

Sparsity Methods for Systems and Control

Dynamical Systems and Optimal Control

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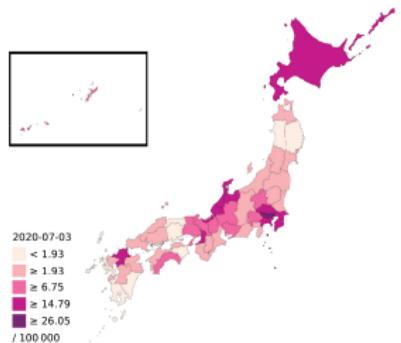
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Dynamical systems

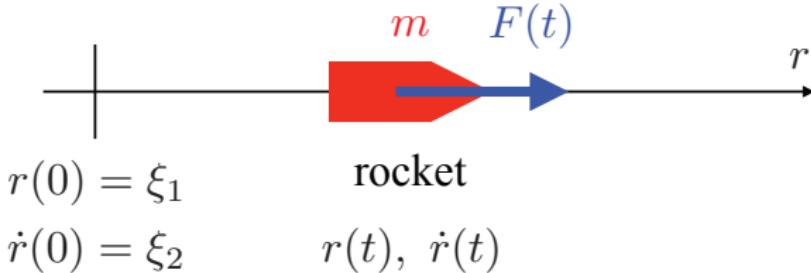
- A dynamical system is a **moving** system.
- industrial products
 - vehicles, airplanes, motors, electric circuits, etc,
- movement of planetary, change of weather, ant swarm, cell movement, fluctuations in stock prices, and spread of virus.
- In Part II, we will learn **sparsity methods for dynamical systems**.



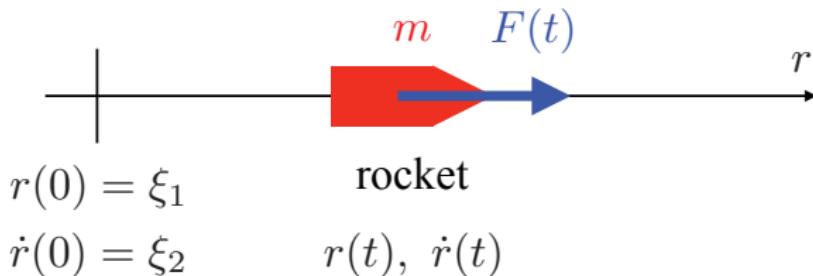
images from wikipedia

Rocket as a dynamical system

- A rocket in the outer space
 - No friction nor gravity acts
- The mass of the rocket is m [kg]
- The position and velocity are $r(t)$ [m] and $v(t) = \dot{r}(t)$ [m/s]
 - The initial condition: $r(0) = \xi_1$, $v(0) = \dot{r}(0) = \xi_2$
- The thrust force is $F(t)$



Differential equation



- The Newton's second law of motion gives

$$m\ddot{r}(t) = F(t), \quad r(0) = \xi_1, \quad \dot{r}(0) = \xi_2.$$

State

- The ordinal differential equation (**ODE**) of the rocket

$$m\ddot{r}(t) = F(t), \quad r(0) = \xi_1, \quad \dot{r}(0) = \xi_2.$$

- Define the **state** $x(t)$ by

$$x(t) \triangleq \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} \triangleq \begin{bmatrix} r(t) \\ \dot{r}(t) \end{bmatrix}.$$

- Then we have

$$\dot{x}(t) = \begin{bmatrix} \dot{r}(t) \\ \ddot{r}(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ m^{-1}F(t) \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}_{\triangleq A} \underbrace{\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}}_{=x(t)} + \underbrace{\begin{bmatrix} 0 \\ m^{-1} \end{bmatrix}}_{\triangleq b} \underbrace{F(t)}_{\triangleq u(t)}$$

State equation

State equation

$$\dot{\boldsymbol{x}}(t) = A\boldsymbol{x}(t) + \boldsymbol{b}u(t), \quad t \geq 0, \quad \boldsymbol{x}(0) = \boldsymbol{\xi} \in \mathbb{R}^d,$$

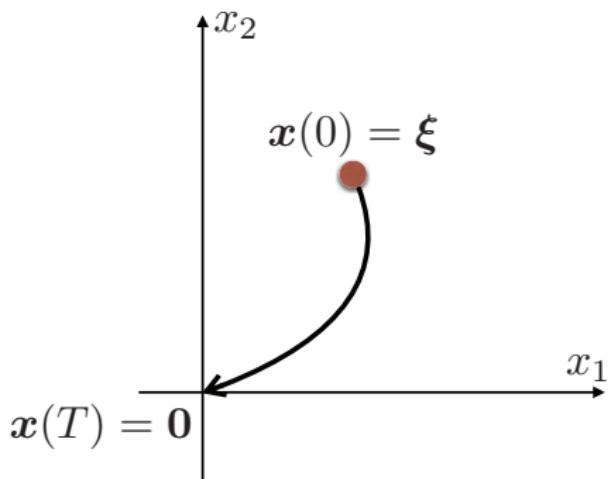
- $\boldsymbol{x}(t)$: state
- $\boldsymbol{\xi} = [\xi_1, \xi_2] = [r(0), \dot{r}(0)]^\top$: initial state
- $u(t)$: control

The solution

$$\boldsymbol{x}(t) = e^{At}\boldsymbol{\xi} + \int_0^t e^{A(t-\tau)}\boldsymbol{b}u(\tau)d\tau, \quad t \geq 0.$$

State transfer problem

- The initial state $x(0) = \xi$ is observed at time $t = 0$.
- Find a control $u(t)$, $0 \leq t \leq T$ that drives the state $x(t)$ from a given initial state ξ to the origin 0 in a given time $T > 0$.



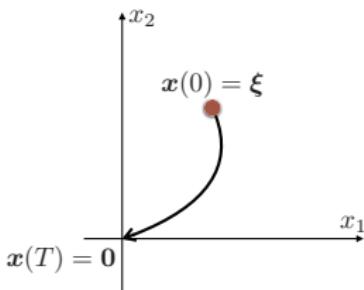
Controllability

- State equation:

$$\dot{x}(t) = Ax(t) + bu(t), \quad t \geq 0, \quad x(0) = \xi \in \mathbb{R}^d, \quad (\star)$$

Controllability

We call the system (\star) is **controllable** if for any initial state $x(0) = \xi \in \mathbb{R}^d$, there exist a time $T > 0$ and control $u(t)$, $0 \leq t \leq T$ such that the state $x(t)$ in (\star) is driven to the origin at time $t = T$, that is $x(T) = 0$.



Controllability

- State equation:

$$\dot{x}(t) = Ax(t) + bu(t), \quad t \geq 0, \quad x(0) = \xi \in \mathbb{R}^d, \quad (\star)$$

Theorem

The dynamical system (\star) is controllable if and only if the following matrix called the **controllability matrix**

$$M \triangleq [b \quad Ab \quad A^2b \quad \dots \quad A^{d-1}b]$$

is **non-singular**.

Controllability

- Suppose a dynamical system is controllable.
- Then for any $\xi \in \mathbb{R}^d$, any $\zeta \in \mathbb{R}^d$, and any $T > 0$, there exist a control $u(t)$, $0 \leq t \leq T$ that achieves

$$x(0) = \xi, \quad x(T) = \zeta.$$

- the shorter the time $T > 0$ is, the larger the magnitude and the shorter the support of $u(t)$ should be.
- The shape of $u(t)$ may approach to something like the Dirac's delta when T approaches to zero.
- This is actually impossible.

T -controllable set

- We usually assume the following limitation on $u(t)$:

$$|u(t)| \leq 1, \quad \forall t \in [0, T].$$

- A control that satisfies this constraint is called an **admissible control**.
- The admissible control can be characterized by the **T -controllable set**.

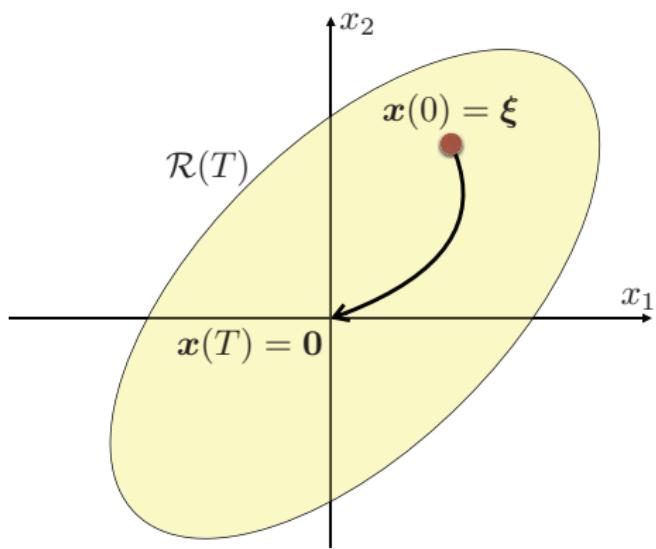
T -Controllable Set

Fix $T > 0$. The set of initial states that can be steered to the origin by some admissible control $u(t)$, $0 \leq t \leq T$ is called the **T -controllable set**. We denote this set by $\mathcal{R}(T)$.

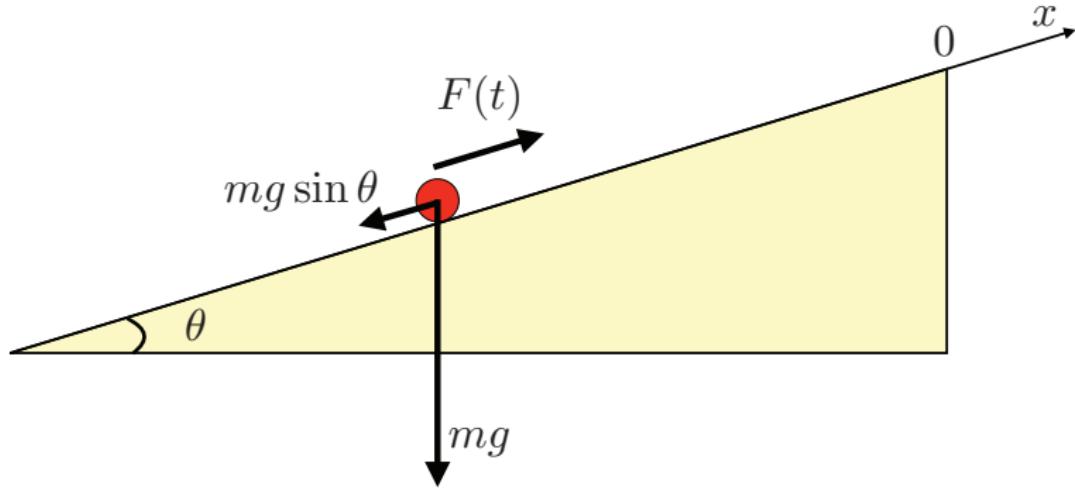
T -controllable set

Theorem

For any $T > 0$, the T -controllable set $\mathcal{R}(T)$ is a bounded, closed, and convex set. Also, if $T_1 < T_2$ then $\mathcal{R}(T_1) \subset \mathcal{R}(T_2)$.



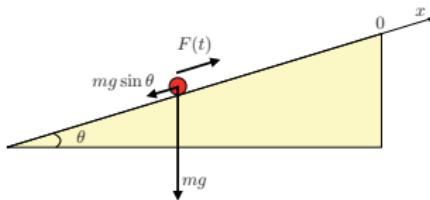
Example



- Move the mass from $x(0) = -\xi$ to $x(T) = 0$ by the force $F(t) \leq 1$.
- ODE

$$m\ddot{x}(t) = F(t) - mg \sin \theta.$$

Example



- Let

$$T^* \triangleq \sqrt{\frac{2m\xi}{1 - mg \sin \theta}}.$$

- We observe that
 - If $T < T^*$ there is **no** admissible control.
 - If $T = T^*$ there is **just one** admissible control $F(t) = 1$, $t \in [0, T^*]$.
 - If $T > T^*$ there is **at least one** admissible control

$$F(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq T^*, \\ mg \sin \theta, & \text{if } T^* < t \leq T, \end{cases}$$

- T^* is called the **minimum time**.

Controllable set

- The minimum time $T^*(\xi)$ from the initial state ξ to the origin is defined as

$$T^*(\xi) \triangleq \inf\{T \geq 0 : \xi \in \mathcal{R}(T)\}.$$

- Is the minimum time finite?
- Define the **controllable set**

$$\mathcal{R} \triangleq \bigcup_{T>0} \mathcal{R}(T).$$

- If $\xi \in \mathcal{R}$, then $T^*(\xi) < \infty$.
- If $\xi \notin \mathcal{R}$, then we write $T^*(\xi) = \infty$.

Controllable set

- Even if the system is controllable, the controllable set \mathcal{R} may not be \mathbb{R}^d .
 - If the system is controllable, and the matrix A is **stable**, that is,

$$\lambda(A) \subset \mathbb{C}_- \triangleq \{z \in \mathbb{C} : \operatorname{Re} z \leq 0\},$$

then $\mathcal{R} = \mathbb{R}^d$.

- If $T_1 < T^*(\xi) < T_2$, then $\mathcal{R}(T_1) \subset \mathcal{R}(T^*(\xi)) \subset \mathcal{R}(T_2) \subset \mathcal{R}$.

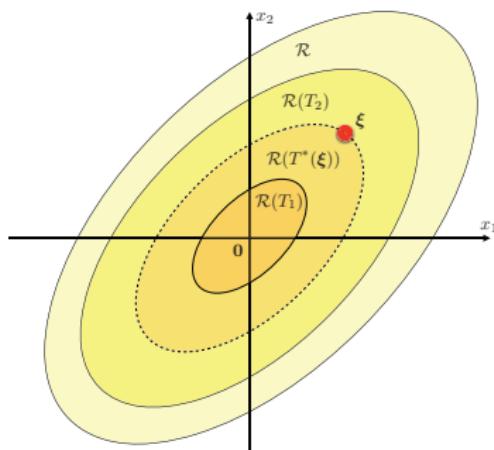


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Feasible control

- Dynamical system

$$\dot{\mathbf{x}}(t) = A\mathbf{x}(t) + \mathbf{b}u(t), \quad t \geq 0, \quad \mathbf{x}(0) = \xi \in \mathbb{R}^d, \quad (\star)$$

- Fix $T > 0$ and assume $\mathbf{x}(0) = \xi \in \mathcal{R}(T)$.
- Then there exists an admissible control $u(t) \in [-1, 1]$ that steers the state from $\mathbf{x}(0)$ to $\mathbf{x}(T) = \mathbf{0}$.
- Such a control is called a **feasible control**.
- Set of all feasible controls is denoted by $\mathcal{U}(T, \xi)$.
- This is given by

$$\mathcal{U}(T, \xi) = \left\{ u \in L^\infty(0, T) : \xi = - \int_0^T e^{-At} \mathbf{b}u(t) dt, |u(t)| \leq 1, \forall t \in [0, T] \right\}.$$

Feasible control

- $\xi \in \mathcal{R}(T) \iff$ there exists an admissible control u such that $u \in \mathcal{U}(T, \xi)$.
- The minimum time $T^*(\xi)$ is represented by

$$T^*(\xi) = \inf\{T \geq 0 : \exists u, u \in \mathcal{U}(T, \xi)\}.$$

- $\xi \in \mathcal{R} \iff T^*(\xi) < \infty \iff$ there exists a final time $T \geq 0$ and an admissible control u such that $u \in \mathcal{U}(T, \xi)$.

Minimum-time control

Minimum-time control

$$\underset{u}{\text{minimize}} \ T \text{ subject to } u \in \mathcal{U}(T, \xi). \quad (\star)$$

- The solution of this optimization is called the **minimum-time control** or **time-optimal control**.

Theorem

Assume $T^*(\xi) < \infty$. Then there exists a minimum-time control $u^* \in \mathcal{U}(T^*(\xi), \xi)$. Moreover, for any $T > T^*(\xi)$, $\mathcal{U}(T, \xi)$ is non-empty.

- $T^*(\xi) < \infty \iff \xi \in \mathcal{R}$

Optimal control problem

Optimal control problem

For the plant modeled by

$$\dot{x}(t) = Ax(t) + bu(t), \quad t \geq 0, \quad x(0) = \xi, \in \mathbb{R}^d,$$

find an admissible control u (i.e. $\|u\|_\infty \leq 1$) that achieves

$$x(T) = \mathbf{0},$$

and minimizes the following cost function

$$J(u) = \int_0^T L(u(t)) dt.$$

- For the minimum-time control we have $L(u) = 1$ (i.e. $J(u) = T$), and T is not fixed.

Hamiltonian

- The **Hamiltonian** for the optimal control problem is defined by

$$H^\eta(x, p, u) \triangleq p^\top(Ax + bu) + \eta L(u)$$

- p is called the **co-state vector**.
- $\eta \in \{0, 1\}$ is called the **abnormal multiplier**.

Pontryagin's Minimum Principle (PMP)

- Assume that an optimal control u^* of the optimal control problem (OPT) exists.
- Let us denote by $x^*(t)$ the **optimal state** with the **optimal control** $u^*(t)$, that is,

$$x^*(t) \triangleq e^{At} \xi + \int_0^t e^{A(t-\tau)} b u^*(\tau) d\tau, \quad \forall t \in [0, T].$$

- Then there exist $\eta \in \{0, 1\}$ and the **optimal costate** $p^*(t)$ that satisfy the following conditions:

Pontryagin's Minimum Principle (PMP)

(non-triviality condition) The abnormal multiplier η and the optimal costate p^* satisfy the **non-triviality condition**:

$$|\eta| + \|p^*\|_\infty > 0.$$

(canonical equation) The following **canonical equations** hold

$$\begin{aligned}\dot{x}^*(t) &= Ax^*(t) + bu^*(t), \\ \dot{p}^*(t) &= -A^\top p^*(t), \quad \forall t \in [0, T].\end{aligned}$$

The differential equation for $p^*(t)$ is called the **adjoint equation**.

(minimum condition) The optimal control $u^*(t)$ minimizes Hamiltonian at each time $t \in [0, T]$. That is,

$$u^*(t) = \arg \min_{|u| \leq 1} H^\eta(x^*(t), p^*(t), u), \quad \forall t \in [0, T].$$

Pontryagin's Minimum Principle (PMP)

(consistency) Hamiltonian satisfies

$$H^\eta(x^*(t), p^*(t), u^*(t)) = c, \quad \forall t \in [0, T],$$

where c is a constant independent of t . If T is not fixed (as in the minimum-time control), then

$$H^\eta(x^*(t), p^*(t), u^*(t)) = 0, \quad \forall t \in [0, T].$$

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Minimum-time control

- Dynamical system

$$\dot{x}(t) = Ax(t) + bu(t), \quad t \geq 0, \quad x(0) = \xi \in \mathbb{R}^d.$$

- Assume $\xi \in \mathcal{R}$. Then $T^*(\xi) < \infty$.
- The minimum-time cost function is

$$J(u) = \int_0^T L(u(t))dt, \quad L(u) = 1.$$

- The Hamiltonian is given by

$$\begin{aligned} H^\eta(x, p, u) &= p^\top(Ax + bu) + \eta L(u) \\ &= p^\top(Ax + bu) + \eta. \end{aligned}$$

Minimum-time control

- From Pontryagin's minimum principle, the optimal control $u^*(t)$ should satisfy

$$u^*(t) = \arg \min_{u \in [-1,1]} H^\eta(x^*(t), p^*(t), u), \quad \forall t \in [0, T^*(\xi)], \quad (\star)$$

- $x^*(t)$: optimal state
- $p^*(t)$: optimal costate
- From (\star) , we have

$$u^*(t) = \arg \min_{u \in [-1,1]} p^*(t)^\top b u = -\text{sgn}(p^*(t)^\top b),$$

$$\text{sgn}(a) = \begin{cases} 1, & a > 0 \\ -1, & a < 0 \end{cases}$$

$$\text{sgn}(a) \in [-1, 1], \quad a = 0.$$

Minimum-time control

- Minimum-time control

$$u^*(t) = -\text{sgn}(\boldsymbol{p}^*(t)^\top \boldsymbol{b}),$$

- If $\boldsymbol{p}^*(t)^\top \boldsymbol{b} = 0$, then $u^*(t)$ cannot be uniquely determined.

Lemma

If (A, \boldsymbol{b}) is controllable, then the function $\boldsymbol{p}^*(t)^\top \boldsymbol{b}$ is not zero for almost all $t \in [0, T^*(\xi)]$.

- If (A, \boldsymbol{b}) is controllable, the the minimum-time control is a piecewise constant function that takes values ± 1 , which is called a **bang-bang control**.

Minimum-time control

- For the minimum-time control problem, we have the following existence and uniqueness theorems.

Theorem (Existence)

If the initial state ξ is in the controllable set \mathcal{R} then a minimum-time control exists.

Theorem (Uniqueness)

Assume that (A, b) is controllable. Then the minimum-time control is (if it exists) unique.

Corollary

Assume that (A, b) is controllable and A is stable. Then for any $\xi \in \mathbb{R}^d$, the minimum-time control $u^ \in \mathcal{U}(\xi)$ uniquely exists.*

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Rocket control problem

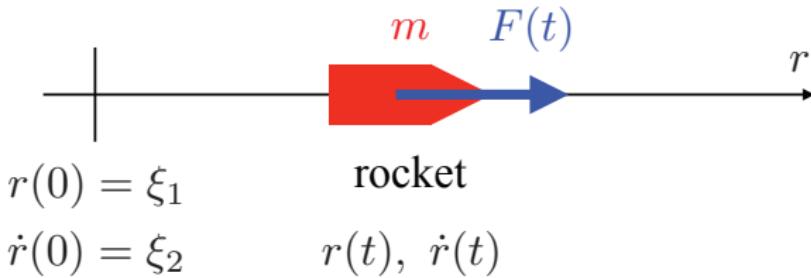
- State equation

$$\dot{x}(t) = Ax(t) + bu(t), \quad t \geq 0, \quad x(0) = \xi,$$

where $x(t) = [r(t), \dot{r}(t)]^\top$, $u(t) = F(t)$, and

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad b = \begin{bmatrix} 0 \\ m^{-1} \end{bmatrix}$$

- Since (A, b) is controllable and A is stable (the eigs are 0, 0), there uniquely exists the minimum-time control $u^*(t)$ for any initial state $\xi \in \mathbb{R}^2$.



Minimum-time control of rocket

- The **Hamiltonian** for the minimum-time control is given by

$$H^\eta(\mathbf{x}, \mathbf{p}, u) = \mathbf{p}^\top (\mathbf{A}\mathbf{x} + \mathbf{b}u) + \eta = p_1x_2 + p_2u + \eta.$$

- The **optimal control** is given by

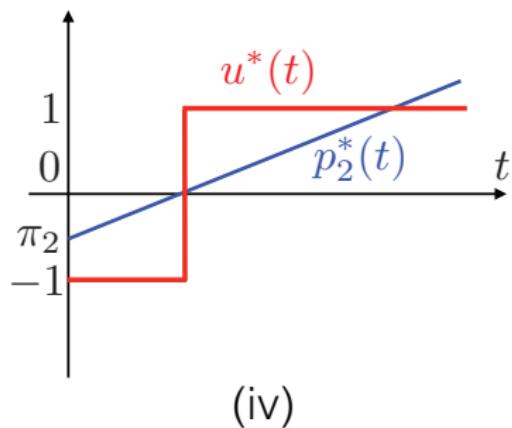
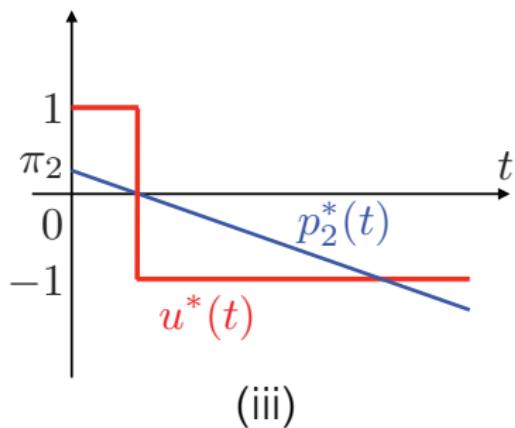
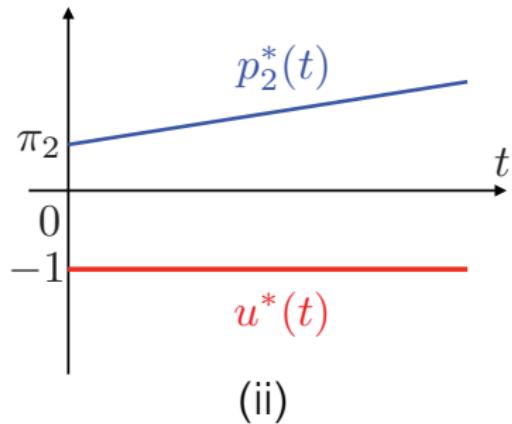
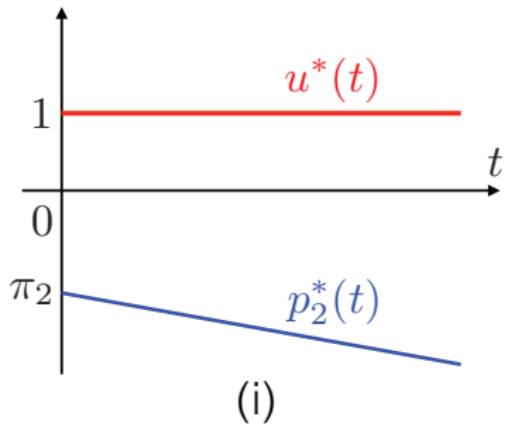
$$u^*(t) = -\text{sgn}(\mathbf{p}^*(t)^\top \mathbf{b}) = -\text{sgn}(p_2^*(t)).$$

where $\mathbf{p}^*(t) = [p_1^*(t), p_2^*(t)]^\top$ is the optimal costate.

- From the canonical equation, $p_2^*(t)$ is a **linear function** given by

$$p_2^*(t) = \pi_2 - \pi_1 t.$$

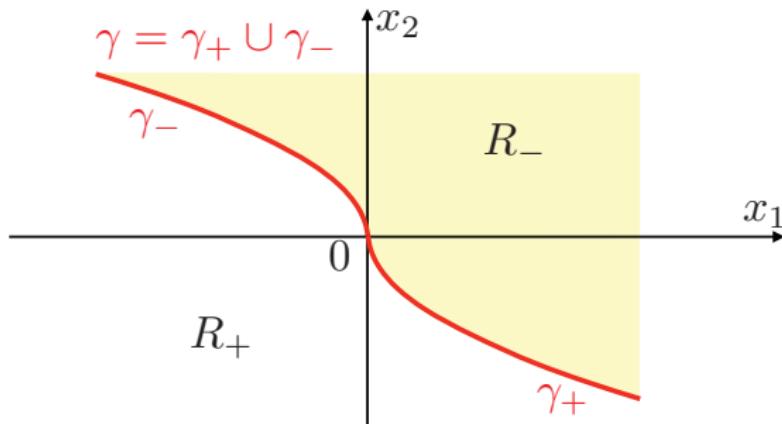
Optimal costate $p_2^*(t)$



Minimum-time control

- The minimum-time control is bang-bang:

$$u^*(t) = \begin{cases} 1, & \text{if } x(t) \in \gamma_+ \cup R_+ \setminus \{\mathbf{0}\}, \\ -1, & \text{if } x(t) \in \gamma_- \cup R_- \setminus \{\mathbf{0}\}, \\ 0, & \text{if } x(t) = \mathbf{0}. \end{cases}$$



- The curve $\gamma = \gamma_+ \cup \gamma_-$ is called the **switching curve**.

Conclusion

- A dynamical system is modeled by a differential equation called the state-space equation.
- We cannot control uncontrollable systems.
- Optimal control is the best control among feasible controls for a controllable system.
- Minimum-time control becomes bang-bang when the system is controllable.