example 12:** A box contains 3 red, 3 blue and 4 white socks. In how many ways can 8 socks be pulled out of the box, one at a time, if order is important?

**Solution:** Let us first see what happens if order isn't important. In this case we count the number of solutions of  $r+b+w = 8$ ,  $0 \leq r, b \leq 3$ ,  $0 \leq w \leq 4$ . To apply Theorem 5, we write  $x = 3 - r$ ,  $y = 3 - b$ ,  $z = 4 - w$ .

Then we have  $x+y+z = 10 - 8 = 2$ , and the number of solutions this has is  $C(3+2-1, 2) = 6$ .

These 6 solutions are  $(1, 0, 1)$ ,  $(0, 1, 1)$ ,  $(1, 1, 0)$ ,  $(2, 0, 0)$ ,  $(0, 2, 0)$ ,  $(0, 0, 2)$ . So, the corresponding solutions for  $(r, b, w)$  are

$(3, 3, 2)$ ,  $(2, 3, 3)$ ,  $(3, 2, 3)$ ,  $(3, 1, 4)$ ,  $(2, 2, 4)$ ,  $(1, 3, 4)$ .

Now, we consider order. From Theorem 2 we know that the number of ways of pulling out 3 red, 3 blue and 2 white socks in some order is  $\frac{8!}{3!3!2!}$ . This number would be the same if you had 2 red, 3 blue and 3 white socks, etc. By this reasoning and considering all different orderings, the number of possibilities is

$$3 \left( \frac{8!}{3!3!2!} \right) + 2 \left( \frac{8!}{3!1!4!} \right) + \frac{8!}{2!2!4!} = 3220.$$

\* \* \*

What we see, via Example 13, is that if we want to find the number of possibilities wherein order matters and repetition is allowed them:

**Step 1:** Find the possibilities when order doesn't matter, using Theorem 5;

**Step 2:** Use Theorem 2, to find the possibilities for each solution obtained in Step 1.

Why don't you try and exercise now?

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E17) How many 6-letter words, not necessarily meaningful can be formed from the letters of CARACAS?

---

Let us now consider why  $C(n, r)$  shows up as the coefficients in the binomial expansions.

## 2.5 BINOMIAL COEFFICIENTS

You must be familiar with expressions like  $a+b$ ,  $p+q$ ,  $x+y$ , all consisting of two terms. This is why they are called binomials. You also know that a **binomial expansion** refers to the expansion of a positive integral power of such a binomial. For instance,  $(a+b)^5 = a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$  is a binomial expansion. Consider coefficients 1, 5, 10, 10, 5, 1 of this expansion. In particular, let us consider the coefficient 10, of  $a^3b^2$  in this expansion. We can get this term by selecting  $a$  from 3 of the binomials and  $b$  from the remaining 2 binomials in the product  $(a+b)(a+b)(a+b)(a+b)(a+b)$ . Now,  $a$  can be chosen in  $C(5, 3)$  ways, i.e., 10 ways. This is the way each coefficient arises in the expansion.

The same argument can be extended to get the coefficients of  $a^r b^{n-r}$  in the expansion of  $(a+b)^n$ . From the  $n$  factors in  $(a+b)^n$ , we have to select  $r$  for  $a$  and the remaining  $(n-r)$  for  $b$ . This can be done in  $C(n, r)$  ways. Thus, the coefficient of  $a^r b^{n-r}$  in the expansion of  $(a+b)^n$  is  $C(n, r)$ .

In view of the fact that  $C(n, r) = C(n, n-r)$ , the coefficients of  $a^r b^{n-r}$  and  $a^{n-r} b^r$  will be the same.  $r$  can only take the values  $0, 1, 2, \dots, n$ . We also see that  $C(n, 0) = C(n, n) = 1$  are the coefficients of  $a^n$  and  $b^n$ . Hence we have established the binomial expansion.

$$(a+b)^n = a^n + C(n, 1) a^{n-1} b + C(n, 2) a^{n-2} b^2 + \dots + C(n, r) a^{n-r} b^r + \dots + b^n.$$

In analogy with ‘binomial’, which is a sum of two symbols, we have ‘multinomial’ which is a sum of two or more (though finite) distinct symbols. Multinomial expansion refers to the expansion of a positive integral power of a multinomial. Specifically we will consider the expansion of  $(a_1 + a_2 + \dots + a_m)^n$ . For the expansion we can use the same technique as we use for the binomial expansion. We consider the  $n$ th power of the multinomial as the product of  $n$  factors, each of which is the same multinomial. Every term in the expansion can be obtained by picking one symbol from each factor and multiplying them. Clearly, any term will be of the form

$a_1^{r_1} a_2^{r_2} \dots a_m^{r_m}$  where  $r_1, r_2, \dots$ , are non-negative integers such that  $r_1 + r_2 + \dots + r_m = n$ . Such a term is obtained by selecting  $a_1$  from  $r_1$  factors,  $a_2$  from  $r_2$  factors **from among the remaining  $(n-r_1)$  factors**, and so on. This can be done in

$C(n, r_1). C(n-r_1, r_2). C(n-r_1-r_2, r_3) \dots C(n-r_1-r_2-\dots-r_{m-1}, r_m)$  ways.

If you simplify this expression, it will reduce to  $\frac{n!}{r_1! r_2! \dots r_m!}$ .

So, we see that the **multinomial expansion** is

$$(a_1 + a_2 + \dots + a_m)^n = \sum \frac{n!}{r_1! r_2! \dots r_m!} a_1^{r_1} a_2^{r_2} \dots a_m^{r_m}$$

where the summation is over all non-negative integers  $r_1, r_2, \dots, r_m$  adding to  $n$ .

The coefficient of  $a_1^{r_1} a_2^{r_2} \dots a_m^{r_m}$  in the expansion of  $(a_1 + a_2 + \dots + a_m)^n$  is  $\frac{n!}{r_1! r_2! \dots r_m!}$ , and

is called a **multinomial coefficient**, in analogy with the binomial coefficient. We represent this by  $C(n; r_1, r_2, \dots, r_m)$ . This is also represented by many authors as

$$\left[ \frac{n}{r_1, r_2, \dots, r_m} \right].$$

For instance, the coefficient of  $x^2 y^2 z^2 t^2 u^2$  in the expansion of  $(x + y + z + t + u)^{10}$  is  $C(10; 2, 2, 2, 2, 2) = 10!/(2!)^5$ .

Let us see an example involving such coefficients.

**Example 13:** What is the sum of the coefficients of all the terms in the expansion of  $(a+b+c)^7$ ?

**Solution:** The required answer is  $\sum \frac{7!}{r! s! t!}$ , where the summation is over all non-

negative integers  $r, s, t$  adding to  $n$ . But it is also the value of  $\sum \frac{7!}{r! s! t!} a^r b^s c^t$  for  $a = b = c = 1$ .

So the answer is  $(1 + 1 + 1)^7 = 3^7$ .

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3, 6, 10, 15, ..., reflects the fact that differences increase by 1 as we move down the diagonal.

Let us now consider some identities involving binomial coefficients.

**Identity 1:**  $C(n, 0) + C(n, 1) + C(n, 2) + \dots + C(n, n-1) + C(n, n) = 2^n$

By setting  $a = b = 1$  in the binomial expansion of  $(a+b)^n$ , we get this identity. In the context of sets, it tells us the number of distinct subset that can be formed from a set with  $n$  elements. Note that the number of subsets containing precisely  $r$  elements is  $C(n, r)$ . Hence the total number of subsets is  $\sum_{r=0}^n C(n, r) = 2^n$ , by the identity. So, this identity tells us that **the number of distinct subsets of a set with  $n$  elements is  $2^n$** .

**Identity 2:**  $C(n, 0) - C(n, 1) + C(n, 2) - \dots + (-1)^n C(n, n) = 0$ .

We get this by setting  $a = 1$ ,  $b = -1$  in the expansion of  $(a+b)^n$ .

Now, adding the two identities, we get

$$2 \sum_{r \text{ even}} C(n, r) = 2^n, \text{ i.e., } \sum_{r \text{ even}} C(n, r) = 2^{n-1}$$

Similarly subtracting the second identity from the first leads us to the equation

$$\sum_{r \text{ odd}} C(n, r) = 2^{n-1}.$$

These two equations tell us that the number of subsets of a set of  $n$  elements with an even number of elements is equal to the number of subsets with an odd number of elements, both being  $2^{n-1}$ .

Why don't you try to prove some identities now?

---

E18) Show that  $C(n, m) C(m, k) = C(n, k) C(n-k, m-k)$ ,  $1 \leq k \leq m \leq n$ .

E19) Prove that  $C(k, k) + C(k+1, k) + C(k+2, k) + \dots + C(n, k) = C(n+1, k+1)$  for all natural numbers  $k \leq n$ .

---

Before ending this section, we just mention another extension of the definition of binomial coefficients. So far, we have defined  $C(n, r)$  for  $n \geq r \geq 0$ . We can extend this definition for any real number  $x$ , and any non-negative integer  $k$ , by

$$C(x, k) = \frac{x(x-1)\dots(x-k+1)}{k!}.$$

This definition coincides with that of  $C(n, k)$ , when  $n$  is a non-negative integer.

So far, in this unit, we have considered various ways of counting different kinds of arrangements. These methods are, not surprisingly, helpful in finding the probability of an event. We shall now discuss this.

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## 2.6 COMBINATORIAL PROBABILITY

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Historically, counting problems have been closely associated with probability. The probability of getting at least 6 heads on 10 flips of a fair coin, the probability of finding a defective bulb in a sample of 25 bulbs if 5 percent of the bulbs from which the sample was drawn are defective — all these probabilities are essentially counting problems. In fact, Pascal's triangle (Fig. 4) was developed by Pascal around 1650 while analysing some gambling probabilities.

Let us start by recalling some basic facts about probability. An **experiment** is a clearly defined procedure that produces one of a given set of outcomes. The set of all outcomes is called **the sample space** of the experiment.

For example, the experiment could be checking the weather to see if it is raining or not on a particular day. The sample space here would be {raining, not raining}.

Given an experiment, we can often associate more than one sample space with it. For instance, suppose the experiment is the tossing of two coins.

- i) If the observer wants to record the number of tails observed as the outcomes, the sample space is {0, 1, 2}.
- ii) If the outcomes are the sequence of heads and tails observed, then the sample space is {HH, HT, TH, TT}.

A subset of the sample space of an experiment is called an **event**. For example, for an experiment consisting of tossing 2 coins, with sample space {HH, HT, TH, TT}, the event that two heads do **not** show up is the subset {HT, TH, TT}.

Suppose  $X$  is a sample space of an experiment with  $N$  outcomes. Then, the events are all the  $2^N$  subsets of  $X$ . The empty set  $\phi$  is called the **impossible event**, and the set  $X$  itself is called the **sure event**.

Now, for the purpose of this course, we will assume that all the outcomes of an experiment are **equally likely**, that is, there is nothing to prefer one case over the other. For example, in the experiment of coin tossing, we assume that the coin is unbiased. This means that 'head' and 'tail' are equally likely in a toss. The toss itself is considered a random mechanism ensuring 'equally likely' outcomes. Of course, there are coins that are 'loaded', which means that one side of the coin may be heavier than the other. But such coins are excluded from our discussion. Also, in our discussions we shall always assume that our **sample space is finite**.

Given this background, we have the following definition.

**Definition:** Then the **probability of the event  $A$** , represented by  $P(A)$ , is  $\frac{n(A)}{n(X)}$ .

For instance, the probability that a card selected from a deck of 52 cards is a spade is  $\frac{13}{52}$ , because  $A$  is the set of 13 spades in the deck.

We represent the number of elements of a finite set  $A$ , i.e., the **cardinality** of  $A$ , by  $n(A)$  or  $|A|$ .

From the definition, we get the following statements:

- i) As  $n(\phi) = 0$ , it follows that  $P(\phi) = 0$ .
- ii) By definition,  $P(X) = \frac{n(X)}{n(X)} = 1$ .
- iii) If  $A$  and  $B$  are two events, then  $n(A \cup B) = n(A) + n(B) - n(A \cap B)$ . Therefore,  $P(A \cup B) = P(A) + P(B) - P(A \cap B)$ .
- iv) (**Addition Theorem in Probability**) : If  $A$  and  $B$  are two mutually exclusive events, then the probability of their union is the sum of the probabilities of  $A$  and  $B$ . i.e., if  $A \cap B = \phi$ , then  **$P(A \cup B) = P(A) + P(B)$** .

[This is a consequence of (i) and (iii) above.]

- v) Suppose  $A$  is an event. Then the probability of  $A^c$  (also denoted by  $A'$ ), the event complementary to  $A$ , or the event 'not  $A$ ' is  $1 - P(A)$ , i.e.,

$$P(A^c) = 1 - P(A).$$

The reason is that the events  $A$  and  $A^c$  are mutually exclusive and exhaustive, i.e.,  $A \cup A^c = X$  and  $P(A) + P(A^c) = 1$ .

- vi) (The generalised addition theorem) : If the events  $A_1, A_2, \dots, A_m$  are pairwise disjoint (i.e., mutually exclusive), then  $P(\bigcup_i A_i) = \sum_i P(A_i)$ .

Let us consider some examples from combinatorial probability.

**Example 14:** A die is rolled once. What are the probabilities of the following events?

- (i) getting an even number,
- (ii) getting at least 2,
- (iii) getting at most 2,
- (iv) getting at least 10.

**Solution:** If we call the events  $A, B, C$  and  $D$ , then we have  $X = \{1, 2, 3, 4, 5, 6\}$ ,  $A = \{2, 4, 6\}$ ,  $B = \{2, 3, 4, 5, 6\}$ ,  $C = \{1, 2\}$ , and  $D = \phi$ .

Hence,  $P(A) = 3/6$ ,  $P(B) = 5/6$ ,  $P(C) = 2/6$ ,  $P(D) = 0$ .

\* \* \*

**Example 15:** A coin is tossed  $n$  times. What is the probability of getting exactly  $r$  heads?

**Solution:** If  $H$  and  $T$  represent head and tail, respectively, then  $X$  consists of sequences of length  $n$  that can be formed using only the letters  $H$  and  $T$ . Therefore,  $n(X) = 2^n$ . The event  $A$  consists of those sequences in which there are precisely  $r$  Hs. So,  $n(A) = C(n, r)$ . Hence, the required probability is  $C(n, r)/2^n$ .

\* \* \*

**Example 16:** Two dice, one red and one white, are rolled. What is the probability that the white die turns up a smaller number than the red die?

**Solution:** If the number on the red die is  $x$  and that on the white die is  $y$ , then  $X$  consists of the 36 pairs  $(x, y)$ , where  $x$  and  $y$  can be any integer from  $\{1, 2, 3, 4, 5, 6\}$ .

For the event  $A$ , we need  $x < y$ . For  $x = 1, 2, 3, 4, 5$ ,  $y$  can be  $x + 1, x + 2, \dots, 6$ , i.e.,  $6 - x$  in number. Thus, by the addition principle,

$$n(A) = \sum_{x=1}^5 (6 - x) = 5 + 4 + 3 + 2 + 1 = 15.$$

Hence,  $P(A) = 15/36 = 5/12$ .

\* \* \*

**Example 17:** If a five-digit number is chosen at random, what is the probability that the product of the digits is 20?

**Solution:** If  $X$  is the collection of all 5-digit numbers, then  $n(X) = 9 \cdot 10^4 = 90000$ . Now, 20 can be factored in only two ways, viz.,  $1.1.1.4.5$  and  $1.1.2.2.5$ , as the product of five factors. Of course, these factors can be permuted to give all possible cases for  $A$ . The numbers 5, 4, 1, 1, 1 can be permuted in  $5!/1!1!1!1! = 20$  ways, and the numbers 5, 2, 2, 1, 1 can be permuted in  $5!/1!2!2! = 30$  ways.

So,  $n(A) = 20 + 30 = 50$ .

Hence,  $P(A) = 50/90000 = 1/1800$ .

\* \* \*

**Example 18:** Suppose A and B are mutually exclusive events such that  $P(A) = 0.3$  and  $P(B) = 0.4$ . What is the probability that

- i) A does not occur?
- ii) A or B occurs?
- iii) Either A or B does not occur?

**Solution:**

- i) This is  $P(A^c) = 0.7$ .
- ii) This is  $P(A \cup B) = 0.7$ .
- iii) This is  $P(A^c \cup B^c) = P[(A \cap B)^c] = P(\phi^c) = P(X) = 1$

\* \* \*

Try some exercises now.

---

E20) A,B,C and D are four candidates for a chairperson's post. Suppose that A is twice as likely to be elected as B, B is thrice as likely as C, and C and D are equally likely to be elected. What is the probability of election of each candidate?

E21) In a ten-question true-false exam, a student must achieve six correct answers to pass. If she selects her answers randomly, what is the probability that she will pass?

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There are several other methods for solving combinatorial problems. These will be taken up in the next two units. Let us now summarise what we have covered in this unit.

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## 2.7 SUMMARY

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In this unit we have discussed some counting techniques. Specifically, we have covered the following points.

1. The multiplication and addition principles for counting the number of ways in which a task can be completed.
2. What a permutation is, the derivation of the formula  $P(n,r) = \frac{n!}{(n-r)!}$ , and its application for solving problems.
3. The number of distinct arrangement of n objects of which  $m_1$  are of Type 1,  $m_2$  are of Type 2, ...,  $m_k$  are of Type k, where  $m_1, m_2 + \dots, m_k = n$ , is
 
$$P(n; m_1, m_2, \dots, m_k) = \frac{n!}{m_1! m_2! \dots m_k!}.$$
4. What a circular permutation is, and that the number of such permutations of n objects, taken all at a time, is  $(n-1)!$
5. What a combination is, the derivation of the formula

$$C(n, r) = \frac{P(n, r)}{r!} = \frac{n!}{(n-r)!r!}, \text{ and its application for solving problems.}$$

6. The proof and applications of the fact that the number of ways of choosing  $r$  objects from a collection of  $n$  objects, with repetition allowed, is  $C(n+r-1, r)$ .
7. Why  $C(n, r)$  is called a binomial coefficients, and its analogue for multinomials.
8. Some identities involving  $C(n, r)$ , including Pascal's formula  
 $C(n+1, r) = C(n, r) + C(n, r-1)$ .
9. The use of counting techniques for finding some discrete probabilities.

## 2.8 SOLUTIONS/ ANSWERS

- E1) For instance, both principles are used to find the number of ways in which 17 files are stored if there are 3 storage locations of 1000 K each and 10 files are of 100 K, 5 of 200 K 2 of 500 K.
- E2) Here we apply the multiplication principle. Each letter has 10 possibilities. Therefore, the total number of words is  $10^4$ .
- E3) Suppose we number the men as 1, 2, 3, ...,  $n$ . Then the first man can be paired with any of the  $n$  women, the second can be paired with any from the remaining  $(n-1)$  women, and so on. Hence, the number of ways of pairing is  $n(n-1)\dots 1 = n!$ .
- E4) By the multiplication principle, the number of integers between 100 and 999 with all digits even is  $4.5.5 = 100$  (Note that the first digit cannot be zero, but the second and third digits can be 0.)
- E5) For a number to be odd the last digit should be odd. So, the last position can be filled up in 5 ways. If the middle position is filled up by 0, then the first position can be filled up in 8 ways. Thus the number of odd numbers with 0 in the middle position and all digits distinct is 40, by the multiplication principle.

If the middle position is filled up by a digit other than 0, then this can be done in 8 ways. Then the first position can be filled up in 7 ways. So, the number of odd numbers with all digits distinct with the middle digit not zero is  $5.8.7 = 280$ .

Thus, by the addition principle the answer is  $40 + 280 = 320$ .

- E6)  $(m+n)! = (m+n)(m+n-1)\dots(m+1)m!$   
 $\Rightarrow (m+n)! - m! = \geq m^n + n! \geq m! [n! + m^n - 1]$   
 $\Rightarrow (m+n)! - m! - n! \geq n! (m! - 1) + m! (m^n - 1) \geq 0$ .
- E7) Without repetitions, the number is  $P(6, 3)$ . For the number to be less than 400, the leftmost digit can only be 2 or 3. The rest of the digits can be filled in  $P(5, 2)$  ways. So, the total number of numbers less than 400 will be  $2P(5, 2)$ . Similarly, the total number of even numbers is  $3P(5, 2)$ .

Note: That the addition principle has been used in both cases.

- E8) A ranking is an ordering of the  $n$  candidates. This can be done in  $P(n, n) = n!$  ways. The total number of rankings in which Sheela is in  $2^{\text{nd}}$  place in  $P(n-1, n-1) = (n-1)!$

E9) In the word 'ASSESES', we have A once, E twice, and S five times. Thus the number of permutations is  $8!/1!2!5! = 168$ .  
In the word 'PATTIVEERANPATTI', R, N and V occur once, P, E and I occur twice, A thrice and T four times. Thus the required number of permutations is  $16!/1!1!1!2!2!2!3!4! = 9.10$ .

E10) By the multiplication principle, the answer is 26.25.24 if the letters cannot be repeated, and 26.26.26 if the letters can be repeated.

E11) The seven men can be seated first. This can be done in  $6!$  ways. The women can sit in between two men. There are seven such places. So, the women can sit in  $P(7,5)$  ways. Hence the answer is  $6! \times P(7,5)$ .

E12) This can be done in  $C(12, 1).C(15,1)$  ways, i.e., 180 ways.

E13) This can be done in  $C(10, 3)$  ways, i.e., 120 ways.

E14) The total number of possibilities is  $C(20,3).C(30,4) = 31,241,700$ .

E15) Applying Theorem 5, we get  $C(9, 4) = 126$  days.

E16) i) Be careful! This is not an application of Theorem 5. This is only the number of ways of choosing 5 balls out of 8 balls, i.e.  $C(8, 5)$ .

ii) First pick 5 red balls, in  $C(8,5)$  ways. Then pick the remaining 2 arbitrarily. These 2 can be chosen in  $C(2+2-1, 2) = 3$  ways. So, the total number of ways is  $C(8, 5) \times 3$ .

E17) We have 2Cs, 3As, 1R and 1S. If order is not a concern, we consider the solutions of

$$c+a+r+s = 6, 0 \leq c \leq 2, 0 \leq a \leq 3, 0 \leq r, s \leq 1.$$

We convert this to the equivalent problem

$$x+y+z+t = 1, \text{ where } x = 2 - c, y = 3 - a, r = 1 - z, s = 1 - t,$$

The number of solutions of this is  $C(4 + 1 - 1, 1) = 4$ .

There are (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0) and (0, 0, 0, 1).

The corresponding solutions in (c, a, r, s) are (1, 3, 1, 1), (2, 2, 1, 1), (2, 3, 0, 1), (2, 3, 1, 0).

Now order becomes important to us. Applying Theorem 2, the required number is

$$\frac{6!}{1!3!1!1!} + \frac{6!}{2!2!1!1!} + 2\left(\frac{6!}{2!3!1!0!}\right) = 420.$$

E18) The left side counts the ways to select a group of  $m$  people chosen from a set of  $n$  people and then select a subset of  $k$  leaders, say, of this group of  $m$ . This can also be done by selecting the subset of  $k$  leaders from the set of  $n$  people first, and then selecting the remaining  $m - k$  members of the group from the remaining  $n - k$  people. The number of ways in which this can be done is given on the right hand side. Therefore, the identity.

You can also prove this algebraically.

**Basic Combinatorics**

- E19) One can prove this by induction on the variable  $n$ . The base case is trivial, since if  $n = 0$ , then  $k = 0$  as well, and the equation reduces to  $C(0, 0) = C(1, 1)$ , which is true. The induction step is proved by Pascal's formula and the induction hypothesis.
- E20) The relative weightages of A, B, C and D are 6.3, 1, 1, respectively. So,  $P(A) = \frac{6}{11}$ ,  $P(B) = \frac{3}{11}$ ,  $P(C) = \frac{1}{11} = P(D)$ .
- E21) The answer is same as the probability of getting at least 6 heads in 10 tosses of a true coin. Hence, the answer is
- $$C(10, 6)/2^{10} + C(10, 7)/2^{10} + C(10, 8)/2^{10} + C(10, 9)/2^{10} + C(10, 10)/2^{10}$$
- $$= (210 + 120 + 45 + 10 + 1)/1024 = 193/512.$$

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## UNIT 3    BOOLEAN ALGEBRA AND CIRCUITS

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### Structure

- 3.0 Introduction
- 3.1 Objectives
- 3.2 Boolean Algebras
- 3.3 Logic Circuits
- 3.4 Boolean Functions
- 3.5 Summary
- 3.6 Solutions/ Answers

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### 3.0 INTRODUCTION

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This unit is very closely linked with Unit 1. It was C.E.Shannon, the founder of information theory, who observed an analogy between the functioning of switching circuits and certain operations of logical connectives. In 1938 he gave a technique based on this analogy to **express and manipulate** simple switching circuits algebraically. Later, the discovery of some new solid state devices (called **electronic switches** or **logic gates**) helped to modify these algebraic techniques and, thereby, paved a way to solve numerous problems related to digital systems algebraically.



Fig. 1: Claude Shannon

In this unit, we shall discuss the symbolic logic techniques which are required for the algebraic understanding of circuits and computer logic. In Sec. 3.2, we shall introduce you to **Boolean algebras** with the help of certain examples based on objects you are already familiar with. You will see that such algebras are apt for describing operations of logical circuits used in computers.

In Sec. 3.3, we have discussed the linkages between **Boolean expressions** and logic circuits.

In Sec. 3.4, you will read about how to express the overall functioning of a circuit mathematically in terms of certain suitably defined functions called **Boolean functions**. In this section we shall also consider a simple **circuit design problem** to illustrate the applications of the relationship between Boolean functions and circuits.

Let us now consider the objectives of this unit.

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### 3.1 OBJECTIVES

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After reading this unit, you should be able to:

- define and give examples of Boolean algebras, expressions and functions;
- give algebraic representations of the functioning of logic gates;
- obtain and simplify the Boolean expression representing a circuit;
- construct a circuit for a Boolean expression;
- design and simplify some simple circuits using Boolean algebra techniques.

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### 3.2 BOOLEAN ALGEBRAS

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Let us start with some questions: Is it possible to design an electric/electronic circuit without actually using switches(or logic gates) and wires? Can a circuit be redesigned, to get a simpler circuit with the help of pen and paper only?

The answer to both these questions is 'Yes'. What allows us to give this reply is the concept of **Boolean algebras**. Before we start a formal discussion on these types of algebras, let us take another look at the objects treated in Unit 1.



As before, let the letters  $p, q, r, \dots$  denote statements (or propositions). We write  $S$  for the set of all propositions. As you may recall, a tautology  $\mathcal{T}$  (or a contradiction  $\mathcal{F}$ ) is any proposition which is always true (or always false, respectively). By abuse of notation, we shall let  $\mathcal{T}$  denote the set of all tautologies and  $\mathcal{F}$  denote the set of all contradictions. Thus,

$$\mathcal{T} \leq S, \mathcal{F} \leq S.$$

You already know from Unit 1 that, given two propositions  $p$  and  $q$ , both  $p \wedge q$  and  $p \vee q$  are again propositions. And so, by the definition of a binary operation, you can see that both  $\wedge$  (**conjunction**) and  $\vee$  (**disjunction**) are binary operations on the set  $S$ , where we are writing  $\wedge(p, q)$  as  $p \wedge q$  and  $\vee(p, q)$  as  $p \vee q \forall p, q \in S$ .

Again, since  $\sim p$  is also a proposition, the operation  $\sim$  (**negation**) defines a unary function  $\sim: S \rightarrow S$ . Thus, the set of propositions  $S$ , with these operations, acquires an algebraic structure.

As is clear from Sec.1.3, under these three operations, the elements of  $S$  satisfy **associative laws, commutative laws, distributive laws** and **complementation laws**.

Also, by E19 of Unit 1, you know that  $p \vee \mathcal{F} = p$  and  $p \wedge \mathcal{T} = p$ , for any proposition  $p$ . These are called the **identity laws**. The set  $S$  with the three operations and properties listed above is a particular case of an algebraic structure which we shall now define.

**Definition: A Boolean algebra  $B$**  is an algebraic structure which consists of a set  $X$  ( $\neq \emptyset$ ) having two binary operations (denoted by  $\vee$  and  $\wedge$ ), one unary operation (denoted by  $'$ ) and two specially defined elements  $O$  and  $I$  (say), which satisfy the following five laws for all  $x, y, z \in X$ .

- |                                  |   |
|----------------------------------|---|
| <b>B1. Associative Laws:</b>     | $x \vee (y \vee z) = (x \vee y) \vee z,$<br>$x \wedge (y \wedge z) = (x \wedge y) \wedge z$                     |
| <b>B2. Commutative Laws:</b>     | $x \vee y = y \vee x,$<br>$x \wedge y = y \wedge x$   |
| <b>B3. Distributive Laws:</b>    | $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z),$<br>$x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$ |
| <b>B4. Identity Laws:</b>        | $x \vee O = x,$<br>$x \wedge I = x$   |
| <b>B5. Complementation Laws:</b> | $x \wedge x' = O,$<br>$x \vee x' = I.$  |

We write this algebraic structure as  $B = (X, \vee, \wedge, ', O, I)$ , or simply  $B$ , if the context makes the meaning of the other terms clear. The two operations  $\vee$  and  $\wedge$  are called the **join operation** and **meet operation**, respectively. The unary operation  $'$  is called the **complementation**.

From our discussion preceding the definition above, you would agree that the set  $S$  of propositions is a Boolean algebra, where  $\mathcal{T}$  and  $\mathcal{F}$  will do the job of  $I$  and  $O$ , respectively. Thus,  $(S, \wedge, \vee, \sim, \mathcal{F}, \mathcal{T})$  is an example of a Boolean algebra.

We give another example of a Boolean algebra below.

**Example 1:** Let  $X$  be a non-empty set, and  $\mathcal{P}(X)$  denote its power set, i.e.,  $\mathcal{P}(X)$  is the set consisting of all the subsets of the set  $X$ . Show that  $\mathcal{P}(X)$  is a Boolean algebra.

In the NCERT textbook, '+' and '.' are used instead of ' $\vee$ ' and ' $\wedge$ ', respectively.

**Solution:** We take the usual set-theoretic operations of intersection ( $\cap$ ), union ( $\cup$ ), and complementation ( $^c$ ) in  $\mathcal{P}(X)$  as the three required operations. Let  $\emptyset$  and  $X$  play the roles of **O** and **I**, respectively. Then you can verify that all the conditions for  $(\mathcal{P}(X), \cup, \cap, ^c, \Phi, X)$  to be a Boolean algebra hold good.

For instance, the identity laws (B4) follow from two set-theoretic facts, namely, 'the intersection of any subset with the whole set is the set itself' and 'the union of any set with the empty set is the set itself'. On the other hand, the complementation laws (B5) follow from another set of facts from set theory, namely, 'the intersection of any subset with its complement is the empty set' and 'the union of any set with its complement is the whole set'.

\*\*\*

Yet another example of a Boolean algebra is based on **switching circuits**. For this, we first need to elaborate on the functioning of ordinary switches in a mathematical way. In fact, we will present the basic idea which helped the American, C.E.Shannon, to detect the connection between the functioning of switches and Boole's symbolic logic.

You may be aware of the functioning of a simple on-off switch which is commonly used as an essential component in the electric (or electronic) networking systems. A switch is a device which allows the current to flow only when it is placed in the **ON** position, i.e., when the gap is **closed** by a conducting rod. Thus, the **ON** position of a switch is one state of a switch, called a **closed state**. The other state of a switch is the open state, when it is placed in the **OFF** position. So, a switch has two stable states.

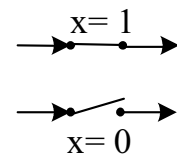


Fig. 2: OFF-ON position

There is another way to talk about the functioning of a switch. We can denote a switch by  $x$ , and use the values 0 and 1 to depict its two states, i.e., to convey that  $x$  is open we write  $x = 0$ , and to convey that  $x$  is closed we write  $x = 1$  (see Fig.2).

These values which denote the state of a switch  $x$  are called the **state-values** (**s.v.**, in short) of that switch.

We shall also write  $x'$  for a switch which is always in a state opposite to  $x$ . So that,  
 **$x$  is open  $\rightarrow x'$  is closed** and  **$x$  is closed  $\rightarrow x'$  is open**.

The switch  $x'$  is called the invert of the switch  $x$ . For example, the switch  $a'$  shown in Fig.3 is an invert of the switch  $a$ .

Table 1: s.v. of  $x'$

$x$	$x'$
0	1
1	0

Table 1 alongside gives the state value of  $x'$  for a given state value of the switch  $x$ . These values are derived from the definition of  $x'$  and our preceding discussion.

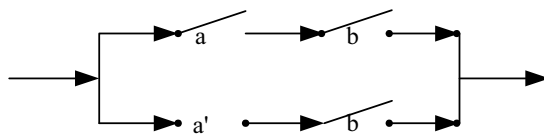


Fig. 3:  $a'$  is the invert of  $a$ .

Note that the variable  $x$  that denotes a switch can only take on 2 values, 0 and 1. Such a variable (which can only take on two values) is called a **Boolean variable**. Thus, if  $x$  is a Boolean variable, so is  $x'$ . Now, in order to design a circuit consisting of several switches, there are two ways in which two switches can be connected: **parallel connections** and **series connections** (see Fig.4).

Do you see a connection between Table 1 above and Table 10, Unit 1 ?

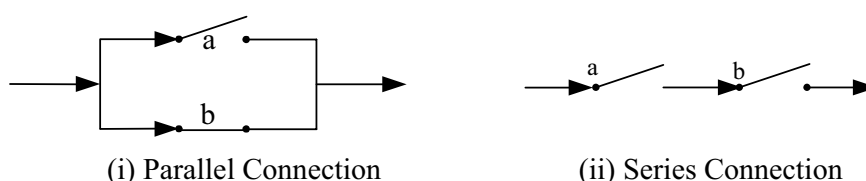


Fig. 4: Two ways of connecting switches

From Fig.4(i) above, you can see that in case of a parallel connection of switches a and b (say), current will flow from the left to the right extreme if **at** least one of the two switches is closed. Note that ‘parallel’ **does not** mean that both the switches are in the same state.

On the other hand, current can flow in a series connection of switches only when **both** the switches a and b are closed (see Fig.4 (ii) ).

Given two switches a and b, we write a **par** b and a **ser** b for these two types of connections, respectively.

In view of these definitions and the preceding discussion, you can see that the state values of the connections a **par** b and a **ser** b, for different pairs of state values of switches a and b, are as given in the tables below.

**Table 2: State values of a par b and a ser b.**

s.v. of a	s.v. of b	s.v. of a <b>par</b> b	s.v. of a	s.v. of b	s.v. of a <b>ser</b> b
0	0	0	0	0	0
0	1	1	0	1	0
1	0	1	1	0	0
1	1	1	1	1	1

We have now developed a sufficient background to give you the example of a Boolean algebra which is based on switching circuits.

**Example 2:** The set  $S = \{0, 1\}$  is a Boolean algebra.

**Solution:** Take **ser** and **par** in place of  $\wedge$  and  $\vee$ , respectively, and inversion ( ' ) instead of  $\sim$  as the three required operations in the definition of a Boolean algebra,. Also take 0 for the element **O** and 1 for the element **I** in this definition. Now, using Tables 1 and 2, you can check that the five laws B1-B5 hold good. Thus,  $(S, \text{par}, \text{ser}, ', 0, 1)$  is a Boolean algebra.

\*\*\*

A Boolean algebra whose underlying set has only two elements is very important in the study of circuits. We call such an algebra a **two-element Boolean algebra**, and denote it by  $\mathcal{B}$ . From this Boolean algebra we can build many more, as in the following example.

**Example 3:** Let  $\mathcal{B}^n = \mathcal{B} \times \mathcal{B} \times \cdots \times \mathcal{B} = \{(e_1, e_2, \dots, e_n) \mid \text{each } e_i = 0 \text{ or } 1\}$ , for  $n \geq 1$ , be the Cartesian product of n copies of  $\mathcal{B}$ . For  $i_k, j_k \in \{0, 1\}$  ( $1 \leq k \leq n$ ), define

$$\begin{aligned} (i_1, i_2, \dots, i_n) \wedge (j_1, j_2, \dots, j_n) &= (i_1 \wedge j_1, i_2 \wedge j_2, \dots, i_n \wedge j_n), \\ (i_1, i_2, \dots, i_n) \vee (j_1, j_2, \dots, j_n) &= (i_1 \vee j_1, i_2 \vee j_2, \dots, i_n \vee j_n), \text{ and} \\ (i_1, i_2, \dots, i_n)' &= (i_1', i_2', \dots, i_n'). \end{aligned}$$

Then  $\mathcal{B}^n$  is a Boolean algebra, for all  $n \geq 1$ .

**Solution:** Firstly, observe that the case  $n = 1$  is the Boolean algebra  $\mathcal{B}$ .

Now, let us write  $0 = (0, 0, \dots, 0)$  and  $1 = (1, 1, \dots, 1)$ , for the two elements of  $\mathcal{B}^n$  consisting of n-tuples of 0's and 1's, respectively. Using the fact that  $\mathcal{B}$  is a Boolean algebra, you can check that  $\mathcal{B}^n$ , with operations as defined above, is a Boolean algebra for every  $n \geq 1$ .

\*\*\*

The Boolean algebras  $\mathcal{B}^n$ ,  $n \geq 1$ , (called **switching algebras**) are very useful for the study of the hardware and software of digital computers.

We shall now state, without proof, some other properties of Boolean algebras, which can be deduced from the five laws (B1-B5).

**Theorem 1:** Let  $\mathcal{B} = (\mathbf{S}, \vee, \wedge, ', \mathbf{O}, \mathbf{I})$  be a Boolean algebra. Then the following laws hold  $\forall x, y \in \mathbf{S}$ .

- a) **Idempotent laws** :  $x \vee x = x, x \wedge x = x$ .
- b) **Absorption laws** :  $x \vee (x \wedge y) = x, x \wedge (x \vee y) = x$ .
- c) **Involution law** :  $(x')' = x$ .
- d) **De Morgan's laws** :  $(x \vee y)' = x' \wedge y', (x \wedge y)' = x' \vee y'$ .

In fact, you have already come across some of these properties for the Boolean algebras of propositions in Unit 1. In the following exercise we ask you to verify them.

- 
- E1) a) Verify the identity laws and absorption laws for the Boolean algebra  $(\mathbf{S}, \wedge, \vee, \sim, \mathcal{T}, \mathcal{F})$  of propositions.  
 b) Verify the absorption laws for the Boolean algebra  $(\mathcal{P}(\mathbf{X}), \cup, \cap, ^c, \Phi, \mathbf{X})$ .
- 

In Theorem 1, you may have noticed that for each statement involving  $\vee$  and  $\wedge$ , there is an analogous statement with  $\wedge$  (instead of  $\vee$ ) and  $\vee$  (instead of  $\wedge$ ). This is not a coincidence, as the following definition and result shows.

**Definition** : If  $p$  is a proposition involving  $\sim, \wedge$  and  $\vee$ , the **dual** of  $p$ , denoted by  $p^d$ , is the proposition obtained by replacing each occurrence of  $\wedge$  (and/or  $\vee$ ) in  $p$  by  $\vee$  (and/or  $\wedge$ , respectively) in  $p^d$ .

For example,  $x \vee (x \wedge y) = x$  is the **dual** of  $x \wedge (x \vee y) = x$ .

Now, the following principle tells us that if a statement is proved true, then we have simultaneously proved that its dual is true.

**Theorem 2 (The principle of duality):** If  $s$  is a theorem about a Boolean algebra, then so is its dual  $s^d$ .

It is because of this principle that the statements in Theorem 1 look so similar.

Let us now see **how to apply Boolean algebra methods to circuit design**.

While expressing circuits mathematically, we identify each circuit in terms of some Boolean variables. Each of these variables represents either a simple switch or an input to some electronic switch.

**Definition:** Let  $\mathcal{B} = (\mathbf{S}, \vee, \wedge, ', \mathbf{O}, \mathbf{I})$  be a Boolean algebra. A **Boolean expression** in variables  $x_1, x_2, \dots, x_k$  (say), each taking their values in the set  $\mathbf{S}$  is defined recursively as follows:

- i) Each of the variables  $x_1, x_2, \dots, x_k$ , as well as the elements  $\mathbf{O}$  and  $\mathbf{I}$  of the Boolean algebra  $\mathcal{B}$  are Boolean expressions.
- ii) If  $\mathbf{X}_1$  and  $\mathbf{X}_2$  are previously defined Boolean expressions, then  $\mathbf{X}_1 \wedge \mathbf{X}_2, \mathbf{X}_1 \vee \mathbf{X}_2$  and  $\mathbf{X}'_1$  are also Boolean expressions.

For instance,  $x_1 \wedge x'_3$  is a Boolean expression because so are  $x_1$  and  $x'_3$ . Similarly, because  $x_1 \wedge x_2$  is a Boolean expression, so is  $(x_1 \wedge x_2) \wedge (x_1 \wedge x'_3)$ .

If  $\mathbf{X}$  is a Boolean expression in  $n$  variables  $x_1, x_2, \dots, x_n$  (say), we write this as  $\mathbf{X} = \mathbf{X}(x_1, \dots, x_n)$ .

In the context of simplifying circuits, we need to reduce Boolean expressions to

simpler ones. 'Simple' means that the expression has fewer connectives, and all the literals involved are distinct. We illustrate this technique now.

**Example 4:** Reduce the following Boolean expressions to a simpler form.

$$(a) X(x_1, x_2) = (x_1 \wedge x_2) \wedge (x_1 \wedge x'_2);$$

$$(b) X(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_3).$$

**Solution:** (a) Here we can write

$$\begin{aligned} (x_1 \wedge x_2) \wedge (x_1 \wedge x'_2) &= ((x_1 \wedge x_2) \wedge x_1) \wedge x'_2 && \text{(Associative law)} \\ &= (x_1 \wedge x_2) \wedge x'_2 && \text{(Absorption law)} \\ &= x_1 \wedge (x_2 \wedge x'_2) && \text{(Associative law)} \\ &= x_1 \wedge \mathbf{O} && \text{(Complementation law)} \\ &= \mathbf{O}. && \text{(Identity law)} \end{aligned}$$

Thus, in its simplified form, the expression given in (a) above is **O**, i.e., a **null expression**.

(b) We can write

$$\begin{aligned} (x_1 \wedge x_2) \vee (x_1 \wedge x'_2 \wedge x_3) \vee (x_1 \wedge x_3) & \\ = [x_1 \wedge \{x_2 \vee (x'_2 \wedge x_3)\}] \wedge (x_1 \wedge x_3) && \text{(Distributive law)} \\ = [x_1 \wedge \{(x_2 \vee x'_2) \wedge (x_2 \vee x_3)\}] \wedge (x_1 \wedge x_3) && \text{(Distributive law)} \\ = [x_1 \wedge \{\mathbf{I} \wedge (x_2 \vee x_3)\}] \wedge (x_1 \wedge x_3) && \text{(Complementation law)} \\ = [x_1 \wedge (x_2 \vee x_3)] \wedge (x_1 \wedge x_3) && \text{(Identity law)} \\ = [(x_1 \wedge x_2) \vee (x_1 \wedge x_3)] \wedge (x_1 \wedge x_3) && \text{(Distributive law)} \\ = [(x_1 \wedge x_2) \wedge (x_1 \wedge x_3)] \vee [(x_1 \wedge x_3) \wedge (x_1 \wedge x_3)] && \text{(Distributive law)} \\ = (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x_3) && \text{(Idemp., & assoc. laws)} \\ = x_1 \wedge [(x_2 \wedge x_3) \vee x_3] && \text{(Distributive law)} \\ = x_1 \wedge x_3 && \text{(Absorption law)} \end{aligned}$$

Thus, the simplified form of the expression given in (b) is  $(x_1 \wedge x_3)$ .

\*\*\*

Now you should find it easy to solve the following exercise.

---

E2) Simplify the Boolean expression

$$\mathbf{X}(x_1, x_2, x_3) = (x_1 \wedge x_2) \vee ((x_1 \wedge x_2) \wedge x_3) \vee (x_2 \wedge x_3).$$


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With this we conclude this section. In the next section we shall give an important application of the concepts discussed here.

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### 3.3 LOGIC CIRCUITS

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If you look around, you would notice many electric or electronic appliances of daily use. Some of them need a simple switching circuit to control the auto-stop (such as in a stereo system). Some would use an auto-power off system used in transformers to control voltage fluctuations. Each circuit is usually a combination of on-off switches, wired together in some specific configuration. Nowadays certain types of **electronic blocks** (i.e., solid state devices such as transistors, resistors and capacitors) are more in use. We call these electronic blocks **logic gates**, or simply, **gates**. In Fig. 5 we have shown a box which consists of some electronic switches (or logic gates), wired together in a specific manner. Each line which is entering the box from the left represents an independent power source (called **input**), where all of them need not supply voltage to the box at a given moment. A single line coming out of the box gives the **final output** of the circuit box. The output depends on the type of input.

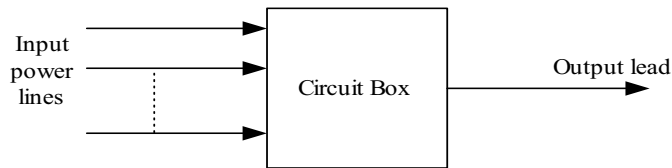


Fig. 5: A Logic circuit

This sort of arrangement of **input power lines**, a **circuit box** and **output lead** is basic to all electronic circuits. Throughout the unit, any such interconnected assemblage of logic gates is referred to as a **logic circuit**.

As you may know, computer hardware are designed to handle only two levels of voltage, both as inputs as well as outputs. These two levels, denoted by 0 and 1, are called **bits** (an acronym for **binary digits**). When the bits are applied to the logic gates by means of **one** or **two** wires (input leads), the output is again in the form of voltages 0 and 1. Roughly speaking, **you may think of a gate to be on or off according to whether the output voltage is at level 1 or 0, respectively**.

Three basic types of logic gates are an **AND-gate**, an **OR-gate** and a **NOT-gate**. We shall now define them one by one.

**Definition :** Let the Boolean variables  $x_1$  and  $x_2$  represent any two bits. An **AND-gate** receives inputs  $x_1$  and  $x_2$  and produces the output, denoted by  $x_1 \wedge x_2$ , as given in Table 3 alongside.

Table 3:Outputs of AND-gate

$x_1$	$x_2$	$x_1 \wedge x_2$
0	0	0
1	1	0
0	1	0
1	1	1

The standard pictorial representation of an **AND-gate** is shown in Fig.6 below.



Fig. 6: Diagrammatic representation of an AND -gate

From the first three rows of Table 3, you can see that whenever the voltage in any one of the input wires of the **AND-gate** is at level 0, then the output voltage of the gate is also at level 0. You have already encountered such a situation in Unit 1. In the following exercise we ask you to draw an analogy between the two situations.

---

E3) Compare Table 3 with Table 2 of Unit 1. How would you relate  $x_1 \wedge x_2$  with  $p \wedge q$ , where  $p$  and  $q$  denote propositions?

---

Let us now consider another elementary logic gate.

**Definition :** An **OR-gate** receives inputs  $x_1$  and  $x_2$  and produces the output, denoted by  $x_1 \vee x_2$ , as given in Table 4. The standard pictorial representation used for the **OR-gate** is as shown in Fig.7.

Table 4: Output of an OR-gate.

$x_1$	$x_2$	$x_1 \vee x_2$
0	0	0
0	1	1
1	0	1
1	1	1



Fig. 7: Diagrammatic representation of an OR-gate

From Table 4 you can see that the situation is the other way around from that in Table 3, i.e., the output voltage of an **OR-gate** is at level 1 whenever the level of voltage in even one of the input wires is 1. What is the analogous situation in the context of propositions? The following exercise is about this.

---

E4) Compare Table 4 with Table 1 of Unit 1. How would you relate  $x_1 \vee x_2$  with  $p \vee q$ , where  $p$  and  $q$  are propositions?

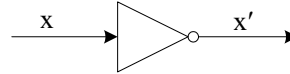
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And now we will discuss an electronic realization of the invert of a simple switch about which you read in Sec. 3.2.

**Definition :** A **NOT-gate** receives bit  $x$  as input, and produces an output denoted by  $x'$ , as given in Table 5. The standard pictorial representation of a **NOT-gate** is shown in Fig. 8 below.

**Table 5: Output of a NOT-gate**

$x$	$x'$
0	1
1	0



**Fig. 8: Diagrammatic representation of NOT-gate**

If you have solved E5 and E6, you would have noticed that Tables 3 and 4 are the same as the truth tables for the logic connectives  $\wedge$  (conjunction) and  $\vee$  (disjunction). Also Table 3 of Unit 1, after replacing T by 1 and F by 0, gives Table 5. This is why the output tables for the three elementary gates are called **logic tables**. You may find it useful to remember these logic tables because they are needed very often for computing the logic tables of logic circuits.

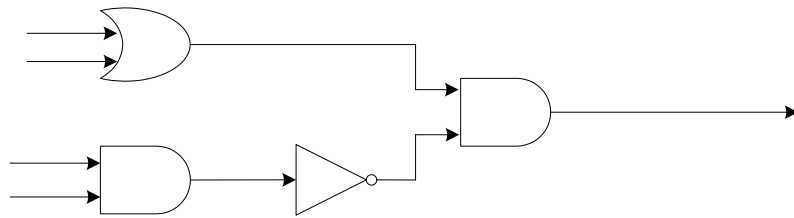
Another important fact that these logic tables will help you prove is given in the following exercise.

---

E5) Let  $\mathcal{B} = \{0, 1\}$  consist of the bits 0 and 1. Show that  $\mathcal{B}$  is a Boolean algebra, i.e., that the bits 0 and 1 form a two-element Boolean algebra.

---

As said before, a logic circuit can be designed using elementary gates, where the output from an **AND-gate**, or an **OR-gate**, or a **NOT-gate** is used as an input to other such gates in the circuitry. The different levels of voltage in these circuits, starting from the input lines, move only in the direction of the arrows as shown in all the figures given below. For instance, one combination of the three elementary gates is shown in Fig.9.



**Fig. 9: A logic circuit of elementary gates.**

Now let us try to see the connection between logic circuits and Boolean expressions. We first consider the elementary gates. For a given pair of inputs  $x_1$  and  $x_2$ , the output in the case of each of these gates is an expression of the form  $x_1 \wedge x_2$  or  $x_1 \vee x_2$  or  $x'$ .

Next, let us look at larger circuits. Is it possible to find an expression associated with a logic circuit, using the symbols  $\wedge$ ,  $\vee$  and  $'$ ? Yes, it is. We will illustrate the technique of finding a Boolean expression for a given logic circuit with the help of some examples. But first, note that the output of a gate in a circuit may serve as an input to some other gate in the circuit, as in Fig. 9. So, to get an expression for a logic circuit the process always moves in the direction of the arrows in the circuitry. With this in mind, let us consider some circuits.

**Example 5:** Find the Boolean expression for the logic circuit given in Fig.9 above.

**Solution:** In Fig.9, there are four input terminals. Let us call them  $x_1$ ,  $x_2$ ,  $x_3$  and  $x_4$ . So,  $x_1$  and  $x_2$  are inputs to an **OR**-gate, which gives  $x_1 \vee x_2$  as an output expression (see Fig. 9(a)).

Similarly, the other two inputs  $x_3$  and  $x_4$ , are inputs to an **AND**-gate. They will give  $x_3 \wedge x_4$  as an output expression. This is, in turn, an input for a **NOT**-gate in the circuit. So, this yields  $(x_3 \wedge x_4)'$  as the output expression. Now, both the expressions  $x_1 \vee x_2$  and  $(x_3 \wedge x_4)'$  are inputs to the extreme right **AND**-gate in the circuit. So, they give  $(x_1 \vee x_2) \wedge (x_3 \wedge x_4)'$  as the final output expression, which represents the logic circuit.

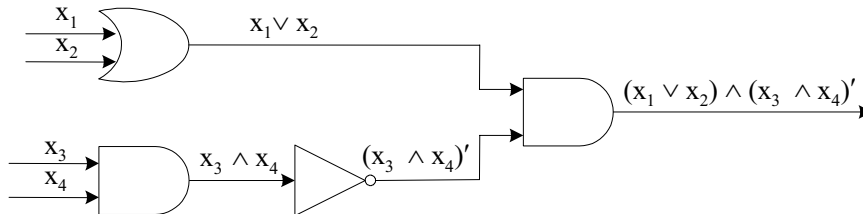


Fig. 9 (a)

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You have just seen how to find a Boolean expression for a logic circuit. For more practice, let us find it for another logic circuit.

**Example 6:** Find the Boolean expression C for the logic circuit given in Fig. 10.

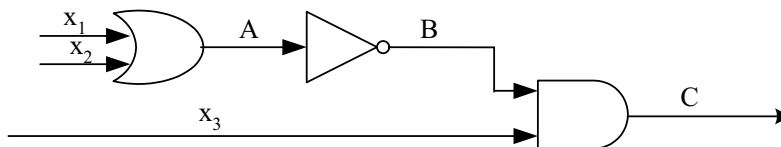


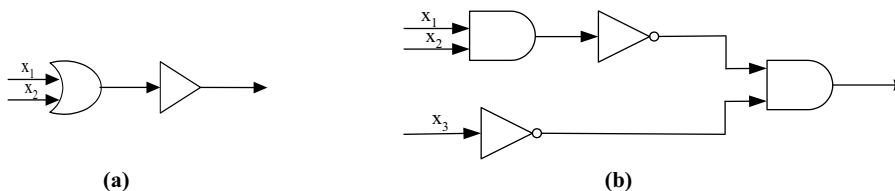
Fig. 10

**Solution:** Here the first output is from an **OR**-gate, i.e., A is  $x_1 \vee x_2$ . This, in turn, serves as the input to a **NOT**-gate attached to it from the right. The resulting bit B is  $(x_1 \vee x_2)'$ . This, and  $x_3$ , serve as inputs to the extreme right **AND**-gate in the circuit given above. This yields an output expression  $(x_1 \vee x_2)' \wedge x_3$ , which is C, the required expression for the circuit given in Fig.10.

\*\*\*

Why don't you try to find the Boolean expressions for some more logic circuits now?

E6) Find the Boolean expression for the output of the logic circuits given below.



So far, you have seen how to obtain a Boolean expression that represents a given circuit. Can you do the converse? That is, can you construct a logic circuit corresponding to a given Boolean expression? In fact, this is done when a circuit



designing problem has to be solved. The procedure is quite simple. We illustrate it with the help of some examples.

**Example 7:** Construct the logic circuit represented by the Boolean expression  $(x'_1 \wedge x_2) \vee (x_1 \vee x_3)$ , where  $x_i (1 \leq i \leq 3)$  are assumed to be inputs to that circuitry.

**Solution:** Let us first see what the portion  $(x'_1 \wedge x_2)$  of the given expression contributes to the complete circuit. In this expression the literals  $x'_1$  and  $x_2$  are connected by the connective  $\wedge$  (**AND**). Thus the circuit corresponding to it is as shown in Fig.11(a) below, by the definitions of **NOT**-gate and **AND**-gate.

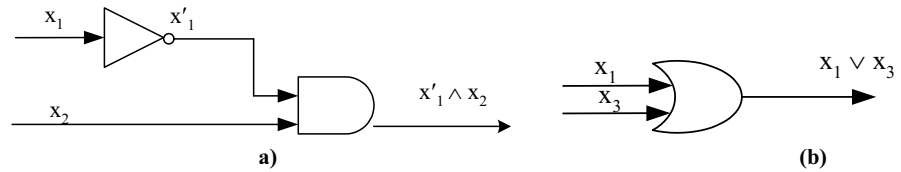


Fig. 11: Logic circuits for the expressions  $x'_1 \wedge x_2$  and  $x_1 \vee x_3$ .

Similarly, the gate corresponding to the expression  $x_1 \vee x_3$  is as shown in Fig.11(b) above. Finally, note that the given expression has two parts, namely,  $x'_1 \wedge x_2$  and  $x_1 \vee x_3$ , which are connected by the connective  $\vee$  (**OR**). So, the two logic circuits given in Fig.11 above, when connected by an **OR**-gate, will give us the circuit shown in Fig. 12 below.

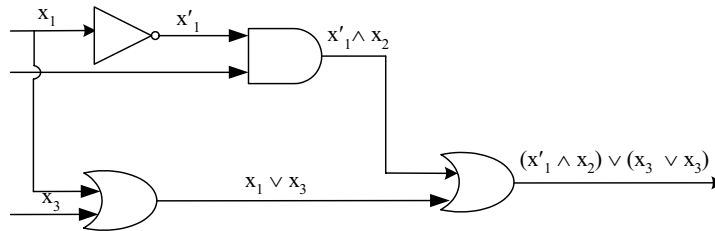


Fig.12: Circuitry for the expression  $(x'_1 \wedge x_2) \vee (x_1 \vee x_3)$

This is the required logic circuit which is represented by the given expression.

\*\*\*

**Example 8:** Given the expression  $(x'_1 \vee (x_2 \wedge x'_3)) \wedge (x_2 \vee x'_4)$ , find the corresponding circuit, where  $x_i (1 \leq i \leq 4)$  are assumed to be inputs to the circuitry.

**Solution:** We first consider the circuits representing the expressions  $x_2 \wedge x'_3$  and  $x_2 \vee x'_4$ . They are as shown in Fig.13(a).

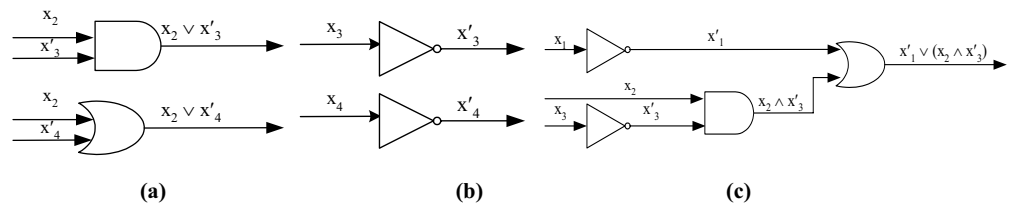


Fig. 13: Construction of a logic circuitry.

Also you know that the literals  $x'_3$  and  $x'_4$  are outputs of the **NOT**-gate. So, these can be represented by logic gates as shown in Fig.13(b). Then the circuit for the part  $x'_1 \vee (x_2 \wedge x'_3)$  of the given expression is as shown in Fig.13(c). You already know how to construct a logic circuit for the expression  $x_2 \vee x'_4$ .

Finally, the two expressions  $(x'_1 \vee (x_2 \wedge x'_3))$  and  $(x_2 \vee x'_4)$  being connected by the connective  $\wedge$  (**AND**), give the required circuit for the given expression as shown in Fig.14.

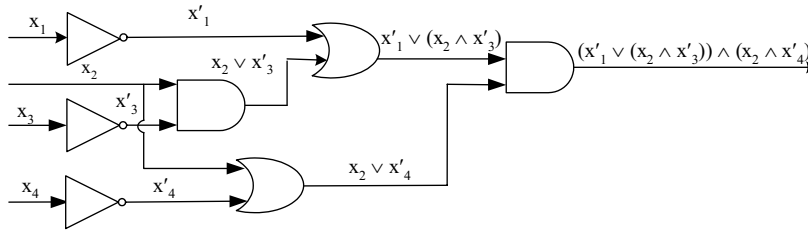


Fig. 14: Circuitry for the expression  $(x'_1 \vee (x_2 \wedge x'_3)) \wedge (x_2 \vee x'_4)$ .

\*\*\*

Why don't you try to solve some exercises now?

E7) Find the logic circuit corresponding to the expression  $x'_1 \wedge (x_2 \vee x'_3)$ .

E8) Construct the logic circuit and obtain the logic table for the expression  $x_1 \vee (x'_2 \wedge x_3)$ .

So far we have established a one-to-one correspondence between logic circuits and Boolean expressions. You may wonder about the utility of this. The mathematical view of a circuit can help us understand the **overall functioning** of the circuit. To understand how, consider the circuit given in Fig.10 earlier.

You may think of the inputs bits  $x_1$ ,  $x_2$ , and  $x_3$  as three variables, each one of which is known to have two values only, namely, 0 or 1, depending upon the level of voltage these inputs have at any moment of time. Then the idea is to evaluate the expression  $(x_1 \vee x_2)' \wedge x_3$ , which corresponds to this circuit, for different values of the 3-tuple  $(x_1, x_2, x_3)$ .

How does this evaluation help us to understand the functioning of the circuit? To see this, consider a situation in which the settings of  $x_1$ ,  $x_2$  and  $x_3$  at a certain stage of the process are  $x_1 = x_3 = 0$  and  $x_2 = 1$ . Then we know that  $x_1 \vee x_2 = 0 \vee 1 = 1$  (see the second row of Table 3 given earlier). Further, using the logic table of a **NOT**-gate, we get  $(x_1 \vee x_2)' = 1' = 0$ . Finally, from Table 3, we get  $(x_1 \vee x_2)' \wedge x_3 = 0 \wedge 1 = 0$ . Thus, the expression  $(x_1 \vee x_2)' \wedge x_3$  has value 0 for the set of values (0, 1, 0) of input bits  $(x_1, x_2, x_3)$ . **Thus, if  $x_1$  and  $x_3$  are closed, while  $x_2$  is open, the circuit remains closed.**

Using similar arguments, you can very easily calculate the other values of the expression  $(x_1 \vee x_2)' \wedge x_3$  in the set

$$\{0,1\}^3 = \{(x_1, x_2, x_3) \mid x_i = 0 \text{ or } 1, 1 \leq i \leq 3\}$$

of values of input bits. We have recorded them in Table 6.

Observe that the row entries in the first three columns of Table 6 represent the different values which the input bits  $(x_1, x_2, x_3)$  may take. Each entry in the last column of the table gives the output of the circuit represented by the expression  $(x_1 \vee x_2)' \wedge x_3$  for the corresponding set of values of  $(x_1, x_2, x_3)$ . For example, if  $(x_1, x_2, x_3)$  is (0,1,0), then the level of voltage in the output lead is at a level 0 (see the third row of Table 6).

You should verify that the values in the other rows are correct.

**Table 6: Logic table for the expression  $(x_1 \vee x_2)' \wedge x_3$ .**

Table 6 is the **logic table** for the circuit given in Fig. 10.

$x_1$	$x_2$	$x_3$	$x_1 \vee x_2$	$(x_1 \vee x_2)$	$(x_1 \vee x_2)' \wedge x_3$
0	0	0	0	1	0
0	0	1	0	1	1
0	1	0	1	0	0
1	0	0	1	0	0
0	1	1	1	0	0
1	1	0	1	0	0
1	0	1	1	0	0
1	1	1	1	0	0

Why don't you try an exercise now?

---

E9) Compute the logic table for the circuit given in E6(b) above.

---

You have seen how the logic table of an expression representing a circuit provides a functional relationship between the state (or level) of voltage in the input terminals and that in the output lead of that logic circuitry. This leads us the concept of Boolean functions, which we will now discuss.

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### 3.4 BOOLEAN FUNCTIONS

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In the last section you studied that an output expression is not merely a device for representing an interconnection of gates. It also defines output values as a function of input bits. This provides information about the overall functioning of the corresponding logic circuit. So, this function gives us a relation between **the inputs to the circuit** and its **final output**.

This is what helps us to understand control over the functioning of logic circuits from a mathematical point of view. To explain what this means, let us reformulate the logic tables in terms of functions of the input bits.

Let us first consider the Boolean expression

$$X(x_1, x_2) = x_1 \wedge x_2',$$

where  $x_1$  and  $x_2$  take values in  $\mathcal{B} = \{0, 1\}$ . You know that all the values of this expression, for different pairs of values of the variables  $x_1$  and  $x_2$ , can be calculated by using properties of the Boolean algebra  $\mathcal{B}$ . For example,

$$0 \wedge 1' = 0 \wedge 0 = 0 \Rightarrow X(0, 1) = 0.$$

Similarly, you can calculate the other values of  $X(x_1, x_2) = x_1 \wedge x_2'$  over  $\mathcal{B}$ .

In this way we have obtained a function  $f: \mathcal{B}_2 \rightarrow \mathcal{B}$ , defined as follows:

$$f(e_1, e_2) = X(e_1, e_2) = e_1 \wedge e_2', \text{ where } e_1, e_2 \in \{0, 1\}.$$

So  $f$  is obtained by replacing  $x_i$  with  $e_i$  in the expression  $X(x_1, x_2)$ . For example, when  $e_1 = 1, e_2 = 0$ , we get  $f(1, 0) = 1 \wedge 0' = 1$ .

More generally, each Boolean expression  $X(x_1, x_2, \dots, x_k)$  in  $k$  variables, where

each variable can take values from the two-element Boolean algebra  $\mathcal{B}$ , defines a function  $f: \mathcal{B}^k \rightarrow \mathcal{B} : f(e_1, \dots, e_k) = X(e_1, \dots, e_k)$ .

Any such function is called a **Boolean function**.

Thus, each Boolean expression over  $\mathcal{B} = \{0, 1\}$  gives rise to a Boolean function.

In particular, corresponding to each circuit, we get a Boolean function.

Therefore, the logic table of a circuit is just another way of representing the Boolean function corresponding to it.

For example, the logic table of an **AND**-gate can be obtained using the function  $\wedge : \mathcal{B}^2 \rightarrow \mathcal{B} : \wedge(e_1, e_2) = e_1 \wedge e_2$ .

To make matters more clear, let us work out an example.

**Example 9:** Let  $f: \mathcal{B}^2 \rightarrow \mathcal{B}$  denote the function which is defined by the Boolean expression  $X(x_1, x_2) = x'_1 \wedge x'_2$ . Write the values of  $f$  in tabular form.

**Solution:**  $f$  is defined by  $f(e_1, e_2) = e'_1 \wedge e'_2$  for  $e_1, e_2 \in \{0, 1\}$ . Using Tables 3, 4 and 5, we have

$$\begin{aligned} f(0, 0) &= 0' \wedge 0' = 1 \wedge 1 = 1, & f(0, 1) &= 0' \wedge 1' = 1 \wedge 0 = 0, \\ f(1, 0) &= 1' \wedge 0' = 0 \wedge 1 = 0, & f(1, 1) &= 1' \wedge 1' = 0 \wedge 0 = 0. \end{aligned}$$

We write this information in Table 7.

**Table 7: Boolean function for the expression  $x'_1 \wedge x'_2$ .**

$e_1$	$e_2$	$e'_1$	$e'_2$	$f(e_1, e_2) = e'_1 \wedge e'_2$
0	0	1	1	$1 \wedge 1 = 1$
0	1	1	0	$1 \wedge 0 = 0$
1	0	0	1	$0 \wedge 1 = 0$
1	1	0	0	$0 \wedge 0 = 0$

\*\*\*

Why don't you try an exercise now?

---

E10) Find all the values of the Boolean function  $f: \mathcal{B}_2 \rightarrow \mathcal{B}$  defined by the Boolean expression  $(x_1 \wedge x_2) \vee (x_1 \wedge x'_3)$ .

---

Let us now consider the Boolean function  $g: \mathcal{B}_2 \rightarrow \mathcal{B}$ , defined by the expression

$$X(x_1, x_2) = (x_1 \vee x_2)'$$

Then  $g(e_1, e_2) = (e_1 \vee e_2)'$ ,  $e_1, e_2 \in \mathcal{B}$ .

So, the different values that  $g$  will take are

$$\begin{aligned} g(0, 0) &= (0 \vee 0)' = 0' = 1, & g(0, 1) &= (0 \vee 1)' = 1' = 0, \\ g(1, 0) &= (1 \vee 0)' = 1' = 0, & g(1, 1) &= (1 \vee 1)' = 1' = 0. \end{aligned}$$

In tabular form, the values of  $g$  can be presented as in Table 8.

**Table 8: Boolean function of the expression  $(x_1 \vee x_2)'$ .**

$e_1$	$e_2$	$e_1 \vee e_2$	$g(e_1, e_2) = (e_1 \vee e_2)'$
0	0	0	1
0	1	1	0
1	0	1	0
1	1	1	0

By comparing Tables 7 and 8, you can see that  $f(e_1, e_2) = g(e_1, e_2)$  for all

$(e_1, e_2) \in \mathcal{B}^2$ . So  $f$  and  $g$  are the same function.

What you have just seen is that **two (seemingly) different Boolean expressions can have the same Boolean function specifying them**. Note that if we replace the input bits by propositions in the two expressions involved, then we get logically equivalent statements. This may give you some idea of how the two Boolean expressions are related. We give a formal definition below.

**Definition :** Let  $X = X(x_1, x_2, \dots, x_k)$  and  $Y = Y(x_1, x_2, \dots, x_k)$  be two Boolean expressions in the  $k$  variables  $x_1, \dots, x_k$ . We say  **$X$  is equivalent to  $Y$**  over the Boolean algebra  $\mathcal{B}$ , and write  **$X \equiv Y$** , if both the expressions  $X$  and  $Y$  define the same Boolean function over  $\mathcal{B}$ , i.e.,

$$X(e_1, e_2, \dots, e_k) = Y(e_1, e_2, \dots, e_k), \text{ for all } e_i \in \{0, 1\}.$$

So, the expressions to which  $f$  and  $g$  (given by Tables 7 and 8) correspond are equivalent.

Why don't you try an exercise now?

---

E11) Show that the Boolean expressions

$$X = (x_1 \wedge x_2) \vee (x_1 \wedge x_3) \text{ and } Y = x_1 \wedge (x_2 \vee x_3)$$

are equivalent over the two-element Boolean algebra  $\mathcal{B} = \{0, 1\}$ .

---

So far you have seen that given a circuit, we can define a Boolean function corresponding to it. You also know that given a Boolean expression over  $\mathcal{B}$ , there is a circuit corresponding to it. Now, you may ask:

Given a Boolean function  $f: \mathcal{B}^n \rightarrow \mathcal{B}$ , is it always possible to get a Boolean expression which will specify  $f$  over  $\mathcal{B}$ ? The answer is 'yes', i.e., for every function  $f: \mathcal{B}^n \rightarrow \mathcal{B}$  ( $n \geq 2$ ) there is a Boolean expression (in  $n$  variables) whose Boolean function is  $f$  itself.

To help you understand the underlying procedure, consider the following examples.

**Example 10:** Let  $f: \mathcal{B}^2 \rightarrow \mathcal{B}$  be a function which is defined by  
 $f(0, 0) = 1, f(1, 0) = 0, f(0, 1) = 1, f(1, 1) = 1$ .

Find the Boolean expression specifying the function  $f$ .

**Solution:**  $f$  can be represented by the following table.

Input		Output
$x_1$	$x_2$	$f(x_1, x_2)$
0	0	1
1	0	0
0	1	1
1	1	1

We find the Boolean expression according to the following algorithm:

**Step 1:** Identify all rows of the table where the output is 1: these are the 1st, 3rd and 4th rows.

**Step 2:** Combine the variables in each of the rows identified in Step 1 with 'and'. Simultaneously, apply 'not' to the variables with value zero in these rows. So, for the  
 1st row:  $x'_1 \wedge x'_2$ ,

In Boolean algebra terminology this is known as the 'disjunctive normal form' (DNF) of the expression.

3rd row:  $x'_1 \wedge x_2$ ,

4th row:  $x_1 \wedge x'_2$ .

**Step 3:** Combine the Boolean expressions obtained in Step 2 with 'or' to get the compound expression representing  $f$ :

So,  $f(x_1, x_2) = (x'_1 \wedge x'_2) \vee (x'_1 \wedge x_2) \vee (x_1 \wedge x_2)$ .

\*\*\*

You can complete Example 10, by doing the following exercise.

E12) In the previous example, show that  $X(e_1, e_2) = f(e_1, e_2) \forall e_1, e_2 \in \mathcal{B}$ .

E13) By Theorem 2, we could also have obtained the expression of  $f$  in Example 10 in 'conjunctive normal form' (CNF). Please do so.

**An important remark:** To get a Boolean expression for a Boolean function  $h$  (say), we should first see how many points  $v_i$  there are at which  $h(v_i) = 0$ , and how many points  $v_i$  there are at which  $h(v_i) = 1$ . **If the number of values for which the function  $h$  is 0 is less than the number of values at which  $h$  is 1, then we shall choose to obtain the expression in CNF, and not in DNF.** This will give us a shorter Boolean expression, and hence, a simpler circuit. For similar reasons, we will prefer DNF if the number of values at which  $h$  is 0 is more.

Why don't you apply this remark now?

E14) Find the Boolean expressions, in DNF or in CNF (keeping in mind the remark made above), for the functions defined in tabular form below.

$x_1$	$x_2$	$x_3$	$f(x_1, x_2, x_3)$
1	1	1	1
1	1	0	0
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	0
0	0	0	1

(a)

$x_1$	$x_2$	$x_3$	$g(x_1, x_2, x_3)$
1	1	1	1
1	1	0	1
1	0	1	0
1	0	0	1
0	1	1	0
0	1	0	0
0	0	1	1
0	0	0	1

(b)

Boolean functions tell us about the functioning of the corresponding circuit. Therefore, circuits represented by two equivalent expressions should essentially do the same job. We use this fact while redesigning a circuit to create a simpler one. In fact, in such a simplification process of a circuit, we write an expression for the circuit and then evaluate the same (over two-element Boolean algebra  $\mathcal{B}$ ) to get the Boolean function. Next, we proceed to get an equivalent, simpler expression. Finally, the process terminates with the construction of the circuit for this simpler expression. Note that, **as the two expressions are equivalent, the circuit represented by the simpler expression will do exactly the same job as the circuit represented by the original expression.**

Let us illustrate this process by an example in some detail.

**Example 11:** Design a logic circuit capable of operating a central light bulb in a hall by three switches  $x_1, x_2, x_3$  (say) placed at the three entrances to that hall.

**Solution:** Let us consider the procedure stepwise.

**Step 1:** To obtain the function corresponding to the unspecified circuit.

To start with, we may assume that the bulb is off when all the switches are off. Mathematically, this demands a situation where  $x_1 = x_2 = x_3 = 0$  implies  $f(0, 0, 0) = 0$ , where  $f$  is the function which depicts the functional utility of the circuit to be designed.

Let us now see how to obtain the other values of  $f$ . Note that every change in the state of a switch should alternately put the light bulb on or off. Using this fact repeatedly, we obtain the other values of the function  $f$ .

Now, if we assign the value  $(1, 0, 0)$  to  $(x_1, x_2, x_3)$ , it brings a single change in the state of the switch  $x_1$  only. So, the light bulb must be on. This can be written mathematically in the form  $f(1, 0, 0) = 1$ . Here the value 1 of  $f$  stands for the on state of the light bulb.

Then, we must have  $f(1, 1, 0) = 0$ , because there is yet another change, now in the state of switch  $x_2$ .

You can verify that the other values of  $f(x_1, x_2, x_3)$  are given as in Table 9.

**Table 9: Function of a circuitry for a three-point functional bulb.**

$x_1$	$x_2$	$x_3$	$f(x_1, x_2, x_3)$
0	0	0	0
1	0	0	1
1	1	0	0
1	1	1	1
0	1	0	1
0	1	1	0
0	0	1	1
1	0	1	0

**Step 2: To obtain a Boolean expression which will specify the function  $f$ .** Firstly, note that the number of 1's in the last column of Table 9 are fewer than the number of 0's. So we shall obtain the expression in DNF (instead of CNF).

By following the stepwise procedure of Example 10, you can see that the required Boolean expression is given by

$$X(x_1, x_2, x_3) = (x_1 \wedge x'_2 \wedge x_3) \vee (x'_1 \wedge x_2 \wedge x'_3) \vee (x'_1 \wedge x'_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3)$$

At this stage we can directly jump into the construction of the circuit for this expression (using methods discussed in Sec.3.3). But why not try to get a simpler circuit?

**Step 3 : To simplify the expression  $X(x_1, x_2, x_3)$  given above.** Firstly, observe that

$$\begin{aligned} (x_1 \wedge x'_2 \wedge x_3) \vee (x_1 \wedge x_2 \wedge x_3) &= x_1 \wedge [(x'_2 \wedge x_3) \vee (x_2 \wedge x_3)] \\ &= x_1 \wedge [(x'_2 \vee x_2) \wedge x_3] \\ &= x_1 \wedge (1 \wedge x_3) \\ &= x_1 \wedge x_3, \end{aligned}$$

by using distributive, complementation and identity laws (in that order).

Similarly, you can see that

$$(x'_1 \wedge x'_2 \wedge x_3) \vee (x_1 \wedge x_3) = (x'_2 \vee x_1) \wedge x_3.$$

We thus have obtained a simpler (and equivalent) expression, namely,

$$X(x_1, x_2, x_3) = (x'_1 \wedge x_2 \wedge x'_3) \vee [(x'_2 \vee x_1) \wedge x_3],$$

whose Boolean function is same as the function  $f$ . (Verify this!)

**Step 4: To design a circuit for the expression obtained in Step 3.**

Now, the logic circuit corresponding to the simpler (and equivalent) expression

obtained in Step 3 is as shown in Fig.15.

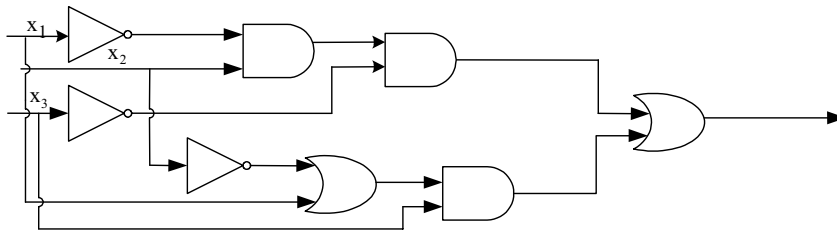


Fig. 15: A circuit for the expression  $(x_1' \wedge x_2 \wedge x_3') \vee ((x_2' \vee x_1) \wedge x_3)$

So, in 4 steps we have designed a 3-switch circuit for the hall.

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We can't claim that the circuit designed in the example above is the simplest circuit. How to get that is a different story and is beyond the scope of the present course.

Why don't you try an exercise now?

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E15) Design a logic circuit to operate a light bulb by two switches,  $x_1$  and  $x_2$  (say).

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We have now come to the end of our discussion on applications of logic. Let us briefly recapitulate what we have discussed here.

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### 3.5 SUMMARY

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In this unit, we have considered the following points.

1. The definition and examples of a Boolean algebra. In particular, we have discussed the two-element Boolean algebra  $\mathcal{B} = \{0, 1\}$ , and the switching algebras  $\mathcal{B}_n$ ,  $n \geq 2$ .
2. The definition and examples of a Boolean expression.
3. The three elementary logic gates, namely, **AND**-gate, **OR**-gate and **NOT**-gate; and the analogy between their functioning and operations of logical connectives.
4. The method of construction of a logic circuit corresponding to a given Boolean expression, and vice-versa.
5. How to obtain the logic table of a Boolean expression, and its utility in the understanding of the overall functioning of a circuit.
6. The method of simplifying a Boolean expression.
7. The method of construction of a Boolean function  $f: \mathcal{B}^n \rightarrow \mathcal{B}$ , corresponding to a Boolean expression, and the concept of **equivalent** Boolean expressions.
8. Examples of the use of Boolean algebra techniques for constructing a logic circuit which can function in a specified manner.

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### 3.6 SOLUTIONS/ ANSWERS

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E1) a) In E19 of Unit 1, you have already verified the Identity laws. Let us proceed to show that the propositions  $p \vee (p \wedge q)$  and  $p$  are logically equivalent. It suffices to show that the truth tables of both these propositions are the same. This follows from the first and last columns of the following table.



p	q	$p \wedge q$	$p \vee (p \wedge q)$
F	F	F	F
F	T	F	F
T	F	F	T
T	T	T	T

Similarly, you can see that the propositions  $p \wedge (p \vee q)$  and  $p$  are equivalent propositions. This establishes the absorption laws for the Boolean algebra  $(S, \wedge, \vee, ', \mathcal{T}, \mathcal{F})$ .

- b) Let  $A$  and  $B$  be two subsets of the set  $X$ . Since  $A \cap B \subseteq A$ ,  $(A \cap B) \cup A = A$ . Similarly, as  $A \subseteq A \cup B$ , we have  $(A \cup B) \cap A = A$ . Thus, both the forms of the absorption laws hold good for the Boolean algebra  $(\mathcal{P}(X), \cup, \cap, ^c, X, \emptyset)$ .

- E2) We can write

$$\begin{aligned} X(x_1, x_2, x_3) &= ((x_1 \wedge x_2) \vee ((x_1 \wedge x_2) \wedge x_3)) \vee (x_2 \wedge x_3) \\ &= (x_1 \wedge x_2) \vee (x_2 \wedge x_3) && \text{(by Absorption law)} \\ &= x_2 \wedge (x_1 \vee x_3) && \text{(by Distributive law)} \end{aligned}$$

This is the simplest form of the given expression.

- E3) Take the propositions  $p$  and  $q$  in place of the bits  $x_1$  and  $x_2$ , respectively. Then, when 1 and 0 are replaced by T and F in Table 3 here, we get the truth table for the proposition  $p \wedge q$  (see Table 2 of Unit 1). This establishes the analogy between the functioning of the **AND**-gate and the conjunction operation on the set of propositions.

- E4) Take the propositions  $p$  and  $q$  in place of the bits  $x_1$  and  $x_2$ , respectively. Then, when 1 and 0 are replaced by T and F in Table 4 here, we get the truth table for the proposition  $p \vee q$  (see Table 1 of Unit 1). This establishes the analogy between the functioning of the **OR**-gate and the disjunction operation on the set of propositions.

- E5) Firstly, observe that the information about the outputs of the three elementary gates, for different values of inputs, can also be written as follows:

$$0 \wedge 0 = 0, 0 \wedge 1 = 1 \wedge 0 = 0, 1 \wedge 1 = 1; \quad \text{(see Table 3)}$$

$$0 \vee 0 = 0, 0 \vee 1 = 1 \vee 0 = 1 \vee 1 = 1; \text{ and (see Table 4)}$$

$$0' = 1, 1' = 0. \quad \text{(see Table 5)}$$

Clearly, then both the operations  $\wedge$  and  $\vee$  are the binary operations on  $\mathcal{B}$  and  $'$  :  $\mathcal{B} \rightarrow \mathcal{B}$  is a unary operation. Also, we may take 0 for **O** and 1 for **I** in the definition of a Boolean algebra.

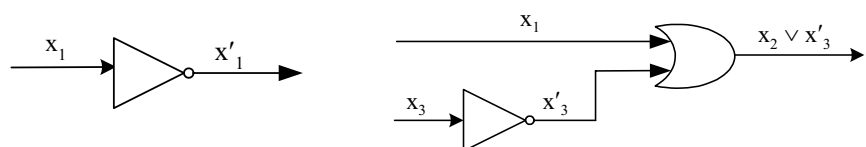
Now, by looking at the logic tables of the three elementary gates, you can see that all the five laws B1-B5 are satisfied. Thus,  $\mathcal{B}$  is a Boolean algebra.

- E6) a) Here  $x_1$  and  $x_2$  are inputs to an **OR**-gate, and so, we take  $x_1 \vee x_2$  as input to the **NOT**-gate next in the chain which, in turn, yields  $(x_1 \vee x_2)'$  as the required output expression for the circuit given in (a).

- b) Here  $x_1$  and  $x_2$  are the inputs to an **AND**-gate. So, the expression  $x_1 \wedge x_2$  serves as an input to the **NOT**-gate, being next in the chain.

This gives the expression  $(x_1 \wedge x_2)'$  which serves as one input to the extreme right **AND**-gate. Also, since  $x_3'$  is another input to this **AND**-gate (coming out of a **NOT**-gate), we get the expression  $(x_1 \wedge x_2)' \wedge x_3'$  as the final output expression which represents the circuit given in (b).

- E7) You know that the circuit representing expressions  $x_1$  and  $x_2 \vee x_3'$  are as shown in Fig.16 (a) and (b) below.



(a) (b)

Fig. 16

Thus, the expression  $x'_1 \vee (x_2 \vee x'_3)$ , being connected by the symbol  $\wedge$ , gives the circuit corresponding to it as given in Fig.17 below.

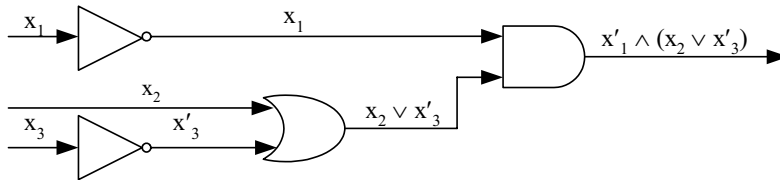


Fig. 17: A logic circuit for the expression  $x'_1 \wedge (x_2 \vee x'_3)$

E8) You can easily see, by following the arguments given in E9, that the circuit represented by the expression  $x_1 \vee (x'_2 \wedge x_3)$  is as given in Fig.18.

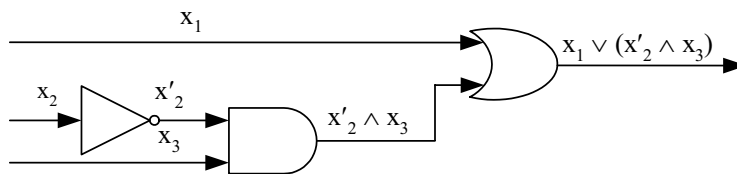


Fig. 18

The logic table of this expression is as given below.

$x_1$	$x_2$	$x_3$	$x'_2$	$x'_2 \wedge x_3$	$x_1 \vee (x'_2 \wedge x_3)$
0	0	0	1	0	0
0	0	1	1	1	1
0	1	0	0	0	0
1	0	0	1	0	1
0	1	1	0	0	0
1	1	0	0	0	1
1	0	1	1	1	1
1	1	1	0	0	1

E9) Since the output expression representing the circuit given in E8(b) is found to be  $(x_1 \wedge x_2)' \wedge x'_3$ , the logic table for this circuit is as given below.

$x_1$	$x_2$	$x_3$	$x_1 \wedge x_2$	$(x_1 \wedge x_2)'$	$x'_3$	$(x_1 \wedge x_2)' \wedge x'_3$
0	0	0	0	1	1	1
0	0	1	0	1	0	0
0	1	0	0	1	1	1
1	0	0	0	1	1	1
0	1	1	0	1	0	0
1	1	0	1	0	1	0
1	0	1	0	1	0	0
1	1	1	1	0	0	0

E10) Because the expression  $(x_1 \wedge x_2) \vee (x_1 \wedge x'_3)$  involves three variables, the

corresponding Boolean function,  $f$  (say) is a three variable function, i.e.  $f: B_3 \rightarrow B$ . It is defined by

$$f(e_1, e_2, e_3) = (e_1 \wedge e_2) \vee (e_1 \wedge e'_3), e_1, e_2 \text{ and } e_3 \in B.$$

Now, you can verify that the values of  $f$  in tabular form are as given in the following table.

$e_1$	$e_2$	$e_3$	$e_1 \wedge e_2$	$e'_3$	$e_1 \wedge e'_3$	$f(e_1, e_2, e_3) = (e_1 \wedge e_2) \vee (e_1 \wedge e'_3)$
0	0	0	0	1	0	0
0	0	1	0	0	0	0
0	1	0	0	1	0	0
1	0	0	0	1	1	1
0	1	1	0	0	0	0
1	1	0	1	1	1	1
1	0	1	0	0	0	0
1	1	1	1	0	0	1

E11)

To show that the Boolean expressions  $X$  and  $Y$  are equivalent over the two-element Boolean algebra  $B = \{0, 1\}$ , it suffices to show that the Boolean functions  $f$  and  $g$  (say) corresponding to the expressions  $X$  and  $Y$ , respectively, are the same. As you can see, the function  $f$  for the expression  $X$  is calculated in E10 above.

Similarly, you can see that the Boolean function  $g$  for the expression  $Y$  in tabular form is as given below.

$x_1$	$x_2$	$x_3$	$x'_3$	$x_2 \vee x'_3$	$G(x_1, x_2, x_3) = X_1 \wedge (x_2 \vee x'_3)$
0	0	0	1	1	0
0	0	1	0	0	0
0	1	0	1	1	0
1	0	0	1	1	1
0	1	1	0	1	0
1	1	0	1	1	1
1	0	1	0	0	0
1	1	1	0	1	1

Comparing the last columns of this table and the one given in E10 above, you can see that  $f(e_1, e_2, e_3) = g(e_1, e_2, e_3) \forall e_1, e_2, e_3 \in B = \{0, 1\}$ . Thus,  $X$  and  $Y$  are equivalent.

E12) Firstly, let us evaluate the given expression  $X(x_1, x_2)$  over the two-element Boolean algebra  $B = \{0, 1\}$  as follows:

$$\begin{aligned} X(0, 0) &= (0' \wedge 0') \vee (0' \wedge 0) \vee (0 \wedge 0) \\ &= (1 \wedge 1) \vee (1 \wedge 0) \vee (0 \wedge 0) \\ &= 1 \vee 0 \vee 0 = 1 = f(0, 0); \end{aligned}$$

$$\begin{aligned} X(1, 0) &= (1' \wedge 0') \vee (1' \wedge 0) \vee (1 \wedge 0) \\ &= (0 \wedge 1) \vee (0 \wedge 0) \vee (1 \wedge 0) \\ &= 0 \vee 0 \vee 0 = 0 = f(1, 0); \end{aligned}$$

$$\begin{aligned} X(0, 1) &= (0' \wedge 1') \vee (0' \wedge 1) \vee (0 \wedge 1) \\ &= (1 \wedge 0) \vee (1 \wedge 1) \vee (0 \wedge 1) \\ &= 0 \vee 1 \vee 0 = 1 = f(0, 1); \end{aligned}$$

$$\begin{aligned} X(1, 1) &= (1' \wedge 1') \vee (1' \wedge 1) \vee (1 \wedge 1) \\ &= (0 \wedge 0) \vee (0 \wedge 1) \vee (1 \wedge 1) \end{aligned}$$

$$= 0 \vee 0 \vee 1 = 1 = f(1, 1).$$

It thus follows that  $X(e_1, e_2) = f(e_1, e_2) \forall e_1, e_2 \in B = \{0, 1\}$ .

E13) **Step 1:** Identify all rows of the table where output is 0: This is the 2nd row.

**Step 2:** Combine  $x_1$  and  $x_2$  with 'or' in these rows, simultaneously applying 'not' to  $x_i$  if its value is 0 in the row: So, for the 2nd row the expression we have is  $x_1 \vee x_2$ .

**Step 3:** Combine all the expressions obtained in Step 2 with 'and' to get the CNF form representing  $f$ : In this case there is only 1 expression.

So  $f$  is represented by  $x_1 \vee x_2$  in CNF.

E14) a) Observe from the given table that, among the two values 0 and 1 of the function  $f(x_1, x_2, x_3)$ , the value 1 occurs the least number of times. Therefore, by the remark made after E 13, we would prefer to obtain the Boolean expression in DNF. To get this we will use the stepwise procedure adopted in Example 10.

Accordingly, the required Boolean expression in DNF is given by

$$X(x_1, x_2, x_3) = (x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge x'_2 \wedge x'_3) \vee (x'_1 \wedge x'_2 \wedge x'_3).$$

b) By the given table, among the two values 0 and 1 of the function the points  $v_i$  at which  $g(v_i) = 0$  are fewer than the points  $v_i$  at which  $g(v_i) = 1$ . So we would prefer to obtain the corresponding Boolean expression in CNF.

Applying the stepwise procedure in the solution to E13, the required Boolean expression (in CNF) is given by

$$X(x_1, x_2, x_3) = (x'_1 \vee x_2 \vee x'_3) \wedge (x_1 \vee x'_2 \vee x'_3) \wedge (x_1 \vee x'_2 \vee x_3).$$

E15) Let  $g$  denote the function which depicts the functional utility of the circuit to be designed. We may assume that the light bulb is off when both the switches  $x_1$  and  $x_2$  are off, i.e., we write  $g(0, 0) = 0$ .

Now, by arguments used while calculating the entries of Table 9, you can easily see that all the values of the function  $g$  are as given below:

$$g(0, 0) = 0, g(0, 1) = 1, g(1, 0) = 1, g(1, 1) = 0.$$

Thus, proceeding as in the previous exercise, it can be seen that the Boolean expression (in DNF), which yields  $g$  as its Boolean function, is given by the expression

$$X(x_1, x_2) = (x'_1 \wedge x_2) \vee (x_1 \wedge x'_2),$$

because  $g(0, 1) = 1$  and  $g(1, 0) = 1$ .

Finally, the logic circuit corresponding to this Boolean expression is shown in Fig. 19.

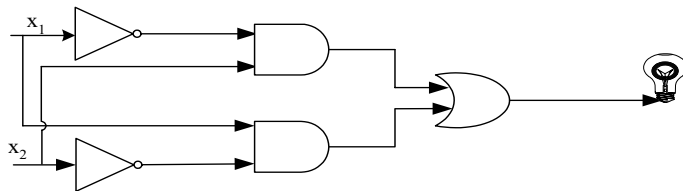


Fig. 19