Sparsity Methods for Systems and Control

Algorithms for Convex Optimization

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ℓ^1 optimization by CVX

ℓ^1 Optimization

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \ \|x\|_1 \quad \text{subject to} \ \ \Phi x = y.$$

The MATLAB CVX code

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cvx_begin
    variable x(n)
    minimize( norm(x, 1) )
    subject to
        y == Phi * x
cvx_end
```

- Useful for a small or middle scale problems
- Not that useful for
  - large-scale problems like image processing
  - real-time applications for control system
- You need to build an efficient algorithm by yourself for your specific problem.

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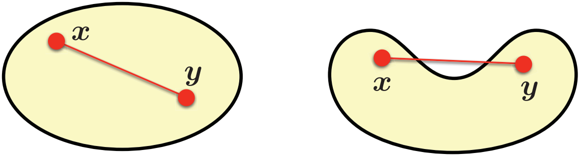
Convex set

Let C be a subset of $\mathbb{R}^n$. C is said to be a convex set if the following inclusion

$$t\boldsymbol{x} + (1-t)\boldsymbol{y} \in C$$

holds for any vectors $\boldsymbol{x}, \boldsymbol{y} \in C$ and for any real number $t \in [0, 1]$.

- Convex and non-convex sets



Effective domain

- The effective domain of a function $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is defined by

$$\mathrm{dom}(f) \triangleq \{\boldsymbol{x} \in \mathbb{R}^n : f(\boldsymbol{x}) < \infty\}.$$

- Indicator function

$$f(\boldsymbol{x}) = \begin{cases} 0, & \text{if } \|\boldsymbol{x}\|_2 \le 1, \\ \infty, & \text{if } \|\boldsymbol{x}\|_2 > 1. \end{cases}$$

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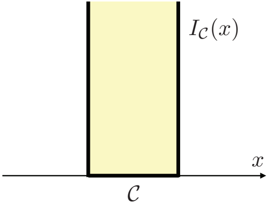

Convex function

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Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a proper function. The function f is said to be a convex function if the following inequality

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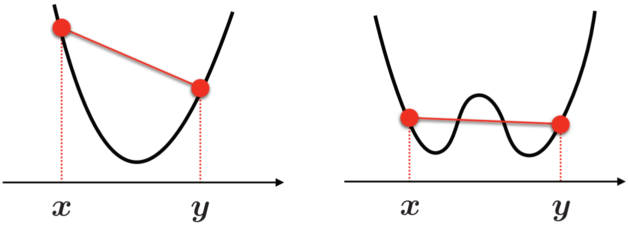
holds for any vectors $\boldsymbol{x}, \boldsymbol{y} \in \mathrm{dom}(f)$ and for any real number $t \in [0,1]$.

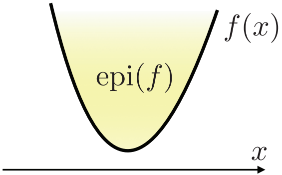
- Convex and non-convex functions



Epigraph

- The epigraph epi(f) of function f is defined by

$$\mathrm{epi}(f) \triangleq \left\{ (\boldsymbol{x}, t) \in \mathbb{R}^n \times \mathbb{R} : \boldsymbol{x} \in \mathrm{dom}(f),\, f(\boldsymbol{x}) \leq t \right\}.$$



- Proper, convex, and closed function

function f	epigraph epi(f)
convex	convex set
closed	closed set
proper	non-empty

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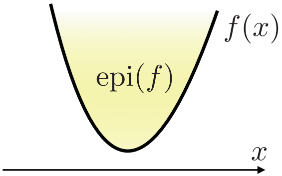

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Convex optimization problem

Convex optimization problem

Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a proper, closed, and convex function, and $C \subset \mathbb{R}^n$ be a closed convex set. Then, a convex optimization problem is a problem to find a vector $\boldsymbol{x}^* \in \mathbb{R}^n$ that minimizes the function $f(\boldsymbol{x})$ over the set $C \subset \mathbb{R}^n$. The problem is briefly written as

$$\underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}} \;\; f(\boldsymbol{x}) \;\; \text{subject to} \;\; \boldsymbol{x} \in C.$$

- The function $f(x)$ is called a cost function or an objective function.
- The set C is called a constraint set or a feasible set.
- The entries of C is called feasible solutions.
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Notation

- Minimum value:

$$\min_{x \in C} f(x).$$

- Minimizer (set):

$$\arg\min_{x \in C} f(x) \triangleq \left\{ x^* \in C : f(x^*) \leq f(x), \ \ \forall x \in C \cap \mathrm{dom}(f) \right\}.$$

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$\min_{\boldsymbol{x} \in C} f(\boldsymbol{x})$ minimum value

$\arg\min_{\boldsymbol{x} \in C} f(\boldsymbol{x})$ minimizer (set)

Global/local minimizers

- Local minimizer: there exists an open set $\mathcal{B}$ that contains a feasible solution $\bar{x} \in C \cap \mathrm{dom}(f)$ such that

$$f(x) \geq f(\bar{x}), \quad \forall x \in \mathcal{B} \cap C.$$

- Global minimizer: a feasible solution $x^* \in C$ that satisfies

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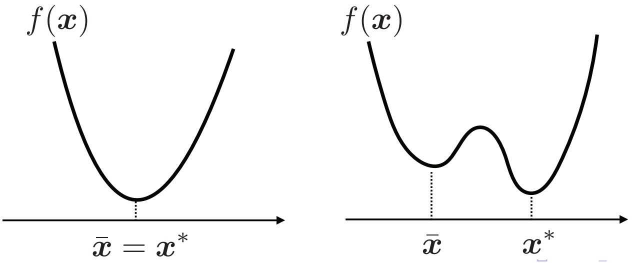

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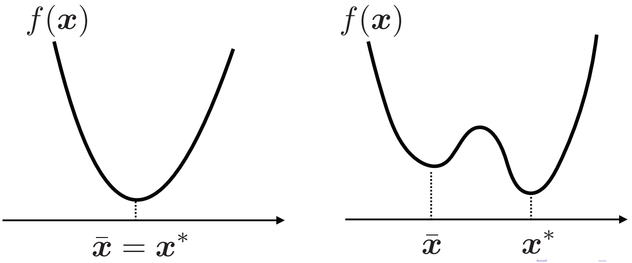

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Theorem

For the convex optimization problem,

$$\underset{x\in\mathbb{R}^n}{\text{minimize}} \ f(x) \ \text{ subject to } \ x \in C.$$

any local minimizer is (if it exists) a global minimizer, and the set of global minimizers is a convex set.

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Strictly and strongly convex functions

- Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a proper function.
- The function f is said to be a strictly convex function if for any $x, y \in \mathrm{dom}(f) \subset \mathbb{R}^n$ with $x \neq y$ and any $t \in (0, 1)$,

$$f\big(tx + (1-t)y\big) < tf(x) + (1-t)f(y)$$

- The function f is said to be a strongly convex function if there exists $\beta > 0$ such that for any $x, y \in \mathrm{dom}(f) \subset \mathbb{R}^n$ and any $t \in [0, 1]$,

$$f\big(tx + (1-t)y\big) \leq tf(x) + (1-t)f(y) - t(1-t)\frac{\beta}{2}\|x - y\|_2^2$$

The constant β is called a *modulus*.

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Strongly convex functions

Theorem

Assume $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ is a proper, closed, and strongly convex function with modulus $\beta > 0$. Then f has the unique minimizer $\boldsymbol{x}^* \in \mathrm{dom}(f)$. That is, for all $\boldsymbol{x} \in \mathrm{dom}(f)$ such that $\boldsymbol{x} \neq \boldsymbol{x}^*$,

$$f(\boldsymbol{x}) > f(\boldsymbol{x}^*).$$

Moreover, for any $\boldsymbol{x} \in \mathrm{dom}(f)$, we have

$$f(\boldsymbol{x}) \geq f(\boldsymbol{x}^*) + \frac{\beta}{2}\|\boldsymbol{x} - \boldsymbol{x}^*\|_2^2.$$

- This is an important property of strongly convex functions.
- This is used to define the proximal operator (see next Section).

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Proximal operator

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Let $f : \mathbb{R}^n \to \mathbb{R} \cup \{\infty\}$ be a proper, closed, and convex function. The proximal operator $\mathrm{prox}_{\gamma f}$ with parameter $\gamma > 0$ is defined by

$$\mathrm{prox}_{\gamma f}(v) \triangleq \underset{x \in \mathrm{dom}(f)}{\arg\min} \left\{ f(x) + \frac{1}{2\gamma} \|x - v\|_2^2 \right\}.$$

- $\gamma = \infty$: Minimizer of $f(z)$:

$$\mathrm{prox}_{\gamma f}(v) = \underset{x \in \mathrm{dom}(f)}{\arg\min} f(x)$$

- $\gamma = 0$: Projection onto $\mathrm{dom}(f)$:

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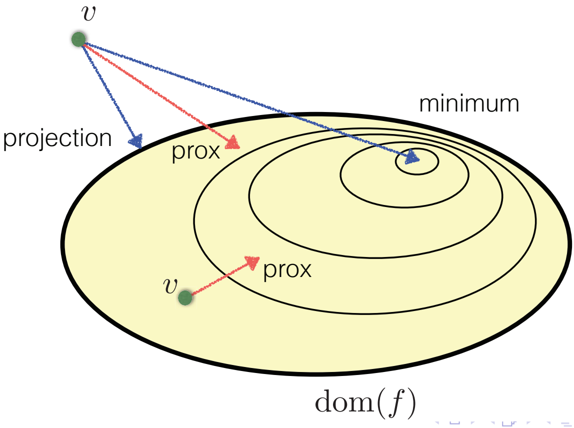

Proximal operator

The "crossing the street" problem.

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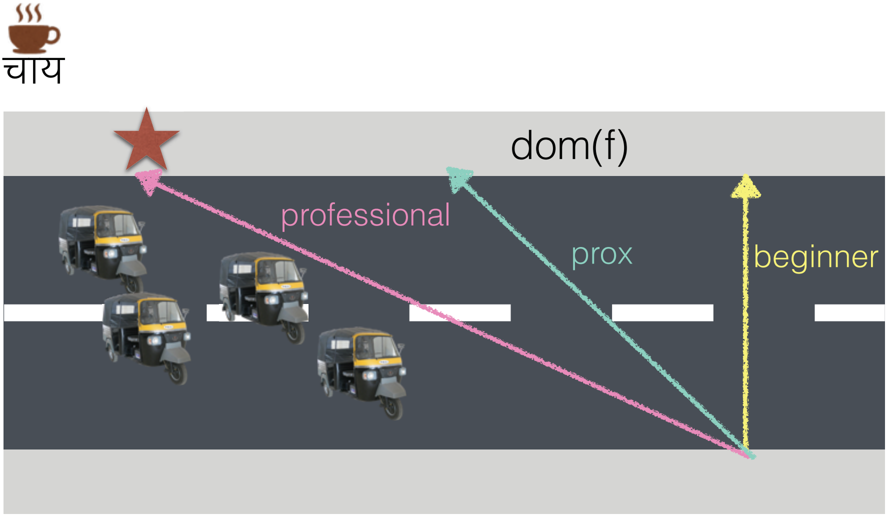

Proximal algorithm

Proximal algorithm

Initialization: give an initial vector $\boldsymbol{x}[0]$ and positive numbers $\gamma_0, \gamma_1, \gamma_2, \ldots$

Iteration: for $k = 0, 1, 2, \ldots,$ do

$$x[k+1] = \mathrm{prox}_{\gamma_k f}(x[k]) = \underset{x \in \mathrm{dom}(f)}{\arg\min} \Big\{ f(x) + \frac{1}{2\gamma_k} \|x - x[k]\|_2^2 \Big\}.$$

- The algorithm minimizes the strongly convex function

$$g_k(x) \triangleq f(x) + \frac{1}{2\gamma_k} \|x - x[k]\|_2^2$$

at each step k, which is an approximation of f that may not be strongly convex.

Proximal algorithm

Proximal algorithm

Initialization: give an initial vector $\boldsymbol{x}[0]$ and positive numbers $\gamma_0, \gamma_1, \gamma_2, \dots$
Iteration: for $k = 0, 1, 2, \dots,$ do

$$x[k+1] = \mathrm{prox}_{\gamma_k f}(x[k]) = \underset{x \in \mathrm{dom}(f)}{\arg\min} \Big\{ f(x) + \frac{1}{2\gamma_k} \|x - x[k]\|_2^2 \Big\}.$$

- The algorithm minimizes the strongly convex function

$$g_k(x) \triangleq f(x) + \frac{1}{2\gamma_k} \|x - x[k]\|_2^2$$

at each step k, which is an approximation of f that may not be strongly convex.

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Convergence theorem of proximal algorithm

Theorem

Suppose that the parameter sequence $\{\gamma_k\}$ satisfies $\gamma_k > 0$ for all k and

$$\sum_{k=0}^{\infty} \gamma_k = \infty.$$

Then, the vector sequence $\{x[k]\}$ generated by the proximal algorithm

$$x[k+1] = \mathrm{prox}_{\gamma_k f}(x[k])$$

converges to one of the minimizers of f for any initial vector $x[0]$.

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Proximable functions

- Proximal operator

$$\mathrm{prox}_{\gamma f}(v) \triangleq \mathop{\arg\min}_{x \in \mathrm{dom}(f)} \left\{ f(x) + \frac{1}{2\gamma} \|x - v\|_2^2 \right\}.$$

  needs to be obtained in a closed form to derive an efficient algorithm.
- A proximable function is a function that has a closed-form proximal operator.
- The following functions are proximable:
  - quadratic functions (including the ℓ^2 norm)
  - indicator functions
  - the ℓ^1 norm

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Quadratic function

- The quadratic function

$$f(\boldsymbol{x}) = \frac{1}{2}\boldsymbol{x}^\top \Phi \boldsymbol{x} - \boldsymbol{y}^\top \boldsymbol{x},$$

  where Φ is a positive-definite matrix.

- The proximal operator is given by

$$\begin{aligned} \mathrm{prox}_{\gamma f}(\boldsymbol{v}) &= \arg\min_{\boldsymbol{x}\in\mathbb{R}^n} \left\{ \frac{1}{2}\boldsymbol{x}^\top \Phi \boldsymbol{x} - \boldsymbol{y}^\top \boldsymbol{x} + \frac{1}{2\gamma}(\boldsymbol{x}-\boldsymbol{v})^\top(\boldsymbol{x}-\boldsymbol{v}) \right\} \\ &= \left(\Phi + \frac{1}{\gamma}I\right)^{-1}\left(\boldsymbol{y} + \frac{1}{\gamma}\boldsymbol{v}\right). \end{aligned}$$

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Indicator function

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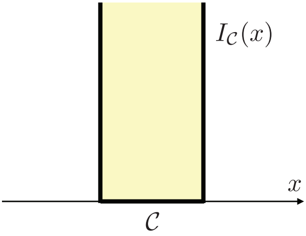
For a subset C in $\mathbb{R}^n$, the indicator function is defined by

$$I_C(\boldsymbol{x}) = \begin{cases} 0, & \boldsymbol{x} \in C \\ \infty, & \boldsymbol{x} \notin C \end{cases}$$

- C: non-empty, closed, and convex $\Rightarrow I_C(\boldsymbol{x})$: a proper, closed, and convex function
- Draw the epigraph of I_C.



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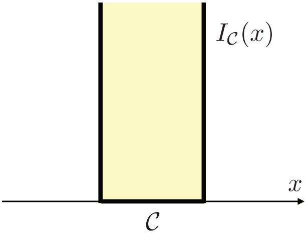

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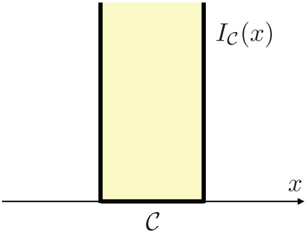

Indicator function

- The proximal operator of $I_C(\boldsymbol{x})$ is given by

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\operatorname{prox}_{\gamma I_C}(\boldsymbol{v}) &= \underset{\boldsymbol{x}\in\mathbb{R}^n}{\arg\min} \left\{ I_C(\boldsymbol{x}) + \frac{1}{2\gamma}\|\boldsymbol{x}-\boldsymbol{v}\|_2^2 \right\} \\
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ℓ^1 norm

- The proximal operator of the ℓ^1 norm $\|\boldsymbol{x}\|_1$ has a closed form

$$\mathrm{prox}_{\gamma\|\cdot\|_1}(\boldsymbol{v}) = S_\gamma(\boldsymbol{v}),$$

where $S_\gamma : \mathbb{R}^n \to \mathbb{R}^n$ is the soft-thresholding operator defined by

$$[S_\gamma(\boldsymbol{v})]_i = \begin{cases} v_i - \gamma, & v_i \geq \gamma, \\ 0, & -\gamma < v_i < \gamma, \\ v_i + \gamma, & v_i \leq -\gamma. \end{cases}$$

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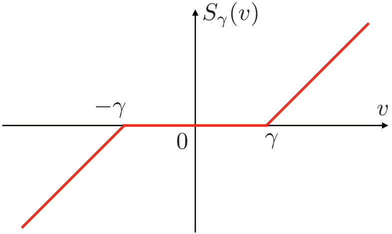

Table of Contents

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$$\underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}}\ \|\boldsymbol{x}\|_1 + I_C(\boldsymbol{x}).$$

- $\|x\|_1 + I_C(x)$ is proper, closed, and convex but not proximable.
  - The proximal algorithm cannot be directly applied.
  - Both functions,

$$f_1(x) \triangleq \|x\|_1, \quad f_2(x) \triangleq I_C(x)$$

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Douglas-Rachford splitting algorithm

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Initialization: give an initial vector $z[0]$ and a parameter $\gamma > 0$
Iteration: for $k = 0, 1, 2, \ldots$ do

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\begin{aligned}
x[k+1] &= S_\gamma(z[k])\\
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Douglas-Rachford splitting for ℓ^1 optimization

- ℓ^1 optimization: $\text{minimize}_{x \in \mathbb{R}^n} \ \|x\|_1 + I_C(x)$
  - $C = \{x \in \mathbb{R}^n : \Phi x = y\}$.
- $f_1(x) = \|x\|_1$ and $f_2(x) = I_C(x)$ are proximable.

$$\begin{aligned}
\text{prox}_{\gamma f_1}(v) &= S_\gamma(v), \\
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$$x[k+1] = \underbrace{\mathrm{prox}_{\gamma f_2}\bigl(x[k] - \gamma \nabla f_1(x[k])\bigr)}_{\triangleq \phi(x[k])}.$$

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$$\begin{aligned}\phi(x) &= \arg\min_{z\in\mathbb{R}^n}\Bigl\{f_2(z) + \frac{1}{2\gamma}\bigl\|z - (x - \gamma\nabla f_1(x))\bigr\|_2^2\Bigr\}\\ &= \arg\min_{z\in\mathbb{R}^n}\Bigl\{\underbrace{f_1(x) + \nabla f_1(x)^\top (z - x)}_{\triangleq \tilde{f}_1(z;x)} + f_2(z) + \frac{1}{2\gamma}\|z - x\|_2^2\Bigr\}.\end{aligned}$$

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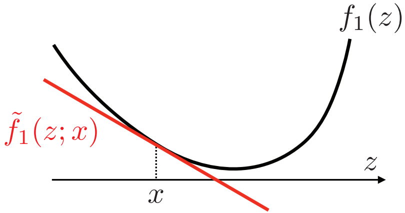

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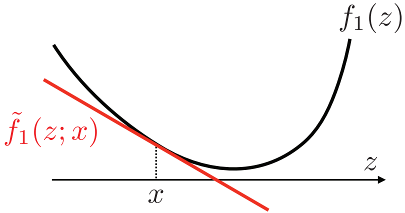

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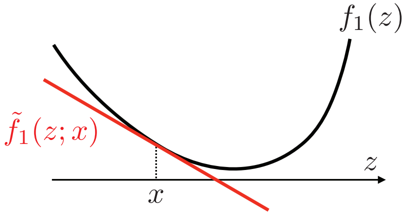

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Convergence analysis

Theorem

Assume the gradient ∇f_1 is Lipschitz continuous over $\mathbb{R}^n$ with Lipschitz constant L. Assume also that the step size γ satisfies

$$\gamma \leq \frac{1}{L}.$$

Then the sequence $\{x[k]\}$ generated by the proximal gradient algorithm converges to an optimal solution x^* at the rate of $O(1/k)$.

- The gradient ∇f_1 is Lipschitz continuous over $\mathbb{R}^n$ with Lipschitz constant L if

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Convergence of ISTA

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Fast ISTA (FISTA)

- Accelerated algorithm of ISTA = FISTA (Fast ISTA)
- The convergence rate is $O(1/k^2)$.

FISTA

Initialization: give initial vectors $x[0]$, $z[0]$, initial number $t[0]$, and parameter $\gamma > 0$
Iteration: for $k = 0, 1, 2, \ldots$ do

$$x[k+1] = S_{\gamma\lambda}\left(z[k] - \gamma\Phi^\top(\Phi z[k] - y)\right),$$

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- Accelerated algorithm of ISTA = FISTA (Fast ISTA)
- The convergence rate is $O(1/k^2)$.

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$$\underset{\boldsymbol{x}\in\mathbb{R}^n}{\text{minimize}} \ \frac{1}{2}\|\Phi\boldsymbol{x}-\boldsymbol{y}\|_2^2+\lambda\|\Psi\boldsymbol{x}\|_1,$$

- The first update in ADMM:

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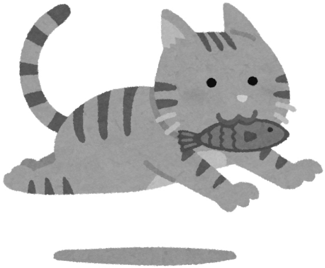
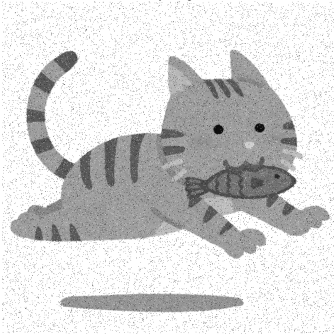
Application: Image denoising

- Remove noise from an image.
- Preserve edges at the same time.
  - Applying a low-pass filter does not work very well.

Original image



Noisy image



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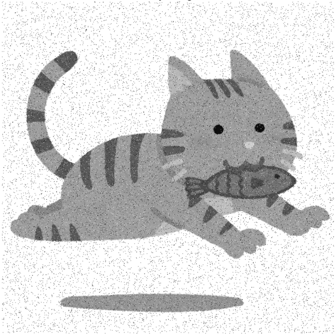

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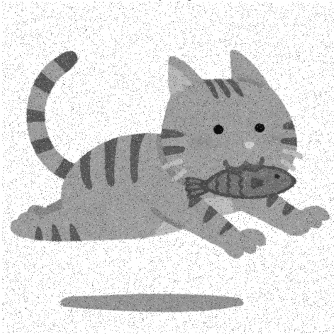

Total variation denoising

- $Y \in \mathbb{R}^{n \times m}$: a noisy image
- Pull out each column vector, say $y \in \mathbb{R}^n$, and solve the following optimization problem, one by one:

$$\underset{x \in \mathbb{R}^n}{\text{minimize}} \ \|x - y\|_2^2 + \lambda \sum_{i=1}^{n} |x_{i+1} - x_i|.$$

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- $\|x - y\|_2^2$: proximity to the data
- $\sum_{i=1}^{n} |x_{i+1} - x_i|$: total variation to preserve edges
- $\lambda > 0$: weight

Total variation denoising

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ADMM for total variation denoising

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- Define $\Phi = I$ and

$$\Psi = \begin{bmatrix} -1 & 1 & 0 & \dots & 0 \\ 0 & -1 & 1 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & -1 & 1 \\ 0 & \dots & 0 & 0 & -1 \end{bmatrix}.$$

- Generalized LASSO

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ADMM for total variation denoising

- The weight λ should be carefully chosen.
- $\lambda = 50$

ADMM for total variation denoising

- The weight λ should be carefully chosen.
- $\lambda = 100$

Restored image

ADMM for total variation denoising

- The weight λ should be carefully chosen.
- $\lambda = 200$

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