MATH 408/508, Winter 2020, Assignment 4

Please submit your solutions on eClass before 3pm on Tuesday, March 24, in the format described on pages 15–18 of the lecture notes.

Use `rng('default')` before each question part with pseudo-random numbers.

The total is 40 points: 17 points for Q1 and 23 points for Q2.

1. In this exercise, we consider a frequently used model for continuous-time interest rates. We assume that the interest rate ρ_t at time t (in years) satisfies the SDE

$$\mathrm{d}\rho_t = \alpha(\beta - \rho_t)\,\mathrm{d}t + \gamma\,\mathrm{d}B_t,$$

where B is a Brownian motion and α, β, γ and ρ_0 are constants.
Remark: This stochastic process is called an Ornstein-Uhlenbeck process. When applied to interest rates, it is called the Vasicek model.

(a) *[4 points]* Write a MATLAB function `Q1a(alpha,beta,gamma,rho0,T,Npaths, Nsteps)` which generates a matrix of sample paths for ρ.

(b) *[4 points]* \$ 1 received at time T has a current value of $E\big[\exp\big(-\int_0^T \rho_t\,\mathrm{d}t\big)\big]$. Write a MATLAB function `Q1b(alpha,beta,gamma,rho0,T,Npaths,Nsteps)` to find an approximate value and the 95%-confidence interval of $E\big[\exp\big(-\int_0^T \rho_t\,\mathrm{d}t\big)\big]$.
Hint: use the approximation $\int_0^T \rho_t\,\mathrm{d}t \approx \sum_{j=1}^{\texttt{Nsteps}}(t_j - t_{j-1})\rho_{t_{j-1}}$ for $t_j = jT/\texttt{Nsteps}$.

In order to apply the function `Q1b`, we need values for α, β, γ and ρ_0. In practice, this is achieved by calibrating the model to market data, which we will do here based on the latest data from the Bank of Canada available in eClass in the file `dataQ1A4.m`.

(c) *[4 points]* Write a MATLAB function `Q1c(parameters)` which calculates the sum of square differences between model and actual interest rates. The input `parameters` is here a two-dimensional vector. Inside the function `Q1c`:

- set `alpha` and `beta` equal to the two elements of `parameters`, `gamma = 0`, `Npaths = 1` and `rho0 = 0.015` (meaning 1.5 % overnight interest rate),
- choose `T` and `Nsteps` such that they correspond to the data in `dataQ1A4.m`,
- call `dataQ1A4.m` and `Q1a`, calculate for each `t = T/Nsteps, 2T/Nsteps, ...,T` the difference between model interest rate (output of `Q1a`) and actual interest rate (given in `dataQ1A4`), take squares and sum these squares up.

(d) *[5 points]* Write a MATLAB script which

- finds the minimizers and the corresponding value of `Q1c(parameters)`; use as starting values $[1, 0.01]$ for the minimization,
- calculates price and confidence interval using `Q1b` with the following parameters: `alpha` and `beta` are minimizers of `Q1c`; `gamma = sqrt(b/Nsteps)` (one can show that this is a good approximation for γ) with `b` equal to the value of `Q1c` in the minimum; `Npaths = 100`; `T`, `Nsteps`, `rho0` as in Q1(c).

2. The file `dataQ2A4.m` contains stock price information and call prices of Coca-Cola Company. In this problem, you will compare option pricing using local volatility models to the Black-Scholes formula. We assume that the risk-free interest rate is zero.

   (a) *[3 points]* Write a MATLAB script that calculates first the implied volatility for each strike price and maturity, and then the mean σ_{BS} of the implied volatility over the given strike prices for call options with 6 weeks time to maturity. For call options with 6 weeks to maturity, calculate the difference between the market prices and the Black-Scholes prices with σ_{BS} for all strikes in a vector `delta1`.

   (b) *[3 points]* Modify the function `Pathsvarsigma` from the lecture notes to **two** new functions such that the local volatility function is given by

$$\sigma(S_t) = \max(\,0.47S_t/S_0 - 0.33, 0.86 - 0.72S_t/S_0\,), \tag{1}$$

$$\sigma(S_t, T) = \mathrm{e}^{-7T}\max(\,1.2S_t/S_0 - 1.18, 2.24 - 2.22S_t/S_0\,) + \left(1 - \mathrm{e}^{-7T}\right)\sigma_L, \tag{2}$$

   where $\sigma_L = 0.22$ is the long term volatility. Note that σ in (2) depends on T (an input of `Pathsvarsigma`) and not on t (current time).

   (c) *[3 points]* Considering still call options with 6 weeks time to maturity, write a MATLAB script that calculates two vectors `delta2` and `delta3` which contain the differences between the market prices and those calculated using (1) and (2), respectively, based on a Monte Carlo simulation with 5,000 time steps and 1,000 paths.

   (d) *[3+3+2 points]* Create a single plot showing the three vectors `delta1`, `delta2` and `delta3` as a function of the strike price. Use appropriate labels, title and legend. Please answer:

      i. *[3 points]* What are the meaning and interpretation of this plot?

      ii. *[2 points]* Which local volatility model, (1) or (2), gives more accurate call prices for options with 6 weeks time to maturity?

   (e) *[3 points]* Proceed analogously to 2(a),(c) and (d), but now create two new plots, one for call options with 2 weeks time to maturity and the other for call options with 18 weeks time to maturity. Use appropriate labels, titles and legends.

   (f) *[3 points]* Based on the plots in 2(e), do you come to the same conclusion as in 2(d)ii.? Explain the reason behind why the conclusion is the same or why it is different.