Sparsity Methods for Systems and Control

Maximum Hands-off Control

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L^0 norm of a function

- The support of a function $u(t)$, $t \in [0, T]$:

$$\operatorname{supp}(u) \triangleq \{t \in [0, T] : u(t) \neq 0\}.$$

- The L^0 norm of a function $u(t)$:

$$\|u\|_0 \triangleq \mu(\operatorname{supp}(u)),$$

  - $\mu(S)$ is the Lebesgue measure (i.e. the length) of a subset $S \subset [0, T]$.

- L^0 norm: the total length of time durations on which the signal takes nonzero values.

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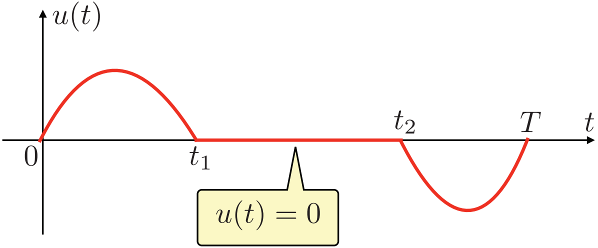
Example: L^0 norm of a function

- The L^0 norm

$$\|u\|_0 = \mu(\mathrm{supp}(u)) = t_1 + (T - t_2) = T - (t_2 - t_1).$$



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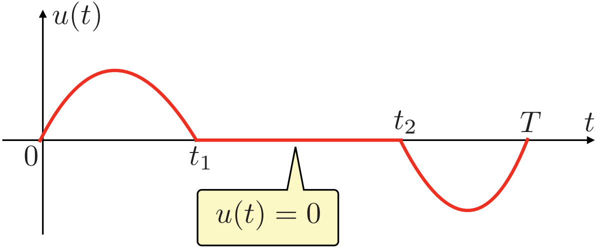

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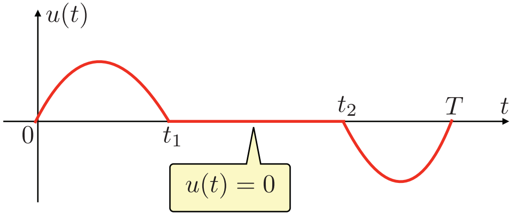
Practical benefits of sparsity in control

- Let us consider the sparse control signal $u(t)$, $t \in [0, T]$.
- Actuators as electric motors needs energy to generate power.
- If the control $u(t)$ is sparse, we can stop energy supply to the actuator over the time interval $[t_1, t_2]$.
- Such a control is called a hands-off control.
  - This is also known as coasting.
- We can also reduce CO or CO2 emissions, noise, and vibrations.



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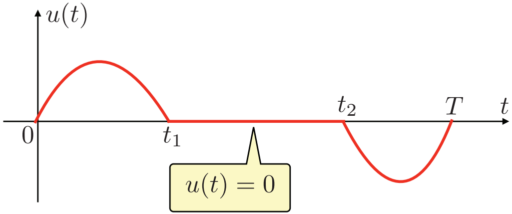

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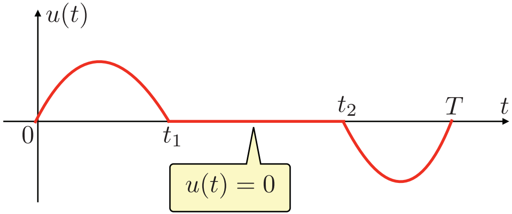

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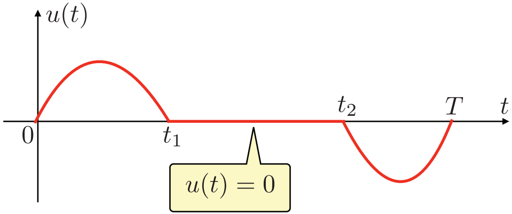

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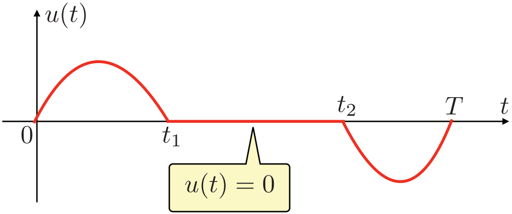

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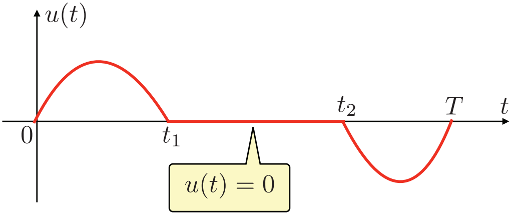

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L^0-optimal control problem

For the linear time-invariant system

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad t \geq 0, \quad \boldsymbol{x}(0) = \boldsymbol{\xi} \in \mathbb{R}^d,$$

find a control $\{u(t) : t \in [0, T]\}$ with $T > 0$ that minimizes

$$J_0(u) = \|u\|_0 = \int_0^T |u(t)|^0 dt$$

subject to

$$\boldsymbol{x}(T) = \boldsymbol{0},$$

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$$\|u\|_\infty \leq 1.$$

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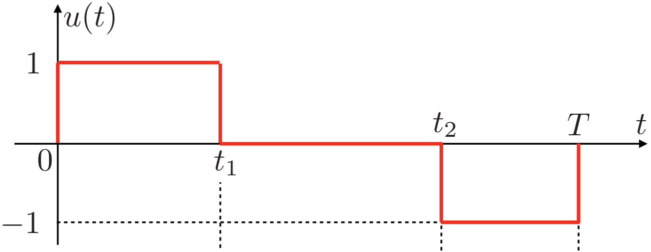
Bang-off-bang control

Theorem

If $(A, \boldsymbol{b})$ is controllable and A is nonsingular, then the L^1 optimal control $u(t)$ takes ± 1 or 0 for almost all $t \in [0, T]$ (if it exists).

- A control that takes ± 1 or 0 is called a bang-off-bang control.



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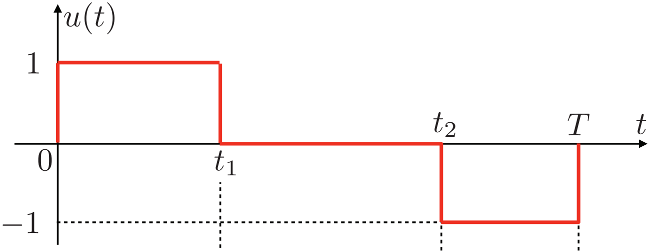


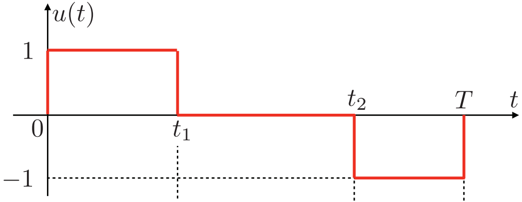
Equivalence between L^0 and L^1 optimal controls

Theorem

Assume that there exists an L^1-optimal control that is bang-off-bang. Then it is also L^0 optimal.

Theorem

Assume that there exists at least one L^1-optimal control. Assume also that $(A, \boldsymbol{b})$ is controllable and A is non-singular. Then there exists at least one L^0-optimal control, and the set of L^0-optimal controls is equivalent to the set of L^1-optimal controls.



Conclusion

- Maximum hands-off control is described as L^0-optimal control.
- Under the assumption of non-singularity, L^0-optimal control is equivalent to L^1-optimal control.
- Maximum hands-off control is a ternary signal that takes values of ± 1 and 0. Such a ternary control is called a bang-off-bang control.