SIMATS

School of Engineering

Applied Mathematics

Science & Humanities

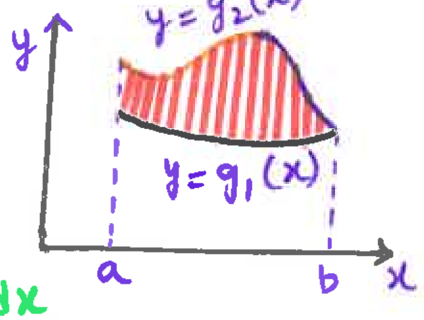
Saveetha Institute of Medical And Technical Sciences,Chennai.

MULTIPLE INTEGRATION AND VECTOR INTEGRATION

DOUBLE INTEGRALS

* $D=\{(x,y)/a\le x\le b,\ g_1(x)\le y\le g_2(x)\}$

$$\iint_D f(x,y)\,dA=\int_a^b\int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$



* Area of D $=\iint_D dA$

$$A=\int_a^b[g_2(x)-g_1(x)]\,dx$$

Problems in Cartesian Co-ordinate

constant Limit :-

EXAMPLE :-

$$\int_0^3\int_0^2 e^{x+y}\,dy\,dx$$

Sol:- Let $I=\int_0^3\int_0^2 e^x\cdot e^y\,dy\,dx$

$\Rightarrow \int_0^3 e^x dx\cdot\int_0^2 e^y dy \Rightarrow [e^x]_0^3\cdot[e^y]_0^2$

$=[e^3-e^0]\cdot[e^2-e^0]$ $\quad\therefore e^0=1$

$=[e^3-1]\cdot[e^2-1]$

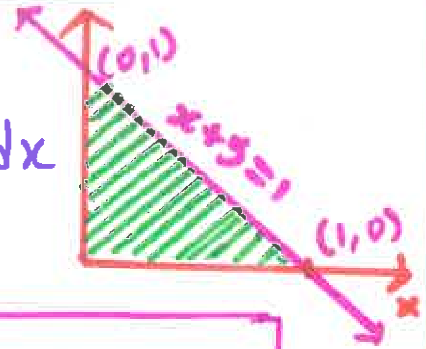
EXERCISE PROBLEM :-

1. Evaluate $\int_0^2\int_0^1 4xy\,dx\,dy$

Variable Limit :-

EXAMPLE :-

$$\int_0^1\int_0^{1-x} y\,dy\,dx$$



Sol:-

$I=\int_0^1\int_0^{1-x} y\,dy\,dx$

$=\int_0^1\left[y^2/2\right]_0^{1-x}dx$

$=\int_0^1\left[\frac{(1-x)^2}{2}-0\right]dx$

$=\int_0^1\frac{(1-x)^2}{2}(dx)$

$=\frac{1}{2}\left[\frac{(1-x)^3}{3}(-1)\right]_0^1$

$=-\frac{1}{6}\left[(1-1)^3-(1-0)^3\right]$

$=-\frac{1}{6}[0-1^3]$

$=-\frac{1}{6}[-1]=\frac{1}{6}$

$\therefore \int_0^1\int_0^{1-x} y\,dy\,dx=\frac{1}{6}$

EXERCISE PROBLEM :-

1. Evaluate $\int_0^1\int_x^1 \frac{y\,dx\,dy}{x^2+y^2}$

Area as a Double Integral

EXAMPLE :-

Find the Area which bounded by $y=x$ and $y=x^2$.

Sol:-

$x: 0$ to 1

$y: x^2$ to x

Required Area.

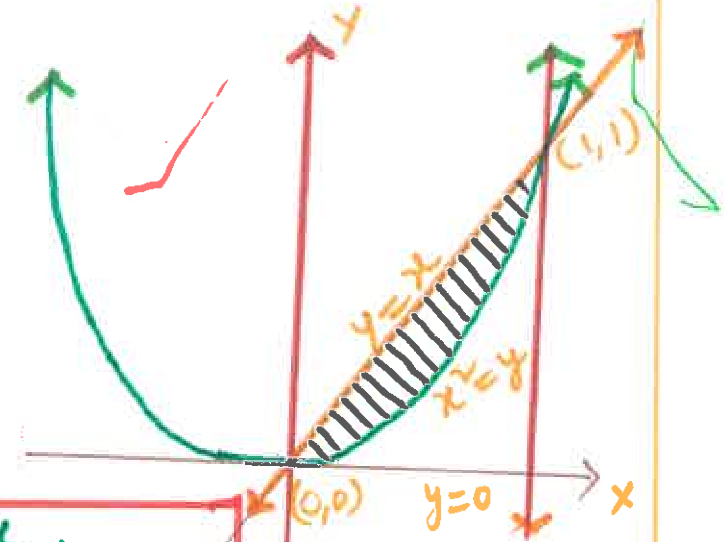
$A=\iint dx\,dy$

$=\int_0^1\int_{x^2}^{x} dy\,dx \Rightarrow \int_0^1[y]_{x^2}^{x}dx$

$=\int_0^1[x-x^2]\,dx \Rightarrow \left[\frac{x^2}{2}-\frac{x^3}{3}\right]_0^1$

$=\left[\frac{1}{2}-\frac{1}{3}\right]-0$

$=\frac{3-2}{6}\Rightarrow\frac{1}{6}$ sq.units



EXERCISE PROBLEM :-

1. Evaluate $\int_0^1\int_0^x dx\,dy$ and also sketch the region of integration roughly.
2. Evaluate $\int_0^1\int_{x^2}^{2-x} dx\,dy$ and also sketch the region of integration roughly.

GREEN'S THEOREM

Statement :- If $u, v, \frac{\partial u}{\partial y}, \frac{\partial v}{\partial x}$ are continuous & one-valued functions in the region R enclosed by the curve C, then

$$\boxed{\int_C u\,dx + v\,dy = \iint_R \left[\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right] dx\,dy}$$

PROBLEM :-

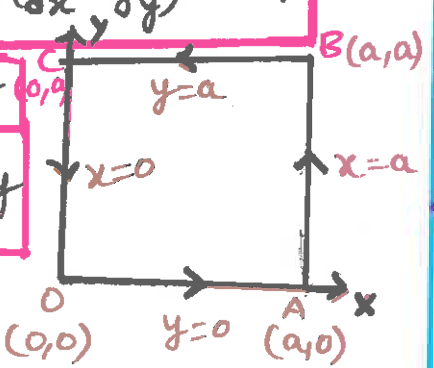
1. Verify Green's Theorem in the plane for $\int_C x^2 dx + xy\,dy$, where C is the curve in the XY plane given by $x=0, y=0, x=a, y=a$ $(a>0)$.

Sol :-

To Prove $\int_C \{u\,dx + v\,dy\} = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

Here $u = x^2$ $\quad v = xy$

$\therefore \frac{\partial u}{\partial y} = 0 \quad \frac{\partial v}{\partial x} = y$



LHS : Evaluate $\int_C x^2 dx + xy\,dy$

We shall take C in four different segments viz

(i) along OA $(y=0)$, (ii) along AB $(x=a)$
(iii) along BC $(y=a)$ (iv) along CO $(x=0)$

(ie) $\int_C = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$

(i) Along OA $(y=0)$

$$\int_C x^2 dx + xy\,dy = \int_{OA} x^2 dx \quad [\because y=0, dy=0]$$

$$= \int_0^a x^2 dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

$$\therefore \int x^2 dx + xy\,dy = \frac{a^3}{3}$$

(ii) Along AB $(x=a)$

$$\int_C x^2 dx + xy\,dy = \int_{AB}(x^2 dx + xy\,dy) = \int_{AB}(0 + ay\,dy)$$

$[\because x=0, dx=0]$

$$= a\int_0^a y\,dy = a\left[\frac{y^2}{2}\right]_0^a = \frac{a^3}{2}$$

$$\therefore \int_C x^2 dx + xy\,dy = \frac{a^3}{2}$$

(iii) Along BC $(y=a)$

$$\int_C (x^2 dx + xy\,dy) = \int_{BC} (x^2 dx + xy\,dy)$$

$$= \int_{BC} x^2 dx + 0 \quad [\because y=a, dy=0]$$

$$= \int_a^0 x^2 dx = 0 - \frac{a^3}{3} = -\frac{a^3}{3} \Rightarrow \int_C (x^2 dx + xy\,dy) = -\frac{a^3}{3}$$

(iv) Along CO $(x=0)$

$$\int_C (x^2 dx + xy\,dy) = \int_{CO} (x^2 dx + xy\,dy) = \int_{CO} (0+0)$$

$$[\because x=0, dx=0]$$

$$\int_C (x^2 dx + xy\,dy) = 0$$

$$\therefore \int_C (x^2 dx + xy\,dx) = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

$$= \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0$$

$$\int_C (x^2 dx + xy\,dx) = \frac{a^3}{2} \Rightarrow \text{L.H.S}$$

RHS : Evaluate $\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \iint_R y\,dx\,dy$$

$$\Rightarrow \int_0^a \int_0^a y\,dx\,dy \Rightarrow \int_0^a y\,[x]_0^a\,dy$$

$$\Rightarrow a\int_0^a y\,dy \Rightarrow a\left[\frac{y^2}{2}\right]_0^a \Rightarrow a \cdot \frac{a^2}{2}$$

$$\iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy = \frac{a^3}{2} \Rightarrow \text{R.H.S}$$

$\therefore$ LHS = RHS

$$\int_C x^2 dx + xy\,dx = \iint_R \left(\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y}\right) dx\,dy$$

$\therefore$ Hence the Green's Theorem is verified.

EXERCISE PROBLEM :-

1. Verify Green's Theorem in a plane with respect to $\int[(x^2-y^2)dx + 2xy\,dy]$, where C is the boundary of the rectangle in the xoy-plane bounded by the lines $x=0, x=a, y=0$ and $y=b$.
2. Verify Green's theorem in a plane for $\int_C [(3x^2-8y^2)dx + (4y-6xy)\,dy]$, where C is the boundary of the region defined by the lines $x=0, y=0$ and $x+y=1$.

STOKE'S THEOREM

Statement :- The surface integral of the normal component of the curl of a vector function F over an open surface s is equal to the line integral of the tangential component of F around the closed curve c bounding S.

$$\boxed{\oint_c \vec{F}\cdot d\vec{r} = \iint_s \nabla\times\vec{F}\cdot\hat{n}\,ds}$$

PROBLEM :-

Verify stoke's theorem for the function $\vec{F} = x^2\vec{i} + xy\vec{j}$ integrated round the square in the $z=0$ plane whose sides are along the lines $x=0$, $y=0$, $x=a$, $y=a$.

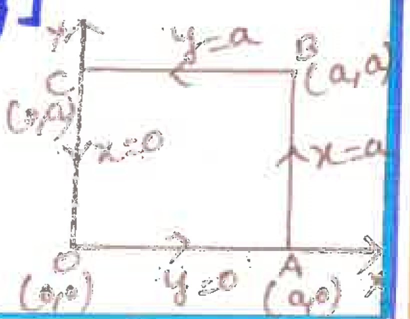
Sol :-

stoke's theorem is

$$\oint_c \vec{F}\cdot d\vec{r} = \iint_s \nabla\times\vec{F}\cdot\hat{n}\,ds$$

Given $\vec{F} = x^2\vec{i} + xy\vec{j}$

[The unit outward normal vector is $\vec{k}$ i.e., $\hat{n} = \vec{k}$]



$$\nabla\times\vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \partial/\partial x & \partial/\partial y & \partial/\partial z \\ x^2 & xy & 0 \end{vmatrix}$$

$$\boxed{\nabla\times\vec{F} = y\vec{k}}$$

$$RHS = \iint_s \nabla\times\vec{F}\cdot\hat{n}\,ds \qquad \text{Here } \hat{n} = \vec{k}$$

$$= \iint_s (y\vec{k})\cdot\vec{k}\,dx\,dy$$

$$= \iint_s y\,dx\,dy$$

$$= \int_0^a\int_0^a y\,dx\,dy = \int_0^a \left[\frac{y^2}{2}\right]_0^a dx$$

$$= \frac{a^2}{2}[x]_0^a \qquad \boxed{\therefore \iint \nabla\times\vec{F}\cdot\hat{n}\,ds = \frac{a^3}{2}}$$

Given $\vec{F} = x^2\vec{i} + xy\vec{j}$

$$d\vec{r} = dx\vec{i} + dy\vec{j} + dz\vec{k}$$

$$\oint_c \vec{F}\cdot d\vec{r} = x^2dx + xy\,dy$$

$$LHS = \oint_c \vec{F}\cdot d\vec{r} = \int_c x^2dx + xy\,dy$$

$$\int_c = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO}$$

Along OA (y=0)

$$\int_{OA} = \int_{OA} x^2dx + xy\,dy \quad [y=0,\ dy=0]$$

$$= \int_{OA} x^2dx = \int_0^a x^2dx = \left[\frac{x^3}{3}\right]_0^a = \frac{a^3}{3}$$

$$\boxed{\int_{OA} = \frac{a^3}{3}}$$

Along AB (x=a)

$$\int_{AB} = \int_{AB} x^2dx + xy\,dy \qquad [x=a;\ dx=0]$$

$$= \int_{AB} ay\,dy$$

$$= \int_0^a ay\,dy = \frac{a^3}{2} \qquad \boxed{\therefore \int_{AB} = \frac{a^3}{2}}$$

Along BC (y=a)

$$\int_{BC} = \int_{BC} x^2dx + xy\,dy \quad [y=a;\ dy=0] \Rightarrow \int_a^0 x^2dx$$

$$= \int_{BC} x^2dx$$

$$\Rightarrow \left[\frac{x^3}{3}\right]_a^0 = -\frac{a^3}{3} \qquad \boxed{\therefore \int_{BC} = -\frac{a^3}{3}}$$

Along CO (x=0)

$$\int_{CO} = \int_{CO} x^2dx + xy\,dy \Rightarrow \int_{CO} = 0 \quad [\because x=0;\ dx=0]$$

$$\int_c = \int_{OA} + \int_{AB} + \int_{BC} + \int_{CO} \Rightarrow \frac{a^3}{3} + \frac{a^3}{2} - \frac{a^3}{3} + 0$$

$$\int_c = \frac{a^3}{2} \qquad \boxed{\Rightarrow \oint_c \vec{F}\cdot d\vec{r} = \frac{a^3}{2}}$$

L.H.S = R.H.S

$$\boxed{\oint_c \vec{F}\cdot d\vec{r} = \iint_s \nabla\times\vec{F}\cdot\hat{n}\,ds}$$

Hence Stoke's Theorem is verified.

GAUSS DIVERGENCE THEOREM

STATEMENT :- If S is a closed surface enclosing a region of space with volume V and if $\vec{F}$ is a vector point function having continuous first order derivative in V then

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

EXAMPLE :-

verify divergence theorem for $\vec{F}=(2x-z)\vec{i}+x^2y\vec{j}-(xz^2)\vec{k}$ taken over the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$ and $z=1$

Sol :- Given

$$\vec{F}=(2x-z)\vec{i}+(x^2y)\vec{j}-(xz^2)\vec{k}$$

$$\text{div}\,\vec{F}=\frac{\partial}{\partial x}(2x-z)+\frac{\partial}{\partial y}(x^2y)-\frac{\partial}{\partial z}(xz^2)$$

$$= 2+x^2-2xz$$

By divergence theorem

$$\iint \vec{F}\cdot\hat{n}\,ds = \iiint_V \nabla\cdot\vec{F}\,dv$$

RHS $\iiint_V \text{div}\,\vec{F}\,dv = \int_0^1\int_0^1\int_0^1 (2+x^2+2xz)\,dx\,dy\,dz$

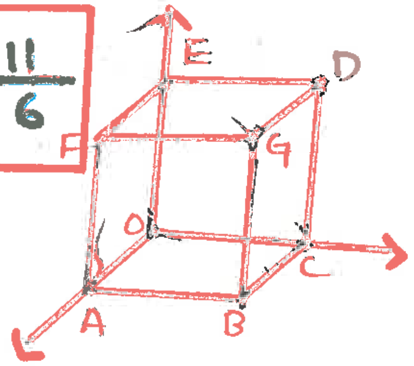
$$= \int_0^1\int_0^1 \left(2x+\frac{x^3}{3}+(x^2z)\right)_0^a dy\,dz$$

$$= \int_0^1\int_0^1 \left(2+\frac{1}{3}-z\right) dy\,dz$$

$$= \int_0^1\int_0^1 \left(\frac{7}{3}-z\right) dy\,dz$$

$$= \int_0^1 \left(\frac{7z}{3}-\frac{z^2}{2}\right)_0^1 dy = \int_0^1 \left(\frac{7}{3}-\frac{1}{2}\right) dy$$

$$= \frac{11}{6}\int_0^1 dy \Rightarrow \frac{11}{6}$$



L.H.S :-

$$\iint_S \vec{F}\cdot\hat{n}\,ds = \iint_{S_1}+\iint_{S_2}+\iint_{S_3}+\iint_{S_4}+\iint_{S_5}+\iint_{S_6}$$

FROM Fig :

Surface	Equation	$\hat{n}$	ds
OABC(S_1)	$z=0$	$-\vec{k}$	$dx\,dy$
DEFG(S_2)	$z=1$	$\vec{k}$	$dx\,dy$
OAFE(S_3)	$y=0$	$-\vec{j}$	$dx\,dz$
BCDG(S_4)	$y=1$	$\vec{j}$	$dx\,dz$
OCDE(S_5)	$x=0$	$-\vec{i}$	$dy\,dz$
ABGF(S_6)	$x=1$	$\vec{i}$	$dy\,dz$

$\iint_{S_1} \vec{F}\cdot\hat{n}\,ds = \iint_{S_1} xz^2\,ds$, at $z=0$

$\Rightarrow 0$

$\iint_{S_2} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (-xz^2)\,dx\,dy$ at $z=1$

$$= \int_0^1\int_0^1 -x\,dx\,dy \Rightarrow -1/2$$

$\iint_{S_3} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 -x^2y\,dx\,dz \Rightarrow 0$ at $y=0$

$$\iint_{S_4} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 x^2y\,dx\,dz \Rightarrow 1/3$$

$\iint_{S_5} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (2x-z)\,dy\,dz$

$$= \int_0^1\int_0^1 (0-z)\,dy\,dz \Rightarrow -1/2$$

$\iint_{S_6} \vec{F}\cdot\hat{n}\,ds = \int_0^1\int_0^1 (2x-z)\,dy\,dz \Rightarrow 3/2$

$$\therefore \iint_S \vec{F}\cdot\hat{n}\,ds = 0-\frac{1}{2}+0-\frac{1}{3}+\frac{1}{2}+\frac{3}{2}$$

$\Rightarrow 11/6$ $\quad\therefore$ LHS = RHS

Hence the divergence theorem is verified.

EXERCISE PROBLEM :-

1. Verify the G.D.T for $\vec{F}=4xz\vec{i}-y^2\vec{j}+yz\vec{k}$ over the cube bounded by $x=0$, $x=1$, $y=0$, $y=1$, $z=0$, $z=1$.

2. verify G.D.T for $\vec{F}=(x^2-yz)\hat{i}+(y^2-zx)\vec{j}+(z^2-xy)\vec{k}$ taken over the rectangular parallelopiped $0\le x\le a$, $0\le y\le b$, $0\le z\le c$.

PERIODIC FUNCTION

- If $f(x)$ is periodic then

$$f(x+p)=f(x)$$

STANDARD & GENERAL FOURIER SERIES IN $(c, c+2l)$

$$f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n \cos\left(\frac{n\pi x}{l}\right)+\sum_{n=1}^{\infty} b_n \sin\left(\frac{n\pi x}{l}\right)$$

where

$$a_0=\frac{1}{l}\int_c^{c+2l} f(x)\,dx$$

$$a_n=\frac{1}{l}\int_0^{c+2l} f(x)\cos\left(\frac{n\pi x}{l}\right)dx$$

$$b_n=\frac{1}{l}\int_c^{c+2l} f(x)\sin\left(\frac{n\pi x}{l}\right)dx$$

DIRICHLET'S CONDITIONS

- If $f(x)$ can be developed as a fourier series to satisfy the following conditions:

i) $f(x)$ is periodic, single valued and finite.

ii) $f(x)$ has a finite number of finite discontinous in any one period and has no infinite discontinuity.

iii) $f(x)$ has at the most a finite number of maximum and minimum

PROBLEM:

Q obtain the fourier series for $f(x)=\left(\frac{\pi-x}{2}\right)^2$ in $0<x<2\pi$

Sol^n Given $\boxed{f(x)=\frac{(\pi-x)^2}{4},\ 0<x<2\pi}$

The fourier series for $f(x)$ is

$$\boxed{f(x)=\frac{a_0}{2}+\sum_{n=1}^{\infty} a_n\cos nx+\sum_{n=1}^{\infty} b_n \sin nx}$$

$$\therefore\ a_0=\frac{1}{\pi}\int_0^{2\pi} f(x)\,dx=\frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}dx$$

$$a_0=\frac{1}{4\pi}\left[\frac{(\pi-x)^3}{-3}\right]_0^{2\pi}$$

$$a_0=-\frac{1}{12\pi}\left[-\pi^3-\pi^3\right]=\frac{\pi^2}{6}$$

$$a_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cos nx\,dx$$

$$=\frac{1}{\pi}\int_0^{2\pi}\frac{(\pi-x)^2}{4}\cos nx\cdot dx$$

$$\Rightarrow \frac{1}{4\pi}\left[(\pi-x)^2\frac{\sin nx}{n}-2(\pi-x)(-1)\cdot\left(-\frac{\cos nx}{n^2}\right)+(2)\left(\frac{-\sin nx}{n^3}\right)\right]_0^{2\pi}$$

$$\Rightarrow \frac{1}{4\pi}\left[(\pi-x)^2\frac{\sin nx}{n}-2(\pi-x)\frac{\cos nx}{n^2}-\frac{2\sin nx}{n^3}\right]_0^{2\pi}$$

$$\Rightarrow \frac{1}{4\pi}\left[0+\frac{2\pi}{n^2}-0-\left(0-\frac{2\pi}{n^2}-0\right)\right]$$

$$\Rightarrow \frac{1}{4\pi}\cdot\frac{4\pi}{n^2}=\frac{1}{n^2},\ n=1,2,3\ldots$$

$$\boxed{b_n=\frac{1}{\pi}\int_0^{2\pi} f(x)\cdot\sin nx\,dx}$$

$$=\frac{1}{\pi}\int_0^{2\pi}(\pi-x)^2/4\,\sin nx\,dx$$

$$\Rightarrow 0\left[\because \int_0^{2a} f(x)\,dx=0 \text{ if } f(2a-x)=-f(x)\right]$$

$\therefore$ The fourier series is

$$f(x)=\pi^2/12+\sum_{n=1}^{\infty} 1/n^2\cos nx$$

$$\Rightarrow \left(\frac{\pi-x}{2}\right)^2=\frac{\pi^2}{12}+\sum_{n=1}^{\infty} 1/n^2\cos nx.$$

Exercise Problem

- Expand $x(2\pi-x)$ as fourier series in $(0,2\pi)$.
- Expand $f(x)=x^2$ as fourier series in $(0,2\pi)$

Fourier Series

1. Odd fn $f(-x) = -f(x)$

even fn $f(-x) = f(x)$

1. Find the Half range Sine fourier series for $f(x) = x(\pi - x)$ in $(0,\pi)$

Sol^n $f(x) = \sum_{n=1}^{\infty} b_n \sin nx$

$b_n = \frac{2}{\pi}\int_0^{\pi} x(\pi - x)\sin nx\,dx$

$= \frac{2}{\pi}\int_0^{\pi} (\pi x - x^2)\sin nx\,dx$

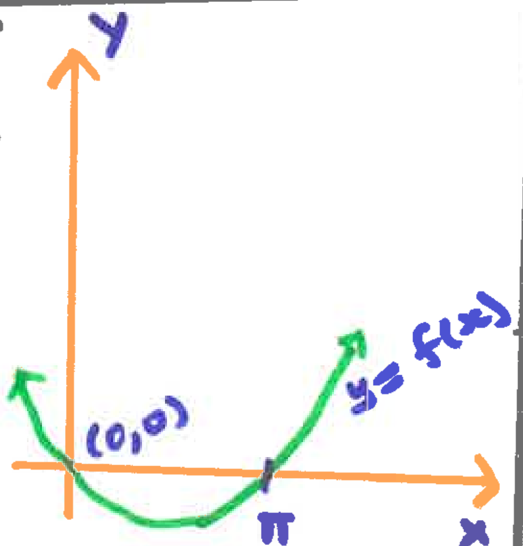
$= \frac{2}{\pi}\left[(\pi x - x^2)\left(-\frac{\cos nx}{n}\right) - (\pi - 2x)\left(-\frac{\sin nx}{n^2}\right) + (-2)\left(\frac{\cos nx}{n^3}\right)\right]_0^{\pi}$

$= \frac{4}{n^3\pi}\left[1 - (-1)^n\right]$

$b_n = \begin{cases} 0, & n \text{ is even} \\ \frac{8}{n^3\pi}, & n \text{ is odd} \end{cases}$

$f(x) = \sum_{n=odd}^{\infty} \frac{8}{n^3\pi}\sin nx$

$= \frac{8}{\pi}\sum_{n=odd}^{\infty} \frac{\sin nx}{n^3}$



2. Find the half range cosine fourier series of $f(x) = x$ $(0,\pi)$

Sol^n $f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} x\,dx$

$= \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi}$

$= \frac{2}{\pi}\left[\frac{\pi^2}{2} - 0\right] \Rightarrow \pi$

$a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$\Rightarrow \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx$

$\Rightarrow \frac{2}{\pi}\left[\frac{x\sin nx}{n} - (1)\cdot\left(-\frac{\cos nx}{n^2}\right)\right]_0^{\pi}$

$\Rightarrow \frac{2}{n\pi}\left[\frac{(-1)^n}{n} - \frac{1}{n}\right]$

$a_n = \begin{cases} 0, & n \text{ is even} \\ -\frac{4}{n^2\pi}, & n \text{ is odd} \end{cases}$

$f(x) = \frac{\pi}{2} + \sum_{n=odd}^{\infty} -\frac{4}{n^2\pi}\cos nx$

$\Rightarrow \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=odd}^{\infty} \frac{1}{n^2}\cos nx.$

3. Find the Fourier Series for $f(x) = |x|$ in $(-\pi, \pi)$ & deduce $\frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots\dots = \pi^2/8$

Given $f(x) = |x|$, $-\pi < x < \pi$

So $f(-x) = |-x| = |x| = f(x)$

$b_n = 0$

$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$

$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} |x|\,dx$

$\Rightarrow \frac{2}{\pi}\left[\frac{x^2}{2}\right]_0^{\pi} = \pi$

& $a_n = \frac{2}{\pi}\int_0^{\pi} f(x)\cos nx\,dx$

$= \frac{2}{\pi}\int_0^{\pi} |x|\cos nx\,dx$

$\Rightarrow \frac{2}{\pi}\int_0^{\pi} x\cos nx\,dx = \frac{2}{\pi}\left[x\left(\frac{\sin nx}{n}\right) - 1\cdot\left(-\cos nx/n^2\right)\right]_0^{\pi}$

$\frac{2}{\pi}\left[\frac{x\sin nx}{n} + \frac{\cos nx}{n^2}\right]_0^{\pi}$

$= \frac{2}{\pi}\left[\frac{(-1)^n}{n^2} - \frac{1}{n^2}\right]$

$\Rightarrow \frac{2}{\pi n^2}\left[(-1)^n - 1\right]$

$\therefore a_n = 0$, n is even

$\therefore a_n = \frac{2}{\pi n^2}(-2) = -\frac{4}{\pi n^2}$, $n = 1,3,5,\dots$, n is odd

$f(x) = \frac{\pi}{2} + \sum_{n=odd} -\frac{4}{\pi n^2}\cos nx$

$= \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=odd} \frac{\cos nx}{n^2}$

$x = 0$

$\therefore \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \dots = \frac{\pi^2}{8}$

PARSEVAL'S IDENTITY

$f(x)$ be period 2π then $\frac{1}{2\pi}\int_{-\pi}^{\pi}[F(x)]^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{1}^{\infty}[a_n^2 + b_n^2]$

EXAMPLE : Find the Fourier series x^2 in $(-\pi, \pi)$. Use parsevals identity to prove $\frac{\pi^4}{90} = 1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \dots$

Solu... Given $f(x) = x^2$, $-\pi < x < \pi$

$f(-x) = (-x)^2 = x^2 = f(x)$ $\therefore b_n = 0$

$$F(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos nx$$

$$a_0 = \frac{2}{\pi}\int_0^{\pi} f(x)\,dx = \frac{2}{\pi}\int_0^{\pi} x^2 dx$$

$$= \frac{2}{\pi}\left[\frac{x^3}{3}\right]_0^{\pi} = \frac{2\pi^3}{3\pi} = \frac{2\pi^2}{3}$$

$$a_n = \frac{2}{\pi}\left[x^2\frac{\sin nx}{n} + \frac{2x\cos nx}{n^2} - \frac{2}{n^3}\sin nx\right]_0^{\pi}$$

$$= \frac{2}{\pi}\left[\frac{2\pi}{n^2}(-1)^n\right] = \frac{4}{n^2}(-1)^n.$$

$n = 1, 2, 3$

$$F(x) = \frac{\pi^2}{3} + \sum_{n=1}^{\infty}\frac{4}{n^2}(-1)^n \cos nx$$

$$= x^2 = \frac{\pi^2}{3} + 4\left[-\frac{\cos x}{1^2} + \frac{\cos 2x}{2^2} - \frac{\cos 3x}{3^2} + \frac{\cos 4x}{4^2}\dots\right]$$

$$= x^2 = \frac{\pi^2}{3} - 4\left[\frac{\cos x}{1^2} - \frac{\cos 2x}{2^2} + \frac{\cos 3x}{3^2} - \frac{\cos 4x}{4^2}\dots\right]$$

$$a_0 = \frac{2\pi^2}{3},\quad a_n = \frac{4}{n^2}(-1)^n,\quad b_n = 0$$

PARSEVAL'S THEOREM :-

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}(f(x))^2 dx = \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{n}(a_n^2 + b_n^2)$$

$$= \frac{a_0^2}{4} + \frac{1}{2}\sum_{n=1}^{\infty} a_n^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi}(x^2)^2 dx$$

$$= \frac{2}{2\pi}\int_0^{\pi} x^4 dx = \frac{1}{\pi}\left[\frac{x^5}{5}\right]_0^{\pi} = \frac{\pi^5}{5\pi} = \frac{\pi^4}{5}$$

$$\therefore \frac{1}{4}\frac{4\pi^4}{9} + \frac{1}{2}\sum_{n=1}^{\infty}\frac{16}{n^4} = \frac{\pi^4}{5}$$

$$= 8\sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{5} - \frac{\pi^4}{4} = \frac{4\pi^4}{45}$$

$$\Rightarrow \sum_{n=1}^{\infty}\frac{1}{n^4} = \frac{\pi^4}{90} \;//$$

COMPLEX FORM OF FOURIER SERIES

1. Find the complex form of the Fourier series of e^{ax}

Given: $f(x) = e^{ax}$ in $-6 < x < 6$

FORMULA : $F(x) = \sum_{n=-\infty}^{\infty} C_n e^{\frac{in\pi x}{l}}$

$C_n = \frac{1}{}\int f(x)\, e^{\frac{in\pi x}{l}} dx$

$$C_n = \frac{1}{2l}\int_{-l}^{l} e^{ax} e^{\frac{-in\pi x}{l}} dx$$

$$= \frac{1}{2l}\int_{-l}^{l} e^{\left(a - \frac{in\pi}{l}\right)} dx = \frac{1}{2l} e^{\left(\frac{al - in\pi}{l}\right)x} dx$$

$$= \frac{1}{2(al - in\pi)}\left[e^{\left(\frac{al - in\pi}{l}\right)x}\right]$$

$$= \frac{1}{2(al - in\pi)}\left[e^{\frac{al - in\pi}{(l)} l} - e^{\left(\frac{al - in\pi}{l}\right)(-l)}\right]$$

$$= \frac{1}{2(al - in x)}\left[e^{al - in\pi} - e^{al + in\pi}\right]$$

$$= \frac{1}{2l(al - in\pi)}\left[e^{al} e^{-\pi} - e^{-al} e^{in\pi}\right]$$

$e^{in\pi} = \cos n\pi + i\sin n\pi = (-1)^n + i0 = (-1)^n$

$e^{-in\pi} = \cos n\pi - i\sin n\pi = (-1)^n - i0 = (-1)^n$

$$= \frac{1}{2(al - in\pi)}\left[e^{al}(-1)^n - e^{-al}(-1)^n\right]$$

$$= \frac{(-1)^n}{2(al - in\pi)}\left[e^{al} - e^{-al}\right] = \frac{(-1)^n}{al - in\pi}\left(\frac{e^{al} - e^{-al}}{2}\right)$$

$$C_n = \frac{(-1)^n}{al - in\pi}\sinh al$$

$$F(x) = \sum_{n=-\infty}^{\infty}\frac{(-1)^n}{al - in\pi}\sinh al\; e^{\frac{in\pi x}{l}}$$

$$= \sinh al\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{al - in\pi} e^{\frac{in\pi x}{l}}$$

$$= \sinh al\sum_{n=-\infty}^{\infty}\frac{(-1)^n}{al - in\pi}\cdot\frac{al + in\pi}{al + in\pi} e^{\frac{in\pi x}{l}}$$

$$= \sinh al\sum_{n=-\infty}^{\infty}\frac{(-1)^n(al + in\pi)}{a^2l^2 + n^2\pi^2} e^{\frac{in\pi x}{l}}$$

Ex :-

Find the complex form of the Fourier series $f(x) = \cos ax$ in $-\pi < x < \pi$

Harmonic Analysis

The process of finding Euler Constant for a tabular function is known as Harmonic Analysis.

Find the fourier Series upto the Second harmonic for $y = f(x)$ in $(0, 2\pi)$ defined by the Table of Values given below.

x	0	$\pi/3$	$2\pi/3$	π	$4\pi/3$	$5\pi/3$	2π
y	1	1.4	1.9	1.7	1.5	1.2	1.0

Solution: F.S $y = \frac{a_0}{2} + a_1\cos x + a_2\cos 2x + b_1\sin x + b_2\sin 2x$

x	y	$y\cos x$	$y\sin x$	$y\cos 2x$	$y\sin 2x$
0°	1.0	1	0	1	0
$\pi/3 = 60^\circ$	1.4	0.7	1.2124	-0.7	1.2124
$\frac{2\pi}{3} = 120^\circ$	1.9	-0.95	1.6454	-0.95	-1.6454
$\pi = 180^\circ$	1.7	-1.7	0	1.7	0
$4\pi/3 = 240^\circ$	1.5	-0.75	-1.299	-0.75	1.299
$\frac{5\pi}{3} = 300^\circ$	1.2	0.6	-1.0392	-0.6	-1.0392
	$\Sigma y = 8.7$	$\Sigma y\cos x = -1.1$	$\Sigma y\sin x = 0.5196$	$\Sigma y\cos 2x = -0.3$	$\Sigma y\sin 2x = -0.1732$

y is a repetition so leave lost Value.

$n = 6$

$a_0 = 2\left[\frac{\Sigma y}{n}\right] = 2.9$, $a_1 = 2\left[\frac{\Sigma y\cos x}{n}\right] = -0.37$

$a_2 = 2\left[\frac{\Sigma y\cos 2x}{n}\right] = -0.1$, $b_1 = 2\left[\frac{\Sigma y\sin x}{n}\right] = 0.17$

$b_2 = 2\left[\frac{\Sigma y\sin 2x}{n}\right] = -0.06$

$y = 1.45 + (-0.37\cos x + 0.17\sin x) + (-0.1\cos 2x - 0.06)\sin 2x)$.

Find an Empirical formula of the form $f(x) = a_0 + a_1\cos x + b_1\sin x$ for the following data given that $f(x)$ is periodic with Period 2π.

x in degrees	0	60	120	180	240	300	360
$y = f(x)$	40.0	31.0	-13.7	20	3.7	-21.0	40.0

Solution:

y is a repetition So leave lost value

x	y	$y\cos x$	$y\sin x$
0°	40.0	40.00	0
60°	31.0	15.50	26.846
120°	-13.7	6.85	-11.864
180°	20.0	-20.00	0
240°	3.7	-1.85	-3.204
300°	-21.0	-10.50	18.186
	$\Sigma y = 60$	$\Sigma y\cos x = 30$	$\Sigma y\sin x = 29.964$

$a_0 = 2\left[\frac{\Sigma y}{n}\right] = 20$ $\quad a_1 = 2\left[\frac{\Sigma y\cos x}{n}\right] = 10$

$b_1 = 2\left[\frac{\Sigma y\sin x}{n}\right] = 9.988$

$\therefore f(x) = 20 + 10\cos x + 9.988\sin x$

The values of x and the Corresponding values of $f(x)$ over a period T are given below, Show that $f(x) = 0.75 + 0.37\cos\theta + 1.004\sin\theta$ where $\theta = \frac{2\pi x}{T}$

x	0	$T/6$	$T/3$	$T/2$	$2T/3$	$5T/6$	T
$f(x) = y$	1.98	1.30	1.05	1.30	-0.88	-0.25	1.98

Solution:-

y is a repetition So leave lost value.

$f(x) = F(\theta) = \frac{a_0}{2} + a_1\cos\theta + b_1\sin\theta$, where $\theta = \frac{2\pi x}{T}$

x	$\theta = \frac{2\pi x}{T}$	y	$y\cos\theta$	$y\sin\theta$
0	$0 = 0^\circ$	1.98	1.98	0
$T/6$	$\pi/3 = 60^\circ$	1.30	0.65	1.1258
$T/3$	$\frac{2\pi}{3} = 120^\circ$	1.05	-0.525	0.9093
$T/2$	$\pi = 180^\circ$	1.30	-1.3	0
$2T/3$	$\frac{4\pi}{3} = 240^\circ$	-0.88	0.44	0.762
$5T/6$	$\frac{5\pi}{3} = 300^\circ$	-0.25	-0.125	0.2165
		$\Sigma y = 4.5$	$\Sigma y\cos\theta = 1.12$	$\Sigma y\sin\theta = 3.0136$

$a_0 = 2/6\,\Sigma y = 1.5$, $a_1 = \frac{2\Sigma y\cos\theta}{n} = 0.3733$,

$b_1 = 2/6(3.0136) = 1.0045$

$f(x) = 0.75 + 0.373\cos\theta + 1.0045\sin\theta$

where $\theta = \frac{2\pi x}{T}$, $n = 6$

LAPLACE TRANSFORMS :-

$$L[f(t)] = \int_0^\infty e^{-st} f(t)\,dt$$

Laplace Transform of elementary functions :-

S.No	$L[f(t)] = f(s)$	$L^{-1}[f(s)] = f(t)$
1.	$L[1] = 1/s$	$L^{-1}\left[\frac{1}{s}\right] = 1$
2.	$L[t] = 1/s^2,\ s>0$	$L^{-1}\left[\frac{1}{s^2}\right] = t$
3.	$L(t^n) = \frac{n!}{s^{n+1}},\ n=1,2,3\ldots$	$L^{-1}\left[\frac{1}{s^{n+1}}\right] = \frac{t^n}{n!},\ n=1,2,3\ldots$
4.	$L(t^\alpha) = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}}$ α is a real number >-1	$L^{-1}\left[\frac{1}{s^{\alpha+1}}\right] = \frac{t^\alpha}{\Gamma(\alpha+1)}$
5.	$L(e^{at}) = \frac{1}{s-a},\ s>a$	$L^{-1}\left[\frac{1}{s-a}\right] = e^{at}$
6.	$L[e^{-at}] = \frac{1}{s+a},\ s>-a$	$L^{-1}\left[\frac{1}{s+a}\right] = e^{-at}$
7.	$L(\sin at) = \frac{a}{s^2+a^2},\ s>0$	$L^{-1}\left[\frac{1}{s^2+a^2}\right] = \frac{\sin at}{a}$
8.	$L[\cos at] = \frac{s}{s^2+a^2},\ s>0$	$L^{-1}\left[\frac{s}{s^2+a^2}\right] = \cos at$
9.	$L[\sinh at] = \frac{a}{s^2-a^2},\ s>\lvert a\rvert$	$L^{-1}\left[\frac{1}{s^2-a^2}\right] = \frac{\sinh at}{a}$
10.	$L[\cosh at] = \frac{s}{s^2-a^2},\ s>\lvert a\rvert$	$L^{-1}\left[\frac{s}{s^2-a^2}\right] = \cosh at$

Basic properties :-

i) $L[af(t)+bg(t)] = aL[f(t)]+bL[g(t)]$

ii) If $L[f(t)] = f(s)$ then

i) $L[e^{at}f(t)] = f(s-a)$ if $s-a>\infty$

ii) $L[e^{-at}f(t)] = f(s+a)$ if $s+a>\infty$

Derivaties :-

$L[f(t)] = F(s)$ then $L[t^n f(t)] = (-1)^n \frac{d^n}{ds^n}(f(s)),\ n=1,2,3\ldots$

Problems :-

1. Find $L[e^{-at}\cos bt]$

Sol: $L[e^{-at}\cos bt] = [L[\cos bt]]_{s\to(s+a)}$

$= \left[\frac{s}{s^2+b^2}\right]_{s\to(s+a)}$

$= \frac{s+a}{(s+a)^2+b^2}.$

2. Find $L[e^{at}\sinh bt]$.

Sol :- $L[e^{at}\sinh bt] = [L[\sinh bt]]_{s\to(s-a)}$

$= \left[\frac{b}{s^2-b^2}\right]_{s\to(s-a)}$

$= \frac{b}{(s-a)^2-b^2}.$

3. Find $L[t\sin at]$

Sol:- $L[t\sin at] = -\frac{d}{ds}L[\sin at]$

$= -\frac{d}{ds}\left[\frac{a}{s^2+a^2}\right]$

$= -\left[\frac{-2as}{(s^2+a^2)^2}\right]$

$= \frac{2as}{(s^2+a^2)^2}$

4. Find $L(t\cos at)$

Sol:- $L(t\cos at) = -\frac{d}{ds}L(\cos at)$

$= -\frac{d}{ds}\left[\frac{s}{s^2+a^2}\right]$

$= -\left[\frac{s^2+a^2-2s^2}{(s^2+a^2)^2}\right]$

$= -\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right]$

$= \frac{s^2-a^2}{(s^2+a^2)^2}.$

5. Find $L[te^{-2t}\sin t]$

Sol :- $L(te^{-2t}\sin t) = \frac{-d}{ds}(L(e^{-2t}\sin t))$

$= -\frac{d}{ds}[(L[\sin t])_{s\to s+2}]$

$= -\frac{d}{ds}\left[\left[\frac{1}{s^2+1}\right]_{s\to s+2}\right]$

$= -\frac{d}{ds}\left[\frac{1}{(s+2)^2+1}\right]$

$= \frac{2(s+2)}{[(s+2)^2+1]^2}.$

Exercise problems :-

1. Find $L[\sinh t\cdot\sin t]$
2. Find $L[t^2\cdot e^{-2t}]$
3. Find $L[t\cdot e^{-t}\cos 2t]$
4. Find $L[t\sin 2t\cdot\sin 3t]$

Unit step function :-

$$u(t-a) = \begin{cases} 0 & \text{for } t < a \\ 1 & \text{for } t > a \end{cases}$$

This is the unit step function of $t=a$

1. Give the L.T of the unit step function.

$$L[u(t-a)] = \int_0^\infty e^{-st} u(t-a)\, dt$$

$$\int_0^a e^{-st} 0\, dt + \int_a^\infty e^{-st} (1)\, dt$$

$$\frac{e^{-as}}{s}$$

Unit Impulse function :-

$$\delta(t-a) = \begin{cases} 0, & t \neq a \\ \infty, & t = a \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(t-a)\, dt = 1$$

$$L[\delta(t-a)] = e^{-as}$$

$$L[\delta(t)] = e^0 = 1$$

Note :- $\int f(t)\delta(t-a)\, dt = f(a)$

Laplace transform of periodic functions :-

$$L[f(t)] = \frac{1}{1-e^{-Ts}} \int_0^T e^{-st} f(t)\, dx$$

1. Find the Laplace transform of the periodic function.

$$f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a-t, & a < t \le 2a \end{cases} \text{ and } f(t+2a) = f(t).$$

Sol :- Given $f(t) = \begin{cases} t, & 0 \le t \le a \\ 2a-t, & a < t \le 2a \end{cases}$ and $f(t)$ is $2a$.

$$L[f(t)] = \frac{\int_0^T e^{-st} f(t)dt}{1-e^{-sT}} = \frac{\int_0^{2a} e^{-st} f(t)\, dt}{1-e^{-2as}}$$

$$= \frac{\int_0^a e^{-st} f(t)dt + \int_a^{2a} e^{-st} f(t)\, dt}{1-e^{-2as}}$$

$$\Downarrow$$

$$= \frac{\int_0^a e^{-st} t\, dt + \int_a^{2a} e^{-st} (2a-t)\, dt}{1-e^{-2\infty}}$$

$$= \frac{1}{1-e^{-2as}} \left\{ \left[\frac{te^{-st}}{-s} - 1 \cdot \frac{e^{-st}}{(-s)^2} \right]_0^a + \left[(2a-t)\frac{e^{-st}}{-s} - \frac{(-1)\cdot e^{-st}}{(-s)^2} \right]_a^{2a} \right\}$$

$$\Downarrow$$

$$= \frac{1}{1-e^{-2as}} \left\{ \left[-\left(\frac{a \cdot e^{-\infty}}{s} + \frac{e^{-\infty}}{s^2} \right) + \left(0 + \frac{e^0}{s^2} \right) \right] + \left[\left(0 + \frac{e^{-2as}}{s^2} \right) - \left(-\frac{ae^{-\infty}}{s} + \frac{e^{-\infty}}{s^2} \right) \right] \right\}$$

$$= \frac{1}{1-e^{-2as}} \left[\frac{1}{s^2} + \frac{e^{-2as}}{s^2} - \frac{2e^{-sa}}{s^2} \right]$$

$$\Downarrow$$

$$= \frac{1}{1-e^{-2as}} \quad \frac{(1-e^{-as})^2}{s^2}$$

$$\Downarrow$$

$$= \frac{(1-e^{-as})^2}{s^2(1-e^{-as})(1+e^{-as})}$$

$$\Downarrow$$

$$= \frac{1-e^{-as}}{s^2(1+e^{-as})}$$

$$\Downarrow$$

$$= \frac{1}{s^2} \tanh \frac{as}{2}.$$

Exercise problem :-

1. Find the laplace transform of a square wave function of period a solved as $f(t) = \begin{cases} 1 & \text{when } 0 < t \le a/2 \\ -1 & \text{when } a/2 < t < a. \end{cases}$

2. Find the laplace transform of a square wave function $f(t)$ given by $f(t) = \begin{cases} k & 0 \le t \le a \\ -k & a \le t \le 2a \end{cases}$ and $f(t+2a) = f(t)$.

3. Find the laplace transform of the periodic function.

$$f(t) = \begin{cases} \cos t, & 0 < t < \pi \\ \pi < t < 2\pi \end{cases}, \quad f(t+2\pi) = f(t).$$

INVERSE LAPLACE TRANSFORM

PARTIAL FRACTION METHOD

① Find $L^{-1}\left[\frac{s+2}{s(s+4)(s+9)}\right]$

Given $F(s) = \frac{s+2}{s(s+2)(s+9)}$

$$\frac{s+2}{s(s+2)(s+9)} = \frac{A}{s} + \frac{B}{s+4} + \frac{C}{s+9}$$

$$\frac{1}{(s+a)^3} = \frac{A}{(s+a)} = \frac{B}{(s+a)^2} = \frac{C}{(s+a)^3}$$

$$s+2 = A(s+4)(s+9) + Bs(s+9) + Cs(s+4) \rightarrow ①$$

put $s=0$ in ① we get $A = \frac{1}{18}$

put $s=-4$ in ① we get $B = \frac{1}{10}$

put $s=-9$ in ① we get $C = \frac{-7}{45}$

$$F(s) = \frac{1}{18}\left(\frac{1}{s}\right) + \frac{1}{10}\left(\frac{1}{s+4}\right) + \frac{-7}{45}\left(\frac{1}{s+9}\right)$$

$$L^{-1}[F(s)] = \frac{1}{18}L^{-1}\left[\frac{1}{s}\right] + \frac{1}{10}L^{-1}\left[\frac{1}{s+4}\right] - \frac{7}{45}L^{-1}\left[\frac{1}{s+9}\right]$$

$$f(t) = \frac{1}{18} + \frac{1}{10}e^{-4t} - \frac{7}{45}e^{-9t}$$

PROBLEMS:

① Find inverse Laplace transform of $\frac{1}{s(s+a)(s+b)}$

② Find inverse Laplace transform of $\frac{1}{s(s+1)(s+2)}$

③ Find $L^{-1}\left[\frac{s^2+1}{s(s+2)(s+3)}\right]$

② Find $L^{-1}\left[\frac{5s+3}{(s-1)(s^2+2s+5)}\right]$

Let $F(s) = \frac{5s+3}{(s-1)(s^2+2s+5)}$

$$F(s) = \frac{A}{s-1} + \frac{Bs+C}{s^2+2s+5}$$

$$5s+3 = A(s^2+2s+5) + (Bs+C)(s-1) \rightarrow ①$$

Put $s=1$ in ① we get $A=1$

Put $s=0$ in ① $\Rightarrow C=2$

Equating co-efficient of s^2 $\Rightarrow A+B=0 \Rightarrow B=-1$

$$F(s) = \frac{1}{s-1} + \frac{-s+2}{s^2+2s+5}$$

$$= \frac{1}{s-1} - \frac{s+1-3}{(s+1)^2+4}$$

$$L^{-1}[F(s)] = L^{-1}\left[\frac{1}{s-1}\right] - L^{-1}\left[\frac{s+1-3}{(s+1)^2+4}\right]$$

$$f(t) = e^{t} - e^{-t}L^{-1}\left[\frac{s-3}{s^2+4}\right]$$

$$f(t) = e^{t} - e^{-t}\left(\cos 2t - \frac{3}{2}\sin 2t\right)$$

$$f(t) = \frac{1}{2}\left[2e^{t} - e^{-t}(2\cos 2t - 3\sin 2t)\right]$$

① Find $L^{-1}\left[s/(s^2+2s+2)(s+1)\right]$

//

CONVOLUTION THEOREM

If $L[f(t)] = F(s)$

$L[g(t)] = G(s)$

then $L^{-1}[F \cdot G] = f(t) * g(t)$

where $f(t)*g(t) = \int_0^t f(u)g(t-u)\,du$

PROBLEM: Find inverse Laplace Transform of $\frac{1}{s(s^2-a^2)}$ using convolution theorem.

$F(s) = \frac{1}{s}$ $\quad G(s) = \frac{1}{s^2-a^2}$

$f(t) = 1$ $\quad g(t) = \frac{1}{a}\sinh at$

$$L^{-1}[F(s) \cdot G(s)] = 1 * \frac{1}{a}\sinh at$$

$$= \frac{1}{a}\int_0^t \sinh au \cdot du$$

$$= \frac{1}{a}\left[\frac{\cosh au}{a}\right]_0^t$$

$$= \frac{1}{a^2}\left[\cosh at - \cosh 0\right]$$

$$= \frac{1}{a^2}(\cosh at - 1)$$

PROBLEM: Find inverse Laplace Transform of $F(s) = \frac{s^2}{(s+a^2)^2}$ using Convolution theorem.

INITIAL VALUE THEOREM

If $L[f(t)] = F(s)$

$$\underset{t\to 0}{Lt}\, f(t) = \underset{s\to\infty}{Lt}\, sF(s)$$

FINAL VALUE THEOREM

$L[f(t)] = F(s)$

$$\underset{t\to\infty}{Lt}\, f(t) = \underset{s\to 0}{Lt}\, sF(s)$$

Problem 1:-

Initial value theorem for

$f(t) = e^{-t}\sin t$

Sol:- $\lim\limits_{t\to 0} f(t) = \lim\limits_{s\to\infty} sF(s)$

$f(t) = e^{-t}\sin t$

$F(s) = L[f(t)] = L[e^{-t}\sin t] = \dfrac{1}{(s+1)^2+1}$

L.H.S : $\lim\limits_{t\to 0} f(t) = \lim\limits_{t\to 0} e^{-t}\sin t = e^{0}\cdot 0 = 0$

R.H.S $= \lim\limits_{s\to\infty} sF(s) = \lim\limits_{s\to\infty} \dfrac{s}{(s+1)^2+1}$

$= \lim\limits_{s\to\infty} \dfrac{1}{2(s+1)} = 0$

L.H.S = R.H.S

Problem 2:-

final value theorem for $f(t)$ function

$f(t) = 1+e^{-t}(\sin t+\cos t)$

Sol:- $\lim\limits_{t\to\infty} f(t) = \lim\limits_{s\to 0} sF(s)$

$f(t) = 1+e^{-t}(\sin t+\cos t)$

$\lim\limits_{t\to\infty} f(t) = \lim\limits_{t\to\infty}\left[1+e^{-t}(\sin t+\cos t)\right]$

$= 1+0 \Rightarrow 1$

$F(s) = L[f(t)]$

$= L[1+e^{-t}(\sin t+\cos t)]$

$= L[1]+L[e^{-t}\sin t]+L[e^{-t}\cos t]$

$= \dfrac{1}{s} + \dfrac{1}{(s+1)^2+1} + \dfrac{s+1}{(s+1)^2+1}$

$\lim\limits_{s\to 0} sF(s) = \lim\limits_{s\to 0} s\left[\dfrac{1}{s} + \dfrac{1}{(s+1)^2+1} + \dfrac{s+1}{(s+1)^2+1}\right]$

$\lim\limits_{s\to 0}\left[1 + \dfrac{s}{(s+1)^2+1} + \dfrac{s^2+s}{(s+1)^2+1}\right] = 1$

$\lim\limits_{t\to\infty} f(t) = \lim\limits_{s\to 0} sF(s)$

1) Verify the initial and final value theorems for $f(t) = e^{-t}(t+2)^2$

Solution of Linear Ordinary Differential Equations of secondary Order with constant coefficient

Problems :-

Solve $\dfrac{d^2y}{dt^2} - 4\dfrac{dy}{dt} + 3y = e^{-t}$

$y(0)=1$ and $y'(0)=0$

Sol:- Given equation $y''-4y'+3y=e^{-t}$

and $y'(0)=1,\ y'(0)=0$

$\Rightarrow$ Taking Laplace transform on both sides of equation(1)

$L[y''] - 4L[y'] + 3L[y] = L[e^{-t}]$

$\Rightarrow s^2L[y] - sy[0] - y'(0) - 4\{sL[y]-y(0)\} + 3L[y] = \dfrac{1}{s+1}$

$= [s^2-4s+3]L[y] = s+\dfrac{1}{s+1}-4$

$L[y] = \dfrac{s^2-3s-3}{(s-3)(s-1)(s+1)} \Rightarrow y = L^{-1}\left[\dfrac{s^2-3s-3}{(s-3)(s-1)(s+1)}\right]$

Let $\dfrac{s^2-3s-3}{(s-3)(s-1)(s+1)} = \dfrac{A}{s-3} + \dfrac{B}{s-1} + \dfrac{C}{s+1}$

$L^{-1}\left[\dfrac{s^2-3s-3}{(s-3)(s-1)(s+1)}\right] = L^{-1}\left[-\dfrac{3}{8}\cdot\dfrac{1}{(s-3)} + \dfrac{5}{4}\cdot\dfrac{1}{(s-1)} + \dfrac{1}{8}\dfrac{1}{(s+1)}\right]$

$= -\dfrac{3}{8}e^{3t} + \dfrac{5}{4}e^{t} + \dfrac{1}{8}e^{-t}$

$= \dfrac{1}{8}\left[e^{-t} - 3e^{3t} + 10e^{t}\right]$

Exercise Problem:-

1) $y''+4y = \sin at,\ y(0)=0,\ y'(0)=0$

FOURIER TRANSFORM

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} F(s)\, e^{-isx}\, ds$$

FOURIER COSINE TRANSFORM

$$F_c(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \cos sx\, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_c(s) \cos sx\, ds$$

FOURIER SINE TRANSFORM

$$F_s(s) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} f(x) \sin sx\, dx$$

$$f(x) = \sqrt{\frac{2}{\pi}} \int_{0}^{\infty} F_s(s) \sin sx\, ds$$

Properties

1. FOURIER TRANSFORM IS LINEAR

$$F[af(x)+bg(x)] = a\,F[f(x)] + b\,F[g(x)]$$

FOURIER TRANSFORMS

2. Shifting Theorem

$$F[f(x)] = F(s)$$

$$F[f(x-a)] = e^{isa} F(s)$$

3. Change of Scale Property

$$\boxed{\begin{aligned} &F\{f(x)\} = F(s) \\ &F\{f(ax)\} = \frac{1}{|a|} F\left(\frac{s}{a}\right),\ a \neq 0 \end{aligned}}$$

4. Modulation Theorem

$$F\{f(x)\} = F(s)$$

$$F\{f(x)\cos ax\} = \frac{1}{2}\left[F(s-a) + F(s+a)\right]$$

Problem

1. S.T. the FOURIER Transform of

$$f(x) = \begin{cases} a^2 - x^2, & |x| < a \\ 0, & |x| > a > 0 \end{cases}$$ is

$$2\sqrt{\frac{2}{\pi}} \left[\frac{\sin as - as\cos as}{s^3}\right].$$

Fourier transform is

$$F[f(x)] = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\, e^{isx}\, dx$$

$$F(s) = \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2)\, e^{isx}\, dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-a}^{a} (a^2 - x^2) [\cos sx + i \sin sx]\, dx$$

$$= \frac{2}{\sqrt{2\pi}} \int_{0}^{a} (a^2 - x^2) \cos sx\, dx + 0$$

$$= \sqrt{\frac{2}{\pi}} \left\{ (a^2 - x^2)\left[\frac{\sin sx}{s}\right] - (-2x)\left[\frac{-\cos sx}{s^2}\right] + (-2)\left[\frac{-\sin sx}{s^3}\right] \right\}_0^a$$

$$= \frac{4}{\sqrt{2\pi}} \left[\frac{\sin as - as\cos as}{s^3}\right]$$

$$= 2\sqrt{\frac{2}{\pi}} \left[\frac{\sin as - as\cos as}{s^3}\right]$$

Hence Proved.

Ex:

1. Find $F_c[e^{-ax}]$, $a > 0$

2. Find $F_s[e^{-ax}]$, $a > 0$

Convolution

$$(f*g)(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(t)\,g(x-t)\,dt$$

Parseval's Identity

$$\int_{-\infty}^{\infty}|f(x)|^2dx=\int_{-\infty}^{\infty}|F(s)|^2ds$$

find the fourier transform of $f(x)$ if $f(x)=\begin{cases}1-|x|, & |x|<1\\ 0, & |x|>1\end{cases}$?

Hence deduce that

i) $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^2dt=\pi/2$ ii) $\int_0^{\infty}\left(\frac{\sin t}{t}\right)^4dt=\pi/3$

Soln

$$F(s)=F[f(x)]=\frac{1}{2\pi}\int_{-\infty}^{\infty}f(x)\,e^{isx}dx$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-1}^{1}(1-|x|)(\cos sx+i\sin sx)dx$$

$$=\frac{1}{\sqrt{2\pi}}2\cdot\int_0^1(1-|x|)\cos sx\,dx+0$$

$$=\frac{1}{\sqrt{2\pi}}\left[(1-x)\frac{\sin sx}{s}(-1)\left(\frac{-\cos sx}{s^2}\right)\right]_0^1$$

$$=\sqrt{\frac{2}{\pi}}\left[\frac{1-\cos s}{s^2}\right]$$

$$=\sqrt{\frac{2}{\pi}}\cdot\frac{2\sin^2(s/2)}{s^2}$$

By Inverse fourier transform

$$f(x)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}f(s)\,e^{-isx}ds$$

$$=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\frac{2\sin^2(s/2)}{s^2}e^{-isx}ds$$

$$f(x)=\frac{2}{\pi}\,2\cdot\int_0^{\infty}\frac{\sin^2(s/2)}{s^2}\cos sx\,ds$$

put $x=0$

$f(x)=1-|x|$
$f(0)=1-0$
$=1$

$$f(0)=\frac{4}{\pi}\int_0^{\infty}\frac{\sin^2(s/2)}{s^2}ds$$

$$1=\frac{4}{\pi}\int_0^{\infty}\frac{\sin^2(s/2)}{s^2}ds$$

put $t=s/2$

$$\int_0^{\infty}\frac{\sin^2t}{4t^2}2dt=\pi/4$$

$$\int_0^{\infty}\frac{\sin^2t}{t^2}dt=\pi/2$$

By Parsevals Identity

$$\int_{-\infty}^{\infty}|f(x)|^2dx=\int_{-\infty}^{\infty}|f(s)|^2ds \longrightarrow (1)$$

$$\int_{-\infty}^{\infty}|f(x)|^2dx=\int_{-1}^{1}[1-|x|]^2dx$$

$$=2\int_0^1(1-|x|)^2dx \quad (|x|=x \text{ in } (0,1))$$

$$2\left[\frac{(1-x)^3}{3(-1)}\right]_0^1=\frac{-2}{3}\left[(1-x)^3\right]_0^1$$

$$=\frac{-2}{3}[0-1]=\frac{2}{3}\longrightarrow (2)$$

$$|f(s)|^2=\frac{2}{\pi}\left[\frac{1-\cos s}{s^2}\right]$$

$$=\frac{2}{\pi}\left[\frac{2\sin^2(s/2)}{s^2}\right]^2$$

$$=\frac{8}{\pi}\frac{\sin^4(s/2)}{s^4}$$

$$\int_{-\infty}^{\infty}|F(s)|^2ds=\frac{8}{\pi}\int_{-\infty}^{\infty}\frac{\sin^4(s/2)}{s^4}ds$$

$$=\frac{16}{\pi}\int_0^{\infty}\frac{\sin^4(s/2)}{s^4}ds$$

$t=s/2 \quad 2t=s \quad 2dt=ds$

$s\to0 \quad t\to0 \quad s\to\infty \quad t\to\infty$

$$=\frac{16}{\pi}\int_0^{\infty}\frac{\sin^4t}{(2t)^4}2dt=\frac{2}{\pi}\int_0^{\infty}\frac{\sin^4t}{t^4}dt$$

$$(1)\Rightarrow\frac{2}{\pi}\int_0^{\infty}\frac{\sin^4t}{t^4}dt=\frac{2}{3}=\int_0^{\infty}\frac{\sin^4t}{t^4}dt \longrightarrow (3)$$

$$=\frac{\pi}{3}$$

Z - TRANSFORMS

Formula :-

$$Z[f(n)] = \sum_{n=0}^{\infty} f(n) z^{-n}$$

Z-transform of Elementary Properties

$f(n)$	$Z[f(n)] = F(z)$
$\frac{1}{n-1}$	$\frac{1}{z}\log\left(\frac{z}{z-1}\right), n>1$
$\frac{1}{n!}$	$e^{1/z}$
$\frac{1}{(n+1)!}$	$ze^{1/z} - z$
K	$K\left[\frac{z}{z-1}\right]$
$(-1)^n$	$\frac{z}{z+1}$
$\left(\frac{1}{a}\right)^n$	$\frac{z}{z-\frac{1}{a}} = \frac{az}{az-1}$
e^{an}	$\frac{z}{z-e^{a}}$
$\cos n\theta$	$\frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}, \lvert z\rvert>1$
$\sin\theta$	$\frac{z\sin\theta}{z^2-2z\cos\theta+1}, \lvert z\rvert>1$
$r^n\cos n\theta$	$\frac{z(z-r\cos\theta)}{z^2-2zr\cos\theta+r^2}$
$r^n\sin\theta$	$\frac{zr\sin\theta}{z^2-2zr\cos\theta+r^2}$
$a^n f(n)$	$F(z/a)$
$a^n f(t)$	$F(z/a)$
$a^n \frac{1}{n}$	$e^{a/z}$

PROPERTIES

If $Z[f(n)] = F(z)$

Then $Z[nf(n)] = -z\frac{d}{dz}[F(z)]$

(ii) $Z[a^{-n}f(n)] = F(az)$

(iii) $Z[f(n+2)] = z^2\left[F(z) - f(0) - \frac{f(1)}{z}\right]$

(iv) $Z[f(n-1)] = z^{-1}F(z)$

Problems :-

1) Solve $Z\left[\frac{1}{n+1}\right] = z\log\left[\frac{z}{n-1}\right]$

Sol: $Z[f(n)] = \sum_{n=0}^{\infty} f(n)z^{-n}$

Given $f(n) = \frac{1}{n+1}$

$\therefore Z\left[\frac{1}{n+1}\right] = \sum_{n=0}^{\infty}\frac{1}{n+1}z^{-n} \Rightarrow 1 + \frac{1}{2}z^{-1} + \frac{1}{3}z^{-2} \dots$

$= z\left[z^{-1} + \frac{z^{-2}}{2} + \frac{z^{-3}}{3} + \dots\right] \Rightarrow z\left[(z^{-1}) + \frac{(z^{-1})^2}{2} + \frac{(z^{-1})^3}{3} \dots\right]$

$= z[-\log(1-z^{-1})]$ if $|z^{-1}| < 1$

$= -z\log\left(1 - \frac{1}{z}\right)$

$\Rightarrow z\log\frac{z}{z-1}$ if $|z| > 1$

2) Find $Z[(n+1)(n+2)]$

$Z[(n+1)(n+2)] = Z[n^2+3n+2]$

$= Z[n^2] + 3Z[n] + 2Z[1]$

$Z[n^2] = Z[n.n]$

$= -z\frac{d}{dz} \Rightarrow -z\frac{d}{dz}\left[\frac{z}{(z-1)^2}\right]$

$= -z\left\{\frac{(z-1)^2.1 - z.2(z-1)}{(z-1)^4}\right\}$

$= -z(z-1)\left\{\frac{z-1-2z}{(z-1)^4}\right\}$

$\Rightarrow Z(n^2) = \frac{-z(-z-1)}{(z-1)^3} = \frac{z(z+1)}{(z-1)^3}$

$\therefore Z[(n+1)(n+2)] = \frac{z(z+1)}{(z-1)^3} + 3.\frac{z}{(z-1)^2} + 2.\frac{z}{z-1}$

3) Prove that $Z[\cos n\theta]$

Sol: $Z[a^n] = \frac{z}{z-a}$

Put $\boxed{a = e^{i\theta}}$ $\therefore a^n = e^{in\theta} = \cos n\theta + i\sin n\theta$

$\therefore Z[\cos n\theta + i\sin n\theta] = \frac{z}{z-e^{i\theta}}, |z|>1$

$\Rightarrow Z[\cos n\theta] + iZ[\sin n\theta] = \frac{z}{z-(\cos\theta + i\sin\theta)}$

$= \frac{z}{(z-\cos\theta) - i\sin\theta} \Rightarrow \frac{z[(z-\cos\theta) + i\sin\theta]}{(z-\cos\theta)^2 + \sin^2\theta}$

$= \frac{z(z-\cos\theta) + iz\sin\theta}{z^2-2z\cos\theta+1}$

$Z[\cos n\theta] + iZ[\sin n\theta] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} + \frac{iz\sin\theta}{z^2-2z\cos\theta+1} \rightarrow (1)$

$a = e^{-i\theta}$, $a^n = e^{-in\theta} = \cos n\theta - i\sin n\theta$

$Z[\cos n\theta] - iZ[\sin n\theta] = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1} - \frac{iz\sin\theta}{z^2-2z\cos\theta+1} \rightarrow (2)$

$(1)+(2) = \frac{z(z-\cos\theta)}{z^2-2z\cos\theta+1}$

$(1)-(2) = \frac{z\sin\theta}{z^2-2z\cos\theta+1}$ if $|z|>1$

Inverse z-transforms :-

Partial fraction Method :-

1. find $z^{-1}\left[\dfrac{z(z^2-z+2)}{(z+1)(z-1)^2}\right]$

Sol:- let $F(z) = \dfrac{z(z^2-z+2)}{(z+1)(z-1)^2}$

$$\frac{F(z)}{z} = \frac{z^2-z+2}{(z+1)(z-1)^2} = \frac{A}{z+1} + \frac{B}{z-1} + \frac{C}{(z-1)^2}$$

$$z^2-z+2 = A(z-1)^2 + B(z+1)(z-1) + C(z+1)$$

$A = 1$

$B = 0$

$C = 1$

$$\frac{F(z)}{z} = \frac{1}{z+1} + \frac{0}{z-1} + \frac{1}{(z-1)^2}$$

$$\frac{F(z)}{z} = \frac{1}{z+1} + \frac{1}{(z-1)^2}$$

$$F(z) = \frac{z}{z+1} + \frac{z}{(z-1)^2}$$

$z\{(f(n))\} = \dfrac{z}{z+1} + \dfrac{z}{(z-1)^2}$ $(\because F(z) = Z(f(n)))$

$$f(n) = z^{-1}\left[\frac{z}{z+1}\right] + z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$= z^{-1}\left[\frac{z}{z-(-1)}\right] + z^{-1}\left[\frac{z}{(z-1)^2}\right]$$

$$f(n) = (-1)^n + n.$$

Convolution theorem :-

If $z[f(n)] = F(z)$ and $z[g(n)] = G(z)$.

$$z^{-1}[F(z)G(z)] = f(n) * g(n) = \sum_{m=0}^{n} f(m)\, g(n-m).$$

1. find $z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right]$ using convolution theorem.

Sol:- $z^{-1}\left[\dfrac{z^2}{(z-a)(z-b)}\right] = z^{-1}\left[\dfrac{z}{z-a} \cdot \dfrac{z}{z-b}\right]$

$$= z^{-1}\left[\frac{z}{z-a}\right] * z^{-1}\left[\frac{z}{z-b}\right],$$

$$= a^n * b^n.$$

$$= \sum_{m=0}^{n} a^m . b^{n-m}$$

$$= \sum_{m=0}^{n} a^m . b^n . b^{-m}$$

$$= b^n \sum_{m=0}^{n} \left(\frac{a}{b}\right)^m$$

$$= b^n \left[1 + \frac{a}{b} + \left(\frac{a}{b}\right)^2 + \cdots + \left(\frac{a}{b}\right)^n\right]$$

$$= b^n \left[\frac{\left(\frac{a}{b}\right)^{n+1} - 1}{a/b - 1}\right] \qquad \because a\left[\frac{r^n-1}{r-1}\right]$$

$$= \frac{b^n\left[a^{n+1} - b^{n+1}\right]}{b^{n+1}\left(\frac{a-b}{b}\right)} = \frac{a^{n+1} - b^{n+1}}{a-b} \qquad n = 0, 1, 2, 3 \ldots$$

Exercise problem :-

1. find $z^{-1}\left[\dfrac{z^3}{(z-2)^2(z-3)}\right]$

Solution of difference equations :-

1. Solve $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$ with $y_0 = y_1 = 0$ using z-transform

Sol:- Given $y_{n+2} + 6y_{n+1} + 9y_n = 2^n$

$$z[y_{n+2}] + 6z[y_{n+1}] + 9z[y_n] = z[2^n]$$

$$z^2\left[f(z) - y_0 - \frac{y_1}{z}\right] + 6z[f(z) - y_0]$$

$$+ 9f(z) = \frac{z}{z-2}$$

Given $y_0 = y_1 = 0$

$$z^2[f(z) - 0 - 0] + 6z[f(z) - 0] + 9[f(z)]$$

$$= \frac{z}{z-2}$$

$$f(z) = \frac{z}{(z-2)^2(z^2+6z+9)}$$

$$f(z) = \frac{1}{(z-2)(z+3)^2}$$

let $\dfrac{1}{(z-2)(z+3)^2} = \dfrac{A}{z-2} + \dfrac{B}{z+3} + \dfrac{C}{(z+3)^2}$

$a = \frac{1}{25}$ $\quad$ $c = -\frac{1}{5}$

$$\frac{f(z)}{z} = \frac{1}{25(z-2)} - \frac{1}{25(z+3)} - \frac{1}{5(z+3)^2}$$

$$f(z) = \frac{1}{25} \cdot \frac{z}{z-2} - \frac{1}{25} \frac{z}{z+3} - \frac{1}{5} \cdot \frac{z}{(z+3)^2}$$

$$Z(y_n) = \frac{1}{25} \cdot \frac{z}{z-2} - \frac{1}{25} \frac{z}{z+3} - \frac{1}{5} \frac{z}{(z+3)^2}$$

$$y_n = \frac{1}{25} z^{-1}\left[\frac{z}{z-2}\right] - \frac{1}{25} z^{-1}\left[\frac{z}{z+3}\right] - \frac{1}{5} z^{-1}\left[\frac{z}{(z+3)^2}\right] = \frac{1}{25} \cdot 2^n - \frac{1}{25}(-3)^n - \frac{1}{5}\left[\frac{-1}{3}\right] n(-3)^n$$

$$\left[\frac{-3z}{(z-(-3))^2}\right].$$

Engineer to Excel

SIMATS
SCHOOL OF ENGINEERING

Approved by AICTE | IET-UK Accreditation

Saveetha Nagar, Thandalam, Chennai - 602 105, TamilNadu, India