Quantitative Decision Making

Study Material

Contents

CHAPTER 1

Formulating and Solving Linear Programs

1.1 An Example of a Furniture Manufacturing Unit

Imagine that you own a small furniture manufacturing unit in Ahmedabad. Your unit manufactures plain wooden tables and simple wooden chairs. You have 15 employees in your unit, and you pay them monthly wages of Rs.5,000 each. You pay a rent of Rs.100,000 per month for the premises that you rent. You receive a fixed quantity of wood at your unit every day, according to a long term contract that you have signed. If you manufacture only tables on a given day, then the wood is enough to manufacture 400 tables, and if you decide to manufacture only chairs, then it is enough to manufacture 700 chairs. The employees in your unit can manufacture 500 tables on any given day if they manufacture only tables, and 600 chairs a day if they manufacture only chairs. The market can take up to 300 tables a day but you can sell as many chairs as you can produce. The contribution[1] that each table brings to the profit is Rs.70 per unit, while a chair generates a contribution of Rs.50 per unit. Your aim is to produce an appropriate number of tables and chairs to maximize your daily total contribution.

In order to decide the mix of tables and chairs that you produce each day, you need to analyze the problem carefully. One line of argument can be that

[1] The *contribution to profit* of an item is defined as the excess of the revenue over the variable manufacturing cost for the item.

since a table generates more contribution than a chair, the unit should manufacture as many tables as the market can absorb (i.e. 300 tables), and then use the remaining your resources to manufacture chairs. Another line of argument can be that since chairs require less resources than tables, the unit should manufacture only chairs. In that way the total contribution will be higher, even though the contribution per unit is less. You need to decide which of the two considerations, if either, is a most profitable option.

Let us analyze the first option. The unit's stock of wood is sufficient to manufacture a maximum of 400 tables per day and its labor resources are sufficient to manufacture a maximum of 500 tables. However, the market can absorb only 300 tables. So after manufacturing 300 tables, the unit can use its remaining resources to manufacture 175 chairs. See that this is within the capacity of your employees. Let us call this Mix A. At the given contribution rates, this mix of products yields a daily contribution of Rs.$(300 \times 70 + 175 \times 50)$ = Rs.29,750.00 and completely uses up the wood available to you each day.

Next, let us analyze the second option. The unit's stock of wood is sufficient to manufacture a maximum of 700 chairs per day, and its labor capacity is sufficient to manufacture a maximum of 600 chairs. So as per this consideration, the unit should manufacture 600 chairs. This would exhaust the labor capacity available to the unit, and hence tables will not be manufactured. Let us call this product mix Mix B. This product mix would generate a contribution of Rs. (600×50) = Rs. 30,000.00. Hence the positive effect of trading off the higher production rates of chairs against the higher contribution of tables yields a higher contribution in this situation.

The question now is whether Mix B is the best mix available to you. While manufacturing according to Mix B, ypu see that you have used up your labor resources, but have a seventh of your stock of wood unused at the end of each day. (You had stock to manufacture 700 chairs but you only manufactured 600 chairs.) So, can you use your resources more efficiently? Starting off with Mix B, let us assume that you manufacture one less chair. This frees up 1/600-th of your labor capacity, which along with your excess stock of wood allows you to manufacture 5/6-th of a table. This change in product mix leads to an excess utilization of $(5/6) \times (1/400) - (1/700) = 11/16800$-th of your stock of wood, and leads to an increase in contribution worth Rs.$((5 \times 70)/6 - 50)$ = Rs.8.33. This shows that Mix B is not the best possible mix and you can obtain a better mix by substituting a chair with 5/6-th of a table. So the next natural question is: How much of such substitutions can you make? Obviously, the substitution did not change the total amount of labor resources that you utilized, but uses 11/16800-th more of your daily stock of wood than that you used for Mix B. You have 1/7-th of your daily stock in excess, so that you could reduce the number of chairs that you produce by $(1/7)/(11/16800) = 2400/11$

and produce $(5/6) \times (2400/11) = 2000/11$ tables. This leads to a product mix with 2000/11 (i.e., 181 9/11) tables and 4200/11 (i.e., 381 9/11) chairs which generates a contribution of Rs.31,818.18 per day. It is easy to check in a similar way that no further trade-offs between the number of tables and chairs you produce will increase the contribution generated. So manufacturing 181 9/11 tables and 381 9/11 chairs is a best, i.e., optimal daily product mix for your unit.

Notice that while taking decisions, we do not consider the wages you pay to the unit's employees, nor do we consider the rent you pay. This is because these are not "relevant costs", i.e., costs directly related to the production of tables and chairs. The cost of wood that you obtain from the lumber company is also not relevant, since both the quantity and price of wood that you receive is fixed through a long term contract, and does not change with your product mix. In this example, all relevant costs are taken into account while computing the contributions from tables and chairs.

What are relevant costs?

1.2 Formulation of Mathematical Programs

In the last section we saw that the construction of optimal product mixes using ad-hoc arguments is quite complicated, even for simple problems. It would be impossible to follow such processes if there are a larger number of products and restrictions, which is usually the case in practice. Fortunately, there are well-developed methods to deal with such problems. In this section, we will see how we can represent the problems in a universally accepted manner — called a mathematical programming formulation, and in the last two sections of this chapter we will briefly outline a tool to solve these problems.

A mathematical programming formulation for a practical problem consists of three parts.

Mathematical programming terminology.

In the first part, we identify decisions that are to be taken in the decision making schenario and code them using variables called "decision variables". Notice that when we talk about making decisions, we talk about things that we are allowed to change. A vector of the decision variables represents a generic solution to the problem. Assigning values to the decision variables in a solution yields particular solutions to the problem. In our example, we can code the decision regarding the number of tables and chairs you produce daily with decision variables T and C, and a product mix (T,C) as a solution to the problem. The units in which we measure our decision variables need to be clear to us.

In the second part, we specify a function that evaluates solutions. This function of the decision variables is called an "objective function". We also specify the nature of the best (i.e., optimal) solution to the problem in terms of the objective function by specifying whether we want to maximize or minimize the objective function. This specification is called the objective of the mathematical program.

In the third and final part of the formulation we define "constraints" that restrict the options we have for solutions to be considered in the problem. Each constraint is an inequality or an equation in the decision variables introduced in the first part, and in the mathematical program, they represent restrictions on the decision variables brought about by the nature of the problem. Examples of constraints in our problem are the restrictions on availability of wood and labor, and the market capacity for absorbing tables that we produce. It is conventional to represent constraints as inequalities or equations, whose left hand side consists of a function of the decision variables and whose right hand side is a constant.

A solution is called "feasible", if the values assigned to the decision variables in that solution are such that *all* constraints in the model are satisfied, otherwise it is called "infeasible". A constraint that holds as an equality at a given solution is said to be "tight" at that solution. Constraints that are not tight are said to have a "slack" or "surplus" depending on whether their left hand side is less than or greater than their right hand side at that solution. "Solving a mathematical program" refers to the process of finding a feasible solution to mathematical program that maximizes (or minimizes, as the case may be) the given objective function. A solution for which the value of the objective function is the best possible among feasible solutions (i.e., the largest for maximization problems and smallest for minimization problems) is called an "optimal solution". A mathematical programming problem in which the objective function and *all* the constraints are linear is called a "linear programming problem". Linear programming problems are widely studied, since they occur widely in practice, and have very nice properties which allows us to solve them very efficiently.

What is a linear program?

The problem described in Section 1.1 can be expressed as a linear programming model. The decision that you need to make in the problem is to determine the number of tables and chairs that you to manufacture every day in your unit. Let us represent that decision in terms of two variables: T denoting the number of tables that you want to manufacture each day, and C denoting the number of chairs. These two variables are the decision variables in this model.The vector (T, C) is a generic solution to the problem. The contribution from a solution is given by the objective function $z = 70T + 50C$, and the objective of model is to maximize the value of z. There are several constraining factors that restrict your manufacturing possibilities.

- First, the stock of wood available each day is enough to manufacture either 400 tables or 700 chairs. This means that each table uses up 1/400th of your daily stock of wood, and each chair uses up 1/700th of it. Thus, if you manufacture T tables and C chairs, you use up $T/400 + C/700$ part of your daily stock of wood. This obviously cannot exceed the stock of wood you have for a day, so that you are constrained to obey the inequality $T/400 + C/700 \leq 1$.

- Similar considerations restrict you to the inequality $T/500 + C/600 \leq 1$ while considering the daily capability of the employees you have.

- The market demand restricts you to producing at most 300 tables each day, hence your manufacturing plan has to obey the restriction $T \leq 300$.

- Lastly, you must ensure that neither T nor C assumes negative values. These constraints are called non-negativity constraints. While they look trivial, they are important in a mathematical program, since the model that we are describing is a mathematical construct and will not be linked to the physical problem as far as the solving agent (e.g., a computer program) is concerned.

The linear program corresponding to the example thus looks like Figure 1.1.

Decision Variables

T**:** number of tables produced by the unit per day; and
C**:** number of chairs produced by the unit per day.

Model (Objective function and constraints)

Maximize

Contribution $z = 70T + 50C$

Subject to

$$
\begin{array}{lll}
T/400 + C/700 & \leq 1 & \text{(Stock of wood)} \\
T/500 + C/600 & \leq 1 & \text{(Labor capacity)} \\
T & \leq 300 & \text{(Market restriction on tables)} \\
T,\ C & \geq 0 & \text{(Non-negativity)}
\end{array}
$$

Figure 1.1: Linear programming formulation for the example in Section 1.1

Two other examples of creating linear programming models for practical problems are given below.

Example 1.1: A manufacturer manufacturers tin cans. Each can has a cylindrical main body shaped out of a rectangular piece of sheet metal, and two circular pieces that form the two ends of the can. He manufacturers the main body and ends by stamping out sheets of metal. He gets metal sheets in three sizes, A, B, and C. The numbers of rectangular and circular pieces that can be obtained from each sheet of the three sizes are:

Size	Number of rectangular pieces	Number of circular pieces
A	2	8
B	3	6
C	3	3

Each week, the manufacturer receives a supply of 300 sheets of size A, 150 sheets of size B, and 250 sheets of size C. Each sheet requires 5 minutes to stamp out. Once the stamped out pieces are obtained, assembling the pieces into cans require 3 minutes per can. The manufacturer can sell as many cans as he can produce. Any excess rectangular or circular pieces produced can be kept in inventory. Cans sell at Rs. 4 per can. With a working week of 40 hours, the manufacturer wants to determine his optimal production plan so that he can maximize his weekly revenues.

The Model: The output of our model for the production plan should tell the manufacturer how many sheets of each size he should stamp out every week. So we define the following decision variables:

x_A**:** number of sheets of size A that are to be stamped out each week;
x_B**:** number of sheets of size B that are to be stamped out each week; and
x_C**:** number of sheets of size C that are to be stamped out each week.

The following variables can also be defined in order to make the formulation more readable.

R**:** number of rectangular pieces that are generated each week;
C**:** number of circular pieces that are generated each week; and
K**:** number of cans that are produced each week.

The objective of the model is to maximize revenue. In terms of our decision variables it can be written as

$$\text{Maximize } 4K.$$

Next we define the constraints of the model.

The weekly supply of sheets of size A, B, and C are 300, 250, and 150 respectively. These enforce the following constraints in the model:

$$x_A \leq 300,$$
$$x_B \leq 150, \text{ and}$$
$$x_C \leq 250.$$

The number of cans that can be produced each week cannot exceed the number of rectangular pieces stamped out each week or half the number of circular pieces stamped out each week. Hence we have the following constraints:

$$K - R \leq 0, \text{ and}$$
$$2K - C \leq 0.$$

The numbers of rectangular and circular pieces produced depend on the numbers of sheets of sizes A, B, and C that are used every week. These relations are represented by the following constraints:

$$R - 2x_A - 3x_B - 3x_C = 0, \text{ and}$$
$$C - 8x_A - 6x_B - 3x_C = 0.$$

The manufacturing setup has a 40 hour working week, so that manufacturing operations can be carried out for a maximum of 2400 minutes. Given the time required to stamp out sheets and to assemble cans, the time constraint can be represented as follows:

$$5x_A + 5x_B + 5x_C + 3K \leq 2400.$$

In addition all the decision variables defined in this example need to be non-negative.

Since any excess of rectangular or circular pieces are kept in the inventory, we do not bother about it in the model. Of course, in practice, one would consider the inventory of rectangular and circular pieces from the previous month while deciding on the production plan.

Combining all this, the linear programming model to solve the manufacturer's problem is given in Figure 1.2.

Decision Variables

x_A**:** number of sheets of size A that are to be stamped out each week;
x_B**:** number of sheets of size B that are to be stamped out each week;
x_C**:** number of sheets of size C that are to be stamped out each week;
R**:** number of rectangular pieces that are generated each week;
C**:** number of circular pieces that are generated each week; and
K**:** number of cans that are produced each week.

Model (Objective function and constraints)

Maximize

Contribution $z = 3K$

Subject to

x_A	≤ 300	(Availability of size A sheets)
x_B	≤ 150	(Availability of size B sheets)
x_C	≤ 250	(Availability of size C sheets)
$K - R$	≤ 0	(Rectangular pieces in cans)
$2K - C$	≤ 0	(Circular pieces in cans)
$R - 2x_A - 3x_B - 3x_C$	$= 0$	(Stamping out of rectangular pieces)
$C - 8x_A - 6x_B - 3x_C$	$= 0$	(Stamping out of circular pieces)
$5x_A + 5x_B + 5x_C + 3K$	≤ 2400	(Time constraint)
x_A, x_B, x_C, R, C, K	≥ 0	(Non-negativity)

Figure 1.2: Linear programming formulation for Example 1.1

The Solution: Solving the model, the following optimal solution is obtained: $z = 2042.7$, $x_A = 10.1$, $x_B = 150$, $x_C = 13.5$, $K = 510.7$, and $C = 1021.4$. In the context of the example, this means that the manufacturer should procure and stamp 10.1 size A sheets, 150 size B sheets, and 13.5 size C sheets each week on average. This would provide him with 510.7 rectangular pieces and 1021.4 circular pieces per week on average, which can be made into 510.7 cans per week on average, which sell for Rs. 2042.70. No extra rectangular or circular pieces are produced in the process. ■

Note that the number of decision variables that we have used in the formulation in Example 1.1 above is larger than what is strictly required. For model maintenance purposes, it is better to use more decision variables and keep the program readable, rather than use fewer decision variables and make the program compact but undecipherable.[2]

Also note that even though the number of sheets stamped, the number of components, and the number of cans produced need to be integers in practice, the formulation only requires them to be non-negative. This is because fractional values for these variables can be explained easily; the suggestion to produce $K = 510.7$ cans per week in an optimal solution translates to producing a total of 5107 cans in ten weeks. It can also be rounded down to 510 cans per week for a good enough solution.

Example 1.2: A small bank offers three types of loans: housing loans at 8.50% interest, education loans at 13.75% interest rates, and loans to senior citizens at 12.25% interest. Further, it needs to adhere to certain policy restrictions. These restrictions require the bank to ensure that

Condition 1: housing loans make up between 25% and 60% of the total loan amount disbursed; and

Condition 2: the amount of loans disbursed to senior citizens should be at least one third of the total amount disbursed as loans.

In a particular year, its lending capacity is Rs.25,000,000. The bank would like to disburse loans so as to maximize its earnings from interest paid.

The Model: Our model should help the bank to find the amounts that it can loan out under each of the three types of loans. Thus we define three decision variables:

H**:** millions loaned out as housing loans;
E**:** millions loaned out as education loans; and
S**:** millions loaned out as loans to senior citizens.

The objective of the bank is to maximize revenue from interests, i.e., to

$$\text{Maximize } 0.085H + 0.1375E + 0.1225S.$$

The lending capacity of the bank is Rs.25,000,000. This results in the constraint

$$H + E + S \leq 25.$$

[2] If you are using Microsoft Excel Solver, it allows you to define and solve models with up to 200 variables. With Premier Solver add-in from Frontline Systems, models with 2000 variables can be solved efficiently. Solvers associated with GAMS, AMPL, etc. can handle millions of variables.

We do *not* write this as an equality constraint. The reason for this is that Rs.25,000,000 is the lending capacity, and we do not want the bank to lend the full amount if the conditions turn out to be unfavorable.

The first condition imposed by the bank's policy requires the housing loans to be between 25% and 60% of the total loan amount. This can be ensured by including the constraints

$$H \geq 0.25(H+E+S), \text{ i.e., } 0.75H - 0.25E - 0.25S \geq 0,$$

and

$$H \leq 0.6(H+E+S), \text{ i.e., } 0.4H - 0.6E - 0.6S \leq 0.$$

The second condition ensures that loans to senior citizens should be at least one third of the total loans disbursed, i.e.

$$\frac{S}{H+E+S} \geq \frac{1}{3}.$$

This constraint is not linear; however, it can be converted by cross-multiplication to the linear constraint $3S \geq H+E+S$ which can be simplified to $H+E-2S \leq 0$.

Of course, the amounts H, E, and S need to be non-negative.

Thus, the linear programming model that the bank needs to formulate to plan its loan disbursement policy is the one shown in Figure 1.3.

The Solution: The optimal solution to the problem is: $z = 2.98$, $H = 6.25$, $E = 10.42$, and $S = 8.33$. Notice that since housing loans yield the lowest rate of interest, they have been put at the minimum permitted level, i.e., 25% by the bank. Among the other two, loans to senior citizens yields less interest. Hence they have been kept at their minimum possible level, i.e., 1/3rd of the total allocation. ∎

1.3 Properties of Linear Programs

In the last section, we mentioned that linear programs have certain nice properties that allow us to solve them quite efficiently. In this section, we will elaborate on some of those properties. A reader interested in other properties of linear programs is advised to look up books by Papadimitriou and Steiglitz[3] or by Bazaraa, Jarvis, and Sherali[4].

[3] C.H. Papdimitriou and K. Steiglitz, Combinatorial Optimization: Algorithms and Complexity, Prentice-Hall Inc., Engelwood Cliffs, N.J., 1982.

[4] M.S. Bazaraa, J.J. Jarvis, and H.D. Sherali, Linear Programming and Network Flows (2nd edition), John Wiley & Sons, New York, 1990.

Decision Variables

H**:** millions loaned out as housing loans;
E**:** millions loaned out as education loans; and
S**:** millions loaned out as loans to senior citizens.

Model (Objective function and constraints)

Maximize

Revenue $z = 0.085H + 0.1375E + 0.1225S$

Subject to

$$
\begin{array}{lll}
H + E + S & \leq 25 & \text{(Lending capacity constraint)} \\
0.75H - 0.25E - 0.25S & \geq 0 & \text{(Condition 1)} \\
0.4H - 0.6E - 0.6S & \leq 0 & \text{(Condition 1)} \\
H + E - 2S & \leq 0 & \text{(Condition 2)} \\
H, E, S & \geq 0 & \text{(Non-negativity)}
\end{array}
$$

Figure 1.3: Linear programming formulation for Example 1.2

We begin by stating the obvious fact that a solution to a linear program can be expressed as a vector of the values attained by the decision variables in the solution. For example, if we represent solutions in the example in Section 1.1 using a vector whose first component represents the number of tables manufactured and the second component represents the number of chairs manufactured, then Mix A would be represented by the vector $(300, 175)$. This kind of representation allows us to represent feasible solutions to linear programs with two decision variables on an X-Y plot, thus allowing a geometrical solution method for such problems. We will see this method in Section 1.4.

Let us look at the representation of the set of feasible solutions, commonly called the "feasible region", to linear programs. It will be easy to visualize the set in problems with two decision variables; so we will choose the problem Section 1.1 as an illustration. We depict the two decision variables along two axes on a plane, the horizontal axis representing the variable T, and the vertical axes representing the variable C. Note that it is also perfectly all right to represent C on the horizontal axis and T on the vertical axis. The non-negativity constraints restrict us to the first quadrant. Plotting the other inequalities in the set of constraints mentioned in the previous section yields the set of feasible solutions shown in Figure 1.4, in which the shaded region denotes the set of feasible solutions to the problem, and the lines mark the boundaries of each of the constraints.

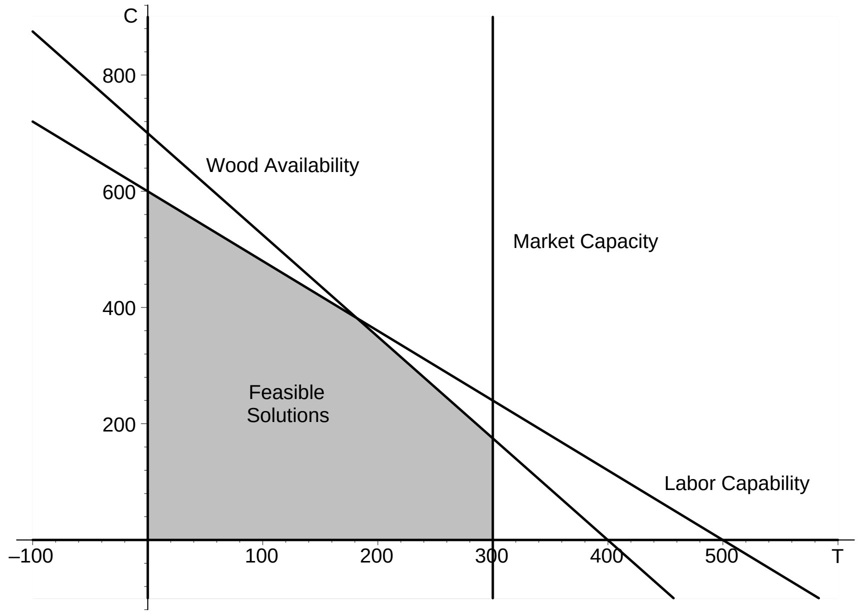


Figure 1.4: The set of feasible solutions to the example problem

Feasible solutions form a convex set.

The set of feasible solutions is a "convex" set. Mathematically, this means that for any pair (T_1, C_1) and (T_2, C_2) of feasible solutions, all solutions of the form $(\alpha T_1 + (1-\alpha)T_2, \alpha C_1 + (1-\alpha)C_2)$ are also feasible, when $0 \leq \alpha \leq 1$. Solutions of the form $(\alpha T_1 + (1-\alpha)T_2, \alpha C_1 + (1-\alpha)C_2)$ lie on a straight line joining (T_1, C_1) and (T_2, C_2), and where the solution is on that line is decided by the value of α. $\alpha = 1$ implies that the solution is at the point (T_1, C_1) and $\alpha = 0$ implies that the solution is at the point (T_2, C_2). Then a set being convex means that if we draw a straight line segment between *any* two feasible solutions, then the whole line segment would lie in the feasible region. In addition to being convex, the feasible set is also a polyhedron, i.e., a figure bounded by lines in two dimensions (see Figure 1.4), planes in three dimensions and hyperplanes in more than three dimensions.

The fact that the set of feasible solutions is convex, and that the objective function is linear allows us to narrow down our search for an optimal solution tremendously. To illustrate this, we return to our example about manufacturing furniture. Figure 1.5 shows the set of solutions for which the objective function values are 15,000 (the lower bold line) and the set for which the objective function values are 25,000 (the upper bold line). These sets (i.e.,

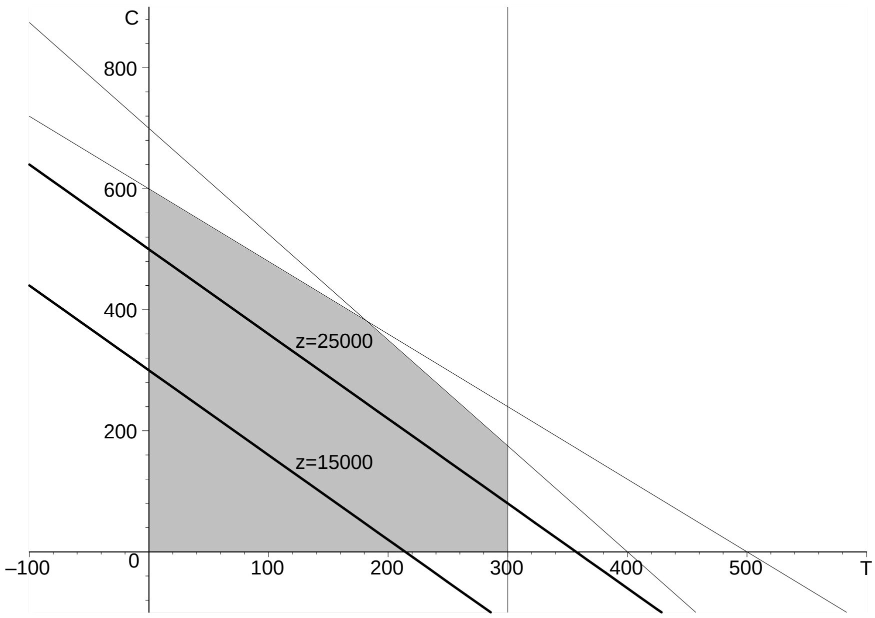


Figure 1.5: Solutions with same objective function values for the example problem

parallel lines) are obtained by setting z, i.e. $70T + 50C$ to 15,000 and 25,000 respectively.

Except the case when we are optimizing a constant, we can show that a point in the interior of the feasible region i.e., not on the boundary of the feasible region, cannot represent an optimal solution. If we are optimizing a constant, then all feasible solutions are optimal. A point being in the interior of the feasible region means that, starting at that point, we can move in *any* direction for at least a *very small* distance and still remain inside the feasible region. Now, since the objective function is linear, there always exists a direction of improvement of the objective function at any point, and this direction is perpendicular, or more correctly, normal to the lines (or planes, or hyperplanes) representing the sets of solutions with the same objective function values. For example, refer to Figure 1.6. Consider the solution $T = 100$, $C = 200$, (represented by the point A in the figure) as a solution in the interior of the feasible region. The objective function value of this solution is Rs.17,000. The bold line in the figure represents the set of solutions for which the objective function value is Rs.17,000. The direction of improvement is shown by the arrow in the figure. Since A lies in the interior of the feasible region, we can move along the direction of improvement to some point B which has a better objective function value, and is also feasible. The existence of a

A point in the interior of the feasible region cannot be optimal.

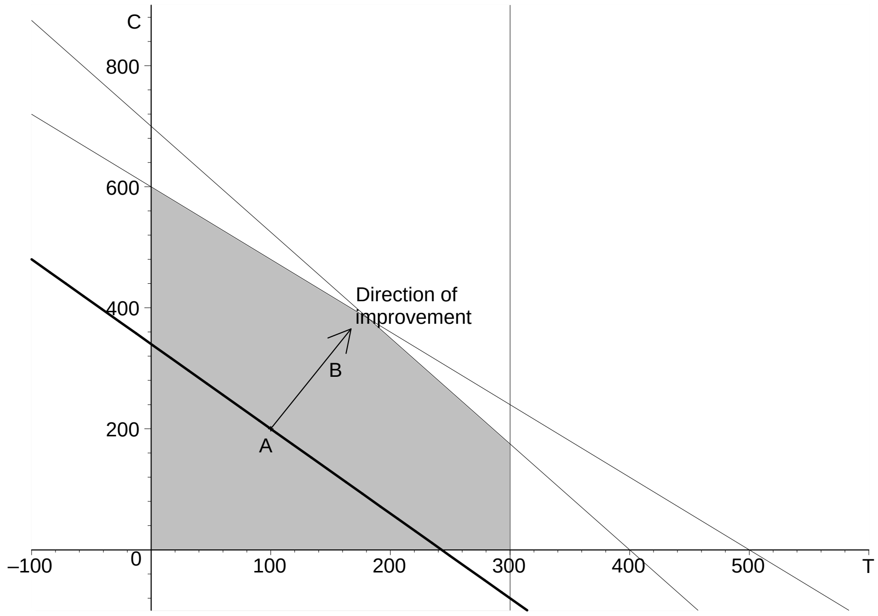


Figure 1.6: Direction of improvement for the example problem

better feasible solution proves that A cannot be optimal.

This fact seems obvious when we consider the practical situation. Suppose we suggest that you manufacture 100 tables and 200 chairs. For this production plan you are using up $100/400+200/700 = 15/28$th portion of your daily stock of wood, and $100/500+200/600 = 8/15$th of your labor capability. In addition, the market can absorb more tables and chairs. Since some portion of *all* your resources are free due to this plan, and the market can absorb more of your products, surely you can do better than the current plan.

We cannot use the same argument for solutions on the boundary of the feasible region, since for such solutions, it is possible that *any* movement along the direction of improvement would be impossible and would render a solution infeasible. Thus, we can restrict our search for optimal solutions to the boundary of the feasible region in a linear program. However, we can do even better! Consider a linear program with two decision variables, X and Y, and a maximization objective. Each solution at the boundary of the feasible region is either a corner point solution, or a *convex combination* of two adjacent corner point solutions. Consider a solution point (x, y) that is a convex combination of two adjacent corner points, say (x_1, y_1) and (x_2, y_2). This means that there exists an $\alpha_0 \in [0,1]$ such that $x = \alpha_0 x_1 + (1-\alpha_0)x_2$ and $y = \alpha_0 y_1 + (1-\alpha_0)y_2$. Geometrically, this also means that (x, y) is on the boundary of the feasible region. Let the objective function for the problem be $cx + dy$.

Without any loss of generality, assume that $cx_1 + dy_1 \geq cx_2 + dy_2$, i.e., (x_1, y_1) is a better solution than (x_2, y_2) for a maximization problem. The objective function value at the point (x, y) is given by the expression $cx + dy$.

Now note that

$$\begin{aligned} cx + dy =& c(\alpha_0 x_1 + (1-\alpha_0)x_2) + d(\alpha_0 y_1 + (1-\alpha_0)y_2) \\ =& \alpha_0(cx_1 + dy_1) + (1-\alpha_0)(cx_2 + dy_2) \\ \leq& \alpha_0(cx_1 + dy_1) + (1-\alpha_0)(cx_1 + dy_1), \text{ by our assumption} \\ =& cx_1 + dy_1. \end{aligned}$$

This means that the corner point solution (x_1, y_1) is not worse than the solution (x, y). Observe that if we had assumed that $cx_1 + dy_1 \leq cx_2 + dy_2$, then a similar argument would have shown that the corner point solution (x_2, y_2) is not worse than the solution (x, y). So we see that for linear programs with two decision variables, we only need to evaluate corner point solutions to obtain an optimal solution.

Similar arguments follow through for linear programs with more than two decision variables, so that we obtain what is perhaps the most attractive property of linear programming problems: If a linear programming problem has one optimal solution, then the optimal solution is a corner point solution. If there are more than one optima, then there are an infinite number of them, but at least one of them is a corner point solution. In other words, if there is an optimal solution that is not a corner point, then at least one adjacent corner point also represents an optimal solution.

One of the optimal solutions to a linear program is a corner point solution.

Based on this property, instead of looking at an infinite number of feasible solutions in search of an optimal solution; we can restrict ourselves to corner point solutions only, which are at least countably finite.

1.4 Solving Linear Programs

There are two different approaches to solve linear programs; the graphical approach and the approach using computer software like Microsoft® Excel Solver, Premium Solver, LINDO, LINGO, GAMS, AMPL, etc. While all these softwares implement complicated algorithms solving linear programs, our interest is not in the algorithms, but the significance of the results that they return.

The graphical approach to solving linear programs.

We first look at the graphical approach to solve linear programs. This approach is mostly pedagogical, and can be used only for linear programs with two decision variables. It is based on the fact that the feasible region of a linear program with two decision variables can be depicted to scale on paper, and that at least one optimal solution of any linear program is at a corner point.

Two closely related graphical methods are known, and we will describe both of them in the following paragraphs.

In the first method, we draw the feasible region for the linear program, and note down the solutions corresponding to the corner points. We then compute the values of the objective function for all the corner point solutions and output the solution for which the objective function value is the best (i.e., the maximum for maximization problems, and the minimum for minimization problems) as an optimal solution.

Consider for example, the problem described in Section 1.1. The feasible region for the problem is shown in Figure 1.7. The solutions corresponding

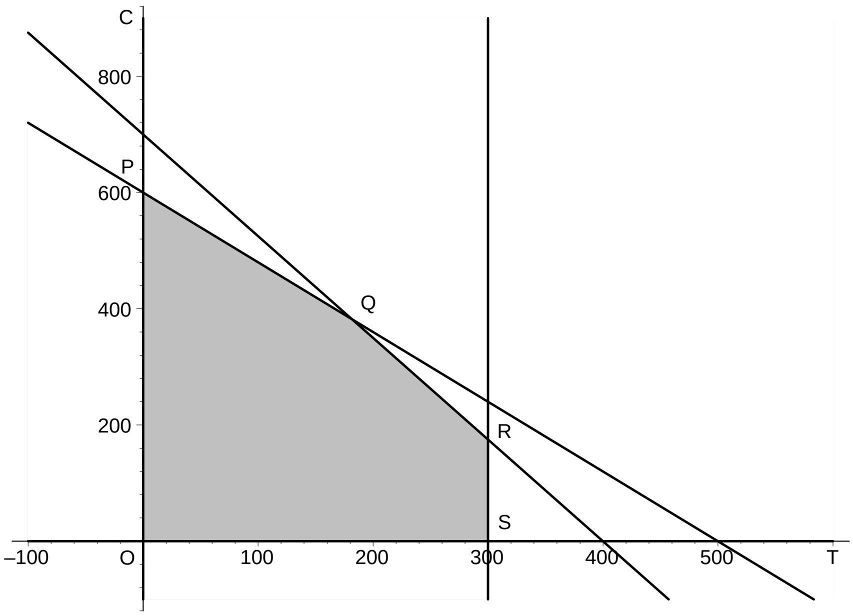


Figure 1.7: Feasible region of the problem in Section 1.1 with corners labeled

to the corner points of the feasible region, and the objective function values at each corner point solution are shown in Table 1.1. Since our objective is to maximize contribution, we choose the optimal solution as the one corresponding to point Q, i.e., a daily production plan to produce 181 9/11 tables and 381 9/11 chairs. The optimal daily contribution from this plan is Rs. 31,818.18.

Evaluating all basic feasible solutions is tedious, especially when there are many constraints, so an alternate graphical method is sometimes used. In this method, we draw the feasible region of the linear program as usual. Then we find a feasible objective function value c (usually from any feasible solution) to start the method, and draw a line with the equation $z = c$, where z is the

Table 1.1: Corner points in Figure 1.7 and their properties

Corner Point	Solution T	Solution C	Objective function value
O	0	0	Rs. 0.00
P	0	600	Rs. 30,000.00
Q	$181\frac{9}{11}$	$381\frac{9}{11}$	Rs. 31,818.18
R	300	200	Rs. 31,000.00
S	300	0	Rs. 21,000.00

objective function of the linear program. We also obtain the direction of improvement of the objective function. In case of the problem in the example in Section 1.1, when $c = 17,000$, the diagram at this stage looks like Figure 1.6 on page 14. We then slide the line $z = c$ parallel to the original line along the direction of improvement until it just touches the feasible region. This situation depicted in Figure 1.8. The solution corresponding to the corner point

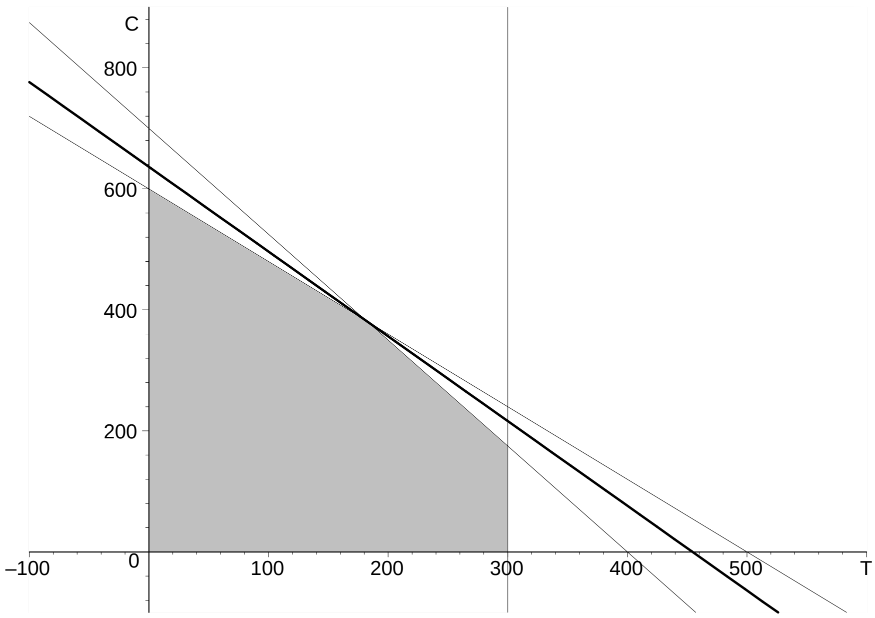


Figure 1.8: Illustration of the graphical method

at which the line now touches the feasible region is an optimal solution. We leave it to the reader to figure out whether the line can touch more than one corner point at this stage, and if it does, what inferences can be drawn from it.

As mentioned earlier, both variants of the graphical method work when a linear program has two decision variables. However most problems that occur in practice are much larger; often with thousands to millions of decision variables. Clearly graphical methods do not work in such situations. There, one uses computer algorithms to obtain optimal solutions. Traditional algorithms, like the simplex method work by jumping from corner point to corner point until an optimal solution is found. Newer methods, like interior point or barrier methods work on slightly different principles. The exact descriptions of these algorithms are beyond the scope of this book. So for the remainder of this section, we will use Microsoft® Excel to solve linear programs.

Using Microsoft® Excel Solver.

Linear programs can be solved in Microsoft® Excel using its Solver add-in.[5] Before invoking Solver, we need to create a worksheet with data that Solver can use. For our example from Section 1.1, we can set up the worksheet as shown in Figure 1.9. Two cells B2 and C2 have been used to store the final values of the decision variables T and C. In cells B3 and C3, we have entered the contribution per table and chair respectively. The cell C4 has a formula for the the objective function value (i.e., contribution) of the solution represented by the cells B2 and C2. This is computed using the formula '=SUMPRODUCT(B2:C2,B3:C3)'. We next enter the constraints in the worksheet. It is always a good idea to label each constraint. In our case, rows 7, 8, and 9 contain the constraints for wood availability, labor capacity, and market capacity for tables respectively. We have entered the function corresponding to the left hand side of the constraints in column E, marked LHS, and written the formula alongside. We next enter the values on the right hand side of the constraints in column D, marked RHS. Normally we would have the LHS column appear before the RHS column. We break from convention here so that we can write down the SUMPRODUCT function used in the formulae for LHS for illustrative purposes only. Notice that we have not yet entered the non-negativity constraints in our model.

We are now ready to invoke Solver by choosing the **Tools** menu and choosing the **Solver** option. Solver opens a dialog box labeled **Solver Parameters** similar to the one shown in Figure 1.10. We first input the cell reference at which the objective function value is computed (in our case, cell C4), and click the radio button below it to specify that we want to maximize the objective function. As is clear from the figure, the other choices that Solver allows you are to minimize an objective function, or to find a solution that sets the objective function to a pre-specified value. Next, in the box labeled "By chang-

[5] Solver is not normally included in the default installation of Microsoft Office; to check if it is loaded on your computer, open Microsoft Excel and check the drop down menu under **Tools**. If **Solver** does not appear in the menu, then you have to separately install it. To do so, go to **Tools→Add-ins...**, check the box next to **Solver Add-in** and click **OK**. Solver should load and appear on the drop down menu under **Tools** the next time you look for it. (You may need the Microsoft® Office CD during the installation process.)

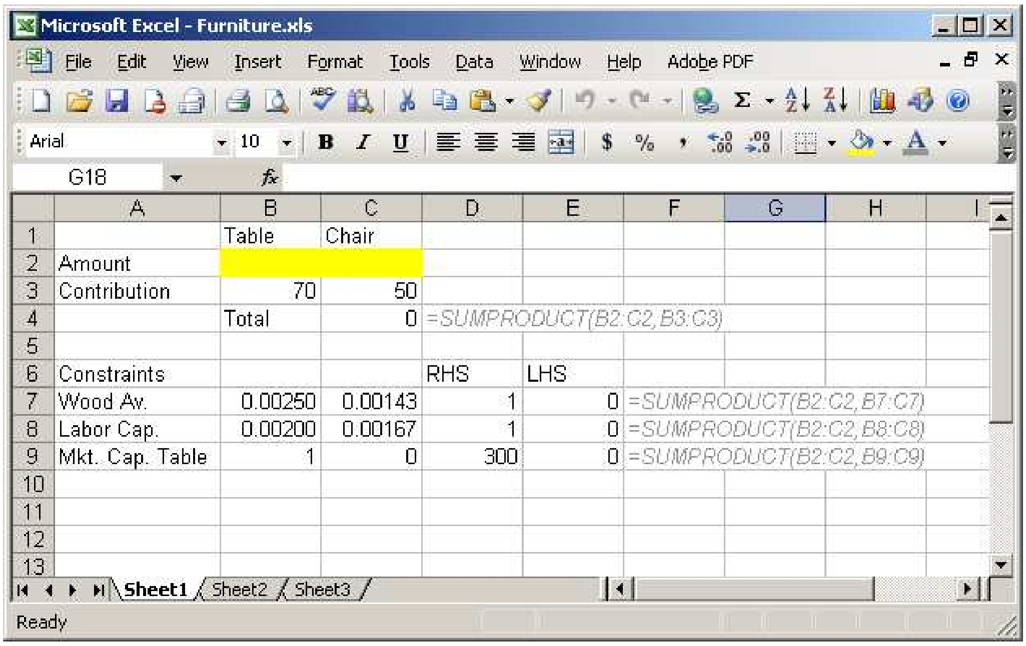


Figure 1.9: The Microsoft® Excel sheet set up before calling Solver

ing cells", we specify the cells in which the decision variables of the model are stored. Finally, we add constraints to the model by clicking on the "Add" button and adding the constraints.

Notice that by default, the variables have been marked non-negative. If this is not desired, we must uncheck the checkbox below the "Subject to the Constraints" box.

Next we must specify to Solver that we want to solve a linear program. By default Solver expects the model to be non-linear and suggests that we use "GRG Nonlinear" as the solution method. We indicate the model to be linear by changing the method to "Simplex LP". Once this is done, Solver has all inputs it needs to solve the model. So we go back to the **Solution Parameters** screen by clicking "Solve" at the bottom of the **Solver Parameters** screen.

On clicking the "Solve" button, Solver solves the model, and shows a solution status dialog box, labeled **Solver Results** (see Figure 1.11). This says that Solver has found an optimal solution. Once this dialog box is seen, there is nothing further to be done, and the **Solver Results** dialog box can be removed by clicking on the "OK" button on this box.

When the **Solver Results** dialog box is removed, the values of the decision

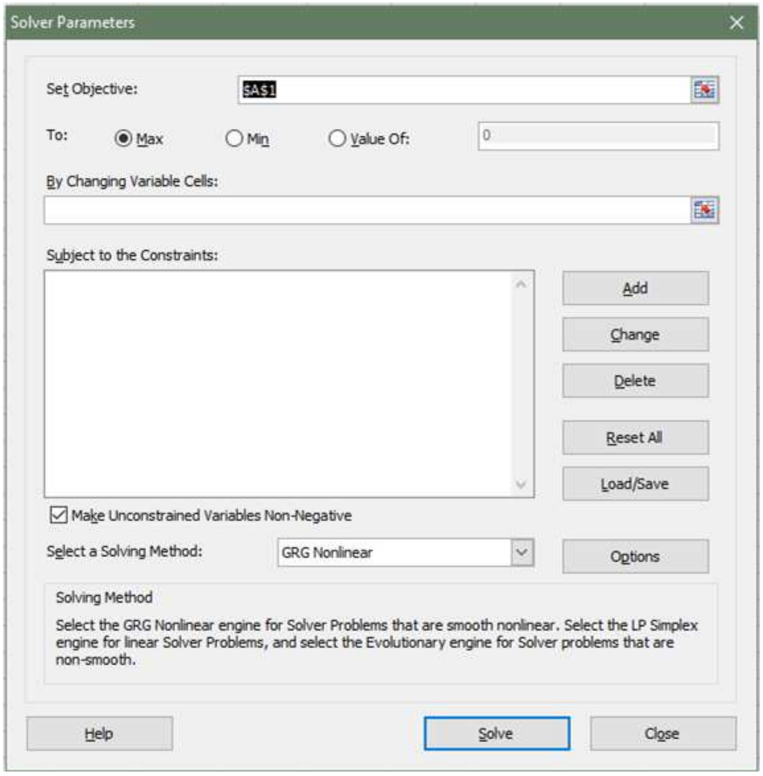


Figure 1.10: Microsoft® Excel Solver parameters

variables and the optimal solution obtained by Solver appear in the appropriate cells (in our case, cells B2, C2 and C4 respectively). After the solution process is over, the worksheet looks like Figure 1.12. Notice that the solution shown in the worksheet is the same as the one that was arrived at in the introductory section.

*The report in the **Solver Results** dialog box is inconspicuous and often ignored, but needs to be checked every time a linear program is solved using Solver. If the model does not have a finite optimal solution (i.e., is unbounded) or has no feasible solution (i.e., infeasible), the cells containing the decision variables and the objective function value in the worksheet would have intermediate values assigned to them by Solver. It is not necessary that these values correspond to the best solution that Solver found (i.e., the least infeasible solution). Ignoring the message in the **Solver Results** dialog box can mislead the modeler in such cases.*

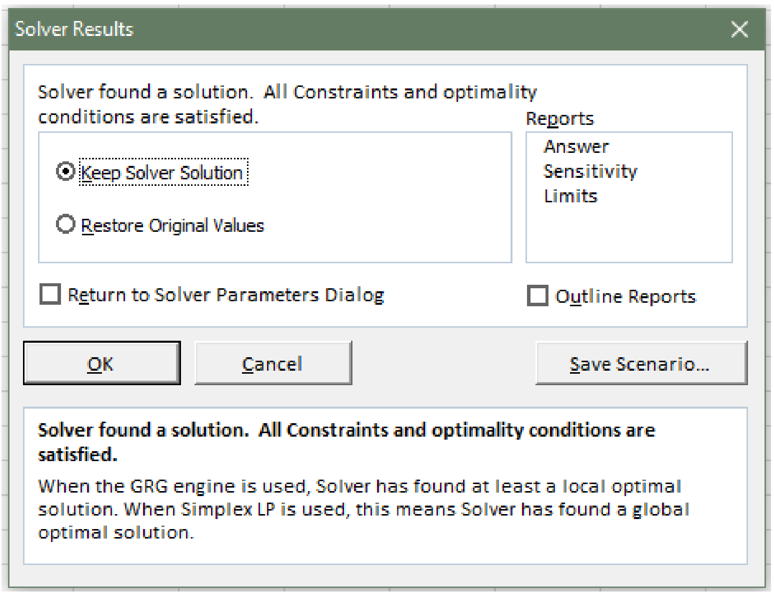


Figure 1.11: Microsoft® Excel solution dialog box after running Solver

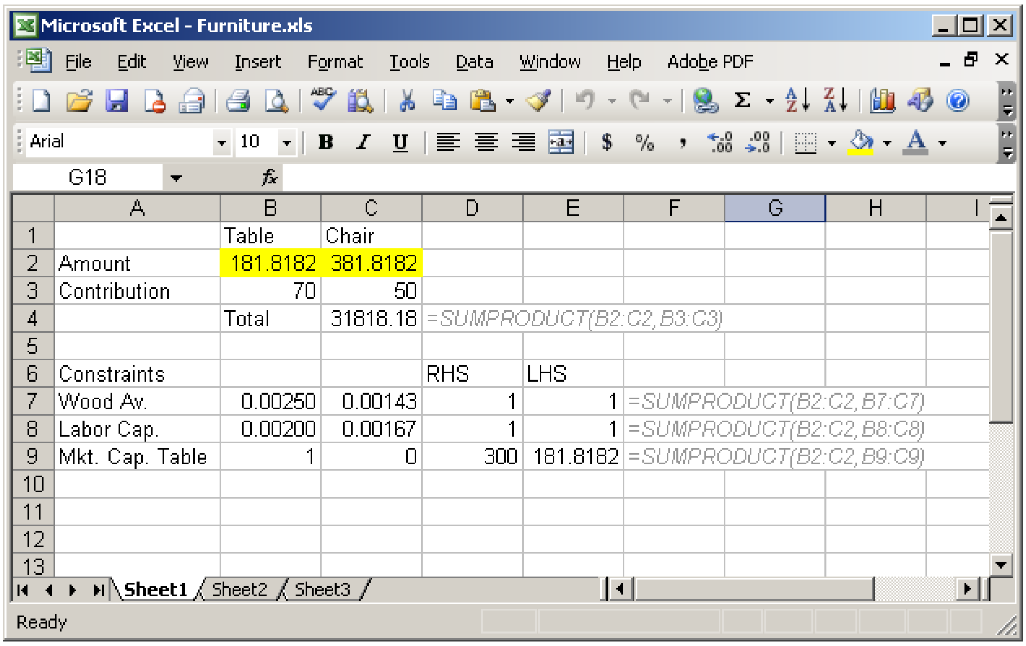


Figure 1.12: The Microsoft® Excel sheet with the solution

1.5 Additional Terminology

This chapter introduces linear programming problems, their properties, and some approaches to solve them. However, while introducing the subject at this level, we did not need to introduce several terms that are quite commonly used in connection to linear programming. In this section, we informally describe some of these terms (in lexicographic order).

Basic variables: Consider the problem discussed in Section 1.1. In this problem, there are two decision variables, and three constraints (excluding the non-negativity constraints). There are several corner point solutions in this problem, and at each corner point solution, some decision variables assume values which are not at their lower bounds (which is zero in this case) or at their upper bounds (which is infinite in this case). These decision variables are said to be basic variables corresponding to the particular corner point.

For example, at the corner point solution marked Q in Figure 1.8 on page 17, both T and C assume non-zero values. So T and C are both basic variables corresponding to the corner point solution at Q. At the corner point solution marked P, only C assumes a non-zero value, while T assumes a zero value, i.e., it is at its lower bound. So C is a basic variable and T is a non-basic variable corresponding to the corner point solution at P.

Binding and Non-Binding Constraints: A constraint that is satisfied exactly, i.e., holds as an equality at a solution is said to be binding at that particular solution. Otherwise a constraint is said to be non-binding at that solution. For example, in Figure 1.8, at the solution marked by the point Q, the constraints on the availability of wood and the capacity of the labor force are the only constraints that are binding. For the point marked S, the constraint on the market demand for tables and the non-negativity constraint of the decision variable C are the only binding constraint. All constraints are non-binding for a solution in the interior of the feasible region.

Degenerate solution: In a linear program with n decision variables, n linearly independent constraints are sufficient to define a corner point solution. However, it may so happen that, including non-negativity constraints, more than n constraints pass through a corner point. Such corner points are then called degenerate. For example, consider the following linear program.

Decision Variables

$X1, X2$: Two decision variables.

Model (Objective function and constraints)

Maximize

$$z = x_1 + x_2$$

Subject to

$$\begin{aligned} x_1 \quad\quad &\le 1 \text{ (Constraint 1)} \\ x_2 &\le 1 \text{ (Constraint 2)} \\ x_1 + x_2 &\le 2 \text{ (Constraint 3)} \\ x_1,\ x_2 &\ge 0 \text{ (Non-negativity)} \end{aligned}$$

In this program, there are two decision variables, x_1 and x_2, and all three constraints pass through the corner point $(1,1)$. (Draw the feasible region to convince yourself of this fact). The point $(1,1)$ is therefore a degenerate solution in this program. Given a linear program with a degenerate solution, it is always possible to remove a constraint which does not affect the feasible region. However, the constraint to be removed needs to be chosen with care. For example, removing constraint 2 in the program above changes the feasible region, while constraint 3 is clearly redundant. In practice it is hard to identify which constraint is redundant.

Infeasible linear program: An infeasible linear program is one in which there are no feasible solutions. This happens when one set of constraints preclude another set of constraints in the linear program. An example of an infeasible linear program is

Decision Variables
$X1, X2$: Two decision variables.

Model (Objective function and constraints)

Maximize

$$z = x_1 + x_2$$

Subject to

$$\begin{aligned} x_1 + x_2 &\le 2 \text{ (Constraint 1)} \\ x_1 + x_2 &\ge 4 \text{ (Constraint 2)} \\ x_1,\ x_2 &\ge 0 \text{ (Non-negativity)} \end{aligned}$$

Draw the constraints to satisfy yourself that this program is infeasible.

Redundant Constraints: In a mathematical program, there may be constraints whose removal does not change the feasible region. Such constraints are called redundant constraints.

For illustration, in the example given in Section 1.1, let there be an additional market constraint that not more than 1000 chairs can be sold on any given day. It is easy to see that the constraint would not affect the feasible region for the linear program in any way. This constraint would then be called redundant.

Relaxation of a mathematical program: A model is said to be relaxed if one or more constraints are removed from the model. A relaxation of a model enlarges the set of solutions that are feasible for the model. Hence the solution to the relaxation is "better" (i.e., larger for maximization problems and smaller for minimization problems), or at least, "no worse" than the original solution.

Unbounded linear program: An unbounded linear program is one in which the set of constraints are unable to stop an objective function from attaining arbitrarily high values in a maximization problem, and from attaining arbitrarily low values in a minimization problem. An example of an unbounded linear program with maximization objective is

Decision Variables
$X1, X2$: Two decision variables.

Model (Objective function and constraints)

Maximize

$$z = x_1 + x_2$$

Subject to

$$\begin{aligned} 2x_1 + x_2 &\geq 5 \text{ (Constraint 1)} \\ x_1 + x_2 &\geq 4 \text{ (Constraint 2)} \\ x_1,\ x_2 &\geq 0 \text{ (Non-negativity)} \end{aligned}$$

Graph the constraints to satisfy yourself that this program is unbounded above.

CHAPTER 2

POST-OPTIMALITY ANALYSIS IN LINEAR PROGRAMS

2.1 Another Furniture Example

Consider a manufacturing setup very similar to the setup described in Section 1.1. You are still a manufacturer of tables and chairs in Ahmedabad. You receive enough of wood at your unit every day to manufacture 400 tables if you manufacture only tables, or 700 chairs if you are manufacture only chairs. The employees in your unit can manufacture 500 tables on any given day if they manufacture only tables, and 600 chairs a day if they manufacture only chairs. Each table that you produce needs to be fitted with a reading lamp. Your vendor supplies you with 300 reading lamps each day on a long term contract. The contribution to profit from each table is Rs. 70 and that from each chair is Rs. 50. Your aim is to maximize the total contribution from your unit.[1] Your daily production plan is obtained by solving the linear programming model shown in Figure 2.1. Your optimal production plan is to manufacture 181 9/11 tables and 381 9/11 chairs every day, on average.

Now suppose that a rival company wants to buy your resources, and asks you to quote a price for each of your resources. Note that the rival company may use your resources in whatever way it may see fit. It is not interested in

[1]We make these changes with respect to the example in Section 1.1 so that the changed problem leads to more meaningful interpretations of the concepts that we introduce in this chapter.

Decision Variables

T**:** number of tables produced by the unit per day; and
C**:** number of chairs produced by the unit per day.

Model (Objective function and constraints)

Maximize

Contribution $z = 70T + 50C$

Subject to

$T/400 + C/700$	≤ 1	(Stock of wood)
$T/500 + C/600$	≤ 1	(Labor capacity)
T	≤ 300	(Stock of reading lamps)
T, C	≥ 0	(Non-negativity)

Figure 2.1: Linear programming formulation for the example in Section 2.1

your business or markets. Let us assume that this company has full and correct information about your production capabilities, your inventory, and the contributions that you obtain from each table and each chair. This scenario occurs when the acquirer is a dominant player and the target is in a weak position.

Warning: This approach is not guaranteed to work!

An intuitive approach to determining prices is the following. Let us try to determine the price of your labor capabilities. We know that with your labor working at full capacity, you make a contribution of Rs. 31,818.18 per day. Now let us reduce the labor capacity by 20%. The labor capacity constraint in the model in Figure 2.1 becomes

$$T/500 + C/600 \leq 1.0 - 0.2 = 0.8$$

while the other constraints remain unchanged. If we solve this changed model, then we get an optimal contribution of Rs. 27,000.00. This implies that 20% of your labor force is capable of generating a contribution of Rs. $(31,818.18 - 27,000.00)$ = Rs. 4818.18, which in turn implies that the price of your labor resource is Rs. $4818.18/0.2$ = Rs. 24,090.91. We could follow a similar process for computing the prices of your daily stock of wood and reading lamps.

There is a problem with this approach. Let us say that we reduce, not 20% of the labor capacity, but only 10% of it. In that case, the labor capacity constraint in the model of Figure 2.1 changes to

$$T/500 + C/600 \leq 1.0 - 0.1 = 0.9,$$

and the optimal contribution from this changed model is Rs. 29,909.09. From this data, the price of your labor capacity turns out to be Rs. $(31,818.18 - 29,909.09)/0.1 =$ Rs. 19,090.91, which is different from the figure of Rs. 24,090.91 that we had earlier.

Let us understand why this difference occurs. Figure 2.2 shows a portion of the feasible region for the original problem. Q marks the corner point so-

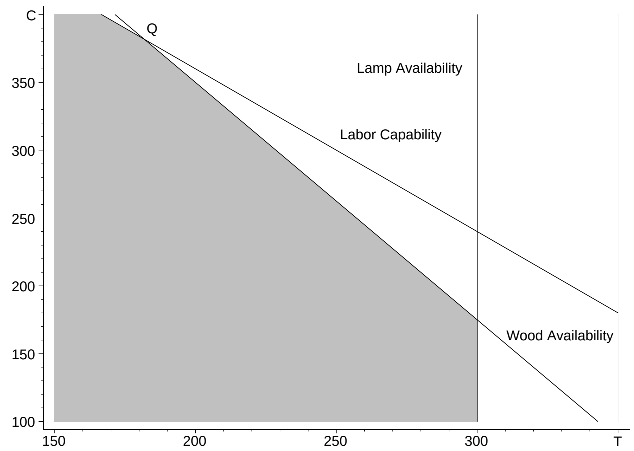


Figure 2.2: The original feasible region and optimum

lution which is optimal for the linear program in Figure 2.1. Notice that labor capacity and wood availability are binding for this solution. When the labor capacity reduces by 10%, the labor capacity constraint shifts and the optimal solution also changes (see the point Q' in Figure 2.3). The change is purely due to the reduced labor capacity, and wood availability and labor capacity still determine the optimal product mix. Therefore, the loss of contribution in this case is solely due to the reduction in labor capacity, and the price of labor computed by reducing labor capacity by 10% (i.e., Rs. 19,090.91) is a true measure of the price of labor at the optimal solution of the original problem.

However, if the labor capacity is reduced by 20%, the feasible region looks like Figure 2.4. The optimal corner point solution in this case is the point Q" in this figure. Notice, that with the labor capacity reduced to this level, there is not enough labor to fully use the wood available, and the number of reading lamps available becomes important. Therefore, the price that we obtained by reducing the labor capacity by 20% was not the true price of labor capacity at the optimal solution because it included the effect of the new binding constraint.

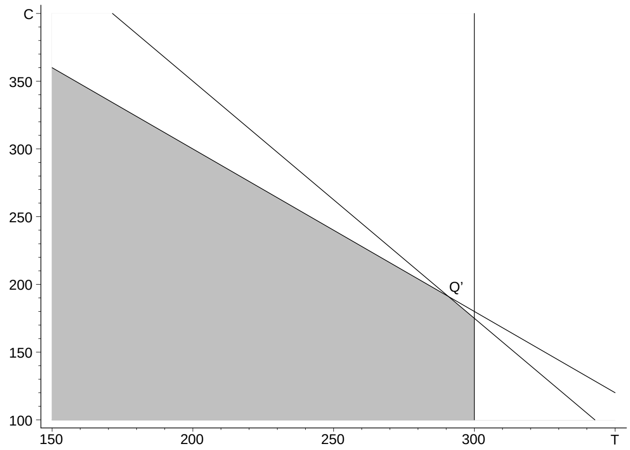


Figure 2.3: The feasible region and optimum with the labor capacity reduced by 10%

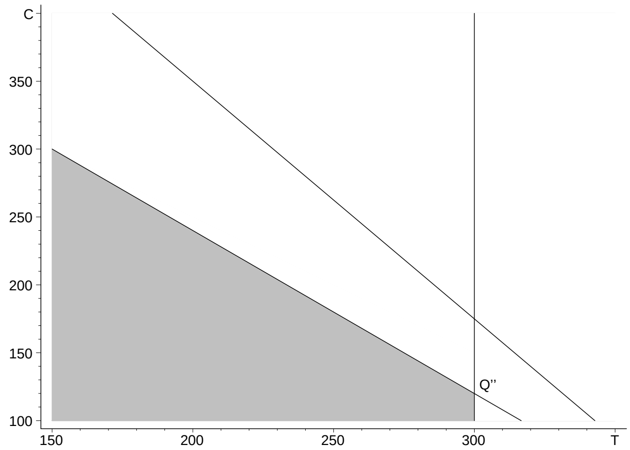


Figure 2.4: The feasible region and optimum with 20% of labor capacity sold off

Since we improve our chances of getting the correct price of a resource by making small increments, we are really trying to find out the change in contribution through a sufficiently small change in the resource level. Thus these prices are marginal prices. A natural question is whether, without solving the original problem, we can suggest any quantity of reduction in the availability of a resource through which we can guarantee that we will obtain the true prices of the resource by following the method described above. Unfortunately, the answer is 'no'. In the worst case, when an optimal solution is degenerate, there may not exist any positive amount of reduction which would work. Also, *the option of using extremely small increments to compute the prices is not advisable, if a computer is used to compute shadow prices. Computers are finite precision machines, and dividing very small quantities by very small quantities leads to large rounding errors.*

We therefore need a method of finding shadow prices for different constraints in a linear program, which has two important features:

1. the method should be guaranteed to work on all linear programs, and should not ask the user to supply details like sufficiently small increments for the value on the right hand side of the constraint; and
2. the shadow prices it outputs should match the shadow prices obtained from the intuitive method described in this section, had the modeler supplied appropriately small increments for the value on the right hand side of the constraint. We insist on this criterion because, apart from the fact that the intuitive method fails for problems in which the step size is inappropriately stated, it is a sensible method of obtaining the price of a resource.

We describe such an approach in the next section.

2.2 Shadow Prices and The Dual Program

Since the rival company has all the information about your manufacturing setup that it needs, let us try to develop a model to decide the prices of your resources from the rival company's point of view. We define the following variables for prices that it should be willing to pay for your resources:

p_w**:** price for the daily stock of wood;
p_l**:** price for the daily labor capacity; and
p_r**:** price of a reading lamp.

What are shadow prices?

We call these prices "shadow prices". They are marginal prices of the resources.

Notice that each of the three resources are never used in isolation. So for example, wood can be used in three different ways: (1) it can be used to manufacture a table, (2) it can be used to manufacture a chair, or (3) it can lie unused. If it is used to manufacture a table, then it is part of a bundle consisting of 1/400-th of the daily stock of wood, 1/500-th of the daily availability of labor, and 1 reading lamp. This bundle generates Rs.70 in contribution for your company. So you will sell this bundle to the rival company instead of producing tables only if the price you get for the bundle is not less than Rs.70. In other words, you will insist that

$$p_w/400 + p_l/500 + p_r \geq 70.$$

If the wood was used to manufacture a chair, it will be part of a bundle consisting of 1/700-th of the daily stock of wood and 1/600-th of the daily availability of labor. This bundle of resources generates a contribution of Rs.50. So you will sell this bundle to the rival company instead of producing chairs only if

$$p_w/700 + p_l/600 \geq 50.$$

If it was lying unused, you would settle for any non-negative price for the wood, i.e., you will insist that

$$p_w \geq 0.$$

Labor similarly has three uses; it could be used as (1) a part of a bundle consisting of 1/400-th of the daily stock of wood, 1/500-th of the daily availability of labor, and 1 reading lamp required to produce a table, or (2) a part of a bundle consisting of 1/700-th of the daily stock of wood and 1/600-th of the daily availability of labor required to produce a chair, or (3) it may be unused. Using an argument similar to the one above, any price that you will agree for labor would have to satisfy the conditions

$$\begin{aligned} p_w/400 + p_l/500 + p_r &\geq 70, \\ p_w/700 + p_l/600 &\geq 50, \text{ and} \\ p_l &\geq 0. \end{aligned}$$

Using a similar argument, you will agree to sell off reading lamps only if the following conditions were satisfied.

$$\begin{aligned} p_w/400 + p_l/500 + p_r &\geq 70, \\ p_w/700 + p_l/600 &\geq 50, \text{ and} \\ p_r &\geq 0. \end{aligned}$$

Thus any set of non-negative prices satisfying the conditions for pricing of bundles used for manufacturing tables and chairs will be fine with you.

Now your rival would want to minimize the amount of money that they pay you. Remember, that the prices are marginal prices, so the rival will want to minimize the price they pay for a very small fraction, say ε part of your resources. The price of this fraction is $\varepsilon(p_w + p_l + 300p_r)$ which your rival

Decision Variables

p_w**:** price for the daily stock of wood;
p_l**:** price for the daily labor capacity; and
p_r**:** price for the daily stock of reading lamps.

Model (Objective function and constraints)

Minimize

Amount payable $z = p_w + p_l + 300p_r$

Subject to

$$
\begin{array}{lll}
p_w/400 + p_l/500 + p_r & \geq 70 & \text{(For tables)} \\
p_w/700 + p_l/600 & \geq 50 & \text{(For chairs)} \\
p_w,\ p_l,\ p_r & \geq 0 & \text{(Non-negativity)}
\end{array}
$$

Figure 2.5: Linear programming model developed by the company buying out your resources

wants to minimize. Now since ε is a constant, this is equivalent to minimizing $p_w + p_l + 300p_r$. Thus the rival company obtains the linear programming model shown in Figure 2.5. Notice that solving this program allows the company to ascertain the prices of all three resources at once.

The optimal solution to this model is $z = 31{,}818.18$, $p_w = 12{,}727.27$, $p_l = 19{,}090.91$, and $p_r = 0$. Check that it meets both the criteria that we set ourselves on page 29. Also notice that the value of the objective function of the optimal solution to this problem is the same as the contribution from your optimal product mix.

The total payout of Rs. 31,818.18 by the rival company is the same as the total contribution that you would make by manufacturing and selling tables and chairs. Since this is the least that you expect from the resource sale, the rival company cannot pay any less.

ϟϟ *We have defined shadow prices here as the price per unit at which the manufacturer is willing to* sell *an infinitesimal portion of the resource being priced. While determining the amount of resource to sell, we ensure that the change does not alter the set of constraints that are binding at the optimal solution. This is not the only interpretation of shadow prices. We can also define shadow price as the price per unit that the manufacturer needs to pay to* buy *an* *infinitesimal portion of the resource being priced. These two prices are equal when the optimal solution is not degenerate. Otherwise there is a problem.*

Consider our example in Section 2.1. Here the optimal solution is not degenerate, since for this two variable problem, only two constraints are binding at this point. We can easily check that the maximum price you pay to have 1% more wood is exactly the same as the minimum price that you are willing to quote for selling 1% of your stock of wood.

In the same problem context, let us say that on average you receive 181 9/11 lamps each day rather than 300. Even in this changed scenario, your optimal product mix remains unaltered, i.e., each day you produce 181 9/11 tables and 381 9/11 chairs. Assume now that you want to sell some reading lamps. If you sell reading lamps, then your production plan becomes infeasible, and the optimal product mix in such a case would bring less contribution than earlier. Hence the shadow price of selling reading lamps is non-zero. (It is actually Rs. 10.00 per reading lamp.) Next assume that you are planning to buy some reading lamps. If you buy any reading lamps, your original optimal solution remains unchanged, since the wood and labor availability restrict you from manufacturing a better mix of tables and chairs. Since your contribution will not increase with your buying new lamps, you will not be willing to pay anything for buying new reading lamps. Hence the shadow price of buying reading lamps is zero, which is different from the shadow price we computed earlier.

Caveat emptor!!

Different linear program solution packages differ in the way they interpret shadow prices. So we need to read the documentation before we interpret the shadow prices provided in the output of a linear programming solution package. Some packages output the price of selling an infinitesimal quantity of the resource, while others output the price of buying that quantity of resource.

What are primal and dual problems?

The pair of linear programs shown in Figure 2.1 and Figure 2.5 are called *a primal-dual pair*, and each of the programs is called a *dual* of each other. Either of the two can be called a primal problem, and once a problem has been labeled a primal problem, the other problem in the pair automatically becomes its dual.

The following example describing what is commonly referred to as the diet problem and its dual illustrates a well-known primal-dual pair of linear programs.

Example 2.1: An army officer is assigned the task of buying meat, bread, and fruit in sufficient quantity to feed a garrison. The primary aim of the officer is to ensure that each person in the garrison has sufficient amount of protein, carbohydrate, and fat in his diet. A portion of meat costs Rs. 25 and provides half the daily requirement of protein, one tenth the daily requirement of carbohydrates, and one third the daily requirement of fat for a person. A portion of bread costs Rs. 10 and provides one tenth the daily requirement of protein and fat, and half the daily requirement of carbohydrates for a person. A portion of fruit costs Rs. 15 and only provides a quarter of a person's daily requirement of carbohydrates. The officer wants to minimize the cost of food for the garrison.

At the same time, a pharmaceutical company is planning to sell nutritional supplements to the garrison. These supplements come as protein pills, carbohydrate pills, and fat pills. Each of the pills contain enough nutrition to supplement a day's requirement for a person. The company wants to find out the maximum price that it can charge for each of these pills from the army officer.

These two problems together form a primal-dual pair. First consider the army officer's problem. He would solve his problem if he finds the minimum cost at which he can provide a day's quota of nutrients to one member of the garrison. If the army officer develops a linear program to solve his problem, then this program would have decision variables for the number of portions of meat, fruit and bread that he should buy for each person in the garrison; and would minimize the cost of buying food portions subject to constraints which ensure that the person's daily requirements of protein, carbohydrate, and fat are met. Such a linear program is given in Figure 2.6.

Next consider the pharmaceutical company's problem. It wants to maximize the revenues that it would get by selling the pills to the army officer. So the linear program that it would solve would have decision variables for the minimum prices that it would like to quote for the pills, and would maximize its revenue from supplying one of each type of pills to each member of the garrison subject to the constraint that the price of obtaining the nutrients in one portion of each type of food does not exceed the price that the army officer pays for that portion. Hence its model would be the one shown in Figure 2.7.

These two models form a primal-dual pair. Each of the models is a dual of the other one. ∎

The following points are valid for *every* primal-dual pair of linear programs.

Strong law of duality.

1. If a primal and a dual problem both admit optimal solutions with finite objective function values, then the objective function value of an optimal solution to the primal equals the objective function value of an optimal solution to the dual. This is called the *strong law of duality*. If holds even if there are multiple optimal solutions to either of the primal and dual problems. This is because, even though there may be multi-

Decision Variables

m**:** number of portions of meat to be bought for each person;
b**:** number of portions of bread to be bought for each person; and
f**:** number of portions of fruit to be bought for each person.

Model (Objective function and constraints)

Minimize

Cost per person $z = 25m + 10b + 15f$

Subject to

$$
\begin{aligned}
&m/2 + b/10 && \geq 1 \text{ (Protein requirement)}\\
&m/10 + b/2 + f/4 && \geq 1 \text{ (Carbohydrate requirement)}\\
&m/3 + b/10 && \geq 1 \text{ (Fat requirement)}\\
&m,\ f,\ b && \geq 0 \text{ (Non-negativity)}
\end{aligned}
$$

Figure 2.6: The army officer's linear programming model

Decision Variables

p_p**:** price of a protein supplement pill;
p_c**:** price of a carbohydrate supplement pill; and
p_f**:** price of a fat supplement pill.

Model (Objective function and constraints)

Maximize

Revenues $r = p_p + p_c + p_f$

Subject to

$$
\begin{aligned}
&p_p/2 \quad + p_c/10 + p_f/3 && \leq 25 \text{ (For a portion of meat)}\\
&p_p/10 + p_c/2 + p_f/10 && \leq 10 \text{ (For a portion of bread)}\\
&\qquad\quad p_c/4 && \leq 15 \text{ (For a portion of fruit)}\\
&p_p,\ p_c,\ p_f && \geq 0 \quad \text{(Non-negativity)}
\end{aligned}
$$

Figure 2.7: The pharmaceutical company's linear programming model

ple optimal solutions to either of the problems, the objective function values of each of the optimal solutions would be identical.

2. If the primal problem has an optimal solution with a finite objective function value, then the dual necessarily has an optimal solution with a finite objective function value. If the primal is unbounded, the dual is infeasible, and vice versa. Both the primal and the dual problems may be infeasible.

3. In addition to the strong law of duality, there is a *weak law of duality*. This law states that if the primal problem has a maximization objective, then the objective function value of every solution to the primal problem will be at most as large as the objective function value of any solution to the dual problem. If the primal has a minimization objective, then the objective function value of every solution to the primal problem will be at least as large as the objective function value of any solution to the dual problem. Notice that although the strong law holds only for optimal solutions, the weak law holds for all feasible solutions.

Weak law of duality.

4. The shadow prices of all the resources are obtained simultaneously when using this approach.

5. The shadow price of a particular resource depends not only on the objective function and the stock of the resource, but also on the stocks of other resources.

6. The shadow price of a resource remains constant as long as the set of constraints that are binding for the optimal solution remains unchanged. In case there is any change in this set, shadow prices may change, and should be recomputed.

Let us look at the models in Figures 2.1 and 2.5 that form a primal-dual pair. Ignoring the non-negativity constraints in both the programs, we see that the coefficients on the right-hand sides of the constraints in each of the problems form the coefficients of the objective function of the other. We also note that the matrix of the coefficients of the variables on the left hand side of both the programs are simply transposes of each other. These observations are handy when we want to construct the dual to any linear program without looking at the economic interpretation behind it, as we shall see in the next section.

There is another way to motivate the dual of a linear program that looks *at a mathematical viewpoint of obtaining "good" bounds on the objective function value of an optimal solution to the primal problem. It also explains why the shadow prices are also referred to as "multipliers". Consider the furniture problem in Section 2.1 again.*

Without actually solving the problem, how can we guess an upper bound to the largest contribution that we can achieve for the linear program in Figure 2.1? The constraint on the availability of wood is $1/400T + 1/700C \leq 1$*. Multiplying this with 35000, we obtain the constraint*

$$87.5T + 50C \leq 35000.$$

Clearly, such a multiplication does not alter the constraint in any way. Since T and C are both non-negative, $87.5T + 50C \geq 70T + 50C$*, which implies that the value of the objective function cannot exceed 35000 inside the feasible region. If we do a similar operation on the labor capacity constraint, i.e., multiply it with 35000, we obtain the constraint*

$$70T + 58.33C \leq 35000.$$

So here too, we conclude that the value of the objective function cannot exceed 35000 inside the feasible region.

For the constraint on reading lamps, the multiplier is infinite since the coefficient of C in this constraint is zero. This leads us to a trivial conclusion that the maximum value of the objective function within the feasible region cannot exceed ∞*.*

Collectively, the three conclusions above means that the upper bound to the objective function value over the feasible region is smallest of the three bounds, i.e., $\min(35000, 35000, \infty)$ *or Rs. 35,000.*

We can do better than this bound. If we simply add the two constraints on wood availability and labor capacity, we obtain the constraint $11/2800T + 11/3000C \leq 2$*. (Draw this constraint along with the other constraints in the example to convince yourself that it does not change the feasible region for the problem.) Multiplying this new constraint with 16153.85, we obtain the constraint*

$$72.69T + 50C \leq 32307.69,$$

which provides us an upper bound for the largest value of the objective function over the feasible region of Rs. 32,307.69, which is better that what we have found so far.

How do we combine the constraints in the linear program to generate an inequality which would yield the best, i.e., smallest upper bound to the maximum value that the objective function can achieve over the feasible region? To answer this question, let us formulate another linear programming model.

We define non-negative variables W, L, and R (called multipliers) with which to multiply the three constraints on wood, labor, and reading lamps, so that we get the constraint of our choice, which is

$$W(T/400 + C/700) + L(T/500 + C/600) + R(T) \leq W + L + 300R$$

or equivalently, after rearranging the terms,

$$(W/400 + L/500 + R)T + (W/700 + L/600)C \leq W + L + 300R. \qquad (2.1)$$

Note that W, L, or R cannot be negative, since they would then invert the inequalities. Our objective in this program is to make the value of the right hand side of the inequality (2.1) as small as possible. So we would like to

$$\text{Minimize } W + L + 300R.$$

For the right hand side of (2.1) to be an upper bound, the coefficients of T and C on the left hand side of (2.1) should at least be equal to 70 and 50 respectively. We ensure this with the following constraints:

$$\begin{aligned} 1/400W + 1/500L + R &\geq 70 \text{ and} \\ 1/700W + 1/600L &\geq 50. \end{aligned}$$

Thus, our model for determining the multipliers for the three constraints which would minimize the value of the right hand side of the combined constraint is the linear program shown in Figure 2.8.

Notice that this program is identical to the program that we have in Figure 2.5 with p_w, p_l, and p_r replaced with W, L, and R.

Every linear program has a unique dual linear program associated with it. In the next section we will show how to construct the dual in the general case.

2.3 Complementary Slackness Conditions

At the optimal solution to the linear programming problem described in Section 2.1, we found that your stock of wood and your labor capacity are fully utilized, while you have excess stock of reading lamps. We express this by saying that your reading lamp constraint has a slack at the optimal solution, which means that this constraint does not hold as an equality at that solution. The difference between the number of reading lamps that you have and the number of reading lamps that you use, i.e., $(300 - 181\,9/11) = 118\,2/11$ is called the slack for the reading lamp constraint at the optimal solution.

Definition of slack

Decision Variables

W**:** multiplier for the wood availability constraint;
L**:** multiplier for the labor capacity constraint; and
R**:** multiplier for the reading lamp constraint.

Model (Objective function and constraints)

Minimize

Amount payable $z = W + L + 300R$

Subject to

$$\begin{array}{lll} W/400 + L/500 + R & \geq 70 & \text{(For tables)} \\ W/700 + L/600 & \geq 50 & \text{(For chairs)} \\ W, L, R & \geq 0 & \text{(Non-negativity)} \end{array}$$

Figure 2.8: Linear programming model to obtain optimal values of the multipliers for the constraints

Recall that the shadow prices of the three resources, that we found by solving the dual program in Figure 2.5 are Rs. 12727.27, Rs. 19090.91, and Rs. 0.00 for wood, labor, and reading lamps, respectively. Notice that the shadow prices corresponding to the resources that had no slack are non-zero, while the shadow price corresponding to the resource that had a slack is zero. This is quite intuitive; for a resource that is being used up completely during production, you would suffer a loss of contribution if you gave away even a little of the resource, and would not give it away for free, while if some part of a resource is not being used during production, then you could part with the unutilized amount for free. This relation between the slack of a resource at an optimal solution and its shadow price is formally expressed in terms of the complementary slackness conditions as follows.

Complementary slackness conditions.

Let a linear programming problem be feasible and have an optimal solution $(x_1^\star, x_2^\star, \ldots, x_n^\star)$. Let $(v_1^\star, v_2^\star, \ldots, v_m^\star)$ be an *optimal* solution to the dual of this problem. Then the following results hold.

1. If the ith constraint in the primal is not binding (i.e. has a non-zero slack), then $v_i^\star = 0$.

2. If $v_i^\star > 0$ in the dual, then the ith constraint in the primal problem is binding.

In other words, if the s_i is the slack on the ith constraint in the primal, and v_i is the dual variable associated with this constraint, then

$$s_i \times v_i^\star = 0.$$

Although the statement of the complementary slackness conditions appears quite simple, the following points are worth noting:

1. **They are valid only when the primal problem is feasible and has an optimal solution.** If the primal problem is infeasible, then it does not have an optimal solution. In this case, computing the slack of any constraint at an optimal solution is meaningless, and the complementary slackness theorem cannot be applied. If the primal problem is unbounded, i.e., is feasible but does not have a finite optimum, the dual problem is infeasible, and does not have an optimal solution. So complementary slackness cannot be applied.

2. **They are valid only for optimal solutions.** If a primal solution is a corner point solution that is not optimal, the dual solution obtained by applying complementary slackness conditions would result in a solution that is not feasible for the dual program. This observation leads to an elegant way of checking whether a corner point solution to a linear program is indeed optimal.

3. **The implications in the complementary slackness theorem are one-sided.** Even though a non-binding primal constraint implies that the corresponding dual variable is zero, a zero dual variable does not imply a non-binding primal constraint. Similarly, even though a positive dual variable implies a binding constraint, a binding constraint does not imply a non-zero dual variable. If the optimal solution is degenerate, it can lead to situations in which the slack on a primal constraint and the value of the dual variable associated with it are both zero at their respective optima.

This is important and slightly tricky.

We now illustrate how complementary slackness conditions can be used to check whether a corner point solution is indeed optimal (This was mentioned in the second point above).

Example 2.2: Consider the problem situation given in Section 2.1. Assume that someone claims that the product mix of 300 tables and 175 chairs is optimal. We will use complementary slackness conditions to refute this claim.

Observe first that the proposed mix is a corner point solution, since at this solution the wood and reading lamp availability constraints are tight.

Next we will assume that this mix is optimal, so that complementary slackness conditions apply to this solution. Let us then use complementary slackness conditions then to predict some results for the optimal solution to the dual program shown in Figure 2.5.

1. The slacks for the wood and reading lamp availability constraints are both zeros, so we cannot say anything about the values of p_w and p_r in an optimal solution to the dual. However, since the slack for the labor capacity constraint is positive (its value is 13/120), complementary slackness conditions allow us to say that the value of p_l in an optimal solution to the dual is zero.

2. Since the dual of the dual program is the primal program, the dual variables associated with the two constraints in the dual program of Figure 2.5 are T and C respectively. Since the optimal values of T and C are both non-zero, we can conclude from complementary slackness conditions that the slacks associated with both the dual constraints are zeros at the dual optimal solution. Hence at the optimal solution, both the dual constraints hold as equalities.

Thus the optimal solution $(p_w^\star, p_l^\star, p_r^\star)$ to the dual problem can be characterized as follows

$$\begin{aligned}
p_l^\star \quad\quad\quad\quad &= 0,\\
p_w^\star/400 + p_r^\star &= 70,\\
p_w^\star/700 \quad\quad &= 50, \text{ and}\\
p_w^\star,\ p_r^\star \quad\quad &\geq 0.
\end{aligned}$$

This characterization does not admit a solution. So we have a situation in which the primal problem is feasible and has a finite optimum, but the dual problem is infeasible. This is simply not possible according to the laws of duality. Hence we can conclude that our original assumption was invalid, and the mix of 300 tables and 175 chairs is not optimal. ∎

Example 2.3: We will now show that complementary slackness conditions indeed verify an optimal solution to a linear program. Let us consider the mix of 181 9/11 tables and 381 9/11 chairs. We will show that this leads to a feasible dual solution based on the complementary slackness principles. Assuming that this solution is optimal, we use complementary slackness conditions to arrive at the following conditions:

1. $p_r = 0$; and
2. both the constraints in the dual hold as equalities at the optimal solution to the dual.

This leads us to the following characterization of an optimal solution $(p_w^\star, p_l^\star, p_r^\star)$ to the dual problem:

$$\begin{aligned} p_r^\star &= 0, \\ p_w^\star/400 + p_l^\star/500 &= 70, \\ p_w^\star/700 + p_l^\star/600 &= 50, \text{ and} \\ p_w^\star, p_l^\star &\geq 0. \end{aligned}$$

This characterization admits the feasible solution, $(12727.27, 19090.91, 0)$. So our assumption about the optimality of the mix of 181 9/11 tables and 381 9/11 chairs is true. This solution has an objective function value of $12727.27 + 19090.91 + 300 \times 0 = 31818.18$, which is identical to the objective function value of the primal solution that we are assuming to be optimal. Therefore, by the strong law of duality, our assumption is correct, i.e., the mix of 181 9/11 tables and 381 9/11 chairs is indeed optimal. ∎

2.4 Reduced Costs

Let us look at the diet problem in Example 2.1 on page 32. The linear program to solve the army officer's problem is shown in Figure 2.6 on page 34. On solving this problem, we find the optimal strategy of the officer is to buy 2.55 portions of meat, 1.49 portions of bread and no fruits for each person in the garrison every day. The cost of this diet is Rs. 78.73 per person. The reason for not buying fruits is clear; they are too costly for the amount of nutrients that they provide.

A natural question that arises is the following: By what amount should the cost of fruits be reduced so that it becomes viable to buy them in an optimal solution? The answer to this question is a quantity called the reduced cost of the variable f in the linear program in Figure 2.6. Formally stated,

What is reduced cost?

> The reduced cost of a decision variable in a linear program is the quantity by which the coefficient of that variable in the objective

function in the linear program needs to be reduced so that the decision variable may assume a non-zero value in some optimal solution to the program.

Obviously, if a decision variable is at a non-zero level in an optimal solution, its objective function coefficient does not have to be reduced at all to include it in an optimal solution. Hence all variables whose values are not zero in an optimal solution have zero reduced cost.

If the optimal solution is unique[2] then reduced cost of variables with zero values in the optimal solution would have different signs depending on whether the linear program has a minimization objective or a maximization objective. If a variable has a zero value in a unique optimal solution in a linear program with minimization objective, then its objective function coefficient is too big, and needs to be reduced by a strictly positive amount for it to be at a non-zero value in an optimal solution. So reduced costs of non-basic variables in a linear program with a unique optimal solution to a minimization problem are strictly positive. On the other hand, in a linear program with maximization objective which has a unique optimal solution, a decision variable has zero value if its coefficient in the objective function is too small. In such cases the coefficient of this variable in the objective function needs to be increased by a strictly positive amount for it to attain a non-zero value in an optimal solution. An increase can be considered to be a decrease by a negative quantity, so that the reduced cost of a decision variable with zero value in a unique optimal solution to a linear program with maximization objective is strictly negative.

If there are multiple optimal solution for a linear program, then it is possible for both the value of a decision variable and its reduced cost to be zeros. For example, in a linear program with two non-negative decision variables x_1 and x_2, a objective of minimizing $x_1 + x_2$, and a single constraint $x_1 + x_2 \geq 1$, an optimal solution is $(x_1, x_2) = (1, 0)$, but the reduced cost of x_2 is zero, since $(x_1, x_2) = (0, 1)$ is also optimal. To summarize,

1. the reduced costs of decision variables in a linear program with minimization objective are non-negative, while those for decision variables in a linear program with maximization objective are non-positive;

2. the reduced costs of all decision variables that have a non-zero value in an optimal solution to a linear program are zeros; and

[2] Note that if the optimal solution to the primal problem is unique, then the optimal solution to the dual problem is non-degenerate.

3. in linear programs with degenerate optimal solutions, it is possible for both the value of a decision variable and its reduced cost to be zeros.

If points 2 and 3 in the summary look very much like complementary slackness conditions, it is for a good reason. The reduced cost of a variable is nothing but the slack on the associated constraint in its dual program, with appropriate sign.

Computing the reduced cost of an decision variable in a linear program is quite straight-forward. To see this, let us continue using the linear problem in Example 2.1. As we saw earlier in this section, the army officer will not procure any of fruit in an optimal plan. We can check that the optimal solution to the problem is unique. So the reduced cost for fruits is non-zero.

In order to find the value of the reduced cost, we first solve the dual of the linear program, i.e., the program shown in Figure 2.7 on page 34. The values of the decision variables at the optimal solution to this program is $p_p = 0$, $p_c = 5.32$, and $p_f = 73.4$. One portion of fruit provides a quarter of a person's daily requirement of carbohydrates, and so is worth only Rs. 5.32/4 = Rs. 1.33. Hence, the cost of a portion of fruit has to be reduced by Rs. (15.00 − 1.33) = Rs. 13.67 for it to be considered in an optimal solution to the army officer's problem. So the reduced cost for the variable f is Rs. 13.67.

The following example illustrates the calculation of reduced costs in a linear program with maximization objective.

Example 2.4: A company can make three types of mixtures, called mix A, mix B, and mix C. Each 60g packet of mix A contains 30g of peanuts, 20g of raisins, and 10g of cashews, and contributes Rs. 4 to profits. Each 60g packet of mix B contains 20g of peanuts, 30g of raisins, and 10g of cashews, and contributes Rs. 6 to profits. Each 60g packet of mix C contains 20g of peanuts, 10g of raisins and 30g of cashews, and contributes Rs. 5 to profits. The company can make and seal a total of 2000 packets each day. It gets a daily supply of 34 kilos of peanuts, 74 kilos of raisins, and 51 kilos of cashews through a long term contract. Its objective is to maximize its contribution to profits. Based on a consultant's advise, the company concentrates on mix B packets only, and makes 1700 packets each day, which generates a daily contribution of Rs. 10,200. The company now want to find out what changes if any in the contributions of mixes A and C would lead to a more balanced product mix.

The Consultant's Model: It is straight-forward to reconstruct the consultant's model. It she used decision variables x_A, x_B, and x_C to denote the numbers of packets of mix A, mix B, and mix C to be produced each day, her linear programming model would be the one shown in Figure 2.9.

Decision Variables

x_A**:** number of packets of mix A to produce each day;
x_B**:** number of packets of mix B to produce each day; and
x_C**:** number of packets of mix C to produce each day.

Model (Objective function and constraints)

Maximize

Contribution $z = 4x_A + 6x_B + 5x_C$

Subject to

$$
\begin{array}{lll}
30x_A + 20x_B + 20x_C & \leq 34000 & \text{(Stock of peanuts)} \\
20x_A + 30x_B + 10x_C & \leq 74000 & \text{(Stock of raisins)} \\
10x_A + 10x_B + 30x_C & \leq 51000 & \text{(Stock of cashews)} \\
x_A + x_B + x_C & \leq 2000 & \text{(Packaging capacity)} \\
x_A,\ x_B,\ x_C & \geq 0 & \text{(Non-negativity)}
\end{array}
$$

Figure 2.9: The consultant's model

Solving this model we find that an optimal solution to it is indeed $x_A = x_C = 0$, $x_B = 1700$, with an objective function value of Rs. 10,200. Solving the dual of this problem, we find that the only non-zero shadow price corresponds to the constraint on the stock of cashews and is Rs. 0.30 .

Our Solution to the Company's Problem: The company basically wants to find out the reduced costs for mixes A and C. We can compute them by using the shadow prices obtained by solving the dual of the consultant's model.

For mix A, the contribution obtained per packet is Rs. 4.00, while the worth of the resources going into each packet is Rs. $(0.3 \times 30 + 0.00 \times 20 + 0.00 \times 10 + 0.00 \times 1)$ = Rs. 9.00. Therefore the reduced cost for x_A is Rs. $(4.00 - 9.00)$ = – Rs. 5.00. So the price of each mix A packet needs to be increased by Rs. 5.00 for it to be considered in an optimal product mix.

For mix C, the contribution obtained per packet is Rs. 5.00, while the worth of the resources going into each packet is Rs. $(0.03 \times 20 + 0.00 \times 30 + 0.05 \times 10 + 0.00 \times 1)$ = Rs. 6.00. Therefore the reduced cost for x_C is Rs. $(5.00 - 6.00)$ = – Rs. 1.00. So the price of each mix C packet needs to be increased by Rs. 1.00

for it to be considered in an optimal product mix. ∎

2.5 Sensitivity Analysis

Our post-optimality analysis has so far concentrated on finding shadow prices in linear programs. Another kind of analysis answers the question of finding the maximum amount by which we can change any of the coefficients in a linear program such that the constraints that determine an optimal solution before the change still remain the same. This is known as sensitivity analysis. The range of values within which the optimal solution is determined by the same set of binding constraints is called the tolerance range, and the maximum increase and decrease that are allowed for the coefficient to remain within the tolerance range are called the upper and lower tolerances respectively.

What is sensitivity analysis?

Note that while doing sensitivity analysis, we allow only one of the problem coefficients to change, while others are maintained at their original values. If more than one coefficient change, then the analysis becomes complicated, and is beyond the scope of this book.

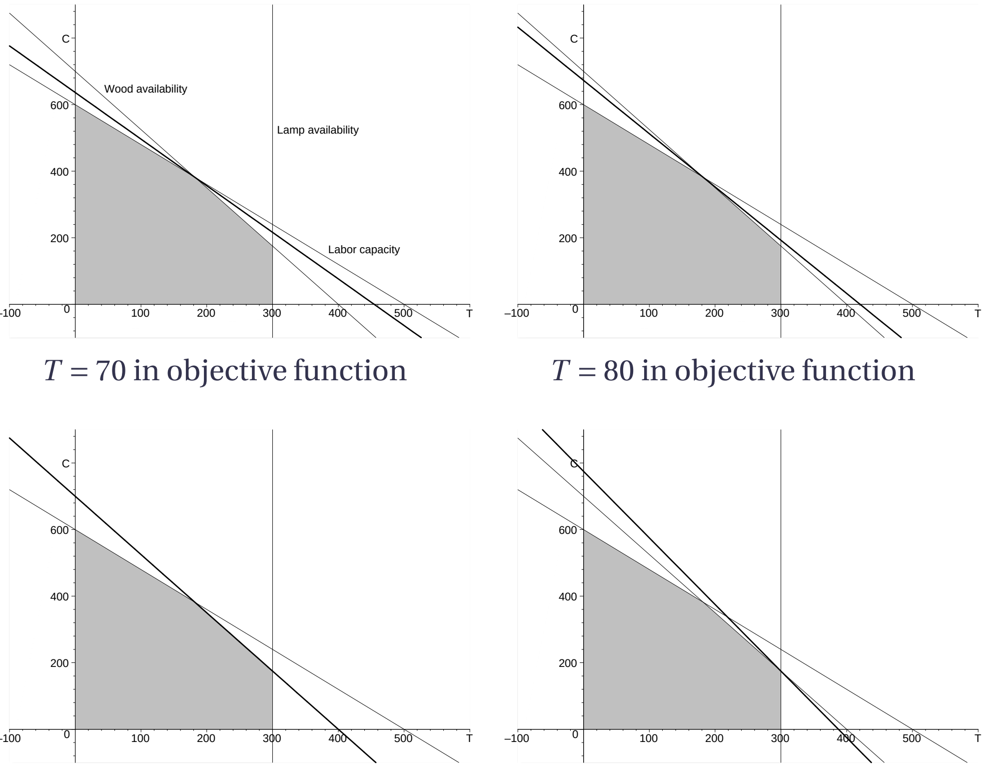
Coefficients of linear programming models can be classified into three classes:

1. coefficients in the objective function;
2. values on the right hand side of the constraints; and
3. coefficients in the left hand side of the constraints.

In the remaining portion of this section, we will illustrate the process of carrying out sensitivity analysis for all these three classes of coefficients. To do so, we will use the model that we have developed for the problem in Section 2.1, i.e., for the linear program in Figure 2.1 on page 26.

Sensitivity analysis for objective function coefficients

Assume that in the linear program in Figure 2.1, the coefficient of T in the objective function changes from 70 to 80 to 87 1/2 to 100. The effect of these changes is shown in Figure 2.10. The feasible region is shaded in each case, and the objective function line through the optimal solution is shown using the thicker line. Notice that the feasible region remains unchanged in all the four cases, since the constraints do not change. Increasing the coefficient of T



$T = 70$ in objective function

$T = 80$ in objective function

$T = 87\ 1/2$ in objective function

$T = 100$ in objective function

Figure 2.10: Effect of changing the objective function in a linear program

in the objective function rotates the objective function line clockwise, so that when $T = 80$, the objective function line through the optimal solution is much closer to the wood availability constraint than when $T = 70$. When $T = 87\ 1/2$, the objective function line through the optimal solution coincides with the wood availability constraint. If the coefficient of T is increased any further, then the objective function line through the same point passes through the interior of the feasible region; so that the solution that was previously optimal no longer remains optimal. At these values of the coefficient of T the adjacent corner point becomes optimal as is seen in the case where the coefficient of T is 100 in Figure 2.10. The labor capacity constraint no longer determines the position of the optimal solution, but the constraint on the availability of reading lamps does. So we can conclude that the coefficient of T can be increased by a maximum of $87\ 1/2 - 70 = 17\ 1/2$ without affecting the set of constraints that determine the optimal solution. So the upper tolerance of the coefficient of T in the objective function is $17\ 1/2$ units.

If the coefficient of T in the objective function is reduced, then the objective function line rotates in the counter-clockwise direction. Using an analysis very similar to that above, we can conclude that the maximum allowable decrease in the coefficient of T before the set of constraints determining the optimal solution changes is 10 units, when the objective function line through the optimal solution coincides with the labor capacity constraint. If the decrease in the coefficient of T is by more than 10 units, then the labor capacity constraint and the C-axis corresponding to the non-negativity constraint $C \geq 0$ determine the position of the optimal solution. Hence the lower tolerance for this coefficient is 10 units.

The tolerance range for the coefficient of T in the objective function is the interval $[80-10, 80+7\,1/2]$, i.e., $[70, 87\,1/2]$.

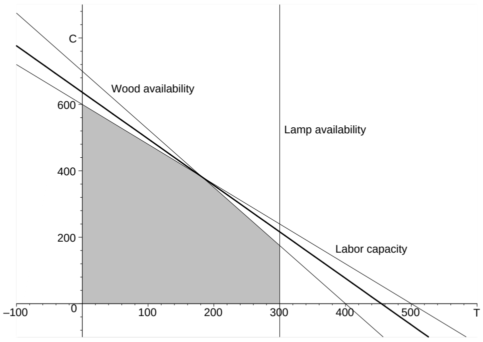
A similar analysis of the effect of changing the coefficient of C would show that its upper tolerance is 8 1/3 units, its lower tolerance is 10 units, and its tolerance range is the interval $[40, 58\,1/3]$.

Note that within the allowable range for the objective function coefficients, the optimal solution, i.e., the numbers of tables and chairs to be produced each day did not change, although the contribution to profits from the same product mix changed with changing objective functions.

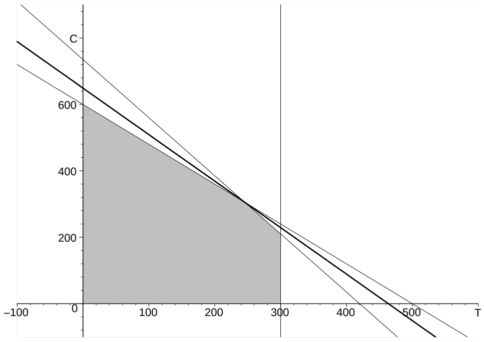
Sensitivity analysis for values on the right hand side of the constraints

Assume that the right hand side of the wood availability constraint in the program in Figure 2.1 changes from 1 to 1 1/20 to 1 13/140 to 1 3/20. The effect of these changes is shown in Figure 2.11. Notice that the slope of the objective function line remains unchanged, since the coefficients in the objective function are not changed. However, the feasible region increases in size because of this change.

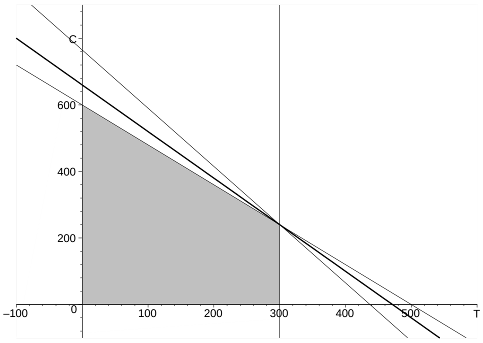
As the right hand side of the wood availability constraint increases from 1 to 1 1/20, it moves toward the top right corner parallel to the original constraint line, and the optimal solution changes, favoring the production of more tables and fewer chairs. When the right hand side of the constraint increases to 1 13/140, the wood availability constraint passes through the intersection of the labor capacity constraint and the reading lamp availability constraint. At this point the optimal solution to the problem is degenerate. When the right hand side of the wood availability constraint increases beyond 1 13/140, to 1 3/20 for example, we see that the wood availability constraint becomes redundant, and it no longer determines the optimal solution. We see this phenomenon in the bottom left diagram in Figure 2.11. This implies that, if the the constraints on labor capacity and reading lamp availability remain unchanged, it does not make sense for you as the manufacturer to increase your



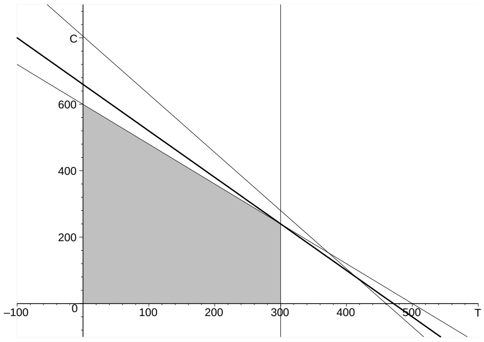
RHS = 1 in the
wood availability constraint



RHS = 1 1/20 in the
wood availability constraint



RHS = 1 13/140 in the
wood availability constraint



RHS = 1 3/20 in the
wood availability constraint

Figure 2.11: Effect of changing the right hand side of the wood availability constraint in the linear program of Figure 2.1

daily stock of wood by more than 13/140th of the current stock. If the daily wood availability reduces, then the wood availability constraint moves toward the bottom left parallel to the original constraint line. If the right hand side reduces by more than 1/7, then the labor capacity constraint is no longer one of the constraints that determine the optimal solution. Thus the upper tolerance for the right hand side of the wood availability constraint is 13/140, the lower tolerance is 1/7, and the tolerance range for this value is $[1-1/7, 1+13/140]$, i.e., $[6/7, 1\ 13/140]$.

We can do a similar analysis for the right hand side of the labor availability constraint to show that its upper tolerance is 1/6, and its lower tolerance is 13/120. This of course means that the labor capacity constraint and the wood availability constraint continue to determine the optimal solution as long as the labor availability constraint remains within 107/120th of the current availability and 7/6th of the current availability, while all other constraints in the problem remain unaltered.

Note that although the set of binding constraints do not change within the allowable limits of the right hand sides of the constraints, the optimal product mix changes when the value on the right hand side of a constraint changes. This is in contrast to the situation when the objective function coefficients changes.

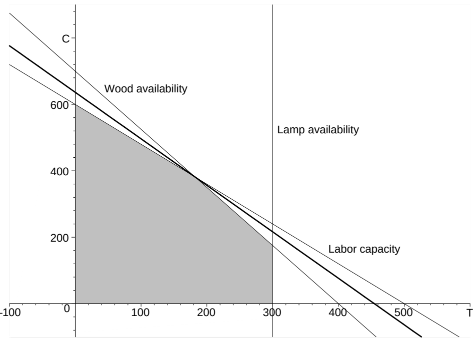
Sensitivity analysis for coefficients in the left hand side of the constraints

Finally let us look at the effect of changing the coefficient of C in the left hand side of the wood availability constraint in the linear program in Figure 2.1. Assume that the coefficient of C in the left hand side of the wood availability constraint increases from the current value of 1/700 to 1/625 to 1/600 to 1/575. The shapes of the feasible region and the objective function lines through the optimal solution in the four situations are shown in Figure 2.12. We see that as this coefficient increases, the wood availability constraint rotates counterclockwise, pivoting around the point at which the constraint meets the T axis. As it rotates, the optimal corner point solution shifts toward the C axis. We can observe this in the diagram on the top right of Figure 2.12 in which the coefficient of C is 1/625. When the coefficient becomes 1/600, the wood availability constraint meets the C axis at the point where the labor capacity constraint meets it. At this point, the optimal solution becomes degenerate. This is shown in the bottom left diagram of Figure 2.12. If the coefficient of C in the wood availability constraint increases further, then the labor capacity constraint becomes redundant, and no longer determines the optimal product mix; see the bottom right diagram of Figure 2.12. Hence the maximum allowable increase in the coefficient of C in the wood availability constraint is 1/600 – 1/700 = 1/4200, which means that the upper tolerance for this coefficient is 1/4200.

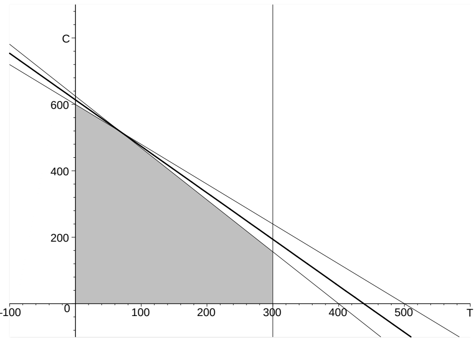
If the coefficient of C decreases, the wood availability constraint rotates in a clockwise direction pivoting around the point where it meets the T axis. When the amount of decrease is by more than 3/5600, it becomes redundant and the labor capacity and reading lamp availability constraint determine the optimal product mix. Hence the lower tolerance for this coefficient is 3/5600.

Here too, although the binding constraints do not change within the allowable limits, the optimal product mix changes when one of the coefficient on the left hand side of a constraint changes. This is again in contrast to the situation when the objective function coefficients changes.

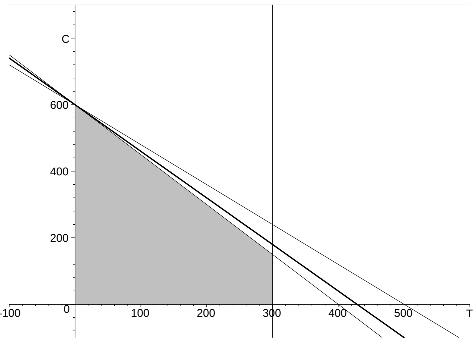
The discussion above shows us how to perform sensitivity analysis by the



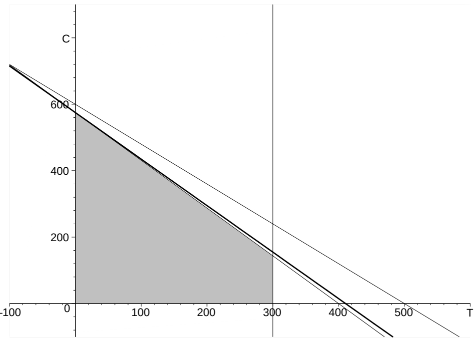
Coefficient of C in the wood availability constraint is $1/700$



Coefficient of C in the wood availability constraint is $1/625$



Coefficient of C in the wood availability constraint is $1/600$



Coefficient of C in the wood availability constraint is $1/575$

Figure 2.12: Effect of changing the coefficient of C in the left hand side of the wood availability constraint in the linear program in Figure 2.1

graphical method for linear programs with two decision variables. Sensitivity analysis for coefficients of linear programs with more than two decision variables is quite complex, and is beyond our scope. Fortunately, many commonly encountered linear program solution software, including Microsoft® Excel Solver, provide most of these values in their output.

The following points are worth noting when we perform sensitivity analysis, and interpret the results from it.

1. Sensitivity analysis works under the assumption that only the coefficient under consideration is changed and all other coefficients are held constant at their initial values. Therefore it is incapable of analyzing the effect of simultaneous changes in more than one program coefficients.

2. When one of the coefficients of a linear programming problem changes,

the optimal solution or its objective function value or both can change. The only thing that will remain unchanged if the coefficient is within the tolerance range is the set of constraints that are binding at the optimal solution.

3. The measurement units of the decision variables in the constraints of the model are important when interpreting post-optimality results. For example, we can rewrite the wood availability constraint in the original problem of Section 2.1 as $7T + 4C \leq 4200$. If represented in this form, the tolerances computed would be different from the tolerance limits that we obtained earlier in this section. However, we must realize that the values of tolerances computed in the earlier model were in units of 'a day's stock of wood', while the tolerances computed with the scaled constraint are in units of '1/4200th of a day's stock of wood'. The numerical values of the tolerances in the latter case would therefore be exactly 4200 times those in the original model.

2.6 Post-Optimality Analysis Using Microsoft® Excel Solver

In the previous sections, we have shown how to obtain shadow prices, and reduced costs by solving the dual of a linear program. We have also shown how to obtain tolerances using a graphical approach. However, most common solution softwares output shadow prices as a by-product of solving the primal linear program. Microsoft® Excel Solver also displays them if asked to. To let Microsoft® Excel Solver display shadow prices, reduced costs and tolerances, we need to click on the "Sensitivity" option in the Reports section of the Solver Results dialog box, as shown in Figure 2.13, before clicking on the "OK" button. Solver then opens a new worksheet and outputs the post-optimality analysis results on that sheet. It calls this worksheet the Sensitivity Report. As an illustration, we have shown the sensitivity report corresponding to the problem in Section 2.1 in Figure 2.14.

The sensitivity report in Microsoft® Excel Solver has two tables, one labeled "Adjustable Cells" dealing with the objective function coefficients, and the other labeled "Constraints" dealing with values on the right hand side of the constraint (in)equalities. The first table has seven columns. The first and second columns describe the position of the decision variable (called adjustable cell in Solver parlance) and its name, and the third column specifies its value in the optimal solution. The fourth column reports the reduced cost of the coefficient of that decision variable in the objective function. Notice

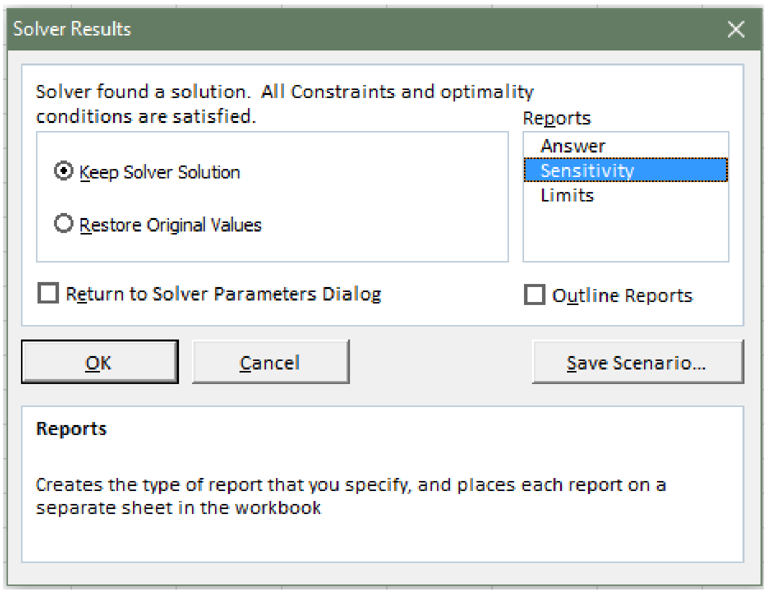


Figure 2.13: Solver Results dialog box for sensitivity analysis

that in Figure 2.14 the reduced costs of the coefficients of both the decision variables are zero, since both the decision variables assume non-zero values in the optimal solution to that problem. The fifth column shows the coefficients of the decision variables in the objective function in the model. The last two columns of the table provides the upper and lower tolerance values for the coefficient of the decision variables in the objective function. See that the values for the coefficient of T in the original problem, derived on page 46 are the ones shown in Figure 2.14.

What shadow price does Microsoft® Excel Solver output?

The second table in the sensitivity report deals with the constraints. This table also has seven columns, and as in the previous case, the first three columns define each inequality, and the value the left hand side of the inequality assumes in the optimal solution to the problem. The fourth column shows the shadow price for each constraint. In Microsoft® Excel Solver, the shadow price output in the sensitivity report is the rate of increase in the objective function value when the right hand side of the constraint is increased marginally. It is independent of whether the objective is of a maximization type or a minimization type, or whether the constraint is of greater-than type, less-than type or even an equality. The shadow price in the sensitivity report should thus be interpreted appropriately. The last two columns in the table show the upper and lower tolerances for the values on the right hand side of the constraints. Microsoft® Excel Solver calls them “Allowable Increase” and “Allowable Decrease” respectively. Notice that the tolerances for the wood availability con-

Microsoft Excel 11.0 Sensitivity Report
Worksheet: [Furniture.xls]Sheet1
Report Created: 4/14/2006 4:33:48 PM

Adjustable Cells

Cell	Name	Final Value	Reduced Cost	Objective Coefficient	Allowable Increase	Allowable Decrease
B2	Amount Table	181.8181818	0	70	17.5	10
C2	Amount Chair	381.8181818	0	50	8.333333333	10

Constraints

Cell	Name	Final Value	Shadow Price	Constraint R.H. Side	Allowable Increase	Allowable Decrease
E7	Wood Av. LHS	1	12727.27273	1	0.092857143	0.142857143
E8	Labor Cap. LHS	1	19090.90909	1	0.166666667	0.108333333
E9	Mkt. Cap. Table LHS	181.8181818	0	300	1E+30	118.1818182

Figure 2.14: Solver sensitivity report for the problem in Section 2.1

straint quoted here match the values we obtained on page 48 through graphical analysis.

Notice, that Microsoft® Excel Solver does not output sensitivity analysis results for the coefficients on the left hand side of the constraint (in)equalities. However, more advanced linear programming solvers output these values too.

CHAPTER 3

Comparing Decisions Under Uncertainty and Risk

3.1 Introduction

In this chapter we try to compare different decisions when the outcomes of the decisions are not completely under the decision maker's control. In particular, we consider situations in which the environment of the decision maker is indifferent to the decision taken by the decision maker. So, the material that we consider in this chapter does not naturally hold when the decision maker's decision is being scrutinized by a competitor, who will take a decision which will maximize the competitor's benefit, possibly to the detriment of the original decision maker.

Consider for example, a furniture manufacturer who manufactures wooden tables and chairs, and whose objective is to maximize the contribution to profits from these two products. Tables require 3 m^2 of wood and 3 labor-hours to produce and earn a contribution of Rs.1000 per unit. Chairs on the other hand require 2m^2 of wood and 1 labor-hour to produce and earn a contribution of Rs.600 per unit. The manufacturer has a supplier of wood, who promises to supplies her up to 2000 m^2 of wood per month on demand at a cost of Rs.150 per m^2. In addition to wood, she requires other material, such as paint, varnish, and fabric to create the tables and chairs. These are abundantly available as long as she orders an adequate quantity well in advance. She has 10 people working for her on fixed salary, each of whom puts in eight hours of labor each working day. She assumes a total of 20 working days in a month.

If the supplier does indeed supply her with 2000 m^2 of wood each month, and the people turn up for work each day, then she can use a linear programming model to optimize her product mix. This mix is to produce 400 tables and 400 chairs each month and earns a contribution of Rs.6,30,000. If anyone suggests an alternate product mix, of say 500 tables and 100 chairs, she can convince herself that her product mix is better simply by computing the contribution from the other product mix and observing that it is lower than that for her product mix.

In a realistic scenario, the manufacturer must face uncertainties that are not resolved at the time of planning. For example, the supplier of wood may not be able to supply the required amount of wood for her manufacturing process. She may face absenteeism among her employees. As a simplified situation, assume that her supplier supplies her with either 2000 m^2 of wood a month, or if the supplier's stocks are low, only 1500 m^2 of wood. In some months her employees put in 1600 labor-hours per month, and in others when absenteeism is high they put in 1200 labor hours. Table 3.1 shows the optimal product mixes for the manufacturer if she knew the conditions that she would face the following month accurately when determining her product mix.

Table 3.1: Optimal mix under different scenarios

Scenario	Wood Availability	Labor Availability	Optimal Tables	Optimal Chairs	Revenue
I	2000 m^2	1600 labor-days	400	400	Rs.340000
II	2000 m^2	1200 labor-days	133	800	Rs.313333
III	1500 m^2	1600 labor-days	500	0	Rs.275000
IV	1500 m^2	1200 labor-days	300	300	Rs.255000

The manufacturer's problem of finding an optimal production strategy now becomes more complicated. For the time being, we will assume that she chooses to adopt a product mix which is optimal in at least one of the four scenarios. Since some of the product mixes are infeasible for some of the scenarios and since she does not know which scenario she is in at the time of her decision, she also needs to decide whether to manufacture the tables first or the chairs first. She thus has seven decision alternatives to choose from (since in one situation she does not manufacture chairs at all). Once she decides on the product mix that she aims for, she orders enough of the other material (paint, varnish, and fabric) that she needs for her product mix. Table 3.2 shows the contributions to profits for different product mixes under different scenarios. Such a table in which the payoffs for each strategy-scenario pair are listed is called a payoff table.

What is a payoff table?

Table 3.2: The manufacturer's payoffs under different scenarios

			Scenarios			
Strategy	Prod. Mix	Order	I	II	III	IV
A	(400,400)	Tables first	Rs.340000	Rs.220000	Rs.265000	Rs.220000
B	(400,400)	Chairs first	Rs.340000	Rs.266667	Rs.248333	Rs.248333
C	(133,800)	Tables first	Rs.313333	Rs.313333	Rs.238333	Rs.238333
D	(133,800)	Chairs first	Rs.313333	Rs.313333	Rs.225000	Rs.225000
E	(500, 0)		Rs.275000	Rs.220000	Rs.275000	Rs.220000
F	(300,300)	Tables first	Rs.255000	Rs.255000	Rs.255000	Rs.255000
G	(300,300)	Chairs first	Rs.255000	Rs.255000	Rs.255000	Rs.255000

Notice from the payoff table that each strategy is evaluated in terms of not one but four payoffs, corresponding to the four scenarios that may arise. So comparing two strategies is not in terms of comparing two numbers, but in terms of comparing two vectors of numbers. This is not easy; for example, Strategy A is better off than Strategy B in Scenario 3, but is worse off in Scenarios 2 and 4. So the task of comparing strategies is the task of combining the entries of the vectors of numbers corresponding to the strategies into single numbers which can then be compared. While doing this, an idea about the relative likelihood of the different scenarios is helpful. However, such an idea may or may not be available. In this context, it is useful to realize that uncertain decision making situations arise in a spectrum. On one end of the spectrum are those situations in which decision scenarios either occur completely at random, or due to processes which have so many contributing factors that it is not realistic to assume that we will have enough information to ascertain the likelihood of different scenarios. Such decision making situations are called situations of (deep) uncertainty. Such situations arise for example, when one wants to predict the weather, say seven days in advance. On the other end of the spectrum, there are situations in which the random process giving rise to the scenarios are understood perfectly, and simple probabilistic methods are useful to determine the likelihood of scenarios. Such situations occur, for example, when one is betting on the throw of fair dice. These situations are called situations of stochastic uncertainty or risk. Most business scenarios fall somewhere in between; some but not all the important determinants of a phenomenon are known, and collecting enough data about the determinants will allow the decision maker to have a rough idea about the likelihood of particular scenarios. These are situations in which management

Levels of uncertainty.

tools such as market research become useful. In any scientific endeavor, the idea is to start from the deep uncertainty end of the spectrum and move the situation to the risk end of the spectrum through a better understanding of the scenario and/or data collection.

The techniques used to make decisions about situations in different parts of the spectrum are different, and are in general a mixture of techniques used for deep uncertainty and risk. In the remainder of the chapter we explain some of the techniques used for situations in the two ends of the spectrum.

3.2 Decision Making Under Deep Uncertainty

As mentioned earlier, when a decision maker makes decisions under uncertainty, they do not know the probabilities with which each of the scenarios occur. Hence they cannot have recourse to probabilities while making decisions. There are three main techniques for making decisions under uncertainty, of which the first two make rather extreme risk profile assumptions.

The maximax approach

In this approach, the decision maker evaluates a decision strategy by the highest payoff that it can yield and ignores payoff values from all other scenarios. For instance, in the example above, a decision maker following a maximax approach will assign a payoff of Rs.340000 to Strategy A, although in scenarios II through IV, the payoffs from the strategy are lower. Hence a decision maker who follows a maximax strategy is an extreme optimist. In the example in the introductory section, the payoffs that the decision maker assigns to each of the strategies is given in Table 3.3.

Table 3.3: Payoffs according to a maximax (or optimistic) approach

	Scenarios				Payoff
Strategy	I	II	III	IV	assigned
A	**Rs.340000**	Rs.220000	Rs.265000	Rs.220000	Rs.340000
B	**Rs.340000**	Rs.266667	Rs.248333	Rs.248333	Rs.340000
C	**Rs.313333**	**Rs.313333**	Rs.238333	Rs.238333	Rs.313333
D	**Rs.313333**	**Rs.313333**	Rs.225000	Rs.225000	Rs.313333
E	**Rs.275000**	Rs.220000	**Rs.275000**	Rs.220000	Rs.275000
F	**Rs.255000**	**Rs.255000**	**Rs.255000**	**Rs.255000**	Rs.255000
G	**Rs.255000**	**Rs.255000**	**Rs.255000**	**Rs.255000**	Rs.255000

The decision maker then proceeds to choose that strategy for which the payoff that they have assigned is the highest. In this example, they will be indifferent between choosing either Strategy A or Strategy B.

Over time, decision makers who choose to follow the maximax strategy can make large payoffs if observed scenarios are favorable in the long run. However, if they are not, then such decision makers are liable to lose large payoffs, and unless they have enough money in reserve, can be wiped out of the market.

The maximin approach

This approach is diametrically opposite to the previous approach. If a decision maker follows this approach, then they evaluate a decision strategy by the minimum payoff that it can yield, ignoring the other payoff values from the strategy in other scenarios. In the example above for instance, such a decision maker will assign a payoff of Rs.220000 to Strategy A, although in Scenarios I and III the payoffs from this strategy are higher. Hence a decision maker adopting a maximin decision making approach is an extreme pessimist. In the example, the payoffs that the decision maker assigns to each decision strategy is given in Table 3.4. The decision maker then proceeds to choose that strategy

Table 3.4: Payoffs according to a maximin (or pessimistic) approach

	Scenarios				Payoff
Strategy	I	II	III	IV	assigned
A	Rs.340000	**Rs.220000**	Rs.265000	**Rs.220000**	Rs.220000
B	Rs.340000	Rs.266667	**Rs.248333**	**Rs.248333**	Rs.248333
C	Rs.313333	Rs.313333	**Rs.238333**	**Rs.238333**	Rs.238333
D	Rs.313333	Rs.313333	**Rs.225000**	**Rs.225000**	Rs.225000
E	Rs.275000	**Rs.220000**	Rs.275000	**Rs.220000**	Rs.220000
F	**Rs.255000**	**Rs.255000**	**Rs.255000**	**Rs.255000**	Rs.255000
G	**Rs.255000**	**Rs.255000**	**Rs.255000**	**Rs.255000**	Rs.255000

for which the payoff that they have assigned is the highest. In this example, they will be indifferent between choosing either Strategy F or Strategy G.

Over time, decision makers who choose to follow the maximin strategy will make small payoffs in each period. They will tend to avoid any strategy that yields a negative payoff in the worst case.

The min-max regret approach

The two decision strategies described above suffer from the drawback that they deal with decision makers who are either extremely risk prone or extremely risk averse. Decision makers do not typically show such extreme behavior. The approach that we discuss now addresses this drawback.

A decision maker takes their decision before they know which scenario will unfold. In case their decision is optimal in the scenario that unfolds, the decision maker would have taken the best decision that could have been taken. If that was not the case, then the payoffs from the decision maker's decision is suboptimal, and they could have obtained a higher payoff than what they has obtained through their decision. The difference between the payoff that the decision maker could have achieved had they known the scenario that would unfold and the payoff that they made through their decision is the decision maker's regret given a particular decision and a scenario. For example, suppose that a decision maker decides to adopt Strategy A, and the scenario that unfolds is Scenario III. The best strategy for Scenario II is Strategy C or D, which would yield a payoff of Rs.313333. The payoff that the decision maker obtains through Strategy A is Rs.22000. So the regret associated with Strategy A when the realized scenario is Scenario II is Rs.(313333 − 220000) = Rs.93333. The regrets corresponding to each strategy-scenario pair is given in Table 3.5.

What is regret?

Table 3.5: Regrets for different strategy-scenario pairs

	Scenarios				Maximum
Strategy	I	II	III	IV	Regret
A	Rs.0	Rs.93333	Rs.10000	Rs.35000	Rs.93333
B	Rs.0	Rs.46666	Rs.26667	Rs.6667	Rs.46666
C	Rs.26667	Rs.0	Rs.36667	Rs.16667	Rs.36667
D	Rs.26667	Rs.0	Rs.50000	Rs.30000	Rs.50000
E	Rs.65000	Rs.93333	Rs.0	Rs.35000	Rs.93333
F	Rs.85000	Rs.58333	Rs.20000	Rs.0	Rs.85000
G	Rs.85000	Rs.58333	Rs.20000	Rs.0	Rs.85000

For each decision strategy therefore, there are four regret values, one corresponding to each scenario. Since the decision maker does not have any information about the likelihood of any of the scenarios being realized, the decision maker chooses the largest of these regrets to represent the regret corresponding to a particular strategy. This value is shown in the last column of Table 3.5 and gives the maximum amount by which the payoff from a particular strategy will be off from the optimal payoff if that strategy is adopted.

Obviously, the decision maker would like to minimize the maximum regret, and chooses that strategy for which the value of the maximum regret is the minimum. In this example therefore, the decision maker will choose Strategy C if they adopt the min-max regret approach.

If one adopts a decision making approach using the min-max regret strategy then one can incorporate a reason for sub-optimality of a decision which is not accounted for in the other two approaches. Consider for example the regret associated with Strategy F in Scenario I. In Strategy F the decision maker decides to produce 300 tables and 300 chairs. In Scenario I, the amounts of wood and labor available are 2000 m^2 and 1600 labor-hours respectively, so that even after producing the 600 articles, she has 500 m^2 of wood and 400 labor-hours remaining unutilized. Her regret in this case is therefore due to a loss in opportunity to create more tables and chairs. (She cannot produce more of tables and chairs, since the other material that she needs for further production, like paint, varnish, and fabric are not available with her.) This opportunity loss factor is not incorporated in either the maximax approach or the maximin approach.

3.3 Decision Making Under Risk

When a decision maker is taking decisions under risk (as opposed to uncertainty) they have a fair idea about the chances of each of the scenarios being realized. In other words, they know the probabilities of occurrence of various scenarios. Under this condition, it is easier to combine the vector of payoffs under different scenarios that each decision strategy entails into a single number which can represent the effectiveness of a decision strategy.

Expected value approach

Consider a decision strategy for a decision maker which gives rise to a series of payoffs corresponding to different scenarios that can unfold. If the decision maker knows the probability of the different scenarios, then the decision maker can obtain a weighted sum of the payoffs from different scenarios in which each payoff is weighted by the probability of the corresponding scenario. This weighted sum is called the expected value of the distribution of payoffs. For example, if Scenarios I through IV occurred with probabilities 0.4, 0.3, 0.1, and 0.2 respectively, then the expected payoff associated with Strategy A is

$$0.4 \times \text{Rs.}340000 + 0.3 \times \text{Rs.}220000 + 0.1 \times \text{Rs.}265000 + 0.2 \times \text{Rs.}220000 = \text{Rs.}272500.$$

Similar calculations show that the expected payoffs of the other six strategies are Rs.290500, Rs.290833, Rs.286833, Rs.247500, Rs.255000, and Rs.255000 respectively. The decision maker chooses the scenario that maximizes expected payoff, i.e., chooses Strategy C whose expected payoff is Rs.290833.

When taking decision under risk, the most common strategy used is that of maximizing expected payoffs (or minimizing expected costs for cost minimization problems). A part of the reason for doing so is that expected values is one of the best known measures for handling uncertain situations, and is thus well-understood by all parties involved in decision making.

Why not expected regret?

Using expected regret values would appear to be a natural method of choosing an appropriate strategy. However, this is not used in practice, as the prescription of this strategy matches that of expected payoff maximization.

To see why this is the case, let us consider comparing two strategies A and B. Suppose that there are k scenarios, with the probability of Scenario j occurring as p_j. Let the payoffs for the two strategies in each of the k scenarios be $a_1, a_2, \ldots, a_k$ and $b_1, b_2, \ldots, b_k$ respectively. Further let the maximum payoffs possible in each of the scenarios be $m_1, m_2, \ldots, m_k$. Note that these values are independent of the strategy chosen by a decision maker.

Now suppose we prefer Strategy A over Strategy B using the expected payoff maximization approach. Denoting the expected payoffs of Strategies A and B with $EP(A)$ and $EP(B)$ respectively, we have

$$EP(A) > EP(B), \text{ i.e., } \sum_{j=1}^{k} a_j p_j > \sum_{j=1}^{k} b_j p_j.$$

Now the regret for Strategy A under the j-th scenario is $(m_j - a_j)$ while that for Strategy B is $(m_j - b_j)$. So the expected regret $ER(A)$ for Strategy A is

$$ER(A) = \sum_{j=1}^{k} (m_j - a_j) p_j = \sum_{j=1}^{k} m_j p_j - EP(A)$$

and that for Strategy B is

$$ER(B) = \sum_{j=1}^{k} m_j p_j - EP(B).$$

Since $EP(A) > EP(B)$, it follows that $ER(A) < ER(B)$ which implies that if the decision maker's objective was to minimize expected regret, they would have chosen Strategy A over Strategy B.

So we see that the strategy which has the highest expected value has the lowest expected regret, and hence decisions using the expected value criterion match those using the expected regret criterion. Since expected values are easier to understand and explain than expected regrets, the expected regret criterion is not used in practice.

CHAPTER 4

Multistage Decision Making Under Risk

4.1 Introduction

Consider the example provided in Section 3.1. In the example, the decision maker has to make her decision in one time point; the product mix that she needs to aim for, and whether to produce tables first, or whether to produce chairs first. In the majority of decision making scenarios, the decision making is more complex. It involves making multiple decisions at different points in time, and decisions made earlier have an effect on the payoffs of decisions made later. In this chapter we study such multi-stage decision making situations.

Assumptions.

We make two assumptions in this chapter. The first is that these situations are of stochastic uncertainty, and each uncertain event is associated with a probability of encountering it. The second assumption is that we evaluate different strategies of the decision maker in terms of the expected value criterion described in Section 3.3. Although these assumptions are generally accepted in practice, it is possible to consider situations in which they are not appropriate. The methods for dealing with such violations are not covered in this chapter.

In order to understand multi-stage decision making processes, consider the following decision problem faced by the production manager of a company X. The company produces a product which is seeing large increases in market demand. The company's current production facility is not enough to meet the demand for the product. In addition to the option of asking the production staff to work overtime, the production manager has two more options to consider. The first option is to subcontract the manufacturing of additional units of the product to another company, Y. The manager feels that there is an 80% chance that Y will be able to fill the additional demand reliably. In that case, X will make a profit of Rs.12 lakhs over what it is currently making over the life-cycle of the product. If Y cannot fill the demand reliably (i.e., their performance is erratic), then X loses Rs.4 lakhs due to various contractual obligations. The second option is for the production manager to approach the company's board asking for permission to augment the production facility. The manager feels that the chance of approval is 70%. If the board approves the request, the manager expects an additional profit of Rs.14 lakhs over the life-cycle of the product. If the board rejects the request, the production manager can either continue with his current facility and pay overtime to his staff, or subcontract the production to company Y. In case he goes for overtime payment, he expects to earn an additional profit of Rs.5 lakhs over the life-cycle of the product. If he subcontracts production to Y at this stage, he expects the additional profit to be Rs.10 lakhs if Y fills the product reliably, and the expected loss to be Rs.5 lakhs if they do not. The decision making problem is one of figuring out what the production manager should do.

A solution to a decision problem must specify a course of action in all scenarios, and so even though the immediate decision that the production manager needs to take is to choose between going for overtime, or for subcontracting, or for approaching the board with a proposal for capacity augmentation, solution alternatives for the manager are the following.

Strategy A: Proceed with same capacity and overtime payment.

Strategy B: Subcontract to company Y.

Strategy C: Approach the board with a proposal to augment capacity, and if they reject the proposal go for paying overtime.

Strategy D: Approach the board with a proposal to augment capacity, and if they reject the proposal, subcontract to company Y.

At the end of the life-cycle of the product, the production manager will find himself in one of the following scenarios.

Scenario I: No proposal was made, and the production staff worked overtime. (Additional profit was Rs.5 lakhs.)

Scenario II: No proposal was made, company Y's service was used, and company Y delivered reliably. (Additional profit was Rs.12 lakhs.)

Scenario III: No proposal was made, company Y's service was used, and company Y delivered erratically. (Loss of profit was Rs.4 lakhs.)

Scenario IV: A proposal was made and accepted, so that the production facility was augmented. (Additional profit was Rs.14 lakhs.)

Scenario V: A proposal was made and rejected, and the production staff worked overtime. (Additional profit was Rs.5 lakhs.)

Scenario VI: A proposal was made and rejected, company Y's service was used, and company Y delivered reliably. (Additional profit was Rs.10 lakhs.)

Scenario VII: A proposal was made and rejected, company Y's service was used, and company Y delivered erratically. (Loss of profit was Rs.5 lakhs.)

Note that every decision scenario will not be observed for each of the decision strategies. For example, if the production manager decides to adopt strategy B, he will only see one of Scenarios B and C. If we construct a payoff table for this problem, the table will be

	Scenario						
Strategy	I	II	III	IV	V	VI	VII
A	5						
B		12	−4				
C				14	5		
D				14		10	−5

In addition, the probability of occurrence of each of the scenarios will be different for each of the decision strategies.

Hence there is a need for a more elegant way of representing and solving multi-stage decision problems. We will see that method in the next section.

4.2 Representing multi-stage decision problems

Since the payoff table is a cumbersome way of representing multi-stage problems, one uses a more elegant "graphical" method to represent such problems. The representation is over time; the left hand side of the diagram representing the present while the right hand side representing the future. Each sequence of decisions and events that make up the realization of the effects of a decision strategy is represented as a path in the diagram from the present to the future.

A diagram that shows all possible realizations in a decision problem is one which starts at a single point on the left corresponding to the present situation (before the decision making process starts) and then fans out to the right, with paths representing different possible realizations. For the decision problem that we consider, this diagram is shown in Figure 4.1. Each scenario represents

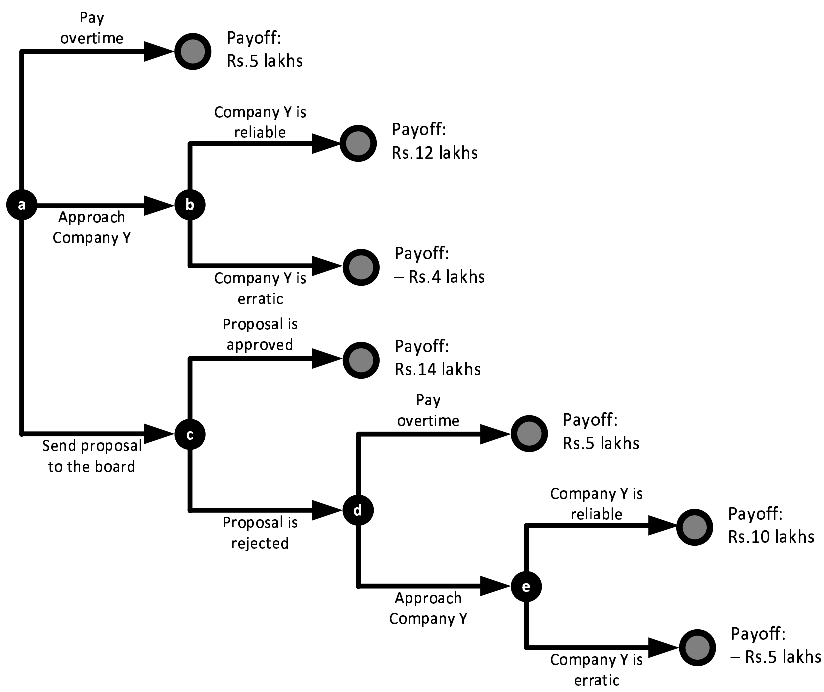


Figure 4.1: A diagram showing all realizations for the problem

a state at the end of the problem reached by a path (i.e., a realization) from the left to the right. The payoff to be achieved in the scenario is also marked in the diagram.

At every junction in the tree (represented by letters from 'a' through 'e') there is a possibility of choosing one of multiple realizations. There is however a difference in the way paths are followed at each junction. In junctions 'a' and 'd' it is up to the decision maker, the production manager in this case, to choose which of the paths he wishes to take. In the other three junctions, the choice of paths is through a random process, and the decision maker cannot guide the choice. In the problem situations that we consider, the choice of paths happen with pre-specified probabilities. If we add these bits of information, i.e., the distinction between junctions where the decision maker can choose paths and where he cannot, and the probabilities with which paths are chosen at junctions where the decision maker cannot choose the paths, we obtain an enhanced version of the diagram in Figure 4.1 called a decision tree. By convention, we represent junctions in which the decision maker can choose paths with squares and call them decision nodes, while the other junctions are represented by circles and are called chance nodes or event nodes. The probabilities of paths being chosen at each chance node is also added to the diagram to form the decision tree. Figure 4.2 shows the decision tree for the production manager's problem.

What is a decision tree?

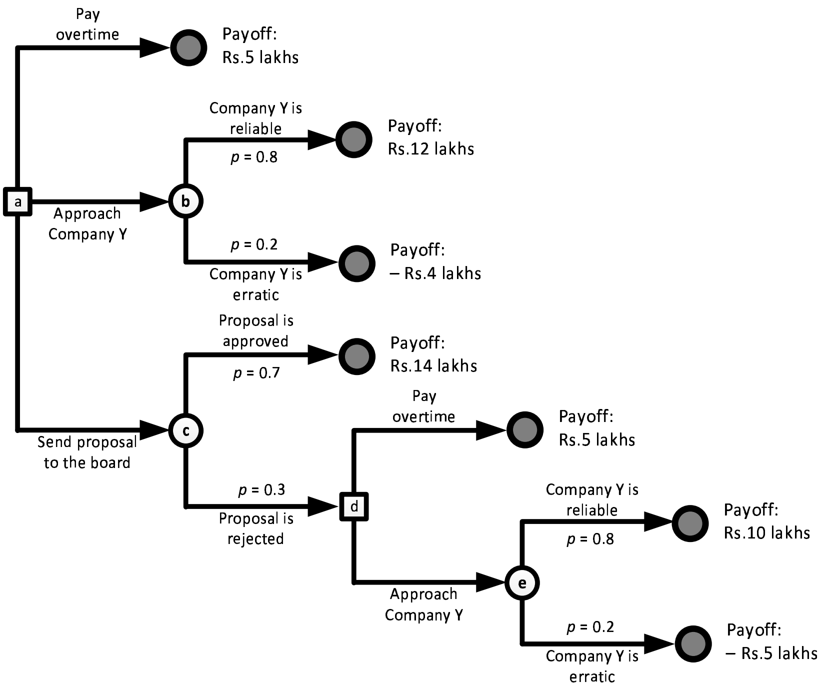


Figure 4.2: The decision tree for the problem

Note that a decision tree is merely a graphical representation of a multi-stage decision problem with all data for the problem represented conveniently.

In the next section we will see how to analyze decision trees and choose an optimal decision strategy.

4.3 Solving multi-stage decision problems

Given the decision tree representation of the production manager's problem, we now solve the problem and arrive at the optimal decision for the manager. Recall that by our assumption, the production manager uses the expected value criterion for comparing decision alternatives and choose a decision that maximizes the expected payoff.

Any decision strategy for this problem answers at least one of two questions; (1) the immediate question of whether to pay overtime, subcontract, or make a proposal to the board (as depicted by decision node 'a' in the decision tree of Figure 4.2, and (2) the question of what to do if a proposal is made and rejected by the board (as depicted by decision node 'd'. Of course for some decision strategies (such as strategy A and strategy B) the second question does not arise, but since we do not know beforehand whether one among strategies A and B would be optimal, we need to answer both the questions. In other words, a solution to a decision tree prescribes an optimal choice at each decision node, whether it is reached or not.

Now consider the optimal choice at decision node 'a'. Since the production manager is an expected value maximizer, the choice that he will make at node 'a' is the one among the three alternatives that will earn him the maximum expected payoff. However, we do not know the expected payoff in two of the three choices (they lead to chance nodes 'b' and 'c'), and so to arrive at this decision, we need to compute the expected payoffs at 'b' and 'c'. It is obvious therefore that our analysis of the decision tree should start at the right hand side of the tree and roll back towards the left, making appropriate decisions whenever possible. Consider node 'e' for example. This is a chance node, and hence the expected payoff at 'e' is Rs.$(10 \times 0.8 - 5 \times 0.2)$ lakhs i.e., Rs.7 lakhs. Similarly, the expected payoff at chance node 'b' is Rs.$(12 \times 0.8 - 4 \times 0.2)$ lakhs or Rs.8.8 lakhs. Once we calculate the expected payoff at 'e', we are in a position to determine the decision at node 'd'. At node 'd' the choice is between paying overtime and earning a payoff of Rs.5 lakhs and of approaching company Y and earning an expected payoff of Rs.7 lakhs. For an expected payoff maximizer, the choice of course is the latter, and hence if the production manager makes a proposal to the board and the board rejects it, it is optimal for the production manager to approach company Y. Having made this decision, we know that should the decision maker reach node 'd' then his expected payoff is Rs.7 lakhs (the maximum of Rs.5 lakhs and Rs.7 lakhs). This allows us to compute the expected payoff at chance node 'c' as Rs.$(14 \times 0.7 + 7 \times 0.3)$ lakhs or Rs.11.9 lakhs. Having computed the expected payoff at node 'c', we can combine this information with the expected payoff at node 'b' and the given payoff at the other node to ascertain that at node 'a', the optimal strategy is to send the proposal of capacity augmentation to the board. And combining the optimal decisions at decision nodes 'a' and 'c', we come to the conclusion that the optimal strategy for the production manager is strategy C, i.e., to send in a proposal for capacity augmentation to the board, and in case the board rejects the proposal, to approach company Y with a subcontracting offer. The expected payoff for this strategy is Rs.11.9 lakhs.

At this point we should be clear about out interpretation of the expected payoff value of Rs.11.9 lakhs. This figure is an expected value. So this means that if the production manager faced the same decision problem a large number (tending to an infinite number) of times, applied strategy C every time, and computed the average of all the payoffs he received, the average payoff would be Rs.111.9 lakhs. Every time he takes the decision, he stands to earn Rs.14 lakhs with probability 0.7, Rs.10 lakhs with probability $0.3 \times 0.8 = 0.24$, and lose Rs.5 lakhs with probability $0.3 \times 0.2 = 0.06$.

4.4 Sensitivity analysis

While computing an optimal decision strategy for the production manager, we have chosen a 0.7 probability value for a proposal made to the board being accepted, and a 0.8 probability value of company Y delivering the product reliably. In any practical scenario, it is almost certain that these probabilities do not have these exact values, although we expect them to be close to the chosen values. An important part of analyzing decision trees is to find out how robust the optimal decision is to changes in these parameters, or in other words, for what ranges of these probability values does the optimal decision strategy remain optimal. Notice that we do not make any claim about the expected payoff from the optimal strategy, but merely ask for the strategy not to be worse than any other strategy in expected value terms. Such an analysis, when we allow the value of exactly one of the parameters to change, is called sensitivity analysis. In the remainder of this section, we demonstrate how to perform sensitivity analysis on decision trees.

Consider any probability value associated with the production manager's decision problem. This could either be the probability that a proposal of capacity expansion made to the board is accepted, or the probability that company Y delivers reliably. Let us denote the chosen probability value by p. Now, since there are two decision nodes 'a' and 'd' in the decision tree in Figure 4.2, any change in the optimal strategy will be reflected as a change in the optimal decision at one or both of these decision nodes. Now let us suppose that submitting a proposal of capacity expension to the board remains the optimal decision at node 'a' if $p \in \left[p_l^a, p_h^a\right]$, and approaching company Y remains the optimal decision at node 'd' if $p \in \left[p_l^d, p_h^d\right]$. Then the range of p for which strategy C remains optimal is

$$\text{range} = \left[\max\left\{p_l^a, p_l^d\right\}, \min\left\{p_h^a, p_h^d\right\}\right].$$

Consider the probability value $p = 0.7$ for a proposal on capacity expansion made to the board being accepted. When we find the allowable range of p, we consider the other probability (that of company Y delivering reliably) to remain unchanged. In that case, there is no change in the expected payoff at node 'e' and the decision at node 'd' remains unchanged whatever be the value of p. Hence we can conclude that the optimal decision at node 'd' for Strategy C remains unchanged if $p \in [0,1]$. The expected payoff at node 'b' is not affected by changes in p and remains unchanged at Rs.8.8 lakhs. The expected payoff at node 'c', calculated as Rs.$14p+7(1-p)$ lakhs = Rs.$7p+7$ lakhs obviously changes with the value of p. The decision at node 'a' will remain unchanged when this payoff does not fall below max{Rs.5 lakhs, Rs.8.8 lakhs} = Rs.8.8 lakhs. So the decision remains unchanged when $7p+7 \geq 8.8$ or $p \geq 0.257$. Therefore the optimal decision at node 'a' for Strategy C remains unchanged if $p \in [0.257,1]$. Combining these values we see that Strategy C remains unchanged when the probability for a proposal made to the board being accepted is in the range [0.257,1].

Next consider the probability value $r = 0.8$ of company Y delivering reliably. Similar to the earlier case, when we consider changes in this value, we assume that the probability that the board accepts a proposal for capacity expansion remains fixed at 0.7. When the value of r changes, the expected payoffs at nodes 'b' and 'e' obviously change. In terms of r, the expected payoff at node 'b' is Rs.$12r-4(1-r)$ lakhs = Rs.$16r-4$ lakhs and that at node 'e' is Rs.$15r-5$ lakhs. In the optimal strategy before the change, i.e., in Strategy C, the decision at node 'd' is to approach company Y. So as the value of r changes, the expected payoff at node 'd' also changes to Rs.$15r-5$ lakhs. The expected payoff at node 'c' changes since the expected payoff at node 'd' changes, and becomes Rs.$11.2+0.3(15r-5)$ lakhs = Rs.$4.5r+9.7$ lakhs. The optimal decision at at node 'd' before the value of r changes remains optimal as long as the expected payoff at node 'e' does not fall below Rs.5 lakhs, i.e., $15r-5 \geq 5$, i.e., $r \in [0.667,1]$. Similarly, the optimal decision at node 'a' before the value of r changes remains optimal as long as $4.5r+9.7 \geq \max\{5, 16r-4\}$. As this never happens, the optimal decision at node 'a' remains optimal when

$r \in [0,1]$. Therefore, strategy C remains unchanged when the probability of company Y acting reliably is in the range [0.667,1].

It is important to reiterate a few points at this stage. First, when performing sensitivity analysis, the objective is to find the range of probability values for which a complete decision strategy remains optimal. Hence it is not sufficient to account only for changes at any one decision node. Note that the decision at a decision node can change only when the payoffs for at least one of the alternatives at that node changes. So if a probability value changes, then decisions "downstream" to the position where that probability value occurs in the decision tree do not change. Also, any decision "upstream" to that position can change only if there are no intermediate decision nodes that filter out the effect of the change (by choosing a decision alternative which does not lie on the same path.)

4.5 Value of information

Recall that the decision maker's criterion for evaluating alternatives is one of maximizing expected payoffs, which means that implicitly they averaging returns from a large number of identical situations. However, since the present decision situation is not likely to be repeated identically several times, the decision maker can benefit with any expert advice that pertains to the situation at hand. For example, in the decision problem that we consider in this chapter, the decision maker is more concerned about the board's response to this particular capacity expansion proposal rather than to the board's response for capacity expansion proposals in general. Also, he is more concerned about the ability of company Y to deliver this particular product reliably rather than that of company Y's general ability to deliver products reliably. Therefore, the decision maker (the production manager in our case) will assign a value to expert advice about their current situation. In this section, we describe a method to compute the worth of such advice. We first consider the utopian case of an expert who is perfect, i.e., never makes mistakes, and then consider more practical fallible experts. In both cases, we compute the worth of the advice as the increase in expected payoffs that the decision maker achieves with the expert's help.

Expected value of perfect information

Consider a perfect expert. In our example, suppose that this expert is infallible at predicting whether or not company Y will deliver the product reliably. How much is the advise of such an expert worth to the production manager?

Note that the advise of the expert is worth something to the production manager *before* he takes the decision of whether or not to approach company Y. After the decision has been taken the expert's advice is of no practical use since the production manager cannot take advantage of the advice to make better decisions. Also note that since the expert is always right, and company Y delivers reliably 80% of the time, the expert will say that company Y will deliver reliably with a probability of 0.8. However, in a given situation they will respond with a definite YES or a definite NO, and not make probabilistic statements. Also note that since the expert is always right, the decision maker will not second-guess the expert, and will take the expert's decision as the truth while taking decisions. (This point seems trivial in the present case, but will be more consequential when we consider fallible experts.)

This is an important point to keep in mind.

We know that without the expert's advice, the optimal strategy for the production manager is strategy C which yields an expected payoff of Rs.11.9 lakhs (see Section 4.3). We also know that in case the expert says that company Y will deliver reliably, the manager does not need to consider the option of paying overtime, since the payoff from approaching company Y will be more than that from paying overtime. Additionally we know that if the expert says that company Y will not deliver reliably, then the manager does not need to consider the option of approaching them, since the payoff from paying overtime is higher. So when there is an option of asking the expert, the decision tree for the decision process is the one shown in Figure 4.3.

Solving the decision tree we find that the expected payoff if the decision maker decides to consult the expert is Rs.12.5 lakhs. In this diagram, the optimal strategy is to consult the expert (at decision node 'a') and to approach the board with a proposal no matter what the expert says (at decision nodes 'c' and 'd'). If the board rejects the proposal, they should approach company Y if the expert says that they will be reliable in their deliveries, and pay overtime otherwise. The reason that the expert's advice earns the production manager a higher expected payoff is that the expert prevents the production manager from ending up in scenarios in which he would lose money.

Since the expected payoff with access to the expert is Rs.12.5 lakhs while that without access to the expert is Rs.11.9 lakhs, the worth of the expert's opinion is calculated to be Rs.12.5 lakhs - Rs.11.9 lakhs = Rs.0.6 lakhs. This figure is an expected payoff figure obtained by assuming perfect information from the expert. So this figure is called the Expected Value of Perfect Information (EVPI).

What is EVPI?

Expected value of sample information

Next consider an imperfect expert. The imperfection of the expert can lead the decision maker to two types of mistakes. The first type of mistake is when

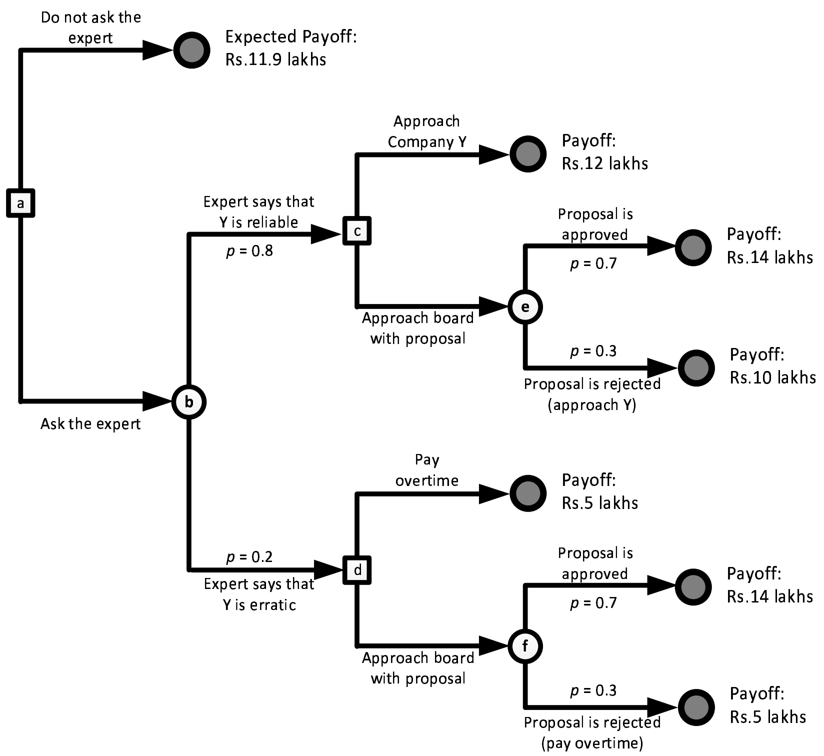


Figure 4.3: The decision tree with access to perfect information

an expert states in error that company Y will deliver reliably when it actually delivers erratically. In such situations, if the production manager follows the expert, then he makes a monetary loss. The second type of mistake is when the expert erroneously states that company Y will be erratic in their delivery when they actually deliver reliably. In this situation, if the production manager follows the expert's advice, then he will prefer paying overtime to approaching company Y and will settle for a lower payoff, thus incurring opportunity losses. Both these errors clearly do not occur for a perfect expert, simply because such experts are infallible. These errors are also the reason why an imperfect expert's advice cannot be worth more than that of a perfect expert.

Why should one take advice from an imperfect expert when we know that an imperfect expert makes mistakes and can advise a decision maker into situations in which the decision maker suffers monetary or opportunity losses? One takes such advice because when the imperfect expert is correct in their decision, they can prevent a decision maker from taking decisions that lead to losses. One expects the latter type of situations to be more frequent than situations in which the expert makes mistakes, and in the expected value sense, a decision maker is better off with an imperfect expert's advice than without it. It is also this reason that suggests that the decision maker follows the expert's advice even though there is a chance that the expert had made a mistake in judgment. The decision tree with the option of asking an imperfect expert is shown in Figure 4.4. Note that we have not determined the probability with which such an expert will say that company Y will deliver reliably.

In order to compute the worth of an imperfect expert's advice, we need to find out the probability p with which the expert will say that company Y will deliver reliably. Data on the expert's past decisions can help us find the value of p using Bayes' rule. Consider for instance in our example, we have an expert who when she says that company Y will deliver reliably is correct 90% of the time, and when she says that they will deliver erratically is correct 80% of the time. The joint probability of company Y delivering reliably and the expert saying that they will is $0.9p$ and the joint probability of company Y delivering reliably and the expert saying that they will not is $(1-0.8)(1-p)$. So in terms of p, the probability that company Y will deliver reliably is $0.9p + 0.2(1-p) = 0.7p + 0.2$. We know this value is actually 0.8, so that $0.7p - 0.2 = 0.8$ or $p = 6/7$. This means that such an expert will say that company Y will deliver reliably $6/7$-th of the time.

Solving the decision tree we find that the expected payoff if the decision maker decides to consult the expert is Rs.12.2 lakhs. The optimal strategy for the production manager in this situation is to ask the expert at node 'a' and submit a proposal to the board at nodes 'e', 'f', 'g', and 'h'. If the board rejects the proposal, the production manager should approach company Y if the expert says that they are reliable, and pay overtime otherwise. In the diagram, the expert's erroneous advice causes the production manager to face a possible monetary loss at chance node 'k', and an opportunity loss at chance nodes 'l' and 'm'.

Since the expected payoff with access to the imperfect expert is Rs.12.2 lakhs while that without access to the expert is Rs.11.9 lakhs, the worth of the expert's opinion is calculated to be Rs.12.2 lakhs - Rs.11.9 lakhs = Rs.0.3 lakhs. As with EVPI, this figure is an expected payoff figure. In practice, since imperfect information is most often obtained through sampling studies, this expected value is called Expected Value of Sample Information (EVSI).

What is EVSI?

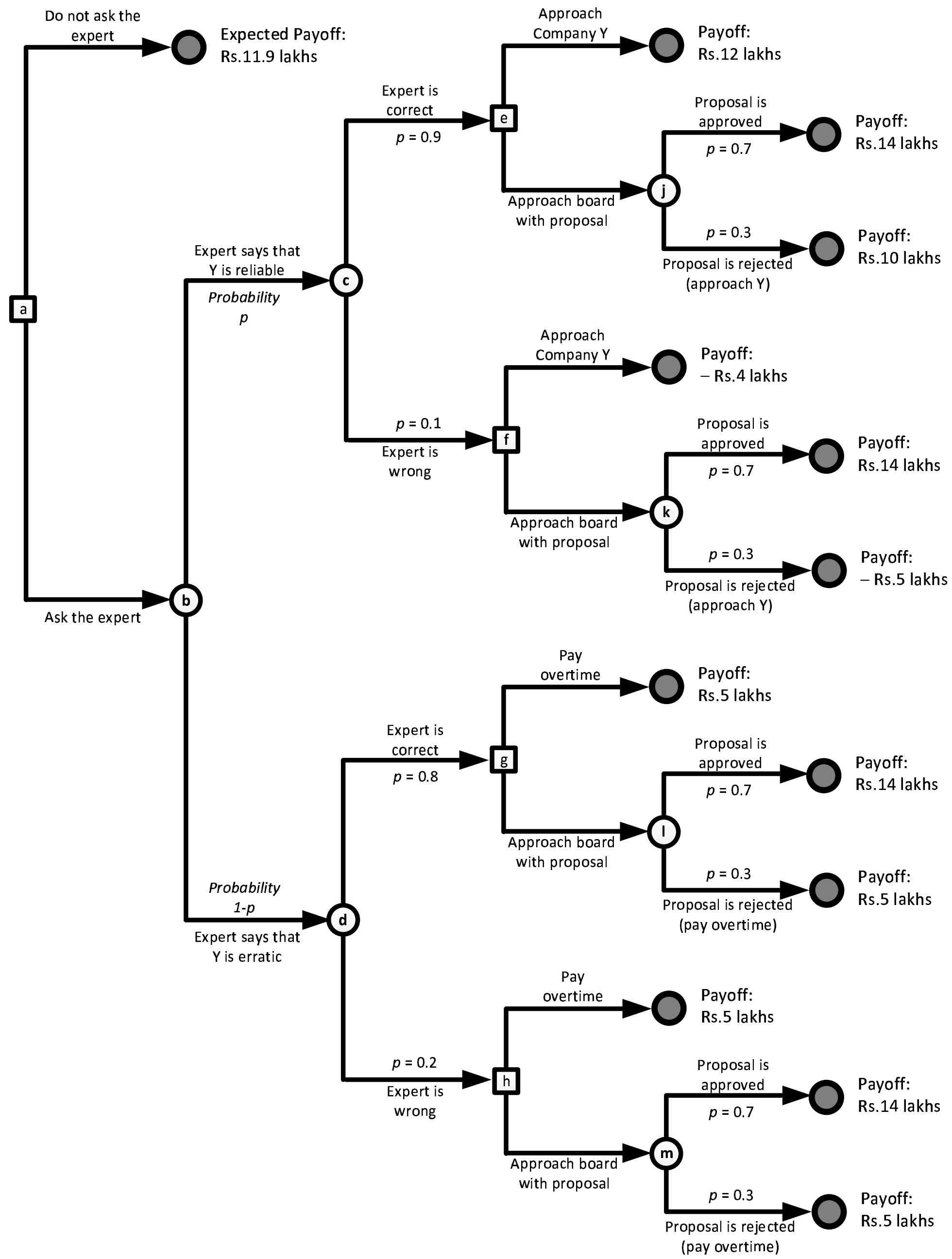


Figure 4.4: The decision tree with access to imperfect information